A GAME THEORETICAL INVESTIGATION OF THE INTERNATIONAL DEBT OVERHANG

by

Jacek Prokop

Dissertation submitted to the Faculty of the

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Economics

APPROVED:

M. Kaneko, Chairman

H. Cremer

J. Crémer

H. Haller

Y. Ioannides

July, 1991

Blacksburg, Virginia

A GAME THEORETICAL INVESTIGATION OF THE INTERNATIONAL DEBT OVERHANG

by

Jacek Prokop

Committee Chairman: Mamoru Kaneko
Economics

(ABSTRACT)

The problem called debt overhang has recently been observed in international financial relations between a sovereign country and foreign commercial banks. The term "debt overhang" expresses the situation where a sovereign country has borrowed money from foreign banks and has been unable to fulfill the scheduled repayments for some time. We formulate this problem as a noncooperative game with the lender banks as players where each decides either to sell its loan exposure to the debtor country at the present price of debt on the secondary market, or to wait and keep its exposure.

We propose two approaches: a one-period approach (Chapter II), and a direct dynamic approach (Chapter III). In the one-period approach, we consider a representative period, while in the dynamic approach, the whole dynamics is directly considered. Both approaches are consistent and complementary in that the first approach considers the effect of a large number of banks, and the second approach captures the dynamic nature of the problem.

In the one-period approach, we consider the behavior of many banks. In the model with n lender banks, there are many pure and mixed strategy Nash equilibria. However we show that in any equilibrium, the resulting secondary market price remains almost the same as the present price when the number of banks is large. In addition, we discuss the structure of the set of Nash equilibria.

The second approach is a direct dynamic formalization of the same problem with two creditor banks. We show that in the dynamic game there exist three types of subgame perfect equilibria with the property called the time continuation. We consider the relationships between the equilibria of the dynamic game and those of the one-period approach and show that the one-period

approach does not lose much of the dynamic nature of the problem. In every equilibrium, each bank waits in every period with high probability, and this probability is close to 1 when the interest rate is small. If the price function of debt is approximated by some homogeneous function for large values of debt, then the central equilibrium probability becomes almost stationary in the long run. The stationary probability is relatively high as long as the interest rate is low.

Finally, in Chapter IV, we consider the duration of debt overhang with two lender banks. We show that the equilibrium duration of debt overhang converges to a constant when the length of a subperiod tends to zero. The constant is large when the degree of homogeneity of the price function is high. When the degree of the homogeneity is low, the constant is close to $\ln 2 / \ln \beta^2$, where β is the annual interest factor.

These results as a whole are interpreted as a tendency for the problem of debt overhang to persist over a long time.

Acknowledgements

I want to thank Professor Mamoru Kaneko for his guidance in the completion of this dissertation. His suggestions, patience, and encouragement were very helpful throughout the entire process of writting.

I thank the members of my committee, Professors Helmuth Cremer, Jacques Crémer, Hans Haller, and Yannis Ioannides for their comments and suggestions. I also benefited from the lectures by Professors Amoz Kats and Hervé Moulin.

My thanks go also to my earlier teachers, especially Professors Jozef Nowicki and Dariusz Rosati who first introduced me to economic theory.

Finally, I dedicate this dissertation to my grandmother Franciszka Zuszman and my parents, Zofia and Jan Prokop. Without their infinite love, I would not have completed this endeavor.

Table of Contents

Chapter I

Introduction

The lending by commercial banks to foreign countries has been present in the world economy for a long time. Recently, however, we have observed new features in the international lending. In the 1970s and early 1980s many less developed countries (LDCs) borrowed very large amounts of money from foreign commercial banks. A number of these countries have experienced difficulties in meeting the payments schedules to which they had originally agreed. Various observers have begun to speak of a debt crisis of an unprecedented scale, one which might shake the financial system of the world.

The trend towards increased debt started when OPEC quadrupled the world price of oil in 1973. On one hand, many LDCs relied on imported oil, and their balance of trade turned sharply to a deficit. On the other hand, commercial banks were flush with the OPEC nations' huge dollar deposits generated by the increased earnings from oil sales, so called petrodollars. The banks helped to *recycle* the deposits of their OPEC customers into loans to the deficit LDCs. At the time the loans were made, the developing country borrowers appeared to be reasonable credit risks. The prices of their export commodities were high, and no one forecast the global recession that occured in the early 1980s. Creditor banks fully expected the loans to be repaid out of the borrowers' export earnings.

A doubling of energy prices in 1979 led to a further increase in LDC debt. The severe world recession that began in 1981 reduced demand for the exports of many LDCs. At the same time, the interest payments on their debt increased due to a rising dollar interest rate. Furthermore, wasteful government spending and lavish consumption splurges occured in a number of debtor countries. As a result of these factors many LDCs could not make their payments.

The lender banks had no choice but to reschedule the debt repayments. In summer 1982, Mexico (after Brazil the world's largest LDC debtor) notified its lenders that it could no longer meet previously scheduled payments on its external debt. Since Brazil, Argentina, and other debtors were practically in the same situation as Mexico, the lenders feared that Mexico's default might be followed by others. To avoid the chain reaction of defaults which could cause a widespread banking crisis, the lender banks were virtually forced to reschedule the debts.

In 1984, the debt situation became a little more optimistic. The world economy experienced exceptional growth which also improved the position of the debtor countries. Since then, there was much less fear of a collapse of the international financial system. However, the debt problem still remained, and in the case of many countries it became even worse than in early 1980s. The situation of these debtor countries who are unable to maintain their repayments is called debt overhang.

To understand the nature of the debt overhang it is necessary to keep in mind specific features of international lending that are different from domestic lending. Unlike the case of domestic lending, international loan contracts are not enforcable in court, as a sovereign country cannot be taken to court by a foreign bank. Nonetheless, there are three main costs to the debtor country who unilaterally refuses to meet its debt obligations.[1]

The first cost of a country's default on debts is the seizure of assets. The creditors of a sovereign defaulter may be able to persuade their governments to seize any of the debtor's assets located

[1] See, for example, Krugman and Obstfeld (1988).

in their jurisdiction. These could include the foreign reserves of the defaulting country's central bank, foreign assets owned by the defaulting country's private citizens, or even goods in international trade owned by the debtor and crossing creditors' borders.

The second possible cost of a default is the exclusion from future borrowing. A country that has defaulted would be excluded from the international capital market, at least for several years. Once a country has already defaulted on previous debts, prospective lenders will be unwilling to believe promises that it will abide by the terms of new loan contracts. Further, even if a sovereign defaulter did succeed in getting a loan abroad, its existing creditors would try to seize the new funds. Therefore a defaulting LDC would no longer be able to draw on foreign savings to develop profitable investment opportunities, and all domestic investment would have to be financed from the meager supply of domestic savings. In addition, the country would lose the flexibility to borrow abroad for smoothing the consumption and investment in the face of temporary fluctuations in its real income. Sharper booms and busts would impose economic costs and might also threaten country's political stability.

The third and most serious cost of default is the reduction of the gains from international trade. As a consequence of the first two, sovereign defaulters could find their ability to engage in international trade severely curtailed. The debtor-country goods involved in international trade would be subject to seizure whenever they crossed a creditor's border. Furthermore, a defaulter's exclusion from the international capital market might leave it unable to obtain trade credits abroad or even to maintain checking accounts in foreign banks (since these accounts could be seized). Since the LDC's dependence on international trade is very high, the defaulter's inability to trade would be very costly.

Although the international loan contracts are not enforcable in court, the existence of costs of a sovereign's default may guarantee the implementation of the implicit contractual agreement between a sovereign country and a foreign bank. Since the costs of a country's default are usually higher than the gains from complying with the loan contract, it is in the interest of the debtor to be willing to repay the debts. The debtor's "willingness to repay" is the factor which supports international lending without the presence of any court.[2]

Nevertheless, the economic situation of a debtor country may prevent the full repayment of debts even when the country has the willingness to repay. This has been a characteristic feature of the debt crisis in the 1980s. The difficult economic situation of indebted LDCs caused the appearance of a secondary market for debts. On the market, the sovereign debts are traded by lender banks and other financial institutions, and each dollar of debt is priced much below one.

For instance, in the case of Argentina the price of one dollar of debt was 64¢ in 1986 and 26¢ in 1988; and in the case of Mexico, the price was 60¢ in 1986 and 52¢ in 1988.[3] Trade on the secondary market means that if a lender sells its loan exposure to some other financial institution at the price, say 26¢, then the lender obtains 26% of the face value of its loan exposure and gives up the remaining 74%, while the other institution will take over the right to the total loan.

The secondary market has undergone some changes since its initiation.[4] While at the beginning, the debtor country was excluded from trade, currently it could be a possible buyer of its debts on the secondary market in addition to the lender banks and other financial institutions. The trade between the debtor country and a lender bank at the price, say 26¢ again, on the secondary market is regarded as a 74% forgiveness, since the lender recoups 26% of its money and the country is not indebted to this particular lender any more.

Since it is in the interest of the debtor country to resolve its debt overhang problem, the country could use the secondary market to buy back its debts. The debtor country might be able to afford the buyback, because the secondary market prices are already very low.[5] For such a buyback to take place, however, it is necessary that the lender banks decide to sell their exposures

---

2 See, for example, Eaton, Gersovitz and Stiglitz (1986).

3 See, for example, Huizinga (1989, p.37).

4 For historical and institutional description of the secondary markets for debts, see, for example, Sachs and Huizinga (1987), Huizinga (1989, pp.5-8), and Hajivassiliou (1989).

5 For example, in 1987/88 Bolivia bought its debts from commercial banks at 6¢ and 11¢ per $1.

to the debtor country. The recent history shows that only few debtors partially succeeded in buying back their debts. Therefore the debt overhang problem remains almost unchanged.

Many proposals for the resolution of the debt overhang problem have been considered. (cf. Versluysen (1989)). Some of them, for example the Baker and Brady Plans, have been implemented to a certain extent, but did not solve the problem.[6] Many other proposals still remain in the discussion stage. For any kind of resolution, however, it is first necessary to understand the nature of the debt overhang.

We can find several attempts to investigate some problems related to the phenomenon of debt overhang. Most of the literature on the debt overhang has been dedicated to the problem of bargaining over debt reduction between the debtor countries and their lender banks. For example, Bulow and Rogoff (1986) have analyzed a model in which renegotiation of the loan contracts is an essential part of the lending process. They use an alternating offers model of bargaining developed by Rubinstein (1982) to explain negotiated partial defaults and repeated rescheduling. Fernandez and Kaaret (1988) have examined the effect that reputational considerations may have on negotiations. They model a situation where one country borrowed money from two banks, one big and one small. Adopting the Nash bargaining solution, they investigated possible agreements on the debt reduction. Another reputational approach has been provided by Armendariz de Aghion (1990) in a bargaining model with two countries and one lender bank. She discusses the commercial banks' reputation for being tough bargainers in the renegotiation process. Thus these authors consider bargaining between a borrowing country or countries and a lender bank or banks over the size of debt repayments, given the existing situation of debt overhang.

The bargaining over debt reduction between a debtor country and lender banks certainly is important. However, in many cases, a debtor country has borrowed money from many foreign banks at the same time. For example, in 1985 Bolivia held loans from about 125, Argentina from 370, and Mexico from 700 banks.[7] In the case where the number of participating lender banks is large, it is hard to expect free collective (or multilateral) bargaining to take place, because it would

---

[6] See, for example, Kenen (1990), or Sachs (1990).

[7] See, for example, Bouchet (1987), p.75.

involve high communication and transaction costs and each bank has some other options, e.g., not to participate in the bargaining, but wait for a change induced by the bargaining of others. This does not necessarily exclude the possibility of bilateral bargaining between a country and an individual bank (or a small number of countries and banks). Typical models of bargaining should be regarded as such a case. Nevertheless, when the number of banks is large, the opportunity of selling debts on the secondary market gives an important outside option to each creditor.

In this dissertation, we investigate the nature of the debt overhang problem from a game theoretical viewpoint. We view the existence of debt overhang as a result of a game of banks to recoup their money from the debtor country through the secondary market transactions. In the game, each bank has to decide either to sell its loan exposure to the country at the secondary market price or to wait and keep its exposure.

There are some incentives for the banks to wait as well as to sell. When many banks sell their loan exposures and the country's debt becomes smaller by paying the discounted amount of the loan exposures, the secondary market price of the next period may become higher. If this is the case, the banks who wait may receive a higher payoff by the increased price, which implies that the banks may have some incentives to wait. When no bank sells, the total outstanding debt increases with the accrued interest. In this case, the price of debt may fall, which implies that there are some incentives for the banks to sell. These two opposite tendences balance in equilibrium.

Therefore we have a dynamic situation which influences the behavior of many creditor banks. In modelling that situation, we face a trade-off between the direct dynamic representation and the large number of banks. To give a faithfull game theoretical description of the problem, we propose two approaches.

The first one is a one-period or static approach to capture the effect of a large number of banks. In the second approach, we model the dynamics directly, but restrict the number of banks to two, sacrificing the insight into the behavior of a large number of banks given by the static approach. Thus both approaches are complementary, and they should be interpreted as a whole.

The one-period approach is presented in Chapter II. We formulate the problem of debt overhang as a noncooperative game of n lender banks where each decides either to sell its loan exposure to the debtor country at the present price of debt on the secondary market, or to wait and sell its exposure in the next period. There are many pure and mixed strategy Nash equilibria in this game. However, we show that in any Nash equilibrium, the resulting secondary market price remains almost the same as the present price when the number of banks is large. Equivalently, the total outstanding debt of the country remains almost unchanged. When the distribution of loan exposures is relatively equal, our result gives the prediction that the proportion of banks selling is approximated by the interest rate. When the loan exposures are unequal, one of the results states that in a mixed strategy equilibrium, the banks with smaller loan exposures have higher probabilities of selling than the banks with larger loan exposures. Therefore the previous prediction is adjusted so that the proportion of banks selling may exceed the interest rate.

The dynamic approach is described in Chapter III. In this chapter, we specify the problem as an infinite horizon game with two banks as players where each bank decides in every period either to sell its loan exposure to the debtor country at the present secondary market price, or to wait and keep its exposure to the next period. We show that the dynamic game has three types of subgame perfect equilibria with the property called time continuation. By "time continuation" we mean that the game of the present period can be viewed as the result of some previous periods so that the extensions of the equilibria of the present game give positive realization probabilities to the present game. The three types of equilibria are called central, alternating and mutating. Each equilibrium is very similar to the unique central equilibrium in the sense that the probabilities of a bank waiting are determined by similar formulas and take similar values. Therefore the average attainable payoffs are almost uniquely determined.

We compare the equilibria of the dynamic game of Chapter III with those of the one-period game of Chapter II. To make this comparison, we construct a sequence of the mixed strategy equilibria for the one-period games describing the situation of banks in the sequence of the respective periods. This sequence of equilibria coincides with the central equilibrium of the dynamic game. That is, we have a decomposition property of the dynamic game into the games of one period, since all equilibria of the dynamic game are similar to the unique central equilibrium. Thus the one-period approach of Chapter II does not lose much of the dynamic nature of the problem.

In Chapter IV, we modify the dynamic game of Chapter III to investigate the duration of the debt overhang. Again, we restrict our considerations to only two banks, since the dynamics with many banks are too complex. The modified game has the same type of equilibria as the original dynamic game of Chapter III. Since lender banks are instanteneous decision makers, the length of a period of thegame should be regarded as short or almost zero. In Chapter IV, we show that if the price function is homogeneous, then the expected equilibrium duration of the debt overhang becomes almost constant when the length of a period tends to zero. The constant limit duration of the debt overhang with two lender banks is long when the degree of homogeneity of the price function is high. When the degree of the homogeneity is low, the constant is close to $\ln 2 / \ln \beta^2$, where β is the annual interest factor. Therefore the problem of debt overhang may exist for a long time.

Chapter II

A Game Theoretical Approach to the International Debt Overhang

This chapter has been written jointly with professor Mamoru Kaneko, Department of Economics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061.

1. Introduction

The "debt overhang" has recently been observed in international financial relations between sovereign countries and foreign commercial banks. The term "debt overhang" expresses the situation where a sovereign country has borrowed money from foreign banks and has been unable to fulfill the scheduled repayments for some period. The existence of the debt overhang is a serious problem for the debtor country, which keeps the country in a depressed economic situation and prevents it from growing. The debtor country should reduce its debt to have an access to the international financial market, which is necessary for its economic growth. Many proposals for resolution of the debt overhang have been discussed (cf. Versluysen (1989)).[8] For any kind of resolution, however, first it is necessary to understand the nature of the debt overhang. The present chapter investigates this problem from the game theoretical viewpoint.

Before describing the details of our game theoretical investigation, we explain the economic background of the problem of debt overhang. The debt overhang is closely related to the presence of the secondary market for debts. The secondary market has emerged as a result of the countries' economic inability to make full repayments. On the market, loan exposures are traded by lender banks and other financial institutions; and each dollar of debt is priced much below one. For instance, in the case of Bolivia, the price of one dollar of debt was 5¢ in 1985 and 6¢ in 1986; and in the case of Peru, it was 19¢ in 1986 and 6¢ in 1988. Trade on the secondary market means that if a lender sells its loan exposure to some other financial institution at the price, say 5¢, then the lender obtains 5% of the lent money and gives up remaining 95%, but the other institution will take over the right to the loan.

It is especially important in this dissertation to remark that currently, the debtor country could be a possible buyer of debts on the secondary market in addition to banks and other financial institutions; while at the beginning of the secondary market, the debtor was excluded from trade.[9] The trade between a debtor country and a lender bank (debt buyback) at the price, say 5¢ again, on the secondary market is regarded as 95% forgiveness, since the lender recoups 5% of its loan, and the country is not indebted any more. Lender banks have given up the possibility of recouping

---

8

There are several attempts to investigate some problems related to the phenomenon of debt overhang. They focus mainly on the bargaining over debt reduction between the country and the banks. For example, Fernandez and Kaaret (1988) considered a situation where one country borrowed money from two banks, one big and one small. Adopting the Nash bargaining solution, they investigate possible agreements on debt reduction. Fernandez and Rosenthal (1989) and Bulow and Rogoff (1986) provided different bargaining models with one country and one bank. Thus those authors considered bargaining over repayments between a borrowing country and a lending bank or banks, given the existing situation of debt overhang.

9

For historical and institutional description of the secondary markets for debts, see, for example, Sachs and Huizinga (1987), Huizinga (1989, pp.5-8), and Hajivassiliou (1989).

their total amounts of loan exposures since it has been practically impossible to expect the total repayments. Although countries have had possibilities to buy back their debts at discounted prices, many of them have not succeeded in reducing their indebtedness significantly.

In addition to the above economic background, we mention a few important empirical facts on the debt overhang. In many cases, a debtor country has borrowed money from many foreign banks at the same time. For example, in 1985 Mexico held loans from about 700 banks, Argentina from 370, and Venezuela from 460 banks (Bouchet, 1987, p.75). However, there has been some tendency for the number of lender banks to decline. Nevertheless, many debtor countries have not reduced their indebtedness because of the compounded interests.

Keeping in mind the background of the problem and the above empirical facts, we explain our endurance competition game and the main result. We consider a situation with one country and its creditor banks in a short period. We formulate the problem as a one-shot game with creditor banks as players. The debtor country is treated as a part of the environment. We also assume the existence of a price function which gives the secondary market price of the country's debt.

In our game, each bank has to decide either to sell its loan exposure to the country at the present secondary market price or to wait and keep its exposure. If a bank sells its exposure, it obtains a payoff equal to the value of the exposure determined by the present secondary market price of debt. If a bank waits, its payoff is assumed to be the present value of the loan exposure determined by the resulting secondary market price.

On one hand, if many banks sell their loan exposures and the country's debt becomes smaller by paying the discounted amount of the loan exposures, then the secondary market price of the next period may become higher. If this is the case, the banks who wait may receive a higher payoff by the increased price, which implies that there are some incentives for the banks to wait. On the other hand, if only few banks sell, the total outstanding debt may increase with the accrued interest. In this case, the price of debt may fall, which implies that there are some incentives for the banks to sell. These two opposite tendences balance in equilibrium.

We consider the Nash equilibrium concept to represent the strategically stable behavior of banks. In fact, there are many pure and mixed strategy Nash equilibria. However, independently of a choice of a Nash equilibrium, we can draw a definite conclusion on the behavior of the secondary market price. It says that the resulting secondary market price of debt remains almost the same as the present price for a large number of banks. Equivalently the total outstanding debt of the country remains almost unchanged. When the distribution of loan exposures is relatively equal, this theorem gives the prediction that the proportion of banks selling is approximated by the interest rate. When the loan exposures are unequal, one of our results states that in a mixed strategy equilibrium, banks with smaller loan exposures have higher probabilities of selling than banks with larger loan exposures. Therefore the previous prediction is adjusted so that the proportion of banks selling may exceed the interest rate. Section 3 discusses these results.

Although the nature of our problem is dynamic, we focus on the behavior of banks in one period. In Chapter III, we formulate this problem as a dynamic game with an infinite horizon in the case of two banks, and show that the set of subgame perfect equilibria for the game is quite limited. In each period, some subgame perfect equilibria give the same outcome as those of our one-period approach and the others also give very similar outcomes. Thus we have the decomposition property of the whole dynamics into the one-period problems, which means that we do not lose much of the dynamic nature of the debt overhang problem by our one-period approach. The one-period and the dynamic approaches are regarded as mutually complementary in that the one-period approach is simpler and enables us to discuss complex problems, and that the dynamic approach captures a long-run behavior but it is too complicated to consider some problems such as effects of a large number of banks.

This chapter is organized as follows. Section 2 gives a description of the endurance competition game and the structure of the set of pure strategy Nash equlibria. In Section 3, the main limit theorem is given, which states that in any (pure or mixed strategy) Nash equilibrium, the secondary market price of debt is almost constant for a large number of banks. We also discuss comparative statics on mixed strategy equilibria. In Section 4, we consider the structure of the set of mixed strategy equilibria.

2. Endurance Competition Game

In an endurance competition game G, we consider decision making of banks in one particular period. The economic situation of the game G is described as a triple (N, $\{D_i\}$, P). The symbol N denotes the set of banks 1, 2, ..., n, who have lent some amount of money to a foreign country, and D_i denotes the present loan exposure of bank i. The symbol P denotes a real-valued continuous function on $[0, +\infty)$, which gives the secondary market price P[d] when the total outstanding debt is d. The present secondary market price is given as P[D], where $D = \sum_{i \in N} D_i$. One additional element is the market interest factor, denoted β (= 1 + the interest rate) > 1. We assume that

$$\text{(2.1)} \qquad P[D] > P[\beta D] \quad \text{and} \quad P[D] \leq P[0]$$

The first inequality of (2.1) states that if the country does not buy back any debt in the present period, then the total outstanding debt increases to βD by the accrued interest and its market price declines. The second inequality states that if all banks sell their loan exposures at the present price P[D], then in the next period the price of (an arbitrarily small) debt is higher than or equal to the present price P[D].

Throughout the chapter, we assume that the price function P is fixed, but the set N of banks and their loan exposures $\{D_i\}$ may vary. Therefore we denote our game G by (N, $\{D_i\}$). In this game, the banks are players and the country is treated as a part of the environment.

Each bank $i \in N$ has two pure strategies 0 - to sell its exposure at the present price $P[D]$ and 1 - to wait and postpone the decision to the next period. We denote the strategy space {0, 1} of player i by S_i. Then $S_1 \times \ldots \times S_n$ is the outcome space. When each bank i chooses its strategy s_i ($i \in N$) and the market transactions are completed, the price of the next period becomes

$$\text{(2.2)} \qquad P[\beta d], \quad \text{where } d = \sum_{j \in N} s_j D_j.$$

If bank i keeps its loan exposure D_i by the next period, then the loan exposure D_i increases to βD_i, and the new price is $P[\beta d]$. Thus the present value of the exposure which will be sold in the next period becomes

$$\frac{1}{\beta}(\beta D_i\, P[\beta d]) = D_i\, P[\beta d]. \tag{2.3}$$

We assume that this present value $D_i\, P[\beta d]$ is the payoff to bank i if it does not sell its loan exposure in the present period. We also assume that if bank i sells D_i in the present period, then its payoff is simply $D_i P[D]$. We do not take into account the possibilities of the banks' revenues by postponing selling its exposure after the next period.[10] Thus the payoff function of bank i is given as

$$h_i(s) = h_i(s_1,\ldots,s_n) = \begin{cases} D_i P[\beta \sum_{j\in N} s_j D_j] & \text{if } s_i = 1 \\ D_i P[D] & \text{if } s_i = 0. \end{cases} \tag{2.4}$$

If all lender banks sell their exposures to the country, then the country must pay the total of $DP[D]$. We assume that the country is able to afford these repayments, which implies that the current price $P[D]$ is small enough for the repayments or the total amount of debts is not so large to prevent the total repayment.[11] In fact, the price function $P[\bullet]$ may also depend upon the country's disposable income (i.e. income left after repayments). Since it is assumed that repayments are made according to the current market price $P[D]$, the disposable income is automatically determined by the market transactions. The price function $P[\bullet]$ is interpreted as determined by taking this consideration into account.

10

This possibility is taken into account in the dynamic description of this problem with two banks in the next chapter. However, we show that this does not substantially change the structure of the equilibria.

11

Formally we need this assumption. However, practically, we need the assumption that the country is able to afford to pay a little more than the payment given by Theorem 1.

In addition to pure strategies, we allow the banks to play mixed strategies. Denote the set of mixed strategies of bank i by T_i , i.e. $T_i = \{p_i : 0 \le p_i \le 1\}$ for $i \in N$, where p_i is the probability of waiting by bank i. Define the expected payoff to bank i by

$$(2.5) \quad H_i(p) = p_i \left[\sum_{S \subseteq N - \{i\}} \prod_{\substack{j \notin S \\ j \in N - \{i\}}} p_j \prod_{j \in S} (1 - p_j) D_i P[\beta(D - \sum_{j \in S} D_j)] \right] + (1 - p_i)\, D_i\, \mathrm{P}[D]$$

$$\text{for } \mathrm{p} = (p_1, \ldots, p_n) \text{ in } T_1 \times \ldots \times T_n.$$

The first term of (2.5) is the expected payoff to bank i from waiting and the second term $(1 - p_i)D_i\mathrm{P}[D]$ is the expected payoff from selling.

We apply the Nash equilibrium concept as a solution of our endurance competition game. A strategy n-tuple $\hat{p} = (\hat{p}_1, \ldots, \hat{p}_n)$ is called a Nash equilibrium if for all $i \in N$,

$$H_i(\hat{p}) \ge H_i(\hat{p}_{-i}, p_i) \quad \text{for all } \; p_i \in T_i\,,$$

where $\hat{p}_{-i} = (\hat{p}_1, \ldots, \hat{p}_{i-1}, \hat{p}_{i+1}, \ldots, \hat{p}_n)$.

In the case of two banks, under the condition

$$\mathrm{P}[\beta D] < \mathrm{P}[D] < \mathrm{P}[\beta D_i] \text{ for } i = 1, 2,$$

our endurance competition game G has three equilibria:

$$(\hat{p}_1, \hat{p}_2) = (1, 0); \quad (\hat{p}_1, \hat{p}_2) = (0, 1)\ ; \ \text{ and } \ \hat{p}_i = \frac{P[\beta D_j] - P[D]}{P[\beta D_j] - P[\beta D]} \text{ for } i, j = 1, 2 \ \ (i \ne j).$$

These three equilibria are given by subgame perfect equilibria in the dynamic formulation with infinite horizon given in the next chapter, i.e., the above equilibria occur in each period in the subgame perfect equilibria. In Chapter III, the pure strategy equilibria are eliminated for the reason of a continuation of the situation. We find some additional subgame perfect equilibria whose outcomes are almost the same as the third equilibrium. Thus the one-period approach does not lose the dynamic nature of the problem.

First we describe the set of pure strategy Nash equilibria of the endurance competition game.

Theorem 1. Assume condition (2.1). Let s be a pure strategy n-tuple, and let $S = \{i \in N: s_i = 0\}$. Then s is a Nash equilibrium (in mixed strategies) if and only if

$$(2.6) \qquad P[\beta(D - \sum_{i \in S} D_i)] \geq P[D] \geq P[\beta(D - \sum_{i \in S - \{j\}} D_i)] \text{ for all } j \in S.$$

Proof. Let s be a Nash equilibrium. If $s_i = 1$ for all $i \in N$, then each player has an incentive to change its strategy 1 to 0, since P[D] > P[βD] by (2.1), which is impossible. Hence $s_k = 0$ for some k. Since player k weakly prefers selling ($s_k = 0$) to waiting, we have $P[D] \geq P[\beta(D - \sum_{i \in S - \{k\}} D_i)]$. If $s_i = 0$ for all $i \in N$, then S = N, which implies $P[\beta(D - \sum_{i \in N} D_i)] = P[0] \geq P[D]$ by (2.1). Suppose $s_j = 1$ for some j. Since player j weakly prefers waiting ($s_j = 1$) to selling, we have $P[\beta(D - \sum_{i \in S} D_i)] \geq P[D]$.

Conversely, suppose (2.6). This together with (2.1) implies $S \neq \emptyset$. Then no player in S or in N-S has an incentive to change its strategy since P[D] $\geq$ P[$\beta(D - \sum_{i \in S - \{j\}} D_i)$] and P[$\beta(D - \sum_{i \in S} D_i)$] $\geq$ P[D], respectively. //

First we consider some implications of Theorem 1 under the assumption that

(2.7) P[d] is a decreasing function.

Under this assumption, condition (2.6) is equivalent to

$$(2.6^*) \qquad \beta(D - \sum_{i \in S} D_i) \leq D \leq \beta(D - \sum_{i \in S - \{j\}} D_i) \quad \text{for all } j \in S.$$

This states that in a pure strategy equilibrium, the remaining debt $\beta(D - \sum_{i \in S} D_i)$ with accrued interest differs from the original total debt D by at most $\min_{j \in S} \beta D_j$. The existence of a pure strategy equilibrium follows from (2.6*). Indeed, if $S = \emptyset$, $\beta(D - \sum_{i \in S} D_i) = \beta D > D$. Therefore we can find a minimal S with the property $\beta(D - \sum_{i \in S} D_i) \leq D$. This S satisfies the right inequality. Thus the pure strategy n-tuple s with $s_i = 0$ if $i \in S$ and $s_i = 1$ otherwise is a Nash equilibrium. Actually, there are many pure strategy Nash equilibria. Under the additional condition

$$(2.8) \qquad D > \beta \sum_{i \neq j} D_i \qquad \text{for all } j \in N,$$

– this condition could be true for a small number of banks – there are exactly n pure strategy Nash equilibria. Each of them is represneted as $s^i = (1, ..., 1, \overset{i}{0}, 1, ..., 1)$ for some i, since the set S in Theorem 1 is $\{i\}$. In the general case, the structure of the set of equilibria is more complicated.

We return to the general case. Since $D - \sum_{i \in S} D_i$ is represented as $\sum_{i \in N} s_i D_i$, condition (2.6) is represented as

$$(2.6^{**}) \qquad P[\beta(\sum_{i \in N} s_i D_i)] \geq P[D] \geq P[\beta(\sum_{i \in N} s_i D_i + D_j)] \qquad \text{for all } j \text{ with } s_j = 0.$$

This means that in a pure strategy equilibrium, the resulting price is not smaller than the present price, but the difference between them is bounded by $P[\beta \sum_{i \in N} s_i D_i] - P[\beta(\sum_{i \in N} s_i D_i + D_j)]$. If D_j is small, then the new price does not differ much from the present price. To state this observation more explicitly, we introduce a sequence $\{G^\nu\} = \{(N^\nu, \{D_i^\nu\})\}$ of the endurance competition games with

$$(2.9) \qquad |N^\nu| \to \infty \quad \text{as} \quad \nu \to \infty;$$ [12]

$$(2.10) \qquad \text{for some } K, \ D_i^\nu \leq \frac{K}{|N^\nu|} \qquad \text{for all } i \in N^\nu \text{ and } \nu \geq 0.$$

Denote $D^\nu = \sum_{i \in N^\nu} D_i^\nu$, and note that D^ν is bounded by K. Then the above observation is formulated as follows.

Theorem 2. Assume condition (2.1) for each G^ν. For any sequence $\{s^\nu\}$ where each s^ν is an arbitrary pure strategy Nash equilibrium in G^ν, we have

$$(2.11) \qquad \lim_{\nu \to \infty} |P[\beta \sum_{i \in N} s_i^\nu D_i^\nu] - P[D^\nu]| = 0.$$

3. Behavior of Nash Equilibria for a Large Number of Banks

In Section 2, we proved that in pure strategy equilibria, the resulting secondary market price of debt remains almost the same as the present price when the number of banks is large. This

[12] |S| stands for the cardinality of S.

section extends this result to mixed strategy equilibria. In Section 4, we will show that there are many mixed strategy Nash equilibria. Nevertheless, in any mixed strategy equilibria, the resulting price of debt remains almost the same as the present price with arbitrarily high probability for a large number of banks. In fact, we can state this result for any mixed and pure strategy equilibria.

To state the result, we introduce random variables describing the outcomes of strategies. For a strategy n-tuple $p = \{p_i\}_{i \in N}$, we define $X = \{X_i\}_{i \in N}$ by

$$(3.1) \qquad X_i = \begin{cases} D_i & \text{if the realization of bank } i\text{'s mixed strategy } p_i^{\nu} \text{ is to wait;} \\ 0 & \text{otherwise.} \end{cases}$$

That is, $Pr(X_i = D_i) = p_i$ and $Pr(X_i = 0) = 1 - p_i$. These random variables are independent because of the basic assumption that the strategy choices are independent.[13] The resulting secondary market price of debt in the game G is given by $P[\beta \sum_{i \in N} X_i]$, which is also a random variable.

Now we can state the main result of the chapter.

Theorem 3. Let $\{G^{\nu}\} = \{(N^{\nu}, \{D^{\nu}\})\}$ be a sequence of endurance games with conditions (2.1), (2.9) and (2.10), and let $\{p^{\nu}\}$ be a sequence of Nash equilibria for G^{ν}'s. Let $\{X^{\nu}\}$ be the sequence of random variables where each X^{ν} is defined with p^{ν} by (3.1). Then for any $\varepsilon > 0$,

$$(3.2) \qquad \lim_{\nu \to \infty} Pr\Big(|P[\beta \sum_{i \in N^{\nu}} X_i^{\nu}] - P[D^{\nu}]| \le \varepsilon\Big) = 1.$$

In fact, this convergence is uniform on the choice of a Nash equilibrium p^{ν} for each G^{ν}. That is, it holds that for any $\varepsilon > 0$ and $\delta > 0$, there is a ν_0 such that $Pr\Big(|P[\beta \sum_{i \in N^{\nu}} X_i(p)] - P[D^{\nu}]| \le \varepsilon\Big) \ge 1 - \delta$ for all Nash equilibria p of the game G^{ν} and all $\nu \ge \nu_0$. Here $\{X_i(p)\}_{i \in N^{\nu}}$ are random variables defined by (3.1) with a Nash equilibrium p.

13

Each random variable X_i is a function from $S = S_1 \times \ldots \times S_n$ to $\{0, D_i\}$ with $X_i(s) = D_i$ if $s_i = 1$ and $X_i(s) = 0$ if $s_i = 0$. Also, the space S has the probability measure μ determined by $\mu(\{s\}) = (\prod_{s_i = 1} p_i)(\prod_{s_i = 0} (1 - p_i))$ for all $s \in S$. Since it is not necessary to return to this basic probability space in this chapter, we work on the random variables without referring to the probability space.

This theorem states that the probability of the distance between the prices of the next and present periods to be less than or equal to ε converges to 1 when the number of banks becomes large. When every p^ν in $\{p^\nu\}$ is a pure strategy Nash equilibrium, then the assertion of Theorem 3 is equivalent to Theorem 2. Indeed, Theorem 2 clearly implies (3.2). Conversely, when every p^ν in $\{p^\nu\}$ is a pure strategy equilibrium, (3.2) means that for any $\varepsilon > 0$ there is a ν_0 such that $|\ \mathrm{P}[\beta \sum_{i \in N^\nu} X_i^\nu] - \mathrm{P}[D^\nu]| \leq \varepsilon$ for all $\nu \geq \nu_0$, which is (2.11). If every p^ν in $\{p^\nu\}$ is a mixed strategy equilibrium, i.e., $0 < p_i^\nu < 1$ for at least one player i, condition (2.1) is not necessary for Theorem 3 (see the proof of Theorem 3).

In Nash equilibria, the price of debt remains almost unchanged when the number of banks is large. On one hand, the total debt may increase through accrued interest, and, on the other hand, the total debt may decrease through the country's buyback of some debt. Theorem 3 says that these two effects balance and the total outstanding debt does not change much. The proportion of debt bought back is approximately

$$(3.3) \qquad (D^\nu - \frac{D^\nu}{\beta})/D^\nu = \frac{\beta - 1}{\beta}.$$

Since the interest factor is close to 1, this proportion is also close to the interest rate $\beta - 1$. The country succeeds in buying back a part of debt tantamount to the interest on the total outstanding debt discounted at the present secondary market price. Consequently, the total outstanding debt remains almost the same and *a fortiori* the secondary market price is almost unchanged.

When the distribution of loan exposures is relatively equal, the proportion of banks selling is also approximately $(\beta - 1)/\beta$. However, if the loan exposures are unequal, then we could expect a larger number of banks to sell their exposures. This follows from the next theorem which is proved under condition (2.7). The theorem states that in a mixed strategy equilibrium, a bank with a smaller loan exposure has a greater tendency to sell than one with a larger loan exposure.

Theorem 4. Suppose condition (2.7). Let $(\hat{p}_1, \ldots, \hat{p}_n)$ be a Nash equilibrium. Then it holds that for any i, j with $0 < \hat{p}_i,\ \hat{p}_j < 1$,

$$(3.5) \qquad D_i \geq D_j \ \text{ if and only if } \ \hat{p}_i \geq \hat{p}_j.$$

Since banks with smaller exposures have higher probabilities of selling, the number of banks selling has an increasing tendency when the loan exposures are more unequally distributed, since the proportion of debt bought back is almost constant. This phenomenon could be observed in the real world. For example, in the case of Bolivia, all American banks with larger debt exposures have kept their loans but some banks with small exposures have sold theirs, which caused the price increase from 6¢ in 1986 to 11¢ in 1988.[14]

The consequence of the almost unchaged price of debt is compatible with the functioning of the secondary market. The country buys back the almost constant portion of the total outstanding debt. This does not prevent the creditor banks and other banks from trading on the secondary market. For the secondary market to function, there must be (at least potentially) some trade of loan exposures. Since the present price is almost the same as the future price, keeping a loan exposure is almost equally profitable (unprofitable) as keeping the money corresponding to the secondary market value of the exposure as a deposit on the money market. Therefore banks may extensively trade the loan exposures among themselves, and their trade does not affect the price. On the other hand, the trade between the creditor banks and the country affects the price of debt. Therefore the creditor banks as a whole sell to the country the proportion of debt approximately equal to (3.3) in equilibrium, to have the almost constant price of debt on the secondary market.

Unfortunately, the prediction that the secondary market prices are almost constant is not confirmed, looking at the data of prices in 1980s. For some countries the secondary market prices seemed to be constant, for some they looked increasing, and for some others decreasing. In totality the tendency of decreasing prices have been regarded as slightly dominant (cf. Hajivassiliou (1989)). In our game, we regard one period as short, i.e., one or two months. Therefore the corresponding decreasing tendency could also be small, which means that the disagreement is not significant. Nevertheless, for the long run there remains some discrepancy between our predection and the evidence.

---

[14] cf. Bulow and Rogoff (1988).

Many factors could be thought of as causing the discrepancy between our prediction and the evidence. Some unexpected political events may be regarded as one of them. If we disregard the political disturbance, there is a subtle issue in the interpretation of the empirical evidence. When the decreasing tendency is dominant and is expected by each bank, no trade among banks occurs. Consequently, the secondary market does not function at all, which implies that the secondary market price cannot be quoted. However, this is not true in the real world. Thus the dominant tendency of decreasing prices is viewed as a result of some unexpected disturbances.

Proof of Theorem 3. We show the claim of Theorem 3 under the assumption that every p^ν has at least one player who plays a completely mixed strategy. If this is done, then this together with Theorem 2 implies Theorem 3. Indeed, suppose the claim is proved under this assumption. If only a finite number of p^ν in $\{p^\nu\}$ have mixed strategies, then Theorem 2 is applied and we obtain the claim. If only a finite number of p^ν in $\{p^\nu\}$ are pure strategy equilibria, then the above claim is applicable. Consider the case where the sequence $\{p^\nu\}$ is divided into subsequences $\{p^{\lambda_\nu}\}$ and $\{p^{\mu_\nu}\}$ so that p^{λ_ν} is a pure strategy Nash equilibrium for each ν and p^{μ_ν} is a mixed strategy Nash equilibrium for each ν. There is a ν_1 by Theorem 2 such that $Pr\Big(|P[\beta \sum_{i\in N^{\lambda_\nu}} s_i X_i^{\lambda_\nu}] - P[D^{\lambda_\nu}]| \le \varepsilon\Big) = 1$ for all $\nu \ge \nu_1$, and also for any $\delta > 0$ there is a ν_2 by the above claim such that $Pr\Big(|P[\beta \sum_{i\in N^{\mu_\nu}} X_i^{\mu_\nu}] - P[D^{\mu_\nu}]| \le \varepsilon\Big) \ge 1-\delta$ for all $\nu \ge \nu_2$. Therefore $Pr\Big(|P[\beta \sum_{i\in N^\nu} s_i X_i^\nu] - P[D^\nu]| \le \varepsilon\Big)$ $\ge 1-\delta$ for all $\nu \ge \max(\lambda^{\nu_1}, \mu^{\nu_2})$. This means $\lim_{\nu\to\infty} Pr\Big(|P[\beta \sum_{i\in N^\nu} s_i X_i^\nu] - P[D^\nu]| \le \varepsilon\Big) = 1$.

In the following, we assume that for every ν, $0 < p_i^\nu < 1$ for some $i \in N^\nu$.

Let us prepare several notions and some lemmas. Since $Pr(X_i^\nu = D_i^\nu) = p_i^\nu$ and $Pr(X_i^\nu = 0) = 1 - p_i^\nu$, the mean and variance of X_i^ν are given as

$$(3.6) \qquad E(X_i^\nu) = p_i^\nu D_i^\nu \text{ ; and } V(X_i^\nu) = E((X_i^\nu - p_i^\nu D_i^\nu)^2) = (D_i^\nu)^2 p_i^\nu (1-p_i^\nu) < \left(\frac{K}{|N^\nu|}\right)^2.$$

Denote $\sum_{i\in N^\nu} X_i^\nu$ by S^ν. Then

$$(3.7) \qquad E(S^\nu) = \sum_{i\in N^\nu} p_i^\nu D_i^\nu \text{ ; and } V(S^\nu) = \sum_{i\in N^\nu} V(X_i^\nu) < |N^\nu|\left(\frac{K}{|N^\nu|}\right)^2 = \frac{K^2}{|N^\nu|}.$$

Lemma 3.1. $\lim_{\nu\to\infty} Pr(|S^\nu - E(S^\nu)| \le \varepsilon) = 1$ for any $\varepsilon > 0$, *Proof.* Applying Chebyshev's inequality (Feller (1957, p.219)) to S^v, we have

$$Pr\left(|S^v - E(S^v)| > t\sqrt{V(S^v)}\right) \leq \frac{1}{t^2} \text{ for any } t > 0.$$

It follows from (3.7) that

$$Pr\left(|S^v - E(S^v)| > t\frac{K}{\sqrt{|N^v|}}\right) \leq \frac{1}{t^2} \text{ for any } t > 0.$$

Putting $t = |N^v|^{1/4}$, we have

$$Pr\left(|S^v - E(S^v)| > \frac{K}{|N^v|^{1/4}}\right) \leq \frac{1}{\sqrt{|N^v|}}.$$

Choose a v_0 so that $\frac{K}{|N^v|^{1/4}} < \varepsilon$ for all $v \geq v_0$. Then we have

$$Pr(|S^v - E(S^v)| > \varepsilon) < \frac{1}{\sqrt{|N^v|}} \text{ for all } v \geq v_0.$$

Therefore we have

$$Pr(|S^v - E(S^v)| \leq \varepsilon) \geq 1 - \frac{1}{\sqrt{|N^v|}} \to 1 \text{ as } v \to \infty. \qquad //$$

Lemma 3.2. For any $\varepsilon > 0$, $\lim_{v\to\infty} Pr(|P[\beta S^v] - P[\beta E(S^v)]| \leq \varepsilon) = 1$.

Proof. Since the price function P is continuous on $[0, +\infty)$, P is uniformly continuous on its relevant domain $[0, \beta K]$. Therefore there is a $\delta > 0$ such that $|S^v - E(S^v)| \leq \delta \Rightarrow |P[\beta S^v] - P[\beta E(S^v)]| \leq \varepsilon$. Thus it follows from Lemma 3.1 that $Pr(|P[\beta S^v] - P[\beta E(S^v)]| \leq \varepsilon) \geq Pr(|S^v - E(S^v)| \leq \delta) \to 1$ as $v \to \infty$. //

Lemma 3.3. 1) $\lim_{v\to\infty} |E(P[\beta S^v]) - P[\beta E(S^v)]| = 0$; 2) $\lim_{v\to\infty} |E(P[\beta S^v]) - P[D^v]| = 0$; 3) $\lim_{v\to\infty} |P[\beta E(S^v)] - P[D^v]| = 0$.

Proof. 1) Let ε be an arbitrary positive number. Since P is a uniformly continuous function on the relevant domain $[0, \beta K]$, there is a $\delta > 0$ such that $|S^v - E(S^v)| \leq \delta \Rightarrow |P[\beta S^v] - P[\beta E(S^v)]| \leq \frac{\varepsilon}{2}$. From Lemma 3.1, there is a v_1 such that for all $v \geq v_1$,

$$Pr(|S^v - E(S^v)| > \delta) < \frac{\varepsilon}{2M}, \text{ where } M = \max_{0 \leq d \leq \beta K} \mathrm{P}[d].$$

The difference $|P[\beta S^v] - P[\beta E(S^v)]|$ is less than $\frac{\varepsilon}{2}$ if $|S^v - E(S^v)| \leq \delta$ and is less than M if $|S^v - E(S^v)| \geq \delta$. Therefore we have

$$|E(P[\beta S^v]) - P[\beta E(S^v)]| = |E(P[\beta S^v] - P[\beta E(S^v)])|$$

$$\leq \left(\frac{\varepsilon}{2}\right)Pr(|S^v - E(S^v)| \leq \delta) + MPr(|S^v - E(S^v)| > \delta)$$

$\leq \frac{\varepsilon}{2} + M \times \frac{\varepsilon}{2M} = \varepsilon$ for all $v \geq v_1$.

2) Recall that in each equilibrium p^v, at least one player i plays a completely mixed strategy p_i^v $(0 < p_i^v < 1)$. We rewrite $E(P[\beta S^v])$ as

$$E(P[\beta S^v]) = \sum_{R \subseteq N} \left(\prod_{j \in R} p_j^v\right) \left(\prod_{j \notin R}(1 - p_j^v)\right) P[\beta \sum_{j \in R} D_j^v] \tag{3.8}$$

$$= p_i^v \left[\sum_{R \subseteq N - \{i\}} \left(\prod_{j \in R} p_j^v\right) \left(\prod_{\substack{j \notin R \\ j \neq i}}(1 - p_j^v)\right) P[\beta \sum_{j \in R \cup \{i\}} D_j^v] \right] + (1 - p_i^v) \left[\sum_{R \subseteq N - \{i\}} \left(\prod_{j \in R} p_j^v\right) \left(\prod_{\substack{j \notin R \\ j \neq i}}(1 - p_j^v)\right) P[\beta \sum_{j \in R} D_j^v] \right].$$

Notice that there is the one-one correspondence between the terms of the first and second brackets, and that each term of the first bracket differs from the corresponding term of the second in that the first has the additional βD_i^v in P[•]. Since $0 < p_i^v < 1$, the player i is indifferent between his choices of waiting and selling. Since the expected payoff from waiting is given as the first bracket and the expected payoff from selling is P[D^v], the first bracket of (3.8) is equal to P[D^v]. Since $\max_{j \in N^v} D_j^v \leq \frac{K}{|N^v|} \to 0$ as $v \to \infty$, the difference between the first and second brackets of (3.8) converges to 0 as $v \to \infty$. As was mentioned above, the first bracket is always the same as $P[D^v]$. This means $\lim_{v \to \infty} |E(P[\beta S^v]) - P[D^v]| = 0$.

3) It follows from 1) and 2) that for any $\varepsilon > 0$, there is a v_0 such that for all $v \geq v_0$,

$|E(P[\beta S^v]) - P[\beta E(S^v)]| < \frac{\varepsilon}{2}$ and $|E(P[\beta S^v]) - P[D^v]| < \frac{\varepsilon}{2}$.

Therefore we have $|P[\beta E(S^v)] - P[D^v]| \leq |P[\beta E(S^v)] - E(P[\beta S^v])| + |E(P[\beta S^v]) - P[D^v]|$

$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for all $v \geq v_0$. //

Finally we prove the assertion of the theorem. From Lemma 3.3, for any $\varepsilon > 0$, there is a v_0 such that for all $v \geq v_0$, $|P[\beta E(S^v)] - P[D^v]| < \frac{\varepsilon}{2}$. Thus, using Lemma 3.2, we have,

$$Pr(|P[\beta S^v] - P[D^v]| \leq \varepsilon) \geq Pr(|P[\beta S^v] - P[\beta E(S^v)]| + |P[\beta E(S^v)] - P[D^v]| \leq \varepsilon)$$

$$\geq Pr\left(|P[\beta S^v] - P[\beta E(S^v)]| \leq \frac{\varepsilon}{2}\right) \to 1 \text{ as } v \to \infty. \quad //$$

Proof of Theorem 4. Since $0 < \hat{p}_i, \hat{p}_j < 1$, banks i and j are indeferrent between waiting and selling. Therefore the expected payoffs from waiting and selling are the same. This is expressed as

$$(3.9)\qquad \sum_{R\subseteq N-\{k\}} \left(\prod_{\substack{m\in N \\ m\notin R\cup\{k\}}} \hat{p}_m \right) \left(\prod_{m\in R} (1-\hat{p}_m) \right) P[\beta(D - \sum_{m\in R} D_m)] = P[D] \;\text{ for } k = i,\, j.$$

The left hand side for $k = i$ is written as

$$\hat{p}_j \left[\sum_{R\subseteq N-\{i,j\}} \left(\prod_{\substack{m\notin N-R \\ m\neq i,j}} \hat{p}_m \right) \left(\prod_{m\in R} (1-\hat{p}_m) \right) P[\beta(D - \sum_{m\in R} D_m)] \right]$$

$$+ (1-\hat{p}_j) \left[\sum_{R\subseteq N-\{i,j\}} \left(\prod_{\substack{m\notin N-R \\ k\neq i,j}} \hat{p}_m \right) \left(\prod_{m\in R} (1-\hat{p}_m) \right) P[\beta(D - \sum_{m\in R\cup\{j\}} D_m)] \right].$$

Each term in the second bracket corresponds to one in the first bracket with the same R, and each is bigger than the cooresponding one by (2.7). Therefore the whole sum in the second bracket is bigger than that in the first one. When $D_i = D_j$, the left hand side of (3.9) for bank j is obtained from the above formula by replacing $\hat{p}_j$ by $\hat{p}_i$. In this case, if $\hat{p}_i \neq \hat{p}_j$, then the value of the obtained formula for bank i is also different from the value of the formula for j. But these two must be the same as P[D]. Hence $D_i = D_j$ implies $\hat{p}_i = \hat{p}_j$.

Now it is sufficient to prove that $D_i > D_j$ implies $\hat{p}_i > \hat{p}_j$. When $D_i > D_j$, the left hand side of (3.9) for bank j is obtained from the above formula by replacing $\hat{p}_j$ by $\hat{p}_i$ and D_j by D_i in P[$D - \sum_{m\in R\cup\{j\}} D_m$] in the second bracket. In this case, the value of the new second bracket is greater than that of the original one. Hence to keep the left hand side of (3.9) equal to P[D] for i, the probability coefficient $\hat{p}_i$ for the first bracket must be higher than $\hat{p}_j$. //

4. The Structure of the Set of Nash Equilibria.

To state and prove the limit theorem of the preceding section, we did not need to investigate the structure of the set of all Nash equilibria. The limit theorem holds independently of the choice of a Nash equilibrium and is proved without using specific equilibria. From this result one might expect that the number of mixed strategy Nash equilibria would be small. However, the fact is quite different from this expectation. A number of mixed strategy equilibria may exist in addition to the pure strategy Nash equilibria described by Theorem 1. The limit Theorem holds commonly for all Nash equilibria. In this section, we investigate the structure of the set of mixed strategy Nash equilibria.

First we have the following theorem.

Theorem 5. Suppose conditions (2.7) and (2.8). Let T be a subset of N with $|T| \geq 2$. Then there exists a Nash equilibrium $\hat{p}$ such that

(4.1) $\quad 0 < p_i < 1$ for all $i \in T$; and $p_i = 1$ for all $i \in N - T$.

Theorem 5 says that each choice T with $|T| \geq 2$ gives a mixed strategy equilibrium. Thus there are at least $2^n - n - 1$ mixed strategy equilibria, since condition $|T| \geq 2$ excludes the possibility of T being a singleton or empty. As we mentioned after Theorem 1, there are n pure strategy Nash equilibria under conditions (2.7) and (2.8). Thus the total number of equilibria is at least $2^n - 1$.

Condition (2.8) holds for a relatively small n. Thus Theorem 5 gives the structure of mixed strategy Nash equilibria in the case of a relatively small number of banks. The next theorem gives the structure of mixed strategy equilibria for any number of banks under the assumption of identical loan exposures, i.e.,

(4.2) $\quad D_1 = D_2 = \ldots = D_n > 0.$

Theorem 6. Suppose conditions (2.7) and (4.2). Let T be a subset of N with $D > \beta(D - \frac{|T| - 1}{n} D)$. Then there exists a unique Nash equilibrium $p = (p_1, \ldots, p_n)$ such that $0 < p_i < 1$ for all $i \in T$ and $p_i = 1$ for all $i \in N - T$.

Under condition (2.8), $D > \beta(D - \frac{|T| - 1}{n} D)$ implies $|T| \geq 2$. In this case, Theorem 6 implies that the total number of Nash equilibria becomes exactly $2^n - 1$.

Consider the case without condition (2.8). Let $t = |T|$ be the smallest integer with $D > \beta(D - \frac{|T| - 1}{n} D)$. Theorem 6 gives exactly $\sum_{k=t}^{n} \binom{n}{k}$ number of mixed strategy equilibria. Every subset T of N with $|T| = t - 1$ satisfies condition (2.6*), which implies that each T gives one pure strategy Nash equlibrium Thus the total number of Nash equilibria is given as $\sum_{k=t-1}^{n}\binom{n}{k}$. This number is approximately 2^n for large n and $\beta < 2$. More formally,

Theorem 7. Under the assumption of Theorem 6 and $\beta < 2$,

$$\lim_{n\to\infty} \frac{|\text{the set of all Nash equilibria}|}{2^n} = 1.$$

The results of this section show that there are approximately 2^n number of Nash equilibria in the endurance competition games, independent of the total number n of banks.

Proof of Theorem 5. We define the following functions:

$$(4.3)\qquad \bar{f}_i(p) = \sum_{R\subseteq N-\{i\}} \Big(\prod_{j\notin R\cup\{i\}} p_j\Big)\Big(\prod_{j\in R}(1-p_j)\Big)\, P[\beta(D-\sum_{j\in R} D_j)] \quad \text{and} \quad f_i(p) = \bar{f}_i(p) - P[D]$$

for $p\in[0, 1]^n$ and $i\in N$. Our objective is to find a point $\hat{p}$ in $[0, 1]^n$ such that $f_i(\hat{p}) = 0$, $0 < \hat{p}_i < 1$ for all $i\in T$ and $\hat{p}_j = 1$ for all $j\in N-T$. Suppose the existence of such a $\hat{p}$ is proved. In this case, each bank i in T is indifferent between waiting and selling, so $\hat{p}_i$ is a best response. By the following lemma, $f_j(\hat{p}) > 0$ for all j in N-T, which implies that $\hat{p}_j = 1$ is a (unique) best response to $\hat{p}$ for all $j\in N-T$. Hence $\hat{p}$ is a Nash equilibrium with the property (4.1).

Lemma 4.1. For any $p\in[0, 1]^n$ and i, j in N, if $p_i < p_j = 1$, then $f_i(p) < f_j(p)$.

Proof. The value $\bar{f}_j(p)$ is described as

$$\bar{f}_j(p) = p_i\left[\sum_{R\subseteq N-\{i,j\}} \Big(\prod_{\substack{k\in N-R\\ k\neq i,j}} p_k\Big)\Big(\prod_{k\in R}(1-p_k)\Big)\, P[\beta(D-\sum_{k\in R} D_k)]\right]$$

$$+ (1-p_i)\left[\sum_{R\subseteq N-\{i,j\}} \Big(\prod_{\substack{k\in N-R\\ k\neq i,j}} p_k\Big)\Big(\prod_{k\in R}(1-p_k)\Big)\, P[\beta(D-\sum_{k\in R\cup\{i\}} D_k)]\right].$$

Each element in the second bracket corresponds to one element of the first bracket with respect to R, and each element in the second bracket is larger than the corresponding one. This implies that the second bracket has a greater value than the first one. The value $\bar{f}_i(p)$ is obtained from $\bar{f}_j(p)$ by replacing p_i by p_j, and D_i by D_j in the second bracket. However, since $p_j = 1$, the second term of $\bar{f}_i(p)$ disappears, and only the small part, the first bracket, remains. This means that $\bar{f}_j(p) > \bar{f}_i(p)$.
//

Now we prove the existence of a point $\hat{p}$ in $[0, 1]^n$ such that $f_i(\hat{p}) = 0$, $0 < \hat{p}_i < 1$ for all $i \in T$ and $\hat{p}_j = 1$ for all j in N-T. Since it is sufficient to look for such a point $\hat{p}$ in the restricted set $\{p \in [0, 1]^n : p_j = 1 \text{ for all } j \in N - T\}$, we do not worry about $p_j = 1$ for all $j \in N - T$. Therefore we assume for notational simplicity that T = N.

Now we consider the following mapping:

$$\phi_i(p) = p_i + f_{i-1}(p) \quad \text{for all i in N and} \quad p \in [0, 1]^n,$$

where $i - 1$ is interpreted as n if i = 1.[15] We would like to apply Brouwer's fixed point theorem to this mapping $\phi = (\phi_1, \ldots, \phi_n)$, but the image of ϕ may not be included in $[0, 1]^n$. Therefore we modify this mapping ϕ by the retraction mapping r: $R^n \to R^n$:

$$r_i(x) = \begin{cases} 0 & \text{if } x_i \leq 0 \\ x_i & \text{if } 0 \leq x_i \leq 1 \\ 1 & \text{if } 1 \leq x_i \end{cases}$$

for all $x \in R^n$ and i = 1, ..., n. Then the composite mapping $\psi = r \cdot \phi$ is a continuous mapping from $[0, 1]^n$ to $[0, 1]^n$. Now, by Brouwer's fixed point theorem, there exists a fixed point $\hat{p}$ in $[0, 1]^n$, i.e., $\psi(\hat{p}) = r \cdot \phi(\hat{p}) = \hat{p}$.

If $\hat{p}$ satisfies $0 < \hat{p}_i < 1$ for all i in N, then, by the definition of the retraction mapping r, $\psi(\hat{p}) = \hat{p}$, that is, $\hat{p}_i + f_{i-1}(\hat{p}) = \hat{p}_i$ for all i in N, which means that $\hat{p}$ satisfies condition (4.1). Now we prove that $0 < \hat{p}_i < 1$ for all i in N.

Claim 1. i) If $\hat{p}_i = 1$, then $f_{i-1}(\hat{p}) \geq 0$; and ii) if $\hat{p}_i < 1$, then $f_{i-1}(\hat{p}) \leq 0$

∵) i) Suppose $f_{i-1}(\hat{p}) < 0$. Then, by the definition of ϕ, $1 = \hat{p}_i = r_i \cdot \phi_i(\hat{p}) = r_i(\hat{p}_i + f_{i-1}(\hat{p})) < 1$, a contradiction.

ii) Similar. //

Claim 2. $\hat{p} \neq (1,\ldots,1)$ and $\hat{p} \neq (0,\ldots,0)$.

---

15 If we define $\phi_i(p)$ by $p_i + f_i(p)$ for i = 1, ..., n, then we can only prove the existence of a Nash equilibrium, which cannot be guaranted to have the property required in Theorem 3. The mere existence of a Nash equilibrium was already obtained by Theorem 2.

$\because$) If $\hat{p} = (1,...,1)$, then $f_i(\hat{p}) = P[\beta D] - P[D] < 0$ for i = 1,...,n by (2.7), but $f_i(\hat{p}) \geq 0$ for i = 1,...,n by Claim 1, which is a contradiction. If $\hat{p} = (0,...,0)$, then $f_i(\hat{p}) = P[\beta D_i] - P[D] > 0$ for all i = 1,...,n by (2.8), but $f_i(p) \leq 0$ for i = 1,...,n by Claim 1, a contradiction. //

Claim 3. $0 < \hat{p}_i < 1$ for all i = 1,...,n.

$\because$) Suppose that $\hat{p}_i = 0$ for some i in N. Then, by Claim 1, $f_{i-1}(\hat{p}) \leq 0$ However it follows from (2.7), (2.8), (4.3) and $\hat{p}_i = 0$ that $\bar{f}_j(\hat{p}) \geq P[\beta(D - D_i)] > P[D]$, i.e., $f_j(\hat{p}) > 0$ for all $j \neq i$. This is a contradiction.

Finally we prove that $\hat{p}_i < 1$ for all $i \in N$. Suppose, on the contrary, that $\hat{p}_i < 1$, $\hat{p}_{i+1} = \hat{p}_{i+2} = ... = \hat{p}_{i+k} = 1$ and $\hat{p}_{i+k+1} < 1$ for some i in N.[16] By Claim 1 and $\hat{p}_{i+1} = 1$, $f_i(\hat{p}) \geq 0$. By applying Lemma 4.1 to i and i+k, we have $f_{i+k}(\hat{p}) > f_i(\hat{p}) \geq 0$. However it follows from Claim 1 and $\hat{p}_{i+k+1} < 1$ that $f_{i+k}(\hat{p}) \leq 0$, a contradiction. //

Proof of Theorem 6. Since the game is symmetric, we can assume without loss of generality that $T = \{1, 2, ..., t\}$. Suppose that $(\hat{p}_1, ..., \hat{p}_n)$ is a Nash equilibrium with the property $0 < \hat{p}_i < 1$ for all $i \in T$ and $\hat{p}_i = 1$ for all $i \in N - T$. Then $\hat{p}_i = \hat{\alpha}$ for all $i \in T$ by Theorem 4. Since each $i \in T$ is indifferent between waiting ($s_i = 1$) and selling ($s_i = 0$),

$$\text{(4.4)} \qquad \sum_{k=0}^{t-1} \binom{t-1}{k} \hat{\alpha}^{t-k-1}(1-\hat{\alpha})^k P[\beta(D - \frac{k}{n} D)] = P[D].$$

Conversely, if (4.4) holds, then every bank $i \in T$ is indifferent between waiting and selling, and every bank $i \in N - T$ prefers waiting to selling, i.e., $\hat{p}_i = 1$, since

$$\begin{aligned}
&\sum_{k=0}^{t} \binom{t}{k} \hat{\alpha}^{t-k}(1-\hat{\alpha})^k P[\beta(D - \frac{k}{n} D)] \\
&= \hat{\alpha}\left[\sum_{k=0}^{t-1} \binom{t-1}{k} \hat{\alpha}^{t-k-1}(1-\hat{\alpha})^k P[\beta(D - \frac{k}{n} D)]\right] \\
&\quad + (1-\hat{\alpha})\left[\sum_{k=0}^{t-1} \binom{t-1}{k} \hat{\alpha}^{t-k-1}(1-\hat{\alpha})^k P[\beta(D - \frac{k+1}{n} D)]\right] > P[D].
\end{aligned}$$

---

[16] If i+m > n, then i+m is interpreted as i+m-n.

Thus it satisfies to prove the unique existence of $\hat{\alpha}$ satisfying (4.4).

Let $g(\alpha) = \sum_{k=0}^{t-1} \binom{t-1}{k} \alpha^{t-k-1}(1-\alpha)^k P[\beta(D - \frac{k}{n}D)]$ for $\alpha \in [0, 1]$. Then $g(0) =$ $P[\beta(D - \frac{t-1}{n}D)] > P[D]$ by the assumption of the theorem, and $g(1) = P[\beta D] < P[D]$. Therefore, by the Intermediate-Value Theorem, there is an $\hat{\alpha}$ in (0, 1) such that $g(\hat{\alpha}) = 0$. This $\hat{\alpha}$ is a solution of (4.4).

The uniqueness of such an $\hat{\alpha}$ is verified by checking the negative sign of the derivative of $g(\alpha)$ for any α $(0 < \alpha < 1)$,

$$g'(\alpha) = \sum_{k=0}^{t-1} \binom{t-1}{k} [(t-k-1)\,\alpha^{t-k-2}(1-\alpha)^k - k\alpha^{t-k-1}(1-\alpha)^{k-1}]\, P[\beta(D - \frac{k}{n}D)]$$

$$\leq - \sum_{k=1}^{t-1} \frac{(t-1)!}{(t-k-1)!\,(k-1)!}\, \alpha^{t-k-1}(1-\alpha)^{k-1} P[\beta(D - \frac{k}{n}D)]$$

$$+ \sum_{k=0}^{t-2} \frac{(t-1)!}{(t-k-2)!\,k!}\, \alpha^{t-k-2}(1-\alpha)^k P[\beta(D - \frac{k}{n}D)]$$

$$= \sum_{k=1}^{t-1} \frac{(t-1)!}{(t-k-1)!\,(k-1)!}\, \alpha^{t-k-1}(1-\alpha)^{k-1}\Big[- P[\beta(D - \frac{k}{n}D)] + P[\beta(D - \frac{k-1}{n}D)]\Big] < 0.$$

Proof of Theorem 7. First note that $D > \beta(D - \frac{t-1}{n}D)$ is equivalent to $\frac{t-1}{n} > 1 - \frac{1}{\beta} \equiv \alpha$. Since $\beta < 2$, we have $\alpha < \frac{1}{2}$. Hence we have to show that

$$\lim_{n \to \infty} \frac{\sum_{k/n > \alpha} \binom{n}{k}}{2^n} \to 1.$$

Define a family of independent random variables $\{X_k^n\}_{k=1}^n$ by

$$X_k^n = \begin{cases} 1 & \text{with probability } \frac{1}{2} \\ 0 & \text{with probability } \frac{1}{2} \end{cases}$$

for $k = 1, 2, \ldots, n$, and define $S^n = \sum_{k=1}^{n} X_k^n / n$. Then $E(S^n) = \frac{1}{2}$ and $V(S^n) = \sum_{k=1}^{n} V(X_k^n)/n^2 = \frac{1}{4n}$. Let ε be a positive number with $\varepsilon \leq \frac{1}{2} - \alpha$. Then it follows from Chebyshev's inequality that

$$Pr\Big(|S^n - \frac{1}{2}| \geq \varepsilon\Big) \leq \frac{V(S^n)}{\varepsilon^2} = \frac{1}{4n\varepsilon^2},$$

that is,

$$Pr\left(|S^n - \frac{1}{2}| < \varepsilon\right) \geq 1 - \frac{1}{4n\varepsilon^2}.$$

Since $\dfrac{\sum_{k/n>\alpha}\binom{n}{k}}{2^n} = \sum_{k/n>\alpha}\binom{n}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{n-k} = Pr(S^n > \alpha)$, and since $\alpha \leq \frac{1}{2} - \varepsilon$, we have

$$\frac{\sum_{k/n>\alpha}\binom{n}{k}}{2^n} = Pr(S^n > \alpha) \geq Pr\left(\frac{1}{2} - \varepsilon < S^n < \frac{1}{2} + \varepsilon\right) \geq 1 - \frac{1}{4n\varepsilon^2} \rightarrow 1 \text{ as } n \rightarrow \infty. \quad //$$

Chapter III

Dynamics of International Debt Overhang with Two Lender Banks

1. Introduction

The international debt overhang is a situation of a sovereign country who has borrowed money from foreign banks and has been unable to fulfill the scheduled repayments. The debt overhang is a serious problem for the debtor country, which keeps the country in a depressed economic situation and prevents from growing. To understand the problem of debt overhang, in Chapter II, we give a game theoretical model of debt overhang where lender banks are players and decide whether or not to sell their loan exposures to the debtor country at the discounted price on the secondary market. The analysis shows that there is a great tendency for the present situation of debt overhang to remain unchanged. In Chapter II, we focus on the decision making of lender banks in a short period and formulate the situation as a one-shot game. However, the problem of debt overhang is dynamic in nature. In this chapter we examine whether the one-shot approach captures the dynamics, considering a dynamic formulation of the one-period model and its equilibria.

The secondary market for debts is an important element which accompanies the problem of debt overhang. The appearance of the secondary market for debts has been caused by the country's inability to repay the debts in full. On this market, loan exposures are traded at a discounted price.[17] The existence of the secondary market creates a possibility for the debtor country to buy back its debts at a discounted price. However, there has been some tendency for the debt overhang to persist even in the presence of the secondary market. As in Chapter II, we assume that the secondary market is represented as a price function.

We formulate the problem as an infinite horizon game with two banks as players. Since the modelling of the dynamics of debt overhang with many banks is too complex, we consider the two bank case. The game is played as follows. In every period of the game each bank decides either to sell its loan exposure to the debtor country at the current secondary market price or to wait and postpone this decision to the next period. If both banks wait in a period, then they face the same decision problem in the following period. When a bank sells its loan exposure to the debtor country at the current secondary market price, it receives a payoff according to the current secondary market price and leaves the game. After both banks sell their exposures to the debtor country, the game ends.

In our dynamic game, there are three types of subgame perfect equilibria with the time continuation property. By the time continuation property we mean that the game of the present period can be viewed as the result of the game of some previous period so that the extensions of the equilibria of the present game give a positive realization probabilities to the present game. The three types of equilibria are called central, alternating and mutating. Each equilibrium is very similar to the unique central equilibrium in the sense that the probabilities of a bank waiting are determined by similar formulas and take similar values. Therefore, the average attainable payoffs are almost uniquely determined.

---

[17] See, for example, Sachs and Huizinga (1987), and Hajivassiliou (1989).

We can compare the equilibria of the dynamic game and the one-period game of the previous chapter. To make this comparison, we construct a sequence consisting of the mixed strategy equilibria for the one-period games describing the situation of banks in the sequence of the respective periods. This sequence of equilibria coincides with the central equilibrium of our dynamic game.[18] That is, we have a decomposition property of the dynamic game into the games of one period. Thus the one-period approach of Chapter II does not lose much of the dynamic nature of the problem.

Our dynamic game may be regarded as a repeated game with a constituent game of the one-period approach. As long as no bank sells its loan exposure until period t, a similar one-period game is played in period $t+1$. However, the Folk Theorem[19] -- that almost all payoffs of the one-period game are attainable by an equilibrium of the repeated game -- does not hold in our game since one player's deviation necessarily implies its exit from the game and does not permit a punishment by the other bank. On the contrary to the Folk Theorem, the average attainable payoffs are almost uniquely determined in our game.

We describe some observations of the behavior of banks over time following from our characterization of equilibria. In any equilibrium each bank waits with relatively high probability in every period, and when the interest rate is small, the probability of a bank waiting is close to 1. This can be interpreted as a tendency for the situation of debt overhang to remain almost unchanged.

We also investigate the strategies of the banks when the time passes. We show that if the price function is approximated by some homogeneous function for large values of debt, then the probability of waiting in a period in the central equilibrium becomes a stationary probability in the long run. The stationary probability of a bank waiting in every period is relatively high as long as the interest rate is low. This suggests a difficulty of a resolution of the debt overhang solely through the secondary market transactions.

[18] In Chapter II, there are also two pure strategy Nash equilibria in the two bank case. The sequences consisting of them become subgame perfect equilibria in our dynamic approach. We, however, eliminate them by the time continuation property.

[19] See, for example, Aumann (1982).

The chapter is organized as follows. In Section 2 we formulate the model called the dynamic endurance game. In Section 3 we present the structure of equilibria. The comparative dynamics (statics) and limit results are given in Section 4. In Section 5, we give the proof of the main theorem. Section 6 contains conclusions.

2. Dynamic Endurance Game

We consider the following dynamic situation: a country has debt obligations $D_1 > 0$ and $D_2 > 0$ to foreign banks 1 and 2, respectively. The country has fallen behind with service payments for some periods, and the current situation of the country does not allow for full repayments. The existence of the secondary market for debts is assumed, where the country is ready to buy its debts at the secondary market price. We assume that the price of debt on the secondary market depends upon the current total outstanding debts. This price is expressed by the function $P(D)$: $R_+ \rightarrow R_+$ with the property:

(2.1) P(D) is a decreasing function of the total outstanding debt D and P(D) $\rightarrow$ 0 as $D \rightarrow \infty$.

This P(D) is country-specific and also depends upon the choice of the present period.

We will focus our attention on the decisions of banks over time. The present period is called 0. In every period each bank has two possible choices either to sell its exposure or to wait and keep it to the next period. Each bank's strategies in every period are s and w, where s denotes selling the loan exposure to the debtor country at the current secondary market price, and w denotes waiting and postponing the decision to the next period.

Each bank discounts the future revenues by the interest rate ($r > 0$). We denote the interest factor $1 + r$ by β. We assume that

(2.2) $\quad \beta^2 D_i < D^0$ for $i = 1, 2,$

where $D^0 = D_1 + D_2$. This assumption says that the distribution of loan exposures of banks is relatively equal and the interest rate is relatively small so that the size of one bank's loan exposure with compounded interests from two periods is smaller than the initial debt of the country.

If bank i keeps its loan exposure D_i until period t ($t \geq 0$), then its exposure increases by the accrued interests to $\beta^t D_i$. The total outstanding debt D^t in period t becomes $\beta^t D_i$ if the other bank has already sold, or $\beta^t D^0$ otherwise. The secondary market price in period t is given as $P(D^t)$. If bank i sells in period t, then the present value of repayment is given as

$$\frac{1}{\beta^t}\beta^t D_i P(D^t) = D_i P(D^t).$$

If bank i and the other bank wait in period t, then bank i does not get any payoff in this period, but it will face the same decision problem in the next period.

We assume that

(2.3) after one bank sells its loan exposure, the other bank sells its exposure immediately in the next period.

It is possible that while one bank sells its exposure in some period, the other bank keeps its exposure for several periods after that. In this case, however, keeping the loan exposure for several periods is not an optimal behavior. Indeed, if the bank postpones selling its exposure, the secondary market price of debt will decrease because of the accrued interest. Suppose bank j sells its exposure in period t and bank i does not. Then the secondary market price in period t + 1 is $P(\beta^{t+1}D_i)$, and the present value of loan exposure is $D_i P(\beta^{t+1}D_i)$. If bank i keeps its loan exposure to period t + 2, then the price falls to $P(\beta^{t+2}D_i)$ and the present value is $D_i P(\beta^{t+2}D_i)$. The optimal behavior of bank i is to sell the exposure in period t + 1. Thus we can assume (2.3).

There are two cases in which the game terminates. The first case is that both banks wait until period $t-1$ and both sell in period t. The second case is that both banks wait until period $t-1$ and one bank sells its loan exposure in period t, and the other bank waits in period t and sells its exposure in period $t+1$. The payoff to bank i ($i = 1, 2$) in the game $\Gamma(0, D^0)$ is defined by

$$
(2.4) \qquad
\begin{array}{ll}
D_iP(\beta^t D^0) & \text{if both banks wait until period } t-1 \text{ and bank } i \text{ sells} \\
 & \text{in period } t, \\
D_iP(\beta^{t+1} D_i) & \text{if both banks wait until period } t-1 \text{ and bank } j\ (j \neq i) \\
 & \text{sells in period } t \text{ and bank } i \text{ waits in period } t.
\end{array}
$$

Our game $\Gamma(0, D^0)$ is described in Figure 1. In the game tree, the payoffs to the banks are given in three branches. In the first branch both banks sell their exposures in period 0. In the second, bank 1 sells in period 1 and bank 2 sells in period 2. The third case is that bank 2 sells its exposure in period 2 and bank 1 sells its exposure in period 3.

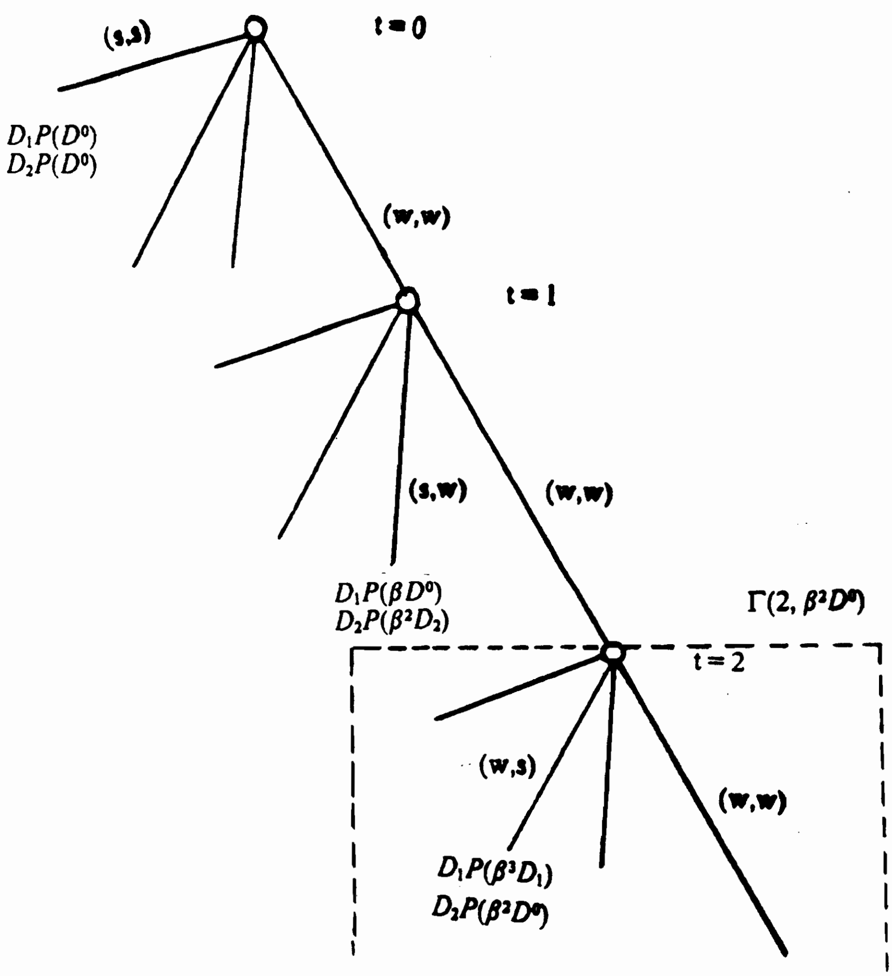

Figure 1.

We allow each bank to use behavior strategies. Since each bank's decision is made in period t only when both banks have waited until period t by assumption (2.3), a behavior strategy of bank $i = 1, 2$ in the game $\Gamma(0, D^0)$ is represented as a sequence $b_i^0 = (p_i^0, p_i^1, ...)$, where p_i^t is a probability of bank i's waiting in period t ($t = 0, 1, ...$) if both banks keep their loan exposures until period t. Denote the set of all behavior strategies of bank i by B_i^0. A behavior strategy combination for the game $\Gamma(0, D^0)$ is a vector $b^0 = (b_1^0, b_2^0) = ((p_1^0, p_1^1, ...), (p_2^0, p_2^1, ...))$.

The expected payoff to bank $i = 1, 2$ for a behavior strategy combination b^0 is the sum of

(i) the expected payoff from selling the exposure in period t ($t = 0, 1, ...$) under the assumption that the other bank does not sell earlier – $D_iP(\beta^t D^0)(1 - p_i^t)\prod_{k=0}^{t-1} p_i^k p_j^k$; and

(ii) the expected payoff from selling its loan exposure in period $t + 1$ under the assumption that the other bank sells in period t – $D_iP(\beta^{t+1}D_i)p_i^t(1 - p_j^t)\prod_{k=0}^{t-1} p_i^k p_j^k$.

Thus the expected payoff from the game $\Gamma(0, D^0)$ under the strategy combination b^0 is given by

$$H_i^0(b^0) = \sum_{t=0}^{\infty} D_iP(\beta^t D^0)(1 - p_i^t)\prod_{k=0}^{t-1} p_i^k p_j^k + \sum_{t=0}^{\infty} D_iP(\beta^{t+1}D_i)p_i^t(1 - p_j^t)\prod_{k=0}^{t-1} p_i^k p_j^k. \tag{2.5}$$

We use the convention $\prod_{k=0}^{-1} p_i^k p_j^k = 1$.

We have described the dynamic endurance game of lender banks $\Gamma(0, D^0)$. In the next section we investigate the decisions of banks in the dynamic endurance game.

3. The Structure of Equilibria

To investigate the decisions of banks in the game $\Gamma(0, D^0)$, we adopt the concept of the subgame perfect equilibrium point of the extensive game (Selten (1975)). To define a subgame perfect equilibrium, we have to consider subgames of the game $\Gamma(0, D^0)$. Here every subtree constitutes a subgame. Thus the subgame which starts at any period t of the game $\Gamma(0, D^0)$ is denoted by

$\Gamma(t, \beta^t D^0)$. The strategy for the subgame $\Gamma(t, \beta^t D^0)$ induced by $b_i^0 = (p_i^0, p_i^1, \ldots)$ is a vector obtained by dropping the first t entries of the vector b_i^0, i.e. $b_i^t = (p_i^t, p_i^{t+1}, \ldots)$. Let B_i^t be the set of all induced behavior strategies of bank i for the subgame $\Gamma(t, \beta^t D^0)$.

Denote by $H_i^t(b^t)$ the expected payoff to bank i from the subgame $\Gamma(t, \beta^t D^0)$ under the induced behavior strategy combination $b^t = (b_1^t, b_2^t)$.

A behavior strategy combination $\hat{b}^t = (\hat{b}_1^t, \hat{b}_2^t)$ is a Nash equilibrium of the subgame $\Gamma(t, \beta^t D^0)$ iff for i = 1, 2,

$$H_i^t(\hat{b}^t) \geq H_i^t(\hat{b}^t/b_i^t) \quad \text{for all } b_i^t \in B_i^t, \tag{3.1}$$

where $\hat{b}^t/b_i^t$ denotes a strategy combination $\hat{b}^t$ with the replacement of $\hat{b}_i^t$ by b_i^t. A subgame perfect equilibrium of the game $\Gamma(0, D^0)$ is a behavior strategy combination $\hat{b}^0 = (\hat{b}_1^0, \hat{b}_2^0)$ which induces a Nash equilibrium on every subgame of the game $\Gamma(0, D^0)$.

In addition to the subgame perfection, we require the equilibrium to have the time continuation property. To define the time continuation property, we introduce a retrospective extension of the game $\Gamma(0, D^0)$ as a game $\Gamma(t, \beta^t D^0)$ for $t = -1, -2, \ldots$, so that the game $\Gamma(0, D^0)$ is a subgame of the game $\Gamma(t, \beta^t D^0)$. A subgame perfect equlibrium $\hat{b}^0$ is said to have the time continuation property iff for any $t = -1, -2, \ldots$ there is a subgame perfect equilibrium $\hat{b}^t$ in the retrospective extension $\Gamma(t, \beta^t D^0)$ of $\Gamma(0, D^0)$ such that $\hat{b}^t$ induces $\hat{b}^0$ and the realization probability of the subgame $\Gamma(0, D^0)$ is positive.

The time continuation property states that the present game situation results as a continuation of the past history. The game $\Gamma(0, D^0)$ is a result of previous decisions of banks. Therefore the present game is a subgame of the game of any preceding period. If the realization probability of the game $\Gamma(0, D^0)$ is zero in an equilibrium for $\Gamma(t, \beta^t D^0)$ then the present situation would be different from $\Gamma(0, D^0)$. However, we assume that the game $\Gamma(0, D^0)$ is reached. Therefore it is compatible with the consideration of $\Gamma(0, D^0)$ to assume that the realization probability is positive.[20]

---

[20] The time continuation property is a concept independent from the time consistency in the macroeconomic literature. The time consistency property is, instead, implied by the subgame perfection.

The structure of equilibria is described by the following theorem.

Theorem 1. The endurance game of two banks has three types of subgame perfect equilibria satisfying the time continuation property, which are called central, alternating, and mutating. In the unique central equilibrium $\hat{b}^0 = (\hat{b}_1^0, \hat{b}_2^0) = ((\hat{p}_1^0, \hat{p}_1^1, ...), (\hat{p}_2^0, \hat{p}_2^1, ...))$, bank i waits in every period t $(t = 0, 1, ...)$ with probability

$$\hat{p}_i^t = \frac{P(\beta^{t+1}D_j) - P(\beta^t D^0)}{P(\beta^{t+1}D_j) - P(\beta^{t+1}D^0)} \quad \text{for} \quad i, j = 1,2 \text{ and } i \neq j. \tag{3.3}$$

There are two alternating equilibria $\tilde{b}^0 = (\tilde{b}_1^0, \tilde{b}_2^0) = ((\tilde{p}_1^0, \tilde{p}_1^1, ...), (\tilde{p}_2^0, \tilde{p}_2^1, ...))$ in which banks i and j wait with probabilities

$$\tilde{p}_i^t = \frac{P(\beta^{t+1}D_j) - P(\beta^{t-1}D^0)}{P(\beta^{t+1}D_j) - P(\beta^{t+1}D^0)} \text{ and } \tilde{p}_j^t = 1 \text{ if t is even;} \tag{3.4}$$

and

$$\tilde{p}_i^t = 1 \text{ and } \tilde{p}_j^t = \frac{P(\beta^{t+1}D_i) - P(\beta^{t-1}D^0)}{P(\beta^{t+1}D_i) - P(\beta^{t+1}D^0)} \text{ if t is odd.} \tag{3.5}$$

In a mutating equilibrium $\bar{b}^0 = (\bar{b}_1^0, \bar{b}_2^0) = ((\bar{p}_1^0, \bar{p}_1^1, ...), (\bar{p}_2^0, \bar{p}_2^1, ...))$ banks wait in every period t until some period $\tau\ (\tau \geq -2)$[21] with probabilities given by (3.3), in period $\tau+1$ they wait with probabilities

$$\bar{p}_i^{\tau+1} = \frac{P(\beta^{\tau+2}D_j) - P(\beta^{\tau+1}D^0)}{P(\beta^{\tau+2}D_j) - P(\beta^{\tau+2}D^0)}, \ \bar{p}_j^{\tau+1} = \frac{P(\beta^{\tau+2}D_i) - P(\beta^{\tau+1}D^0)}{P(\beta^{\tau+2}D_i) - P(\beta^{\tau+3}D^0) + \bar{p}_j^{\tau+2}[P(\beta^{\tau+3}D_i) - P(\beta^{\tau+3}D^0)]}, \tag{3.6}$$

in period $\tau + 2$ they wait with probabilities

$$\bar{p}_i^{\tau+2} = 1, \ \bar{p}_j^{\tau+2} \in [\frac{P(\beta^{\tau+3}D_i) - P(\beta^{\tau+1}D^0)}{P(\beta^{\tau+3}D_i) - P(\beta^{\tau+3}D^0)}, \ \frac{P(\beta^{\tau+3}D_i) - P(\beta^{\tau+2}D^0)}{P(\beta^{\tau+3}D_i) - P(\beta^{\tau+3}D^0)}], \tag{3.7}$$

and in period $t\ (t \geq \tau + 3)$, they wait with probabilities given by (3.4) when $t = \tau + 3, \tau + 5, \ldots$, and with probabilities given by (3.5) when $t = \tau + 4, \tau + 6, \ldots$.

---

[21] When $\tau = -2$, $\bar{p}_1^0$ and $\bar{p}_2^0$ are given by (3.7) and (3.6) is irrelevant. When $\tau = -1$, $\bar{p}_1^0$ and $\bar{p}_2^0$ are given by (3.6).

In the central equilibrium, bank i waits in period t with probability given by (3.3). In an alternating equilibrium banks i and j wait with probabilities given by (3.4) in every even period, and with probabilities given by (3.5) in every odd period. Thus each bank alternates its strategy between waiting for sure and waiting with probability $\tilde{p}_i^t$ given in (3.4). In a mutating equilibrium banks behave initially according to the central equilibrium strategies and in some future period their central equilibrium strategies mutate into the alternating equilibrium strategies. The strategies of transitory periods in a mutating equilibrium are given by (3.6) and (3.7).

The central equilibrium differs from the alternating equilibrium in that each bank sells with some probability in the former, and each bank alternates between waiting for sure and selling with some probability in the latter. Nevertheless, the central and alternating equilibria are similar in the sense that the probabilities of waiting in each of them are determined by similar formulas and take similar values as will be shown in the example below. The mutating equilibrium is a combination of the central and the alternating equilibria. The central equlibrium mutates into the alternating equilibrium but not the other way around.

In every equilibrium of our dynamic game the probability of a bank waiting in each period is relatively high as long as the interest factor is low. This suggests that there is a tendency for the situation of the debt overhang to remain unchanged, no matter what equilibrium strategies the banks use.

Although we found three types of equilibria in our dynamic endurance game, the strategies of banks in every equilibrium are close to the strategies of the unique central equilibrium. The central equilibrium of the dynamic game gives the local equilibrium strategies which coincide with the equilibrium of a one-period endurance game investigated in Chapter II. In other words, the central equilibrium can be constructed as a sequence of mixed strategy equilibria of the one-shot games of Chapter II. This link between the dynamic and one-period formulations allows us to use the one-period approach without losing much of the dynamic nature of the problem.

The following example illustrates the claim of Theorem 1.

Example 3.1. Let $D_1 = D_2 = \frac{1}{2} D^0 = \frac{1}{2}$ and $\beta = 1.1$. Assume that the price function is given by

$$P(D) = \frac{90}{D+1} \quad .$$

The central equilibrium strategy of each bank is to wait in period t with probability

$$\hat{p}_i^t = \frac{9}{11} \frac{(1.1)^{t+1} + 1}{(1.1)^t + 1} \quad .$$

The table below shows some values for the local strategies.

Table 1.

t	0	1	2	3	4	8	16	32
p_i^t	.859	.861	.863	.865	.867	.874	.885	.896

In this example the central equilibrium probability of a bank waiting in period t is high. It increases with the time t and converges to .9 as t becomes large.

The alternating equilibrium strategy of banks i and j is to wait with probabilities

$$\tilde{p}_i^t = \frac{79}{121} \frac{(1.1)^{t+1} + 1}{(1.1)^{t-1} + 1}, \quad \tilde{p}_j^t = 1 \qquad \text{in every even period;}$$

$$\tilde{p}_i^t = 1, \quad \tilde{p}_j^t = \frac{79}{121} \frac{(1.1)^{t+1} + 1}{(1.1)^{t-1} + 1} \qquad \text{in every odd period.}$$

The table below shows some values for the local strategies in an alternating equilibrium.

Table 2.

t	0	1	2	3	4	5	8	9
$\tilde{p}_i^t$	.718	1	.725	1	.731	1	.744	1
p_j^t	1	.721	1	.728	1	.734	1	.746

In this example the alternating equilibrium probability of a bank waiting in period t is high, too. The lower probability of waiting increases with the time t and converges to .79 as t becomes large.

An example of a mutating equilibrium is presented in Figure 2.

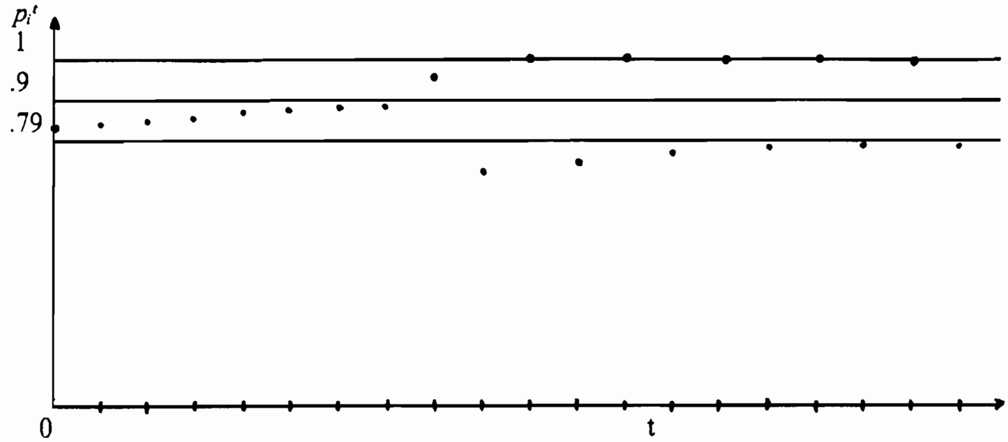

Figure 2.

In a mutating equilibrium a bank waits in every period with probability close to .9, and from some period on starts switching its local strategy either waiting for sure or waiting with probability close to .79. Thus the probability of waiting in each period is high in every equilibrium. We will characterize the behavior of equilibria more precisely in Section 4.

4. The Behavior of Equilibria

The following comparative statics result is true.

Theorem 2. Let $\hat{b}^0 = (\hat{b}_1{}^0, \hat{b}_2{}^0) = ((\hat{p}_1{}^0, \hat{p}_1{}^1, ...), (\hat{p}_2{}^0, \hat{p}_2{}^1, ...))$ be the central equilibrium. Then for $i,j = 1,2$ $(i \neq j)$ and all $t = 0, 1, ...,$ it holds

(4.1) $\qquad \hat{p}_i{}^t \leq \hat{p}_j{}^t$ if and only if $D_i \leq D_j$.

Proof of Theorem 2. From equations given by (3.3), we have that

$$\hat{p}_i{}^t - \hat{p}_j{}^t = \frac{[P(\beta^{m+1}D_j) - P(\beta^{m+1}D_i)][P(\beta^m D_0) - P(\beta^{m+1}D_0)]}{[P(D_j) - P(D_0)][P(D_i) - P(D_0)]} \quad \text{for } j \neq i.$$

From the above equality it follows that

$\hat{p}_i{}^t \leq \hat{p}_j{}^t$ if and only if $D_i \leq D_j$. //

This theorem says that in the central equilibrium a bank with a bigger loan exposure waits longer in every period than the bank with a smaller one. The reasoning for Theorem 2 is as follows. Each bank has in every period two (pure) alternative choices: to wait or to sell. In an equilibrium mixed local strategy, these two choices give the same expected payoffs to each bank, since otherwise a pure local strategy would be chosen. Bank *i*, evaluating its expected payoff from waiting, takes into account bank *j*'s probability of selling and vice versa. If $D_i \le D_j$, then bank *i*'s evaluation is more positively affected by D_j than bank *j*'s by D_i. Therefore, to have the same expected values for the two alternative choices, bank *i*'s probability of waiting becomes lower than bank *j*'s.

The result of Theorem 2 gives the same prediction about the behavior of banks in a period as the prediction derived from the one-period model of Chapter II. In Chapter II, we showed in a one-period model that a bank with a higher loan exposure has a higher probability of waiting than the other one. Since the sequence of mixed strategy equilibria for the one-period games coincides with the central equilibrium of the dynamic game, our Theorem 2 is obtained immediately. The behavior of lender banks in the Bolivian buyback constitutes the best illustration of these results. All American banks with larger loan exposures have kept their loans but some banks with small exposures have sold theirs.

In the previous section, we observed that the probability of a bank waiting in each period in every equilibrium is relatively high as long as the interest factor is low. Now, we consider the limit behavior of the probability of a bank waiting when the interest factor is close to 1. If the function $P(\bullet)$ is continuous at D^0, then from equations (3.3)-(3.7) we obtain that in every equilibrium of the endurance competition game the probability of a bank waiting in each period converges to 1 as β converges to 1. Thus we have the following property of each equilibrium in the game $\Gamma(0, D^0)$.

Theorem 3. Let $b^0 = (b_1^0, b_2^0) = ((p_1^0, p_1^1, \ldots), (p_2^0, p_2^1, \ldots))$ be any equilibrium for the game $\Gamma(0, D^0)$ with the interest factor β. If $P(\bullet)$ is continuos at D^0, then for each $t \ge 0$ $p_i^t \to 1$ as $\beta \to 1$.

Theorem 3 means that when the interest rate is small (the interest factor is close to 1), each bank has strong incentives to wait and keep its loan exposure in every period along an equilibrium path for our dynamic game.

Now, we would like to characterize more precisely the changes of the probability of waiting over time. If the price function $P(\bullet)$ is homogeneous, i.e. $P(kD) = k^n P(D)$ where n is a negative number, than from (3.3) the central equilibrium probability of waiting by bank i in period t is constant and equals to

$$\frac{P(\beta D_j) - P(D^0)}{P(\beta D_j) - P(\beta D^0)}.$$

This observation can be generalized in the following way. If the price function P(D) is approximated by a homogeneous function F(D) for the large values of D, then the central equilibrium probability of waiting by bank i in period t becomes almost constant as t increases. To see this, we assume that there is a function F(D): $R_+ \to R_+$ homogeneous of degree n (i.e. $F(kD) = k^n F(D)$, where n is a negative number), such that $P(D)/F(D) \to 1$ as $D \to \infty$. Then from (3.3),

$$\hat{p}_i^t = \frac{P(\beta^{t+1}D_j) - P(\beta^t D^0)}{P(\beta^{t+1}D_j) - P(\beta^{t+1}D^0)} = \frac{F(\beta^{t+1}D_j)/F(\beta^{t+1}D^0) - P(\beta^t D^0)F(\beta^{t+1}D_j)/\beta^n F(\beta^t D^0)P(\beta^{t+1}D_j)}{F(\beta^{t+1}D_j)/F(\beta^{t+1}D^0) - P(\beta^{t+1}D^0)F(\beta^{t+1}D_j)/F(\beta^{t+1}D^0)P(\beta^{t+1}D_j)}.$$

Thus we obtain

$$\hat{p}_i^t \to \frac{F(D_j)/F(D^0) - 1/\beta^n}{F(D_j)/F(D^0) - 1} = \frac{F(\beta D_j) - F(D^0)}{F(\beta D_j) - F(\beta D^0)} \quad \text{as} \quad t \to \infty.$$

We can summarize the above result in the following theorem.

Theorem 4. Let $\hat{b}^0 = (\hat{b}_1^0, \hat{b}_2^0) = ((\hat{p}_1^0, \hat{p}_1^1, \ldots), (\hat{p}_2^0, \hat{p}_2^1, \ldots))$ be the central equilibrium for the game $\Gamma(0, D^0)$ given by Theorem 1. If there exists a homogeneous function $F(D)$ such that $P(D)/F(D) \to 1$ as $D \to \infty$, then

$$\hat{p}_i^t \to \frac{F(\beta D_j) - F(D^0)}{F(\beta D_j) - F(\beta D^0)} \quad \text{as} \quad t \to \infty.$$

This theorem says that a sufficient condition for the central equilibrium probability of a bank waiting in period t to become constant when t increases, is a homogeneity of the price function for the large values of debt. We have not succeeded in finding any interesting necessary conditions for this type of stationarity of banks′ behavior to occur.

The following example illustrates the claims of Theorems 2 and 4.

Example 4.1.

Let $D_1 = \frac{1}{3}D^0$, $D_2 = \frac{2}{3}D_0$, $D^0 = 1$, $\beta = 1.1$. Let us also assume that the price function is

$$P(D) = \frac{90}{D+1} \quad .$$

The central equilibrium local strategies are

$$\hat{p}_1{}^t = \frac{8}{11}\frac{(1.1)^{t+1}+1}{(1.1)^t+1} \text{ and } \hat{p}_2{}^t = \frac{19}{22}\frac{(1.1)^{t+1}+1}{(1.1)^t+1}.$$

As predicted by Theorem 2, $\hat{p}_1{}^t < \hat{p}_2{}^t$ for every t, and as predicted by Theorem 4,

$$\hat{p}_1{}^t \rightarrow .8 \text{ and } \hat{p}_2{}^t \rightarrow \frac{19}{20} \text{ as } t \rightarrow \infty.$$

5. Proof of Theorem 1

For the proof we need a precise definition of expected payoffs from the subgame $\Gamma(t, \beta^t D^0)$. An expected payoff to bank i from the subgame $\Gamma(t, \beta^t D^0)$ under b^0 is defined analogously to (2.3) by

$$H_i^t(b^t) = \sum_{\tau=t}^{\infty}(1-p_i{}^\tau)D_iP(\beta^\tau D^0)\prod_{k=t}^{\tau-1}p_1{}^kp_2{}^k + \sum_{\tau=t}^{\infty}p_i{}^\tau(1-p_j{}^\tau)D_iP(\beta^{\tau+1}D_i)\prod_{k=t}^{\tau-1}p_1{}^kp_2{}^k \tag{5.1}$$

for all $t = 0, 1, \ldots$.

We may use the following form of the payoff function

$$H_i^t(b^t) = (1-p_i{}^t)D_iP(\beta^tD^0) + p_i{}^t(1-p_j{}^t)D_iP(\beta^{t+1}D_i) + p_1{}^tp_2{}^tH_i^{t+1}(b^{t+1}), \tag{5.2}$$

which comes from

$$\begin{aligned} H_i^t(b^t) &= (1-p_i{}^t)D_iP(\beta^tD^0) + p_i{}^t(1-p_j{}^t)D_iP(\beta^{t+1}D_i) \\ &+ p_i{}^tp_j{}^t\left[\sum_{\tau=t+1}^{\infty}(1-p_i{}^\tau)D_iP(\beta^\tau D^0)\prod_{k=t+1}^{\tau-1}p_1{}^kp_2{}^k + \sum_{\tau=t+1}^{\infty}p_i{}^\tau(1-p_j{}^\tau)D_iP(\beta^{\tau+1}D_i)\prod_{k=t+1}^{\tau-1}p_1{}^kp_2{}^k\right]. \end{aligned}$$

5.1. The Necessary Conditions for a Subgame Perfect Equilibrium with the Time Continuation Property

In this section, we prove that any subgame perfect equilibrium with the time continuation property must be either central, or alternating, or mutating given by Theorem 1.

We first prove that if in a subgame perfect equilibrium of $\Gamma(t, \beta^t D^0)$ for some $t \leq -1$ a local strategy combination (0, 1) is played in period $m \geq 0$, then the same local strategy combination is played in period $m-1$.

Lemma 1. Let t and m be integers with $t \leq -1$ and $m \geq 0$, and let $b^t = (b_1{}^t, b_2{}^t) = ((p_1{}^t, p_1{}^{t+1}, \ldots), (p_2{}^t, p_2{}^{t+1}, \ldots))$ be a subgame perfect equilibrium for the game $\Gamma(t, \beta^t D^0)$. Then if $(p_i{}^m, p_j{}^m) = (0, 1)$, then $(p_i{}^{m-1}, p_j{}^{m-1}) = (0, 1)$.

By applying Lemma 1 repeatedly to $m-1$, ..., 1, 0, the local strategy combination (0, 1) is played in every period from $t = -1$ until $t = m$. Thus if a local strategy combination (0, 1) is played in some period $m \geq 0$, then the subgame perfect equilibrium b^t for a retrospective extension $\Gamma(t, \beta^t D^0)$ of $\Gamma(0, D^0)$ gives a realization probability 0 to the game $\Gamma(0, D^0)$. It means that b^0 induced by b^t does not satisfy the time continuation property.

Proof of Lemma 1. Without loss of generality we set $i = 1$ and $j = 2$. Thus the expected payoffs to banks 1 and 2 from the subgame $\Gamma(m, \beta^m D^0)$ under $\hat{b}^t$ are

$$H_1{}^m((p_1{}^m, p_1{}^{m+1}, \ldots), (\hat{p}_2{}^m, \hat{p}_2{}^{m+1}, \ldots)) = D_1 P(\beta^m D^0)$$

(5.3) and

$$H_2{}^m((\hat{p}_1{}^m, \hat{p}_1{}^{m+1}, \ldots), (\hat{p}_2{}^m, \hat{p}_2{}^{m+1}, \ldots)) = D_2 P(\beta^{m+1} D_2),$$

respectively.

Consider the subgame $\Gamma(m-1, \beta^{m-1} D^0)$. In this game, we assume that the banks play $b^m = (b_1{}^m, b_2{}^m)$ after period m, and then consider optimal behavior of the banks in period $m-1$. Thus we have one period game of period $m-1$, whose payoffs are given in the following table.

Table 3.

1 \ 2	s	w
s	$D_1P(\beta^{m-1}D^0)$ $D_2P(\beta^{m-1}D^0)$	$D_1P(\beta^{m-1}D^0)$ $D_2P(\beta^{m}D_2)$
w	$D_1P(\beta^{m}D_1)$ $D_2P(\beta^{m-1}D^0)$	$D_1P(\beta^{m}D^0)$ $D_2P(\beta^{m+1}D_2)$

If the strategy combination $b^{m-1} = ((p_1^{m-1}, b_1^m), (p_2^{m-1}, b_2^m))$ is a Nash equilibrium for the subgame $\Gamma(m-1, \beta^{m-1}D^0)$, then the strategies p_1^{m-1} and p_2^{m-1} constitute a Nash equilibrium of the game discribed by Table 3. We can calculate that the game of Table 3 has the unique Nash equilibrium $(p_1^{m-1}, p_2^{m-1}) = (0, 1)$. //

We look for necessary conditions for a behavior strategy combination to be a subgame perfect equilibrium with the time continuation property. Consider a subgame perfect equilibrium b^0 satisfying the time continuation property. By definition, b^0 induces a Nash equilibrium on every subgame of $\Gamma(0,D^0)$. Take subgames $\Gamma(t, \beta^t D^0)$ and $\Gamma(t+1, \beta^{t+1}D^0)$ for any $t = 0, 1, \ldots$. We want to find a relation between the equilibrium local strategy combination p^t and b^{t+1}. When b^{t+1} is fixed, the payoffs from the subgame $\Gamma(t, \beta^t D^0)$ depend only on the local strategies in period t, and are shown in the following table.

Table 4.

1 \ 2	s	w	
s	$D_1P(\beta^t D^0)$ $D_2P(\beta^t D^0)$	$D_1P(\beta^t D^0)$ $D_2P(\beta^{t+1}D_2)$	(i)
w	$D_1P(\beta^{t+1}D_1)$ $D_2P(\beta^t D^0)$ $\leftarrow (ii)$	$H_1^{t+1}(b^{t+1})$ $\rightarrow$ $H_2^{t+1}(b^{t+1})$	

If the strategy combination $b^t = ((p_1{}^t, \hat{p}_1{}^{t+1}), (\hat{p}_2{}^t, \hat{p}_2{}^{t+1}))$ is a Nash equilibrium of the subgame $\Gamma(t, \beta^t D^0)$, then the strategies $\hat{p}_1{}^t$ and $\hat{p}_2{}^t$ constitute a Nash equilibrium of the matrix game M(t) discribed by Table 4.

We look for all possible equilibria in the matrix game M(t). We have to consider the following seven cases. The classification is made by comparing (i) payoffs for bank 1 in the second column and (ii) payoffs for bank 2 in the lowest row.

1^o $D_iP(\beta^t D^0) > H_i{}^{t+1}(b^{t+1})$ and $D_jP(\beta^t D^0) < H_j{}^{t+1}(b^{t+1})$. There exists only one equilibrium $p_i{}^t = 0$ and $p_j{}^t = 1$ in the game M(t). The equilibrium payoffs from the game $\Gamma(t, \beta^t D^0)$ are $H_i{}^t(b^t) = D_iP(\beta^t D^0)$ and $H_j{}^t(b^t) = D_jP(\beta^{t+1} D_j)$.

2^o $D_1P(\beta^t D^0) > H_1{}^{t+1}(b^{t+1})$ and $D_2P(\beta^t D^0) > H_2{}^{t+1}(b^{t+1})$. There exist three equilibria in the game M(t). Two of them coincide with the equilibria of 1^o, and the third one is a mixed strategy equilibrium given by

$$(5.4) \qquad p_i{}^t = \frac{D_jP(\beta^{t+1} D_j) - D_jP(\beta^t D^0)}{D_jP(\beta^{t+1} D_j) - H_j{}^{t+1}(b^{t+1})} \quad \text{for } i,j = 1, 2 \ (i \neq j).$$

The equilibrium payoff to bank i from the game $\Gamma(t, \beta^t D^0)$ when the mixed strategies are played is $H_i{}^t(\hat{b}^t) = D_iP(\beta^t D^0)$.

3^o $D_1P(\beta^t D^0) > H_1{}^{t+1}(b^{t+1})$ and $D_2P(\beta^t D^0) = H_2{}^{t+1}(b^{t+1})$. There exist two types of equilibria in the game M(t):

(i) $\qquad p_1{}^t = 0$ and $p_2{}^t = 1$;

(ii) $\qquad p_1{}^t = 1$ and $p_2{}^t \in [0, \dfrac{D_1P(\beta^{t+1} D_1) - D_1P(\beta^t D^0)}{D_1P(\beta^{t+1} D_1) - H_1{}^{t+1}(b^{t+1})}]$.

The equilibrium payoffs from the game $\Gamma(t, \beta^t D^0)$ are, respectively,

(i) $\qquad H_1{}^t(b^t) = D_1P(\beta^t D^0)$ and $H_2{}^t(b^t) = D_2P(\beta^{t+1} D_2)$;

(ii) $\qquad H_1{}^t(b^t) = (1 - \hat{p}_2{}^t)D_1P(\beta^{t+1} D_1) + \hat{p}_2^t H_1{}^{t+1}(b^{t+1})$ and $H_2{}^t(b^t) = D_2P(\beta^t D^0)$.

4^o If $D_1P(\beta^tD^0) = H_1^{t+1}(b^{t+1})$ and $D_2P(\beta^tD^0) > H_2^{t+1}(b^{t+1})$. There exist two types of equilibria in the game M(t):

(i) $p_1^t = 1$ and $p_2^t = 0$,

(ii) $p_1^t \in [0, \dfrac{D_2P(\beta^{t+1}D_1) - D_2P(\beta^tD^0)}{D_2P(\beta^{t+1}D_1) - H_2^{t+1}(b^{t+1})}]$ and $\hat{p}_2^t = 1$.

The equilibrium payoffs from the game $\Gamma(t, \beta^tD^0)$ are, respectively,

(i) $H_1^t(b^t) = D_1P(\beta^{t+1}D_1)$ and $H_2^t(b^t) = D_2P(\beta^tD^0)$;

(ii) $H_1^t(b^t) = D_1P(\beta^tD^0)$ and $H_2^t(b^t) = (1 - p_1^t)D_2P(\beta^{t+1}D_2) + p_1^tH_2^{t+1}(b^{t+1})$.

5^o $D_iP(\beta^tD^0) < H_i^{t+1}(b^{t+1})$ and $D_jP(\beta^tD^0) = H_j^{t+1}(b^{t+1})$. There exists only one type of equilibria in the game M(t), namely

$p_i^t = 1$ and $p_j^t \in [0, 1]$.

The equilibrium payoffs from the game $\Gamma(t, \beta^tD^0)$ are $H_i^t(b^t) = (1 - p_j^t)D_iP(\beta^{t+1}D_i) + p_j^tH_i^{t+1}(b^{t+1})$ and $H_j^t(b^t) = D_jP(\beta^tD^0)$.

6^o If $D_1P(\beta^tD^0) = H_1^{t+1}(b^{t+1})$ and $D_2P(\beta^tD^0) = H_2^{t+1}(b^{t+1})$. There exist two types of equilibria:

(i) $p_1^t \in [0, 1]$ and $p_2^t = 1$,

(ii) $p_1^t = 1$ and $p_2^t \in [0, 1]$.

The equilibrium payoffs from the game $\Gamma(t, \beta^tD^0)$ are, respectively,

(i) $H_1^t(b^t) = D_1P(\beta^tD^0)$ and $H_2^t(b^t) = (1 - p_1^t)D_2P(\beta^{t+1}D_2) + p_1^tD_2P(\beta^tD^0)$;

(ii) $H_1^t(b^t) = (1 - p_2^t)D_1P(\beta^{t+1}D_1) + p_2^tD_1P(\beta^tD^0)$ and $H_2^t(b^t) = D_2P(\beta^tD^0)$.

7^o $D_1P(\beta^tD^0) < H_1^{t+1}(b^{t+1})$ and $D_2P(\beta^tD^0) < H_2^{t+1}(b^{t+1})$. There exists only one equilibrium $p_1^t = 1$ and $p_2^t = 1$ in the game M(t).

The equilibrium payoffs from the game $\Gamma(t, \beta^tD^0)$ are $H_1^t(b^t) = H_1^{t+1}(b^{t+1})$ and $H_2^t(b^t) = H_2^{t+1}(b^{t+1})$.

From the conclusions following Lemma 1, we know that the pairs (0, 1) and (1, 0) cannot be local strategies of any subgame perfect equilibrium with the time continuation property for the game $\Gamma(0, D^0)$. Thus, we can immediately exclude 1^o as impossible. By the same argument, we can also exclude equilibrium (i) in 3^o and 4^o.

We observe also that 5^o, and 6^o cannot take place for any t. This is because in any of the above cases the equilibrium payoffs $H_1^t(b^t)$, $H_2^t(b^t)$ do not satisfy the conditions 5^o, and 6^o.

We show that 7^o is not possible, either. Observe that if the payoffs $H_1^{t+1}(b^{t+1})$, $H_2^{t+1}(b^{t+1})$ satisfy one of the conditions $1^o - 6^o$, then the equilibrium payoffs $H_1^t(b^t)$, $H_2^t(b^t)$ do not satisfy the conditions 7^o. Thus, if the equilibrium payoffs $H_1^t(b^t)$, $H_2^t(b^t)$ would satisfy assumptions of 7^o, then the payoffs $H_1^{t+1}(b^{t+1})$, $H_2^{t+1}(b^{t+1})$ could only satisfy the assumptions of 7^o. By induction, the payoffs $H_1^\tau(b^\tau)$, $H_2^\tau(b^\tau)$ for all $\tau \geq t$ could only satisfy the assumptions of 7^o. But then, the players would play an equilibrium local strategy combination (1, 1) from the time t on, and it would hold that $H_1^t(b^t) = H_1^\tau(b^\tau)$ and $H_2^t(b^t) = H_2^\tau(b^\tau)$ for any τ ($\tau \geq t$). However, the strategy combination $b^t = ((1, 1, ...), (1, 1, ...))$ is not a Nash equilibrium for the game $\Gamma(t, \beta^t D^0)$, because for τ sufficiently big $D_1P(\beta^{\tau+1}D_1) < D_1P(\beta^t D^0)$, and because $H_1^t(b^t) = H_1^\tau(b^\tau) < D_1P(\beta^{\tau+1}D_1)$, it is better for the player 1 to sell in period t instead of waiting.

Summarizing, the local strategy combinations which could occur in any subgame perfect equilibrium must satisfy: 2^o, 3^o (*ii*), or 4^o (*ii*).

Observe the following regularities. If the payoffs $H_1^{t+1}(b^{t+1})$, $H_2^{t+1}(b^{t+1})$ satisfy conditions 4^o, then the payoffs $H_1^t(b^t)$, $H_2^t(b^t)$ satisfy the conditions

$$2^o \quad \text{when} \quad p_1^t \in \left(\frac{D_2P(\beta^{t+1}D_2) - D_2P(\beta^{t-1}D^0)}{D_2P(\beta^{t+1}D_2) - H_2^{t+1}(b^{t+1})}, \ \frac{D_2P(\beta^{t+1}D_2) - D_2P(\beta^t D^0)}{D_2P(\beta^{t+1}D_2) - H_2^{t+1}(b^{t+1})} \right]; \text{ and}$$

$$3^o \quad \text{when} \quad p_1^t = \frac{D_2P(\beta^{t+1}D_2) - D_2P(\beta^{t-1}D^0)}{D_2P(\beta^{t+1}D_2) - H_2^{t+1}(b^{t+1})}.$$

If the payoffs $H_1^{t+1}(b^{t+1})$, $H_2^{t+1}(b^{t+1})$ satisfy conditions 3^o, then the equilibrium payoffs $H_1^t(b^t)$, $H_2^t(b^t)$ satisfy the conditions

2^o when $p_2^t \in \left(\frac{D_1P(\beta^{t+1}D_1) - D_1P(\beta^{t-1}D^0)}{D_1P(\beta^{t+1}D_1) - H_1^{t+1}(b^{t+1})}, \frac{D_1P(\beta^{t+1}D_1) - D_1P(\beta^t D^0)}{D_1P(\beta^{t+1}D_1) - H_1^{t+1}(b^{t+1})} \right]$; and

3^o when $p_2^t = \frac{D_1P(\beta^{t+1}D_1) - D_1P(\beta^{t-1}D^0)}{D_1P(\beta^{t+1}D_1) - H_1^{t+1}(b^{t+1})}$.

If the payoffs $H_1^{t+1}(b^{t+1})$, $H_2^{t+1}(b^{t+1})$ satisfy conditions 2^o, then the equilibrium payoffs $H_1^t(b^t)$, $H_2^t(b^t)$ satisfy the conditions 2^o as well.

Therefore we can identify exactly three types of candidates for a subgame perfect equilibria of the game $\Gamma(0, D^0)$ satisfying the time continuation property. The first type of candidates are strategy combinations which in all games $\Gamma(t, \beta^t D^0)$ give the payoffs satisfying the conditions 2^o. There is only one such strategy combination and it coincides with the central equilibrium given by (3.3). The second type of candidates are strategy combinations which in the game $\Gamma(t, \beta^t D^0)$ give the payoffs satisfying the assumptions of 3^o when t is even (odd), and the assumptions of 4^o when t is odd (even). There are two such strategy combinations and they coincide with alternating equilibria given by (3.4) and (3.5). The third type of candidates are strategy combinations which in the game $\Gamma(t, \beta^t D^0)$ give the payoffs satisfying the assumptions of 2^o for all t smaller than or equal to $\tau + 2$ $(\tau \geq -2)$, and the assumptions of 3^o for $t > \tau + 2$ and t even (odd), and the assumptions of 4^o for $t > \tau + 2$ and t odd (even). This set of strategy combinations coincides with the set of mutating equilibria given by Theorem 1.

Thus we showed that any subgame perfect equilibrium of the game $\Gamma(0, D^0)$ satisfying the time continuation property must be central, alternating or mutating.

5.2. The Sufficient Conditions for a Subgame Perfect Equilibrium with the Time Continuation Property

It remains to show that the strategy combinations called central, alternating and mutating are indeed the subgame perfect equilibria with the time continuation property. First, we prove the following lemmas.

Lemma 2. For any t and any $b_i^t \in B_i^t$ it holds

$$H_i^t(b_i^t, \hat{b}_j^t) - D_iP(\beta^t D^0) = [H_i^{\tau+1}(b_i^{\tau+1}, \hat{b}_j^{\tau+1}) - D_iP(\beta^{\tau+1}D^0)]\prod_{k=t}^{\tau} p_i^k \hat{p}_j^k \quad \text{for all } \tau \geq t.$$

Proof. From (5.2) and (3.3), we have

$$H_i^t(b_i^t, \hat{b}_j^t) - D_i P(\beta^t D^0) = (1 - p_i^t) D_i P(\beta^t D^0) + p_i^t (1 - \hat{p}_j^t) D_i P(\beta^{t+1} D_i) + p_i^t \hat{p}_j^t H_i^{t+1}(b_i^{t+1}, \hat{b}_j^{t+1}) - D_i P(\beta^t D^0)$$

$$= p_i^t \hat{p}_j^t H_i^{t+1}(b_i^{t+1}, \hat{b}_j^{t+1}) - p_i^t \hat{p}_j^t D_i P(\beta^{t+1} D^0) + p_i^t \hat{p}_j^t D_i P(\beta^{t+1} D^0) - p_i^t D^0 P(\beta^t D^0) + p_i^t D_i P(\beta^{t+1} D_i) - p_i^t \hat{p}_j^t D_i P(\beta^{t+1} D_i)$$

$$= \left[H_i^{t+1}(b_i^{t+1}, \hat{b}_j^{t+1}) - D^0 P(\beta^{t+1} D^0)\right] p_i^t \hat{p}_j^t - p_i^t D^0 P(\beta^t D^0) - p_i^t \hat{p}_j^{t+1}[D_i P(\beta^{t+1} D_i) - D^0 P(\beta^{t+1} D^0)] + p_i^t D_i P(\beta^{t+1} D_i)$$

$$= [H_i^{t+1}(b_i^{t+1}, \hat{b}_j^{t+1}) - D_i P(\beta^{t+1} D^0)] p_i^t \hat{p}_j^t - p_i^t D^0 P(\beta^t D^0) - p_i^t \frac{P(\beta^{t+1} D_i) - P(\beta^t D^0)}{P(\beta^{t+1} D_i) - P(\beta^{t+1} D^0)} [D_i P(\beta^{t+1} D_i) - D^0 P(\beta^{t+1} D^0)] + p_i^t D_i P(\beta^{t+1} D_i)$$

$$= [H_i^{t+1}(b_i^{t+1}, \hat{b}_j^{t+1}) - D_i P(\beta^{t+1} D^0)] p_i^t \hat{p}_j^t,$$

i.e., Lemma 2 is true for $\tau = t$. Repeating similar argument, we obtain the claim of Lemma 2. //

Lemma 3. For any t, any b^t and $\tau > t$, it holds

1) $$[H_i^{\tau+1}(b^{\tau+1}) - D_i P(\beta^{\tau+1} D^0)] \prod_{k=t}^{\tau} p_1^k p_2^k \to 0 \quad \text{as} \quad \tau \to \infty;$$

2) $$[H_i^{\tau+1}(b^{\tau+1}) - D_i P(\beta^{\tau} D^0)] \prod_{k=t}^{\tau} p_1^k p_2^k \to 0 \quad \text{as} \quad \tau \to \infty.$$

Proof. The expected payoff to bank i from the game $\Gamma(\tau + 1, \beta^{\tau+1} D^0)$ cannot be higher than the payoff when bank j sells in period $\tau + 1$ and bank i waits in period $\tau + 1$ and sells in period $\tau + 2$, i.e., $H_i^{\tau+1}(b^{\tau+1}) \leq D_i P(\beta^{\tau+2} D_i)$ for $\tau = 0, 1, \dots$. By (2.1) $P(\beta^{\tau+2} D_i) \to 0$ as $\tau \to \infty$. Thus, since the payoff to bank i is nonnegative, we have $H_i^{\tau+1}(b^{\tau+1}) \to 0$ as $\tau \to \infty$. Because $\prod_{k=t}^{\tau} p_1^k p_2^k \leq 1$ for any $\tau \geq t$, and, by (2.1) $P(\beta^{\tau+1} D^0) \to 0$ and $P(\beta^{\tau} D^0) \to 0$ as $\tau \to \infty$, we have the claim of Lemma 3. //

Here, we prove that the behavior strategy combination $\hat{b}^0$ called central is a subgame perfect equilibrium of the game $\Gamma(0, D^0)$ with the time continuation property. From Lemma 2 and Lemma 3 we have that for every t

$$H_i^t(b_i^t, \hat{b}_j^t) = D_iP(\beta^t D^0) \text{ for any } b_i^t \in B_i^t. \tag{5.5}$$

By (5.5) $H_i^t(\hat{b}^t) = D_iP(\beta^t D^0)$, thus

$$H_i^t(b_i^t, \hat{b}_j^t) \leq H_i^t(\hat{b}_i^t, \hat{b}_j^t) \text{ for every t and any } b_i^t \in B_i^t,$$

i.e., bank i has no incentives to deviate from its strategy $\hat{b}_i^t$ for every t. Thus the strategy combination $\hat{b}^0$ called central induces a Nash equilibrium on every subgame of the game $\Gamma(0, D^0)$, i.e., $\hat{b}^0$ is a subgame perfect equilibrium of the game $\Gamma(0, D^0)$.

The central equilibrium $\hat{b}^0$ satisfies the time continuation property, because for every $t \leq -1$ the strategy combination $\hat{b}^t$ is a subgame perfect equilibrium of the retrospective extension $\Gamma(t, \beta D^0)$ of the game $\Gamma(0, D^0)$ and the realization probability $\prod_{\tau=t}^{-1} \hat{p}_1^\tau \hat{p}_2^\tau$ of the game $\Gamma(0, D^0)$ is positive.

The following lemma will be used to prove that the behavior strategy combination $\tilde{b}^0$ called alternating is a subgame perfect equilibrium of the game $\Gamma(0, D^0)$.

Lemma 4. For any $b_i^t \in B_i^t$ it holds

1) for even t

$$H_i^t(b_i^t, \tilde{b}_j^t) - D_iP(\beta^t D^0) = \left| \begin{array}{l} [H_i^{\tau+1}(b_i^{\tau+1}, \tilde{b}_j^{\tau+1}) - D_iP(\beta^\tau D^0)] \prod_{k=t}^{\tau} p_i^k \tilde{p}_j^k \\ \quad + \sum_{s=(t+2)/2}^{\tau/2} [D_iP(\beta^{2s-1} D^0) - D_iP(\beta^{2s-2} D^0)](1 - p_i^{2s-1}) \prod_{k=t}^{2s-2} p_i^k \tilde{p}_j^k \\ \quad \text{for } \tau = t+2, t+4, \ldots, \\ \\ [H_i^{\tau+1}(b_i^{\tau+1}, \tilde{b}_j^{\tau+1}) - D_iP(\beta^{\tau+1} D^0)] \prod_{k=t}^{\tau} p_i^k \tilde{p}_j^k \\ \quad + \sum_{s=t/2}^{(\tau-1)/2} [D_iP(\beta^{2s+1} D^0) - D_iP(\beta^{2s} D^0)](1 - p_i^{2s+1}) \prod_{k=t}^{2s} p_i^k \tilde{p}_j^k \\ \quad \text{for } \tau = t+1, t+3, \ldots; \end{array} \right.$$

2) for odd t

$$
H_i^t(b_i^t, \tilde{b}_j^t) - D_iP(\beta^{t-1}D^0) = \left|
\begin{array}{l}
[H_i^{\tau+1}(b_i^{\tau+1}, \tilde{b}_j^{\tau+1}) - D_iP(\beta^\tau D^0)]\prod_{k=t}^{\tau} p_i^k\tilde{p}_j^k \\
\quad + \sum_{s=(t+1)/2}^{\tau/2} [D_iP(\beta^{2s-1}D^0) - D_iP(\beta^{2s-2}D^0)](1-p_i^{2s-1})\prod_{k=t}^{2s-2} p_i^k\tilde{p}_j^k \\
\quad \text{for } \tau = t+1, t+3, \dots,^{22} \\
\\
[H_i^{\tau+1}(b_i^{\tau+1}, \tilde{b}_j^{\tau+1}) - D_iP(\beta^{\tau+1}D^0)]\prod_{k=t}^{\tau} p_i^k\tilde{p}_j^k \\
\quad + \sum_{s=(t-1)/2}^{(\tau-1)/2} [D_iP(\beta^{2s+1}D^0) - D_iP(\beta^{2s}D^0)](1-p_i^{2s+1})\prod_{k=t}^{2s} p_i^k\tilde{p}_j^k \\
\quad \text{for } \tau = t+2, t+4, \dots .
\end{array}
\right.
$$

Proof. 1) Let t be even. From (5.2) and (3.4), we have

$$
H_i^t(b_i^t, \tilde{b}_j^t) - D_iP(\beta^t D^0) = (1-p_i^t)D_iP(\beta^t D^0) + p_i^t\tilde{p}_j^t H_i^{t+1}(b_i^{t+1}, \tilde{b}_j^{t+1}) - D_iP(\beta^t D^0)
$$

$$
= \left[H_i^{t+1}(b_i^{t+1}, \tilde{b}_j^{t+1}) - D_iP(\beta^t D^0)\right]p_i^t\tilde{p}_j^t.
$$

Conducting further substitution by using (5.2) and (3.5) for t + 1, we have

$$
H_i^t(b_i^t, \tilde{b}_j^t) - D_iP(\beta^t D^0) =
$$

$$
\left[(1-p_i^{t+1})D_iP(\beta^{t+1}D^0) + p_i^{t+1}(1-\tilde{p}_j^{t+1})D_iP(\beta^{t+2}D_i) + p_i^{t+1}\tilde{p}_j^{t+1}H_i^{t+2}(b_i^{t+2}, \tilde{b}_j^{t+2}) - D_iP(\beta^t D^0)\right]p_i^t\tilde{p}_j^t
$$

$$
= H_i^{t+2}(b_i^{t+2}, \tilde{b}_j^{t+2})p_i^t\tilde{p}_j^t p_i^{t+1}\tilde{p}_j^{t+1} - D_iP(\beta^{t+2}D^0)p_i^t\tilde{p}_j^t p_i^{t+1}\tilde{p}_j^{t+1} + D_iP(\beta^{t+2}D^0)p_i^t\tilde{p}_j^t p_i^{t+1}\tilde{p}_j^{t+1}
$$
$$
+ D_iP(\beta^{t+1}D^0)p_i^t\tilde{p}_j^t(1-p_i^{t+1}) + D_iP(\beta^{t+2}D_i)p_i^t\tilde{p}_j^t p_i^{t+1}(1-\tilde{p}_j^{t+1}) - D_iP(\beta^t D^0)p_i^t\tilde{p}_j^t
$$

$$
= \left[H_i^{t+2}(b_i^{t+2}, \tilde{b}_j^{t+2}) - D_iP(\beta^{t+2}D^0)\right]p_i^t\tilde{p}_j^t p_i^{t+1}\tilde{p}_j^{t+1} + D_iP(\beta^{t+1}D^0)p_i^t\tilde{p}_j^t(1-p_i^{t+1})
$$
$$
- [D_iP(\beta^{t+2}D_i) - D_iP(\beta^{t+2}D^0)]p_i^t\tilde{p}_j^t p_i^{t+1}\tilde{p}_j^{t+1} + D_iP(\beta^{t+2}D_i)p_i^t\tilde{p}_j^t p_i^{t+1} - D_iP(\beta^t D^0)p_i^t\tilde{p}_j^t
$$

$$
= \left[H_i^{t+2}(b_i^{t+2}, \tilde{b}_j^{t+2}) - D_iP(\beta^{t+2}D^0)\right]p_i^t\tilde{p}_j^t p_i^{t+1}\tilde{p}_j^{t+1} + D_iP(\beta^{t+1}D^0)p_i^t\tilde{p}_j^t(1-p_i^{t+1})
$$
$$
- [D_iP(\beta^{t+2}D_i) - D_iP(\beta^{t+2}D^0)]p_i^t\tilde{p}_j^t p_i^{t+1}\frac{P(\beta^{t+2}D_i) - P(\beta^t D^0)}{P(\beta^{t+2}D_i) - P(\beta^{t+2}D^0)}
$$
$$
+ D_iP(\beta^{t+2}D_i)p_i^t\tilde{p}_j^t p_i^{t+1} - D_iP(\beta^t D^0)p_i^t\tilde{p}_j^t
$$

---

22 We use the convention $\prod_{k=t}^{t-1} p_i^k\tilde{p}_j^k = 1$.

$$= \left[H_i^{t+2}(b_i^{t+2}, \tilde{b}_j^{t+2}) - D_iP(\beta^{t+2}D^0)\right]p_i^t\tilde{p}_j^t p_i^{t+1}\tilde{p}_j^{t+1} + D_iP(\beta^{t+1}D^0)p_i^t\tilde{p}_j^t(1 - p_i^{t+1})$$
$$+ D_iP(\beta^t D^0)p_i^t\tilde{p}_j^t p_i^{t+1} - D_iP(\beta^t D^0)p_i^t\tilde{p}_j^t$$

$$= \left[H_i^{t+2}(b_i^{t+2}, \tilde{b}_j^{t+2}) - D_iP(\beta^{t+2}D^0)\right]p_i^t\tilde{p}_j^t p_i^{t+1}\tilde{p}_j^{t+1} + [D_iP(\beta^{t+1}D^0) - D_iP(\beta^t D^0)]p_i^t\tilde{p}_j^t(1 - p_i^{t+1}),$$

i.e., Lemma 4.1) is true for $\tau = t + 1$.

Using (5.2) and (3.4) for t + 2, we obtain

$$H_i^t(b_i^t, \tilde{b}_j^t) - D_iP(\beta^t D^0) = \left[H_i^{t+3}(b_i^{t+3}, \tilde{b}_j^{t+3}) - D_iP(\beta^{t+2}D^0)\right]p_i^t\tilde{p}_j^t p_i^{t+1}\tilde{p}_j^{t+1}p_i^{t+2}\tilde{p}_j^{t+2}$$
$$+ [D_iP(\beta^{t+1}D^0) - D_iP(\beta^t D^0)]p_i^t\tilde{p}_j^t(1 - p_i^{t+1}),$$

i.e., Lemma 4.1) is true for $\tau = t + 2$. Repeating similar calculations, we obtain the claim of Lemma 4.1).

2) Analogously, we obtain the claim of Lemma 4.2). //

Here we prove that the behavior strategy combination $\tilde{b}^0$ called alternating is a subgame perfect equilibrium of the game $\Gamma(0, D^0)$ with the time continuation property. For a complete proof it is necessary to show that banks i and j do not have any incentives to deviate from their strategies in $\tilde{b}^t = (\tilde{b}_i^t, \tilde{b}_j^t)$ for t = 0, 1, ... Since the proof for the bank j goes allong the same lines, we show only that bank i does not have any incentives to deviate from its strategy $\tilde{b}_i^t$ for t = 0, 1, ...

Since $\tilde{p}_i^{2s-1} = 1$ for s = 1, 2, ... , it follows from Lemma 3 and Lemma 4 that for even t,

(5.6) $$H_i^t(\tilde{b}^t) = D_iP(\beta^t D^0),$$

and for odd t,

(5.7) $$H_i^t(\tilde{b}^t) = D_iP(\beta^{t-1}D^0).$$

From Lemma 4.1) we have that for even t and any $b_i^t \in B_i^t$

$$H_i^t(b_i^t, \tilde{b}_j^t) - D_i P(\beta^t D^0) \leq [H_i^{\tau+1}(b_i^{\tau+1}, \tilde{b}_j^{\tau+1}) - D_i P(\beta^{\tau+1} D^0)] \prod_{k=t}^{\tau} p_i^k \tilde{p}_j^k \quad \text{for } \tau = t+1, t+3, \dots .$$

Using Lemma 3.2) we obtain

$$H_i^t(b_i^t, \tilde{b}_j^t) - D_i P(\beta^t D^0) \leq 0 \quad \text{for even t and any } b_i^t \in B_i^t.$$

Applying (5.6) we have

$$H_i^t(b_i^t, \tilde{b}_j^t) \leq H_i^t(\tilde{b}^t) \text{ for even t and any } b_i^t \in B_i^t,$$

i.e., bank i has no incentives to deviate from its strategy $\tilde{b}_i^t$ for every even t.

In the same way, we prove that bank i has no incentives to deviate from its strategy $\tilde{b}_i^t$ for every odd t. From Lemma 4.2) we have that for odd t and any $b_i^t \in B_i^t$

$$H_i^t(b_i^t, \tilde{b}_j^t) - D_i P(\beta^{t-1} D^0) \leq [H_i^{\tau+1}(b_i^{\tau+1}, \tilde{b}_j^{\tau+1}) - D_i P(\beta^{\tau} D^0)] \prod_{k=t}^{\tau} p_i^k \tilde{p}_j^k \quad \text{for } \tau = t+1, t+3, \dots .$$

Using Lemma 3.2) we obtain

$$H_i^t(b_i^t, \tilde{b}_j^t) - D_i P(\beta^{t-1} D^0) \leq 0 \quad \text{for odd t and any } b_i^t \in B_i^t.$$

Applying (5.7) we have

$$H_i^t(b_i^t, \tilde{b}_j^t) \leq H_i^t(\tilde{b}^t) \quad \text{for odd t and any } b_i^t \in B_i^t,$$

i.e., bank i has no incentives to deviate from its strategy $\tilde{b}_i^t$ for every odd t. Thus the strategy combination $\tilde{b}^0$ called alternating induces a Nash equilibrium on every subgame of the game $\Gamma(0, D^0)$, i.e., $\tilde{b}^0$ is a subgame perfect equilibrium of the game $\Gamma(0, D^0)$.

The alternating equilibrium $\tilde{b}^0$ satisfies the time continuation property, because for every $t \leq -1$ the strategy combination $\tilde{b}^t$ is a subgame perfect equilibrium of the retrospective extension $\Gamma(t, \beta D^0)$ of the game $\Gamma(0, D^0)$ and the realization probability $\prod_{\tau=t}^{-1} \tilde{p}_1^\tau \tilde{p}_2^\tau$ of the game $\Gamma(0, D^0)$ is positive.

Now, we prove that a strategy combination $\bar{b}^0$ called mutating is a subgame perfect equilibrium of the game $\Gamma(0, D^0)$. By definition of $\bar{b}^0$, $\bar{b}^t = \tilde{b}^t$ for $t = \tau + 3,\ \tau + 4, \dots$. Since the alternating equilibrium $\tilde{b}^0$ is subgame perfect, the mutating equilibrium $\bar{b}^0$ induces a Nash equilibrium on the subgames $\Gamma(\tau+3, \beta^{\tau+3}D^0)$, $\Gamma(\tau+4, \beta^{\tau+4}D^0)$, ... It remains to consider the subgames $\Gamma(0, D^0), \Gamma(1, \beta D^0), \ldots, \Gamma(\tau+2, \beta^{\tau+2}D^0)$.

First, we consider the subgames $\Gamma(\tau+1, \beta^{\tau+1}D^0)$ and $\Gamma(\tau+2, \beta^{\tau+2}D^0)$. Since the local strategies of banks in periods $\tau+1$ and $\tau+2$ are asymmetric, we have to look at the incentives of each bank to deviate separately.

Consider the payoffs to bank i from the game $\Gamma(\tau+2, \beta^{\tau+2}D^0)$. From (5.2) we have

$$H_i^{\tau+2}(b_i^{\tau+2}, \bar{b}_j^{\tau+2}) = (1-p_i^{\tau+2})D_iP(\beta^{\tau+2}D^0) + p_i^{\tau+2}(1-\bar{p}_j^{\tau+2})D_iP(\beta^{\tau+3}D_i) + p_i^{\tau+2}\bar{p}_j^{\tau+2}H_i^{\tau+3}(b_i^{\tau+3}, \bar{b}_j^{\tau+3}).$$

Because $H_i^{\tau+3}(b_i^{\tau+3}, \bar{b}_j^{\tau+3}) = H_i^{\tau+3}(b_i^{\tau+3}, \tilde{b}_j^{\tau+3})$ for all $b_i^{\tau+3} \in B_i^{\tau+3}$, thus using (3.7) we have

$$H_i^{\tau+2}(\bar{b}^{\tau+2}) = (1-\bar{p}_j^{\tau+2})D_iP(\beta^{\tau+3}D_i) + \bar{p}_j^{\tau+2}H_i^{\tau+3}(\tilde{b}^{\tau+3}).$$

Since also $H_i^{\tau+3}(b_i^{\tau+3}, \tilde{b}_j^{\tau+3}) \leq H_i^{\tau-3}(\tilde{b}^{\tau+3})$ for all $b_i^{\tau+3} \in B_i^{\tau+3}$, we have

$$H_i^{\tau+2}(b_i^{\tau+2}, \bar{b}_j^{\tau+2}) \leq (1-p_i^{\tau+2})D_iP(\beta^{\tau+2}D^0) + p_i^{\tau+2}(1-\bar{p}_j^{\tau+2})D_iP(\beta^{\tau+3}D_i) + p_i^{\tau+2}\bar{p}_j^{\tau+2}H_i^{\tau+3}(\tilde{b}^{\tau+3})$$

$$= D_iP(\beta^{\tau+2}D^0) + p_i^{\tau+2}\Big[-D_iP(\beta^{\tau+2}D^0) + D_iP(\beta^{\tau+3}D_i) - \bar{p}_j^{\tau+2}(D_iP(\beta^{\tau+3}D_i) - H_i^{\tau+3}(\tilde{b}^{\tau+3}))\Big].$$

The term in the last square bracket is positive, because $H_i^{\tau+3}(\tilde{b}^{\tau+3}) = D_iP(\beta^{\tau+3}D^0)$ by (5.6) for $\tau+3$, and $\bar{p}_j^{\tau+2}$ is given by (3.7). Thus

$$H_i^{\tau+2}(b_i^{\tau+2}, \bar{b}_j^{\tau+2}) \leq D_iP(\beta^{\tau+2}D^0) + 1 \times \Big[-D_iP(\beta^{\tau+2}D^0) + D_iP(\beta^{\tau+3}D_i) - \bar{p}_j^{\tau+2}(D_iP(\beta^{\tau+3}D_i) - H_i^{\tau+3}(\tilde{b}^{\tau+3}))\Big]$$

$$= (1-\bar{p}_j^{\tau+2})D_iP(\beta^{\tau+3}D_i) + \bar{p}_j^{\tau+2}H_i^{\tau+3}(\tilde{b}^{\tau+3}).$$

Therefore

$$H_i^{\tau+2}(b_i^{\tau+2}, \bar{b}_j^{\tau+2}) \leq H_i^{\tau+2}(\bar{b}^{\tau+2}) \text{ for all } b_i^{\tau+2} \in B_i^{\tau+2},$$

i.e., bank i has no incentives to deviate from its strategy $\bar{b}_i^{\tau+2}$ in the game $\Gamma(\tau+2, \beta^{\tau+2}D^0)$.

Now, consider the payoff to bank j from the game $\Gamma(\tau+2, \beta^{\tau+2}D^0)$. From (5.2) and (3.7) we have

$$H_j^{\tau+2}(\bar{b}_i^{\tau+2}, b_j^{\tau+2}) = (1-p_j^{\tau+2})D_jP(\beta^{\tau+2}D^0) + p_j^{\tau+2}H_j^{\tau+3}(\bar{b}_i^{\tau+3}, b_j^{\tau+3}).$$

Because $H_j^{\tau+3}(\bar{b}_i^{\tau+3}, b_j^{\tau+3}) = H_j^{\tau+3}(\tilde{b}_i^{\tau+3}, b_j^{\tau+3})$ for all $b_j^{\tau+3} \in B_j^{\tau+3}$, thus using (3.7) we have

$$H_j^{\tau+2}(\bar{b}^{\tau+2}) = (1-\bar{p}_j^{\tau-2})D_jP(\beta^{\tau+2}D^0) + \bar{p}_j^{\tau+2}H_j^{\tau+3}(\tilde{b}^{\tau+3}).$$

Substituting $H_j^{\tau+3}(\tilde{b}^{\tau+3}) = D_jP(\beta^{\tau+2}D^0)$ which follows from (5.7) for $\tau+3$, we obtain

$$H_j^{\tau+2}(\bar{b}^{\tau+2}) = D_jP(\beta^{\tau+2}D^0).$$

Since also $H_j^{\tau+3}(\tilde{b}_i^{\tau+3}, b_j^{\tau+3}) \leq H_j^{\tau-3}(\tilde{b}^{\tau+3})$ for all $b_j^{\tau+3} \in B_j^{\tau+3}$, we have

$$H_j^{\tau+2}(\bar{b}_i^{\tau+2}, b_j^{\tau+2}) \leq (1 - p_j^{\tau+2})D_jP(\beta^{\tau+2}D^0) + p_j^{\tau+2}H_j^{\tau+3}(\tilde{b}^{\tau+3}) = D_jP(\beta^{\tau+2}D^0).$$

Therefore

$$H_j^{\tau+2}(\bar{b}_i^{\tau+2}, b_j^{\tau+2}) \leq H_j^{\tau+2}(\bar{b}^{\tau+2}) \text{ for all } b_j^{\tau+2} \in B_j^{\tau+2},$$

i.e., bank j has no incentives to deviate from its strategy $\bar{b}_j^{\tau+2}$ in the game $\Gamma(\tau + 2, \beta^{\tau+2}D^0)$.

Consider the payoff to bank i from the game $\Gamma(\tau + 1, \beta^{\tau+1}D^0)$. From (5.2) we have that for any $b_i^{\tau+1} \in B_i^{\tau+1}$

$$H_i^{\tau+1}(b_i^{\tau+1}, \bar{b}_j^{\tau+1}) = (1 - p_i^{\tau+1})D_iP(\beta^{\tau+1}D^0) + p_i^{\tau+1}(1 - \bar{p}_j^{\tau+1})D_iP(\beta^{\tau+2}D_i) + p_i^{\tau+1}\bar{p}_j^{\tau+1}H_i^{\tau+2}(b_i^{\tau+2}, \bar{b}_j^{\tau+2}).$$

Because $H_i^{\tau+2}(\bar{b}^{\tau+2}) = (1 - \bar{p}_j^{\tau+2})D_iP(\beta^{\tau+3}D_i) + \bar{p}_j^{\tau+2}D_iP(\beta^{\tau+3}D^0)$, thus

$$H_i^{\tau+1}(\bar{b}^{\tau+1}) = (1 - \bar{p}_i^{\tau+1})D_iP(\beta^{\tau+1}D^0) + \bar{p}_i^{\tau+1}(1 - \bar{p}_j^{\tau+1})D_iP(\beta^{\tau+2}D_i) + \bar{p}_i^{\tau+1}\bar{p}_j^{\tau+1}H_i^{\tau+2}(\bar{b}^{\tau+2})$$

$$= (1 - \bar{p}_i^{\tau+1})D_iP(\beta^{\tau+1}D^0) + \bar{p}_i^{\tau+1}(1 - \bar{p}_j^{\tau+1})D_iP(\beta^{\tau+2}D_i)$$
$$+ \bar{p}_i^{\tau+1}\bar{p}_j^{\tau+1}[D_iP(\beta^{\tau+3}D_i) - \bar{p}_j^{\tau+2}(D_iP(\beta^{\tau+3}D_i) - D_iP(\beta^{\tau+3}D^0))]$$

$$= (1 - \bar{p}_i^{\tau+1})D_iP(\beta^{\tau+1}D^0) + \bar{p}_i^{\tau+1}D_iP(\beta^{\tau+2}D_i)$$
$$- \bar{p}_i^{\tau+1}\bar{p}_j^{\tau+1}[D_iP(\beta^{\tau+2}D_i) - D_iP(\beta^{\tau+3}D_i) + \bar{p}_j^{\tau+2}(D_iP(\beta^{\tau+3}D_i) - D_iP(\beta^{\tau+3}D^0))].$$

Substituting (3.6) we obtain

$$H_i^{\tau+1}(\bar{b}^{\tau+1}) = D_iP(\beta^{\tau+1}D^0).$$

Since also $H_i^{\tau+2}(b_i^{\tau+2}, \bar{b}_j^{\tau+2}) \leq H_i^{\tau+2}(\bar{b}^{\tau+2})$ for all $b_i^{\tau+1} \in B_i^{\tau+1}$, we have

$$H_i^{\tau+2}(b_i^{\tau+2}, \bar{b}_j^{\tau+1}) \leq (1 - p_i^{\tau+1})D_iP(\beta^{\tau+1}D^0) + p_i^{\tau+1}(1 - \bar{p}_j^{\tau+1})D_iP(\beta^{\tau+2}D_i) + p_i^{\tau+1}\bar{p}_j^{\tau+1}H_i^{\tau+2}(\bar{b}^{\tau+2})$$

$$= (1 - p_i^{\tau+1})D_iP(\beta^{\tau+1}D^0) + p_i^{\tau+1}(1 - \bar{p}_j^{\tau+1})D_iP(\beta^{\tau+2}D_i)$$
$$+ p_i^{\tau+1}\bar{p}_j^{\tau+1}[D_iP(\beta^{\tau+3}D_i) - \bar{p}_j^{\tau+2}(D_iP(\beta^{\tau+3}D_i) - D_iP(\beta^{\tau+3}D^0))]$$

$$= (1 - p_i^{\tau+1})D_iP(\beta^{\tau+1}D^0) + p_i^{\tau+1}D_iP(\beta^{\tau+2}D_i)$$
$$- p_i^{\tau+1}\bar{p}_j^{\tau+1}[D_iP(\beta^{\tau+2}D_i) - D_iP(\beta^{\tau+3}D_i) + \bar{p}_j^{\tau+2}(D_iP(\beta^{\tau+3}D_i) - D_iP(\beta^{\tau+3}D^0))].$$

Substituting (3.6) into the last expression we obtain

$$H_i^{\tau+1}(b_i^{\tau+1}, \bar{b}_j^{\tau+1}) \leq D_iP(\beta^{\tau+1}D^0) \text{ for all } b_i^{\tau+1} \in B_i^{\tau+1}.$$

Thus

$$H_i^{\tau+1}(b_i^{\tau+1}, \bar{b}_j^{\tau+1}) \le H_i^{\tau+1}(\bar{b}^{\tau+1}) \quad \text{for all } b_i^{\tau+1} \in B_i^{\tau+1},$$

i.e., bank i has no incentives to deviate from its strategy $\bar{b}_i^{\tau+1}$ in the game $\Gamma(\tau+1, \beta^{\tau+1}D^0)$.

Now, consider the payoff to bank j from the game $\Gamma(\tau+1, \beta^{\tau+1}D^0)$. From (5.2) we have

$$\begin{aligned} H_j^{\tau+1}(\bar{b}_i^{\tau+1}, b_j^{\tau+1}) &= (1-p_j^{\tau+1})D_jP(\beta^{\tau+1}D^0) + p_j^{\tau+1}(1-\bar{p}_i^{\tau+1})D_jP(\beta^{\tau+2}D_j) \\ &\quad + \bar{p}_i^{\tau+1}p_j^{\tau+1}H_j^{\tau+2}(\bar{b}_i^{\tau+2}, b_j^{\tau+2}). \end{aligned}$$

Because $H_j^{\tau+2}(\bar{b}^{\tau+2}) = D_jP(\beta^{\tau+2}D^0)$, thus

$$\begin{aligned} H_j^{\tau+1}(\bar{b}^{\tau+1}) &= (1-\bar{p}_j^{\tau+1})D_jP(\beta^{\tau+1}D^0) + \bar{p}_j^{\tau+1}(1-\bar{p}_i^{\tau+1})D_jP(\beta^{\tau+2}D_j) + \bar{p}_i^{\tau+1}\bar{p}_j^{\tau+1}D_jP(\beta^{\tau+2}D^0) \\ &= D_jP(\beta^{\tau+1}D^0) - \bar{p}_j^{\tau+1}[D_jP(\beta^{\tau+1}D^0) - D_jP(\beta^{\tau+2}D_j) + \bar{p}_i^{\tau+1}(D_jP(\beta^{\tau+2}D_j) - D_jP(\beta^{\tau+2}D^0))]. \end{aligned}$$

Substituting (3.6) we obtain

$$H_j^{\tau+1}(\bar{b}^{\tau+1}) = D_jP(\beta^{\tau+1}D^0).$$

Since also $H_j^{\tau+2}(\bar{b}_i^{\tau+2}, b_j^{\tau+2}) \le H_j^{\tau+2}(\bar{b}^{\tau+2})$ for all $b_j^{\tau+1} \in B_j^{\tau+1}$, we have

$$\begin{aligned} H_j^{\tau+1}(\bar{b}_i^{\tau+1}, b_j^{\tau+1}) &\le (1-p_j^{\tau+1})D_jP(\beta^{\tau+1}D^0) + p_j^{\tau+1}(1-\bar{p}_i^{\tau+1})D_jP(\beta^{\tau+2}D_j) + \bar{p}_i^{\tau+1}p_j^{\tau+1}H_j^{\tau+2}(\bar{b}^{\tau+2}) \\ &\le (1-p_j^{\tau+1})D_jP(\beta^{\tau+1}D^0) + p_j^{\tau+1}(1-\bar{p}_i^{\tau+1})D_jP(\beta^{\tau+2}D_j) + \bar{p}_i^{\tau+1}p_j^{\tau+1}D_jP(\beta^{\tau+2}D^0) \\ &= D_jP(\beta^{\tau+1}D^0) - p_j^{\tau+1}[D_jP(\beta^{\tau+1}D^0) - D_jP(\beta^{\tau+2}D_j) + \bar{p}_i^{\tau+1}(D_jP(\beta^{\tau+2}D_j) - D_jP(\beta^{\tau+2}D^0))]. \end{aligned}$$

Substituting (3.6) into the last expression we obtain

$$H_j^{\tau+1}(\bar{b}_i^{\tau+1}, b_j^{\tau+1}) \le D_jP(\beta^{\tau+1}D^0) \quad \text{for all } b_j^{\tau+1} \in B_j^{\tau+1},$$

Thus

$$H_j^{\tau+1}(\bar{b}_i^{\tau+1}, b_j^{\tau+1}) \le H_j^{\tau+1}(\bar{b}^{\tau+1}) \qquad \text{for all } b_j^{\tau+1} \in B_j^{\tau+1},$$

i.e., bank j has no incentives to deviate from its strategy $\bar{b}_j^{\tau+1}$ in the game $\Gamma(\tau+1, \beta^{\tau+1}D^0)$.

Consider the payoff to bank $i = 1, 2$ from the game $\Gamma(t, \beta^tD^0)$ for $t = 0, 1, ..., \tau$. By definition of $\bar{b}^0 = ((\bar{p}_1^0, \bar{p}_1^1, ...),(\bar{p}_2^0, \bar{p}_2^1, ...))$, $\bar{p}_i^t = \hat{p}_i^t$ for $t = 0, 1, ..., \tau$. It follows from Lemma 2 that

$$H_i^t(b_i^t, \bar{b}_j^t) - D_iP(\beta^tD^0) = [H_i^{\tau+1}(b_i^{\tau+1}, \bar{b}_j^{\tau+1}) - D_iP(\beta^{\tau+1}D^0)]\prod_{k=t}^{\tau} p_i^k\hat{p}_j^k.$$

Because $H_i^{\tau+1}(\bar{b}^{\tau+1}) - D_iP(\beta^{\tau+1}D^0) = 0$, thus

$$H_i^t(\bar{b}^t) = D_iP(\beta^tD^0).$$

Since $H_i^{\tau+1}(b_i^{\tau+1}, \bar{b}_j^{\tau+1}) \le H_i^{\tau+1}(\bar{b}^{\tau+1})$ for all $b_i^{\tau+1} \in B_i^{\tau+1}$, we also have

$$H_i^t(b_i^t, \bar{b}_j^t) - D_iP(\beta^tD^0) \le 0 \quad \text{for } t = 0, 1, ..., \tau \text{ and for all } b_i^t \in B_i^t.$$

Thus we obtain

$$H_i^t(b_i^t, \bar{b}_j^t) \leq H_i^t(\bar{b}^t) \quad \text{for } t = 0, 1, \ldots, \tau \text{ and for all } b_i^t \in B_i^t,$$

i.e., bank i has no incentives to deviate from its strategy $\bar{b}_i^t$ in the game $\Gamma(t, \beta^t D^0)$ for $t = 0, 1, \ldots, \tau$.

Therefore the strategy combination $\bar{b}^0$ called mutating induces a Nash equilibrium on every subgame of the game $\Gamma(0, D^0)$, i.e., $\bar{b}^0$ is a subgame perfect equilibrium of the game $\Gamma(0, D^0)$.

A mutating equilibrium $\bar{b}^0$ satisfies the time continuation property, because for every $t \leq -1$ the strategy combination $\bar{b}^t$ is a subgame perfect equilibrium of the retrospective extension $\Gamma(t, \beta D^0)$ of the game $\Gamma(0, D^0)$ and the realization probability $\prod_{\tau=t}^{-1} \bar{p}_1^\tau \bar{p}_2^\tau$ of the game $\Gamma(0, D^0)$ is positive. //

6. Conclusions

In this chapter, we consider the problem of debt overhang, formulating the problem as an infinite horizon game with two banks as players. We find that there exist three types of subgame perfect equilibria with the time continuation property which are called central, alternating and mutating, respectively (Theorem 1). However, the strategies of banks in every equilibrium do not differ much from the strategies of the unique central equilibrium. Therefore the average attainable payoffs are almost uniquely determined.

Our game constitutes a dynamic version of the one-shot game of lender banks given in Chapter II. In Chapter II, we focused on the behavior of a large number of banks in a short period. Since the underlying story is dynamic, an important extension of the previous analysis is a direct dynamic approach to the banks' behavior. Sacrificing the insight into the behavior of a large number of banks obtained in Chapter II, we investigate the long-run behavior of banks in the presence of the secondary market for debts.

The central equilibrium gives the local equilibrium strategies in each period which coincide with the equilibrium of the one-period game of Chapter II. This link between the two formulations allows us to use the one-period approach without losing the dynamic nature of the problem, i.e., we obtain a decomposition property of our dynamic game into the one-period games. Both approaches are complementary in that the one-period model is static but enables us to discuss the effects of a large number of banks, and the dynamic model helps us to understand the long-run behavior of banks but it is too complex to consider the behavior of many banks.

The Folk theorem does not hold for our repeated game, since a deviation of one bank causes its exit from the game and takes away a possibility of punishment by the other bank. In each equilibrium of the dynamic game, the average attainable payoff is almost uniquely determined on the contrary to the Folk Theorem.

In every equilibrium each bank waits in every period with a relatively high probability, and when the interest factor is close to 1, the probability of waiting is close to 1, as well (Theorem 3). We also show that when the price function is approximated by some homogeneous function for the large values of debt, the probability of a bank waiting in period t becomes almost constant as t increases (Theorem 4). The constant is close to 1 as long as the interest factor is relatively low. It suggests that the situation of debt overhang may remain unchanged over time.

In addition, we show that along the central equilibrium path, in every period a bank with a higher loan exposure has a higher probability of waiting than the other one (Theorem 2).

Chapter IV

Duration of Debt Overhang with Two Lender Banks

1. Introduction

Since 1982 the problem of debt overhang has been present in the financial relations between the less developed countries and the foreign commercial banks. The debt overhang is a serious problem for the debtor countries as well as for the banks. On one hand, it keeps the debtor country in a depressed economic situation and prevents it from growing; and on the other hand, it deteriorates the financial situations of the lender banks. The problem of debt overhang has been accompanied with the secondary market for debts where loan exposures of banks are traded at a discounted price. The existence of the secondary market creates a possibility for the debtor country to buy back its debts at a discounted price, and it gives an opportunity for the lender banks to sell their "bad" loans and recoup at least part of their money. However, it is observed that the debt overhang continues to exist even in the presence of the secondary market.

To understand the problem of debt overhang, in Chapters II and III, we give game theoretical models of debt overhang where lender banks are players and decide whether or not to sell their loan exposures to the debtor country at the discounted price on the secondary market. These analyses show that there is a great tendency for the present situation of the debt overhang to remain almost unchanged. It remains to investigate the duration of the debt overhang, i.e., how long it takes for the problem of debt overhang to disappear. The present chapter seeks to answer this question.

We work on the problem formulated in Chapter III, which is an infinite horizon game with two lender banks as players. Since lender banks are very cautious and simultaneously instanteneous decision makers, the length of a period of the game should be regarded as short or almost zero. To describe this feature of a short period and to consider the effect of the length of a period on the duration of debt overhang, we devide each annual period of the game into n subperiods in which the decision making takes place. In every subperiod of the game each bank decides either to sell its loan exposure to the debtor country at the current secondary market price or to wait and postpone this decision to the next subperiod. By increasing the number of subperiods n, we capture the situation with very short periods of decision making. This treatment of the varying length of a period can be done in the framework of Chapter III.

First, under the assumption of homogeneity of the price function, we show that the equilibrium probability of a bank waiting in every subperiod is close to one when the degree of homogeneity is high. When the degree of homogeneity is low, the equilibrium probability of a bank waiting in every subperiod is bounded away from one, but it still remains quite high.

Next, we focus directly on the duration of debt overhang in equilibrium. We show that if the price function is homogeneous, then the expected duration of debt overhang in the central and alternating equilibria becomes almost constant when the length of a subperiod tends to zero (n tends to infinity). In this case, the constant limit duration of debt overhang is the same in both equilibria. The constant limit duration of debt overhang is long when the degree of homogeneity of the price function is high. When the degree of homogeneity is low, the constant is close to $\ln 2/\ln \beta^2$, where β is the annual interest factor. We interpret these results as a possibility for the debt overhang to persist for a long time.

Finally, we consider the duration of debt overhang in an example with a price function which is almost homgeneous for the large values of debt. In the example, the expected equilibrium duration of debt overhang still remains quite long when the length of a subperiod tends to zero. This may suggest that our conclusions about the long duration of debt overhang could be extended on a class of price functions larger than the homogeneous one.

2. Dynamic Endurance Game and Its Equilibria

We consider a country who has debt obligations of equal size to two foreign banks. The country has been unable to maintain the service payments on the outstanding debts for some periods. We assume the existence of the secondary market for debts, where the country is ready to buy its debts at the secondary market price. The price of debt on the secondary market is assumed to depend on the current total outstanding debts. This price is expressed by the function $P(D)\colon R_{+} \rightarrow R_{+}$ with the property:

(2.1) P(D) is a decreasing function of the total outstanding debt D and P(D) $\rightarrow$ 0 as $D \rightarrow \infty$.

This P(D) is country-specific and also depends upon the choice of the present period.

The present period is called 0 and the length of each original period is one year. We devide each period into n subperiods of equal length in which decisions of banks are made. In every subperiod each bank has two possible choices either to sell its loan exposure or to wait and keep it to the next subperiod. Each bank's pure strategies in every subperiod are s and w, where s denotes selling the loan exposure to the debtor country at the current secondary market price, and w denotes waiting and postponing the decision to the next subperiod.

Each bank discounts the future revenues by the interest rate $(r > 0)$. We denote the annual interest factor $1 + r$ by β. We assume that

(2.2) $$\beta^2 < 2.$$

Since the banks make their decisions in each subperiod, we need the interest factor for a subperiod. The interest factor for a subperiod $(1 + r_n) = \beta_n$ must give the accumulated interest from n subperiods equal to the annual interest rate, i.e., $(1 + r_n)^n = 1 + r$, or

(2.3) $$\beta_n = \beta^{1/n}.$$

If bank i = 1,2 keeps its loan exposure $D^0/2$ until subperiod t ($t \geq 0$), then its exposure increases by the accrued interests to $\beta_n{}^t D^0/2$. The total outstanding debt D^t in subperiod t becomes $\beta_n{}^t D^0/2$ if the other bank has already sold, or $\beta_n{}^t D^0$ otherwise. The secondary market price in subperiod t is given as $P(D^t)$. If bank i sells in subperiod t, then the present value of repayment is given as

$$\frac{1}{\beta_n{}^t}\beta_n{}^t\frac{1}{2}D^0P(D^t) = \frac{1}{2}D^0P(D^t).$$

If bank i and the other bank wait in subperiod t, then bank i does not get any payoff in this subperiod, but it will face the same decision problem in the next subperiod.

We assume that

(2.4) after one bank sells its loan exposure, the other bank sells its exposure immediately in the next subperiod.

It is possible that while one bank sells its exposure in some subperiod, the other bank keeps its exposure for several subperiods after that. In this case, however, keeping the loan exposure for several subperiods is not an optimal behavior. Indeed, if the bank postpones selling its exposure, the secondary market price of debt will decrease because of the accrued interest. Suppose bank j sells its exposure in subperiod t and bank i does not. Then the secondary market price in subperiod t + 1 is $P(\beta_n{}^{t+1}D^0/2)$, and the present value of loan exposure is $\frac{1}{D^0}P(\beta_n{}^{t+1}D^0/2)$. If bank i keeps its loan exposure to subperiod t + 2, then the price falls to $P(\beta_n{}^{t+2}D^0/2)$ and the present value is $D_iP(\beta_n{}^{t+2}D^0/2)$. The optimal behavior of bank i is to sell the exposure in subperiod t + 1. Thus we can assume (2.4) without loss of generality.

There are two cases in which the game terminates. The first case is that both banks wait until subperiod $t-1$ and both sell in subperiod t. The second case is that both banks wait until subperiod $t-1$ and one bank sells its loan exposure in subperiod t, and the other bank waits in subperiod t and sells its exposure in subperiod $t+1$. The payoff to bank i = 1, 2 in the game $\Gamma_n(0, D^0)$ is defined by

$$
(2.5)\qquad
\begin{array}{ll}
\frac{1}{2} D^0 P(\beta_n{}^t D^0) & \text{if both banks wait until subperiod } t-1 \text{ and bank } i \text{ sells in subperiod } t, \\[2ex]
\frac{1}{D^0} P(\beta_n{}^{t+1} D^0/2) & \text{if both banks wait until subperiod } t-1 \text{ and bank } j \ (j \neq i) \text{ sells in subperiod } t \text{ and bank } i \text{ waits in subperiod } t.
\end{array}
$$

Our game $\Gamma_n(0, D^0)$ is described in Figure 1. In the game tree, the payoffs to the banks are given in three branches. In the first branch both banks sell their exposures in subperiod 0. In the second, bank 1 sells in subperiod 1 and bank 2 sells in subperiod 2. The third case is that bank 2 sells its exposure in subperiod 2 and bank 1 sells its exposure in subperiod 3.

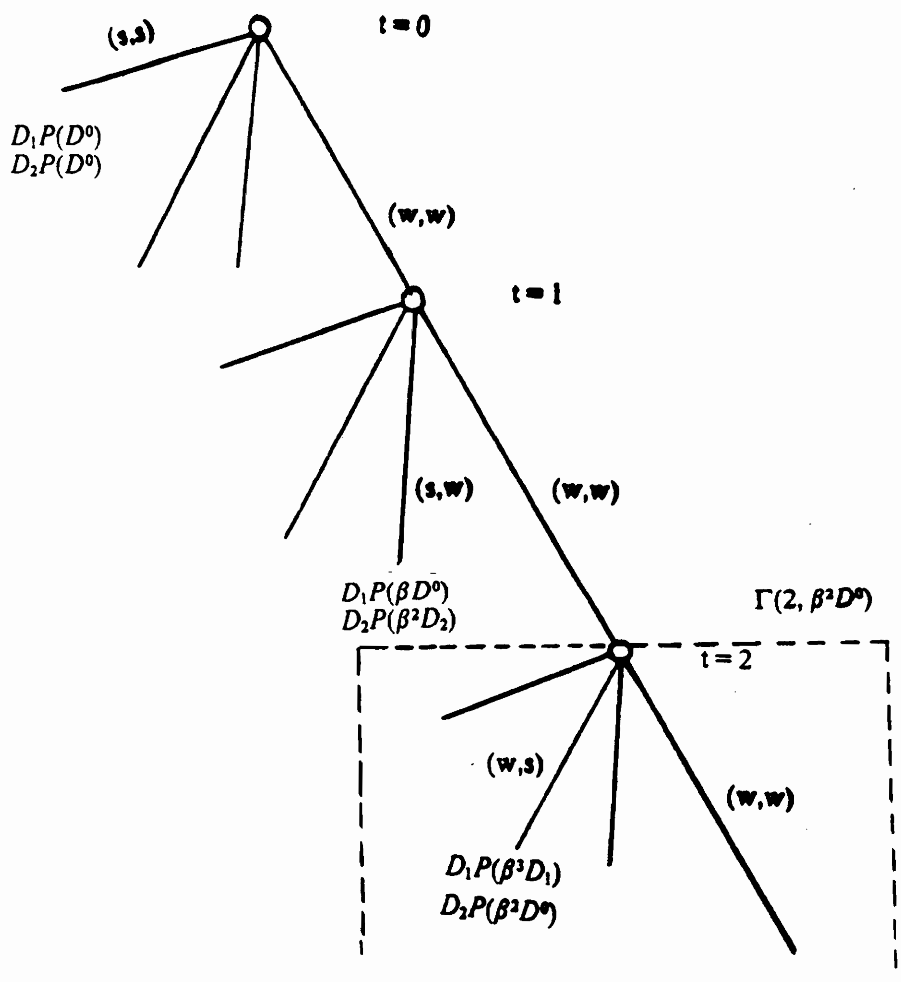

Figure 1.

We allow each bank to use behavior strategies. Since each bank's decision is made in subperiod t only when both banks have waited until subperiod t by assumption (2.4), a behavior strategy of bank $i = 1, 2$ in the game $\Gamma_n(0, D^0)$ is represented as a sequence $b_i^0 = (p_i^0, p_i^1, \ldots)$, where p_i^t is a probability of bank i's waiting in subperiod t $(t = 0, 1, \ldots)$ if both banks keep their loan exposures until subperiod t. Denote the set of all behavior strategies of bank i by B_i^0. A behavior strategy combination for the game $\Gamma_n(0, D^0)$ is a vector $b^0 = (b_1^0, b_2^0) = ((p_1^0, p_1^1, \ldots), (p_2^0, p_2^1, \ldots))$.

The expected payoff to bank $i = 1, 2$ for a behavior strategy combination b^0 is the sum of

(i) the expected payoff from selling the exposure in subperiod t $(t = 0, 1, \ldots)$ under the assumption that the other bank does not sell earlier $-$ $\frac{1}{2} D^0 P(\beta_n^t D^0)(1 - p_i^t) \prod_{k=0}^{t-1} p_i^k p_j^k$; and

(ii) the expected payoff from selling its loan exposure in subperiod $t + 1$ under the assumption that the other bank sells in period t $-$ $\frac{1}{2} D_i P(\beta_n^{t+1} D^0/2) p_i^t (1 - p_j^t) \prod_{k=0}^{t-1} p_i^k p_j^k$.

Thus the expected payoff from the game $\Gamma_n(0, D^0)$ under the strategy combination b^0 is given by

$$H_i^0(b^0) = \sum_{t=0}^{\infty} \frac{1}{2} D^0 P(\beta_n^t D^0)(1 - p_i^t) \prod_{k=0}^{t-1} p_i^k p_j^k + \sum_{t=0}^{\infty} \frac{1}{2} D^0 P(\beta_n^{t+1} D^0/2) p_i^t (1 - p_j^t) \prod_{k=0}^{t-1} p_i^k p_j^k. \tag{2.6}$$

We use the convention $\prod_{k=0}^{-1} p_i^k p_j^k = 1$.

We have described the dynamic endurance game of lender banks $\Gamma_n(0, D^0)$. To investigate the decisions of banks in the game $\Gamma_n(0, D^0)$, we adopt the concept of the subgame perfect equilibrium of the extensive game (Selten (1975)). To define a subgame perfect equilibrium, we have to consider subgames of the game $\Gamma_n(0, D^0)$. Here every subtree constitutes a subgame. Thus the subgame which starts at any subperiod t of the game $\Gamma_n(0, D^0)$ is denoted by $\Gamma_n(t, \beta^t D^0)$. The strategy for the subgame $\Gamma_n(t, \beta_n^t D^0)$ induced by $b_i^0 = (p_i^0, p_i^1, \ldots)$ is a vector obtained by dropping the first t entries of the vector b_i^0, i.e. $b_i^t = (p_i^t, p_i^{t+1}, \ldots)$. Let B_i^t be the set of all induced behavior strategies of bank i for the subgame $\Gamma_n(t, \beta^t D^0)$.

Denote by $H_i^t(b^t)$ the expected payoff to bank i from the subgame $\Gamma_n(t, \beta_n^t D^0)$ under the induced behavior strategy combination $b^t = (b_1^t, b_2^t)$.

A behavior strategy combination $\hat{b}^t = (\hat{b}_1{}^t, \hat{b}_2{}^t)$ is a Nash equilibrium of the subgame $\Gamma_n(t, \beta_n{}^t D^0)$ iff for i = 1, 2,

$$H_i{}^t(\hat{b}^t) \geq H_i{}^t(\hat{b}^t/b_i{}^t) \quad \text{for all } b_i{}^t \in B_i{}^t, \tag{2.7}$$

where $\hat{b}^t/b_i{}^t$ denotes a strategy combination $\hat{b}^t$ with the replacement of $\hat{b}_i{}^t$ by $b_i{}^t$. A subgame perfect equilibrium of the game $\Gamma_n(0,D^0)$ is a behavior strategy combination $\hat{b}^0 = (\hat{b}_1{}^0, \hat{b}_2{}^0)$ which induces a Nash equilibrium on every subgame of the game $\Gamma_n(0, D^0)$.

In addition to the subgame perfection, we require the equilibrium to have the time continuation property. To define the time continuation property, we introduce a retrospective extension of the game $\Gamma_n(0, D^0)$ as a game $\Gamma_n(t, \beta^t D^0)$ for $t = -1, -2, \ldots$, so that the game $\Gamma_n(0, D^0)$ is a subgame of the game $\Gamma_n(t, \beta^t D^0)$. A subgame perfect equlibrium $\hat{b}^0$ is said to have the time continuation property iff for any $t = -1, -2, \ldots$ there is a subgame perfect equilibrium $\hat{b}^t$ in the retrospective extension $\Gamma_n(t, \beta_n{}^t D^0)$ of $\Gamma_n(0, D^0)$ such that $\hat{b}^t$ induces $\hat{b}^0$ and the realization probability of the subgame $\Gamma_n(0, D^0)$ is positive.

The time continuation property states that the present game situation results as a continuation of the past history. The game $\Gamma_n(0, D^0)$ is a result of previous decisions of banks. Therefore the present game is a subgame of the game of any preceding period. If the realization probability of the game $\Gamma_n(0, D^0)$ is zero in an equilibrium for $\Gamma_n(t, \beta_n{}^t D^0)$ then the present situation would be different from $\Gamma_n(0, D^0)$. However, we assume that the game $\Gamma_n(0, D^0)$ is reached. Therefore it is compatible with the consideration of $\Gamma_n(0, D^0)$ to assume that the realization probability is positive.[23]

Technically speaking our game is the same as the game of Chapter III except for the different interest factor. Thus applying Theorem 1 of Chapter III to the game $\Gamma_n(0, D^0)$ with interest factor β_n, we obtain that our game has three types of subgame perfect equilibria satisfying the time continuation property: central, alternating, and mutating. In the unique central equilibrium

[23] The time continuation property is a concept independent from the time consistency in the macroeconomic literature. The time consistency property is, instead, implied by the subgame perfection.

$\hat{b}^0 = (\hat{b}_1^0, \hat{b}_2^0) = ((\hat{p}_1^0, \hat{p}_1^1, ...), (\hat{p}_2^0, \hat{p}_2^1, ...))$, bank i = 1, 2 waits in every period t ($t = 0, 1, ...$) with probability

$$(2.8) \qquad \hat{p}_i^t = \hat{p}^t = \frac{P(\beta_n^{t+1}D^0/2) - P(\beta_n^t D^0)}{P(\beta_n^{t+1}D^0/2) - P(\beta_n^{t+1}D^0)}.$$

There are two alternating equilibria $\tilde{b}^0 = (\tilde{b}_1^0, \tilde{b}_2^0) = ((\tilde{p}_1^0, \tilde{p}_1^1, ...), (\tilde{p}_2^0, \tilde{p}_2^1, ...))$ in which banks i and j wait with probabilities

$$(2.9) \qquad \tilde{p}_i^t = \tilde{p}^t = \frac{P(\beta_n^{t+1}D^0/2) - P(\beta_n^{t-1}D^0)}{P(\beta_n^{t+1}D^0/2) - P(\beta_n^{t+1}D^0)} \text{ and } \tilde{p}_j^t = 1 \text{ if t is even; and}$$

$$(2.10) \qquad \tilde{p}_i^t = 1 \text{ and } \tilde{p}_j^t = \tilde{p}^t = \frac{P(\beta_n^{t+1}D^0/2) - P(\beta_n^{t-1}D^0)}{P(\beta_n^{t+1}D^0/2) - P(\beta_n^{t+1}D^0)} \text{ if t is odd.}$$

In a mutating equilibrium banks behave initially according to the central equilibrium strategies and in some future subperiod their central equilibrium strategies mutate into the alternating equilibrium strategies. Since the mutating equilibrium is a combination of the central and the alternating equilibria, we focus on the behavior of banks in the central and alternating equilibria.

In Chapter III, we show under the assumption of continuity of the price function that in every equilibrium of the dynamic game the probability of a bank waiting in each subperiod is close to 1 when the interest factor is low. Here, we look at the changes in the probability of a bank waiting when the shape of the price function changes. Throughout Section 3, we assume that

(2.11) the price function P(•) is homogeneous of degree k,

i.e., $P(\lambda D) = \lambda^k P(D)$, where $\lambda \in R$ and $k < 0$. Since the degree of homogeneity k determines the shape of the price function, we investigate the changes of the probabilities $\hat{p}^t$ and $\tilde{p}^t$ of bank waiting in subperiod t as the degree of homogeneity changes.

Under the assumption (2.11), the probabilities of a bank waiting in each subperiod t in the central and alternating equilibria are independent of t, and equal

$$(2.12) \qquad \hat{p}^t = \frac{P(\beta_n D^0/2) - P(D^0)}{P(\beta_n D^0/2) - P(\beta_n D^0)} = \frac{(\beta_n D^0/2)^k - (D^0)^k}{(\beta_n D^0/2)^k - (\beta_n D^0)^k} = \frac{1 - (\beta_n/2)^{-k}}{1 - (1/2)^{-k}} \text{ and}$$

$$(2.13) \qquad \tilde{p}^t = \frac{P(\beta_n^2 D^0/2) - P(D^0)}{P(\beta_n^2 D^0/2) - P(\beta_n^2 D^0)} = \frac{(\beta_n^2 D^0/2)^k - (D^0)^k}{(\beta_n^2 D^0/2)^k - (\beta_n^2 D^0)^k} = \frac{1 - (\beta_n^2/2)^{-k}}{1 - (1/2)^{-k}},$$

respectively. We have the following global behavior of the central and alternating equilibrium probabilities of waiting in each subperiod.

Theorem 1. If the price function is homogeneous of degree $k < 0$, then for every $t = 0, 1, \ldots,$

(i) $\hat{p}^t \to 1$ and $\tilde{p}^t \to 1$ as $k \to -\infty$;

(ii) $\hat{p}^t \to 1 - (\ln \beta^{1/n} / \ln 2)$ and $\tilde{p}^t \to 1 - (\ln \beta^{2/n} / \ln 2)$ as $k \to 0$.

Proof. (i) It follows directly from (2.12), (2.13) and the assumption (2.2).

(ii) From (2.12) and (2.13) by the de L'Hospital rule, we have

$$\lim_{k \to 0} \hat{p}^t = \lim_{k \to 0} \frac{1 - (\beta_n/2)^{-k}}{1 - (1/2)^{-k}} = \lim_{k \to 0} \frac{(\beta_n/2)^{-k} \ln(\beta_n/2)}{(1/2)^{-k} \ln(1/2)} = \frac{\ln(\beta_n/2)}{\ln(1/2)} = 1 - (\ln \beta_n / \ln 2); \text{ and}$$

$$\lim_{k \to 0} \tilde{p}^t = \lim_{k \to 0} \frac{1 - (\beta_n^2/2)^{-k}}{1 - (1/2)^{-k}} = \lim_{k \to 0} \frac{(\beta_n^2/2)^{-k} \ln(\beta_n^2/2)}{(1/2)^{-k} \ln(1/2)} = \frac{\ln(\beta_n^2/2)}{\ln(1/2)} = 1 - (\ln \beta_n^2 / \ln 2).$$

Using equality (2.3), we obtain claim (ii). //

Claim (i) of the theorem says that the central and alternating probabilities of a bank waiting in each subperiod are close to 1 as the degree of homogeneity of the price function becomes high. Claim (ii) says that when the degree of homogeneity becomes low the probability of a bank waiting is close to $1 - (\ln \beta_n / \ln 2)$ in the central equilibrium, and is close to $1 - (\ln \beta_n^2 / \ln 2)$ in the alternating equilibrium. Since $1 < \beta < 2^{1/2}$, the central equilibrium probability of a bank waiting in each subperiod is greater than $1/2$. The probability of a bank waiting in each subperiod in the alternating equilibrium is greater than $1/2$ when the interest factor for a subperiod is smaller than 1.19 (interest rate for a subperiod is smaller than 19%).

The relatively high probability of a bank waiting in each subperiod in every equilibrium of the game $\Gamma_n(0, D^0)$ suggests that the situation of debt overhang may continue to exist for some time. In the next section, we estimate the duration of the debt overhang.

3. The Duration of Debt Overhang

We consider the space of events that the problem of debt overhang disappears in subperiod $t = 0, 1, \dots$. On this space, we define a random variable $\hat{X}_n$ by

(3.1) $\hat{X}_n = \frac{1}{n}t$ if the debt overhang disappears in subperiod $t = 0, 1, \dots$ in the central equilibrium.

There are three ways in which the debt overhang disappears in subperiod t in the central equilibrium. First is when both banks wait until subperiod $t - 1$ and both sell their exposures to the debtor country in subperiod t. Second and third way is when both banks wait until subperiod $t - 1$ and one bank sells its loan exposure to the debtor country in subperiod $t - 1$ and the other bank waits in subperiod $t - 1$ and sells its loan exposure in subperiod t. That is,

$$(3.2) \quad Pr(\hat{X}_n = \frac{1}{n}t) = \begin{cases} (1 - \hat{p}^0)^2 & \text{when } t = 0 \\ \prod_{\tau=0}^{t-1}(\hat{p}^\tau)^2 (1 - \hat{p}^t)^2 + 2\prod_{\tau=0}^{t-2}(\hat{p}^\tau)^2 (1 - \hat{p}^{t-1})\, \hat{p}^{t-1} & \text{when } t \geq 1. \end{cases}$$

We use the convention that $\prod_{\tau=0}^{-1}(\hat{p}^\tau)^2 = 1$.

Since $\hat{p}^\tau$ is the same for $\tau = 0, 1, \dots$, it is not difficult to check that $\sum_{t=0}^{\infty} Pr(\hat{X}_n = \frac{1}{n}t) = 1$.

Similarly, we define a random variable $\tilde{X}_n$ by

(3.3) $\tilde{X}_n = \frac{1}{n}t$ if the debt overhang disappears in subperiod $t = 0, 1, \dots$ in the alternating equilibrium.

There is only one way in which the debt overhang disappears in subperiod t in the alternating equilibrium. It happens when both banks wait until subperiod $t - 1$ and the bank who plays a mixed local strategy in subperiod $t - 1$ sells its loan exposure to the debtor country in subperiod $t - 1$ and the other bank sells its loan exposure in subperiod t. That is, (3.4) $$Pr(\tilde{X}_n = \frac{1}{n} t) = \begin{cases} 0 & \text{when } t = 0 \\ \prod_{\tau=0}^{t-2} \tilde{p}^{\tau} (1 - \tilde{p}^{t-1}) & \text{when } t \geq 1. \end{cases}$$

Since $\tilde{p}^{\tau}$ is the same for $\tau = 0, 1, ...,$ it is not difficult to check that $\sum_{t=0}^{\infty} Pr(\tilde{X}_n = \frac{1}{n} t) = 1.$

Now we can state the main results of our paper.

Theorem 2. If the price function is homogeneous of degree $k < 0$ and the size of a subperiod is $\frac{1}{n}$, then

(i) $$E(\hat{X}_n) \rightarrow \frac{2^{-k} - 1}{2(-k) \ln \beta} \quad \text{as} \quad n \rightarrow \infty;$$

(ii) $$E(\tilde{X}_n) \rightarrow \frac{2^{-k} - 1}{2(-k) \ln \beta} \quad \text{as} \quad n \rightarrow \infty.$$

Theorem 2 says that if the price function is homogeneous of degree k, the expected duration of debt overhang in the central and alternating equilibria becomes almost constant when the size of a subperiod becomes small. The constant limit expected duration of debt overhang is the same for both equilibria and equals to $\frac{2^{-k} - 1}{2(-k) \ln \beta}$. Denote it by $E(X_{\infty})$.

Theorem 3. If the price function is homogeneous of degree $k < 0$, then

(i) $E(X_{\infty}) \rightarrow \infty$ as $k \rightarrow -\infty$;

(ii) $E(X_{\infty}) \rightarrow \frac{\ln 2}{\ln \beta^2}$ as $k \rightarrow 0$.

The above theorem says that if the price function is homogeneous of degree $k < 0$, the expected duration of debt overhang in the case of short subperiods of decision making by banks becomes large when the degree of homogeneity increases and becomes close to $\ln 2/ \ln \beta^2$ when the degree of homogeneity decreases.

Observe that $E(X_{\infty})$ is a decreasing function of k. Therefore, it follows from Theorem 3 and the assumption (2.2) that the expected duration of debt overhang in the case of frequent decision making by banks is greater than one year.

Before proving the above theorems, we first give the following lemma.

Lemma 1. For $x \in (0, 1)$, it holds $\sum_{t=1}^{\infty} t\,(x)^{t-1} = \dfrac{1}{(1-x)^2}$ (Leja (1979, p. 149)).

Proof of Theorem 2. (i) We have

$$(3.5)\qquad E(\hat{X}_n) = \frac{1}{n}\,0\,(1-\hat{p}^0)^2 + \frac{1}{n}\sum_{t=1}^{\infty} t\left[\prod_{\tau=0}^{t-1}(\hat{p}^\tau)^2\,(1-\hat{p}^t)^2 + 2\prod_{\tau=0}^{t-2}(\hat{p}^\tau)^2\,(1-\hat{p}^{t-1})\,\hat{p}^{t-1}\right].$$

Since $\hat{p}^\tau$ is the same for $\tau = 0, 1, \ldots$, we may drop the superscript and denote $\hat{p}^\tau = \hat{p}$. Now, we obtain

$$E(\hat{X}_n) = \frac{1}{n}\sum_{t=1}^{\infty} t\left[((\hat{p})^2)^t\,(1-\hat{p})^2 + 2\,((\hat{p})^2)^{t-1}\,(1-\hat{p})\,\hat{p}\right] = \frac{1}{n}\left[(\hat{p})^2\,(1-\hat{p}^t)^2 + 2\,\hat{p}\,(1-\hat{p})\right]\sum_{t=1}^{\infty} t\,((\hat{p})^2)^{t-1}.$$

It follows from Lemma 1 for $x = (\hat{p})^2 \in (0, 1)$ that $\sum_{t=1}^{\infty} t\,((\hat{p})^2)^{t-1} = \dfrac{1}{(1-(\hat{p})^2)^2}$. Thus

$$E(\hat{X}_n) = \frac{1}{n}\left[(\hat{p})^2\,(1-\hat{p})^2 + 2\,\hat{p}\,(1-\hat{p})\right]\frac{1}{(1-(\hat{p})^2)^2} = \frac{1}{n}\,\frac{\hat{p}^2}{1+(\hat{p})^2} + \frac{2\,\hat{p}}{n\,(1-\hat{p})\,(1+\hat{p})^2}.$$

We take the limit of the last expression when $n \to \infty$. Since $\hat{p} \to 1$ as $n \to \infty$, we obtain

$$(3.6)\qquad \lim_{n\to\infty} E(\hat{X}_n) = 0 + \frac{1}{2\,\lim_{n\to\infty} n(1-\hat{p})}.$$

We have the following lemma.

Lemma 2. $n(1-\hat{p}) \to \dfrac{(-k)\,\ln\beta}{2^{-k}-1}$ as $n \to \infty$.

Proof. From (2.12) and (2.3), we have

$$\lim_{n\to\infty} n(1-\hat{p}) = \lim_{n\to\infty} n\,\frac{(\beta^{1/n})^{-k}-1}{2^{-k}-1} = \frac{1}{2^{-k}-1}\lim_{n\to\infty}\frac{(\beta^{1/n})^{-k}-1}{1/n}.$$

Applying the de L'Hospital rule, we obtain

$$\lim_{n\to\infty} n(1-\hat{p}) = \frac{1}{2^{-k}-1}\lim_{n\to\infty}\frac{\beta^{-k/n}\,(-1/n^2)\,(-k)\,\ln\beta}{(-1/n^2)} = \frac{(-k)\,\ln\beta}{2^{-k}-1}. \quad //$$

From (3.6) and Lemma 2, we have claim (i) of Theorem 2.

(ii) We have

$$(3.7)\qquad E(\tilde{X}_n) = \frac{1}{n}\,0\times 0 + \frac{1}{n}\sum_{t=1}^{\infty} t\prod_{\tau=0}^{t-2}\tilde{p}^\tau\,(1-\tilde{p}^{t-1}).$$

Since $\tilde{p}^{\tau}$ is the same for $\tau = 0, 1, \ldots$, we may drop the superscript and denote $\tilde{p}^{\tau} = \tilde{p}$. Now, we obtain

$$E(\tilde{X}_n) = \frac{1}{n} \sum_{t=1}^{\infty} t\,(\tilde{p})^{t-1}\,(1 - \tilde{p}).$$

It follows from Lemma 1 for $x = \tilde{p} \in (0, 1)$ that $\sum_{t=1}^{\infty} t\,(\tilde{p})^{t-1} = \frac{1}{(1-\tilde{p})^2}$. Thus

$$E(\tilde{X}_n) = \frac{1}{n} \frac{1}{(1-\tilde{p})^2} (1 - \tilde{p}) = \frac{1}{n\,(1-\tilde{p})}. \tag{3.8}$$

We have the following lemma.

Lemma 3. $n(1-\tilde{p}) \to \dfrac{2\,(-k)\,\ln\beta}{2^{-k}-1}$ as $n \to \infty$.

Proof. From (2.13) and (2.3), we have

$$\lim_{n\to\infty} n(1-\tilde{p}) = \lim_{n\to\infty} n \frac{(\beta^{2/n})^{-k}-1}{2^{-k}-1} = \frac{1}{2^{-k}-1} \lim_{n\to\infty} \frac{(\beta^{2/n})^{-k}-1}{1/n}.$$

Applying the de L'Hospital rule, we obtain

$$\lim_{n\to\infty} n(1-\tilde{p}) = \frac{1}{2^{-k}-1} \lim_{n\to\infty} \frac{\beta^{-2k/n}\,(-1/n^2)\,(-2k)\,\ln\beta}{(-1/n^2)} = \frac{2\,(-k)\,\ln\beta}{2^{-k}-1}. \quad //$$

From (3.8) and Lemma 3, we have claim (ii) of Theorem 2. //

Proof of Theorem 3. By the de L'Hospital rule, we have

$$\lim_{k\to-\infty} \frac{2^{-k}-1}{2\,(-k)\ln\beta} = \frac{1}{2\ln\beta} \lim_{k\to-\infty} \frac{(-2^{-k})\ln 2}{(-1)} = \infty.$$

Thus we obtain claim (i) of Theorem 3. Similarily,

$$\lim_{k\to 0} \frac{2^{-k}-1}{2\,(-k)\ln\beta} = \frac{1}{2\ln\beta} \lim_{k\to 0} \frac{(-2^{-k})\ln 2}{(-1)} = \frac{\ln 2}{\ln\beta^2}.$$

Thus we have claim (ii) of Theorem 3. //

4. An Example with a Price Function Homogeneous in the Limit

A natural step in the investigation of the duration of debt overhang is to consider a larger class of price functions. Unfortunately, we did not succeed in obtaining complete results for any other class of functions. Instead, we present the duration of debt overhang in an example with a price function homogeneous in the limit.

Example 4.1. Let $D^0 = 1$ and $\beta = 1.1$. Assume that the price function is given by

$$P(D) = \frac{90}{D+1} \quad .$$

The central equilibrium strategy of each bank is to wait in subperiod $t = 0, 1, \ldots$ with probability

$$\hat{p}^t = \left(\frac{2}{\beta^{1/n}} - 1\right) \frac{(\beta^{1/n})^{t+1} + 1}{(\beta^{1/n})^t + 1}. \tag{4.1}$$

The alternating equilibrium strategy of banks i and j is to wait in each subperiod t with probabilities

$$\begin{aligned} &\tilde{p}_i^t = \tilde{p}^t = \left(\frac{2}{\beta^{2/n}} - 1\right) \frac{(\beta^{1/n})^{t+1} + 1}{(\beta^{1/n})^{t-1} + 1}, \quad \tilde{p}_j^t = 1 \qquad \text{in every even period;} \\ &\tilde{p}_i^t = 1, \quad \tilde{p}_j^t = \tilde{p}^t = \left(\frac{2}{\beta^{2/n}} - 1\right) \frac{(\beta^{1/n})^{t+1} + 1}{(\beta^{1/n})^{t-1} + 1} \qquad \text{in every odd period.} \end{aligned} \tag{4.2}$$

From (3.2) and (4.1), we calculate the limit expected duration of the debt overhang in the central equilibrium when the length of a subperiod tends to zero,

$$E(\hat{X}_\infty) \equiv \lim_{n \to \infty} E(\hat{X}_n) = \frac{17}{36 \ln \beta}.$$

For $\beta = 1.1$, $E(\hat{X}_\infty) = 4.95$, i.e., the expected duration of the debt overhang in the central equilibrium is close to 5 as the length of a subperiod tends to zero.

From (3.4) and (4.2), we calculate the limit expected duration of the debt overhang in the alternating equilibrium when the length of a subperiod tends to zero,

$$E(\tilde{X}_\infty) \equiv \lim_{n \to \infty} E(\tilde{X}_n) = \frac{17}{48 \ln \beta}.$$

For $\beta = 1.1$, $E(\tilde{X}_\infty) = 3.72$, i.e., the expected duration of the debt overhang in the alternating equilibrium is close to 4 as the length of a subperiod tends to zero.

Now, we can make a comparison between the case of the homogeneous price function and the price function which is almost homogeneous for the large values of debt. Observe that for the large values of D, the price function P(D) is close to the function $F(D) = \frac{90}{D}$, which is homogeneous of degree -1. By Theorem 2, for $k = -1$, we obtain

$$E(\hat{X}_\infty) = E(\tilde{X}_\infty) = \frac{1}{2 \ln \beta},$$

which for $\beta = 1.1$ equals to 5.25. Thus in our example with the approximately homogeneous price function, the limit duration of the debt overhang is similar to the limit duration of the debt overhang when the price function is homogeneous. In all presented cases, under often decision making by banks, the limit duration of the debt overhang with two banks is quite long.

5. Conclusion

In this chapter, we investigated the duration of debt overhang using the dynamic framework for the banks behavior given in Chapter III. We modified the game of Chapter III to allow the banks to make many decisions within the period of one year. We assumed that each annual period of the game is devided into n subperiods in which the decision making takes place. The modified game has the same types of equilibria as the game of Chapter III. We showed that if the price function is homogeneous, then the expected duration of the debt overhang in the central and alternating equilibria becomes almost constant when the length of a subperiod tends to zero. In this case, the constant limit expected duration of debt overhang is the same in each equilibrium. The constant limit duration of debt overhang is long when the degree of homogeneity of the price function is high. When the degree of homogeneity is low, the constant is close to $\ln 2/ \ln \beta^2$, where β is the annual interest factor. We interpreted these results as a possibility for the debt overhang to exist for a relatively long time.

We attempted to relax the assumption of the homogeneity of the price function, but without a major success. Instead, we presented an example of the debt overhang with a price function which becomes almost homogeneous for the large values of debt. In the example, the expected equilibrium duration of the debt overhang still remains quite long when the banks make the decisions about selling or waiting frequently. This suggests that our conclusions about the long duration of debt overhang could be extended on the cases of the non-homogeneous price functions. To make a definite conclusion, however, further research is necessary.

Bibliography

Armendariz de Aghion, B., "International Debt. An Explanation of the Commercial Banks' Lending Behavior After 1982", *Journal of International Economics*, 28 (1990), 173-186.

Aumann, R.J., "Survey of Repeated Games", *Essays in Game Theory and Mathematical Economics*, in Honor of Oskar Morgenstern, Bibliografischers Institut (1981), 11-42.

Bouchet, M. H., *The Political Economy of International Debt*, Quorum Books, New York (1987).

Bulow, J., and K. Rogoff, "A Constant Recontracting Model of Sovereign Debt", NBER Working Paper No. 2088 (1986).

Bulow, J., and K. Rogoff, "Sovereign Debt is to Forgive to Forget?", mimeo (1988).

Bulow, J., and K. Rogoff, "The Buyback Boondoggle", *Brookings Papers on Economic Activity*, (1988:2).

Bulow, J., and K. Rogoff, "Sovereign Debt Repurchases: No Cure for Overhang", mimeo (1990).

Cohen, D., "Is the Discount on the Secondary Market a Case for LDC Debt Relief?", International Economics Department, The World Bank, Working Papers (1988).

Crawford, V. P., "International Lending, Long-Term Credit Relationships, and Dynamic Contract Theory", *Princeton Studies in International Finance*, No. 59 (1987).

Diwan, I., "Linking Development, Trade, and Debt Strategies in Highly Indebted Countries", International Economics Department, The World Bank, Working Papers (1988).

Diwan, I., and S. Claessens, "An Analysis of Debt-Reduction Schemes Initiated by Debtor Countries", International Economics Department, The World Bank, Working Papers (1989).

Eaton, J., and M. Gersovitz, "Debt with Potential Repudiation: Theoretical and Empirical Analysis", *Review of Economic Studies*, 48 (1981), 289-309.

Eaton, J., and M. Gersovitz, "Poor-Country Borrowing in Private Financial Markets and the Repudiation Issue", *Princeton Studies in International Finance*, No. 47 (1981).

Eaton, J., M. Gersovitz, and J. E. Stiglitz, "The Pure Theory of Country Risk", *European Economic Review*, 30 (1986).

Felix, D., and J. Caskey, "Baker to Brady to Chance? Tinkering with the Latin American Debt Crisis", Working Paper #140 (1989).

Feller, W., *An Introduction to Probability Theory and Its Applications*, Vol.I, John Wiley & Sons, 1st ed. (1950), 2nd ed. (1957).

Fernandez, R., and J. Glazer, "Why Haven't Debtor Countries Formed a Cartel?", mimeo (1988).

Fernandez, R., and J. Glazer, "The Scope for Collusive Behavior among Debtors in a Model of Sovereign-Debt Renegotiation with Costly Penalties", mimeo (1989).

Fernandez, R., and D. Kaaret, "Bank Size, Reputation, and Debt Renegotiation", NBER Working Paper No. 2704 (1988).

Fernandez, R., and R. Rosenthal, "Strategic Models of Sovereign-Debt Renegotiations", *Review of Economic Studies*, (1989).

Fudenberg, D., and E. Maskin, "The Folk Theorem in Repeated Games with discounting and with Incomplete Information", *Econometrica*, 54 (1986), 533-554.

Gale, D., and M. Hellwig, "Incentive Compatible Debt Contracts: The One Period Problem", *Review of Economic Studies*, 52 (1985), 647-663.

Gale, D., and M. Hellwig, "Repudiation and Renegotiation: The Case of Sovereign Debt", *International Economic Review*, 30 (1989), 3-31.

Hajivassiliou, V., "Do the Secondary Markets Believe in Life after Debt?", Cowles Foundation Discussion Paper No. 911 (1989).

Hellwig, M., "A Model of Borrowing and Lending with Bankruptcy", *Econometrica*, 45 (1977), 308-313.

Huizinga, H., "How Does the Debt Crisis Affected Commercial Banks?" International Economics Department, The World Bank, Working Papers (1989).

Jaffe, D., and T. Russell, "Imperfect Information, Uncertainty, and Credit Rationing", *Quarterly Journal of Economics*, 90 (1976), 651-666.

Kenen, P. B., "Organizing Debt Relief: The Need for a New Institution", *Journal of Economic Perspectives*, Volume 4, Number 1, Winter 1990, 7-18.

Kohlberg, E., and J.-F. Mertens, "On the Strategic Stability of Equilibria", *Econometrica*, 54 (1986), 1003-1037.

Krugman, P., "Financing vs. Forgiving a Debt Overhang", *Journal of Development Economics*, 29 (1988), 253-268.

Krugman, P., and M. Obstfeld, *International Economics. Theory and Policy*, Scott, Foresman and Company (1988).

Krugman, P., "Reducing Developing Country Debt", *Revista de Analisis Economico*, 4 (1989), 3-18.

Leja, F., *Rachunek Rozniczkowy i Calkowy*, PWN, Warszawa (1979).

Rosenthal, R. W., "On the Incentives Associated with Sovereign Debt", mimeo (1989).

Rubinstein, A., "Perfect Equilibrium in a Bargaining Model", *Econometrica*, 50 (1982), 97-109.

Sachs, J., "Theoretical Issues in the International Borrowing", *Princeton Studies in*

International Finance, No. 54 (1984).

Sachs, J., "Efficient Debt Reduction", International Economics Department, The World Bank, Working Papers (1989).

Sachs, J., "New Approaches to the Latin American Debt Crisis", *Princeton Studies in International Finance*, No. 174 (1989).

Sachs, J., "A Strategy for Efficient Debt Reduction", *Journal of Economic Perspectives*, 4 (1990), 19-29.

Sachs, J., and H. Huizinga, "U.S. Commercial Banks and the Developing-Country Debt Crisis", *Brookings Papers on Economic Activity* (1987:2).

Selten, R., "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games", *Internatinal Journal of Game Theory*, 4 (1975).

Stiglitz, J., and A. Weiss, "Credit Rationingin Market with Imperfect Information", *American Economic Review*, 71 (1981), 393-411.

Stiglitz, J., and A. Weiss, "Incentive Effects of Terminations: Applications to the Credit and Labor Markets", *American Economic Review*, 73 (1983), 912-927.

Versluysen, E. L., "A Review of Alternative Debt Strategies", International Economics Department, The World Bank, Working Papers (1989).

Vita

Jacek Prokop was born November 5, 1961, in Lublin, Poland. He graduated Konopnicka high school in Zamość in May 1980. He received his M. A. in Economics from the Department of Foreign Trade at the Central School of Planning and Statistics, Warsaw, Poland in February 1985. Since October 1984, he has been employed as an Assistant Professor at the Department of Foreign Trade in the Central School of Planning and Statistics, Warsaw. He entered Virginia Polytechnic Institute and State University in September 1987, and received his Ph. D. in July 1991.

Jacek Prokop