

**ANALYSIS, FINITE ELEMENT  
APPROXIMATION, AND COMPUTATION OF  
OPTIMAL AND FEEDBACK FLOW  
CONTROL PROBLEMS**

by

Hyung-Chun Lee

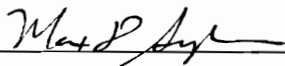
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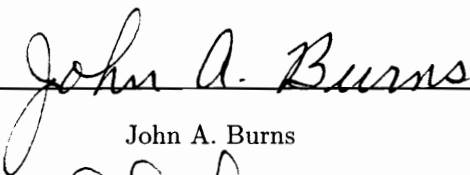
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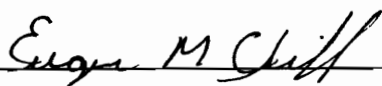
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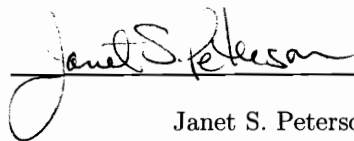
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July, 1994

Blacksburg, Virginia

# ANALYSIS, FINITE ELEMENT APPROXIMATION, AND COMPUTATION OF OPTIMAL AND FEEDBACK FLOW CONTROL PROBLEMS

by

Hyung-Chun Lee

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## (ABSTRACT)

The analysis, finite element approximation, and numerical simulation of some control problems associated with fluid flows are considered.

First, we consider a coupled solid/fluid temperature control problem. This optimization problem is motivated by the desire to remove temperature peaks, *i.e.*, “hot spots”, along the bounding surface of containers of fluid flows. The heat equation of the solid container is coupled to the energy equation for the fluid. Control is effected by adjustments to the temperature of the fluid at the inflow boundary. We give a precise statement of the mathematical model, prove the existence and uniqueness of optimal solutions, and derive an optimality system. We study a finite element approximation and provide rigorous error estimates for the error in the approximate solution of the optimality system. We then develop and implement an iterative algorithm to compute the approximate solution.

Second, a computational study of the feedback control of the magnitude of the lift in flow around a cylinder is presented. The uncontrolled flow exhibits an unsymmetric Karman vortex street and a periodic lift coefficient. The size of the oscillations in the lift is reduced through an active feedback control system. The control used is the injection and suction of fluid through orifices on the cylinder; the amount of fluid injected or sucked is determined, through a simple feedback law, from pressure measurements at stations along the surface

of the cylinder.

The results of some computational experiments are given; these indicate that the simple feedback law used is effective in reducing the size of the oscillations in the lift.

Finally, some boundary value problems which arise from a feedback control problem are considered. We give a precise statement of the mathematical problems and then prove the existence and uniqueness of solutions to the boundary value problems for the Laplace and Stokes equations by studying the boundary integral equation method.

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# Chapter 1

## INTRODUCTION

We consider the analysis, finite element approximation, and numerical simulation of some control problems associated with fluid flows. First, we study the analysis, finite element approximation, and computation of a coupled solid/fluid temperature control problem. Second, we study the feedback control of Karman vortex shedding. Finally, we consider some boundary value problems which arise from a feedback control problem.

### 1.1 Analysis, Finite Element Approximation, and Computation of a Coupled Solid/Fluid Temperature Control Problem

The control of fluid motions for the purpose of achieving some desired objective is crucial to many technological applications. The optimal control of fluid flows is a rapidly developing subject which has been addressed recently by several authors. This interest is quickly expanding so that, at this time, flow control is becoming a very active and successful area of inquiry. Optimal control theory of distributed parameter systems has been studied by Lions ([81], [82] and [83]); he addresses the question of existence of an optimal control and the derivation of necessary conditions. Gunzburger, Hou and Svobodny ([54], [55], [56], [57], [58] and [59]) consider various optimal control problems in fluid mechanics and study mathematical and numerical problems such as the existence of optimal controls, necessary first order optimality conditions, the discretization of these problems by finite

element methods, and convergence and error estimates for the discrete problems. Many other publications provide analyses of various aspects of flow control problems; for examples, Abergel and Temam ([1] and [2]), Arumugan and Pironneau [7], Bristeau *et al* [18], Cuvelier ([29] and [30]), Fursikov ([46], [47] and [48]), Ito and Desai [70], Li and Zhou [75], and Sritharan *et al* ([37], [105] and [106]).

In Chapter 2, we consider a coupled solid/fluid temperature control problem. This optimization problem is motivated by the desire to remove temperature peaks, *i.e.*, “hot spots”, along the bounding surface of containers of fluid flows.

The major steps in the study of optimal fluid control and/or fluid optimization problems are as follows.

1. Give a precise mathematical statement of the problem; in particular, one must specify the function classes in which one will seek optimal states and controls, as well as provide a description of the optimization objective and the constraints.
2. Show that optimal states and controls exist, and provide information about their regularity.
3. From the first-order necessary conditions holding at an extremum of the objective functional, and from the constraints, derive a system of equations, and perhaps inequalities, that optimal controls and states satisfy. This system we call the *optimality system*.
4. Devise schemes for the approximate solution of the optimality system.
5. Deduce error estimates for the approximate solutions.



6. Develop computer codes implementing the algorithms devised for approximating solutions of the optimality system.

Clearly, from a practical standpoint, the fourth and sixth of these studies are most important. However, here we treat all six steps for a coupled solid/fluid temperature control problem which we now describe.

We suppose that the regular bounded domain  $\Omega$  in  $R^2$  is made up of two sub-domains  $\Omega_1$  and  $\Omega_2$  separated by an interface  $\Gamma_w$ . The state variables, *i.e.*, the velocity  $\mathbf{u}$ , pressure  $p$ , and temperature  $T$ , and the control  $g$  are required to satisfy the Navier-Stokes equations

$$-\nu\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega_2,$$

the incompressibility constraint

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega_2,$$

with, for simplicity, the boundary condition

$$\mathbf{u} = \mathbf{h} \quad \text{on } \partial\Omega_2,$$

and the energy equations

$$-\kappa_1\Delta T = \bar{Q}_1 \quad \text{in } \Omega_1,$$

$$-\kappa_2\Delta T + (\mathbf{u} \cdot \nabla)T = \bar{Q}_2 + 2\nu(\nabla\mathbf{u} + \nabla\mathbf{u}^T) : (\nabla\mathbf{u} + \nabla\mathbf{u}^T) \quad \text{in } \Omega_2,$$

with the boundary conditions

$$\begin{aligned} T &= g \quad \text{on } \Gamma_c, \\ \frac{\partial T}{\partial n} &= 0 \quad \text{on } \partial\Omega/\Gamma_c \end{aligned}$$

where  $\Gamma_c$  is a part of  $\partial\Omega$  on which control is to be effected. The data functions  $\mathbf{f}$ ,  $\mathbf{h}$ ,  $\bar{Q}_1$  and  $\bar{Q}_2$  are assumed to be known. The constant  $\nu$  is the kinematic viscosity coefficient of the fluid, and the constants  $\kappa_1$  and  $\kappa_2$  are the thermal conductivity coefficients in the solid and fluid respectively; see [103] for details.

We wish to find an optimal control in order to meet a specified objective. For example, given a velocity field  $\mathbf{u}$ , we would seek a temperature field  $T$  and a control  $g$  such that the functional

$$\mathcal{J}(T, g) = \frac{1}{2\gamma} \int_{\Gamma_\sigma} |T - T_d|^2 d\Gamma + \frac{\kappa_2 \delta}{2} \int_{\Gamma_c} (|g|^2 + |\nabla_s g|^2) d\Gamma$$

is minimized subject to the energy equations with boundary conditions, where  $\nabla_s$  denotes the surface gradient operator and  $T_d$  is some desired temperature distribution, *e.g.*, something close to the average temperature along  $\Gamma_\sigma$  for the uncontrolled system. The non-negative parameters  $\gamma$  and  $\delta$  can be used to change the relative importance of the two terms appearing in the definition of  $\mathcal{J}$  as well as to act as a penalty parameter.

In section 2.1, we give a precise statement of the mathematical model we have introduced above. In sections 2.2 and 2.3, we prove the existence and uniqueness of optimal solutions and derive an optimality system, *i.e.*, a set of equations from which the optimal control and state may be determined. In section 2.4, we study a finite element approximation and provide rigorous error estimates for the approximate solution of the optimality system. Finally, in sections 2.5 and 2.6, we develop and implement an iterative algorithm to compute the approximate solution.

## 1.2 Feedback Control of Karman Vortex Shedding

The phenomenon of vortex shedding behind bluff bodies is familiar since the days of Leonardo da Vinci, and has been studied systematically at least since the days of Strouhal [107]. Many important contributions have been made, and a partial list includes Karman [72], Karman and Rubach [73], Fage and Johansen ([35] and [36]), Kovasznay [74], Roshko [100], Tritton [112], Abernathy and Kronauer [3], Berger [14] and Gerrard [50]. Rayleigh [98] first pointed out that formation of the vortex street is related to an instability of the cylinder wake. Later, Kovasznay [74] attributed the vortex street formation entirely to the wake instability, based on experimental observations. The mechanism by which the wake instability leads to a vortex street formation has been illustrated by Abernathy and Kronauer [3]. They studied the instability of two infinite vortex sheets initially at a fixed distance apart using a numerical technique developed earlier by Rosenhead [99].

The understanding of vortex shedding from bluff bodies has advanced greatly in recent years. Provansal, Mathis and Boyer [97] and others showed experimentally that vortex shedding was the result of an initially linear wake instability. The current interest in considering the absolute and convective nature of instabilities in wakes followed from concepts developed in the field of plasma physics and has been applied to free shear flows by Huerre and Monkewitz ([68] and [69]). It is now widely accepted that the observed Karman vortex street is the nonlinear limit cycle of a self-excited global instability - the behavior has been related to the existence of a finite region of absolute instability in the near wake. Schumm [102] has presented experimental support for this by showing that transient behavior follows very closely that predicted by the Landau equation.

The Karman vortex street in the wake of a circular cylinder has been widely studied

during this century, and many of these studies have considered methods of altering the wake structure. This problem is of practical as well as academic importance because, particularly at higher Reynolds numbers, the shedding of vortices from alternate sides of bluff bodies is associated with strong periodic transverse forces that can damage structures. It was shown early that relatively small changes to the geometry such as adding a splitter plate in the near wake or surrounding the cylinder with a shroud can reduce the transverse forcing significantly; see, *e.g.*, Price [95]. More recently, other methods of preventing or affecting vortex shedding have been demonstrated: steady or periodic suction or bleeding from the recirculation zone can prevent the vortex street from forming, as in various Reynolds number ranges can heating the cylinder, locating a small secondary cylinder in the near wake, imparting a large-amplitude transverse oscillation to the cylinder at an appropriate frequency or imparting a constant or periodically varying angular rotation to the cylinder; see, Wood [118], Bearman [12], Berger [15], Strykowski and Sreenivasan [108], Williams and Amato [116], Tokumaru and Dimotakis [111], Lecordier, Hamma and Paranthoen [78], and the review in Huerre and Monkewitz [69]. All of these are open-loop control methods, control being achieved without the use of sensors. While there has been an enormous amount of effort put in the open loop control of vortex shedding, very little research has been undertaken in the area of feedback control of vortex shedding.

Recent years have seen the active control of various unstable fluid mechanical and other system. The approach is generally to introduce feedback into the system with one or more sensors and actuators so that the combined system is stable. Instabilities that have been stabilized using active control include aerofoil flutter (Huang [66]), compressor surge (Ffowcs Williams and Huang [40]), reheat buzz (Langhorne, Dowling and Hooper [77]), the Rijke tube (Dines [33]; Heckl [63]), and flow over cavities (Huang and Weaver [67]).

The possibility for feedback control of global oscillations was investigated analytically by Monkewitz [86], with reference to Berger's earlier experiments. The conclusion was that stable flows could readily be destabilized by feedback, but that self-excited systems might only be able to be stabilized over a small range of Reynolds numbers above the onset of vortex shedding, because at higher Reynolds numbers the feedback necessary to stabilize the most unstable mode would destabilize higher modes. Monkewitz, Berger and Schumm [87] repeated Berger's original experiment and confirmed that the variation of response at the sensor location with gain in the feedback loop agreed with their theoretical predictions. Fflocs Williams and Zhao [41] considered independently the potential for suppressing vortex shedding using active control. They suggested that if vortex shedding was the limit cycle of an initially linear instability then active control, suppressing instability, should prevent vortex shedding. They performed experiments to test this hypothesis and observed that at Reynolds numbers between 400 and  $10^4$  control significantly reduced the vortex shedding frequency component. In addition they presented results indicating that the vortex street was suppressed significantly throughout the wake. All of these forcing mechanism can also be used as effective and potential means for the feedback control of vortex shedding as well as the open loop control.

In Chapter 3, we study the efficacy of a simple feedback control law for the reduction of the magnitude of the lift. The context of our study is the full, nonlinear partial differential equations for plane, unsteady, incompressible, viscous flow. The control mechanism used to attempt to the reduce the size of the lift oscillations is the injection and suction of fluid through orifices on the surface of the cylinder. The amount of fluid injected or sucked through the orifices is determined, using a simple feedback law, from the pressure "measured" at various stations on the cylinder. Thus, in the language of feedback control,

the *sensor* determines the pressure at various stations on the cylinder and the *actuator* injects or sucks fluid through orifices also on the cylinder. Other sensing and actuating mechanisms could also be used, *e.g.*, the vorticity or tangential stress on the cylinder for sensing and rotation or shape modification of the cylinder for actuating.

Once a feedback law has been chosen, the computational simulation of the resulting flow is essentially no more difficult to accomplish than is the similar task for an uncontrolled problem. Of course, arbitrarily chosen feedback laws may not be effective in meeting the stated goal of reducing the size of the oscillations of the lift. However, there is evidence that appropriate controls can be applied to achieve a reduction in the amplitude of the oscillations of the lift. For example, it has been observed that large forces, *i.e.*, lift and drag, result from the small viscous forces acting in the vicinity of the surface of cylinder. This implies that small control mechanisms applied to the boundary layer can be effectively used to modify the generation of vortices and unsteady lift forces. This concept has been demonstrated by, *e.g.*, Ffowcs Williams and Zhao [41].

The type of feedback laws used in our study are based on the observation that differences in the pressure on the top and bottom halves of the cylinder should give an indication of the asymmetric behavior of the lift. Such pressure differences are then used to determine how much fluid to inject or suck through the orifices. No attempt is made to systematically design an “optimal” feedback law, *i.e.*, one that in some sense does the best possible job in meeting the objective. However, the computational results we have obtained indicate that the simple feedback law we use here is quite effective in reducing the size of the oscillations in the lift.

In section section 3.1, we formulate the feedback control problem. In sections 3.2, 3.3, and 3.3, we provide computational results for the uncontrolled problem and for two different

arrangements of controls.

### 1.3 Some Boundary Value Problems Associated with Feedback Control

In Chapter 4, we will study some boundary value problems associated with feedback control theory. Feedback control involves constructing a control  $\phi$  as a function of the state variables  $u$  or some observation of  $u$ . Let us consider the following problem as an example:

$$-\Delta u = f \quad \text{in } \Omega, \tag{1.3.1}$$

$$u = b \quad \text{on } \Gamma_s, \tag{1.3.2}$$

$$u = b + F\left(\frac{\partial u}{\partial n}\Big|_{\Gamma_s}\right)g \quad \text{on } \Gamma_c, \tag{1.3.3}$$

where  $\Omega$  is a nonempty simply connected domain in  $R^N$ ,  $N = 2$  or  $3$ , with a smooth boundary  $\partial\Omega = \Gamma$ ;  $\Gamma_s$  and  $\Gamma_c$ , on which the sensors and actuators are located respectively, are portions of  $\Gamma$ . For simplicity we let  $\bar{\Gamma} = \bar{\Gamma}_s \cup \bar{\Gamma}_c$  and  $\Gamma_s \cap \Gamma_c = \phi$ . In (1.3.1)-(1.3.3),  $f$  denotes a given source,  $b$  and  $g$  given functions defined on boundary, and  $\partial/\partial n$  the normal derivative to  $\Gamma$ . Also,  $F(\cdot)$  is a given functional. The function  $g$  has a compact support in  $\Gamma_c$ . Thus, in this example we have a control  $\phi = F(\partial u/\partial n|_{\Gamma_s})g$ .

Since the above problem (1.3.1)-(1.3.3) is not an elliptic problem in the usual sense, we cannot use the general theory of elliptic equations. However, by using potential theory, we can prove the existence and uniqueness of the solution to problem (1.3.1)-(1.3.3). Specifically, we will use the boundary integral equation method which is closely related to the classical Green's function method.

In section 4.1, we give a precise statement of the mathematical problems we consider. We then study the existence and uniqueness of the boundary value problems for the Laplace and Stokes equations in sections 4.2 and 4.3, respectively.

## Chapter 2

# ANALYSIS, FINITE ELEMENT APPROXIMATION, AND COMPUTATION OF A COUPLED SOLID/FLUID TEMPERATURE CONTROL PROBLEM

### 2.1 Mathematical Modeling

We suppose that the regular bounded domain  $\Omega$  in  $R^2$  is made up of two sub-domains  $\Omega_1$  and  $\Omega_2$  separated by an interface  $\Gamma_w$ , with the result that  $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma_w$  (see Figure 2.1). The solid material occupies a sub-domain  $\Omega_1$  having a boundary  $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_w$  and the fluid flow occupies a domain  $\Omega_2$  having a boundary  $\Gamma_c \cup \Gamma_o \cup \Gamma_w \cup \Gamma_4$ . We have an inflow boundary  $\Gamma_c$ , an outflow boundary  $\Gamma_o$ , and a solid wall  $\Gamma_w$ . The geometry of all these boundary segments is prescribed, as is the inflow velocity  $\mathbf{u}_c$ . At the outflow, one can impose one's favorite outflow boundary conditions. On the walls, we have the no slip boundary conditions for the velocity. Control is to be effected through adjusting the temperature along the boundary  $\Gamma_c$ . The heat-flux is specified along the boundary  $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_o$ . We assume that the flow is incompressible and convection driven so that buoyancy effects can be neglected, and thus temperature effects on the mechanical properties of the flow, *i.e.*, the velocity and pressure, are negligible. We are interested in controls such that we get a desired temperature along  $\Gamma_w$  or a portion  $\Gamma_\sigma \subset \Gamma_w$ , and thus we assume that the flow is stationary. Other combinations of control and controlled surfaces are also possible.



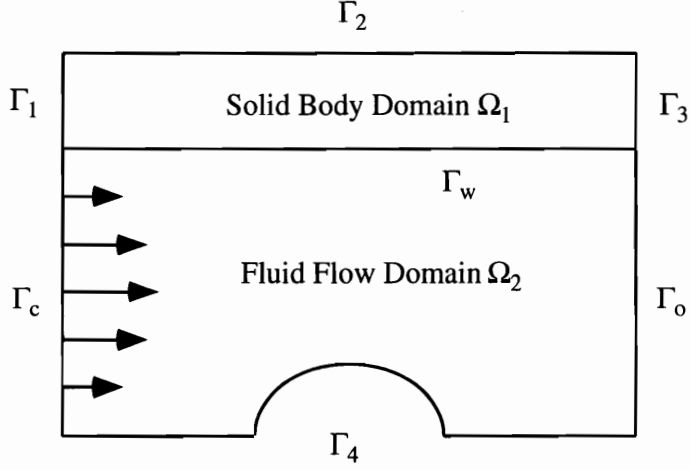


Figure 2.1: The domain  $\Omega$

As a result of our assumptions about the flow, the state variables, *i.e.*, the velocity  $\mathbf{u}$ , pressure  $p$ , temperature  $T$ , and control  $g$  are required to satisfy the Navier-Stokes equations

$$-\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega_2, \quad (2.1.1)$$

the incompressibility constraint

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega_2, \quad (2.1.2)$$

with, for simplicity, the boundary conditions

$$\mathbf{u} = \mathbf{h} \quad \text{on } \Gamma_c, \quad (2.1.3)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_w \cup \Gamma_4, \quad (2.1.4)$$

$$\frac{\partial \mathbf{u}}{\partial n} = \mathbf{0} \quad \text{on } \Gamma_o, \quad (2.1.5)$$

and the energy equations

$$-\kappa_1 \Delta T = \bar{Q}_1 \quad \text{in } \Omega_1, \quad (2.1.6)$$

$$-\kappa_2 \Delta T + (\mathbf{u} \cdot \nabla) T = \bar{Q}_2 + 2\nu(\nabla \mathbf{u} + \nabla \mathbf{u}^T) : (\nabla \mathbf{u} + \nabla \mathbf{u}^T) \quad \text{in } \Omega_2, \quad (2.1.7)$$

with the boundary conditions

$$T = g \quad \text{on } \Gamma_c, \quad (2.1.8)$$

$$\frac{\partial T}{\partial n} = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_o. \quad (2.1.9)$$

The data functions  $\mathbf{f}$ ,  $\mathbf{h}$ ,  $\bar{Q}_1$  and  $\bar{Q}_2$  are assumed to be known. The constant  $\nu$  is the kinematic viscosity coefficient of the fluid, and the constants  $\kappa_1$  and  $\kappa_2$  are the thermal conductivity coefficients in the solid and fluid, respectively; see [103] for details.

Note that as a result of our assumptions about the flow, the mechanical equations (2.1.1)-(2.1.5) uncouple from the thermal equations (2.1.6)-(2.1.9). Indeed, (2.1.6)-(2.1.9) may be solved for  $\mathbf{u}$  and  $p$  without regard of the temperature  $T$ . Thus, in the present context, the velocity field  $\mathbf{u}$ , which is determined by solving (2.1.1)-(2.1.5), merely acts as a coefficient function and in the source term in (2.1.7).

We now define the optimal control problem. Our objective is, given a velocity field  $\mathbf{u}$ , to seek a temperature field  $T$  and a control  $g$  such that the functional

$$\mathcal{J}(T, g) = \frac{1}{2\gamma} \int_{\Gamma_\sigma} |T - T_d|^2 d\Gamma + \frac{\kappa_2 \delta}{2} \int_{\Gamma_c} (|g|^2 + |\nabla_s g|^2) d\Gamma \quad (2.1.10)$$

is minimized subject to (2.1.6)-(2.1.9), where  $\nabla_s$  denotes the surface gradient operator and  $T_d$  is some desired temperature distribution, *e.g.*, something close to the average temperature along  $\Gamma_\sigma$  for the uncontrolled system. The non-negative parameters  $\gamma$  and  $\delta$  can be used to change the relative importance of the two terms appearing in the definition of  $\mathcal{J}$  as well as to act as a penalty parameter. Incidentally, the appearance of the control  $g$  in the  $\mathcal{J}$  is necessary because we are not imposing any *a priori* limits on the size of this control. It should be noted that the minimization of (2.1.10) can be interpreted as an approximate

“controllability” result in the sense that one can control the solution to be “close”, *i.e.*, in an  $L^2$ -norm sense, to any desired distribution  $T_d$ .

Under the realistic assumption that  $\mathbf{u} \cdot \mathbf{n} = 0$  on  $\Gamma_w \cup \Gamma_4$  and  $\mathbf{u} \cdot \mathbf{n} \geq 0$  on  $\Gamma_o$ , we prove the existence and uniqueness of optimal solutions and derive an optimality system, *i.e.*, a set of equations from which the optimal control and state may be determined. Also, a finite element method is used to compute an approximate solution of the optimality system. We have also developed and implemented an iterative algorithm to compute the approximate solution.

We close this section by introducing some of the notation used below. Throughout,  $C$  will denote a positive constant whose meaning and value changes with context. Also,  $H^s(\mathcal{D})$ ,  $s \in \mathbb{R}$ , denotes the standard Sobolev space of order  $s$  with respect to the set  $\mathcal{D}$ , where  $\mathcal{D}$  is either the domain  $\Omega$ , or its boundary  $\Gamma$ , or part of that boundary. Of course,  $H^0(\mathcal{D}) = L^2(\mathcal{D})$ . Dual spaces will be denoted by  $(\cdot)^*$ .

Of particular interest will be the space

$$H^1(\Omega) = \{S \in L^2(\Omega) : \frac{\partial S}{\partial x_k} \in L^2(\Omega) \text{ for } k = 1, 2\} \quad (2.1.11)$$

and the subspace

$$H_D^1(\Omega) = \{S \in H^1(\Omega) : S = 0 \text{ on } \Gamma_c\}. \quad (2.1.12)$$

For functions defined on  $\Gamma_c$ , we will use the subspace

$$W(\Gamma_c) = \{g \in H^1(\Gamma_c) : g = 0 \text{ at } \bar{\Gamma}_c \cap \bar{\Gamma}_1\}. \quad (2.1.13)$$

Norms of functions belonging to  $H^s(\Omega)$ ,  $H^s(\Gamma)$  and  $H^s(\Gamma_c)$  are denoted by  $\|\cdot\|_{s,\Omega}$ ,  $\|\cdot\|_{s,\Gamma}$  and  $\|\cdot\|_{s,\Gamma_c}$ , respectively. Of particular interest are the  $L^2(\Omega)$ -norm  $\|\cdot\|_{0,\Omega}$ , the semi-norm

$$|T|_{1,\Omega}^2 = \sum_{j=1}^2 \left\| \frac{\partial T}{\partial x_j} \right\|_{0,\Omega}^2 \quad (2.1.14)$$

and norm

$$\|T\|_{1,\Omega}^2 = |T|_{1,\Omega}^2 + \|T\|_{0,\Omega}^2 \quad (2.1.15)$$

defined for functions belonging to  $H^1(\Omega)$ . Also, we are interested in the semi-norm  $|\cdot|_{1,\Gamma_c}$  defined by

$$|g|_{1,\Gamma_c}^2 = \int_{\Gamma_c} |\nabla_s g|^2 d\Gamma_c \quad (2.1.16)$$

and the norm

$$\|g\|_{1,\Gamma_c}^2 = |g|_{1,\Gamma_c}^2 + \|g\|_{0,\Gamma_c}^2 \quad (2.1.17)$$

defined for functions belonging to  $H^1(\Gamma_c)$  and  $W(\Gamma_c)$ .

We define, for  $(TS) \in L^1(\Omega)$ ,

$$(T, S)_\Omega = \int_{\Omega} TS d\Omega \quad (2.1.18)$$

and, for  $(pq) \in L^1(\Gamma)$ ,

$$(p, q)_\Gamma = \int_{\Gamma} pq d\Gamma. \quad (2.1.19)$$

Thus, the inner product in  $L^2(\Omega)$  is denoted by  $(\cdot, \cdot)_\Omega$ , that in  $L^2(\Gamma)$  by  $(\cdot, \cdot)_\Gamma$ . The notation of (2.1.18)-(2.1.19) will also be employed to denote the pairing between Sobolev spaces and their duals.

We will use the bilinear forms, for  $i = 1, 2$ ,

$$a_i(T, S) = \kappa_i \int_{\Omega_i} \nabla T \cdot \nabla S d\Omega \quad \forall T, S \in H^1(\Omega) \quad (2.1.20)$$

and

$$a(T, S) = \sum_{i=1}^2 \kappa_i \int_{\Omega_i} \nabla T \cdot \nabla S d\Omega = a_1(T, S) + a_2(T, S), \quad (2.1.21)$$

and the trilinear form

$$c(\mathbf{u}, T, S)_{\Omega_2} = \int_{\Omega_2} (\mathbf{u} \cdot \nabla T) S d\Omega \quad \forall \mathbf{u} \in \mathbf{H}^1(\Omega_2) \text{ and } \forall T, S \in H^1(\Omega_2). \quad (2.1.22)$$

These forms are continuous in the sense that there exist constants  $c_i > 0$  and  $c_c > 0$  such that, for  $i = 1, 2$

$$|a_i(T, S)| \leq c_i \|T\|_{1, \Omega_i} \|S\|_{1, \Omega_i} \quad \forall T, S \in H^1(\Omega), \quad (2.1.23)$$

$$|c(\mathbf{u}, T, S)_{\Omega_2}| \leq c_c \|\mathbf{u}\|_{1, \Omega_2} \|T\|_{1, \Omega_2} \|S\|_{1, \Omega_2} \quad (2.1.24)$$

$$\forall \mathbf{u} \in \mathbf{H}^1(\Omega_2) \text{ and } \forall T, S \in H^1(\Omega),$$

and thus

$$\begin{aligned} |a(T, S)| &\leq |a_1(T, S)| + |a_2(T, S)| \\ &\leq c_1 \|T\|_{1, \Omega_1} \|S\|_{1, \Omega_1} + c_2 \|T\|_{1, \Omega_2} \|S\|_{1, \Omega_2} \\ &\leq (c_1 + c_2) \|T\|_{1, \Omega} \|S\|_{1, \Omega} \quad \forall T, S \in H^1(\Omega). \end{aligned} \quad (2.1.25)$$

Moreover, we have the coercivity property, for  $i = 1, 2$ , there exist constants  $C_i > 0$  such that

$$a_i(T, T) \geq C_i \|T\|_{1, \Omega_i}^2 \quad \forall T \in H^1(\Omega) \quad (2.1.26)$$

and thus

$$a(T, T) = a_1(T, T) + a_2(T, T) \geq \sum_{i=1}^2 C_i \|T\|_{1, \Omega_i}^2 \geq \min(C_1, C_2) \|T\|_{1, \Omega}^2. \quad (2.1.27)$$

## 2.2 The Optimization Problem and the Existence of Optimal Solutions

We now give a precise statement of the optimization problem we consider. We will assume the domain  $\Omega$  is in  $R^2$  and consists of two subdomains  $\Omega_1$  and  $\Omega_2$  such that  $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma_w$ . Let  $g \in W(\Gamma_c)$  denote the boundary control and let  $T \in H^1(\Omega)$  denote the state, *i.e.*, the temperature field. The state and control variables are constrained to

satisfy the system (2.1.6)-(2.1.9) , which we recast into the following weak form: find  $(T, t) \in H^1(\Omega) \times H^{-1/2}(\Gamma_c)$  such that

$$a(T, S) + c(\mathbf{u}, T, S)_{\Omega_2} - (t, S)_{\Gamma_c} = (Q, S)_{\Omega} \quad \forall S \in H^1(\Omega) \quad (2.2.1)$$

and

$$(T, R)_{\Gamma_c} - (g, R)_{\Gamma_c} = 0 \quad \forall R \in H^{-1/2}(\Gamma_c), \quad (2.2.2)$$

where we have introduced the simplifying notation

$$Q = \begin{cases} \bar{Q}_1 & \text{in } \Omega_1, \\ \bar{Q}_2 + 2\nu(\nabla \mathbf{u} + \nabla \mathbf{u}^T) : (\nabla \mathbf{u} + \nabla \mathbf{u}^T) & \text{in } \Omega_2. \end{cases}$$

One may show that, in the distributional sense,

$$t = \kappa_2 \nabla T \cdot \mathbf{n}|_{\Gamma_c}. \quad (2.2.3)$$

In (2.2.1)-(2.2.2), we introduced the Lagrange multiplier  $t$  to enforce the boundary condition. This will be very useful in the proof of error estimates for finite element approximations.

First, we show that for each possible control function  $g$ , there is a unique corresponding state function  $(T, t)$ .

**Lemma 2.2.1** *For every  $g \in W(\Gamma_c)$ , there exists a unique  $(T, t) \in H^1(\Omega) \times H^{-1/2}(\Gamma_c)$  such that (2.2.1)-(2.2.2) are satisfied. Moreover, there exists a constant  $C = C(\Omega)$  such that*

$$\|T\|_{1,\Omega} + \|t\|_{-1/2,\Gamma_c} \leq C(\|Q\|_{-1,\Omega} + \|g\|_{1,\Gamma_c}). \quad (2.2.4)$$

*Proof.* For given  $g \in W(\Gamma_c)$ , (2.2.1)-(2.2.2) is equivalent to

$$a(T, S) + c(\mathbf{u}, T, S)_{\Omega_2} = (Q, S)_{\Omega} \quad \forall S \in H_D^1(\Omega), \quad (2.2.5)$$

$$T = g \quad \text{on } \Gamma_c \quad (2.2.6)$$

and

$$t = \kappa_2 \nabla T \cdot \mathbf{n}|_{\Gamma_c} \quad (2.2.7)$$

in the distributional sense. By the virtue of the trace theorem, let  $\hat{T}$  in  $H^1(\Omega)$  satisfy  $\hat{T} = g$  on  $\Gamma_c$  and examine the following problem: find  $T$  in  $H^1(\Omega)$  such that

$$\begin{aligned} T - \hat{T} &\in H_D^1(\Omega), \\ a(T - \hat{T}, S) + c(\mathbf{u}, T - \hat{T}, S)_{\Omega_2} & \\ &= (Q, S)_\Omega - a(\hat{T}, S) - c(\mathbf{u}, \hat{T}, S)_{\Omega_2} \quad \forall S \in H_D^1(\Omega). \end{aligned} \quad (2.2.8)$$

Let  $\tilde{T} = T - \hat{T} \in H_D^1(\Omega)$ . From the assumption of  $\mathbf{u}$ , *i.e.*,  $\mathbf{u} \cdot \mathbf{n} = 0$  on  $\Gamma_w \cup \Gamma_4$  and  $\mathbf{u} \cdot \mathbf{n} \geq 0$  on  $\Gamma_o$ , we have that

$$\begin{aligned} c(\mathbf{u}, \tilde{T}, \tilde{T})_{\Omega_2} &= \frac{1}{2} \int_{\partial\Omega_2} (\mathbf{u} \cdot \mathbf{n}) \tilde{T}^2 d\Gamma \\ &= \frac{1}{2} \int_{\Gamma_o} (\mathbf{u} \cdot \mathbf{n}) \tilde{T}^2 d\Gamma \\ &\geq 0. \end{aligned} \quad (2.2.9)$$

Thus, we have

$$a(\tilde{T}, \tilde{T}) + c(\mathbf{u}, \tilde{T}, \tilde{T})_{\Omega_2} \geq \min(C_1, C_2) \|\tilde{T}\|_{1,\Omega}^2. \quad (2.2.10)$$

Therefore, by the Lax-Milgram theorem there is a unique  $\tilde{T} \in H_D^1(\Omega)$  and there is unique  $T = \tilde{T} + \hat{T} \in H^1(\Omega)$  and the estimate

$$\|T\|_{1,\Omega} \leq C(\|Q\|_{-1,\Omega} + \|g\|_{1/2,\Gamma_c}) \leq C(\|Q\|_{-1,\Omega} + \|g\|_{1,\Gamma_c}) \quad (2.2.11)$$

holds. From the trace theorem and the theory of partial differential equations (see [8]), we have

$$\|t\|_{-1/2,\Gamma_c} \leq C(\|T\|_{1,\Omega_2} + \|Q_2\|_{-1,\Omega_2}) \leq C(\|T\|_{1,\Omega} + \|Q\|_{-1,\Omega}), \quad (2.2.12)$$

where  $Q_2 = \bar{Q}_2 + 2\nu(\nabla \mathbf{u} + \nabla \mathbf{u}^T) : (\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ . Then, (2.2.4) follows from above two estimates.  $\square$

The *admissibility set*  $\mathcal{U}_{ad}$  is defined by

$$\mathcal{U}_{ad} = \{(T, g) \in H^1(\Omega) \times W(\Gamma_c) : \mathcal{J}(T, g) < \infty, \text{ and there exists a } t \in H^{-1/2}(\Gamma_c) \text{ such that (2.2.1)-(2.2.2) is satisfied}\}. \quad (2.2.13)$$

Then,  $(\hat{T}, \hat{g}) \in \mathcal{U}_{ad}$  is called an *optimal solution* if there exists  $\epsilon > 0$  such that

$$\mathcal{J}(\hat{T}, \hat{g}) \leq \mathcal{J}(T, g) \quad \forall (T, g) \in \mathcal{U}_{ad} \text{ satisfying } \|T - \hat{T}\|_1 + \|g - \hat{g}\|_{1, \Gamma_c} \leq \epsilon. \quad (2.2.14)$$

We now show the existence and uniqueness of optimal solutions.

**Theorem 2.2.2** *There exists a unique optimal solution  $(\hat{T}, \hat{g}) \in \mathcal{U}_{ad}$ .*

*Proof.* We first claim that  $\mathcal{U}_{ad}$  is not empty. Let  $g \equiv 0$  and then let  $(\tilde{T}, \tilde{g}) \in H^1(\Omega) \times W(\Gamma_c)$  be a solution of (2.2.1)-(2.2.2); note that with  $g = 0$ , (2.2.1)-(2.2.2) is equivalent to

$$a(\tilde{T}, S) + c(\mathbf{u}, \tilde{T}, S)_{\Omega_2} = (Q, S)_{\Omega} \quad \forall S \in H_D^1(\Omega),$$

$$\tilde{T} = 0 \quad \text{on } \Gamma_c,$$

and

$$\tilde{t} = \kappa_2 \nabla \tilde{T} \cdot \mathbf{n}|_{\Gamma_c} \quad (2.2.15)$$

in the distributional sense. By Lemma 2.2.1,  $\tilde{T}$ , and therefore  $\tilde{t}$ , exists and  $(\tilde{T}, 0) \in \mathcal{U}_{ad}$ .

Now let  $\{T^{(n)}, g^{(n)}\}$  be a sequence in  $\mathcal{U}_{ad}$  such that

$$a(T^{(n)}, S) + c(\mathbf{u}, T^{(n)}, S)_{\Omega_2} - (t^{(n)}, S)_{\Gamma_c} = (Q, S)_{\Omega} \quad \forall S \in H^1(\Omega), \quad (2.2.16)$$



$$(T^{(n)}, R)_{\Gamma_c} - (g^{(n)}, R)_{\Gamma_c} = 0 \quad \forall R \in H^{-1/2}(\Gamma_c), \quad (2.2.17)$$

and

$$\lim_{n \rightarrow \infty} \mathcal{J}(T^{(n)}, g^{(n)}) = \inf_{(T, g) \in \mathcal{U}_{ad}} \mathcal{J}(T, g),$$

for some  $t^{(n)} \in H^{-1/2}(\Gamma_c)$ . Then, using (2.1.10) and (2.2.13), we have that  $\{\|g^{(n)}\|_{1, \Gamma_c}\}$  is uniformly bounded which in turn yields that  $\{\|T^{(n)}\|_1\}$  and  $\{\|t^{(n)}\|_{-1/2, \Gamma_c}\}$  are uniformly bounded. We may then extract subsequences such that

$$\begin{aligned} g^{(n)} &\rightharpoonup \hat{g} \text{ in } W(\Gamma_c) \\ T^{(n)} &\rightharpoonup \hat{T} \text{ in } H^1(\Omega) \\ t^{(n)} &\rightharpoonup \hat{t} \text{ in } H^{-1/2}(\Gamma_c) \\ T^{(n)} &\rightarrow \hat{T} \text{ in } L^2(\Omega) \\ T^{(n)}|_{\Gamma_\sigma} &\rightarrow \hat{T}|_{\Gamma_\sigma} \text{ in } L^2(\Gamma_\sigma) \end{aligned}$$

for some  $(\hat{T}, \hat{g}) \in H^1(\Omega) \times W(\Gamma_c)$ . The last two convergence results above follow from the compact imbeddings  $H^1(\Omega) \subset L^2(\Omega)$  and  $H^{1/2}(\Gamma_\sigma) \subset L^2(\Gamma_\sigma)$ . We may then easily pass to the limit in (2.2.16)-(2.2.17) to determine that  $(\hat{T}, \hat{g}, \hat{t})$  satisfies (2.2.1)-(2.2.2). Now, by the weak lower semicontinuity of  $\mathcal{J}(\cdot, \cdot)$ , we conclude that  $(\hat{T}, \hat{g})$  is an optimal solution, *i.e.*,

$$\mathcal{J}(\hat{T}, \hat{g}) = \inf_{(T, g) \in \mathcal{U}_{ad}} \mathcal{J}(T, g).$$

Thus, we have shown that an optimal solution belonging to  $\mathcal{U}_{ad}$  exists. Finally, the uniqueness of the optimal solution follows from the convexity of the functional and the linearity of the constraint equations.  $\square$

Let

$$L = - \sum_{i=1}^2 \frac{\partial}{\partial x_i} \left( \sum_{j=1}^2 a_{ij} \frac{\partial}{\partial x_j} \right) + \sum_{i=1}^2 c_i \frac{\partial}{\partial x_i}$$

be a differential operator of the second order in divergence form on an open set  $\Omega$  of  $R^2$ .

We introduce the *bilinear differential operator associated with  $L$*

$$L(T, S) = \sum_{i,j=1}^2 a_{ij} \frac{\partial T}{\partial x_i} \frac{\partial S}{\partial x_j} + \sum_{i=1}^2 c_i \frac{\partial T}{\partial x_i} S. \quad (2.2.18)$$

Now, setting

$$a_{11} = a_{22} = \begin{cases} \kappa_1 & \text{on } \Omega_1, \\ \kappa_2 & \text{on } \Omega_2 \end{cases} \quad (2.2.19)$$

$$a_{12} = a_{21} = 0 \quad \text{on } \Omega, \quad (2.2.20)$$

$$c_i = \begin{cases} 0 & \text{on } \Omega_1, \\ u_i & \text{on } \Omega_2, \end{cases} \quad (2.2.21)$$

where  $\mathbf{u} = (u_1, u_2)$ , we have the following theorems.

**Theorem 2.2.3** *Let  $T$  be the solution of (2.2.5)-(2.2.6) and let  $T_1$  and  $T_2$  be the restrictions of  $T$  to  $\Omega_1$  and  $\Omega_2$ , then  $T_1$  and  $T_2$  are solutions of the transmission problem*

$$a_1(T_1, S_1) = (Q_1, S_1)_{\Omega_1} \quad \forall S_1 \in H^1(\Omega_1), \quad (2.2.22)$$

$$a_2(T_2, S_2) + c(\mathbf{u}, T_2, S_2) = (Q_2, S_2)_{\Omega_2} \quad \forall S_2 \in H_D^1(\Omega_2), \quad (2.2.23)$$

$$T_2 = g \quad \text{on } \Gamma_c, \quad (2.2.24)$$

$$T_1 = T_2 \quad \text{on } \Gamma_w, \quad (2.2.25)$$

$$\kappa_1 \frac{\partial T_1}{\partial n} + \kappa_2 \frac{\partial T_2}{\partial n} = 0 \quad \text{on } \Gamma_w. \quad (2.2.26)$$

*Proof.* For the proof, see [32].  $\square$

**Theorem 2.2.4** *If we suppose the restrictions of  $Q$  in (2.2.5)-(2.2.6) to  $\Omega_1$  and  $\Omega_2$  are  $C^\infty(\Omega_1 \cup \Gamma_w)$  and  $C^\infty(\Omega_2 \cup \Gamma_w)$ , respectively, and  $g \in C^\infty$ , then every solution in  $H^1(\Omega)$  of*

(2.2.5)-(2.2.6) has its restrictions  $T_1, T_2$  to  $\Omega_1, \Omega_2$  belong to  $C^\infty(\Omega_1 \cup \Gamma_w), C^\infty(\Omega_2 \cup \Gamma_w)$ , respectively, and therefore in particular, the result applies to the solutions of the transmission problem (2.2.22)-(2.2.26).

*Proof.* For the proof, see ([31], Proposition 9, p592 ).  $\square$

We define the space  $\tilde{H}^s(\Omega)$  for  $s \geq 1$  by

$$\tilde{H}^s(\Omega) = \{T \in H^1(\Omega) : \|T\|_{\tilde{H}^s(\Omega)} < \infty\}, \quad (2.2.27)$$

where

$$\|T\|_{\tilde{H}^s(\Omega)}^2 = \|T\|_{H^1(\Omega)}^2 + \sum_{i=1}^2 \|T\|_{H^s(\Omega_i)}^2. \quad (2.2.28)$$

**Theorem 2.2.5** *Let  $T \in H^1(\Omega)$  be the solution of the problem (2.2.1)-(2.2.2). Then, we have*

$$\|T\|_{\tilde{H}^2(\Omega)} + \|t\|_{H^{1/2}(\Gamma_w)} \leq C\|Q\|_{H^0(\Omega)}. \quad (2.2.29)$$

*Proof.* For the proof, see ([9], Theorem 8.5.1 ).  $\square$

### 2.3 First-Order Necessary Conditions for the Optimal Solution and an Optimal System

We now proceed to derive the first-order optimality conditions associated with problem (2.2.14). The optimal control problem (2.2.14) is equivalent to the following minimization problem: find  $g \in W(\Gamma_c)$  such that  $\mathcal{K}(g) := \mathcal{J}(T(g), g)$  is minimized where  $T(g) \in H^1(\Omega)$  is defined as solution of (2.2.1)-(2.2.2). By studying the Gâteaux derivative of the functional  $\mathcal{K}(g)$ , we can obtain the first-order necessary conditions for the optimal solution  $(\hat{T}, \hat{g})$  in a straightforward manner. Let  $\hat{g}$  be a solution of the minimization problem  $\min_{g \in W(\Gamma_c)} \mathcal{K}(g)$ ,

then for every  $z \in W(\Gamma_c)$  we have

$$\forall \lambda \in \mathbb{R}, \quad \mathcal{K}(\hat{g} + \lambda z) \geq \mathcal{K}(\hat{g}) \quad (2.3.1)$$

due to the definition of  $\hat{g}$ . In particular, we have,

$$\forall \lambda > 0, \quad \frac{\mathcal{K}(\hat{g} + \lambda z) - \mathcal{K}(\hat{g})}{\lambda} \geq 0 \quad (2.3.2)$$

and,

$$\forall \lambda < 0, \quad \frac{\mathcal{K}(\hat{g} + \lambda z) - \mathcal{K}(\hat{g})}{\lambda} \leq 0 \quad (2.3.3)$$

which implies that the Gâteaux derivative of  $\mathcal{K}(\hat{g})$

$$\frac{d\mathcal{K}(\hat{g})}{dg} = \frac{d\mathcal{J}(T(\hat{g}), \hat{g})}{dg} = 0. \quad (2.3.4)$$

**Lemma 2.3.1** *The mapping  $g \rightarrow T(g)$ , from  $W(\Gamma_c)$  into  $H^1(\Omega)$ , has a Gâteaux derivative  $(dT(g)/dg) \cdot z$  in every direction  $z \in W(\Gamma_c)$ . Furthermore,  $(dT(g)/dg) \cdot z = V(z)$  is the solution of*

$$a(V, S) + c(\mathbf{u}, V, S)_{\Omega_2} - (\eta, S) = 0 \quad \forall S \in H^1(\Omega), \quad (2.3.5)$$

$$(V, R)_{\Gamma_c} = (z, R)_{\Gamma_c} \quad \forall R \in H^{-1/2}(\Gamma_c). \quad (2.3.6)$$

*Proof.* It is immediate from the linearity of (2.2.1)-(2.2.2).  $\square$

Now, we derive an optimality system from the first-order necessary condition (2.3.4). For each fixed  $g$ , the derivative  $d\mathcal{K}(g)/dg \cdot z$  for every direction  $z \in W(\Gamma_c)$  may be easily computed

$$\begin{aligned} \frac{d\mathcal{K}(g)}{dg} \cdot z &= \kappa_2 \delta \int_{\Gamma_c} (\nabla g \cdot \nabla z + gz) \, d\Gamma + \frac{1}{\gamma} \int_{\Gamma_\sigma} (T - T_d) V \, d\Gamma \\ &= \kappa_2 \delta (\nabla g, \nabla z)_{\Gamma_c} + \kappa_2 \delta (g, z)_{\Gamma_c} + \frac{1}{\gamma} (T - T_d, V)_{\Gamma_\sigma}, \\ &\quad \forall z \in W(\Gamma_c), \end{aligned} \quad (2.3.7)$$

where for each  $z \in W(\Gamma_c)$ ,  $V \in H^1(\Omega)$  is the solution of (2.3.5)-(2.3.6).

Let  $(T, t) \in H^1(\Omega) \times H^{-1/2}(\Gamma_c)$  be the solution of (2.2.1)-(2.2.2) and let  $(\Phi, \tau) \in H^1(\Omega) \times H^{-1/2}(\Gamma_c)$  be defined as the solution of the adjoint problem

$$a(Z, \Phi) + c(\mathbf{u}, Z, \Phi)_{\Omega_2} + (\tau, Z)_{\Gamma_c} = \frac{1}{\gamma}(Z, T - T_d)_{\Gamma_\sigma} \quad \forall Z \in H^1(\Omega), \quad (2.3.8)$$

$$(W, \Phi)_{\Gamma_c} = 0 \quad \forall W \in H^{-1/2}(\Gamma_c). \quad (2.3.9)$$

Setting  $S = \Phi$  in (2.3.5)-(2.3.6) and  $Z = V$  in (2.3.8)-(2.3.9), we have that

$$(\tau, z)_{\Gamma_c} = \frac{1}{\gamma}(V, T - T_d)_{\Gamma_\sigma}. \quad (2.3.10)$$

Thus, from the necessary condition (2.3.4), we see that the optimal value of the control  $g$  satisfies

$$(\nabla g, \nabla z)_{\Gamma_c} + (g, z)_{\Gamma_c} = -\frac{1}{\kappa_2 \delta}(\tau, z)_{\Gamma_c} \quad \forall z \in W(\Gamma_c). \quad (2.3.11)$$

Collecting the above results, we obtain the optimal system

$$a(T, S) + c(\mathbf{u}, T, S)_{\Omega_2} - (t, S)_{\Gamma_c} = (Q, S)_\Omega \quad \forall S \in H^1(\Omega), \quad (2.3.12)$$

$$(T, R)_{\Gamma_c} - (g, R)_{\Gamma_c} = 0 \quad \forall R \in H^{-1/2}(\Gamma_c), \quad (2.3.13)$$

$$a(Z, \Phi) + c(\mathbf{u}, Z, \Phi)_{\Omega_2} + (\tau, Z)_{\Gamma_c} = \frac{1}{\gamma}(Z, T - T_d)_{\Gamma_\sigma} \quad \forall Z \in H^1(\Omega), \quad (2.3.14)$$

$$(W, \Phi)_{\Gamma_c} = 0 \quad \forall W \in H^{-1/2}(\Gamma_c) \quad (2.3.15)$$

and

$$(\nabla g, \nabla z)_{\Gamma_c} + (g, z)_{\Gamma_c} = -\frac{1}{\delta \kappa_2}(\tau, z)_{\Gamma_c} \quad \forall z \in W(\Gamma_c). \quad (2.3.16)$$

Integration by parts may be used to show that the system (2.3.12)-(2.3.16) constitutes a weak formulation of the boundary value problem

$$-\kappa_1 \Delta T = \bar{Q}_1 \quad \text{in } \Omega_1, \quad (2.3.17)$$

$$-\kappa_2 \Delta T + (\mathbf{u} \cdot \nabla) T = \bar{Q}_2 + 2\nu(\nabla \mathbf{u} + \nabla \mathbf{u}^T) : (\nabla \mathbf{u} + \nabla \mathbf{u}^T) \quad \text{in } \Omega_2, \quad (2.3.18)$$

$$T = g \quad \text{on } \Gamma_c, \quad (2.3.19)$$

$$-\kappa_1 \Delta \Phi = 0 \quad \text{in } \Omega_1, \quad (2.3.20)$$

$$-\kappa_2 \Delta \Phi - (\mathbf{u} \cdot \nabla) \Phi = 0 \quad \text{in } \Omega_2, \quad (2.3.21)$$

$$\Phi = 0 \quad \text{on } \Gamma_c, \quad (2.3.22)$$

$$\frac{\partial \Phi}{\partial n} = \frac{1}{\kappa_2 \gamma} (T - T_d) \quad \text{on } \Gamma_\sigma \quad (2.3.23)$$

and

$$-\Delta_s g + g = \frac{1}{\delta} \nabla \Phi \cdot \mathbf{n}|_{\Gamma_c} \quad \text{on } \Gamma_c. \quad (2.3.24)$$

## 2.4 Finite Element Approximation and Error Estimates

A finite element discretization of the optimality system (2.3.12)-(2.3.16) is defined as follows. One may choose families of finite dimensional subspaces  $V_1^h \subset H^1(\Omega_1)$ ,  $V_2^h \subset H^1(\Omega_2)$  such that  $V_1^h|_{\Gamma_w} = V_2^h|_{\Gamma_w}$ . These families are parameterized by the parameter  $h$  that tends to zero; commonly, this parameter is chosen to be some measure of the grid size in a subdivision of  $\Omega$  into finite elements. Let  $V^h = V_1^h \cup V_2^h$  and  $O^h = V^h|_{\Gamma_c}$ , *i.e.*,  $O^h$  consists of the restriction, to the boundary  $\Gamma_c$ , of the functions, belonging to  $V^h$ . For all choices of conforming finite element spaces, we then have that  $V^h \subset H^1(\Omega)$  and  $O^h \subset H^{-1/2}(\Gamma_c)$ . Again, for all choices of conforming finite element spaces  $V^h$  we have that  $O^h \subset H^1(\Gamma_c)$ . Let  $N_0^h = O^h \cap W(\Gamma_c)$ . For the subspaces  $V_1^h$ ,  $V_2^h$ ,  $O^h$  and  $N_0^h$ , we assume the approximation properties: there exist an integer  $k$  and a constant  $C$ , independent of  $h$ ,  $T_1$ ,  $T_2$ ,  $t$  and  $g$ , such that

$$\inf_{T_1^h \in V_1^h} \|T_1 - T_1^h\|_1 \leq Ch^m \|T_1\|_{m+1, \Omega_1} \quad \forall T_1 \in H^{m+1}(\Omega_1), \quad 1 \leq m \leq k, \quad (2.4.1)$$

$$\inf_{T_2^h \in V_2^h} \|T_2 - T_2^h\|_1 \leq Ch^m \|T_2\|_{m+1, \Omega_2} \quad \forall T_2 \in H^{m+1}(\Omega_2), \quad 1 \leq m \leq k, \quad (2.4.2)$$

$$\inf_{t^h \in O^h} \|t - t^h\|_{-1/2, \Gamma_c} \leq Ch^m \|t\|_{m-1/2, \Gamma_c} \quad \forall t \in H^{m-1/2}(\Gamma_c), \quad 1 \leq m \leq k \quad (2.4.3)$$

and

$$\inf_{g^h \in N_0^h} \|g - g^h\|_{s, \Gamma_c} \leq Ch^{m-s+1/2} \|g\|_{m+1/2, \Gamma_c} \quad \forall g \in W(\Gamma_c), \quad 1 \leq m \leq k, \quad 0 \leq s \leq 1. \quad (2.4.4)$$

A finite element algorithm for determining approximations of the solution of the optimality system (2.3.12)-(2.3.16) is as follows: seek  $T^h \in V^h$ ,  $t^h \in O^h$ ,  $g^h \in N_0^h$ ,  $\Phi^h \in V^h$  and  $\tau^h \in O^h$  such that

$$a(T^h, S^h) + c(\mathbf{u}, T^h, S^h)_{\Omega_2} - (t^h, S^h)_{\Gamma_c} = (Q, S^h) \quad \forall S^h \in V^h, \quad (2.4.5)$$

$$(T^h, R^h)_{\Gamma_c} - (g^h, R^h)_{\Gamma_c} = 0 \quad R^h \in O^h, \quad (2.4.6)$$

$$\kappa_2 \delta(g^h, K^h)_{\Gamma_c} + \kappa_2 \delta(\nabla_s g^h, \nabla_s K^h)_{\Gamma_c} = -(K^h, \tau^h)_{\Gamma_c} \quad \forall K^h \in N_0^h, \quad (2.4.7)$$

$$a(Z^h, \Phi^h) + c(\mathbf{u}, Z^h, \Phi^h)_{\Omega_2} + (Z^h, \tau^h)_{\Gamma_c} = \frac{1}{\gamma} (Z^h, T^h - T_d)_{\Gamma_\sigma} \quad \forall Z^h \in V^h \quad (2.4.8)$$

and

$$(W^h, \Phi^h)_{\Gamma_c} = 0 \quad \forall W^h \in O^h. \quad (2.4.9)$$

The major task in this section is to obtain error estimates for the finite element approximations. It turns out to be convenient to apply the *Brezzi-Rappaz-Raviart* theory [17], even though our problem is linear. Before doing this, let us introduce the *Brezzi-Rappaz-Raviart* theory which concerns the approximation of a class of nonlinear problems. The nonlinear problems considered in [17] and [27] are of type

$$F(\lambda, \psi) \equiv \psi + TG(\lambda, \psi) = 0 \quad (2.4.10)$$

where  $T \in \mathcal{L}(Y; X)$ ,  $G$  is a  $C^2$  mapping from  $\Lambda \times X$  into  $Y$ ,  $X$  and  $Y$  are Banach spaces and  $\Lambda$  is a compact interval of  $R$ . We say that  $\{(\lambda, \psi(\lambda)) : \lambda \in \Lambda\}$  is a branch of solutions of

(2.4.10) if  $\lambda \rightarrow \psi(\lambda)$  is a continuous function from  $\Lambda$  into  $X$  such that  $F(\lambda, \psi(\lambda)) = 0$ . The branch is called a *nonsingular branch* if we also have that  $D_\psi F(\lambda, \psi(\lambda))$  is an isomorphism from  $X$  into  $X$  for all  $\lambda \in \Lambda$ . (Here,  $D_\psi F(\cdot, \cdot)$  denotes the Frechet derivative of  $F(\cdot, \cdot)$  with respect to the second argument.)

Approximations are defined by introducing a subspace  $X^h \subset X$  and an approximating operator  $T^h \in \mathcal{L}(Y; X^h)$ . Then we seek  $\psi^h \in X^h$  such that

$$F^h(\lambda, \psi^h) \equiv \psi^h + T^h G(\lambda, \psi^h) = 0. \quad (2.4.11)$$

We will assume that there exists another Banach space  $Z$ , contained in  $Y$ , with continuous imbedding, such that

$$D_\psi G(\lambda, \psi) \in \mathcal{L}(X; Z) \quad \forall \lambda \in \Lambda \quad \text{and} \quad \psi \in X. \quad (2.4.12)$$

Concerning the operator  $T^h$ , we assume the approximation properties

$$\lim_{h \rightarrow 0} \|(T^h - T)y\|_X = 0 \quad \forall y \in Y \quad (2.4.13)$$

and

$$\lim_{h \rightarrow 0} \|(T^h - T)\|_{\mathcal{L}(Z; X)} = 0. \quad (2.4.14)$$

Note that (2.4.12) and (2.4.14) imply that the operator  $D_\psi G(\lambda, \psi) \in \mathcal{L}(X; X)$  is compact. Moreover, (2.4.14) follows from (2.4.13) whenever the imbedding  $Z \subset Y$  is compact.

We now state the first results of [17] and [27] that will be used in the sequel. In the statement of the theorem,  $D_\psi^2 G$  represents any are all second Frechet derivatives of  $G$ .

**Theorem 2.4.1** *Let  $X$  and  $Y$  be Banach spaces and  $\Lambda$  a compact subset of  $R$ . Assume that  $G$  is a  $C^2$  mapping from  $\Lambda \times X$  into  $Y$  and that  $D^2 G$  is bounded on all sets of  $\Lambda \times X$ . Assume that (2.4.12)-(2.4.14) hold and that  $\{(\lambda, \psi(\lambda)) : \lambda \in \Lambda\}$  is a branch of nonsingular*



solutions of (2.4.10). Then there exists a neighborhood  $\mathcal{O}$  of the origin in  $X$  and, for  $h \leq h_0$  small enough, a unique  $C^2$  function  $\lambda \rightarrow \psi^h(\lambda) \in X^h$  such that  $\{(\lambda, \psi^h(\lambda)) : \lambda \in \Lambda\}$  is a branch of nonsingular solutions of (2.4.11) and  $\psi^h(\lambda) - \psi(\lambda) \in \mathcal{O}$  for all  $\lambda \in \Lambda$ . Moreover, there exists a constant  $C > 0$ , independent of  $h$  and  $\lambda$ , such that

$$\|\psi^h(\lambda) - \psi(\lambda)\|_X \leq C\|(T^h - T)G(\lambda, \psi(\lambda))\|_X \quad \forall \lambda \in \Lambda. \quad \square \quad (2.4.15)$$

For the second result, we have to introduce two other Banach spaces  $H$  and  $W$ , such that  $W \subset X \subset H$ , with continuous imbeddings, and assume that

$$\begin{aligned} &\text{for all } w \in W, \text{ the operator } D_\psi G(\lambda, w) \text{ may be} \\ &\text{extended as a linear operator of } \mathcal{L}(H; Y), \end{aligned} \quad (2.4.16)$$

the mapping  $w \rightarrow D_\psi G(\lambda, w)$  being continuous from  $W$  onto  $\mathcal{L}(Y; H)$ .

We also suppose that

$$\lim_{h \rightarrow 0} \|(T^h - T)\|_{\mathcal{L}(Y; H)} = 0. \quad (2.4.17)$$

Then we may state the following additional result.

**Theorem 2.4.2** *Assume the hypotheses of Theorem 2.4.1 and also assume that (2.4.16) and (2.4.17) hold. Assume in addition that for each  $\lambda \in \Lambda$ ,  $\psi(\lambda) \in W$  and the function  $\lambda \rightarrow \psi(\lambda)$  is continuous from  $\Lambda$  into  $W$  and for each  $\lambda \in \Lambda$ ,  $D_\psi F(\lambda, \psi(\lambda))$  is an isomorphism of  $H$ . Then, for  $h \leq h_1$  sufficiently small, there exists a constant  $C$ , independent of  $h$  and  $\lambda$ , such that*

$$\begin{aligned} \|\psi^h(\lambda) - \psi(\lambda)\|_H &\leq C\|(T^h - T)G(\lambda, \psi(\lambda))\|_H + \|\psi^h(\lambda) - \psi(\lambda)\|_X^2 \\ &\quad \forall \lambda \in \Lambda. \quad \square \end{aligned} \quad (2.4.18)$$

We now introduce some spaces and operators, and verify the requirements of the *Brezzi-Rappaz-Raviart* theory. In the following discussion, the constants  $\kappa_1$ ,  $\kappa_2$  and  $\delta$  will be held fixed. Thus, the system (2.3.12)-(2.3.16) and (2.4.5)-(2.4.9) depend on the single parameter  $\gamma$ .

Let  $X = H^1(\Omega) \times H^{-1/2}(\Gamma_c) \times W(\Gamma_c) \times H^1(\Omega) \times H^{-1/2}(\Gamma_c)$ ,  $Y = (H^1(\Omega))^* \times H^{-1/2}(\Gamma_c) \times (H^1(\Omega))^*$  and  $Z = L^2(\Omega) \times L^2(\Gamma_c) \times L^2(\Gamma_\sigma)$ . Let the operator  $\mathcal{B} \in \mathcal{L}(Y, X)$  be defined as follows:  $\mathcal{B}(\tilde{Q}, \tilde{\Theta}, \tilde{P}) = (\tilde{T}, \tilde{t}, \tilde{g}, \tilde{\Phi}, \tilde{\tau})$  for  $(\tilde{Q}, \tilde{\Theta}, \tilde{P}) \in Y$  and  $(\tilde{T}, \tilde{t}, \tilde{g}, \tilde{\Phi}, \tilde{\tau}) \in X$  if and only if

$$a(\tilde{T}, S) + c(\mathbf{u}, \tilde{T}, S)_{\Omega_2} - (\tilde{t}, S)_{\Gamma_c} = (\tilde{Q}, S)_\Omega \quad \forall S \in H^1(\Omega), \quad (2.4.19)$$

$$(\tilde{T}, R)_{\Gamma_c} - (\Theta, R)_{\Gamma_c} = 0 \quad \forall R \in H^{-1/2}(\Gamma_c), \quad (2.4.20)$$

$$\kappa_2 \delta(\tilde{g}, K)_{\Gamma_c} + \kappa_2 \delta(\nabla_s \tilde{g}, \nabla_s K)_{\Gamma_c} = -(K, \tilde{\tau})_{\Gamma_c} \quad \forall K \in W(\Gamma_c), \quad (2.4.21)$$

$$a(Z, \tilde{\Phi}) + c(\mathbf{u}, Z, \tilde{\Phi})_{\Omega_2} + (Z, \tilde{\tau})_{\Gamma_c} = (Z, \tilde{P})_\Omega \quad \forall Z \in H^1(\Omega) \quad (2.4.22)$$

and

$$(W, \tilde{\Phi})_{\Gamma_c} = 0 \quad \forall W \in H^{-1/2}(\Omega). \quad (2.4.23)$$

Note that this system is weakly coupled. First, one may separately solve the problems (2.4.19)-(2.4.20) for  $\tilde{T}$  and  $\tilde{t}$  and (2.4.22)-(2.4.23) for  $\tilde{\Phi}$  and  $\tilde{\tau}$ ; then, one may solve the Laplacian problem (2.4.21) for  $\tilde{g}$ .

Let  $X^h = V^h \times O^h \times N_0^h \times V^h \times O^h$ . Analogously, the operator  $\mathcal{B}^h \in \mathcal{L}(Y; X^h)$  is defined as follows:  $\mathcal{B}^h(\tilde{Q}, \tilde{\Theta}, \tilde{P}) = (\tilde{T}^h, \tilde{t}^h, \tilde{g}^h, \tilde{\Phi}^h, \tilde{\tau}^h)$  for  $(\tilde{Q}, \tilde{\Theta}, \tilde{P}) \in Y$  and  $(\tilde{T}^h, \tilde{t}^h, \tilde{g}^h, \tilde{\Phi}^h, \tilde{\tau}^h) \in X^h$  if and only if

$$a(\tilde{T}^h, S^h) + c(\mathbf{u}, \tilde{T}^h, S^h)_{\Omega_2} - (\tilde{t}^h, S^h)_{\Gamma_c} = (\tilde{Q}, S^h)_\Omega \quad \forall S^h \in V^h, \quad (2.4.24)$$

$$(\tilde{T}^h, R^h)_{\Gamma_c} - (\Theta, R^h)_{\Gamma_c} = 0 \quad \forall R^h \in O^h, \quad (2.4.25)$$

$$\kappa_2 \delta(\tilde{g}^h, K^h)_{\Gamma_c} + \kappa_2 \delta(\nabla_s \tilde{g}^h, \nabla_s K^h)_{\Gamma_c} = -(K^h, \tilde{\tau}^h)_{\Gamma_c} \quad \forall K^h \in N_0^h, \quad (2.4.26)$$

$$a(Z^h, \tilde{\Phi}^h) + c(\mathbf{u}, Z^h, \tilde{\Phi}^h)_{\Omega_2} + (Z^h, \tilde{\tau}^h)_{\Gamma_c} = (Z^h, \tilde{P}^h)_{\Omega} \quad \forall Z^h \in V^h \quad (2.4.27)$$

and

$$(W^h, \tilde{\Phi}^h)_{\Gamma_c} = 0 \quad \forall W^h \in O^h. \quad (2.4.28)$$

The system (2.4.24)-(2.4.28) is weakly coupled in the same sense as the system (2.4.19)-(2.4.23).

**Theorem 2.4.3** *The second order elliptic problem (2.4.19)-(2.4.23) has a unique solution belonging to  $X$ . Assume that (2.4.1)-(2.4.4) hold. Then the discrete second order elliptic problem (2.4.24)-(2.4.28) has a unique solution belonging to  $X^h$ . Let  $(\tilde{T}, \tilde{t}, \tilde{g}, \tilde{\Phi}, \tilde{\tau})$  and  $(\tilde{T}^h, \tilde{t}^h, \tilde{g}^h, \tilde{\Phi}^h, \tilde{\tau}^h)$  denote the solutions of (2.4.19)-(2.4.23) and (2.4.24)-(2.4.28), respectively. Then we have that*

$$\|\tilde{T} - \tilde{T}^h\|_1 + \|\tilde{t} - \tilde{t}^h\|_{-1/2, \Gamma_c} + \|\tilde{g} - \tilde{g}^h\|_{1, \Gamma_c} + \|\tilde{\Phi} - \tilde{\Phi}^h\|_1 + \|\tilde{\tau} - \tilde{\tau}^h\|_{-1/2, \Gamma_c} \rightarrow 0 \quad \text{as } h \rightarrow 0 \quad (2.4.29)$$

*In addition, if  $(\tilde{T}, \tilde{t}, \tilde{g}, \tilde{\Phi}, \tilde{\tau}) \in \tilde{H}^2(\Omega) \times H^{1/2}(\Gamma_c) \times H^2(\Gamma_c) \times \tilde{H}^2(\Omega) \times H^{1/2}(\Gamma_c)$ , then*

$$\begin{aligned} \|\tilde{T} - \tilde{T}^h\|_1 + \|\tilde{t} - \tilde{t}^h\|_{-1/2, \Gamma_c} + \|\tilde{g} - \tilde{g}^h\|_{1, \Gamma_c} + \|\tilde{\Phi} - \tilde{\Phi}^h\|_1 + \|\tilde{\tau} - \tilde{\tau}^h\|_{-1/2, \Gamma_c} \\ \leq Ch(\|\tilde{T}\|_{\tilde{H}^2(\Omega)} + \|\tilde{\Phi}\|_{\tilde{H}^2(\Omega)}). \end{aligned} \quad (2.4.30)$$

*Proof.* First, it follows from 2.2.1 that the two second order elliptic problems (2.4.19)-(2.4.20) and (2.4.22)-(2.4.23) each have a unique solution  $(\tilde{T}, \tilde{t})$  and  $(\tilde{\Phi}, \tilde{\tau})$  belonging to  $H^1(\Omega) \times H^{-1/2}(\Gamma_c)$ , respectively. From the Babuška's theory, the discrete second order elliptic problems (2.4.24)-(2.4.25) and (2.4.27)-(2.4.28) each have a unique solution  $(\tilde{T}^h, \tilde{t}^h)$  and  $(\tilde{\Phi}^h, \tilde{\tau}^h)$  belonging to  $V^h \times O^h$ , respectively. Moreover, we have that

$$\|\tilde{T} - \tilde{T}^h\|_1 + \|\tilde{t} - \tilde{t}^h\|_{-1/2, \Gamma_c} \rightarrow 0 \quad (2.4.31)$$

and

$$\|\tilde{\Phi} - \tilde{\Phi}^h\|_1 + \|\tilde{\tau} - \tilde{\tau}^h\|_{-1/2, \Gamma_c} \rightarrow 0 \quad (2.4.32)$$

as  $h \rightarrow 0$ , and if in addition  $(\tilde{T}, \tilde{t}) \in \tilde{H}^2(\Omega) \times H^{1/2}(\Gamma_c)$  and  $(\tilde{\Phi}, \tilde{\tau}) \in \tilde{H}^2(\Omega) \times H^{1/2}(\Gamma_c)$ , we have that

$$\|\tilde{T} - \tilde{T}^h\|_1 + \|\tilde{t} - \tilde{t}^h\|_{-1/2, \Gamma_c} \leq Ch \|\tilde{T}\|_{\tilde{H}^2(\Omega)}, \quad (2.4.33)$$

$$\|\tilde{\Phi} - \tilde{\Phi}^h\|_1 + \|\tilde{\tau} - \tilde{\tau}^h\|_{-1/2, \Gamma_c} \leq Ch \|\tilde{\Phi}\|_{\tilde{H}^2(\Omega)}. \quad (2.4.34)$$

Note that the problem (2.4.21) is a well-known equation. Thus, we have that the problems (2.4.21) and (2.4.26) both have unique solutions and that

$$\|\tilde{g}\|_{2, \Gamma_c} \leq C \|\tilde{\tau}\|_{1/2, \Gamma_c} \leq C \|\tilde{\Phi}\|_{\tilde{H}^2(\Omega)}, \quad (2.4.35)$$

$$\|\tilde{g} - \tilde{g}^h\|_{1, \Gamma_c} \leq C (\|\tilde{g} - \hat{g}^h\|_{1, \Gamma_c} + \|\tilde{\tau} - \tilde{\tau}^h\|_{-1/2, \Gamma_c}) \quad \forall \hat{g}^h \in N_0^h. \quad (2.4.36)$$

Using (2.4.4), (2.4.31) and (2.4.32) we then have that

$$\|\tilde{g} - \tilde{g}^h\|_{1, \Gamma_c} \rightarrow 0 \quad \text{as } h \rightarrow 0, \quad (2.4.37)$$

and using (2.4.4), (2.4.33) and (2.4.34), we conclude that

$$\|\tilde{g} - \tilde{g}^h\|_{1, \Gamma_c} \leq Ch \|\tilde{\Phi}\|_{\tilde{H}^2(\Omega)}. \quad (2.4.38)$$

Then, (2.4.31), (2.4.32) and (2.4.37) yield (2.4.29), and (2.4.33), (2.4.34) and (2.4.38) yield (2.4.30).  $\square$

Let  $\Lambda$  denote a compact subset of  $R_+$ . We define the operator  $\mathcal{G}$  from  $\Lambda \times X$  to  $Y$  as follows:  $\mathcal{G}(\gamma, (T, t, g, \Phi, \tau)) = (\tilde{Q}, \tilde{\Theta}, \tilde{P})$  for every  $(\tilde{Q}, \tilde{\Theta}, \tilde{P}) \in Y$  and  $(\gamma, (T, t, g, \Phi, \tau)) \in \Lambda \times X$  if and only if

$$(\tilde{Q}, S)_\Omega = -(\tilde{Q}_1, S)_{\Omega_1} - (Q_2, S)_{\Omega_2} \quad \forall S \in H^1(\Omega), \quad (2.4.39)$$

$$(\tilde{\Theta}, R)_{\Gamma_c} = (-g, R)_{\Gamma_c} \quad \forall R \in H^{-1/2}(\Gamma_c) \quad (2.4.40)$$

and

$$(\tilde{P}, Z)_{\Omega} = -\frac{1}{\gamma}(Z, T - T_d)_{\Gamma_\sigma} \quad \forall Z \in H^1(\Omega), \quad (2.4.41)$$

where  $Q_2 = \bar{Q}_2 + 2\nu(\nabla \mathbf{u} + \nabla \mathbf{u}^T) : (\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ . The operator  $\mathcal{G}$  is obviously of class  $C^\infty$ . The derivative of  $\mathcal{G}$  with respect to  $(T, t, g, \Phi, \tau)$ , which we denote by  $\mathcal{G}_X(\gamma, (T, t, g, \Phi, \tau))$ , can be defined as follows:

$$\mathcal{G}_X(\gamma, (T, t, g, \Phi, \tau)) = (0, -g, -\frac{1}{\gamma}T) \quad (2.4.42)$$

for every  $(\gamma, (T, t, g, \Phi, \tau)) \in \Lambda \times X$ . Furthermore,  $\mathcal{G}(\gamma, (T, t, g, \Phi, \tau)) \in \mathcal{L}(X, Z)$ . Since  $\Lambda$  is a compact interval in  $R_+$  and the constant  $\kappa_2$  is fixed, we see that  $\mathcal{G}$  and its first and second Frechet derivatives and all locally bounded maps.

It is easily seen that the optimality system (2.3.12)-(2.3.16) is equivalent to

$$(T, t, g, \Phi, \tau) + \mathcal{B}\mathcal{G}(\gamma, (T, t, g, \Phi, \tau)) = 0 \quad (2.4.43)$$

and that the discrete optimality system (2.4.5)-(2.4.9) is equivalent to

$$(T^h, t^h, g^h, \Phi^h, \tau^h) + \mathcal{B}^h\mathcal{G}(\gamma, (T^h, t^h, g^h, \Phi^h, \tau^h)) = 0. \quad (2.4.44)$$

Now, having verified the hypotheses of the *Brezzi-Rappaz-Raviart* theory, we may use that theory to deduce the estimate

$$\begin{aligned} \|T - T^h\|_1 + \|t - t^h\|_{-1/2, \Gamma_c} + \|g - g^h\|_{1, \Gamma_c} + \|\Phi - \Phi^h\|_1 + \|\tau - \tau^h\|_{-1/2, \Gamma_c} \\ \leq Ch(\|T\|_{\tilde{H}^2(\Omega)} + \|\Phi\|_{\tilde{H}^2(\Omega)}). \end{aligned} \quad (2.4.45)$$

We also have, from Theorem 2.5, applied to (2.3.12)-(2.3.16), the regularity estimates

$$\begin{aligned} \|T\|_{\tilde{H}^2(\Omega)} + \|t\|_{1/2, \Gamma_c} + \|g\|_{1, \Gamma_c} + \|\Phi\|_{\tilde{H}^2(\Omega)} + \|\tau\|_{1/2, \Gamma_c} \\ \leq C(\|Q\|_0 + \|T_d\|_{1/2, \Gamma_\sigma}). \end{aligned} \quad (2.4.46)$$

The combination of (2.4.45) and (2.4.46) results in the following error estimates.

**Theorem 2.4.4** *Let  $(T, t, \Phi, \tau)$  be the solution of (2.3.12)-(2.3.16) and let  $(T^h, t^h, \Phi^h, \tau^h)$  be the solution of (2.4.5)-(2.4.9). Assume that  $T, \Phi \in \tilde{H}^2(\Omega)$ ; also assume that (2.4.1)-(2.4.4) hold. Then,*

$$\begin{aligned} \|T - T^h\|_1 + \|t - t^h\|_{-1/2, \Gamma_c} + \|g - g^h\|_{1, \Gamma_c} + \|\Phi - \Phi^h\|_1 + \|\tau - \tau^h\|_{-1/2, \Gamma_c} \quad (2.4.47) \\ \leq Ch(\|Q\|_0 + \|T_d\|_{1/2, \Gamma_\sigma}), \end{aligned}$$

where  $C$  is independent of  $h, T, \Phi$ .

We note that higher order estimates are possible if  $T$  is smooth in each sub-domain  $\Omega_1$  and  $\Omega_2$ . For example, if  $(T, t, g, \Phi, \tau) \in \tilde{H}^m(\Omega) \times H^{m-3/2}(\Gamma_c) \times H^{m-1/2}(\Gamma_c) \times \tilde{H}^m(\Omega) \times H^{m-3/2}(\Gamma_c)$  where  $m \geq 3$  and if the finite element spaces  $V_1^h$  and  $V_2^h$  are chosen to be piecewise quadratic polynomials on a triangular mesh such that  $V_1^h = V_2^h$  on  $\Gamma_w$ , then we have

$$\|T - T^h\|_1 + \|t - t^h\|_{-1/2, \Gamma_c} + \|g - g^h\|_{1, \Gamma_c} + \|\Phi - \Phi^h\|_1 + \|\tau - \tau^h\|_{-1/2, \Gamma_c} = O(h^2).$$

## 2.5 Numerical Algorithm

Let us consider the gradient method for the following minimization problem: find  $g \in W(\Gamma_c)$  such that  $\mathcal{K}(g) := \mathcal{J}(T(g), g)$  is minimized where  $T(g) \in H^1(\Omega)$  is defined as solution of (2.2.1)-(2.2.2).

The classical Simple Gradient Algorithm proceeds as follows:

$$\begin{aligned} \text{Given } & g^{(0)}; \\ \text{define } & g^{(n+1)} = g^{(n)} - \frac{1}{\kappa_2 \delta} \frac{d\mathcal{K}(g^{(n)})}{dg} \quad \text{recursively.} \quad (2.5.1) \end{aligned}$$

Recall from section 2.3 that for each fixed  $g$ , the derivative  $d\mathcal{K}(g)/dg \cdot z$  may be computed

$$\frac{d\mathcal{K}(g)}{dg} \cdot z = \kappa_2 \delta(-\Delta_s g + g, z) + \frac{1}{\gamma} (T - T_d, V)_{\Gamma_\sigma} \quad \forall z \in W(\Gamma_c), \quad (2.5.2)$$

where for each  $z \in W(\Gamma_c)$ ,  $V \in H^1(\Omega)$  is the solution of

$$a(V, S) + c(\mathbf{u}, V, S)_{\Omega_2} = 0 \quad \forall S \in H_D^1(\Omega), \quad (2.5.3)$$

$$V = z \quad \text{on} \quad \Gamma_c. \quad (2.5.4)$$

From (2.3.13), we see that

$$\left(-\kappa_2 \frac{\partial \Phi}{\partial n}, z\right)_{\Gamma_c} = \frac{1}{\gamma} (V, T - T_d)_{\Gamma_\sigma}. \quad (2.5.5)$$

Thus, (2.5.1) may be replaced by

$$\begin{aligned} & \text{for } n = 0, 1, 2, \dots, \\ \text{set } g^{(n+1)} &= g^{(n)} - \frac{1}{\kappa_2 \delta} \left( \kappa_2 \delta(-\Delta_s g^{(n)} + g^{(n)}) - \kappa_2 \frac{\partial \Phi^{(n)}}{\partial n} \Big|_{\Gamma_c} \right) \\ &= \Delta_s g^{(n)} + \frac{1}{\delta} \frac{\partial \Phi^{(n)}}{\partial n} \Big|_{\Gamma_c}, \end{aligned} \quad (2.5.6)$$

where  $\Phi^{(n)}$  is determined from  $g^{(n)}$  through the relations

$$a(T^{(n)}, S) + c(\mathbf{u}, T^{(n)}, S) = (Q, S) \quad \forall S \in H_D^1(\Omega), \quad (2.5.7)$$

$$T^{(n)} = g^{(n)} \quad \text{on} \quad \Gamma_c, \quad (2.5.8)$$

and

$$a(Z, \Phi^{(n)}) + c(\mathbf{u}, Z, \Phi^{(n)}) = \frac{1}{\gamma} (Z, T^{(n)} - T_d)_{\Gamma_\sigma} \quad \forall Z \in H_D^1(\Omega), \quad (2.5.9)$$

$$\Phi^{(n)} = 0 \quad \text{on} \quad \Gamma_c. \quad (2.5.10)$$

Therefore, we have that the gradient algorithm results in the following iteration:

Choose  $g^{(1)}$ ;

for  $n = 1, 2, 3, \dots$ , solve for  $T^{(n)}$  and  $\Phi^{(n)}$  from

$$a(T^{(n)}, S) + c(\mathbf{u}, T^{(n)}, S) = (Q, S) \quad \forall S \in H_D^1(\Omega),$$

$$T^{(n)} = g^{(n)} \quad \text{on } \Gamma_c,$$

and

(2.5.11)

$$a(Z, \Phi^{(n)}) + c(\mathbf{u}, Z, \Phi^{(n)}) = \frac{1}{\gamma}(Z, T^{(n)} - T_d)_{\Gamma_\sigma} \quad \forall Z \in H_D^1(\Omega),$$

$$\Phi^{(n)} = 0 \quad \text{on } \Gamma_c,$$

then solve for  $g^{(n+1)}$  from

$$g^{(n+1)} = \Delta_s g^{(n)} + \frac{1}{\delta} \frac{\partial \Phi^{(n)}}{\partial n} \Big|_{\Gamma_c}.$$

The convergence of the algorithm (2.5.11) is a direct consequence of the following lemma.

**Lemma 2.5.1** *Let  $\mathcal{K}$  be a real-valued functional on a Hilbert space  $X$  with norm  $\|\cdot\|_X$  and scalar product  $(\cdot, \cdot)_X$ . Suppose that there exist two constants  $m$  and  $M$  such that*

i)  $\mathcal{K}$  has a local minimum at a point  $\bar{x}$  and is of class  $C^2$  in an open ball  $B$  centered at  $\bar{x}$ ,

ii)  $\forall u \in B, \quad \forall (x, y) \in X \times X, \quad \mathcal{K}''(u) \cdot (x, y) \leq M \|x\|_X \|y\|_X,$

iii)  $\forall u \in B, \quad \forall x \in X, \quad \mathcal{K}''(u) \cdot (x, x) \geq m \|x\|_X^2.$

Let  $\mathbf{R}$  denote the Riesz map, i.e.  $\langle f, x \rangle = (\mathbf{R}f, x)_X$  for all  $x \in X$  and all  $f \in X^*$ . Choose  $x^{(0)} \in B$  and choose a sequence  $\{\rho_n\}$  such that  $0 < \rho_* \leq \rho_n \leq \rho^* < 2m/M^2$ . Then, the sequence  $\{x^{(n)}\}$  defined by

$$x^{(n)} = x^{(n-1)} - \rho_n \mathbf{R} \mathcal{K}'(x^{(n-1)}) \quad \text{for } n = 1, 2, \dots, \quad (2.5.12)$$



converges to  $\bar{x}$ . Furthermore, if  $B = X$  and  $\bar{x}$  is a global minimum, then the gradient algorithm converges to  $\bar{x}$  for any initial value  $x^{(0)}$ .

*Proof.* For the proof, see, e.g., [26]  $\square$

**Theorem 2.5.2** *Let  $(T^{(n)}, \Phi^{(n)}, g^{(n)})$  be the solution of (2.5.11) and  $(T, \Phi, g)$  the solution of (2.3.12)-(2.3.16). Then, if  $\gamma\delta$  is sufficiently large,  $g^{(n)} \rightarrow g$  and thus,  $T^{(n)} \rightarrow T$  in  $H^1(\Omega)$  and  $\Phi^{(n)} \rightarrow \Phi$  in  $H_D^1(\Omega)$  as  $n \rightarrow \infty$ .*

*Proof.* In (2.5.11), we have the fixed parameter  $\rho = 1/(\kappa_2\delta)$ . For each  $g \in W(\Gamma_c)$ , the second Frechet-derivative  $\mathcal{K}''(g) \cdot (z, w)$  may be computed by

$$\mathcal{K}''(g) \cdot (z, w) = \kappa_2\delta(\nabla w, \nabla z)_{\Gamma_c} + \kappa_2\delta(w, z)_{\Gamma_c} + \frac{1}{\gamma}(U, V)_{\Gamma_\sigma}, \quad (2.5.13)$$

where  $U \in H^1(\Omega)$  and  $V \in H^1(\Omega)$  are the solution of

$$a(U, S) + c(\mathbf{u}, U, S) = 0 \quad \forall S \in H_D^1(\Omega), \quad (2.5.14)$$

$$U = w \quad \text{on } \Gamma_c, \quad (2.5.15)$$

and of

$$a(V, S) + c(\mathbf{u}, V, S) = 0 \quad \forall S \in H_D^1(\Omega), \quad (2.5.16)$$

$$U = z \quad \text{on } \Gamma_c. \quad (2.5.17)$$

One can easily have that  $\|U\|_1 \leq C\|w\|_{1,\Gamma_c}$  and  $\|V\|_1 \leq C\|z\|_{1,\Gamma_c}$ , where the value of the constant  $C$  depends only on  $\Omega$ . Then,

$$\begin{aligned} \mathcal{K}''(g) \cdot (z, w) &\leq \kappa_2\delta\|w\|_{1,\Gamma_c}\|z\|_{1,\Gamma_c} + \frac{C}{\gamma}\|w\|_{1,\Gamma_c}\|z\|_{1,\Gamma_c} \\ &= \left(\kappa_2\delta + \frac{C}{\gamma}\right)\|w\|_{1,\Gamma_c}\|z\|_{1,\Gamma_c} \end{aligned} \quad (2.5.18)$$

and

$$\begin{aligned}\mathcal{K}''(g) \cdot (z, z) &= \kappa_2 \delta \|z\|_{1, \Gamma_c}^2 + \frac{1}{\gamma} \int_{\Gamma_\sigma} |V|^2 d\Gamma \\ &\geq \kappa_2 \delta \|z\|_{1, \Gamma_c}^2.\end{aligned}\tag{2.5.19}$$

Setting  $M = \kappa_2 \delta + C/\gamma$  and  $m = \kappa_2 \delta$ , we have, if  $\gamma \delta > C/((\sqrt{2} - 1)\kappa_2)$ ,  $2m/M^2 > \rho = 1/(\kappa_2 \delta)$ . The other hypotheses in the Lemma 2.5.1 are easily shown to be valid. Hence, from that lemma, we obtain that

$$g^{(n)} \rightarrow g \quad \text{in } W(\Gamma_c) \quad \text{as } n \rightarrow \infty.\tag{2.5.20}$$

The desired convergence results follow from the a priori estimate (2.2.4).  $\square$

Of course, the gradient algorithm (2.5.11) is applied to the discrete equations. Then, we have two contribution to the errors in the computational solution, the discretization error  $T - T^h$  and the iteration error  $T^h - T^{h(n)}$ . From practical point of view, it is difficult to calculate  $\Delta_s g^{(n)}$  in the last equation of (2.5.11). By using (2.3.24), we can replace (2.5.11) by the following iteration:

Choose  $g^{(1)}$  and  $\Phi^{(0)}$ ;

for  $n = 1, 2, 3, \dots$ , solve for  $T^{(n)}$  and  $\Phi^{(n)}$  from

$$a(T^{(n)}, S) + c(\mathbf{u}, T^{(n)}, S) = (Q, S) \quad \forall S \in H_D^1(\Omega),$$

$$T^{(n)} = g^{(n)} \quad \text{on } \Gamma_c,$$

and (2.5.21)

$$a(Z, \Phi^{(n)}) + c(\mathbf{u}, Z, \Phi^{(n)}) = \frac{1}{\gamma} (Z, T^{(n)} - T_d)_{\Gamma_\sigma} \quad \forall Z \in H_D^1(\Omega),$$

$$\Phi^{(n)} = 0 \quad \text{on } \Gamma_c,$$

then solve for  $g^{(n+1)}$  from

$$g^{(n+1)} = g^{(n)} - \frac{1}{\delta} \left( \frac{\partial \Phi^{(n-1)}}{\partial n} \Big|_{\Gamma_c} - \frac{\partial \Phi^{(n)}}{\partial n} \Big|_{\Gamma_c} \right).$$

## 2.6 Numerical Experiments

**Test 1 :** We consider that the domain  $\Omega$  is the unit square  $(0, 1) \times (0, 1) \subset \mathbb{R}^2$ , sub-domain  $\Omega_1 = (0, 1) \times (0.75, 1)$  and sub-domain  $\Omega_2 = (0, 1) \times (0, 0.75)$ . Let  $\Gamma_\sigma = (0.075, 1) \times \{0.75\} \subset \Gamma_w = (0, 1) \times \{0.75\}$  and  $\Gamma_c = \{0\} \times (0, 0.75)$  (see Figure 2.1 without the bump on the bottom boundary).

The finite element spaces  $V_1^h$  and  $V_2^h$  are chosen to be piecewise quadratic elements on a triangular mesh such that  $V_1^h = V_2^h$  on  $\Gamma_w$ . We use the mesh size  $h = 1/12$  for all computation. Of course, calculations with varying mesh sizes were performed and these agreed with the theoretical error estimates; thus, we do not report them here.

Now, we consider the following problem

$$-\Delta T = 6.0 \quad \text{on } \Omega_1, \tag{2.6.1}$$

$$-2\Delta T + (\mathbf{u} \cdot \nabla)T = 0 \quad \text{on } \Omega_2, \tag{2.6.2}$$

$$T = 1 + g \quad \text{on } \Gamma_c, \tag{2.6.3}$$

$$\frac{\partial T}{\partial n} = 0 \quad \text{on } \partial\Omega \setminus \Gamma_c, \tag{2.6.4}$$

where the velocity  $\mathbf{u}$  is the solution of the Navier-Stokes equations

$$-\Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{0} \quad \text{in } \Omega_2, \tag{2.6.5}$$

the incompressibility constraint

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega_2, \tag{2.6.6}$$

and the boundary condition

$$\mathbf{u} = \mathbf{h} \quad \text{on } \Gamma_c, \quad (2.6.7)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_w \cup \Gamma_4, \quad (2.6.8)$$

$$\frac{\partial u_1}{\partial n} = 0 \quad \text{and} \quad u_2 = 0 \quad \text{on } \Gamma_o, \quad (2.6.9)$$

where  $\mathbf{u} = (u_1, u_2)$  and  $\mathbf{h} = (1.5y - 2y^2, 0)$ . To get approximate solutions for the Navier-Stokes solutions, we use the Taylor-Hood finite element on the domain  $\Omega_2$ . Actually, we have simple solutions  $\mathbf{u} = (1.5y - 2y^2, 0)$  of the above Navier-Stokes problems.

Note that since all data in (2.6.1)-(2.6.9) are sufficiently smooth in each domain  $\Omega_i$  for  $i = 1, 2$ , we may assume that  $T \in \tilde{H}^s(\Omega)$ ,  $s \geq 3$  by the Theorem 2.4. Thus we may use piecewise quadratic elements for the finite element approximation for obtaining fast convergence with respect to  $h$ .

When  $g = 0$  in (2.6.3), we say that the problem (2.6.1)-(2.6.9) is an uncontrolled problem. The numerical solution of the uncontrolled problem is shown in Figure 2.2 and Figure 2.3 in which one can see that the temperature is above 2.0 on  $(0.3, 1) \times \{0.75\}$  and even higher in the domain  $(0.3, 1) \times (0.75, 1)$ .

Now, we try to get the desired temperature distribution along  $\Gamma_\sigma$ . One can choose any reasonable desired temperature  $T_d$  on  $\Gamma_\sigma$ ; we choose  $T_d = 1.2$  on  $\Gamma_\sigma$ ; thus we have

$$\mathcal{J}(T, g) = \frac{1}{2\gamma} \int_{\Gamma_\sigma} |T - 1.2|^2 d\Gamma + \delta \int_{\Gamma_c} (|g|^2 + |\nabla_s g|^2) d\Gamma. \quad (2.6.10)$$

For the various choices of the parameters  $\gamma$  and  $\delta$  appearing in the functional (2.6.10), computations were performed. In this thesis, we report the numerical results for the cases

1.  $\gamma = \delta = 1$ ,
2.  $\gamma = 0.01$  and  $\delta = 0.003$ .

Table 2.1: Numerical results for Test 1.

	$\ T - 1.2\ _{0,\Gamma_\sigma}^2$	$\ g\ _{\Gamma_c}^2$	$\mathcal{J}(T, g)$
$\gamma = 1, \delta = 1$	$2.88669 \times 10^{-2}$	15.63322	15.64765
$\gamma = 0.01, \delta = 0.003$	$1.93423 \times 10^{-3}$	78.80300	0.333120

The costs are shown in the Table 2.1.

In Figure 2.4-2.7, we plot the surfaces of the temperature  $T$  and adjoint state  $\Phi$  for each case. If one chooses the relatively small  $\gamma\delta$ , then one can have the relatively small value of  $\|T - 1.2\|_{0,\Gamma_\sigma}$ .

Further reinforcement of our conclusions can be obtained from Figure 2.8 and 2.9 in which are found contour plots of the temperature  $T$  and adjoint state  $\Phi$ .

In Figure 2.10, we plot the approximate optimal control  $g^h$  on the boundary  $\Gamma_c$ . In Figure 2.11, we compare the temperature distribution on  $\Gamma_\sigma$  in the uncontrolled case with the optimal temperature distributions in the controlled cases. In examining these figures one should keep in mind that we only trying to match the temperature on  $(0.075, 1) \times \{0.75\}$ .

**Test 2 :** We solve the problem (2.6.1)-(2.6.9) with  $\mathbf{h} = (1.5y - 2y^2, 0)$  on the domain  $\Omega$  which has a bumped boundary (see Figure 2.1 and 2.12). We assume that all parameters and data are the same as in Test 1. To get the approximate solutions for the Navier-Stokes equations, we also use Taylor-Hood finite elements on the domain  $\Omega_2$ .

We report the numerical results for the cases

1.  $\gamma = \delta = 1$ ,
2.  $\gamma = 0.01$  and  $\delta = 0.002$ .

The costs are shown in the Table 2.2. We get the almost same results as in Test 1 except

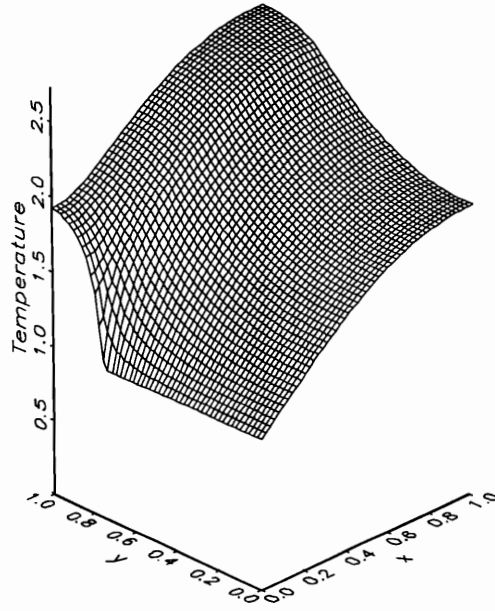


Figure 2.2: Temperature surface plot for the uncontrolled problem.

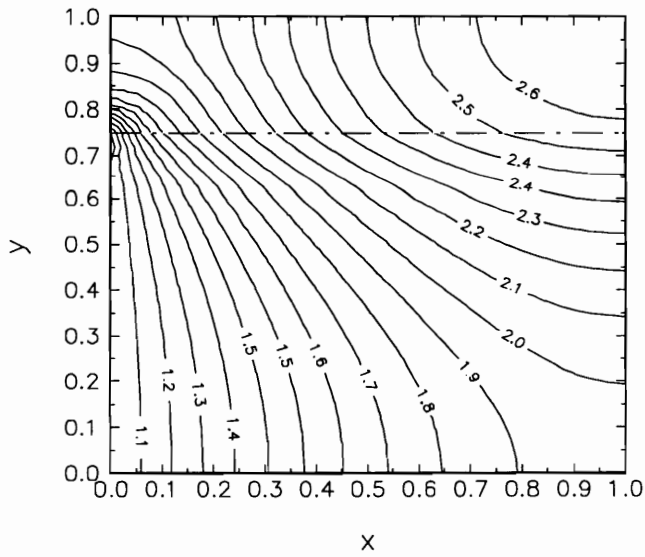


Figure 2.3: Temperature contour plot for the uncontrolled problem.

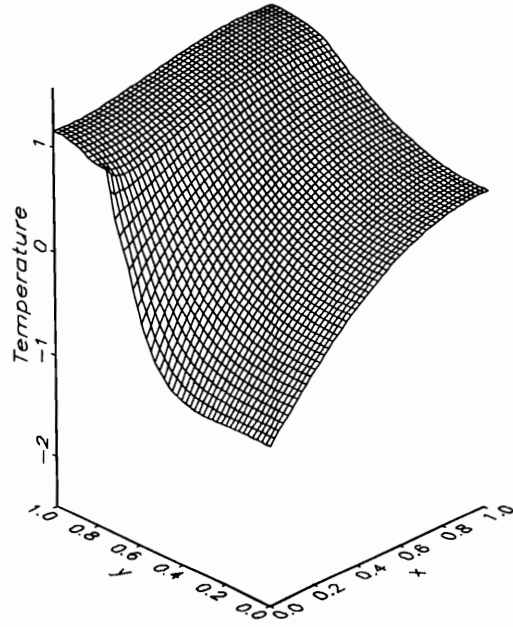


Figure 2.4: Surface plot for the temperature  $T$  ( $\gamma = \delta = 1$ ).

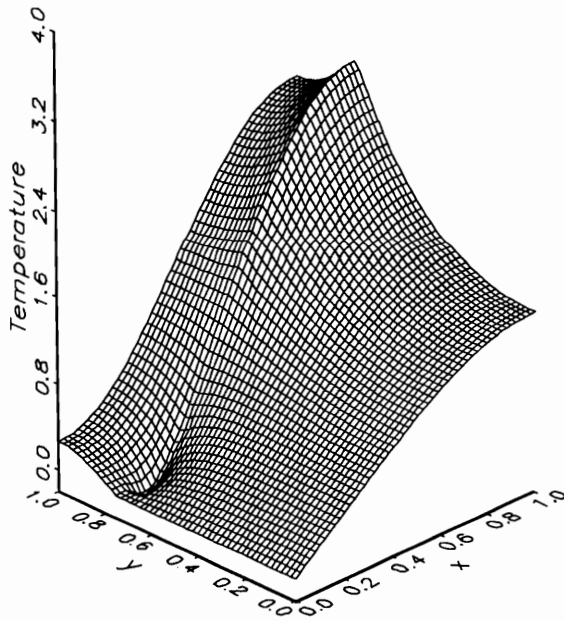


Figure 2.5: Surface plot for the adjoint state  $\Phi$  ( $\gamma = \delta = 1$ ).

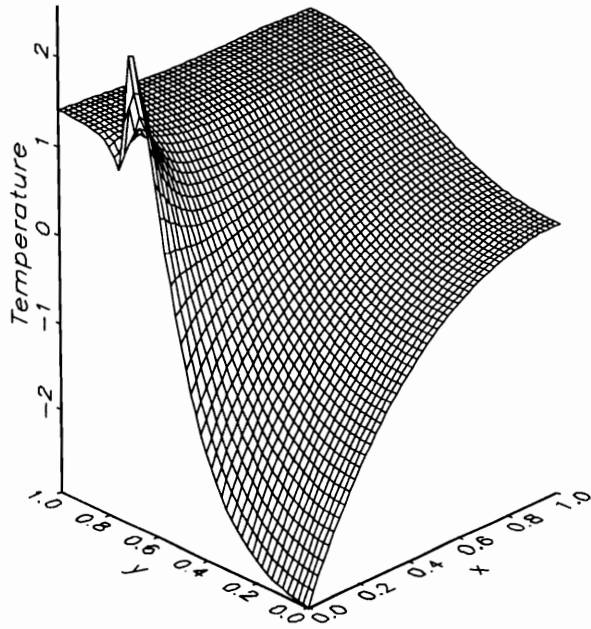


Figure 2.6: Surface plot for the temperature  $T$  ( $\gamma = 0.01$ ,  $\delta = 0.003$ ).

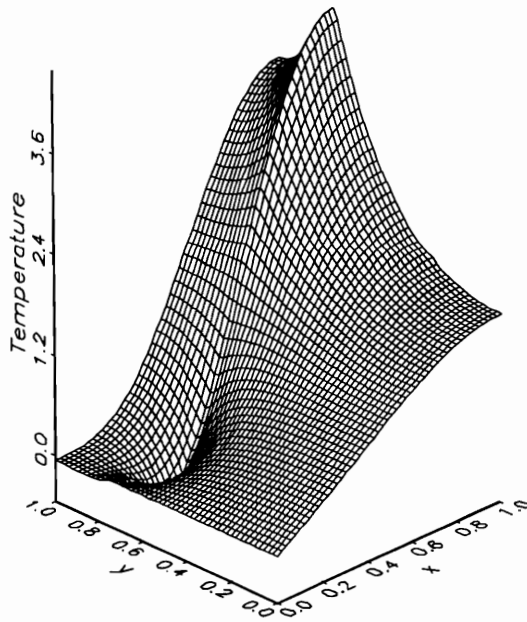


Figure 2.7: Surface plot for the adjoint state  $\Phi$  ( $\gamma = 0.01$ ,  $\delta = 0.003$ ).



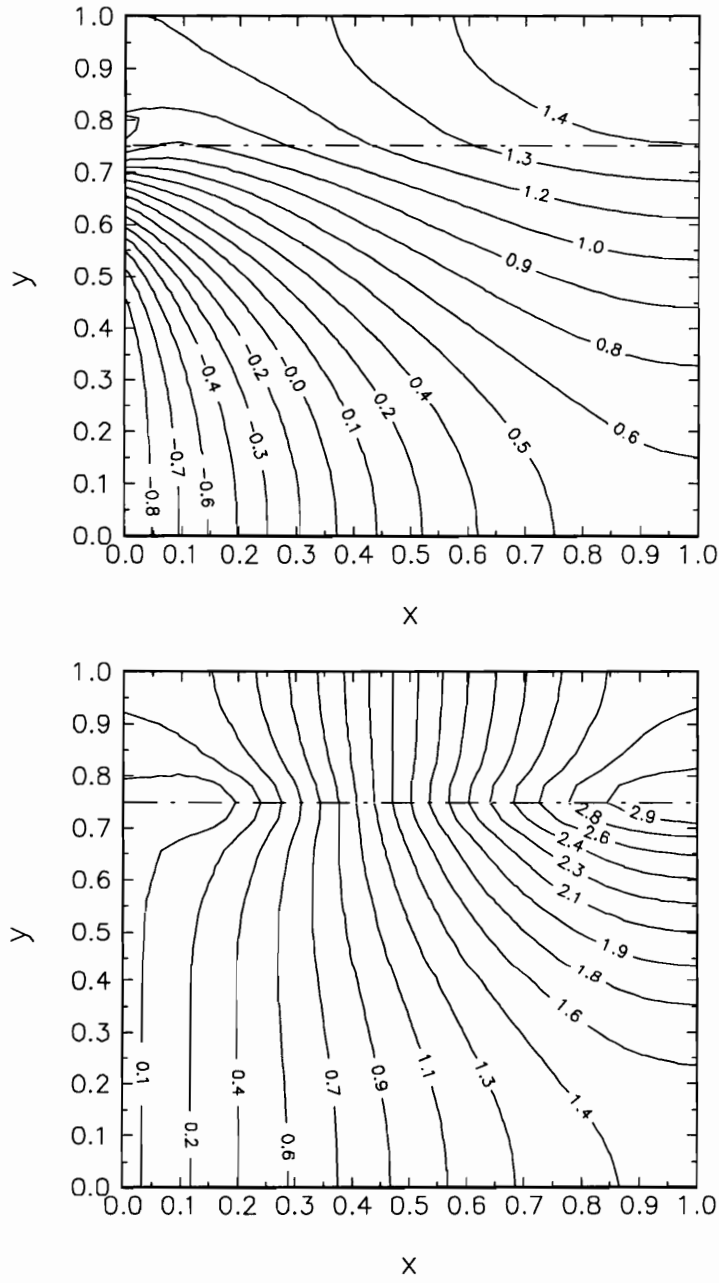


Figure 2.8: Contour plots for the temperature  $T$ (top) and adjoint state  $\Phi$ (bottom) ( $\gamma = \delta = 1$ ).

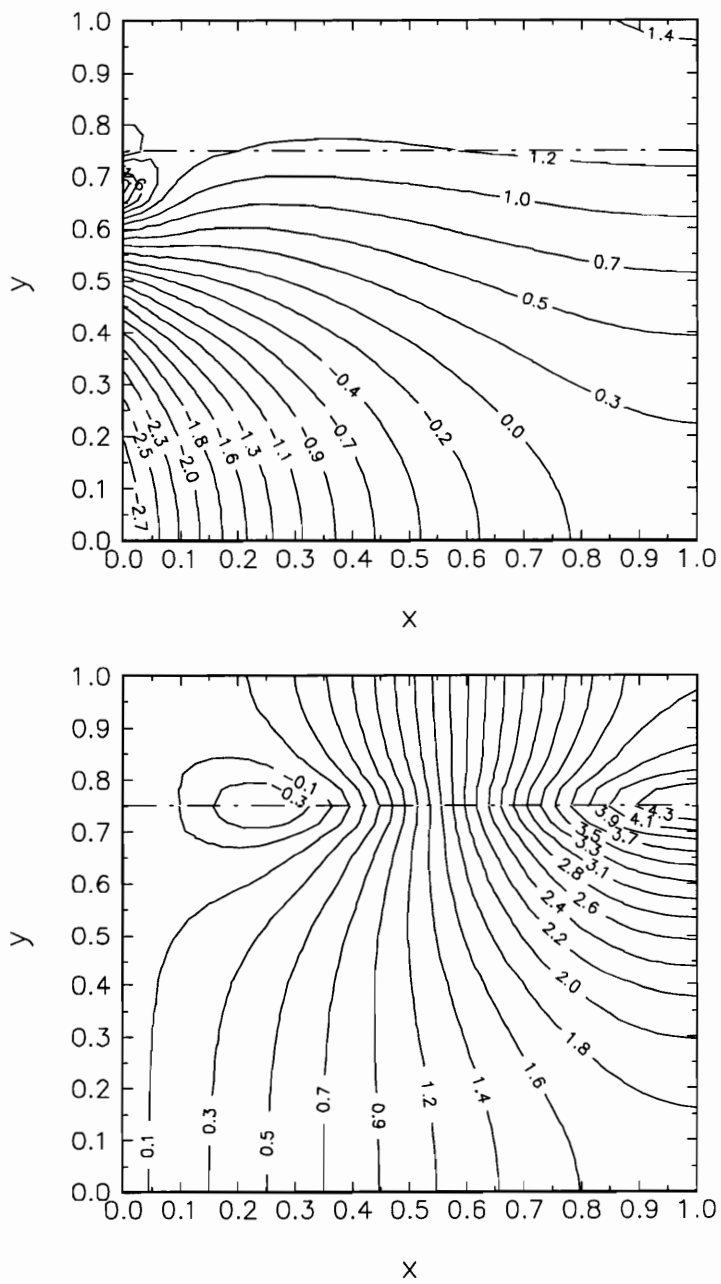


Figure 2.9: Contour plots for the temperature  $T$ (top) and adjoint state  $\Phi$ (bottom) ( $\gamma = 0.01$ ,  $\delta = 0.003$ ).

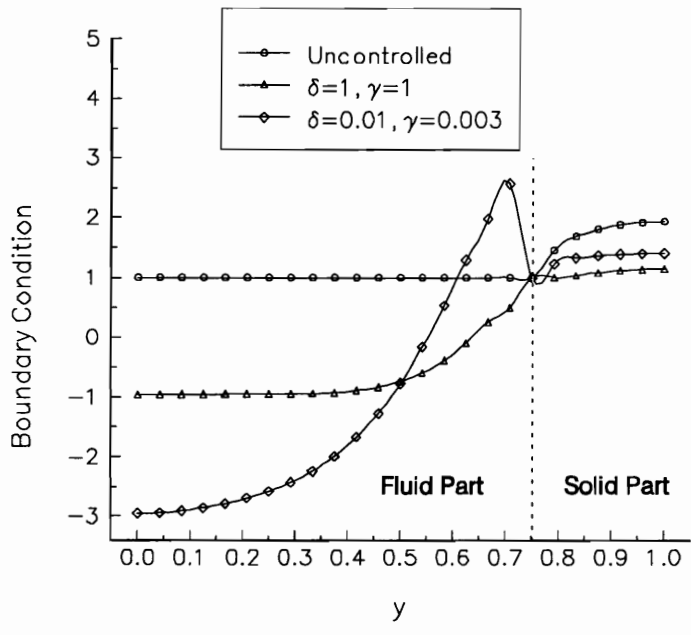


Figure 2.10: Optimal controls on  $\Gamma_c$ .

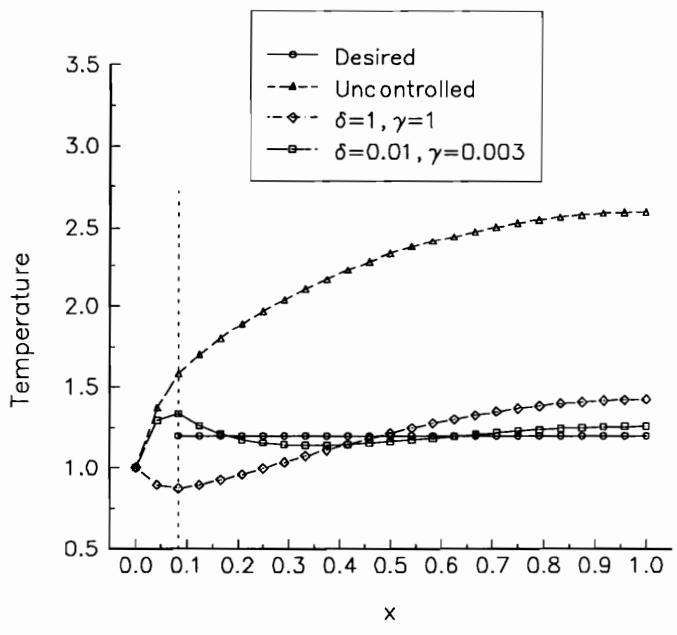


Figure 2.11: Temperature distributions on  $\Gamma_w$ .

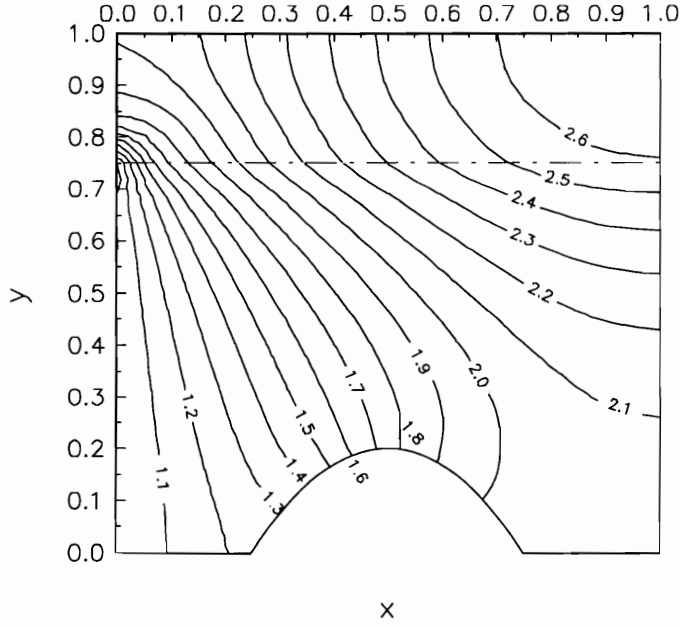


Figure 2.12: Temperature contour plot for the uncontrolled problem.

Table 2.2: Numerical results for Test 2.

	$\ T - 1.2\ _{0,\Gamma_\sigma}^2$	$\ g\ _{\Gamma_c}^2$	$\mathcal{J}(T, g)$
$\gamma = 1, \delta = 1$	$3.77877 \times 10^{-2}$	18.65837	18.67727
$\gamma = 0.01, \delta = 0.002$	$3.13342 \times 10^{-3}$	117.14224	0.390956

that we need a little more control  $g$  on  $\Gamma_c$ . Thus, even though the fluid flow is moderately complicated, given any  $\epsilon > 0$ , we can have  $\gamma$  and  $\delta$  such that  $\|T - T_d\|_{0,\Gamma_\sigma} < \epsilon$  when  $\gamma\delta$  is sufficiently small.

In Figure 2.12, we plot the temperature contour for the uncontrolled problem. In Figure 2.13-2.14, we have the contour plots of the temperature  $T$  and adjoint state  $\Phi$  for each cases. Finally, Figure 2.15-2.16 display the approximate optimal control  $g^h$  along  $\Gamma_c$  and the temperature distributions on  $\Gamma_\sigma$ , respectively.

*Remarks* : For the case  $\gamma = \delta = 1$ , it was found that 10 – 15 iterations were sufficient to get the optimal control  $g$ . Since  $\nu = 1$  and maximum velocity is 1, the control  $g$  affects the temperature distribution on  $\Gamma_\sigma$  very weakly. For the case that  $\gamma\delta$  is small, for example  $\gamma\delta < 0.1$ , our gradient method does not converge. Thus, we need to adjust the iteration step size. In such a case, we need a significant number of iterations. Thus, one may look for an efficient iteration algorithm. But the good news is that the iteration algorithm requires only one LU factorization and the same number of back and forward substitution as the iteration number, *i.e.*, a comparable number of floating point operations relative to that required for solving the full coupled system (2.4.5)-(2.4.9). Of course we assume that  $h$  is sufficiently small.

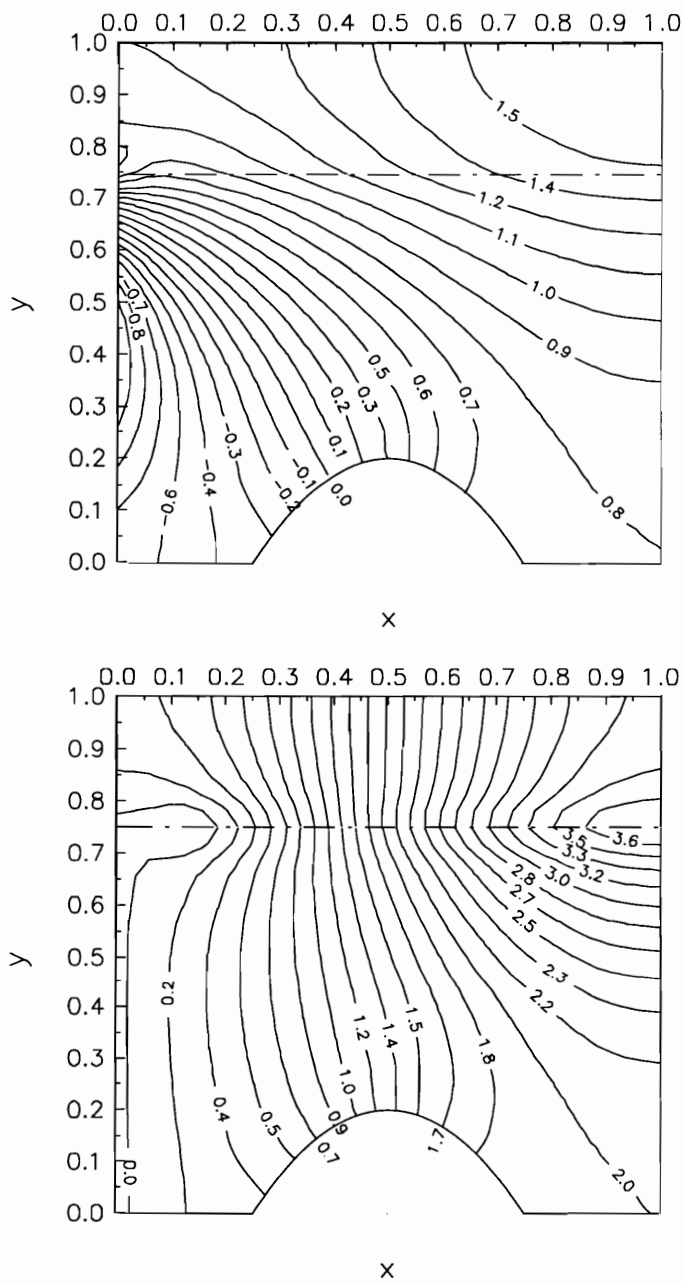


Figure 2.13: Contour plots for the temperature  $T$ (top) and adjoint state  $\Phi$ (bottom) ( $\gamma = \delta = 1$ ).

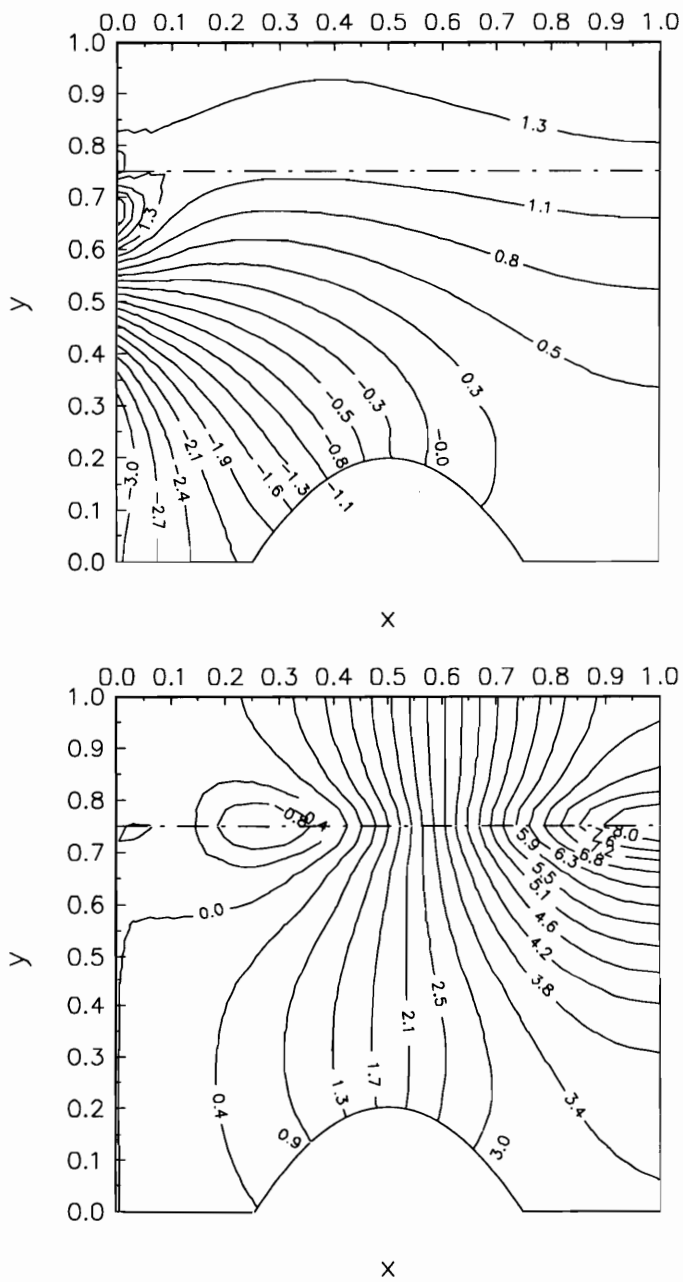


Figure 2.14: Contour plots for the temperature  $T$ (top) and adjoint state  $\Phi$ (bottom) ( $\gamma = 0.01$ ,  $\delta = 0.002$ ).

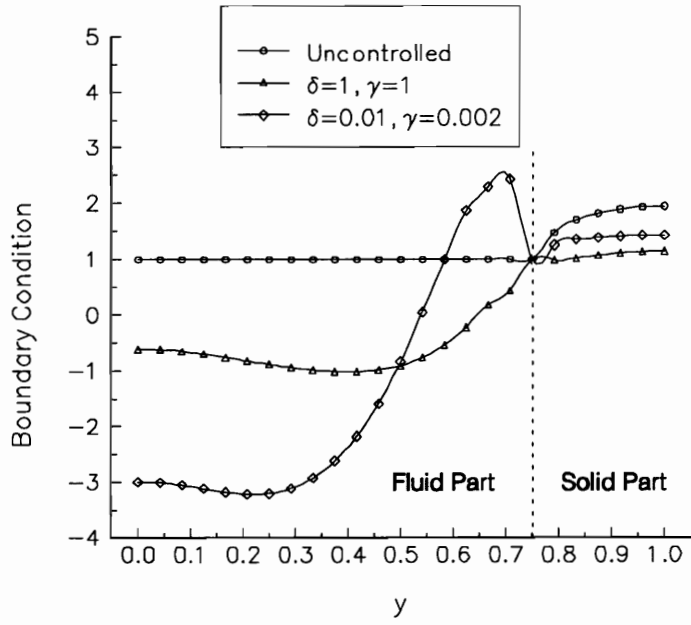


Figure 2.15: Optimal controls on  $\Gamma_c$ .

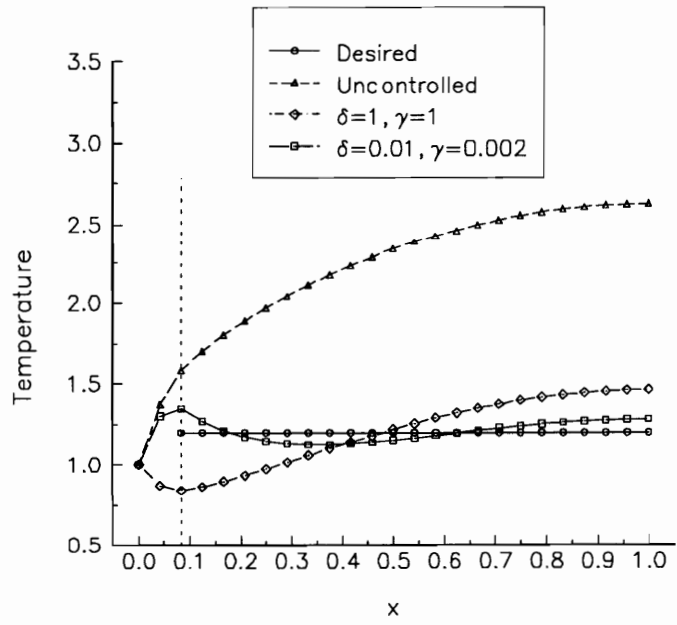


Figure 2.16: Temperature distributions on  $\Gamma_w$ .



# Chapter 3

## FEEDBACK CONTROL OF KARMAN VORTEX SHEDDING

### 3.1 The Computational Problem

Let  $B$  denote a circular cylinder and let  $\Gamma$  denote its boundary. The flow at infinity is assumed to be uniform and in the direction of the  $x$ -axis. We nondimensionalize with respect to the diameter of the cylinder and the velocity at infinity. Thus, in nondimensional variables, the cylinder has diameter unity and the flow at infinity has speed unity. The domain exterior to  $B$  is denoted by  $\Omega$ . Let  $\mathbf{u}(\mathbf{x}, t)$  and  $p(\mathbf{x}, t)$  denote the velocity and kinematic pressure (pressure divided by density), respectively. The governing equations, in nondimensional form, for the unsteady, incompressible, viscous flow past the cylinder  $B$  are the continuity equation

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \quad \text{and for } t > 0 \quad (3.1.1)$$

and the Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \frac{1}{Re} \nabla^2 \mathbf{u} \quad \text{in } \Omega \quad \text{and for } t > 0, \quad (3.1.2)$$

where  $Re$  denotes the Reynolds number based on the (dimensional) cylinder diameter and speed at infinity. Initial conditions are imposed on the velocity. Throughout, we start the flow impulsively from rest so that

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{0} \quad \text{in } \Omega. \quad (3.1.3)$$

On  $\Gamma$ , the surface of the cylinder, we prescribe the velocity. For the uncontrolled flow about the cylinder, the cylinder surface is a solid wall throughout so that in this case

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma \quad \text{and} \quad \text{for } t > 0. \quad (3.1.4)$$

When controls are applied, this boundary condition becomes inhomogeneous on the portions of  $\Gamma$  covered by the injection and suction orifices. We defer discussion of such boundary conditions until feedback controls are introduced in sections 3.3 and 3.4.

Since  $\Omega$  is an exterior domain, one should also specify conditions that govern the behavior of the flow at large distances from the cylinder. We do not state these here since in the computational simulations a bounded flow domain will be used. Of course, one must choose an artificial computational domain and specify boundary conditions on the boundary of that domain. The combination of computational domain and computational boundary conditions should be chosen so that the resulting flow in the computational domain accurately approximates the flow in the exterior domain  $\Omega$ .

The origin of the  $(x, y)$  coordinate system is located at the center of the cylinder and the cylinder has a unit diameter. The computational domain we use is the rectangle  $-5 \leq x \leq 15$  and  $-5 \leq y \leq 5$  with the cylinder excluded. The geometry of the domain is sketched in Figure 3.1. The exterior boundary of the computational domain consists of the four sides of the rectangle; again, see Figure 3.1. At the inflow boundary  $\Gamma_i$  the velocity is set to the uniform value at infinity, *i.e.*,  $u = 1$  and  $v = 0$ , where  $u$  and  $v$  denote the  $x$  and  $y$  components of the velocity vector, respectively. At the outflow  $\Gamma_o$  a vanishing “stress” boundary condition is imposed. Specifically, if  $\mathbf{t} = -pn + (1/Re)\partial\mathbf{u}/\partial n$ , then on  $\Gamma_o$  we set  $t_1 = t_2 = 0$ , where  $t_1$  and  $t_2$  respectively denote the  $x$  and  $y$  components of  $\mathbf{t}$ . The vector  $\mathbf{t}$  is not actually the stress vector; however, it has been found that imposing such an outflow

condition is often more effective than requiring the true stress to vanish. Moreover, it is a more convenient boundary condition to use with the particular form of the viscous term appearing in (3.1.2). On the top  $\Gamma_t$  and bottom  $\Gamma_b$  of the rectangle we impose the mixed conditions  $v = 0$  and  $t_1 = 0$ . Summarizing the above discussion, on the exterior boundary of the computational domain we specify

$$u = 1 \quad \text{and} \quad v = 0 \quad \text{on } \Gamma_i \quad \text{and} \quad \text{for } t > 0, \quad (3.1.5)$$

$$t_1 = 0 \quad \text{and} \quad t_2 = 0 \quad \text{on } \Gamma_o \quad \text{and} \quad \text{for } t > 0, \quad (3.1.6)$$

and

$$v = 0 \quad \text{and} \quad t_1 = 0 \quad \text{on } \Gamma_b \quad \text{and} \quad \Gamma_t \quad \text{and} \quad \text{for } t > 0. \quad (3.1.7)$$

The boundary conditions are also indicated on the sketch of the computational domain given in Figure 3.1. For the uncontrolled case, the problem we wish to discretize is then defined by (3.1.1)-(3.1.7), where in (3.1.1)-(3.1.3)  $\Omega$  now denotes the computational domain sketched in Figure 3.1.

The spatial discretization is effected using the Taylor-Hood finite element pair based on a triangulation of the computational flow domain. A typical triangulation is depicted in Figure 3.2; it consists of 1735 triangles and 3735 (velocity) nodes. The velocity (pressure) field is approximated using continuous piecewise quadratic (linear) polynomials; the same grid is used for both fields.

The semi-implicit single-step Euler method is employed for the time discretization. This time-stepping algorithm is defined as follows: given  $\mathbf{u}^{m-1}$ , the pair  $(\mathbf{u}^m, p^m)$  is obtained by solving the linear system

$$\nabla \cdot \mathbf{u}^m = 0 \quad (3.1.8)$$

and

$$\frac{\mathbf{u}^m - \mathbf{u}^{m-1}}{\delta} + (\mathbf{u}^{m-1} \cdot \nabla) \mathbf{u}^m + \nabla p^m = \frac{1}{Re} \nabla^2 \mathbf{u}^m, \quad (3.1.9)$$

where  $\delta$  denotes the time step. Of course, (3.1.8) and (3.1.9) are supplemented by the appropriate boundary and initial conditions. This scheme has been analyzed in [110], where the unconditional stability is proved.

### 3.2 Results for the Uncontrolled Problem

Using the computational scheme described in previous section, we have simulated the uncontrolled problem at  $Re = 60$  and  $Re = 80$ . To break the symmetry in the problem and trigger the vortex shedding, a perturbation is applied for a short time interval. The transient solution develops until a periodic state is reached at approximately  $t = 70$  and  $t = 90$  for  $Re = 60$  and  $Re = 80$ , respectively. These phases of the solution are perfectly captured by the lift coefficient  $C_L$  defined by

$$C_L = \int_0^{2\pi} \tilde{t}_2(\theta) d\theta, \quad (3.2.1)$$

where  $\tilde{t}_2$  is the  $y$  component of the true stress vector and  $\theta$  is the angle along the surface of the cylinder measured from the leading edge. The evolution in time of  $C_L$  shown in Figure 3.3 exhibits an oscillation having a period  $T \approx 7.0$  units in time for  $Re = 60$  and  $T \approx 6.5$  for  $Re = 80$ , leading to a Strouhal number  $St \approx 0.14$  for  $Re = 60$  and  $St \approx 0.15$  for  $Re = 80$ . These values are in agreement with experimental results and other numerical simulations; see, e.g., [19], [100], [104], and the references cited therein.

We have already mentioned that we will sense the pressure at stations on the surface of the cylinder, see Figure 3.4. It is therefore instructive to examine the oscillatory behavior of that pressure distribution. In Figure 3.6 is given the pressure distribution along the surface

of the cylinder at different times during one periods of oscillation of the lift coefficient, *i.e.*, in a time interval between successive sheddings of vortices from the same location on the cylinder. The zero angle corresponds to the forward stagnation point. As is expected, the pressure distribution on the cylinder is asymmetric. This is especially evident at the angles  $\pm\pi/2$ , *i.e.*, the top and bottom of the cylinder.

The oscillatory nature of the pressure is also evident from Figure 3.5 in which the pressure at some specific locations on the cylinder is graphed vs. time. Note that the period of the oscillations are independent of the location, although the magnitudes are not.

Instead of the pressure, one could also sense the vorticity on the surface of the cylinder. As can be inferred from Figure 3.7, the vorticity distribution along the surface of the cylinder deviates from an antisymmetric (about the  $x$  - axis) configuration.

### 3.3 Control by a Pair of Suction/Blowing Orifices

Perhaps the most “natural” arrangement of orifices is a pair of blowing/suction slots located symmetrically on the back-side of the cylinder. We have tried such arrangements with slots of length  $\pi/16$  centered about the angles  $\pm 11\pi/32$ ,  $\pm 15\pi/32$ , or  $\pm 19\pi/32$  from the leading edge of the cylinder. The separation points on the cylinder surface at a Reynolds number of 60 wander in the vicinity of  $\pm 2\pi/3$  as the vortices are shed. When the blowing/suction slots are located slightly ahead of the separation points of the uncontrolled wake, they proved to be a strong actuator which is in agreement with [96]. We place the sensors at symmetric locations on the front-side of the cylinder. From Figures 3.5 and 3.6 one may conclude that the specific position of the sensors is not overly important. We place the sensors at  $\pm 3\pi/32$  from the leading edge.

We first determine the blowing/suction velocity through the orifices as dictated by the

following feedback law:

$$\mathbf{u} = \min \{ \alpha(p(\theta_1) - p(\theta_2)), \beta \} \mathbf{g}(\theta), \quad (3.3.1)$$

where  $\mathbf{g}(\theta)$  is a given function,  $\theta_1$  and  $\theta_2$  are the positions of the sensors on the cylinder,  $\alpha$  is the feedback coefficient, and  $\beta > 0$  is a parameter that limits the size of the control. The function  $\mathbf{g}$  can be used to specify the flow profile at the orifices. Our computational results show that for any value of  $\alpha$  the system quickly becomes unstable, *i.e.*, at every time step each orifice switches from blowing to suction to blowing, etc.

Next, we try another approach called *digital control* which is based on the *sample-and-hold* technique. The values  $p(\theta_1)$  and  $p(\theta_2)$  in (3.3.1) are measured only at discrete instants or *sampling times*. The control law is updated at each time  $k\Delta t$  on the basis of the sampled values  $p(\theta_1)$  and  $p(\theta_2)$ . According to our experience, a choice of  $\Delta t$  in the range  $2\delta \leq \Delta t \leq 5\delta$  resulted in a stable configuration. If we set  $\Delta t = 5\delta$ ,  $\alpha = 3$ , and  $\beta = 0.1$ , we found that the magnitude of the oscillations in the lift coefficient was reduced by a factor of one half. If  $\alpha = 4$ , the lift coefficient was shifted upwards which implies that one orifice stays in a blowing state for long periods of time and in a sucking state for only a short period. If  $\beta > 0.1$ , the Karman vortex shedding was enhanced. In Figure 3.8, the lift coefficients are given for the cases of  $\alpha = 3.0$ ,  $\beta = 0.1$ , and  $\alpha = 4.0$ ,  $\beta = 0.1$ . In Figures 3.9-3.11, the results of a particular computations are given. These correspond to the case  $\alpha = 3.0$ ,  $\beta = 0.1$ .

We next add some damping to the system to obtain the *proportional-derivative* feedback law:

$$\mathbf{u} = \min \left\{ \alpha \left( (p(\theta_1) - p(\theta_2)) + \gamma \frac{\partial}{\partial t} (p(\theta_1) - p(\theta_2)) \right), \beta \right\} \mathbf{g}(\theta), \quad (3.3.2)$$

where again this relation gives the velocity at the orifices. We have simulated this case for

several choices of  $\alpha$ ,  $\beta$ , and  $\gamma$ . In most cases, the Karman vortex shedding was enhanced.

### 3.4 A Pair of Suction Slots with a Single Blowing Slot

To the pair of orifices used in section 3.3, we add an orifice at  $\theta = \pi$ , *i.e.*, at the trailing edge of the cylinder. Specifically, fluid is sucked through two slots centered at  $\pm 23\pi/32$  and is blown through the slot centered at  $\pi$ . We again sense the pressure at two locations  $\theta_1$  and  $\theta_2$  on the front-face of the cylinder. The feedback control law is chosen as follows:

$$\mathbf{u} = \min \{ \alpha |p(\theta_1) - p(\theta_2)|, \beta \} \mathbf{g}(\theta). \quad (3.4.1)$$

We have performed computational simulations for numerous cases but report only the case of  $\alpha = 80$ ,  $\beta = 2.0$ . In this case, the net mass flow through the three orifices was zero, *i.e.*,  $\int_{\partial B} \mathbf{u} \cdot \mathbf{n} \, dS = 0$  so that the amount of fluid blown through the slot located on the  $x$ -axis was the same as the total amount of fluid sucked through the two off-axis slots. The results of this case are given in Figures 3.13-3.16. From these results one can conclude that the simple feedback law (3.4.1) can be quite effective in reducing the size of the oscillations in the lift.

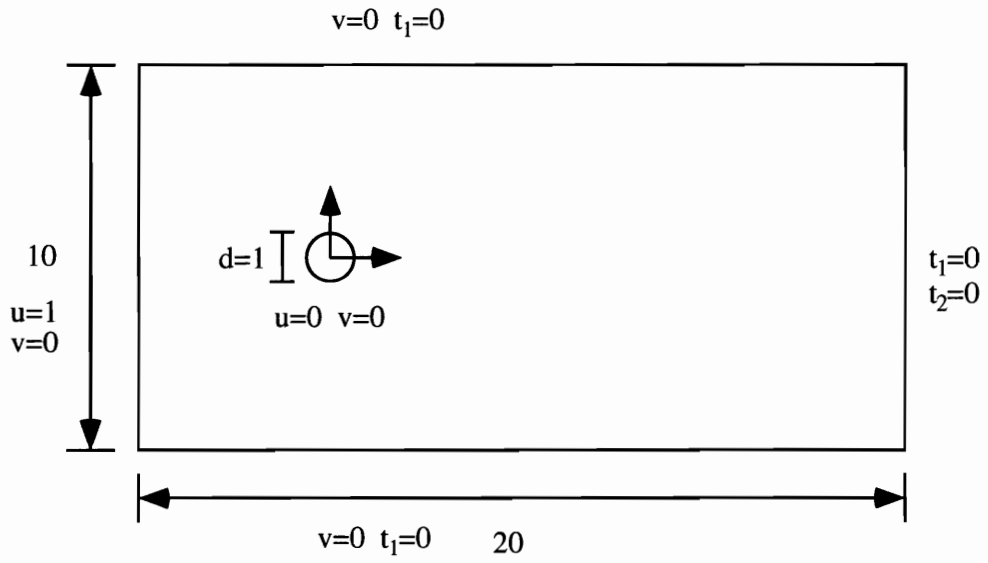


Figure 3.1: The Computational Domain  $\Omega$  for simulating flow past a circular cylinder

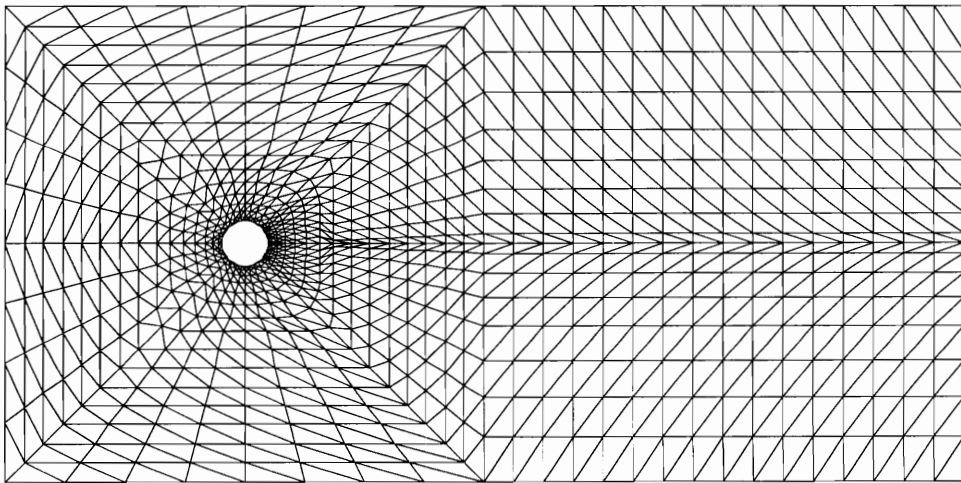


Figure 3.2: The finite-element grid for simulating flow past a circular cylinder



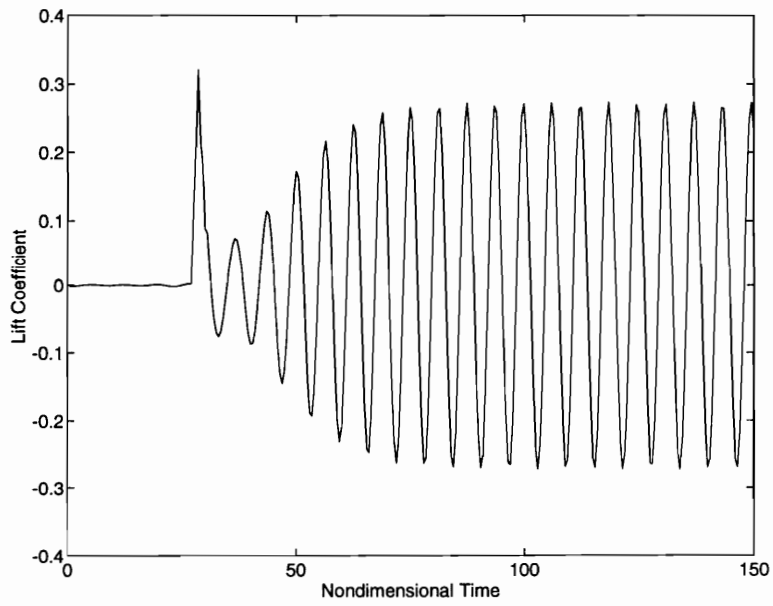
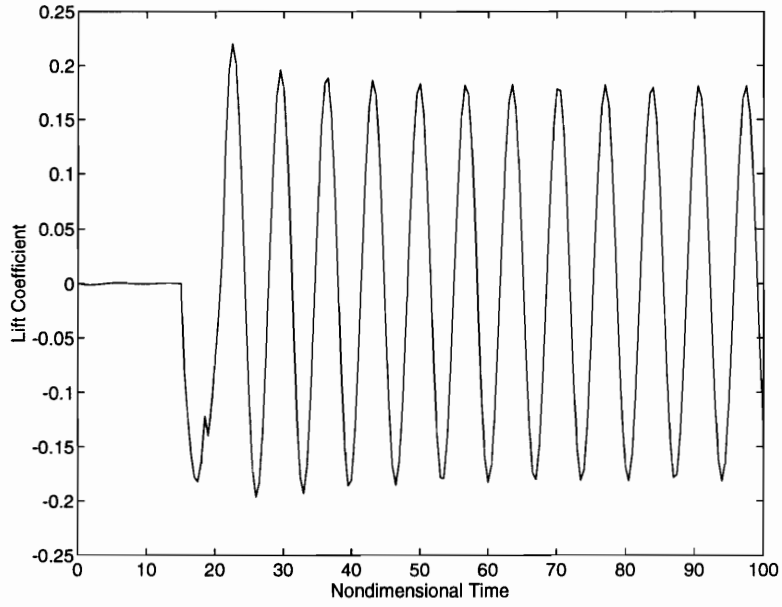


Figure 3.3: The evolution in time of  $C_L$  for  $R = 60$  (top) and  $R = 80$  (bottom)

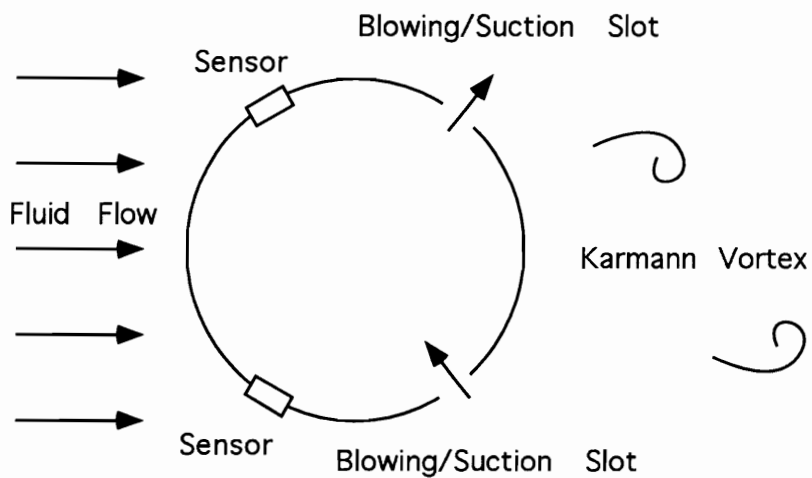


Figure 3.4: Locations of sensors and actuators

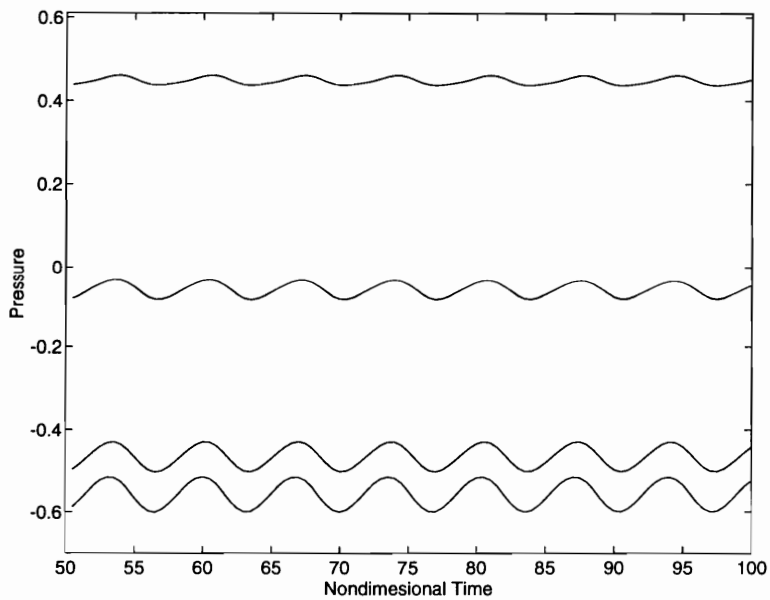


Figure 3.5: The evolution in time of pressure at (from top to bottom)  $\theta = \pi/8, \pi/4, 3\pi/8,$  and  $\pi/2$  for  $Re = 60$ .

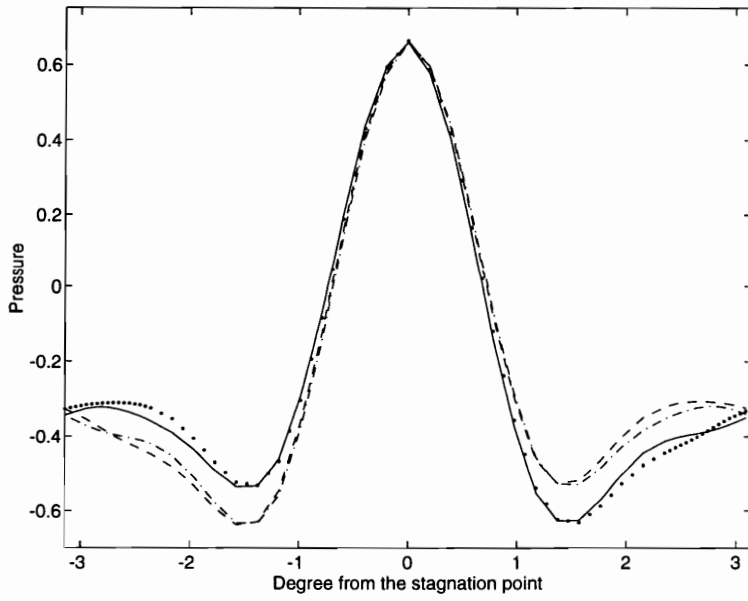
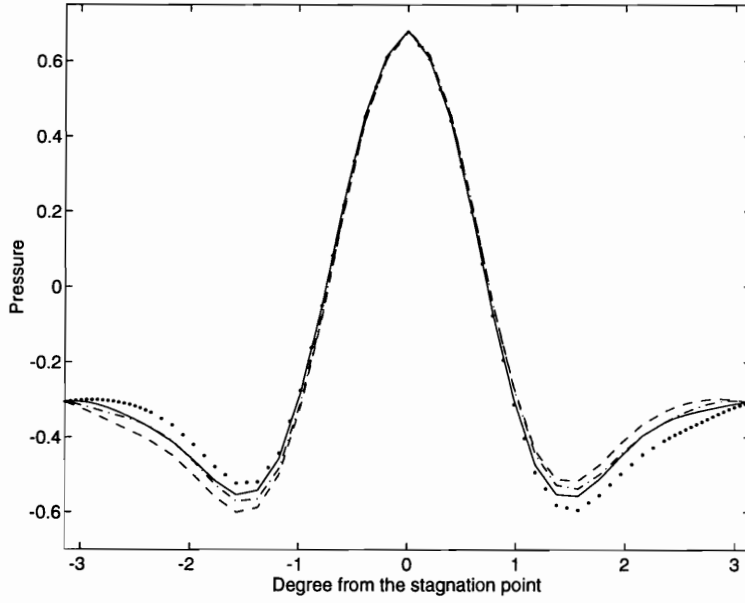


Figure 3.6: Pressure distributions along the cylinder surface at different times for  $Re = 60$  (top) and  $Re = 80$  (bottom).

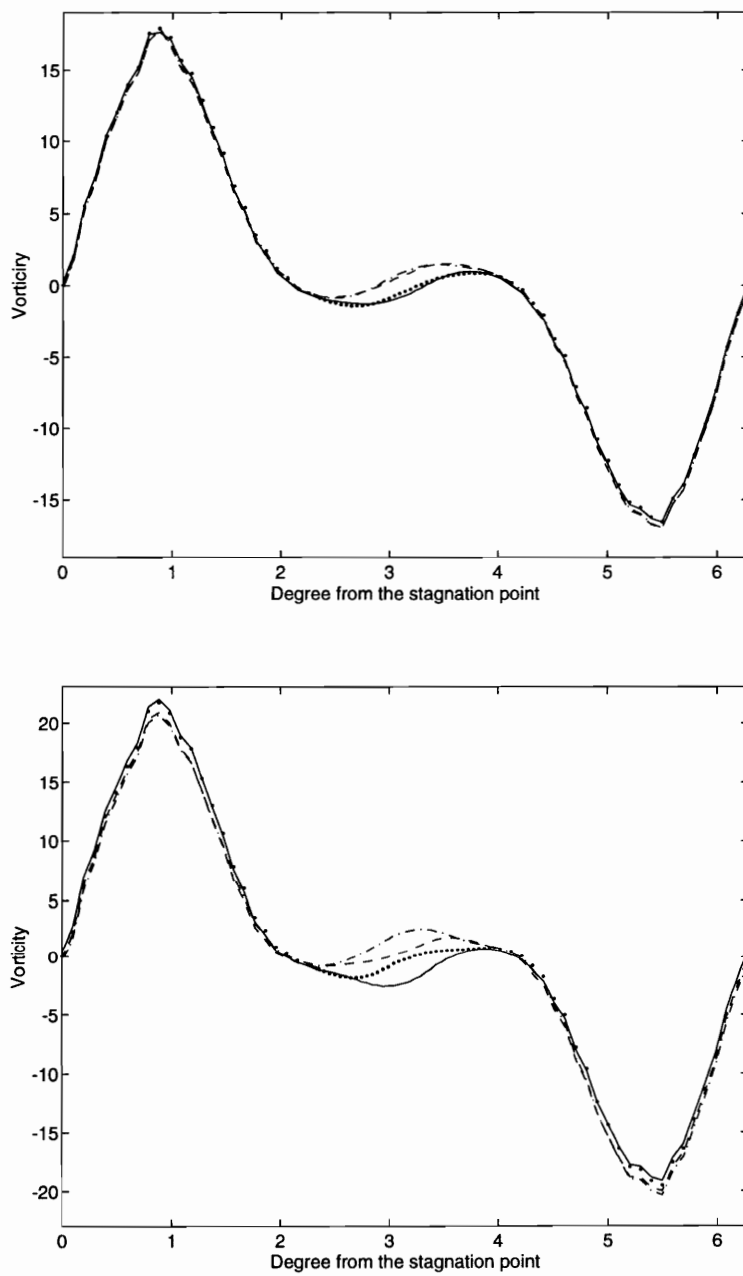


Figure 3.7: Vorticity distributions along the cylinder surface at different times for  $Re = 60$  (top) and  $Re = 80$  (bottom).

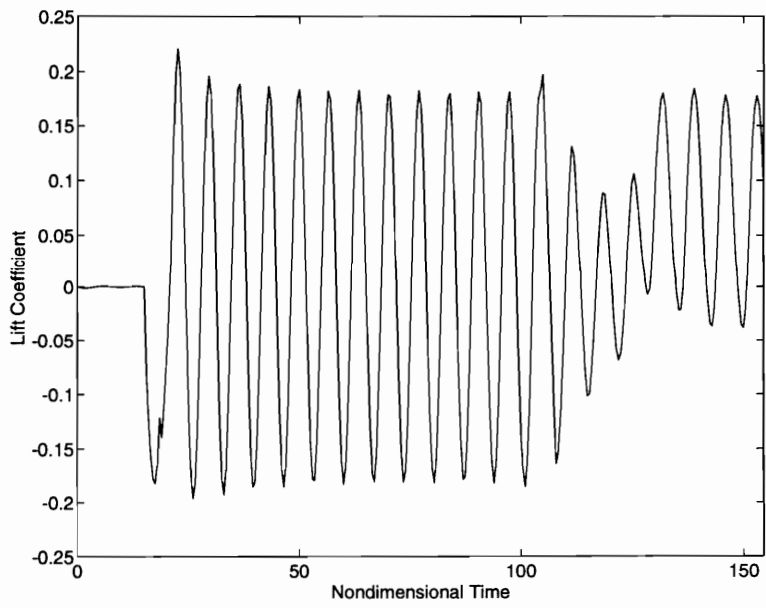
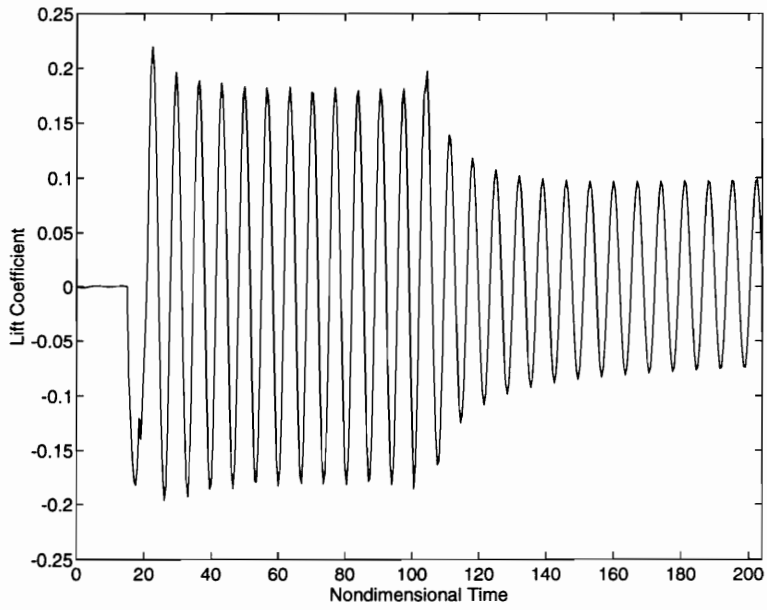


Figure 3.8: The evolution of the lift coefficient for  $\alpha = 3.0$  (top) and  $\alpha = 4.0$  (bottom).

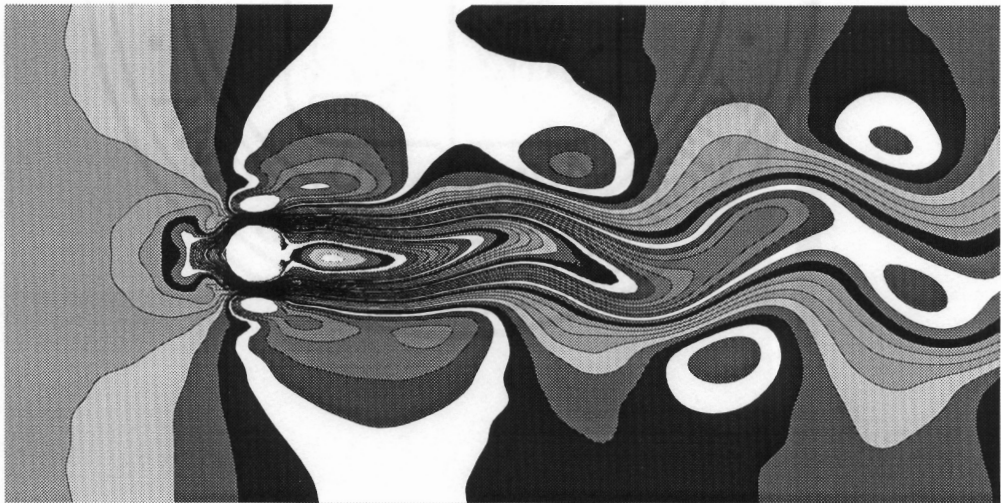
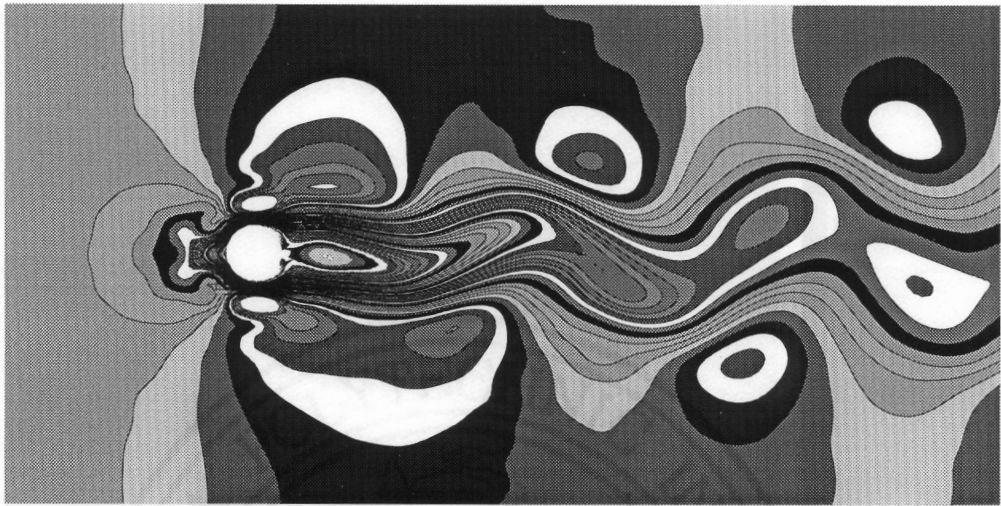


Figure 3.9: The horizontal velocity for uncontrolled (top) and controlled (bottom) flows around cylinder.

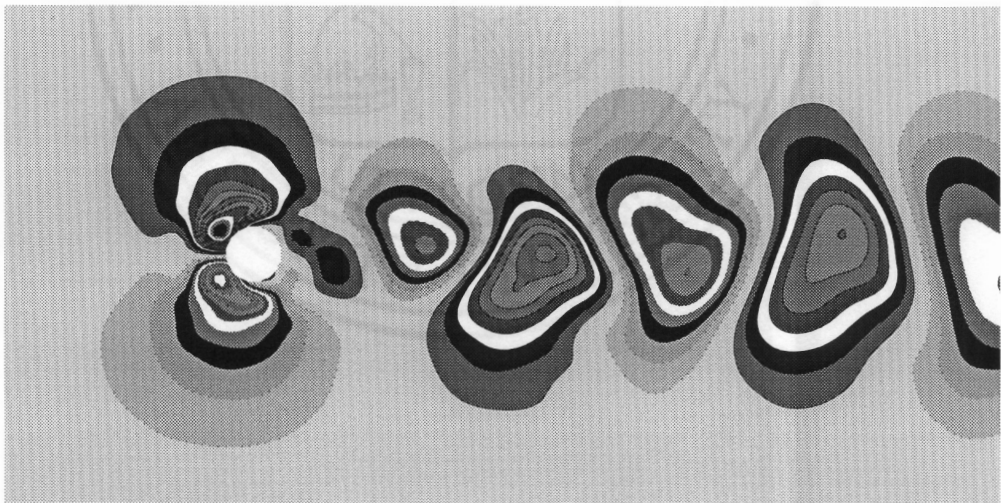
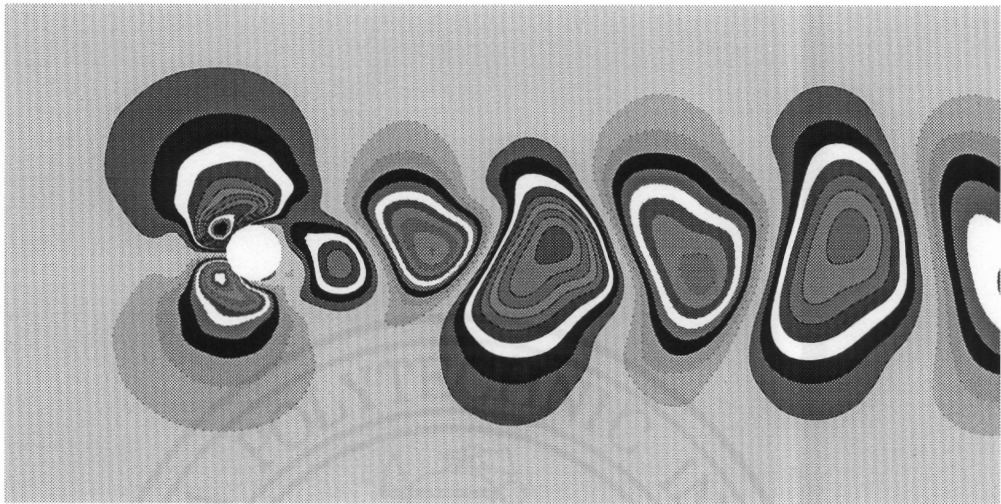


Figure 3.10: The vertical velocity for uncontrolled (top) and controlled (bottom) flows around cylinder.

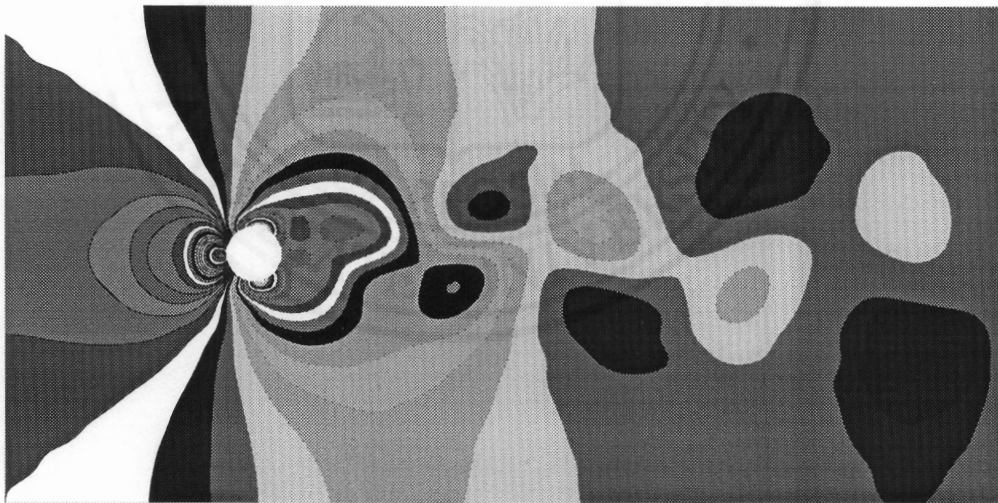
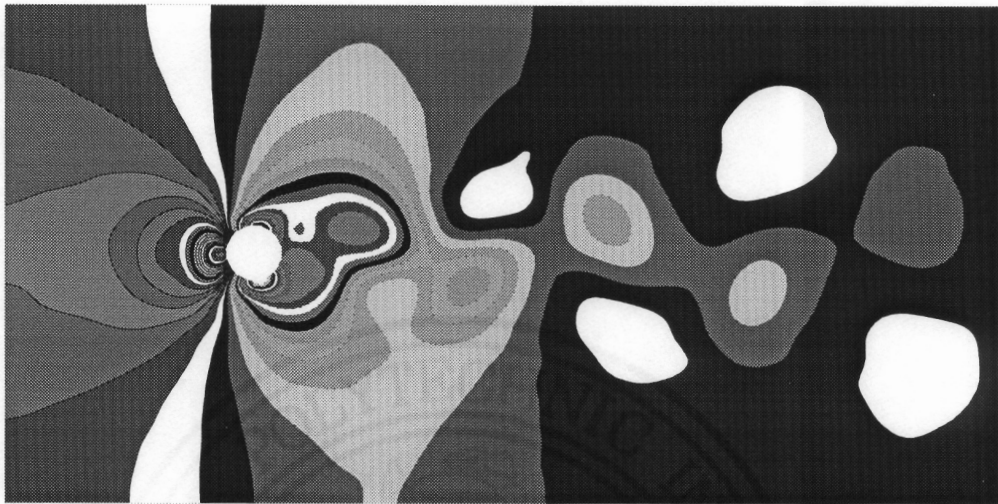


Figure 3.11: The pressure for uncontrolled (top) and controlled (bottom) flows around cylinder.



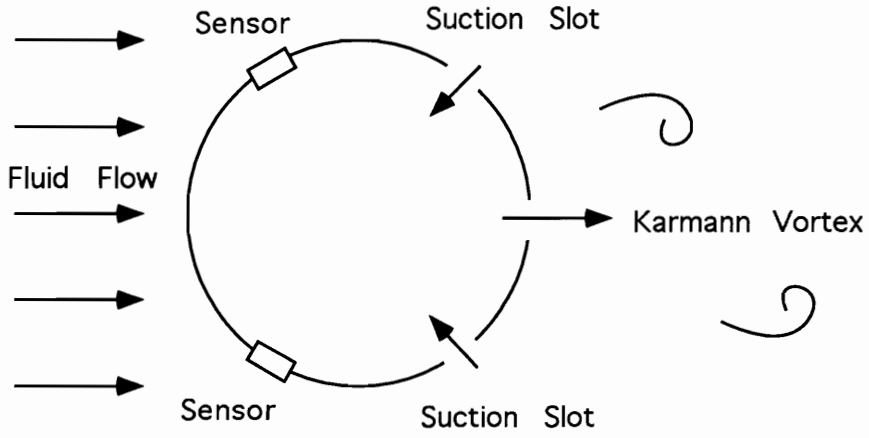


Figure 3.12: Locations of sensors and actuators

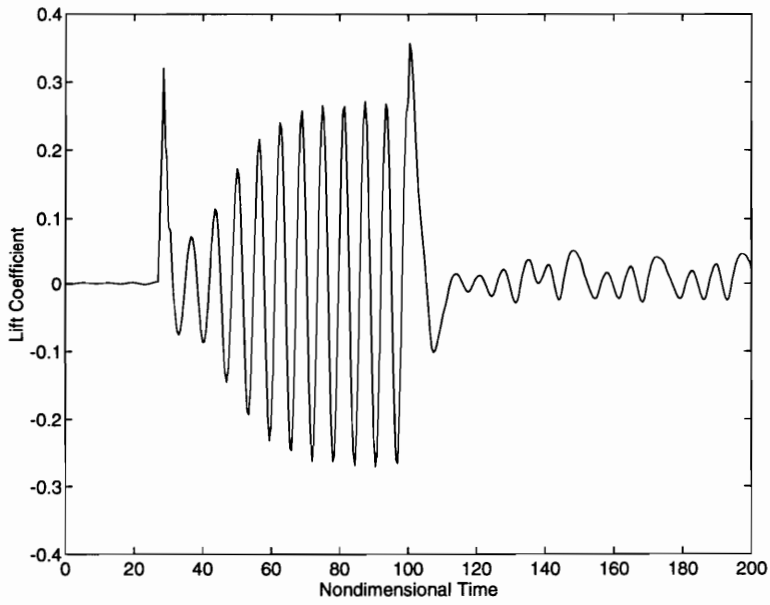


Figure 3.13: The evolution of the lift coefficient.

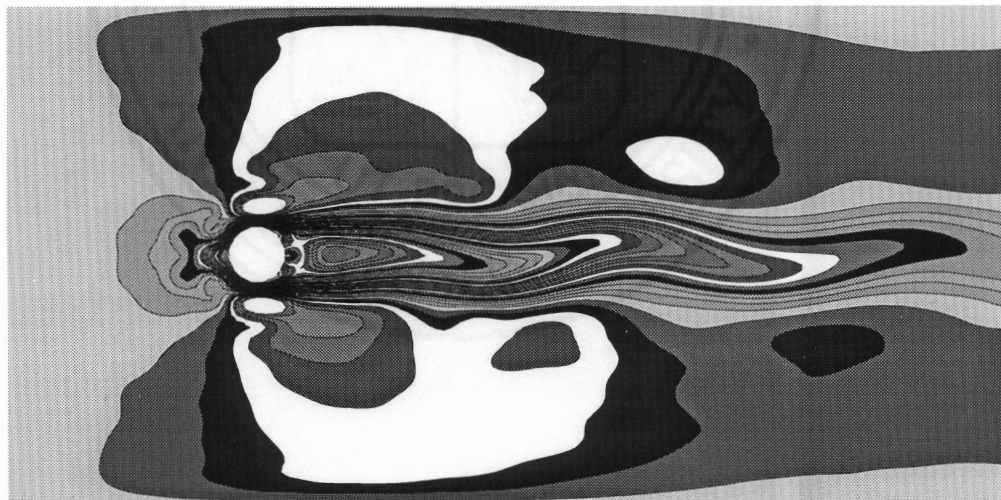
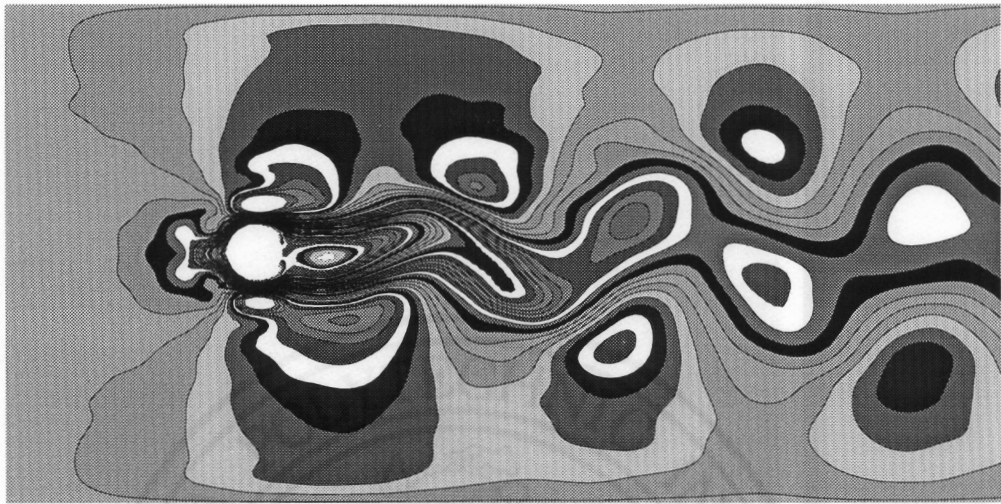


Figure 3.14: The horizontal velocity for uncontrolled (top) and controlled (bottom) flows around cylinder.

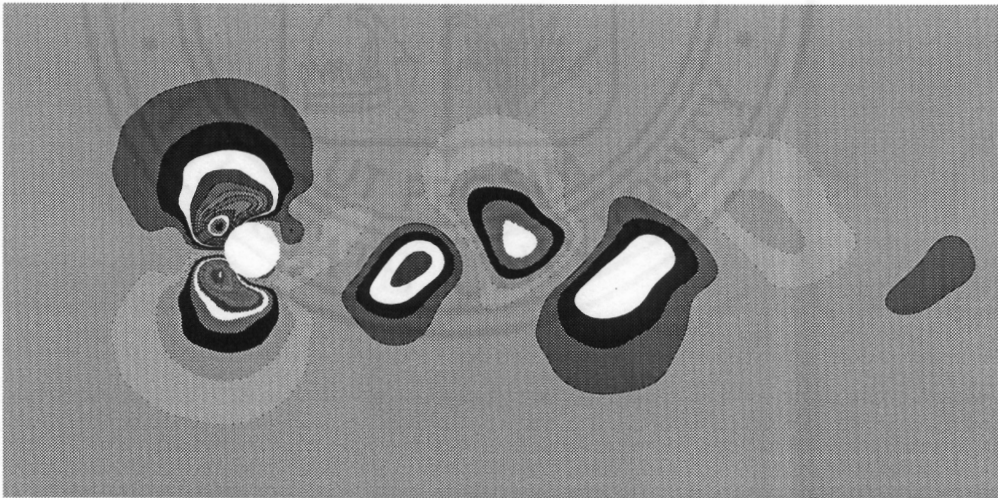
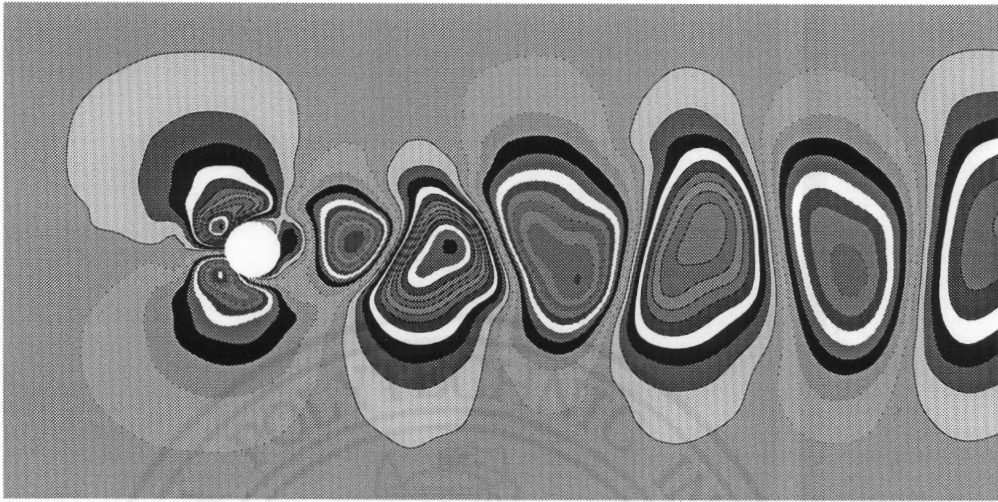


Figure 3.15: The vertical velocity for uncontrolled (top) and controlled (bottom) flows around cylinder.

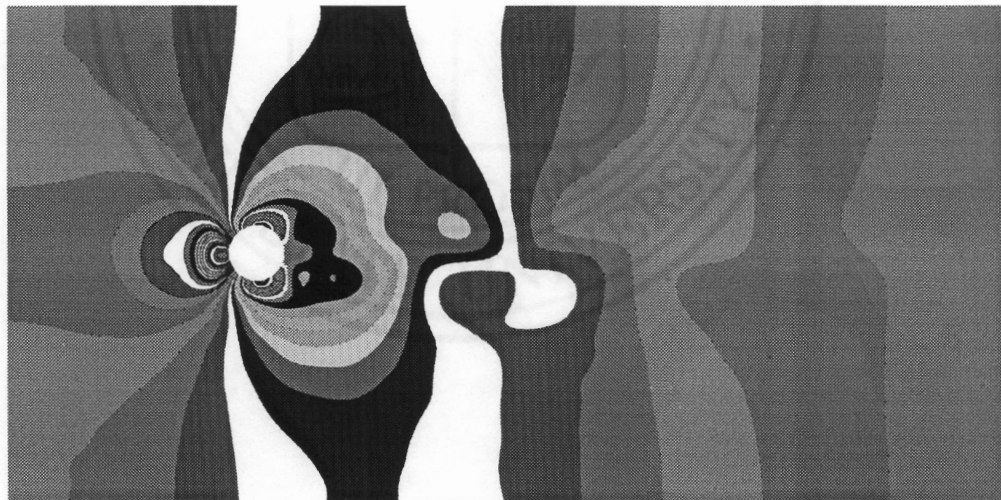
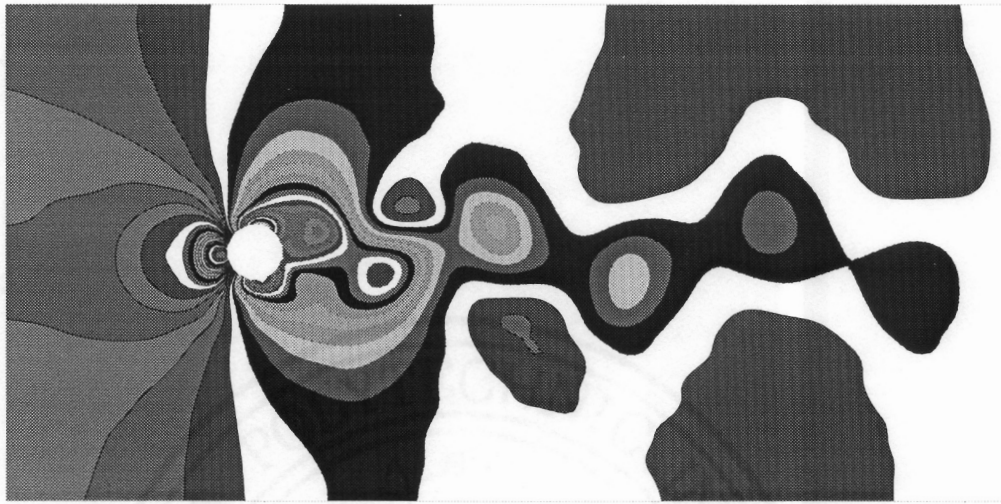


Figure 3.16: The pressure for uncontrolled (top) and controlled (bottom) flows around cylinder.

## Chapter 4

# SOME BOUNDARY VALUE PROBLEMS ASSOCIATED WITH FEEDBACK CONTROL

### 4.1 Mathematical Modeling

We now examine some boundary value problems that are motivated by feedback control theory. A typical optimal control problem is the following: find the best controller  $\phi$  such that some observation  $\gamma = Fu$  achieves a desired value  $\gamma_d$  or is at least as close as possible to  $\gamma_d$ , where  $F$  is a general linear or nonlinear operator which may involve integrals of  $u$  and/or derivatives of  $u$ , where  $u$  is the state of the system.

The mathematical formulation of the problem is the following: find  $\phi$  which minimizes a cost functional  $J$  subject to the state equations. The methods of the calculus of variations give us some characterizations of the best  $\phi$  through the adjoint state and also some algorithms to reach the best (optimal) control. Feedback theory involves constructing  $\phi$  as a function of the state variables  $u$  or some observation of  $u$ . Let us consider the following boundary value problem as an example:

$$-\Delta u = f \quad \text{in } \Omega, \tag{4.1.1}$$

$$u = b \quad \text{on } \Gamma_s, \tag{4.1.2}$$

$$u = b + F\left(\frac{\partial u}{\partial n}\Big|_{\Gamma_s}\right)g \quad \text{on } \Gamma_c, \tag{4.1.3}$$

where  $\Omega$  is a nonempty simply connected domain in  $R^N$ ,  $N = 2$  or  $3$ , with a smooth

boundary  $\partial\Omega = \Gamma$ ;  $\Gamma_s$  and  $\Gamma_c$ , on which the sensors and actuators are located, respectively, are portions of  $\Gamma$ . For simplicity we let  $\bar{\Gamma} = \bar{\Gamma}_s \cup \bar{\Gamma}_c$  and  $\Gamma_s \cap \Gamma_c = \emptyset$ . In (4.1.1)-(4.1.3),  $f$  denotes a given source and  $b$  and  $g$  given functions defined on boundary such that  $b \in H^{1/2}(\Gamma)$  and  $g \in H^{1/2}(\Gamma)$ . The function  $g$  has compact support on  $\Gamma_c$ .  $F$  is a functional on  $H^{-1/2}(\Gamma_s)$ .

Since the above problem (4.1.1)-(4.1.3) is not an elliptic problem in the usual sense, we cannot use the general theory of elliptic equations for solving this problem. However, by using potential theory, we can prove the existence and uniqueness of the solution to problem (4.1.1)-(4.1.3). In particular, we will use the boundary integral equation method which is closely related to the classical Green's function method. They are both based on the concepts of adjoint operators, Dirac delta functions, and fundamental solutions. In the classical Green's function method, we apply the definition of the adjoint operator to a special function, called the Green's function, which satisfies certain suitable boundary conditions. The result of this process is an explicit expression for the solution. The Green's function is easily obtained for simple problems. However, for geometries of practical interest, obtaining an expression for the Green's function may be just as hard as the original problem. On the other hand, one may examine what happens if the above mentioned boundary conditions on the Green's function are relaxed. In this case, a particular solution, known as the fundamental solution, is much easier to obtain than the Green's function and typically coincides with the free-space Green's function. The disadvantage is that, instead of an explicit solution, one obtains an integral equation on the boundary of the domain. This method is known as the boundary integral equation method.

We shall study the existence and uniqueness of the boundary value problem for the Laplace and Stokes equations in sections 4.2 and 4.3, respectively. Here, we introduce some

of the notions used in subsequent sections. Let  $H^s(\mathcal{D})$ ,  $s \in \mathbb{R}$ , be the standard Sobolev space of order  $s$  with respect to the set  $\mathcal{D}$ , where  $\mathcal{D}$  is either the domain  $\Omega \subseteq \mathbb{R}^N$ , or its boundary  $\Gamma$ , or part of that boundary. Recall that  $H^0(\mathcal{D}) = L^2(\mathcal{D})$ . Let the space  $H_D^m(\mathcal{D})$  be the closure in the  $H^m(\mathcal{D})$  norm of the functions in  $H^m(\mathcal{D})$  which have compact support in  $\Gamma_c$ . We close this subsection by introducing some theorems which will be useful later. The following theorems can be found in [8] and in the references cited there. Throughout,  $C$  will be a generic constant with different values on different places.

**Theorem 4.1.1** *Let  $u \in H^k(\Omega)$ ,  $k > 1/2$ . Then there exists a trace of the function  $u$  on  $\partial\Omega$  and*

$$\|u\|_{H^{k-1/2}(\partial\Omega)} \leq C \|u\|_{H^k(\Omega)} \quad (4.1.4)$$

where  $C$  does not depend on  $u$ .  $\square$

**Theorem 4.1.2** *Let  $u \in H^k(\Omega)$ ,  $k > 3/2$ . Then there exists a trace  $\partial u / \partial n$  on  $\partial\Omega$  and*

$$\left\| \frac{\partial u}{\partial n} \right\|_{H^{k-3/2}(\partial\Omega)} \leq C \|u\|_{H^k(\Omega)} \quad (4.1.5)$$

where  $C$  does not depend on  $u$ .  $\square$

For the case  $k \leq 3/2$ , we have the following theorem.

**Theorem 4.1.3** *Let  $\mathcal{T}(\Omega) \subset H^1(\Omega)$  be the space of all functions which satisfy the equation*

$$-\Delta u = 0 \quad (4.1.6)$$

in the weak sense. Then, for  $u \in \mathcal{T}(\Omega)$ , we have  $\partial u / \partial n \in H^{-1/2}(\partial\Omega)$  and

$$\left\| \frac{\partial u}{\partial n} \right\|_{H^{-1/2}(\partial\Omega)} \leq C \|u\|_{H^1(\Omega)} \quad (4.1.7)$$

where  $C$  does not depend on  $u$ .  $\square$

## 4.2 The Laplace Equation

Let  $\Omega$  be an open, bounded and nonempty simply connected domain in  $R^N$ ,  $N = 2$  or  $3$ , with smooth boundary  $\partial\Omega = \Gamma$ . The boundary  $\Gamma$  consists of  $\Gamma_c$  and  $\Gamma_s$  such that  $\Gamma = \bar{\Gamma}_s \cup \bar{\Gamma}_c$  and  $\Gamma_s \cap \Gamma_c = \emptyset$ . Let  $\Omega^c$  denote the exterior domain of  $\Gamma$ . Throughout this section, we will assume  $\psi \in L^2(\Omega)$ ,  $\phi \in H^{1/2}(\Gamma)$ , and  $g \in H_D^{1/2}(\Gamma)$  whenever we do not specify the function spaces.

### 4.2.1 The Dirichlet Type Boundary Value Problem

Let us consider the interior and exterior inhomogeneous Dirichlet type boundary value problem

$$\Delta u(\mathbf{x}) = \psi(\mathbf{x}) \quad \text{in } \Omega \text{ (resp. } \Omega^c), \quad (4.2.1)$$

$$u(\mathbf{x}) = \phi(\mathbf{x}) \quad \text{on } \Gamma_s, \quad (4.2.2)$$

$$u(\mathbf{x}) = F\left(\frac{\partial u}{\partial n}\Big|_{\Gamma_s}\right)g(\mathbf{x}) + \phi(\mathbf{x}) \quad \text{on } \Gamma_c, \quad (4.2.3)$$

where  $F$  is a bounded linear functional on  $H^{-1/2}(\Gamma_s)$ . We denote points in  $R^N$  by  $\mathbf{x} = (x_1, \dots, x_N)$ .

Using potential theory, we are going to prove the existence and uniqueness of solutions of the problem (4.2.1)-(4.2.3). First of all, we have to determine the fundamental solution of the Laplace equation. We consider the problem : find a function  $E(\mathbf{x}, \mathbf{y})$  depending on  $\mathbf{x}$  and  $\mathbf{y}$  such that

$$\Delta E(\mathbf{x} - \mathbf{y}) = -\delta(\mathbf{x} - \mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in R^N. \quad (4.2.4)$$

We easily find that,

$$E(\mathbf{x} - \mathbf{y}) = \begin{cases} -\frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{y}|, & N = 2, \\ \frac{1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{y}|}, & N = 3, \end{cases}$$



satisfies (4.2.4).  $E$  is defined everywhere except for  $\mathbf{x} = \mathbf{y}$ , where it is singular. This  $E$  is unique up to a harmonic function. We know that the Dirac delta function is not a point-defined function in the classical sense, but rather a generalized function or a distribution;  $E$  itself should be regarded as a generalized function.

In the integration approach, we may need to reduce the partial differential equation itself to a homogeneous one. We let

$$\begin{aligned} v(\mathbf{x}) &= V(\psi)(\mathbf{x}) \\ &= -(E * \psi)(\mathbf{x}) \\ &= -\int_{\Omega} E(\mathbf{x} - \mathbf{y})\psi(\mathbf{y}) d\mathbf{y}. \end{aligned}$$

Then in  $\Omega$ ,  $v$  satisfies

$$\Delta v(\mathbf{x}) = \psi(\mathbf{x}). \quad (4.2.5)$$

The operator  $V$  is linear, producing a volume potential  $v$ . Now, define

$$\tilde{u}(\mathbf{x}) = u(\mathbf{x}) - v(\mathbf{x}). \quad (4.2.6)$$

Then  $\tilde{u}$  satisfies the homogeneous equation

$$\Delta \tilde{u}(\mathbf{x}) = \Delta u(\mathbf{x}) - \Delta v(\mathbf{x}) = 0. \quad (4.2.7)$$

On  $\Gamma$ ,  $\tilde{u}$  satisfies

$$\tilde{u}(\mathbf{x}) = \phi(\mathbf{x}) - v(\mathbf{x})|_{\Gamma} \equiv \tilde{\phi}(\mathbf{x}) \quad \text{on } \Gamma_s, \quad (4.2.8)$$

$$\tilde{u}(\mathbf{x}) = \left[ F\left(\frac{\partial u}{\partial n}\Big|_{\Gamma_s}\right) - F\left(\frac{\partial v}{\partial n}\Big|_{\Gamma_s}\right) \right] g(\mathbf{x}) + \phi(\mathbf{x}) - v(\mathbf{x})|_{\Gamma} \quad (4.2.9)$$

$$= F\left(\frac{\partial \tilde{u}}{\partial n}\Big|_{\Gamma_s}\right) g(\mathbf{x}) - \tilde{\phi}(\mathbf{x}) \quad \text{on } \Gamma_c. \quad (4.2.10)$$

Rewriting  $\tilde{u}$  as  $u$  and  $\tilde{\phi}$  as  $\phi$ , we have the following Dirichlet type boundary value problems:

$$\Delta u(\mathbf{x}) = 0 \quad \text{in } \Omega \ (\Omega^c), \quad (4.2.11)$$

$$u(\mathbf{x}) = \phi(\mathbf{x}) \quad \text{on } \Gamma_s, \quad (4.2.12)$$

$$u(\mathbf{x}) = F\left(\frac{\partial u}{\partial n}\Big|_{\Gamma_s}\right)g(\mathbf{x}) + \phi(\mathbf{x}) \quad \text{on } \Gamma_c. \quad (4.2.13)$$

The single-layer potential solution for the problem (4.2.11)- (4.2.13) is based on the ansatz

$$u(\mathbf{x}) = \int_{\Gamma} E(\mathbf{x} - \mathbf{y})f(\mathbf{y}) dS_{\mathbf{y}}, \quad \mathbf{x} \in R^2 \quad \text{or} \quad R^3, \quad (4.2.14)$$

for some density function  $f$  on  $\Gamma$  and then solving for  $f$  after applying the boundary conditions.

Let us first consider the case  $F \equiv 0$ , *i.e.*, the inhomogeneous Dirichlet problems,

$$\Delta \tilde{u} = 0 \quad \text{in } \Omega \ (\Omega^c), \quad (4.2.15)$$

$$\tilde{u} = \phi \quad \text{on } \Gamma. \quad (4.2.16)$$

In principle, the two-dimensional Laplace equation  $\Delta u(\mathbf{x}) = 0$  subject to various boundary conditions can be treated in the same way as the three-dimensional case using boundary integral equation methods. Nevertheless, the theories in  $R^2$  and  $R^3$  are slightly different because of the characteristics of their fundamental solutions. So, we shall study the existence and uniqueness of the problems (4.2.11)-(4.2.13) and (4.2.15)-(4.2.16) for two- and three-dimensional cases separately.

**Definition 4.2.1** *Let  $\Omega$  be a bounded domain in  $R^N$ ,  $N = 2$  or  $3$ . A function  $f$  on  $\Omega^c$  is said to be regular at  $\infty$  if*

$$(i) \quad f \in H_{loc}^1(\Omega^c). \quad (4.2.17)$$

$$(ii) \quad f(\mathbf{x}) = o(1) \quad \text{and} \quad \nabla f(\mathbf{x}) = o(|\mathbf{x}|^{-1}) \quad \text{for large } |\mathbf{x}|. \quad (4.2.18)$$

A function  $f(\mathbf{x})$  is said to be generalized regular at  $\infty$  if  $f(\mathbf{x}) - c$  is regular at  $\infty$  for some constant  $c$ .  $\square$

Now, we start with *three-dimensional* case. First of all, let us introduce the following theorem.

**Theorem 4.2.2** (Single-layer representation for the Dirichlet boundary value problem in  $R^3$ ) Let  $\Omega \subseteq R^3$  be bounded with smooth boundary  $\partial\Omega$ . The mapping

$$\mathcal{L}_1(\tilde{f})(\mathbf{x}) \equiv \int_{\Gamma} E(\mathbf{x} - \mathbf{y}) \tilde{f}(\mathbf{y}) dS_{\mathbf{y}}, \quad \mathbf{x} \in \Gamma \quad (4.2.19)$$

is a linear continuous mapping isomorphically from  $H^r(\partial\Omega)$  onto  $H^{r+1}(\partial\Omega)$  for any  $r \in R$ , satisfying

$$\dim \mathcal{N}(\mathcal{L}_1) = 0, \quad \dim \text{Coker}(\mathcal{L}_1) = 0. \quad (4.2.20)$$

Its adjoint operator  $\mathcal{L}_1^*$  maps  $H^{-(r+1)}(\partial\Omega)$  isomorphically onto  $H^{-r}(\partial\Omega)$  and is given by

$$\mathcal{L}_1^* = \mathcal{L}_1, \quad (4.2.21)$$

with

$$\dim \mathcal{N}(\mathcal{L}_1^*) = 0, \quad \dim \text{Coker}(\mathcal{L}_1^*) = 0. \quad (4.2.22)$$

Consequently, the solution  $\tilde{u}(\mathbf{x})$  (resp. regular at  $\infty$ ) to

$$\Delta \tilde{u}(\mathbf{x}) = 0 \quad \text{in } \Omega \text{ (resp. } \Omega^c), \quad (4.2.23)$$

$$\tilde{u}(\mathbf{x}) = \phi(\mathbf{x}) \in H^r(\Gamma), \quad \text{on } \Gamma, \quad r \in R, \quad (4.2.24)$$

is uniquely given by

$$\tilde{u}(\mathbf{x}) = \int_{\Gamma} E(\mathbf{x} - \mathbf{y}) \tilde{f}(\mathbf{y}) dS_{\mathbf{y}} \in H^{r+1/2}(\Omega) \text{ (resp. } H_{loc}^{r+1/2}(\Gamma)), \quad \forall \mathbf{x} \in \Omega, \quad (4.2.25)$$

where the potential density  $\tilde{f} \in H^{r-1}(\Gamma)$  uniquely solves the boundary integral equation

$$\mathcal{L}_1(\tilde{f}) = \phi. \quad (4.2.26)$$

*Proof.* For the proof, see [23].  $\square$

Note that the above theorem is valid for both interior and exterior problems. By using Theorem 4.2.2, we have the following result which is the existence and uniqueness theorem for the *interior* Dirichlet type boundary value problem of (4.2.11)-(4.2.13) in the *three-dimensional* case.

**Theorem 4.2.3** *Let  $\tilde{f}$  and  $\hat{f}$  be the solution of the boundary integral equations*

$$\mathcal{L}_1(\tilde{f}) = \phi \quad \text{and} \quad \mathcal{L}_1(\hat{f}) = g, \quad (4.2.27)$$

*respectively. If*

$$F\left(\left[\frac{1}{2}\hat{f}(\mathbf{x}) + \int_{\Gamma} \frac{\partial E(\mathbf{x}-\mathbf{y})}{\partial n_{\mathbf{x}}} \hat{f}(\mathbf{y}) dS_{\mathbf{y}}\right]\Big|_{\Gamma_s}\right) \neq 1, \quad (4.2.28)$$

*then the Dirichlet boundary value problem*

$$\Delta u(\mathbf{x}) = 0 \quad \text{in } \Omega \subset R^3, \quad (4.2.29)$$

$$u(\mathbf{x}) = \phi(\mathbf{x}) \quad \text{on } \Gamma_s, \quad (4.2.30)$$

$$u(\mathbf{x}) = F\left(\frac{\partial u}{\partial n}\Big|_{\Gamma_s}\right)g(\mathbf{x}) + \phi(\mathbf{x}) \quad \text{on } \Gamma_c, \quad (4.2.31)$$

*has a unique solution which is given by*

$$u(\mathbf{x}) = \int_{\Gamma} E(\mathbf{x}-\mathbf{y})f(\mathbf{y}) dS_{\mathbf{y}}, \quad \mathbf{x} \in \bar{\Omega}, \quad (4.2.32)$$

*where*

$$f(\mathbf{x}) = \tilde{f}(\mathbf{x}) + \hat{f}(\mathbf{x}) \frac{F\left(\left[\frac{1}{2}\tilde{f}(\mathbf{x}) + \int_{\Gamma} \frac{\partial E(\mathbf{x}-\mathbf{y})}{\partial n_{\mathbf{x}}} \tilde{f}(\mathbf{y}) dS_{\mathbf{y}}\right]\Big|_{\Gamma_s}\right)}{1 - F\left(\left[\frac{1}{2}\hat{f}(\mathbf{x}) + \int_{\Gamma} \frac{\partial E(\mathbf{x}-\mathbf{y})}{\partial n_{\mathbf{x}}} \hat{f}(\mathbf{y}) dS_{\mathbf{y}}\right]\Big|_{\Gamma_s}\right)}. \quad (4.2.33)$$

*Proof.* By Theorem 4.2.2, given any  $\alpha \in \mathbb{R}$ , the boundary integral equation

$$\mathcal{L}_1(f) = \alpha g + \phi, \quad (4.2.34)$$

has a unique solution  $f = \alpha \hat{f} + \tilde{f}$ .

Taking the normal derivative in (4.2.32), from the jump property of the boundary layer potentials, we get

$$\begin{aligned} \frac{\partial u(\mathbf{x})}{\partial n} &= \frac{1}{2}f(\mathbf{x}) + \int_{\Gamma} \frac{\partial E(\mathbf{x} - \mathbf{y})}{\partial n_{\mathbf{x}}} f(\mathbf{y}) dS_{\mathbf{y}} \\ &= \frac{1}{2}(\alpha \hat{f}(\mathbf{x}) + \tilde{f}(\mathbf{x})) + \int_{\Gamma} \frac{\partial E(\mathbf{x} - \mathbf{y})}{\partial n_{\mathbf{x}}} (\alpha \hat{f}(\mathbf{y}) + \tilde{f}(\mathbf{y})) dS_{\mathbf{y}}, \quad \forall \mathbf{x} \in \Gamma. \end{aligned} \quad (4.2.35)$$

Taking a bounded linear functional  $F$  on the both side of (4.2.35), we have

$$\begin{aligned} F\left(\frac{\partial u(\mathbf{x})}{\partial n}\Big|_{\Gamma_s}\right) &= \frac{\alpha}{2}F(\hat{f}(\mathbf{x})|_{\Gamma_s}) + \frac{1}{2}F(\tilde{f}(\mathbf{x})|_{\Gamma_s}) \\ &+ \alpha F\left(\int_{\Gamma} \frac{\partial E(\mathbf{x} - \mathbf{y})}{\partial n_{\mathbf{x}}} \hat{f}(\mathbf{y}) dS_{\mathbf{y}}\Big|_{\Gamma_s}\right) + F\left(\int_{\Gamma} \frac{\partial E(\mathbf{x} - \mathbf{y})}{\partial n_{\mathbf{x}}} \tilde{f}(\mathbf{y})_{\Gamma_s} dS_{\mathbf{y}}\Big|_{\Gamma_s}\right). \end{aligned}$$

Thus, providing (4.2.28) holds,  $\alpha$  is well defined and

$$\alpha = \frac{F\left(\left[\frac{1}{2}\tilde{f}(\mathbf{x}) + \int_{\Gamma} \frac{\partial E(\mathbf{x} - \mathbf{y})}{\partial n_{\mathbf{x}}} \tilde{f}(\mathbf{y}) dS_{\mathbf{y}}\right]\Big|_{\Gamma_s}\right)}{1 - F\left(\left[\frac{1}{2}\hat{f}(\mathbf{x}) + \int_{\Gamma} \frac{\partial E(\mathbf{x} - \mathbf{y})}{\partial n_{\mathbf{x}}} \hat{f}(\mathbf{y}) dS_{\mathbf{y}}\right]\Big|_{\Gamma_s}\right)}. \quad (4.2.36)$$

By the substitution method and the integral representation, we have shown the existence of the solution  $u$ . The uniqueness of the solution follows from the following Lemma and the linearity of the problem.  $\square$

**Lemma 4.2.4** For any  $g \in H_D^{1/2}(\Gamma)$  such that,

$$F\left(\left[\frac{1}{2}\hat{f}(\mathbf{x}) + \int_{\Gamma} \frac{\partial E(\mathbf{x} - \mathbf{y})}{\partial n_{\mathbf{x}}} \hat{f}(\mathbf{y}) dS_{\mathbf{y}}\right]\Big|_{\Gamma_s}\right) \neq 1, \quad (4.2.37)$$

where  $\hat{f}$  is the solution of the boundary integral equation  $\mathcal{L}_1(\hat{f}) = g$ , the boundary value problem

$$\Delta u = 0 \quad \text{in } \Omega, \quad (4.2.38)$$

$$u = 0 \quad \text{on } \Gamma_s, \quad (4.2.39)$$

$$u = F\left(\frac{\partial u}{\partial n}\Big|_{\Gamma_s}\right)g \quad \text{on } \Gamma_c, \quad (4.2.40)$$

has a unique solution  $u \equiv 0$ .

*Proof.* From Theorem 4.2.3, we only need to show the uniqueness of the solution. Let  $\hat{u}$  be the nontrivial solution of (4.2.38)-(4.2.40), i.e.,  $\hat{u}$  is not identically zero, which implies that

$$F\left(\frac{\partial \hat{u}}{\partial n}\Big|_{\Gamma_s}\right) \neq 0. \quad (4.2.41)$$

Taking the normal derivative and a bounded linear functional  $F$  in  $\hat{u}$ , we have

$$\hat{\alpha} = \hat{\alpha}F\left(\left[\frac{1}{2}\hat{f}(\mathbf{x}) + \int_{\Gamma} \frac{\partial E(\mathbf{x} - \mathbf{y})}{\partial n_{\mathbf{x}}} \hat{f}(\mathbf{y}) dS_{\mathbf{y}}\right]\Big|_{\Gamma_s}\right), \quad (4.2.42)$$

where  $\hat{\alpha} = F(\partial \hat{u} / \partial n|_{\Gamma_s})$ . From (4.2.37),  $\hat{\alpha}$  must be zero which contradicts (4.2.41).  $\square$

The theorem 4.2.3 is also valid for the *exterior* Dirichlet type boundary value problem in *three-dimensions* for a regular solution by setting

$$\frac{\partial u(\mathbf{x})}{\partial n} = -\frac{1}{2}f(\mathbf{x}) + \int_{\Gamma} \frac{\partial E(\mathbf{x} - \mathbf{y})}{\partial n_{\mathbf{x}}} f(\mathbf{y}) dS_{\mathbf{y}}. \quad (4.2.43)$$

In this case  $u \in H_{loc}^1(\Omega^c)$ .

Now, we examine the essential hypothesis (4.2.28) in Theorem 4.2.3 which is  $F(\partial \hat{u} / \partial n) \neq$

1. First, we consider the case that the bounded linear functional  $F$  is nonnegative, i.e.,  $F(u) \geq 0$ , if  $u \geq 0$ . For example,  $F$  is the average of  $u$  on  $\Gamma_s$ . Second, for any  $\epsilon \geq 0$ , we can have  $F(\partial \hat{u} / \partial n) \leq \epsilon$  by adjusting a linear functional  $F$  or a basis function  $g \in H_D^1(\Gamma)$ .

**Proposition 4.2.5** *If  $F$  is nonnegative, i.e.,  $F(u) \geq 0$  if  $u \geq 0$ , and  $g > 0$  in  $H_D^{1/2}(\Gamma)$ , then the boundary value problem*

$$\Delta u(\mathbf{x}) = 0 \quad \text{in } \Omega \subset R^3, \quad (4.2.44)$$

$$u(\mathbf{x}) = \phi(\mathbf{x}) \quad \text{on } \Gamma_s, \quad (4.2.45)$$

$$u(\mathbf{x}) = F\left(\frac{\partial u}{\partial n}\Big|_{\Gamma_s}\right) g(\mathbf{x}) + \phi(\mathbf{x}) \quad \text{on } \Gamma_c. \quad (4.2.46)$$

*has a unique solution.*

*Proof.* From Theorem 4.2.3, we only need to show that

$$F\left(\left[\frac{1}{2}\hat{f}(\mathbf{x}) + \int_{\Gamma} \frac{\partial E(\mathbf{x}-\mathbf{y})}{\partial n_{\mathbf{x}}} \hat{f}(\mathbf{y}) dS_{\mathbf{y}}\right]\Big|_{\Gamma_s}\right) \neq 1. \quad (4.2.47)$$

Since

$$\frac{1}{2}\hat{f}(\mathbf{x}) + \int_{\Gamma} \frac{\partial E(\mathbf{x}-\mathbf{y})}{\partial n_{\mathbf{x}}} \hat{f}(\mathbf{y}) dS_{\mathbf{y}} = \frac{\partial \hat{u}}{\partial n}, \quad (4.2.48)$$

where  $\hat{u}$  is the unique solution of the boundary value problem

$$\Delta \hat{u}(\mathbf{x}) = 0 \quad \text{in } \Omega \subset R^3, \quad (4.2.49)$$

$$\hat{u}(\mathbf{x}) = 0 \quad \text{on } \Gamma_s, \quad (4.2.50)$$

$$\hat{u}(\mathbf{x}) = g(\mathbf{x}) \quad \text{on } \Gamma_c, \quad (4.2.51)$$

one can easily show that  $\hat{u}(\mathbf{x}) \geq 0$  in  $\bar{\Omega}$  and  $\frac{\partial \hat{u}}{\partial n} \leq 0$  in  $\Gamma_s$ . Thus,  $F\left(\frac{\partial \hat{u}}{\partial n}\Big|_{\Gamma_s}\right) \leq 0$ . The proof is completed.  $\square$

**Proposition 4.2.6** *Given any  $\epsilon > 0$  and bounded linear functional  $F$  on  $H^{-1/2}(\Gamma_s)$  such that  $\|F\| \leq M$ , there is a function  $g \in H_D^{1/2}(\Gamma)$  such that*

$$\left|F\left(\frac{\partial \hat{u}}{\partial n}\Big|_{\Gamma_s}\right)\right| \leq \epsilon, \quad (4.2.52)$$

where  $\hat{u}$  is a solution of the Dirichlet problem,

$$\Delta \hat{u} = 0 \quad \text{in } \Omega, \quad (4.2.53)$$

$$\hat{u} = g \quad \text{on } \Gamma. \quad (4.2.54)$$

*Proof.* By the theory of partial differential equation, for any given  $g \in H_D^{1/2}(\Gamma)$ , we have

$$\left\| \frac{\partial \hat{u}}{\partial n} \right\|_{H^{-1/2}(\Gamma_s)} \leq D_1 \|\hat{u}\|_{H^1(\Omega)} \leq D_1 D_2 \|g\|_{H^{1/2}(\Gamma)}. \quad (4.2.55)$$

Since  $g$  has compact support on  $\Gamma_c$ , by letting  $D = D_1 D_2$ , we have

$$\begin{aligned} \left| F \left( \frac{\partial \hat{u}}{\partial n} \Big|_{\Gamma_s} \right) \right| &\leq \|F\| \left\| \frac{\partial \hat{u}}{\partial n} \right\|_{H^{-1/2}(\Gamma_s)} \\ &\leq MD \|g\|_{H^{1/2}(\Gamma_c)}. \end{aligned}$$

Thus, if we choose a function  $g \in H_D^{1/2}(\Gamma)$  such that,

$$\|g\|_{H^{1/2}(\Gamma)} \leq \frac{\epsilon}{MD}, \quad (4.2.56)$$

then, we have

$$\left| F \left( \frac{\partial \hat{u}}{\partial n} \Big|_{\Gamma_s} \right) \right| \leq \epsilon. \quad \square \quad (4.2.57)$$

For the second case, we have a regularity result for the solution  $f$  of the boundary integral equation  $\mathcal{L}_1(f) = \alpha g + \phi$ .

**Theorem 4.2.7** *For any  $g \in H_D^{1/2}(\partial\Omega)$  such that*

$$\left| F \left( \frac{\partial \hat{u}}{\partial n} \Big|_{\Gamma_s} \right) \right| \leq \epsilon < 1, \quad (4.2.58)$$

*we have the estimate*

$$\|f\|_{H^{-1/2}(\partial\Omega)} \leq C \left( \|\phi\|_{H^{1/2}(\Omega)} + \|g\|_{H^{1/2}(\Omega)} \frac{\|F\| \|\phi\|_{H^{1/2}(\partial\Omega)}}{1 - \epsilon} \right) \quad (4.2.59)$$



where  $f \in H^{-1/2}(\partial\Omega)$  and  $\hat{u} \in H^1(\Omega)$  are the solutions of the equations (4.2.34) and (4.2.53)-(4.2.54), respectively.

*Proof.* From the continuity and invertibility of

$$\mathcal{L}_1 : H^{-1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega) \quad (4.2.60)$$

and

$$\mathcal{L}_1^{-1} : H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega), \quad (4.2.61)$$

we have

$$\|\tilde{f}\|_{H^{-1/2}(\partial\Omega)} \leq C\|\phi\|_{H^{1/2}(\partial\Omega)}, \quad \|\hat{f}\|_{H^{-1/2}(\partial\Omega)} \leq C\|g\|_{H^{1/2}(\partial\Omega)}. \quad (4.2.62)$$

Thus, by Theorem 4.1.3, we have

$$\begin{aligned} \|f\|_{H^{-1/2}(\partial\Omega)} &\leq \|\hat{f}\|_{H^{-1/2}(\Omega)}\|\tilde{f}\|_{H^{-1/2}(\Omega)} + \frac{\left|F\left(\frac{\partial\tilde{u}}{\partial n}\Big|_{\Gamma_s}\right)\right|}{\left|1 - F\left(\frac{\partial\tilde{u}}{\partial n}\Big|_{\Gamma_s}\right)\right|} \\ &\leq C_1\|\phi\|_{H^{1/2}(\Omega)} + C_2\|g\|_{H^{1/2}(\Omega)} \frac{\|F\| \|\tilde{u}\|_{H^1(\Omega)}}{1 - \epsilon} \\ &\leq C\left(\|\phi\|_{H^{1/2}(\Omega)} + \|g\|_{H^{1/2}(\Omega)} \frac{\|F\| \|\phi\|_{H^{1/2}(\partial\Omega)}}{1 - \epsilon}\right). \quad \square \end{aligned}$$

We now consider the Dirichlet type boundary value problem in the *two-dimensional* case.

**Theorem 4.2.8** (*Single-layer representation for the Dirichlet boundary value problem in  $R^2$  with  $O(\ln|\mathbf{x}|)$  growth*) We have

$$\mathcal{L}'_1 : R \oplus H^r(\Gamma) \rightarrow R \oplus H^{r+1}(\Gamma), \quad r \in R, \quad \text{isomorphically,} \quad (4.2.63)$$

where

$$\mathcal{L}'_1 \left( \begin{bmatrix} a \\ f \end{bmatrix} \right) (\mathbf{x}) = \begin{bmatrix} \int_{\Gamma} f(\mathbf{y}) dS_{\mathbf{y}} \\ \int_{\Gamma} E(\mathbf{x} - \mathbf{y})f(\mathbf{y}) dS_{\mathbf{y}} + a \end{bmatrix}, \quad \mathbf{x} \in \Gamma,$$

with adjoint

$$\mathcal{L}'_1{}^* : R \oplus H^{-(r+1)}(\Gamma) \rightarrow R \oplus H^{-r}(\Gamma), \quad r \in R, \quad \text{isomorphically,} \quad (4.2.64)$$

where

$$\mathcal{L}'_1{}^* \left( \begin{bmatrix} b \\ \tau \end{bmatrix} \right) (\mathbf{x}) = \begin{bmatrix} \int_{\Gamma} \tau(\mathbf{y}) dS_{\mathbf{y}} \\ \int_{\Gamma} E(\mathbf{x} - \mathbf{y}) \tau(\mathbf{y}) dS_{\mathbf{y}} + b \end{bmatrix}, \quad \mathbf{x} \in \Gamma.$$

The canonical restriction or extension of  $\mathcal{L}'_1{}^*$  to  $R \oplus H^r(\Gamma)$  makes  $\mathcal{L}'_1{}^* = \mathcal{L}'_1$ , i.e.,  $\mathcal{L}'_1$  is self-adjoint. The solution  $u(\mathbf{x})$  to the Dirichlet boundary value problem

$$\Delta u(\mathbf{x}) = 0, \quad \text{in } \Omega \text{ (resp. } \Omega^c), \quad (4.2.65)$$

$$u(\mathbf{x}) = \phi(\mathbf{x}) \quad \text{on } \Gamma_s, \quad (4.2.66)$$

$$u(\mathbf{x}) = A \ln |\mathbf{x}| + O(1), \quad \text{for large } |\mathbf{x}|, \quad \text{with } A \text{ given,} \quad (4.2.67)$$

is uniquely given by

$$u(\mathbf{x}) = \int_{\Gamma} E(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) dS_{\mathbf{y}} + a \in H^{r+1/2}(\Omega) \text{ (resp. } H_{loc}^{r+1/2}(\Omega^c)) \quad (4.2.68)$$

where  $(a, f) \in R \oplus H^r(\Gamma)$  uniquely solves the boundary integral equation

$$\mathcal{L}'_1 \left( \begin{bmatrix} a \\ f \end{bmatrix} \right) = \begin{bmatrix} A \\ \psi \end{bmatrix},$$

for some (arbitrary) given  $A \in R$ .

*Proof.* For the proof, see [23].  $\square$

Note that the above theorem is also valid for both interior and exterior problems. By using Theorem 4.2.8, we have the following result which is the existence and uniqueness theorem for the *interior* Dirichlet type boundary value problem of the problem (4.2.11)-(4.2.13) in the *two-dimensional* case.

**Theorem 4.2.9** Let  $(\tilde{f}, \tilde{a})$  and  $(\hat{f}, \hat{a})$  be the solution of the boundary integral equations

$$\mathcal{L}'_1 \left( \begin{bmatrix} \tilde{a} \\ \tilde{f} \end{bmatrix} \right) = \begin{bmatrix} A \\ \psi \end{bmatrix}, \quad \text{and} \quad \mathcal{L}'_1 \left( \begin{bmatrix} \hat{a} \\ \hat{f} \end{bmatrix} \right) = \begin{bmatrix} A \\ g \end{bmatrix},$$

respectively. If

$$F \left( \left[ \frac{1}{2} \hat{f}(\mathbf{x}) + \int_{\Gamma} \frac{\partial E(\mathbf{x}-\mathbf{y})}{\partial n_{\mathbf{x}}} \hat{f}(\mathbf{y}) dS_{\mathbf{y}} \right] \Big|_{\Gamma_s} \right) \neq 1, \quad (4.2.69)$$

then the boundary value problem

$$\Delta u(\mathbf{x}) = 0, \quad \text{in } \Omega \subset R^2, \quad (4.2.70)$$

$$u(\mathbf{x}) = \phi, (\mathbf{x}) \quad \text{on } \Gamma_s, \quad (4.2.71)$$

$$u(\mathbf{x}) = F \left( \frac{\partial u}{\partial n} \Big|_{\Gamma_s} \right) g(\mathbf{x}) + \phi(\mathbf{x}), \quad \text{on } \Gamma_c, \quad (4.2.72)$$

has a unique solution which is given by

$$u(\mathbf{x}) = \int_{\Gamma} E(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) dS_{\mathbf{y}} + a \in H^{r+1/2}(\Omega), \quad (4.2.73)$$

where

$$f(\mathbf{x}) = \tilde{f}(\mathbf{x}) + \hat{f}(\mathbf{x}) \frac{F \left( \left[ \frac{1}{2} \tilde{f}(\mathbf{x}) + \int_{\Gamma} \frac{\partial E(\mathbf{x}-\mathbf{y})}{\partial n_{\mathbf{x}}} \tilde{f}(\mathbf{y}) dS_{\mathbf{y}} \right] \Big|_{\Gamma_s} \right)}{1 - F \left( \left[ \frac{1}{2} \hat{f}(\mathbf{x}) + \int_{\Gamma} \frac{\partial E(\mathbf{x}-\mathbf{y})}{\partial n_{\mathbf{x}}} \hat{f}(\mathbf{y}) dS_{\mathbf{y}} \right] \Big|_{\Gamma_s} \right)}, \quad (4.2.74)$$

and

$$a = \tilde{a} + \hat{a} \frac{F \left( \left[ \frac{1}{2} \tilde{f}(\mathbf{x}) + \int_{\Gamma} \frac{\partial E(\mathbf{x}-\mathbf{y})}{\partial n_{\mathbf{x}}} \tilde{f}(\mathbf{y}) dS_{\mathbf{y}} \right] \Big|_{\Gamma_s} \right)}{1 - F \left( \left[ \frac{1}{2} \hat{f}(\mathbf{x}) + \int_{\Gamma} \frac{\partial E(\mathbf{x}-\mathbf{y})}{\partial n_{\mathbf{x}}} \hat{f}(\mathbf{y}) dS_{\mathbf{y}} \right] \Big|_{\Gamma_s} \right)}. \quad (4.2.75)$$

*Proof.* By Theorem 4.2.8, given  $\forall \alpha \in R$ , the boundary integral equation

$$\mathcal{L}'_1 \left( \begin{bmatrix} a \\ f \end{bmatrix} \right) = \begin{bmatrix} 2A \\ \psi + \alpha g \end{bmatrix},$$

has unique solutions

$$\begin{pmatrix} a \\ f \end{pmatrix} = \begin{pmatrix} \tilde{a} + \alpha \hat{a} \\ \tilde{f} + \alpha \hat{f} \end{pmatrix}. \quad (4.2.76)$$

Taking the normal derivative in

$$u(\mathbf{x}) = \int_{\partial\Omega} E(\mathbf{x} - \mathbf{y})f(\mathbf{y}) dS_{\mathbf{y}} + a, \quad (4.2.77)$$

we get

$$\begin{aligned} \frac{\partial u(\mathbf{x})}{\partial n} &= \frac{1}{2}f(\mathbf{x}) + \int_{\Gamma} \frac{\partial E(\mathbf{x} - \mathbf{y})}{\partial n_{\mathbf{x}}} f(\mathbf{y}) dS_{\mathbf{y}} \\ &= \frac{1}{2}(\alpha \hat{f}(\mathbf{x}) + \tilde{f}(\mathbf{x})) + \int_{\Gamma} \frac{\partial E(\mathbf{x} - \mathbf{y})}{\partial n_{\mathbf{x}}} (\alpha \hat{f}(\mathbf{y}) + \tilde{f}(\mathbf{y})) dS_{\mathbf{y}}, \quad \forall \mathbf{x} \in \Gamma. \end{aligned} \quad (4.2.78)$$

Taking a bounded linear functional  $F$  on the both side of (4.2.78), we have

$$\begin{aligned} F\left(\frac{\partial u(\mathbf{x})}{\partial n}\Big|_{\Gamma_s}\right) &= \frac{\alpha}{2}F(\hat{f}(\mathbf{x})|_{\Gamma_s}) + \frac{1}{2}F(\tilde{f}(\mathbf{x})|_{\Gamma_s}) \\ &+ \alpha F\left(\int_{\Gamma} \frac{\partial E(\mathbf{x} - \mathbf{y})}{\partial n_{\mathbf{x}}} \hat{f}(\mathbf{y}) dS_{\mathbf{y}}\Big|_{\Gamma_s}\right) + F\left(\int_{\Gamma} \frac{\partial E(\mathbf{x} - \mathbf{y})}{\partial n_{\mathbf{x}}} \tilde{f}(\mathbf{y})_{\Gamma_s} dS_{\mathbf{y}}\Big|_{\Gamma_s}\right). \end{aligned}$$

Thus, providing (4.2.69) holds,  $\alpha$  is well defined and

$$\alpha = \frac{F\left(\left[\frac{1}{2}\tilde{f}(\mathbf{x}) + \int_{\Gamma} \frac{\partial E(\mathbf{x} - \mathbf{y})}{\partial n_{\mathbf{x}}} \tilde{f}(\mathbf{y}) dS_{\mathbf{y}}\right]\Big|_{\Gamma_s}\right)}{1 - F\left(\left[\frac{1}{2}\hat{f}(\mathbf{x}) + \int_{\Gamma} \frac{\partial E(\mathbf{x} - \mathbf{y})}{\partial n_{\mathbf{x}}} \hat{f}(\mathbf{y}) dS_{\mathbf{y}}\right]\Big|_{\Gamma_s}\right)}. \quad (4.2.79)$$

By the substitution method and the integral representation, we have shown the existence of the solution  $u$ . The uniqueness of the solution can be proved in a similar way to Theorem 4.2.3.  $\square$

Also, Theorem 4.2.9 is valid for the *exterior* Dirichlet problem in *two-dimensions* for a regular solution by setting

$$\frac{\partial u(\mathbf{x})}{\partial n} = -\frac{1}{2}f(\mathbf{x}) + \int_{\Gamma} \frac{\partial E(\mathbf{x} - \mathbf{y})}{\partial n_{\mathbf{x}}} f(\mathbf{y}) dS_{\mathbf{y}}. \quad (4.2.80)$$

In this case  $u \in H_{loc}^1(\Omega^c)$ . Similarly, we can prove Theorem 4.2.9 for the case of  $O(1)$  growth.

## 4.2.2 The Neumann Type Boundary Value Problem

Let us consider the Neumann Type boundary value problems

$$\Delta u(\mathbf{x}) = 0 \quad \text{in } \Omega \ (\Omega^c), \quad (4.2.81)$$

$$\frac{\partial u(\mathbf{x})}{\partial n} = \phi(\mathbf{x}) \quad \text{on } \Gamma_s, \quad (4.2.82)$$

$$\frac{\partial u(\mathbf{x})}{\partial n} = F(u(\mathbf{x})|_{\Gamma_s})g(\mathbf{x}) + \phi(\mathbf{x}) \quad \text{on } \Gamma_c. \quad (4.2.83)$$

where  $g(\mathbf{x}) \in H_D^{-1/2}(\Gamma)$ ,  $\phi(\mathbf{x}) \in H^{-1/2}(\Gamma)$ , and  $F$  is a bounded linear functional on  $H^{1/2}(\Gamma_s)$ .

First, we consider the Neumann boundary value problems

$$\Delta u(\mathbf{x}) = 0 \quad \text{in } \Omega \ (\Omega^c), \quad (4.2.84)$$

$$\frac{\partial u(\mathbf{x})}{\partial n} = \phi(\mathbf{x}) \in H^r(\partial\Omega), \quad r \in R, \quad (4.2.85)$$

where  $n$  is unit exterior normal on  $\partial\Omega$ . From the theory of the elliptic partial differential equations, we know that there is a solution  $u \in H^{r+3/2}(\Omega)$  satisfying (4.2.84)-(4.2.85), and

$$\inf \|u\|_{H^{r+3/2}(\Omega)} \leq C \|\phi\|_{H^r(\partial\Omega)} \quad (4.2.86)$$

for some  $C > 0$  independent of  $\phi$ , with the infimum taken over all such  $u$  satisfying (4.2.84)-(4.2.85), if and only if the compatibility condition

$$\int_{\partial\Omega} \phi(\mathbf{x}) \, dS = 0 \quad (4.2.87)$$

is satisfied.

**Theorem 4.2.10** (*Single-layer representation for the interior Neumann boundary value problem*) Let  $\Omega \subset R^N$ ,  $N = 2$  or  $3$ , be bounded with smooth boundary  $\Gamma$ . The mapping

$$\mathcal{L}_2(f)(\mathbf{x}) = \int_{\Gamma} \frac{\partial E(\mathbf{x} - \mathbf{y})}{\partial n_{\mathbf{x}}} f(\mathbf{y}) \, dS_{\mathbf{y}} + \frac{1}{2} f(\mathbf{x}), \quad \mathbf{x} \in \Gamma, \quad (4.2.88)$$

is Fredholm:

$$\mathcal{L}_2 : H^r(\Gamma) \rightarrow H^r(\Gamma) \quad (4.2.89)$$

for any  $r \in R$ , with

$$\dim \mathcal{N}(\mathcal{L}_2) = 1, \quad \dim \text{Coker}(\mathcal{L}_2) = 1, \quad (4.2.90)$$

hence zero index. Its adjoint operator

$$\mathcal{L}_2^* : H^{-r}(\Gamma) \rightarrow H^{-r}(\Gamma) \quad (4.2.91)$$

is given by

$$\mathcal{L}_2^*(\tau)(\mathbf{x}) = \int_{\Gamma} \frac{\partial E(\mathbf{x} - \mathbf{y})}{\partial n_{\mathbf{x}}} \tau(\mathbf{y}) dS_{\mathbf{y}} + \frac{1}{2} \tau(\mathbf{x}), \quad \mathbf{x} \in \Gamma, \quad (4.2.92)$$

with

$$\dim \mathcal{N}(\mathcal{L}_2^*) = 1, \quad \dim \text{Coker}(\mathcal{L}_2^*) = 1, \quad (4.2.93)$$

$$\mathcal{N}(\mathcal{L}_2^*) = \{c | c \in R, \text{ on } \Gamma\} (= \text{Coker}(\mathcal{L}_2)), \quad (4.2.94)$$

$$\text{Coker}(\mathcal{L}_2^*) = \begin{cases} \text{span}\{\lambda_0(\cdot) | \lambda_0(\cdot) \text{ satisfies } \mathcal{L}_1(\lambda_0) \equiv 1 \text{ on } \Gamma\} (= \mathcal{N}(\mathcal{L}_2)), & \text{for } N=3, \\ \text{span}\{\lambda_0(\cdot) | \lambda_0(\cdot) \neq 0 \text{ satisfies } \int_{\Gamma} E_2(\mathbf{x} - \mathbf{y}) \lambda_0(\mathbf{y}) dS_{\mathbf{y}} = c \in R \text{ on } \Gamma, \\ \text{for some } c \in R, c \text{ may be } 0\} (= \mathcal{N}(\mathcal{L}_2)), & \text{for } N=2. \end{cases}$$

$\mathcal{L}_2^*$  satisfies

$$\mathcal{L}_2^* : H^s(\Gamma) \rightarrow H^s(\Gamma) \quad \text{continuously for any } s \in R. \quad (4.2.95)$$

Consequently, the solution  $\tilde{u}(\mathbf{x})$  to the interior Neumann problem,

$$\Delta \tilde{u}(\mathbf{x}) = 0 \quad \text{in } \Omega, \quad (4.2.96)$$

$$\frac{\partial \tilde{u}(\mathbf{x})}{\partial n} = \phi(\mathbf{x}) \in H^r(\Gamma), \quad \text{on } \Gamma, \quad (4.2.97)$$

can be given in the form

$$\tilde{u}(\mathbf{x}) = \int_{\Gamma} E(\mathbf{x} - \mathbf{y}) \tilde{f}(\mathbf{y}) dS_{\mathbf{y}} + c \in H^{r+3/2}, \quad \mathbf{x} \in \Omega, \quad (4.2.98)$$

for some  $\tilde{f} \in H^r(\Gamma)$  and for any given constant  $c \in R$ , where  $\tilde{f}$  is a solution to the boundary integral equation,

$$\mathcal{L}_2(\tilde{f}) = \phi, \quad (4.2.99)$$

provided that the compatibility condition

$$\langle \phi, 1 \rangle_{L^2(\Gamma)} = \int_{\Gamma} \phi(\mathbf{x}) dS_{\Gamma} = 0 \quad (4.2.100)$$

is satisfied, i.e.,  $\phi \perp \text{Coker}(\mathcal{L}_2)$ .

*Proof.* For the proof, see [23].  $\square$

The operator  $\mathcal{L}_2$  is not invertible, since  $\mathcal{N}(\mathcal{L}_2)$  has dimension 1. This causes the nonuniqueness of the solution  $f$  to the boundary integral equation (4.2.99) which is inconvenient both theoretically and numerically. But from the above Theorem, we have complete information about  $\mathcal{N}(\mathcal{L}_2)$ , i.e.,  $\mathcal{L}_2$  is a Fredholm operator with zero index on  $H^r(\partial\Omega)$ ,  $r \in R$ . It is a pseudodifferential operator of order 0. The integral operator on the RHS of (4.2.88) is a compact operator on  $H^r(\partial\Omega)$ . Further,

$$\mathcal{N}(\mathcal{L}_2) = \text{span}\{k \in C^\infty(\partial\Omega) | \mathcal{L}_1 k = 1, \text{ on } \partial\Omega\}, \quad (4.2.101)$$

$$\text{Coker}(\mathcal{L}_2) = \mathcal{N}(\mathcal{L}_2^*) = \text{span}\{1 \text{ on } \partial\Omega\}, \quad (4.2.102)$$

$$(\mathcal{L}_2^* \tau)(\mathbf{x}) = \frac{1}{2} \tau(\mathbf{x}) + \int_{\partial\Omega} \frac{\partial}{\partial n_{\mathbf{x}}} E(\mathbf{x}, \mathbf{y}) \tau(\mathbf{y}) dS_{\mathbf{y}}, \quad \mathbf{x} \in \partial\Omega, \quad (4.2.103)$$

and  $\mathcal{L}_2^* : H^{-r}(\partial\Omega) \rightarrow H^{-r}(\partial\Omega)$ ,  $r \in R$ , is also Fredholm with zero index. In order to fix the solution of (4.2.84)-(4.2.87) uniquely, we consider appending the extra condition such that  $\tilde{u}(\mathbf{x}_0) = 0$  without loss of generality.

First, note that  $\tilde{u}$  is defined pointwise at  $\mathbf{x}_0 \in \partial\Omega$ , provided that  $\phi \in H^r(\partial\Omega)$  with  $r > N/2 - 1$  and thus  $\tilde{u} \in H^{3/2+r}(\Omega)$ . Also, its trace satisfies

$$\tilde{u}|_{\partial\Omega} \in H^{3/2+r-1/2}(\partial\Omega) = H^{1+r}(\partial\Omega) \subset C^0(\partial\Omega), \quad (4.2.104)$$

for  $r > N/2 - 1$ , by the Sobolev imbedding theorem. Corresponding to the single-layer representation (4.2.98),  $\tilde{u}(\mathbf{x}_0) = 0$  implies that

$$\int_{\partial\Omega} E(\mathbf{x}_0, \mathbf{y}) \tilde{f}(\mathbf{y}) dS_{\mathbf{y}} = 0. \quad (4.2.105)$$

So we let

$$\int_{\partial\Omega} E(\mathbf{x}_0, \mathbf{y}) \tilde{f}(\mathbf{y}) dS_{\mathbf{y}} \equiv \int_{\partial\Omega} \tilde{f}(\mathbf{y}) \tau^*(\mathbf{y}) dS_{\mathbf{y}} = \langle \tilde{f}, \tau^* \rangle_{L^2}. \quad (4.2.106)$$

Let  $k \in \mathcal{N}(\mathcal{L}_2)$  be its basis element satisfying

$$\int_{\partial\Omega} E(\mathbf{x}, \mathbf{y}) k(\mathbf{y}) dS_{\mathbf{y}} = 1, \quad \forall \mathbf{x} \in \partial\Omega. \quad (4.2.107)$$

Then we let the augmented system be

$$\begin{bmatrix} 0 & \tau^* \\ 1 & \mathcal{L}_2 \end{bmatrix} \begin{bmatrix} a \\ \tilde{f} \end{bmatrix} = \begin{bmatrix} 0 \\ \phi \end{bmatrix}, \quad \text{in } R \oplus H^r(\partial\Omega),$$

*i.e.*,

$$\langle \tilde{f}, \tau^* \rangle = \int_{\partial\Omega} \tilde{f}(\mathbf{y}) \tau^*(\mathbf{y}) dS_{\mathbf{y}} = 0, \quad (4.2.108)$$

$$a + (\mathcal{L}_2 \tilde{f})(\mathbf{x}) = \phi(\mathbf{x}). \quad (4.2.109)$$

The above system is invertible if and only if

$$\langle k, \tau^* \rangle_{H^r(\partial\Omega) \times H^{-r}(\partial\Omega)} = \langle k, \tau^* \rangle_{L^2(\partial\Omega)} \neq 0, \quad (4.2.110)$$

*i.e.*,

$$\int_{\partial\Omega} E(\mathbf{x}_0, \mathbf{y}) k(\mathbf{y}) dS_{\mathbf{y}} \neq 0, \quad (4.2.111)$$

But, from (4.2.107), we know that (4.2.111) is always satisfied. We conclude the following.



**Proposition 4.2.11** *The Neumann boundary value problem*

$$\Delta \tilde{u}(\mathbf{x}) = 0, \quad (4.2.112)$$

$$\frac{\partial \tilde{u}(\mathbf{x})}{\partial n} = \phi(\mathbf{x}) \in H^r(\partial\Omega), \quad r > \frac{N}{2}, \quad (4.2.113)$$

with the compatibility condition  $\int_{\partial\Omega} \phi \, dS_{\mathbf{x}}$  has a unique solution  $\tilde{u} \in H^{r+3/2}(\partial\Omega)$  also satisfying the condition

$$\tilde{u}(\mathbf{x}_0) = 0, \quad \text{with } \mathbf{x}_0 \in \partial\Omega. \quad (4.2.114)$$

Furthermore, the augmented boundary integral equation system (4.2.108)-(4.2.109) based on the single-layer potential has a unique solution  $(a, \tilde{f}) = (0, \tilde{f}) \in R \times H^r(\partial\Omega)$  satisfying

$$\langle \tilde{f}, \tau^* \rangle = 0, \quad (4.2.115)$$

where  $\tau^*$  is defined by (4.2.106).  $\square$

From the above Proposition, we have the following result. Of course, we can prove it in a similar way to Theorem 4.2.3.

**Theorem 4.2.12** *Let  $\phi \in H^r(\partial\Omega)$  and  $g \in H_D^r(\partial\Omega)$ , where  $r > N/2 - 1$ . Let  $\tilde{f}$  be the solution to the boundary integral equation,*

$$\mathcal{L}_2(\tilde{f}) = \phi \quad (4.2.116)$$

with extra condition  $\tilde{u}(\mathbf{x}_0) = 0$ ,  $\mathbf{x}_0 \in \Gamma_s$ , i.e.,

$$\langle \tilde{f}, \tau^* \rangle = 0. \quad (4.2.117)$$

and let  $\hat{f}$  be the solution to the boundary integral equation,

$$\mathcal{L}_2(\hat{f}) = g, \quad \langle \hat{f}, \tau^* \rangle = 0. \quad (4.2.118)$$

If

$$F\left(\left[\int_{\Gamma} E(\mathbf{x}-\mathbf{y})\hat{f}(\mathbf{y}) dS_{\mathbf{y}}\right]\Big|_{\Gamma_s}\right) \neq 1, \quad (4.2.119)$$

then the boundary value problem,

$$\Delta u(\mathbf{x}) = 0 \quad \text{in } \Omega, \quad (4.2.120)$$

$$\frac{\partial u(\mathbf{x})}{\partial n} = \phi(\mathbf{x}) \quad \text{on } \Gamma_s, \quad (4.2.121)$$

$$\frac{\partial u(\mathbf{x})}{\partial n} = F(u|_{\Gamma_s})g + \phi(\mathbf{x}) \quad \text{on } \Gamma_c, \quad (4.2.122)$$

has a unique solution which is given by

$$u(\mathbf{x}) = \int_{\Gamma} E(\mathbf{x}-\mathbf{y})f(\mathbf{y}) dS_{\mathbf{y}}, \quad \mathbf{x} \in \Omega, \quad (4.2.123)$$

where

$$f(\mathbf{x}) = \tilde{f}(\mathbf{x}) + \hat{f}(\mathbf{x}) \frac{F\left(\left[\int_{\Gamma} E(\mathbf{x}-\mathbf{y})\tilde{f}(\mathbf{y}) dS_{\mathbf{y}}\right]\Big|_{\Gamma_s}\right)}{1 - F\left(\left[\int_{\Gamma} E(\mathbf{x}-\mathbf{y})\hat{f}(\mathbf{y}) dS_{\mathbf{y}}\right]\Big|_{\Gamma_s}\right)}, \quad (4.2.124)$$

provided that the compatibility condition

$$\langle \phi, 1 \rangle_{L^2(\Gamma)} = 0 \quad (4.2.125)$$

and

$$\langle g, 1 \rangle_{L^2(\Gamma)} = 0, \quad (4.2.126)$$

are satisfied and also satisfying the extra condition  $u(\mathbf{x}_0) = 0$ ,  $\mathbf{x}_0 \in \Gamma_s$ .  $\square$

Similarly, we can prove the existence and the uniqueness for the exterior Neumann problem in the two- and three-dimensional cases. Of course, we assume that the solution  $u$  is regular at  $\infty$ . We just introduce the following basic theorems which are useful to study the existence and the uniqueness for the exterior Neumann problem in the two- and three-dimensional cases.

**Theorem 4.2.13** *There is a unique solution  $u$ , regular at  $\infty$ , to the exterior Neumann problem in  $R^3$*

$$\Delta \tilde{u}(\mathbf{x}) = 0 \quad \text{in } \Omega^c, \quad (4.2.127)$$

$$\frac{\partial \tilde{u}(\mathbf{x})}{\partial n} = \phi(\mathbf{x}) \quad \text{on } \Gamma, \quad (4.2.128)$$

given by

$$\tilde{u}(\mathbf{x}) = \int_{\Gamma} E(\mathbf{x} - \mathbf{y}) \tilde{f}(\mathbf{y}) dS_{\mathbf{y}} \in H_{loc}^{r+3/2}(\Omega^c) \quad (4.2.129)$$

for unique  $\tilde{f} \in H^r(\Gamma)$ , where  $\tilde{f}$  uniquely solves the boundary integral equation

$$\int_{\Gamma} \frac{\partial E(\mathbf{x} - \mathbf{y})}{\partial n_{\mathbf{x}}} \tilde{f}(\mathbf{y}) dS_{\mathbf{y}} - \frac{1}{2} \tilde{f}(\mathbf{x}) = \phi(\mathbf{x}). \quad \square \quad (4.2.130)$$

Note that for the exterior Neumann boundary value problem, the compatibility condition

$$\int_{\Gamma} \phi dS_{\Gamma} = 0, \quad (4.2.131)$$

does not hold in general, because

$$0 = \int_{\Omega^c \cap \{|\mathbf{x}| \leq D\}} \Delta u d\mathbf{x} = \int_{|\mathbf{x}|=D} \frac{\partial u}{\partial r} dS - \int_{\Gamma} \phi dS, \quad (4.2.132)$$

thus  $\int_{\Gamma} \phi dS$  is balanced by the flux at  $\infty$  when  $D \uparrow \infty$ . If it is known *a priori* that  $\frac{\partial u}{\partial r} = O(|r|^{-2})$  then  $\phi$  must satisfy (4.2.131).

**Theorem 4.2.14** *There is a unique solution  $u$  to the exterior Neumann problem in  $R^2$ ,*

$$\Delta \tilde{u} = 0 \quad \text{in } \Omega^c, \quad (4.2.133)$$

$$\frac{\partial \tilde{u}}{\partial n} = \phi \quad \text{on } \Gamma, \quad (4.2.134)$$

and

$$\tilde{u} - \left( \frac{1}{2\pi} \int_{\Gamma} \phi(\mathbf{y}) dS_{\mathbf{y}} \right) \ln |\mathbf{x}| = O(|\mathbf{x}|^{-1}), \quad \text{large } |\mathbf{x}|, \quad (4.2.135)$$

given by

$$\tilde{u}(\mathbf{x}) = \int_{\Gamma} E(\mathbf{x} - \mathbf{y}) \tilde{f}(\mathbf{y}) dS_{\mathbf{y}} \in H_{loc}^{r+3/2}(\Omega^c) \quad (4.2.136)$$

where  $\tilde{f}$  is the unique solution to

$$\int_{\Gamma} \tilde{f}(\mathbf{y}) dS_{\mathbf{y}} = - \int_{\Gamma} \phi(\mathbf{y}) dS_{\mathbf{y}}, \quad (4.2.137)$$

$$-\frac{1}{2}\tilde{f}(\mathbf{x}) + \int_{\Gamma} \frac{\partial E(\mathbf{x} - \mathbf{y})}{\partial n_{\mathbf{x}}} \tilde{f}(\mathbf{y}) dS_{\mathbf{y}} = \phi(\mathbf{x}). \quad \square \quad (4.2.138)$$

### 4.3 Stokes Problem

Let us consider the Dirichlet problem of steady-state Stokes equations:

$$-\nu \Delta \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \quad \text{or} \quad \Omega^c, \quad (4.3.1)$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \quad \text{or} \quad \Omega^c, \quad (4.3.2)$$

$$\mathbf{u} = \mathbf{b} \quad \text{on } \Gamma_s, \quad (4.3.3)$$

$$\mathbf{u} = \mathbf{b} + F((\mathbf{u}, p)|_{\Gamma_s}) \mathbf{h} \quad \text{on } \Gamma_c, \quad (4.3.4)$$

where  $\Omega$  is a simply connected domain in  $R^N$ ,  $N = 2$  or  $3$ , with a piecewise-smooth boundary  $\partial\Omega = \Gamma$ ;  $\Gamma_s$  and  $\Gamma_c$  are portions of  $\Gamma$  such that  $\bar{\Gamma} = \bar{\Gamma}_s \cup \bar{\Gamma}_c$  and  $\Gamma_s \cap \Gamma_c = \emptyset$ ,  $\Omega^c = R^N - \bar{\Omega}$ . Throughout, we assume that  $N$  is the space dimension, *i.e.*,  $N = 2$  or  $3$ . In (4.3.1)-(4.3.4),  $\nu$  denotes the (constant) kinematic viscosity,  $\mathbf{f}$  a given body force and  $\mathbf{b}$  and  $\mathbf{h}$  given velocity fields defined on boundary such that  $\mathbf{b} \in \mathbf{H}^{1/2}(\Gamma)$  and  $\mathbf{h} \in \mathbf{H}^{1/2}(\Gamma)$ .  $\mathbf{h}$  has compact support in  $\Gamma_c$ . In (4.3.4),  $F$  is a bounded linear functional on  $\mathbf{H}^{1/2}(\Gamma_s)$  which is not a simple point-wise function of  $\mathbf{u}$  and/or  $p$ . It may involve complex or non-local operations such as differentiation or integration of  $\mathbf{u}$  and  $p$ .

Using potential theory, particularly the boundary integral equation method, we shall prove the existence and uniqueness of solutions to the problem (4.3.1)-(4.3.4). Most theories of hydrodynamical potentials are similar to those of ordinary potentials.

First of all, we determine the fundamental singular solution of the linearized Navier-Stokes system, or, more exactly, the tensor made up of the solutions corresponding to forces directed along the various coordinate axis. Thus, we consider the problem

$$\begin{aligned} -\nu\Delta\mathbf{u}^k(\mathbf{x},\mathbf{y}) + \nabla p^k(\mathbf{x},\mathbf{y}) &= \delta(\mathbf{x}-\mathbf{y})\mathbf{e}^k, \\ \nabla \cdot \mathbf{u}^k &= 0, \end{aligned} \quad (4.3.5)$$

where  $k = 1, \dots, N$ ,  $\mathbf{e}^k$  is a unit vector directed along the  $k$ -th coordinate axis, and  $\delta(\mathbf{x}-\mathbf{y})$  is the Dirac delta function. All differentiations are carried out with respect to the variable  $\mathbf{x}$ , and the point  $\mathbf{y}$  plays the role of a parameter. The system is supplemented by the requirement that  $\mathbf{u}^k$  and  $p^k$  approach zero as  $|\mathbf{x}| \rightarrow \infty$ .

The fundamental singular solution of the problem (4.3.5), is given by Ladyzhenskaya [76]:

$$u_j^k(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi\nu} \left[ \delta_{kj} \ln \frac{1}{|\mathbf{x}-\mathbf{y}|} + \frac{(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^2} \right], \quad (4.3.6)$$

$$p^k(\mathbf{x}) = -\frac{1}{2\pi} \frac{\partial}{\partial x_k} \ln \frac{1}{|\mathbf{x}-\mathbf{y}|}, \quad (4.3.7)$$

for  $N = 2$  and

$$u_j^k(\mathbf{x},\mathbf{y}) = \frac{1}{8\pi\nu} \left[ \frac{\delta_{kj}}{|\mathbf{x}-\mathbf{y}|} + \frac{(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^3} \right], \quad (4.3.8)$$

$$p^k(\mathbf{x}) = -\frac{x_k-y_k}{4\pi|\mathbf{x}-\mathbf{y}|^3}, \quad (4.3.9)$$

for  $N = 3$ .

From the fundamental singular solutions (4.3.6)-(4.3.7) for  $N = 2$ , (4.3.8)-(4.3.9) for  $N = 3$  and the equation (4.3.5), in the argument  $\mathbf{y}$ , the functions  $\mathbf{u}^k(\mathbf{x},\mathbf{y})$  and  $p^k(\mathbf{x},\mathbf{y})$

satisfy the adjoint system

$$-\nu\Delta_{\mathbf{y}}\mathbf{u}^k - \nabla_{\mathbf{y}}p^k = \delta(\mathbf{x} - \mathbf{y})\mathbf{e}^k, \quad (4.3.10)$$

$$\nabla_{\mathbf{y}} \cdot \mathbf{u}^k = 0. \quad (4.3.11)$$

The solutions  $\mathbf{u}^k = (u_1^k, u_2^k)$  and  $p^k$  allow us to construct the volume potentials

$$\mathbf{U}(\mathbf{x}) = \int_{\Omega} \mathbf{u}^k(\mathbf{x}, \mathbf{y}) f_k(\mathbf{y}) d\mathbf{y}, \quad (4.3.12)$$

$$P(\mathbf{x}) = \int_{\Omega} p^k(\mathbf{x}, \mathbf{y}) f_k(\mathbf{y}) d\mathbf{y}, \quad (4.3.13)$$

which satisfy the nonhomogeneous Navier-Stokes system

$$-\nu\Delta\mathbf{U} + \nabla P = \mathbf{f}(\mathbf{x}), \quad (4.3.14)$$

$$\nabla \cdot \mathbf{U} = 0. \quad (4.3.15)$$

By using the volume potentials (4.3.12)-(4.3.13), we can reduce the linearized Navier-Stokes system (4.3.1)-(4.3.4) to a homogeneous one. Thus, we only consider problems (4.3.1)-(4.3.4) with  $\mathbf{f} \equiv \mathbf{0}$ ,

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{0} \quad \text{in } \Omega \quad \text{or} \quad \Omega^c, \quad (4.3.16)$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \quad \text{or} \quad \Omega^c, \quad (4.3.17)$$

$$\mathbf{u} = \mathbf{b} \quad \text{on } \Gamma_s, \quad (4.3.18)$$

$$\mathbf{u} = \mathbf{b} + F((\mathbf{u}, p)|_{\Gamma_s}) \mathbf{h} \quad \text{on } \Gamma_c, \quad (4.3.19)$$

In order to derive an equivalent variational formulation for problem (4.3.1)-(4.3.4), we need an integral representation for the solution for problem (4.3.1)-(4.3.4). Before giving a formal definition of the single layer potentials, we write the Green's formulas corresponding to the Navier-Stokes system. These formulas are obtained by integrating by parts, and are

valid for any smooth solenoidal vectors  $\mathbf{u}$ ,  $\mathbf{v}$  and  $p$ ,  $q$ . They are most simply verified by using the identity

$$\frac{\partial}{\partial x_k} [T_{ik}(\mathbf{u})v_i] = \frac{\nu}{2} \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) + \left( \nu \Delta u_i - \frac{\partial p}{\partial x_i} \right) v_i, \quad (4.3.20)$$

in which

$$T_{ik}(\mathbf{u}) = -\delta_i^k p + \nu \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) \quad (4.3.21)$$

is the *stress tensor* corresponding to the flow  $\mathbf{u}$ ,  $p$ . Integrating (4.3.20) over  $\Omega$ , we obtain

$$\int_{\Omega} \left( \nu \Delta u_i - \frac{\partial p}{\partial x_i} \right) v_i \, d\mathbf{x} = - \int_{\Omega} \frac{\nu}{2} \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) \, d\mathbf{x} + \int_{\Gamma} T_{ik}(\mathbf{u}) v_i n_k \, dS, \quad (4.3.22)$$

i.e.,

$$2\nu \sum_{i,j=1}^N \int_{\Omega} e_{ij}(\mathbf{u}) e_{ij}(\mathbf{v}) \, d\mathbf{x} + \int_{\Omega} \mathbf{v} \cdot (\nu \Delta \mathbf{u} - \nabla p) \, d\mathbf{x} = \int_{\Gamma} \mathbf{n} \cdot (\mathcal{T}(\mathbf{u})) \mathbf{v} \, dS_{\mathbf{x}}, \quad (4.3.23)$$

where  $\mathbf{n} = (n_1, \dots, n_N)$ , is the exterior (with respect to  $\Omega$ ) normal to  $\Gamma$ ,  $\mathcal{T}(\mathbf{u}) = [T_{ij}(\mathbf{u})]_{i,j=1}^N$  is the stress tensor, and  $e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$  is the deformation tensor. Interchanging  $\mathbf{u}$  and  $\mathbf{v}$ , and introducing together with  $p$  an arbitrary smooth function  $q$ , we obtain from (4.3.23) the formula

$$\int_{\Omega} \left[ (\nu \Delta \mathbf{v} - \nabla q) \mathbf{u} - \mathbf{v} (\nu \Delta \mathbf{u} + \nabla p) \right] \, d\mathbf{x} = \int_{\Gamma} \left[ \mathbf{n} \cdot (\mathcal{T}(\mathbf{v}) \mathbf{u}) - \mathbf{n} \cdot (\mathcal{T}(\mathbf{u}) \mathbf{v}) \right] \, dS_{\mathbf{x}}. \quad (4.3.24)$$

From the analytic point of view, two-dimensional problems are more difficult to handle than three-dimensional owing to the behavior of the solution at infinity. We mainly study the boundary value problems (4.3.16)-(4.3.19) for the case of two-dimensional space. Similarly, we can analyze the three-dimensional case.

Let us consider the following interior and exterior Stokes equations:

$$-\nu \Delta \mathbf{u} + \nabla p = \mathbf{0} \quad \text{in } \Omega \quad \text{or} \quad \Omega^c, \quad (4.3.25)$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \quad \text{or} \quad \Omega^c, \quad (4.3.26)$$

$$\mathbf{u} = \mathbf{b} \quad \text{on } \Gamma. \quad (4.3.27)$$

where  $\Omega$  is a simply connected domain in  $R^2$  with a piecewise-smooth boundary  $\Gamma = \partial\Omega$ , and  $\Omega^c = R^2/\bar{\Omega}$ .  $\mathbf{u} = (u_1, u_2)$  is the velocity,  $p$  is its pressure and  $\nu > 0$  is the kinematic viscosity.

We will use the following Sobolev spaces defined in Ladyzhenskaya [76], Lions and Magenes [84], Nedelec [88]:

$$H^1(\Omega) = \{u(\mathbf{x}) \in L^2(\Omega) : \nabla u \in (L^2(\Omega))^2\}, \quad (4.3.28)$$

$$W_0^1(\Omega^c) = \left\{u(\mathbf{x}) : \frac{u}{\sqrt{1+|\mathbf{x}|^2} \ln(2+|\mathbf{x}|^2)} \in L^2(\Omega^c), \nabla u \in (L^2(\Omega^c))^2\right\}, \quad (4.3.29)$$

$$\mathbf{W}(R^2) = \{\mathbf{u}(\mathbf{x}) \in (W_0^1(R^2))^2 : \nabla \cdot \mathbf{u} = 0, \quad \forall \mathbf{x} \in \Omega \cup \Omega^c\}, \quad (4.3.30)$$

$$H^s(\Gamma) = \text{the standard Sobolev space defined on } S \text{ with index } r, \quad (4.3.31)$$

$$U_2(\Gamma) = \{\mathbf{u} \in (H^s(\Gamma))^2 : \int_{\Gamma} \mathbf{u} \cdot \mathbf{n} \, dS = 0\} \quad (s \geq 1/2), \quad (4.3.32)$$

$$T_2(\Gamma) = \{\mathbf{t} \in (H^{s-1}(\Gamma))^2 \cap (H^{-1/2}(\Gamma))^2 : \int_{\Gamma} \mathbf{t} \cdot \mathbf{c} \, dS = 0\} \quad (s \geq 1/2), \quad (4.3.33)$$

and  $E$  represents an equivalent relation:

$$\mathbf{t} \sim \mathbf{t}', \quad \text{if and only if} \quad \mathbf{t}(\mathbf{x}) - \mathbf{t}'(\mathbf{x}) = c \mathbf{n}(\mathbf{x}), \quad \text{for } c \in R. \quad (4.3.34)$$

Using equation (4.3.24) and the fundamental singular solution (4.3.6)-(4.3.7), we have

**Lemma 4.3.1** *Assume  $\mathbf{b} \in U_2(\Gamma)$ . Then, the problem (4.3.25)-(4.3.27) has a unique solution which can be expressed as*

$$\mathbf{u}(\mathbf{x}) = \int_{\Gamma} \mathbf{u}^k(\mathbf{x} - \mathbf{y}) t_k(\mathbf{y}) \, dS_{\mathbf{y}} + \mathbf{c}, \quad \mathbf{x} \in R^2, \quad (4.3.35)$$

$$p(\mathbf{x}) = \int_{\Gamma} p^k(\mathbf{x} - \mathbf{y}) t_k(\mathbf{y}) \, dS_{\mathbf{y}}, \quad \mathbf{x} \in R^2/\Gamma, \quad (4.3.36)$$



where the density  $\mathbf{t}(\mathbf{y}) = (\mathcal{T}(\mathbf{u})\mathbf{n})|_{interior} - (\mathcal{T}(\mathbf{u})\mathbf{n})|_{exterior}$  represents the jump of  $\mathcal{T}(\mathbf{u})\mathbf{n}$ , the normal stress, across  $\Gamma$ .  $\mathbf{c} \in R^2$  is a constant vector.  $\square$

In equation (4.3.23), we write out the corresponding equations for the interior domain  $\Omega$  and the exterior domain  $\Omega^c$ , and then sum up these two formulae together. We have

$$2\nu \sum_{i,j=1}^N \int_{R^2} e_{ij}(\mathbf{u})e_{ij}(\mathbf{v}) d\mathbf{y} = \int_{\Gamma} \mathbf{t}(\mathbf{y}) \cdot \mathbf{v} dS_{\mathbf{y}}, \quad \forall \mathbf{v} \in \mathbf{W}(R^2). \quad (4.3.37)$$

Since  $\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{v} = 0$ , we have

$$\nu \int_{R^2} \nabla \mathbf{u} \cdot \nabla \mathbf{v} d\mathbf{y} = \int_{\Gamma} \mathbf{t}(\mathbf{y}) \cdot \mathbf{v} dS_{\mathbf{y}}, \quad \forall \mathbf{v} \in \mathbf{W}(R^2). \quad (4.3.38)$$

Choosing  $\mathbf{v} = \mathbf{c} \in R^2$  in equation (4.3.38) yields

$$\int_{\Gamma} \mathbf{t}(\mathbf{y}) \cdot \mathbf{c} dS_{\mathbf{y}} = 0, \quad \forall \mathbf{c} \in R^2. \quad (4.3.39)$$

Thus, if  $(\mathbf{u}, p)$  is the solution of problem (4.3.25)-(4.3.27), then  $\mathbf{t}(\mathbf{y})$  must satisfy the constraint (4.3.39). Since the pressure  $p$  can only be uniquely determined up to a constant in  $\Omega$ , from the expressions of  $\mathcal{T}(\mathbf{u}, p)$  and  $\mathbf{t}(\mathbf{x})$ ,  $\mathbf{t}(\mathbf{x})$  can be determined up to a vector  $c \mathbf{n}$ . From these, we have

**Lemma 4.3.2** *The integral equations of the first kind*

$$b_j(\mathbf{x}) = \sum_{i=1}^2 \int_{\Gamma} u_j^i(\mathbf{x} - \mathbf{y}) t_i(\mathbf{y}) dS_{\mathbf{y}} + c_j, \quad \forall \mathbf{x} \in \Gamma, \quad j = 1, 2 \quad (4.3.40)$$

define a continuous bijective mapping from  $U_2(\Gamma)$  to  $T_2(\Gamma)/E$ .  $\square$

From the equation (4.3.40), we can state the equivalent variational formulation as follows: find  $\mathbf{t} \in T_2(\Gamma)/E$  such that

$$A(\mathbf{t}, \mathbf{s}) = \langle \mathbf{b}, \mathbf{s} \rangle, \quad \forall \mathbf{s} \in T_2(\Gamma)/E, \quad (4.3.41)$$

where

$$A(\mathbf{t}, \mathbf{s}) = \sum_{j,k=1}^2 \int_{\Gamma} \int_{\Gamma} u_j^k(\mathbf{x} - \mathbf{y}) t_j(\mathbf{y}) s_k(\mathbf{x}) dS_{\mathbf{y}} dS_{\mathbf{x}}, \quad (4.3.42)$$

$$\langle \mathbf{b}, \mathbf{s} \rangle = \sum_{k=1}^2 \int_{\Gamma} b_k(\mathbf{x}) s_k(\mathbf{x}) dS_{\mathbf{x}}. \quad (4.3.43)$$

**Theorem 4.3.3**  $A(\cdot, \cdot)$  defined in the equation (4.3.42) is a symmetric, bounded and coercive bilinear form on  $T_2(\Gamma)/E \times T_2(\Gamma)/E$ . That is, there exist positive constants  $C_1, C_2$ , such that for all  $\mathbf{t}, \mathbf{s} \in T_2(\Gamma)/E$ ,

$$A(\mathbf{t}, \mathbf{s}) = A(\mathbf{s}, \mathbf{t}), \quad (4.3.44)$$

$$A(\mathbf{t}, \mathbf{t}) \geq C_1 \|\mathbf{t}\|_{T_2(\Gamma)/E}^2, \quad (4.3.45)$$

$$|A(\mathbf{t}, \mathbf{s})| \leq C_2 \|\mathbf{t}\|_{T_2(\Gamma)/E} \|\mathbf{s}\|_{T_2(\Gamma)/E}. \quad (4.3.46)$$

Given any  $\mathbf{b} \in U_2(\Gamma)$ , the variational problem (4.3.41) has a unique solution  $\tilde{\mathbf{t}} \in T_2(\Gamma)/E$  such that

$$\|\tilde{\mathbf{t}}\|_{T_2(\Gamma)/E} \leq C \|\mathbf{b}\|_{U_2(\Gamma)}, \quad (4.3.47)$$

where  $C$  depends only on  $\Gamma$ .

*Proof.* For the proof, see [114].  $\square$

Now, let us consider an equivalent variational formulation for problem (4.3.16)-(4.3.19): find  $\mathbf{t} \in T_2(\Gamma)/E$  such that

$$A(\mathbf{t}, \mathbf{s}) = \langle \mathbf{b}, \mathbf{s} \rangle + \langle \alpha \mathbf{h}, \mathbf{s} \rangle_{\Gamma_c}, \quad \forall \mathbf{s} \in T_2(\Gamma)/E, \quad (4.3.48)$$

where  $\alpha = F((\mathbf{u}, p)|_{\Gamma_s})$ . Note that the above variational problem is not elliptic since  $\alpha$  is a function of  $\mathbf{t}$  through  $\mathbf{u}$ .  $F((\mathbf{u}, p)|_{\Gamma_s})$  is a bounded linear functional of velocity  $\mathbf{u}$

and pressure  $p$  and not a simple point-wise function. It may involve complex or non-local operations such as differentiation or integration of  $\mathbf{u}$  and  $p$ . First, we consider the case of  $F$  is a functional of only  $p$ . Using the substitution method, we have the following result.

**Theorem 4.3.4** *Let  $\hat{\mathbf{t}}$  and  $\tilde{\mathbf{t}}$  be the solutions of the following equations*

$$A(\hat{\mathbf{t}}, \mathbf{s}) = \langle \mathbf{h}, \mathbf{s} \rangle, \quad \forall \mathbf{s} \in T_2(\Gamma)/E \quad (4.3.49)$$

and

$$A(\tilde{\mathbf{t}}, \mathbf{s}) = \langle \mathbf{b}, \mathbf{s} \rangle, \quad \forall \mathbf{s} \in T_2(\Gamma)/E, \quad (4.3.50)$$

respectively. If  $F$  is a bounded linear functional of  $p$  and

$$F\left(\int_{\Gamma} p^k(\mathbf{x} - \mathbf{y}) \hat{t}_k(\mathbf{y}) dS_{\mathbf{y}}|_{\Gamma_s}\right) \neq 1 \quad (4.3.51)$$

for interior and exterior problems, then the problem (4.3.48) has a unique solution  $\mathbf{t} \in T_2(\Gamma)/E$  such that

$$\mathbf{t} = \tilde{\mathbf{t}} + \hat{\mathbf{t}} \frac{F(\int_{\Gamma} p^k(\mathbf{x} - \mathbf{y}) \tilde{t}_k(\mathbf{y}) dS_{\mathbf{y}}|_{\Gamma_s})}{1 - F(\int_{\Gamma} p^k(\mathbf{x} - \mathbf{y}) \hat{t}_k(\mathbf{y}) dS_{\mathbf{y}}|_{\Gamma_s})}. \quad (4.3.52)$$

*Proof.* From Theorem 4.3.3, given  $\forall \alpha \in R$ , the variational problem,

$$A(\mathbf{t}, \mathbf{s}) = \langle \mathbf{b} + \alpha \mathbf{h}, \mathbf{s} \rangle, \quad \forall \mathbf{s} \in T_2(\Gamma)/E, \quad (4.3.53)$$

has a unique solution

$$\mathbf{t} = \tilde{\mathbf{t}} + \alpha \hat{\mathbf{t}}. \quad (4.3.54)$$

But

$$\begin{aligned} \alpha &= F(p|_{\Gamma_s}) \\ &= F\left(\int_{\Gamma} p^k(\mathbf{x} - \mathbf{y}) t_k(\mathbf{y}) dS_{\mathbf{y}}|_{\Gamma_s}\right) \\ &= F\left(\int_{\Gamma} p^k(\mathbf{x} - \mathbf{y}) \tilde{t}_k(\mathbf{y}) dS_{\mathbf{y}}|_{\Gamma_s}\right) + \alpha F\left(\int_{\Gamma} p^k(\mathbf{x} - \mathbf{y}) \hat{t}_k(\mathbf{y}) dS_{\mathbf{y}}|_{\Gamma_s}\right). \end{aligned}$$

Thus, providing (4.3.51) holds,  $\alpha$  is well defined and

$$\alpha = \frac{F(\int_{\Gamma} p^k(\mathbf{x} - \mathbf{y}) \tilde{t}_k(\mathbf{y}) dS_{\mathbf{y}}|_{\Gamma_s})}{1 - F(\int_{\Gamma} p^k(\mathbf{x} - \mathbf{y}) \hat{t}_k(\mathbf{y}) dS_{\mathbf{y}}|_{\Gamma_s})}, \quad (4.3.55)$$

Substituting (4.3.55) to (4.3.54), we have a solution

$$\mathbf{t} = \tilde{\mathbf{t}} + \hat{\mathbf{t}} \frac{F(\int_{\Gamma} p^k(\mathbf{x} - \mathbf{y}) \tilde{t}_k(\mathbf{y}) dS_{\mathbf{y}}|_{\Gamma_s})}{1 - F(\int_{\Gamma} p^k(\mathbf{x} - \mathbf{y}) \hat{t}_k(\mathbf{y}) dS_{\mathbf{y}}|_{\Gamma_s})}. \quad (4.3.56)$$

Now, let us show the uniqueness of the solution. Let  $\mathbf{t}_a$  and  $\mathbf{t}_b$  be two different solutions of (4.3.48), then, by the linearity of our problem,  $\mathbf{t}_a - \mathbf{t}_b$  satisfies

$$A(\mathbf{t}_a - \mathbf{t}_b, \mathbf{s}) = \langle (\alpha_a - \alpha_b) \mathbf{h}, \mathbf{s} \rangle, \quad \forall \mathbf{s} \in T_2(\Gamma)/E, \quad (4.3.57)$$

where

$$\alpha_a = F\left(\int_{\Gamma} p^k(\mathbf{x} - \mathbf{y}) (t_a)_k(\mathbf{y}) dS_{\mathbf{y}}|_{\Gamma_s}\right) \quad (4.3.58)$$

and

$$\alpha_b = F\left(\int_{\Gamma} p^k(\mathbf{x} - \mathbf{y}) (t_b)_k(\mathbf{y}) dS_{\mathbf{y}}|_{\Gamma_s}\right). \quad (4.3.59)$$

But

$$\alpha_a - \alpha_b = (\alpha_a - \alpha_b) F\left(\int_{\Gamma} p^k(\mathbf{x} - \mathbf{y}) \hat{t}_k(\mathbf{y}) dS_{\mathbf{y}}|_{\Gamma_s}\right). \quad (4.3.60)$$

From (4.3.51), we have that  $\alpha_a = \alpha_b$  and thus  $\mathbf{t}_a = \mathbf{t}_b$  which contradicts the assumption  $\mathbf{t}_a \neq \mathbf{t}_b$ . Thus, the proof is completed.  $\square$

In the same way, we have the following result for the case that  $F$  is a functional of the normal stress  $\mathcal{T}(\mathbf{u})\mathbf{n}$ .

**Theorem 4.3.5** *Let  $\tilde{\mathbf{t}}$  and  $\hat{\mathbf{t}}$  be the solutions of the following equations*

$$A(\hat{\mathbf{t}}, \mathbf{s}) = \langle \mathbf{h}, \mathbf{s} \rangle, \quad \forall \mathbf{s} \in T_2(\Gamma)/E, \quad (4.3.61)$$

and

$$A(\tilde{\mathbf{t}}, \mathbf{s}) = \langle \mathbf{h}, \mathbf{s} \rangle, \quad \forall \mathbf{s} \in T_2(\Gamma)/E, \quad (4.3.62)$$

respectively. If  $F$  is a bounded linear functional of the normal stress  $\mathcal{T}(\mathbf{u})\mathbf{n}$  and

$$F\left(\left[\int_{\Gamma} T_{ij}(\mathbf{u}^k(\mathbf{x}, \mathbf{y}))_{\mathbf{x}} n_k(\mathbf{x}) t_j(\mathbf{y}) dS_{\mathbf{y}} \pm \frac{1}{2} t_i(\mathbf{x})\right]\Big|_{\Gamma_s}\right) \neq 1, \quad (4.3.63)$$

for interior and exterior problems, then the variational problem,

$$A(t_i, s_i) = \langle b_i, s_i \rangle + \langle \alpha_i h_i, s_i \rangle_{\Gamma_c}, \quad i=1,2, \quad \forall \mathbf{s} = (s_1, s_2) \in T_2(\Gamma)/E, \quad (4.3.64)$$

where

$$\alpha_i = F\left(\left[\int_{\Gamma} T_{ij}(\mathbf{u}^k(\mathbf{x}, \mathbf{y}))_{\mathbf{x}} n_k(\mathbf{x}) t_j(\mathbf{y}) dS_{\mathbf{y}} \pm \frac{1}{2} t_i(\mathbf{x})\right]\Big|_{\Gamma_s}\right) \quad (4.3.65)$$

has a unique solution  $\mathbf{t} \in T_2(\Gamma)/E$  and

$$t_i = \tilde{t}_i + \hat{t}_i \frac{F\left(\left[\int_{\Gamma} T_{ij}(\mathbf{u}^k(\mathbf{x}, \mathbf{y}))_{\mathbf{x}} n_k(\mathbf{x}) \tilde{t}_j(\mathbf{y}) dS_{\mathbf{y}} \pm \frac{1}{2} \tilde{t}_i(\mathbf{x})\right]\Big|_{\Gamma_s}\right)}{1 - F\left(\left[\int_{\Gamma} T_{ij}(\mathbf{u}^k(\mathbf{x}, \mathbf{y}))_{\mathbf{x}} n_k(\mathbf{x}) \hat{t}_j(\mathbf{y}) dS_{\mathbf{y}} \pm \frac{1}{2} \hat{t}_i(\mathbf{x})\right]\Big|_{\Gamma_s}\right)}. \quad (4.3.66)$$

The subscript  $\mathbf{x}$  on  $T_{ij}(\mathbf{u}^k(\mathbf{x}, \mathbf{y}))$  shows that the differentiation in  $T_{ij}$  is carried out with respect to  $\mathbf{x}$ .  $\square$

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