

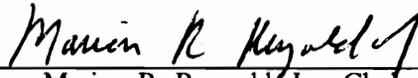
**Multivariate Control Charts for the Mean Vector and Variance-Covariance Matrix with Variable
Sampling Intervals**

by

Gyo-Young Cho

Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
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Statistics

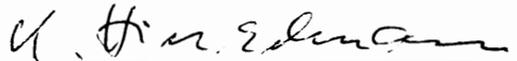
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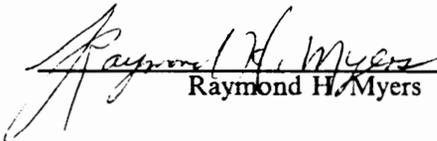
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(ABSTRACT)

When using control charts to monitor a process it is frequently necessary to simultaneously monitor more than one parameter of the process. Multivariate control charts for monitoring the mean vector, for monitoring variance-covariance matrix and for simultaneously monitoring the mean vector and the variance-covariance matrix of a process with a multivariate normal distribution are investigated. A variable sampling interval (VSI) feature is considered in these charts.

Two basic approaches for using past sample information in the development of multivariate control charts are considered. The first approach, which is called the combine-accumulate approach, reduces each multivariate observation to a univariate statistic and then accumulates over past samples. The second approach, which is called the accumulate-combine approach, accumulates past sample information for each parameter and then forms a univariate statistic from the multivariate accumulations.

Multivariate control charts are compared on the basis of their average time to signal (ATS) performance. The numerical results show that the multivariate control charts based on the accumulate-combine approach are more efficient than the corresponding multivariate control charts based on the combine-accumulate approach in terms of ATS. Also VSI charts are more efficient than corresponding FSI charts.

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Table of Contents

Introduction	1
Literature Review	4
Notation and Properties	11
Review of FSI and VSI Univariate Control Charts	16
4.1 Univariate Shewhart Control Charts	16
4.2 Univariate CUSUM Control Charts	20
4.3 Univariate EWMA Control Charts	28
Multivariate Control Charts for the Mean Vector	30
5.1 Introduction	30
5.2 Shewhart Control Charts for the Mean Vector	32
5.3 Multivariate CUSUM Charts for the Mean Vector	32
5.4 Multivariate EWMA Charts for the Mean Vector	34
5.5 Numerical Results for Charts for the Mean Vector	36

Multivariate Control Charts for the Variance-Covariance Matrix	49
6.1 Introduction	49
6.2 Shewhart Charts for the Covariance Matrix	52
6.3 CUSUM Charts for the Covariance Matrix	52
6.4 Multivariate EWMA Charts for the Covariance Matrix	53
6.4.1 Multivariate EWMA Charts Based on the Combine-accumulate Approach.	54
6.4.2 A Multivariate EWMA Chart Based on the Accumulate-combine Approach	55
6.5 Numerical Results	58
Multivariate Control Charts for the Mean Vector and Variance-Covariance Matrix	68
7.1 Introduction	68
7.2 Shewhart Charts for the Mean Vector and Covariance Matrix	70
7.3 CUSUM Charts for the Mean Vector and Covariance Matrix	70
7.4 Multivariate EWMA Charts for the Mean vector and Covariance Matrix	71
7.4.1 Multivariate EWMA Charts Based on the Combine-accumulate Approach	72
7.4.2 Multivariate EWMA Charts Based on the Accumulate-combine Approach	73
7.4.3 Separate Charts for Each of the Means and Variances	81
7.4.4 Multivariate EWMA Charts Based on both MEWMA and EWMA using (7.5)	83
7.4.5 MEWMA Charts	84
7.5 Diagnostic Procedures	85
7.6 Numerical Results for the Mean vector and Covariance Matrix	86
Guidelines for Design of VSI Charts, a Summary and Topics for Future Research	122
8.1 Guidelines for Design of VSI Charts	122
8.2 Summary of Results	123
8.3 Topics for Future Research	125
BIBLIOGRAPHY	126

Vita **131**

List of Tables

Table 5.1	Values of h and g for CUSUM Charts for Various Values of p and k when the ATS at $\underline{\mu} = \underline{\mu}_0$ is 200	38
Table 5.2	ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector ($p = 2$)	39
Table 5.3	ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector ($p = 3$)	39
Table 5.4	ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector ($p = 4$)	40
Table 5.5	ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector ($p = 5$)	40
Table 5.6	ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector ($p = 6$)	41
Table 5.7	ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector ($p = 7$)	41
Table 5.8	ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector ($p = 8$)	42
Table 5.9	ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector ($p = 9$)	42

Table 5.10 ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector
 ($p = 10$) 43

Table 5.11 ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector
 ($p = 20$) 43

Table 5.12 Steady State ATS Values for Matched FSI and VSI CUSUM Charts for the Mean
 Vector ($p = 2$) 44

Table 5.13 Steady State ATS Values for Matched FSI and VSI CUSUM Charts for the Mean
 Vector ($p = 3$) 44

Table 5.14 Steady State ATS Values for Matched FSI and VSI CUSUM Charts for the Mean
 Vector ($p = 4$) 45

Table 5.15 Steady State ATS Values for Matched FSI and VSI CUSUM Charts for the Mean
 Vector ($p = 5$) 45

Table 5.16 Steady State ATS Values for Matched FSI and VSI CUSUM Charts for the Mean
 Vector ($p = 8$) 46

Table 5.17 Steady State ATS Values for Matched FSI and VSI CUSUM Charts for the Mean
 Vector ($p = 10$) 46

Table 5.18 Steady State ATS Values for Matched FSI and VSI CUSUM Charts for the Mean
 Vector ($p = 20$) 47

Table 5.19 Values of h and g for MEWMA Charts for Various Values of p and λ when the
 ATS at $\underline{\mu} = \underline{\mu}_0$ is approximately 200..... 47

Table 5.20 ATS Values for FSI MEWMA Charts for the Mean Vector 48

Table 5.21 ATS Values for VSI MEWMA Charts for the Mean Vector 48

Table 6.1	Values of h and g for Various Values of λ and $p = 2-5$ when the ATS at $\Sigma = \Sigma_0$ is approximately 200	60
Table 6.2	ATS Values for Matched FSI and VSI EWMA Charts Using (6.4) for the Covariance Matrix ($p = 2, \rho = 0.5$)	61
Table 6.3	ATS Values for Matched FSI and VSI EWMA Charts Using (6.4) for the Covariance Matrix ($p = 2, \rho = 0.0$)	62
Table 6.4	ATS Values for Matched FSI and VSI EWMA Charts Using (6.4) for the Covariance Matrix ($p = 2, \rho = 0.8$)	62
Table 6.5	ATS Values for Matched FSI and VSI EWMA Charts Using (6.8) for the Covariance Matrix ($p = 2, \rho = 0.5$)	63
Table 6.6	ATS Values for Matched FSI and VSI EWMA Charts Using (6.8) for the Covariance Matrix ($p = 2, \rho = 0.0$)	64
Table 6.7	ATS Values for Matched FSI and VSI EWMA Charts Using (6.8) for the Covariance Matrix ($p = 2, \rho = 0.8$)	65
Table 6.8	ATS Values for Three Matched FSI and VSI EWMA Charts for the Covariance Matrix ($p = 5, \rho = 0.5$)	66
Table 7.1	Values h and g for Various Values of λ and $p = 2-5$ when the ATS at $\underline{\mu} = \underline{\mu}_0$ and $\Sigma = \Sigma_0$ is approximately 200	90
Table 7.2	ATS Values for Matched FSI and VSI MEWMA Charts ($p = 2$)	91
Table 7.3	ATS Values for Matched FSI and VSI MEWMA Charts ($p = 2, \rho = 0.5$)	92
Table 7.4	ATS Values for Matched FSI and VSI MEWMA Charts ($p = 2, \rho = 0.0$)	93

Table 7.5	ATS Values for Matched FSI and VSI MEWMA Charts ($p = 2, \rho = 0.8$)	94
Table 7.6	ATS Values for Matched FSI and VSI EWMA Using (7.1) Charts ($p = 2$)	95
Table 7.7	ATS Values for Matched FSI and VSI EWMA Using (7.1) Charts ($p=2, \rho=0.5$) . . .	96
Table 7.8	ATS Values for Matched FSI and VSI EWMA Using (7.1) Charts ($p=2, \rho=0.0$) . . .	97
Table 7.9	ATS Values for Matched FSI and VSI EWMA Using (7.1) Charts ($p=2, \rho=0.8$) . . .	98
Table 7.10	ATS Values for Matched FSI and VSI EWMA Charts Using (7.5) ($p=2, \rho=0.0$) . . .	99
Table 7.11	ATS Values for Matched FSI and VSI EWMA Charts Using (7.5) ($p=2, \rho=0.5$) . .	102
Table 7.12	ATS Values for Matched FSI and VSI EWMA Charts Using (7.5) ($p=2, \rho=0.8$) . .	105
Table 7.13	ATS Values for the FSI and VSI Separate EWMA Charts ($p = 2, \rho = 0.0$)	108
Table 7.14	ATS Values for the FSI and VSI Separate EWMA Charts ($p = 2, \rho = 0.5$)	111
Table 7.15	ATS Values for Matched FSI and VSI Separate EWMA Charts ($p=2, \rho=0.8$) . . .	114
Table 7.16	ATS Values for FSI and VSI Charts Based on Both MEWMA and EWMA Using (7.5) ($p = 2, \lambda = 0.05$ and $\rho = 0.5$)	117
Table 7.17	ATS Values for FSI and VSI EWMA Charts for the Mean Vector and Variance- Covariance Matrix ($p = 5, \rho = 0.5$)	118

Chapter 1

Introduction

Control charts are used to monitor quality variables from a process to detect changes in the parameters of the distribution of these variables. A control chart is maintained by taking samples from a process and plotting in time order on the control chart the relevant statistic computed from the samples. Control limits are chosen so that if the process is in control, nearly all of the control statistics will fall within the control limits. As long as the control statistic plots within the control limits, the process is assumed to be in control, and no action is necessary. However, when a statistic plots outside of the control limits, the chart then signals and rectifying action is taken to detect the cause of changes and bring the process back into control.

The standard Shewhart control chart uses the information about the process contained in the current sample and, unless runs rules are used, does not formally use the information in past samples. This chart is easy to construct and good at detecting large shifts from the target value. The cumulative sum (CUSUM) control chart has been used as an alternative to the Shewhart control chart. It directly incorporates all of the information in the sequence of sample values by plotting the cumulative sums of the deviations of the sample values from the target value. The CUSUM chart is efficient in detecting small and moderate shifts from the target value. The exponentially

weighted moving average (EWMA) chart uses a weighted average of observations where the most recent observation is assigned the greatest weight, and all previous observations have weights decreasing in exponential progression from the most recent back to the first. Like the Shewhart chart, the EWMA control chart is easy to use and interpret.

The standard practice in maintaining the control chart is to take samples from the process at fixed-length sampling intervals, say, every hour. However, if a control statistic is close to but not actually outside the control limits, one might want to take the next sample in less than the regular time unit to see if a process change has occurred. Or, if the control statistic is very close to the target value, a time longer than the regular time unit may be used to decrease the amount of samples taken unnecessarily. Whenever the control statistic falls outside the control limits, the chart signals. This logical extension of fixed sampling interval (FSI) control charts is called variable sampling interval (VSI) control charts.

There are many situations in which the simultaneous control of two or more related quality characteristics is necessary. Quality control problems in which several related variables are of interest are sometimes called multivariate quality control problems. The monitoring of p quality characteristics can be done by using p quality control charts or by using a single control chart which uses an appropriate univariate statistic for controlling multivariate data.

Most work on multivariate control charts has been concerned with monitoring the multivariate mean vector. The objective of this dissertation is to investigate multivariate control charts for monitoring the mean vector, for monitoring the variance-covariance matrix and for simultaneously monitoring the mean vector and covariance matrix. It will be assumed that the process quality variables have a multivariate normal distribution. The VSI feature will be considered in each case. Multivariate control charts using the VSI idea have been studied by Hui (1980) and Chengalur-Smith et al. (1990) but this work was only for Shewhart control charts. We will be looking at multivariate Shewhart, CUSUM and EWMA charts.

Chapter 2 gives a review of the literature that is related to the dissertation. Chapter 3 gives the notation and properties of the FSI and VSI procedures. Chapter 4 gives the properties of the FSI and VSI univariate Shewhart, CUSUM and EWMA charts. The Markov chain approach and the

integral equation approach for evaluating properties of control charts is also discussed in Chapter 4.

In Chapter 5, we discuss ways to use past sample information in multivariate control charts and present and evaluate the performance of FSI and VSI multivariate Shewhart, CUSUM and EWMA control charts for monitoring the mean vector. In Chapter 6, we develop three different univariate statistics for monitoring the covariance matrix and discuss the FSI and VSI multivariate Shewhart, CUSUM and EWMA charts for each proposed statistic. In Chapter 7, we study three different statistics for simultaneously monitoring the mean vector and covariance matrix and discuss the FSI and VSI multivariate Shewhart, CUSUM and EWMA charts for each proposed statistic. Chapter 8 gives guidelines for the design of VSI EWMA charts, a summary and topics for future research

Chapter 2

Literature Review

Shewhart (1931) first proposed control charts. If the objective is to control the process mean μ , the Shewhart \bar{X} chart uses the sample mean as the control statistic and usually has control limits set at $\mu_0 \pm 3$ standard errors, where μ_0 is the target value of the mean. This chart is easy to implement and is good at detecting large shifts from the target value. If the objective is to control the process variability or dispersion σ^2 , the S^2 -chart or R-chart can be used. The S^2 -chart uses the sample variance as the control statistic. The R-chart uses the sample range as the control statistic. Page (1955) investigated the properties of warning lines used within the control limits of a Shewhart chart in order to incorporate the information from the past few samples. He suggested that if r out of the last N points fell between the warning and action lines, an action should be taken. He showed that the Shewhart control chart with warning lines is more efficient in detecting small shifts than the standard Shewhart chart since the control chart with warning lines uses information from the last few samples instead of using only the last sample. Weiler (1953) and Moore (1958) suggested that the chart should also signal if r consecutive points fall above (below) the target value. Page (1962) showed that the control charts with warning lines are generally better than Moore's

runs rules. Champ and Woodall (1987) developed a Markov chain model for the case in which several runs rules are used simultaneously.

A more recent and often more efficient procedure is the cumulative sum (CUSUM) control chart. The CUSUM chart directly incorporates all of the information in the sequence of sample values by using a control statistic which is a cumulative sum of statistics computed from each sample. A control chart procedure based on cumulative sums was originally introduced by Page (1954). Page (1954,1961) and Barnard (1959) developed the CUSUM procedure as a sequential likelihood ratio test for testing the hypothesis that the process mean is equal to the target value against the alternative that it is not.

In general, the exact determination of the run length distribution of the CUSUM chart is difficult. Ewan and Kemp (1960) used integral equations for the approximations of the run length distribution and recommended approximating the run length distribution by a geometric distribution when the average run length (ARL) is large. Van Dobben de Bruyn (1968) recommended using Monte Carlo methods to estimate the run length distribution when the ARL is not large. A nomogram that is useful in the design of CUSUM control charts was proposed by Goel and Wu (1971). Johnson and Bagshaw (1974), Bagshaw and Johnson (1975), and Reynolds (1975) gave asymptotic methods based on approximation of the CUSUM chart by a Wiener process.

Brook and Evans (1972) used a Markov chain approach to get the run length distribution for a discrete one-sided CUSUM chart and to approximate the run length distribution for a continuous one sided CUSUM chart by discretizing the continuous state space so that the CUSUM is restricted to a finite set of values. A method for approximating the run length distribution of one sided CUSUM procedures for continuous random variables was given by Woodall (1983). This method is more accurate in the continuous case than that of Brook and Evans (1972) since it takes advantage of numerical quadrature techniques. Bounds for the run length distribution of both the one-sided and two-sided CUSUM schemes were derived by Waldmann (1986). Yashchin (1985) obtained the expression for the Laplace transform of the run length distribution and developed a method of analysis for general two-sided CUSUM schemes. Woodall (1984) used a Markov chain approach to determine the ARL for a two-sided CUSUM chart.

Lucas and Crosier (1982a) showed that a fast initial response (FIR) feature is useful for CUSUM control schemes because processes are more likely to be away from the target value when a control scheme is initiated due to start-up problems or because of ineffective control action after the previous out-of-control signal. Lucas (1982) introduced a combined Shewhart-CUSUM chart that provides protection against both large and small shifts in the process. Lucas and Crosier (1982b) developed a more robust CUSUM procedure that provides extra protection against outliers. A new two-sided CUSUM quality control scheme using two one-sided CUSUM schemes was proposed by Crosier (1986). Champ and Woodall (1987) showed that the runs rules cause the Shewhart \bar{X} chart to be more sensitive to small shifts in the mean, but not as sensitive as the CUSUM chart. Properties of control charts are usually developed under the assumption that the observations from the process are independent, but closely spaced observations from many types of processes will exhibit serial correlation. Several studies have shown that serial correlation has a major influence on the ARL of the CUSUM procedure (Goldsmith and Whitfield (1961), Johnson and Bagshaw (1974), Bagshaw and Johnson (1975), and Kartha and Abraham (1979)).

Roberts (1959) first introduced the exponentially weighed moving average (EWMA) control scheme in which the most recent point is assigned a weight r , $0 < r \leq 1$, and every other point is assigned a fraction $(1-r)$ of the weight of its immediate successor. Like the Shewhart control scheme, an EWMA control scheme is easy to implement and interpret. Roberts (1959) showed by using simulation that the EWMA is useful for detecting small shifts in the mean of a process. The EWMA was shown to have optimal properties in some forecasting and control applications by Box, Jenkins, and MacGregor (1974), and Muth (1960). By writing the current EWMA as the previous EWMA plus a fraction of difference between the current observation and the previous EWMA, Hunter (1986) showed that the EWMA can be viewed as a one-step-ahead forecast for the process. Robinson and Ho (1987) numerically evaluated the ARL's of EWMA control schemes using an Edgeworth series expansion. Crowder (1987) evaluated the properties of EWMA's by formulating and solving a system of integral equations. Lucas and Saccucci (1990) used a Markov chain approach to evaluate the run length properties of the EWMA scheme used to monitor the mean of a normally distributed process. The results showed that the properties of EWMA's are

very close to those of CUSUM schemes. They also considered several enhancements to EWMA control schemes such as a FIR feature, a combined Shewhart/EWMA, and a robust EWMA. These enhancements work as well for EWMA control schemes as they do for CUSUM control schemes.

There are many situations in which the simultaneous control of two or more related quality characteristics is necessary. The original work in multivariate quality control was introduced by Hotelling (1947). Jackson (1956) and Ghare and Torgerson (1968) presented control ellipses for the case where means of two variables are being controlled simultaneously. Jackson and Morris (1957) extended Hotelling's procedure for use with principal components when the dispersion matrix is near singular. Jackson (1959) extended the method of principal components for the bivariate case to the multivariate case. A detailed description of the method of principal components and its relationship to quality control was given by Jackson (1981a, 1981b, 1981c). Jackson and Bradley (1961a, 1961b) showed that a sequential test can be constructed to achieve specified Type I and Type II error probabilities and that the sequential test terminates with probability one. When the covariance is known, they developed a sequential χ^2 - test; when the covariance matrix is estimated from the sample, they developed a sequential T^2 - test. They also developed a sequential procedure for monitoring process dispersion parameters. Jackson and Mudholkar (1979) proposed alternative procedures for the residual analysis. Hui (1980) examined the monitoring and diagnostic procedures in multicharacteristic quality control, maintaining the efficient use of resources in a production process, and a VSI inspection policy. Reynolds and Ghosh (1981) suggested several procedures of the Shewhart type for monitoring the mean and variance simultaneously. They also discussed extensions to the multivariate case. The problem of monitoring a process that can be defective on more than one quality attribute was examined by Marcucci (1982). Alt (1984) discussed multivariate control charts for the mean vector and variance-covariance matrix. Jackson (1985) discussed multivariate control charts and some of the techniques currently available. A Shewhart-type scheme for simultaneous control of the standard deviations of two variables with a joint bivariate normal distribution was constructed by Tuprah and Woodall (1986).

Woodall and Ncube (1985) suggested extending the univariate CUSUM procedure to the case of multivariate means. Woodall and Ncube (1985) obtained the run length distribution for independent and dependent quality characteristics. On the basis of sequential probability ratio tests, a method of constructing CUSUM control charts for multivariate data was derived by Alwan (1986). Healy (1987) showed that CUSUM formed as a sum of the T^2 statistics is the appropriate sequential theory test for an inflation of the covariance matrix. Crosier (1988) proposed two heuristically derived multivariate CUSUM procedures that have ARL's that depend on the noncentrality parameter. A multivariate extension of the EWMA control chart was presented by Lowry et al. (1989). By using simulation, Lowry et al. (1989) showed that the ARL performance of the multivariate EWMA (MEWMA) chart is better than that of other existing multivariate control charts in detecting a shift in the mean vector of a multivariate normal distribution. Two new multivariate CUSUM charts for controlling a multivariate normal process were presented by Pignatiello and Runger (1990).

In a number of contexts efficient quality control procedures have been developed by allowing the sampling interval or time between samples to vary depending on what is being observed from the data. Variable sampling rates have been used in the context of acceptance sampling. In acceptance sampling, a sample from each lot is inspected, and on the basis of the data provided by the sample it is decided whether the lot as a whole shall be accepted or rejected. There are several ways that variable sampling rates are used in lot acceptance sampling, for example when the sample is sequential, when rejected lots are 100 % inspected, and when skip-lot plans are used.

Continuous sampling plans were developed for use when it is not convenient to group items into lots for one reason or another. A continuous sampling plan, which was proposed by Dodge (1943) and later labeled CSP-1, inspects all or a fraction of items in the sequence of their production depending on the observed quality. Wald and Wolfowitz (1945) discussed a similar sampling inspection plan that was more effective in quickly detecting a long sequence of low quality items. Lieberman and Solomon (1955) introduced a multi-level sampling plan which was designed to correct for the abrupt change between partial inspection and 100 % inspection and switching could only occur between adjacent levels. Three generalizations of the Lieberman-Solomon multi-level

plan were given by Derman, Littauer, and Solomon (1957). These plans made it possible to switch back more than one level at a time, producing plans with an increased sensitivity to large process shifts.

Acceptance sampling plans and continuous sampling plans are designed for accepting or rejecting product that has either been produced or is at an intermediate stage of production. However, the control charts are designed to be used to continuously monitor the production process to quickly detect any deterioration in quality so that the problem that caused the deterioration in quality can be found and eliminated.

A sampling procedure which uses variable sampling intervals for monitoring the amount of dissolved oxygen in streams was introduced by Arnold (1970). He used the Markovian structure of the scheme and developed the expected sample size required for various sampling plans. Crigler (1973) derived an economically optimal sampling policy using Arnold's model. The results were extended by Crigler and Arnold (1979, 1986). Smeach and Jernigan (1977) extended Arnold's work and derived equations for the variance of the sample size. They also developed simple approximate formulae for the expected sample size and its variance that could greatly enhance the implementation of this sampling procedure. Hui (1980) and Hui and Jensen (1980) extended this problem to the multivariate case. They also extended the model of Arnold (1970) for the specific purpose of monitoring a production process, and extracted more information about the process from this extension. In particular, their model allows for adjusting the process when it is out of control, but they did not explicitly consider the properties of the procedure when the process is out of control. They used Markov processes to obtain the expected values and the variances of the number of samples and the numbers of various types of adjustments to the process during any given time period.

The basic idea of using the variable sampling interval (VSI) is that the sampling interval should be short if there is some indication of a change in the process and long if there is no indication of a change. Reynolds and Arnold (1989) developed general expressions for properties of a VSI Shewhart chart such as the average time to signal and the average number of samples to signal when the process is in control and when it is out of control. Other properties of the VSI Shewhart charts had been studied by Hui (1980). Reynolds and Arnold (1989) and Reynolds (1989)

showed that the optimal VSI chart uses only the shortest possible interval and the longest possible interval from a range of possible sample intervals. A chart is said to be optimal if it minimizes the time required to detect a shift in a process parameter subject to a given false alarm rate and a given average sampling rate. This optimality result applies to Shewhart charts and to any other chart which can be modeled as a Markov chain. They showed that the VSI chart will detect changes in a process much faster than the corresponding FSI chart. Reynolds et al. (1988) considered details of the application of the VSI feature to the Shewhart \bar{X} chart for monitoring the process mean and showed that it was more efficient than the corresponding FSI \bar{X} chart. Amin (1987) and Reynolds et al. (1990) considered VSI CUSUM charts and Cui and Reynolds (1988) considered VSI Shewhart \bar{X} charts with runs rules. Saccucci, Amin and Lucas (1990) investigated the properties of the univariate EWMA chart and showed that VSI EWMA charts have similar properties to VSI CUSUM charts. Chengalur-Smith et al. (1989) considered VSI Shewhart control charts for simultaneously monitoring the mean and variance from a univariate normal distribution. Chengalur-Smith et al. (1990) considered the use of VSI Shewhart charts for the simultaneous monitoring of several means from a multivariate normal distribution.

Chapter 3

Notation and Properties

Suppose that the process of interest has p quality characteristics represented by the random vector $\underline{X} = (X_1, \dots, X_p)$, $p = 2, 3, \dots$ and the family of the distributions for \underline{X} is indexed by a set of parameters θ , for example $\theta = (\underline{\mu}, \Sigma)$ where $\underline{\mu}$ is the mean vector and Σ is the variance-covariance matrix. Let θ_0 be the target value for θ and assume that any deterioration in quality is reflected by changes in θ . The process operates over time and thus θ is actually a function of time, but the above notation is used for simplicity.

A control chart is used to detect any departure by θ from the target value θ_0 . Let the sample of n observations taken at the i^{th} sampling point be represented by $\underline{X}_i = (\underline{X}_{i1}, \dots, \underline{X}_{in})$, where $\underline{X}_{ij} = (X_{ij1}, \dots, X_{ijp})$ is the j^{th} observation vector among n observation vectors taken at the i^{th} sampling point. Thus, \underline{X}_i is an $1 \times np$ vector. It will be assumed that the observation vectors within and between samples are independent. Even though most control charts make this assumption, one should note that this is perhaps not very realistic, since production processes are inherently time-dependent.

The operation of a control chart in detecting process changes can be described simply in terms of a control statistic and two disjoint regions, the signal region and the in-control region. If the

control statistic computed from a sample falls in the signal region, the process is deemed to be out of control. Otherwise, the process is allowed to continue to the next sample. For a FSI-control chart, the random number of samples required to produce a signal is called the run length and the expected value of run length is called the average run length (ARL). For an FSI control chart, the expected time to signal can be obtained by multiplying the ARL by the fixed sampling interval, so the ARL can be thought of as the expected time to signal where a time unit is taken as the fixed sampling interval. A good control chart should quickly detect changes in the process while producing few false alarms. Thus, when the process is in control the ARL should be large so that the frequency of false alarm is low, and when the process changes the ARL should be small so that the change will be detected quickly.

It is known that using two sampling intervals is optimal in VSI control charts. Formulas for properties of VSI control charts can be given for more than two sampling intervals, but for simplicity, two sampling intervals will be considered here in the formulas and numerical results. Assume that the VSI control chart uses two sampling intervals d_1 and d_2 such that $l_1 \leq d_1 < d_2 \leq l_2$, where l_1 is the shortest time required to take a sample and l_2 is the longest time that is reasonable to allow the process to run without sampling. Let $d(w)$ be the sampling interval to be used when the control statistic $W = w$ is observed, and let $s(w)$ be a signal function such that

$$s(w) = \begin{cases} 1 & \text{if the chart signals} \\ 0 & \text{otherwise} \end{cases}$$

For VSI control charts the sampling interval is a random variable and thus the time required for the chart to signal is not a constant multiple of the run length. Thus separate measures for the time to signal and number of samples are needed to evaluate the performance of a VSI control chart. Following the same notation as introduced by Reynolds et al. (1988) the time to signal (TS) is defined to be the length of time required for the chart to signal and the average time to signal

(ATS) is defined to be the expected value of the time to signal. Also the number of samples to signal (NSS) is defined to be the number of samples required for the chart to signal and the average number of samples to signal (ANSS) is defined to be the expected value of the NSS. Let

N = number of samples to signal

T = time to signal

R_i = length of sampling interval used before the i^{th} sample is taken

Thus, the time to signal can be expressed as

$$T = \sum_{i=1}^N R_i \quad (3.1)$$

The ATS is $E(T)$ and using (3.1) and Wald's identity, the ATS can be written in terms of $E(N)$ and $E(R_i)$. For Shewhart charts it is relatively easy to evaluate $E(N)$ and $E(R_i)$

Reynolds and Arnold (1989) showed that using two sampling intervals spaced as far apart as possible is optimal for VSI charts. Following Reynolds et al.(1990), when two sampling intervals d_1 and d_2 are used, let

ρ_1 = the long run proportion of sampling intervals which are d_1

$1 - \rho_1$ = the long run proportion of sampling intervals which are d_2

Then, the ATS can be written as

$$ATS = (ANSS)(d)$$

where

$$d = d_1\rho_1 + d_2(1 - \rho_1) \quad (3.2)$$

For CUSUM and EWMA charts methods for finding the ANSS and ρ_1 can be developed. For a VSI chart d can be interpreted as the average sampling interval and for FSI charts d will be used as the fixed sampling interval.

The ATS when the process goes out of control is computed under the simplifying assumption that the process goes out of control at time zero. A more realistic model is that the process may begin in control and then shift at a random time to an out-of-control state, where the time of shift may fall between samples. In this case, the detection time that is of interest is the time from the process shift to the point at which the chart signals. The expected value of the time from the process shift until a signal is called a steady state ATS. To develop a model let

T^* = steady state time to signal

Y = time from the process shift until the next sample (3.3)

Z = time from the next sample after the process shift until a signal

N = number of samples after the shift until a signal

It is clear that

$$T^* = Y + Z \tag{3.4}$$

Methods for finding $E(Y)$ and $E(Z)$ will be given in later Chapters.

In order to evaluate the performance of the VSI chart it seems reasonable to compare it to the corresponding FSI chart. As long as the VSI chart and the FSI chart have the same parameters which determine when the charts signal, then both procedures will have the same ANSS. If two regions I_1 and I_2 and the corresponding sampling intervals d_1 and d_2 are selected so that the two charts have the same ATS when $\theta = \theta_0$ then both charts will have the same average sampling rate when $\theta = \theta_0$. If the expected sampling interval of the VSI chart is equal to the sampling interval of the FSI chart when $\theta = \theta_0$, then both charts will have the same ATS at $\theta = \theta_0$. When the VSI and FSI charts are matched in this way, the performance of the two charts can be evaluated by com-

puting the ATS values at various values of $\theta_1 \neq \theta_0$ to determine which chart detects changes in θ in the process more quickly.

Chapter 4

Review of FSI and VSI Univariate Control Charts

4.1 Univariate Shewhart Control Charts

4.1.1 FSI Shewhart Charts

The Shewhart control chart is one of the most widely used control charts for monitoring the parameters of a process. Consider the case where the process of interest has only one quality characteristic X whose distribution is represented by the density $f(x | \theta)$, where θ is a parameter. Let $\underline{X}_i = (X_{i1}, \dots, X_{in})$ be the random sample of n observations taken at sampling point i . A standard Shewhart control chart for θ has a target value θ_0 and control limits which are usually set at $\theta_0 \pm \gamma$ standard errors of the control statistic, where γ is frequently taken to be 3. If the control statistic computed at the i^{th} sample falls the outside of the control limits, the chart then signals. For successive random samples of size n , the control chart can be viewed as repeated tests of significance of the form $H_0: \theta = \theta_0$ vs. $H_1: \theta \neq \theta_0$. The standard Shewhart chart, although simple to construct and understand, uses only the information in the current sample and is thus relatively inefficient in detecting small or moderate changes in the parameter.

The ATS is used as the major criterion for evaluating and comparing control charts. When the ATS is reasonably large, the distribution of the time to signal will be roughly exponential and thus other properties of the distribution such as the standard deviation can be obtained approximately from the ATS. To get the properties of the FSI Shewhart control charts, let

$$q = \text{the probability that a control statistic falls outside the control limits} \quad (4.1)$$

Since N has a geometric distribution with parameter q , when the process does not change the ANSS is

$$E(N) = \frac{1}{q}$$

Thus, the ANSS is the reciprocal of signal probability of the procedure and is independent of the sampling interval. The variance of N is

$$\text{Var}(N) = \frac{1-q}{q^2}$$

For simplicity it is assumed that the process starts at time zero and that the fixed sampling interval is d . The ATS can be expressed as

$$E(T) = dE(N) = \frac{d}{q}$$

and the variance of T is

$$\text{Var}(T) = d^2 \text{Var}(N) = d^2 \frac{(1-q)}{q^2}$$

4.1.2 VSI Shewhart Charts

In the standard Shewhart chart, the length of the time interval between samples is fixed, but in the VSI Shewhart chart, the interval between samples \underline{X}_i and \underline{X}_{i+1} will depend on the value of \underline{X}_i . Thus, when the current sample $\underline{X}_i = \mathbf{x}$ is observed, a signal function and a sampling interval function can be expressed as $s(\mathbf{x})$ and $d(\mathbf{x})$ respectively. Let the in-control region $\{\mathbf{x}; s(\mathbf{x}) = 0\}$ be partitioned into η regions I_1, \dots, I_η , where I_i is the region in which the sampling interval d_i is used. Thus the sampling interval used between \underline{X}_i and \underline{X}_{i+1} is $d(\underline{X}_i)$.

As in the FSI Shewhart chart, the ANSS is

$$E(N) = \frac{1}{q},$$

where q is defined as in (4.1). The distribution of R_i is the conditional distribution of $d(\mathbf{x})$ given that the chart does not signal at the previous sample. It is assumed that the first interval is a fixed constant, say d_0 , which could be d_1 , d_2 , or d (or any other value). Using Wald's identity and equation (3.1), the ATS can be expressed as

$$E(T) = d_0 + E(N - 1) E(R_i) = d_0 + (ANSS - 1) E(R_i),$$

assuming that the process parameters remain constant. Reynolds and Arnold (1989) showed that the expected value of R_i is

$$E(R_i) = \sum_{j=1}^{\eta} \frac{d_j p_j}{(1 - q)}, \quad (4.2)$$

where $p_j = P(d(\underline{X}_i) = d_j) = P(\underline{X}_i \in I_j)$ and $\sum_{i=1}^{\eta} p_i = 1 - q < 1$. Thus the ATS is

$$E(T) = d_0 + \sum_{j=1}^{\eta} \frac{d_j p_j}{q} \quad (4.3)$$

Note that q and p_j are functions of θ . When $\theta = \theta_0$, the ATS is the expected time to the first false alarm, but when $\theta \neq \theta_0$, the ATS is the expected time to detect this deviation from target.

Reynolds and Arnold (1989) showed that using two sampling intervals spaced as far apart as possible is optimal for the VSI Shewhart chart. If two sampling intervals are used and $E(R_i) = 1$ when $\theta = \theta_0$, fixing the in-control ATS and specifying the two sampling intervals determines the probabilities of sampling intervals and then I_1 and I_2 . Alternatively specifying I_1 and I_2 determines the probabilities of the two sampling intervals and then d_1 and d_2 .

4.1.3 Steady State Time to Signal

If the ATS is computed for some $\theta \neq \theta_0$ then equation (4.3) assumes that $\theta = \theta_1$ from time zero. But in practice the process may start out with $\theta = \theta_0$ and then shift from θ_0 to θ_1 at some random time in the future, where the time of shift may fall between samples. An expression for the steady state time to signal T^* is given by (3.4) and the distribution of Z in (3.3) is the same as $\sum_{i=1}^{N-1} R_i$ where the distribution of each R_i is the conditional distribution of $d(X_i)$ given no signal when the process has shifted. Hence

$$E(T^*) = E(Y) + E(Z) = E(Y) + E(N - 1) E(R_i)$$

The distribution of N is still geometric with parameter q and an expression for $E(R_i)$ is given by (4.2). In order to determine $E(Y)$ it is assumed that when the shift falls in an interval between samples, the position in the interval is uniformly distributed over the interval, and the probability of falling in an interval of particular length depends on this length and on the frequency with which intervals of this length occur when the process is in control. Using such a model, Reynolds and Arnold (1988) showed that the steady state ATS is

$$E(T^*) = \sum_{j=1}^{\eta} d_j^2 \frac{P_{0j}}{2 \sum_{j=1}^{\eta} d_j P_{0j}} + \frac{1}{q_1} \sum_{j=1}^{\eta} d_j P_{1j}$$

where $q_1 = P[s(\underline{X}_j) = 1 | \theta_1]$, $P_{0j} = P[d(\underline{X}_j) = d_j, s(\underline{X}_j) = 0 | \theta_0]$ and $P_{1j} = P[d(\underline{X}_j) = d_j, s(\underline{X}_j) = 0 | \theta_1]$. For a FSI Shewhart chart, the steady state ATS reduces to ANSS - $\frac{d}{2}$.

4.2 Univariate CUSUM Control Charts

4.2.1 FSI CUSUM Charts

The standard Shewhart control chart uses only the information in the current sample, but the CUSUM chart is an attempt to make use of all information. It is well known that the CUSUM chart is much more efficient than the standard Shewhart chart in detecting small and moderate shifts in the process. A standard CUSUM chart for θ is maintained by taking samples from the process at fixed time intervals and plotting a control statistic based on a cumulative sum of scores computed for each sample.

The CUSUM chart for detecting a shift from the target value θ_0 to a specified alternative θ_1 can be considered as a sequence of independent tests where each test is a sequential probability ratio test (SPRT) for testing $H_0: \theta = \theta_0$ vs. $H_1: \theta = \theta_1$, and each test has a lower boundary at 0 and upper boundary at, say b . If the test statistic crosses below 0 (accept H_0) a new test is started with the next sample. Tests are applied sequentially until a test crosses b (reject H_0). This sequence of SPRT's is equivalent to using the CUSUM statistic

$$Y'_j = \max\{Y'_{j-1} + z_j, 0\}$$

where

$$z_j = \ln \frac{f(X_j | \theta_1)}{f(X_j | \theta_0)}$$

The chart signals whenever $Y_j > b$.

As an example consider the problem of controlling the normal mean μ when σ_0^2 is known. For detecting positive shifts in μ the CUSUM statistic that is usually used at the j^{th} sample can be written as

$$\hat{Y}_j = \max\{\hat{Y}_{j-1} + (Z_j - k), 0\}$$

where $Z_j = \frac{\sqrt{n}(\bar{X}_j - \mu_0)}{\sigma_0}$. For adding the VSI feature to the CUSUM chart, Reynolds et al. (1990) proposed the modified CUSUM statistic.

$$Y_j = \max\{Y_{j-1}, 0\} + (Z_j - k) \quad (4.4)$$

where $Y_0 = w$ ($w \geq 0$). The CUSUM control chart signals whenever $Y_j \geq h$ (which is equivalent to $\hat{Y}_j \geq h$) where h is called the decision interval. The difference between \hat{Y}_j and Y_j is that \hat{Y}_j is never negative while Y_j records negative values. Except for recording of negative CUSUM values, \hat{Y}_j and Y_j are equivalent. These negative values may be needed to specify the sampling interval. The parameter k which is called the reference value is determined by the shift in the mean which the CUSUM is designed to detect. The value of k is usually taken as $k = \frac{\sqrt{n}(\mu_1 - \mu_0)}{2\sigma}$, where μ_1 represents the value of μ that should be detected quickly. The parameter h is chosen to give a specified ANSS and ATS to signal when $\mu = \mu_0$.

4.2.2 VSI CUSUM Charts

For VSI control charts, Amin (1987), Reynolds et al. (1988), Reynolds and Arnold (1989), and Reynolds (1988a) showed that the use of two sampling intervals spaced as apart as possible is optimal. Intuitively, a short sampling interval d_1 should be used when the CUSUM statistic Y_j is close to the signal region; a long sampling interval d_2 should be used when the CUSUM statistic Y_j is close to the target value. For detecting positive shifts in θ , suppose that

$$\begin{aligned} d_1 \text{ is used when } Y_j \in (g, h] \\ d_2 \text{ is used when } Y_j \in (-\infty, g] \end{aligned} \tag{4.5}$$

where $-\infty < g \leq h$ and Y_j is given by (4.4). Thus g is the boundary between the regions specifying d_1 and d_2 .

4.2.3. The Markov Chain Approach

One method for numerically evaluating the properties of the CUSUM chart is based on a Markov chain approximation. Brook and Evans (1972) developed a Markov chain approach for a one-sided FSI CUSUM chart. The Markov chain approach can be utilized to calculate the exact ANSS for a CUSUM chart with a discrete state space and yields good approximations when the state space is continuous. In the continuous case, the continuous state space of the CUSUM statistic is partitioned into a finite number of discrete class intervals and the probability distribution of the CUSUM is discretized.

A modification of the Brook and Evans model (1972) which is appropriate for the VSI CUSUM chart will be considered. Suppose that the in-control region $C = (-\infty, h)$ for the CUSUM statistic Y_j is partitioned into r subintervals E_1, \dots, E_r where each subinterval corresponds to a state of the Markov chain, one absorbing state A corresponding to the region $[h, \infty)$. Y_j is discretized into a statistic such that $Y_j \in E_i$ corresponds to the discrete version of Y_j equal to the

midpoint of E_i . We will be using Y_j for both continuous and discrete versions of the control statistic. Let

b_i = the sampling interval used when $Y_j \in E_i$

$\underline{b} = (b_1, \dots, b_r)'$

$p_{ij}(k) = P(Y_{k+1} = j | Y_k = i)$ for $ij = 1, 2, \dots, r+1$ and $k = 0, 1, 2, \dots$

Then the transition matrix $P = [p_{ij}]$ has the following forms:

$$P = \begin{bmatrix} Q & (I - Q)\underline{1} \\ \underline{Q}' & 1 \end{bmatrix}$$

where Q is the $r \times r$ transition matrix corresponding to the transient states, \underline{Q} is an $r \times 1$ vector of 0's and $\underline{1}$ is an $r \times 1$ vector of 1's. Define the fundamental matrix M as

$$M = (I - Q)^{-1} = [m_{ij}],$$

where m_{ij} is the expected number of times that the process is in transient state j before being absorbed, given that the process starts in state i . For each possible value $j \in \{1, 2, \dots, r\}$ for Y_0 , let

N_j = number of samples to signal when $Y_0 = j$

T_j = time to signal given that $Y_0 = j$

$\underline{N} = (N_1, \dots, N_r)'$ and $\underline{T} = (T_1, \dots, T_r)'$

Then, $E(N_i) = \sum_{j=1}^r m_{ij}$ is the ANSS when the process starts in state i . The vector of ANSS values is

$$E(\underline{N}) = M \underline{1}$$

and the variance vector of the NSS is

$$\text{var}(\underline{N}) = (2M - I) E(\underline{N}) - (E(\underline{N}))^{(2)}$$

where $(E(\underline{N}))^{(2)}$ is a vector whose i^{th} component is the square of the i^{th} component of $E(\underline{N})$. Following Amin (1987), $E(T_i) = \sum_{j=1}^r m_j b_j$ is the ATS when the chart starts in state i . The ATS vector is

$$E(\underline{T}) = M \underline{b}$$

and the variance vector of \underline{T} is

$$\text{var}(\underline{T}) = MB E(2M - I) - (M\underline{b})^{(2)}$$

where B is a diagonal matrix with elements b_1, \dots, b_r and $(M\underline{b})^{(2)}$ is a vector whose i^{th} component is the square of the i^{th} component of $M\underline{b}$.

Let $ATS(r)$ be the ATS calculated using r states. Similar to the procedure of Brook and Evans (1972), the continuous-state ATS can be approximated as the least squares intercept A of the quadratic equation in the reciprocal of r^2 ,

$$ATS(r) = A + \frac{B}{r^2} + \frac{C}{r^4}$$

The approximation A is sometimes called the asymptotic ATS.

4.2.4 The Steady State ATS

As in the VSI Shewhart chart, a steady state ATS is needed for the VSI CUSUM chart as a measure of detection time for shifts that occur after the chart has been running for some time. In order to determine the distribution of the steady state time to signal (T^*) the distribution of the point of the shift within the sampling interval and the distribution of the CUSUM statistic at the time of the shift must be determined. A stationary distribution can be obtained by assuming that

no false alarms occur before the shift and that the time to shift is long enough for the control statistic to reach a stationary distribution.

The conditional stationary distribution, say $\underline{\pi} = (\pi_1, \dots, \pi_r)$, is the normalized left eigenvector corresponding to the largest eigenvalue of Q (Darroch and Seneta 1965). Following the development in Reynolds et al.(1990), assume that the probability of a shift falling in an interval of a particular length is proportional to the product of the length of this interval and the frequency with which this interval is used when the Markov chain is in its steady state. The probability, say α_i , that the Markov chain is in state i at the sample immediately before the shift is

$$\alpha_i = \frac{\pi_i b_i}{\sum_{j=1}^r \pi_j b_j}$$

It is also assumed that the distribution of the shift within an interval is uniform within this interval. Thus the expected time from the shift to the next sample is $\frac{b_i}{2}$ if the Markov chain is in state i immediately before the shift. If the Markov chain is in state j at the first sample after the shift, the additional random time required to signal is T_j . Therefore the ATS when the Markov chain was in state i immediately before the shift is

$$\frac{b_i}{2} + \sum_{j=1}^r p_{ij} E(T_j),$$

where p_{ij} is as defined earlier. Taking a weighted average, the steady state ATS is

$$E(T^*) = \underline{\alpha} \left[\frac{\underline{b}}{2} + Q E(T) \right] = \underline{\alpha} \left[\frac{I}{2} + Q M \right] \underline{b} = \underline{\alpha} \left[M - \frac{I}{2} \right] \underline{b}$$

where $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$ Also, Reynolds (1988b) showed that

$$V(T^*) = \alpha \left[(2M - 1) \left(\frac{B}{2} - \frac{b\alpha}{4} \right) (2M - 1) - \frac{B}{6} \right] b$$

4.2.5 The Integral Equation Approach

Integral equations can be used to approximate the ATS when the control statistic is a continuous random variable. Page (1954) developed the integral equations for the FSI CUSUM chart. Since Z_i in (4.4) is continuous, the control statistic Y_i in (4.4) will be continuous. For continuous Markov process let $f(y,u)$ be the one-step transition density from a point y at time S_i to a point u at time S_{i+1} , where it is assumed that the process is time homogeneous when θ is constant. Let

$$A(y) = ATS \text{ given that } Y_0 = y.$$

Reynolds (1988b) showed that the ATS satisfies the integral equation

$$A(y) = d(y) + \int_{-\infty}^h A(u) f(y,u) du, \quad (4.6)$$

where $d(y)$ is the sampling interval function. Following the development in Reynolds (1988b), if f and F represent the density and cdf, respectively, of $(Z - k)$ for applying the integral equation (4.6) to the CUSUM chart (4.4), then

$$f(y,u) = \begin{cases} f(u - y) & y \geq 0 \\ f(u) & y < 0 \end{cases}$$

Suppose that we use two sampling intervals d_1 and d_2 and the CUSUM chart uses the decision rule (4.5). Reynolds (1988b) showed that the ATS satisfies the following integral equations.

For the case $0 \leq y \leq h$ and $0 < g < h$,

$$A(y) = l(y) + A(0) F(-y) + \int_0^g A(u) f(u-y) du + \int_g^h A(u) f(u-y) du,$$

where $l(y)$ is

$$l(y) = \begin{cases} d_1 & 0 \leq g < y \leq h \\ d_2 & 0 \leq y \leq g \leq h \\ d_1(1 - F(g-y)) + d_2 F(g-y) & 0 \leq y \leq g \leq h \end{cases}$$

If $g \geq 0$ then

$$A(y) = A(0) \quad \text{for all } y \leq 0$$

If $g < 0$ then

$$A(y) = \begin{cases} (d_2 - d_1) + A(0) & y \leq g < 0 \\ A(0) & g < y \leq 0 \end{cases}$$

Thus, $A(y)$ for $y < 0$ can be obtained from $A(0)$.

Suppose that the integrals are approximated using Gaussian quadrature with m points. Let a_{1j} and y_{1j} , $j = 1, 2, \dots, m$, be the weights and roots, respectively, for the interval $(0, g)$, and let a_{2j} and y_{2j} , $j = 1, 2, \dots, m$, be the weights and roots, respectively, for the interval (g, h) . Let $\tilde{A}(\tilde{y})$ be the approximate solution. Then $\tilde{A}(\tilde{y})$ satisfies

$$\tilde{A}(0) = d_2 + \tilde{A}(0) F(0) + \sum_{i=1}^2 \sum_{j=1}^m a_{ij} \tilde{A}(y_{ij}) f(y_{ij}) \quad (4.7)$$

and for $i = 1, 2$, and $j = 1, 2, \dots, m$

$$\tilde{A}(y_{ij}) = l(y_{ij}) + \tilde{A}(0) F(-y_{ij}) + \sum_{k=1}^2 \sum_{l=1}^m a_{kl} \tilde{A}(y_{kl}) f(y_{kl} - y_{ij}) \quad (4.8)$$

Solving the $2m + 1$ equations given by (4.7) and (4.8) gives $\tilde{A}(0), \tilde{A}(y_{11}), \dots, \tilde{A}(y_{2m})$. Then for any $0 \leq y \leq h$,

$$\tilde{A}(y) = l(y) + \tilde{A}(0) F(-y) + \sum_{i=1}^2 \sum_{j=1}^m a_{ij} \tilde{A}(y_{ij}) f(y_{ij} - y) \quad (4.9)$$

If $g \leq 0$ then there is no need to break the integral on $(0, h)$ into two parts and thus in this case there are only $m + 1$ equations to be solved.

4.3 Univariate EWMA Control Charts

4.3.1 FSI EWMA Charts

The exponentially weighted moving average (EWMA) chart for θ is based on the statistic

$$Y_i = (1 - \lambda) Y_{i-1} + \lambda Z_i \quad (4.10)$$

$i = 1, 2, \dots$, where $Y_0 = \theta_0$ and $0 < \lambda \leq 1$ and Z_i is a function of the current sample \underline{X}_i . For example, an EWMA chart for the mean μ is based on the statistic

$$Y_i = (1 - \lambda) Y_{i-1} + \lambda \bar{X}_i$$

where $Y_0 = \mu_0$ and $\bar{X}_i = \sum_{j=1}^n \frac{X_{ij}}{n}$. Equation (4.10) can be written as

$$Y_i = (1 - \lambda)^i Y_0 + \lambda \sum_{j=0}^{i-1} (1 - \lambda)^j Z_{i-j}$$

If Z_1, Z_2, \dots are independent with common mean θ and variance σ_Z^2 , it is easy to show that the mean of Y_i is $E(Z_i)$ and the variance of Y_i is

$$\sigma_{Y_i}^2 = \frac{\lambda}{2-\lambda} [1 - (1-\lambda)^{2i}] \sigma_Z^2$$

Unless λ is small, the variance quickly converges to its asymptotic value

$$\sigma_Y^2 = \frac{\lambda}{2-\lambda} \sigma_Z^2$$

Thus the control limits of the EWMA chart for the mean are often set at $\theta_0 \pm \gamma \sigma_Y$, where γ is frequently taken to be 3. In general, the EWMA control chart signals whenever $|Y_i| > h$, where $h > 0$.

4.3.2 VSI EWMA Charts

For adding the VSI feature to the FSI EWMA chart, suppose that we use only two sampling intervals d_1 and d_2 with $d_1 < d_2$. The in-control region $(-h, h)$ is partitioned into three regions $I_{-1} = (-h, -g)$, $I_1 = [g, h)$ and $I_2 = [-g, g)$ such that if $Y_i \in I_{-1}$ or I_1 then use a short interval d_1 and if $Y_i \in I_2$ then use a long interval d_2 . The parameters h and g are chosen to give a specified ATS when the process is in control. The properties of EWMA control charts can be evaluated by using the Markov chain approach or the integral equation approach. For the FSI EWMA chart, a detailed discussion of the Markov chain approach is given by Lucas and Saccucci (1990). VSI EWMA charts for the mean were investigated by Saccucci, Amin, and Lucas (1990). The properties of EWMA charts can be evaluated using the Markov chain approach or the integral equation approach similar to what was done for CUSUM charts.

Chapter 5

Multivariate Control Charts for the Mean Vector

5.1 Introduction

There are various approaches to constructing control charts for multivariate data. For monitoring the mean using the Shewhart chart we could have a separate \bar{X} –chart for each parameter and then look at the joint properties of these p charts, or construct a univariate statistic such as

$$Z_i^2 = n(\bar{X}_i - \underline{\mu}_0)\Sigma_0^{-1}(\bar{X}_i - \underline{\mu}_0)', \quad (5.1)$$

and use one chart in which Z_i^2 is plotted, where \bar{X}_i is the sample means vector of the n observations on occasion i, $\underline{\mu}_0$ is the target means vector, and Σ_0 is the known variance-covariance matrix.

CUSUM and EWMA charts are more efficient than Shewhart charts in detecting small changes in θ because the CUSUM and EWMA charts accumulate past sample information. For multivariate CUSUM and EWMA charts, there are several ways to use the past sample information. The two basic approaches are defined as follows:

The "combine-accumulate" approach : combine the multivariate data into a univariate statistic

and then accumulate over past samples.

The "accumulate-combine" approach : first accumulate past sample information
for each parameter and then

(1) use separate charts for each parameter

or

(2) combine the separate accumulations into a univariate statistic.

Under the combine-accumulate approach, the multivariate CUSUM (or EWMA) procedure reduces each multivariate sample to a univariate statistic and then forms a CUSUM (or EWMA) of the univariate statistics. For example, to monitor the p components of $\underline{\mu}$, \underline{X}_i could be reduced to a univariate statistic Z_i^2 and then past sample information could be accumulated by forming an EWMA of $Z_i^2, Z_{i-1}^2, Z_{i-2}^2, \dots$. Under the accumulate-combine approach, after the past sample information is accumulated for each parameter, we could (1) use p separate CUSUM (or EWMA) charts and look at joint properties or (2) form a univariate CUSUM (or EWMA) statistic from the multivariate accumulations.

Constructing p separate Shewhart, CUSUM, or EWMA charts for p -variate data, one for each parameter, is common in industry and it is easy to identify which of the p quality characteristics caused the signal when the control chart indicates that the process is out of control. One disadvantage to this method is that properties are difficult to find when the quality characteristics are not mutually independent. For multivariate data, it is relatively easy to evaluate the properties of a single Shewhart, EWMA, or CUSUM chart which uses the combine-accumulate approach to form a univariate control statistic. Properties of charts based on the accumulate-combine approach are usually more difficult to evaluate. One disadvantage to reducing multivariate data to a univariate statistic is that when the process is out of control, it may be difficult to determine which combination of variables caused the signal.

5.2 Shewhart Control Charts for the Mean Vector

Suppose that the objective is to monitor $\underline{\mu}$ for the multivariate normal distribution when the target value for the mean vector, $\underline{\mu}_0$, is given and the covariance matrix is known as Σ_0 . Assume that any deterioration in quality is reflected by a change in $\underline{\mu}$. The likelihood ratio test statistic (Anderson 1958) of $H_0: \underline{\mu} = \underline{\mu}_0$ vs. $H_1: \underline{\mu} \neq \underline{\mu}_0$ is given by (5.1). For arbitrary values of $\underline{\mu}$, Z_i^2 has a non central chi-squared distribution with p degrees of freedom and noncentrality parameter

$$\tau^2 = n(\underline{\mu} - \underline{\mu}_0)\Sigma_0^{-1}(\underline{\mu} - \underline{\mu}_0)'$$

The null hypothesis will be rejected if $Z_i^2 > \chi^2_{1-\alpha}(p)$. Thus, the likelihood ratio test statistic Z_i^2 can be the Shewhart control statistic for $\underline{\mu}$ and the control limits would be set at $\{0, \chi^2_{1-\alpha}(p)\}$. If a control statistic plots above the control limit, the process mean is deemed to be out of control and assignable causes of variation are sought. Chengalur-Smith et al. (1990) worked on the VSI Shewhart control charts for the mean vector.

5.3 Multivariate CUSUM Charts for the Mean Vector

The most direct and obvious method of replacing the multivariate Shewhart chart by a CUSUM procedure is to form a CUSUM of the scalars Z_j^2 ($j = 1, 2, \dots$). As in the univariate case, the FSI and VSI multivariate CUSUM statistic for $\underline{\mu}$ at the j^{th} sample is

$$Y_j = \max\{Y_{j-1}, 0\} + (Z_j^2 - k), \quad (5.2)$$

where $Y_0 = w$ ($w \geq 0$), $k > 0$, and Z_j^2 is defined in (5.1). The multivariate CUSUM control chart signals whenever $Y_j \geq h$. For the VSI multivariate CUSUM charts, suppose that

d_1 is used when $Y_j \in (g, h]$

d_2 is used when $Y_j \in (-k, g]$

where $-k < g \leq h$. The properties of the multivariate CUSUM chart can be evaluated by using the Markov chain approach or integral equations.

Charts which accumulate past sample information for each parameter and then form a univariate CUSUM statistic from the multivariate data were considered by Crosier (1988) and Pignatiello and Runger (1990). The multivariate CUSUM chart (MCUSUM) proposed by Crosier (1988) is based on the following statistics. Let

$$C_j = \{n(\underline{S}_{j-1} + \bar{X}_j - \underline{\mu}_0) \Sigma_o^{-1} (\underline{S}_{j-1} + \bar{X}_j - \underline{\mu}_0)'\}^{1/2}$$

and

$$\underline{S}_j = \begin{cases} \underline{Q}' & \text{if } C_j \leq k \\ (\underline{S}_{j-1} + \bar{X}_j - \underline{\mu}_0) \left(1 - \frac{k}{C_j}\right) & \text{if } C_j > k \end{cases}$$

$i = 1, 2, \dots$, where $\underline{S}_0 = \underline{Q}'$ and $k > 0$. Let

$$Y_j = \{n\underline{S}_j \Sigma_o^{-1} \underline{S}_j'\}^{1/2}$$

$j = 1, 2, \dots$, Crosier's MCUSUM scheme signals when $Y_j > h$, where $h > 0$.

The multivariate CUSUM chart, MC1, proposed by Pignatiello and Runger (1990) is based on the following statistics. Let

$$D_j = \sum_{l=j-l_j+1}^j (\bar{X}_l - \underline{\mu}_0)$$

and

$$MC1_j = \max\{0, (nD_j \Sigma_o^{-1} D_j')^{1/2} - kl_j\}$$

where $k > 0$ and

$$l_j = \begin{cases} l_{j-1} + 1 & \text{if } MC1_{j-1} > 0 \\ 1 & \text{otherwise} \end{cases}$$

An out-of-control signal is given when $MC1_j > h$ where $h > 0$. Pignatiello and Runger(1990) found that an MC1 chart based on the accumulate-combine approach does much better than a chart (MC2) which forms a univariate statistic and then accumulates.

5.4 Multivariate EWMA Charts for the Mean Vector

Consider first the chart which reduces the multivariate data to Z_i^2 and then forms an EWMA:

$$Y_i = (1 - \lambda) Y_{i-1} + \lambda Z_i^2 \quad i = 1, 2, \dots$$

where $Y_0 = 0$, $0 < \lambda \leq 1$, and Z_i^2 is given by (5.1). The EWMA chart based on the control statistic Y_i signals that the process is out-of-control when $Y_i \geq h$. As in the univariate VSI EWMA chart, suppose that

d_1 is used when $Y_i \in (g, h]$

d_2 is used when $Y_i \in (0, g]$

The ATS of this chart can be obtained using a Markov chain approach or integral equations.

Another EWMA chart can be constructed by forming a univariate statistic from a multivariate EWMA statistic. The vectors of EWMA's are defined as

$$\underline{Y}_i = (I - \Lambda) \underline{Y}_{i-1} + \Lambda \bar{X}_i' \quad i = 1, 2, \dots,$$

where $\underline{Y}_0 = \underline{\mu}_0$, $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_p)$, $0 < \lambda_j \leq 1$, $j = 1, 2, \dots, p$. The control statistic is

$$T_i^2 = (\underline{Y}_i - \underline{\mu}_0) \Sigma_{\underline{Y}_i}^{-1} (\underline{Y}_i - \underline{\mu}_0)' \quad (5.3)$$

This multivariate extension of the EWMA chart is referred to as the MEWMA chart by Lowry et al. (1989).

Under the assumption that $\lambda_1 = \lambda_2 = \dots = \lambda_p = \lambda$, Lowry et al. (1989) showed that the distribution of T_i^2 depends on $\underline{\mu}$ and Σ through the noncentrality parameter τ . The MEWMA vectors can then be written as

$$\underline{Y}_i = (1 - \lambda) \underline{Y}_{i-1} + \lambda \bar{X}_i' \quad i = 1, 2, \dots \quad (5.4)$$

and the covariance matrix of \underline{Y}_i is

$$\Sigma_{\underline{Y}_i} = \frac{\lambda}{2 - \lambda} [1 - (1 - \lambda)^{2i}] \frac{\Sigma}{n}$$

which is derived by Lowry et al. (1989). Also they showed that the ANSS performance of the MEWMA chart depends on the mean vector $\underline{\mu}$ and covariance matrix Σ only through the value of the noncentral parameter τ . The MEWMA chart signals that the process is out-of-control whenever $T_i^2 \geq h$.

As in the univariate VSI EWMA chart, suppose that

d_1 is used when $T_i^2 \in (g, h]$

d_2 is used when $T_i^2 \in (0, g]$

The MEWMA ATS can be obtained using simulation.

5.5 Numerical Results for Charts for the Mean Vector

When comparing VSI and FSI charts, some kind of standard for comparison is necessary. The charts are matched for ANSS and ATS when the process is in control. This enables the performance to be evaluated when the process has shifted away from its target value. For convenience the unit of time was chosen as the sampling interval of the FSI chart so that $d = 1$. When $d = 1$ the ANSS and ATS of the FSI chart have the same value. By using the relationships between FSI and VSI sampling intervals, two sampling intervals d_1 and d_2 for the VSI charts can be chosen so that two charts have the same ATS when the process is in control. In our computation, the ANSS in control was fixed to be 200 and all of the VSI charts used $d_1 = 0.1$ and $d_2 = 1.9$.

5.5.1 Multivariate CUSUM Charts for the Mean Vector

In order to calculate the ATS for the VSI multivariate CUSUM charts with control statistic given by (5.2), the parameters h , g and k should be determined first. By using $m = 400$ in the approximate solution (4.9) to the integral equation (4.6), the parameters h and g for four different k -values were determined so that the ANSS at $\underline{\mu} = \underline{\mu}_0$ is 200. Table 5.1 gives the values of h and g for various values of p and k when the ATS at $\underline{\mu} = \underline{\mu}_0$ is 200. The ATS and steady state ATS values for the FSI chart were calculated by using Markov chains with $r = 100$; values for the VSI chart were calculated using Markov chains with $r = 200$. For $p = 2, 10$ and 20 , the ATS values for matched FSI and VSI multivariate CUSUM charts are given in Tables 5.2-5.11, respectively. Tables 5.12-5.18 give the steady state ATS values for $p = 2, 3, 4, 5, 8, 10$, and 20 , respectively. The results in Tables 5.2-5.18 show that the ATS values for large k are much larger for small shifts and only slightly smaller for large shifts when compared to the ATS values for small k . From Tables 5.2-5.18, $k = p + 1$ seems to work well over a range of different shifts. If we are concerned primarily

well small shift, $k = p + 0.5$ works well. Also, Tables 5.2-5.18 show that the VSI CUSUM charts for $\underline{\mu}$ are more efficient than the corresponding FSI CUSUM charts.

5.5.2 Multivariate EWMA (MEWMA) Charts for the Mean Vector

The MEWMA ATS values and parameters h and g were obtained using 10,000 simulations, where the sample size used for each sample observation was one. Table 5.19 gives the values of h and g for $p = 2$ and $\lambda = 0.05, 0.1, 0.3$ and for $p = 3-5$ and $\lambda = 0.05$ when the ATS at $\underline{\mu} = \underline{\mu}_0$ is approximately 200. For $p = 2$ and 5, Tables 5.20-5.21 give ATS values for matched FSI and VSI MEWMA charts for $\underline{\mu}$. The standard errors are given in parentheses.

As shown in Tables 5.20 and 5.21, for $p = 2$ smaller values of λ are more effective in detecting all shifts in the mean vector. The VSI MEWMA chart has smaller ATS values than corresponding FSI MEWMA charts. Tables 5.2, 5.5, 5.20, and 5.21 show that the MEWMA seems to be much better than the multivariate CUSUM. This may be because the CUSUM chart uses Y_j in (5.2) which reduces information to a univariate statistic before accumulating. This suggests that it may be better to accumulate information in the CUSUM chart before reduction to a univariate statistic. Crosier (1988), and Pignatiello and Runger (1990) investigated multivariate CUSUM charts based on the accumulate-combine approach. Lowry, Woodall, Champ, and Rigdon (1989) showed that the MEWMA procedure performs better than the multivariate CUSUM procedures of Crosier (1988), and Pignatiello and Runger (1988) in terms of the ARL. Recommendations for the selection and design of VSI charts for the mean vector will be given in Chapter 8.

Table 5.1: Values of h and g for CUSUM Charts for Various Values of p and k when the ATS at $\mu = \mu_0$ is 200

p	$k = p + 0.5$	$k = p + 1.0$	$k = p + 1.5$	$k = p + 2.0$
2	13.4621 0.5342	10.2324 -0.8231	8.5913 -1.6531	7.5031 -2.3220
3	17.4075 1.2865	13.0612 -0.3626	10.8589 -1.3720	9.4290 -2.1345
4	20.9317 1.1078	15.6823 0.0343	12.9450 -1.0764	11.1957 -1.9298
5	24.1993 2.8120	18.1482 0.0476	14.9293 -0.7782	12.8698 -1.7164
6	27.2542 3.4929	20.4950 0.8945	16.8320 -0.4788	14.4779 -1.4982
7	30.1362 4.1451	22.7435 1.3136	18.6689 -0.1799	16.0348 -1.2774
8	32.8734 4.7732	24.9079 1.7255	20.4495 0.0482	17.5493 -1.0549
9	35.4871 5.3795	26.9989 2.1305	22.1811 0.3385	19.0272 -0.8316
10	37.9939 5.9679	29.0248 2.5288	23.8689 0.6269	20.4729 -0.6077
20	59.1303 11.0832	46.7630 6.2133	39.0027 3.3999	33.6415 1.4870

The top number in each cell is h

The bottom number in each cell is g

Table 5.2: ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector ($p = 2$)

shift τ^2	$h = 13.4621$ $g = 0.5341$ $k = 2.5$		$h = 10.2324$ $g = -0.8231$ $k = 3.0$		$h = 8.5913$ $g = -1.6532$ $k = 3.5$		$h = 7.5031$ $g = -2.3220$ $k = 4.0$	
	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI
.0	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00
.5	47.21	33.27	54.23	39.58	60.20	46.36	64.71	51.43
1.0	22.10	13.08	24.84	13.71	28.28	16.91	31.39	19.85
1.5	13.89	7.83	14.77	6.53	16.59	7.90	18.52	9.43
2.0	10.09	5.72	10.22	3.82	11.18	4.39	12.37	5.17
2.5	7.94	4.62	7.75	2.55	8.26	2.77	9.00	3.17
3.0	6.56	3.96	6.25	1.84	6.50	1.91	6.97	2.11
3.5	5.61	3.51	5.24	1.41	5.35	1.40	5.65	1.50
4.0	4.92	3.20	4.53	1.12	4.55	1.08	4.74	1.13
4.5	4.39	2.96	4.00	0.91	3.97	0.86	4.09	0.88
5.0	3.97	2.79	3.59	0.76	3.53	0.71	3.60	0.71

Table 5.3: ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector ($p = 3$)

shift τ^2	$h = 17.4075$ $g = 1.2865$ $k = 3.5$		$h = 13.0612$ $g = -0.3626$ $k = 4.0$		$h = 10.8539$ $g = -1.3720$ $k = 4.5$		$h = 9.4290$ $g = -2.1345$ $k = 5.0$	
	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI
.0	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00
.5	55.82	40.32	62.60	47.32	69.08	54.64	74.24	60.43
1.0	27.04	16.36	29.68	17.36	33.49	20.10	37.13	24.53
1.5	17.13	9.72	27.82	8.49	19.80	10.05	22.04	11.91
2.0	12.45	6.99	12.35	5.03	13.35	5.66	14.70	6.61
2.5	9.79	5.57	9.37	3.38	9.84	3.59	10.67	4.07
3.0	8.08	4.70	7.54	2.46	7.73	2.49	8.23	2.73
3.5	6.90	4.12	6.31	1.89	6.35	1.84	6.65	1.95
4.0	6.03	3.70	5.44	1.51	5.39	1.42	5.56	1.47
4.5	5.37	3.40	4.48	1.24	4.69	1.14	4.78	1.15
5.0	4.85	3.16	4.29	1.03	4.15	0.94	4.19	0.92

Table 5.4: ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector ($p = 4$)

shift τ^2	$h = 20.9317$ $g = 1.1078$ $k = 4.5$		$h = 15.6823$ $g = 0.0343$ $k = 5.0$		$h = 12.9450$ $g = -1.0764$ $k = 5.5$		$h = 11.1957$ $g = -1.9298$ $k = 6.0$	
	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI
.0	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00
.5	62.45	46.80	69.10	54.58	75.58	60.84	81.01	66.98
1.0	31.22	19.63	33.82	22.11	37.74	24.45	41.67	28.35
1.5	19.95	11.64	20.53	11.95	22.56	11.98	24.99	14.07
2.0	14.54	8.30	14.28	7.88	15.27	6.83	16.72	7.91
2.5	11.43	6.54	10.84	5.92	11.27	4.37	12.14	4.91
3.0	9.43	5.47	8.72	4.81	8.85	3.05	9.36	3.31
3.5	8.05	4.75	7.29	4.13	7.26	2.26	7.55	2.37
4.0	7.03	4.23	6.28	3.67	6.15	1.75	6.30	1.70
4.5	6.24	3.85	5.52	3.33	5.34	1.41	5.41	1.40
5.0	5.63	3.55	4.93	3.08	4.73	1.16	4.74	1.13

Table 5.5: ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector ($p = 5$)

shift τ^2	$h = 24.1993$ $g = 2.8120$ $k = 5.5$		$h = 18.1482$ $g = 0.0476$ $k = 6.0$		$h = 14.9293$ $g = -0.7782$ $k = 6.5$		$h = 12.8698$ $g = -1.7164$ $k = 7.0$	
	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI
.0	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00
.5	68.03	51.86	74.37	59.52	80.75	65.85	86.25	72.15
1.0	34.93	22.40	37.44	24.94	41.41	27.49	45.49	31.64
1.5	22.51	13.32	22.99	13.60	25.04	13.75	27.59	16.03
2.0	16.44	9.46	16.06	8.95	17.03	7.94	18.55	9.12
2.5	12.94	7.41	12.21	6.67	12.59	5.12	13.49	5.71
3.0	10.68	6.16	9.82	5.39	9.89	3.59	10.40	3.87
3.5	9.10	5.32	8.20	4.57	8.11	2.67	8.39	2.79
4.0	7.94	4.71	7.06	4.03	6.87	2.08	7.00	2.11
4.5	7.05	4.26	6.20	3.63	5.96	1.67	6.00	1.65
5.0	6.35	3.91	5.54	3.34	5.27	1.38	5.25	1.34

Table 5.6: ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector ($p = 6$)

shift τ^2	$h = 27.2542$ $g = 3.4929$ $k = 6.5$		$h = 20.4950$ $g = 0.8945$ $k = 7.0$		$h = 16.8320$ $g = -0.4788$ $k = 7.5$		$h = 14.4779$ $g = -1.4982$ $k = 8.0$	
	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI
.0	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00
.5	72.79	56.33	78.81	63.75	85.04	70.05	90.50	76.39
1.0	38.26	25.00	40.68	27.54	44.65	30.23	48.81	34.55
1.5	24.84	14.93	25.25	15.16	27.30	15.41	29.92	17.83
2.0	18.20	10.25	17.72	9.97	18.67	9.00	20.23	10.27
2.5	14.34	8.25	13.49	7.39	13.84	5.85	14.75	6.48
3.0	11.84	6.82	10.85	5.93	10.88	4.12	11.38	4.41
3.5	10.08	5.86	9.08	5.01	8.93	3.08	9.18	3.19
4.0	8.80	5.18	7.80	4.38	7.56	2.40	7.66	2.42
4.5	7.81	4.66	6.85	3.94	6.56	1.94	6.57	1.91
5.0	7.03	4.26	6.12	3.60	5.79	1.60	5.74	1.54

Table 5.7: ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector ($p = 7$)

shift τ^2	$h = 30.1362$ $g = 4.1451$ $k = 7.5$		$h = 22.7435$ $g = 1.3136$ $k = 8.0$		$h = 18.6689$ $g = -0.1799$ $k = 8.5$		$h = 16.0348$ $g = -1.2774$ $k = 9.0$	
	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI
.0	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00
.5	76.93	60.29	82.65	67.45	88.70	73.67	94.09	79.99
1.0	41.28	27.41	43.61	29.94	47.56	32.72	51.75	37.16
1.5	27.00	16.44	27.34	16.65	29.40	16.98	32.06	19.50
2.0	19.84	11.63	19.27	10.96	20.21	10.02	21.80	11.35
2.5	15.64	9.05	14.70	8.10	15.02	6.55	15.94	7.22
3.0	12.91	7.46	11.84	6.47	11.82	4.64	12.32	4.94
3.5	11.01	6.39	9.90	5.44	9.70	3.47	9.95	3.59
4.0	9.60	5.63	8.51	4.74	8.22	2.72	8.30	2.73
4.5	8.52	5.05	7.47	4.23	7.12	2.20	7.11	2.15
5.0	7.66	4.60	6.67	3.85	6.29	1.82	6.21	1.75

Table 5.8: ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector ($p = 8$)

shift τ^2	$h = 32.8734$ $g = 4.7732$ $k = 8.5$		$h = 24.9079$ $g = 1.7255$ $k = 9.0$		$h = 20.4495$ $g = 0.0482$ $k = 9.5$		$h = 17.5493$ $g = -1.0549$ $k = 10.0$	
	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI
.0	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00
.5	80.58	63.84	86.03	70.74	91.89	77.86	97.18	83.12
1.0	44.04	29.66	46.30	32.17	50.21	36.43	54.40	39.53
1.5	29.02	17.89	29.30	18.06	31.35	20.05	34.04	21.07
2.0	21.37	12.64	20.74	11.91	21.67	12.67	23.28	12.40
2.5	16.86	9.82	15.86	8.79	16.15	8.95	17.07	7.94
3.0	13.93	8.08	12.78	6.99	12.73	6.88	13.22	5.46
3.5	11.88	6.90	10.69	5.86	10.45	5.62	10.68	3.98
4.0	10.36	6.06	9.19	5.08	8.85	4.80	8.91	3.03
4.5	9.19	5.43	8.07	4.53	7.67	4.23	7.63	2.40
5.0	8.27	4.94	7.19	4.11	6.77	3.81	6.67	1.95

Table 5.9: ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector ($p = 9$)

shift τ^2	$h = 35.4871$ $g = 5.3795$ $k = 9.5$		$h = 26.9989$ $g = 2.1305$ $k = 10.0$		$h = 22.1811$ $g = 0.3385$ $k = 10.5$		$h = 19.0272$ $g = -0.8316$ $k = 11.0$	
	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI
.0	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00
.5	83.86	67.05	89.04	73.71	94.72	80.68	99.89	85.88
1.0	46.59	31.78	48.78	34.27	52.64	38.54	56.81	41.71
1.5	30.90	19.28	31.14	19.42	33.18	21.44	35.89	22.55
2.0	22.82	13.62	22.14	12.83	23.05	13.61	24.67	13.40
2.5	18.04	10.56	16.96	9.46	17.23	9.62	18.15	8.64
3.0	14.90	8.67	13.69	7.51	13.59	7.37	14.08	5.97
3.5	12.70	7.40	11.45	6.27	11.17	6.00	11.38	4.36
4.0	11.08	6.48	9.84	5.42	9.46	5.10	9.51	3.34
4.5	9.83	5.80	8.64	4.81	8.20	4.48	8.14	2.64
5.0	8.84	5.26	7.70	4.36	7.24	4.03	7.11	2.15

Table 5.10: ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector ($p = 10$)

shift τ^2	$h = 37.9939$ $g = 5.9679$ $k = 10.5$		$h = 29.0248$ $g = 2.5288$ $k = 11.0$		$h = 23.8689$ $g = 0.6269$ $k = 11.5$		$h = 20.4729$ $g = -0.6077$ $k = 12.0$	
	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI
.0	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00
.5	86.81	70.00	91.76	76.40	97.26	83.21	102.31	88.35
1.0	48.96	33.79	51.09	36.24	54.89	40.51	59.03	43.74
1.5	32.67	20.60	32.88	20.73	34.91	22.77	37.62	23.95
2.0	24.20	14.57	23.47	13.73	24.37	14.52	26.00	14.36
2.5	19.14	11.28	18.01	10.11	18.26	10.26	19.19	9.32
3.0	15.83	9.25	14.54	8.01	14.43	7.86	14.91	6.47
3.5	13.49	7.88	12.18	6.67	11.86	6.38	12.07	4.74
4.0	11.77	6.89	10.47	5.76	10.05	5.41	10.08	3.63
4.5	10.44	6.15	9.20	5.10	8.71	4.73	8.64	2.88
5.0	9.39	5.58	8.19	4.60	7.69	4.24	7.55	2.35

Table 5.11: ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector ($p = 20$)

shift τ^2	$h = 59.1303$ $g = 11.0832$ $k = 20.5$		$h = 46.7630$ $g = 6.2133$ $k = 21.0$		$h = 39.0027$ $g = 3.3999$ $k = 21.5$		$h = 33.6415$ $g = 1.4870$ $k = 22.0$	
	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI
.0	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00
.5	106.42	90.22	109.78	94.83	113.88	100.22	117.88	105.29
1.0	66.36	49.36	68.00	51.43	71.21	55.38	74.89	59.77
1.5	46.42	31.57	46.47	31.59	48.30	33.68	50.90	36.57
2.0	35.16	22.61	34.28	21.56	35.07	22.39	36.70	24.07
2.5	28.13	17.49	26.78	15.98	26.90	16.08	27.82	16.96
3.0	23.39	14.27	21.83	12.61	21.55	12.30	21.99	12.68
3.5	20.01	12.07	18.37	10.40	17.86	9.90	17.98	9.98
4.0	17.48	10.49	15.84	8.87	15.20	8.28	15.11	8.19
4.5	15.52	9.29	13.91	7.76	13.20	7.13	12.99	6.94
5.0	13.96	8.36	12.41	6.92	11.66	6.29	11.37	6.05

Table 5.12: Steady State ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector ($p = 2$)

shift τ^2	$k = 2.5$		$k = 3.0$		$k = 3.5$		$k = 4.0$	
	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI
.0	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00
.5	44.03	31.94	52.56	40.25	59.08	47.09	63.85	52.20
1.0	19.85	11.86	23.53	14.42	27.33	17.66	30.62	20.63
1.5	12.10	6.68	13.65	7.28	15.74	8.67	17.80	10.21
2.0	8.57	4.60	9.22	4.58	10.39	5.17	11.68	5.96
3.0	5.36	2.88	5.40	2.62	5.79	2.70	6.33	2.91
4.0	3.88	2.15	3.76	1.91	3.89	1.88	4.14	1.93
5.0	3.04	1.75	2.87	1.55	2.90	1.51	3.01	1.51
8.0	1.83	1.24	1.67	1.14	1.62	1.11	1.61	1.10
10.0	1.44	1.11	1.30	1.05	1.24	1.02	1.23	1.02

Table 5.13: Steady State ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector ($p = 3$)

shift τ^2	$k = 3.5$		$k = 4.0$		$k = 4.5$		$k = 5.0$	
	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI
.0	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00
.5	51.44	38.71	60.28	47.90	67.56	55.32	73.12	61.16
1.0	23.99	14.96	27.91	18.02	32.24	21.72	36.16	25.28
1.5	14.74	8.43	16.36	9.19	18.72	10.80	21.17	12.68
2.0	10.47	5.76	11.08	5.76	12.27	6.42	13.89	7.38
3.0	6.55	3.54	6.49	3.22	6.89	3.27	7.50	3.52
4.0	4.74	2.59	4.52	2.28	4.63	2.21	4.89	2.26
5.0	3.71	2.08	3.45	1.82	3.44	1.73	3.55	1.72
8.0	2.23	1.40	2.00	1.26	1.92	1.20	1.90	1.18
10.0	1.55	1.21	1.47	1.12	1.47	1.08	1.44	1.06

Table 5.14: Steady State ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector ($p = 4$)

shift τ^2	$k = 4.5$		$k = 5.0$		$k = 5.5$		$k = 6.0$	
	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI
.0	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00
.5	57.06	44.93	66.17	53.29	73.66	61.47	79.61	67.68
1.0	27.45	18.02	31.61	20.92	36.20	25.14	40.50	29.09
1.5	17.02	10.18	18.73	10.81	21.25	12.70	23.96	14.82
2.0	12.11	6.93	12.74	6.79	14.11	7.57	15.78	8.67
3.0	7.59	4.21	7.48	3.75	7.88	3.81	8.53	4.09
4.0	5.49	3.04	5.21	2.62	5.29	2.53	5.55	2.58
5.0	4.30	2.40	3.98	2.06	3.94	1.95	4.04	1.93
8.0	2.58	1.56	2.30	1.37	2.19	1.29	2.15	1.26
10.0	2.04	1.32	1.79	1.19	1.68	1.14	1.63	1.11

Table 5.15: Steady State ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector ($p = 5$)

shift τ^2	$k = 5.5$		$k = 6.0$		$k = 6.5$		$k = 7.0$	
	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI
.0	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00
.5	61.73	49.74	70.88	58.15	78.45	66.44	84.58	72.82
1.0	30.50	20.60	34.82	23.69	39.58	28.15	44.11	32.35
1.5	19.07	11.70	20.86	12.42	23.51	14.45	26.38	16.76
2.0	13.61	7.95	14.26	7.81	15.69	8.66	17.47	9.87
3.0	8.54	4.79	8.39	4.29	8.79	4.34	9.48	4.64
4.0	6.18	3.44	5.85	2.97	5.91	2.85	6.18	2.89
5.0	4.84	2.70	4.47	2.30	4.40	2.16	4.49	2.13
8.0	2.91	1.71	2.59	1.48	2.45	1.38	2.40	1.34
10.0	2.30	1.42	2.01	1.26	1.87	1.20	1.81	1.16

Table 5.16: Steady State ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector ($p = 8$)

shift τ^2	$k = 8.5$		$k = 9.0$		$k = 9.5$		$k = 10.0$	
	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI
.0	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00
.5	72.11	61.03	81.08	69.10	88.57	76.53	94.75	83.70
1.0	37.95	27.30	42.56	30.71	47.59	35.20	52.42	40.17
1.5	24.26	15.80	26.29	16.71	29.18	18.87	32.36	21.75
2.0	17.47	10.73	18.21	10.63	19.80	11.54	21.80	13.11
3.0	11.01	6.40	10.82	5.80	11.24	5.80	12.00	6.21
4.0	7.99	4.53	7.57	3.94	7.58	3.74	7.86	3.80
5.0	6.25	3.51	5.78	3.00	5.65	2.77	5.72	2.72
8.0	3.76	2.13	3.35	2.81	3.15	1.66	3.06	1.59
10.0	2.97	1.72	2.61	1.49	2.41	1.38	2.31	1.32

Table 5.17: Steady State ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector ($p = 10$)

shift τ^2	$k = 10.5$		$k = 11.0$		$k = 11.5$		$k = 12.0$	
	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI
.0	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00
.5	77.19	66.79	85.99	74.57	93.32	81.79	99.42	88.88
1.0	41.95	31.09	46.69	34.62	51.78	39.21	56.68	44.34
1.5	27.17	18.22	29.34	19.24	32.34	21.55	35.63	24.60
2.0	19.67	12.40	20.50	12.33	22.17	13.34	24.26	15.04
3.0	12.45	7.37	12.26	6.73	12.69	6.74	13.50	7.19
4.0	9.04	5.19	8.59	4.54	8.59	4.32	8.87	4.38
5.0	7.08	4.00	6.57	3.43	6.41	3.18	6.47	3.11
8.0	4.26	2.39	3.81	2.03	3.57	1.85	3.46	1.76
10.0	3.36	1.92	2.96	1.64	2.74	1.51	2.62	1.43

Table 5.18: Steady State ATS Values for Matched FSI and VSI CUSUM Charts for the Mean Vector ($\rho = 20$)

shift τ^2	$k = 20.5$		$k = 21.0$		$k = 21.5$		$k = 22.0$	
	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI
.0	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00
.5	92.87	85.54	100.94	92.13	107.57	98.31	113.11	103.73
1.0	56.00	45.36	61.04	49.06	66.12	53.67	70.96	58.34
1.5	38.08	28.02	40.77	29.45	44.05	32.10	47.56	35.23
2.0	28.20	19.40	29.49	19.58	31.43	20.90	33.80	22.79
3.0	18.16	11.53	18.13	10.84	18.71	10.93	19.69	11.48
4.0	13.26	8.06	12.82	7.26	12.85	6.99	13.19	7.04
5.0	10.41	6.16	9.85	5.40	9.65	5.06	9.70	4.93
8.0	6.28	3.57	5.73	3.04	5.41	2.76	5.24	2.58
10.0	4.96	2.79	4.47	2.37	4.16	2.14	3.97	2.00

Table 5.19: Values of h and g for MEWMA Charts for Various Values of p and λ when the ATS at $\mu = \mu_0$ is approximately 200

	$p = 2$			$p = 3$	$p = 4$	$p = 5$
	$\lambda = 0.3$	$\lambda = 0.1$	$\lambda = 0.05$	$\lambda = 0.05$	$\lambda = 0.05$	$\lambda = 0.05$
h	10.1480	8.7730	7.6800	9.8050	11.6782	13.4072
g	1.3090	1.3250	1.2550	2.1890	3.1050	4.1440

Table 5.20: ATS Values for FSI MEWMA Charts for the Mean Vector

shift τ	$p = 2$			$p = 5$
	$h = 7.680$ $\lambda = 0.05$	$h = 8.773$ $\lambda = 0.1$	$h = 10.148$ $\lambda = 0.3$	$h = 13.4072$ $\lambda = 0.05$
.0	199.060 (2.134)	200.174 (2.058)	199.979 (1.994)	199.586 (2.138)
.5	20.829 (0.168)	24.671 (0.203)	43.433 (0.408)	27.379 (0.214)
1.0	6.806 (0.047)	7.771 (0.051)	10.621 (0.082)	8.823 (0.058)
1.5	3.634 (0.022)	4.026 (0.023)	4.912 (0.031)	4.609 (0.027)
2.0	2.349 (0.013)	2.603 (0.014)	2.988 (0.016)	2.937 (0.016)
2.5	1.738 (0.008)	1.893 (0.009)	2.100 (0.010)	2.139 (0.010)
3.0	1.385 (0.006)	1.496 (0.007)	1.632 (0.007)	1.672 (0.008)

Table 5.21: ATS Values for VSI MEWMA Charts for the Mean Vector

shift τ	$p = 2$			$p = 5$
	$h = 7.680$ $g = 1.255$ $\lambda = 0.05$	$h = 8.773$ $g = 1.325$ $\lambda = 0.1$	$h = 10.148$ $g = 1.309$ $\lambda = 0.3$	$h = 13.4072$ $g = 4.144$ $\lambda = 0.05$
.0	201.010 (2.252)	199.806 (2.118)	199.819 (1.989)	199.992 (2.305)
.5	11.715 (0.121)	14.645 (0.145)	30.740 (0.300)	15.919 (0.157)
1.0	3.458 (0.031)	3.744 (0.033)	5.008 (0.044)	4.417 (0.040)
1.5	1.903 (0.014)	1.970 (0.015)	2.152 (0.015)	2.331 (0.019)
2.0	1.378 (0.008)	1.421 (0.008)	1.464 (0.008)	1.587 (0.010)
2.5	1.158 (0.004)	1.190 (0.005)	1.212 (0.005)	1.289 (0.006)
3.0	1.066 (0.002)	1.079 (0.003)	1.093 (0.003)	1.139 (0.004)

Chapter 6

Multivariate Control Charts for the Variance-Covariance Matrix

6.1 Introduction

Suppose that the objective is to monitor Σ where the target values Σ_0 and $\underline{\mu}_0$ are known. It is assumed that the in-control process covariance matrix is

$$\Sigma_0 = \begin{bmatrix} \sigma_{011} & \sigma_{012} & \cdot & \cdot & \cdot & \sigma_{01p} \\ \sigma_{021} & \sigma_{022} & \cdot & \cdot & \cdot & \sigma_{02p} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \sigma_{0p1} & \sigma_{0p2} & \cdot & \cdot & \cdot & \sigma_{0pp} \end{bmatrix}$$

We will consider the case in which the primary objective is to detect changes in the variances, not in the correlation coefficients.

Several different control charts for Σ will be presented since different statistics can be used to describe variability. In the univariate case, the S^2 - chart is used to control the variance under the normality assumption. The S^2 - chart signals for large values of S^2 or equivalently for large values of V_i where

$$V_i = \frac{(n-1)S_i^2}{\sigma_0^2},$$

where $S_i^2 = \sum_{j=1}^n (X_{ij} - \bar{X}_i)^2 / (n-1)$ and σ_0^2 is the target value for σ^2 . One possible multivariate version of V_i is

$$V_i = \sum_{j=1}^n (\underline{X}_{ij} - \bar{X}_i) \Sigma_0^{-1} (\underline{X}_{ij} - \bar{X}_i)' = \text{tr}(A_i \Sigma_0^{-1}), \quad (6.1)$$

where

$$A_i = \sum_{j=1}^n (\underline{X}_{ij} - \bar{X}_i)' (\underline{X}_{ij} - \bar{X}_i)$$

When $\Sigma = \Sigma_0$, V_i has a chi-squared distribution with $(n-1)p$ degrees of freedom. Hotelling (1947) proposed the use of the Lawley-Hotelling V_i statistic in monitoring the process variance-covariance matrix. The distribution of V_i was studied by Lawley (1938) and Hotelling (1951).

Hui (1980) studied the use of the sample generalized variance in monitoring the process variance-covariance matrix using the following statistic

$$L_i = \frac{|A_i|}{| \Sigma_0 |} \quad (6.2)$$

It is known that $(n - 1)^{1/2} (L_i - 1)$ is asymptotically normal with mean 0 and variance $2p$ (Anderson (1958)).

Another chart can be constructed by using the likelihood ratio statistic for testing $H_0: \Sigma = \Sigma_0$ vs. $H_1: \Sigma \neq \Sigma_0$. For the i^{th} sample ($i = 1, 2, \dots$), the likelihood ratio statistic is (Anderson, 1958, p.265)

$$\lambda = \left(\frac{e}{n}\right)^{\frac{1}{2} np} |A_i \Sigma_0^{-1}|^{\frac{1}{2} n} \exp\left\{-\frac{1}{2} \text{tr}(\Sigma_0^{-1} A_i)\right\}$$

This likelihood ratio test is not unbiased. If this likelihood ratio statistic is modified by changing the sample size n to the degrees of freedom $N = n - 1$, then the test is unbiased. Let

$$\lambda_1 = \left(\frac{e}{N}\right)^{\frac{1}{2} Np} |A_i \Sigma_0^{-1}|^{\frac{1}{2} N} \exp\left\{-\frac{1}{2} \text{tr}(\Sigma_0^{-1} A_i)\right\}$$

be the modified likelihood ratio statistic, then

$$W_i = -2 \ln \lambda_1 = \text{tr}(\Sigma_0^{-1} A_i) - N \ln |A_i| + N \ln |\Sigma_0| + Np \ln N - Np \quad (6.3)$$

Nagarsenker and Pillai (1973) developed a method for obtaining the exact null distribution of $L = \lambda_1^{2/N}$ in a series form and computed percentage points of L to any degree of accuracy even for small sample size. Thus the statistic W_i can be a control statistic for Σ .

Thus, three different control statistics for Σ are as follows:

1. $V_i = \sum_{j=1}^n (\underline{X}_{ij} - \bar{X}_i) \Sigma_0^{-1} (\underline{X}_{ij} - \bar{X}_i)' = \text{tr}(A_i \Sigma_0^{-1})$
2. $L_i = \frac{|A_i|/(n-1)}{|\Sigma_0|}$
3. $W_i = -2 \ln \lambda_1 = \text{tr}(\Sigma_0^{-1} A_i) - N \ln |A_i| + N \ln |\Sigma_0| + Np \ln N - Np$

In general, if the process shifts from Σ_0 to Σ_1 then it is difficult to obtain the distributions of V_i , L_i and W_i . Thus, in order to evaluate the properties of the charts for Σ it is necessary to use computer simulations.

6.2 Shewhart Charts for the Covariance Matrix

The control limits for a Shewhart chart based on the control statistic L_i would be set from the distribution of L_i . The control limits for a Shewhart chart based on the control statistic V_i would be set at $\{0, \chi^2_{1-\alpha}((n-1)p)\}$. The control limits for a Shewhart chart based on the control statistic W_i would be set by using percentage points of W_i . The VSI versions of these charts can be constructed by using the distributions of these control statistics.

Montgomery and Wadsworth (1972) suggested the use of the logarithm of the sample generalized variance, $\log \left| \frac{A_i}{n-1} \right|$, in monitoring the process variance-covariance matrix.

6.3 CUSUM Charts for the Covariance Matrix

As in the multivariate CUSUM charts for μ , the multivariate CUSUM charts for Σ are as follows: For the statistic V_i given by (6.1), the FSI and VSI multivariate CUSUM control statistic for Σ at the j^{th} sample is

$$Y_{1j} = \max \{Y_{1j-1}, 0\} + (V_j - k)$$

The multivariate CUSUM chart signals whenever $Y_{1j} > h_1$. For the statistic L_i given by (6.2), the FSI and VSI multivariate CUSUM control statistic for Σ at the j^{th} sample is

$$Y_{2j} = \max \{Y_{2j-1}, 0\} + (L_j - k)$$

The multivariate CUSUM chart signals whenever $Y_{2j} > h_2$. For the statistic W_i given by (6.3), the FSI and VSI multivariate CUSUM control statistic for Σ at the j^{th} sample is

$$Y_{3j} = \max \{Y_{3j-1}, 0\} + (W_j - k)$$

The multivariate CUSUM chart signals whenever $Y_{3j} > h_3$

For the VSI multivariate CUSUM charts for Σ , suppose that

d_1 is used when $Y_{kj} \in (g_k, h_k]$

d_2 is used when $Y_{kj} \in (-k, g_k]$

$k = 1, 2, 3$ where $-k < g_k \leq h_k$. The parameters g_k and h_k are chosen to give specified ANSS and ATS values when the process is in control. These parameters can be obtained by using a Markov chain approach or integral equations.

6.4 Multivariate EWMA Charts for the Covariance Matrix

We will discuss multivariate EWMA charts based on the combine-accumulate approach, and multivariate EWMA charts based on the accumulate-combine approach for monitoring the variance-covariance matrix.

6.4.1 Multivariate EWMA Charts Based on the Combine-accumulate

Approach.

Multivariate EWMA charts based on the combine-accumulate approach can be constructed by combining multivariate data into a univariate statistic and then forming a univariate EWMA statistic. For the statistic V_i given by (6.1), the EWMA chart is based on the control statistic

$$Y_{2i} = (1 - \lambda) Y_{2i-1} + \lambda V_i \quad (6.4)$$

where $Y_{10} = 0$, $0 < \lambda \leq 1$. This chart signals when $Y_{2i} > h_1$. For the statistic L_i given by (6.2), the EWMA chart is based on the control statistic

$$Y_{2i} = (1 - \lambda) Y_{2i-1} + \lambda L_i$$

where $Y_{20} = 0$, $0 < \lambda \leq 1$. This chart signals when $Y_{2i} > h_2$. For the statistic W_i given by (6.3), the EWMA chart is based on the control statistic

$$Y_{3i} = (1 - \lambda) Y_{3i-1} + \lambda W_i$$

where $Y_{30} = 0$, $0 < \lambda \leq 1$. This chart signals when $Y_{3i} > h_3$.

As in the VSI multivariate CUSUM charts for $\underline{\mu}$, suppose that

d_1 is used when $Y_{ki} \in (g_k, h_k]$

d_2 is used when $Y_{ki} \in (0, g_k]$

$k = 1, 2, 3$ where $0 < g_k \leq h_k$. The parameters g_1 , g_2 , g_3 , h_1 , h_2 , and h_3 can be obtained by using Markov chain approach or integral equations.

6.4.2 A Multivariate EWMA Chart Based on the Accumulate-combine

Approach

Multivariate EWMA charts based on the accumulate-combine approach can be constructed by forming a univariate statistic from a multivariate EWMA statistic. In the univariate case, an EWMA chart for σ^2 can be constructed by using the statistic

$$Y_k = (1 - \lambda)Y_{k-1} + \lambda \sum_{j=1}^n \left(\frac{X_{kj} - \bar{X}_{k.}}{\sigma_0} \right)^2 \quad (6.5)$$

By repeated substitution in (6.5), it can be shown that

$$Y_k = (1 - \lambda)^k Y_0 + \sum_{i=1}^k \lambda (1 - \lambda)^{k-i} \sum_{j=1}^n \left(\frac{X_{ij} - \bar{X}_{i.}}{\sigma_0} \right)^2 \quad (6.6)$$

$k = 1, 2, \dots$ and $0 < \lambda \leq 1$.

In the multivariate case, we define vectors of EWMA's

$$\underline{Y}_k = \begin{bmatrix} Y_{k1} \\ Y_{k2} \\ \cdot \\ \cdot \\ \cdot \\ Y_{kp} \end{bmatrix} = \begin{bmatrix} (1 - \lambda)^k Y_{10} + \sum_{i=1}^k \lambda (1 - \lambda)^{k-i} \left[\sum_{j=1}^n \left(\frac{X_{ij1} - \bar{X}_{i.1}}{\sigma_{01}} \right)^2 - (n - 1) \right] \\ (1 - \lambda)^k Y_{20} + \sum_{i=1}^k \lambda (1 - \lambda)^{k-i} \left[\sum_{j=1}^n \left(\frac{X_{ij2} - \bar{X}_{i.2}}{\sigma_{02}} \right)^2 - (n - 1) \right] \\ \cdot \\ \cdot \\ \cdot \\ (1 - \lambda)^k Y_{p0} + \sum_{i=1}^k \lambda (1 - \lambda)^{k-i} \left[\sum_{j=1}^n \left(\frac{X_{ijp} - \bar{X}_{i.p}}{\sigma_{0p}} \right)^2 - (n - 1) \right] \end{bmatrix} \quad (6.7)$$

$k = 1, 2, \dots$ where $Y_0 = 0$ and $0 < \lambda_l \leq 1$, $l = 1, 2, \dots, p$. A multivariate EWMA chart for Σ is based on the statistic

$$T_k^2 = \underline{Y}_k' \Sigma_{\underline{Y}_k}^{-1} \underline{Y}_k \quad (6.8)$$

where $\Sigma_{\underline{Y}_k}$ is the variance-covariance matrix of \underline{Y}_k which will be given in (6.14) in Theorem 1 below. Vector \underline{Y}_k is one possible multivariate extension of Y_k in (6.6). In general the distribution of \underline{Y}_k is difficult to obtain, but the asymptotic distribution will be obtained for the case in which the process is in control. To simplify notation, let

$$Z_{il} = \sum_{j=1}^n \left(\frac{X_{ijl} - \bar{X}_{l,l}}{\sigma_{0l}} \right)^2 - (n-1), \text{ and } \underline{Z}_l = (Z_{l1}, \dots, Z_{lp})' \quad \text{for } l = 1, 2, \dots, p, i = 1, 2, \dots \quad (6.9)$$

then the multivariate EWMA vectors can be expressed as

$$\underline{Y}_k = (1 - \lambda) \underline{Y}_{k-1} + \lambda \underline{Z}_k \quad (6.10)$$

By repeated substitution in (6.10), it can be shown that

$$\underline{Y}_k = \sum_{i=1}^k \lambda (1 - \lambda)^{k-i} \underline{Z}_i \quad (6.11)$$

It is easy to show that

$$E(Z_{il} | \mu_l = \mu_{0l}, \sigma_l = \sigma_{il}) = (n-1) \left[\left(\frac{\sigma_{il}}{\sigma_{0l}} \right)^2 - 1 \right] \quad l = 1, 2, \dots, p \quad (6.12)$$

Thus, under the assumption that $\underline{\mu} = \underline{\mu}_0$ and $\Sigma = \Sigma_1$, the expected value of the random vector \underline{Z}_l denoted by $\underline{\mu}_Z$ is

$$\underline{\mu}_Z = (n-1) \left[\left(\frac{\sigma_{11}}{\sigma_{01}} \right)^2 - 1, \left(\frac{\sigma_{12}}{\sigma_{02}} \right)^2 - 1, \dots, \left(\frac{\sigma_{1p}}{\sigma_{0p}} \right)^2 - 1 \right] \quad (6.13)$$

If $\Sigma = \Sigma_0$, then $\underline{\mu}_Z = \underline{0}$

THEOREM 6.1

The variance-covariance matrix of \underline{Y}_k when a process is in control and $\underline{Y}_0 = \underline{0}$ is

$$\Sigma_{\underline{Y}_k} = \left\{ \frac{\lambda[1 - (1 - \lambda)^{2k}]}{(2 - \lambda)} \right\} \Sigma_Z \quad (6.14)$$

$$\text{where, } \Sigma_Z = 2(n - 1)R^{(2)} \quad (6.15)$$

where $R^{(2)}$ is used to denote the matrix whose (l, l') th element is the 2nd power of the (l, l') th component of R which is the correlation matrix of $\underline{X} = (X_1, \dots, X_p)$.

PROOF. The proof is similar to that of THEOREM 7.1 except $n-1$ is used instead of n .

The following theorem gives the asymptotic distribution of \underline{Y}_k given by (6.11) when some conditions are satisfied.

THEOREM 6.2

Let p -component vectors $\underline{X}_1, \underline{X}_2, \dots$ be independently identically distributed according to $N_p(\underline{\mu}, \Sigma)$. Then $\{\Sigma_{\underline{Y}_k}^{-1/2} \underline{Y}_k, k \geq 1\}$ converges in distribution to a multivariate normal distribution with mean vector $\underline{0}$ and variance-covariance matrix I_p as $k \rightarrow \infty, \lambda \rightarrow 0$ and $k \lambda \rightarrow 1$

PROOF. The proof is similar to that of THEOREM 7.2 except $n-1$ is used instead of n .

COROLLARY 6.2

$\{T_k^2, k \geq 1\}$ converges in distribution to a chi-squared distribution with p degrees of freedom as $k \rightarrow \infty, \lambda \rightarrow 0$ and $k \lambda \rightarrow 1$

PROOF. The proof is similar to that of the COROLLARY 7.2 except $n-1$ is used instead of n .

The multivariate EWMA chart signals that the process is out-of-control whenever $T_k^2 \geq h$.

For the VSI multivariate EWMA chart, suppose that

d_1 is used when $T_k^2 \in (g, h)$

d_2 is used when $T_k^2 \in (0, g]$

The ATS and parameters h and g can be obtained by using simulations.

6.5 Numerical Results

In this Section the following control procedures are compared on the basis of their ATS performance.

1. Multivariate EWMA charts with control statistic given by (6.4)
2. Multivariate EWMA charts with control statistic given by (6.8)

The performance of the charts for monitoring a variance-covariance matrix depends on the value of Σ . It was not possible to investigate all of the different ways in which Σ could change, but the following types of shifts were considered:

(V1) all variances and covariances change by a constant factor i.e. $\Sigma_1 = c \Sigma_0$

(V2) one variance increased to $ci \sigma_{0ii}$ and the other variances remain on target

(V3) approximately half of the variances and covariances change by a constant and there are no shifts in the rest

In order to compare a VSI chart with a FSI chart, the standard for comparison is the same as that used in Section (5.5). The sample size used for each sample observation was five. It is assumed that the correlation coefficient ρ is the same for all variables. Table 6.1 gives the values of h and g for $p = 2-5$ and λ when the ATS at $\Sigma = \Sigma_0$ is approximately 200. In Tables 6.2-6.8, the standard errors are shown in parentheses. and ci represents the constant factor by which σ_{0ii} changes. ATS values for the EWMA charts based on the combined-accumulate approach depend only on τ and c when $\Sigma_1 = c\Sigma_0$.

For the multivariate EWMA chart using (6.4) based on the combine-accumulate approach, the parameters h and g were obtained to give specified ANSS and ATS values when the process is in control. These parameters were obtained using Markov chains with $r = 100$, and the ATS values for various shifts in Σ were calculated by using 10,000 simulations. For $p = 2$, and three different correlation coefficients $\rho = 0.0, 0.5, 0.8$, Tables 6.2-6.4 give FSI and VSI ATS values. The results in Tables 6.2-6.4 show that except for a very small shift in Σ , larger values of λ are more effective in detecting shifts in the variance-covariance matrix. The VSI multivariate EWMA chart has smaller ATS values than the corresponding FSI multivariate EWMA chart except for $\lambda = 0.05$ and 0.1 .

For the multivariate EWMA chart using (6.8) based on the accumulate-combine approach, ATS values and parameters h and g were calculated by using 10,000 simulations. For $p = 2$, and three different correlation coefficients $\rho = 0.0, 0.5, 0.8$, Tables 6.5-6.7 give FSI and VSI ATS values. As shown in Tables 6.5-6.7, smaller values of λ are more effective in detecting all shifts in Σ for $p = 2$. Tables 6.2-6.7 show that the multivariate EWMA chart using (6.8) seems to be much better in detecting all shifts in Σ than the multivariate EWMA chart using (6.4). For $p = 5$, Table 6.8 give FSI and VSI ATS values for two different EWMA procedures mentioned above. The common correlation coefficient is given by $\rho = 0.5$.

The results in Table 6.2-6.8 show that The multivariate EWMA chart based on the accumulate-combine approach is better than the multivariate EWMA chart based on the combine-accumulate approach in terms of ATS. For $p = 2, 5$ and two different EWMA procedures, the results showed that ATS values for $\rho = -0.5$, and -0.8 are very close to those for $\rho = 0.5$, and 0.8 , respectively.

Table 6.1: Values of h and g for Various Values of λ and $\rho = 2-5$ when the ATS at $\Sigma = \Sigma_0$ is approximately 200

p	ρ	λ	<i>EWMA</i> using (6.4)	<i>EWMA</i> using (6.8)
2	0.0	0.30	12.8840 7.7161	14.0154 1.2730
		0.10	10.0348 7.6788	9.4105 1.2790
		0.05	9.0620 5.5198	7.7675 1.2390
2	0.5	0.30	12.8840 7.7161	14.3502 1.2560
		0.10	10.0348 7.6788	9.5307 1.2710
		0.05	9.0620 7.5198	7.8020 1.2210
2	0.8	0.30	12.8840 7.7161	14.3502 1.2560
		0.10	10.0348 7.6788	9.5307 1.2710
		0.05	9.0620 7.5198	7.8020 1.2210
3	0.5	0.10	14.4509 11.5882	
		0.05		10.0917 2.1340
4	0.5	0.10	18.8011 15.5040	
		0.05		12.1742 3.0670
5	0.5	0.10	23.1094 19.4234	
		0.05		14.0632 3.9810

The top number in each cell is h

The bottom number in each cell is g

Table 6.2: ATS Values for Matched FSI and VSI EWMA Charts Using (6.4) for the Covariance Matrix ($p = 2, \rho = 0.5$)

shift	$\lambda = 0.05$		$\lambda = 0.10$		$\lambda = 0.30$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
$c = 1.00$	200.000	200.000	200.000	200.000	200.000	200.000
$c = 1.21$	50.848 (0.156)	60.522 (0.137)	36.746 (0.180)	35.189 (0.124)	33.363 (0.271)	22.316 (0.161)
$c = 1.44$	30.885 (0.069)	41.284 (0.083)	19.667 (0.066)	22.030 (0.063)	13.342 (0.087)	9.139 (0.048)
$c = 1.69$	22.482 (0.045)	31.513 (0.061)	13.728 (0.039)	16.538 (0.046)	7.951 (0.043)	6.010 (0.029)
$c = 1.96$	17.645 (0.034)	25.292 (0.050)	10.585 (0.029)	13.161 (0.037)	5.683 (0.027)	4.591 (0.022)
$c = 2.25$	14.444 (0.028)	20.911 (0.042)	8.572 (0.023)	10.868 (0.031)	4.430 (0.020)	3.781 (0.018)
$c = 2.56$	12.851 (0.025)	18.671 (0.039)	7.585 (0.017)	9.669 (0.028)	3.877 (0.108)	3.372 (0.016)
$cl = 1.21$	77.011 (0.329)	82.373 (0.239)	63.823 (0.404)	56.383 (0.279)	66.471 (0.596)	51.262 (0.440)
$cl = 1.44$	47.921 (0.139)	58.403 (0.134)	34.045 (0.162)	33.565 (0.117)	29.052 (0.235)	19.886 (0.141)
$cl = 1.69$	35.458 (0.087)	46.330 (0.099)	23.180 (0.089)	25.072 (0.076)	16.384 (0.116)	11.401 (0.067)
$cl = 1.96$	27.940 (0.063)	38.206 (0.079)	17.534 (0.058)	20.434 (0.060)	11.144 (0.070)	8.160 (0.044)
$cl = 2.25$	23.100 (0.050)	32.251 (0.068)	14.152 (0.044)	17.071 (0.051)	8.226 (0.048)	6.412 (0.033)
$cl = 2.56$	18.960 (0.040)	27.142 (0.059)	11.497 (0.035)	14.240 (0.044)	6.248 (0.033)	5.114 (0.026)

Table 6.3: ATS Values for Matched FSI and VSI EWMA Charts Using (6.4) for the Covariance Matrix ($p = 2, \rho = 0.0$)

shift	$\lambda = 0.05$		$\lambda = 0.10$		$\lambda = 0.30$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
c1 = 1.21	77.943 (0.336)	82.969 (0.243)	64.785 (0.415)	57.050 (0.283)	67.667 (0.612)	52.291 (0.453)
c1 = 1.44	48.973 (0.148)	59.331 (0.136)	34.892 (0.166)	34.252 (0.120)	30.538 (0.248)	20.750 (0.149)
c1 = 1.69	36.464 (0.090)	47.415 (0.101)	24.035 (0.093)	25.784 (0.079)	17.336 (0.125)	11.916 (0.071)
c1 = 1.96	29.070 (0.067)	39.420 (0.081)	18.326 (0.062)	21.047 (0.062)	11.964 (0.077)	8.548 (0.046)
c1 = 2.25	24.194 (0.052)	33.685 (0.070)	14.876 (0.047)	17.795 (0.052)	8.862 (0.052)	6.736 (0.035)
c1 = 2.56	20.596 (0.044)	29.240 (0.062)	12.547 (0.038)	15.364 (0.046)	7.014 (0.038)	5.627 (0.028)

Table 6.4: ATS Values for Matched FSI and VSI EWMA Charts Using (6.4) for the Covariance Matrix ($p = 2, \rho = 0.8$)

shift	$\lambda = 0.05$		$\lambda = 0.10$		$\lambda = 0.30$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
c1 = 1.21	73.906 (0.308)	80.205 (0.228)	60.264 (0.374)	53.804 (0.259)	61.324 (0.543)	47.036 (0.397)
c1 = 1.44	44.316 (0.128)	55.184 (0.126)	30.542 (0.137)	31.152 (0.105)	24.520 (0.194)	17.051 (0.117)
c1 = 1.69	31.609 (0.077)	42.246 (0.092)	20.233 (0.073)	22.685 (0.069)	13.347 (0.089)	9.755 (0.055)
c1 = 1.96	24.265 (0.054)	33.859 (0.073)	14.953 (0.049)	17.947 (0.055)	8.790 (0.052)	6.881 (0.036)
c1 = 2.25	19.415 (0.043)	27.677 (0.062)	11.764 (0.037)	14.645 (0.046)	6.445 (0.035)	5.361 (0.028)
c1 = 2.56	14.296 (0.032)	20.736 (0.049)	8.531 (0.026)	10.890 (0.036)	4.446 (0.022)	3.927 (0.021)

Table 6.5: ATS Values for Matched FSI and VSI EWMA Charts Using (6.8) for the Covariance Matrix ($p = 2, \rho = 0.5$)

shift	$\lambda = 0.05$		$\lambda = 0.10$		$\lambda = 0.30$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
c = 1.00	200.558 (2.214)	199.270 (2.299)	200.502 (2.094)	199.998 (2.135)	200.228 (2.035)	199.762 (2.043)
c = 1.21	24.945 (0.260)	19.367 (0.229)	27.844 (0.281)	22.507 (0.251)	38.632 (0.383)	33.480 (0.347)
c = 1.44	8.659 (0.085)	5.990 (0.068)	9.835 (0.093)	6.621 (0.072)	13.496 (0.127)	9.368 (0.096)
c = 1.69	4.720 (0.043)	3.337 (0.035)	5.267 (0.047)	3.500 (0.035)	6.911 (0.060)	4.252 (0.041)
c = 1.96	3.097 (0.026)	2.286 (0.022)	3.438 (0.028)	2.361 (0.022)	4.392 (0.036)	2.646 (0.024)
c = 2.25	2.330 (0.018)	1.775 (0.015)	2.558 (0.020)	1.833 (0.015)	3.119 (0.024)	1.956 (0.018)
c = 2.56	1.901 (0.013)	1.509 (0.011)	2.052 (0.015)	1.551 (0.012)	2.428 (0.017)	1.614 (0.012)
c1 = 1.21	37.399 (0.385)	29.779 (0.351)	42.487 (0.422)	35.674 (0.390)	61.724 (0.614)	55.716 (0.572)
c1 = 1.44	13.541 (0.129)	9.471 (0.107)	15.273 (0.143)	10.772 (0.116)	22.353 (0.215)	16.927 (0.173)
c1 = 1.69	7.381 (0.068)	5.085 (0.055)	8.249 (0.074)	5.465 (0.057)	11.403 (0.103)	7.570 (0.075)
c1 = 1.96	4.865 (0.043)	3.456 (0.035)	5.396 (0.046)	3.650 (0.036)	6.978 (0.060)	4.403 (0.042)
c1 = 2.25	3.564 (0.029)	2.627 (0.025)	3.939 (0.032)	2.741 (0.025)	4.918 (0.041)	3.077 (0.028)
c1 = 2.56	2.849 (0.023)	2.145 (0.019)	3.110 (0.025)	2.217 (0.020)	3.789 (0.030)	2.412 (0.021)

Table 6.6: ATS Values for Matched FSI and VSI EWMA Charts Using (6.8) for the Covariance Matrix ($p = 2, \rho = 0.0$)

shift	$\lambda = 0.05$		$\lambda = 0.10$		$\lambda = 0.30$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
$c = 1.00$	199.456 (2.182)	200.004 (2.288)	199.637 (2.070)	199.941 (2.120)	199.531 (2.018)	200.003 (2.040)
$c = 1.21$	21.652 (0.220)	16.497 (0.194)	24.391 (0.240)	19.176 (0.213)	34.406 (0.332)	29.361 (0.298)
$c = 1.44$	7.573 (0.073)	5.214 (0.057)	8.472 (0.077)	5.630 (0.059)	11.742 (0.109)	7.923 (0.080)
$c = 1.69$	4.112 (0.036)	2.930 (0.030)	4.595 (0.039)	3.065 (0.031)	5.965 (0.051)	3.588 (0.034)
$c = 1.96$	2.718 (0.022)	2.016 (0.018)	3.009 (0.024)	2.097 (0.018)	3.790 (0.030)	2.297 (0.020)
$c = 2.25$	2.416 (0.019)	1.853 (0.016)	2.620 (0.020)	1.906 (0.016)	3.192 (0.025)	2.043 (0.017)
$c = 2.56$	2.074 (0.015)	1.640 (0.013)	2.237 (0.016)	1.671 (0.013)	2.716 (0.020)	1.753 (0.013)
$c1 = 1.21$	38.427 (0.395)	31.516 (0.369)	42.903 (0.431)	36.657 (0.404)	60.711 (0.594)	55.717 (0.562)
$c1 = 1.44$	14.012 (0.136)	10.121 (0.118)	15.778 (0.150)	11.332 (0.124)	22.489 (0.215)	17.650 (0.180)
$c1 = 1.69$	7.671 (0.072)	5.409 (0.060)	8.499 (0.077)	5.860 (0.062)	11.717 (0.108)	8.030 (0.081)
$c1 = 1.96$	5.043 (0.045)	3.683 (0.039)	5.586 (0.048)	3.852 (0.039)	7.165 (0.063)	4.648 (0.046)
$c1 = 2.25$	3.701 (0.031)	2.732 (0.027)	4.055 (0.034)	2.839 (0.027)	5.067 (0.042)	3.248 (0.030)
$c1 = 2.56$	2.918 (0.024)	2.239 (0.021)	3.180 (0.025)	2.316 (0.021)	3.863 (0.031)	2.510 (0.022)

Table 6.7: ATS Values for Matched FSI and VSI EWMA Charts Using (6.8) for the Covariance Matrix ($\rho = 2, \rho = 0.8$)

shift	$\lambda = 0.05$		$\lambda = 0.10$		$\lambda = 0.30$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
$c = 1.00$	200.052 (2.197)	198.690 (2.289)	201.711 (2.131)	201.220 (2.172)	200.830 (2.044)	200.663 (2.054)
$c = 1.21$	28.528 (0.305)	23.204 (0.281)	31.513 (0.328)	26.442 (0.302)	41.983 (0.422)	36.865 (0.386)
$c = 1.44$	10.123 (0.104)	7.126 (0.084)	11.410 (0.111)	7.910 (0.088)	15.456 (0.149)	11.148 (0.116)
$c = 1.69$	5.437 (0.053)	3.815 (0.041)	6.048 (0.056)	4.063 (0.042)	7.908 (0.072)	5.054 (0.050)
$c = 1.96$	3.545 (0.032)	2.581 (0.026)	3.968 (0.035)	2.731 (0.027)	4.991 (0.043)	3.057 (0.028)
$c = 2.25$	2.625 (0.022)	1.997 (0.018)	2.887 (0.024)	2.058 (0.018)	3.547 (0.029)	2.223 (0.019)
$c = 2.56$	2.118 (0.016)	1.666 (0.014)	2.295 (0.018)	1.709 (0.014)	2.737 (0.021)	1.786 (0.014)
$c1 = 1.21$	27.927 (0.276)	20.171 (0.236)	33.019 (0.326)	25.001 (0.275)	52.016 (0.513)	43.931 (0.447)
$c1 = 1.44$	9.862 (0.090)	6.390 (0.070)	11.325 (0.100)	7.164 (0.074)	17.006 (0.159)	11.205 (0.113)
$c1 = 1.69$	5.434 (0.047)	3.600 (0.036)	6.107 (0.051)	3.848 (0.037)	8.259 (0.072)	4.818 (0.046)
$c1 = 1.96$	3.632 (0.030)	2.526 (0.024)	4.032 (0.032)	2.639 (0.024)	5.206 (0.042)	3.009 (0.026)
$c1 = 2.25$	2.743 (0.021)	1.967 (0.017)	3.001 (0.023)	2.049 (0.017)	3.697 (0.028)	2.221 (0.018)
$c1 = 2.56$	2.222 (0.016)	1.697 (0.014)	2.410 (0.017)	1.744 (0.014)	2.844 (0.021)	1.818 (0.014)

Table 6.8: ATS Values for Three Matched FSI and VSI EWMA Charts for the Covariance Matrix ($p = 5, \rho = 0.5$)

shift	<i>EWMA using (6.4)</i>		<i>EWMA using (6.8)</i>	
	$\lambda = 0.10$		$\lambda = 0.05$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
c = 1.00	200.000	200.000	200.402 (2.269)	199.428 (2.353)
c = 1.21	27.849 (0.087)	32.032 (0.070)	18.562 (0.202)	12.969 (0.162)
c = 1.44	16.149 (0.034)	21.330 (0.040)	5.997 (0.062)	3.876 (0.044)
c = 1.69	11.626 (0.021)	16.103 (0.029)	3.223 (0.030)	2.164 (0.021)
c = 1.96	9.121 (0.016)	12.846 (0.023)	2.147 (0.017)	1.550 (0.012)
c = 2.25	7.468 (0.013)	10.612 (0.020)	1.654 (0.011)	1.228 (0.008)
c = 2.56	6.314 (0.011)	8.958 (0.018)	1.383 (0.008)	1.158 (0.005)
c1 = 1.21	90.194 (0.615)	80.812 (0.426)	46.265 (0.485)	35.662 (0.414)
c1 = 1.44	51.934 (0.267)	50.209 (0.171)	16.063 (0.151)	10.734 (0.119)
c1 = 1.69	35.733 (0.139)	38.429 (0.101)	8.525 (0.076)	5.701 (0.060)
c1 = 1.96	27.196 (0.087)	31.992 (0.075)	5.572 (0.048)	3.847 (0.038)
c1 = 2.25	22.132 (0.062)	27.479 (0.061)	4.027 (0.033)	2.950 (0.028)
c1 = 2.56	18.647 (0.048)	24.079 (0.053)	3.185 (0.025)	2.413 (0.022)

Table 6.8: (continued)

c1-3 = 1.21	40.811 (0.174)	41.820 (0.116)	22.734 (0.233)	15.553 (0.181)
c1-3 = 1.44	22.836 (0.061)	27.963 (0.059)	7.474 (0.071)	4.839 (0.053)
c1-3 = 1.69	16.373 (0.036)	21.589 (0.042)	4.052 (0.036)	2.705 (0.027)
c1-3 = 1.96	12.789 (0.025)	17.511 (0.034)	2.680 (0.021)	1.895 (0.016)
c1-3 = 2.25	10.454 (0.020)	14.602 (0.028)	2.031 (0.014)	1.532 (0.011)
c1-3 = 2.56	8.812 (0.016)	12.478 (0.025)	1.665 (0.011)	1.340 (0.009)

Chapter 7

Multivariate Control Charts for the Mean Vector and Variance-Covariance Matrix

7.1 Introduction

Suppose that the objective is to monitor both $\underline{\mu}$ and Σ , where the target values Σ_0 and $\underline{\mu}_0$ are known. There are several ways of monitoring $\underline{\mu}$ and Σ simultaneously. In practice, the usual procedure is probably to use separate charts for each individual mean and variance.

The control procedure using separate charts for each individual parameter may not be optimal for detecting simultaneous changes in $\underline{\mu}$ and Σ . As an alternative to using separate charts for each individual parameter, Reynolds and Ghosh (1981) proposed using a single chart based on the following statistic.

$$D_l = \sum_{j=1}^n (\underline{X}_{lj} - \underline{\mu}_0) \Sigma_0^{-1} (\underline{X}_{lj} - \underline{\mu}_0)'$$

When $\underline{\mu} = \underline{\mu}_0$ and $\Sigma = \Sigma_0$, D_i has a chi-squared distribution with np degrees of freedom. This statistic thus includes information about $\underline{\mu}$ and Σ . This procedure has the practical advantage that only one chart needs to be plotted. The statistic D_i can be interpreted as the sum of control statistics that have been proposed for $\underline{\mu}$ and Σ .

$$D_i = n(\bar{X}_i - \underline{\mu}_0)\Sigma_0^{-1}(\bar{X}_i - \underline{\mu}_0)' + \text{tr}(A_i\Sigma_0^{-1}) = Z_i^2 + V_i$$

The chart for $\underline{\mu}$ signals for large values of Z_i^2 and the chart for Σ signals for large values of V_i . Thus D_i combines the information in Z_i^2 and V_i through the sum

$$D_i = Z_i^2 + V_i$$

Note that Z_i^2 and V_i are independent and when $\underline{\mu} = \underline{\mu}_0$ and $\Sigma = \Sigma_0$, Z_i^2 is $\chi^2(p)$ and V_i is $\chi^2(n-1)p$.

Another single chart to simultaneously monitor both $\underline{\mu}$ and Σ can be constructed using the likelihood ratio statistic for testing $H_0: \underline{\mu} = \underline{\mu}_0$ and $\Sigma = \Sigma_0$ vs. $H_1: \underline{\mu} \neq \underline{\mu}_0$ or $\Sigma \neq \Sigma_0$. For the i^{th} sample ($i = 1, 2, \dots$), it is expressed as (Anderson, 1958)

$$\lambda = \left(\frac{e}{n}\right)^{\frac{1}{2}np} |A_i \Sigma_0^{-1}|^{\frac{1}{2}n} \exp\left[-\frac{1}{2} \text{tr} \Sigma_0^{-1}\{A_i + n(\bar{X}_i - \underline{\mu}_0)'(\bar{X}_i - \underline{\mu}_0)\}\right]$$

Let

$$U_i = -2 \ln \lambda = \text{tr}\{(\Sigma_0^{-1}A_i) + n(\bar{X}_i - \underline{\mu}_0)' \Sigma_0^{-1}(\bar{X}_i - \underline{\mu}_0)\} - n \ln |A_i| + n \ln |\Sigma_0| + np \ln n - np$$

The exact null distribution of λ was obtained as a chi-square series or as a beta series by Nagarsenker and Pillai (1974). From these, percentage points of λ can be computed to any degree of accuracy even for small sample sizes. Thus, the statistic U_i can be used as a control statistic for $\underline{\mu}$ and Σ . Thus three different control statistics for $\underline{\mu}$ and Σ are as follows:

1. $D_i = \sum_{j=1}^n (X_{ij} - \underline{\mu}_0)\Sigma_0^{-1}(X_{ij} - \underline{\mu}_0)'$
2. $U_i = \text{tr}\{(\Sigma_0^{-1}A_i) + n(\bar{X}_i - \underline{\mu}_0)' \Sigma_0^{-1}(\bar{X}_i - \underline{\mu}_0)\} - n \ln |A_i| + n \ln |\Sigma_0| + np \ln n - np$
3. (Z_i^2, V_i)

In general, if a change in Σ occurs then it is difficult to obtain the distributions of D_i and U_i . Thus, simulations are needed to evaluate the properties of the charts for $\underline{\mu}$ and Σ .

7.2 Shewhart Charts for the Mean Vector and Covariance Matrix

The control limits for a Shewhart chart based on the control statistic D_i would be set at $\{0, \chi^2_{1-\alpha}(np)\}$. The control limits for a Shewhart chart based on the control statistic U_i would be set by using percentage points of W_i . A Shewhart chart based on the control statistic (Z_i^2, V_i) uses separate charts for $\underline{\mu}$ and Σ (Z_i^2 for $\underline{\mu}$, and V_i for Σ) and signals if either the chart for $\underline{\mu}$ or Σ signals. The signal probabilities of the charts for $\underline{\mu}$ and Σ are denoted by $\alpha_{\underline{\mu}}$ and α_{Σ} when $\underline{\mu} = \underline{\mu}_0$ and $\Sigma = \Sigma_0$. Since Z_i^2 and V_i are independent under the assumption of multivariate normality, the overall probability is $\alpha = 1 - (1 - \alpha_{\underline{\mu}})(1 - \alpha_{\Sigma})$. To achieve an overall signal probability of α , the control limits for $\underline{\mu}$ can be set at $\{0, \chi^2_{1-\alpha_{\underline{\mu}}}(p)\}$ and the control limits for Σ can be set at $\{0, \chi^2_{1-\alpha_{\Sigma}}(n-1)p\}$, where $\chi^2_{\alpha}(p)$ denotes the α quantile of the $\chi^2(p)$ distribution. The VSI version of these charts can be constructed by using the distributions of these control statistics.

7.3 CUSUM Charts for the Mean Vector and Covariance Matrix

As in the multivariate CUSUM charts for $\underline{\mu}$ or Σ , the multivariate CUSUM charts for $\underline{\mu}$ and Σ are as follows: For the case of using D_i , the FSI and VSI multivariate CUSUM control statistic for $\underline{\mu}$ and Σ at the j^{th} sample is

$$Y_{1j} = \max \{Y_{1j-1}, 0\} + (D_j - k)$$

The multivariate CUSUM chart signals whenever $Y_{1j} > h_1$. For the case of using U_i , the FSI and VSI multivariate CUSUM control statistic for $\underline{\mu}$ and Σ at the j^{th} sample is

$$Y_{2j} = \max \{Y_{2j-1}, 0\} + (U_j - k)$$

For the VSI multivariate CUSUM charts for $\underline{\mu}$ and Σ , suppose that

d_1 is used when $Y_{kj} \in (g_k, h_k]$

d_2 is used when $Y_{kj} \in (-k, g_k]$

$k = 1, 2$ where $-k < g_k \leq h_k$. As stated in the VSI multivariate CUSUM charts for Σ , the parameters g_k and h_k can be obtained.

7.4 Multivariate EWMA Charts for the Mean vector and Covariance Matrix

We will discuss multivariate EWMA charts based on the combine-accumulate approach, multivariate EWMA charts based the accumulate-combine approach, and separate EWMA control charts for each of the means and variances.

7.4.1 Multivariate EWMA Charts Based on the Combine-accumulate

Approach

Multivariate EWMA charts based on the combine-accumulate approach can be constructed by combining multivariate data into a univariate statistic and then forming a univariate EWMA statistic. As in the MEWMA charts for Σ , for the case of using D_i , the EWMA chart is based on the control statistic

$$Y_{1i} = (1 - \lambda) Y_{1i-1} + \lambda D_i \quad (7.1)$$

where $Y_{10} = 0, 0 < \lambda \leq 1$. This chart signals when $Y_{1i} > h_1$. For the case of using U_i , the EWMA chart is based on the control statistic

$$Y_{2i} = (1 - \lambda) Y_{2i-1} + \lambda U_i$$

where $Y_{20} = 0, 0 < \lambda \leq 1$. This chart signals when $Y_{2i} > h_2$.

For the VSI multivariate EWMA charts for $\underline{\mu}$ and Σ , Suppose that

d_1 is used when $Y_{ki} \in (g_k, h_k]$

d_2 is used when $Y_{ki} \in (0, g_k]$

As stated in the VSI multivariate EWMA charts for Σ the parameters g_k and h_k can be obtained to give a specified ANSS and ATS values when the process is in control by using a Markov chain approach or integral equations.

7.4.2 Multivariate EWMA Charts Based on the Accumulate-combine

Approach

Multivariate EWMA charts based on the accumulate-combine approach can be constructed by forming a univariate statistic from a multivariate EWMA statistic. In the univariate case, Reynolds and Ghosh (1981) proposed a single chart based on the statistic

$$D_i = \sum_{j=1}^n \left(\frac{X_{ij} - \mu_0}{\sigma_0} \right)^2$$

which is the statistic D_i considered in Section 7.1 in the special case of $p=1$. When $\mu = \mu_0$ and $\sigma = \sigma_0$, C_i has a chi-squared distribution with n degrees of freedom. This sum statistic includes information about μ and σ . The statistic D_i can be interpreted as the sum of statistics used by the \bar{X} and S^2 - charts.

$$D_i = \sum_{j=1}^n \left(\frac{X_{ij} - \mu_0}{\sigma_0} \right)^2 = \frac{n(\bar{X}_i - \mu_0)^2}{\sigma_0^2} + \frac{(n-1)S_i^2}{\sigma_0^2} = Z_i^2 + V_i$$

Note that Z_i^2 and V_i are independent and when $\mu = \mu_0$ and $\sigma = \sigma_0$, Z_i^2 is $\chi^2(1)$ and V_i is $\chi^2(n-1)$.

An EWMA chart for μ and σ can be constructed by using the statistic

$$Y_k = (1 - \lambda)Y_{k-1} + \lambda \sum_{j=1}^n \left(\frac{X_{kj} - \mu_0}{\sigma_0} \right)^2 \quad (7.2)$$

By repeated substitution in (7.2), it can be shown that

$$Y_k = (1 - \lambda)^k Y_0 + \sum_{l=1}^k \lambda (1 - \lambda)^{k-l} \sum_{j=1}^n \left(\frac{X_{lj} - \mu_0}{\sigma_0} \right)^2 \quad (7.3)$$

$k = 1, 2, \dots$ and $0 < \lambda \leq 1$.

In the multivariate case, we define vectors of EWMA's

$$\underline{Y}_k = \begin{bmatrix} Y_{k1} \\ Y_{k2} \\ \cdot \\ \cdot \\ \cdot \\ Y_{kp} \end{bmatrix} = \begin{bmatrix} (1-\lambda)^k Y_{10} + \sum_{i=1}^k \lambda(1-\lambda)^{k-i} \left[\sum_{j=1}^n \left(\frac{X_{ij1} - \mu_{01}}{\sigma_{01}} \right)^2 - n \right] \\ (1-\lambda)^k Y_{20} + \sum_{i=1}^k \lambda(1-\lambda)^{k-i} \left[\sum_{j=1}^n \left(\frac{X_{ij2} - \mu_{02}}{\sigma_{02}} \right)^2 - n \right] \\ \cdot \\ \cdot \\ \cdot \\ (1-\lambda)^k Y_{p0} + \sum_{i=1}^k \lambda(1-\lambda)^{k-i} \left[\sum_{j=1}^n \left(\frac{X_{ijp} - \mu_{0p}}{\sigma_{0p}} \right)^2 - n \right] \end{bmatrix} \quad (7.4)$$

$k = 1, 2, \dots$, where $Y_0 = 0$ and $0 < \lambda \leq 1, l = 1, 2, \dots, p$.

A multivariate EWMA chart for μ and Σ can be based on the control statistic

$$T_k^2 = \underline{Y}_k' \Sigma_{\underline{Y}_k}^{-1} \underline{Y}_k \quad (7.5)$$

where $\Sigma_{\underline{Y}_k}$ is the variance-covariance matrix of \underline{Y}_k which will be given in (7.12) in THEOREM 7.1 below. The vector \underline{Y}_k is one possible multivariate extension of Y_k in (7.3). In general the distribution of \underline{Y}_k is difficult to obtain, but the asymptotic distribution will be obtained for the case in which the process is in control. To simplify notation, let

$$Z_{il} = \sum_{j=1}^n \left(\frac{X_{ijl} - \mu_{0l}}{\sigma_{0l}} \right)^2 - n, \text{ and } \underline{Z}_l = (Z_{l1}, \dots, Z_{lp})' \text{ for } l = 1, 2, \dots, p, i = 1, 2, \dots \quad (7.6)$$

then the multivariate EWMA vectors can be expressed as

$$\underline{Y}_k = (1-\lambda)\underline{Y}_{k-1} + \lambda\underline{Z}_k \quad (7.7)$$

By repeated substitution in (7.7), it can be shown that

$$\underline{Y}_k = \sum_{i=1}^k \lambda(1-\lambda)^{k-i} \underline{Z}_i \quad (7.8)$$

It is easy to show that

$$E(Z_{il} | \mu_l = \mu_{il}, \sigma_l = \sigma_{il}) = n \left[\frac{\sigma_{il}^2 + (\mu_{il} - \mu_{0l})^2}{\sigma_{0l}^2} - 1 \right] \quad l = 1, 2, \dots, p \quad (7.9)$$

If $\sigma_{il} = \sigma_{0l}$, then the right hand side of (7.9) is

$$n \left[\frac{(\mu_{il} - \mu_{0l})^2}{\sigma_{0l}^2} \right] \quad l = 1, 2, \dots, p \quad (7.10)$$

Thus, under the assumption that $\underline{\mu} = \underline{\mu}_1 = (\mu_{11}, \dots, \mu_{p1})'$ and $\Sigma = \Sigma_0$, the expected value of the random vector \underline{Z}_i denoted by $\underline{\mu}_Z$ is

$$\underline{\mu}_Z = \left[\frac{(\mu_{11} - \mu_{01})^2}{\sigma_{01}^2}, \frac{(\mu_{12} - \mu_{02})^2}{\sigma_{02}^2}, \dots, \frac{(\mu_{1p} - \mu_{0p})^2}{\sigma_{0p}^2} \right] \quad (7.11)$$

THEOREM 7.1

The variance-covariance matrix of \underline{Y}_k when a process is in control and $\underline{Y}_0 = \underline{0}$ is

$$\Sigma_{\underline{Y}_k} = \left\{ \frac{\lambda[1 - (1-\lambda)^{2k}]}{(2-\lambda)} \right\} \Sigma_Z \quad (7.12)$$

where, $\Sigma_Z = (2n) R^{(2)}$

where $R^{(2)}$ is used to denote the matrix whose (l, l') th component is the 2nd power of the (l, l') th component of R which is the correlation matrix of $\underline{X} = (X_1, \dots, X_p)$.

PROOF. It is easy to show that

$$\Sigma_{\underline{Y}_k} = \sum_{i=1}^k Cov[\lambda(1-\lambda)^{k-i} \underline{Z}_i] = \left\{ \frac{\lambda[1-(1-\lambda)^{2k}]}{(2-\lambda)} \right\} \Sigma_{\underline{Z}}$$

When a process is in control, the mean vector and variance-covariance matrix of \underline{Z}_i is derived as follows: recall that $Z_{il} + n = \sum_{j=1}^n \left(\frac{X_{ijl} - \mu_{0l}}{\sigma_{0l}} \right)^2$ has a chi-squared distribution with n degrees of freedom because X_{i1}, \dots, X_{in} is a random sample taken at the i^{th} sampling point from a multivariate normal distribution with mean vector $\underline{\mu}$ and variance-covariance matrix Σ . Thus

$$E(Z_{il}) = 0$$

$$\text{Var}(Z_{il}) = 2n \quad \text{for } l = 1, 2, \dots, p, \quad i = 1, 2, \dots$$

Now

$$\begin{aligned} Cov[Z_{il}, Z_{lr}] &= Cov[Z_{il} + n, Z_{lr} + n] \\ &= Cov\left[\sum_{j=1}^n \left(\frac{X_{ijl} - \mu_{0l}}{\sigma_{0l}} \right)^2, \sum_{j=1}^n \left(\frac{X_{ijr} - \mu_{0r}}{\sigma_{0r}} \right)^2 \right] \\ &= n Cov\left[\left(\frac{X_{ijl} - \mu_{0l}}{\sigma_{0l}} \right)^2, \left(\frac{X_{ijr} - \mu_{0r}}{\sigma_{0r}} \right)^2 \right] \end{aligned}$$

For simplicity, let

$$U = \frac{X_{ijl} - \mu_{0l}}{\sigma_{0l}}, \quad \text{and} \quad V = \frac{X_{ijr} - \mu_{0r}}{\sigma_{0r}}$$

Then, it can be expressed that $(U, V) \sim N_2(0, 0, 1, 1, \rho_{u,v})$. Thus

$$\begin{aligned} Cov(Z_{il}, Z_{lr}) &= n Cov(U^2, V^2) \\ &= n[E(U^2 V^2) - E(U^2) E(V^2)] \end{aligned}$$

By using the moment generating function of the bivariate normal distribution, it can be shown that

$$E(U^2V^2) = 1 + 2\rho^2_{u,v}$$

Therefore

$$\begin{aligned} E(\underline{Z}_l) &= \underline{0} \\ \Sigma_{\underline{Z}} &= \text{Cov}(\underline{Z}_l) = (2n)R^{(2)} \end{aligned} \tag{7.13}$$

The following theorem gives the asymptotic distribution of \underline{Y}_k given by (7.7) when the process is in control.

THEOREM 7.2

Let the p -component vectors $\underline{X}_1, \underline{X}_2, \dots$ be independently identically distributed according to $N_p(\underline{\mu}_0, \Sigma_0)$. Then $\{\Sigma_{\underline{Y}_k}^{-1/2} \underline{Y}_k, k \geq 1\}$ converges in distribution to a multivariate normal distribution with mean vector $\underline{0}$ and variance-covariance matrix I_p as $k \rightarrow \infty, \lambda \rightarrow 0$ and $k\lambda \rightarrow 1$

PROOF. Recall that \underline{Y}_k is

$$\underline{Y}_k = (1 - \lambda)\underline{Y}_{k-1} + \lambda\underline{Z}_k \quad k = 1, 2, \dots$$

where $\underline{Y}_0 = \underline{0}$, and $0 < \lambda \leq 1$. \underline{Y}_k can also be expressed as

$$\underline{Y}_k = \sum_{i=1}^k \lambda(1 - \lambda)^{k-i} \underline{Z}_i$$

For $k \geq 1$, let

$$A_k = \frac{1}{k} \sum_{i=1}^k \text{Cov}(\lambda(1 - \lambda)^{k-i} \underline{Z}_i)$$

B_k is the symmetric, positive definite matrix satisfying $B_k^2 = A_k^{-1}$

$\gamma_k =$ smallest eigenvalue of A_k

By the corollaries 18.2 and 18.3 of Bhattacharya and Rao (1975), if

$$\Theta_k(\lambda) = k^{-3/2} \sum_{i=1}^k E \|B_k \lambda (1-\lambda)^{k-i} \underline{Z}_i\|^3 \longrightarrow 0 \quad \text{as } k \rightarrow \infty, \lambda \rightarrow 0 \text{ and } k\lambda \rightarrow 1$$

then

$$k^{-1/2} B_k \sum_{i=1}^k \lambda (1-\lambda)^{k-i} \underline{Z}_i \xrightarrow{d} N_p(Q, I) \quad \text{as } k \rightarrow \infty, \lambda \rightarrow 0 \text{ and } k\lambda \rightarrow 1$$

The inequality given in equation (17.63) of Bhattacharya and Rao (1975) is

$$\|B_k \underline{Z}_i\| \leq \gamma_k^{-1/2} \|\underline{Z}_i\| \quad , 1 \leq i \leq k$$

and this gives

$$\|B_k \underline{Z}_i\|^3 \leq \gamma_k^{-3/2} \|\underline{Z}_i\|^3 \quad , 1 \leq i \leq k$$

Thus

$$\begin{aligned} \Theta_k(\lambda) &= k^{-3/2} \sum_{i=1}^k E \|B_k \lambda (1-\lambda)^{k-i} \underline{Z}_i\|^3 \\ &= k^{-3/2} \lambda^3 \sum_{i=1}^k (1-\lambda)^{3(k-i)} E \|B_k \underline{Z}_i\|^3 \\ &\leq k^{-3/2} \lambda^3 \sum_{i=1}^k (1-\lambda)^{3(k-i)} \gamma_k^{-3/2} E \|\underline{Z}_i\|^3 \end{aligned} \quad (7.14)$$

Now

$$A_k = \frac{1}{k} \Sigma_{Y_k} = \frac{1}{k} \left\{ \frac{\lambda [1 - (1-\lambda)^{2k}]}{(2-\lambda)} \right\} \Sigma_Z$$

Let $\gamma =$ smallest eigenvalue of Σ_Z , then the smallest eigenvalue of A_k can be expressed as

$$\gamma_k = \left\{ \frac{1}{k} \frac{\lambda[1 - (1 - \lambda)^{2k}]}{(2 - \lambda)} \right\} \gamma$$

Thus, the right hand side of inequality (7.14) is less than or equal to

$$\left\{ \left(\frac{\lambda(1 - (1 - \lambda)^{2k})}{(2 - \lambda)} \right) \gamma \right\}^{-3/2} \lambda^3 \sum_{i=1}^k (1 - \lambda)^{3(k-i)} E \|Z_{ii}\|^3 \quad (7.15)$$

By using the inequality given by Chung (1974, p.48), it is easy to show that

$$\begin{aligned} E \|Z_l\|^3 &= E \left[\sum_{i=1}^p Z_{li}^2 \right]^{3/2} \\ &\leq p^{1/2} \sum_{i=1}^p E |Z_{li}|^3 \\ &= p^{3/2} E |Z_{li}|^3 \quad , l = 1, 2, \dots, p \end{aligned}$$

Let $m_3 = E |Z_{li}|^3 < \infty, l = 1, 2, \dots, p$. Thus, the quantity (7.15) is less than or equal to

$$\left(\frac{p}{\gamma} \right)^{3/2} m_3 \frac{\lambda^{1/2} (1 - (1 - \lambda)^{3k})}{(\lambda^2 - 3\lambda + 3)} \left[\frac{(2 - \lambda)}{1 - (1 - \lambda)^{2k}} \right]^{3/2} \longrightarrow 0 \text{ as } k \rightarrow \infty, \lambda \rightarrow 0 \text{ and } k\lambda \rightarrow 1$$

Therefore

$$k^{-1/2} B_k \sum_{i=1}^k \lambda(1 - \lambda)^{k-i} \underline{Z}_i = \Sigma_{Y_k}^{-1/2} Y_k \xrightarrow{d} N_p(0, I) \text{ as } k \rightarrow \infty, \lambda \rightarrow 0 \text{ and } k\lambda \rightarrow 1$$

COROLLARY 7.2

$\{T_k^2, k \geq 1\}$ converges in distribution to a chi-squared distribution with p degrees of freedom as $k \rightarrow \infty, \lambda \rightarrow 0$ and $k\lambda \rightarrow 1$

PROOF. Recall that the control statistic T_k^2 is

$$T_k^2 = \underline{Y}_k' \underline{\Sigma}_{\underline{Y}_k}^{-1} \underline{Y}_k$$

which can be expressed as

$$T_k^2 = (\underline{\Sigma}_{\underline{Y}_k}^{-1/2} \underline{Y}_k)' (\underline{\Sigma}_{\underline{Y}_k}^{-1/2} \underline{Y}_k)$$

By THEOREM 7.2 and the corollary of Serfling (1980, p.25),

$$T_k^2 = \underline{Y}_k' \underline{\Sigma}_{\underline{Y}_k}^{-1} \underline{Y}_k \xrightarrow{d} \chi^2(p) \quad \text{as } k \rightarrow \infty, \lambda \rightarrow 0 \text{ and } k \lambda \rightarrow 1$$

The multivariate EWMA chart signals that the process is out-of-control whenever $T_k^2 \geq h$.

For the VSI multivariate EWMA chart, suppose that

d_1 is used when $T_k^2 \in (g, h)$

d_2 is used when $T_k^2 \in (0, g]$

The ATS and parameters h and g can be obtained by using simulations.

Since the distribution of the vector \underline{Y} given by (5.4) is multivariate normal, the ATS performance of the MEWMA chart for $\underline{\mu}$ depends on the mean vector $\underline{\mu}$ and variance-covariance matrix $\underline{\Sigma}$ only through the value of the noncentrality parameter τ . In THEOREM 7.2, we proved that the asymptotic distribution of \underline{Y}_k given by (7.8) is multivariate normal under conditions given by THEOREM 7.2. The parameters h and g for (7.5) which give a specified large ATS are close to h and g for the MEWMA based on numerical results under conditions given by THEOREM 7.2. This result means that the h and g values for the MEWMA charts can be used as approximate h and g values for the multivariate EWMA chart using (7.5) under large in-control ATS values and conditions given by THEOREM 7.2. These numerical results will be given in Section 7.6.

7.4.3 Separate Charts for Each of the Means and Variances

Consider a procedure which uses separate EWMA charts for each of the means and variances ($2p$ EWMA charts, one for each μ_i and one for each σ_i^2 , $i = 1, 2, \dots, p$) and signals if any of the individual charts signal. An FSI EWMA chart for each μ_i gives an out-of-control signal as soon as

$$Y_{it}^{(\mu)} = (1 - \lambda) Y_{i(t-1)}^{(\mu)} + \lambda \frac{\sqrt{n} (\bar{X}_{i,t} - \mu_{0i})}{\sigma_{0i}} \quad | > h_i^{(\mu)}$$

An FSI EWMA chart for each σ_i^2 gives an out-of-control signal as soon as

$$Y_{it}^{(\sigma)} = (1 - \lambda) Y_{i(t-1)}^{(\sigma)} + \lambda \frac{(n-1) S_{it}^2}{\sigma_{0i}^2} \quad > h_i^{(\sigma)}$$

$$i = 1, 2, \dots, \text{ where } S_{it}^2 = \sum_{j=1}^n \frac{(X_{ijt} - \bar{X}_{i,t})^2}{n-1}, \quad 0 < \lambda \leq 1, \text{ and } i = 1, 2, \dots, p$$

For the VSI EWMA charts for each μ_i , suppose that

$$d_1 \text{ is used when } | Y_{it}^{(\mu)} | \in (g_i^{(\mu)}, h_i^{(\mu)})$$

$$d_2 \text{ is used when } | Y_{it}^{(\mu)} | \in (0, g_i^{(\mu)}]$$

For the VSI EWMA charts for each σ_i^2 , suppose that

$$d_1 \text{ is used when } Y_{it}^{(\sigma)} \in (g_i^{(\sigma)}, h_i^{(\sigma)})$$

$$d_2 \text{ is used when } Y_{it}^{(\sigma)} \in (0, g_i^{(\sigma)}]$$

If the desired ANSS and ATS values of each EWMA chart are specified, then the parameters $g_i^{(\mu)}$, $g_i^{(\sigma)}$, $h_i^{(\mu)}$, and $h_i^{(\sigma)}$ can be obtained by using the Markov chain or integral equation approach.

If the VSI feature is to be applied using $2p$ separate EWMA charts, there are several sampling interval functions that are viable. It is assumed that the sampling interval used is the minimum of

the sampling intervals assigned by the $2p$ individual charts. The performance of the separate charts depends on the value of $\underline{\mu}$ and $\underline{\Sigma}$. Therefore various ways in which the shifts in means and variance-covariance matrix might take place, will be considered in Section 7.5.

If $\lambda = 1$, then each EWMA chart is a Shewhart chart. If $\alpha = P$ (signal for the separate shewhart charts), then the probability that any of the $2p$ Shewhart charts signal is denoted by α_{2p} where $\alpha_{2p} = 1 - (1 - \alpha)^{(1/2p)}$ when the p variables are independent. Thus, let

ATS (ind) = ATS for each chart

ATS (joint) = ATS for the separate charts used simultaneously.

Then, the ATS (ind) is approximately the product of $2p$ and ATS (joint) when a process is in control. If p variables are correlated and a process is in control then the ATS (ind) can be approximately calculated to get a given ATS (joint) by the Bonferroni's inequality. In the EWMA chart, it may be difficult to calculate an exact ATS (ind) to get a fixed ATS (joint) when a process is in control. Under certain conditions, it is possible to calculate an exact ATS (ind) to get a fixed ATS (joint) by using simulations when a process is in control. For simplicity, when a process is in control it is assumed that

- (1) all individual EWMA charts used have the same ATS.
- (2) all individual EWMA charts for μ and σ have the same control limits, respectively.
- (3) all individual VSI EWMA charts for μ and σ have the same g_1 and g_2 , respectively.

Under the above conditions, the ATS (joint) will be estimated.

7.4.4 Multivariate EWMA Charts Based on both MEWMA and EWMA using (7.5)

Consider a procedure which uses separate multivariate EWMA charts for $\underline{\mu}$ and Σ (2 EWMA charts, one for $\underline{\mu}$ and one for Σ) and signals if either of these two EWMA charts signals. From the numerical results of Chapters 5 and 7, the MEWMA chart and multivariate EWMA chart using (7.5) perform well in detecting shifts in the mean vector and variance-covariance matrix, respectively. Thus two separate EWMA charts based on the accumulate-combine approach, which are the MEWMA chart for $\underline{\mu}$ and multivariate EWMA chart using (7.5) for Σ , are used to simultaneously monitor the mean vector and variance-covariance matrix.

The MEWMA chart for $\underline{\mu}$ gives an out-of-control signal as soon as

$$T_i^2 = (\underline{Y}_i - \underline{\mu}_0) \Sigma_{\underline{Y}_i}^{-1} (\underline{Y}_i - \underline{\mu}_0)' > h_1$$

where \underline{Y}_k was given by (5.4). The multivariate EWMA chart for Σ gives an out-of-control signal as soon as

$$T_k^2 = \underline{Y}_k' \Sigma_{\underline{Y}_k}^{-1} \underline{Y}_k > h_2$$

where \underline{Y}_k was given by (7.7).

For the VSI MEWMA chart for $\underline{\mu}$, suppose that

d_1 is used when $\underline{Y}_i \in (g_1, h_1)$

d_2 is used when $\underline{Y}_i \in (0, g_1]$

For the VSI multivariate EWMA chart using (7.5) for Σ , suppose that

d_1 is used when $\underline{Y}_k \in (g_2, h_2)$

d_2 is used when $\underline{Y}_k \in (0, g_2]$

For simplicity, when a process is in control it is assumed that these two EWMA charts mentioned above have the same ANSS and ATS. When using the VSI feature, there are several possible decision rules that could be used. If shifts in $\underline{\mu}$ are more likely than shifts in Σ , the sampling interval could depend only on the chart for $\underline{\mu}$. If the sampling interval is to depend on both charts then the minimum of the two sampling intervals assigned by separate charts for $\underline{\mu}$ and Σ could be used. The ATS and parameters $h1, h2, g1$ and $g2$ can be obtained by using simulations.

7.4.5 MEWMA Charts

In the univariate case, the \bar{X} -chart signals for large values of $|\bar{X}_i - \mu_0|$ which is equivalent to signaling for large values of Z^2 where

$$Z^2 = \frac{n(\bar{X} - \mu_0)^2}{\sigma_0^2}$$

has a chi-square distribution with 1 degree of freedom. If μ changes, the distribution of Z^2 depends only on the noncentrality parameter $\lambda = \frac{n(\mu - \mu_0)^2}{\sigma_0^2}$. But if σ^2 increases then the distribution of Z^2 changes as a function of $\frac{\sigma^2}{\sigma_0^2}$. Z^2 is quite sensitive to changes in μ but is also somewhat sensitive to changes in σ^2 . Control charts are frequently used to monitor means without directly monitoring corresponding variances. In practice there is usually no way to guarantee that the variance will remain constant and thus it would be interesting to determine how well Z^2 works for detecting σ^2 . One possible multivariate version of Z^2 is the chi-squared statistic

$$Z_i^2 = n(\underline{\bar{X}}_i - \underline{\mu}_0)' \Sigma_0^{-1} (\underline{\bar{X}}_i - \underline{\mu}_0)$$

which is the same as (5.1) used for monitoring $\underline{\mu}$. The statistic Z_i^2 has a central chi-squared distribution with p degrees of freedom when $\Sigma = \Sigma_0$. Although the statistic Z_i^2 is designed for moni-

toring $\underline{\mu}$ it should show some sensitivity to changes in Σ . Thus the control statistic T_i^2 given by (5.3) for the MEWMA chart based on the accumulate-combine approach to monitor $\underline{\mu}$ will be evaluated for the case in which both means and variances can change.

7.5 Diagnostic Procedures

Diagnostic procedures are needed to help identify the cause of changes in parameters after a chart signals. For the separate EWMA charts for each parameter, it is easy to investigate which of the quality variables caused the signal. For other EWMA charts for $\underline{\mu}$ and Σ mentioned previously, it may be hard to set up a specific rule as a formal diagnostic procedure for deciding whether there has been a change in each parameter.

Let's consider an informal procedure. If we get a signal by the MEWMA or EWMA chart given by (7.1) or (7.5), then look at individual EWMA's to suggest where the problem may be. For each one we may not need extremely strong evidence to suggest that we look at that parameter. To identify any potential parameters that may have changed first standardize individual variables. Then compute and plot in order of magnitude

EWMA for μ_i

EWMA for σ_i $i=1, 2, \dots, p$

Suppose we have two cutoff values at 3 and 2 standard deviations of the EWMA statistic for deciding the level of evidence of changes in parameters. The cut off values at 3 and 2 standard deviations of the EWMA statistic stand for strong and some evidence of changes in parameters, respectively. Thus, the search for the cause of the signal can be undertaken based on the quality variables identified according to the decision rule mentioned above.

7.6 Numerical Results for the Mean vector and Covariance Matrix

In this Section the following control procedures are compared on the basis of their ATS performance.

1. The MEWMA chart
2. The Multivariate EWMA chart with control statistic given by (7.1)
3. The Multivariate EWMA chart with control statistic given by (7.5)
4. Separate EWMA charts for each of the means and variances.
5. The MEWMA and the EWMA using (7.5) simultaneously.

In order to cover various ways in which the shifts could take place, the following types of shifts given noncentrality parameter τ were considered:

Shift in Means

(M1) One mean shifted and the others remain constant

(M2) Equal shifts in all the means

(M3) Equal shifts in the positive direction in approximately half the means and equal shifts in the negative direction in the rest

(M4) Equal shifts in the positive direction in approximately half the means and no shifts in the rest

Shift in Variance-Covariance Matrix

Three different types of shifts (V1), (V2) and (V3) were given in Section (6.5).

Shift in Means and Variance-Covariance Matrix

The shifts in both $\underline{\mu}$ and Σ that were considered are (M1,V1), (M1,V2), (M2,V2), (M3,V2) and (M4,V3).

In order to compare VSI with FSI charts, the standard for comparison is the same as that used in Section (5.5). Except for the MEWMA chart, the sample size used for each sample observation was five. For $p=2-5$, Table 7.1 gives the values of h and g for various values of λ and ρ when the ATS at $\underline{\mu} = \underline{\mu}_0$ and $\Sigma = \Sigma_0$ is approximately 200. In Tables 7.2-7.17, the standard errors are shown in parentheses, and +, - and 0 in parentheses refer to the positive, negative and no mean shifts for the corresponding component, respectively. ATS values for the MEWMA and EWMA charts based on the combine-accumulate approach depend only on τ and c when $\Sigma_1 = c\Sigma_0$. Other EWMA charts except for the MEWMA and EWMA chart based on the combine-accumulate approach depend on $\underline{\mu}$ and Σ . For the separate EWMA charts, the parameters $h_i^{(\mu)}$, $g_i^{(\mu)}$, $h_i^{(\sigma)}$, $h_i^{(\rho)}$ were determined so that the ANSS at $\underline{\mu} = \underline{\mu}_0$ and $\Sigma = \Sigma_0$ is 200. These parameters were calculated by using the Markov chain approach with $r=50$ and integral equation procedures for the individual EWMA charts. The ATS values for the separate EWMA charts used together were calculated by using 10,000 simulations.

For the MEWMA and Multivariate EWMA charts using (7.5) based on the accumulate-combine approach, the parameters h and g were also determined so that the ANSS at $\underline{\mu} = \underline{\mu}_0$ and $\Sigma = \Sigma_0$ is the same as that of the separate EWMA charts. These parameters and the ATS for various shifts in $\underline{\mu}$ and Σ were calculated by using 10,000 simulations.

For $p=2$, Tables 7.2-7.5 give ATS values for matched FSI and VSI MEWMA charts for $\underline{\mu}$, Σ , and $\underline{\mu}$ and Σ . As shown in Table 7.2-7.5, MEWMA charts perform well in detecting all shifts in $\underline{\mu}$, and $\underline{\mu}$ and Σ , but not in Σ . The ATS values for $\underline{\mu}$ are almost the same as those for $\underline{\mu}$ and Σ . This means that the ATS values for $\underline{\mu}$ and Σ are almost determined by shifts in the mean vector. Smaller values of λ are more effective in detecting all shifts in the mean vectors. This result is the reverse

of the MEWMA chart for Σ . The VSI MEWMA charts are more efficient than the corresponding FSI MEWMA charts.

For $p=2$ and three different correlation coefficients $\rho = 0.0, 0.5, 0.8$, the ATS values for matched FSI and VSI multivariate EWMA charts using (7.1) based on the combine-accumulate approach are given in Table 7.6-7.9. The results in Tables 7.6-7.9 show that multivariate EWMA charts using (7.1) based on the combine-accumulate approach do not perform well in detecting all shifts in $\underline{\mu}$, Σ , and $\underline{\mu}$ and Σ compared to the other two EWMA charts based on the accumulate-combine approach. But this EWMA chart is better than the MEWMA chart for Σ .

For $p=2$ and three different correlation coefficients $\rho = 0.0, 0.5, 0.8$, the ATS values for matched FSI and VSI multivariate EWMA charts using (7.5) based on the accumulate-combine approach are given in Tables 7.10-7.12. The results in Tables 7.10-7.12 show that multivariate EWMA charts using (7.5) based on the accumulate-combine approach perform well in detecting all shifts in Σ , and $\underline{\mu}$ and Σ , but not in $\underline{\mu}$. The ATS values for Σ are almost the same as those for $\underline{\mu}$ and Σ . This means that the ATS values for $\underline{\mu}$ and Σ are almost determined by shifts in the variance-covariance matrix. Smaller values of λ are more effective in detecting all shifts in mean vector and variance-covariance matrix. The VSI multivariate EWMA charts using (7.5) based on the accumulate-combine approach are more efficient than the corresponding FSI charts. Also this EWMA chart is better than the EWMA chart based on the combine-accumulate approach.

For $p=2$ and three different correlation coefficients $\rho = 0.0, 0.5, 0.8$, the ATS values for matched FSI and VSI separate EWMA charts are given in Tables 7.13-7.15. The results in Tables 7.13-7.15 show that separate EWMA charts do not perform well in detecting all shifts in $\underline{\mu}$ and Σ . Smaller values of λ are more effective in detecting smaller shifts and larger values of λ are more effective in detecting larger shifts.

For $p=2$, $\lambda = 0.05$ and $\rho = 0.5$, the ATS values for matched FSI and VSI procedures based on both the MEWMA and the EWMA using (7.5) simultaneously are given in Table 7.16. The results in Table 7.16 show that using these two charts simultaneously gives good performance in detecting all shifts in $\underline{\mu}$, Σ , and $\underline{\mu}$ and Σ . This chart is the best in detecting shifts in $\underline{\mu}$ and Σ among these five different EWMA charts. The VSI chart is more effective than the corresponding FSI

chart. The results show that for five different EWMA procedures mentioned above, ATS values for $\rho = -0.5, -0.8$ are almost the same as those for $\rho = 0.5, 0.8$, respectively.

For $p = 5$ and $\rho = 0.5$, Table 7.17 gives matched FSI and VSI ATS values for these five different EWMA procedures mentioned above. The results are the same as those of $p = 2$. Tables 7.2-7.10 show that among these five different EWMA charts the MEWMA chart is best in detecting all shifts in $\underline{\mu}$, and the multivariate EWMA chart using (7.5) based on the accumulate-combine approach is best for Σ , and both the MEWMA and EWMA using (7.5) simultaneously is best in detecting shifts in $\underline{\mu}$ and Σ among these five different EWMA charts. There is no definite winner among these five different EWMA charts but the following general recommendations can be given. If we are interested mainly in changes in $\underline{\mu}$ then use the MEWMA chart. If we are interested mainly in changes in Σ then use the multivariate EWMA chart using (7.5). If we are interested in changes in both $\underline{\mu}$ and Σ use both the MEWMA chart and the EWMA charts using (7.5).

Table 7.1 shows that h and g values for MEWMA charts with $\lambda = 0.05$ are very close to those of multivariate EWMA charts with $\lambda = 0.05$ using (7.5). If we use larger in-control ATS and sample size than what we used, the difference will be smaller. This means that if we use small λ and large in-control ATS and large sample size, the h and g values for the MEWMA chart can be used as approximate h and g values for the multivariate EWMA chart using (7.5). So the asymptotic distribution is useful for designing control charts. The design recommendations will be given in Chapter 8.

Table 7.1: Values h and g for Various Values of λ and $p=2-5$ when the ATS at $\mu = \mu_0$ and $\Sigma = \Sigma_0$ is approximately 200

p	ρ	λ	<i>MEWMA</i>	<i>Separate EWMA's</i>		<i>EWMA using(7.1)</i>	<i>EWMA using(7.5)</i>	<i>both MEWMA and (7.5)</i>	
2	0.0	0.30	10.1480	1.3302	8.7756	15.3656	13.4376		
			1.3090	0.5846	5.0902	9.6952	1.2237		
		0.10	8.7730	0.6782	6.0548	12.2535	9.3192		
			1.3250	0.3156	4.5030	9.6323	1.2463		
		0.05	7.6800	0.4370	5.2055	11.1765	7.7599		
			1.2550	0.2426	4.1762	9.4715	1.2220		
2	0.5	0.30	10.1480	1.3286	8.7689	15.3656	13.6861		
			1.3090	0.5662	5.0154	9.6952	1.2840		
		0.10	8.7730	0.6740	6.0353	12.2535	9.3950		
			1.3250	0.3065	4.4704	9.6323	1.2690		
		0.05	7.6800	0.4342	5.1924	11.1765	7.8020	9.2854	9.6327
			1.2550	0.2290	4.1528	9.4715	1.2210	2.2596	2.2547
2	0.8	0.30	10.1480	1.3147	8.6877	15.3656	13.6861		
			1.3090	0.5273	4.8504	9.6952	1.2840		
		0.10	8.7730	0.6636	5.9858	12.2535	9.3950		
			1.3250	0.2845	4.3890	9.6323	1.2690		
		0.05	7.6800	0.4261	5.1542	11.1765	7.8020		
			1.2550	0.2083	4.1079	9.4715	1.2210		
3	0.5	0.10	0.7038		6.1756	17.7181			
			0.3472		4.6166	14.5247			
3	0.5	0.05	9.8050				10.0515		
			2.1890				2.1240		
4	0.5	0.10	0.7240		6.2704	23.1094			
			0.3755		4.7142	19.4234			
4	0.5	0.05	11.6872				12.0593		
			3.1050				3.0380		
5	0.5	0.10	0.7389		6.3403	28.4540			
			0.3958		4.7822	24.2104			
5	0.5	0.05	13.4072				13.9860	15.4070	16.3663
			4.1440				3.9580	5.7556	5.8435

The top number in each cell is h

The bottom number in each cell is g

Table 7.2: ATS Values for Matched FSI and VSI MEWMA Charts ($p = 2$)

<i>shifts</i>	$h = 10.148$ $g = 1.309$ $\lambda = 0.3$		$h = 8.773$ $g = 1.325$ $\lambda = 0.1$		$h = 7.680$ $g = 1.225$ $\lambda = 0.05$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
.0	199.979 (1.994)	199.819 (1.989)	200.174 (2.058)	199.802 (2.118)	199.060 (2.134)	200.015 (2.252)
$\tau = 0.5$	43.433 (0.408)	30.740 (0.300)	24.671 (0.203)	14.645 (0.145)	20.829 (0.168)	11.715 (0.121)
$\tau = 1.0$	10.621 (0.082)	5.008 (0.044)	7.771 (0.051)	3.744 (0.033)	6.806 (0.047)	3.458 (0.031)
$\tau = 1.5$	4.912 (0.031)	2.152 (0.015)	4.026 (0.023)	1.970 (0.014)	3.634 (0.022)	1.903 (0.014)
$\tau = 2.0$	2.988 (0.016)	1.464 (0.008)	2.603 (0.014)	1.421 (0.008)	2.349 (0.013)	1.378 (0.008)
$\tau = 2.5$	2.100 (0.010)	1.212 (0.005)	1.893 (0.009)	1.190 (0.005)	1.738 (0.008)	1.158 (0.004)
$\tau = 3.0$	1.632 (0.007)	1.093 (0.003)	1.496 (0.006)	1.079 (0.003)	1.385 (0.006)	1.066 (0.002)
$c = 1.21$	90.554 (0.918)	77.513 (0.802)	100.868 (1.055)	91.503 (1.016)	105.800 (1.192)	97.474 (1.201)
$c = 1.44$	47.441 (0.487)	38.460 (0.401)	57.611 (0.625)	48.858 (0.580)	60.882 (0.748)	54.101 (0.738)
$c = 1.69$	28.839 (0.295)	21.477 (0.229)	36.298 (0.411)	29.521 (0.373)	40.328 (0.519)	32.382 (0.479)
$c = 1.96$	19.425 (0.199)	13.407 (0.146)	24.352 (0.286)	18.803 (0.255)	26.736 (0.365)	21.383 (0.337)
$c = 2.25$	13.800 (0.143)	9.457 (0.104)	17.922 (0.216)	12.768 (0.179)	18.368 (0.263)	14.377 (0.237)
$c = 2.56$	10.522 (0.082)	6.716 (0.076)	13.120 (0.051)	9.303 (0.133)	13.585 (0.047)	10.233 (0.176)

Table 7.3: ATS Values for Matched FSI and VSI MEWMA Charts ($p = 2, \rho = 0.5$)

<i>shifts</i>	$\lambda = 0.3$		$\lambda = 0.1$		$\lambda = 0.05$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
$c1 = 1.21$	126.410 (1.241)	117.520 (1.178)	133.003 (1.386)	128.959 (1.407)	137.868 (1.529)	133.658 (1.590)
$c1 = 1.44$	80.123 (0.803)	70.988 (0.731)	91.439 (0.967)	85.353 (0.960)	98.363 (1.149)	92.307 (1.171)
$c1 = 1.69$	52.978 (0.533)	45.120 (0.472)	64.895 (0.702)	58.512 (0.681)	70.870 (0.855)	64.692 (0.859)
$c1 = 1.96$	36.422 (0.374)	30.008 (0.324)	46.861 (0.520)	41.132 (0.497)	53.247 (0.665)	47.419 (0.655)
$c1 = 2.25$	27.000 (0.276)	21.618 (0.234)	35.007 (0.398)	30.124 (0.377)	39.908 (0.518)	34.887 (0.506)
$c1 = 2.56$	20.663 (0.208)	16.125 (0.174)	27.233 (0.318)	23.074 (0.298)	30.946 (0.415)	26.712 (0.401)
$\tau = 0.5$ $c = 1.21$	29.663 (0.277)	20.443 (0.203)	21.237 (0.184)	13.060 (0.138)	18.370 (0.163)	10.908 (0.123)
$\tau = 1.0$ $c = 1.44$	8.184 (0.065)	4.374 (0.041)	6.820 (0.053)	3.635 (0.035)	6.220 (0.050)	3.375 (0.034)
$\tau = 1.5$ $c = 1.69$	4.166 (0.030)	2.164 (0.018)	3.700 (0.027)	2.041 (0.017)	3.314 (0.025)	1.956 (0.017)
$\tau = 2.0$ $c = 1.96$	2.728 (0.018)	1.536 (0.010)	2.460 (0.016)	1.518 (0.010)	2.251 (0.015)	1.465 (0.010)
$\tau = 2.5$ $c = 2.25$	2.038 (0.012)	1.299 (0.005)	1.864 (0.011)	1.284 (0.005)	1.727 (0.010)	1.252 (0.004)
$\tau = 3.0$ $c = 2.56$	1.640 (0.009)	1.172 (0.005)	1.547 (0.009)	1.166 (0.005)	1.447 (0.007)	1.146 (0.005)
$\tau = 0.5$ $c1 = 1.21$	34.047 (0.329)	24.561 (0.254)	22.932 (0.198)	14.448 (0.152)	19.712 (0.171)	11.951 (0.131)
$\tau = 1.0$ $c1 = 1.44$	9.120 (0.074)	5.009 (0.047)	7.373 (0.054)	4.004 (0.038)	6.652 (0.051)	3.744 (0.037)
$\tau = 1.5$ $c1 = 1.69$	4.564 (0.032)	2.461 (0.020)	3.988 (0.028)	2.272 (0.020)	3.635 (0.026)	2.178 (0.019)
$\tau = 2.0$ $c1 = 1.96$	2.981 (0.019)	2.719 (0.012)	2.694 (0.018)	1.655 (0.012)	2.473 (0.017)	1.607 (0.012)
$\tau = 2.5$ $c1 = 2.25$	2.204 (0.013)	1.399 (0.008)	2.017 (0.012)	1.368 (0.008)	1.886 (0.011)	1.337 (0.008)
$\tau = 3.0$ $c1 = 2.56$	1.777 (0.010)	1.242 (0.006)	1.663 (0.009)	1.230 (0.006)	1.564 (0.009)	1.209 (0.006)

Table 7.4: ATS Values for Matched FSI and VSI MEWMA Charts ($p = 2, \rho = 0.0$)

<i>shifts</i>	$\lambda = 0.3$		$\lambda = 0.1$		$\lambda = 0.05$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
$c1 = 1.21$	128.593 (1.280)	119.535 (1.213)	133.929 (1.395)	129.849 (1.417)	139.241 (1.545)	134.924 (1.605)
$c1 = 1.44$	83.493 (0.835)	73.910 (0.761)	93.432 (1.006)	87.043 (0.995)	100.178 (1.151)	93.826 (1.171)
$c1 = 1.69$	56.427 (0.569)	47.983 (0.502)	68.306 (0.740)	61.454 (0.713)	74.074 (0.889)	67.415 (0.889)
$c1 = 1.96$	40.008 (0.404)	32.835 (0.347)	50.324 (0.552)	44.072 (0.525)	55.558 (0.692)	49.390 (0.680)
$c1 = 2.25$	29.592 (0.302)	23.579 (0.254)	37.895 (0.422)	32.448 (0.400)	42.681 (0.547)	37.229 (0.530)
$c1 = 2.56$	22.897 (0.231)	17.787 (0.191)	29.872 (0.343)	25.059 (0.319)	33.450 (0.441)	28.759 (0.426)
$\tau = 0.5$ $c1 = 1.21$	34.350 (0.331)	24.774 (0.257)	22.967 (0.198)	14.449 (0.152)	19.726 (0.170)	11.998 (0.131)
$\tau = 1.0$ $c1 = 1.44$	9.117 (0.074)	5.002 (0.047)	7.356 (0.054)	4.017 (0.038)	6.648 (0.051)	3.733 (0.037)
$\tau = 1.5$ $c1 = 1.69$	4.590 (0.032)	2.468 (0.021)	4.012 (0.028)	2.278 (0.019)	3.624 (0.026)	2.169 (0.019)
$\tau = 2.0$ $c1 = 1.96$	2.972 (0.019)	2.719 (0.012)	2.683 (0.017)	1.655 (0.012)	2.472 (0.016)	1.605 (0.012)
$\tau = 2.5$ $c1 = 2.25$	2.198 (0.013)	1.402 (0.009)	2.021 (0.012)	1.373 (0.008)	1.878 (0.011)	1.340 (0.008)
$\tau = 3.0$ $c1 = 2.56$	1.767 (0.010)	1.240 (0.006)	1.646 (0.009)	1.227 (0.006)	1.556 (0.009)	1.210 (0.006)

Table 7.5: ATS Values for Matched FSI and VSI MEWMA Charts ($p = 2, \rho = 0.8$)

<i>shifts</i>	$\lambda = 0.3$		$\lambda = 0.1$		$\lambda = 0.05$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
$c1 = 1.21$	119.294 (1.182)	110.894 (1.123)	127.098 (1.330)	123.312 (1.356)	133.724 (1.489)	129.726 (1.549)
$c1 = 1.44$	68.545 (0.690)	60.768 (0.633)	82.049 (0.886)	72.631 (0.883)	89.251 (1.042)	84.103 (1.068)
$c1 = 1.69$	41.225 (0.424)	35.237 (0.378)	54.067 (0.597)	49.136 (0.586)	60.036 (0.744)	55.252 (0.755)
$c1 = 1.96$	27.282 (0.277)	22.636 (0.242)	36.284 (0.416)	32.221 (0.405)	40.987 (0.535)	37.013 (0.534)
$c1 = 2.25$	19.371 (0.195)	15.663 (0.169)	26.309 (0.312)	23.030 (0.301)	29.596 (0.402)	26.352 (0.399)
$c1 = 2.56$	14.505 (0.149)	11.526 (0.127)	19.589 (0.238)	16.914 (0.228)	21.727 (0.306)	19.152 (0.305)
$\tau = 0.5$ $c1 = 1.21$	33.362 (0.322)	24.131 (0.249)	22.793 (0.199)	14.404 (0.152)	19.629 (0.171)	11.883 (0.131)
$\tau = 1.0$ $c1 = 1.44$	8.988 (0.073)	4.991 (0.046)	7.354 (0.055)	4.035 (0.038)	6.618 (0.052)	3.709 (0.037)
$\tau = 1.5$ $c1 = 1.69$	4.548 (0.032)	2.467 (0.020)	3.993 (0.028)	2.289 (0.020)	3.637 (0.026)	2.186 (0.019)
$\tau = 2.0$ $c1 = 1.96$	2.997 (0.020)	2.738 (0.013)	2.722 (0.018)	1.667 (0.012)	2.512 (0.017)	1.616 (0.012)
$\tau = 2.5$ $c1 = 2.25$	2.229 (0.014)	1.408 (0.008)	2.052 (0.013)	1.388 (0.009)	1.914 (0.012)	1.356 (0.008)
$\tau = 3.0$ $c1 = 2.56$	1.813 (0.010)	1.248 (0.006)	1.701 (0.010)	1.238 (0.006)	1.612 (0.009)	1.217 (0.006)

Table 7.6: ATS Values for Matched FSI and VSI EWMA Using (7.1) Charts ($p = 2$)

<i>shifts</i>	$h = 15.3656$ $g = 9.6952$ $\lambda = 0.3$		$h = 12.2535$ $g = 9.6323$ $\lambda = 0.1$		$h = 11.1765$ $g = 9.4715$ $\lambda = 0.05$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
.0	200.000	200.000	200.000	200.000	200.000	200.000
$\tau = 0.5$	64.434 (0.585)	48.495 (0.419)	61.737 (0.381)	54.997 (0.254)	75.668 (0.298)	83.505 (0.221)
$\tau = 1.0$	13.038 (0.080)	8.803 (0.041)	19.659 (0.059)	22.642 (0.056)	31.460 (0.060)	43.263 (0.075)
$\tau = 1.5$	5.345 (0.021)	4.488 (0.018)	10.388 (0.022)	13.312 (0.029)	17.757 (0.027)	25.993 (0.040)
$\tau = 2.0$	3.128 (0.010)	2.807 (0.012)	6.577 (0.013)	8.636 (0.019)	11.387 (0.016)	16.993 (0.026)
$\tau = 2.5$	2.194 (0.006)	1.890 (0.010)	4.597 (0.008)	5.993 (0.013)	7.940 (0.011)	11.807 (0.018)
$\tau = 3.0$	1.689 (0.005)	1.283 (0.006)	3.431 (0.006)	4.370 (0.010)	5.884 (0.008)	8.619 (0.013)
$c = 1.21$	28.838 (0.229)	19.195 (0.129)	34.125 (0.151)	33.849 (0.105)	48.534 (0.132)	60.917 (0.127)
$c = 1.44$	11.718 (0.071)	8.350 (0.040)	18.576 (0.056)	21.712 (0.056)	29.912 (0.059)	41.501 (0.074)
$c = 1.69$	7.102 (0.035)	5.757 (0.025)	13.077 (0.034)	16.269 (0.040)	21.928 (0.039)	31.659 (0.056)
$c = 1.96$	5.130 (0.022)	4.436 (0.019)	10.093 (0.024)	12.995 (0.032)	17.246 (0.030)	25.337 (0.045)
$c = 2.25$	4.030 (0.016)	3.609 (0.016)	8.206 (0.020)	10.703 (0.028)	14.133 (0.025)	20.934 (0.038)
$c = 2.56$	3.360 (0.013)	3.076 (0.014)	6.872 (0.016)	9.057 (0.024)	11.914 (0.021)	17.786 (0.034)

Table 7.7: ATS Values for Matched FSI and VSI EWMA Using (7.1) Charts ($p = 2$, $\rho = 0.5$)

<i>shifts</i>	$\lambda = 0.3$		$\lambda = 0.1$		$\lambda = 0.05$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
$c1 = 1.21$	59.252 (0.526)	45.450 (0.380)	59.566 (0.364)	53.107 (0.239)	73.966 (0.286)	81.929 (0.219)
$c1 = 1.44$	25.587 (0.200)	17.381 (0.113)	31.807 (0.134)	32.475 (0.100)	46.164 (0.123)	58.827 (0.122)
$c1 = 1.69$	14.650 (0.098)	10.281 (0.054)	21.729 (0.074)	24.785 (0.069)	34.229 (0.075)	46.555 (0.090)
$c1 = 1.96$	9.776 (0.057)	7.518 (0.056)	16.601 (0.050)	20.115 (0.054)	27.221 (0.055)	38.402 (0.073)
$c1 = 2.25$	7.250 (0.038)	6.041 (0.028)	13.489 (0.038)	16.819 (0.045)	22.495 (0.043)	32.418 (0.062)
$c1 = 2.56$	5.880 (0.028)	5.085 (0.024)	11.257 (0.031)	14.414 (0.040)	19.056 (0.037)	27.767 (0.054)
$\tau = 0.5$ $c = 1.21$	25.318 (0.198)	16.615 (0.105)	30.914 (0.127)	31.684 (0.095)	45.270 (0.116)	57.518 (0.117)
$\tau = 1.0$ $c = 1.44$	9.201 (0.050)	6.922 (0.031)	15.851 (0.044)	19.225 (0.047)	26.035 (0.048)	36.820 (0.065)
$\tau = 1.5$ $c = 1.69$	5.383 (0.024)	4.610 (0.020)	10.466 (0.025)	13.462 (0.033)	17.862 (0.031)	26.179 (0.046)
$\tau = 2.0$ $c = 1.96$	3.749 (0.015)	3.364 (0.015)	7.658 (0.018)	10.066 (0.026)	13.226 (0.023)	19.672 (0.036)
$\tau = 2.5$ $c = 2.25$	2.864 (0.011)	2.633 (0.013)	5.962 (0.014)	7.856 (0.021)	10.283 (0.018)	15.313 (0.029)
$\tau = 3.0$ $c = 2.56$	2.332 (0.009)	2.116 (0.011)	4.812 (0.012)	6.282 (0.018)	8.264 (0.015)	12.289 (0.025)
$\tau = 0.5$ $c1 = 1.21$	48.661 (0.423)	34.865 (0.276)	50.111 (0.283)	46.183 (0.186)	64.815 (0.227)	74.851 (0.182)
$\tau = 1.0$ $c1 = 1.44$	16.380 (0.111)	11.329 (0.063)	23.415 (0.083)	26.199 (0.073)	36.495 (0.082)	48.993 (0.096)
$\tau = 1.5$ $c1 = 1.69$	8.329 (0.044)	6.664 (0.031)	14.877 (0.042)	18.301 (0.048)	24.559 (0.047)	35.115 (0.065)
$\tau = 2.0$ $c1 = 1.96$	5.421 (0.025)	4.723 (0.022)	10.571 (0.027)	13.613 (0.036)	17.988 (0.034)	26.372 (0.050)
$\tau = 2.5$ $c1 = 2.25$	3.957 (0.017)	3.609 (0.017)	8.014 (0.020)	10.512 (0.029)	13.812 (0.026)	20.536 (0.040)
$\tau = 3.0$ $c1 = 2.56$	3.072 (0.013)	2.872 (0.015)	6.363 (0.017)	8.395 (0.025)	10.940 (0.021)	16.332 (0.034)

Table 7.8: ATS Values for Matched FSI and VSI EWMA Using (7.1) Charts ($\rho = 2$, $\rho = 0.0$)

<i>shifts</i>	$\lambda = 0.3$		$\lambda = 0.1$		$\lambda = 0.05$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
$c1 = 1.21$	61.041 (0.546)	46.655 (0.392)	60.339 (0.370)	53.649 (0.243)	74.603 (0.291)	82.536 (0.223)
$c1 = 1.44$	26.812 (0.213)	18.053 (0.118)	32.734 (0.141)	33.006 (0.102)	47.098 (0.127)	59.579 (0.125)
$c1 = 1.69$	15.474 (0.105)	10.711 (0.057)	22.488 (0.078)	25.358 (0.070)	35.247 (0.078)	47.547 (0.091)
$c1 = 1.96$	10.363 (0.060)	7.809 (0.037)	17.374 (0.053)	20.692 (0.055)	28.208 (0.057)	39.688 (0.074)
$c1 = 2.25$	7.784 (0.041)	6.341 (0.029)	14.129 (0.039)	17.545 (0.045)	23.556 (0.045)	33.837 (0.064)
$c1 = 2.56$	6.307 (0.031)	5.297 (0.024)	11.909 (0.032)	15.137 (0.041)	20.098 (0.038)	29.195 (0.056)
$\tau = 0.5$ $c1 = 1.21$	49.648 (0.437)	35.393 (0.281)	50.545 (0.287)	46.586 (0.189)	65.392 (0.228)	75.215 (0.183)
$\tau = 1.0$ $c1 = 1.44$	16.846 (0.117)	11.555 (0.065)	23.910 (0.086)	26.506 (0.074)	37.080 (0.084)	49.524 (0.097)
$\tau = 1.5$ $c1 = 1.69$	8.620 (0.046)	6.746 (0.031)	15.191 (0.043)	18.624 (0.048)	25.042 (0.048)	35.677 (0.066)
$\tau = 2.0$ $c1 = 1.96$	5.600 (0.026)	4.815 (0.022)	10.811 (0.028)	13.899 (0.037)	18.408 (0.033)	26.940 (0.050)
$\tau = 2.5$ $c1 = 2.25$	4.058 (0.017)	3.682 (0.017)	8.232 (0.021)	10.810 (0.030)	14.514 (0.026)	20.994 (0.041)
$\tau = 3.0$ $c1 = 2.56$	3.156 (0.013)	2.922 (0.015)	6.535 (0.017)	8.637 (0.025)	11.259 (0.021)	16.783 (0.034)

Table 7.9: ATS Values for Matched FSI and VSI EWMA Using (7.1) Charts ($p = 2$, $\rho = 0.8$)

<i>shifts</i>	$\lambda = 0.3$		$\lambda = 0.1$		$\lambda = 0.05$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
$c1 = 1.21$	54.868 (0.484)	41.560 (0.345)	55.964 (0.330)	51.038 (0.224)	71.004 (0.269)	79.553 (0.208)
$c1 = 1.44$	21.759 (0.164)	15.064 (0.093)	28.766 (0.116)	30.374 (0.093)	42.742 (0.109)	55.581 (0.115)
$c1 = 1.69$	11.864 (0.073)	8.926 (0.045)	18.976 (0.061)	22.316 (0.062)	30.620 (0.066)	42.553 (0.083)
$c1 = 1.96$	7.750 (0.042)	6.450 (0.032)	14.226 (0.042)	17.636 (0.048)	23.611 (0.047)	33.967 (0.067)
$c1 = 2.25$	5.835 (0.029)	5.105 (0.024)	11.187 (0.031)	14.391 (0.041)	18.930 (0.038)	27.643 (0.056)
$c1 = 2.56$	4.610 (0.022)	4.232 (0.021)	9.157 (0.025)	11.939 (0.036)	15.626 (0.032)	23.114 (0.049)
$\tau = 0.5$ $c1 = 1.21$	45.181 (0.388)	32.637 (0.258)	47.893 (0.264)	44.271 (0.175)	62.681 (0.214)	73.328 (0.175)
$\tau = 1.0$ $c1 = 1.44$	14.625 (0.096)	10.459 (0.056)	21.822 (0.075)	24.901 (0.070)	34.507 (0.076)	46.884 (0.092)
$\tau = 1.5$ $c1 = 1.69$	7.400 (0.038)	6.171 (0.029)	13.714 (0.038)	17.098 (0.045)	22.883 (0.044)	32.900 (0.063)
$\tau = 2.0$ $c1 = 1.96$	4.875 (0.022)	4.341 (0.021)	9.624 (0.026)	12.517 (0.035)	16.490 (0.031)	24.269 (0.047)
$\tau = 2.5$ $c1 = 2.25$	3.588 (0.015)	3.308 (0.017)	7.251 (0.019)	9.589 (0.028)	12.484 (0.024)	18.673 (0.039)
$\tau = 3.0$ $c1 = 2.56$	2.798 (0.012)	2.630 (0.014)	5.733 (0.016)	7.624 (0.024)	9.798 (0.020)	14.655 (0.033)

Table 7.10: ATS Values for Matched FSI and VSI EWMA Charts Using (7.5) ($p = 2, \rho = 0.0$)

<i>shifts</i>	$h = 13.4376$ $g = 1.224$ $\lambda = 0.3$		$h = 9.3192$ $g = 1.244$ $\lambda = 0.1$		$h = 7.7599$ $g = 1.222$ $\lambda = 0.05$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
.0	200.370 (2.017)	199.924 (2.037)	199.812 (2.062)	200.376 (2.167)	199.455 (2.158)	200.028 (2.306)
one mean						
$\tau = 0.5$	146.864 (1.480)	141.321 (1.444)	133.403 (1.399)	129.674 (1.412)	131.227 (1.449)	128.476 (1.509)
$\tau = 1.0$	59.677 (0.591)	52.309 (0.536)	41.523 (0.409)	33.832 (0.370)	36.428 (0.368)	28.500 (0.333)
$\tau = 1.5$	19.867 (0.188)	13.689 (0.140)	13.286 (0.117)	8.582 (0.090)	11.727 (0.106)	7.773 (0.086)
$\tau = 2.0$	7.940 (0.067)	4.437 (0.042)	5.813 (0.047)	4.523 (0.034)	5.211 (0.044)	3.398 (0.034)
$\tau = 2.5$	4.098 (0.031)	2.231 (0.018)	3.213 (0.023)	2.051 (0.017)	2.902 (0.022)	1.982 (0.017)
$\tau = 3.0$	2.516 (0.016)	1.520 (0.010)	2.078 (0.013)	1.469 (0.010)	1.922 (0.012)	1.449 (0.010)
two means (+ +)						
$\tau = 0.5$	156.764 (1.526)	145.710 (1.492)	140.760 (1.464)	138.251 (1.490)	141.673 (1.567)	142.691 (1.649)
$\tau = 1.0$	73.035 (0.725)	67.018 (0.683)	55.544 (0.570)	49.139 (0.545)	51.090 (0.539)	43.934 (0.519)
$\tau = 1.5$	28.655 (0.277)	22.266 (0.229)	19.445 (0.183)	13.653 (0.149)	17.144 (0.167)	11.897 (0.136)
$\tau = 2.0$	12.044 (0.108)	7.371 (0.073)	8.319 (0.073)	5.172 (0.054)	7.326 (0.067)	4.829 (0.052)
$\tau = 2.5$	5.921 (0.048)	3.195 (0.028)	4.455 (0.036)	2.742 (0.025)	3.963 (0.033)	2.631 (0.025)
$\tau = 3.0$	3.467 (0.025)	1.918 (0.015)	2.741 (0.019)	1.776 (0.014)	2.485 (0.018)	1.746 (0.014)
two means (+ -)						
$\tau = 0.5$	151.185 (1.532)	146.073 (1.497)	141.672 (1.479)	139.168 (1.506)	142.812 (1.550)	146.073 (1.630)
$\tau = 1.0$	73.282 (0.734)	67.198 (0.686)	55.245 (0.571)	48.981 (0.545)	50.831 (0.538)	43.900 (0.524)
$\tau = 1.5$	28.770 (0.279)	22.309 (0.231)	19.433 (0.184)	13.802 (0.154)	17.231 (0.168)	12.118 (0.141)
$\tau = 2.0$	12.063 (0.109)	7.390 (0.074)	8.404 (0.074)	5.174 (0.054)	7.447 (0.068)	4.858 (0.053)
$\tau = 2.5$	5.944 (0.049)	3.193 (0.029)	4.428 (0.036)	2.711 (0.025)	3.945 (0.033)	2.621 (0.025)
$\tau = 3.0$	3.474 (0.025)	1.899 (0.015)	2.745 (0.019)	1.783 (0.014)	2.480 (0.018)	1.743 (0.014)

Table 7.10: (Continued)

<i>shifts</i>	$\lambda = 0.3$		$\lambda = 0.1$		$\lambda = 0.05$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
$c = 1.21$	30.076 (0.289)	23.976 (0.244)	21.083 (0.204)	15.410 (0.171)	18.714 (0.186)	13.494 (0.159)
$c = 1.44$	9.798 (0.088)	6.108 (0.061)	7.180 (0.064)	6.559 (0.048)	6.393 (0.059)	4.280 (0.046)
$c = 1.69$	4.977 (0.024)	1.907 (0.015)	3.918 (0.019)	1.792 (0.014)	3.515 (0.018)	1.767 (0.014)
$c = 1.96$	3.183 (0.024)	1.907 (0.015)	2.601 (0.019)	1.792 (0.014)	2.379 (0.018)	1.767 (0.014)
$c = 2.25$	2.323 (0.016)	1.528 (0.010)	1.975 (0.013)	1.474 (0.010)	1.835 (0.012)	1.452 (0.010)
$c = 2.56$	1.860 (0.012)	1.325 (0.008)	1.626 (0.010)	1.295 (0.008)	1.538 (0.009)	1.283 (0.008)
$c1 = 1.21$	54.051 (0.536)	46.911 (0.482)	37.884 (0.377)	30.721 (0.341)	33.901 (0.345)	26.373 (0.311)
$c1 = 1.44$	19.037 (0.179)	13.685 (0.139)	13.611 (0.127)	9.142 (0.099)	12.121 (0.115)	8.354 (0.095)
$c1 = 1.69$	9.707 (0.088)	6.105 (0.061)	7.260 (0.063)	4.653 (0.047)	6.508 (0.059)	4.377 (0.047)
$c1 = 1.96$	5.952 (0.050)	3.603 (0.034)	4.749 (0.039)	3.137 (0.030)	4.311 (0.037)	3.056 (0.031)
$c1 = 2.25$	4.227 (0.033)	2.605 (0.023)	3.455 (0.027)	2.381 (0.022)	3.174 (0.026)	2.332 (0.022)
$c1 = 2.56$	3.255 (0.024)	2.068 (0.017)	2.736 (0.021)	1.970 (0.017)	2.520 (0.019)	1.928 (0.017)
$\tau = 0.5$ $c = 1.21$	25.427 (0.247)	19.520 (0.202)	17.850 (0.172)	12.653 (0.141)	15.892 (0.167)	11.138 (0.129)
$\tau = 1.0$ $c = 1.44$	6.950 (0.060)	4.098 (0.040)	5.323 (0.046)	3.385 (0.034)	4.811 (0.043)	3.272 (0.034)
$\tau = 1.5$ $c = 1.69$	3.313 (0.025)	1.970 (0.016)	2.703 (0.020)	1.831 (0.015)	2.473 (0.019)	1.796 (0.015)
$\tau = 2.0$ $c = 1.96$	2.083 (0.014)	1.398 (0.008)	1.791 (0.011)	1.358 (0.008)	1.675 (0.010)	1.340 (0.008)
$\tau = 2.5$ $c = 2.25$	1.558 (0.009)	1.186 (0.005)	1.401 (0.007)	1.173 (0.006)	1.342 (0.007)	1.168 (0.006)
$\tau = 3.0$ $c = 2.56$	1.306 (0.006)	1.097 (0.004)	1.218 (0.005)	1.090 (0.004)	1.182 (0.004)	1.085 (0.004)

Table 7.10: (Continued)

<i>shifts</i>	$\lambda = 0.3$		$\lambda = 0.1$		$\lambda = 0.05$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
one mean and corresponding variance						
$\tau = 0.5$ $c1 = 1.21$	41.092 (0.400)	34.249 (0.349)	28.363 (0.276)	21.546 (0.237)	24.941 (0.247)	18.430 (0.216)
$\tau = 1.0$ $c1 = 1.44$	10.734 (0.096)	6.902 (0.069)	7.981 (0.070)	5.143 (0.054)	7.152 (0.065)	4.864 (0.053)
$\tau = 1.5$ $c1 = 1.69$	4.718 (0.038)	2.833 (0.026)	3.809 (0.030)	2.559 (0.024)	3.462 (0.028)	2.469 (0.023)
$\tau = 2.0$ $c1 = 1.96$	2.779 (0.020)	1.780 (0.013)	2.354 (0.017)	1.701 (0.013)	2.178 (0.016)	1.666 (0.013)
$\tau = 2.5$ $c1 = 2.25$	1.955 (0.012)	1.398 (0.009)	1.724 (0.011)	1.367 (0.009)	1.638 (0.010)	1.353 (0.009)
$\tau = 3.0$ $c1 = 2.56$	1.551 (0.008)	1.223 (0.006)	1.416 (0.007)	1.207 (0.006)	1.364 (0.007)	1.197 (0.006)
two means (+) and all variances						
$\tau = 0.5$ $c = 1.21$	25.515 (0.249)	19.599 (0.203)	17.946 (0.171)	12.683 (0.140)	16.025 (0.157)	11.244 (0.131)
$\tau = 1.0$ $c = 1.44$	7.002 (0.061)	4.168 (0.040)	5.382 (0.046)	3.428 (0.034)	4.828 (0.043)	3.286 (0.034)
$\tau = 1.5$ $c = 1.69$	3.373 (0.026)	2.004 (0.016)	2.729 (0.021)	1.870 (0.015)	2.498 (0.019)	1.826 (0.016)
$\tau = 2.0$ $c = 1.96$	2.119 (0.014)	1.412 (0.009)	1.815 (0.014)	1.368 (0.009)	1.702 (0.011)	1.356 (0.009)
$\tau = 2.5$ $c = 2.25$	1.576 (0.009)	1.194 (0.006)	1.412 (0.007)	1.175 (0.006)	1.354 (0.007)	1.166 (0.005)
$\tau = 3.0$ $c = 2.56$	1.314 (0.006)	1.094 (0.004)	1.223 (0.005)	1.083 (0.004)	1.183 (0.005)	1.079 (0.004)
two means (+-) and all variances						
$\tau = 0.5$ $c = 1.21$	25.555 (0.243)	19.616 (0.198)	17.895 (0.170)	12.737 (0.141)	16.111 (0.159)	11.349 (0.132)
$\tau = 1.0$ $c = 1.44$	7.092 (0.062)	4.185 (0.041)	5.339 (0.046)	3.404 (0.034)	4.844 (0.043)	3.296 (0.034)
$\tau = 1.5$ $c = 1.69$	3.372 (0.025)	1.992 (0.016)	2.739 (0.021)	1.871 (0.015)	2.512 (0.019)	1.835 (0.015)
$\tau = 2.0$ $c = 1.96$	2.129 (0.014)	1.418 (0.009)	1.822 (0.012)	1.384 (0.009)	1.706 (0.011)	1.368 (0.009)
$\tau = 2.5$ $c = 2.25$	1.585 (0.009)	1.205 (0.006)	1.421 (0.008)	1.193 (0.006)	1.360 (0.007)	1.181 (0.006)
$\tau = 3.0$ $c = 2.56$	1.321 (0.006)	1.111 (0.004)	1.230 (0.005)	1.100 (0.004)	1.194 (0.005)	1.096 (0.004)

Table 7.11: ATS Values for Matched FSI and VSI EWMA Charts Using (7.5) ($p = 2, \rho = 0.5$)

shifts	$h = 13.6861$ $g = 1.284$ $\lambda = 0.3$		$h = 9.3950$ $g = 1.269$ $\lambda = 0.1$		$h = 7.8020$ $g = 1.221$ $\lambda = 0.05$	
	FSI	VSI	FSI	VSI	FSI	VSI
.0	199.426 (2.024)	200.046 (2.048)	200.298 (2.100)	199.225 (2.142)	200.562 (2.209)	199.219 (2.300)
one mean						
$\tau = 0.5$	162.348 (1.643)	162.928 (1.664)	149.971 (1.568)	148.204 (1.628)	148.771 (1.634)	151.005 (1.740)
$\tau = 1.0$	80.687 (0.800)	75.870 (0.773)	56.737 (0.572)	49.588 (0.549)	50.124 (0.517)	42.173 (0.499)
$\tau = 1.5$	29.391 (0.282)	23.164 (0.234)	19.133 (0.178)	13.590 (0.149)	16.860 (0.159)	11.385 (0.128)
$\tau = 2.0$	11.963 (0.106)	7.550 (0.074)	8.330 (0.069)	5.132 (0.052)	7.317 (0.065)	4.771 (0.050)
$\tau = 2.5$	6.014 (0.047)	3.274 (0.030)	4.389 (0.033)	2.780 (0.026)	3.988 (0.032)	2.665 (0.026)
$\tau = 3.0$	3.465 (0.025)	1.978 (0.015)	2.180 (0.020)	1.824 (0.014)	2.538 (0.018)	1.811 (0.015)
two means (+ +)						
$\tau = 0.5$	138.384 (1.401)	136.109 (1.374)	124.696 (1.316)	126.594 (1.386)	125.384 (1.389)	127.012 (1.478)
$\tau = 1.0$	80.687 (0.800)	50.162 (0.515)	56.737 (0.572)	33.823 (0.376)	50.124 (0.517)	29.026 (0.343)
$\tau = 1.5$	18.429 (0.176)	13.970 (0.143)	12.945 (0.119)	8.988 (0.098)	11.476 (0.110)	7.762 (0.087)
$\tau = 2.0$	7.626 (0.065)	4.696 (0.045)	5.700 (0.048)	3.648 (0.036)	5.067 (0.044)	3.523 (0.037)
$\tau = 2.5$	6.014 (0.047)	2.300 (0.019)	3.154 (0.024)	2.074 (0.018)	2.850 (0.022)	2.063 (0.018)
$\tau = 3.0$	2.495 (0.017)	1.975 (0.015)	2.091 (0.014)	1.824 (0.014)	1.895 (0.012)	1.819 (0.015)
two means (+ -)						
$\tau = 0.5$	173.301 (1.750)	173.251 (1.766)	168.587 (1.773)	167.709 (1.820)	169.868 (1.869)	169.307 (1.962)
$\tau = 1.0$	117.044 (1.178)	114.568 (1.170)	102.014 (1.075)	98.952 (1.092)	99.688 (1.088)	95.943 (1.131)
$\tau = 1.5$	67.492 (0.691)	62.291 (0.651)	50.633 (0.521)	44.168 (0.493)	46.493 (0.502)	38.707 (0.470)
$\tau = 2.0$	35.949 (0.357)	29.297 (0.305)	24.500 (0.245)	17.737 (0.200)	21.425 (0.219)	14.887 (0.176)
$\tau = 2.5$	19.214 (0.186)	12.858 (0.133)	12.812 (0.119)	7.833 (0.082)	11.130 (0.108)	6.999 (0.078)
$\tau = 3.0$	10.962 (0.099)	5.926 (0.057)	7.372 (0.064)	4.134 (0.041)	6.354 (0.058)	3.818 (0.040)

Table 7.11: (Continued)

<i>shifts</i>	$\lambda = 0.3$		$\lambda = 0.1$		$\lambda = 0.05$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
$c = 1.21$	34.141 (0.341)	28.379 (0.292)	24.055 (0.235)	18.559 (0.206)	21.712 (0.222)	15.863 (0.183)
$c = 1.44$	11.203 (0.104)	7.544 (0.077)	8.320 (0.076)	5.323 (0.056)	7.453 (0.071)	4.894 (0.055)
$c = 1.69$	5.714 (0.048)	3.397 (0.032)	4.429 (0.038)	2.847 (0.028)	4.038 (0.035)	2.758 (0.027)
$c = 1.96$	3.653 (0.029)	2.158 (0.018)	2.968 (0.024)	1.983 (0.017)	2.670 (0.021)	1.947 (0.017)
$c = 2.25$	2.634 (0.019)	1.672 (0.012)	2.243 (0.016)	1.594 (0.012)	2.054 (0.015)	1.571 (0.012)
$c = 2.56$	2.110 (0.014)	1.421 (0.009)	1.813 (0.012)	1.380 (0.009)	1.688 (0.011)	1.361 (0.009)
$c1 = 1.21$	54.012 (0.530)	47.923 (0.489)	37.232 (0.366)	29.948 (0.329)	32.450 (0.323)	24.870 (0.289)
$c1 = 1.44$	18.663 (0.179)	13.524 (0.139)	12.963 (0.116)	8.740 (0.092)	11.550 (0.107)	7.848 (0.087)
$c1 = 1.69$	9.245 (0.081)	5.903 (0.058)	6.937 (0.060)	4.479 (0.045)	6.234 (0.055)	4.195 (0.043)
$c1 = 1.96$	5.796 (0.049)	3.538 (0.033)	4.572 (0.038)	3.017 (0.029)	4.103 (0.035)	2.863 (0.028)
$c1 = 2.25$	4.075 (0.032)	2.542 (0.022)	3.338 (0.026)	2.307 (0.021)	3.055 (0.024)	2.219 (0.020)
$c1 = 2.56$	3.147 (0.023)	2.025 (0.016)	2.634 (0.020)	1.889 (0.016)	2.447 (0.018)	1.850 (0.016)
$\tau = 0.5$ $c = 1.21$	30.014 (0.296)	24.358 (0.249)	21.172 (0.206)	15.900 (0.176)	18.935 (0.192)	13.507 (0.158)
$\tau = 1.0$ $c = 1.44$	8.698 (0.079)	5.468 (0.054)	6.512 (0.057)	4.106 (0.042)	5.802 (0.054)	3.868 (0.042)
$\tau = 1.5$ $c = 1.69$	4.099 (0.033)	2.394 (0.021)	3.247 (0.027)	2.167 (0.020)	2.953 (0.024)	2.120 (0.019)
$\tau = 2.0$ $c = 1.96$	2.496 (0.018)	1.592 (0.011)	2.133 (0.015)	1.524 (0.011)	1.946 (0.013)	1.503 (0.011)
$\tau = 2.5$ $c = 2.25$	1.851 (0.012)	1.284 (0.007)	1.608 (0.010)	1.278 (0.007)	1.506 (0.009)	1.251 (0.007)
$\tau = 3.0$ $c = 2.56$	1.496 (0.008)	1.153 (0.005)	1.341 (0.007)	1.147 (0.005)	1.284 (0.006)	1.125 (0.005)

Table 7.11: (Continued)

<i>shifts</i>	$\lambda = 0.3$		$\lambda = 0.1$		$\lambda = 0.05$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
one mean and corresponding variance						
$\tau = 0.5$ $c_1 = 1.21$	43.972 (0.433)	37.675 (0.386)	29.615 (0.290)	22.716 (0.251)	25.683 (0.255)	18.747 (0.218)
$\tau = 1.0$ $c_1 = 1.44$	11.819 (0.105)	7.785 (0.076)	8.438 (0.074)	5.535 (0.058)	7.617 (0.069)	5.152 (0.056)
$\tau = 1.5$ $c_1 = 1.69$	5.235 (0.043)	3.185 (0.029)	4.132 (0.033)	2.759 (0.026)	3.751 (0.031)	2.660 (0.025)
$\tau = 2.0$ $c_1 = 1.96$	3.099 (0.023)	1.973 (0.015)	2.575 (0.019)	1.856 (0.015)	2.382 (0.018)	1.804 (0.015)
$\tau = 2.5$ $c_1 = 2.25$	2.164 (0.014)	1.517 (0.010)	1.882 (0.012)	1.463 (0.010)	1.769 (0.011)	1.436 (0.010)
$\tau = 3.0$ $c_1 = 2.56$	1.704 (0.010)	1.293 (0.007)	1.531 (0.009)	1.263 (0.007)	1.462 (0.008)	1.251 (0.007)
two means (+) and all variances						
$\tau = 0.5$ $c = 1.21$	26.804 (0.262)	21.925 (0.284)	19.303 (0.187)	14.148 (0.156)	17.238 (0.173)	12.034 (0.141)
$\tau = 1.0$ $c = 1.44$	7.237 (0.063)	4.506 (0.044)	5.597 (0.049)	3.590 (0.036)	4.995 (0.046)	3.432 (0.036)
$\tau = 1.5$ $c = 1.69$	3.382 (0.026)	2.081 (0.017)	2.793 (0.022)	1.910 (0.017)	2.555 (0.020)	1.890 (0.017)
$\tau = 2.0$ $c = 1.96$	2.144 (0.014)	1.443 (0.009)	1.849 (0.012)	1.411 (0.009)	1.715 (0.011)	1.394 (0.009)
$\tau = 2.5$ $c = 2.25$	1.611 (0.009)	1.214 (0.006)	1.425 (0.008)	1.206 (0.006)	1.358 (0.007)	1.183 (0.006)
$\tau = 3.0$ $c = 2.56$	1.332 (0.006)	1.161 (0.005)	1.229 (0.005)	1.153 (0.005)	1.189 (0.005)	1.132 (0.005)
two means (+-) and all variances						
$\tau = 0.5$ $c = 1.21$	30.904 (0.307)	25.630 (0.269)	22.363 (0.220)	16.779 (0.188)	19.802 (0.202)	14.302 (0.170)
$\tau = 1.0$ $c = 1.44$	9.443 (0.085)	5.939 (0.060)	7.004 (0.062)	4.497 (0.046)	6.223 (0.058)	4.161 (0.045)
$\tau = 1.5$ $c = 1.69$	4.542 (0.037)	2.650 (0.024)	3.556 (0.029)	2.317 (0.021)	3.198 (0.027)	2.233 (0.021)
$\tau = 2.0$ $c = 1.96$	2.775 (0.020)	1.701 (0.012)	2.288 (0.017)	1.608 (0.012)	2.111 (0.016)	1.571 (0.012)
$\tau = 2.5$ $c = 2.25$	2.003 (0.013)	1.362 (0.008)	1.716 (0.011)	1.321 (0.008)	1.604 (0.010)	1.297 (0.008)
$\tau = 3.0$ $c = 2.56$	1.584 (0.009)	1.192 (0.006)	1.412 (0.007)	1.170 (0.006)	1.349 (0.007)	1.159 (0.005)

Table 7.12: ATS Values for Matched FSI and VSI EWMA Charts Using (7.5) ($p = 2, \rho = 0.8$)

shifts	$h = 13.6861$ $g = 1.284$ $\lambda = 0.3$		$h = 9.3950$ $g = 1.269$ $\lambda = 0.1$		$h = 7.8020$ $g = 1.221$ $\lambda = 0.05$	
	FSI	VSI	FSI	VSI	FSI	VSI
.0	199.952 (2.024)	200.933 (2.048)	201.051 (2.092)	200.022 (2.142)	200.233 (2.102)	199.200 (2.209)
one mean						
$\tau = 0.5$	174.346 (1.752)	173.372 (1.757)	168.749 (1.756)	166.600 (1.786)	168.086 (1.848)	165.793 (1.930)
$\tau = 1.0$	113.156 (1.134)	107.394 (1.094)	90.176 (0.954)	83.294 (0.928)	82.961 (0.893)	75.071 (0.894)
$\tau = 1.5$	55.144 (0.542)	47.005 (0.480)	37.047 (0.369)	29.320 (0.326)	32.089 (0.330)	24.339 (0.291)
$\tau = 2.0$	25.031 (0.238)	18.254 (0.186)	16.404 (0.154)	11.220 (0.124)	14.421 (0.139)	9.855 (0.115)
$\tau = 2.5$	12.211 (0.110)	7.752 (0.078)	8.471 (0.075)	5.479 (0.059)	7.582 (0.071)	5.091 (0.057)
$\tau = 3.0$	6.893 (0.059)	4.072 (0.039)	5.219 (0.044)	3.315 (0.033)	4.639 (0.041)	3.138 (0.032)
two mean (+ +)						
$\tau = 0.5$	134.258 (1.351)	134.208 (1.369)	121.541 (1.266)	120.663 (1.308)	121.965 (1.317)	120.510 (1.396)
$\tau = 1.0$	50.503 (0.505)	46.309 (0.478)	37.094 (0.373)	31.595 (0.350)	33.462 (0.345)	27.185 (0.321)
$\tau = 1.5$	17.129 (0.160)	12.527 (0.128)	12.106 (0.111)	8.161 (0.088)	10.685 (0.101)	7.388 (0.083)
$\tau = 2.0$	7.141 (0.062)	4.275 (0.041)	5.334 (0.045)	3.434 (0.033)	4.733 (0.041)	3.240 (0.033)
$\tau = 2.5$	3.773 (0.029)	2.174 (0.017)	2.990 (0.023)	1.990 (0.017)	2.709 (0.021)	1.932 (0.017)
$\tau = 3.0$	2.384 (0.016)	1.491 (0.010)	1.972 (0.013)	1.427 (0.009)	1.823 (0.012)	1.403 (0.009)
two mean (+ -)						
$\tau = 0.5$	183.768 (1.856)	183.943 (1.862)	181.058 (1.897)	178.988 (1.927)	179.758 (1.976)	177.987 (2.059)
$\tau = 1.0$	142.870 (1.441)	138.428 (1.413)	132.307 (1.402)	128.074 (1.407)	133.426 (1.499)	129.949 (1.551)
$\tau = 1.5$	96.708 (0.974)	89.885 (0.921)	84.747 (0.905)	79.455 (0.894)	85.865 (0.974)	80.833 (0.994)
$\tau = 2.0$	61.636 (0.623)	54.556 (0.565)	51.716 (0.558)	46.040 (0.534)	51.187 (0.595)	45.284 (0.580)
$\tau = 2.5$	38.827 (0.387)	32.451 (0.337)	31.938 (0.350)	26.468 (0.317)	30.599 (0.359)	24.895 (0.329)
$\tau = 3.0$	25.080 (0.256)	19.524 (0.210)	20.326 (0.221)	15.352 (0.185)	18.822 (0.219)	13.964 (0.183)

Table 7.12: (Continued)

<i>shifts</i>	$\lambda = 0.3$		$\lambda = 0.1$		$\lambda = 0.05$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
$c = 1.21$	37.268 (0.375)	32.200 (0.338)	27.824 (0.284)	22.340 (0.252)	25.241 (0.264)	19.457 (0.234)
$c = 1.44$	12.934 (0.121)	9.001 (0.092)	9.738 (0.092)	6.459 (0.070)	8.721 (0.086)	5.905 (0.067)
$c = 1.69$	6.575 (0.058)	4.087 (0.039)	5.191 (0.046)	3.404 (0.035)	4.682 (0.043)	3.221 (0.034)
$c = 1.96$	4.228 (0.035)	2.565 (0.023)	3.367 (0.028)	2.301 (0.021)	3.052 (0.027)	2.210 (0.021)
$c = 2.25$	2.993 (0.023)	1.916 (0.016)	2.508 (0.019)	1.794 (0.015)	2.300 (0.018)	1.740 (0.015)
$c = 2.56$	2.362 (0.017)	1.581 (0.011)	2.013 (0.014)	1.516 (0.011)	1.873 (0.013)	1.481 (0.011)
$cl = 1.21$	44.421 (0.445)	36.560 (0.383)	28.184 (0.273)	20.285 (0.223)	24.076 (0.235)	16.638 (0.192)
$cl = 1.44$	13.735 (0.125)	8.636 (0.086)	9.386 (0.080)	5.771 (0.059)	8.328 (0.073)	5.306 (0.057)
$cl = 1.69$	6.776 (0.056)	3.832 (0.035)	5.118 (0.041)	3.164 (0.029)	4.611 (0.038)	3.007 (0.029)
$cl = 1.96$	4.267 (0.033)	2.450 (0.020)	3.409 (0.026)	2.225 (0.019)	3.101 (0.024)	2.135 (0.019)
$cl = 2.25$	3.072 (0.022)	1.872 (0.014)	2.557 (0.018)	1.762 (0.014)	2.360 (0.017)	1.715 (0.013)
$cl = 2.56$	2.437 (0.017)	1.578 (0.011)	2.084 (0.014)	1.518 (0.011)	1.948 (0.013)	1.481 (0.010)
$\tau = 0.5$ $c = 1.21$	34.788 (0.350)	29.554 (0.312)	25.825 (0.263)	20.363 (0.231)	23.395 (0.246)	17.791 (0.214)
$\tau = 1.0$ $c = 1.44$	11.103 (0.103)	7.398 (0.076)	8.296 (0.078)	5.453 (0.059)	7.442 (0.073)	5.065 (0.057)
$\tau = 1.5$ $c = 1.69$	5.281 (0.046)	3.178 (0.030)	4.185 (0.037)	2.760 (0.027)	3.784 (0.034)	2.651 (0.027)
$\tau = 2.0$ $c = 1.96$	3.244 (0.026)	2.012 (0.016)	2.673 (0.021)	1.867 (0.016)	2.450 (0.020)	1.816 (0.016)
$\tau = 2.5$ $c = 2.25$	2.320 (0.017)	1.533 (0.011)	1.977 (0.014)	1.467 (0.010)	1.827 (0.013)	1.430 (0.010)
$\tau = 3.0$ $c = 2.56$	1.819 (0.012)	1.302 (0.007)	1.590 (0.010)	1.272 (0.007)	1.505 (0.009)	1.259 (0.007)

Table 7.12: (Continued)

<i>shifts</i>	$\lambda = 0.3$		$\lambda = 0.1$		$\lambda = 0.05$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
one mean and corresponding variance						
$\tau = 0.5$ $c_1 = 1.21$	39.091 (0.389)	31.404 (0.326)	24.794 (0.238)	17.437 (0.190)	21.272 (0.204)	14.427 (0.164)
$\tau = 1.0$ $c_1 = 1.44$	10.531 (0.093)	6.390 (0.063)	7.563 (0.065)	4.658 (0.047)	6.722 (0.059)	4.305 (0.045)
$\tau = 1.5$ $c_1 = 1.69$	4.912 (0.039)	2.893 (0.025)	3.859 (0.031)	2.558 (0.023)	3.510 (0.029)	2.453 (0.023)
$\tau = 2.0$ $c_1 = 1.96$	3.055 (0.023)	1.929 (0.015)	2.551 (0.018)	1.820 (0.015)	2.365 (0.017)	1.767 (0.014)
$\tau = 2.5$ $c_1 = 2.25$	2.212 (0.015)	1.529 (0.011)	1.926 (0.013)	1.478 (0.010)	1.808 (0.012)	1.450 (0.010)
$\tau = 3.0$ $c_1 = 2.56$	1.778 (0.011)	1.332 (0.008)	1.600 (0.009)	1.303 (0.008)	1.528 (0.009)	1.290 (0.008)
two means (+) and all variances						
$\tau = 0.5$ $c = 1.21$	29.140 (0.287)	24.157 (0.253)	21.336 (0.211)	16.293 (0.184)	19.369 (0.199)	14.290 (0.170)
$\tau = 1.0$ $c = 1.44$	7.990 (0.071)	5.097 (0.051)	6.118 (0.056)	3.999 (0.042)	5.503 (0.053)	3.807 (0.041)
$\tau = 1.5$ $c = 1.69$	3.643 (0.029)	2.275 (0.019)	2.986 (0.024)	2.076 (0.018)	2.739 (0.023)	2.009 (0.018)
$\tau = 2.0$ $c = 1.96$	2.274 (0.016)	1.528 (0.010)	1.938 (0.013)	1.476 (0.010)	1.810 (0.012)	1.438 (0.010)
$\tau = 2.5$ $c = 2.25$	1.668 (0.010)	1.251 (0.007)	1.491 (0.008)	1.228 (0.007)	1.418 (0.008)	1.210 (0.006)
$\tau = 3.0$ $c = 2.56$	1.374 (0.007)	1.124 (0.004)	1.268 (0.006)	1.108 (0.004)	1.231 (0.005)	1.101 (0.004)
two means (+-) and all variances						
$\tau = 0.5$ $c = 1.21$	36.051 (0.358)	31.078 (0.322)	27.062 (0.273)	21.638 (0.245)	24.608 (0.256)	18.794 (0.227)
$\tau = 1.0$ $c = 1.44$	12.290 (0.114)	8.493 (0.087)	9.346 (0.088)	6.246 (0.068)	8.264 (0.081)	5.656 (0.065)
$\tau = 1.5$ $c = 1.69$	6.104 (0.053)	3.764 (0.036)	4.814 (0.043)	3.178 (0.032)	4.355 (0.040)	3.030 (0.032)
$\tau = 2.0$ $c = 1.96$	3.855 (0.032)	2.342 (0.020)	3.140 (0.026)	2.113 (0.019)	2.848 (0.024)	2.048 (0.018)
$\tau = 2.5$ $c = 2.25$	2.758 (0.021)	1.755 (0.013)	2.313 (0.017)	1.667 (0.013)	2.131 (0.016)	1.623 (0.013)
$\tau = 3.0$ $c = 2.56$	2.160 (0.015)	1.470 (0.010)	1.867 (0.012)	1.430 (0.010)	1.747 (0.011)	1.402 (0.010)

Table 7.13: ATS Values for the FSI and VSI Separate EWMA Charts ($\rho = 2, \rho = 0.0$)

shifts	h1 = 1.3302	h2 = 8.7756	h1 = 0.6882	h2 = 6.0548	h1 = 0.4370	h2 = 5.2055
	g1 = 0.5846	g2 = 5.0914	g1 = 0.3156	g2 = 4.5030	g1 = 0.2426	g2 = 4.1726
	$\lambda = 0.3$		$\lambda = 0.1$		$\lambda = 0.05$	
	FSI	VSI	FSI	VSI	FSI	VSI
.0	199.954 (1.947)	199.980 (1.937)	200.666 (1.825)	199.527 (1.777)	199.890 (1.647)	200.446 (1.490)
one mean						
$\tau = 0.5$	58.134 (0.540)	46.692 (0.434)	33.940 (0.238)	27.624 (0.193)	30.467 (0.172)	32.058 (0.193)
$\tau = 1.0$	13.474 (0.098)	8.349 (0.061)	11.038 (0.051)	8.730 (0.047)	11.942 (0.044)	12.008 (0.053)
$\tau = 1.5$	6.001 (0.033)	3.673 (0.023)	6.446 (0.022)	5.153 (0.023)	7.438 (0.021)	7.371 (0.027)
$\tau = 2.0$	3.785 (0.016)	2.347 (0.014)	4.601 (0.013)	3.653 (0.015)	5.450 (0.013)	5.335 (0.018)
$\tau = 2.5$	2.787 (0.010)	1.726 (0.010)	3.606 (0.009)	2.835 (0.012)	4.343 (0.009)	4.164 (0.013)
$\tau = 3.0$	2.251 (0.007)	1.379 (0.007)	3.003 (0.007)	2.259 (0.010)	3.633 (0.007)	3.436 (0.009)
two mean (+)						
$\tau = 0.5$	65.677 (0.610)	52.059 (0.481)	38.819 (0.277)	29.652 (0.204)	34.277 (0.195)	34.047 (0.200)
$\tau = 1.0$	17.231 (0.132)	9.713 (0.072)	13.252 (0.061)	9.389 (0.049)	14.201 (0.052)	13.228 (0.058)
$\tau = 1.5$	7.596 (0.044)	4.053 (0.024)	7.786 (0.027)	5.647 (0.025)	8.973 (0.025)	8.399 (0.030)
$\tau = 2.0$	4.737 (0.021)	2.579 (0.014)	5.595 (0.016)	4.116 (0.016)	6.649 (0.016)	6.214 (0.020)
$\tau = 2.5$	3.446 (0.013)	1.919 (0.010)	4.416 (0.011)	3.248 (0.012)	5.328 (0.011)	4.958 (0.015)
$\tau = 3.0$	2.742 (0.009)	1.514 (0.008)	3.672 (0.008)	2.684 (0.010)	4.479 (0.008)	4.123 (0.012)
two mean (+)						
$\tau = 0.5$	66.096 (0.615)	52.359 (0.489)	39.259 (0.286)	29.839 (0.210)	34.406 (0.199)	34.051 (0.201)
$\tau = 1.0$	17.332 (0.134)	9.747 (0.072)	13.353 (0.062)	9.453 (0.049)	14.215 (0.052)	13.342 (0.057)
$\tau = 1.5$	7.680 (0.044)	4.068 (0.025)	7.831 (0.027)	5.671 (0.025)	8.998 (0.025)	8.421 (0.030)
$\tau = 2.0$	4.715 (0.021)	2.572 (0.014)	5.601 (0.016)	4.083 (0.016)	6.679 (0.016)	6.243 (0.020)
$\tau = 2.5$	3.438 (0.012)	1.921 (0.010)	4.403 (0.011)	3.246 (0.012)	5.340 (0.011)	4.941 (0.014)
$\tau = 3.0$	2.756 (0.009)	1.525 (0.008)	3.680 (0.008)	2.695 (0.010)	4.481 (0.009)	4.114 (0.011)

Table 7.13: (Continued)

<i>shifts</i>	$\lambda = 0.3$		$\lambda = 0.1$		$\lambda = 0.05$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
$c = 1.21$	47.479 (0.424)	31.733 (0.273)	45.251 (0.281)	34.487 (0.175)	55.939 (0.248)	54.275 (0.195)
$c = 1.44$	18.145 (0.138)	9.880 (0.067)	22.355 (0.098)	18.504 (0.076)	32.226 (0.104)	34.881 (0.116)
$c = 1.69$	9.954 (0.065)	5.459 (0.032)	14.846 (0.055)	13.121 (0.052)	22.889 (0.066)	25.969 (0.085)
$c = 1.96$	6.626 (0.038)	3.185 (0.022)	11.114 (0.037)	10.166 (0.042)	17.712 (0.048)	20.435 (0.068)
$c = 2.25$	4.940 (0.026)	2.982 (0.017)	8.804 (0.029)	8.188 (0.034)	14.321 (0.038)	16.684 (0.056)
$c = 2.56$	3.929 (0.020)	2.440 (0.014)	7.257 (0.023)	6.780 (0.029)	11.951 (0.031)	14.075 (0.048)
$c1 = 1.21$	74.485 (0.687)	59.865 (0.546)	65.447 (0.476)	53.872 (0.346)	71.909 (0.388)	70.233 (0.300)
$c1 = 1.44$	29.764 (0.251)	20.174 (0.162)	30.323 (0.166)	25.901 (0.123)	39.868 (0.152)	43.478 (0.154)
$c1 = 1.69$	15.773 (0.120)	10.063 (0.070)	19.271 (0.086)	17.679 (0.076)	27.537 (0.089)	31.897 (0.108)
$c1 = 1.96$	10.112 (0.070)	6.550 (0.042)	14.135 (0.057)	13.600 (0.057)	21.121 (0.063)	25.340 (0.084)
$c1 = 2.25$	7.251 (0.045)	4.857 (0.030)	11.139 (0.042)	11.111 (0.047)	16.996 (0.049)	20.928 (0.070)
$c1 = 2.56$	5.669 (0.033)	3.896 (0.024)	9.133 (0.034)	9.298 (0.040)	14.191 (0.041)	17.675 (0.060)
one mean and $\tau = 0.5$ $c = 1.21$	28.783 (0.245)	17.469 (0.140)	25.204 (0.147)	19.420 (0.106)	27.607 (0.142)	28.405 (0.150)
$\tau = 1.0$ $c = 1.44$	8.889 (0.058)	4.953 (0.031)	10.312 (0.048)	8.259 (0.0345)	11.834 (0.049)	4.953 (0.062)
$\tau = 1.5$ $c = 1.69$	4.872 (0.026)	2.897 (0.017)	6.343 (0.025)	5.166 (0.027)	7.494 (0.027)	7.571 (0.035)
$\tau = 2.0$ $c = 1.96$	3.363 (0.015)	2.056 (0.012)	4.633 (0.017)	3.704 (0.020)	5.541 (0.018)	2.056 (0.024)
$\tau = 2.5$ $c = 2.25$	2.600 (0.011)	1.640 (0.009)	3.669 (0.012)	2.873 (0.015)	4.435 (0.014)	4.355 (0.019)
$\tau = 3.0$ $c = 2.56$	3.124 (0.009)	1.387 (0.007)	3.071 (0.010)	2.355 (0.013)	7.717 (0.011)	3.575 (0.015)

Table 7.13: (Continued)

<i>shifts</i>	$\lambda = 0.3$		$\lambda = 0.1$		$\lambda = 0.05$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
one mean and corresponding variance						
$\tau = 0.5$ $c_1 = 1.21$	35.659 (0.314)	25.155 (0.216)	27.718 (0.176)	22.456 (0.137)	28.604 (0.156)	30.191 (0.170)
$\tau = 1.0$ $c_1 = 1.44$	10.049 (0.070)	6.276 (0.043)	10.556 (0.051)	8.829 (0.050)	11.916 (0.050)	12.469 (0.064)
$\tau = 1.5$ $c_1 = 1.69$	5.260 (0.029)	3.414 (0.022)	6.424 (0.026)	5.420 (0.030)	7.511 (0.027)	7.694 (0.036)
$\tau = 2.0$ $c_1 = 1.96$	3.569 (0.017)	2.367 (0.015)	4.665 (0.017)	3.884 (0.021)	5.552 (0.018)	5.606 (0.025)
$\tau = 2.5$ $c_1 = 2.25$	2.729 (0.012)	1.855 (0.011)	3.693 (0.013)	2.980 (0.016)	4.438 (0.014)	4.398 (0.019)
$\tau = 3.0$ $c_1 = 2.56$	3.223 (0.009)	1.537 (0.009)	3.087 (0.010)	2.441 (0.014)	3.719 (0.011)	3.607 (0.015)
two means (+) and all variances						
$\tau = 0.5$ $c = 1.21$	65.677 (0.610)	52.059 (0.481)	38.819 (0.277)	29.652 (0.204)	34.277 (0.195)	34.047 (0.200)
$\tau = 1.0$ $c = 1.44$	9.502 (0.062)	5.078 (0.031)	11.246 (0.051)	8.525 (0.045)	13.252 (0.054)	12.860 (0.063)
$\tau = 1.5$ $c = 1.69$	5.206 (0.028)	2.969 (0.017)	7.040 (0.028)	5.349 (0.028)	8.470 (0.030)	8.118 (0.036)
$\tau = 2.0$ $c = 1.96$	3.604 (0.017)	2.119 (0.013)	5.144 (0.018)	3.894 (0.020)	6.297 (0.020)	5.978 (0.025)
$\tau = 2.5$ $c = 2.25$	2.779 (0.012)	1.665 (0.009)	4.100 (0.014)	3.039 (0.016)	5.052 (0.015)	4.724 (0.019)
$\tau = 3.0$ $c = 2.56$	2.270 (0.009)	1.407 (0.007)	3.416 (0.011)	2.473 (0.013)	4.238 (0.012)	3.912 (0.016)
two means (-) and all variances						
$\tau = 0.5$ $c = 1.21$	29.799 (0.256)	17.873 (0.144)	26.749 (0.158)	19.661 (0.106)	29.372 (0.149)	29.130 (0.149)
$\tau = 1.0$ $c = 1.44$	9.639 (0.065)	5.102 (0.031)	11.397 (0.051)	8.658 (0.046)	13.294 (0.054)	12.913 (0.062)
$\tau = 1.5$ $c = 1.69$	5.240 (0.028)	2.951 (0.018)	7.065 (0.027)	5.403 (0.028)	8.512 (0.030)	8.181 (0.036)
$\tau = 2.0$ $c = 1.96$	3.609 (0.017)	2.106 (0.013)	5.152 (0.018)	3.870 (0.020)	6.329 (0.020)	6.015 (0.025)
$\tau = 2.5$ $c = 2.25$	2.773 (0.012)	1.675 (0.010)	4.086 (0.014)	3.024 (0.016)	5.057 (0.015)	4.743 (0.020)
$\tau = 3.0$ $c = 2.56$	2.273 (0.009)	1.408 (0.007)	3.417 (0.011)	2.461 (0.013)	4.238 (0.012)	3.901 (0.016)

Table 7.14: ATS Values for the FSI and VSI Separate EWMA Charts ($p = 2, \rho = 0.5$)

shifts	h1 = 1.3286	h2 = 8.7689	h1 = 0.6740	h2 = 6.0353	h1 = 0.4342	h2 = 5.1924
	g1 = 0.5662	g2 = 5.0154	g1 = 0.3065	g2 = 4.4704	g1 = 0.2290	g2 = 4.1528
	$\lambda = 0.3$		$\lambda = 0.1$		$\lambda = 0.05$	
	FSI	VSI	FSI	VSI	FSI	VSI
.0	199.714 (1.951)	200.423 (1.960)	199.391 (1.848)	200.455 (1.820)	200.180 (1.641)	199.440 (1.497)
one mean						
$\tau = 0.5$	74.187 (0.685)	60.933 (0.565)	42.746 (0.317)	33.834 (0.238)	36.798 (0.223)	36.203 (0.220)
$\tau = 1.0$	18.390 (0.148)	11.099 (0.087)	13.564 (0.069)	10.170 (0.055)	14.176 (0.057)	13.412 (0.062)
$\tau = 1.5$	7.769 (0.048)	4.393 (0.027)	7.636 (0.028)	5.843 (0.027)	8.640 (0.027)	8.214 (0.032)
$\tau = 2.0$	4.671 (0.022)	2.754 (0.016)	5.368 (0.016)	4.132 (0.017)	6.292 (0.016)	5.912 (0.021)
$\tau = 2.5$	3.366 (0.013)	2.013 (0.011)	4.170 (0.011)	3.229 (0.013)	4.983 (0.011)	4.616 (0.015)
$\tau = 3.0$	2.653 (0.009)	1.577 (0.008)	3.452 (0.008)	2.623 (0.011)	4.142 (0.009)	3.788 (0.012)
two mean (+)						
$\tau = 0.5$	51.012 (0.460)	39.613 (0.359)	31.283 (0.218)	24.688 (0.174)	28.845 (0.162)	28.107 (0.173)
$\tau = 1.0$	12.729 (0.094)	7.454 (0.055)	10.882 (0.049)	8.096 (0.045)	11.969 (0.043)	10.976 (0.050)
$\tau = 1.5$	5.969 (0.032)	3.365 (0.021)	6.525 (0.022)	4.847 (0.022)	7.617 (0.021)	6.930 (0.026)
$\tau = 2.0$	3.844 (0.016)	2.207 (0.013)	4.723 (0.013)	3.502 (0.015)	5.654 (0.013)	5.092 (0.017)
$\tau = 2.5$	2.864 (0.010)	1.639 (0.009)	3.746 (0.009)	2.745 (0.012)	4.540 (0.010)	4.061 (0.013)
$\tau = 3.0$	2.333 (0.007)	1.324 (0.006)	3.146 (0.007)	2.216 (0.010)	3.824 (0.007)	3.385 (0.009)
two mean (+)						
$\tau = 0.5$	103.065 (0.990)	86.172 (0.828)	60.716 (0.481)	44.544 (0.327)	49.232 (0.303)	44.169 (0.252)
$\tau = 1.0$	34.133 (0.301)	18.671 (0.155)	19.970 (0.108)	12.589 (0.063)	19.645 (0.077)	16.700 (0.071)
$\tau = 1.5$	14.052 (0.100)	6.141 (0.037)	11.174 (0.043)	7.307 (0.031)	12.292 (0.037)	10.505 (0.037)
$\tau = 2.0$	7.823 (0.043)	3.522 (0.018)	7.816 (0.024)	5.268 (0.020)	9.033 (0.023)	7.796 (0.024)
$\tau = 2.5$	5.282 (0.023)	2.545 (0.013)	6.037 (0.016)	4.165 (0.015)	7.183 (0.016)	6.210 (0.018)
$\tau = 3.0$	4.009 (0.015)	1.991 (0.010)	4.962 (0.012)	3.444 (0.012)	6.003 (0.012)	5.207 (0.014)

Table 7.14: (Continued)

<i>shifts</i>	$\lambda = 0.3$		$\lambda = 0.1$		$\lambda = 0.05$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
$c = 1.21$	48.980 (0.444)	33.478 (0.293)	46.621 (0.296)	35.980 (0.190)	56.598 (0.258)	54.238 (0.205)
$c = 1.44$	18.963 (0.149)	10.548 (0.075)	22.783 (0.103)	19.016 (0.080)	32.777 (0.110)	34.653 (0.120)
$c = 1.69$	10.406 (0.069)	5.754 (0.035)	15.237 (0.058)	13.481 (0.055)	23.303 (0.069)	25.835 (0.087)
$c = 1.96$	6.956 (0.038)	3.972 (0.022)	11.377 (0.037)	10.393 (0.042)	18.030 (0.048)	20.341 (0.068)
$c = 2.25$	5.168 (0.028)	3.114 (0.018)	8.987 (0.031)	8.332 (0.036)	14.546 (0.040)	16.623 (0.058)
$c = 2.56$	4.516 (0.020)	2.762 (0.014)	8.053 (0.023)	7.495 (0.029)	13.008 (0.031)	14.953 (0.048)
$cl = 1.21$	41.564 (0.366)	29.608 (0.256)	32.122 (0.210)	25.253 (0.151)	33.029 (0.180)	32.829 (0.178)
$cl = 1.44$	22.355 (0.184)	14.293 (0.111)	22.236 (0.117)	18.500 (0.095)	27.247 (0.123)	28.095 (0.135)
$cl = 1.69$	13.402 (0.098)	8.408 (0.057)	16.454 (0.074)	14.590 (0.069)	22.165 (0.086)	23.917 (0.106)
$cl = 1.96$	9.210 (0.061)	5.869 (0.038)	12.778 (0.052)	11.891 (0.055)	18.342 (0.063)	20.617 (0.085)
$cl = 2.25$	6.815 (0.043)	4.489 (0.028)	10.353 (0.040)	9.978 (0.045)	15.419 (0.050)	17.851 (0.071)
$cl = 2.56$	5.416 (0.032)	3.708 (0.023)	8.667 (0.033)	8.543 (0.039)	13.211 (0.041)	15.532 (0.061)
one mean and $\tau = 0.5$ $c = 1.21$	all variances 32.959 (0.281)	20.339 (0.165)	28.476 (0.169)	21.453 (0.115)	31.426 (0.159)	30.565 (0.154)
$\tau = 1.0$ $c = 1.44$	10.424 (0.071)	5.568 (0.034)	11.854 (0.055)	9.218 (0.048)	13.807 (0.060)	13.370 (0.067)
$\tau = 1.5$ $c = 1.69$	5.605 (0.031)	3.188 (0.019)	7.279 (0.030)	5.716 (0.030)	8.617 (0.033)	8.316 (0.039)
$\tau = 2.0$ $c = 1.96$	3.828 (0.019)	2.241 (0.013)	5.295 (0.020)	4.109 (0.021)	6.343 (0.022)	6.057 (0.013)
$\tau = 2.5$ $c = 2.25$	2.915 (0.013)	1.756 (0.010)	4.173 (0.015)	3.227 (0.017)	5.059 (0.016)	4.744 (0.021)
$\tau = 3.0$ $c = 2.56$	2.438 (0.010)	1.523 (0.008)	3.493 (0.012)	2.634 (0.014)	4.244 (0.013)	3.973 (0.018)

Table 7.14: (Continued)

<i>shifts</i>	$\lambda = 0.3$		$\lambda = 0.1$		$\lambda = 0.05$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
one mean and	corresponding variance					
$\tau = 0.5$ $c1 = 1.21$	41.564 (0.366)	29.608 (0.256)	32.122 (0.210)	25.253 (0.151)	33.029 (0.180)	32.829 (0.178)
$\tau = 1.0$ $c1 = 1.44$	12.069 (0.089)	7.240 (0.050)	12.402 (0.062)	10.000 (0.055)	14.018 (0.062)	13.894 (0.072)
$\tau = 1.5$ $c1 = 1.69$	6.188 (0.037)	3.830 (0.024)	7.464 (0.032)	6.127 (0.034)	8.690 (0.034)	8.578 (0.042)
$\tau = 2.0$ $c1 = 1.96$	4.138 (0.021)	2.655 (0.017)	5.378 (0.021)	4.373 (0.024)	6.380 (0.023)	6.218 (0.029)
$\tau = 2.5$ $c1 = 2.25$	3.122 (0.015)	2.039 (0.012)	4.228 (0.016)	3.419 (0.019)	5.086 (0.017)	4.859 (0.022)
$\tau = 3.0$ $c1 = 2.56$	2.531 (0.011)	1.690 (0.010)	3.512 (0.012)	2.769 (0.015)	4.247 (0.013)	4.047 (0.019)
two means (+)	+) and all variances					
$\tau = 0.5$ $c = 1.21$	26.705 (0.223)	16.366 (0.130)	23.747 (0.140)	18.385 (0.106)	25.989 (0.135)	25.652 (0.144)
$\tau = 1.0$ $c = 1.44$	8.405 (0.055)	4.733 (0.030)	9.897 (0.046)	7.768 (0.044)	11.529 (0.048)	11.028 (0.058)
$\tau = 1.5$ $c = 1.69$	4.693 (0.025)	2.749 (0.017)	6.178 (0.025)	4.790 (0.027)	7.403 (0.027)	6.922 (0.033)
$\tau = 2.0$ $c = 1.96$	3.261 (0.015)	1.986 (0.012)	4.533 (0.016)	3.460 (0.019)	5.517 (0.018)	5.085 (0.023)
$\tau = 2.5$ $c = 2.25$	2.537 (0.011)	1.585 (0.009)	3.633 (0.012)	2.688 (0.015)	4.448 (0.013)	4.016 (0.018)
$\tau = 3.0$ $c = 2.56$	2.108 (0.009)	1.350 (0.007)	3.037 (0.010)	2.164 (0.012)	3.724 (0.010)	3.300 (0.014)
two means (+)	-) and all variances					
$\tau = 0.5$ $c = 1.21$	37.527 (0.331)	23.008 (0.193)	32.920 (0.201)	23.359 (0.122)	36.935 (0.183)	33.808 (0.158)
$\tau = 1.0$ $c = 1.44$	12.506 (0.088)	6.178 (0.038)	14.436 (0.064)	10.308 (0.050)	17.181 (0.069)	15.511 (0.071)
$\tau = 1.5$ $c = 1.69$	6.694 (0.038)	3.392 (0.020)	9.014 (0.035)	6.444 (0.031)	11.058 (0.039)	9.825 (0.042)
$\tau = 2.0$ $c = 1.96$	4.480 (0.022)	2.421 (0.014)	6.559 (0.023)	4.688 (0.022)	8.160 (0.026)	7.187 (0.029)
$\tau = 2.5$ $c = 2.25$	3.390 (0.015)	1.874 (0.011)	5.151 (0.017)	3.635 (0.017)	6.507 (0.020)	5.708 (0.023)
$\tau = 3.0$ $c = 2.56$	2.853 (0.012)	1.616 (0.009)	4.355 (0.014)	3.062 (0.015)	5.476 (0.016)	4.781 (0.019)

Table 7.15: ATS Values for Matched FSI and VSI Separate EWMA Charts ($p = 2, \rho = 0.8$)

<i>shifts</i>	h1 = 1.3174	h2 = 8.6877	h1 = 0.6636	h2 = 5.9858	h1 = 0.4261	h2 = 5.1542
	g1 = 0.5273	g2 = 4.8504	g1 = 0.2845	g2 = 4.3890	g1 = 0.2083	g2 = 4.1079
	$\lambda = 0.3$		$\lambda = 0.1$		$\lambda = 0.05$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
.0	199.714 (1.947)	200.054 (1.951)	199.409 (1.818)	200.524 (1.800)	200.588 (1.675)	200.587 (1.563)
one mean						
$\tau = 0.5$	114.098 (1.086)	104.130 (0.979)	71.499 (0.589)	57.658 (0.444)	58.576 (0.405)	54.292 (0.333)
$\tau = 1.0$	38.632 (0.349)	25.852 (0.228)	23.382 (0.146)	15.728 (0.088)	22.209 (0.108)	18.964 (0.092)
$\tau = 1.5$	16.360 (0.126)	8.338 (0.057)	12.408 (0.060)	8.382 (0.040)	13.111 (0.050)	8.338 (0.048)
$\tau = 2.0$	8.951 (0.057)	4.330 (0.025)	8.338 (0.033)	5.885 (0.026)	9.269 (0.030)	8.063 (0.031)
$\tau = 2.5$	5.838 (0.031)	2.998 (0.016)	6.283 (0.021)	4.507 (0.019)	7.223 (0.021)	6.341 (0.023)
$\tau = 3.0$	4.377 (0.020)	2.295 (0.012)	5.034 (0.015)	3.643 (0.015)	5.928 (0.015)	5.143 (0.018)
two mean (+ +)						
$\tau = 0.5$	46.290 (0.417)	35.861 (0.329)	28.967 (0.199)	22.959 (0.165)	27.042 (0.150)	25.350 (0.161)
$\tau = 1.0$	11.677 (0.084)	6.707 (0.049)	10.227 (0.046)	7.480 (0.043)	11.167 (0.041)	9.799 (0.046)
$\tau = 1.5$	3.300 (0.017)	2.048 (0.013)	5.169 (0.020)	4.125 (0.024)	6.727 (0.026)	6.267 (0.035)
$\tau = 2.0$	2.775 (0.013)	1.766 (0.011)	4.165 (0.016)	3.164 (0.019)	5.173 (0.018)	4.629 (0.024)
$\tau = 2.5$	2.385 (0.011)	1.511 (0.009)	3.440 (0.012)	2.504 (0.015)	4.183 (0.014)	3.640 (0.018)
$\tau = 3.0$	2.043 (0.009)	1.343 (0.007)	2.914 (0.010)	2.042 (0.012)	3.536 (0.010)	2.973 (0.014)
two mean (+ -)						
$\tau = 0.5$	144.176 (1.370)	129.390 (1.229)	98.504 (0.845)	77.844 (0.630)	79.598 (0.566)	67.379 (0.407)
$\tau = 1.0$	71.413 (0.667)	47.373 (0.433)	38.064 (0.258)	20.785 (0.109)	32.229 (0.156)	23.661 (0.100)
$\tau = 1.5$	35.185 (0.310)	15.446 (0.120)	19.697 (0.099)	10.676 (0.044)	19.329 (0.069)	14.441 (0.052)
$\tau = 2.0$	18.780 (0.144)	6.523 (0.036)	13.006 (0.050)	7.451 (0.028)	13.965 (0.041)	10.618 (0.034)
$\tau = 2.5$	11.651 (0.077)	4.107 (0.019)	9.714 (0.031)	5.769 (0.020)	10.931 (0.028)	8.398 (0.025)
$\tau = 3.0$	8.027 (0.043)	3.096 (0.014)	7.811 (0.022)	4.806 (0.016)	9.048 (0.020)	7.061 (0.020)

Table 7.15: (Continued)

<i>shifts</i>	$\lambda = 0.3$		$\lambda = 0.1$		$\lambda = 0.05$	
	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>	<i>FSI</i>	<i>VSI</i>
$c = 1.21$	52.579 (0.470)	36.780 (0.328)	48.910 (0.328)	38.193 (0.220)	59.133 (0.228)	55.294 (0.227)
$c = 1.44$	20.724 (0.166)	11.903 (0.089)	23.748 (0.115)	19.870 (0.089)	34.048 (0.122)	35.053 (0.166)
$c = 1.69$	11.182 (0.077)	6.249 (0.040)	15.776 (0.065)	13.927 (0.061)	24.050 (0.075)	26.020 (0.094)
$c = 1.96$	7.465 (0.046)	4.320 (0.027)	11.829 (0.044)	10.711 (0.047)	18.575 (0.055)	20.342 (0.075)
$c = 2.25$	5.544 (0.032)	3.343 (0.020)	9.418 (0.033)	8.587 (0.038)	14.976 (0.043)	16.737 (0.062)
$c = 2.56$	4.425 (0.024)	2.720 (0.017)	7.756 (0.027)	7.166 (0.033)	12.534 (0.036)	14.074 (0.053)
$cl = 1.21$	75.238 (0.705)	60.764 (0.568)	63.858 (0.467)	52.971 (0.355)	71.027 (0.380)	67.080 (0.302)
$cl = 1.44$	29.557 (0.257)	19.977 (0.167)	29.443 (0.160)	24.974 (0.121)	38.686 (0.147)	40.580 (0.152)
$cl = 1.69$	15.388 (0.117)	9.699 (0.069)	18.744 (0.083)	16.963 (0.074)	26.963 (0.088)	29.885 (0.107)
$cl = 1.96$	9.808 (0.066)	6.188 (0.040)	13.748 (0.054)	13.062 (0.057)	20.748 (0.062)	23.974 (0.084)
$cl = 2.25$	7.097 (0.044)	4.583 (0.029)	10.853 (0.041)	10.599 (0.046)	16.709 (0.049)	19.615 (0.070)
$cl = 2.56$	5.511 (0.032)	3.703 (0.023)	8.862 (0.033)	8.856 (0.040)	13.931 (0.040)	16.576 (0.060)
one mean and $\tau = 0.5$ $c = 1.21$	all variances 41.674 (0.375)	26.871 (0.234)	36.481 (0.236)	26.729 (0.146)	41.204 (0.208)	37.641 (0.181)
$\tau = 1.0$ $c = 1.44$	14.107 (0.103)	7.413 (0.049)	16.087 (0.078)	11.765 (0.058)	19.366 (0.086)	17.416 (0.084)
$\tau = 1.5$ $c = 1.69$	7.584 (0.047)	3.942 (0.023)	10.055 (0.044)	7.344 (0.036)	12.299 (0.050)	10.976 (0.051)
$\tau = 2.0$ $c = 1.96$	5.074 (0.028)	2.760 (0.016)	7.262 (0.029)	5.351 (0.026)	8.958 (0.034)	7.942 (0.036)
$\tau = 2.5$ $c = 2.25$	3.827 (0.019)	2.148 (0.013)	5.667 (0.021)	4.130 (0.021)	7.108 (0.026)	6.205 (0.029)
$\tau = 3.0$ $c = 2.56$	3.073 (0.014)	1.766 (0.010)	4.659 (0.017)	3.387 (0.017)	5.884 (0.021)	5.089 (0.023)

Table 7.15: (Continued)

shifts	$\lambda = 0.3$		$\lambda = 0.1$		$\lambda = 0.05$	
	FSI	VSI	FSI	VSI	FSI	VSI
one mean and corresponding variance						
$\tau = 0.5$ $c1 = 1.21$	53.447 (0.487)	39.485 (0.357)	42.236 (0.289)	32.416 (0.195)	44.578 (0.245)	41.111 (0.203)
$\tau = 1.0$ $c1 = 1.44$	17.353 (0.136)	10.068 (0.073)	17.362 (0.091)	13.182 (0.068)	20.159 (0.095)	18.658 (0.095)
$\tau = 1.5$ $c1 = 1.69$	8.680 (0.058)	4.962 (0.031)	10.535 (0.048)	8.221 (0.042)	12.674 (0.055)	11.723 (0.058)
$\tau = 2.0$ $c1 = 1.96$	5.726 (0.033)	3.364 (0.020)	7.576 (0.033)	5.906 (0.031)	9.185 (0.038)	8.501 (0.042)
$\tau = 2.5$ $c1 = 2.25$	4.227 (0.022)	2.583 (0.016)	5.887 (0.024)	4.631 (0.025)	7.261 (0.028)	6.675 (0.033)
$\tau = 3.0$ $c1 = 2.56$	3.373 (0.017)	2.098 (0.012)	4.857 (0.019)	3.810 (0.021)	5.991 (0.022)	5.509 (0.027)
two means (+) and all variances						
$\tau = 0.5$ $c = 1.21$	26.273 (0.220)	16.673 (0.138)	23.093 (0.140)	17.981 (0.109)	25.048 (0.134)	23.792 (0.142)
$\tau = 1.0$ $c = 1.44$	8.311 (0.055)	4.681 (0.032)	9.538 (0.045)	7.344 (0.044)	10.950 (0.046)	10.017 (0.055)
$\tau = 1.5$ $c = 1.69$	3.300 (0.017)	2.048 (0.013)	5.169 (0.020)	4.125 (0.024)	6.727 (0.026)	6.267 (0.035)
$\tau = 2.0$ $c = 1.96$	3.186 (0.015)	1.933 (0.012)	4.346 (0.016)	3.230 (0.018)	5.209 (0.017)	5.583 (0.022)
$\tau = 2.5$ $c = 2.25$	2.491 (0.011)	1.562 (0.009)	3.475 (0.012)	2.497 (0.015)	4.196 (0.013)	3.625 (0.017)
$\tau = 3.0$ $c = 2.56$	2.043 (0.009)	1.343 (0.007)	2.914 (0.010)	2.042 (0.012)	3.536 (0.010)	2.973 (0.007)
two means (+-) and all variances						
$\tau = 0.5$ $c = 1.21$	44.728 (0.402)	29.194 (0.254)	40.364 (0.261)	28.712 (0.156)	46.197 (0.226)	40.488 (0.177)
$\tau = 1.0$ $c = 1.44$	16.299 (0.125)	8.201 (0.055)	18.542 (0.087)	12.889 (0.058)	23.499 (0.095)	19.858 (0.085)
$\tau = 1.5$ $c = 1.69$	8.752 (0.056)	4.265 (0.025)	11.889 (0.049)	8.217 (0.037)	15.379 (0.057)	12.730 (0.053)
$\tau = 2.0$ $c = 1.96$	5.803 (0.033)	2.932 (0.016)	8.640 (0.032)	5.924 (0.027)	11.368 (0.039)	9.624 (0.037)
$\tau = 2.5$ $c = 2.25$	4.356 (0.022)	2.285 (0.013)	6.836 (0.024)	4.643 (0.021)	9.026 (0.030)	7.295 (0.029)
$\tau = 3.0$ $c = 2.56$	4.356 (0.022)	2.285 (0.013)	6.836 (0.024)	4.643 (0.021)	9.026 (0.030)	7.295 (0.029)

Table 7.16: ATS Values for FSI and VSI Charts based on both MEWMA and EWMA Using (7.5) ($p = 2, \lambda = 0.05$ and $\rho = 0.5$)

		$h1 = 9.2854$ $g1 = 2.2596$			$h2 = 9.6327$ $g2 = 2.2547$			
shifts	FSI	VSI	shifts	FSI	VSI	shifts	FSI	VSI
τ	200.630 (2.192)	199.902 (2.292)		200.630 (2.192)	199.902 (2.292)		200.630 (2.192)	199.902 (2.292)
M			VV			MV		
$\tau = 0.5$	24.528 (0.197)	14.606 (0.147)	$c = 1.21$	24.892 (0.253)	17.703 (0.209)	$\tau = 0.5$ $c1 = 1.21$	17.474 (0.152)	10.516 (0.109)
$\tau = 1.0$	7.877 (0.053)	4.272 (0.038)	$c = 1.44$	8.339 (0.079)	5.234 (0.059)	$\tau = 1.0$ $c1 = 1.44$	5.732 (0.046)	3.365 (0.032)
$\tau = 1.5$	4.112 (0.024)	2.300 (0.018)	$c = 1.69$	4.432 (0.039)	2.857 (0.029)	$\tau = 1.5$ $c1 = 1.69$	3.073 (0.022)	1.958 (0.016)
$\tau = 2.0$	2.631 (0.014)	1.582 (0.010)	$c = 1.96$	2.873 (0.024)	1.956 (0.018)	$\tau = 2.0$ $c1 = 1.96$	2.080 (0.013)	1.484 (0.010)
$\tau = 2.5$	1.898 (0.009)	1.257 (0.006)	$c = 2.25$	2.157 (0.016)	1.558 (0.012)	$\tau = 2.5$ $c1 = 2.25$	1.619 (0.009)	1.258 (0.007)
$\tau = 3.0$	1.508 (0.007)	1.113 (0.004)	$c = 2.56$	1.924 (0.013)	1.421 (0.010)	$\tau = 3.0$ $c1 = 2.56$	1.370 (0.007)	1.144 (0.005)
MM (+ +)			V			MM (+ +)VV		
$\tau = 0.5$	24.367 (0.195)	14.481 (0.142)	$c1 = 1.21$	38.006 (0.387)	28.809 (0.337)	$\tau = 0.5$ $c = 1.21$	14.017 (0.128)	8.277 (0.090)
$\tau = 1.0$	7.788 (0.053)	4.244 (0.037)	$c1 = 1.44$	13.254 (0.121)	8.754 (0.095)	$\tau = 1.0$ $c = 1.44$	4.531 (0.037)	2.683 (0.025)
$\tau = 1.5$	4.040 (0.024)	2.282 (0.018)	$c1 = 1.69$	6.963 (0.061)	4.561 (0.049)	$\tau = 1.5$ $c = 1.69$	2.465 (0.018)	1.646 (0.013)
$\tau = 2.0$	2.574 (0.014)	1.571 (0.010)	$c1 = 1.96$	4.545 (0.038)	3.055 (0.030)	$\tau = 2.0$ $c = 1.96$	1.705 (0.011)	1.269 (0.007)
$\tau = 2.5$	1.872 (0.009)	1.254 (0.006)	$c1 = 2.25$	3.332 (0.027)	2.333 (0.022)	$\tau = 2.5$ $c = 2.25$	1.367 (0.007)	1.128 (0.005)
$\tau = 3.0$	1.493 (0.007)	1.110 (0.004)	$c1 = 2.56$	2.625 (0.020)	1.911 (0.017)	$\tau = 3.0$ $c = 2.56$	1.212 (0.005)	1.064 (0.003)
MM (+ -)			MVV			MM (+ -)VV		
$\tau = 0.5$	24.631 (0.201)	14.487 (0.145)	$\tau = 0.5$ $c = 1.21$	14.275 (0.129)	8.369 (0.090)	$\tau = 0.5$ $c = 1.21$	14.393 (0.132)	8.468 (0.092)
$\tau = 1.0$	7.913 (0.053)	4.249 (0.038)	$\tau = 1.0$ $c = 1.44$	4.679 (0.039)	2.721 (0.026)	$\tau = 1.0$ $c = 1.44$	4.717 (0.039)	2.740 (0.026)
$\tau = 1.5$	4.099 (0.025)	2.278 (0.018)	$\tau = 1.5$ $c = 1.69$	2.540 (0.018)	1.642 (0.012)	$\tau = 1.5$ $c = 1.69$	2.596 (0.019)	1.655 (0.013)
$\tau = 2.0$	2.629 (0.014)	1.570 (0.010)	$\tau = 2.0$ $c = 1.96$	1.755 (0.011)	1.287 (0.007)	$\tau = 2.0$ $c = 1.96$	1.801 (0.011)	1.298 (0.008)
$\tau = 2.5$	1.930 (0.009)	1.260 (0.006)	$\tau = 2.5$ $c = 2.25$	1.406 (0.007)	1.135 (0.005)	$\tau = 2.5$ $c = 2.25$	1.428 (0.008)	1.144 (0.005)
$\tau = 3.0$	1.526 (0.007)	1.113 (0.004)	$\tau = 3.0$ $c = 2.56$	1.364 (0.007)	1.141 (0.005)	$\tau = 3.0$ $c = 2.56$	1.269 (0.006)	1.085 (0.004)

Table 7.17: ATS Values for FSI and VSI EWMA Charts for the Mean Vector and Variance-Covariance Matrix ($p = 5, \rho = 0.5$)

shift τ	<i>MEWMA</i>		<i>Separate EWMA's</i>		<i>EWMA using(7.1)</i>		<i>EWMA using(7.5)</i>		<i>both (7.5) & MEWMA</i>			
	$\lambda = 0.05$		$\lambda = 0.1$		$\lambda = 0.1$		$\lambda = 0.05$		$\lambda = 0.05$			
	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI		
$\tau = 0.0$	199.586 (2.138)	199.992 (2.305)	199.870 (1.835)	199.843 (1.788)	200.00	200.000	199.730 (2.226)	200.055 (2.322)	200.155 (2.190)	200.159 (2.301)		
one mean $\tau = .5$	27.379 (0.214)	15.919 (0.157)	66.685 (0.534)	57.265 (0.417)	137.335 (1.238)	156.562 (0.921)	173.143 (1.930)	167.762 (1.968)	31.720 (0.261)	19.569 (0.191)		
$\tau = 1.0$	8.823 (0.058)	4.417 (0.040)	19.105 (0.105)	15.805 (0.081)	89.748 (0.591)	79.191 (0.370)	81.087 (0.864)	68.001 (0.798)	10.004 (0.067)	5.463 (0.048)		
$\tau = 1.5$	4.609 (0.027)	2.331 (0.019)	10.063 (0.041)	8.671 (0.038)	50.516 (0.243)	47.721 (0.139)	27.582 (0.268)	18.660 (0.209)	5.154 (0.030)	2.827 (0.022)		
$\tau = 2.0$	2.937 (0.016)	1.587 (0.010)	6.860 (0.023)	6.059 (0.024)	35.695 (0.109)	32.425 (0.077)	11.804 (0.106)	7.400 (0.079)	3.230 (0.017)	1.856 (0.013)		
$\tau = 2.5$	2.139 (0.010)	1.289 (0.006)	5.223 (0.015)	4.669 (0.017)	23.249 (0.057)	28.579 (0.053)	6.240 (0.052)	3.941 (0.039)	2.319 (0.011)	1.421 (0.008)		
$\tau = 3.0$	1.672 (0.008)	1.139 (0.004)	4.239 (0.011)	3.792 (0.013)	17.911 (0.035)	23.481 (0.039)	3.789 (0.029)	2.505 (0.023)	1.793 (0.008)	1.205 (0.005)		
all means $\tau = .5$	++++)		27.379 (0.214)	15.919 (0.157)	33.735 (0.237)	27.798 (0.184)	137.335 (1.238)	156.562 (0.921)	129.366 (1.469)	125.300 (1.513)	31.439 (0.265)	19.450 (0.193)
$\tau = 1.0$	8.823 (0.058)	4.417 (0.040)	11.889 (0.053)	9.505 (0.048)	89.748 (0.591)	79.191 (0.370)	40.117 (0.438)	32.061 (0.395)	9.709 (0.067)	5.333 (0.047)		
$\tau = 1.5$	4.609 (0.027)	2.331 (0.019)	7.172 (0.024)	5.808 (0.023)	50.516 (0.243)	47.721 (0.139)	12.748 (0.128)	8.537 (0.099)	4.941 (0.031)	2.802 (0.023)		
$\tau = 2.0$	2.937 (0.016)	1.587 (0.010)	5.207 (0.014)	4.291 (0.015)	35.695 (0.109)	32.425 (0.077)	5.491 (0.051)	3.619 (0.039)	3.101 (0.018)	1.826 (0.013)		
$\tau = 2.5$	2.139 (0.010)	1.289 (0.006)	4.132 (0.010)	3.434 (0.011)	23.249 (0.057)	28.579 (0.053)	3.007 (0.025)	2.085 (0.019)	2.192 (0.012)	1.408 (0.008)		
$\tau = 3.0$	1.672 (0.008)	1.139 (0.004)	3.465 (0.007)	2.905 (0.009)	17.911 (0.035)	23.481 (0.039)	1.975 (0.014)	1.465 (0.010)	1.696 (0.008)	1.193 (0.005)		
all means $\tau = .5$	+++--)		27.379 (0.214)	15.919 (0.157)	97.700 (0.815)	79.961 (0.618)	137.335 (1.238)	156.562 (0.921)	182.340 (2.073)	172.069 (2.136)	32.563 (0.273)	20.130 (0.197)
$\tau = 1.0$	8.823 (0.058)	4.417 (0.040)	35.229 (0.227)	22.956 (0.116)	89.748 (0.591)	79.191 (0.370)	140.761 (1.602)	135.697 (1.640)	9.968 (0.066)	5.415 (0.047)		
$\tau = 1.5$	4.609 (0.027)	2.331 (0.019)	18.528 (0.087)	12.146 (0.047)	50.516 (0.243)	47.721 (0.139)	91.268 (1.049)	83.429 (1.032)	5.176 (0.031)	2.851 (0.023)		
$\tau = 2.0$	2.937 (0.016)	1.587 (0.010)	12.377 (0.044)	8.566 (0.029)	35.695 (0.109)	32.425 (0.077)	52.991 (0.613)	43.914 (0.566)	3.289 (0.018)	1.875 (0.013)		
$\tau = 2.5$	2.139 (0.010)	1.289 (0.006)	9.357 (0.029)	6.799 (0.022)	23.249 (0.057)	28.579 (0.053)	30.291 (0.339)	21.811 (0.275)	2.361 (0.012)	1.436 (0.008)		
$\tau = 3.0$	1.672 (0.008)	1.139 (0.004)	7.544 (0.020)	5.671 (0.017)	17.911 (0.035)	23.481 (0.039)	17.758 (0.192)	11.369 (0.139)	1.825 (0.008)	1.218 (0.005)		

Table 7.17: (Continued)

shift τ	$\lambda = 0.05$		$\lambda = 0.1$		$\lambda = 0.1$		$\lambda = 0.05$		$\lambda = 0.05$	
	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI	F SI	VSI
three means (+ + +00)										
$\tau = .5$	27.379 (0.214)	15.919 (0.157)	64.636 (0.514)	54.317 (0.395)	137.335 (1.238)	156.562 (0.921)	168.659 (1.893)	164.627 (1.945)	31.703 (0.265)	19.407 (0.189)
$\tau = 1.0$	8.823 (0.058)	4.417 (0.040)	21.120 (0.122)	16.230 (0.085)	89.748 (0.591)	79.191 (0.370)	85.797 (0.932)	76.064 (0.899)	9.975 (0.066)	5.396 (0.047)
$\tau = 1.5$	4.609 (0.027)	2.331 (0.019)	11.511 (0.049)	9.135 (0.040)	50.516 (0.243)	47.721 (0.139)	33.305 (0.344)	24.444 (0.286)	5.134 (0.031)	2.851 (0.023)
$\tau = 2.0$	2.937 (0.016)	1.587 (0.010)	7.908 (0.028)	6.458 (0.026)	35.695 (0.109)	32.425 (0.077)	14.310 (0.139)	9.389 (0.107)	3.232 (0.018)	1.857 (0.013)
$\tau = 2.5$	2.139 (0.010)	1.289 (0.006)	6.054 (0.018)	5.051 (0.019)	23.249 (0.057)	28.579 (0.053)	7.482 (0.068)	4.825 (0.052)	2.318 (0.012)	1.433 (0.008)
$\tau = 3.0$	1.672 (0.008)	1.139 (0.004)	4.955 (0.013)	4.164 (0.014)	17.911 (0.035)	23.481 (0.039)	4.479 (0.038)	2.961 (0.030)	1.804 (0.008)	1.216 (0.005)
$c = 1.21$	82.943 (0.956)	72.455 (0.930)	41.128 (0.240)	31.745 (0.143)	26.400 (0.073)	31.213 (0.061)	16.155 (0.170)	10.778 (0.132)	18.795 (0.196)	11.672 (0.143)
$c = 1.44$	41.039 (0.529)	31.516 (0.467)	20.360 (0.082)	17.450 (0.064)	15.569 (0.029)	21.023 (0.035)	5.271 (0.052)	3.307 (0.036)	5.763 (0.057)	3.243 (0.036)
$c = 1.69$	21.780 (0.309)	15.052 (0.252)	13.486 (0.046)	12.378 (0.046)	11.304 (0.018)	15.900 (0.026)	2.769 (0.024)	1.855 (0.017)	2.991 (0.027)	1.786 (0.016)
$c = 1.96$	12.713 (0.193)	8.023 (0.144)	10.039 (0.032)	9.463 (0.036)	8.880 (0.014)	12.708 (0.021)	1.901 (0.014)	1.388 (0.010)	1.964 (0.015)	1.327 (0.008)
$c = 2.25$	8.224 (0.130)	8.947 (0.091)	7.947 (0.024)	7.625 (0.030)	7.296 (0.011)	10.499 (0.018)	1.486 (0.009)	1.187 (0.006)	1.515 (0.009)	1.148 (0.005)
$c = 2.56$	5.425 (0.087)	3.173 (0.057)	6.552 (0.020)	6.265 (0.025)	6.172 (0.010)	8.870 (0.016)	1.277 (0.006)	1.096 (0.004)	1.291 (0.006)	1.067 (0.003)
$c1 = 1.21$	158.065 (1.719)	156.117 (1.821)	86.105 (0.678)	76.056 (0.541)	85.183 (0.549)	73.472 (0.338)	40.502 (0.409)	29.462 (0.335)	48.859 (0.499)	36.336 (0.418)
$c1 = 1.44$	122.643 (1.369)	117.799 (1.429)	36.693 (0.216)	33.022 (0.163)	48.486 (0.227)	47.110 (0.139)	13.541 (0.124)	8.645 (0.093)	15.932 (0.142)	10.052 (0.107)
$c1 = 1.69$	95.474 (1.105)	89.323 (1.131)	21.883 (0.103)	21.363 (0.090)	33.697 (0.120)	37.206 (0.088)	7.241 (0.062)	4.772 (0.048)	8.251 (0.070)	5.221 (0.053)
$c1 = 1.96$	73.591 (0.876)	67.377 (0.886)	15.549 (0.063)	16.086 (0.064)	25.916 (0.074)	31.158 (0.065)	4.710 (0.039)	3.232 (0.031)	5.208 (0.043)	3.396 (0.033)
$c1 = 2.25$	50.437 (0.700)	50.541 (0.699)	11.993 (0.046)	12.921 (0.052)	21.169 (0.053)	26.994 (0.054)	3.453 (0.027)	2.478 (0.023)	3.765 (0.030)	2.572 (0.024)
$c1 = 2.56$	44.212 (0.568)	38.988 (0.566)	9.750 (0.036)	10.793 (0.043)	17.940 (0.041)	23.714 (0.046)	2.696 (0.020)	2.021 (0.017)	2.940 (0.022)	2.088 (0.018)

Table 7.17: (Continued)

shift τ	$\lambda = 0.05$		$\lambda = 0.1$		$\lambda = 0.1$		$\lambda = 0.05$		$\lambda = 0.05$	
	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI	F SI	VSI
c1-3 = 1.21	110.489 (1.245)	102.485 (1.263)	51.427 (0.336)	41.054 (0.217)	38.416 (0.151)	40.041 (0.097)	19.669 (0.196)	12.773 (0.147)	23.117 (0.231)	14.742 (0.170)
c1-3 = 1.44	164.609 (0.772)	55.246 (0.742)	23.816 (0.106)	20.840 (0.080)	21.830 (0.052)	27.458 (0.051)	6.472 (0.059)	4.030 (0.043)	7.278 (0.066)	4.229 (0.045)
c1-3 = 1.69	39.905 (0.509)	31.761 (0.465)	15.457 (0.056)	14.523 (0.053)	15.767 (0.030)	21.311 (0.037)	3.470 (0.029)	2.269 (0.021)	3.793 (0.023)	2.311 (0.022)
c1-3 = 1.96	25.286 (0.350)	19.024 (0.307)	11.357 (0.039)	11.125 (0.042)	12.376 (0.022)	17.327 (0.030)	2.330 (0.018)	1.639 (0.013)	2.490 (0.019)	1.640 (0.013)
c1-3 = 2.25	16.678 (0.244)	11.860 (0.204)	8.921 (0.029)	8.990 (0.034)	10.157 (0.017)	14.432 (0.025)	1.790 (0.012)	1.363 (0.009)	1.879 (0.013)	1.345 (0.009)
c1-3 = 2.56	11.759 (0.178)	8.113 (0.147)	7.330 (0.023)	7.490 (0.029)	8.595 (0.015)	12.293 (0.022)	1.490 (0.008)	1.216 (0.006)	1.546 (0.009)	1.198 (0.006)
one mean $\tau = 0.5$ and all variances $c = 1.21$	20.470 (0.184)	12.066 (0.136)	32.288 (0.183)	24.624 (0.110)	25.485 (0.068)	30.519 (0.059)	15.418 (0.162)	10.156 (0.123)	13.153 (0.127)	7.115 (0.081)
$\tau = 1.0$ $c = 1.44$	6.530 (0.055)	3.354 (0.036)	14.011 (0.060)	11.646 (0.050)	14.630 (0.026)	19.931 (0.033)	4.753 (0.046)	2.967 (0.032)	4.090 (0.035)	2.219 (0.020)
$\tau = 1.5$ $c = 1.69$	3.363 (0.026)	1.854 (0.016)	8.683 (0.033)	7.436 (0.032)	10.325 (0.016)	14.648 (0.024)	2.434 (0.020)	1.662 (0.014)	2.191 (0.016)	1.389 (0.009)
$\tau = 2.0$ $c = 1.96$	2.205 (0.016)	1.371 (0.009)	6.281 (0.022)	5.430 (0.023)	7.920 (0.012)	11.363 (0.019)	1.671 (0.011)	1.280 (0.008)	1.540 (0.009)	1.144 (0.005)
$\tau = 2.5$ $c = 2.25$	1.679 (0.010)	1.192 (0.006)	4.910 (0.016)	4.275 (0.018)	6.359 (0.010)	9.123 (0.016)	1.323 (0.007)	1.110 (0.004)	1.247 (0.006)	1.056 (0.003)
$\tau = 3.0$ $c = 2.56$	1.378 (0.007)	1.101 (0.004)	4.050 (0.013)	3.440 (0.015)	5.263 (0.008)	7.524 (0.014)	1.164 (0.004)	1.050 (0.003)	1.121 (0.004)	1.023 (0.002)
one mean $\tau = 0.5$ and corresponding one variance $c1 = 1.21$	25.853 (0.214)	16.048 (0.167)	46.505 (0.332)	39.050 (0.241)	74.251 (0.447)	64.959 (0.270)	33.246 (0.331)	23.229 (0.262)	22.760 (0.196)	13.450 (0.135)
$\tau = 1.0$ $c1 = 1.44$	8.550 (0.064)	4.720 (0.047)	16.313 (0.085)	14.459 (0.072)	36.928 (0.139)	39.264 (0.096)	9.594 (0.085)	6.183 (0.065)	7.240 (0.056)	4.103 (0.038)
$\tau = 1.5$ $c1 = 1.69$	4.601 (0.032)	2.613 (0.024)	9.505 (0.042)	8.718 (0.042)	23.818 (0.063)	29.303 (0.058)	4.626 (0.039)	3.160 (0.030)	3.743 (0.027)	2.301 (0.020)
$\tau = 2.0$ $c1 = 1.96$	3.023 (0.020)	2.840 (0.015)	6.701 (0.027)	6.244 (0.030)	17.472 (0.039)	23.211 (0.044)	2.884 (0.022)	2.130 (0.019)	2.481 (0.016)	1.664 (0.012)
$\tau = 2.5$ $c1 = 2.25$	2.260 (0.014)	1.494 (0.010)	5.199 (0.020)	4.859 (0.023)	13.603 (0.027)	18.855 (0.036)	2.099 (0.014)	1.636 (0.012)	1.871 (0.011)	1.366 (0.008)
$\tau = 3.0$ $c1 = 2.56$	1.846 (0.011)	1.322 (0.008)	4.265 (0.015)	3.934 (0.019)	10.996 (0.021)	15.576 (0.031)	1.676 (0.010)	1.377 (0.009)	1.541 (0.008)	1.218 (0.006)

Table 7.17: (Continued)

shift τ	$\lambda = 0.05$		$\lambda = 0.1$		$\lambda = 0.1$		$\lambda = 0.05$		$\lambda = 0.05$	
	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI
all means (+ + + +) and all variances										
$\tau = 0.5$ $c = 1.21$	20.470 (0.184)	12.066 (0.136)	23.804 (0.134)	18.733 (0.095)	25.485 (0.068)	30.519 (0.059)	13.385 (0.140)	8.735 (0.106)	12.700 (0.124)	7.002 (0.079)
$\tau = 1.0$ $c = 1.44$	6.530 (0.055)	3.354 (0.036)	10.126 (0.044)	8.446 (0.042)	14.630 (0.026)	19.931 (0.033)	3.831 (0.036)	2.496 (0.026)	3.823 (0.034)	2.183 (0.021)
$\tau = 1.5$ $c = 1.69$	3.363 (0.026)	1.854 (0.016)	6.390 (0.024)	5.337 (0.026)	10.325 (0.016)	14.648 (0.024)	1.999 (0.015)	1.463 (0.011)	2.042 (0.015)	1.363 (0.009)
$\tau = 2.0$ $c = 1.96$	2.205 (0.016)	1.371 (0.009)	4.685 (0.016)	3.897 (0.019)	7.920 (0.012)	11.363 (0.019)	1.420 (0.008)	1.172 (0.006)	1.442 (0.008)	1.131 (0.005)
$\tau = 2.5$ $c = 2.25$	1.679 (0.010)	1.192 (0.006)	3.749 (0.012)	3.055 (0.015)	6.359 (0.010)	9.123 (0.016)	1.188 (0.005)	1.066 (0.003)	1.199 (0.005)	1.052 (0.003)
$\tau = 3.0$ $c = 2.56$	1.378 (0.007)	1.101 (0.004)	3.139 (0.010)	2.490 (0.013)	5.263 (0.008)	7.524 (0.014)	1.088 (0.003)	1.026 (0.002)	1.090 (0.003)	1.021 (0.002)
all means (+ + + --) and all variances										
$\tau = 0.5$ $c = 1.21$	20.470 (0.184)	12.066 (0.136)	35.010 (0.198)	25.976 (0.114)	25.485 (0.068)	30.519 (0.059)	15.484 (0.163)	10.216 (0.125)	13.369 (0.129)	7.214 (0.080)
$\tau = 1.0$ $c = 1.44$	6.530 (0.055)	3.354 (0.036)	16.483 (0.067)	12.777 (0.051)	14.630 (0.026)	19.931 (0.033)	4.863 (0.047)	3.067 (0.033)	4.145 (0.037)	2.230 (0.021)
$\tau = 1.5$ $c = 1.69$	3.363 (0.026)	1.854 (0.016)	10.538 (0.037)	8.322 (0.033)	10.325 (0.016)	14.648 (0.024)	2.500 (0.021)	1.692 (0.014)	2.206 (0.016)	1.387 (0.009)
$\tau = 2.0$ $c = 1.96$	2.205 (0.016)	1.371 (0.009)	7.700 (0.025)	6.137 (0.024)	7.920 (0.012)	11.363 (0.019)	1.710 (0.011)	1.290 (0.008)	1.540 (0.009)	1.147 (0.005)
$\tau = 2.5$ $c = 2.25$	1.679 (0.010)	1.192 (0.006)	6.114 (0.019)	4.849 (0.020)	6.359 (0.010)	9.123 (0.016)	1.352 (0.007)	1.127 (0.005)	1.254 (0.006)	1.061 (0.003)
$\tau = 3.0$ $c = 2.56$	1.378 (0.007)	1.101 (0.004)	5.018 (0.015)	3.963 (0.016)	5.263 (0.008)	7.524 (0.014)	1.182 (0.005)	1.055 (0.003)	1.119 (0.004)	1.024 (0.002)
three means (+ + + 00) and all variances										
$\tau = 0.5$ $c = 1.21$	22.952 (0.199)	13.754 (0.150)	35.944 (0.224)	28.003 (0.142)	36.166 (0.134)	38.535 (0.091)	17.499 (0.170)	11.369 (0.130)	15.509 (0.139)	8.675 (0.093)
$\tau = 1.0$ $c = 1.44$	6.530 (0.055)	3.354 (0.036)	15.025 (0.070)	12.547 (0.058)	19.828 (0.044)	25.530 (0.046)	5.293 (0.048)	3.367 (0.033)	4.870 (0.040)	2.721 (0.025)
$\tau = 1.5$ $c = 1.69$	3.363 (0.026)	1.854 (0.016)	9.284 (0.038)	8.020 (0.038)	13.775 (0.016)	19.010 (0.024)	2.760 (0.022)	1.889 (0.016)	2.593 (0.019)	1.610 (0.012)
$\tau = 2.0$ $c = 1.96$	2.205 (0.016)	1.371 (0.009)	6.709 (0.025)	5.831 (0.027)	10.475 (0.012)	14.833 (0.019)	1.861 (0.012)	1.416 (0.009)	1.783 (0.011)	1.269 (0.007)
$\tau = 2.5$ $c = 2.25$	1.679 (0.010)	1.192 (0.006)	5.261 (0.019)	4.562 (0.021)	8.325 (0.014)	11.948 (0.022)	1.458 (0.008)	1.217 (0.006)	1.410 (0.007)	1.126 (0.004)
$\tau = 3.0$ $c = 2.56$	1.378 (0.007)	1.101 (0.004)	4.323 (0.015)	3.751 (0.018)	6.849 (0.012)	9.837 (0.019)	1.257 (0.006)	1.115 (0.005)	1.223 (0.005)	1.062 (0.003)

Chapter 8

Guidelines for Design of VSI Charts, a Summary and Topics for Future Research

8.1 Guidelines for Design of VSI Charts

A design procedure for univariate FSI EWMA charts was discussed by Lucas and Saccucci (1990). The design of a FSI EWMA chart requires the specification of the usual parameters h , λ , n and d . When d is specified, the false alarm rate will depend on the in-control ANSS, and the parameter h is usually chosen to give a specified in-control ANSS.

Reynolds et al.(1990) discussed methods for designing VSI CUSUM charts. One relatively simple approach to the design of a VSI EWMA chart for a particular application is to choose h , λ and n in essentially the same way as would be done for a FSI EWMA chart. The parameter d in the FSI chart becomes the average sampling rate in the VSI chart. Once d has been specified, the parameters d_1 , d_2 and g can be chosen under the constraint (3.2). In VSI charts the value of d_1 should be as small as is practical. If d_1 is chosen as small as possible and d is specified, then the value of ρ_1 determines the value of d_2 through the relation (3.2), where d_2 is chosen according to

the size of shift to be detected. For fixed d and d_1 , reducing ρ_1 reduces d_2 . For example, if $d = 1.0$, $d_1 = 0.1$ and $\rho_1 = 0.5$ then $d_2 = 1.9$.

Note that the numerical results in previous chapters are based on the unit of time being chosen as the length of the fixed sampling interval so that in effect $d = 1$. To convert these results to the case where $d \neq 1$ it is necessary to multiply d_1 , d_2 , and the ATS by d . For example, if the matching FSI chart uses 2 hour sampling intervals then $d_1 = 0.1$ and $d_2 = 1.9$ from the previous results would correspond to 0.2 hours and 2.8 hours, respectively.

8.2 Summary of Results

In the univariate EWMA control scheme, Lucas and Saccucci (1990) showed that small values of λ provide more protection against small shifts in the process and the large values of λ provide more protection against large shifts in the process. In our numerical studies, the same results happened in the multivariate EWMA charts based on the combine-accumulate approach, but smaller values of λ in multivariate EWMA charts based on the accumulate-combine approach provide more protection against all shifts in $\underline{\mu}$ and/or Σ . Also multivariate EWMA charts based on the accumulate-combine approach are more efficient than multivariate EWMA charts based on the combined-accumulate approach. Thus small values of λ may be best for practical applications.

Multivariate CUSUM and MEWMA charts for the mean vector have been investigated in Chapter 5. The numerical results show that multivariate CUSUM charts using the statistic Z_t^2 based on the combine-accumulate approach for monitoring $\underline{\mu}$ did not do well compared to the MEWMA chart based on the accumulate-combine approach.

For monitoring Σ some numerical results using the multivariate EWMA charts based on the combine-accumulate approach and multivariate EWMA charts based on the accumulate-combine approach for monitoring Σ have been presented in Chapter 6. Some numerical results show that

the multivariate EWMA charts using (6.4) based on the combine-accumulate approach did not do well compared to multivariate EWMA charts using (6.8) based on the accumulate-combine approach. This means that the multivariate EWMA charts based on the accumulate-combine approach are better than multivariate EWMA charts based on the combine-accumulate approach for Σ .

For monitoring $\underline{\mu}$ and Σ , some statistics which are based on the combine-accumulate approach and the accumulate-combine approach have been studied in Chapter 7. Some numerical results show that the MEWMA and multivariate EWMA charts using (7.5) are very good in detecting shifts in $\underline{\mu}$ and Σ . But, the ATS values for the MEWMA charts and multivariate EWMA charts using (7.5) are almost determined by shifts in $\underline{\mu}$ and Σ , respectively. Using both the MEWMA chart and the EWMA chart using (7.5) based on the accumulate-combine approach is more effective than either the MEWMA or the EWMA using (7.5) in detecting shifts in $\underline{\mu}$ and Σ . Also performance of this separate EWMA chart is almost as good as the MEWMA for $\underline{\mu}$ and the EWMA using (7.5) for Σ .

In summary, if interest is primarily in shifts in $\underline{\mu}$ then use the MEWMA chart. The required values of the parameters h and g are given in Table 5.19. If interest is primarily in shifts in Σ then use the multivariate EWMA charts based on (7.5). The required values of the parameters h and g are given in Table 7.1. If interest is in shifts in both $\underline{\mu}$ and Σ , then use both the MEWMA for $\underline{\mu}$ and the multivariate EWMA chart based on (7.5) for Σ . The required values of parameters h and g are given in Table 7.1. The results show that multivariate EWMA charts based on the accumulate-combine scheme are more effective than multivariate EWMA charts based on the combine-accumulate approach in terms of ATS. Also, for EWMA charts based on the accumulate-combine approach, smaller values of λ are more effective in detecting all shifts in $\underline{\mu}$, Σ and both $\underline{\mu}$ and Σ . Also VSI charts are more efficient than corresponding FSI charts. There is no definite winner among these five different EWMA charts for monitoring the mean vector and variance-covariance matrix.

8.3 Topics for Future Research

For the MEWMA chart, the multivariate EWMA chart using (7.5) and the multivariate EWMA chart based on both the MEWMA and EWMA using (7.5), the required values of the parameters h and g for small values of λ and $p > 5$ are needed to be calculated for using these EWMA charts in practice.

Regarding properties of the charts obtained so far, it was assumed that the observed random vectors are independent. For some processes, this assumption may not met and, as in the univariate case, ATS properties of the charts can be greatly affected and out-of-control signals can become meaningless. For autocorrelated multivariate observations, using the theory of multiple time series is required.

It was also assumed in this dissertation that the shift in any of the parameters remains constant over time. For some processes, there may be a continuous drift in the parameters. Then the properties of the VSI chart for a drift in any of the parameters need to be studied.

Lucas and Saccucci (1990) showed that a FIR feature is especially useful for the univariate EWMA control schemes designed with small values of λ . This feature gives extra protection against problems that might arise during start-up of the process, since starting the EWMA chart closer to the signal region reduces the time required to reach the signal region. Since small values of λ may be best for the multivariate EWMA charts based on the accumulate-combine approach, a FIR feature in the multivariate EWMA charts needs to be studied.

Another interesting problem would be a combined Shewhart/EWMA chart. Although EWMA control schemes can be designed to detect small shifts quickly, Lucas and Saccucci (1990) showed that a combined Shewhart EWMA gives improved properties when both large and small shifts are to be detected. This is achieved in the multivariate EWMA chart by adding a multivariate Hotelling's χ^2 limit to the multivariate EWMA chart so that an out-of-control signal is given if the multivariate EWMA statistic is outside the control limit or if the current observation is outside the Hotelling's χ^2 limit.

Chapter 9

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