A Duality Approach To Spline Approximation

by

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Abstract

This dissertation discusses a new approach to spline approximation. A periodic spline approximation $f_{M,m,N}(x) = \sum_{k=1}^{N} \alpha_k \phi_{M,k}(x)$ to a periodic function $f(x)$ is determined by requiring $< \phi_{m,j}, f - f_{M,m,N} > \ = 0$ for $j = 1, \ldots, N$, where the $\phi_{L,k}$'s are the unique periodic spline basis functions of order L. Error estimates, examples and some relationships to wavelets are given for the case $M - m = 2\mu$. The case $M - m = 2\mu + 1$ is briefly discussed but not completely explored.

With All My Love

This Dissertation is Dedicated to my Parents

In Honor of my Mother

SHIRLEY ANN ARNOLD BONAWITZ

And in Loving Memory of my Father

HARVEY WAYNE BONAWITZ

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Chapter 1

Introduction

Often one wishes to approximate a function $f(x)$ by another function $f_a(x)$. One way to approximate $f(x)$ is by letting $f_a(x)$ be, possibly, a different polynomial on different intervals $[\xi_{i-1}, \xi_i]$, $i = 0, 1, \ldots, N$. The values ξ_i are called nodes or knots. Such a function $f_a(x)$ is referred to as a piecewise polynomial. If each polynomial section of $f_a(x)$ has degree no greater than $M - 1$ and $f_a(x)$ has $M - 2$ continuous derivatives, then $f_a(x)$ is called a **spline of order** M. This approximation technique resembles a physical process that has been used by drafters. By attaching weights to a thin rod, called a spline, the drafters cause the rod to pass through a given set of points. The resulting curve traced out by the spline not only passes through the given points, but it also smooths out as much as possible between the points. This drafters' technique has led to today's study of spline approximation. For additional information see Ahlberg, Nilson and Walsh [1] and Johnson and Riess [8].

One way to construct a spline approximate $f_a(x)$ to $f(x)$ is to require $f_a(x)$ to equal $f(x)$ on some given set of values x_i, $i = 0, 1, \ldots, N$. The values x_i are called interpolation points. In this case, $f_a(x)$ is called a **point interpolate** of $f(x)$. Schoenberg [12] has shown that if M is even and the interpolation points to the nodal points ξ_i then $f_a(x)$ is unique. Similarly, Schoenberg has also shown that if M is odd and the interpolation points x_i are the midpoints of each interval $[\xi_{i-1}, \xi_i]$ then again $f_a(x)$ is unique. For details see Schoenberg [12]. Additional information can be found in Ahlberg, Nilson and Walsh [1]. From Prenter [11] and Schultz [14] we have that even order (odd degree) splines have an error estimate of

$$\| f^{(j)}(x) - f_a^{(j)}(x) \| \leq c_j \, h^{m-j} \, \| f^{(m)}(x) \| .$$

Here $\| \cdot \|$ is the L_2 norm, h is the length of the longest interval and $f \in C^m[a, b]$.

For ease of computation, the order of the spline approximation $f_a(x)$ is usually kept low. Also, if the nodal points and interpolation points are identical, even order splines are generally used. Specifically, much work has been done in the analysis and construction of splines of order 4. Such splines are called cubic splines. Further information on cubic splines can be obtained from Johnson and Riess [8].

Frequently one selects the nodal points ξ_i to be equally spaced. This allows one to easily compute a set of splines, called B-splines, that form a basis from which all other splines can be constructed. The function $\phi_{M,k}(x)$, $k = 1, 2, \ldots, N$ represents the basis functions for splines of order M, Powell [10]. We then have a unique representation

$$f_a(x) = \sum_{k=1}^{N} \alpha_k \phi_{M,k}(x).$$

With this notation we can describe another approach to finding a spline approximate $f_a(x)$ to $f(x)$. The **Galerkin** approximate is determined by requiring $< \phi_{M,j}, f - f_a > = 0$, $j = 1, 2, \ldots, N$ where $< f, g >$ is the L_2 inner product

$< \phi_{M,j}, f - f_a > = 0, \quad j = 1, 2, \ldots, N$ where $< f, g >$ is the L_2 inner product $< f, g > = \int f(x)g(x)dx$. For more information see Botha and Pinder [2]. This method is also called the **least squares** approximation because it yields the best approximate with respect to the L_2 or square norm , deBoor [6].

The B-spline basis functions of order $L+1$ can be obtained from the B-spline basis functions of order L. One construction method is

$$\phi_{L+1,k}(x) = N \int_{x_{k-1}}^{x} [\phi_{L,k}(s) - \phi_{L,k+1}(s)] \, ds.$$

For additional ways to construct $\phi_{L+1,k}$ see Schoenberg [12], Prenter [11] and Chui [4]. In this notation, $\phi_{0,k}(x) = \delta(x - x_k) = \delta_k$.

Using this definition of $\phi_{0,k}$. we can define another approach to approximating $f(x)$ by a spline function $f_a(x)$. The **collocation method**, see Botha and Pinder [2], for constucting $f_a(x)$ is determined by requiring

$$< \delta_k . f - f_a > \; = \; 0, \quad k = 1, 2, \ldots, N$$

with $f_a(x) = \sum_{j=1}^{N} \alpha_j \phi_{M,j}(x)$. That is,

$$\begin{aligned} < \delta_k, f - f_a > &= \int [f(x) - f_a(x)]\delta(x - x_k) \, dx && (1.1) \\ &= f(x_k) - f_a(x_k) && (1.2) \\ &= 0. && (1.3) \end{aligned}$$

Hence, the method of collocation is equivalent to requiring $f(x_k) = f_a(x_k)$, which is

These last two examples of collocation and Galerkin methods suggest the method of approximation that will be examined in this dissertation. We are searching for a periodic spline function $f_a(x) = f_{M,m,N}(x) = \sum_{j=1}^{N} \alpha_j \phi_{M,j}(x)$ to a periodic function $f(x)$. To calculate $f_{M,m,N}$ we will require

$$< \phi_{m,k}, f - f_{M,m,N} > \; = 0, \quad k = 1, 2, \ldots, N.$$

Notice, if $m = M$ this method produces the least squares or Galerkin approximation. Also, as we have seen, if $m = 0$ and M is even, this method produces the point interpolate of f using even order (odd degree) splines and interpolation at the nodal points.

The intermediate cases $0 < m < M$ will be developed in this document. We first show that if $0 < m < M$ the approximate $f_{M,m,N}$ is well-defined. We then focus on the cases $M - m = 2\mu$. In the cases $M = 4, m = 0, 2, 4$ and $M = 5, m = 1, 3, 5$ a few sample functions $f(x)$ are approximated and the results are plotted. Also using the restriction $M - m = 2\mu$ a connection is made with wavelets. For more information on wavelets see Daubechies [5] and Meyer [9]. This new method of approximation leads to the construction of biorthogonal wavelets which are computed and the results are plotted. Lastly, it is shown that this method, with $M - m = 2\mu$ has an error estimate of

$$\| f^{(j)}(x) - f^{(j)}_{M,m,N}(x) \| \; \leq \; 2^{\mu+k-j}(k+1)^k h^k \; \| f^{(\mu+k)}(x) \|$$

with $j = 0, 1, \ldots, (\mu - 1)$ and $k = K + 1$ where either $m + \mu = 2K$ or $m + \mu = 2K + 1$.

Although this is rather a rough estimate, if M is even and $m = 0$, this estimate is comparible to the known estimates for odd degree splines, at least in the case $j = 0$. However, the derivative estimates are not as tight as those already known for odd degree splines.

Chapter 2

More Background Information On Splines

In the following material we will let I denote the interval $[0, 1)$ with the point 0 and 1 identified with each other, thus creating a periodic domain. We will also assume that N is a positive integer and construct the uniform partition of I determined by the nodes

$$x_k = \frac{k}{N}, \quad k = 0, 1, 2, \ldots, N-1.$$

Similarly, for any $x \in I$, we let $I^x = [x, x+1)$, with $x+1$ identified with x. In agreement with the periodic structure of I we identify x_k and x_j if $k-j$ is a multiple of N; thus, x_0 is the same as x_N. We will follow the same convention in dealing with other quantities comparably indexed. Hence, if $f(x)$ is a function defined on I and we let $f_k = f(x_k)$. then f_k is identified with f_{k+mN} for any integers k and m.

We denote by $S_{M,N}$ the vector space of periodic spline functions which are piecewise polynomials of maximum degree $M-1$ over each subinterval $I_k = [x_k, x_{k+1})$ with the j^{th} derivative, $j = 0, 1, \ldots, M-2$, being continuous at each of the nodal points x_k, $M, N = 1, 2, \ldots$. Further, we define $S_{0,N}$ to be the vector space of distributions, each element of which is a linear combination of the Dirac measures δ_k with support at the nodal points $\{x_k\}$ for $k = 0, 1, 2, \ldots, N-1$. We will see that this is a very natural extension from positive values of M to the value $M = 0$. Also, we will always assume that for all values of M being considered, $N \geq M + 2$.

We define $\phi_{M,k}$, $k = 0, 1, 2, \ldots, N-1$, $M > 0$, to be the unique spline basis [1], [14] for $S_{M,N}$ consisting of spline functions of order M with support confined to the interval $[x_k, x_{k+M}]$ and such that

$$\int_I \phi_{M,k}(x)\, dx = \int_{x_k}^{x_{k+M}} \phi_{M,k}(x)\, dx = 1.$$

When $M = 0$ $\phi_{0,k}$ is just δ_k, the (unit) Dirac measure with support $\{x_k\}$. Also, for example, $\phi_{1,k}$ is the step function which has the value N on the interval I_k and is equal to 0 outside the interval I_k.

Theorem 1 *For $M = 0, 1, 2, \ldots,$ $k = 0, 1, \ldots, N-1,$ and $x \in [x_{k-1}, x_{k-1} + 1) = I^{x_{k-1}}$ we have*

$$\phi_{M+1,k}(x) = N \int_{x_{k-1}}^{x} [\phi_{M,k}(s) - \phi_{M,k+1}(s)]\, ds.$$

In the case $M = 0$ this integral is interpreted in agreement with the theory of distributions. In addition, the lower limit x_{k-1} could be replaced by any ξ not in the interval $[x_k, x_{k+M+1}]$. Furthermore, it should also be noted that an equivalent statement appears in Chui, Chapter 1 [4].

Proof. Recall that $\phi_{0,k}$ is δ_k, the unit Dirac measure with support $\{x_k\}$. Thus

$\phi_{0,k+1}$ is the unit Dirac measure δ_{k+1} with support $\{x_{k+1}\}$. Hence,

$$\begin{aligned}\int_{x_{k-1}}^{x} [\phi_{0,k}(s) - \phi_{0,k+1}(s)]\, ds &= \int_{x_{k-1}}^{x} [\delta_k - \delta_{k+1}]\, ds \\ &= \begin{cases} 0 & x \in [x_{k-1}, x_k) \\ 1 & x \in [x_k, x_{k+1}) \\ 0 & x \in [x_{k+1}, x_{k+2}] \end{cases}\end{aligned}$$

Integrating again over the interval I, and noting that the interval of support is contained in $[x_{k-1}, x_{k+2}]$, we have

$$\begin{aligned}\int_I \int_{x_{k-1}}^{x} [\phi_{0,k}(s) - \phi_{0,k+1}(s)]\, ds\, dx &= \int_{x_{k-1}}^{x_{k+2}} \int_{x_{k-1}}^{x} [\phi_{0,k}(s) - \phi_{0,k+1}(s)]\, ds\, dx \\ &= \int_{x_{k-1}}^{x_k} 0\, dx + \int_{x_k}^{x_{k+1}} 1\, dx + \int_{x_{k+1}}^{x_{k+2}} 0\, dx \\ &= [c_1 - c_1] + [x_{k+1} - x_k] + [c_2 - c_2] \\ &= x_{k+1} - x_k = \frac{1}{N}.\end{aligned}$$

Thus, if the above integral is multiplied by N we have normalized the integral. That is, the function $f(x)$ where

$$f(x) = N \int_{x_{k-1}}^{x} [\phi_{0,k}(s) - \phi_{0,k+1}(s)]\, ds$$

is a polynomial of order 1 (degree 0 or constant) on the interval $[x_k, x_{k+1}]$ and has integral equal to 1. Thus $f(x)$ meets our description of the spline basis function $\phi_{1,k}(x)$. Hence, the formula

$$\phi_{M+1,k}(x) = N \int_{x_{k-1}}^{x} [\phi_{M,k}(s) - \phi_{M,k+1}(s)]\, ds \tag{2.1}$$

is valid when going from $\phi_{0,k}$ to $\phi_{1,k}$.

Next, assume that formula (2.1) is valid for $M = 0, 1, 2, \ldots, L-1$. To complete our proof by induction we must show that we can obtain $\phi_{L+1,k}(x)$ from formula (2.1). Since this formula is true for $M = 0, 1, 2, \ldots, L-1$ we know that

$$\phi_{L,k}(x) = N \int_{x_{k-1}}^{x} [\phi_{L-1,k}(s) - \phi_{L-1,k+1}(s)]\, ds$$

is a spline basis function of order L. Hence, we have

$$\int_I \phi_{L,k}(x)\, dx = 1.$$

Define the function $f(x)$ as follows:

$$f(x) = N \int_{x_{k-1}}^{x} [\phi_{L,k}(s) - \phi_{L,k+1}(s)]\, ds.$$

Since the support of $\phi_{L,k}$ is $[x_k, x_{k+L}]$ and the support of $\phi_{L,k+1}$ is $[x_{k+1}, x_{k+L+1}]$, the support of $f(x)$ is contained in the interval $[x_{k-1}, x_{k+L+1}]$. Next, consider

$$\begin{aligned}
\int_I f(x)\, dx &= \int_{x_{k-1}}^{x_{k+L+1}} f(x)\, dx \\
&= \int_{x_{k-1}}^{x_{k+L+1}} N \int_{x_{k-1}}^{x} [\phi_{L,k}(s) - \phi_{L,k+1}(s)]\, ds\, dx \\
&= N \int_{x_{k-1}}^{x_{k+L+1}} \int_{x_{k-1}}^{x} [\phi_{L,k}(s) - \phi_{L,k+1}(s)]\, ds\, dx.
\end{aligned}$$

Exchanging the order of integration yields

$$\begin{aligned}
\int_{x_{k-1}}^{x_{k+L+1}} f(x)\, dx &= N \int_{x_{k-1}}^{x_{k+L+1}} \int_{s}^{x_{k+L+1}} [\phi_{L,k}(s) - \phi_{L,k+1}(s)]\, dx\, ds \\
&= N \int_{x_{k-1}}^{x_{k+L+1}} \{[\phi_{L,k}(s) - \phi_{L,k+1}(s)]\, (x_{k+L+1} - s)\}\, ds.
\end{aligned}$$

Since

$$
\begin{aligned}
\left[\phi_{L,k}(s) - \phi_{L,k+1}(s)\right](x_{k+L+1} - s) \quad &= \quad x_{k+L+1}\phi_{L,k}(s) - x_{k+L+1}\phi_{L,k+1}(s) \\
&\qquad - s\phi_{L,k}(s) + s\phi_{L,k+1}(s)
\end{aligned}
$$

we have

$$
\begin{aligned}
\int_{x_{k-1}}^{x_{k+L+1}} f(x)\,dx \quad &= \quad N x_{k+L+1}\int_{x_{k-1}}^{x_{k+L+1}} \phi_{L,k}(s)\,ds - N x_{k+L+1}\int_{x_{k-1}}^{x_{k+L+1}} \phi_{L,k+1}(s)\,ds \\
&\qquad - N\int_{x_{k-1}}^{x_{k+L+1}} s\phi_{L,k}(s)\,ds + N\int_{x_{k-1}}^{x_{k+L+1}} s\phi_{L,k+1}(s)\,ds \\
&= \quad N\cdot x_{k+L+1}\cdot 1 - N\cdot x_{k+L+1}\cdot 1 \\
&\qquad - N\int_{x_{k-1}}^{x_{k+L+1}} s\phi_{L,k}(s)\,ds + N\int_{x_{k-1}}^{x_{k+L+1}} s\phi_{L,k+1}(s)\,ds \\
&= \quad -N\int_{x_{k-1}}^{x_{k+L+1}} s\phi_{L,k}(s)ds + N\int_{x_{k-1}}^{x_{k+L+1}} s\phi_{L,k+1}(s)ds.
\end{aligned}
$$

Continuing, we note that $\phi_{L,k+1}(s)$ is the function $\phi_{L,k}(s)$ shifted to the right by a distance of $\frac{1}{N}$. That is, $\phi_{L,k+1}(s) = \phi_{L,k}(s - \frac{1}{N})$. Hence,

$$
N\int_{x_{k-1}}^{x_{k+L+1}} s\phi_{L,k+1}(s)\,ds = N\int_{x_{k-1}}^{x_{k+L+1}} s\phi_{L,k}(s - \frac{1}{N})\,ds.
$$

Using the change of variable $t = s - \frac{1}{N}$ we have

$$
\begin{aligned}
N\int_{x_{k-1}}^{x_{k+L+1}} s\phi_{L,k-1}(s)\,ds \quad &= \quad N\int_{x_{k-2}}^{x_{k+L}} (t + \frac{1}{N})\phi_{L,k}(t)\,dt \\
&= \quad N\int_{x_{k-2}}^{x_{k+L}} t\phi_{L,k}(t)\,dt + \int_{x_{k-2}}^{x_{k+L}} \phi_{L,k}(t)\,dt \\
&= \quad N\int_{x_{k-2}}^{x_{k+L}} t\phi_{L,k}(t)\,dt + 1.
\end{aligned}
$$

Thus

$$\begin{aligned}
\int_{x_{k-1}}^{x_{k+L+1}} f(x)\,dx &= -N\int_{x_{k-1}}^{x_{k+L+1}} s\phi_{L,k}(s)\,ds + N\int_{x_{k-2}}^{x_{k+L}} t\phi_{L,k}(t)\,dt + 1 \\
&= 1 + N\int_{x_k}^{x_{k+L}} t\phi_{L,k}(t)\,dt - N\int_{x_k}^{x_{k+L}} s\phi_{L,k}(s)\,ds \\
&= 1.
\end{aligned}$$

The next to last equality follows from the support of $\phi_{L,k}(x)$ being the interval $[x_k, x_{k+L}]$.

Therefore, since $f(x)$ is a polynomial of degree L on each subinterval, has $L-2$ continuous derivatives, and $\int_I f(x)\,dx = 1$, $f(x)$ must be $\phi_{L+1,k}(x)$, the spline basis function of order $L+1$. Thus, our proof by induction is concluded. ■

Next, consider being given $M > 0$. Let $m = M - 2\mu$, where μ is a non-negative integer, be such that $0 \le m \le M$. Clearly m is even (odd) if and only if M is even (odd).

Given M, m, μ as above, we note that the center point of the interval of support of $\phi_{M,k}$ is the same as the center point of the interval of support of $\phi_{m,k+\mu}$. Thus, these two spline basis functions, though of different orders, are centered relative to each other.

Definition 1 *Let f be a continuous function defined on I. Given M and m as above, we define $P_{M,m,N}\, f = f_{M,m,N}$, where $f_{M,m,N} = \sum_{k=0}^{N-1} \alpha_k \phi_{M,k} \in S_{M,N}$ is the*

M-th order spline function satisfying

$$< \phi_{m,k} \ , \ f - f_{M,m,N} >_m \ = \ 0, \quad k = 0, 1, 2, \ldots, N-1.$$

Here, for any function g continuous on I,

$$< \phi_{m,k} \ , \ g >_m \ = \ \begin{cases} \int_I \phi_{m,k}(x) \, g(x) \, dx & m > 0 \\ g(x_k) & m = 0. \end{cases}$$

We will henceforth suppress N when that integer has an understood value.

We see that $f_{M,m}$ is an M-th order spline approximation to f determined in a particular way, depending on the value of m. If M is even (corresponding to odd degree polynomial pieces) then 0 is a permissible value of m. In this case $f_{M,0}$ is just the point interpolate of f by M-th order splines.

To see why this last statement is true, consider the case M even and $m = 0$. Then to form the approximation $f_{M,0}$ we require the inner product

$$< \phi_{0,k} \ , \ f - f_{M,0} >_0 \ = \ 0.$$

Breaking up the inner product, we have

$$< \phi_{0,k} \ , \ f >_0 \ = \ < \phi_{0,k} \ , \ f_{M,0} >_0 \, .$$

Using the definition of $< \cdot, \cdot >_0$ and recalling that $\phi_{0,k}$ is the Dirac distribution with support $\{x_k\}$, the above equation reduces to

$$f(x_k) = f_{M,0}(x_k)$$

which is precisely the definition of point intepolation. Corresponding to a term used in the numerical solution of equations, we also might refer to $f_{M,0}$ as being the approximation to f obtained via *collocation* [11].

By contrast, $m = M$, which is always admissible since it corresponds to $\mu = 0$, corresponds to the Galerkin or the least squares approximation method with the operator $P_{M,M}$ being simply the orthogonal projection of f onto S_M relative to the inner product in $L^2[0,1]$ [11].

Now, if M is odd (corresponding to even degree polynomial pieces) then $m = 1$ is an admissible value of m. In this case, forming the approximation $f_{M,1}$ requires the equations

$$< \phi_{1,k} \, , \, f >_1 \; = \; < \phi_{1,k} \, , \, f_{M,1} >_1$$

to hold. Calculating the inner product $< \cdot, \cdot >_1$ we have

$$\int_I \phi_{1,k}(x) \; f(x) \; dx \; = \int_I \phi_{1,k}(x) \; f_{M,1}(x) \; dx.$$

Since $\phi_{1,k}$ is equal to N on the interval $[x_k, x_{k+1}]$ and zero elsewhere, the last equation above reduces to

$$\int_{x_k}^{x_{k+1}} N f(x) \; dx = \int_{x_k}^{x_{k+1}} N f_{M,1}(x) \; dx.$$

Dividing both sides by N yields

$$\int_{x_k}^{x_{k+1}} f(x)\,dx = \int_{x_k}^{x_{k+1}} f_{M,1}(x)\,dx.$$

That is, if M is odd and $m = 1$ then we are in essence requiring that the integral of the approximate $f_{M,1}$ and the integral of the actual function f agree on each interval $[x_k, x_{k+1}]$. We shall refer to this as *integral interpolation.*

Recall that S_M is the vector space of M-th order (degree $M-1$) periodic spline functions. Contained in S_M is the subspace S_M^0 consisting of M-th order periodic splines whose integral over the interval $I = [0,1)$ is zero. We will denote by D the differentiation operator applied to differentiable functions defined on $[0,1)$. We observe that

$$D : S_M \to S_{M-1}^0, \quad M = 1, 2, 3, \ldots$$

is onto. Here, S_0 is a space of Dirac distributions with supports confined to the nodal points $\{x_k\}$, $k = 1, 2, \ldots, N$. We define the restriction of D to S_M^0 to be D_0 and observe that

$$D_0 : S_M^0 \to S_{M-1}^0$$

is also onto and thus is invertible. Hence there is an inverse (integration)

$$D_0^{-1} : S_{M-1}^0 \to S_M^0$$

which is also onto.

Next we notice that

$$N[\phi_{M,k} - \phi_{M,k+1}] \equiv -\Delta\phi_{M,k} \in S_M^0. \tag{2.2}$$

This can be seen from observing that, since $\int \phi_{M,k}(x)\ dx = 1$ as well as $\int \phi_{M,k+1}(x)\ dx = 1$, we have

$$\int [\phi_{M,k}(x) - \phi_{M,k+1}(x)]\ dx = 1 - 1 = 0.$$

With (2.2), our recursion formula now reads

$$\phi_{M+1,k} = 1 + D_0^{-1}[-\Delta\phi_{M,k}] \equiv \mathcal{D}_1^{-1}[-\Delta\phi_{M,k}].$$

If we take ξ to be any point in $[0,1)$ lying outside the interval (x_k, x_{k+M+1}) it is easy to see that

$$\phi_{M+1,k}(x) = \mathcal{D}_1^{-1}[-\Delta\phi_{M,k}](x) = \int_\xi^x [-\Delta\phi_{M,k}(s)]\ ds$$

for $x \in [\xi, \xi+1)$. (Note that if ξ were in (x_k, x_{k+M+1}) the second and third elements of the preceding formula would differ only by a constant.) Given any $f \in S_M^0$ which is a linear combination of the $-\Delta\phi_{M,k}$, that is, given

$$f = \sum_{k=0}^{N-1} c_k[-\Delta\phi_{M,k}],$$

we can then define a particular antiderivative of f :

$$\mathcal{D}_1^{-1} f = \sum_{k=0}^{N-1} c_k \mathcal{D}_1^{-1}[-\Delta\phi_{M,k}].$$

The antiderivative $\mathcal{D}_0^{-1} f$ could be defined similarly and will differ from $\mathcal{D}_1^{-1} f$ by a constant function.

Using the fact that

$$\phi_{M+1,k} = \mathcal{D}_1^{-1}[-\Delta\phi_{M,k}],$$

we can see that

$$\phi_{M+2,k} = \mathcal{D}_1^{-1}\Delta\mathcal{D}_1^{-1}\Delta\phi_{M,k}$$

for $k = 0, 1, \ldots, N-1$. Also, we notice that

$$\Delta\mathcal{D}_1^{-1}\Delta\phi_{M,k} = \mathcal{D}_1^{-1}\Delta\Delta\phi_{M,k} = \mathcal{D}_1^{-1}\Delta^2\phi_{M,k}$$

so that now

$$\phi_{M+2,k} = \mathcal{D}_1^{-2}\Delta^2\phi_{M,k}$$

for $k = 0, 1, 2, \ldots, N-1$. Similarly,

$$\phi_{M+\mu,k} = (-1)^\mu\mathcal{D}_1^{-\mu}\Delta^\mu\phi_{M,k}.$$

A special case of this is obtained for $M = 0$. In this case

$$\phi_{\mu,k} = (-1)^\mu\mathcal{D}_1^{-\mu}\Delta^\mu\phi_{0,k}. \tag{2.3}$$

Writing $\mu = \lambda + \nu$ we have

$$\phi_{\mu,k} = \phi_{\lambda+\nu,k} = (-1)^\lambda\mathcal{D}_1^{-\lambda}\Delta^\lambda\phi_{\nu,k} \tag{2.4}$$

for $k = 0, 1, \ldots, N-1$.

Let us summarize the significance of these last two formulae. Applying D^μ to formula (2.3) we see that

$$D^\mu \phi_{\mu,k} = (-1)^\mu \Delta^\mu \phi_{0,k},$$

showing that the μ-th derivative, in the distribution sense, of $\phi_{\mu,k}$ is $(-N)^\mu$ times a Dirac distribution

$$\sum_{j=0}^{\mu} (-1)^j \binom{\mu}{j} \delta_{k+\mu-j}$$

whose coefficients are the same as those of the binomial expansion of $(z-1)^\mu$. Here the $(-N)^\mu$ factor comes from the definition $-\Delta\phi_{M,k} = N[\phi_{M,k} - \phi_{M,k+1}]$.

Formula (2.4) is a generalization; applying D^λ we obtain

$$D^\lambda \phi_{\mu,k} = (-1)^\lambda \Delta^\lambda \phi_{\mu-\lambda,k},$$

showing that for $0 \leq \lambda \leq \mu$ the λ-th derivative of $\phi_{\mu,k}$ is $(-N)^\lambda$ times a linear combination of the $\phi_{\mu-\lambda,k+\lambda-j}$:

$$\sum_{j=1}^{\lambda} (-1)^j \binom{\lambda}{j} \phi_{\mu-\lambda,k+\lambda-j}$$

whose coefficients are also the same as those of the binomial expansion of $(z-1)^\lambda$.

Let f be a periodic M-th order spline, i.e. $f \in S_M$. Then we have

$$f = \sum_{k=1}^{N-1} c_0^k \phi_{M,k},$$

which enables us to identify f with the N-vector c_0 whose components are c_0^k for $k = 0, 1, \ldots, N-1$. Each derivative $f^{(\lambda)}$ of f can then be written as

$$f^{(\lambda)} = \sum_{k=0}^{N-1} c_\lambda^k \phi_{M-\lambda,k}$$

so that, relative to the functions $\phi_{M-\lambda,k}$, $f^{(\lambda)}$ can be identified with the N-vector c_λ with components c_λ^k, $k = 0, 1, \ldots, N-1$. Our previous work now shows that for $0 \leq \lambda \leq M-2$ we have

$$c_{\lambda+1} = -N\mathcal{E}c_\lambda,$$

where $\mathcal{E}$ is the $N \times N$ 'circulant' matrix

$$\mathcal{E} = \begin{bmatrix} -1 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & -1 & 1 \\ 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & -1 \end{bmatrix}.$$

Theorem 2 *Let f be defined and integrable on the interval $I = [0,1]$. Given a positive integer M and a non-negative integer with m, $0 \leq m \leq M$, such that $m = M - 2\mu$, where μ is a non-negative integer. Then there is exactly one function*

$f_{M,m} \in S_M$ *such that*

$$\int_I \phi_{m,k}(x)[f_{M,m}(x) - f(x)]\, dx = 0, \quad k = 0, 1, \ldots, N-1.$$

Proof. The assumed integrability of f shows that

$$b_k = \int_I \phi_{m,k}(x)\, f(x)\, dx$$

is well defined, $k = 0, 1, \ldots, N-1$. It is clearly enough, then, to show that the corresponding homogeneous 'moment' equations

$$\int_I \phi_{m,k}(x)\, f_{M,m}(x)\, dx = 0, \quad k = 0, 1, \ldots, N-1,$$

have $f_{M,m}(x) = 0$ as their only solution. Since $f_{M,m}$ is assumed to lie in S_M, let us write

$$f_{M,m}(x) = \sum_{j=0}^{N-1} \alpha_j\, \phi_{M,j}(x).$$

Now, since we have the 'partition of unity' identity

$$\frac{1}{N} \sum_{k=0}^{N-1} \phi_{m,k}(x) \equiv 1$$

it follows that, on one hand we have,

$$\int_I f_{M,m}(x)\, dx = \int_I f_{M,m}(x) \cdot 1\, dx \tag{2.5}$$

$$= \int_I f_{M,m}(x) \frac{1}{N} \sum_{k=0}^{N-1} \phi_{m,k}(x)\, dx \tag{2.6}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \int_I \phi_{m,k}(x)\, f_{M,m}(x)\, dx \tag{2.7}$$

$$= \quad 0. \tag{2.8}$$

This last equality is due to the fact that we are assuming the moment equations are homogeneous. We also see that

$$\begin{aligned}
\int_I f_{M,m}(x)\, dx &= \int_I \sum_{k=0}^{N-1} \alpha_j\, \phi_{M,j}(x)\, dx \\
&= \sum_{j=0}^{N-1} \alpha_j \int_I \phi_{M,j}(x)\, dx \\
&= \sum_{j=0}^{N-1} \alpha_j \\
&= 0.
\end{aligned}$$

If we denote by $\alpha_{M,m}$ the N-vector with components α_j and let $\Phi_{M,m}$ be the matrix whose k, j-th entry is

$$\int_I \phi_{m,k+\mu}(x)\, \phi_{M,j}(x)\, dx, \tag{2.9}$$

then the moment equations are equivalent to

$$\Phi_{M,m}\, \alpha_{M,m} = 0. \tag{2.10}$$

Integrating equation (2.9) by parts and using the periodicity of the ϕ's we have

$$\begin{aligned}
\int_I \phi_{m,k+\mu}(x)\, \phi_{M,j}(x)\, dx &= D_1^{-1}[\phi_{m,k+\mu}(x)]\, \phi_{M,j}(x)\, \Big|_{x=0}^{x=1} && (2.11)\\
&\quad - \int_I D_1^{-1}[\phi_{m,k+\mu}(x)]\, D[\phi_{M,j}(x)]\, dx && (2.12)\\
&= -\int_I D_1^{-1}[\phi_{m,k+\mu}(x)]\, D[\phi_{M,j}(x)]\, dx. && (2.13)
\end{aligned}$$

Replacing k by $k+1$ in this formula we have

$$\int_I \phi_{m,k+1+\mu}(x)\ \phi_{M,j}(x)\ dx = -\int_I D_1^{-1}[\phi_{m,k+1+\mu}(x)]\ D[\phi_{M,j}(x)]\ dx. \tag{2.14}$$

Subtracting formula (2.13) from formula (2.14), multiplying the difference by N and then using the fact that $-\Delta\phi_{m+\mu,k} \equiv N[\phi_{m,k+\mu}(x) - \phi_{m,k+1+\mu}(x)] \in S_m$, we find that

$$\begin{aligned}
\int_I [\Delta\phi_{m,k+\mu}(x)]\ \phi_{M,j}(x)\ dx &= -\int_I D_1^{-1}[\Delta\phi_{m,k+\mu}(x)]\ D[\phi_{M,j}(x)]\ dx \\
&= \int_I \phi_{m+1,k+\mu}(x)\ [\Delta\phi_{M-1,j}(x)]\ dx.
\end{aligned}$$

Referring to the matrix $\mathcal{E}$ defined earlier, this shows that

$$\mathcal{E}\ \Phi_{M,m} = \Phi_{M-1,m+1}\ \mathcal{E}.$$

Repeating this process μ times and using the fact that, since $m = M - 2\mu$, we have $M - \mu = m + \mu$, yields

$$\mathcal{E}^\mu\ \Phi_{M,m} = \Phi_{M-\mu,m+\mu}\ \mathcal{E}^\mu = \Phi_{m+\mu,m+\mu}\ \mathcal{E}^\mu. \tag{2.15}$$

Clearly, then

$$\begin{aligned}
\Phi_{M,m}\ \alpha_{M,m} = 0 \quad &\longrightarrow \quad \mathcal{E}^\mu\ \Phi_{M,m}\ \alpha_{M,m} = 0 \\
&\longrightarrow \quad \Phi_{m+\mu,m+\mu}\ \mathcal{E}^\mu\ \alpha_{M,m} = 0.
\end{aligned}$$

Since the $\phi_{m+\mu,k}$, $k = 0, 1, \ldots, N-1$, being a basis for $S_{m+\mu}$, are linearly independent, their Gram matrix $\Phi_{m+\mu,m+\mu}$ is nonsingular. Hence, we conclude that

$$\mathcal{E}^\mu\ \alpha_{M,m} = 0. \tag{2.16}$$

Now, once again consider the matrix $\mathcal{E}$. We have $rank(\mathcal{E}) = N - 1$. To see this, recall that

$$\mathcal{E} = \begin{bmatrix} -1 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & -1 & 1 \\ 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & -1 \end{bmatrix}.$$

Determining $rank(\mathcal{E})$ requires reducing $\mathcal{E}$ to upper triangular form [3]. To reduce $\mathcal{E}$ to upper triangular form we first add row 1 to row N. This yields the matrix

$$M_1 = \begin{bmatrix} -1 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & -1 & 1 \\ 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & -1 \end{bmatrix}.$$

Adding row 2 of matrix M_1 to row N of M_1 yields

$$M_2 = \begin{bmatrix} -1 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & -1 & 1 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 & 0 & -1 \end{bmatrix}.$$

Continuing in this way M_3 is M_2 with row 3 added to row N, M_4 is M_3 with row 4 added to row N, etc. After doing this $N-2$ times we have the matrix

$$M_{N-1} = \begin{bmatrix} -1 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & -1 & 1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & -1 \end{bmatrix}.$$

Lastly, adding row $N-1$ to row N we have the matrix

$$M_N = \begin{bmatrix} -1 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & -1 & 1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \end{bmatrix}.$$

From this last matrix we see that $rank(M_N) = rank(\mathcal{E}) = N - 1$.

Next, let $\mathbf{r}_i$ represent the i-th row of the matrix $\mathcal{E}$. Similarly, let $\mathbf{c}_j$ represent the j-th column of $\mathcal{E}$. Then, we see that

$$\mathbf{r}_N = \sum_{i=1}^{N-1} \mathbf{r}_i \text{ and } \mathbf{c}_N = \sum_{j=1}^{N-1} \mathbf{c}_j.$$

Thus,

$$\begin{aligned} \mathcal{E}^2 &= \mathcal{E} \cdot \mathcal{E} \\ &= \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{r}_{N-1} \\ \mathbf{r}_N \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdot & \cdot & \cdot & \mathbf{c}_{N-1} & \mathbf{c}_N \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{r}_{N-1} \\ \sum_{i=1}^{N-1} \mathbf{r}_i \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdot & \cdot & \cdot & \mathbf{c}_{N-1} & \sum_{j=1}^{N-1} \mathbf{c}_j \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{c}_1 & \mathbf{r}_1 \cdot \mathbf{c}_2 & \cdot & \mathbf{r}_1 \cdot \mathbf{c}_{N-1} & \mathbf{r}_1 \cdot (\sum_{j=1}^{N-1} \mathbf{c}_j) \\ \mathbf{r}_2 \cdot \mathbf{c}_1 & \mathbf{r}_2 \cdot \mathbf{c}_2 & \cdot & \mathbf{r}_2 \cdot \mathbf{c}_{N-1} & \mathbf{r}_2 \cdot (\sum_{j=1}^{N-1} \mathbf{c}_j) \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \mathbf{r}_{N-1} \cdot \mathbf{c}_1 & \mathbf{r}_{N-1} \cdot \mathbf{c}_2 & \cdot & \mathbf{r}_{N-1} \cdot \mathbf{c}_{N-1} & \mathbf{r}_{N-1} \cdot (\sum_{j=1}^{N-1} \mathbf{c}_j) \\ (\sum_{i=1}^{N-1} \mathbf{r}_i) \cdot \mathbf{c}_1 & (\sum_{i=1}^{N-1} \mathbf{r}_i) \cdot \mathbf{c}_2 & \cdot & (\sum_{i=1}^{N-1} \mathbf{r}_i) \cdot \mathbf{c}_{N-1} & (\sum_{i=1}^{N-1} \mathbf{r}_i) \cdot (\sum_{j=1}^{N-1} \mathbf{c}_j) \end{bmatrix}$$

Here, $\mathbf{a} \cdot \mathbf{b}$ is the usual dot product.

Now, consider the $(N-1) \times (N-1)$ principal submatrix [7]

$$S = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{c}_1 & \mathbf{r}_1 \cdot \mathbf{c}_2 & \cdot \ \cdot \ \cdot & \mathbf{r}_1 \cdot \mathbf{c}_{N-1} \\ \mathbf{r}_2 \cdot \mathbf{c}_1 & \mathbf{r}_2 \cdot \mathbf{c}_2 & \cdot \ \cdot \ \cdot & \mathbf{r}_2 \cdot \mathbf{c}_{N-1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \mathbf{r}_{N-1} \cdot \mathbf{c}_1 & \mathbf{r}_{N-1} \cdot \mathbf{c}_2 & \cdot \ \cdot \ \cdot & \mathbf{r}_{N-1} \cdot \mathbf{c}_{N-1} \end{bmatrix}.$$

Since the rows $\mathbf{r}_i$, for $i = 1, \ldots, N-1$ are linearly independent and the columns $\mathbf{c}_j$, for $j = 1, \ldots, N-1$ are also linearly independent, the matrix S has rank $(N-1)$. Hence $rank(\mathcal{E}^2) \geq (N-1)$.

Next, notice that the last row of the matrix $\mathcal{E}^2$ is the sum of the first $(N-1)$ rows and that the last column is the sum of the first $(N-1)$ columns. Therefore, $rank(\mathcal{E}^2) = (N-1)$. Thus we have $rank(\mathcal{E}^2) = rank(\mathcal{E})$. This argument can be continued to include all powers of $\mathcal{E}$. That is, for all $p \geq 1$, $rank(\mathcal{E}^p) = rank(\mathcal{E}) =$

$(N-1)$.

Therefore, since $rank(\mathcal{E}^{\mu}) = rank(\mathcal{E}) = (N-1)$ and all of the columns of $\mathcal{E}^{\mu}$ are orthogonal to the N-vector $(1, 1, \cdots, 1)^T$, equation (2.16) together with the earlier result

$$\sum_{j=0}^{N-1} \alpha_j = [1, 1, \cdots, 1] \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{N-1} \end{bmatrix} = 0$$

shows that $\alpha_{M,m} = 0$. Hence, we have that the homogeneous moment equations have only the trivial solution, and the proof that $f_{M,m}$ is unique is concluded. ■

Before moving on we wish to note that equation (2.15) indicates that if $M - m = 2\mu$ then our approximation technique is equivalent to least square approximation of the μ-th derivative of f. This will be further explored in the error estimates.

Next, let us further explore the properties of the matrix $\Phi_{M,m}$.

Theorem 3 *If $M - m$ is even, then the matrix $\Phi_{M,m}$ is symmetric and positive definite.*

Proof. Symmetry is a consequence of the symmetry of the basis functions $\phi_{m,k}$ and $\phi_{M,j}$ about the center point of their supports. Since a symmetric matrix is positive definite if and only if all of its eigenvalues are positive, [7] we need to show that all the eigenvalues of $\Phi_{M,m}$ are positive.

As equation (2.15) tells us,

$$\mathcal{E}^{\mu}\ \Phi_{M,m} = \Phi_{m+\mu,m+\mu}\ \mathcal{E}^{\mu} \tag{2.17}$$

From the partition of unity result we also have that the vector

$$\begin{aligned}
v &= \Phi_{M,m} \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} <\phi_{M,0},\phi_{m+\mu,0}> & <\phi_{M,0},\phi_{m+\mu,1}> & \cdot & <\phi_{M,0},\phi_{m+\mu,N-1}> \\ <\phi_{M,1},\phi_{m+\mu,0}> & <\phi_{M,1},\phi_{m+\mu,1}> & \cdot & <\phi_{M,1},\phi_{m+\mu,N-1}> \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ <\phi_{M,N-1},\phi_{m+\mu,0}> & <\phi_{M,N-1},\phi_{m+\mu,1}> & \cdot & <\phi_{M,N-1},\phi_{m+\mu,N-1}> \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} <\phi_{M,0},\phi_{m+\mu,0}> + <\phi_{M,0},\phi_{m+\mu,1}> + \ldots + <\phi_{M,0},\phi_{m+\mu,N-1}> \\ <\phi_{M,1},\phi_{m+\mu,0}> + <\phi_{M,1},\phi_{m+\mu,1}> + \ldots + <\phi_{M,1},\phi_{m+\mu,N-1}> \\ \cdot \\ \cdot \\ \cdot \\ <\phi_{M,N-1},\phi_{m+\mu,0}> + <\phi_{M,N-1},\phi_{m+\mu,1}> + \ldots + <\phi_{M,N-1},\phi_{m+\mu,N-1}> \end{bmatrix} \\
&= \begin{bmatrix} \sum_{j=0}^{N-1} <\phi_{M,0},\phi_{m+\mu,j}> \\ \sum_{j=0}^{N-1} <\phi_{M,1},\phi_{m+\mu,j}> \\ \cdot \\ \cdot \\ \cdot \\ \sum_{j=0}^{N-1} <\phi_{M,N-1},\phi_{m+\mu,j}> \end{bmatrix} \\
&= \begin{bmatrix} <\phi_{M,0}, \sum_{j=0}^{N-1} \phi_{m+\mu,j}> \\ <\phi_{M,1}, \sum_{j=0}^{N-1} \phi_{m+\mu,j}> \\ \cdot \\ \cdot \\ \cdot \\ <\phi_{M,N-1}, \sum_{j=0}^{N-1} \phi_{m+\mu,j}> \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} < \phi_{M,0}, N > \\ < \phi_{M,1}, N > \\ \cdot \\ \cdot \\ \cdot \\ < \phi_{M,N-1}, N > \end{bmatrix} = \begin{bmatrix} N \\ N \\ \cdot \\ \cdot \\ \cdot \\ N \end{bmatrix} = N \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}.$$

Hence, the vector $\mathbf{v} \equiv (1, 1, \ldots, 1)^T$ is an eigenvector of $\Phi_{M,m}$ corresponding to the eigenvalue N. Now, let λ be an eigenvalue of $\Phi_{M,m}$ with eigenvector $\mathbf{c} = (c_1, c_2, \ldots, c_N)^T$ such that $\mathbf{c}$ is orthogonal to $\mathbf{v}$. Then from equation (2.17) we have

$$\begin{aligned} \lambda(\mathcal{E}^{\mu}\mathbf{c}) &= \mathcal{E}^{\mu}\,(\lambda\mathbf{c}) \\ &= \mathcal{E}^{\mu}\,(\Phi_{M,m}\,\mathbf{c}) \\ &= (\mathcal{E}^{\mu}\,\Phi_{M,m})\,\mathbf{c} \\ &= (\Phi_{m+\mu,m+\mu}\,\mathcal{E}^{\mu})\,\mathbf{c} \\ &= \Phi_{m+\mu,m+\mu},\,(\mathcal{E}^{\mu}\mathbf{c}) \end{aligned}$$

so that $\mathcal{E}^{\mu}\mathbf{c}$ is an eigenvector of $\Phi_{m+\mu,m+\mu}$ corresponding to λ, and orthogonal to the eigenvector $(1, 1, \ldots, 1)^T$ of $\Phi_{m+\mu,m+\mu}$. Since $\Phi_{m+\mu,m+\mu}$ is positive definite, $\lambda > 0$. It follows that all eigenvalues of $\Phi_{M,m}$ are positive; hence, $\Phi_{M,m}$ is positive definite.

■

Theorem 4 *Let f be defined and integrable on the interval $I = [0, 1]$. Also assume that f and its integrals are periodic. Given a positive integer M and a non-negative integer m, $0 \leq m \leq M$ such that $M - m = 2\mu + 1$, where μ is also a non-negative* *integer. Then there is exactly one function $f_{M,m} \in S_M$ such that*

$$\int_I \phi_{m,k}(x)[f_{M,m}(x) - f(x)]\, dx = 0 \quad k = 0, 1, \ldots, N-1.$$

Proof. The proof of this theorem parallels the proof of Theorem 2. In following the proof of Theorem 2, everything is the same until equation (2.10). At this point instead of considering the integral

$$\int_I \phi_{M,k+\mu}(x)\phi_{M,j}(x)\, dx$$

we substitute the integral

$$\begin{cases} \int_I \phi_{M+m,k+\mu}(x)\phi_{0,j}(x)\, dx & m \text{ even} \\ \int_I \phi_{M+m-1,k+\mu}(x)\phi_{1,j}(x)\, dx & m \text{ odd} \end{cases}$$

Continuing the proof of Theorem 2 with this substitution leads to equation (2.15) being replaced by the equation

$$\begin{cases} \mathcal{E}^m \Phi_{M+m,0} = \Phi_{M+m-m,0+m}\mathcal{E}^m = \Phi_{M,m}\mathcal{E}^m & m \text{ even} \\ \mathcal{E}^{m-1}\Phi_{M+m-1,1} = \Phi_{M+m-1-(m-1),1+(m-1)}\mathcal{E}^{m-1} = \Phi_{M,m}\mathcal{E}^{m-1} & m \text{ odd} \end{cases}$$

Now, since $M - m = 2\mu + 1$ we have that $M + m = 2\mu + 2m + 1$. That is, $M + m$ is always odd. Thus we see that if m of even this approximation scheme is equivalent to point interpolation, with odd order splines, on the m-th integral of f. If m is odd this scheme is equivalent to integral interpolation with even order splines. Now since we assumed that $\phi_{m,k}(x)$ was centered relative to $\phi_{M,k}(x)$ we have that this is really midpoint interpolation with odd order splines or integral interpolation using shifted and even order splines. From [14], and [13] we have that midpoint interpolation with odd order splines is well-defined and an adaptation of our previous proof will show that integral interpolation is also well-defined. Therefore, there is exactly one $f_{M,m}$ and our proof in concluded. ■

While we have established that our interpolation scheme is well-defined for any M and m, we have not yet been able to establish that this new matrix $\Phi_{M,m}$, for $M-m$ odd, is positive definite.

In the a late chapter we will look into why the matrix $\Phi_{M,m}$ needs to be positive definite.

Chapter 3

Applications

3.1 Coefficient Matrices

Suppose we wish to approximate a periodic function $f(x)$ using periodic splines of order M. In this situation, we are looking for a function that is a linear combination of the spline basis functions $\phi_{M,j}$; that is, our approximation will be of the form

$$f_{M,m}(x) = \sum_{j=0}^{N-1} \alpha_j \ \phi_{M,j}(x).$$

Further, suppose we also require that our approximation $f_{M,m}$ satisfies the property

$$< \phi_{m,k}, f - f_{M,m} >_m = 0.$$

This requirement results in having

$$< \phi_{m,k}, f_{M,m} >_m \ = \ < \phi_{m,k}, f >_m \quad k = 0, 1, \ldots, N-1,$$

where m is such that $0 \leq m \leq M$ and $m = M - 2\mu$. Then, if we let $A_{M,m}$ represent the matrix whose i,j element $a_{i,j}$ is

$$a_{i,j} = \int_I \phi_{m,i}(x)\, \phi_{M,j}(x)\, dx = < \phi_{m,i}, \phi_{M,j} >_m,$$

as well as letting α and b be the N-vectors

$$\alpha = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{N-1} \end{bmatrix} \qquad b = \begin{bmatrix} < \phi_{m,0}, f > \\ < \phi_{m,1}, f > \\ \cdot \\ \cdot \\ \cdot \\ < \phi_{m,N-1}, f > \end{bmatrix} = \begin{bmatrix} \int_I \phi_{m,0}(x)\, f(x)\, dx \\ \int_I \phi_{m,1}(x)\, f(x)\, dx \\ \cdot \\ \cdot \\ \cdot \\ \int_I \phi_{m,N-1}(x)\, f(x)\, dx \end{bmatrix},$$

finding our approximation $f_{M,m}$ reduces to solving the matrix equation

$$A_{M,m}\, \alpha = b.$$

Consider the case of $M = 4$. In this situation, the acceptable values of m are $m = 4, 2$, and 0 corresponding to $\mu = 0, 1$, and 2 respectively. Here we have

$$A_{4,4} = \frac{N}{5040} \begin{bmatrix} 2416 & 1191 & 120 & 1 & \cdots & 0 & 1 & 120 & 1191 \\ 1191 & 2416 & 1191 & 120 & \cdots & 0 & 0 & 1 & 120 \\ 120 & 1191 & 2416 & 1191 & \cdots & 0 & 0 & 0 & 1 \\ 1 & 120 & 1191 & 2416 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 2416 & 1191 & 120 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 1191 & 2416 & 1191 & 120 \\ 120 & 1 & 0 & 0 & \cdots & 120 & 1191 & 2416 & 1191 \\ 1191 & 120 & 1 & 0 & \cdots & 1 & 120 & 1191 & 2416 \end{bmatrix}.$$

Next we have

$$A_{4,2} = \frac{N}{120}\begin{bmatrix} 26 & 1 & 0 & 0 & \cdots & 0 & 1 & 26 & 66 \\ 66 & 26 & 1 & 0 & \cdots & 0 & 0 & 1 & 26 \\ 26 & 66 & 26 & 1 & \cdots & 0 & 0 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 66 & 26 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 26 & 66 & 26 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 1 & 26 & 66 & 26 \end{bmatrix}.$$

Lastly, we have

$$A_{4,0} = \frac{N}{6}\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 4 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 4 \\ 4 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & 4 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 4 & 1 & 0 \end{bmatrix}.$$

For the case where $M = 5$ acceptable values for m are $m = 5, 3$, and 1 which correspond to $\mu = 0, 1$, and 2 respectively. Thus we have $A_{5,5} = \frac{N}{362880}B$ where $B =$

$$\begin{bmatrix} 156190 & 88234 & 14608 & 502 & 1 & \cdot & 1 & 502 & 14608 & 88234 \\ 88234 & 156190 & 88234 & 14608 & 502 & \cdot & 0 & 1 & 502 & 14608 \\ 14608 & 88234 & 156190 & 88234 & 14608 & \cdot & 0 & 0 & 1 & 502 \\ 502 & 14608 & 88234 & 156190 & 88234 & \cdot & 0 & 0 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & 0 & 0 & \cdot & 156190 & 88234 & 14608 & 502 \\ 502 & 1 & 0 & 0 & 0 & \cdot & 88234 & 156190 & 88234 & 14608 \\ 14608 & 502 & 1 & 0 & 0 & \cdot & 14608 & 88234 & 156190 & 88234 \\ 88234 & 14608 & 502 & 1 & 0 & \cdot & 502 & 14608 & 88234 & 156190 \end{bmatrix}.$$

We also have

$$A_{5,3} = \frac{N}{5040} \begin{bmatrix} 1191 & 120 & 1 & 0 & \cdots & 1 & 120 & 1191 & 2416 \\ 2416 & 1191 & 120 & 1 & \cdots & 0 & 1 & 120 & 1191 \\ 1191 & 2416 & 1191 & 120 & \cdots & 0 & 0 & 1 & 120 \\ 120 & 1191 & 2416 & 1191 & \cdots & 0 & 0 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1191 & 120 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 2416 & 1191 & 120 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 1191 & 2416 & 1191 & 120 \\ 120 & 1 & 0 & 0 & \cdots & 120 & 1191 & 2416 & 1191 \end{bmatrix}.$$

Lastly, we have

$$A_{5,1} = \frac{N}{120} \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 1 & 26 & 66 & 26 \\ 26 & 1 & 0 & 0 & \cdots & 0 & 1 & 26 & 66 \\ 66 & 26 & 1 & 0 & \cdots & 0 & 0 & 1 & 26 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 26 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 66 & 26 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 26 & 66 & 26 & 1 \end{bmatrix}.$$

3.2 Examples

Suppose we wish to approximate a periodic function $f(x)$ using periodic splines of order M. Recall that we are looking for a function that is a linear combination of the spline basis functions $\phi_{M,j}$; that is, our approximation will be of the form

$$f_{M,m}(x) = \sum_{j=0}^{N-1} \alpha_j \, \phi_{M,j}(x).$$

Further, suppose we also require that our approximation $f_{M,m}$ satisfies the property

$$< \phi_{m,k}, f_{M,m} >_m \; = \; < \phi_{m,k}, f >_m \quad k = 0, 1, \ldots, N-1,$$

where m is such that $0 \leq m \leq M$ and $m = M - 2\mu$.

To calculate $\alpha_j, \quad j = 0, 1, \ldots, N-1$, as shown earlier, we must solve the $N \times N$ linear system

$$A_{M,m}\, \alpha = b \tag{3.1}$$

where, $A_{M,m}$ represents the matrix whose i, j element $a_{i,j}$ is

$$a_{i,j} = \int_I \phi_{m,i}(x)\, \phi_{M,j}(x)\, dx,$$

as well as, α and b represent the N-vectors

$$\alpha = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{N-1} \end{bmatrix} \qquad b = \begin{bmatrix} \int_I \phi_{m,0}(x)\, f(x)\, dx \\ \int_I \phi_{m,1}(x)\, f(x)\, dx \\ \cdot \\ \cdot \\ \cdot \\ \int_I \phi_{m,N-1}(x)\, f(x)\, dx \end{bmatrix}.$$

For our first example we will consider the case where $f(x)$ is defined temporarily as the step function

$$f(x) = \begin{cases} 1 & x \in [0, \frac{1}{5}] \\ 2 & x \in (\frac{1}{5}, \frac{2}{5}] \\ 1 & x \in (\frac{2}{5}, 1]. \end{cases}$$

For ease of understanding, let b_m represent the vector b associated with the matrix $A_{M,m}$ mentioned in equation (3.1).

Also, to make the calculations less cumbersome, let us redefine $f(x)$ as

$$f(x) = \begin{cases} 1 & x \in [1, p_1] \\ 2 & x \in (p_1, p_2] \\ 1 & x \in (p_2, 1] \end{cases} \tag{3.2}$$

In the definition, $p_1 = Gr(\frac{N}{5})$ and $p_2 = Gr(\frac{2N}{5})$ where $Gr(x)$ represents the greatest integer function. Then for this function $f(x)$ as defined in (3.2) we have for $m = 0$ the vector b_0 has components

$$b_0^j = \begin{cases} 1 & j = 0, \dots, p_1 \\ 2 & j = (p_1 + 1), \dots, p_2 \\ 1 & j = (p_2 + 1), \dots, (N - 1). \end{cases}$$

For $m = 1$ the vector b_1 has components

$$b_1^j = \begin{cases} 1 & j = 0, \dots, (p_1 - 1) \\ 2 & j = p_1, \dots, (p_2 - 1) \\ 1 & j = p_2, \dots, (N - 1). \end{cases}$$

Also for this function $f(x)$ as defined in (3.2) we have for $m = 2$ the vector b_2 has components

$$b_2^j = \begin{cases} 1 & j = 0, \dots, (p_1 - 2) \\ \frac{3}{2} & j = (p_1 - 1) \\ 2 & j = p_1, \dots, (p_2 - 2) \\ \frac{3}{2} & j = (p_2 - 1) \\ 1 & j = p_2, \dots, (N - 1). \end{cases}$$

If $m = 3$ the vector b_3 has components

$$b_3^j = \begin{cases} 1 & j = 0, \ldots, (p_1 - 3) \\ \frac{7}{6} & j = (p_1 - 2) \\ \frac{11}{6} & j = (p_1 - 1) \\ 2 & j = p_1, \ldots, (p_2 - 3) \\ \frac{11}{6} & j = (p_2 - 2) \\ \frac{7}{6} & j = (p_2 - 1) \\ 1 & j = p_2, \ldots, (N - 1). \end{cases}$$

For $m = 4$ the vector b_4 has components

$$b_4^j = \begin{cases} 1 & j = 0, \ldots, (p_1 - 4) \\ \frac{25}{24} & j = (p_1 - 3) \\ \frac{3}{2} & j = (p_1 - 2) \\ \frac{47}{24} & j = (p_1 - 1) \\ 2 & j = p_1, \ldots, (p_2 - 4) \\ \frac{47}{24} & j = (p_2 - 3) \\ \frac{3}{2} & j = (p_2 - 2) \\ \frac{25}{24} & j = (p_2 - 1) \\ 1 & j = p_2, \ldots, (N - 1). \end{cases}$$

Lastly, using this function $f(x)$ we have for $m = 4$ the vector b_4 has components

$$b_5^j = \begin{cases} 1 & j = 0, \ldots, (p_1 - 5) \\ \frac{121}{120} & j = (p_1 - 4) \\ \frac{147}{120} & j = (p_1 - 3) \\ \frac{213}{120} & j = (p_1 - 2) \\ \frac{239}{120} & j = (p_1 - 1) \\ 2 & j = p_1, \ldots, (p_2 - 5) \\ \frac{239}{120} & j = (p_2 - 4) \\ \frac{213}{120} & j = (p_2 - 3) \\ \frac{147}{120} & j = (p_2 - 2) \\ \frac{121}{120} & j = (p_2 - 1) \\ 1 & j = p_2, \ldots, (N - 1). \end{cases}$$

We are now ready for some computer results. Figures 2.1–2.12, at the end of this section, graphically represent the computer generated data. It should be noted that in the following graphs the jaggedness around the values $x = \frac{1}{5}$ and $x = \frac{2}{5}$ is due to the discontinuity of the function $f(x)$ and not due to the numerical approximation scheme.

mbox

For our next example let us consider the task of approximating the cubic spline function

$$\phi_{4,20,40}(x) = \begin{cases} \frac{N^4}{6}(x - x_k)^3 \\ \frac{N}{6} + \frac{N^2}{2}(x - x_{k+1}) + \frac{N^3}{2}(x - x_{k+1})^2 - \frac{N^4}{2}(x - x_{k+1})^3 \\ \frac{2N}{3} - N^3(x - x_{k+2})^2 + \frac{N^4}{2}(x - x_{k+2})^3 \\ \frac{N}{6} - \frac{N^2}{2}(x - x_{k+3}) + \frac{N^3}{2}(x - x_{k+3})^2 - \frac{N^4}{6}(x - x_{k+3})^3 \\ 0 \end{cases}$$

for $x \in [x_k, x_{k+1})$, $x \in [x_{k+1}, x_{k+2})$, $[x_{k+2}, x_{k+3})$, $[x_{k+3}, x_{k+4}]$ and elsewhere respectively. Here $k = 20$ and $N = 40$.

In the last example, we saw that the more spline functions used in the approximation, the better the accuracy. However, in this example, we use a relatively small number of spline functions. In fact, here we use only ten spline functions in each case. In so doing the subtle differences in each type of approximation can be seen. These results are represented graphically in Figures 2.13–2.18 found at the end of this section. Two graphs that combine the results of Figures 2.13–2.15 and Figures 2.16–2.18, respectively, can be found at the end of the chapter on error estimation.

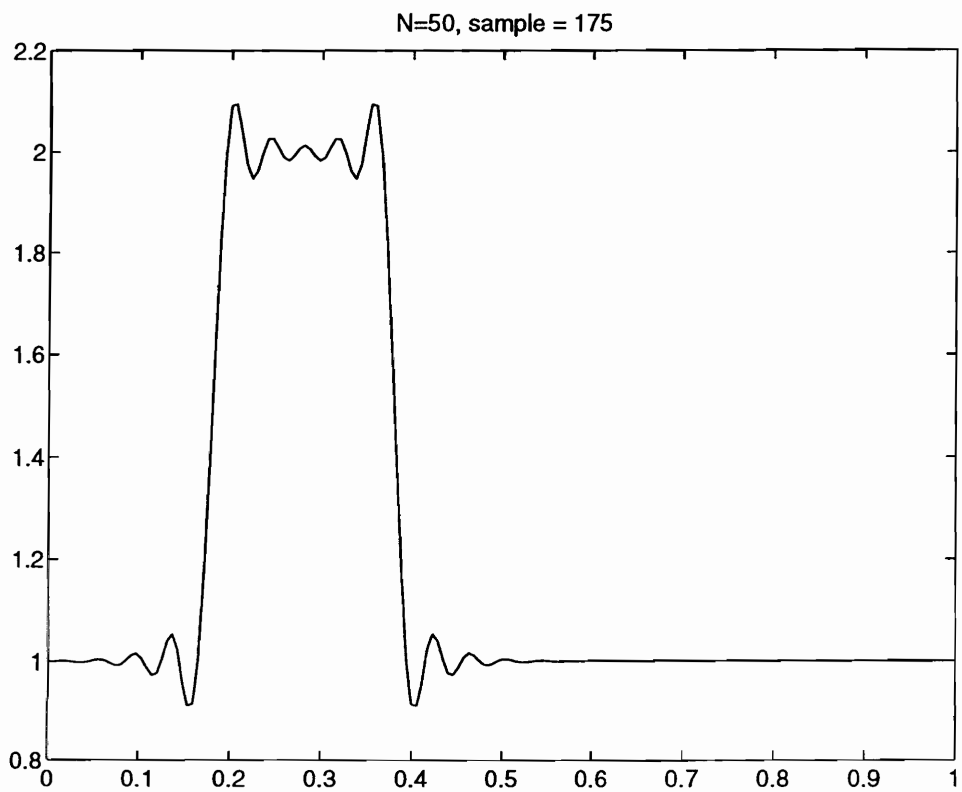

Figure 3.1: Approximation of step function with $M = 4, m = 4$ and $N = 50$

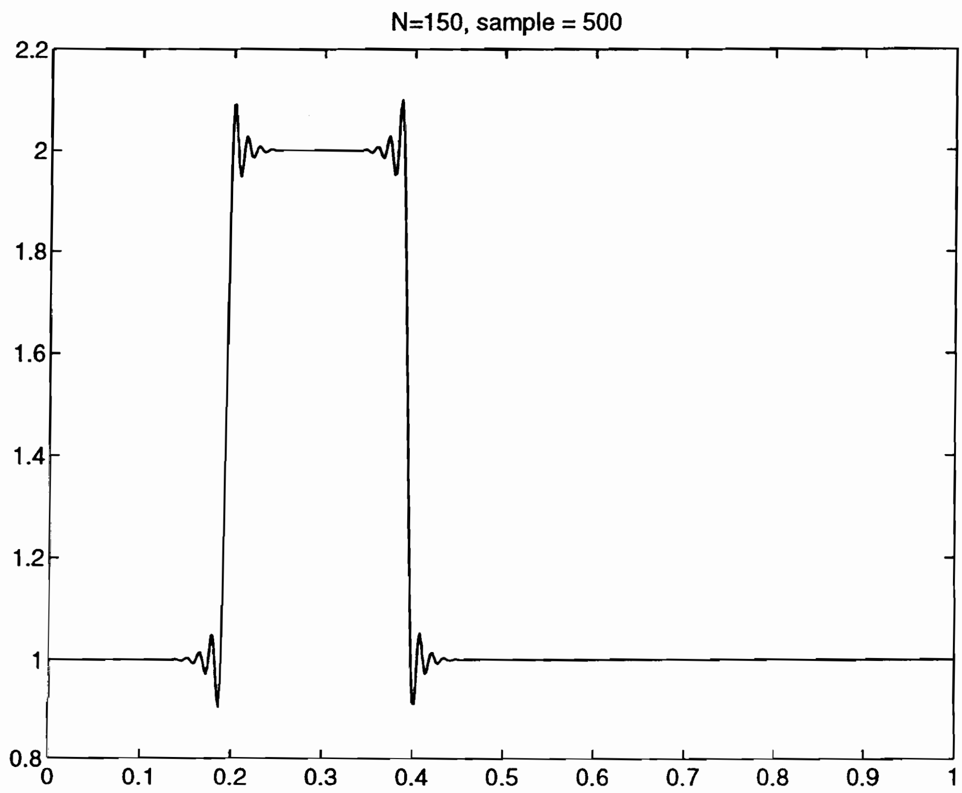

Figure 3.2: Approximation of step function with $M = 4, m = 4$ and $N = 150$

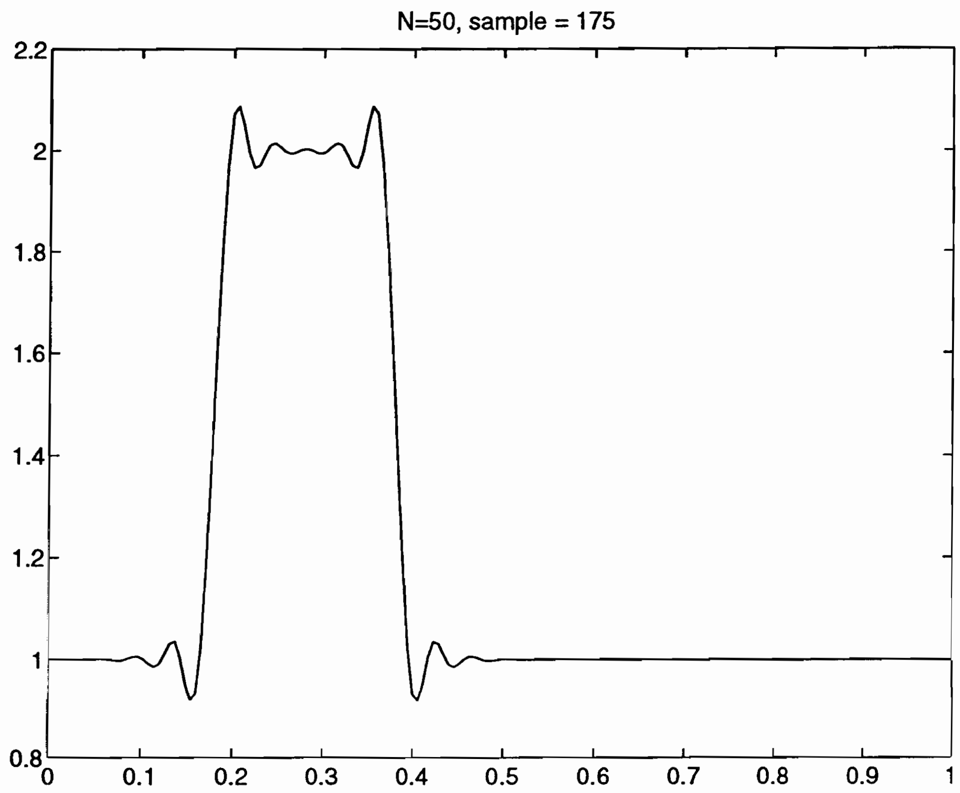

Figure 3.3: Approximation of step function with $M = 4, m = 2$ and $N = 50$

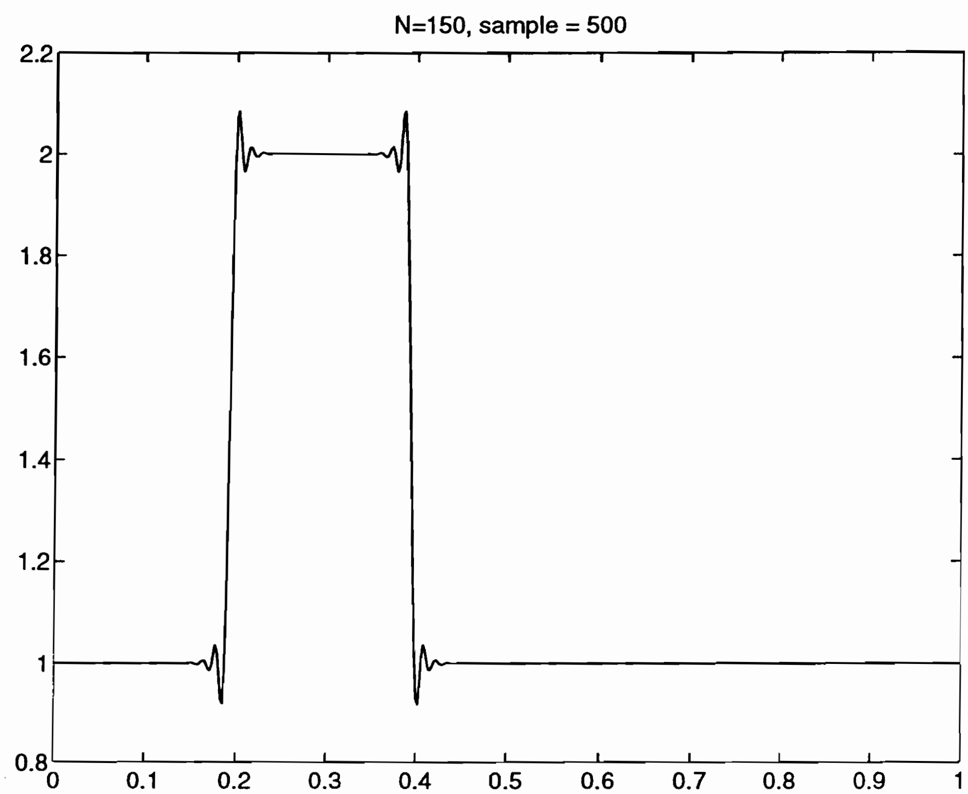

Figure 3.4: Approximation of step function with $M = 4, m = 2$ and $N = 150$

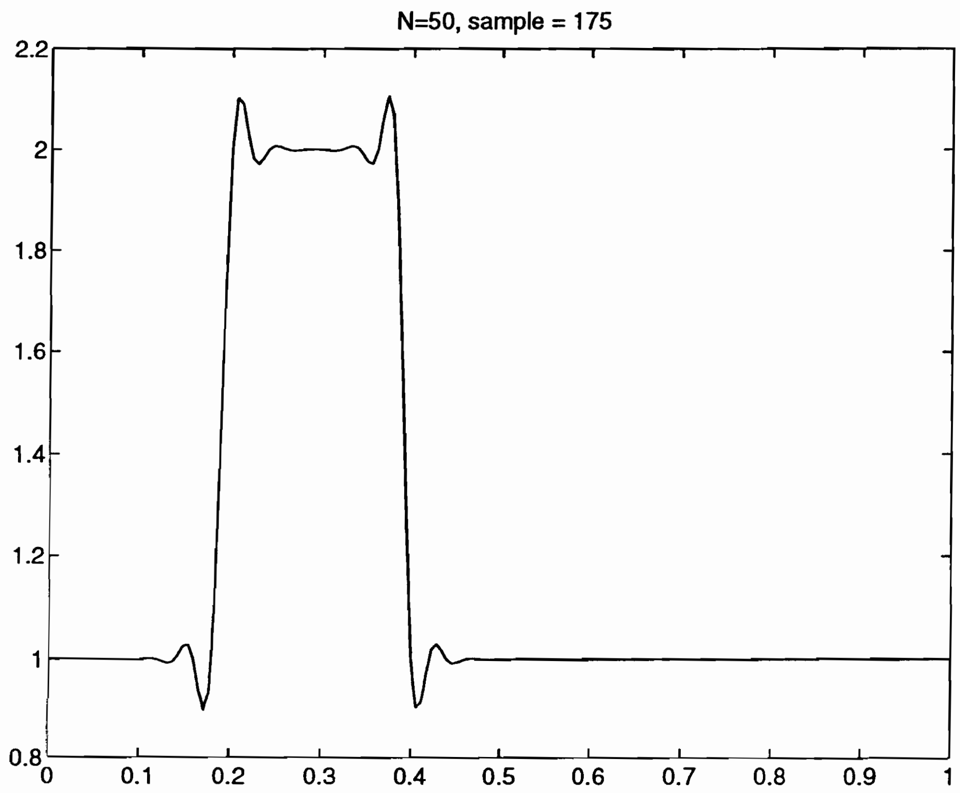

Figure 3.5: Approximation of step function with $M = 4, m = 0$ and $N = 50$

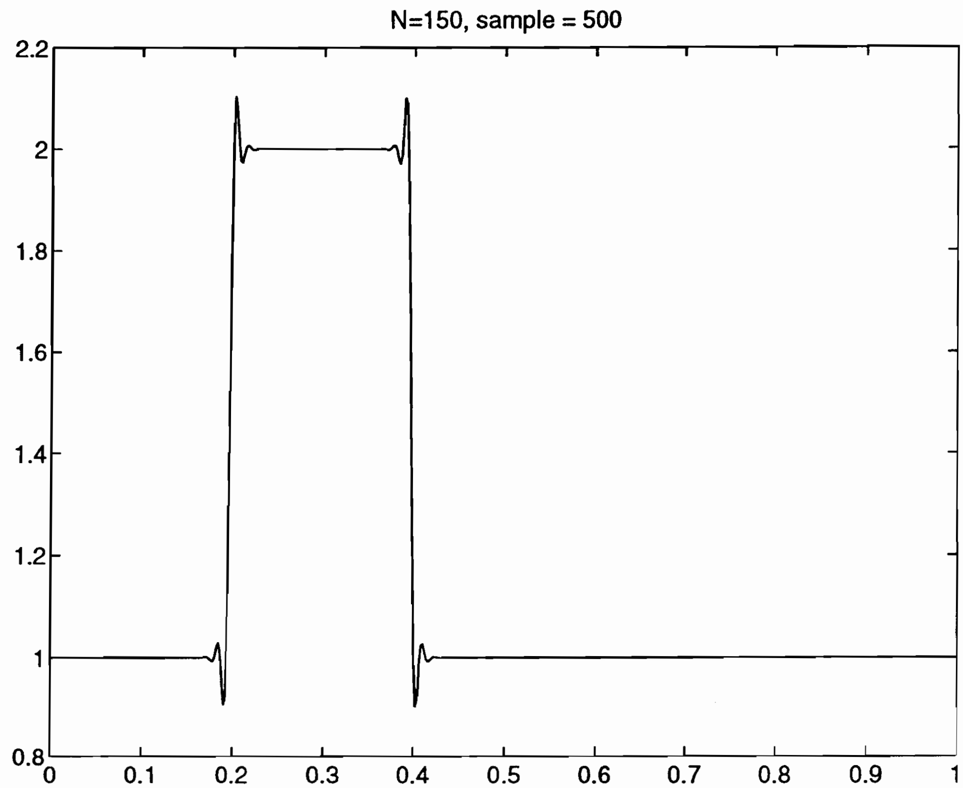

Figure 3.6: Approximation of step function with $M = 4, m = 0$ and $N = 150$

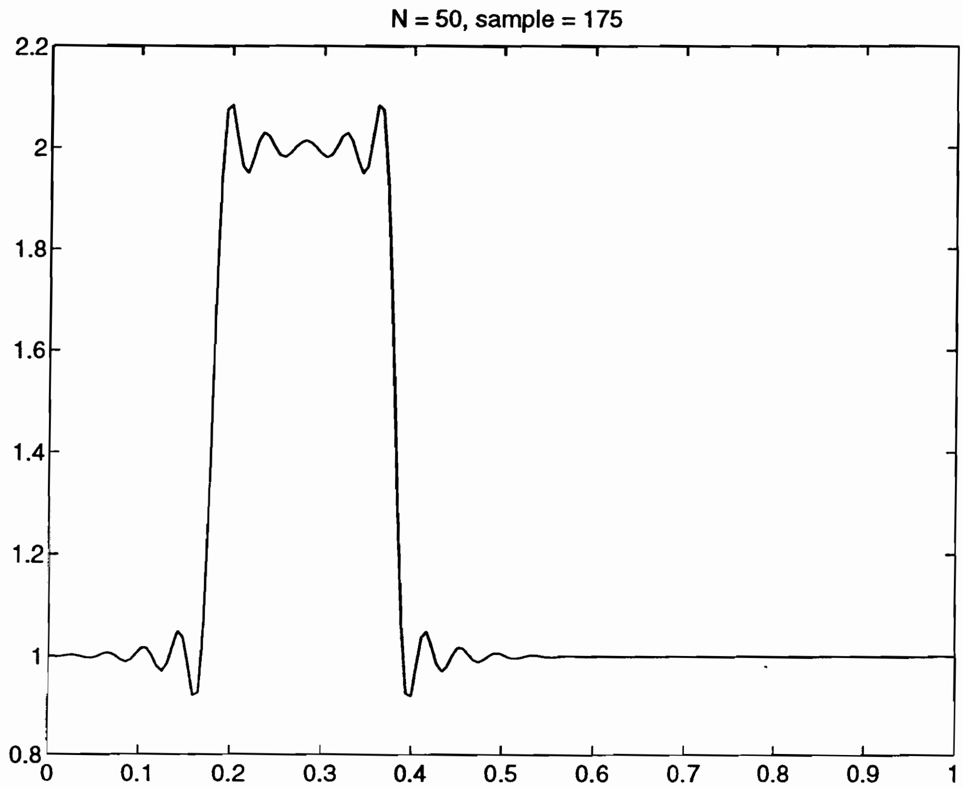

Figure 3.7: Approximation of step function with $M = 5, m = 5$ and $N = 50$

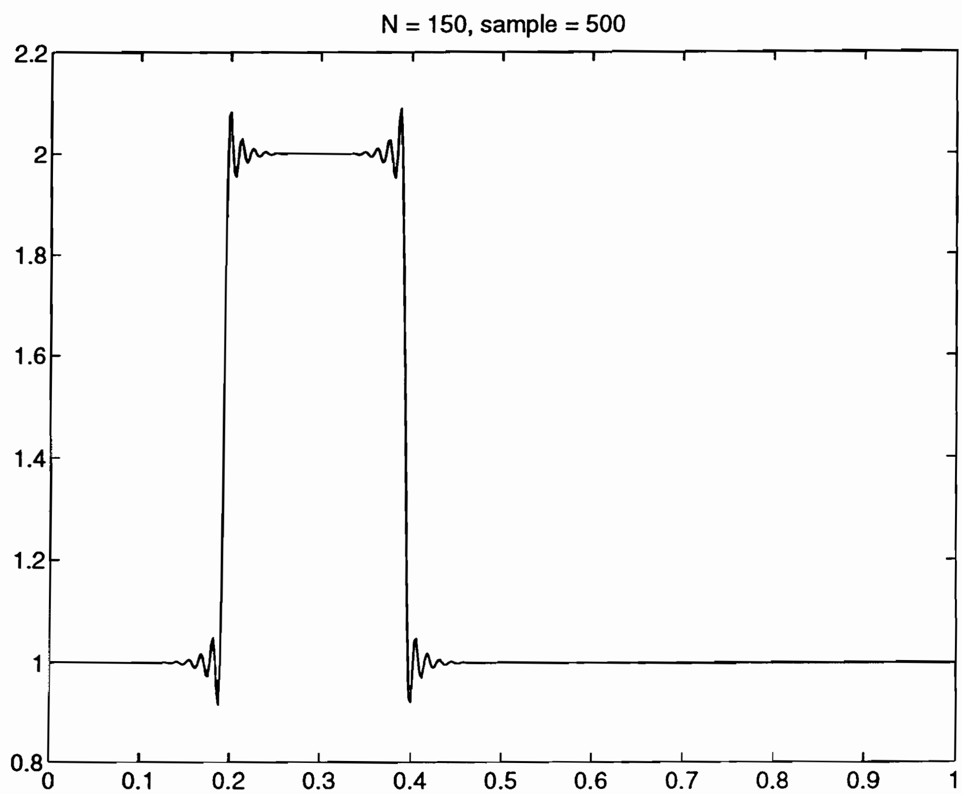

Figure 3.8: Approximation of step function with $M = 5, m = 5$ and $N = 150$

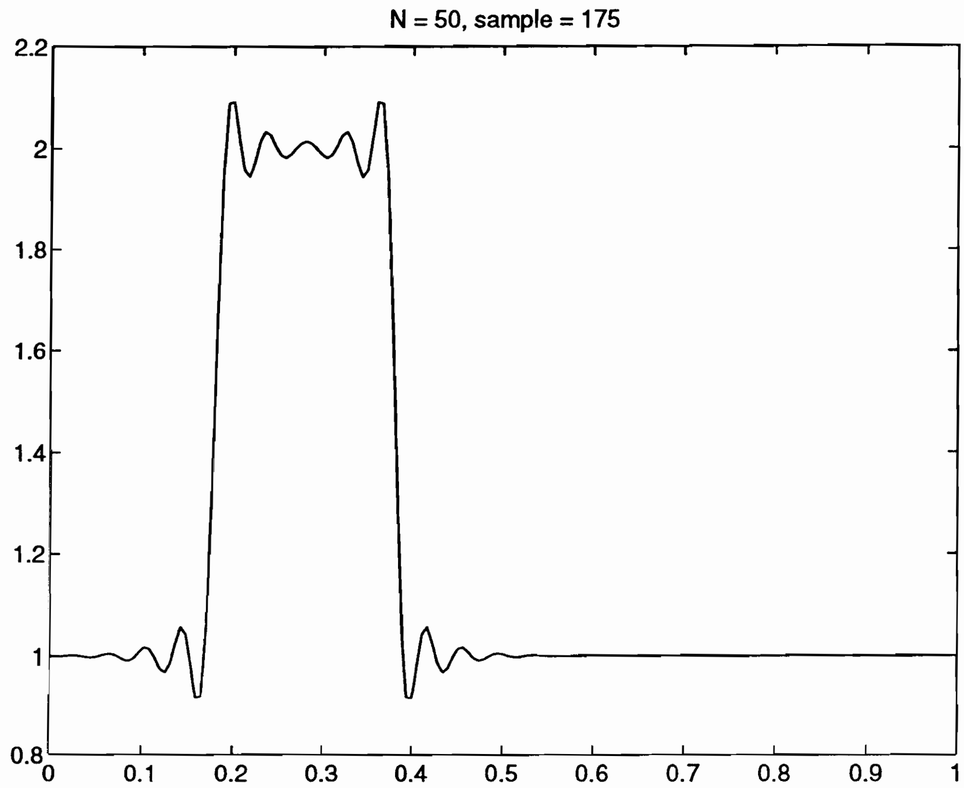

Figure 3.9: Approximation of step function with $M = 5, m = 3$ and $N = 50$

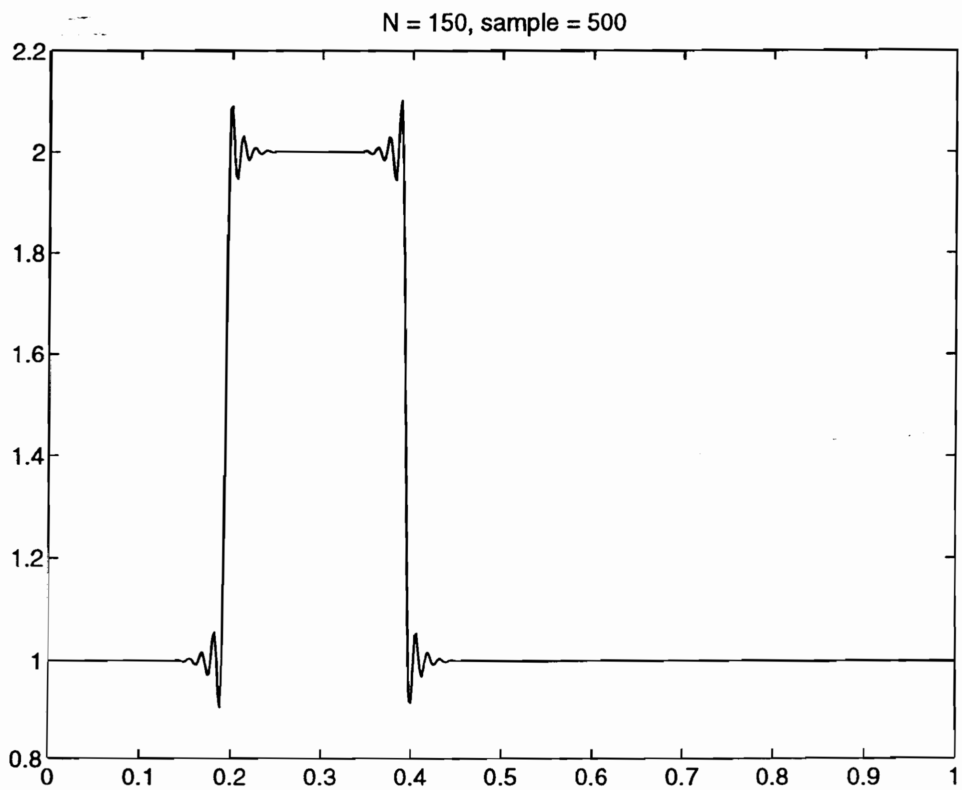

Figure 3.10: Approximation to step function with$M = 5, m = 3$ and $N = 150$

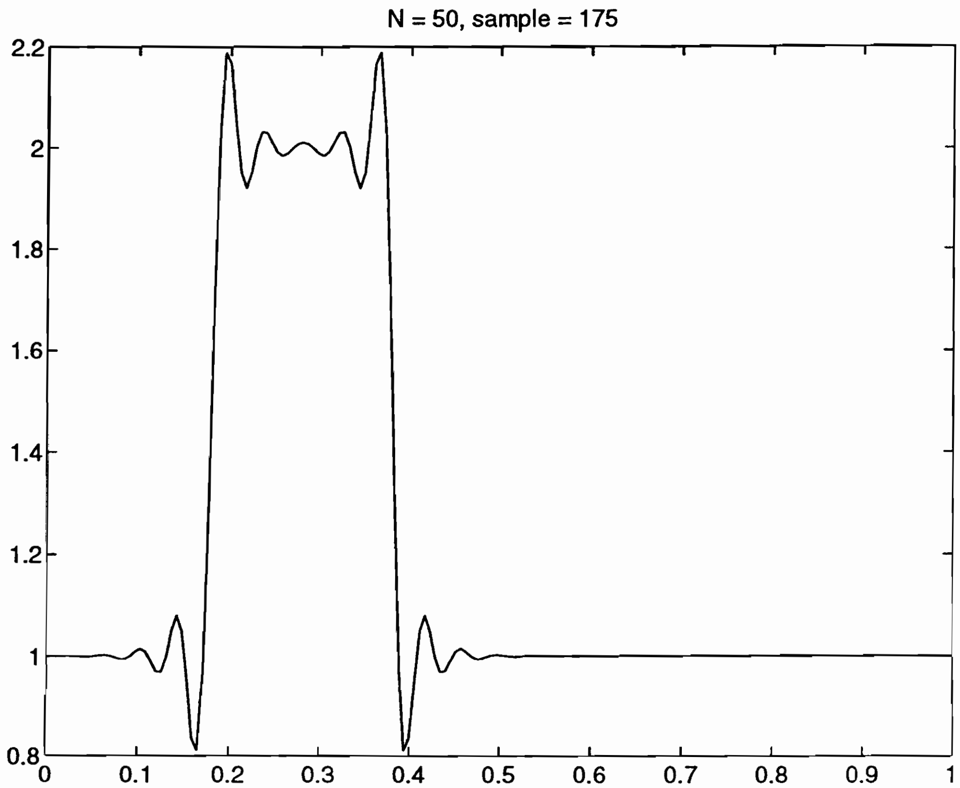

Figure 3.11: Approximation of step function with $M = 5, m = 1$ and $N = 50$

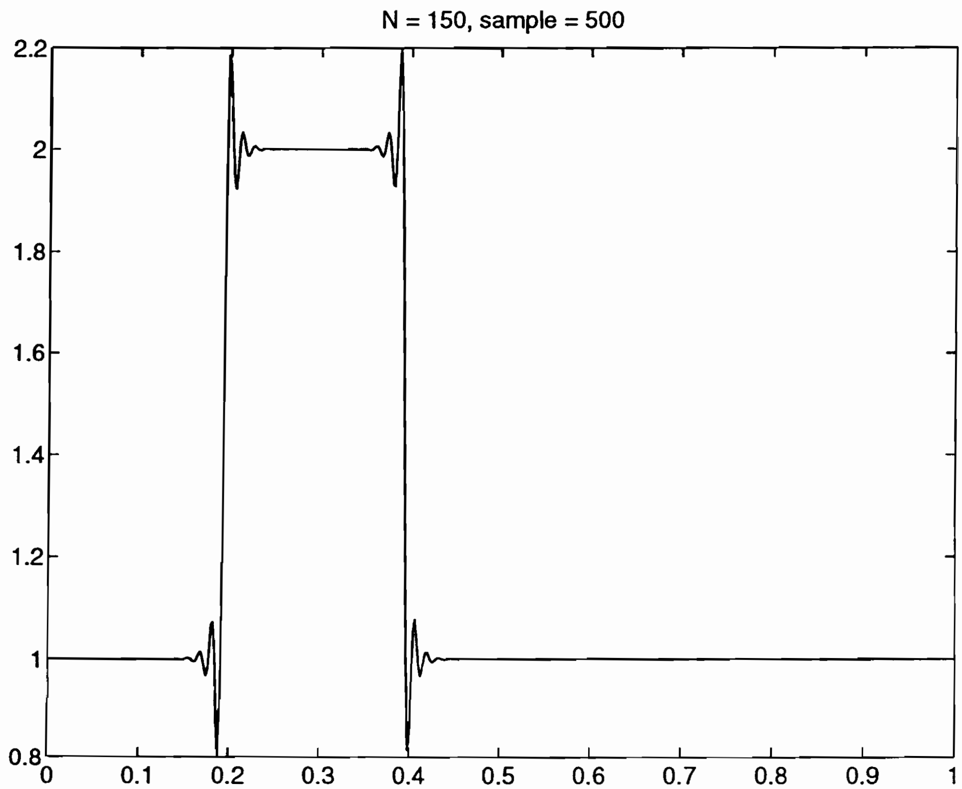

Figure 3.12: Approximation of step function with $M = 5, m = 1$ and $N = 150$

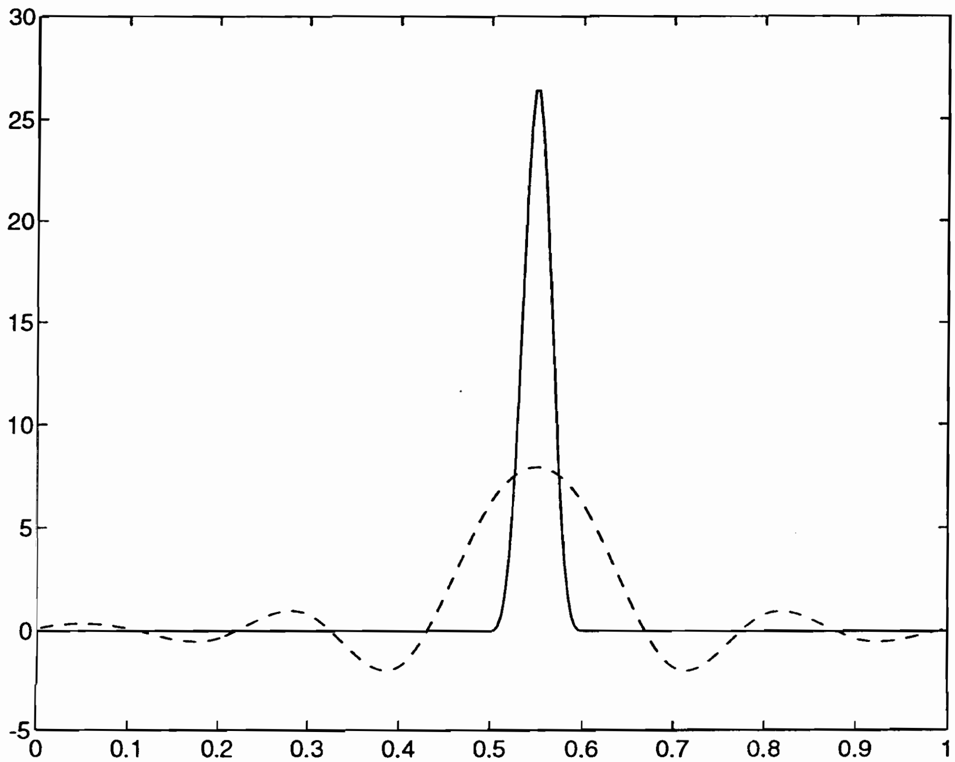

Figure 3.13: Approximation using $M = 4, m = 4$

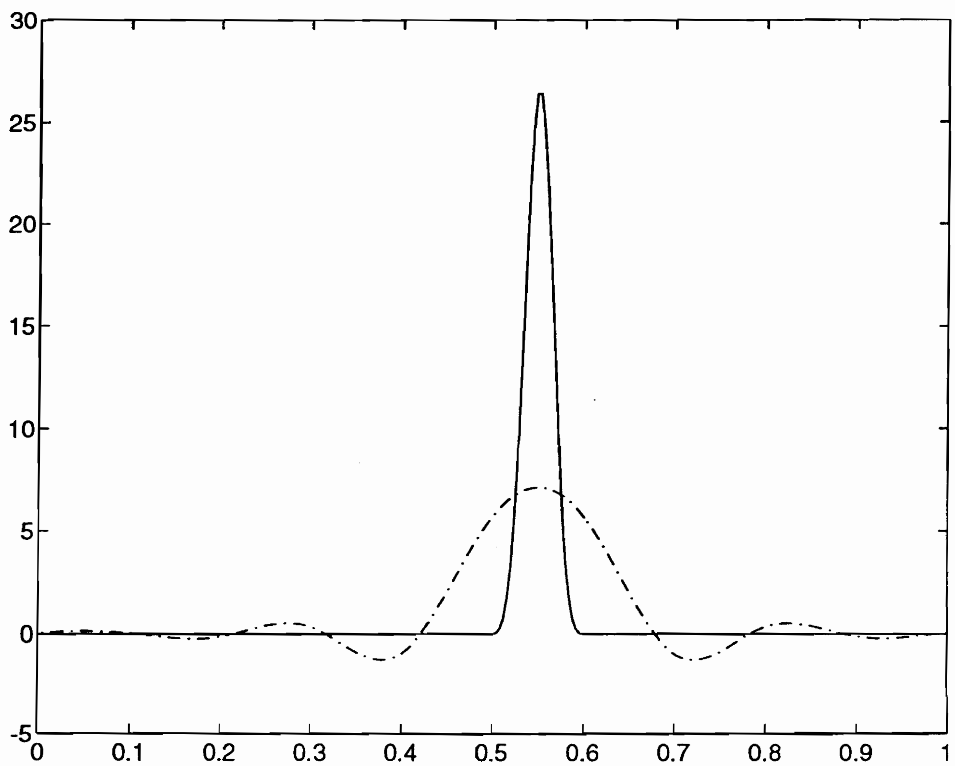

Figure 3.14: Approximation using $M = 4, m = 2$

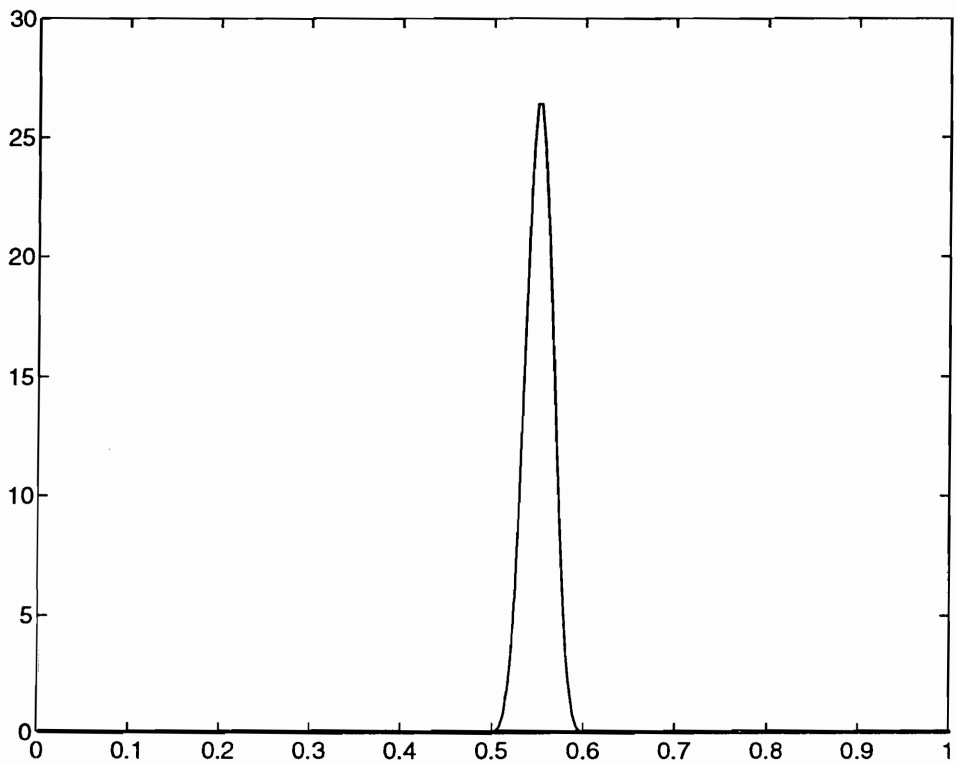

Figure 3.15: Approximation using $M = 4, m = 0$

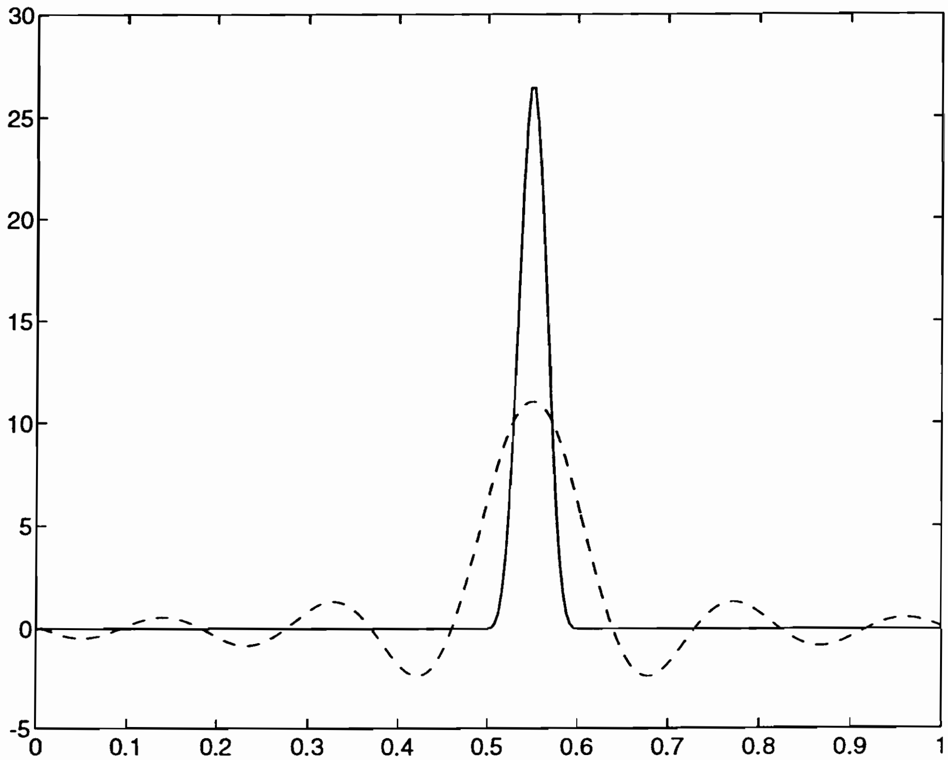

Figure 3.16: Approximation using $M = 5, m = 5$

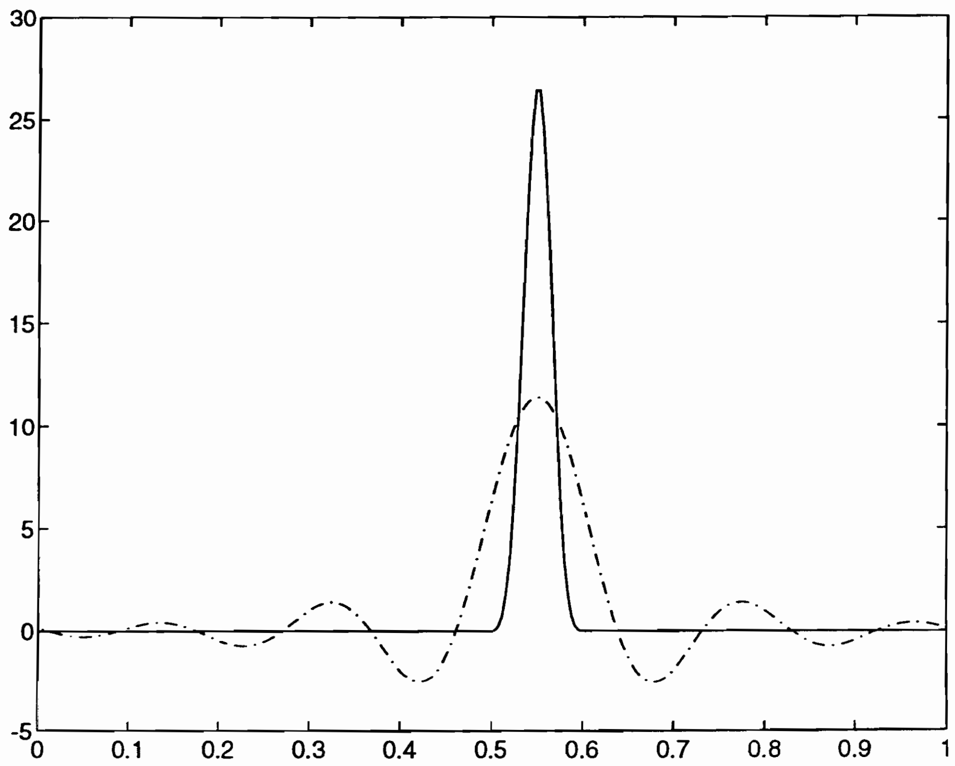

Figure 3.17: Approximation using $M = 5, m = 3$

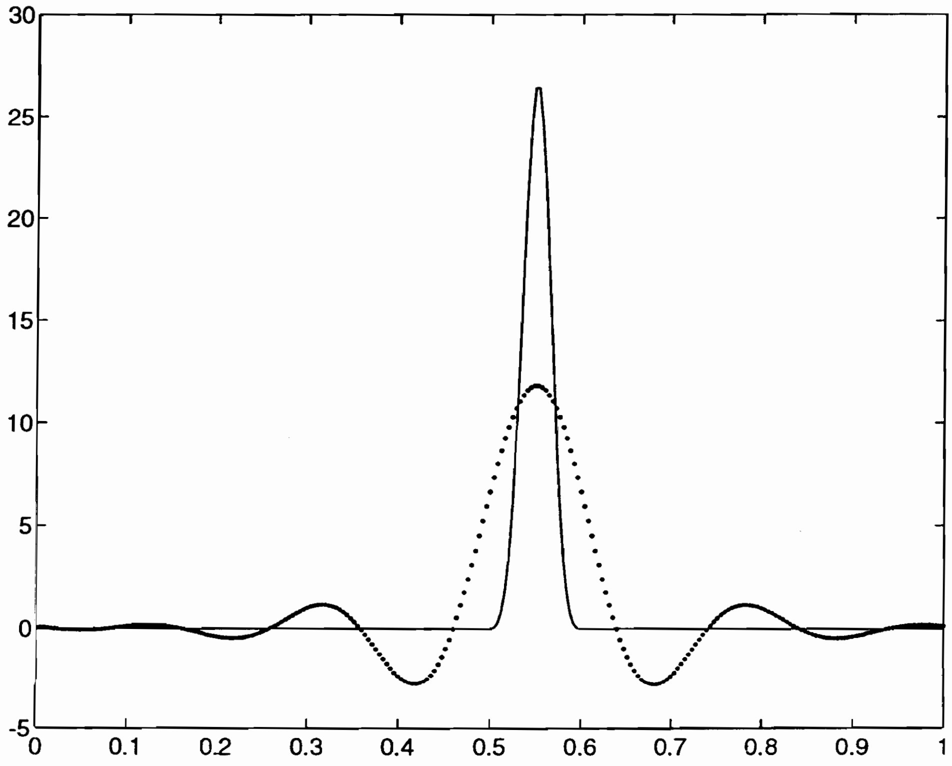

Figure 3.18: Approximation using $M = 5, m = 1$

Chapter 4

Dual Wavelets And Multiresolution Analysis

Let us denote by $S_{m,N}$, or just S_m when N is understood, the vector space of m-th order periodic splines on $[0,1]$, the point $x = 1$ being identified with $x = 0$ in the usual way. For $0 \leq m = M - \lambda$, where λ is a non-negative integer, we let $\phi_{m,k}$ and $\phi_{M,k}$ be the standard b-splines, both centered at x_k (thus the nodes are at the half intervals if $M - m$ is odd), $k = 1, 2, \ldots, N$. As usual, we identify $k = 0$ with $k = N$. We also denote by $\phi_j(x)$ the N dimensional column vector function whose k-th component is the b-spline $\phi_{j,k}(x)$.

We have seen in our earlier work that the matrix

$$\Phi_{M,m} = \int_0^1 \phi_m(x)\phi_M(x)^* \, dx$$

is always nonsingular. When $M - m = \lambda = 2\mu$ is even, we have also seen that this matrix is positive definite. We suspect that this is also true when $M - m$ is odd, but we have not yet proven it. We note also for the record that $\Phi_{M,m}$ is a circulant matrix.

The M, m approximant in $S_{M,N}$ to the periodic function $f(x)$ defined on $[0,1]$ is

$$f_{M,m}(x) = \phi_M(x)^* c, \qquad c = \begin{pmatrix} c^1 \\ c^2 \\ \cdot \\ \cdot \\ \cdot \\ c^N \end{pmatrix}.$$

The vector of coefficients c is determined by

$$\int_0^1 \phi_m(x)\phi_M(x)^*\, dx\; c = \int_0^1 \phi_m(x) f(x)\, dx.$$

That is

$$c = \Phi_{M,m}^{-1} \int_0^1 \phi_m(x) f(x)\, dx = \int_0^1 \psi_{M,m}(x) f(x)\, dx$$

where the components $\psi_{M,m,k}(x)$ of the N dimensional column vector function $\psi_{M,m}(x)$ constitute the *dual basis* in S_m for the original basis in S_M consisting of the components of $\phi_M(x)$. These enjoy the property of *biorthogonality* relative to the $\phi_{M,k}(x)$, expressed by the relation

$$\int_0^1 \psi_{M,m}(x)\phi_M(x)^*\, dx = I_N$$

where I_N is the $N \times N$ identity matrix. That is

$$\int_0^1 \psi_{M,m,j}(x)\phi_{M,k}(x)\, dx = \delta_{jk} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}.$$

When $m = M$ we would use the term *orthonormal* rather that the term biorthogonal.

We are familiar with the fact that the functions $\phi_{M,k}(x)$ are highly localized with respect to x (as localized as piecewise $(M-1)$-st degree polynomials can be without being zero). When the $\psi_{M,m,j}(x)$ are computed they can also be seen to possess a degree of localization – more pronounced as m gets smaller – but they are less localized than the $\phi_{M,k}(x)$. The $\psi_{M,m,j}(x)$ have a very characteristic oscillatory form and they decay exponentially, with exponent depending on N, as x moves away from their center-point x_j.

A compromise between the degree of localization of these dual basis functions can be realized in the following way if we know that the symmetric matrix $\Phi_{M,m}$ is positive definite (as we know when $M-m$ is even). Since $\Phi_{M,m}$ is symmetric and positive definite, so is its inverse $\Phi_{M,m}^{-1}$. For ease of notation, define $P = \Phi_{M,m}^{-1}$. It can be verified that P is also a circulant matrix. As a positive definite symmetric matrix, P has a positive definite square root matrix $P^{\frac{1}{2}}$ which is also a circulant matrix [15]. Since P commutes with $\Phi_{M,m}$, so does $P^{\frac{1}{2}}$. Thus from the relation $\Phi_{M,m}P = I$ and the symmetry of P we have

$$P^{\frac{1}{2}}\Phi_{M,m}P^{\frac{1}{2}} = (P^{\frac{1}{2}})^*\Phi_{M,m}P^{\frac{1}{2}} = I.$$

Since

$$I = P^{\frac{1}{2}}\int_0^1 \phi_m(x)\phi_M(x)^*\,dx P^{\frac{1}{2}} = \int_0^1 \sigma_{M,m}(x)\theta_{M,m}(x)^*\,dx$$

the component functions $\sigma_{M,m,j}(x), \theta_{M,m,k}(x)$ remain dual bases relative to each other in $L^2[0,1]$. The functions $\sigma_{M,m,j}(x)$ are functions in S_m and the functions $\theta_{M,m,k}(x)$ are functions in S_M. It will generally be seen that the $\sigma_{M,m,j}(x)$ are more localized than the $\psi_{M,m}(x)$ while the $\theta_{M,m}(x)$ are less localized than the $\phi_{M,k}(x)$. The reason for this is quite simple; multiplication of a vector ϕ by a circulant matrix A amounts to descrete convolution of the row vector a against ϕ. That is, we have $a * \phi$. In general this is a *de-localizing* or *blurring* process. If p is the basic row vector of P and q is the basic row vector of $P^{\frac{1}{2}}$, then $p = q * q$, from which it follows that p should be less localized than q. Thus we should not be surprised to find that the $\psi_{M,m,j}(x)$, which are translates of $p * \phi_m(x)$, are less localized than the $\sigma_{M,m,j}(x)$, which are translates of $q * \phi_m(x)$. In Daubechies' book the $\psi_{M,m,j}(x)$ and $\phi_{M,k}(x)$ are called biorthogonal *scaling functions* of Battle-Meyer type. These are translation invariant in the sense that $\psi_{M,m,j}(x + N^{-1}) = \psi_{M,m,j-1}(x)$ with a similar relation holding for the $\phi_{M,k}(x)$. The functions $\sigma_{M,m,j}(x)$ and $\theta_{M,m,k}(x)$ have this same property of translation invariance. We might want to call them *balanced dual scaling functions.* Biorthogonal sets of translation invariant scaling functions serve as bases for a dual *wavelet analysis*, or *multiresolution analysis*, of periodic functions.

In order to get to the multiresolution aspect of wavelent analysis, let us suppose that $N = 2^\nu$, where ν is a positive integer. Typically 2^ν is quite large; 1024, 2048, 4096, etc., are standard values used in practice. Correspondingly, let $\mathcal{R}_\nu = R^{2^\nu}$. The standard orthonormal *Haar basis* for $\mathcal{R}_\nu$ consists of the vectors $h_0, h_{10}, h_{20}, h_{21}$

where

$N^{\frac{1}{2}}h_0 =$ all ones

$N^{\frac{1}{2}}h_{10} = 2^{\nu-1}$ ones, followed by $2^{\nu-1}$ negative ones

$N^{\frac{1}{2}}h_{20} = 2^{\nu-2}$ ones, followed by $2^{\nu-2}$ negative ones, followed by $2^{\nu-1}$ zeros

$N^{\frac{1}{2}}h_{21} = 2^{\nu-1}$ zeros, followed by $2^{\nu-2}$ ones, followed by $2^{\nu-2}$ negative ones

$\ldots .$

In general, for $n = 3, \ldots, \nu$ and for $l = 0, 1, \ldots, 2^{n-1}$, h_{nl} consists of a block of $2^{\nu-n}$ ones, followed by a block of $2^{\nu-n}$ negative ones and starting at position $(l-1)2^{\nu-n+1}$ all other elements are zeros. The vector is then multiplied by the scaling factor $1/(2^{\nu-n+1})^{\frac{1}{2}}$.

Thus, except for the normalization factors, for $\nu = 3$ the vectors are

$$
\begin{array}{c}
(\ 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1\ ) \\
(\ 1 \quad 1 \quad 1 \quad 1 \quad -1 \quad -1 \quad -1 \quad -1\ ) \\
(\ 1 \quad 1 \quad -1 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0\ ) \\
(\ 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad -1 \quad -1\ ) \\
(\ 1 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\ ) \\
(\ 0 \quad 0 \quad 1 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0\ ) \\
(\ 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad -1 \quad 0 \quad 0\ ) \\
(\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad -1\ ).
\end{array}
$$

We will denote the components of h_{nl} by h_{nl}^k, $k = 0, 1, \ldots, (2^\nu - 1)$. These vectors clearly form an orthonormal basis for $\mathcal{R}_\nu$, the Haar basis.

Now consider the maps $H_{m\nu}$ and $H_{M\nu}$ from $\mathcal{R}_\nu$ to S_m and S_M, respectively,

$$
\begin{aligned}
H_{m\nu} : h \in \mathcal{R}_\nu &\rightarrow \textstyle\sum_{k=0}^{N-1} h^k \sigma_{M,m,k}; \\
H_{M\nu} : h \in \mathcal{R}_\nu &\rightarrow \textstyle\sum_{k=0}^{N-1} h^k \theta_{M,m,k}.
\end{aligned}
$$

From our construction it is clear that

$$(H_{m\nu}h, H_{M\nu}g)_{L^2[0,1]} = (h, g)_{\mathcal{R}_\nu},$$

or, with * denoting the adjoint of an operator,

$$(h, H^*_{m\nu}H_{M\nu}g)_{\mathcal{R}_\nu} = (h, g)_{\mathcal{R}_\nu},$$

showing that $H^*_{m\nu} = H^{-1}_{M\nu}$. When $m = M$, so that $\sigma_{M,m,k} = \theta_{M,m,k}$, this becomes $H^*_{M\nu} = H^{-1}_{M\nu}$, so that $H_{M\nu}$ is an *isometery* from $\mathcal{R}_\nu$ onto S_M. When $m = M$ this means that the orthonormal basis h_{nl} is carried into an orthonormal basis in S_M; otherwise, $H_{m\nu}$ and $H_{M\nu}$ carry the h_{nl} into dual biorthogonal bases in S_m and S_M respectively. These are *dual wavelet bases (orthonormal basis if* $m = M$) in S_m and S_M. The images of the h_{nl} are wavelets which are progressively more localized as n increases, and simply translate as l varies. They are convolutions of the scaling functions (sometimes called *mother wavelets*) with the Haar vectors. Just as the basic pattern of the Haar vectors is more or less invariant from one n level to another, the wavelet forms also change very little an n varies. As $N \to \infty$ and n stays bounded, the wavelets at different n levels are just rescalings and translations of each other.

More information on wavelets and their uses can be found in Ingrid Daubechies book [5] and in Yves Meyer's book on wavelets [9].

On the following pages can be found pictures of the Haar basis functions for the two cases $M = m = 4$ and $M = 4, m = 2$.

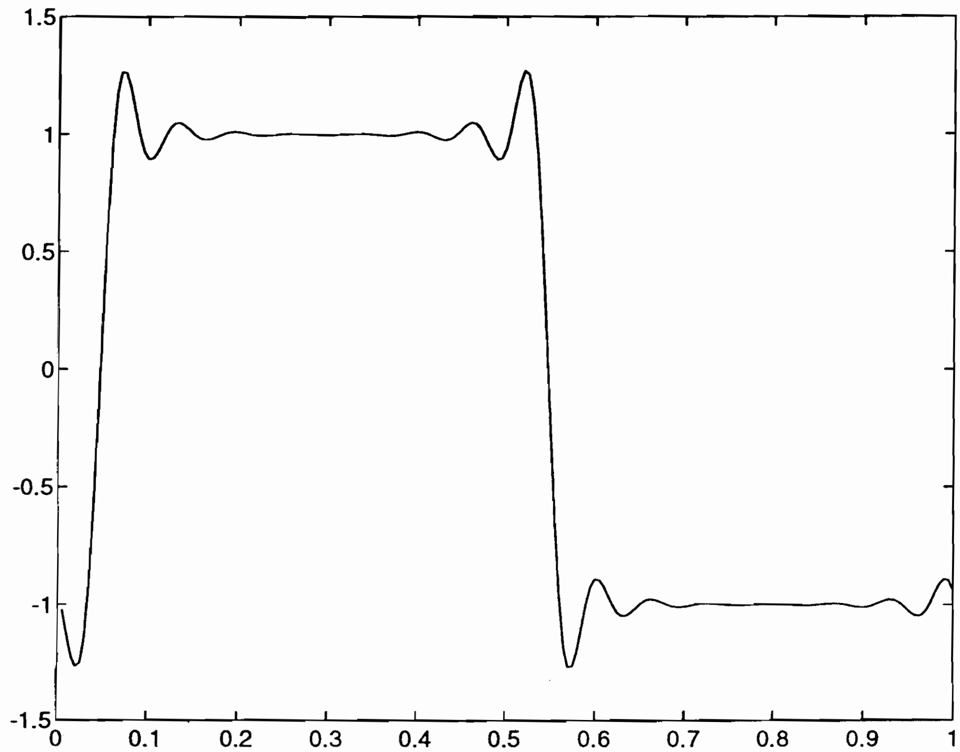

Figure 4.1: Haar basis function for h_{10}, $\sigma_{4,4} = \theta_{4,4}$, and $\nu = 5$

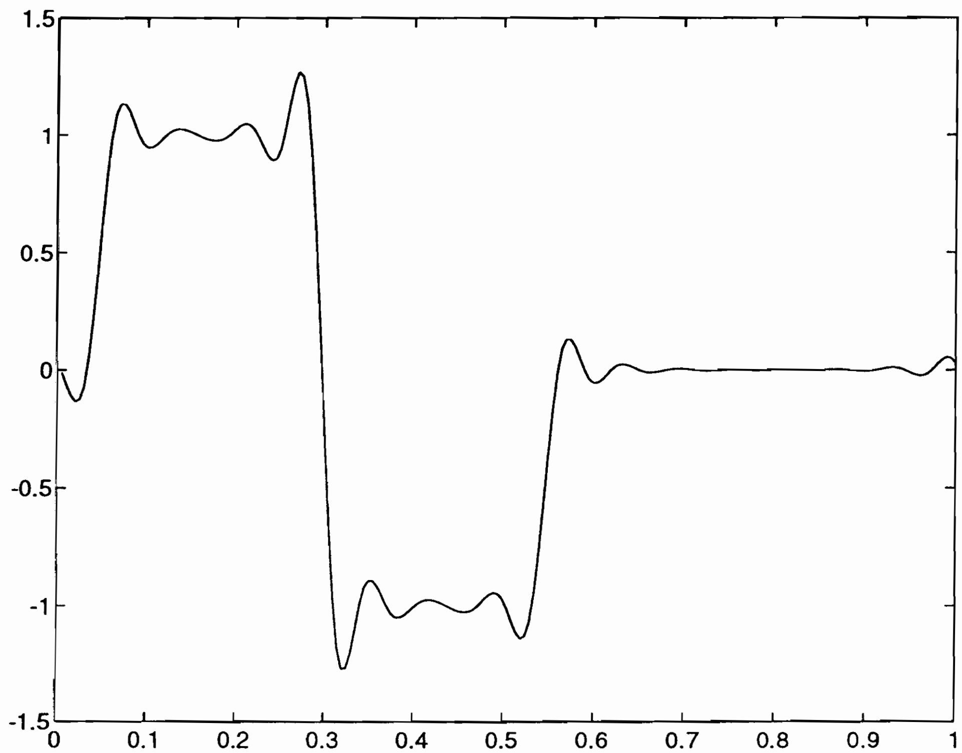

Figure 4.2: Haar basis function for h_{20}, $\sigma_{4,4} = \theta_{4,4}$, and $\nu = 5$

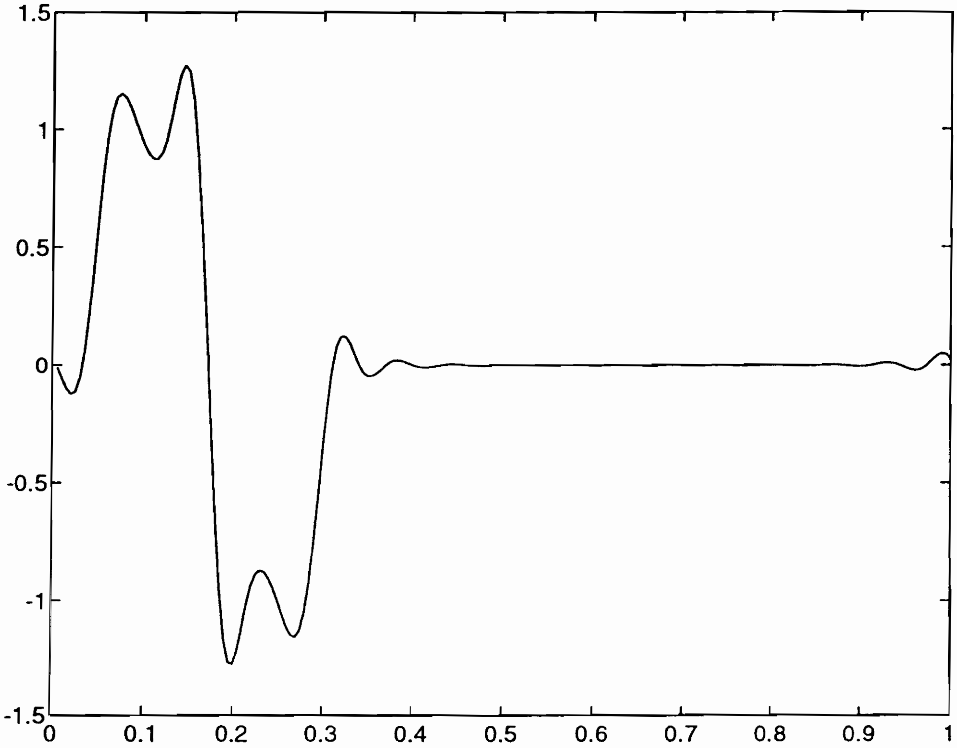

Figure 4.3: Haar basis function for h_{30}, $\sigma_{4,4} = \theta_{4,4}$, and $\nu = 5$

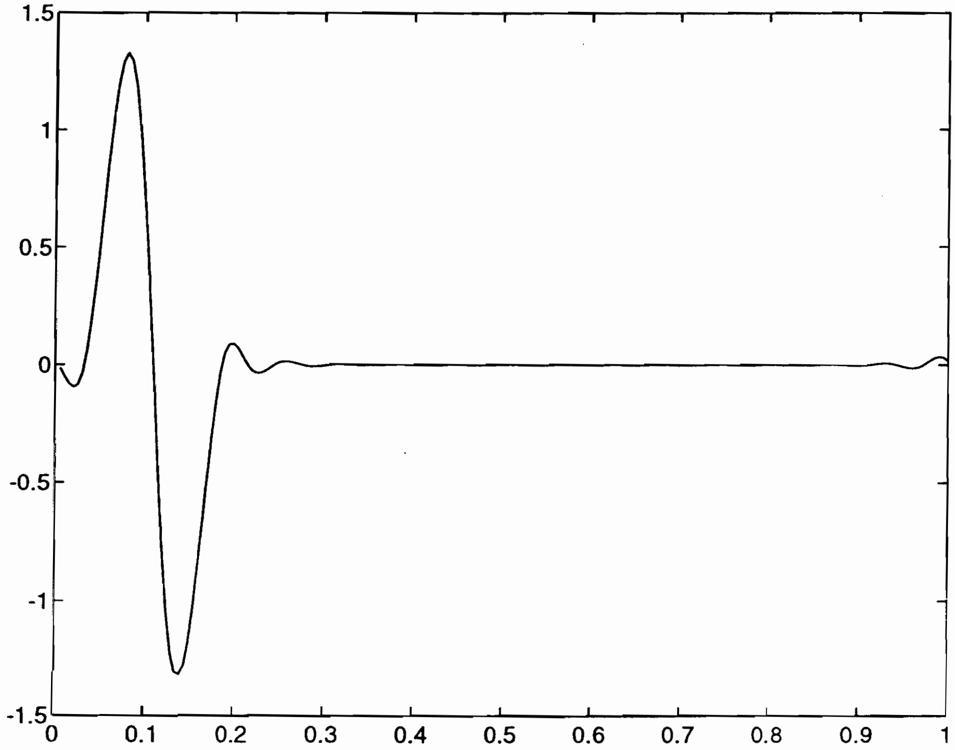

Figure 4.4: Haar basis function for h_{40}, $\sigma_{4,4} = \theta_{4,4}$, and $\nu = 5$

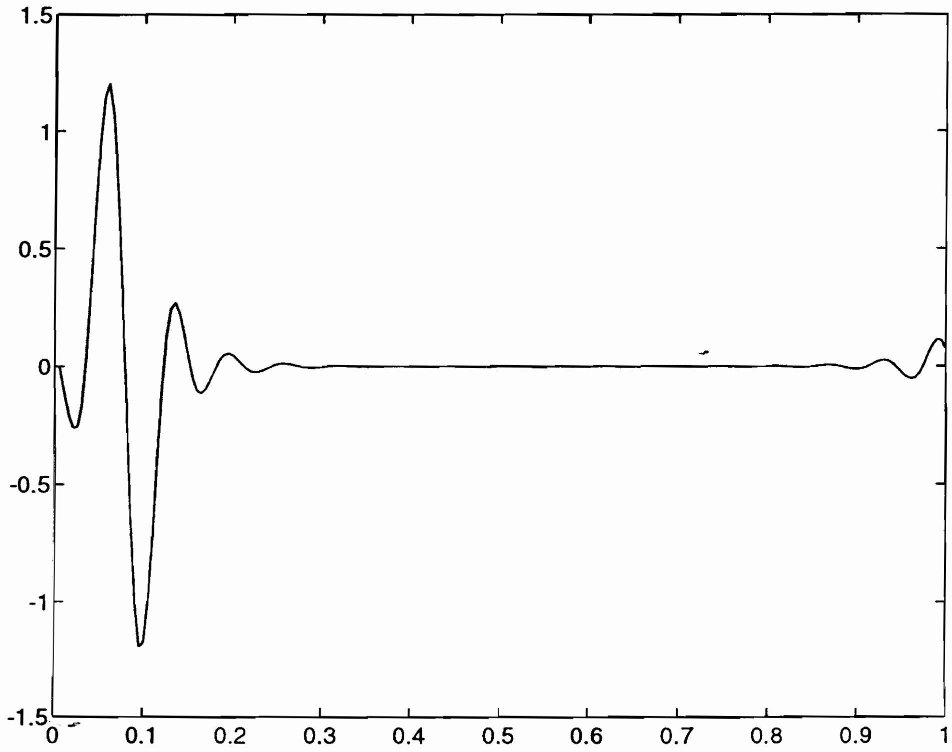

Figure 4.5: Haar basis function for h_{50}, $\sigma_{4,4} = \theta_{4,4}$. and $\nu = 5$

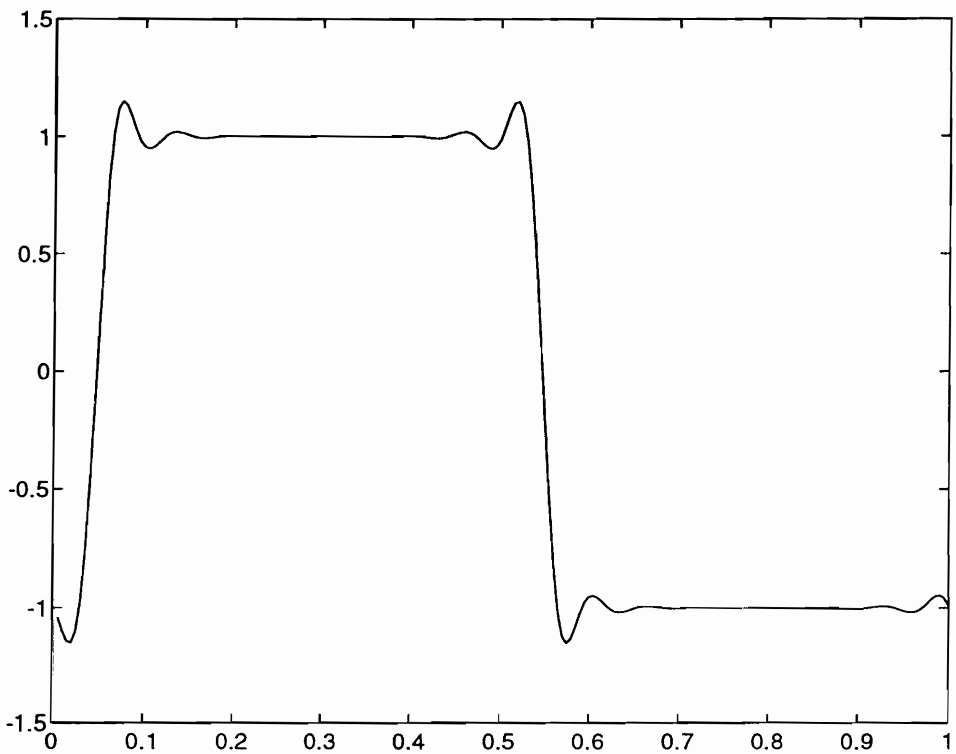

Figure 4.6: Haar basis function for h_{10}, $\theta_{4,2}$, and $\nu = 5$

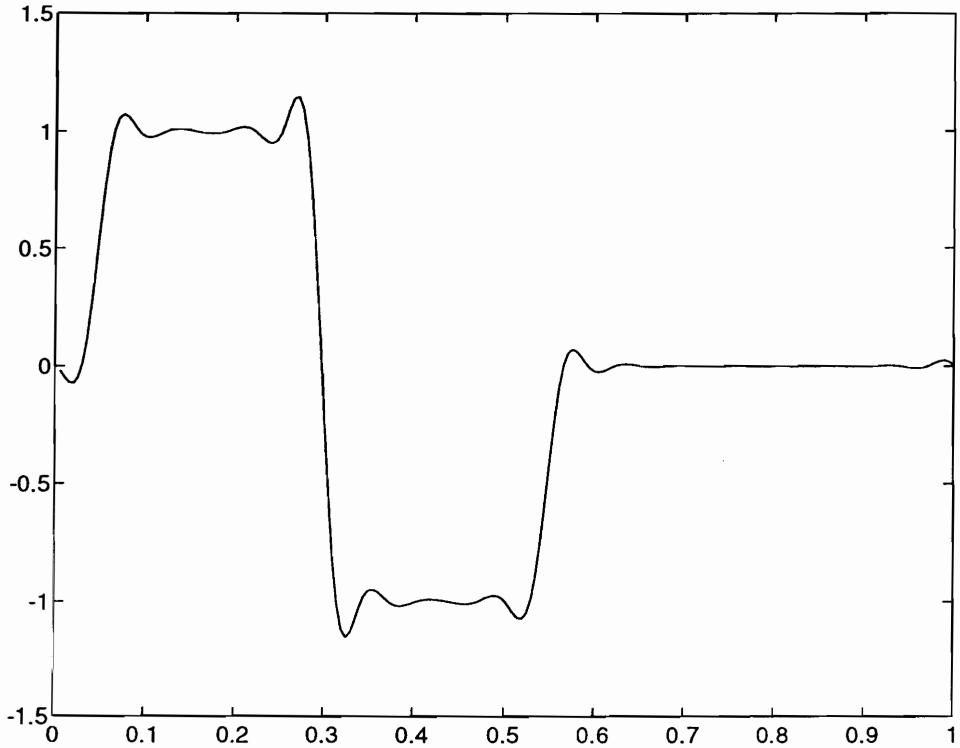

Figure 4.7: Haar basis function for h_{20}, $\theta_{4,2}$, and $\nu = 5$

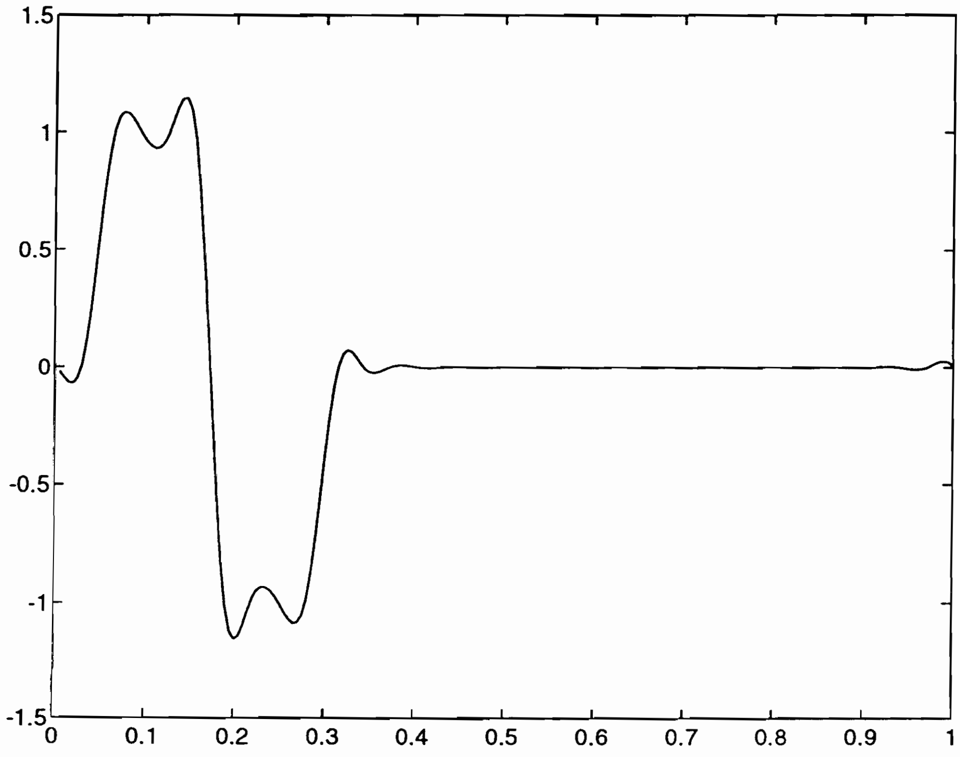

Figure 4.8: Haar basis function for h_{30}, $\theta_{4,2}$, and $\nu = 5$

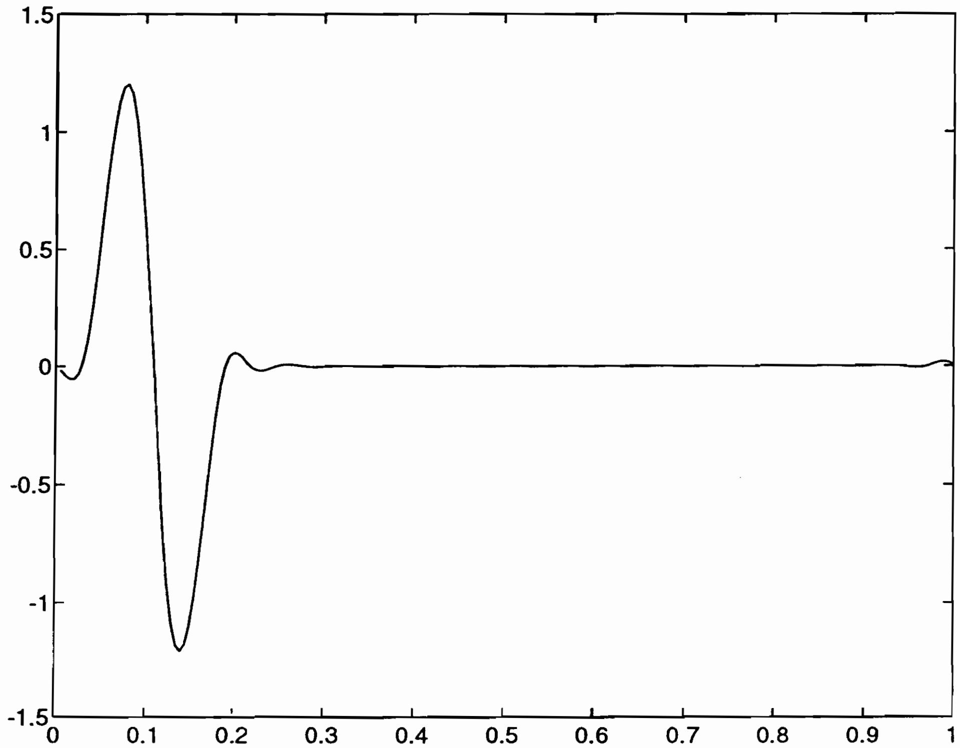

Figure 4.9: Haar basis function for h_{40}, $\theta_{4,2}$, and $\nu = 5$

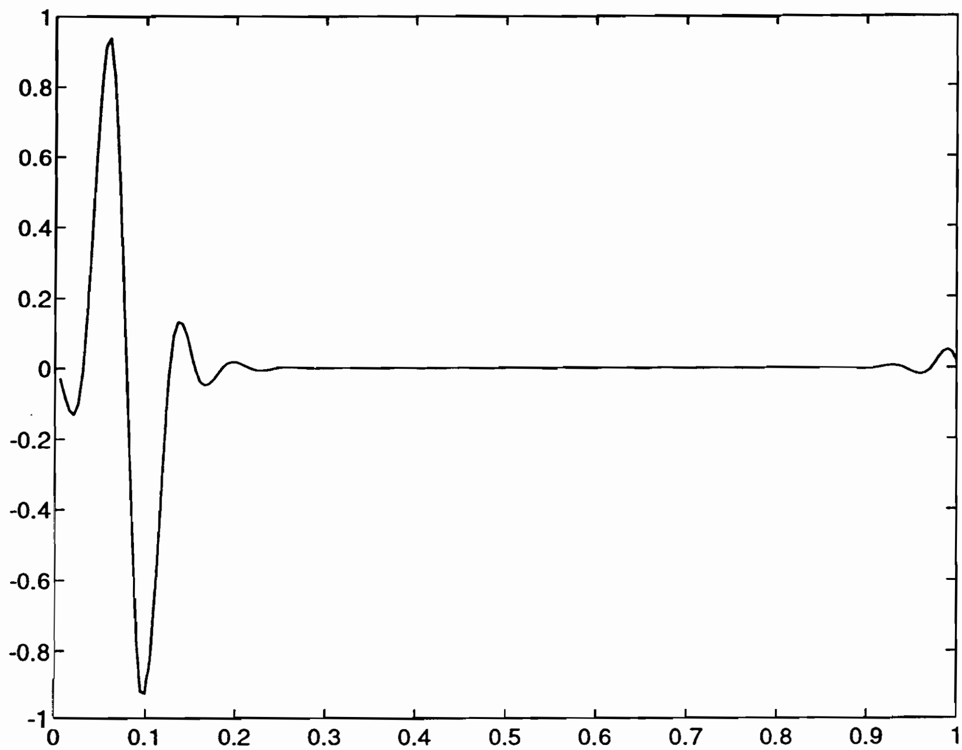

Figure 4.10: Haar basis function for h_{50}, $\theta_{4,2}$, and $\nu = 5$

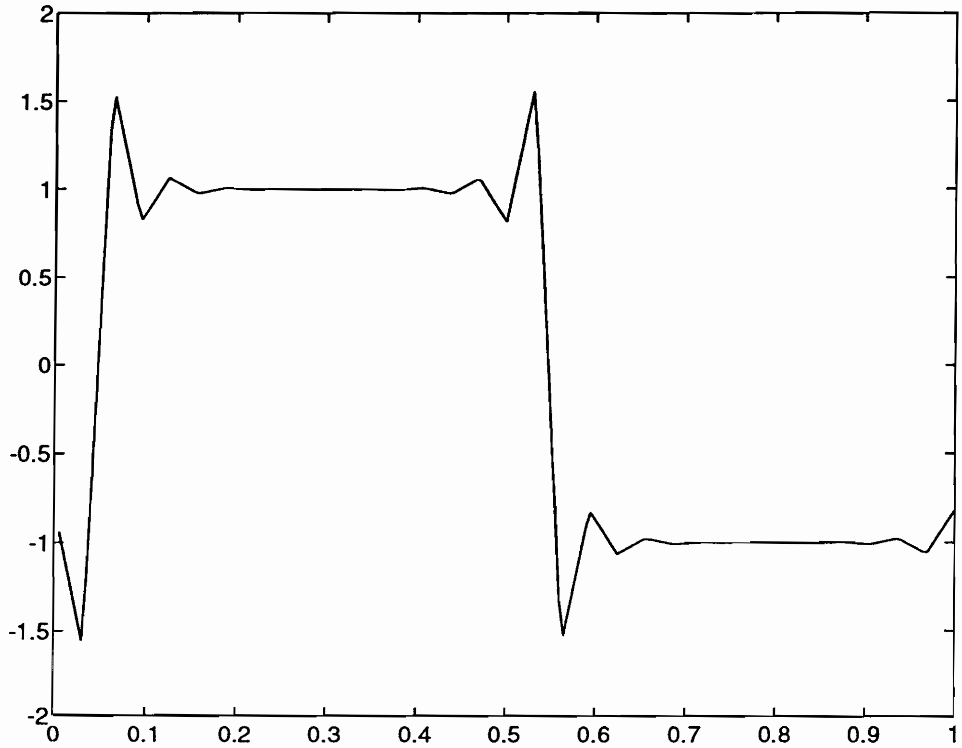

Figure 4.11: Haar basis function for h_{10}, $\sigma_{4,2}$, and $\nu = 5$

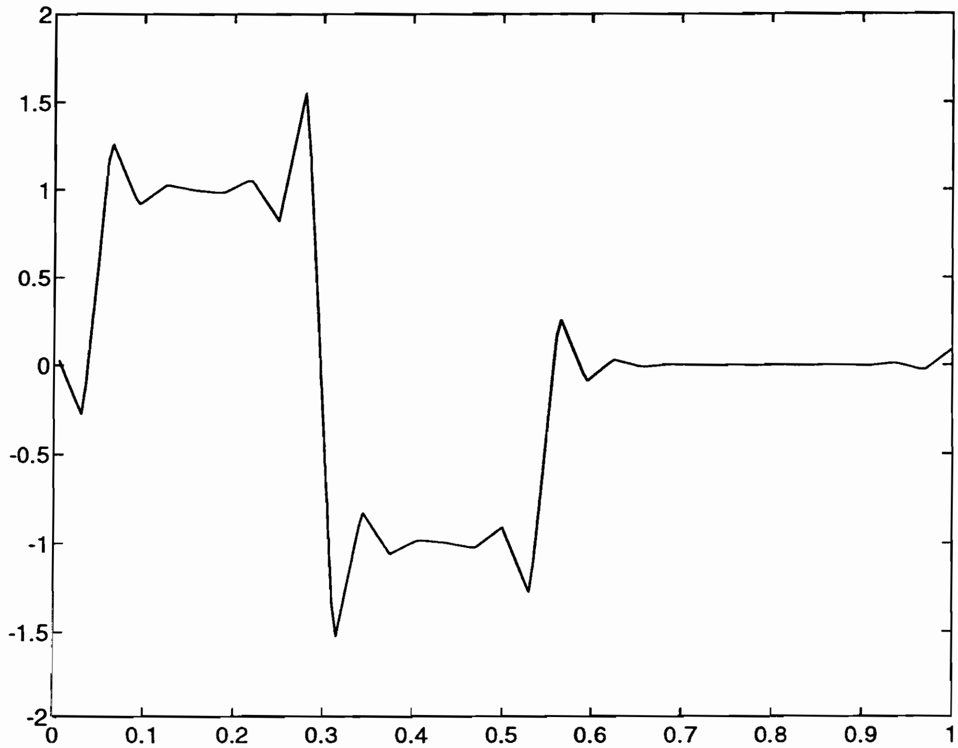

Figure 4.12: Haar basis function for h_{20}, $\sigma_{4,2}$, and $\nu = 5$

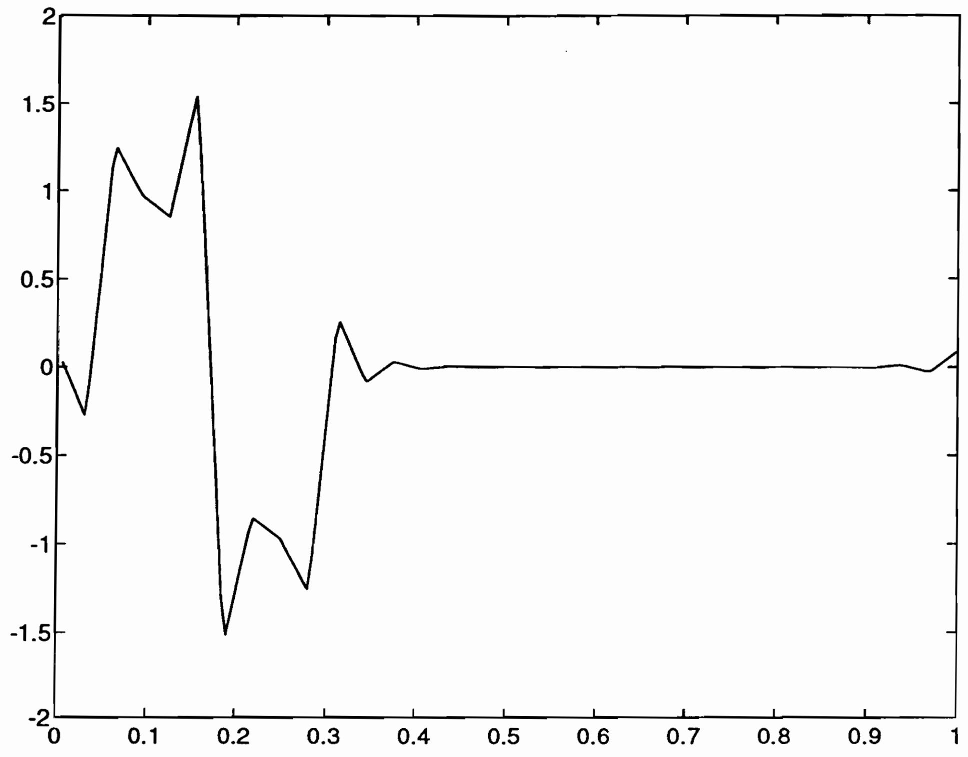

Figure 4.13: Haar basis function for h_{30}, $\sigma_{4,2}$, and $\nu = 5$

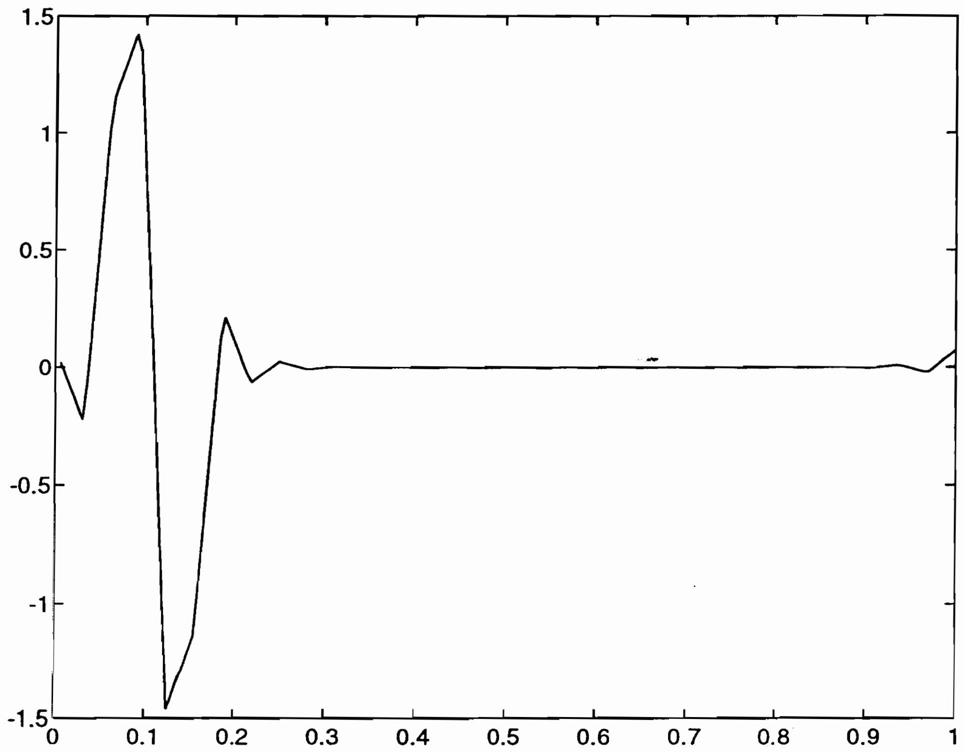

Figure 4.14: Haar basis function for h_{40}, $\sigma_{4,2}$, and $\nu = 5$

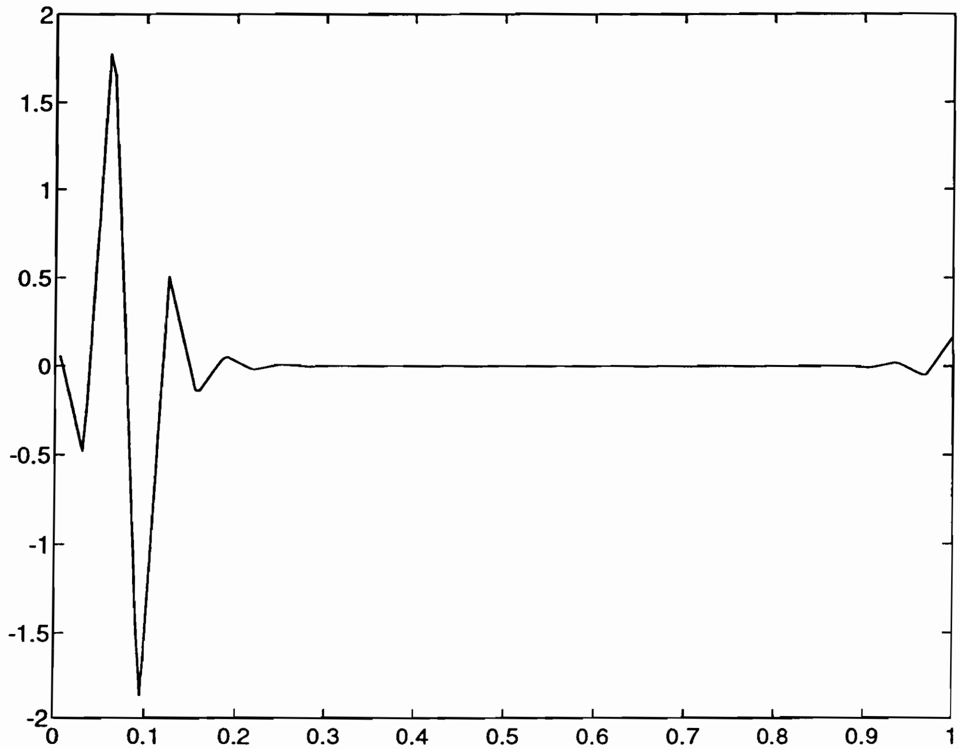

Figure 4.15: Haar basis function for h_{50}, $\sigma_{4,2}$, and $\nu = 5$

Chapter 5

Error Estimates

Now that we have seen some examples, let us explore the closeness of the approximation $f_{M,m,N}$ to the original function f. At the end of the chapter there are two graphs showing the relative closeness of the different approximation schemes for the second example in the last chapter. To mathematically determine the accuracy , we first wish to demonstrate that *the first integral relation* holds.

Theorem 5 *If* $m = M - 2\mu$ *where* $0 \leq m \leq M$ *and* $f \in C^{(\mu)}[0,1]$ *and periodic, then*

$$\int_I [f^{(\mu)}(x) - f^{(\mu)}_{M,m,N}(x)]^2 \, dx = \int_I [f^{(\mu)}(x)]^2 \, dx - \int_I [f^{(\mu)}_{M,m,N}(x)]^2 \, dx. \tag{5.1}$$

Equation (5.1) is referred to as *the first integral relation.*

Proof. For ease of notation let $S(x) = f_{M,m,N}(x)$. Then we first notice that

$$\begin{aligned} \int_I [f^{(\mu)}(x) - S^{(\mu)}(x)]^2 \, dx &= \int_I [f^{(\mu)}(x)]^2 \, dx - \int_I [S^{(\mu)}(x)]^2 \, dx \\ &\quad - 2\int_I S^{(\mu)}(x)[f^{(\mu)} - S^{(\mu)}(x)] \, dx. \end{aligned}$$

To show that the first integral relation is true, we must show that the last integral above equals zero. Integration by parts yields

$$\begin{aligned}\int_I S^{(\mu)}(x)[f^{(\mu)}(x) - S^{(\mu)}(x)]\,dx &= S^{(\mu)}(x)[f^{(\mu-1)}(x) - S^{(\mu-1)}(x)]\,\Big|_0^1 \\ &\quad - \int_I S^{(\mu+1)}(x)[f^{(\mu-1)}(x) - S^{(\mu-1)}(x)]\,dx.\end{aligned}$$

Due to the periodicity of $f(x)$, and hence $S(x)$, the first term in the above equation is zero. Thus

$$\int_I S^{(\mu)}(x)[f^{(\mu)}(x) - S^{(\mu)}(x)]\,dx = -\int_I S^{(\mu+1)}(x)[f^{(\mu-1)}(x) - S^{(\mu-1)}(x)]\,dx.$$

Integrating by parts again yields

$$\begin{aligned}\int_I S^{(\mu)}(x)[f^{(\mu)}(x) - S^{(\mu)}(x)]\,dx &= -S^{(\mu+1)}(x)[f^{(\mu-2)}(x) - S^{(\mu-2)}(x)]\,\Big|_0^1 \\ &\quad + \int_I S^{(\mu+2)}(x)[f^{(\mu-2)}(x) - S^{(\mu-2)}(x)]\,dx.\end{aligned}$$

Again, by periodicity of $f(x)$, and hence $S(x)$, we have that the first term in the last equation is zero. Thus

$$\int_I S^{(\mu)}(x)[f^{(\mu)}(x) - S^{(\mu)}(x)]\,dx = \int_I S^{(\mu+2)}(x)[f^{(\mu-2)}(x) - S^{(\mu-2)}(x)]\,dx.$$

Performing integration by parts a total of μ times, using the periodicity of $f(x)$ and $S(x)$ each time, yields

$$\int_I S^{(\mu)}(x)[f^{(\mu)}(x) - S^{(\mu)}(x)]\,dx = (-1)^{\mu}\int_I S^{(2\mu)}(x)[f(x) - S(x)]\,dx. \tag{5.2}$$

Now recall that by definition $S(x) = \sum \alpha_j \phi_{M,j}(x)$. Thus $S'(x) = \sum \alpha_j^1 \phi_{M-1,j}(x)$. This leads to $S''(x) = \sum \alpha_j^2 \phi_{M-2,j}(x)$. Hence, continuing, we have

$$S^{(2\mu)}(x) = \sum_{j=0}^{N-1} \gamma_j \phi_{M-2\mu,j}(x). \tag{5.3}$$

Now substituting $m = M-2\mu$ in (5.3), and then substituting the resulting expression into equation (5.2) we have

$$\int_I S^{(\mu)}(x)[f^{(\mu)}(x) - S^{(\mu)}(x)]\, dx = (-1)^\mu \sum_{j=0}^{N-1} \gamma_j \int_I \phi_{m,j}(x)[f(x) - S(x)]\, dx.$$

This last integral is zero due to the construction of the approximation $S(x)$. Therefore,

$$\int_I [f^{(\mu)}(x) - S^{(\mu)}(x)]^2\, dx = \int_I [f^{(\mu)}(x)]^2\, dx - \int_I [S^{(\mu)}(x)]^2\, dx$$

and the first integral relation is proven. ■

Theorem 6 *Again, let $S(x) = f_{M,m,N}(x)$. Then the function $S^{(\mu)}(x)$ is the least squares or Galerkin approximation to the function $f^{(\mu)}(x)$.*

Proof. By construction of the function $S(x)$ and using intgration by parts several times we have

$$\begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} = \begin{bmatrix} < \phi_{m,0}, f - S > \\ < \phi_{m,1}, f - S > \\ \cdot \\ \cdot \\ \cdot \\ < \phi_{m,N-1}, f - S > \end{bmatrix}$$

$$= \begin{bmatrix} \int_I \phi_{m,0}(x)[f(x) - S(x)]\,dx \\ \int_I \phi_{m,1}(x)[f(x) - S(x)]\,dx \\ \cdot \\ \cdot \\ \cdot \\ \int_I \phi_{m,N-1}(x)[f(x) - S(x)]\,dx \end{bmatrix}$$

$$= -N\mathcal{E} \begin{bmatrix} \int_I \phi_{m+1,0}(x)[f'(x) - S'(x)]\,dx \\ \int_I \phi_{m+1,1}(x)[f'(x) - S'(x)]\,dx \\ \cdot \\ \cdot \\ \cdot \\ \int_I \phi_{m+1,N-1}(x)[f'(x) - S'(x)]\,dx \end{bmatrix}$$

$$= N^2\mathcal{E}^2 \begin{bmatrix} \int_I \phi_{m+2,0}(x)[f''(x) - S''(x)]\,dx \\ \int_I \phi_{m+2,1}(x)[f''(x) - S''(x)]\,dx \\ \cdot \\ \cdot \\ \cdot \\ \int_I \phi_{m+2,N-1}(x)[f''(x) - S''(x)]\,dx \end{bmatrix}$$

$$= \ldots$$

$$= (-N)^\mu \mathcal{E}^\mu \begin{bmatrix} \int_I \phi_{m+\mu,0}(x)[f^{(\mu)}(x) - S^{(\mu)}(x)]\,dx \\ \int_I \phi_{m+\mu,1}(x)[f^{(\mu)}(x) - S^{(\mu)}(x)]\,dx \\ \cdot \\ \cdot \\ \cdot \\ \int_I \phi_{m+\mu,N-1}(x)[f^{(\mu)}(x) - S^{(\mu)}(x)]\,dx \end{bmatrix}.$$

Now, since $\mathcal{E}^\mu$ has rank $(N-1)$ and each row of $\mathcal{E}$ is orthogonal to the vector $\mathbf{v} = (1, 1, \ldots, 1)^T$ we have that for each k, $k = 0, \ldots, N-1$ the equation

$$\int_I \phi_{m+\mu,k}(x)[f^{(\mu)}(x) - S^{(\mu)}(x)]\,dx = c$$

where c is a constant independent of k. Thus, summing over k, we have

$$\sum_{k=0}^{N-1} \int_I \phi_{m+\mu,k}(x)[f^{(\mu)}(x) - S^{(\mu)}(x)]\, dx = \sum_{k=0}^{N-1} c = cN$$

Hence,

$$\begin{aligned} cN &= \int_I \sum_{k=0}^{N-1} \{\phi_{m+\mu,k}(x)[f^{(\mu)}(x) - S^{(\mu)}(x)]\}\, dx \\ &= \int_I [f^{(\mu)}(x) - S^{(\mu)}(x)] \sum_{k=0}^{N-1} \phi_{m+\mu,k}(x)\, dx \end{aligned}$$

By the partition of unity identity $\sum_{k=0}^{N-1} \phi_{m+\mu,k}(x) = 1$, as well as using the periodicity of $f(x)$ and $S(x)$, we have

$$\begin{aligned} cN &= \int_I [f^{(\mu)}(x) - S^{(\mu)}(x)]\, dx \\ &= [f^{(\mu-1)}(x) - S^{(\mu-1)}(x)] \,|_0^1 \\ &= 0. \end{aligned}$$

That is, $cN = 0$; hence, $c = 0$. Therefore, for $k = 0, \ldots, N-1$ we have

$$\int_I \phi_{m+\mu,k}(x)[f^{(\mu)}(x) - S^{(\mu)}(x)]\, dx = \, < \phi_{m+\mu}, f^{(\mu)} - S^{(\mu)} > \, = 0.$$

Lastly, since

$$S(x) = \sum_{j=0}^{N-1} \alpha_j \phi_{M,j}(x)$$

we have

$$S'(x) = \sum_{j=0}^{N-1} \alpha_j^1 \phi_{M-1,j}(x),$$

and

$$S''(x) = \sum_{j=0}^{N-1} \alpha_j^2 \phi_{M-2,j}(x).$$

Repeating this μ times we see that

$$S^{(\mu)}(x) = \sum_{j=0}^{N-1} \alpha_j^{\mu} \phi_{M-\mu,j}(x).$$

Now, recall that $M - \mu = m + \mu$. Finally, we have

$$< \phi_{m+\mu,k}, f^{(\mu)} - S^{(\mu)} > = 0$$

where

$$S^{(\mu)}(x) = \sum_{j=0}^{N-1} \beta_j \phi_{m+\mu,j}(x).$$

Hence, by definition $S^{(\mu)}(x)$ is the least squares or Galerkin approximation [11] of $f^{(\mu)}(x)$ and our proof is concluded. ∎

This result is suggested by deBoor page 67 [6].

Theorem 7 *For $j = 0, 1, \ldots, \mu - 1$ we have*

$$\| f^{(j)}(x) - S^{(j)}(x) \| \leq 2^{\mu-j} \| f^{(\mu)}(x) - S^{(\mu)}(x) \| .$$

Here we are using the L_2 norm

$$\| q(x) \| = \{ \int_I [q(x)]^2 \, dx \}^{\frac{1}{2}}.$$

Proof. Let $E(x) = f(x) - S(x)$. Then, since $f(x)$ and $S(x)$ are both periodic with period 1, $E(x)$ is also periodic with period 1. Hence, $E(0) = E(1)$. Therefore, by Rolle's Theorem there exists a $c_1 \in [0,1]$ such that $E'(c_1) = 0$. That is, for some value $c_1 \in [0,1]$ we have $f'(c_1) - S'(c_1)$.

Since $E(x)$ is of period 1 and differentible, we have that $E'(x)$ is also of period 1. Thus we have $E'(0) = E'(1)$. Again, by Rolle's Theorem there exists a $c_2 \in [0,1]$ such that $E''(c_2) = 0$. That is, for some value $c_2 \in [0,1]$ we have $f''(c_2) - S''(c_2) = 0$.

In general, for $j = 1 \ldots, M-2$ we have that $E^{(j)}(x)$ is of period 1; hence, $E^{(j)}(0) = E^{(j)}(1)$. Therefore, once again Rolle's Theorem yields the existence of a value $c_j \in [0,1]$ such that $E^{(j)}(c_j) = f^{(j)}(c_j) - S^{(j)}(c_j) = 0$.

Next, let t be an arbitrary, but fixed, value in $[0,1]$. We first consider the fact that

$$\frac{d}{dx}[f^{(\mu-1)}(x) - S^{(\mu-1)}(x)]^2 = 2[f^{(\mu-1)}(x) - S^{(\mu-1)}(x)]\,[f^{(\mu)}(x) - S^{(\mu)}(x)].$$

Integrating from $c_{\mu-1}$ to t we have

$$\begin{aligned}
[f^{(\mu-1)}(t) - S^{(\mu-1)}(t)]^2 &= \int_{c_{\mu-1}}^{t} \frac{d}{dx}[f^{(\mu-1)}(x) - S^{(\mu-1)}(x)]^2\,dx \\
&= 2\int_{c_{\mu-1}}^{t} [f^{(\mu-1)}(x) - S^{(\mu-1)}(x)]\,[f^{(\mu)}(x) - S^{(\mu)}(x)]\,dx \\
&\leq 2\{\int_{c_{\mu-1}}^{t} [f^{(\mu-1)}(x) - S^{(\mu-1)}(x)]^2\,dx\}^{\frac{1}{2}} \\
&\qquad \{\int_{c_{\mu-1}}^{t} [f^{(\mu)}(x) - S^{(\mu)}(x)]^2\,dx\}^{\frac{1}{2}} \\
&\leq 2 \parallel f^{(\mu-1)}(x) - S^{(\mu-1)}(x) \parallel \parallel f^{(\mu)}(x) - S^{(\mu)}(x) \parallel .
\end{aligned}$$

The next to last inequality is a result of Schwarz's inequatity.

Integrating again, this time from $t = 0$ to $t = 1$ we have

$$\begin{aligned}
\int_I [f^{(\mu-1)}(t) - S^{(\mu-1)}(t)]^2 \, dt &= \| f^{(\mu-1)}(x) - S^{(\mu-1)}(x) \|^2 \\
&\leq 2 \int_I \| f^{(\mu-1)}(x) - S^{(\mu-1)}(x) \| \, \| f^{(\mu)}(x) - S^{(\mu)}(x) \| \, dt \\
&\leq 2 \| f^{(\mu-1)}(x) - S^{(\mu-1)}(x) \| \, \| f^{(\mu)}(x) - S^{(\mu)}(x) \| \, .
\end{aligned}$$

Dividing through by $\| f^{(\mu-1)}(x) - S^{(\mu-1)}(x) \|$ yields

$$\| f^{(\mu-1)}(x) - S^{(\mu-1)}(x) \| \leq 2 \| f^{(\mu)}(x) - S^{(\mu)}(x) \| \, .$$

Through a similar argument, it can be shown that

$$\| f^{(\mu-2)}(x) - S^{(\mu-2)}(x) \| \leq 2 \| f^{(\mu-1)}(x) - S^{(\mu-1)}(x) \| \, .$$

Combining these last two results yields

$$\| f^{(\mu-2)}(x) - S^{(\mu-2)}(x) \| \leq 4 \| f^{(\mu)}(x) - S^{(\mu)}(x) \| \, .$$

Continuing in the same fashion, we have for $j = 0, \ldots, \mu - 1$

$$\| f^{(j)}(x) - S^{(j)}(x) \| \leq 2^{\mu-j} \| f^{(\mu)}(x) - S^{(\mu)}(x) \| \, .$$

■

Recall from Theorem 5 that $S^{(\mu)}(x)$ is the least squares approximation to $f^{(\mu)}(x)$. Thus, by definition of least squares

$$\| f^{(\mu)}(x) - S^{(\mu)}(x) \| = \min \| f^{(\mu)}(x) - s(x) \|$$

where $s(x) = \sum_{j=0}^{N-1} \gamma_j \phi_{m+\mu}(x)$ is any spline approximation to $f^{(\mu)}(x)$ [11].

For ease of notation, let $g(x) = f^{(\mu)}(x)$.

If $m + \mu$ is even, which corresponds to odd degree splines $\phi_{m+\mu}(x)$, let $s(x)$ be the approximation to $g(x)$ obtained via point interpolation. If $m + \mu$ is odd, which corresponds to even degree splines $\phi_{m+\mu}(x)$, let $s(x)$ be the approximation to $g(x)$ obtained via integral interpolation. Then we have the following result.

Theorem 8 *If $m + \mu = 2K$ for some K, or $m + \mu = 2K + 1$ for some K, then we have the following form of the first integral relation with $k = K + 1$*

$$\int_I [g^{(k)}(x) - s^{(k)}(x)]^2 \, dx = \int_I [g^{(k)}(x)]^2 \, dx - \int_I [s^{(k)}(x)]^2 \, dx. \tag{5.4}$$

Proof. We first notice that

$$\begin{aligned} \int_I [g^{(k)}(x) - s^{(k)}(x)]^2 \, dx &= \int_I [g^{(k)}(x)]^2 \, dx - \int_I [s^{(k)}(x)]^2 \, dx \\ &\quad - 2 \int_I s^{(k)}(x)[g^{(k)} - s^{(k)}(x)] \, dx. \end{aligned}$$

Integrating this last integral by parts yields

$$\begin{aligned} \int_I s^{(k)}(x)[g^{(k)}(x) - s^{(k)}(x)] \, dx &= s^{(k)}(x)[g^{(k-1)}(x) - s^{(k-1)}(x)] \mid_0^1 \\ &\quad - \int_I s^{(k+1)}(x)[g^{(k-1)}(x) - s^{(k-1)}(x)] \, dx. \end{aligned}$$

Due to the periodicity of $g(x)$, and hence $s(x)$, the first term on the right hand side in the above equation is zero. Thus

$$\int_I s^{(k)}(x)[g^{(k)}(x) - s^{(k)}(x)]\, dx = -\int_I s^{(k+1)}(x)[g^{(k-1)}(x) - s^{(k-1)}(x)]\, dx.$$

Integrating by parts again yields

$$\begin{aligned}\int_I s^{(k)}(x)[g^{(k)}(x) - s^{(k)}(x)]\, dx &= -s^{(k+1)}(x)[g^{(k-2)}(x) - s^{(k-2)}(x)]\,|_0^1 \\ &\quad + \int_I s^{(k+2)}(x)[g^{(k-2)}(x) - s^{(k-2)}(x)]\, dx.\end{aligned}$$

Again, by periodicity of $g(x)$, and hence $s(x)$, we have that the first term on the right hand side in the last equation is zero. Thus

$$\int_I s^{(k)}(x)[g^{(k)}(x) - s^{(k)}(x)]\, dx = \int_I s^{(k+2)}(x)[g^{(k-2)}(x) - s^{(k-2)}(x)]\, dx.$$

Performing integration by parts a total of $k-1$ times, using the periodicity of $g(x)$ and $s(x)$ each time, yields

$$\int_I s^{(k)}(x)[g^{(k)}(x) - s^{(k)}(x)]\, dx = (-1)^{k-1}\int_I s^{(2k-1)}(x)[g'(x) - s'(x)]\, dx.$$

Since $s(x)$ is a piecewise polynomial of order $m+\mu$, each polynomial piece has degree $m+\mu-1$. Thus $s^{(2k-1)}(x)$ has polynomial pieces of degree $m+\mu-1-(2k-1) = m+\mu-2k$.

Now, if $m + \mu = 2k$, that is $m + \mu$ is even, then $s^{(2k-1)}(x)$ has degree $m + \mu - 2k = 2k - 2k = 0$. Hence, $s^{(2k-1)}(x)$ is constant on each subinterval. Thus

$$\begin{aligned}
\int_I s^{(2k-1)}(x)[g'(x) - s'(x)]\, dx &= \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} c_j[g'(x) - s'(x)]\, dx \\
&= \sum_{j=0}^{N-1} c_j \int_{x_j}^{x_{j+1}} [g'(x) - s'(x)]\, dx \\
&= \sum_{j=0}^{N-1} c_j[g(x) - s(x)] \,|_0^1 .
\end{aligned}$$

This last expression is zero due to the fact that, since $m + \mu$ is even, we are letting $s(x)$ be the point interpolate of $g(x)$. Therefore, if $m + \mu$ is even, then equality (5.4) holds.

If $m + \mu = 2k + 1$ then $s^{(2k-1)}(x)$ has polynomial pieces of degree $m + \mu - 2k = 2k + 1 - 2k = 1$. Thus, $s^{(2k-1)}(x)$ is piecewise linear. Since $s^{(2k-1)}(x)$ is continuous we can perform integration by parts once more to get

$$\int_I s^{(2k-1)}(x)[g'(x) - s'(x)]\, dx = s^{(2k-1)}(x)[g(x) - s(x)] \,|_0^1 - \int_I s^{(2k)}(x)[f(x) - s(x)]\, dx.$$

The first quantity on the right hand side is zero due to the periodicity of $g(x)$ and $s(x)$. Also, since $s^{(2k)}(x)$ is piecewise constant, we have

$$\begin{aligned}
\int_I s^{(2k)}(x)[g(x) - s(x)]\, dx &= \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} c_j[g(x) - s(x)]\, dx \\
&= \sum_{j=0}^{N-1} c_j \int_{x_j}^{x_{j+1}} [g(x) - s(x)]\, dx.
\end{aligned}$$

Since we are assuming $m + \mu$ is odd, in this case we let $s(x)$ be the integral interpolate of $g(x)$. Hence, this last quantity is zero.

Therefore, if $m + \mu = 2K$ or $m + \mu = 2K + 1$ and we define $k = K = 1$, we have equation (5.4) holding. This concludes the proof.

■

We use equation (5.4) to prove the next two theorems.

Theorem 9 *Let $g(x)$ and $s(x)$ be as above and $m+\mu = 2K+1$, then with $k = K+1$*

$$\| g(x) - s(x) \| \leq [2(k+1)h]^k \| g^{(k)}(x) \| .$$

Proof. Let $F(x)$ be defined as $F(x) = \int_0^x [g(t) - s(t)]dt$ then $F(x_i) = 0$ for $i = 0, 1, \ldots, N$. By Rolle's Theorem there are values $c_{1,i} \in [x_{i-1}, x_i]$, $i = 1, 2, \ldots, N$ such that for each i $F'(c_{1,i}) = g(c_{1,i}) - s(c_{1,i}) = 0$. Again, by Rolle's Theorem, there are values $c_{2,i} \in [c_{1,i-1}, c_{1,i}]$, $i = 2, 3, \ldots, N$ such that $F''(c_{2,i}) = g'(c_{2,i}) - s'(c_{2,i}) = 0$ for each i. Continuing the process, for $j = 1, 2, \ldots, k$ there exist values $c_{j,i} \in [c_{j-1,i-1}, c_{j-1,i}]$ such that $F^{(j)}(c_{j,i}) = g^{(j-1)}(c_{j,i}) - s^{(j-1)}(c_{j,i}) = 0$.

Next, let t be an arbitrary, but fixed, value in $[0, 1]$. Then for some i we have $t \in [c_{k,i-1}, c_{k,i}]$. We first consider the fact that

$$\frac{d}{dx}[g^{(k-1)}(x) - s^{(k-1)}(x)]^2 = 2[g^{(k-1)}(x) - s^{(k-1)}(x)][g^{(k)}(x) - s^{(k)}(x)].$$

Integrating from $c_{k,i-1}$ to t we have

$$\begin{aligned} [g^{(k-1)}(t) - s^{(k-1)}(t)]^2 &= \int_{c_{k,i-1}}^t \frac{d}{dx}[g^{(k-1)}(x) - s^{(k-1)}(x)]^2 dx \\ &= 2\int_{c_{k,i-1}}^t [g^{(k-1)}(x) - s^{(k-1)}(x)][g^{(k)}(x) - s^{(k)}(x)]dx \end{aligned}$$

$$\leq \quad 2\{\int_{c_{k,i-1}}^{t} [g^{(k-1)}(x) - s^{(k-1)}(x)]^2 dx\}^{\frac{1}{2}}$$
$$\{\int_{c_{k,i-1}}^{t} [g^{(k)}(x) - s^{(k)}(x)]^2 dx\}^{\frac{1}{2}}.$$

This last statement is a result of Schwarz's inequality. Since both integrands on the right hand side are positive we have the following:

$$\begin{aligned}
[g^{(k-1)}(t) - s^{(k-1)}(t)]^2 \quad &\leq \quad 2\{\int_{c_{k,i-1}}^{c_{k,i}} [g^{(k-1)}(x) - s^{(k-1)}(x)]^2 dx\}^{\frac{1}{2}} \\
&\qquad \{\int_{c_{k,i-1}}^{c_{k,i}} [g^{(k)}(x) - s^{(k)}(x)]^2 dx\}^{\frac{1}{2}}
\end{aligned}$$

for $t \in [c_{k,i-1}, c_{k,i}]$.

Integrating again, we have

$$\begin{aligned}
\int_{c_{k,i}}^{t} [g^{(k-1)}(x) - s^{(k-1)}(x)]^2 dx \quad &\leq \quad 2\int_{c_{k,i-1}}^{t} \{\int_{c_{k,i-1}}^{c_{k,i}} [g^{(k-1)}(x) - s^{(k-1)}(x)]^2 dx\}^{\frac{1}{2}} \\
&\qquad \{\int_{c_{k,i-1}}^{c_{k,i}} [g^{(k)}(x) - s^{(k)}(x)]^2 dx\}^{\frac{1}{2}} dt \\
&\leq \quad [2(k+1)h]\, \{\int_{c_{k,i-1}}^{c_{k,i}} [g^{(k-1)}(x) - s^{(k-1)}(x)]^2 dx\}^{\frac{1}{2}} \\
&\qquad \{\int_{c_{k,i-1}}^{c_{k,i}} [g^{(k)}(x) - s^{(k)}(x)]^2 dx\}^{\frac{1}{2}}
\end{aligned}$$

Taking the maximum of the left hand side yields

$$\begin{aligned}
\int_{c_{k,i-1}}^{c_{k,i}} [g^{(k-1)}(x) - s^{(k-1)}(x)]^2 dx \quad &\leq \quad [2(k+1)h]\, \{\int_{c_{k,i-1}}^{c_{k,i}} [g^{(k-1)}(x) - s^{(k-1)}(x)]^2 dx\}^{\frac{1}{2}} \\
&\qquad \{\int_{c_{k,i-1}}^{c_{k,i}} [g^{(k)}(x) - s^{(k)}(x)]^2 dx\}^{\frac{1}{2}}.
\end{aligned}$$

Dividing through by $\{\int_{c_{k,i-1}}^{c_{k,i}}[g^{(k-1)}(x)-s^{(k-1)}(x)]^2dx\}^{\frac{1}{2}}$ yields

$$\{\int_{c_{k,i-1}}^{c_{k,i}}[g^{(k-1)}(x)-s^{(k-1)}(x)]^2dx\}^{\frac{1}{2}} \;\le\; [2(k+1)h]\;\{\int_{c_{k,i-1}}^{c_{k,i}}[g^{(k)}(x)-s^{(k)}(x)]^2dx\}^{\frac{1}{2}}.$$

Now, squaring both sides gives

$$\int_{c_{k,i-1}}^{c_{k,i}}[g^{(k-1)}(x)-s^{(k-1)}(x)]^2dx \;\le\; [2(k+1)h]^2\int_{c_{k,i-1}}^{c_{k,i}}[g^{(k)}(x)-s^{(k)}(x)]^2dx.$$

Summing this last inequality over i yields

$$\int_I[g^{(k-1)}(x)-s^{(k-1)}(x)]^2dx \;\le\; [2(k+1)h]^2\int_I[g^{(k)}(x)-s^{(k)}(x)]^2dx.$$

By taking the squareroot of both sides the following inequality results

$$\| g^{(k-1)}(x)-s^{(k-1)}(x) \| \;\le\; [2(k+1)h]\;\| g^{(k)}(x)-s^{(k)}(x) \| .$$

Through a similar argument, it can be shown that

$$\| g^{(k-2)}(x)-s^{(k-2)}(x) \| \;\le\; [2(k+1)h]\;\| g^{(k-1)}(x)-s^{(k-1)}(x) \| .$$

Thus we have

$$\| g^{(k-2)}(x)-s^{(k-2)}(x) \| \;\le\; [2(k+1)h]^2\;\;\| g^{(k)}(x)-s^{(k)}(x) \|$$

Continuing in this fashion yields

$$\begin{aligned}\| g(x)-s(x) \| &\le [2(k+1)h]^k\;\| g^{(k)}(x)-s^{(k)}(x) \| \\ &\le [2(k+1)h]^k\;\| g^{(k)}(x) \| .\end{aligned}$$

Our proof in now complete. ■

Now that we have an error estimate for the case $m + \mu$ odd, consider the case $m + \mu$ even. For this case we have a similar theorem.

Theorem 10 *Let $g(x)$ and $s(x)$ be as above and $m + \mu = 2K$, then with $k = K + 1$*

$$\| g(x) - s(x) \| \leq [2(k+1)h]^k \| g^{(k)}(x) \| .$$

Proof. In this case, let $E(x)$ be defined as $E(x) = g(x) - s(x)$, then $E(x_i) = 0$ for $i = 0, 1, \ldots, N$. By Rolle's Theorem there are values $c_{1,i} \in [x_{i-1}, x_i]$, $i = 1, 2, \ldots, N$ such that for each i $E'(c_{1,i}) = g'(c_{1,i}) - s'(c_{1,i}) = 0$. Again, by Rolle's Theorem, there are $c_{2,i} \in [c_{1,i-1}, c_{1,i}]$, $i = 2, 3, \ldots, N$ such that $E''(c_{2,i}) = g''(c_{2,i}) - s''(c_{2,i}) = 0$ for some i. Continuing the process, for $j = 1, 2, \ldots, k$ there exist $c_{j,i} \in [c_{j-1,i}, c_{j-1,i+1}]$ such that $E^{(j)}(c_{j,i}) = g^{(j)}(c_{j,i}) - s^{(j)}(c_{j,i}) = 0$.

The rest of the proof is identical to the proof of the previous theorem. Hence, our proof is now complete. ■

Therefore, since $g(x) = f^{(\mu)}(x)$ and $s(x) = S^{(\mu)}(x)$ we have

$$\| f^{(\mu)}(x) - S^{(\mu)}(x) \| \leq [2(k+1)h]^k \; \| f^{(\mu+k)}(x) \|$$

where K is such that $m + \mu = 2K + 1$ or $m + \mu = 2K$ depending on whether $m + \mu$ is odd or even, respectively, and $k = K + 1$.

Thus we have

Theorem 11 *Let $f(x)$ be a $\mu+k$ differentiable function and $S(x) = \sum_{j=0}^{N-1} \alpha_j \phi_{M,j}(x)$ be the approximation to $f(x)$ obtained by requiring $< \phi_m, f - S > \; = \; 0$, where $m = M - 2\mu$ and K is such that $m + \mu = 2K + 1$ or $m + \mu = 2K$ depending on whether $m + \mu$ is odd or even respectively and $k = K + 1$. Then we have that for $j = 0, 1, \ldots, \mu - 1$*

$$\| f^{(j)}(x) - S^{(j)}(x) \| \; \leq \; 2^{\mu+k-j} \, (k+1)^k \, h^k \; \| f^{(\mu+k)}(x) \| \, .$$

Proof. The proof follows directly from the previous theorem. ■

This is rather a rough estimate. One reason for this is the estimate used for the least square approximation of $\| f^{(\mu)} - S^{(\mu)} \|$. However, if M is even and $m = 0$, this estimate is comparible to the known estimates for odd degree splines, at least in the case $j = 0$. However, the derivative estimates are not as tight as those already known for odd degree splines.

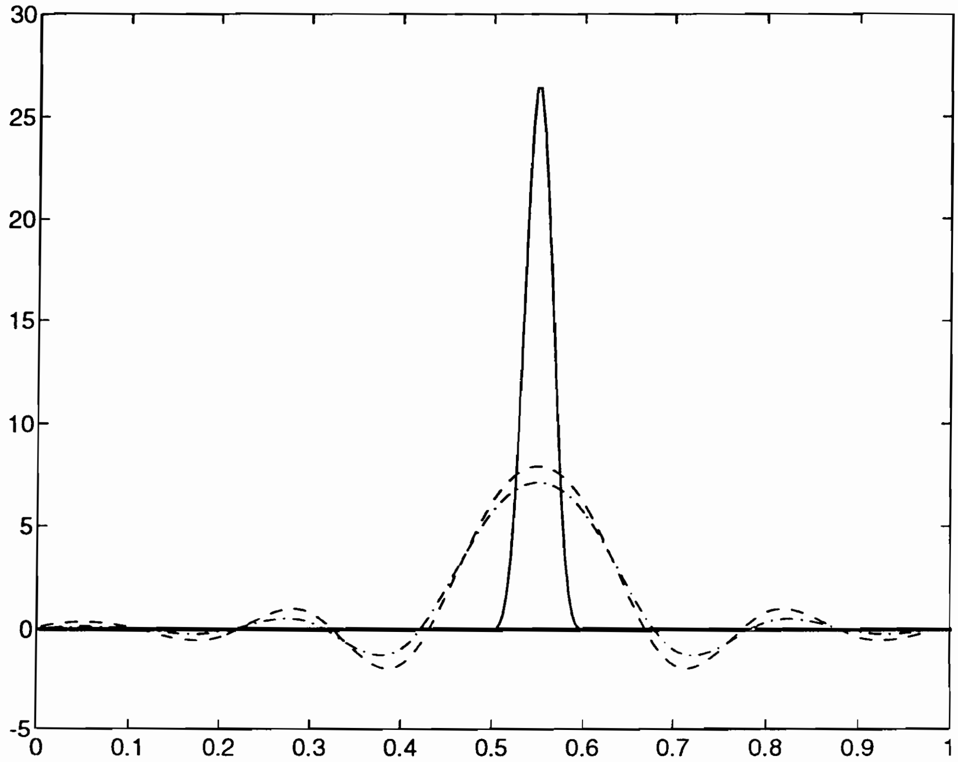

Figure 5.1: Approximation comparison for $M = 4, m = 0, 2, 4$

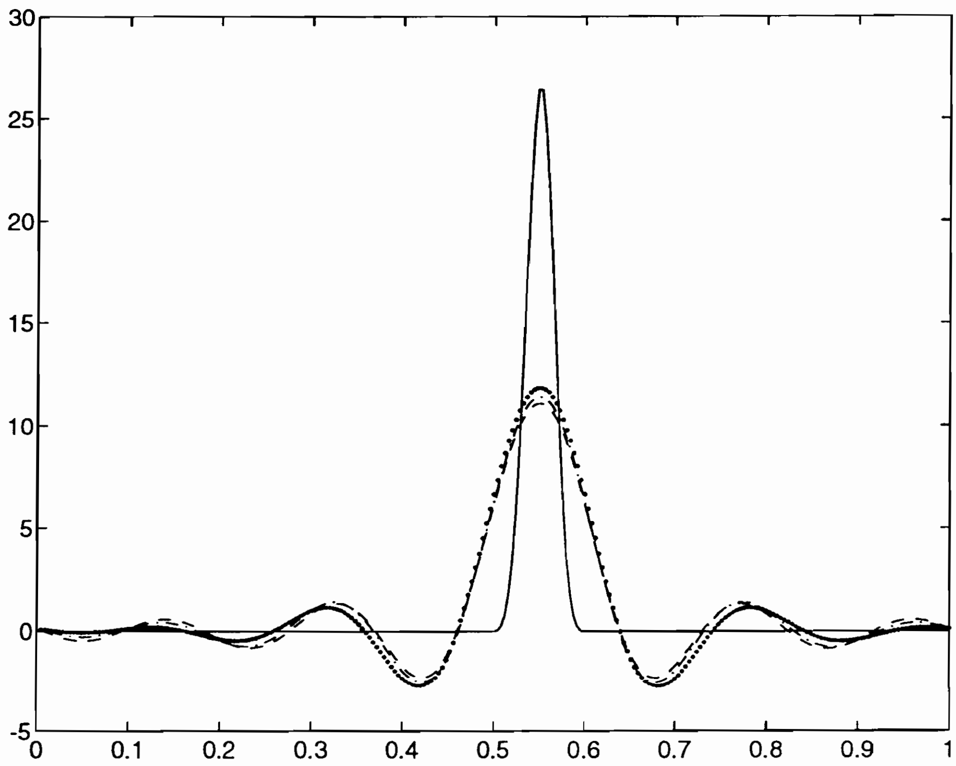

Figure 5.2: Approximation comparison for $M = 5, m = 1, 3, 5$

Chapter 6

Appendix

6.1 Splines Of Order Less Than Six

Recall from chapter 1 that the recursion formula to create the k-th spline of order M on the interval $I = [0, 1)$ is

$$\phi_{M+1,k}(x) = N \int_{x_{k-1}}^{x} [\phi_{M,k}(s) - \phi_{M,k+1}(s)]\, ds. \tag{6.1}$$

Also, recall that $\phi_{0,k}(x)$ is defined to be the Dirac measure δ_k with support $\{x_k\}$. Thus, since we know $\phi_{0,k}(x)$ we can create all the spline basis functions from the recursion formula (6.1). In this chapter, we show that

$$\phi_{1,k}(x) = \begin{cases} N & x \in [x_k, x_{k+1}) \\ 0 & \text{elsewhere.} \end{cases}$$

$$\phi_{2,k}(x) = \begin{cases} N^2(x - x_k) & x \in [x_k, x_{k+1}) \\ N - N^2(x - x_{k+1}) & x \in [x_{k+1}, x_{k+2}] \\ 0 & \text{elsewhere.} \end{cases}$$

$$\phi_{3,k}(x) = \begin{cases} \frac{N^3}{2}(x-x_k)^2 & x \in [x_k, x_{k+1}) \\ \frac{N}{2} + N^2(x-x_{k+1}) - N^3(x-x_{k+1})^2 & x \in [x_{k+1}, x_{k+2}) \\ \frac{N}{2} - N^2(x-x_{k+2}) + \frac{N^3}{2}(x-x_{k+2})^2 & x \in [x_{k+2}, x_{k+3}] \\ 0 & \text{elsewhere.} \end{cases}$$

$$\phi_{4,k}(x) = \begin{cases} \frac{N^4}{6}(x-x_k)^3 & x \in [x_k, x_{k+1}) \\ \frac{N}{6} + \frac{N^2}{2}(x-x_{k+1}) + \frac{N^3}{2}(x-x_{k+1})^2 - \frac{N^4}{2}(x-x_{k+1})^3 & x \in [x_{k+1}, x_{k+2}) \\ \frac{2N}{3} - N^3(x-x_{k+2})^2 + \frac{N^4}{2}(x-x_{k+2})^3 & x \in [x_{k+2}, x_{k+3}) \\ \frac{N}{6} - \frac{N^2}{2}(x-x_{k+3}) + \frac{N^3}{2}(x-x_{k+3})^2 - \frac{N^4}{6}(x-x_{k+3})^3 & x \in [x_{k+3}, x_{k+4}] \\ 0 & \text{elsewhere.} \end{cases}$$

and $\phi_{5,k}(x) =$

$$\begin{cases} \frac{N^5}{24}(x-x_k)^4 \\ \frac{N}{24} + \frac{N^2}{6}(x-x_{k+1}) + \frac{N^3}{4}(x-x_{k+1})^2 + \frac{N^4}{6}(x-x_{k+1})^3 - \frac{N^5}{6}(x-x_{k+1})^4 \\ \frac{11N}{24} + \frac{N^2}{2}(x-x_{k+2}) - \frac{N^3}{4}(x-x_{k+2})^2 - \frac{N^4}{2}(x-x_{k+2})^3 + \frac{N^5}{4}(x-x_{k+2})^4 \\ \frac{11N}{24} - \frac{N^2}{2}(x-x_{k+3}) - \frac{N^3}{4}(x-x_{k+3})^2 + \frac{N^4}{2}(x-x_{k+3})^3 - \frac{N^5}{6}(x-x_{k+3})^4 \\ \frac{N}{24} - \frac{N^2}{6}(x-x_{k+4}) + \frac{N^3}{4}(x-x_{k+4})^2 - \frac{N^4}{6}(x-x_{k+4})^3 + \frac{N^5}{24}(x-x_{k+4})^4 \\ 0 \end{cases}$$

for x in the intervals $[x_k, x_{k+1})$, $[x_{k+1}, x_{k+2})$, $[x_{k+2}, x_{k+3})$, $[x_{k+3}, x_{k+4})$, $[x_{k+4}, x_{k+5}]$ and elsewhere respectively.

As we saw in chapter 1

$$\begin{aligned} o_{1,k}(x) &= N\int_{x_{k-1}}^{x} [\phi_{0,k}(s) - \phi_{0,k+1}(s)]\, ds \\ &= \begin{cases} N & x \in [x_k, x_{k+1}) \\ 0 & \text{elsewhere.} \end{cases} \end{aligned}$$

Now that we have $\phi_{1,k}(x)$ we can determine $\phi_{2,k}(x)$;

$$\phi_{2,k}(x) = N \int_{x_{k-1}}^{x} [\phi_{1,k}(s) - \phi_{1,k+1}(s)] \, ds.$$

Since the support of $\phi_{2,k}(x)$ is the interval $[x_k, x_{k+2}]$, if $x \in [x_k, x_{k+1})$ we have

$$\phi_{2,k}(x) = N \int_{x_k}^{x} [N - 0] \, ds = N^2(x - x_k).$$

If $x \in [x_{k+1}, x_{k+2}]$ we have

$$\begin{aligned}
\phi_{2,k}(x) &= N \int_{x_k}^{x_{k+1}} [\phi_{1,k}(s) - \phi_{1,k+1}(s)] \, ds + N \int_{x_{k+1}}^{x} [\phi_{1,k}(s) - \phi_{1,k+1}(s)] \, ds \\
&= N^2(x_{k+1} - x_k) + N \int_{x_{k+1}}^{x} [0 - N] \, ds \\
&= N - N^2(x - x_{k+1}).
\end{aligned}$$

This last equatity comes from the fact that we are using the regular partition of the interval $[0, 1)$ with $x_k = \frac{k}{N}$ for $k = 0, 1, \ldots, N-1$. Combining the cases for $x \in [x_k, x_{k+1})$ and $x \in [x_{k+1}, x_{k+2}]$ we have

$$\phi_{2,k}(x) = \begin{cases} N^2(x - x_k) & x \in [x_k, x_{k-1}) \\ N - N^2(x - x_{k+1}) & x \in [x_{k+1}, x_{k+2}] \\ 0 & \text{elsewhere.} \end{cases}$$

For $\phi_{3,k}(x)$ we have that the interval of support is $[x_k, x_{k+3}]$. Now if $x \in [x_k, x_{k+1})$ we have that

$$\phi_{3,k}(x) = N \int_{x_k}^{x} [\phi_{2,k}(s) - \phi_{2,k+1}(s)] \, ds$$

$$
\begin{aligned}
&= N\int_{x_k}^{x}[N^2(s-x_k)-0]\,ds\\
&= \frac{N^3}{2}(x-x_k)^2.
\end{aligned}
$$

If $x \in [x_{k+1}, x_{k+2})$ we have

$$
\begin{aligned}
\phi_{3,k}(x) &= N\int_{x_k}^{x_{k+1}}[\phi_{2,k}(s)-\phi_{2,k+1}(s)]\,ds + N\int_{x_{k+1}}^{x}[\phi_{2,k}(s)-\phi_{2,k+1}(s)]\,ds\\
&= \frac{N}{2} + N\int_{x_{k+1}}^{x}[\{N-N^2(s-x_{k+1})\}-\{N^2(s-x_{k+1})\}]\,ds\\
&= \frac{N}{2} + N\int_{x_{k+1}}^{x}[N-2N^2(s-x_{k+1})]\,ds\\
&= \frac{N}{2} + N^2(x-x_{k+1}) - N^3(x-x_{k+1})^2.
\end{aligned}
$$

Lastly, if $x \in [x_{k+2}, x_{k+3}]$ we have

$$
\begin{aligned}
\phi_{3,k}(x) &= N\int_{x_k}^{x_{k+1}}[\phi_{2,k}(s)-\phi_{2,k+1}(s)]\,ds + N\int_{x_{k+1}}^{x_{k+2}}[\phi_{2,k}(s)-\phi_{2,k+1}(s)]\,ds\\
&\quad + N\int_{x_{k+2}}^{x}[\phi_{2,k}(s)-\phi_{2,k+1}(s)]\,ds\\
&= \frac{N}{2} + 0 + N\int_{x_{k+2}}^{x}[0-\{N-N^2(s-x_{k+2})\}]\,ds\\
&= \frac{N}{2} + N\int_{x_{k+2}}^{x}[-N+N^2(s-x_{k+2})]\,ds\\
&= \frac{N}{2} - N^2(x-x_{k+2}) + \frac{N^3}{2}(x-x_{k+2})^2.
\end{aligned}
$$

Therefore,

$$
\phi_{3,k}(x) = \begin{cases}
\frac{N^3}{2}(x-x_k)^2 & x \in [x_k, x_{k+1})\\
\frac{N}{2} + N^2(x-x_{k+1}) - N^3(x-x_{k+1})^2 & x \in [x_{k+1}, x_{k+2})\\
\frac{N}{2} - N^2(x-x_{k+2}) + \frac{N^3}{2}(x-x_{k+2})^2 & x \in [x_{k+2}, x_{k+3}]\\
0 & \text{elsewhere.}
\end{cases}
$$

For $\phi_{4,k}(x)$ the interval of support is $[x_k, x_{k+4}]$. Thus for $x \in [x_k, x_{k+1})$ we have

$$\begin{aligned}\phi_{4,k}(x) &= N\int_{x_k}^{x}[\phi_{3,k}(s) - \phi_{3,k+1}(s)]\,ds \\ &= N\int_{x_k}^{x}[\frac{N^3}{2}(s-x_k)^2 - 0]\,ds \\ &= \frac{N^4}{6}(x-x_k)^3.\end{aligned}$$

If $x \in [x_{k+1}, x_{k+2})$

$$\begin{aligned}\phi_{4,k}(x) &= N\int_{x_k}^{x_{k+1}}[\phi_{3,k}(s) - \phi_{3,k+1}(s)]\,ds + N\int_{x_{k+1}}^{x}[\phi_{3,k}(s) - \phi_{3,k+1}(s)]\,ds \\ &= \frac{N}{6} \\ &\quad +N\int_{x_{k+1}}^{x}[\{\frac{N}{2} + N^2(s-x_{k+1}) - N^3(s-x_{k+1})^2\} - \{\frac{N^3}{2}(s-x_{k+1})^2\}]\,ds \\ &= \frac{N}{6} + N\int_{x_{k+1}}^{x}[\frac{N}{2} + N^2(s-x_{k+1}) - \frac{3}{2}N^3(s-x_{k+1})^2]\,ds \\ &= \frac{N}{6} + \frac{N^2}{2}(x-x_{k+1}) + \frac{N^3}{2}(x-x_{k+1})^2 - \frac{N^4}{2}(x-x_{k+1})^3.\end{aligned}$$

For $x \in [x_{k+2}, x_{k+3})$ we have

$$\begin{aligned}\phi_{4,k}(x) &= N\int_{x_k}^{x_{k+1}}[\phi_{3,k}(s) - \phi_{3,k+1}(s)]\,ds + N\int_{x_{k+1}}^{x_{k+2}}[\phi_{3,k}(s) - \phi_{3,k+1}(s)]\,ds \\ &\qquad + N\int_{x_{k+2}}^{x}[\phi_{3,k}(s) - \phi_{3,k+1}(s)]\,ds \\ &= \frac{N}{6} + \frac{N}{2} + N\int_{x_{k+2}}^{x}[\{\frac{N}{2} - N^2(s-x_{k+2}) + \frac{N^3}{2}(s-x_{k+2})^2\} \\ &\qquad - \{\frac{N}{2} + N^2(s-x_{k+2}) - N^3(s-x_{k+2})^2\}]\,ds \\ &= \frac{2N}{3} + N\int_{x_{k+2}}^{x}[-2N^2(s-x_{k+2}) + \frac{3N^3}{2}(s-x_{k+2})^2]\,ds \\ &= \frac{2N}{3} - N^3(x-x_{k+2})^2 + \frac{N^4}{2}(x-x_{k+2})^3.\end{aligned}$$

Lastly, for $x \in [x_{k+3}, x_{k+4}]$ we have

$$\begin{aligned}
\phi_{4,k}(x) &= N\int_{x_k}^{x_{k+1}}[\phi_{3,k}(s) - \phi_{3,k+1}(s)]\,ds + N\int_{x_{k+1}}^{x_{k+2}}[\phi_{3,k}(s) - \phi_{3,k+1}(s)]\,ds \\
&\quad + N\int_{x_{k+2}}^{x_{k+3}}[\phi_{3,k}(s) - \phi_{3,k+1}(s)]\,ds + N\int_{x_{k+3}}^{x}[\phi_{3,k}(s) - \phi_{3,k+1}(s)]\,ds \\
&= \frac{N}{6} + \frac{N}{2} - \frac{N}{2} + N\int_{x_{k+3}}^{x}[0 - \{\frac{N}{2} - N^2(s - x_{k+3}) + \frac{N^3}{2}(s - x_{k+3})^2\}]\,ds \\
&= \frac{N}{6} + N\int_{x_{k+3}}^{x}[-\frac{N}{2} + N^2(s - x_{k+3}) - \frac{N^3}{2}(s - x_{k+3})^2]\,ds \\
&= \frac{N}{6} - \frac{N^2}{2}(x - x_{k+3}) + \frac{N^3}{2}(x - x_{k+3})^2 - \frac{N^4}{6}(x - x_{k+3})^3.
\end{aligned}$$

Therefore,

$$\phi_{4,k}(x) = \begin{cases}
\frac{N^4}{6}(x - x_k)^3 & x \in [x_k, x_{k+1}) \\
\frac{N}{6} + \frac{N^2}{2}(x - x_{k+1}) + \frac{N^3}{2}(x - x_{k+1})^2 - \frac{N^4}{2}(x - x_{k+1})^3 & x \in [x_{k+1}, x_{k+2}) \\
\frac{2N}{3} - N^3(x - x_{k+2})^2 + \frac{N^4}{2}(x - x_{k+2})^3 & x \in [x_{k+2}, x_{k+3}) \\
\frac{N}{6} - \frac{N^2}{2}(x - x_{k+3}) + \frac{N^3}{2}(x - x_{k+3})^2 - \frac{N^4}{6}(x - x_{k+3})^3 & x \in [x_{k+3}, x_{k+4}] \\
0 & \text{elsewhere.}
\end{cases}$$

To create $\phi_{5,k}(x)$ we first realize that the interval of support for this spline is $[x_k, x_{k+5}]$. Now if $x \in [x_k, x_{k+1})$ we have

$$\begin{aligned}
\phi_{5,k}(x) &= N\int_{x_k}^{x}[\phi_{4,k}(s) - \phi_{4,k+1}(s)]\,ds \\
&= N\int_{x_k}^{x}[\{\frac{N^4}{6}(s - x_k)^3\} - \{0\}]\,ds \\
&= \frac{N^5}{24}(x - x_k)^4.
\end{aligned}$$

If $x \in [x_{k+1}, x_{k+2})$ we see that

$$\begin{aligned}
\phi_{5,k}(x) &= N\int_{x_k}^{x_{k+1}}[\phi_{4,k}(s)-\phi_{4,k+1}(s)]\,ds + N\int_{x_{k+1}}^{x}[\phi_{4,k}(s)-\phi_{4,k+1}(s)]\,ds \\
&= \frac{N}{24} + N\int_{x_{k+1}}^{x}[\{\frac{N}{6}+\frac{N^2}{2}(s-x_{k+1})+\frac{N^3}{2}(s-x_{k+1})^2-\frac{N^4}{2}(s-x_{k+1})^3\} \\
&\qquad -\{\frac{N^4}{6}(s-x_{k+1})^3\}]\,ds \\
&= \frac{N}{24} + N\int_{x_{k+1}}^{x}[\frac{N}{6}+\frac{N^2}{2}(s-x_{k+1})+\frac{N^3}{2}(s-x_{k+1})^2-\frac{2N^4}{3}(s-x_{k+1})^3]\,ds \\
&= \frac{N}{24}+\frac{N^2}{6}(x-x_{k+1})+\frac{N^3}{4}(x-x_{k+1})^2+\frac{N^4}{6}(x-x_{k+1})^3-\frac{N^5}{6}(x-x_{k+1})^4.
\end{aligned}$$

For $x \in [x_{k+2}, x_{k+3})$ we have

$$\begin{aligned}
\phi_{5,k}(x) &= N\int_{x_k}^{x_{k+1}}[\phi_{4,k}(s)-\phi_{4,k+1}(s)]\,ds + N\int_{x_{k+1}}^{x_{k+2}}[\phi_{4,k}(s)-\phi_{4,k+1}(s)]\,ds \\
&\qquad + N\int_{x_{k+2}}^{x}[\phi_{4,k}(s)-\phi_{4,k+1}(s)]\,ds \\
&= \frac{N}{24}+\frac{5N}{12}+N\int_{x_{k+2}}^{x}[\{\frac{2N}{3}-N^3(s-x_{k+2})^2+\frac{N^4}{2}(s-x_{k+2})^3\} \\
&\qquad -\{\frac{N}{6}+\frac{N^2}{2}(s-x_{k+2})+\frac{N^3}{2}(s-x_{k+2})^2-\frac{N^4}{2}(s-x_{k+2})^3\}]\,ds \\
&= \frac{11N}{24} \\
&\qquad +N\int_{x_{k+2}}^{x}[\frac{N}{2}-\frac{N^2}{2}(s-x_{k+2})-\frac{3N^3}{2}(s-x_{k+2})^2+N^4(s-x_{k+2})^3]\,ds \\
&= \frac{11N}{24}+\frac{N^2}{2}(x-x_{k+2})-\frac{N^3}{4}(x-x_{k+2})^2 \\
&\qquad -\frac{N^4}{2}(x-x_{k+2})^3+\frac{N^5}{4}(x-x_{k+2})^4.
\end{aligned}$$

Next, if $x \in [x_{k+3}, x_{k+4})$ we have

$$\begin{aligned}
\phi_{5,k}(x) &= N\int_{x_k}^{x_{k+1}}[\phi_{4,k}(s)-\phi_{4,k+1}(s)]\,ds + N\int_{x_{k+1}}^{x_{k+2}}[\phi_{4,k}(s)-\phi_{4,k+1}(s)]\,ds \\
&\qquad + N\int_{x_{k+2}}^{x_{k+3}}[\phi_{4,k}(s)-\phi_{4,k+1}(s)]\,ds + N\int_{x_{k+4}}^{x}[\phi_{4,k}(s)-\phi_{4,k+1}(s)]\,ds
\end{aligned}$$

$$
\begin{aligned}
&= \frac{N}{24} + \frac{5N}{12} + 0 \\
&\quad + N\int_{x_{k+3}}^{x} [\{\frac{N}{6} - \frac{N^2}{2}(s - x_{k+3}) + \frac{N^3}{2}(s - x_{k+3})^2 - \frac{N}{6}(s - x_{k+3})^3\} \\
&\quad - \{\frac{2N}{3} - N^3(s - x_{k+3})^2 + \frac{N^4}{2}(s - x_{k+3})^3\}]\, ds \\
&= \frac{11N}{24} \\
&\quad + N\int_{x_{k+3}}^{x} [-\frac{N}{2} - \frac{N^2}{2}(s - x_{k+3}) + \frac{3N^3}{2}(s - x_{k+3})^2 - \frac{2N^4}{3}(s - x)^3]\, ds \\
&= \frac{11N}{24} - \frac{N^2}{2}(x - x_{k+3}) - \frac{N^3}{4}(x - x_{k+3})^2 \\
&\quad + \frac{N^4}{2}(x - x_{k+3})^3 - \frac{N^5}{6}(x - x_{k+3})^4.
\end{aligned}
$$

Lastly, for $x \in [x_{k+4}. x_{k+5}]$ we see that

$$
\begin{aligned}
\phi_{5,k}(x) &= N\int_{x_k}^{x_{k+1}} [o_{4,k}(s) - \phi_{4,k+1}(s)]\, ds + N\int_{x_{k+1}}^{x_{k+2}} [\phi_{4,k}(s) - \phi_{4,k+1}(s)]\, ds \\
&\quad + N\int_{x_{k+2}}^{x_{k+3}} [\phi_{4,k}(s) - \phi_{4,k+1}(s)]\, ds + N\int_{x_{k+3}}^{x_{k+4}} [\phi_{4,k}(s) - \phi_{4,k+1}(s)]\, ds \\
&\quad + N\int_{x_{k+4}}^{x} [\phi_{4,k}(s) - \phi_{4,k+1}(s)]\, ds \\
&= \frac{N}{24} + \frac{5N}{12} + 0 + \frac{-5N}{12} + N\int_{x_{k+4}}^{x} [\{0\} \\
&\quad - \{\frac{N}{6} - \frac{N^2}{2}(s - x_{k+4}) + \frac{N^3}{2}(s - x_{k+4})^2 - \frac{N^4}{6}(s - x_{k+4})^3\}]\, ds \\
&= \frac{N}{24} \\
&\quad + N\int_{x_{k-4}}^{x} [-\frac{N}{6} + \frac{N^2}{2}(s - x_{k+4}) - \frac{N^3}{2}(s - x_{k+4})^2 + \frac{N^4}{6}(s - x_{k+4})^3]\, ds \\
&= \frac{N}{24} - \frac{N^2}{6}(x - x_{k+4}) + \frac{N^3}{4}(x - x_{k+4})^2 \\
&\quad - \frac{N^4}{6}(x - x_{k+4})^3 + \frac{N^5}{24}(x - x_{k+4})^4.
\end{aligned}
$$

Finally, we have that $\phi_{5,k}(x) =$

$$\begin{cases} \frac{N^5}{24}(x-x_k)^4 \\ \frac{N}{24}+\frac{N^2}{6}(x-x_{k+1})+\frac{N^3}{4}(x-x_{k+1})^2+\frac{N^4}{6}(x-x_{k+1})^3-\frac{N^5}{6}(x-x_{k+1})^4 \\ \frac{11N}{24}+\frac{N^2}{2}(x-x_{k+2})-\frac{N^3}{4}(x-x_{k+2})^2-\frac{N^4}{2}(x-x_{k+2})^3+\frac{N^5}{4}(x-x_{k+2})^4 \\ \frac{11N}{24}-\frac{N^2}{2}(x-x_{k+3})-\frac{N^3}{4}(x-x_{k+3})^2+\frac{N^4}{2}(x-x_{k+3})^3-\frac{N^5}{6}(x-x_{k+3})^4 \\ \frac{N}{24}-\frac{N^2}{6}(x-x_{k+4})+\frac{N^3}{4}(x-x_{k+4})^2-\frac{N^4}{6}(x-x_{k+4})^3+\frac{N^5}{24}(x-x_{k+4})^4 \\ 0 \end{cases}$$

for x is the interval $[x_k, x_{k+1})$, $[x_{k+1}, x_{k+2})$, $[x_{k+2}, x_{k+3})$, $[x_{k+3}, x_{k+4})$, $[x_{k+4}, x_{k+5}]$ and elsewhere respectively.

6.2 Dual Basis Fuctions

In this section we graph the dual Haar basis functions for $M = 4$ and $m = 2$ on the same axis. This further demonstrates the duality nature and biorthogonality of the Haar basis functions under the approximation scheme described in this dissertation.

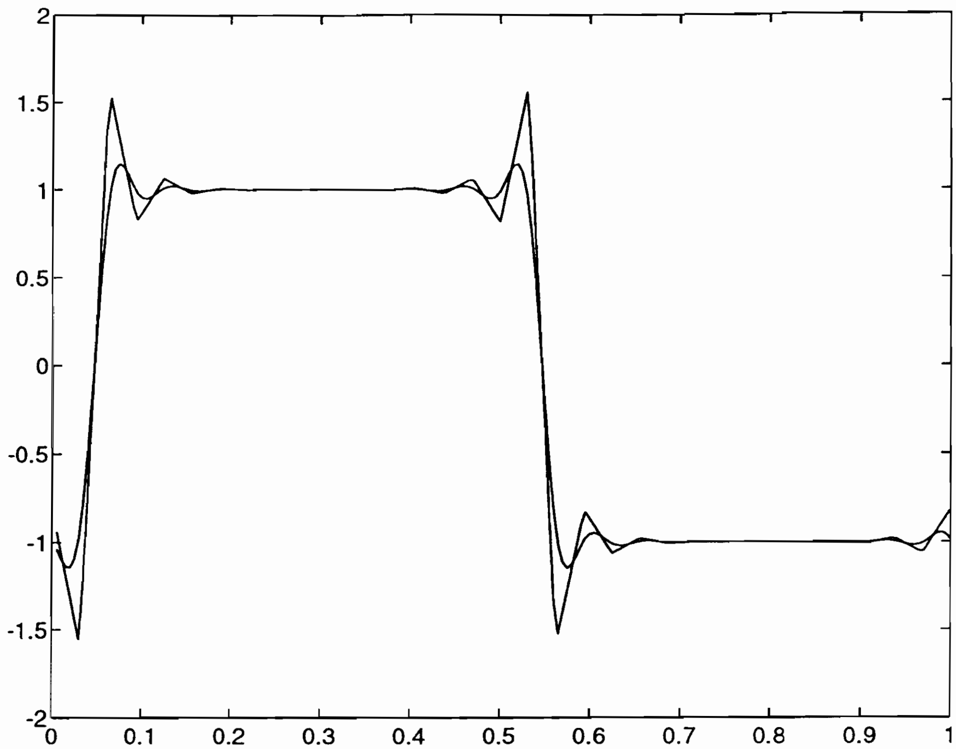

Figure 6.1: Haar basis function for h_{10}, $\sigma_{4,2}$, $\theta_{4,2}$, and $\nu = 5$

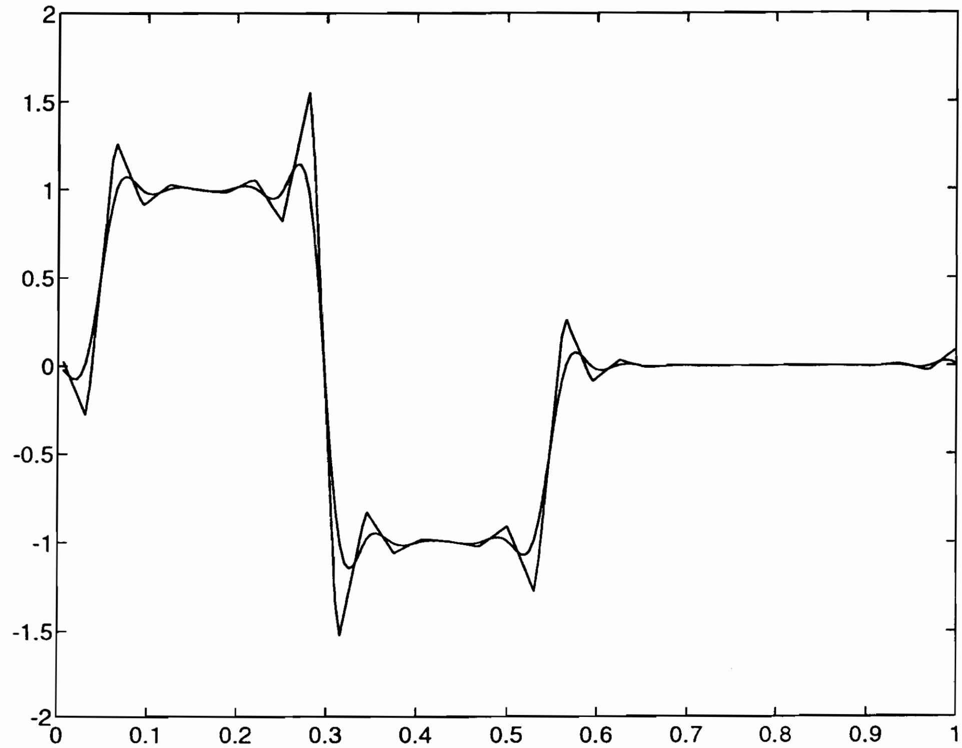

Figure 6.2: Haar basis function for h_{20}, $\sigma_{4,2}$, $\theta_{4,2}$, and $\nu = 5$

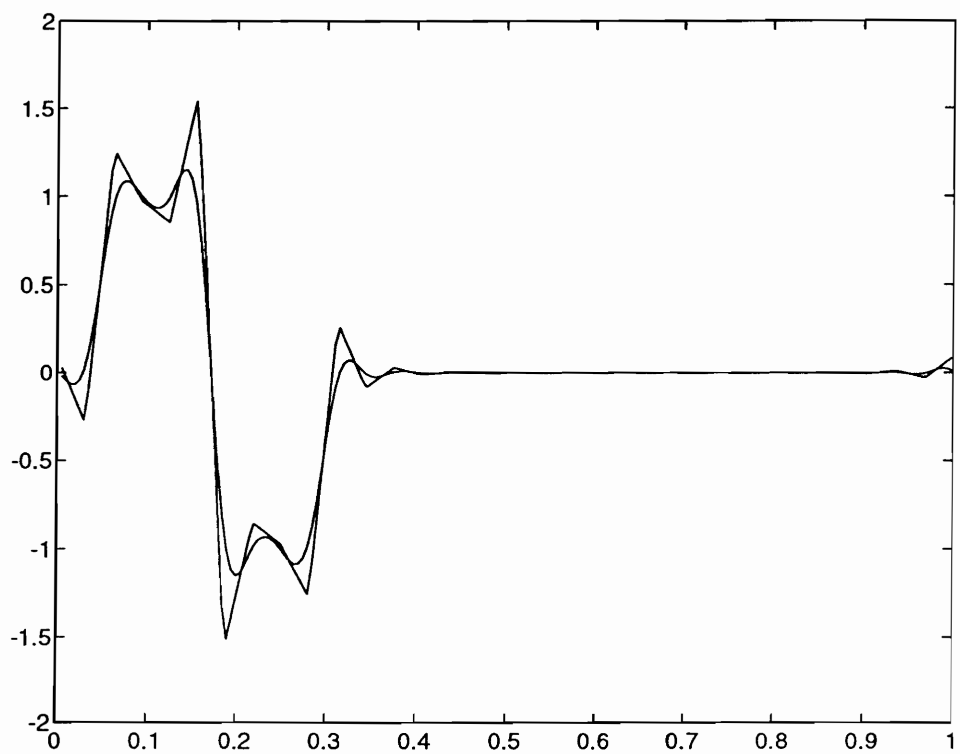

Figure 6.3: Haar basis function for h_{30}, $\sigma_{4,2}$, $\theta_{4,2}$, and $\nu = 5$

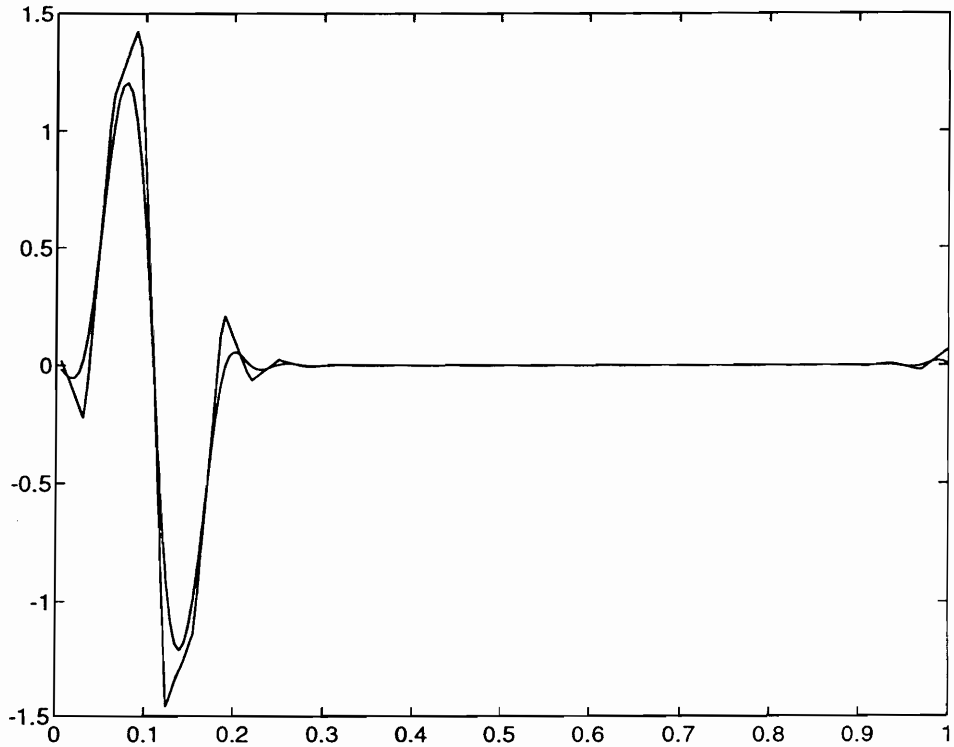

Figure 6.4: Haar basis function for h_{40}, $\sigma_{4,2}$, $\theta_{4,2}$, and $\nu = 5$

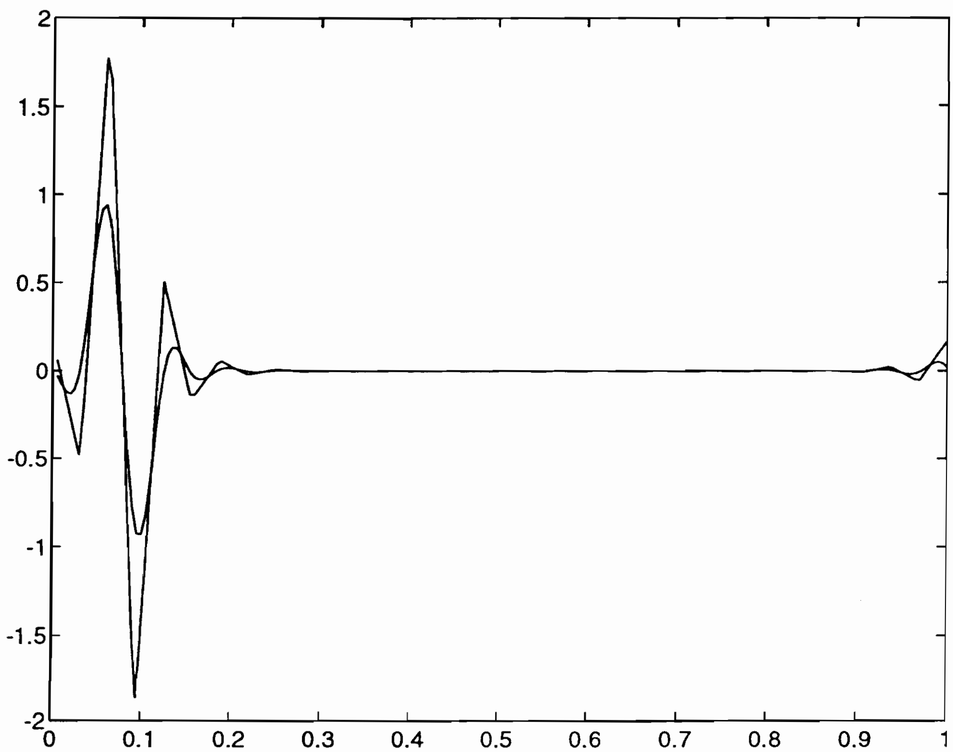

Figure 6.5: Haar basis function for h_{50}, $\sigma_{4,2}$, $\theta_{4,2}$, and $\nu = 5$

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Vita

Elizabeth Ann Bonawitz was born in Lancaster, PA on October 17, 1964 and grew up in Millersville, PA. She graduated from Penn Manor High School in 1982. In 1986 she graduated magna cum laude from Millersville University of PA with a B.A. in mathematics. Elizabeth then went to Virginia Polytechnic Institute and State University where she received her M.S. in mathematics in 1988, and her Ph.D. in mathematics in 1994.

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