ECONOMICALLY OPTIMUM DESIGN OF CUSUM CHARTS WHEN THERE IS A MULTIPLICITY OF ASSIGNABLE CAUSES

by

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CHAPTER I

INTRODUCTION

A control chart is a statistical procedure used for the study and control of repetitive processes. It is desirable to quickly detect any change in the distribution of the observations taken from the process. The observations considered in this paper are the means for samples taken at regular time intervals. The objective of this thesis is to provide a procedure for the design of cumulative sum control charts based on a minimum cost criterion when several assignable causes are to be considered.

Consider the problem of detecting shifts in the process mean caused by the occurrence of assignable causes. A sample of size n is taken from the process at regular time intervals s and a quality measurement is made for each item in the sample. In a Shewhart control chart the mean of these measurements is then computed and plotted against the sample number. A signal that the mean of the process has shifted from a specified control value is given if a sample mean falls more than a specified distance from the control value. When a signal is given an investigation is undertaken to find the assignable cause of the shift. The positions for control limits that are most frequently adopted are at $\mu_o \pm 3\sigma/\sqrt{n}$, where μ_o is the control value, σ is the standard deviation and n is the sample size. Various modifications have been proposed such as using warning lines at $\mu_o \pm 2\sigma/\sqrt{n}$, so that if a given number of consecutive observations fall outside these lines then an investigation would be required.

The Shewhart chart is not very effective in detecting small deviations from the control value. The sensitivity of the control chart to small deviations can be improved by taking smaller control limits. This will, however, result in more frequent false alarms when the process is in control. Using warning lines may also improve its sensitivity. However, these procedures use the information in only the last few observations. A more recent and efficient procedure is the cumulative sum (cusum) control chart suggested by Page (1954). It uses the cumulative sum of deviations from the control value rather than the individual observations. A control chart based on a cusum procedure reacts more promptly to small deviations and the change can be seen by visual inspection much easier than with the Shewhart control chart procedure.

The operation of a one-sided cusum chart for controlling the mean of a process consists of taking samples of size n at regular time intervals s and plotting the cumulative sum $S_m = \sum_{j=1}^{m} (\bar{X}_j - k)$ versus sample number m, where $\bar{X}_j$ is the sample mean for the j^{th} sample and k is a constant usually called the reference value. Let μ_o be the control value. The reference value k is chosen such that when the process mean μ_o is less than k ($k \geq \mu_o$), the mean path of the chart is in a downward direction but if the process mean level changes from μ_o to μ_1 ($\mu_1 \geq k$) the mean path of the chart will change to an upward direction. A change in the direction of S_m is taken as an indication of a change in the level of the process mean which is being controlled. However, changes in direction of the cusum chart will occur due to random variations of the sample mean. It is necessary, therefore, to formulate a procedure which will help to distinguish these changes from changes due to assignable causes. The procedure for deviations in the positive direction signals at the first m for which

$$S_m - \min_{0 \leq r < m} S_r \geq h \qquad (1.1)$$

where $h \geq 0$ is a constant, usually called the decision limit. For deviations in the negative direction the cusum chart signals at the first m for which

$$\max_{0 \leq r < m} \sum_{i=1}^{r} (\bar{X}_i + k') - \sum_{i=1}^{m} (\bar{X}_i + k') \geq h \qquad (1.2)$$

A two-sided procedure for detecting both positive and negative deviations would signal at the first m for which one of the inequalities in (1.1) and (1.2) is true. The sample size n, sampling interval s, reference value k, and decision limit h are the parameters required for operating a cusum chart.

The average run length (ARL) at a given quality level μ, $L(\mu)$, is defined to be the expected number of samples observed before an out-of-control signal occurs. If the process remains in control, the ARL should be large so that the frequency of false alarms is low and if the mean shifts from the control value, the ARL should be small so that the out-of-control condition is quickly detected. The ARL is the criterion most commonly used in evaluating the efficiency of control charts for detecting lack of control. Designing the cusum chart to minimize $L(\mu_1)$, the ARL when the process mean shifts to μ_1, or maximize $L(\mu_o)$, the ARL when the process mean remains in control, $\mu = \mu_o$, does not consider the cost aspect of the process. In practice, a complete solution in determining n, s, h, k would require knowledge of the costs involved in not detecting shifts when they occur, the cost of false alarms, the costs connected with the amount and frequency of sampling, the costs associated with the magnitude of shifts from the control value, etc.. Therefore, it is desirable to use a criterion that reflects the cost and risk associated with the process.

Some work has been done in this area, but only for the Shewhart control chart or for the cusum chart for the simple case where there is only one assignable cause considered. This dissertation is concerned with the design of cusum charts based on a minimum cost criterion when several assignable causes are considered. The various cost and risk factors associated with the process are assumed to be known.

The present study is confined to the design of cusum charts for the control of the mean of a process. The means for samples taken from the process are assumed to be independent and normally distributed with known variance σ^2.

Chapter II is a review of the cusum chart procedure and the literature on the economic design of both Shewhart and cusum control charts.

Chapter III presents a derivation of the cost-loss model which relates the design parameters of the cusum chart and the cost and risk factors of the process.

Chapter IV deals with the calculation of the ARL in the model. The adjustment factor for h in a Brownian motion approximation for the ARL is found by a regression procedure in order to improve the accuracy of this approximation. The values of ARL obtained by this modified Brownian motion approximation are compared with those tabulated in the literature.

Chapter V is concerned with the numerical study of the loss-cost model. Optimum design parameters are obtained and the effects of changes in some cost factors and process parameters on the model are studied for several sets of cost parameters.

CHAPTER II

LITERATURE REVIEW

Cusum charts were first introduced by Page (1954). For detecting positive deviations, he suggested assigning a score d_j to the j^{th} sample and plotting cumulative sums $\sum_{j=1}^{m} d_j$ against m, where d_j is chosen so that the mean path of the chart is downwards when the quality is satisfactory and upwards when it is unsatisfactory. Action is taken after the m^{th} sample if $\sum_{j=1}^{m} d_j - \min_{0 \leq r < m} \sum_{j=1}^{r} d_j \geq h$ for some specified value h.

Ewan and Kemp (1960) mentioned another method of plotting which consists of substracting the reference value k from each sample value $\bar{X}_j$ and plotting the cumulative sums $S_m = \sum_{j=1}^{m} (\bar{X}_j - k)$ versus m. The slope of the path is downwards if the process mean is less than k and upwards if it is greater than k. Schemes for which the cumulative chart slopes downwards for quality which is acceptable and upwards when quality is not acceptable are similar to those described by Page (1954). Action is taken when $S_m - \min_{0 \leq r < m} S_r \geq h$. If the cusum chart is used to control both positive and negative deviations in the process mean, then two one-sided charts are used. One is to detect positive shifts and the other is to detect negative shifts.

Following essentially the same procedure, Barnard (1959) takes $S'_m = \sum_{j=1}^{m} (\bar{X}_j - \mu_o)$ for the cumulative scores where μ_o is the control level. Changes in the process mean are then reflected by changes in the average slope of the cusum graph. He also introduced a graphical procedure using a V-shaped mask. The procedure is to plot S'_m first and then place on the chart a V-shaped mask with an angle θ, its vertex a distance d from the current point and its axis horizontal. If at any time the cumulative sum path cuts either edge of the V-mask an out-of-control signal is given. Kemp (1961) showed that the use of a V-mask is equivalent to the two-sided cusum procedure.

Determining the ARL for a cusum chart is not a simple matter. Page (1954) derived the intergral equation for ARL, L(z), of a cusum test, starting at z,

$$L(z) = 1 + L(0)F(-z) + \int_0^h f(x-z)L(x)dx \qquad 0 \leq z \leq h$$

where F is the distribution function of the values $x=\bar{x}-\mu_o-k$ to be accumulated in the cusum test, f is the probability density function for x and h is the decision limit. This integral equation can not be solved analytically in general. Van Dobben (1968) tabulated values of ARL for the cusum charts controlling the mean of a normal distribution as a function of μ-k and h by using a numerical approximation method to solve the integral equations. They are correct to one unit in the second decimal place. Ewan and Kemp (1960) published a nomogram for finding approximate values of ARL covering the range h = 2.0(1.0)5.0. For each of these values of h, the ARL is given for about 15 values of μ-k. These values are also for the cusum chart where the variable plotted is normally distributed.

Goldsmith and Whitfield (1961) used Monte Carlo methods to obtain a rather more extensive set of graphs of the ARL as a function of μ for 16 different combinations of k and h. The results are also expressed in terms of two empirical formulae. One is for $L(\mu_o)$ and provides a good approximation in general. The other is for $L(\mu_1)$ and is adequate when the magnitude of the mean shift is between 1.5 and 4. They also studied the case where the successive observations were correlated and showed by simulation that the ARL is not too sensitive to the degree of correlation for a fair range of the correlation coefficient.

Goel and Wu (1971) and Chiu (1974) replaced the integral equation by a system of linear algebraic equations which can be solved to obtain an approximation for the ARL. Goel and Wu (1971) showed that the results are reliable to at least the fourth decimal place.

Reynolds (1975) derived an analytical expression for ARL of the cusum chart by using a Brownian motion approximation. It is an asymptotic analytic formula for the ARL obtained by approximating the behavior of the cumulative sum used in the cusum chart procedure by a Brownian motion process in the interval $(0,\infty)$.

The ARL for a two-sided scheme can be derived from the ARLs for the two one-sided schemes of which it is composed. Kemp (1962) has shown that if L_1 is the ARL at μ of a positive procedure with reference value k_1 and decision limit h, and if L_2 is the ARL at μ of a negative procedure with reference value k_2 and decision limit -h, then if L_t is the ARL of the two-sided scheme obtained by using both of these simultaneously, we have

$$\frac{1}{L_t} = \frac{1}{L_1} + \frac{1}{L_2} \quad .$$

The parameters needed in the design of cusum charts are the sample size n, sampling interval s, reference value k and the decision limit h. Ewan and Kemp (1960) showed that a central reference value $k = (\mu_o+\mu_1)/2$

has certain advantages over other choices of k. For a fixed sample size n and a given $L(\mu_o)$, $L(\mu_1)$ is minimized for this value of k. Also for fixed $L(\mu_o)$ and $L(\mu_1)$, the sample size n is minimized when $k = (\mu_o+\mu_1)/2$. They also showed by use of a nomogram that the relationship between the sampling interval and sample size is nearly linear so that the difference in ARL at μ_o and μ_1 between taking one sample every unit of time, or a sample, three times as big, every third unit of time is relatively small. It is therefore better to sample as frequently as practical considerations allow, and to determine the appropriate values of h and n by use of a nomogram.

For the economic design of control charts, Duncan (1956) formulated a model to measure approximately the average net income of a process under surveillance of a Shewhart chart ($\bar{X}$-chart) when the process is subject to random shifts in the process mean. He assumed that the process is not shut down while the search for the assignable cause is in progress, nor is the cost of adjustment or repair and the cost of bringing the process back into a state of control after the assignable cause is discovered charged to the control chart program. Assuming that the occurrence time of the assignable cause is exponentially distributed, he developed a formula expressing the long run average cost per unit time as a function of the parameters n, s, and h. This formula is called the loss-cost function, and an optimum design is the design that minimizes the long run loss-cost per hour. Optimum values of n, h, and s were obtained by a trial and error method.

Duncan (1971) extended the investigation to allow for the occurrence of several assignable causes. He studied two types of model. Model I assumes that once an assignable cause occurs, it continues to affect the process until it is detected by the control chart, and during this period the process is undisturbed by the occurrence of other assignable causes. For Model II the process allows for the second occurrence of an assignable cause following the first occurance. To simplify the analysis it is assumed that the joint effect of the two assignable causes is always to produce the same shift in the process mean regardless of which two assignable causes occur jointly. The occurrence times of the various assignable causes A_j are assumed to be independently and exponentially distributed with means $1/\lambda_j$. Hence, the ratio λ_j/λ (with $\lambda = \Sigma\lambda_j$) is the conditional probability of A_j given the occurrence of an assignable cause. When A_j occurs, it produces a shift in the process mean of $\delta_j\sigma$. Thus a given set of λ_j's may be viewed as determining the "prior probabilities" for the various shifts due to different causes. Duncan considered three different sets of λ_j's. In one set the λ_j are are chosen proportional to $\frac{1}{2}e^{-\delta_j/2}$, and this set is referred to as the "exponential prior distribution" for $\delta_j\sigma$. In a second set the λ_j are taken to be equal, and this set is referred to as the "uniform prior distribution" for $\delta_j\sigma$. In a final set the λ_j are chosen proportional to $(1/\sqrt{2\pi})e^{-z_j^2/2}$ where $z_j = \delta_j/2$ and the set is referred to as the "half-normal prior distribution" for $\delta_j\sigma$. In Duncan's article, the δ_j are taken to run from 0.75 to 6.25 in steps of 0.5 except for the uniform prior distribution when they only run to 3.25. It is noted that the three prior distributions are kept roughly similar in certain respects. For all of them the λ_j and M_j are chosen so that $\Sigma\lambda_jM_j = 1$, where M_j is the cost per hour that is attributed to the occurrence of the assignable cause A_j. The means are all roughly equal to 2.0. The sums of the λ_j's are fairly close to 0.0057 for all the distributions considered. The values of M_j are derived from the assumption that the rate of production is constant and the specification limits are at $\mu_o+3\sigma/\sqrt{n}$. When the process mean changes by $2\sigma/\sqrt{n}$, M_j is arbitrarily given the value $115.0 per hour. The other values of M_j are taken to be proportional to the increase in the percentage of defective items caused by the shift in the mean. Based on some numerical examples, it was found that the optimum designs are about the same for all three prior distributions. These optimum designs are determined by the direct search procedure. Duncan also found that the optimum design yielded by Model II was either identical with that yielded by Model I or very close to it. Contours of the loss-cost surface were plotted. They showed that there was only a single minimum for Model II, and this fell practically at the local minimum of Model I. Duncan further showed by a few numerical examples that a multiple cause model could be approximated by a certain matched single cause model. These matching criteria were selected such that (1) the shifts produced by the single assignable cause (δ_s) is equal to the mean shift for the multiple cause model, i.e. $\delta_s = \Sigma\lambda_j\delta_j/\lambda$, (2) the rate of occurrence of the single cause (λ_s) is equal to the sum of the rates of occurrence of the individual assignable causes in the multiple cause model, i.e. $\lambda_s = \lambda$, and (3) the cost per unit time M_s that is attributed to the occurrence of this single assignable cause is equal to the weighted mean of the M_j, i.e. $M_s = \Sigma\lambda_j M_j/\lambda$.

Gibra (1971) provided a procedure to determine optimum parameters of the $\bar{X}$-chart so as to minimize the long run average cost per unit time subject to the restriction that the mean shifts will be detected and eliminated within a prescribed time interval with some known probability. This procedure is for a single assignable cause model where the occurrence of this assignable cause follows the exponential distribution. The function for long run average cost per unit time is derived and the optimal parameters are determined by a trial and error method.

Chiu and Wetherill (1975) proposed a simplified scheme for the economic design of $\bar{X}$-charts. The principle for the choice of parameters is to minimize the average loss-cost per unit time with a constraint that the probability P of a sampling point falling outside the control limit is equal to some specified value. The plan obtained in this way is found to be close to optimum. They illustrated this by applying the simplified scheme to Duncan's examples with $P \geq 0.90$.

Taylor (1968) gave an approximate formula for the long run average cost per unit time as a function of the parameters of the cusum chart using the V-mask procedure. He assumed that the process is shut down while a search for the assignable cause is made, but the cost of maintaining the control chart is omitted. The time at which the assignable cause occurs is also assumed to be exponentially distributed. To obtain the design parameters, Taylor assumed that the sample size and the sampling interval are known. The half-angle of the V-mask θ is given by $\tan^{-1}(\mu_1-\mu_o)/2$ which is equivalent to choosing $k = \mu_1 - \tan\theta =$

$\frac{1}{2}(\mu_1+\mu_o)$. The only value to be determined is d, the lead distance of the V-mask, which is obtained by using an approximate formula for ARL given by Goldsmith and Whitfield (1961).

Goel and Wu (1973) developed a model which related the design variables of a cusum chart and the cost and risk factors of the process being controlled. The derivation of the model followed Duncan's approach for the economic design of $\bar{X}$-charts when only a single assignable cause is assumed to occur. The values n, h, and s that minimize the long run average cost per unit time are the optimum values of the design variables. In order to obtain the optimum design parameters, a direct search numerical method was employed. The central reference value was used. The calculation of ARL was by the method of solving a system of linear algebraic equations. It was found that as the shift in the process mean is decreased, the optimum procedure requires larger samples, taken less frequently with a smaller decision limit. An increase in the cost of maintaining the control chart and in the expected number of assignable causes per hour results in a reduction of the optimum sampling interval. However, the sample size and the decision limit are relatively insensitive to these factors.

Chiu (1974) studied the economic design of cusum charts. The model allows the process to be shut down when a search for the assignable cause is being carried out, and it includes the time and cost of adjusting the process. The loss-cost function F per hour for this model is formulated. The values of the ARL required in the model are calculated by the method of Goel and Wu (1973) and the central reference value for k is used. The optimum value of the sampling interval s is determined by an analytical expression which is obtained by solving $\partial F/\partial s = 0$ under the assumption that λ and $1/L(\mu_o)$ are small so that terms in λ^2 and $\lambda^2/L(\mu_o)$ can be omitted. A search procedure was used to find the optimum values for n and h. Through the numerical results of 15 cases, he found that any value in the range 1.05-1.20 is a nearly optimum choice for $L(\mu_1)$. To be more specific, he chose to set $L(\mu_1)=1.1$ and proposed a simplified procedure in which $L(\mu_1)=1.1$. It appeared from some numerical studies that this simplified procedure gives control parameters which are close to optimum. The study was confined to a single cause model.

For the acceptance sampling plan design, Schmidt and Bennett (1972) presented an economic model which is based on lot accept-reject decisions for a product containing several attributes. Its primary use is to identify the acceptance numbers and sample sizes for the respective attributes which in combination yield the minimum cost plan. Optimization of the model is achieved through the pattern search technique.

Latimer, Bennett and Schmidt (1973) proposed an economic model for the situation in which lot-by-lot sampling has the dual purpose of controlling the lot mean and determining lot disposition. The design provided for the simultaneous control of a number of mutually independent characteristics. The design variables for the system are each characteristic's sample size, and $\bar{X}$-chart decision limit. The lot size is assumed to be known and constant from lot to lot.

Ailor, Schmidt and Bennett (1975) presented an economic acceptance sampling model for the case where several attributes and several variables are simultaneously subjected to acceptance sampling. The decision to accept or reject each inspection lot is based upon the results of sampling inspection on each of the several variables and attributes. An optimum plan was found by minimizing the expected cost model with respect to the decision variables which are the sample sizes and control limits on the sample means for variables and the sample sizes and acceptance numbers for attributes. Optimization was accomplished by using the pattern search technique.

CHAPTER III

DERIVATION OF THE COST MODEL

3.1 Nature of the Process

Samples $\bar{X}_1$, $\bar{X}_2$, ... of size n are taken at regular time intervals from the output of the process to be controlled. These samples are assumed to be independent and normally distributed with mean μ_o and known variance σ^2/n. Assignable cause A_j, j=1,2,...,p, occurs at random and causes a shift in the process mean of a known magnitude $\delta_j\sigma$, j=1,2,...,p, so that the new mean after cause A_j is $\mu_j=\mu_o+\delta_j\sigma$. It is assumed that the process starts in a state of control. When the process mean shifts to μ_j because of A_j, j=1,2,...,p, the model assumes that the process stays at this level until a lack of control is indicated and adjustments are made to bring the process back to the control level. Without loss of generality, we can assume that $\mu_o=0$ since we can always consider the samples $\bar{X}_i-\mu_o$.

A model is developed which relates the design parameters of a cusum chart (sample size n, sampling interval s, decision limit h, and reference value k) and the loss and risk factors of the process. The derivation of this model follows Duncan's approach for $\bar{X}$-charts (1971). It extends Goel and Wu's model (1973) to the case where there is a multiplicity of assignable causes.

Let Y_i be the net income for i^{th} in-control-out-of-control cycle and Z_i be the length of time for i^{th} cycle. The net income per unit of time for g cycles is defined as $\sum_{i=1}^{g} Y_i / \sum_{i=1}^{g} Z_i$ and the average net income per unit of time is defined as the almost sure limit of this ratio as $g \to \infty$. Without loss of generality, we choose an hour as a unit of time. The cost model that will be derived gives the average net income per hour.

The following assumptions are made about the behavior of the process.

(1) When the process is in the state of control, the time t_j until the occurrence of assignable cause A_j is exponentially distributed with mean $1/\lambda_j$, where $j=1,2,\ldots,p$. The times t_1, t_2, ..., t_p are independently distributed.

(2) When the process has been disturbed by a given assignable cause, it is free from the occurrence of other assignable causes until the process mean has been returned to the control value.

(3) The process level does not change while the sample is being taken.

(4) The process continues to run while overall checking is done and adjustments are made following an out-of-control signal.

(5) The cost of repairs to items inspected is not charged to the process.

(6) The following cost and risk factors associated with the process are assumed to be known.

I_o: the expected income per hour from the process when it is in control.

I_j: the expected income per hour from the process when it is out-of-control because of the occurrence of assignable cause A_j.

M_j: $I_o - I_j$, where $j=1,2,\ldots,p$.

c : the cost of measuring or inspecting an item of the product.

b : the cost per sample of sampling and charting.

W : the cost of looking for an assignable cause when an out-of-control signal appears.

D : the time between the out-of-control signal and the removal of the assignable cause.

en: the time between taking the sample and plotting the sample point.

(7) The cost of maintaining the control chart per hour of operation is equal to (b+cn)/s as it is in Duncan's model.

Let

γ = the proportion of the time in the long run that the process will be in-control.

γ_j= the proportion of the time in the long run the process will be out-of-control because of assignable cause A_j, j=1,2,...,p.

a_f= the expected number of false alarms per hour.

ε = the expected number of times per hour that the process actually goes out-of-control.

The average net income per hour of the process can be written as,

Average net income per hour = Average income per hour -

Average cost per hour.

The average income per hour is divided into two parts:

(a) average income per hour when the process is in control, γI_o,

and

(b) average income per hour when the process is out of control, $\sum_{j=1}^{p} \gamma_j I_j$.

The average cost per hour is also divided into two components:

(a) average cost per hour for out-of-control signals, $(a_f + \varepsilon)W$, and

(b) cost of maintaining the control chart, $(b+cn)/s$.

Therefore, the average net income per hour of operation based on the above assumptions is equal to

$$A = (\gamma I_o + \sum_{j=1}^{p} \gamma_j I_j) - (a_f + \varepsilon)W - (b+cn)/s \quad .$$

We would like to find the optimum solutions for n, h, s, and k that maximizes A, the average net income per hour, for various cost and risk factors. In order to achieve this aim, we need to obtain expressions for the unknown quantities γ, γ_j, a_f and ε in terms of s, λ_j, n, $L(\mu_o)$, $L(\mu_j)$, D and e. These are derived in the following sections.

3.2 Expressions for γ and γ_j

Let $A_{(1)}$ be the first assignable cause to occur and T be the time that $A_{(1)}$ occurs. From assumption (1) the mean time for the assignable cause A_j to occur is $1/\lambda_j$ and the probability density function for t_j is $\lambda_j e^{-\lambda_j t_j}$. Since

$$T = \min\ (\ t_1, t_2, \ldots, t_p\ )$$

and

$$P(\ T \leq t\ ) = 1 - \prod_{j=1}^{p} P(\ t_j \geq t\ )$$

with

$$P(\ t_j \geq t\ ) = \int_t^{\infty} \lambda_j e^{-\lambda_j t_j} dt_j = e^{-\lambda_j t}$$

we have

$$P(\ T \leq t\ ) = 1 - e^{-t \sum_{j=1}^{p} \lambda_j} \quad .$$

Then, the probability density function f(t) of T is

$$f(t) = \sum_{j=1}^{p} \lambda_j e^{-t \sum_{j=1}^{p} \lambda_j}$$

and the expected time that the process will be in control is

$$\int_0^\infty t \sum_{j=1}^{p} \lambda_j e^{-t \sum_{j=1}^{p} \lambda_j} dt = 1 / \sum_{j=1}^{p} \lambda_j = 1/\lambda \qquad (3.2.1)$$

where $\lambda = \sum_{j=1}^{p} \lambda_j$.

Let $N(t_o)$ denote the number of occurrences of any assignable cause in the time interval $(0,t_o)$, then

$$P(\ N(t_o)=k\ ) = (\lambda t_o)^k e^{-\lambda t_o}/k! \quad , \qquad k = 0, 1, 2, \ldots$$

It follows that the expected number of occurrences in $(0,t_o)$ is λt_o. Similarly, let $N_i(t_o)$ denote the number of occurrences of assignable cause A_i in the time interval $(0,t_o)$ then the expected number of occurrences in $(0,t_o)$ is $\lambda_i t_o$.

Now, given an occurrence in $(0,t_o)$, the probability that it is assignable cause A_j that occurs is

$$P(\ A_j \text{ occurs} \mid \text{one occurrence}\ ) = \frac{P(\text{ only } A_j \text{ occurs once })}{P(\text{ one occurrence })}$$

$$= \frac{P(\ N_j(t_o)=1\ ) \prod_{i \neq j}^{p} P(\ N_i(t_o)=0\ )}{P(\ N(t_o)=1\ )} = \frac{\lambda_j t_o e^{-\lambda_j t_o} \prod_{i \neq j} e^{-\lambda_i t_o}}{\lambda t_o e^{-\lambda t_o}} = \lambda_j/\lambda \quad .$$

When the process goes out-of-control because of the assignable cause A_j, the length of time it will stay out-of-control before the assignable cause is removed will be denoted by T^j. Then, the expected value of the time the process will be out-of-control, T_o, is

$$E(T_o) = \sum_{j=1}^{p}(\lambda_j/\lambda)E(T^j) \quad . \tag{3.2.2}$$

Following equation (3.2.1), the expected value of the time the process will be in-control, T_I, is

$$E(T_I) = 1/\lambda \quad . \tag{3.2.3}$$

From (3.1.1) and (3.1.2), it follows that the expected time for one in-control-out-of-control cycle is

$$E(T_I+T_o) = E(T_I) + E(T_o) = 1/\lambda + \sum_{j=1}^{p}(\lambda_j/\lambda)E(T^j) \quad .$$

Let $\{T_{Ii}\}$ be the sequence of in-control times and $\{Z_i\}$ be the sequence of total cycle times. The proportion of time that the process is in control for g cycles is $\sum_{i=1}^{g}T_{Ii}/\sum_{i=1}^{g}Z_i$. The long run proportion of time that the process is in-control, γ, is defined (14) as the almost sure limit

$$\lim_{g\to\infty} \sum_{i=1}^{g}T_{Ii}/\sum_{i=1}^{g}Z_i \quad .$$

By the strong law of large numbers, this limit is equal to

$$E(T_I)/E(T_I+T_o) = 1/\{\ 1 + \sum_{j=1}^{p}\lambda_j E(T^j)\ \} \quad .$$

Similarly, γ_j, the long run proportion of time that the process is out of control because of A_j, is

$$\gamma_j = \lambda_j E(T^j)/\{\ 1 + \sum_{j=1}^{p}\lambda_j E(T^j)\ \}$$

for all $j = 1, 2, \ldots, p$.

3.3 Expression for $E(T_1)$ and $E(T_2)$

Let T_1 denote the time period between the last sample taken before $A_{(1)}$ occurs and the occurrence of $A_{(1)}$. Let T_2 denote the time period (see Fig. 1) between the occurrence of $A_{(1)}$ and the first sample taken after $A_{(1)}$ occurs. If s is the sampling interval then $T_2 = s - T_1$ and

$$E(T_2) = s - E(T_1) \quad .$$

Let q denote the number of samples taken in the in-control period. Then, sq is the time period until $A_{(1)}$ occurs and $T_1 = T_I - sq$.

$$E(q) = \sum_{i=0}^{\infty} i\ P(\text{ taking i samples in the in-control period })$$

$$= \sum_{i=0}^{\infty} i\ P(\text{ any of the causes occurs in the time interval } (is,(i+1)s) \text{ and no cause occurs sooner than this })$$

$$= \sum_{i=0}^{\infty} i \int_{is}^{(i+1)s} \lambda e^{-t\lambda} dt$$

$$= \sum_{i=0}^{\infty} i e^{-is\lambda}(1-e^{-\lambda s})$$

$$= (1-e^{-\lambda s})(-1/s)\frac{\partial}{\partial\lambda}\left(\sum_{i=0}^{\infty} e^{-is\lambda}\right)$$

$$= (1-e^{-\lambda s})(-1/s)\frac{\partial}{\partial\lambda}(1-e^{-\lambda s})^{-1}$$

$$= (1-e^{-\lambda s})(-1/s)(-se^{-\lambda s})/(1-e^{-\lambda s})^2$$

$$= e^{-\lambda s}/(1-e^{-\lambda s}) \quad .$$

Then,

$$E(T_1) = E(T_I - sq) = E(T_I) - sE(q) = 1/\lambda - se^{-\lambda s}/(1-e^{-\lambda s})$$

and

$$E(T_2) = s - E(T_1) = s/(1-e^{-\lambda s}) - 1/\lambda \quad .$$

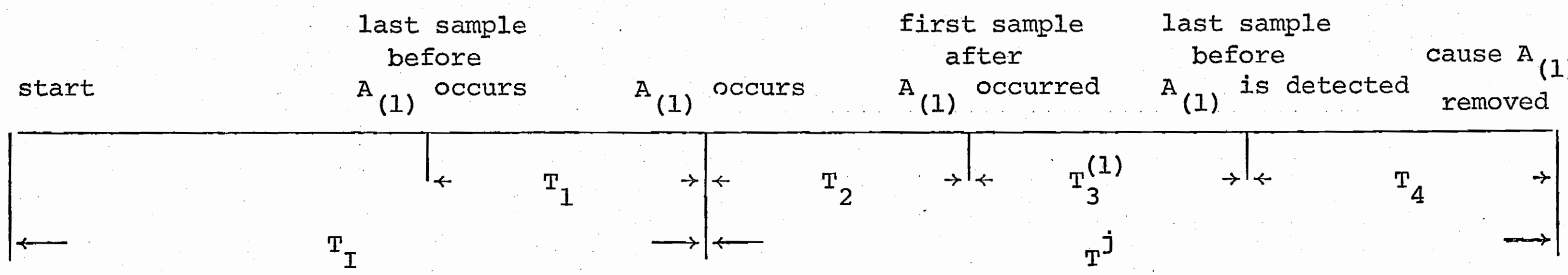

Figure 1. Diagram of the process behavior

3.4 Expression for T_4

Let T_4 denote the time period between the last sample taken before lack of control is detected and the time that the assignable cause is removed. It follows from assumptions (4) and (6) that

$$T_4 = D + en \quad .$$

3.5 Expression for $E(T^j)$

Let T_3^j be the time period between the first sample taken after the occurrence of the assignable cause A_j and the last sample taken prior to its detection. Let L_r^j denote the ARL when the process mean shifts to $\delta_j\sigma$. Taylor (1968) showed that

$$E(T_3^j) \cong (L_r^j - 1)s$$

for $j = 1, 2, \ldots, p$.

Then,

$$E(T^j) = E(T_2 + T_3^j + T_4) = E(T_2) + E(T_3^j) + E(T_4)$$

$$= s/(1-e^{-\lambda s}) - 1/\lambda + (L_r^j - 1)s + D + en \quad .$$

3.6 Expression for ε and a_f

The process goes out of control once per in-control-out-of-control cycle. Then the average number of times per hour the process actually goes out of control for g cycles is

$$g / \sum_{i=1}^{g} Z_i$$

Following Johns and Miller (1963), the expected number of times per hour the process actually goes out of control is

$$\varepsilon = \lim_{n\to\infty} n/\sum_{i=1}^{n} Z_i \qquad \text{a.s.}$$

$$= 1/\{\ 1/\lambda + \sum_{j=1}^{p} (\lambda_j/\lambda) E(T^j)\ \}$$

$$= \lambda/\{\ 1 + \sum_{j=1}^{p} \lambda_j E(T^j)\ \}$$

Goel and Wu (1973) showed that the expected number of false alarms per hour is approximately equal to

$$a_f \cong \frac{1}{L_o s\lambda} \cdot \frac{1}{E(Z)} = 1/\{\ sL_o(1 + \sum_{j=1}^{p} \lambda_j E(T^j))\ \} \quad .$$

3.7 Expression for the Loss-Cost Function

From the derivations 3.1 to 3.6 we have

$$A \doteq \gamma I_o + \sum_{j=1}^{p} \gamma_j I_j - (a_f + \varepsilon)W - (b + cn)/s$$

$$\cong I_o/\{\ 1 + \sum_{j=1}^{p} \lambda_j E(T^j)\ \} + \sum_{j=1}^{p} \{\lambda_j E(T^j) I_j/(\ 1+\sum_{j=1}^{p} \lambda_j E(T^j))\}$$

$$- W\{\ 1/sL_o(1+\sum_{j=1}^{p} \lambda_j E(T^j)) + \lambda/(1+\sum_{j=1}^{p} \lambda_j E(T^j))\}$$

$$-(b+cn)/s \quad .$$

Set

$$B = 1 + \sum_{j=1}^{p} \lambda_j E(T^j) \quad .$$

Then

$$A \cong I_o/B + \sum_{j=1}^{p} \lambda_j E(T^j) I_j/B - (1+\lambda sL_o)W/BsL_o - (b+cn)/s$$

$$= I_o - \sum_{j=1}^{p} \lambda_j E(T^j) M_j/B - (1/sL_o+\lambda)W/B - (b+cn)/s \quad .$$

Let

$$C = \sum_{j=1}^{p} \lambda_j E(T^j) M_j / B + (1/sL_o + \lambda) W/B + (b+cn)/s \tag{3.7.1}$$

Then

$$A = I_o - C \quad .$$

The quantity C is the long run average loss-cost per hour of the process. Since $A = I_o - C$, A is maximized when C is minimized. Therefore, minimizing C is equivalent to maximizing A.

Equation (3.7.1) is the loss-cost model that associates the design parameters of the cusum chart and the cost and risk factors of the process.

CHAPTER IV

BROWNIAN MOTION APPROXIMATION FOR ARL

Let the distribution function of the values $y=\bar{x}-\mu_o-k$ to be accumulated in the cusum procedure be $F(y)$. Then the integral equation for the ARL of a cusum test $L(z)$, starting at point z, is (see Page(1954))

$$L(z) = 1 + L(0)F(-z) + \int_0^h f(x-z)L(x)dx \qquad 0 \leq z \leq h \qquad (4.1)$$

we are usually only interested in the ARL of a test starting at zero; knowledge of $L(0)$ would be sufficient. But, as is evident from the integral equation, $L(z)$ must be known for all values of z in the interval $(0,h)$ to enable one to find $L(0)$. As this equation is not of a form which can be solved analytically in general, it will often be necessary to use numerical or approximation results.

Tables of the ARL for tests controlling the mean of a normal distribution when h = 1.3, 1.5, 2.0, 2.5, 3.0, 4.0, 5.0, 6.0, 8.0, 10.0 are given in Chapter II of Van Dobben De Bruyn (1968). These quantities are correct up to one unit in the second decimal place. Some other available graphs and tables of the ARL for normally distributed observations were mentioned in Chapter II.

A large number of computations of the ARL is required in finding the economic optimum design for the cusum procedure. Tables which are available do not meet this need and numerical methods for solving the integral equation (4.1) are expensive and time consuming. Therefore, it is desirable to have an approximate analytic formula for the ARL.

Reynolds (1975) derived an asymptotic analytic fromula for the ARL by approximating the behavior of the cusum chart procedure by a Brownian motion process on the interval $(0,\infty)$. For the procedure for detecting positive deviations, the ARL when n=1 is

$$ARL^{+} = \begin{cases} \frac{1}{\mu-k}\{h + \frac{\sigma^2}{2(\mu-k)}\left(\operatorname{Exp}\left(\frac{-2(\mu-k)h}{\sigma^2}\right) - 1\right)\} & \mu-k \neq 0 \\ h^2/\sigma^2 & \mu-k = 0 \end{cases} \tag{4.2}$$

and for the procedure for detecting negative deviations, the ARL is

$$ARL^{-} = \begin{cases} -\frac{1}{\mu+k}\{h - \frac{\sigma^2}{2(\mu+k)}\left(\operatorname{Exp}\left(\frac{2(\mu+k)h}{\sigma^2}\right) - 1\right)\} & \mu+k \neq 0 \\ h^2/\sigma^2 & \mu+k = 0 \end{cases} \tag{4.3}$$

where μ is the process mean, σ^2 is the process variance, k is the reference value and h is the decision limit.

Reynolds compared the exact values of ARL^{+} tabulated in Van Dobben De Bruyn (1968) for normal observations with the Brownian motion approximation. He found that this approximation underestimates ARL^{+} and for small values of h the discrepancy in the approximation is serious. A modified Brownian motion approximation is suggested by using $h+\Delta$, for some $\Delta \geq 0$, instead of h in the formula of the approximation. Reynolds suggested that Δ be chosen such that the approximate ARL^{+} at $\mu-k = 0$ is equal to the exact value. However, this does not guarantee that the modified value will be good at values of $\mu-k \neq 0$. Bakir (1976) used the SAS computer package stepwise regression procedure to determine a formula for Δ as a function of $\mu-k$ and h. This regression procedure is easy to use and provides a good fit. The function is as follows (σ^2=1),

$$\Delta = 1.153517 + 0.060216(\mu-k) + 0.056672(\mu-k)^2 - 0.000072h^3 \qquad (4.4)$$

It was found that for $-1.2 \leq \mu-k \leq 4$, and $2 \leq h \leq 10$, the Brownian motion approximation with modified h value, $h+\Delta$, where Δ is calculated by equation (4.4), gives quite satisfactory ARL^+.

Goel and Wu (1971) calculated the ARL by replacing the integral equation (4.1) by systems of linear algebraic equations which can be solved to obtain an approximate ARL. They showed that the results obtained are accurate to the fourth decimal place. A detailed table of the ARL for $h = 0.1(0.1)5.0$ and $\mu-k = -2.0(0.1)2.0$ is tabulated (1971). Chiu (1974) used the same method to obtain values of ARL for $h = 0.2(0.2)2.0$ and $\mu-k = -4.0, -3.0(0.2)3.0, 4.0$. These tables have considerable practical importance in the economic approach to the design of cusum charts.

The values of ARL tabulated in Goel and Wu (1971) were compared with the modified Brownian motion approximation. It was found that the Δ value given in equation (4.4) did not give a desirable approximate value for ARL when $h \leq 2$. In this dissertation, another formula for Δ was determined for the case where $h \leq 2$.

In replacing h by $h+\Delta$, equation (4.2) can be written as,

$$ARL^+ - \frac{1}{\mu-k}\{(h+\Delta) + \frac{\sigma^2}{2(\mu-k)}\left(\text{Exp}\left(\frac{-2(\mu-k)(h+\Delta)}{\sigma^2}\right) - 1\right)\} = 0$$

when $\mu-k \neq 0$, and

$$ARL^+ - \frac{(h+\Delta)^2}{\sigma^2} = 0$$

when $\mu-k = 0$.

Brown's method (1971) was used to solve the above equation for Δ

and values of ARL^+ were taken from Goel and Wu's results. After determining Δ values for each ARL^+, the SAS stepwise regression procedure was used to determine a formula for Δ as a function of μ-k and h. It was found that when

$$\Delta = 1.4105019 - 0.43708249h + 0.24647213h^2 - 0.04498068h^3 + 0.0970047(\mu-k) + 0.11848928(\mu-k)^2 + 0.03525852(\mu-k)^3 - 0.00154907(\mu-k)^5 + 0.00030157(\mu-k)^6 - 0.10060047h(\mu-k) - 0.08267103h(\mu-k)^2 - 0.01452112h(\mu-k)^3 + 0.03555026h^2(\mu-k) + 0.00457078h^2(\mu-k)^3 + 0.00815311h^3(\mu-k)^2 - 0.00008643h^6(\mu-k)^3 - 0.0003074h^6(\mu-k)^4 \quad (4.5)$$

is used, we can obtain a desirable approximate value for ARL when $h \leq 2$. Equation (4.5) was used to calculate the modified Brownian motion approximation and the results were compared with those tabulated in Goel and Wu (1971) and Chiu (1974). The comparisons given in Table 4.1 and Table 4.2 show that this modified approximation gives a very good approximate value for ARL when $\mu-k \geq -2.4$. However, it is seen that for $\mu-k \leq -2.4$ the discrepancy is serious. Yet we may notice that for the λ_j and δ_j considered in this thesis k is about 1.225 and values of μ-k are always larger than -2.4 unless $\mu=0$ which is the case when the process is in-control. The ARL when the process is in-control, L_o, appears in the function C only in the reciprocal form. This discrepancy on the values of L_o will have little effect on C since $1/L_o$ is very small for small values of h and for $\mu-k < -2.4$.

When n>1 the variance for the sample mean becomes σ^2/n. The quantities μ-k, μ+k, h, σ^2 in the equations (4.2)-(4.5) are then

Table 4.1 Modified Brownian Motion Approximation of ARL^+ for $h\sqrt{n}/\sigma$ = 0.4, 0.6, 1.0

$(\mu-k)\sqrt{n}/\sigma$	$h\sqrt{n}/\sigma$ = 0.4 M.B.M.	0.4 S.L.A.E.	0.6 M.B.M.	0.6 S.L.A.E.	1.0 M.B.M.	1.0 S.L.A.E.
-4.0	$>10^5$	$>10^5$	$>10^5$	$>10^5$	$>10^5$	$>10^5$
-2.8	3047	1453	5812	2958	23672	13709
-2.4	455.1	389.8	832.6	734.3	3133	2901
-2.0	121.6	120.9	209.98	210.3	697.15	702.0
-1.6	42.8	43.2	68.74	69.2	193.16	192.6
-1.2	17.70	17.76	26.13	26.15	60.47	60.01
-0.8	8.37	8.38	11.37	11.39	21.61	21.59
-0.4	4.54	4.54	5.37	5.74	9.20	9.22
0.0	2.81	2.81	3.35	3.35	4.75	4.75
0.4	1.95	1.96	2.24	2.23	2.91	2.91
0.8	1.50	1.51	1.67	1.66	2.05	2.04
1.2	1.26	1.26	1.35	1.35	1.59	1.58
1.6	1.13	1.12	1.18	1.18	1.33	1.32
2.0	1.06	1.05	1.08	1.09	1.17	1.17
2.4	1.03	1.02	1.03	1.04	1.08	1.08
2.8	1.03	1.01	1.01	1.01	1.02	1.04
4.0	1.18	1.00	1.12	1.00	1.04	1.00

M.B.M.= ARL^+ obtained by modified Brownian motion approximation

S.L.A.E. = ARL^+ obtained by solving systems of linear algebraic equations

Table 4.2 Modified Brownian Motion Approximation of ARL^+ for $h\sqrt{n}/\sigma$ = 1.2, 1.6, 2.0

$(\mu-k)\sqrt{n}/\sigma$	$h\sqrt{n}/\sigma$ = 1.2		1.6		2.0	
	M.B.M.	S.L.A.E.	M.B.M.	S.L.A.E.	M.B.M.	S.L.A.E.
-2.8	51085	31210	$>10^5$	$>10^5$		
-2.4	6460	6083	31251	29720	$>10^5$	$>10^5$
-2.0	1336	1346	5394	5432	24753	24471
-1.6	336.25	334.4	1084	1081	3757	3768
-1.2	94.18	93.29	236.28	235.0	609.49	613.8
-0.8	30.06	30.02	58.55	58.59	113.56	114.5
-0.4	11.61	11.65	18.25	18.30	27.91	28.02
0.0	5.60	5.61	7.63	7.63	10.02	10.0
0.4	3.30	3.30	4.16	4.15	5.08	5.06
0.8	2.26	2.26	2.74	2.74	3.24	3.24
1.2	1.73	1.73	2.03	2.04	2.37	2.38
1.6	1.42	1.42	1.64	1.64	1.88	1.89
2.0	1.23	1.23	1.39	1.39	1.58	1.58
2.4	1.12	1.12	1.23	1.22	1.37	1.37
2.8	1.05	1.06	1.12	1.12	1.22	1.22
4.0	1.01	1.00	1.00	1.01	0.96	1.02

M.B.M. = ARL^+ obtained by modified Brownian motion approximation

S.L.A.E. = ARL^+ obtained by solving systems of linear algebraic equations replaced by $(\mu-k)\sqrt{n}/\sigma$, $(\mu+k)\sqrt{n}/\sigma$, $h\sqrt{n}/\sigma$ and σ^2/n respectively.

Khan (1978) gave Wald's approximation to the ARL in a cusum procedure for an exponential family of densities. He compared the results with values obtained by Van Dobben De Bruyn in the normal case. From his table we found that using the modified Brownian motion approximation given by equations (4.4) and (4.5) seems to give a better approximation in general.

Since the modified Brownian motion approximation gives sufficiently accurate values and is easy to compute, it was considered to be the most suitable method to use. In the numerical study on the function C, this approximation is then used to obtain the ARLs.

CHAPTER V

NUMERICAL ANALYSIS

5.1 Optimization Procedure

The optimum design parameters are those values of n, h, k and s that minimize the cost function C given by equation (3.7.1). The necessary condition for an interior point in the region of interest (h,s,k) to be the minimum solution of C for a given n is that this value of (h,s,k) satisfies the following equations:

$$
\begin{aligned}
f_1(h,s,k) &= \frac{\partial C}{\partial h} = 0 \\
f_2(h,s,k) &= \frac{\partial C}{\partial s} = 0 \qquad (5.1.1) \\
f_3(h,s,k) &= \frac{\partial C}{\partial k} = 0
\end{aligned}
$$

An analytical solution for these parameters is not possible because the ARLs L_o and L_r^j can not be obtained without knowing n, h, and k. Therefore, a numerical procedure, Brown's method (1971), is employed.

Brown's method handles the functions (5.1.1) one at a time. The basic process is (1) to approximate f_1 by Taylor's expansion about the initial point, then (2) ignore higher order terms and equate this approximation f_1 to zero, and (3) solve for one variable, say h, as a linear combination, $L_1(s,k)$, of the remaining two variables. Let $g_2 \equiv f_2(L_1(s,k),s,k)$ and repeat the basic process and obtain $f_3(L_1\{L_2(k), k\},L_2(k),k)$. Now expanding, linearizing and solving for k, the point k thus obtained can be used as the next approximation to the first component of the solution to functions (5.1.1). Then improved approximations to the other components can be obtained by back solving L_2 and L_1. These calculations are done in the computer using the FORTRAN program taken from Brown's article (1971). The program is terminated whenever $\max(f_1,f_2,f_3) < 10^{-7}$. Brown and Dennis (1971) proved that Brown's method is of second order convergence, that is, the number of correct decimal places is doubled at each step. In addition, it is inexpensive.

Equations (5.1.1) were solved for each n and the overall minimum solution was then found by varying n. In order to assure that the solution is the minimum solution for C, we need to look at the second derivatives of C. This is very tedious since the function C is not in a simple form. We thus investigated and tabulated the function C as in Table 5.17 for several sets of the cost and probability parameters studied. These tabulations served as a check that the values of h, s and k yielded by Brown's method actually correspond to the local minimum in the region of interest.

The partial derivative of C with respect to h yields

$$\frac{\partial C}{\partial h} = -(1/B^2)\sum_{j=1}^{p}\lambda_j E(T^j)M_j\frac{\partial B}{\partial h} + (1/B)\sum_{j=1}^{p}\lambda_j M_j\frac{\partial E(T^j)}{\partial h} - (W/B^2)\left(\frac{1}{L_o s} + \lambda\right)\frac{\partial B}{\partial h}$$
$$- (W/B)(1/sL_o^2)\frac{\partial L_o}{\partial h} \quad .$$

where

$$\frac{\partial E(T^j)}{\partial h} = s(\partial L_r^{\,j}/\partial h) \quad ,$$

$$\frac{\partial B}{\partial h} = \sum_{j=1}^{p}\lambda_j\frac{\partial E(T^j)}{\partial h} = \sum_{j=1}^{p}\lambda_j s(\partial L_r^{\,j}/\partial h) \quad .$$

Thus,

$$\frac{\partial C}{\partial h} = -\frac{1}{B}\sum_{j=1}^{p}\lambda_j s\frac{\partial L_r^j}{\partial h}\left(\frac{1}{B}\sum_{j=1}^{p}\lambda_j M_j E(T^j) + \frac{W}{B}\left(\frac{1}{sL_o} + \lambda\right)\right) + \frac{1}{B}\sum_{j=1}^{p}\lambda_j M_j s\frac{\partial L_r^j}{\partial h}$$

$$- (W/BsL_o^2)\frac{\partial L_o}{\partial h} \quad .$$

If

$$\frac{\partial C}{\partial h} = 0$$

then,

$$\sum_{j=1}^{p}\lambda_j \frac{\partial L_r^j}{\partial h}\left\{\frac{s}{B}\sum_{j=1}^{p}\lambda_j M_j E(T^j) + \frac{W}{B}\left(\frac{1}{L_o} + s\lambda\right)\right\} - \sum_{j=1}^{p}\lambda_j M_j s\frac{\partial L_r^j}{\partial h} + (W/sL_o^2)\frac{\partial L_o}{\partial h} = 0 \tag{5.1.2}$$

The partial derivative of C with respect to s gives

$$\frac{\partial C}{\partial s} = -(1/B^2)\sum_{j=1}^{p}\lambda_j E(T^j)M_j\frac{\partial B}{\partial s} + (1/B)\sum_{j=1}^{p}\lambda_j M_j\frac{\partial E(T^j)}{\partial s} - (W/B^2)(1/sL_o+\lambda)\frac{\partial B}{\partial s}$$

$$-W/s^2BL_o - (b+cn)/s^2 \quad .$$

where

$$\frac{\partial E(T^j)}{\partial s} = (1-e^{-s\lambda}-s\lambda e^{-s\lambda})/(1-e^{-s\lambda})^2 + L_r^j - 1 \quad ,$$

$$\frac{\partial B}{\partial s} = \sum_{j=1}^{p}\lambda_j\frac{\partial E(T^j)}{\partial s} \quad .$$

Thus,

$$\frac{\partial C}{\partial s} = -(1/B^2)\sum_{j=1}^{p}\lambda_j\frac{\partial E(T^j)}{\partial s}\left\{\sum_{j=1}^{p}\lambda_j M_j E(T^j) + W\left(\frac{1}{sL_o} + \lambda\right)\right\} + \frac{1}{B}\sum_{j=1}^{p}\lambda_j M_j\frac{\partial E(T^j)}{\partial s}$$

$$-W/s^2BL_o - (b+cn)/s^2 \quad .$$

If

$$\frac{\partial C}{\partial s} = 0$$

then,

$$\sum_{j=1}^{p} \lambda_j \frac{\partial E(T^j)}{\partial s} \{\frac{1}{B} \sum_{j=1}^{p} \lambda_j M_j E(T^j) + \frac{W}{B}(\frac{1}{sL_o} + \lambda)\} - \sum_{j=1}^{p} \lambda_j M_j \frac{\partial E(T^j)}{\partial s} + W/s^2 L_o$$

$$+ B(b+cn)/s^2 = 0 \quad . \qquad (5.1.3)$$

Finally,

$$\frac{\partial C}{\partial k} = (1/B) \sum_{j=1}^{p} \lambda_j M_j \frac{\partial E(T^j)}{\partial k} - (1/B^2) \sum_{j=1}^{p} \lambda_j M_j E(T^j) \frac{\partial B}{\partial k} - (W/sBL_o^2) \frac{\partial L_o}{\partial k}$$

$$-(W/B^2)(\frac{1}{sL_o} + \lambda) \frac{\partial B}{\partial k}$$

where

$$\frac{\partial E(T^j)}{\partial k} = s \frac{\partial L_r^j}{\partial k}$$

$$\frac{\partial B}{\partial k} = \sum_{j=1}^{p} \lambda_j \frac{\partial E(T^j)}{\partial k} = \sum_{j=1}^{p} \lambda_j s \frac{\partial L_r^j}{\partial k} \quad .$$

Thus,

$$\frac{\partial C}{\partial k} = -(1/B^2) \sum_{j=1}^{p} \lambda_j s \frac{\partial L_r^j}{\partial k} (\sum_{j=1}^{p} \lambda_j M_j E(T^j) + W(\frac{1}{sL_o} + \lambda)) + (1/B) \sum_{j=1}^{p} \lambda_j M_j s \frac{\partial L_r^j}{\partial k} -$$

$$(W/sBL_o^2) \frac{\partial L_o}{\partial k} \quad .$$

If

$$\frac{\partial C}{\partial k} = 0$$

then,

$$\sum_{j=1}^{p}\lambda_j s \frac{\partial L_r^j}{\partial k}\{\frac{1}{B}\sum_{j=1}^{p}\lambda_j M_j E(T^j)+\frac{W}{B}(\frac{1}{sL_o}+\lambda)\}-\sum_{j=1}^{p}\lambda_j M_j s\frac{\partial L_r^j}{\partial k}+(W/sL_o^2)\frac{\partial L_o}{\partial k} = 0$$

In solving equations (5.1.2) - (5.1.4), we need to write $\partial L_r^j/\partial k$, $\partial L_o/\partial k$, $\partial L_r^j/\partial h$ and $\partial L_o/\partial h$ in terms of n, h and μ-k. The modified Brownian motion approximation for ARL when the process mean is at level μ and the process variance is equal to 1 is

$$L_r = ARL^+ = \begin{cases} \{1/(\mu-k)\sqrt{n}\}\{H+(1/2(\mu-k)\sqrt{n})(\mathrm{Exp}(-2(\mu-k)\sqrt{n}H)-1)\} & \mu-k\neq 0 \\ H^2 & \mu-k=0 \end{cases}$$

where $H=h\sqrt{n}+\Delta$ and Δ is calculated by equation (4.5).

Considering $\partial L_r/\partial k$, we have

$$\frac{\partial L_r}{\partial k} = 1/\{(\mu-k)^2\sqrt{n}\}\{(\mu-k)\frac{\partial H}{\partial k}+H\}+\{1/(2n(\mu-k)^4)((\mu-k)^2\{(\mathrm{Exp}(-2(\mu-k)\sqrt{n}H))$$

$$(-2\sqrt{n})((\mu-k)\frac{\partial H}{\partial k}-H)\}+\{\mathrm{Exp}(-2(\mu-k)\sqrt{n}H)-1\}\{2(\mu-k)\}\}$$

$$= \{1/((\mu-k)\sqrt{n})\}(1-\mathrm{Exp}(-2(\mu-k)\sqrt{n}H))\frac{\partial H}{\partial k}+\{H/((\mu-k)^2\sqrt{n})\}$$

$$(1+\mathrm{Exp}(-2(\mu-k)\sqrt{n}H))+\{1/(n(\mu-k)^3)\}(\mathrm{Exp}(-2(\mu-k)\sqrt{n}H)-1) \qquad \text{for } \mu-k\neq 0$$

and

$$\frac{\partial L_r}{\partial k} = 2H\frac{\partial H}{\partial k} \qquad , \text{ for } \mu-k=0$$

where

$$\frac{\partial H}{\partial k} = \frac{\partial \Delta}{\partial k} \qquad .$$

For $h > 2$

$$\frac{\partial \Delta}{\partial k} = -0.060216\sqrt{n} - 0.0566723 \cdot 2 \cdot (\mu-k)n \qquad ,$$

and for $h \leq 2$

$$\begin{aligned}\frac{\partial \Delta}{\partial k} = {} & -0.0970047\sqrt{n}-0.23697856(\mu-k)n-0.1057755(\mu-k)^2\sqrt{n}^3+ \\ & 0.00774535(\mu-k)^4\sqrt{n}^5-0.00180942(\mu-k)^5n^3+0.10060047hn+ \\ & 0.16534206(\mu-k)h\sqrt{n}^3+0.04356336(\mu-k)^2hn^2-0.03555026h^2\sqrt{n}^3- \\ & 0.01371234(\mu-k)^2h^2\sqrt{n}^5-0.01630622(\mu-k)h^3\sqrt{n}^5+ \\ & 0.00285219(\mu-k)^2h^6\sqrt{n}^9+0.00012296(\mu-k)^3h^6n^5 \qquad .\end{aligned}$$

Then if we consider $\partial L_r/\partial h$,

$$\frac{\partial L_r}{\partial h} = \{1/((\mu-k)\sqrt{n})\}\{1-\mathrm{Exp}(-2(\mu-k)\sqrt{n}H)\}(\sqrt{n}+\partial\Delta/\partial h) \qquad \text{for } \mu-k\neq 0$$

and

$$\frac{\partial L_r}{\partial h} = 2H(\sqrt{n}+\partial\Delta/\partial h) \qquad \text{for } \mu-k=0$$

For $h > 2$

$$\frac{\partial \Delta}{\partial h} = -0.00021597h^2\sqrt{n}^3 \qquad ,$$

and for $h \leq 2$

$$\frac{\partial\Delta}{\partial h} = -0.43708249\sqrt{n}+0.49294426hn-0.13494189h^2\sqrt{n}^3-0.10060047(\mu-k)n-$$

$$0.08267103(\mu-k)^2\sqrt{n}^3-0.01452112(\mu-k)^3n^2+0.07110052(\mu-k)h\sqrt{n}^3+$$

$$0.00914156(\mu-k)^3h\sqrt{n}^5+0.02445933(\mu-k)^2h^2\sqrt{n}^5-$$

$$0.00051858(\mu-k)^3h^5\sqrt{n}^9-0.00018444(\mu-k)^4h^5n^5 \quad .$$

5.2 Numerical Examples

The procedure described above can be used to obtain optimum cusum designs for any given set of cost and risk factors. Optimum designs for several examples were determined by this procedure and the results are shown in tables and graphs. A detailed study and discussion of these designs will now be given.

In the numerical examples used, the parameter W, the cost of looking for an assignable cause, is varied from \$25.0 to \$50.0 to \$75.0 to \$100. The parameter c, the inspection cost per unit, is varied from \$0.1 to \$0.15 to \$0.2 to \$0.25 to \$0.3 and the parameter b, the fixed cost per sample, is varied from \$0.50 to \$0.75 to \$1.0 to \$1.25 to \$1.50. The time to find an assignable cause D, is chosen to be equal to 2.0 and e is equal to 0.05. The shifts in mean δ_j and the corresponding values for λ_j and M_j are shown in Table 5.1. The process is assumed to start in-control with mean μ=0. When the cause A_j occurs, μ shifts to $\delta_j\sigma$. When δ_j=2, value for M_j is arbitrarily set equal to \$100.0. Other values of M_j are proportional to the increase in the percent of product outside a 3σ specification limit under the assumption of a normal distribution

Table 5.1 The reference set of λ_j for the prior distributions used in the study.

δ_j	M_j	λ_j		
		N.E.[1]	H.N.[2]	U.F.[3]
1.25	24.023	0.0012	0.0013	0.0011
1.75	65.983	0.0010	0.0011	0.0011
2.25	143.498	0.0008	0.0009	0.0011
2.75	255.413	0.0006	0.0006	0.0011
3.25	381.871	0.0005	0.0004	0.0011
3.75	497.786	0.0004	0.0003	
4.25	571.381	0.0003	0.0002	
4.75	613.261	0.0002	0.0001	
5.25	631.134	0.0001	0.0001	

1) N.E. denotes the negative exponential prior distribution.

2) H.N. denotes the half-normal prior distribution.

3) U.F. denotes the uniform prior distribution.

for the observations with mean μ and variance σ^2.

It is noted that the cost factors in this numerical study are taken from those in Duncan (8) and Goel and Wu (12). Findings based on this numerical study should not be interpreted to be true in general. However, because of the fairly large range of values studied, it seems to be appropriate to assume that they are representative of what is expected to occur in practical applications.

The following discussion uses one set of cost and risk factors as examples. Detailed results are shown in the appendix for reference.

5.3 Effect of Changes in the Prior Distributions of the Assignable Causes

Besides the negative exponential distribution, minimum cost designs for the uniform distribution and the half-normal distribution are also presented. The results are given in Table 5.2. These three distributions are kept roughly similar in certain respects. For all of them $\sum_j \lambda_j M_j = 2$. The mean shifts are also all roughly equal. For the negative exponential distribution the mean is $\delta_j = 2.47$; for the half-normal distribution the mean is $\delta_j = 2.3$; and for the uniform distribution the mean is $\delta_j = 2.25$. The sums of λ_j's for all three distributions are also close. For the negative exponential distribution $\lambda = 0.0051$; for the half-normal distribution $\lambda = 0.005$; and for the uniform distribution $\lambda = 0.0055$. The range of the uniform distribution differs from the others, but equality of ranges can not be attained with the same starting point and equal means. We chose to standardize the latter.

It is noted from Table 5.2 that the optimum design parameters and

Table 5.2 The optimum designs for negative exponential, half-normal and uniform prior distribution

W	b	c	P.D.	n	s	h	k	C
25.0	0.75	0.15	N	2	1.189	0.856	1.094	4.223
			H	2	1.316	0.850	1.019	3.759
			U	3	1.507	0.642	0.916	3.881
	1.50	0.30	N	2	1.687	0.788	1.071	4.955
			H	2	1.879	0.755	0.983	4.418
			U	2	2.020	0.727	0.905	4.506
50.0	0.75	0.15	N	2	1.183	0.949	1.109	4.396
			H	2	1.279	0.953	1.044	3.949
			U	3	1.499	0.707	0.963	4.058
	1.50	0.30	N	2	1.688	0.854	1.097	5.132
			H	2	1.869	0.849	1.021	4.617
			U	3	2.138	0.674	0.917	4.719
75.0	0.75	0.15	N	2	1.175	1.025	1.109	4.551
			H	2	1.279	1.031	1.050	4.116
			U	3	1.492	0.733	0.988	4.223
	1.50	0.30	N	2	1.685	0.904	1.108	5.293
			H	2	1.856	0.903	1.038	4.793
			U	3	2.134	0.691	0.946	4.887
100.0	0.75	0.15	N	2	1.166	1.244	1.014	4.673
			H	2	1.266	1.098	1.048	4.269
			U	3	1.485	0.754	1.005	4.381
	1.50	0.30	N	2	1.680	0.947	1.114	5.446
			H	2	1.843	0.950	1.047	4.956
			U	3	2.129	0.706	0.965	5.048

P.D. denotes the prior distribution.

N denotes the negative exponential prior distribution.

H denotes the half-normal prior distribution.

U denotes the uniform prior distribution.

the cost values are about the same for all three prior distribution. It seems that under the restrictions imposed we can expect that, for a monotonic decreasing distribution function, the minimum cost value is not affected significantly by the shape of the prior distribution.

5.4 Effects of Increase in the λ_j's

To evaluate the effect of increases in the λ_j's, we consider cases where the λ_j's have a negative exponential distribution with λ equal to 0.0051, 0.01 and 0.02. One set of cost and risk factors, W=\$50.0, b=\$1.0 and c=\$0.2 was used for illustration since other sets of cost and risk factors will give similar results (see Appendices).

Table 5.3 shows that as λ increases, the parameter k is almost unchanged, the parameter h decreases slightly, while s decreases noticeably. This means if the frequency of occurrences of assignable causes increases then samples should be taken more often and a slightly smaller decision limit should be used.

5.5 Effect of Changes in Cost Parameters

Effects of changes in cost parameters W, b and c on the minimum cost design are shown in Table 5.4 - Table 5.7. Table 5.4 shows that the only design parameter that is significantly affected by W is the decision limit h. As might have been expected, when the cost of overall adjusting the process increases a wider decision limit should be used.

The cost of maintaining the control chart is determined by b and c. The results shown in Table 5.5 - Table 5.7 show that increases of b or c lead to increases in s and decreases in h. The sample size n decreases

Table 5.3 Variation of s, h, k, n with respect to λ.

(For negative exponential prior distribution with W=$50.0, b=$1.0 and c=$0.2)

	λ		
	0.0051	0.01	0.02
s	1.372	0.961	0.695
h	0.906	0.887	0.859
k	1.106	1.144	1.194
n	2	2	2

Table 5.4 Variation of n, s, h, k with respect to W.

(For negative exponential prior distribution with λ=0.0051, b=\$1.0 and c=\$0.2)

	W				
	25.0	50.0	75.0	100.0	500.0
n	2	2	2	2	3
s	1.376	1.372	1.366	1.360	1.282
h	0.825	0.906	0.968	1.023	0.955
k	1.085	1.106	1.112	1.111	1.183

Table 5.5 Variation of n, s, h, k with respect to b and c (For negative exponential prior distribution with $\lambda=0.0051$ and W=$50.0)

	b=$0.5 c=$0.1	b=$0.75 c=$0.15	b=$1.0 c=$0.2	b=$1.25 c=$0.25	b=$1.50 c=$0.30
n	2	2	2	2	2
s	0.957	1.183	1.372	1.538	1.688
h	1.028	0.949	0.906	0.876	0.854
k	1.107	1.109	1.106	1.101	1.097

Table 5.6 Variation of n, s, h, k with respect to b.

(For negative exponential prior distribution with $\lambda=0.0051$, W=\$50.0 and c=\$0.2.)

	b			
	0.5	1.0	2.5	5.0
n	2	2	2	2
s	1.092	1.372	1.988	2.719
h	0.977	0.906	0.820	0.768
k	1.110	1.106	1.087	1.065

Table 5.7 Variation of n, s, h, k with respect to c.

(For negative exponential prior distribution with λ=0.0051, W=$50.0 and b=$1.0)

	c			
	0.2	1.0	2.5	5.0
n	2	1	1	1
s	1.372	1.696	2.348	3.172
h	0.906	1.244	1.095	0.933
k	1.106	1.134	1.077	0.999

as the sampling cost c increases. These results suggest that the frequency of sampling as well as sample size should be reduced and the smaller decicion limit should be used as the cost of sampling and plotting increase.

5.6 Optimum k Value

For the single assignable cause model, Kemp (1961) has shown that the ARL at the out-of-control level μ_1 is minimized for a fixed ARL at the in-control level $\mu_o=0$, when k is chosen to be equal to $\frac{1}{2}\mu_1$. It is found from previous discussions 5.2 - 5.5 that the k value for the multiple assignable cause model is essentially independent of the values of the cost factors. In this study, we considered k as a variable and used the procedure described in Chapter III to determine optimum design parameters n, h, k for each n. Values of the cost function C were calculated and the overall optimum design parameters, say (s^*,h^*,k^*,n^*), were the ones which gave the minimum cost value, say C^*. It appears that, through all sets of cost and risk favtors, the value of k^* is close to k', where $k'= \sum\lambda_j\delta_j/2\lambda$. Using the same procedure to determine the optimum design parameters (s',h',k',n') when k is fixed to be equal to k', the minimum cost value C' was calculated and compared with C^*. Results for the negative exponential prior distribution with $\lambda=0.0051$ are shown in Table 5.8. The table shows that the cost value C increases only negligibly when the k value is chosen to be equal to k', and the design parameters also agree closely as shown in Table 5.9. Therefore, in practice we can fix $k=k'$ and optimize over the other parameters s, h, and n. This will simplify the optimization procedure.

Table 5.8 Variation on the minimum cost value C with respect to optimum choice of k value. (For negative exponential prior distribution with λ=0.0051)

W	b	c	C'	C*
25.0	0.75	0.15	4.233	4.223
25.0	1.25	0.25	4.751	4.737
50.0	0.75	0.15	4.407	4.396
50.0	1.25	0.25	4.927	4.914
75.0	0.75	0.15	4.565	4.551
75.0	1.25	0.25	5.088	5.074
100.0	0.75	0.15	4.714	4.694
100.0	1.25	0.25	5.240	5.225

5.7 Use of A Single Cause Model

Duncan (1971) suggested that for the $\bar{X}$-chart a single cause model can be taken as an approximation to a multiple cause model by a certain matching criterion which was mentioned in Chapter II. To be more specific for the numerical examples studied, this matching criterion is (1) $\lambda = \Sigma\lambda_j = 0.0051$ (2) $\delta = \Sigma\delta_j\lambda_j/\lambda = 2.45$ (3) $M = \Sigma\lambda_j M_j/\lambda = 217.643$. The central reference value for k is used in this matched single cause model. The result of using these criteria for cusum charts for several sets of cost and risk factors are shown in Table 5.9. It is noted from Table 5.9 that the optimum design parameters yielded by the matched single cause model may differ from those for the multiple cause model, but the minimum cost values are about the same for both models.

Table 5.10 gives the comparisons of minimum cost values among the multiple cause model with k varied, the multiple cause model with fixed k and the matched single cause model. It shows that the multiple cause model with k fixed gives a better approximation than the matched single cause model. However, this difference is negligible in practice. In addition, it is simpler to determine the optimum design parameters with a single cause model. Therefore, in seeking the optimum design for a multiple cause model based on the minimum cost criterion we can obtain an approximate solution by finding the optimum design for the matched single cause model.

5.8 Effect of Changing n, s, h and k on the Values of the Loss-Cost Function C

Table 5.9 Comparisons of the optimum designs among the multiple cause model (M), the matched single cause model (S), and the multiple cause model with k fixed (M') (For negative exponential prior distribution with λ = 0.0051)

W	b	c	Model	n	s	h	k	C
25.0	0.75	0.15	M	2	1.189	0.856	1.094	4.223
			S	2	1.372	0.484	1.225	4.298
			M'	2	1.180	0.717	1.225	4.233
	1.25	0.25	M	2	1.539	0.804	1.077	4.737
			S	1	1.440	0.794	1.225	4.852
			M'	2	1.525	0.654	1.225	4.751
50.0	0.75	0.15	M	2	1.183	0.949	1.109	4.396
			S	2	1.269	0.642	1.225	4.449
			M'	2	1.175	0.818	1.225	4.407
	1.25	0.25	M	2	1.538	0 876	1.101	4.914
			S	2	1.699	0.524	1.225	5.003
			M'	2	1.527	0.743	1.225	4.927
75.0	0.75	0.15	M	2	1.175	1.025	1.109	4.551
			S	2	1.230	0.739	1.225	4.597
			M'	2	1.168	0.886	1.225	4.565
	1.25	0.25	M	2	1.534	0.931	1.111	5.074
			S	2	1.655	0.618	1.225	5.146
			M'	2	1.524	0.803	1.225	5.088
100.0	0.75	0.15	M	2	1.168	1.099	1.098	4.694
			S	2	1.201	0.807	1.225	4.742
			M'	2	1.161	0.936	1.225	4.714
	1.25	0.25	M	2	1.528	0.978	1.114	5.225
			S	2	1.619	0.685	1.225	5.288
			M'	2	1.519	0.849	1.225	5.240

Table 5.10 Comparisons of minimum cost values among multiple cause model with k varied, C*, multiple cause model with k=k'=1.225, C', and matched single cause model, C_s.

W	b	c	C*	C'	C_s
25.0	0.75	0.15	4.223	4.233	4.298
25.0	1.25	0.25	4.737	4.751	4.852
50.0	0.75	0.15	4.396	4.407	4.449
50.0	1.25	0.25	4.914	4.927	5.003
75.0	0.75	0.15	4.551	4.565	4.597
75.0	1.25	0.25	5.074	5.088	5.146
100.0	0.75	0.15	4.694	4.714	4.742
100.0	1.25	0.25	5.225	5.240	5.288

We now investigate the cost surface near the optimum value. Table 5.11, Table 5.12 and Table 5.13 show respectively that changing s, h and k within ±0.3 of the optimum value have very little effect on the cost value C'. Investigations were also made on the cost value C. The same results were obtained. We may note here that only one parameter is varied at a time and the other parameters are held fixed. We conclude that the cost function is fairly flat on the surface (s,h,k). This gives some flexibility in selecting the design in practice where the exact optimum values can not always be attained. However, Table 5.14 shows that the sample size n has a greater influence on the cost value C'. The results shown on the table suggest that the optimum sample size should be taken when it is possible.

5.9 Effect of Changes in δ_j

The results discussed on the previous sections are based on the set of λ_j's and δ_j's listed in Table 5.1 which covers a wide range of δ_j. If we only consider a few small shifts as indicated in Table 5.15, it would be interesting to see how the choice of δ_j affects the design parameters and especially the nature of the loss-cost function. Table 5.16 gives the optimum design for the reference set tabulated in Table 5.15. It is noted that the sample size increases greatly from the optimum sample size obtained by using values in Table 5.1. This agrees with what has been found for the single cause model. The smaller the shift considered the larger the sample size needed. The other design parameters are also far from the optimum design parameters based on Table 5.1. Therefore, the choice of δ_j is important in seeking an optimum design.

Table 5.11 Effect of changes in the reference value on C' in the neighborhood of the optimum value.

	b=\$0.75, c=\$0.15				b=\$1.25, c=\$0.25			
	W=\$25.0,	W=\$50.0,	W=\$75.0,	W=\$100.0	W=\$25.0,	W=\$50.0,	W=\$75.0,	W=\$100.0
k*-0.3	4.357	4.588	4.786	4.967	4.870	5.103	5.307	5.494
k*-0.1	4.238	4.417	4.578	4.729	4.754	4.935	5.100	5.255
k*	4.233	4.407	4.565	4.714	4.751	4.927	5.088	5.240
k*+0.1	4.260	4.438	4.600	4.752	4.783	4.962	5.126	5.281
k*+0.3	4.418	4.631	4.818	4.990	4.960	5.174	5.364	5.540

k* denotes the optimum value of the reference value k.

The entries are the C' values.

Table 5.12 Effect of changes in the decision limit on C' in the neighborhood of the optimum value.

	b=$0.75, c=$0.15				b=$1.25, c=$0.25			
	W=$25.0,	W=$50.0,	W=$75.0,	W=$100.0	W=$25.0,	W=$50.0,	W=$75.0,	W=$100.0
h*-0.3	4.349	4.553	4.731	4.894	4.877	5.086	5.269	5.439
h*-0.1	4.245	4.422	4.582	4.733	4.764	4.943	5.107	5.261
h*	4.233	4.407	4.565	4.714	4.751	4.927	5.088	5.240
h*+0.1	4.241	4.415	4.572	4.720	4.761	4.937	5.098	5.250
h*+0.3	4.301	4.476	4.630	4.775	4.834	5.014	5.176	5.326

h* denotes the optimum value of the decision limit h.

The entries are the C' values.

Table 5.13 Effect of changes in the sampling interval on C' in the neighborhood of the optimum value.

	b=$0.75, c=$0.15				b=$1.25, c=$0.25			
	W=$25.0,	W=$50.0,	W=$75.0,	W=$100.0	W=$25.0,	W=$50.0,	W=$75.0,	W=$100.0
s*-0.3	4.316	4.493	4.653	4.805	4.809	4.986	5.148	5.301
s*-0.1	4.241	4.415	4.573	4.722	4.757	4.933	5.093	5.246
s*	4.233	4.407	4.565	4.714	4.751	4.927	5.088	5.240
s*+0.1	4.239	4.414	4.572	4.721	4.756	4.932	5.093	5.245
s*+0.3	4.282	4.458	4.617	4.767	4.790	4.966	5.128	5.281

s* denotes the optimum value of the sampling interval s.

The entries are the C' values.

Table 5.14 Effect of changes in the sample size on C' in the neighborhood of the optimum value.

	b=$0.75, c=$0.15				b=$1.25, c=$0.25			
n	W=$25.0,	W=$50.0,	W=$75.0,	W=$100.0	W=$25.0,	W=$50.0,	W=$75.0,	W=$100.0
1	4.521	5.043	5.482	5.873	4.910	5.410	5.836	6.220
2	4.233	4.407	4.565	4.714	4.751	4.927	5.088	5.240
3	4.595	4.737	4.871	5.000	5.229	5.368	5.504	5.637
4	5.240	5.340	5.429	5.508	6.064	6.173	6.277	6.375
5	6.156	6.152	6.120	5.101	7.273	7.312	7.334	7.342

Table 5.15 The reference set of λ_j for negative exponential distribution. (With 4 assignable causes considered)

δ_j	λ_j	M_j
0.75	0.0070	6.149
1.25	0.0055	24.023
1.75	0.0043	65.983
2.25	0.0033	143.498

Table 5.16 The optimum designs for the reference set in Table 5.15

W	b	c	λ	n	s	h	k	C
25.0	0.75	0.15	0.02	4	1.722	0.548	0.637	4.354
			0.01	4	2.321	0.566	0.615	2.675
	1.25	0.25	0.02	4	2.276	0.515	0.601	4.805
			0.01	4	3.068	0.529	0.579	3.010
50.0	0.75	0.15	0.02	5	1.825	0.541	0.651	4.902
			0.01	5	2.437	0.565	0.621	2.976
	1.25	0.25	0.02	4	2.262	0.551	0.666	5.367
			0.01	5	3.210	0.531	0.597	3.332
75.0	0.75	0.15	0.02	5	1.829	0.559	0.684	5.400
			0.01	5	2.419	0.595	0.642	3.250
	1.25	0.25	0.02	5	2.421	0.522	0.665	5.872
			0.01	5	3.201	0.553	0.623	3.608
100.0	0.75	0.15	0.02	5	1.839	0.567	0.713	5.881
			0.01	5	2.406	0.617	0.657	3.514
	1.25	0.25	0.02	5	2.439	0.527	0.698	6.351
			0.01	5	3.195	0.570	0.641	3.873

We further studied the nature of the loss-cost function for values in Table 5.15. It was found that the function behaves in a similar manner as that for Table 5.1 and the results given in the previous sections are still applicable.

5.10 Behavior of C as h approaches infinity

Duncan (1971) tabulated values of the loss-cost function C for values of h from 0.4 to 6.9 in steps of 0.5 and values of s from 0.1 to 8.1 in steps of 0.5. Through this tabulation, he found that for some cost parameters the local minimum obtained by a search procedure is higher than the loss-cost at values of h=5.9 and s=8.1. He then studied the limiting behavior of C as $h\to\infty$. It was found that in many cases the local minimum is greater than the loss-cost at s=8.1 and $h\to\infty$ with n and s fixed. He argued that the problem occurred because only one cause was allowed to affect the process mean at one time. Thus, after a small shift, no larger shift can occur until the mean is returned to the control value. He further conjectured that as the model became more realistic these competitors of the local minimum in the region of interest would disappear.

We would also like to investigate the limiting behavior of the function C developed in Chapter III as h approaches infinity with s, k and n constant. As h approaches infinity, using the Brownian motion approximation for ARL, all the L_r^j approach infinity, but for $j\neq 1$

$$\lim_{h\to\infty} L_r^j/L_r^1 = \lim_{h\to\infty}\{(\mu_1-k)/(\mu_j-k)\}\{1-\mathrm{Exp}\ -2(\mu_j-k)h\}/\{1-\mathrm{Exp}\ -2(\mu_1-k)h\} \quad .$$

If $\mu_j - k > 0$ for all $j=1,2,\ldots,p$ then

$$\lim_{h\to\infty} L_r^{\,j}/L_r^{\,1} = (\mu_1-k)/(\mu_j-k)$$

so that

$$\lim_{h\to\infty} C = \lim_{h\to\infty} \left\{ \frac{\sum_{j=1}^{p} \lambda_j M_j E(T^j)/L_r^{\,1}}{B/L_r^{\,1}} + \frac{W/L_r^{\,1}}{B/L_r^{\,1}} \left(\frac{1}{L_o s} + \lambda\right) + \frac{b+cn}{s} \right\}$$

$$= \frac{\sum_{j=1}^{p} \lambda_j M_j/(\mu_j-k)}{\sum_{j=1}^{p} \lambda_j/(\mu_j-k)} + \frac{b+cn}{s} \quad .$$

As $h\to\infty$ the ratio of the ARLs $L_r^{\,j}/L_r^{\,1}$ is constant. The limiting value of C, apart from the cost due to maintaining the control chart, is the weighted mean of the cost M_j. The weight depends on the corresponding ARL and the probability of occurrence for cause A_j.

If $\mu_1 - k < 0$, then

$$\lim_{h\to\infty} L_r^{\,j}/L_r^{\,1} = 0$$

so that

$$\lim_{h\to\infty} C = M_1 + (b+cn)/s \quad .$$

As h approaches infinity the ARL of the smallest shift goes to infinity faster than that of larger shifts and the ratio $L_r^{\,j}/L_r^{\,1}$ becomes zero. The limiting value of C, apart from the cost due to maintaining the control chart, depends entirely on the cost M_1 corresponding to the smallest shift.

Table 5.17 shows that for the reference set of values in Table 5.1

Table 5.17 Tabulation of the loss cost function C. (Negative exponential distribution with values of λ_j listed in Table 5.1 and W=\$75.0, b=\$1.25, c=\$0.25, k=1.111, n=2)

h	S							
	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
0.3	9.960	6.092	5.957	6.349	6.912	7.549	8.223	8.917
0.9	6.858	5.078	5.364	5.935	6.592	7.283	7.987	8.693
1.5	6.591	5.264	5.789	6.557	7.391	8.243	9.095	9.939
2.1	6.767	5.826	6.701	7.795	8.933	10.069	11.186	12.277
2.7	6.964	6.393	7.605	9.005	10.423	11.813	13.161	14.463
3.3	7.163	6.956	8.491	10.178	11.850	13.465	15.014	16.495
3.9	7.363	7.513	9.356	11.310	13.212	15.027	16.748	18.377
4.5	7.562	8.062	10.198	12.399	14.509	16.498	18.367	20.119
5.1	7.760	8.600	11.016	13.445	15.742	17.885	19.878	21.732
5.7	7.957	9.129	11.809	14.449	16.914	19.191	21.291	23.228
6.3	8.153	9.648	12.578	15.413	18.029	20.423	22.611	24.617
6.9	8.346	10.156	13.323	16.337	19.089	21.584	23.848	25.908
7.5	8.539	10.653	14.045	17.224	20.097	22.680	25.008	27.112

and for $n=2$, $k=1.111$ the loss-cost function C has a local minimum in the neighborhood of $s=1.5$, $h=0.9$. It appears from the table that this local minimum is the absolute minimum in the region of interest. For this set of reference values $\mu_j-k>0$ for all $j=1,2,\ldots,9$ and the limiting value of C depends mostly on the weighted mean of M_j weighted by λ_j/μ_j-k. When $n=2$, $s=7.5$, $k=1.111$, $b=\$1.25$ and $c=\$0.25$ the limiting value of C as $h\to\infty$ is equal to \$65.87 which is larger than the local minimum. We may notice from previous sections that the value of C is unlikely to be greater than \$65.87 when the cost parameters are within the range of interest unless λ becomes much larger than 0.02. But in practice λ is usually very small. The local minimum is then smaller than the limiting value and hence there is no competitor of the local minimum in the region of interest when $\mu_j-k>0$ for all $j=1,2,\ldots,p$.

When a smaller shift in the process mean is considered so that $\mu_1-k<0$ and the corresponding M_1 is relatively small, the limiting value of C becomes $M_1+(b+cn)/s$. We suspect in some cases there exist points in the region of interest that yield a lower loss-cost than that of the local minimum.

Consider the cost and probability parameters listed in Table 5.18 which includes shift as small as 0.75σ. Values of the loss-cost function C for values of h from 0.3 to 7.5 in steps of 0.6, values of s from 0.5 to 7.5 in steps of 1.0 and $n=2$, $k=1.0435$ are tabulated in Table 5.19. It show that C has a local minimum in the neighborhood of $s=1.5$ and $h=0.9$. It also shows that the function has a ridge as h increases. A detailed analysis of the costs indicated why this occurred.

Table 5.18 The reference set of λ_j for negative exponential distribution (with 10 assignable causes considered)

δ_j	λ_j	M_j
0.75	0.00246	6.149
1.25	0.00192	24.023
1.75	0.00150	65.983
2.25	0.00116	143.498
2.75	0.00090	255.413
3.25	0.00070	381.871
3.75	0.00050	497.786
4.25	0.00042	571.381
4.75	0.00034	613.261
5.25	0.00026	631.134

Table 5.19 Tabulation of the loss-cost function C. (For negative exponential distribution with values of λ_j listed in Table 5.18 and W=\$75.0, b=\$1.25, c=\$0.25, k=1.0435, n=2)

h	s 0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
0.3	12.696	8.792	9.030	9.824	10.779	11.788	12.811	13.803
0.9	8.902	7.457	8.074	8.917	9.789	10.641	11.459	12.238
1.5	8.528	7.593	8.227	9.021	9.773	10.459	11.080	11.640
2.1	8.797	8.119	8.862	9.585	10.202	10.720	11.157	11.530
2.7	9.094	8.454	8.986	9.445	9.802	10.081	10.304	10.485
3.3	9.369	8.484	8.676	8.853	8.986	9.087	9.165	9.227
3.9	9.580	8.284	8.181	8.167	8.168	8.174	8.179	8.185
4.5	9.701	8.006	7.720	7.605	7.544	7.506	7.480	7.461
5.1	9.741	7.760	7.379	7.218	7.129	7.073	7.034	7.006
5.7	9.734	7.584	7.158	6.976	6.876	6.811	6.767	6.735
6.3	9.712	7.472	7.025	6.833	6.727	6.659	6.613	6.578
6.9	9.691	7.404	6.947	6.751	6.643	6.573	6.526	6.490
7.5	9.675	7.365	6.903	6.705	6.595	6.525	6.477	6.441
∞	9.649	7.316	6.849	6.649	6.538	6.467	6.418	6.382

The cost associated with failing to detect a shift in the process mean as soon as it occurs and cost associated with out-of-control signals for s=1.24 and for various h values are shown in Table 5.20. It is noted that the total cost is dominated by the cost of failing to detect an assignable cause as soon as it occurs. As h increases this part of the cost first increases, but later decreases producing a ridge in the loss-cost function. The reason can be seen from Table 5.21 that as h increases the contributions of the assignable causes producing the smaller shifts tend to dominate the total contribution. In the limit, as can be seen from the limiting value of C, $M_1+(b+cn)/s$, the assignable cause producing the shift in the process mean of 0.75σ contributes almost 100% of this cost component.

The model assumes that an assignable cause that occurs will continue undisturbed by any other causes until it is detected by the control chart. In reality the combination of causes will eventually occur and produce a large shift in the process mean which would continue until detected. With this extension of the model, we may conjecture that the local minimum would be the overall minimum in the region of interest.

Table 5.20 An analysis of costs with h = 0.5(1.0)7.5 (For s=1.24 and values for W, λ_j, b, c, k, n are the same as those for Table 5.19)

h	$\frac{1}{B}\sum_{j=1}^{p}\lambda_j E(T^j)M_j$	$\frac{W}{BsL_o}$	$\frac{1}{B}W\lambda$	C
0.5	5.063	0.892	0.725	8.091
1.5	5.361	0.013	0.667	7.453
2.5	6.313	d	0.528	8.253
3.5	6.726	d	0.318	8.455
4.5	6.614	d	0.140	8.165
5.5	6.392	d	0.050	7.853
6.5	6.254	d	0.002	7.682
7.5	6.191	d	d	7.607

[d]practically zero

Table 5.21 An analysis of the cost $\frac{1}{B}\sum_{j=1}^{10}\lambda_j E(T^j)M_j$ with h=0.5(1.0)7.5, and δ_j=0.75(0.5)5.25 (Values for λ_j, W, b, c, k, n, s are the same as those for Table 5.19)

δ_j	h 0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
0.75	0.140	0.556	1.653	3.402	4.920	5.700	5.999	6.100
1.25	0.210	0.374	0.469	0.392	0.222	0.098	0.038	0.014
1.75	0.322	0.439	0.468	0.354	0.187	0.079	0.030	0.011
2.25	0.465	0.560	0.562	0.409	0.211	0.087	0.032	0.012
2.75	0.605	0.664	0.642	0.455	0.231	0.094	0.035	0.012
3.25	0.696	0.705	0.662	0.461	0.230	0.093	0.034	0.012
3.75	0.664	0.619	0.569	0.390	0.193	0.077	0.028	0.010
4.25	0.681	0.576	0.520	0.351	0.172	0.068	0.024	0.008
4.75	0.661	0.488	0.435	0.291	0.141	0.055	0.020	0.007
5.25	0.618	0.379	0.334	0.221	0.106	0.041	0.015	0.005

CHAPTER VI

CONCLUSION

This research has provided a procedure for the design of cumulative sum charts based a minimum cost criterion when there are multiple assignable causes. A cost model has been developed that relates the design parameters of a cusum chart and the loss and risk factors of the process. Optimum designs were obtained for various sets of cost and risk factors by use of Brown's method. This optimization procedure was found to be fast and easy to use especially when a modified Brownian motion approximation was used for calculating ARLs in the cost model.

The results of the numerical study indicate that the optimum designs were almost the same for exponential, half-normal and uniform prior distributions of the assignable causes. However, general increases in the frequency of occurrences for assignable causes lead to decreases in the sampling interval s and the decision limit h. Also, as the cost of adjusting the process increases a wider decision limit should be used. Samples should be taken less often and the decision limit should be reduced if the cost of sampling and plotting increases. These results agree with what was found for the single cause model in Chiu (1974) and Goel and Wu (1971).

The reference value k seems to be insensitive to the cost and risk factors. The optimum k value was found to be close to 1/2 of the mean shift, i.e. $\Sigma\lambda_j\delta_j/2\lambda$. When p=1, $\Sigma\lambda_j\delta_j/2\lambda = \frac{1}{2}\delta$, and this "central reference value" was shown by Kemp (1961) to be the optimum k value for the single cause model.

The magnitude of the mean shift δ_j was found to have great effect on the optimum solution, but the nature of the loss-cost function was not influenced by the choice of δ_j. Thus, in seeking optimum designs the values of δ_j have to be determined carefully.

In testing the sensitivity of C to changes in s, h, k and n, one parameter was varied while the others were held fixed. It was found that the values of C have noticeable increases above the optimum value only when the sample size n changes; otherwise the cost surface is rather flat in the neighborhood of the optimum value.

Duncan (1971) found that the multiple cause model can be well approximated by a matched single cause model when the $\bar{X}$-chart was used. We found that this is also true for cusum chart procedure. Then in seeking an economic optimum design for a multiple cause situation we can obtain a good approximation by finding the optimum design for a matched single cause model. Models like those proposed by Chiu (1974) or Goel and Wu (1971) for the single cause model become applicable.

The investigation on the limiting property of C when h approaches infinity with s, k and n remaining constant reveals that the function C has a ridge in some cases. It seems that when the limiting value of C is significantly less than the value of the loss-cost at the local minimum there exist some points in the region of interest that yield a lower loss-cost than this local minimum.

There are some recommendations for the future work.

(1) The assumption that only one of the assignable causes can occur before detection simplifies the analysis but is not realistic.

The extension of the model which allows a combination of causes would be desirable. With this extension of the model, we would expect as noted in section 5.10 that the local minimum would be the only minimum in the region of interest.

(2) The cost of looking for an assignable cause when an out-of-control signal appears is assumed to be equal for all the assignable causes in this study. The time between the out-of-control signal and the removal of the assignable cause is assumed to be independent of the assignable causes. Models which include different values of cost and time for various assignable causes would be more realistic.

(3) Chiu (1974) found that the shutdown time for a search for assignable causes is rather important for the economic design unless the process continues in production during a search. The extension of the model which considers the case where the process is shut down during a search for assignable causes would be useful.

(4) The modified Brownian motion approximation for the ARL used in this thesis was derived under the assumption of normality. For non-normal observations the exact values for the ARL are not available. Detailed tables of ARL values for non-normal distributions would be useful for robustness studies and for deriving approximations like the modified Brownian motion approximation. If an approximation could be developed then the optimum design for the observations that are not normally distributed could be found by use of the same optimization procedure indicated in Chapter V.

(5) For the single cause model, Chiu (1974) found a simplified design which gives design parameters close to optimum. A simplified scheme for the multiple assignable cause model that gives an approximate solution to the optimum design would be useful for practical applications.

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Appendix 1 Optimum design for W = $25.0

b	c	P.D.	λ	n	s	h	k	C
0.5	0.1	N.E.	0.02	2	0.487	0.865	1.185	12.460
		N.E.	0.01	2	0.676	0.889	1.139	6.979
		N.E.	0.0051	2	0.967	0.908	1.104	3.896
		H.N.	0.005	2	1.065	0.908	1.034	3.464
		U.F.	0.0055	2	1.226	0.692	0.943	3.587
0.75	0.15	N.E.	0.02	2	0.599	0.819	1.178	13.107
		N.E.	0.01	2	0.831	0.840	1.131	7.446
		N.E.	0.0051	2	1.189	0.856	1.094	4.223
		H.N.	0.005	2	1.316	0.850	1.019	3.759
		U.F.	0.0055	2	1.507	0.674	0.916	3.881
1.0	0.2	N.E.	0.02	1	0.684	1.175	1.155	13.599
		N.E.	0.01	2	0.961	0.810	1.124	7.837
		N.E.	0.0051	2	1.376	0.825	1.085	4.497
		H.N.	0.005	2	1.527	0.816	1.005	4.007
		U.F.	0.0055	2	1.635	0.758	0.951	4.124
1.25	0.25	N.E.	0.02	1	0.777	1.123	1.131	14.011
		N.E.	0.01	2	1.076	0.790	1.118	8.182
		N.E.	0.0051	2	1.539	0.804	1.077	4.737
		H.N.	0.005	2	1.713	0.792	0.993	4.223
		U.F.	0.0055	2	1.842	0.739	0.927	4.325
1.50	0.3	N.E.	0.02	1	0.861	1.085	1.109	14.378
		N.E.	0.01	1	1.186	1.092	1.086	8.451
		N.E.	0.0051	2	1.687	0.788	1.071	4.955
		H.N.	0.005	2	1.879	0.775	0.983	4.418
		U.F.	0.0055	2	2.029	0.727	0.905	4.506

P.D. denotes the prior distribution.

N.E. denotes the negative exponential prior distribution.

H.N. denotes the half-normal prior distribution.

U.F. denotes the uniform prior distribution.

Appendix 2 Optimum design for W = $50.0

b	c	P.D.	λ	n	s	h	k	C
0.5	0.1	N.E.	0.02	2	0.486	0.962	1.195	13.009
		N.E.	0.01	2	0.672	1.002	1.144	7.282
		N.E.	0.0051	2	0.957	1.028	1.107	4.064
		H.N.	0.005	2	1.042	1.033	1.049	3.647
		U.F.	0.0055	3	1.216	0.733	0.987	3.763
0.75	0.15	N.E.	0.02	2	0.599	0.897	1.196	13.659
		N.E.	0.01	2	0.829	0.929	1.146	7.754
		N.E.	0.0051	2	1.183	0.949	1.109	4.396
		H.N.	0.005	2	1.297	0.953	1.044	3.949
		U.F.	0.0055	3	1.499	0.707	0.963	4.058
1.0	0.2	N.E.	0.02	2	0.695	0.859	1.194	14.202
		N.E.	0.01	2	0.961	0.887	1.144	8.147
		N.E.	0.0051	2	1.372	0.906	1.106	4.672
		H.N.	0.005	2	1.511	0.906	1.036	4.201
		U.F.	0.0055	3	1.739	0.692	0.945	4.306
1.25	0.25	N.E.	0.02	2	0.779	0.833	1.191	14.679
		N.E.	0.01	2	1.078	0.858	1.140	8.492
		N.E.	0.0051	2	1.538	0.876	1.101	4.914
		H.N.	0.005	2	1.699	0.873	1.028	4.420
		U.F.	0.0055	3	1.949	0.681	0.929	4.523
1.50	0.3	N.E.	0.02	2	0.855	0.814	1.187	15.109
		N.E.	0.01	2	1.182	0.837	1.136	8.803
		N.E.	0.0051	2	1.688	0.854	1.097	5.132
		H.N.	0.005	2	1.869	0.849	1.021	4.617
		U.F.	0.0055	3	2.137	0.674	0.917	4.719

P.D. denotes the prior distribution.

N.E. denotes the negative exponential prior distribution.

H.N. denotes the half-normal prior distribution.

U.F. denotes the uniform prior distribution.

Appendix 3 Optimum design for W = $75.0

b	c	P.D.	λ	n	s	h	k	C
0.5	0.1	N.E.	0.02	2	0.485	1.039	1.191	13.528
		N.E.	0.01	2	0.667	1.108	1.124	7.557
		N.E.	0.0051	2	0.948	1.163	1.073	4.208
		H.N.	0.005	2	1.025	1.134	1.042	3.805
		U.F.	0.0055	3	1.207	0.764	1.009	3.925
0.75	0.15	N.E.	0.02	2	0.599	0.955	1.203	14.185
		N.E.	0.01	2	0.826	0.998	1.147	8.039
		N.E.	0.0051	2	1.175	1.025	1.109	4.551
		H.N.	0.005	2	1.279	1.031	1.050	4.116
		U.F.	0.0055	3	1.492	0.733	0.988	4.223
1.0	0.2	N.E.	0.02	2	0.969	0.908	1.205	14.730
		N.E.	0.01	2	0.959	0.945	1.150	8.436
		N.E.	0.0051	2	1.366	0.968	1.112	4.830
		H.N.	0.005	2	1.495	0.973	1.048	4.372
		U.F.	0.0055	3	1.732	0.713	0.972	4.472
1.25	0.25	N.E.	0.02	2	0.781	0.877	1.204	15.208
		N.E.	0.01	2	1.076	0.909	1.149	8.783
		N.E.	0.0051	2	1.534	0.931	1.111	5.074
		H.N.	0.005	2	1.685	0.934	1.043	4.594
		U.F.	0.0055	3	1.944	0.700	0.958	4.691
1.50	0.3	N.E.	0.02	2	0.857	0.854	1.202	15.637
		N.E.	0.01	2	1.182	0.884	1.148	9.095
		N.E.	0.0051	2	1.685	0.904	1.108	5.293
		H.N.	0.005	2	1.856	0.904	1.038	4.793
		U.F	0.0055	3	2.134	0.691	0.946	4.887

P.D. denotes the prior distribution.

N.E. denotes the negative exponential prior distribution.

H.N. denotes the half-normal prior distribution.

U.F. denotes the uniform prior distribution.

Appendix 4 Optimum design for W = $100.0

b	c	P.D.	λ	n	s	h	k	C
0.75	0.15	N.E.	0.02	2	0.598	1.004	1.205	14.696
		N.E.	0.01	2	0.822	1.063	1.140	8.311
		N.E.	0.0051	2	1.166	1.244	1.014	4.673
		H.N.	0.005	2	1.266	1.098	1.048	4.269
		U.F.	0.0055	3	1.485	0 759	1.005	4.381
1.0	0.2	N.E.	0.02	2	0.696	0.949	1.211	15.245
		N.E.	0.01	2	0.957	0.995	1.151	8.713
		N.E.	0.0051	2	1.360	1.023	1.111	4.979
		H.N.	0.005	2	1.481	1.029	1.052	4.530
		U.F.	0.0055	3	1.726	0.732	0.989	4.632
1.25	0.25	N.E.	0.02	2	0.781	0.912	1.212	15.723
		N.E.	0.01	2	1.075	0.953	1.154	9.062
		N.E.	0.0051	2	1.528	0.978	1.114	5.225
		H.N.	0.005	2	1.671	0.984	1.050	4.755
		U.F.	0.0055	3	1.938	0.717	0.976	4.851
1.50	0.30	N.E.	0.02	2	0.859	0.886	1.212	16.152
		N.E.	0.01	2	1.181	0.923	1.154	9.375
		N.E.	0.0051	2	1.680	0.947	1.114	5.446
		H.N.	0.005	2	1.843	0.950	1.047	4.956
		U.F.	0.0055	3	2.129	0.706	0.965	5.048

P.D. denotes the prior distribution.

N.E. denotes the negative exponential prior distribution.

H.N. denotes the half-normal prior distribution.

U.F. denotes the uniform prior distribution.

VITA

Margaretha Mei-Ing Hsu was born on February 23, 1946 in the Republic of China. She graduated from the Provincial First Girls' High School in Taipei in June, 1964.

She began her undergraduate studies in Fu-Jen Catholic University and received a B.S. in mathematics in June, 1968.

In September, 1969, she came to the United States and began her graduate work at the Oakland University, Michigan. She received an M.A. in mathematics in December, 1970. She continued her graduate study in mathematics at the Mathematics Department, Virginia Polytechnic Institute and State University until September, 1973, she entered the Statistics Department at V.P.I. & S.U.. She completed the requirements for a PhD in statistics in September, 1978.

Margaretha M. Hsu

ECONOMICALLY OPTIMUM DESIGN OF CUSUM CHARTS

WHEN THERE IS A MULTIPLICITY OF ASSIGNABLE CAUSES

by

Margaretha Mei-Ing Hsu

(ABSTRACT)

This study is concerned with the design of cumulative sum charts based on a minimum cost criterion when there are multiple assignable causes occurring randomly, but with known effect. A cost model is developed that relates the design parameters (i.e. sampling interval, decision limit, reference value and sample size) of a cusum chart and the cost and risk factors of the process to the long run average loss-cost per hour for the process. Optimum designs for various sets of cost and risk factors are found by minimizing the long run average loss-cost per hour of the process with respect to the design parameters of a cusum chart. Optimization is accomplished by use of Brown's method. A modified Brownian motion approximation is used for calculating ARLs in the cost model.

The nature of the loss-cost function is investigated numerically. The effects of changes in the design parameters and in the cost and risk factors are also studied. An investigation of the limiting behavior of the loss-cost function as the decision limit approaches infinity reveals that in some cases there exist some points that yield a lower loss-cost than that of the local minimum obtained by Brown's method. It is conjectured that if the model is extended to include

more realistic assumption about the occurrence of assignable causes then only the local minimum solutions will remain.

This paper also shows that the multiple assignable cause model can be well approximated by a matched single cause model. Then in practice it may be sufficient to find the optimum design for the matched single cause model.