FUNCTIONAL OBSERVATIONS:

A BIOSTATISTICAL STUDY OF GROWTH

by

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TABLE OF CONTENTS

Chapter		Page
I	INTRODUCTION	1
	1.1 Functional Observations	1
	1.2 Specification of Functional Form	4
	1.3 Properties and Uses of Functional Observations	5
II	REVIEW OF LITERATURE	8
	2.1 Review of Literature	8
III	RESEARCH PLAN	11
	3.1 Introduction	11
	3.2 Biological Material and Methods	11
IV	SPECIFICATION OF THE FUNCTIONAL FORM	14
	4.1 General Development and Evaluation of Parametric Families	14
	4.2 Defining the Region of Juvenile Growth . .	33
	4.3 Comparison of Juvenile Growth Between Two Individuals	41
V	MANIPULATION OF THE PARAMETRIC FAMILY, DERIVATION OF GROWTH INFORMATION AND THE PROBABILITY STRUCTURE OF A FUNCTIONAL OBSERVATION	44
	5.1 Introduction	44

TABLE OF CONTENTS (cont.)

Chapter — Page

5.2 Growth Rate 47

5.3 Weight and Age Relationships 54

5.4 Growth Proportionality 56

5.5 Alternative Representations 58

5.6 Probability Structure of a Functional Observation 62

5.7 Discussion of Appropriate Statistical Tests to Compare Realizations of the Functional Observations 88

VI STATISTICAL GENETICS OF GROWTH 90

6.1 Introduction 90

6.2 Functional Observations of the A-C Sample of Chickens 94

6.3 Structural Relationships between Male and Female Stochastic Parameters 111

VII SUMMARY AND SUGGESTED FURTHER RESEARCH 118

7.1 Summary 118

7.2 Suggestions for Further Research 122

ACKNOWLEDGEMENTS 124

BIBLIOGRAPHY 126

VITA 129

TABLES OF CONTENTS (cont.)

	Page
APPENDICES	131
APPENDIX A	132
APPENDIX B	146
APPENDIX C	156
APPENDIX D	168
APPENDIX E	181
APPENDIX F	186

CHAPTER I

INTRODUCTION

1.1 Functional Observations

The purpose of this research is to investigate the usefulness of a concept which we call the functional observation and to relate this concept to a statistical-genetic study of juvenile avian growth.

A functional observation is a mathematical relationship between two or more mathematical variables observed as an entity on each individual of a sample. Thus $Y_i = f_i(x)$ is a function observed for the ith individual relating the mathematical variables y and x for this individual.

Examples of functional observations are becoming more common with the advent of continuously recording devices. With these devices one or more variables are graphically recorded as tracings and the entities observed are parametric functions of a time variable. Thus a recording of deflection, as a function of applied force, for a sample of structural timbers; individual hysteresis loops for a sample of test specimens; salt penetration in brine-soaked hams as a function of time; or the time-dependent temperature change recorded for individual eggs being warmed from ambient temperature to incubator temperature are all realizations of the functional observation type.

A previous study between the Statistics and Poultry Husbandry departments indicated that an individual chicken's juvenile growth curve was a unique characteristic. In Figure 1.1 are data from this study represented graphically. The data were not collected by a continuously recording device but by daily weighings of the individual birds. The scale of the figure is such that the observations associated with individual weighings is obscured; however, it is clear that further refinement of either the weighing procedure or the time spacing of the observations would not materially alter the results as indicated in the figure. Thus the individual curves are graphical representations of continuous functions of time as a mathematical variable. For each individual bird, then, there is a function relating body weight and time and one can visualize the functions represented as being a sample from a population of functions. The juvenile growth curve is thus a concrete example of a functional observation which should aid in discussion of the general concept and suggest problems to be solved, characteristics

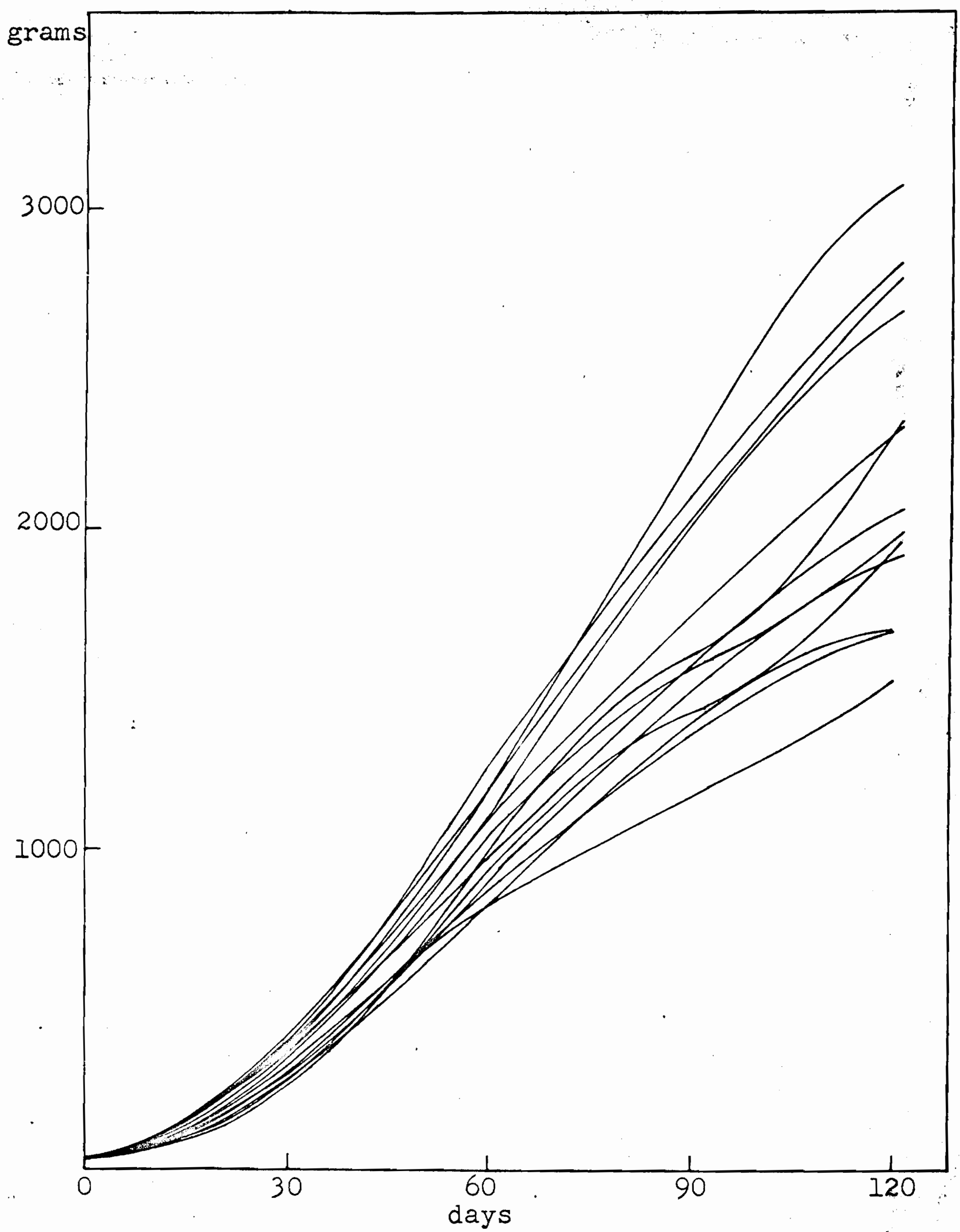

Figure 1.1 **Body weights of individual birds as a function of time**

unique to the functional observation, and the uses of such types of observational information.

1.2 Specification of Functional Form

These examples indicate the importance of specifying a family of functions of sufficient generality to adequately describe the response of each individual in the investigated population. The specification problem is usually complicated by "noise" (error) added to the "signal" (functional observation) due to the measurement device (e.g., continuously recording instrument) or external sources (usually of minor relative importance compared to the signal, but often troublesome to effectively eliminate). In the case of the chicken weight data such sources of error might be ingestion of food or elimination of waste products.

A reasonable requirement on the family of functions is that the random nature of the functions should be incorporated into the functional form. This can be done by requiring the function to be not only a function of one or more mathematical variables, but also of one or more random variables. Then, for an individual "realization" or functional observation, the random variables in the function each have numeric realizations and define uniquely and explicitly a mathematical function of the mathematical variable or variables. If the functional observation is observed by means of an interrupted process, (such as daily weighings), or an imperfect recording device then the realizations of the random variables in the functional observation can only be observed subject to error due to the necessary estimation process that reduces the crude data to functional observations. Thus a name for the random variables in the function might be the stochastic parameters of the function if they must first be estimated for each individual.

The functional form needed for representing each individual must usually be constructed by using the available information about the particular phenomenon under investigation. Sometimes this information is in the form of differential equations and/or conditions that must be met. Generally the model building procedure is a bit tentative in that forms are proposed and tested by fitting until it is felt that the form is adequate for the purposes visualized.

1.3 Properties and Uses of Functional Observations

The formulation of the functional observation as a mathematical function for each individual suggests that information describing the individual can be derived from each observation. Thus, maxima, minima,points of inflection, definite integrals, and a variety of other descriptive quantities can be derived from each function. Further possibilities are indefinite integrals, derivatives, and alternative representations which are themselves functional observations. The usefulness as well as the conditions necessary for the existence of this derived information will depend on the particular form assumed and the conditions under which the functional observations were observed. Thus it is seen that the appropriateness of the model is of major importance.

The availability of such derived information is a property of the functional observation that is of great importance to the experimenter in that it thus summarizes a variety of aspects of the relationship being investigated. This property might even enable subsequent investigators to use functional observations previously generated to examine aspects of the relationship not considered by the original investigator. How to derive such information will be shown in subsequent chapters for the particular functional observation considered.

The uses of functional observations are dependent upon the particular field of research and the natural conditions of the experiment in which they were obtained. This particular research represents a formulation and discussion of chicken growth. The formulation will involve the characterization of each individual's growth as a functional observation, which is a member of a particular parametric family. Various derived descriptive quantities and derived functions will be discussed and interpreted in terms of chicken growth. A statistical-genetics study will be presented which treats each individual's entire growth pattern (functional observation), as a genetic trait. This discussion will also illustrate how the previously mentioned derived information can be used to write the parametric family in terms of interpretable stochastic parameters. The derived properties can also be used as univariate measures in genetic research. The probability structure of a functional observation will be given and used as a theoretical explanation of various phenomena, such as the variance among body weights at a particular age, which are often observed in such experiments.

CHAPTER II

REVIEW OF LITERATURE

2.1 Review of Literature

Wishart (1938) reported a study of the growth of the bacon pig. In his analysis he suggested that, to each individual pig's growth data, a first and second degree polynomial be fitted by least squares. Then he considered the coefficients of the linear and quadratic terms to be growth characteristics which were analyzed in place of the observed age-body weight pairs. He concluded that differences among lots were concentrated among the linear coefficients which were analyzed by an analysis of variance. Thus Wishart recognized that individuals gave distinct responses which should be taken into account to better analyze the data.

Myers (1960) discussed a part of the general framework of functional observations relative to chicken growth. His data were from a sample of Athens-Canadian Randombred chickens and he showed that an individual's growth curve was distinct and could be represented by a smooth curve. He discussed four functions which might be appropriate to generate individual functional observations. He discussed the consequences of assuming that each stochastic parameter was an element of independent uniform and independent gamma distributions and found moment estimates of the mean and variance of Y_{it} conditional on the value of t.

The consideration of individual growth curves as functional observations as entities has received little attention in the literature. Some of the ideas considered in the development which follow are related to non-stationary continuous time stochastic processes. The general results available in this area are of little help in the present problem, however, and it was thought better to reformulate the problem from first principles in order to allow the most appropriate specific problems to be more clearly defined.

Consideration of an individual functional observation of juvenile growth as an inherited trait is believed to be unique to this paper.

There are many papers pertinent to aspects of this study which will be cited and discussed in the sections where their content is relevant.

It is appropriate to recognize that there is a wealth of information concerning population growth curves, considered in the usual regression sense. Some of these will be cited later. There are also many papers dealing with physiological processes and growth which include some mathematical formulations of these processes at the cell level (Weiss and Kavanau, 1957; Kavanau, 1960; and Kavanau, 1961). However, these papers are for the most part irrelevant to this paper.

CHAPTER III

RESEARCH PLAN

3.1 Introduction

At the initiation of this study it was proposed that a realistic and appropriate union would be sought between the statistical formulation of functional observations and the biological system studied, so that statistical theory could be checked by experimental results and these results in turn used to suggest modifications in theory.

3.2 Biological Material and Methods

The work reported by Myers (1960) involved data collected from a sample of Athens-Canadian Randombred (A-C), (Hess, 1962), chickens. A part of the present study is a continuation of the basic problem which he discussed. Therefore the animals used in this study were chosen to be chickens of this stock.

The use of this stock should result in more generally applicable results than might be obtained by using birds from a selected population, since they are from a broad genetic base. Further, complete pedigree and health records were kept for this population.

Biological experimentation was conducted in co-operation with the Department of Poultry Husbandry, Virginia Polytechnic Institute, Blacksburg, Virginia. Actual experimentation was performed at the V.P.I. Poultry Research Center at Blacksburg, Virginia.

Eggs from the A-C stock were hatched March 6, 1962. A sample of 230 chicks representing 42 full-sibships from 10 sires were permanently marked at hatching. Each chick was identified according to its sire, dam, and chick number within a sibship.

This group of birds was reared and later maintained to: (1) Provide a quantity of data which could be used in the statistical studies; and (2) Make available a parental pool with known growth histories for future selection experiments.

The chicks were reared in floor pens throughout. Both sexes occupied the same pen until the birds were approximately six weeks old; they were then sexed and placed in adjacent pens. Both groups were moved from the brooder house to an adult house when they were twelve weeks of age, where they remained until sexually mature.

Both groups (sexes) were inoculated against Newcastle disease when they were two weeks old. Males were dubbed and debeaked when they were thirteen weeks of age. No diseases were apparent, and post-morten examinations were made on all birds that died. Both groups ran out of feed on the 103rd day, a Sunday, and a noticeable drop in body weight followed. They apparently recovered from this shock during the next four to eight days.

An attempt was made to control the environment as much as possible. The same medium-energy ration (Siegel, 1962) was fed free choice and each pen was equipped with automatic watering devices.

Each chick was weighed at hatching and every four days thereafter. Body weight was measured in grams. Weighing was performed between 8 and 10 a.m. by an experimental team of three persons, a band reader, a weigher, and a recorder. The birds were handled as quietly and quickly as possible.

Weights were recorded on punch cards soon after they were taken. Each card contained an identification code and the age of the bird corresponding to that particular weight.

CHAPTER IV

SPECIFICATION OF THE FUNCTIONAL FORM

4.1 General Development and Evaluation of Parametric Families

The development of the idea of functional observations is dependent on the existence of a family of functions whose elements can be taken to represent the response of individual experimental units. Therefore primary interest must be devoted to describing the observations as a parametric family of functions.

The parametric family, η, dictates the form of the functional observation. It is a function of p random variables and q mathematical variables. A functional observation results when η is evaluated for an individual's set of p random variables. This procedure generates a functional observation, a random function, for each experimental unit. The variation among functional observations arises from the different sets of p random variables.

In some experiments it may be unnecessary to know the form which η takes or the realized values of the p random variables involved. Such cases might occur when a continuously recording device has plotted the individual's response with negligible noise (measurement error). Then it may be possible to use the tracing and an analog electronic computer to obtain all the information which is pertinent to the experiment.

Generally the parametric family may be expressed symbolically as

$$\eta(\underline{t}) = \eta(\theta_1,\theta_2,\ldots,\theta_p;\ t_1,t_2,\ldots,t_q) \quad ,$$

where η is a functional form, the $\theta_1,\theta_2,\ldots,\theta_p$ are random variables and $t_1,t_2,\ldots,t_q$ are mathematical variables.

A particular functional observation for the i<u>th</u> individual is then

$$\eta_i(\underline{t}) = \eta(\theta_{i1},\theta_{i2},\ldots,\theta_{ip};\ t_1,t_2,\ldots,t_q) \quad .$$

$\eta_i(\underline{t})$, above, represents the response of the i<u>th</u> individual for given $\underline{t}$ and $\theta_{i1},\theta_{i2},\ldots,\theta_{ip}$, that individual's particular set of random variables.

If an individual's set of random variables is not known, but instead must be determined indirectly, the functional observation may be written as

$$Y_i(\underline{t}) = \eta(\theta_{i1},\theta_{i2},\ldots,\theta_{ip};\ t_1,t_2,\ldots,t_q) + \varepsilon_{i\underline{t}} \quad .$$

Thus a second source of variation among the functional observations is allowed due to estimating the values of the p random variables, which will be called <u>stochastic parameters</u>. Our biological data (weights of A-C chickens) fall in this category. The $\varepsilon_{i_{\underline{t}}}$, considered an additive random measurement error, will be assumed to have the properties,

$$\mathcal{E}(\varepsilon_{i_{\underline{t}}}) = 0$$

$$\mathcal{E}(\varepsilon^2_{i_{\underline{t}}}) = \sigma^2 \quad ,$$

and serial independence. The serial correlation between $Y_i(\underline{t})$ and $Y_i(\underline{t}')$ is thus assumed to be a part of η.

Some criteria must be set down to judge whether a particular parametric family is satisfactory or unsatisfactory. An important criterion is the attainment of goodness of fit to the data, since the functional observation is taken to represent the observed response. The parametric family should be appropriate in that its form should reflect the broad aspects of the process. Thus, since growth is a continuous, monotonic increasing and asymptotic process, the parametric family should reflect these characteristics. It would be desirable for the stochastic parameters in the parametric family to have meaningful biological interpretations and the parametric family to admit the derivation of growth characteristics which reflect aspects of the individual's performance. The parametric family should involve as few stochastic parameters as are necessary to describe the functional observations. Finally the parametric family must be sufficiently general to generate functional observations to represent the diverse responses of all individuals in the visualized population.

The remainder of this section will be devoted to a study of various proposed parametric families. Each parametric form will be developed and maximum likelihood functional observations found for several individuals. On the basis of these results a satisfactory form for the parametric family will be chosen using the criteria previously mentioned.

Chicken growth may be considered as a process which is dependent on age (t) and weight (Y_{it}) at that age. It is possible that the well known differential equation (Hald, 1952)

(4.1) $$\frac{d\eta}{dt} = h(\eta,t)$$

permits, by suitable choice of form, a solution which might be used as the parametric family.

If it is assumed that the function (4.1) is separable into two functions, one involving only η, the other only t, then it can be written as

(4.2) $$\frac{d\eta}{dt} = f(\eta)g(t) \quad ,$$

which can be separated and integrated, giving

$$F(\eta) = G(t) \quad ,$$

where

$$\int \frac{d\eta}{f(\eta)} = F(\eta)$$

and

$$\int g(t)dt = G(t) \quad ,$$

and the constant of integration is absorbed in $G(t)$.

If it is further assumed that growth rate at age t is proportional to body weight at age t and amount of growth remaining, $f(\eta)$ can be written as

$$f(\eta) = \eta(\beta-\eta) \quad ,$$

where β represents final weight of the animal and the constant of proportionality is absorbed in $g(t)$. It follows from (4.2) that

(4.3) $$\eta = \frac{\beta}{1+e^{-\beta G(t)}}$$

is a solution, for $0 < \eta < \beta$, which could equivalently be written as

(4.4) $$G(t)\beta = -\ell n\left(\frac{\beta-\eta}{\eta}\right) \quad .$$

The data collected from the sample of A-C chickens were used to study the form of $G(t)$. Approximate values of β were taken from the data and $G(t)$ evaluated. Several sets of data were studied, and two representative $G(t)\beta$ are given in Figure 4.1.

It was apparent from Figure 4.1 that $G(t)$ was an increasing function of t, negative for small t, becoming positive in the vicinity of $t = 18$ (four day units). Although $G(t)$ differed markedly for different individuals, its general shape seemed quite stable. Generally the curve was approximately linear for t greater than 5 or 6 but dropped off markedly for small values of t.

The general form of $G(t)$ suggests that $g(t)$ in (4.2) might take the form

$$g(t) = \kappa/t \quad ,$$

and (4.2) can be written as

$$\frac{d\eta}{dt} = \frac{\kappa}{t}\,\eta(\beta-\eta) \quad .$$

The differential equation now states that growth rate is directly proportional to weight at age t and remaining growth, but inversely proportional to t.

Integrating $g(t)$ gives

$$G(t) = \kappa\,\ell n\, t + C \quad ,$$

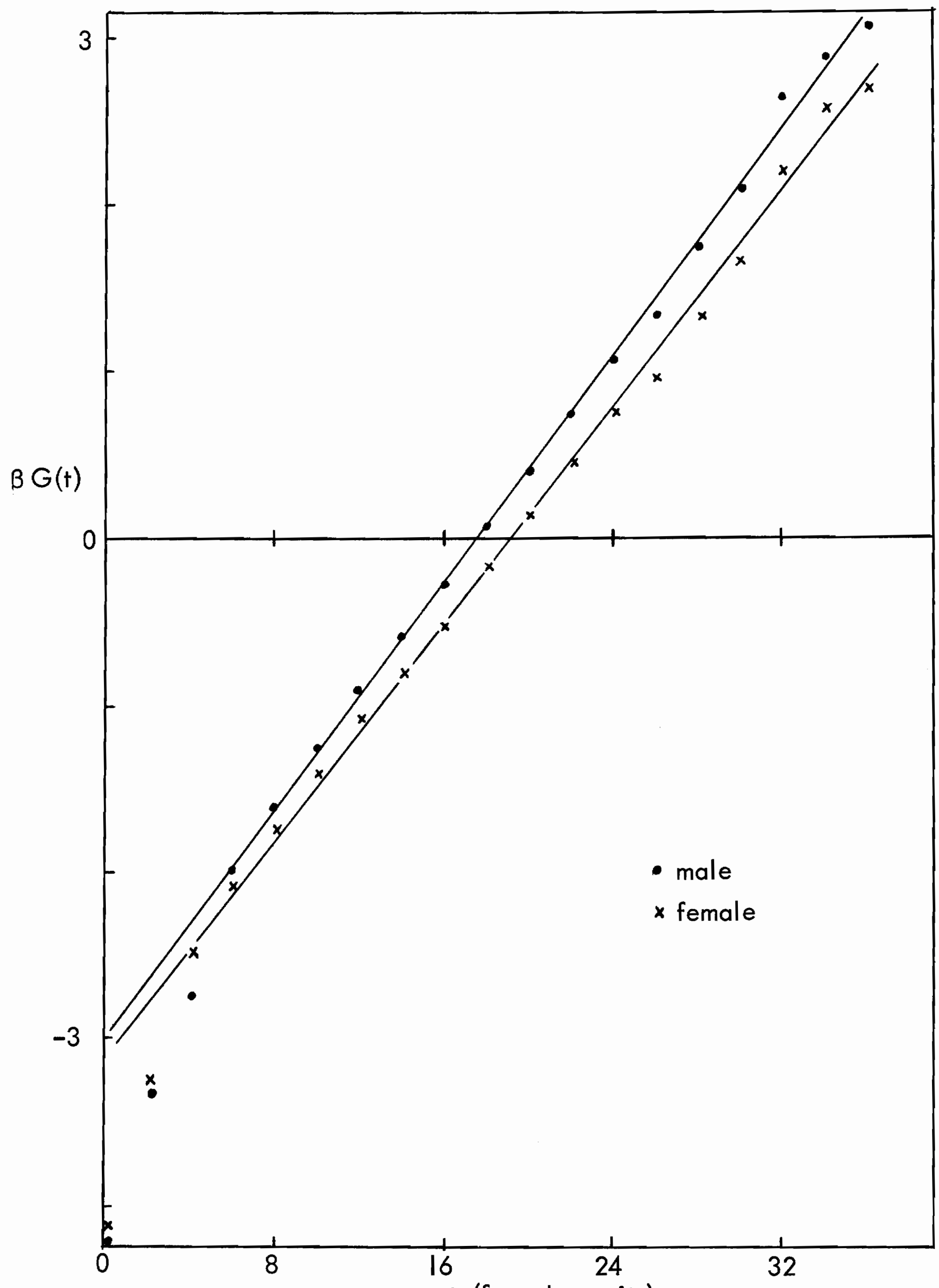

FIGURE 4.1 βG(t) for two individuals

and substituting this result (4.3) gives the parametric family

$$\eta = \frac{\beta}{1+e^{-\beta(\kappa\ \ell n\ t+C)}} \quad .$$

Defining the stochastic parameters in the form as

$$\beta = \beta$$

$$A = e^{-C/\kappa}$$

$$R = \kappa\beta \quad ,$$

gives

$$\eta(t) = \frac{\beta}{1 + (\frac{A}{t})^{R}} \quad .$$

The functional observation for the ith animal is

$$(4.5) \qquad Y_i = \frac{\beta_i}{1 + (\frac{A_i}{t})^{R_i}} + \varepsilon_{it} \ , \quad (t \geq 1) \quad .$$

Data collected from the previously discussed sample of A-C chickens were used to find such functional observations for various individuals.

Maximum likelihood estimates of the stochastic parameters for individual functional observations were found by a modified Newton method for transcendental equations. The ε_{it} were assumed to be NID$(0,\sigma^2)$ and serially independent.

A general form of (4.5), and of forms for functional observations which will be considered subsequently, is

(4.6) $$Y_i(t) = \Psi_i(t) + \varepsilon_{it} \quad .$$

Suppressing the i subscript and summing over t, the log-likelihood equation to be maximized reduces to

$$L = \lambda\Sigma(y-\Psi)^2 + \lambda' \quad ,$$

where λ and λ' do not involve the stochastic parameters and y is the realization of Y. The normal equations for the stochastic parameters, $\theta_1, \theta_2, \ldots, \theta_p$, are

$$\underline{X} = \begin{bmatrix} \frac{\partial L}{\partial \theta_1} \\ \frac{\partial L}{\partial \theta_2} \\ \vdots \\ \frac{\partial L}{\partial \theta_p} \end{bmatrix} = -2\lambda \begin{bmatrix} \Sigma(y_t - \Psi_t)\frac{\partial \Psi}{\partial \theta_1} \\ \Sigma(y_t - \Psi_t)\frac{\partial \Psi}{\partial \theta_2} \\ \vdots \\ \Sigma(y_t - \Psi_t)\frac{\partial \Psi}{\partial \theta_p} \end{bmatrix} \quad ,$$

which can not be solved explicitly because of the non-linear nature of Ψ_t.

The usual Newton procedure is to expand $\underline{X}$ by a Taylor's expansion about suitable starting values $\underline{\theta_0}$ retaining only the first order terms in the corrections, $\underline{\Delta\theta_1}$ and thus obtaining linearly correcting iteration equations. These equations are:

$$\underline{0} = \underline{X}(\underline{\theta}) + L_{\underline{\theta}\,\underline{\theta}'}\,\underline{\Delta\theta} \quad ,$$

where typical elements of $\underline{X}(\underline{\theta})$ and $L_{\underline{\theta}\,\underline{\theta}'}$ respectively are

$$\frac{\partial L}{\partial \theta_i} \quad \text{and} \quad \frac{\partial^2 L}{\partial \theta_i \partial \theta_j} \quad ,$$

and each are evaluated for suitable starting values, $\underline{\theta}_0$. A computational advantage can be realized if we modify the latter elements. Since

$$\frac{\partial^2 L}{\partial \theta_i \partial \theta_j} = -2\lambda \sum (y_t - \Psi_t) \frac{\partial^2 \Psi}{\partial \theta_i \partial \theta_j} + 2\lambda \sum \frac{\partial \Psi}{\partial \theta_i} \cdot \frac{\partial \Psi}{\partial \theta_j}$$

$$= M + N \quad , \text{ say,}$$

M may be neglected if the $(y_t - \Psi_t)$ are small. The terms of $L_{\underline{\theta}_0\underline{\theta}'}$ may then be approximated by

$$(4.7) \qquad v_{ij} = 2\lambda \sum \frac{\partial \Psi}{\partial \theta_i} \cdot \frac{\partial \Psi}{\partial \theta_j} \quad .$$

The computational saving is realized because the quantities in N have previously been computed in evaluating $\underline{X}(\underline{\theta})$, eliminating the evaluation of M completely. This abbreviated procedure gave good results in finding the solution of $\underline{X}$ in this case and in other instances discussed later.

The solution of $\underline{X}$ was found by repeated evaluation of the iteration equation

$$\underline{\theta}_{k+1} = \underline{\theta}_k - V^{-1}\ \underline{X}(\underline{\theta}_k) \quad ,$$

where V has elements v_{ij}, ceasing when the difference on the right-hand side was less than a criterion vector which insured seven significant digits in $\underline{\theta}_k$, the maximum likelihood estimates of the stochastic parameters.

Details of the estimation procedure, using 4.5, which include the normal equations, the iterant matrix and a list of FORTRAN statements to execute the process are given in Appendix A. A brief set of operating instructions are also given for the computer program. This program and several others which will be described later were written for an IBM 1620 digital computer and generally require approximately 40K memory.

The result of using 4.5 to represent individual functional observations is illustrated in Table 4.1. Deviations between the data and the functional observation $(y_t - \hat{Y}_t)$ are given for a representative set of data.

These residuals indicate that the functional observation deviated systematically with respect to sign and magnitude from the data. Therefore the form was considered inadequate for present purposes.

Table 4.1. Deviations from functional observations, for a given set of growth data, from various parametric families, and for two values of τ_1 using the general logistic (4.12).

t	y_t	Equation number of functional observation 4.5	4.10	4.11	4.12	$\tau_1 = 5$	$\tau_1 = 6$
0	43		- .08	.00	- 99.07		
1	62	51.68	- 3.00	6.88	-105.66		
2	111	72.40	13.61	40.43	- 86.56		
3	160	77.03	15.54	69.78	- 72.38		
4	192	50.07	- 19.20	76.90	- 80.75		
5	284	70.03	- 18.61	137.51	- 35.38	-63.07	
6	381	83.50	- 41.01	195.14	8.07	-20.73	-27.38
7	464	73.09	-104.73	229.10	30.00	.41	- 6.38
8	540	47.40	-196.02	244.56	36.60	6.86	.04
9	587	- 14.00	-324.27	217.60	5.58	-23.79	-30.48
10	700	- 14.60	-379.29	241.43	31.55	3.24	- 3.19
11	800	- 32.02	-427.21	235.61	35.40	8.90	2.90
12	912	- 39.94	-436.01	224.36	42.30	18.38	12.99
13	1035	- 38.24	-405.72	206.88	51.78	31.16	26.54
14	1130	- 64.87	-378.55	145.68	25.67	9.02	5.32
15	1252	- 63.96	-304.46	98.71	20.21	8.04	5.37
16	1366	- 69.75	-223.47	35.28	1.96	- 5.44	- 7.00
17	1518	- 35.61	- 93.81	6.66	18.72	16.15	15.73
18	1642	- 27.04	15.24	- 47.45	6.51	8.56	9.25
19	1755	- 26.61	118.31	-104.68	- 15.64	- 9.42	- 7.70
20	1880	- 11.02	236.76	-137.60	- 22.72	-13.03	-10.43
21	2025	27.98	377.44	-135.08	- 4.93	7.36	10.68
22	2155	55.53	504.61	-130.44	4.27	18.20	22.03
23	2245	46.75	592.75	-148.32	- 18.90	- 4.36	- .23
24	2355	61.69	701.53	-129.42	- 13.61	.56	4.77
25	2415	30.36	760.74	-145.12	- 49.37	-36.48	-32.38
26	2505	32.72	850.22	-117.18	- 46.00	-35.17	-31.38
27	2600	43.74	944.88	- 72.49	- 28.63		
28	2680	43.33	1024.65	- 32.93	- 17.60	-12.64	- 9.89
29	2740	26.41	1084.51	- 5.20	- 18.40	-16.95	-14.86
30	2825	37.86	1169.41	54.21	13.37	11.12	12.47
31	2890	32.59	1234.35	98.99	32.04	26.00	26.59
32	2940	15.46	1284.31	133.09	41.92	32.13	31.93
33	2950	- 38.65	1294.28	130.60	17.33	3.89	2.92
34	2980	- 69.85	1324.26	150.83	17.63	.69	- 1.03
35	3000	-108.28	1344.25	163.20	12.21	- 8.02	-10.46

Figure 4.1 indicates G(t) to be approximately linear except for small t. This suggests that g(t) in (4.2) might be taken as

$$g(t) = \kappa \quad ,$$

which gives after integrating

$$G(t) = \kappa t + C \quad ,$$

a linear relationship. The form, $g(t) = \kappa$, postulates that growth rate is directly proportional to weight at age t and remaining growth.

This result gives the parametric family

$$\eta = \frac{\beta}{1+e^{-\beta(\kappa t+C)}} \quad ,$$

which may be written as

$$(4.8) \qquad \eta = \frac{\beta}{1+\alpha\rho^{t}} \quad ,$$

by defining the various constants as

$$\beta = \beta$$

$$\rho = e^{-\beta\kappa}$$

$$\alpha = e^{-\beta C} \quad .$$

This is the well known "general logistic", giving functional observations

$$(4.9) \qquad Y_i = \frac{\beta_i}{1+\alpha_i\rho_i^{t}} + \varepsilon_{it} \; , \quad (t \geq 0) \quad .$$

Stevens (1951) discussed a reciprocal transform of (4.9). Using it, (4.9) can be expressed as

$$(4.10) \qquad W_i = 1/Y_i = C_i + D_i\rho_i^t + \delta_{it} \quad ,$$

where

$$C_i = 1/\beta_i \quad \text{and} \quad D_i = \alpha_i/\beta_i \quad ,$$

and the δ_{it} are assumed by Stevens to be additive random errors, NID$(0,\sigma^2)$. This form has particular appeal computationally and the resulting estimates could be transformed to represent those in (4.9).

A procedure is given in Appendix B to reduce the normal equations to a function only of r, say H(r). The maximum likelihood estimator of ρ was then found using a two-stage procedure. The usual Newton procedure requires the evaluation of H'(r) for each iteration. To avoid evaluating this derivative, H(r) was evaluated for increments of .1 over the range (0,.9) giving an approximation (graphical) to the root of H(r). An approximate value of H'(r) was found by taking the difference in adjacent H(r) which span the graphical intercept, affixing appropriate sign, and multiplying by 10.

The first pass evaluates H(r) and punches results, the second pass utilizes the approximate H'(r) and the starting value found graphically and applies Newton's procedure to find the root to several decimal places. This estimate is then used to find estimates of c and d which are shown in Appendix B to be functions of r.

Functional observations for several individuals' data sets were computed according to (4.10). Details associated with finding the maximum likelihood estimates are given in Appendix B. These include the necessary computational formulae, a list of FORTRAN source statements which execute the procedure and a brief set of operating instructions.

Signed deviations $(y-\hat{Y})$ obtained by using this parametric family for the previously mentioned data given in Table 4.1 are shown in the third column of Table 4.1. The sign and magnitude of these deviations indicate that the form (4.10) was clearly unsatisfactory. Undoubtedly these results are in part due to fitting reciprocals of numbers (y) which range over two orders of magnitude. Thus the reciprocal transformation introduces extreme heteroscedasticity in the deviations assumed of constant variance by Stevens.

A third parametric form was proposed. The parametric family (4.8) can be forced to represent the weight observed at $t = 0$ by reducing the form to two random variables and t. Setting $t = 0$, (4.8) becomes

$$\eta = \frac{\beta}{1 + \alpha} \quad .$$

Solving this for α gives

$$\alpha = \frac{\beta - Y_0}{Y_0} \quad ,$$

which when substituted into (4.8) results in a parametric family involving Y_0, β, ρ, and t,

$$\eta = \frac{\beta Y_0}{Y_0 + (\beta - Y_0)\rho^t} \quad ,$$

giving functional observations

$$\text{(4.11)} \qquad Y_i = \frac{\beta_i Y_{i0}}{Y_{i0} + (\beta_i - Y_{i0})\rho_i^t} + \varepsilon_{it} \; ; \quad t > 0 \; ,$$
$$= Y_{i0} \; . \; . \; . \; . \; . \; . \; . \; . \qquad ; \quad t = 0 \; .$$

Functional observations of this type (4.11) were found by using several individuals' data from the sample of A-C chickens. Maximum likelihood estimates of the stochastic parameters β_i and ρ_i were found using a procedure similar to that used to estimate the random variables in (4.5). The usual assumptions are made for the ε_{it}, i.e., that they are NID$(0,\sigma^2)$ for $t > 0$.

Details involved in finding the estimates are in Appendix C. The normal equations which must be solved, the iteration matrix, and a list of FORTRAN source statements, with brief operating instructions which execute the iterative procedure are given.

The reduction of (4.8) to (4.11), a two stochastic parameter situation, offered a computational advantage. The results of having used it, given in Table 4.1 for the previously used illustrative data, indicate the typical result found for other sets of data. The deviations $(y-\hat{Y})$ are larger and more systematic than those resulting from using (4.5). Similar to the results of having used (4.10), this functional observation is less than the observed weights for large t. The form (4.11) was then deemed unsatisfactory.

The literature is strongly in favor of using the "general logistic" (4.9) to represent the growth of an animal. Pearl (1928), Stevens (1951), Hald (1952), Rao (1958), Richards (1959), and Nelder (1962) are only a few of the authors who have suggested using this form or its variations. No results of having <u>applied</u> the "general logistic" to the growth of <u>chickens</u> using maximum likelihood or least squares procedures have been reported.

The "general logistic" functional observation

$$(4.12) \qquad Y_i = \frac{\beta_i}{1+\alpha_i\rho_i^t} + \varepsilon_{it}$$

may be calculated for the crude data by weighting the deviations inversly proportional to their variances. This procedure reverts to the usual unweighted form when the w_t are all taken to be unity. A computational procedure which solves the more general case is advantageous however, since a set of weights may exist which improve the goodness of fit for all the data sets.

Estimates of the stochastic parameters α_i, β_i, and ρ_i were found using the procedure discussed for (4.5). Computational details are found in Appendix D which include the normal equations for general weights, the iteration matrix and a listing of FORTRAN source statements with operating instructions.

A typical set of deviations $(Y_i - \hat{Y})$ resulting from the use of (4.12) (unweighted) on the illustrative A-C chicken data is presented in Table 4.1. The magnitude of these deviations are less than for the forms previously examined. Systematic deviations are apparent however, and the functional observation was, in particular, overestimating the early weights.

A partial listing of sets of weights used to obtain weighted functional observations is given in Table 4.2. Not

Table 4.2. Some of the sets of weights used to evaluate the general logistic (4.12).

t	w_t (unity unless otherwise indicated)								
0	4	4	7	30	10	30	25	5	12
1	4	4	6	25	9	20		3	10
2	4		5	20	8	10			8
3	4		4	15	7				6
4	4		3	10	6				4
5	4		2	5	5				2
6					4				
7					3				
8					2				
9									
10									
11	2	2							
12	2	2							
13	2	2							
14	2	2							
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21									
22									
23									
24									
25	2	2							
26	2	2							
27	2	2							
28		2							
29									
30									
31	2								
32	2				2				
33	2			5	3	5			
34				10	4	10			
35				15	5	15			

given are the deviations $(Y-\hat{Y})$ from the unweighted case which were also used as weights (see Table 4.1). Although the use of weights improved goodness of fit, no general set of weights was found.

Each of the parametric families utilized all the data. Consistent deviations from the data were noted in each case, reflecting inflexibility in the forms. For the several forms tried, the indication is that some biological stress is being reflected in the data which prohibits the use of one of these forms for all ages.

4.2 Defining the Region of Juvenile Growth

A critical evaluation of the underlying characteristics of growth in chickens explains the non-linearity in G(t) (Figure 4.1) for small t and the failure of (4.12) to represent the early weights.

A suggestive paper by Nelder (1961) in which he discussed the growth of a bacterial culture, is indicative of what will be proposed for chicken growth. In his paper he divided growth of a culture into three phases:

Phase I: characterized by small absolute growth (lag phase).

Phase II: characterized by logarithmic growth (logistic phase).

Phase III: characterized by growth decreasing to zero (stationary phase).

Genetic and physiological studies have shown that prior to approximately three weeks of age, the chick is influenced by distinct factors which are not present at later ages.

Abplanalp and Kosin (1952) reported maternal effects on heritability of body weight for Broad-Breasted Bronze turkeys. Their results indicate that these effects were highest at hatching and diminish thereafter, becoming nearly constant at four weeks. Similar results have been reported for chickens by Bray and Iton (1962).

McDevitt (1956) studied the relationship between egg weight, initial chick weight, and growth rate of several breeds of chickens for a period of eight weeks. None of the stocks involved were A-C chickens, but they are expected to perform in a similar manner. His results indicate the correlation between egg weight and chick weight (for combined males and females) was significant at hatching (.86 for White Plymouth Rocks) and decreased thereafter (.15 at five weeks of age for the same breed).

All of the energy for the animal body is derived from the oxidation of foodstuffs. This energy must be available for heating, cooling, digestion and for growth and repair of the body tissues. Barrott et al., (1938) found that Rhode Island Red chickens experienced quite variable rates of basal metabolism at different ages. They reported that basal metabolism is higher during the early weeks of life, roughly corresponding with the higher relative growth rate during the same period. Further, the thermo-regulator mechanism of the chick is not fully developed until several weeks post-hatching (Hutt and Crawford, 1960), and considerable variation exists among individuals in unselected populations for this trait.

Important changes in matabolic activity occur during the post-incubation period. The yolk is absorbed prior to hatching and is a prime source of foodstuffs for the early post-incubation period. The influence of the yolk on growth at early ages is demonstrated by the presence of yolk material in the abdomen of some chicks as late as seven days of age.

This evidence suggests that a biologically reasonable division can be made in the early part of a chicken's growth curve to define the onset of juvenile growth. The period of juvenile growth can be considered terminated at the onset of ovulation in a majority of the females. Although males produce semen at earlier ages the fecundity of the species can not occur until the females become sexually mature. Thus this age is the logical physiological point to terminate juvenile growth of both sexes. Thus, between these divisions growth should more nearly reflect the individual's juvenile growth characteristics.

Diagrammatically, such a division is illustrated by Figure 4.2. Two divisions are made in a chicken's growth curve at ages τ_1 and τ_2. It is desirable for τ_1 to be as small as possible, making more information (data) available to characterize individual juvenile growth. The upper limit τ_2 can realistically be set on the basis of average female sexual maturity. Thus the theoretical growth curve is divided into three characteristic phases:

Phase I: Characterized by large relative growth compared to that of later ages, a large maternal effect and a high metabolic rate relative to that found for older birds.

Phase II: Characterized by a relatively constant metabolic rate and nearly constant maternal effects.

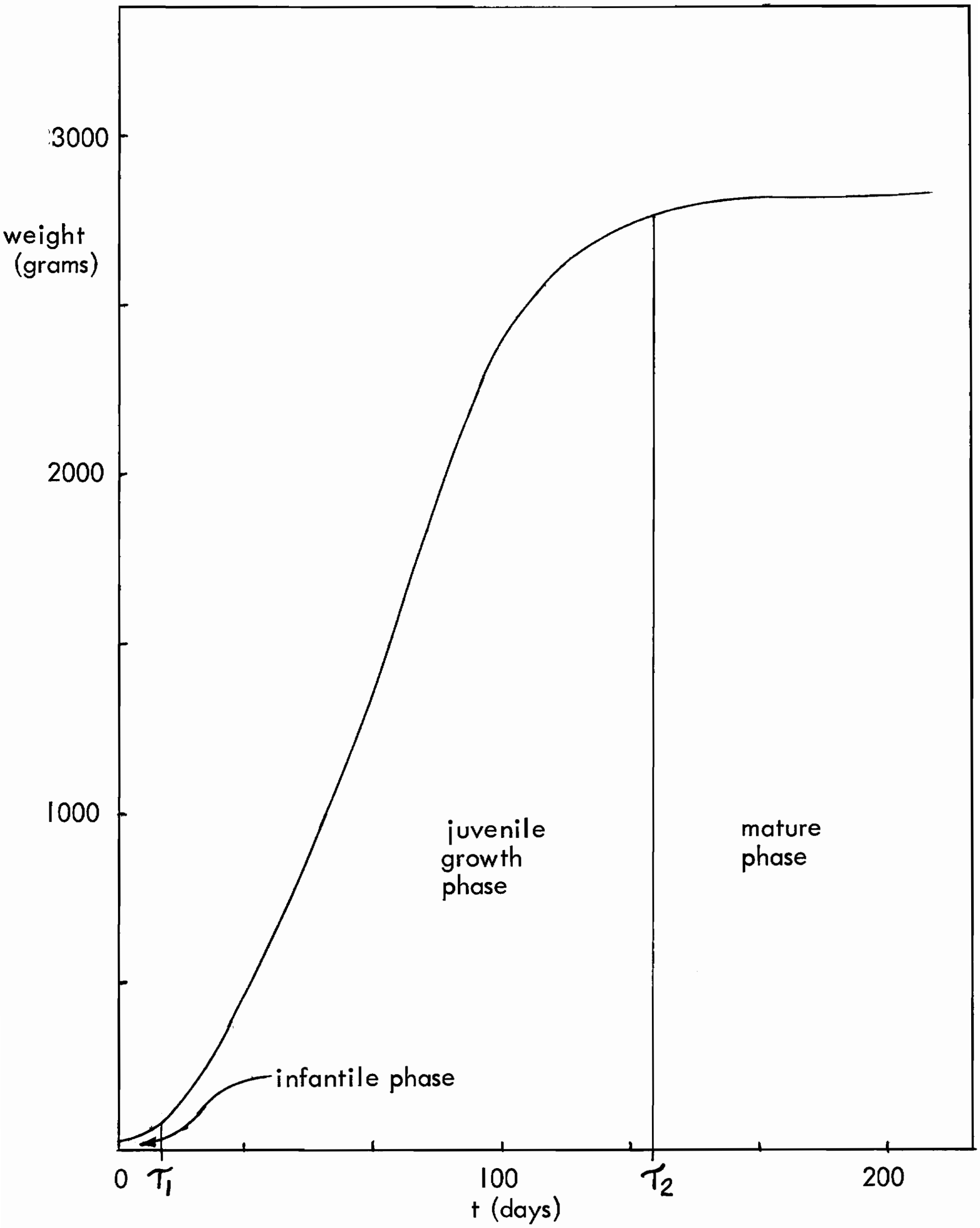

FIGURE 4.2 Diagrammatic growth curve

Phase III: Characterized by reproduction, increased fat deposition, and a small absolute and relative growth rate.

A study was performed, designed to establish an average minimal τ_1. Several sets of data from our A-C chickens were used to study G(t) and establish where its linearity and early non-linearity phases could be divided. Then the "general logistic" was used to characterize these data sets after omitting consecutive weights (beginning with t_0). The deviations from the computed functional observation were examined for consistent deviations and the quantity

$$(4.13) \qquad R^2 = \Sigma(y-\hat{Y})^2/\Sigma(y-\bar{y})^2$$

computed. The index of summation for the sums in (4.13) run from the first admitted observation to 35 (corresponding to 20 weeks of age (τ_2)). The denominator represents the usual sum of squared deviations from the sample mean. The computed values of R^2 were examined as a measure of goodness of fit and an attempt was found to find a relative maximum R^2 after having omitted several early weights.

Table 4.3 illustrates the effect of omitting various numbers of beginning weights. The values of the estimates changed as more beginning observations were omitted; however, the first few weights had the most effect. R^2 is seen to increase, then decrease. Deviations from the estimated functional observations are given in Table 4.1 for $\tau_1 = 5$ and $\tau_1 = 6$. They are seen to differ very little. (Note that y_{i27} is also omitted since the birds experienced an environmental stress at that age.) The non-linear portion of G(t), Figure 4.1, roughly corresponds to ages less than 5 or 6. (Approximating G(t) by a linear function of t necessarily results in poor goodness of fit for such values of t. This is the result found by using (4.12) (see Table 4.1).)

Table 4.3. Stochastic parameters in the general logistic and R^2 when (τ_1) early age weights were eliminated from the data in Table 4.1.

τ_1	a	b	r	R^2
0	21.02	3128	.8401	.9983
1	20.64	3136	.8412	.9985
2	20.17	3145	.8426	.9988
3	19.71	3155	.8440	.9990
4	19.24	3164	.8455	.9991
5	18.63	3178	.8472	.9995
6	18.19	3187	.8488	.9996
7	17.98	3192	.8495	.9996
8	17.89	3193	.8498	.9996
9	17.85	3194	.8499	.9995

Since it is desirable to choose τ_1 as small as possible, it was decided (on the basis of other results similar to those illustrated in Table 4.3) to let $\tau_1 = 5$ which corresponds to 20 days of age. As stated previously, τ_2 was chosen at 35 (20 weeks of age) since many of the females were laying eggs at that age.

The remainder of this study will emphasize the "juvenile growth" of chickens. Unweighted functional observations (4.12) will be estimated for each individual from the parametric family (4.8) omitting y_{i27}. This form represents the observations quite well and the deviations seem to be approximately random about the functional observation (Table 4.1).

Each functional observation is known only in the presence of noise, generated by the life process and measurement error. However, growth is a type of measurement which cannot be observed repeatedly on the same individual, prohibiting repeated sampling and further improvement in the realization. Therefore, subject to the parametric family assumed, the functional observations will be considered to represent each chicken's juvenile growth.

4.3 Comparison of Juvenile Growth Between Two Individuals

The noise about a functional observation can be utilized to compare two functional observations. An appropriate way to compare two growth patterns is to examine their difference in the interval (τ_1, τ_2). If they can be judged significantly different at any t in this interval, the null hypotheses of equality should be rejected. This seems to be a meaningful way to formulate the test since a functional observation represents a continuous response. Two functional observations may be nearly alike for some t, indeed coincident, but deviate from one another markedly for other t, thus indicating a different overall response. Such a test formulated this way, although lacking some optimum properties, is given below.

Let the null hypotheses be H_0: $\mathcal{E}(\hat{Y}_{1t}) = \mathcal{E}(\hat{Y}_{2t})$ and the alternative be H_A: $\mathcal{E}(\hat{Y}_{1t}) \neq \mathcal{E}(\hat{Y}_{2t})$. Under the null hypotheses the $d_t = (\hat{Y}_{1t}-\hat{Y}_{2t})$ are assumed to be normally distributed with mean equal to zero. The test statistic

$$(4.14) \qquad F' = \frac{\max_t d_t^2}{2s_e^2} \quad ,$$

where

$$s_e^2 = \frac{\sum_{t=\tau_1}^{\tau_2} (Y_{1t}-\hat{Y}_{1t})^2 + \sum_{t=\tau_1}^{\tau_2} (Y_{2t}-\hat{Y}_{2t})}{2(n) - 6} ,$$

$$(t=\tau_1,\tau_1+1,\ldots,\tau_1+n-1(=\tau_2)),$$

max d_t is the largest observed difference existing between the functional observations. F' is an approximate test statistic which should be compared with F with 1 and 2 (n)-6 degrees of freedom. Rejection of the null hypotheses occurs when $F' > F_{\alpha/n}$, which gives approximately an α size test.

The size of the test is at most α. This results from an application of Bonferroni's Inequality (Feller, 1957), and assuming the denominator of (4.14) is approximately a $\chi^2(2(n)-6)$. Let P_1 be the probability that at least one d_t is greater than a constant. Let the probability (p_t) that any d_t is greater than a constant be α/n. Then

$$S_1 = \sum_{t=\tau_1}^{\tau_2} p_t = \alpha ,$$

and by Bonferroni's Inequality

$$S_1 - S_2 \leq P_1 \leq S_1 ,$$

or

$$0 < \alpha - S_2 \leq P_1 \leq \alpha ,$$

where S_2 designates the sum of the probabilities of the joint occurrences of any two different d_t being greater than a constant dependent on α. S_2 is not known, however, the test given above should have size approximately equal to α. Otherwise it is a conservative test, rejecting H_0 when it is true with probability less than α.

Application of the test would be facilitated by comparing $\sqrt{F'}$ with tabular values of $t_{\alpha/n}$, d.f. = 2n-6, using the extended tables given by Federighi (1959).

CHAPTER V

MANIPULATION OF THE PARAMETRIC FAMILY, DERIVATION OF GROWTH INFORMATION AND THE PROBABILITY STRUCTURE OF A FUNCTIONAL OBSERVATION

5.1 Introduction

Each functional observation represents a summary of an individual's response in a convenient <u>functional form</u>, a distinct difference from the usual univariate or multivariate observation. This important property makes it possible to derive many individual characteristics (stochastic parameters or functional relationships) which are important in studying each response.

It is possible to express a given functional observation in alternate, equivalent, ways. This will be possible when unique transformations exist, which are functions of the original stochastic parameters. Or more formally, when the original functional observation is in terms of θ_{i1}, θ_{i2}, and θ_{i3} and there exists a transformation

$$\Psi_{i1} = F(\theta_{i1},\theta_{i2},\theta_{i3})$$

$$\Psi_{i2} = G(\theta_{i1},\theta_{i2},\theta_{i3})$$

$$\Psi_{i3} = H(\theta_{i1},\theta_{i2},\theta_{i3}) \quad ,$$

with

$$J\left(\frac{\theta_{i1},\theta_{i2},\ldots,\theta_{ip}}{\Psi_{i1},\Psi_{i2},\ldots,\Psi_{ip}}\right) \neq 0 \quad .$$

Then the functional observation can be expressed in terms of Ψ_{i1}, Ψ_{i2}, and Ψ_{i3}. This method provides a way to express the functional observation in terms of the various derived characteristics mentioned above.

Reformulating the functional observations may be useful for several reasons. The major goal in any experiment is to learn as much as possible about a process. The effects of various characteristics can be evaluated by their role in the functional observation. Further the procedure is suggestive of other characteristics emphasized by the new formulation. This procedure thus permits the experimenter to derive stochastic parameters which have clear interpretations in the system. Non-linear forms frequently involve expressions which are difficult to visualize. The researcher need then only formulate the characteristic desired and derive it from the functional observation. The method permits the reformulation of the functional observation in terms of the more easily interpretable stochastic parameters. Estimates of the new stochastic parameters can then be obtained from the values of the original ones since they are functions of them.

The normal equations which would ordinarily have to be solved for a new set of stochastic parameters will usually be different from those for the original stochastic parameters. Therefore it may be possible to express the functional observations in terms of a set of stochastic parameters which are nearly orthogonal, a useful property in non-linear estimation.

The probability structure of a functional observation involves (except for the errors in estimation) the probability densities assumed for the stochastic parameters and their functional relationship. The functional relationship is given by η and is generally non-linear making the derivation of the probability density of a functional observation quite difficult. However, a certain expression of the parametric family may exist which is particularly amenable (mathematically) to the assumed densities of the stochastic parameters.

The remainder of this section will be devoted to an examination of derived information and the probability structure of functional observations to represent juvenile growth of chickens.

5.2 Growth Rate

Wishart (1938) and Rao (1958) give typical discussions of methods of analyzing growth data for growth rate.

Wishart, in a study of growth rate of the bacon pig, suggested that to each individual's growth data a second degree polynomial be fitted by least squares. The linear and quadratic terms were considered to be growth characteristics and were taken in place of the usual data pairs. He found that differences were concentrated among the linear coefficients which were analyzed by an analysis of variance.

Rao suggests analyzing differences in body weight at various time intervals (the same intervals being used for each individual) to study growth rate. Also he suggests using a linearizing transformation in t if it is necessary. His analysis consisted of an analysis of variance and covariance of the y_0, (beginning weight) and each b, the linear regression coefficient of the weight differences on t or a transform of t = g(t).

Both authors recognize that growth rate is an important growth characteristic which is not uniform for all t. Both suggest that by properly dividing the data or by using a suitable transformation, it may be considered nearly linear.

Growth rate may be studied with the functional observations directly with no further estimation required. Using them a realistic non-linear characterization of growth rate can be found and various properties of growth rate examined.

The functional observation

$$\eta = \frac{\beta}{1+\alpha\rho^{t}} \tag{5.1}$$

has an alternative representation given by the transformations

$$\begin{aligned} \beta &= \beta \\ \tau &= \frac{\ell n\ \alpha}{\gamma} \\ \gamma &= -\ell n\ p \quad . \end{aligned} \tag{5.2}$$

Applying these we have

$$\eta = \frac{\beta}{1+e^{-\gamma(t-\tau)}} \quad ,$$

which is equivalent to the functional observation used previously, (5.1). Since this represents the growth of an individual, growth rate at age t for that individual is

$$R(t) = \frac{d\eta}{} = \frac{\beta\gamma}{(1+e^{x})(1+e^{-x})} \quad , \tag{5.3}$$

where

$$x = \gamma(t-\tau).$$

This is clearly a function of t and the stochastic parameters associated with the individual.

R(t) is always positive, since in this case β, γ, and τ are greater than zero. An examination of (5.3) shows that R(t) is symmetric about τ and that R(t) approaches zero for large positive (or negative) t. Since R(t) is symmetrical about τ and asymptotically zero for large t it must, necessarily, attain its maximum value at

(5.4) $$t = \tau \quad ,$$

(this was reported by Verhagen (1960)).

R(t) evaluated at $t = \tau$ gives

(5.5) $$R = \max_t R(t) = R(\tau) = \frac{\beta\gamma}{4} \quad ,$$

the maximum growth rate of the animal.

The simplicity of this expression and its meaningful biological interpretation suggests that R be used as a member of a set of meaningful stochastic parameters to write η in an alternative form. Further τ and R jointly define age and rate of gain at that age. If the third stochastic parameter was chosen as β, the set would also involve a weight measure.

R(t) could be used to define a parametric family generating growth rate functional observations, thus

$$r_{it} = \frac{\beta_i \gamma_i}{(1+e^{\gamma_i(t-\tau)})(1+e^{-\gamma_i(t-\tau)})}$$

is a non-linear function of time for each individual. Such a representation may be considered useful by some experimenters as a more appropriate form than functional observations of body weight. However, both functional observations involve the same stochastic parameters so that once β_i, γ_i, and τ_i are known either form may be used.

Differentiating R(t) a second time with respect to t, we have

$$R''(t) = \frac{e^{\gamma} \beta\gamma^3[e^{-2\gamma(t-\tau)} - 4e^{-\gamma(t-\tau)} + 1]}{(1+e^{-\gamma(t-\tau)})^4} ,$$

and using the relationships (5.2) and setting R"(t) = 0 gives

$$(\alpha\rho^t)^2 - 4\alpha\rho^t + 1 = 0 \quad .$$

The roots of this equation occur at

$$(5.6) \qquad \begin{aligned} I_1 &= \tau - \frac{\ln(2+\sqrt{3})}{\gamma} \\ I_2 &= \tau + \frac{\ln(2+\sqrt{3})}{\gamma} \quad . \end{aligned}$$

defining the ages when the rate of growth rate changes sign. These two points define a nearly linear portion of the functional observation, see Figure 5.1. Growth rate evaluated at these uniquely defined ages was (using symmetry)

$$(5.7) \qquad R(I_1) = R(I_2) = \frac{\beta\gamma}{6} = \frac{2}{3} R \quad .$$

Thus between these ages growth rate is at least 2/3 R. The length of this period is determined by γ since

$$I_2 - I_1 = \frac{2 \ln(2+\sqrt{3})}{\gamma} \quad ,$$

and varies inversely with γ.

The average rate of gain in weight based on ages I_1 and I_2 gives an indication of linearity of growth rate. If we define $\bar{R}(I_1, I_2)$ as this average growth rate, we have

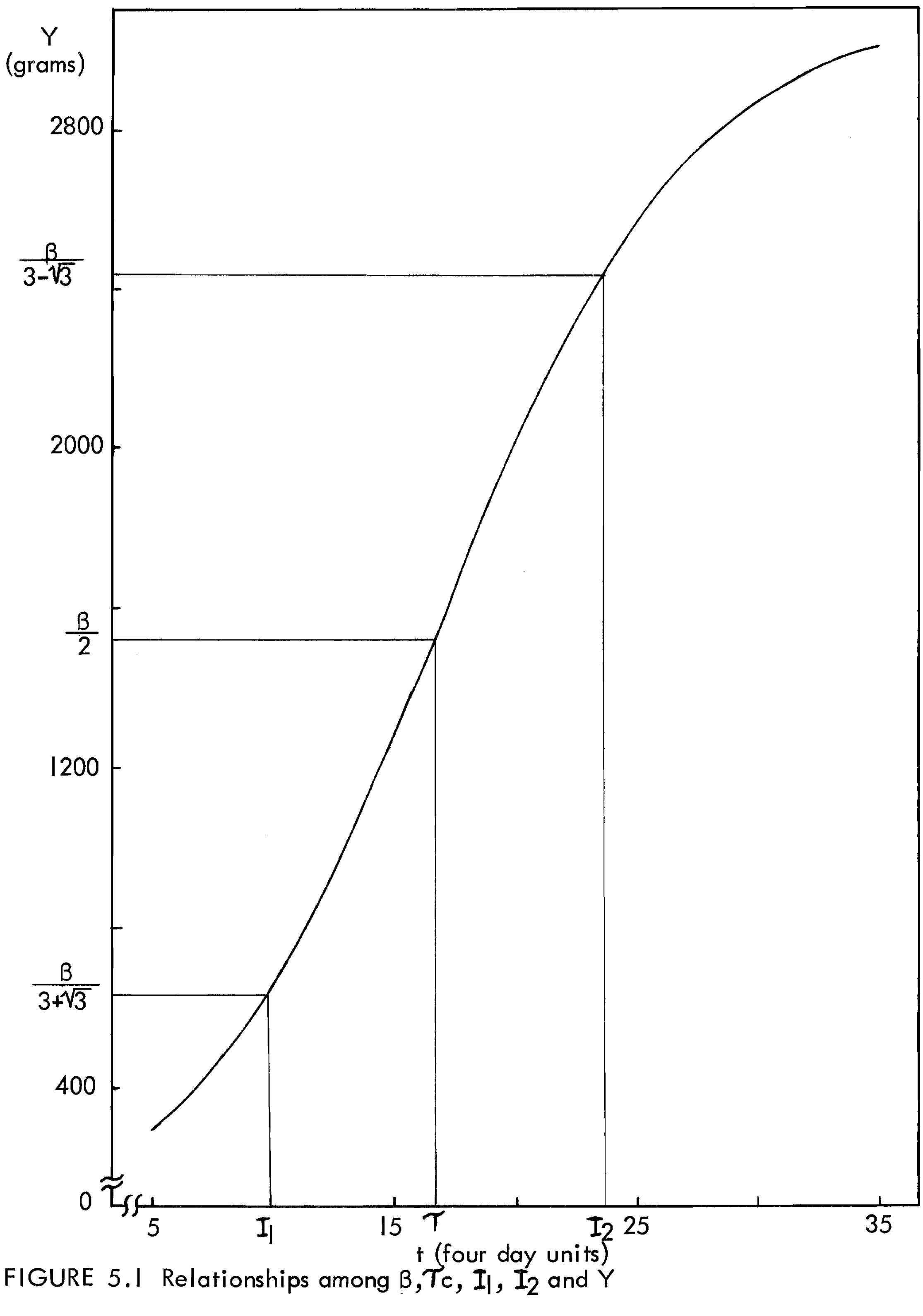

FIGURE 5.1 Relationships among β, τ_c, I_1, I_2 and Y

$$\bar{R}(I_1,I_2) = \frac{Y_{I_2} - Y_{I_1}}{I_2 - I_1}$$

(5.8) $$= \beta[(\frac{1}{3-\sqrt{3}}) - (\frac{1}{3+\sqrt{3}})] \div \frac{2\ \ell n(2+\sqrt{3})}{\gamma}$$

$$= R(2\sqrt{3}/3\ \ell n(2+\sqrt{3})) \quad .$$

(The numeric constant $2\sqrt{3} \div 3\ \ell n(2+\sqrt{3}) \cong .8767$.)

These results show:

1. R(t) is always positive in the case considered in this paper, i.e., (all stochastic parameters in (5.1) greater than zero).
2. Maximum growth rate (R) occurs when the animal's age is τ and has value $\beta\gamma/4$.
3. R(t) is symmetric about τ.
4. At two uniquely defined ages (5.6), equally spaced about τ, growth rate is 2/3 R.
5. The average growth rate $\bar{R}(I_1,I_2)$ differs from R by a constant and is approximately 87.67 per cent of R.
6. This result (5) supports Wishart's work, which indicated that the linear component accounted for most of the variation in growth rate. If the data is chosen for appropriate ages, (within I_2-I_1 and approximately symmetric about τ), the linear estimate may be quite satisfactory.

5.3 Weight and Age Relationships

The functional observations,

$$\hat{Y}_t = \frac{\beta}{1+e^{-\gamma(t-\tau)}}$$

represent a set of asymptotic regressions, similar to those discussed by Stevens (1951) and Turner, et al., (1961). An asymptotic parametric family has appeal when studying a self-limiting process, such as growth.

The stochastic parameter in the numerator of $\hat{Y}_t$ represents the asymptotic value which $\hat{Y}_t$ approaches for large positive t. That is

$$\lim_{t\to\infty} \hat{Y}_t = \beta \quad ,$$

which represents the "asymptotic juvenile weight of an animal". This is meant to imply that juvenile growth of an individual ceases when the animal attains weight β. Of course, life processes (aging, reproduction, fattening, etc.) tend to influence body weight at later ages, hence β does not realistically represent weight for these ages.

Setting $t = \tau$ in (5.2) gives

$$(5.9) \qquad \hat{Y}_\tau = \beta/2 \quad ,$$

which is similar to a result reported by Yoshida and Morimoto (1961). This shows that an individual attains half its final juvenile weight by age τ, Figure 5.1. Since growth rate is symmetrical about τ, it might be supposed that $Y_{2\tau}$ would be equal to β. This is not the case, because $R(2\tau)$ is not equal to zero, the value it attains asymptotically. In fact,

$$\hat{Y}(2\tau) = \frac{\beta}{1+e^{-\gamma\tau}} \quad ,$$

which may be reduced to a more convenient form by using the functional relationship

$$\alpha = e^{\gamma\tau} \quad .$$

Thus

(5.10) $$\hat{Y}(2\tau) = \frac{\beta}{1+(\frac{1}{\alpha})} \quad .$$

Noting that final juvenile weight is β, the amount of growth remaining beyond 2τ is

(5.11) $$\beta - \hat{Y}(2\tau) = \hat{Y}(2\tau)(\frac{1}{\alpha}) \quad ,$$

a fraction of the weight at 2τ. If

(5.12) $$N = \frac{1}{\alpha} \quad ,$$

then N is the proportion of the weight at 2τ remaining to be grown. This measure also reflects the ability of the individual to gain weight in the second τ length of time relative to the first τ period. It suggests that scaling age in τ units may be meaningful in comparing growth per unit age.

Such a natural scaling would be

$$S_t = \frac{t-\tau}{\tau} \quad ,$$

with S_t taking on values:

$$\begin{aligned} S_t &= -1 && \text{corresponding to hatching,} \\ &= 0 && \text{at age } \tau, \\ &= 1 && \text{at } 2\tau, \\ & \text{etc.} \end{aligned}$$

5.4 <u>Growth Proportionality</u>

The general logistic parametric family results from a certain differential equation. This equation

$$\text{(5.13)} \qquad \frac{dY_t}{dt} = \kappa Y_t(\beta - Y_t)$$

assumes that growth rate is directly proportional to the product of weight at age t, (Y_t), and remaining growth, ($\beta - Y_t$). The solution of (5.13) is

$$Y_t = \frac{\beta}{1+e^{-\beta\kappa(t-C)}} \quad ,$$

which may be re-expressed by defining the constants to be

$$\kappa = \frac{\gamma}{\beta}$$

and

$$C = \tau \quad .$$

The result is the usual general logistic,

$$(5.14) \qquad Y_t = \frac{\beta}{1+e^{-\gamma(t-\tau)}} \quad .$$

From the remarks given above, it is seen that the constant of proportionality assumed in (5.13) is

$$\kappa = \frac{\gamma}{\beta} \quad .$$

A second proportionality parameter of growth is ρ. Previously it was shown that

$$\rho = e^{-\gamma} \quad ,$$

which may be substituted into (5.14) giving

$$Y_t = \frac{\beta}{1+\rho^{(t-\tau)}} \quad .$$

For simplicity define:

$RG_t = \beta - Y_t$ = growth remaining at age t, and

$RG_{t+1} = \beta - Y_{t+1}$ = growth remaining at age t+1.

Then the ratio

$$\frac{\frac{RG_{t+1}}{RG_t}}{\frac{Y_{t+1}}{Y_t}} = \frac{\beta - \frac{\beta}{1+\rho^{t-\tau+1}}}{\beta - \frac{\beta}{1+\rho^{t-\tau}}} \div \frac{\frac{\beta}{1+\rho^{t-\tau+1}}}{\frac{\beta}{1+\rho^{t-\tau}}}$$

(5.15)

$$= \frac{\frac{\rho^{t-\tau+1}}{1+\rho^{t-\tau+1}}}{\frac{\rho^{t-\tau}}{1+\rho^{t-\tau}}} \div \frac{1+\rho^{t-\tau}}{1+\rho^{t-\tau+1}}$$

$$= \frac{\rho^{t-\tau+1}}{\rho^{t-\tau}}$$

$$= \rho \ ,$$

or

$$\frac{RG_{t+1}}{RG_t} = \rho \frac{Y_{t+1}}{Y_t} \ .$$

Since the denominator in (5.15) is a ratio of sizes, we will call ρ a "growth-size stochastic parameter".

5.5 Alternative Representations

Alternative representations of the general logistic parametric family have been given throughout the previous sections. There are, however, an infinite number of ways to express this form.

Interest in writing alternative representations of the form is two-fold. First, it permits the expression of the functional observations in terms of meaningful (interpretable) stochastic parameters. Second, an expression may be possible which involves a set of nearly independent stochastic parameters.

The utilization of interpretable stochastic parameters adds an amount of appeal to the functional observations. Their interpretations are also useful in evaluating the differences among the functional observations. Considerable mathematical reasoning is required to evaluate how one functional observation actually represents a different response from another if they are expressed in terms of obscurely defined stochastic parameters.

An orthogonal set of clearly interpretable stochastic parameters may not exist. If a set exists which is nearly orthogonal, however, great advantages can be gained. The associated dispersion matrix (V) in the iterative solution is conditioned according to the correlation structure among them. If this matrix has large diagonal terms, relative to the off-diagonal terms, the number of iterations required in estimating the stochastic parameters should be reduced, representing a considerable computational saving.

A possible procedure when making selections for juvenile growth is to select on the values of the stochastic parameters. Often selections are made in such a way that all configurations of the stochastic parameters, in a factorial arrangement, must be present. Such a procedure will be facilitated if the stochastic parameters are nearly independently distributed for the population of individuals.

A summary of some alternative representations is given below. The set of transformations is also given for each which must be applied to previously defined representations.

Beginning with the family as derived in Chapter IV,

$$\eta = \frac{\beta}{1+\alpha\rho^{t}} \quad , \tag{5.16}$$

the following representations are possible:

	Transformations	Parametric Family
(5.17)	$\beta = \beta$ $\gamma = -\ln\rho$ $\alpha = \alpha$	$\eta = \dfrac{\beta}{1+\alpha e^{-\gamma t}}$
(5.18)	$\beta = \beta$ $\tau = \dfrac{\ln\alpha}{\gamma}$ $\gamma = \gamma$	$\eta = \dfrac{\beta}{1+e^{-\gamma(t-\tau)}}$

	Transformations	Parametric Family
(5.19)	$\beta = \beta$, $N = \frac{1}{\alpha}$, $\tau = \tau$	$\eta = \dfrac{\beta}{1+N^{\frac{t-\tau}{\tau}}}$
(5.20)	$\beta = \beta$	$\eta = \dfrac{\beta}{1+e^{-\frac{4R}{\beta}(t-\tau)}}$
	$\tau = \tau$	or
(5.21)	$R = \frac{\beta\gamma}{4}$	$\eta = \dfrac{4R}{\gamma[1+e^{-\gamma(t-\tau)}]}$
(5.22)	$\beta = \beta$, $N = N$, $S_t = \frac{t-\tau}{\tau}$	$\eta = \dfrac{\beta}{1+N^{S_t}}$
(5.23)	$\beta = \beta$	$\eta = \dfrac{\beta}{1+e^{-\kappa\beta(t-\tau)}}$
	$\kappa = \frac{\gamma}{\beta}$	or
(5.24)	$\tau = \tau$	$\eta = \dfrac{\gamma}{\kappa[1+e^{-\gamma(t-\tau)}]}$

The representation (5.16) has particular appeal computationally because of its simplicity. Derivation of the iteration equation from this form is straightforward. Probably (5.20) involves the most interpretable (in terms of chicken growth) set of stochastic parameters. The remaining representations may have appeal to other investigators.

Later studies of the realizations of the stochastic parameters may indicate a representation which involves a nearly independent set. This topic is deferred to Chapter VI.

5.6 Probability Structure of a Functional Observation

A functional observation is a random function. It is a function of stochastic parameters and a mathematical variable (t). An individual functional observation is uniquely defined by realizations of these stochastic parameters.

Each individual is assigned, by some random process, a set of these stochastic parameters. Their realizations are expressed in the functional observation for that individual. A second individual expresses its randomly obtained set defining another functional observation, etc.

The parent probability densities of the stochastic parameters and the functional relationship determine what the probability density of the functional observation will be. The functional relationship is given by η, the parametric family which is usually a non-linear expression.

The parent probability densities of the stochastic parameters often must be assumed. Realistically they should reflect the properties of the stochastic parameters. In this study, the growth of A-C chickens, the probability densities of the stochastic parameters are not known. However, the individual stochastic parameters are continuous variables which must, for this case, be positive.

Thus the probability density of a functional observation represents a non-linear formulation of the underlying probability densities. The realizations from this probability density are functional observations.

The parametric family which we have used to study chicken growth is

$$\eta = \frac{\beta}{1+\alpha\rho^{t}} \quad .$$

A functional observation results from an individual's random choice of β, α, and ρ, defining the particular realization. By its nature, the probability density of a functional observation ($f(Y)$) involves more than one variable. Therefore its probability density may best be studied by considering its conditional probability density given t. Since the functional observation exists for positive t, its conditional probability density is defined for all positive t.

A functional observation involves all three stochastic parameters for most values of t. However, for some particular values of t, all three variables are not involved. Thus for $t = 0$ a functional observation involves only the stochastic parameter β and α. When $t = \tau$ the functional observation involves only β, which is also the only one involved for infinite t. The functional observation, at these t, is

$$Y_0 = \frac{\beta}{1+\alpha} \quad ,$$

$$Y_\tau = \frac{\beta}{2} \quad ,$$

and

$$Y_\infty = \beta \quad .$$

Therefore it is apparent that the conditional probability density of the functional observation, $f(Y|t)$, must take different forms for different t.

Knowing the probability structure of a functional observation invites studies of a host of theoretical topics. The variable form of the conditional density makes it unusual among distributions. The moment properties of the conditional probability density, $f(Y|t)$, could be examined and various moment regressions studied.

Here the principal interest in developing $f(Y|t)$ is illustrative. The intent is to show a possible probability structure for functional observations and its appropriateness as a description for this type of data.

Probability structures other than the one which follows are possible, if other probability densities of the stochastic parameters are assumed, however, the integrations required limit the feasibility of most alternative formulations. Perhaps in the future a study involving a random mating population could be performed to examine the sampling density of each stochastic parameter.

Assume that the stochastic parameters β_i, α_i, and ρ_i have independent log-normal probability densities. Following the notation of Aitchison and Brown (1957), denote these as follows:

$$\beta_i \sim \Lambda(\mu_\beta, \sigma^2_\beta)$$

(5.25) $$\alpha_i \sim \Lambda(\mu_\alpha, \sigma^2_\alpha)$$

$$\rho_i \sim \Lambda(\mu_\rho, \sigma^2_\rho) \quad ,$$

where log-normal probability density represents the probability density of a statistical variable whose logarithm is normally distributed.

The joint element of probability of β_i, α_i, and ρ_i; dropping the i subscript on the individual's stochastic parameters:

$$h(\beta,\alpha,\rho)d\beta\ d\alpha\ d\rho = \{\frac{1}{(2\pi)^{3/2}\beta\alpha\rho\ \sigma_\beta\sigma_\alpha\sigma_\rho}\}$$

(5.26)

$$\cdot\ \exp-\tfrac{1}{2}[\{\frac{\ell n\beta-\mu_\beta}{\sigma_\beta}\}^2+\{\frac{\ell n\alpha-\mu_\alpha}{\sigma_\alpha}\}^2+\{\frac{\ell n\rho-\mu_\rho}{\sigma_\rho}\}^2]d\beta d\alpha d\rho \quad ,$$

$$0 < \beta < \infty$$

$$0 < \alpha < \infty$$

$$0 < \rho < \infty \quad .$$

Making the change of variable,

$$S = \rho^t$$

$$0 < S < \infty \quad ,$$

which has absolute Jacobian,

$$|J| = \frac{1}{t}\, S^{\frac{1}{t}-1} \quad ,$$

gives

$$h(\beta,\alpha,S|t)d\beta\ d\alpha\ dS = \{\frac{1}{(2\pi)^{3/2}\alpha\beta S\sigma_\beta\sigma_\alpha t\sigma_\rho}\}$$

(5.27)

$$\cdot\ \exp-\tfrac{1}{2}[\{\frac{\ell n\beta-\mu_\beta}{\sigma_\beta}\}^2+\{\frac{\ell n\alpha-\mu_\alpha}{\sigma_\alpha}\}^2+\{\frac{\ell nS-t\mu_\rho}{t\sigma_\rho}\}^2]d\beta d\alpha dS \quad .$$

The previous expression can be reduced further by letting $X = \alpha S$, taking advantage of the convolution of log-normal variables, which gives

(5.28) $$h(\beta,X|t)d\beta\, dX = \frac{1}{2\pi\beta X\sigma_\beta\sigma_X} \cdot \exp-\tfrac{1}{2}\left[\left\{\frac{\ell n\beta-\mu_\beta}{\sigma_\beta}\right\}^2+\left\{\frac{\ell nX-\mu_X}{\sigma_X}\right\}^2\right]d\beta dX \quad ,$$

$$0 < X < \infty \quad .$$

In this expression

$$\mu_X = \mu_\alpha + t\mu_\rho$$

$$\sigma_X^2 = \sigma_\alpha^2 + t^2\sigma^2\rho \quad .$$

(That $X = \alpha S$ is itself log-normally distributed follows from the convolution of the sum of two normal variates, given by Aitchison and Brown (1957). For

$$\ell n\ \alpha \sim N(\mu_\alpha, \sigma_\alpha^2)$$

$$\ell n\ S \sim N(t\mu\rho, t^2\sigma_\rho^2) \quad ,$$

hence

$$\ell n(\alpha S) = \ell n\alpha + \ell nS \sim N(\mu_\alpha+t\mu_\rho, \sigma_\alpha^2+t^2\sigma_\rho^2) \quad ,$$

and

$$X = \alpha S \sim \Lambda(\mu_X, \sigma_X^2).)$$

The quantity

$$W = \frac{1}{1 + X}$$

takes on values from 1 to 0 as X goes from 0 to ∞. The change of variables,

$$Y = \frac{\beta}{1 + X}$$

$$W = \frac{1}{1 + X} \quad ,$$

which have ranges

$$0 < Y < \infty$$

$$0 < W < 1 \quad ,$$

has absolute Jacobian

$$|J| = \frac{1}{W^3} \quad .$$

This change of variable gives

$$(5.29) \qquad f(Y,W|t)dWdY = \{\frac{1}{2\pi\sigma_\beta\sigma_X YW(1-W)}\}$$

$$\cdot \exp - \tfrac{1}{2}[\{\frac{\ell n(Y|W)-\mu_\beta}{\sigma_\beta}\}^2+\{\frac{\ell n(\frac{1-W}{W})-\mu_X}{\sigma_X}\}^2]dWdY \quad .$$

The conditional probability density of Y given t results when W is integrated out. Thus

$$(5.30) \qquad f(Y|t)dY = (2\pi\sigma_X\sigma_\beta Y)^{-1}$$

$$\cdot \int_0^1 \frac{1}{W(1-W)} \exp-\tfrac{1}{2}[\{\frac{\ell n(Y|W)-\mu_\beta}{\sigma_\beta}\}^2+\{\frac{\ell n(\frac{1-W}{W})-\mu_X}{\sigma_X}\}^2]dWdY \quad .$$

This is virtually an intractable integration. (Severo and Olds (1956) encountered a similar problem when studying the power of the test for comparing means of log-normal variables.

Their work required a closed form for the probability density of the sum of n log-normal variates, which they were unable to find.)

Expanding the squares in the exponent of (5.29) and collecting terms not involving W into one factor gives

$$(5.31) \qquad f(Y,W|t)\,dW dY = \left\{\frac{e^{-\frac{1}{2\sigma_\beta^2}(\ell nY-\mu_\beta)^2-\frac{\mu_X^2}{2\sigma_X^2}}}{2\pi\ \sigma_X\sigma_\beta Y}\right\} \cdot \left\{e^{-\frac{1}{2\sigma_X^2}((P-Q)^2-2\mu_X(P-Q))}\right\} \cdot \left\{e^{-\frac{Q}{2\sigma_\beta^2}(-2\ell nY+Q+2\mu_\beta)-(P+Q)}\right\} dW dY \quad ,$$

where

$$P = \ell n(1-W)$$

$$Q = \ell n(W) \quad .$$

Writing

$$P + Q = P - Q + 2Q$$

permits further factorization in the two exponent factors involving W, which become

$$(5.32) \qquad e^{(P-Q)[A-C_1(P-Q)]+Q(B-C_2Q)} \quad .$$

In this expression

$$A = \frac{\mu_X}{\sigma_X^2} - 1$$

$$B = \left(\frac{\ell nY - \mu_\beta}{\sigma_\beta^2}\right) - 2$$

$$C_1 = \frac{1}{2\sigma_X^2}$$

and

$$C_2 = \frac{1}{2\sigma_\beta^2} \quad .$$

Since W has range (0,1), an infinite series based on the definition of the Riemann integral can be written. Divide the interval (0,1) into 2^n subintervals, each of length $1/2^n$. Then

$$(5.33) \qquad f(Y|t)dY = \left\{\frac{e^{-\frac{1}{2\sigma_\beta^2}(\ell nY-\mu_\beta)^2 - \frac{\mu_X^2}{2\sigma_X^2}}}{2\pi\ \sigma_X\sigma_\beta Y}\right\}$$

$$\cdot \left\{\lim_{n\to\infty} \sum_{i=0}^{2^n-1} \frac{1}{2^n}\, e^{(M-N)[A-C_1(M-N)]+N(B-C_2N)}\right\} \quad ,$$

where

$$M = \ell n\left(1 - \frac{i}{2^n}\right)$$

and

$$N = \ell n\left(\frac{i}{2^n}\right) \quad .$$

To investigate the value of the first term of this series, obtained when $i = 0$, consider the following:

Remembering

$$P = \ell n(1-W)$$

and

$$Q = \ell n(W) \quad ,$$

the first term, T_0, is

$$T_0 = \kappa \lim_{W \to 0} e^{(P-Q)[A-C_1(P-Q)]+Q(B-C_2Q)}$$

$$= \kappa \lim_{Q \to -\infty} e^{Q(B-A)-Q^2(C_1+C_2)}$$

$$= \kappa \lim_{Q \to -\infty} \frac{1}{e^{-Q(B-A)+Q^2(C_1+C_2)}}$$

$$= 0 \quad ,$$

since $C_1+C_2 > 0$, if $B-A > 0$, the result is immediate, if $B-A < 0$ the $Q^2(C_1+C_2)$ term will dominate the exponent and T_0 will again be zero.

Evaluation of $f(Y|t)$ is a numerical problem which lends itself well to the use of a digital computer. Since the first term of the series is zero, it can be omitted. The final form to be evaluated is

$$f(Y|t) = \frac{e^{-\frac{1}{2\sigma_\beta^2}(\ell nY-\mu_\beta)^2 - \frac{\mu_X^2}{2\sigma_X^2}}}{2\pi\ \sigma_X\sigma_\beta Y}$$

$$\cdot \sum_{i=1}^{2^n-1} \frac{1}{2^n}\, e^{(M-N)[A-C_1(M-N)]+N(B-C_2N)} \quad ,$$

where n must be chosen to insure accuracy (to a certain number of significant digits) in the ordinates.

The approximate expression of $f(Y|t)$ involves six constants and t. In order to illustrate the form of $f(Y|t)$, a set of these constants,

$$\begin{array}{ccc} \mu_\rho & \mu_\alpha & \mu_\beta \\ \sigma_\rho^2 & \sigma_\alpha^2 & \sigma_\beta^2 \end{array} \quad ,$$

was required. Among our A-C chicken data was a sibship which contained six males. Their individual realizations of β_i, ρ_i, and α_i are available from their functional observations. Since we have assumed β_i, ρ_i, and α_i to be lognormally distributed, the individual realizations were transformed to logarithms (base e) and the means and variances estimated. These are given in Table 5.1.

Table 5.1. Estimates of the mean and variance of the probability densities of $\ell n\ \beta_i$, $\ell n\ \alpha_i$, and $\ell n\ \rho_i$ for six, male, full-sibs.

variable	estimate of mean	estimate of variance
$\ell n\ \beta_i$	7.8279216	.0026948867
$\ell n\ \alpha_i$	3.0390514	.0094343907
$\ell n\ \rho_i$	- .16350167	.0000903183

A digital computer program was written to perform the numerical integration over the indicated range and evaluate the resulting function of Y. Each evaluation gave an ordinate of the conditional density corresponding to a given Y and t. Appendix E contains a tabulation of FORTRAN source statements to evaluate ordinates of $f(Y|t)$. A brief set of operating instructions is also given.

The effect of using various size n (taking a variable number of terms in the numeric integration) was investigated. Table 5.2 contains representative results from this study. Taking n = 8 implies accumulating 255 terms in the numerical integration, while taking n = 10 results in evaluating 1023 terms. The results in Table 5.2 indicate that taking n = 8 as opposed to n = 10 results in ordinates which differ only

Table 5.2. Values of ordinates of $f(Y|t)$ for several Y, t and letting $n = 8$ and $n = 10$.

t	Y	$f(Y\|t)$ $n = 8$	$f(Y\|t)$ $n = 10$
5	220	.00999302	.00999282
7.5	350	.00005373	.00005373
10	500	.00659181	.00659180
15	800	.00258253	.00258253
20	1325	.00239341	.00239340
22.5	1750	.00203456	.00203455
25	2250	.00233865	.00233865
30	2325	.00298570	.00298570
35	2800	.00001475	.00001475

in the sixth decimal place. Taking $n = 8$ gave adequate accuracy for the purposes of this study.

Our particular functional observations represent individual juvenile growth. This has been defined to occur when the animal is between 20 and 140 days of age ($t = 5$ and 35 four day units). $f(Y|t)$ was evaluated for $t = 5$, 7.5, 10, 15, 20, 25, 30, and 35 four day units which cover the juvenile growth period. Approximately twenty-five ordinates, corresponding to different Y, at each t were evaluated. Graphs of these conditional densities are shown in Figures 5.2, 5.3, 5.4, 5.5, 5.6, 5.7, 5.8, and 5.9.

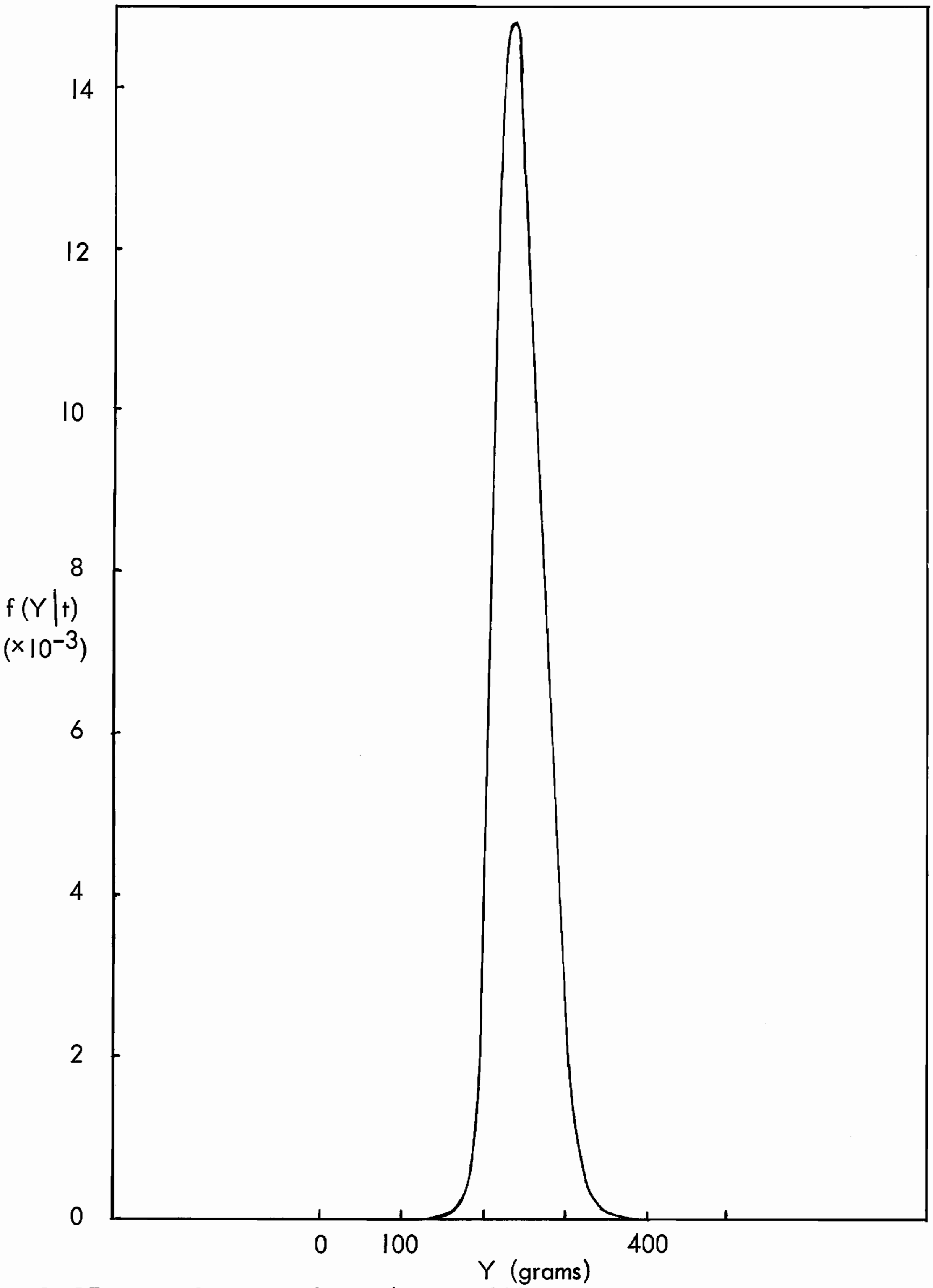

FIGURE 5.2 Conditional distribution of Y given t = 5

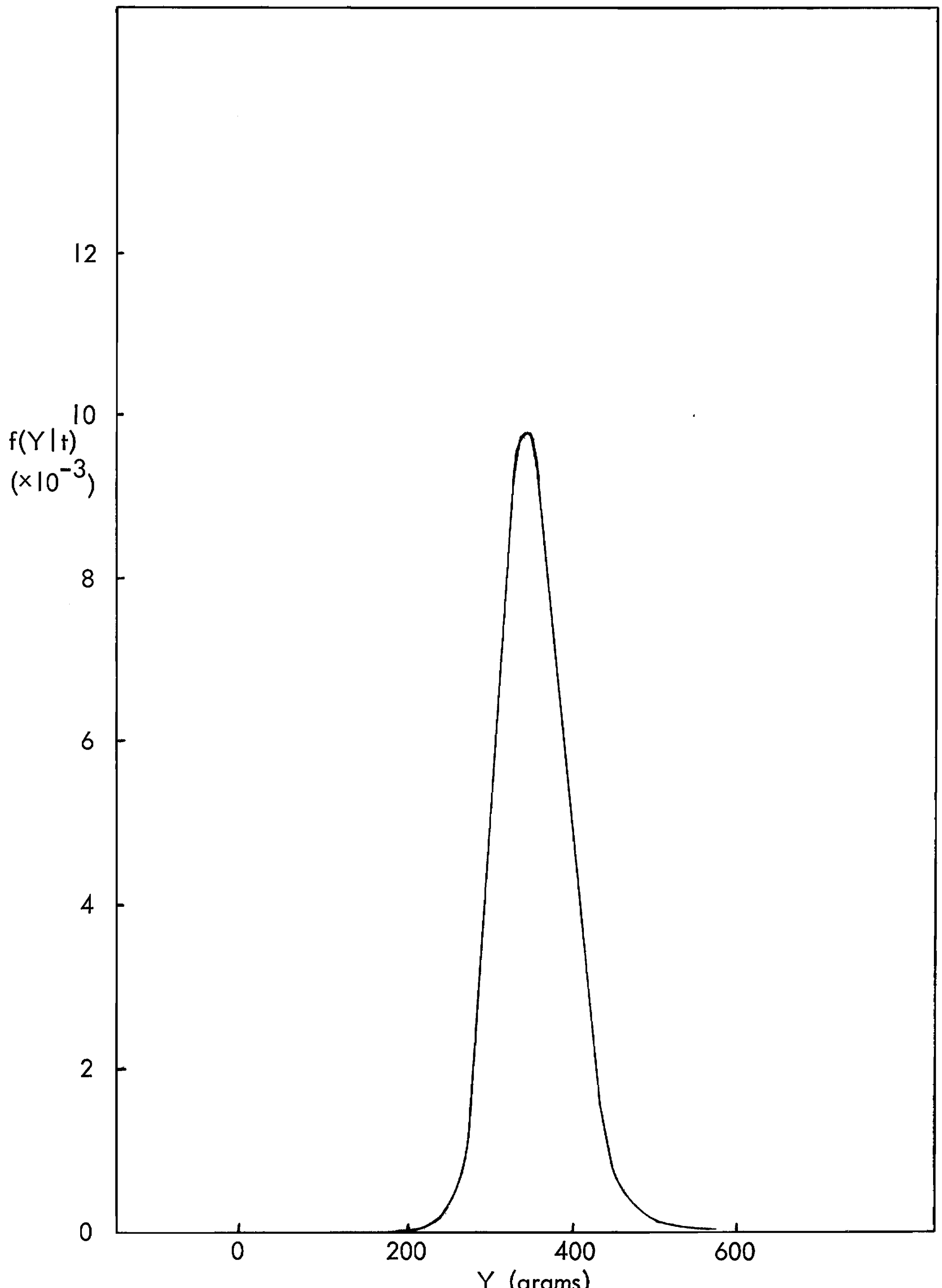

FIGURE 5.3 Conditional distribution of Y given t = 7.5

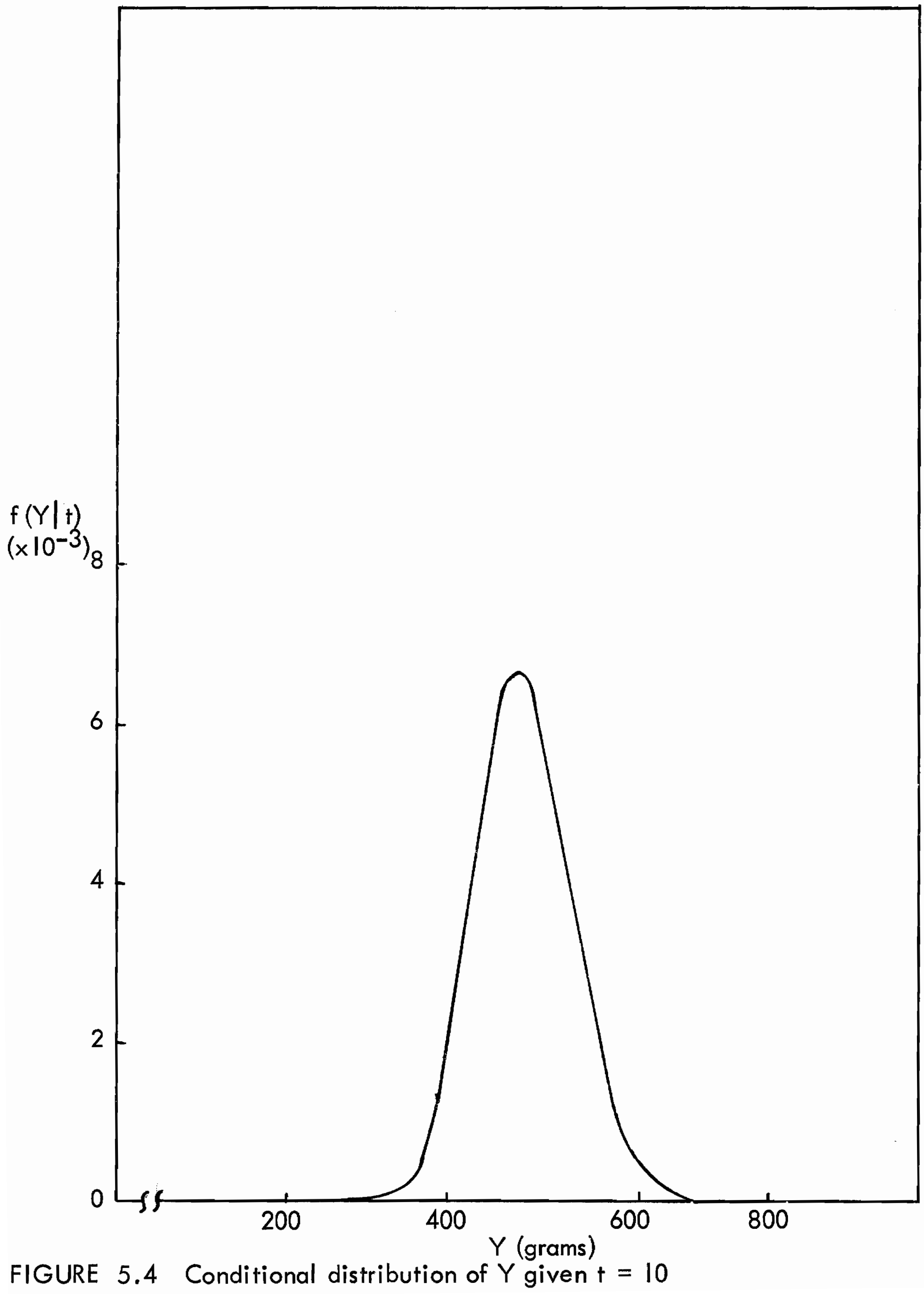

FIGURE 5.4 Conditional distribution of Y given t = 10

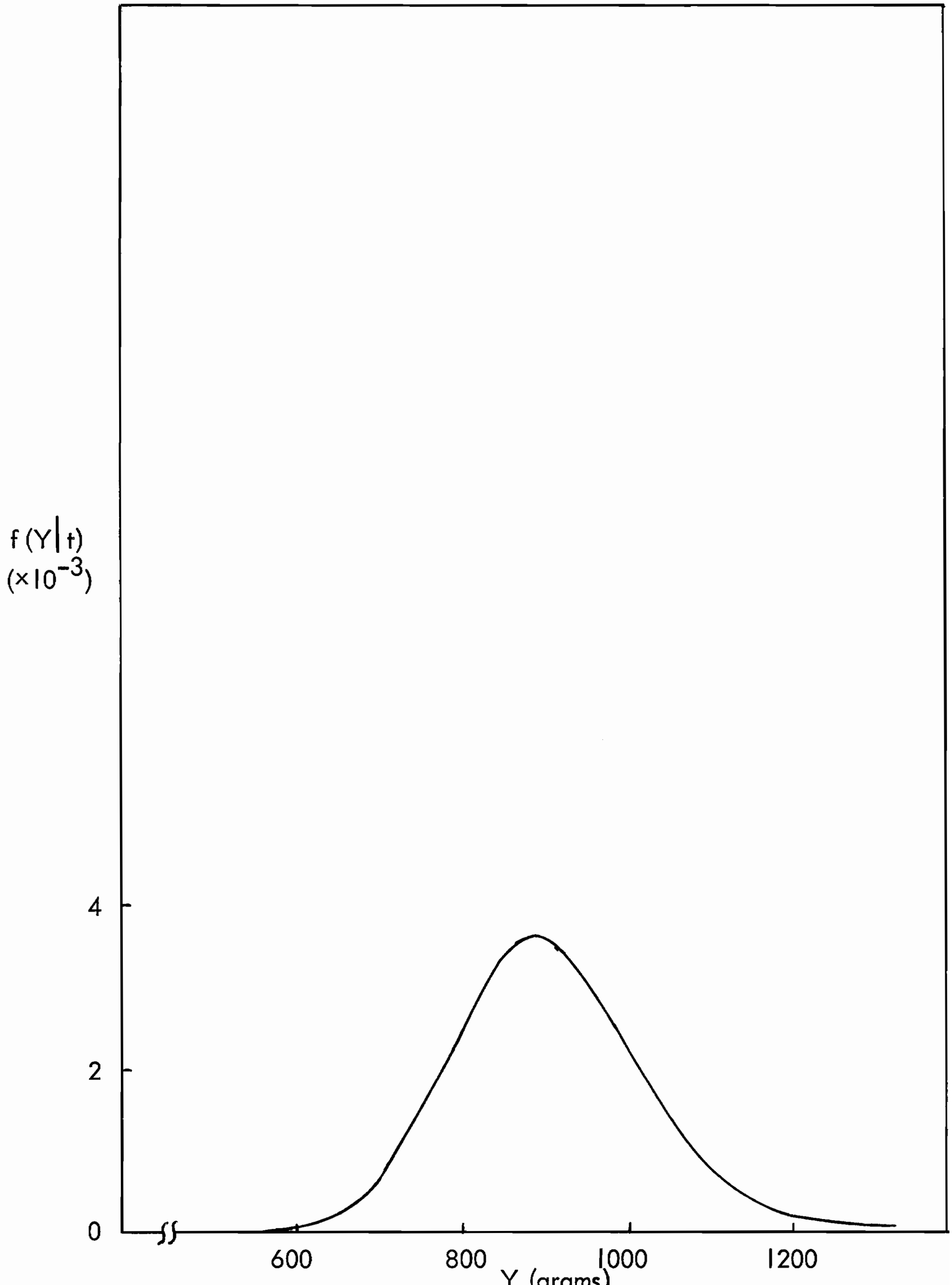

FIGURE 5.5 Conditional distribution of Y given t = 15

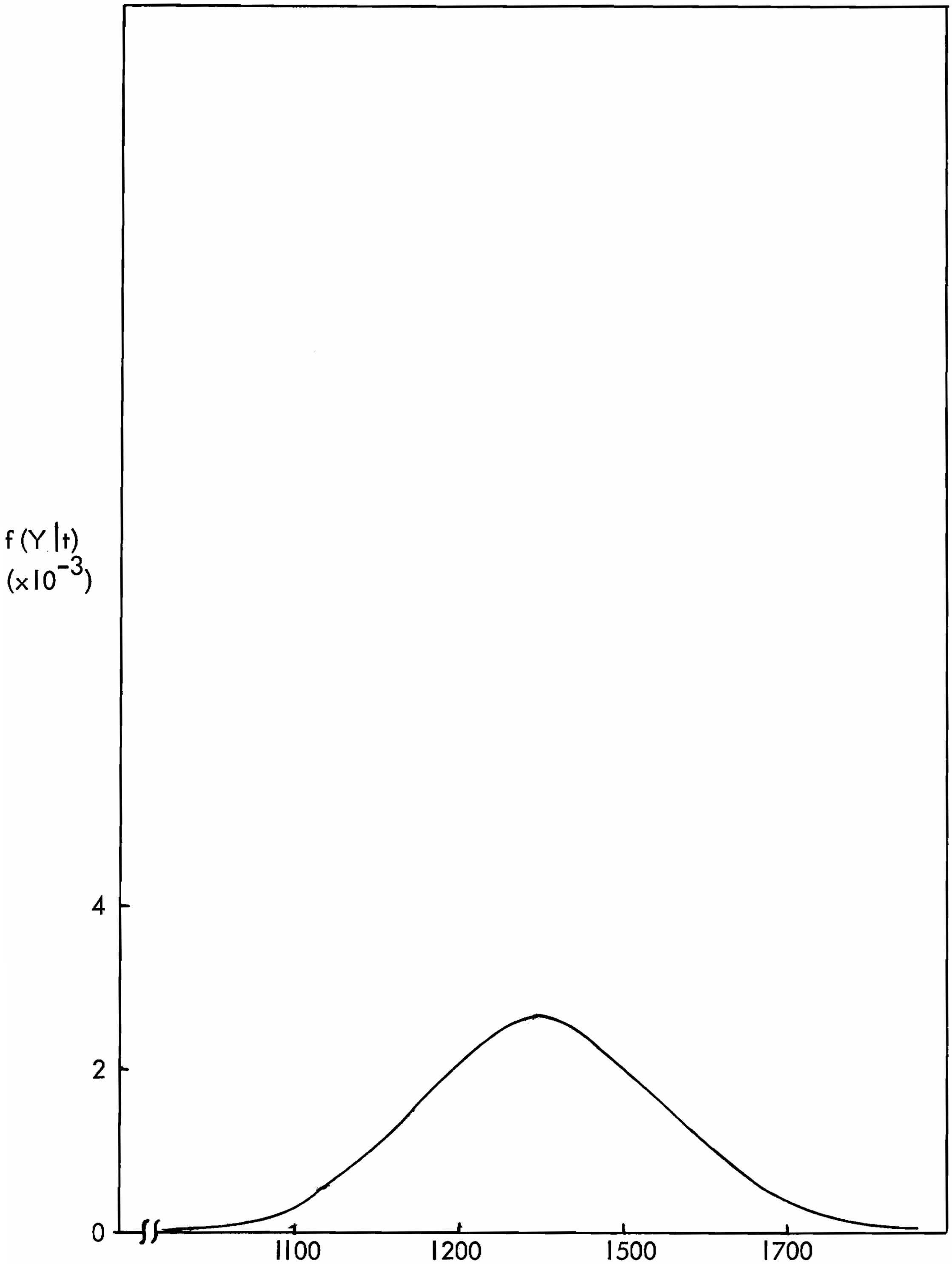

FIGURE 5.6 Conditional distribution of Y given t = 20

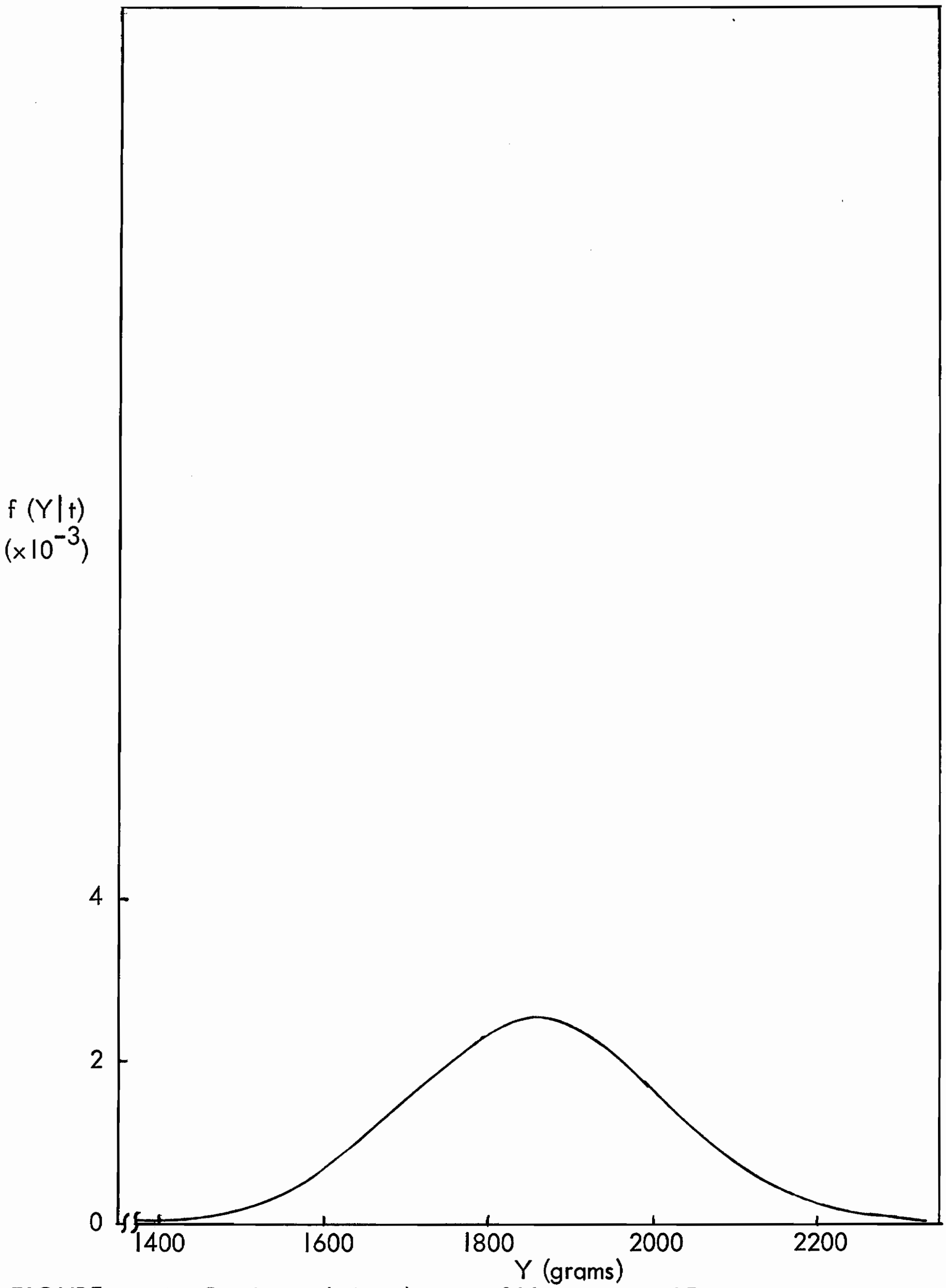

FIGURE 5.7 Conditional distribution of Y given t = 25

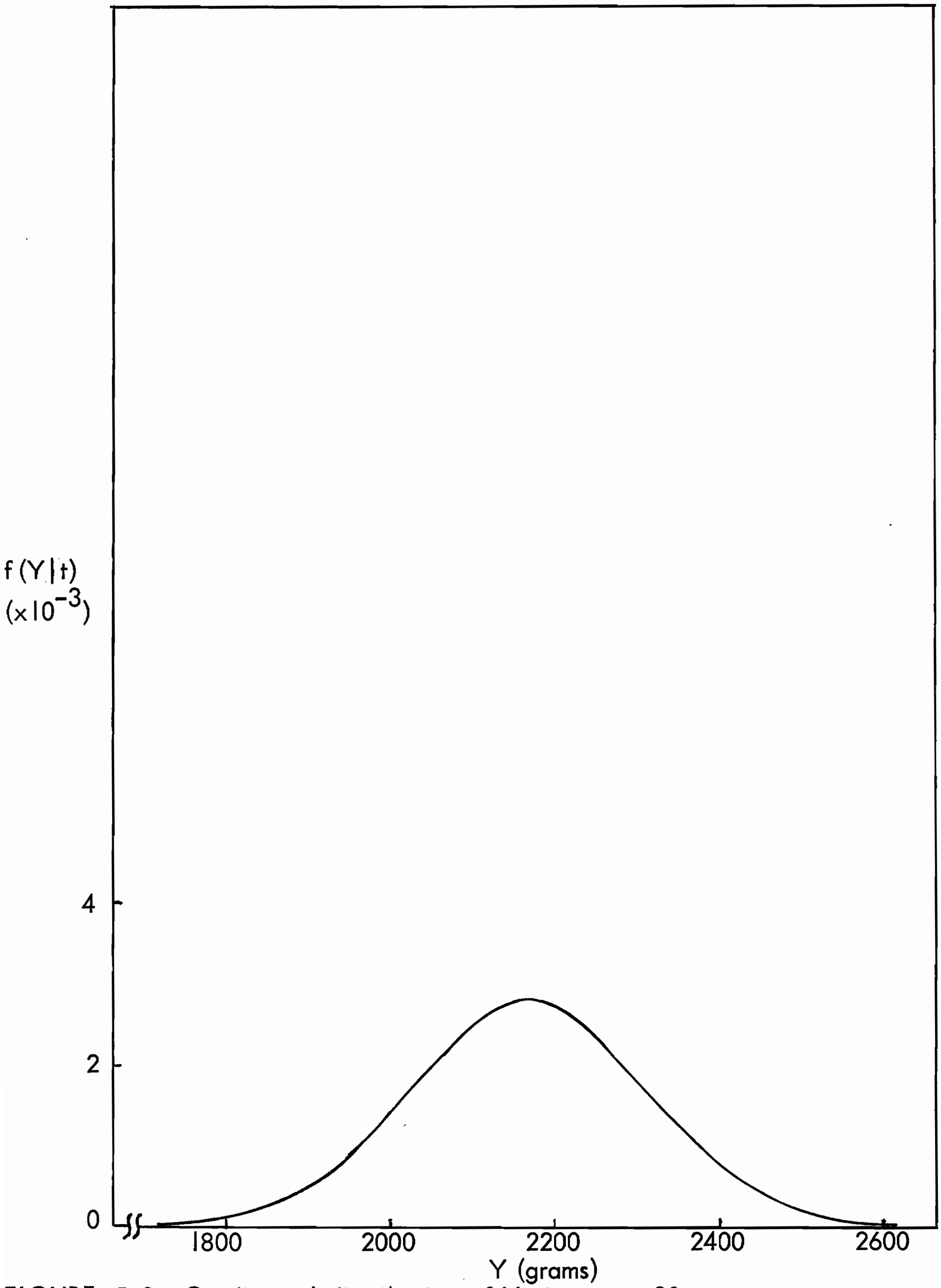

FIGURE 5.8 Conditional distribution of Y given t = 30

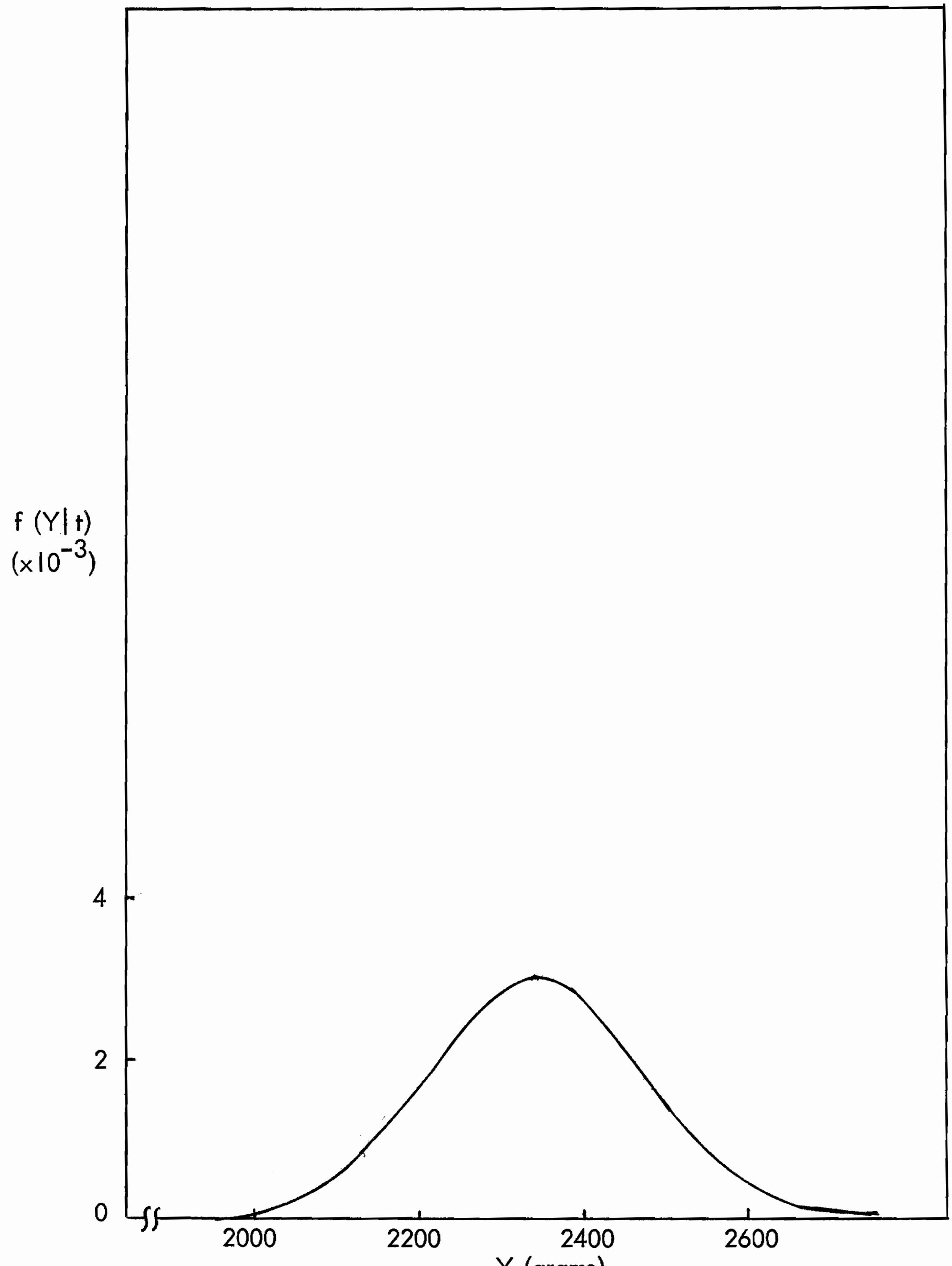

FIGURE 5.9 Conditional distribution of Y given t = 35

A photograph of a three-dimensional model of the probability density of a functional observation is given in Figure 5.10.

A tabulation of modal ordinates and their abscissas for several values of t is given in Table 5.3.

Table 5.3. Modal ordinates and their abscissas of $f(Y|t)$ for several values of t.

t	Y	Modal ordinate
5	242.5	.01478
7.5	347.5	.00984
10	487.5	.00671
15	887.5	.00365
20	1392.5	.00265
22.5	1640.0	.00253
25	1855.0	.00255
30	2170.0	.00279
35	2337.5	.00300

The figures 5.2 through 5.10 and Table 5.3 illustrate several properties of this density which realistically reflect what is observed, see also Figure 1.1, by examining a set of realizations:

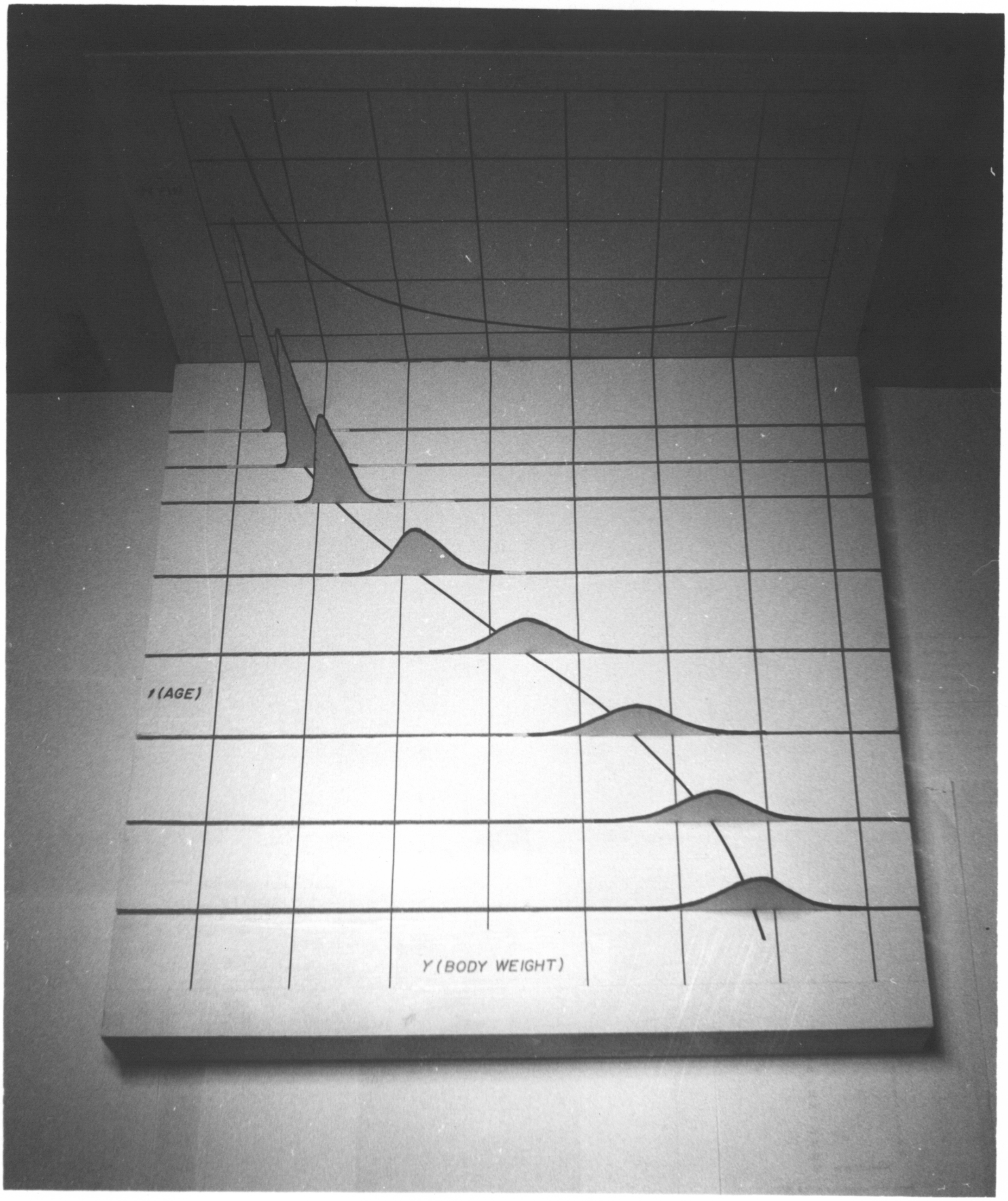

Figure 5.10 Model of f(Y|t)

1. The modal ordinate of $f(Y|t)$ projects a general logistic in the (t,Y) plane (Figure 5.11).
2. The largest modal ordinate of $f(Y|t)$ occurs at $t = 5$ (.01478), the least near $t = 22.5$ (.00253).
3. The dispersion of $f(Y,t)$ increases with t to $t = 22.5$ then decreases (Figure 5.10).
4. The $f(Y|t)$ are nearly symmetrical, although positively skewed.
5. The projection of the modal ordinate in the (t,f) plane is decreasing from its largest value at $t = 5$ (.01478) to approximately $t = 22.5$ (.00253), then increasing to $t = 35$ (.00300), (Figure 5.12).

These results illustrate the uniqueness of the probability structure of a functional observation. Particularly pleasing is its reflection of what is apparently occurring in the biological system. The variance structure at different t is especially interesting. Indeed, the formulation in terms of functional observations represents a new explanation of the frequently observed correlation between body weights of different individuals and t. Thus repeated sampling of functional observations from the probability structure assumed, followed by analyzing the values of the

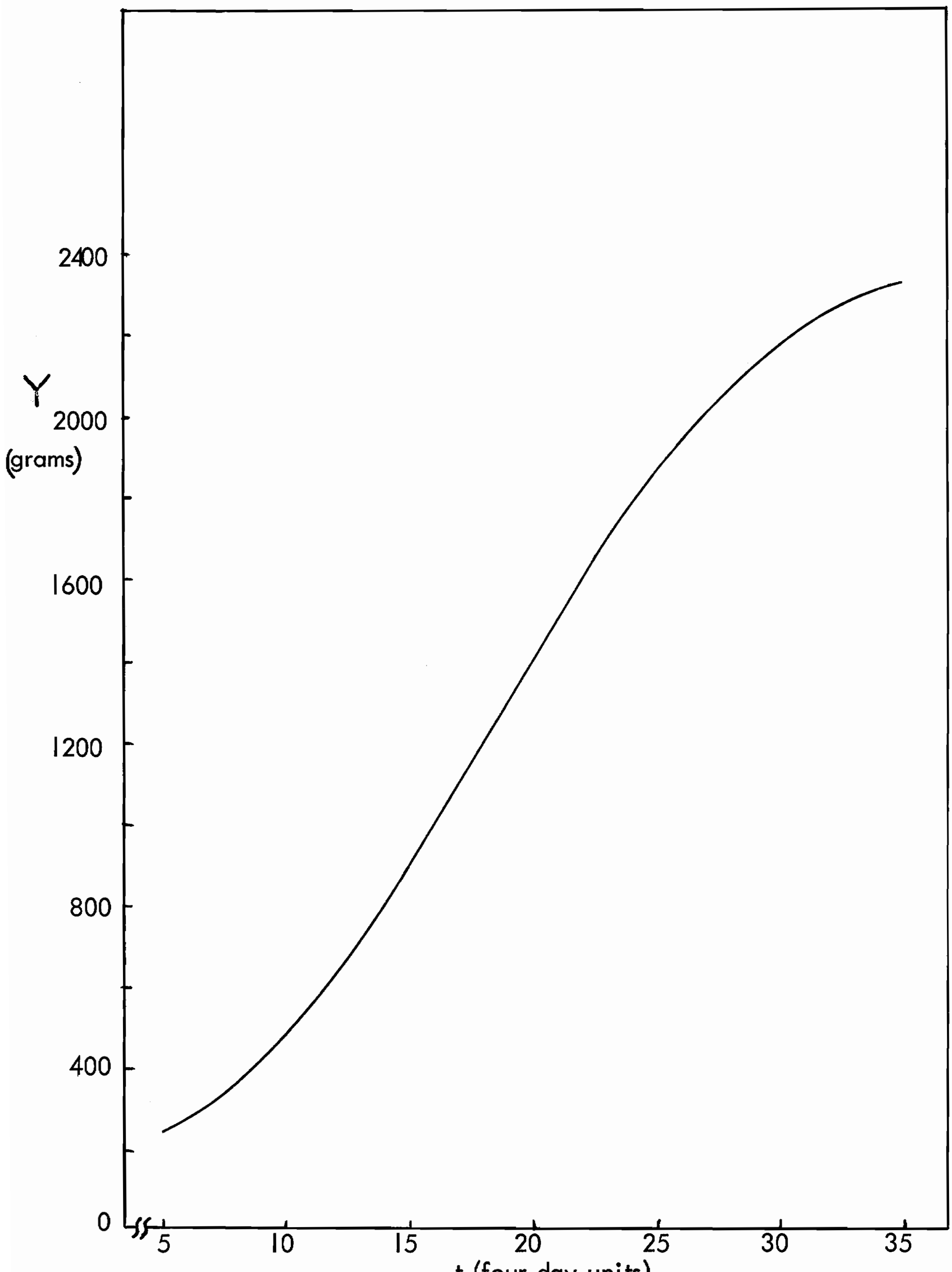

FIGURE 5.11 Projection of modes of $f(Y|t)$ in the (t, Y) plane

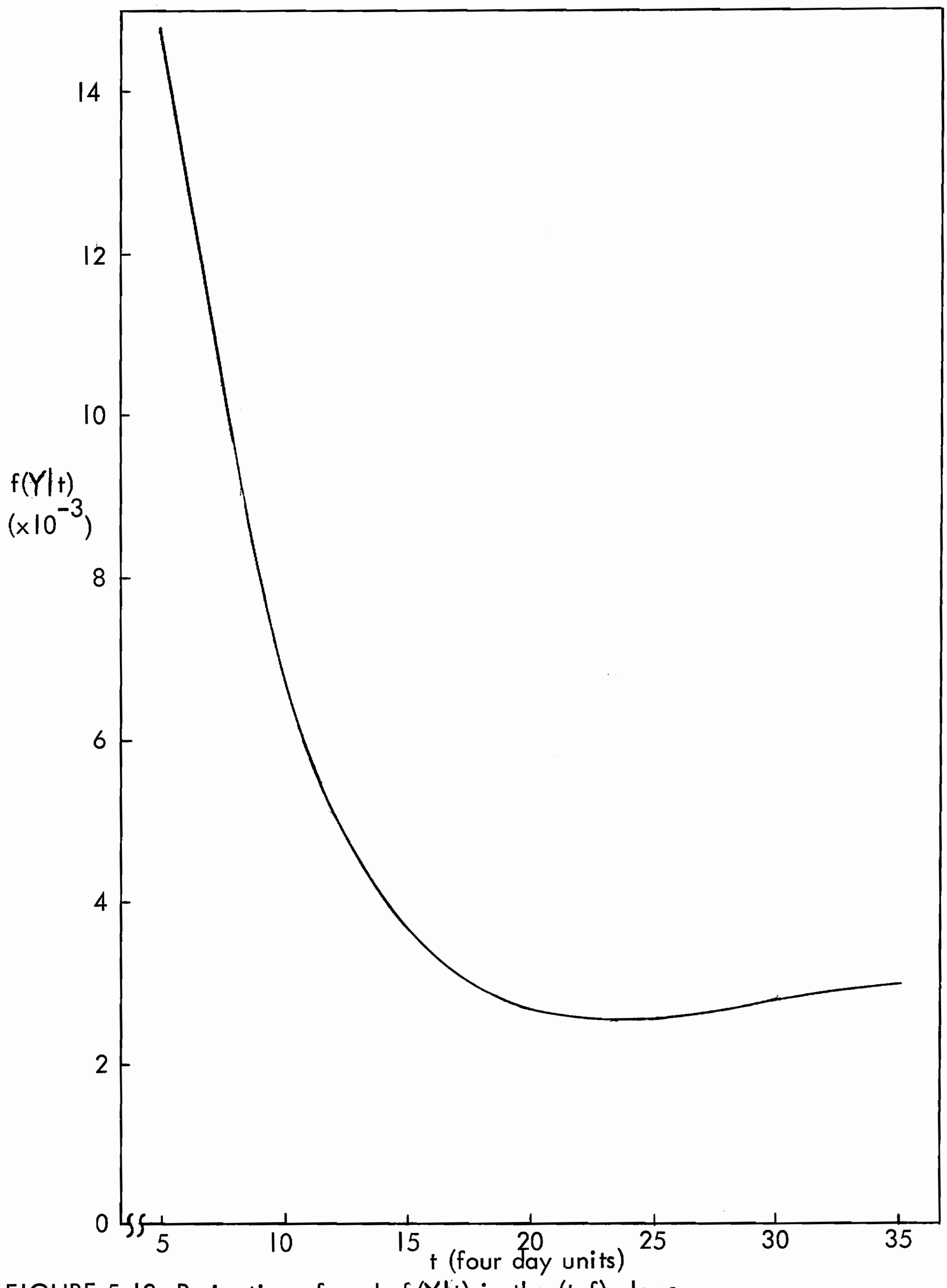

FIGURE 5.12 Projection of mode f (Y|t) in the (t,f) plane

functional observations at various t would generate this phenomenon.

5.7 Discussion of Appropriate Statistical Tests to Compare Realizations of the Functional Observations

A useful property of a functional observation is its summary form. At the beginning of this chapter various quantities were derived from the functional observations and related to chicken growth. Later it was shown that sets of three of these quantities (which have Jacobian of transformation different from zero), could be used to uniquely define, or represent in an alternative form, the functional observation. Earlier an approximate test was given which may be used to compare functional observations.

The appropriate univariate significance test for a single type of stochastic parameter is dependent on the probability density of the realization of the stochastic parameter. In the previous section it was assumed that a, b, and r were each elements of independent log-normal densities. Therefore, to be consistent with that assumption, a natural logarithm transformation of these quantities allows the use of well-developed significance tests for normally distributed variables. As was pointed out earlier, the assumption of parent log-normal probability densities for these variables did not result from examination of a large number of individual realizations (only about 200). This assumption is realistic, however, in depicting the range of the variables and their continuity. Further, the resulting probability density seems to be quite realistic. However, other parent densities could be assumed and appropriate tests made on each stochastic parameter. This judgement remains in the hands of the experimenter.

The use of multivariate techniques for comparing vectors of realizations is also possible. Juvenile growth could be compared by applying these techniques to sets of individual stochastic parameters which uniquely define juvenile growth. Such a procedure is facilitated by assuming each set (or some transform of the set) to be jointly normally distributed. Caution should be extended that different results may occur if a different set (also capable of defining the functional observation) is used to make a second test. The results of two such multivariate tests may not be consistent since the second set will usually be a non-linear transformation of the first.

CHAPTER VI

STATISTICAL GENETICS OF GROWTH

6.1 Introduction

One of the major objectives of this study has been to define a probability structure which would describe individual juvenile growth in a realistic way. This statistical formulation, discussed in previous chapters, is based on the idea of random functions, called functional observations. According to this formulation, each growth pattern of each individual can be treated as an entity.

The probability structure of a functional observation was given in Chapter V. That discussion utilized the random variables (stochastic parameters) and the mathematical variable t involved in the functional observation. The probability structure can be developed for the biological system in the following way. The growth pattern of an individual is peculiar to the environment and the genetic composition of that individual. The genotype of the animal is determined by a random sampling, at the time of fertilization. The expression of this sampling of growth genes, in a given environment, is a unique growth pattern that is characterized by a functional observation involving a unique set of stochastic parameters for each individual. Thus the stochastic parameters are determined by a random process. Their sampling distribution is generated by the possible random combinations of genes at meiosis.

This suggests that an entire growth pattern of an individual is a genetic trait. Such a consideration would permit studies of the inheritance of entire growth patterns, a very realistic procedure since it would allow the use of all the information in the function about the growth of each individual. Individual growth patterns, characterized by their functional observations, can be quite diverse. An examination of Figure 6.1 (in which actual functional observations of growth for four male chickens are given), illustrates this property. Note that each of these functional observations are identified with a capital letter in the upper right-hand corner of that figure.

The following relationships can be found among them:

1. They rank, from heaviest to lightest, with respect to $\hat{\beta}$; B,A,D,C, (3187, 3102, 2548, and 2444).
2. Their $\hat{Y}_{i35}$ rank A,B,C,D.
3. Their $\hat{Y}_{i15}$ rank A,C,B,D.
4. Their $\hat{Y}_{i5}$ rank C,A,D,B.

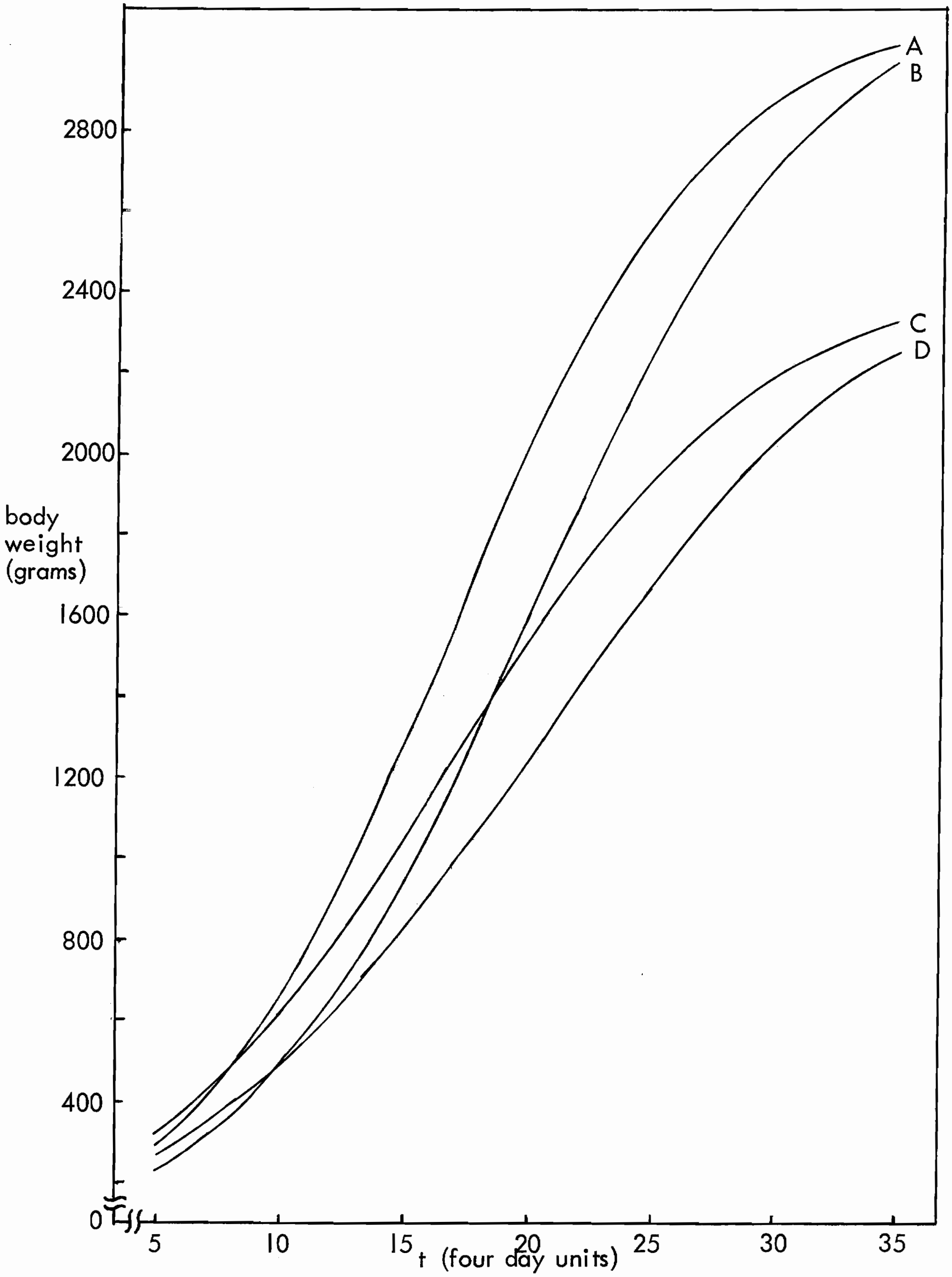

FIGURE 6.1 Functional observations for male chickens

Thus it is shown that the individual (B) which is heaviest at the termination of growth is smallest at twenty days of age. The reverse is true for individual (C). It can be observed that the functional observations cross one another, so that at these ages two individuals would be judged equal with respect to body weight, however at another age they differed considerably. (For example, B and C at $t = 18.7$, B and D at $t = 10.2$ and the same compared at $t = 35$.) It is obvious that knowing the relationship between body weights of two individuals at a specific age does not insure that the same relationship will exist at all ages.

Use of individual body weights at an arbitrary age as a characterization of growth may thus be ambiguous. A better procedure would be to use functional observations which summarize each individual's entire growth pattern.

If each individual's growth pattern is, to a large extent, under genetic control it may be possible to incorporate this characteristic into breeding plans for various growth traits. Further, it appears that since body weight at a specific age is a heritable trait (Siegel, 1962, gives a summary of such statistics), then a trait which incorporates an animal's weight at several ages should be, at least, as highly heritable.

A discussion of the heritability of juvenile growth as measured by functional observations will be given in the next section. The structural relationships between male and female growth patterns will also be discussed and a set of structural equations given which relate them.

6.2 Functional Observations of the A-C Sample of Chickens

Some of the procedures and results already discussed in this paper have been evaluated on the basis of how they reflect body growth of a sample of A-C chickens. These chickens were reared according to procedures discussed in Chapter III. When they were 140 days old there were 98 females and 78 males in the sample. The remainder of this study will involve only these animals.

A functional observation was computed for each individual in the sample. Each functional observation was an element of the parametric family,

$$\eta(t) = \frac{\beta}{1 + \alpha\rho^t} \quad .$$

Six stochastic parameters were also estimated for each individual, they were:

$$\begin{array}{lll} \beta_i & \rho_i & \tau_i \\ \alpha_i & \gamma_i & R_i \quad . \end{array}$$

The functional observations were computed using an IBM 1620 digital computer which used the program described in Appendix D. Although various amounts of time were required to find the solution for the stochastic parameters of different individual data sets, usually convergence to seven significant figures was obtained in two and one-half minutes. This amounted to approximately six iterations for suitable starting values. The starting values were found by comparing data sets to be computed with those already computed. The first few sets of starting values were found by solving

$$200 = \frac{b_i}{1 + a_i}$$

$$y_{18} = \frac{b_i}{2}$$

and trying various values of r_i. These were of course crude, but gave satisfactory results.

A listing of a set of stochastic parameters which uniquely define each functional observation, namely $\hat{\beta}_i$, $\hat{R}_i$, and $\hat{\tau}_i$ are given by pedigree in Appendix F for each individual animal.

An analysis of variance and covariance components (hierarchal classification, Model II (Eisenhart, 1947)) was computed for the stochastic parameters. The analysis performed is similar to that discussed by King and Henderson (1954) for chickens. An outline of the one used here, for unequal subclass numbers, with symbolic expected mean squares and expected mean cross products is given in Table 6.1.

Table 6.1. Outline of analysis of variance used to analyse the individual stochastic parameters, with expected mean squares and products.

Source	DF	E(MS)*	E(MP)**
Sires (S)	s-1	$\sigma^2_E+n_2\sigma^2_{D:S}+n_3\sigma^2_S$	$\rho_{EE'}\sigma_E\sigma_{E'}+n_2\rho_{DD'}\sigma_D\sigma_{D'}+n_3\rho_{SS'}\sigma_S\sigma_{S'}$
Dams (D:S)	d-s	$\sigma^2_E+n_1\sigma^2_{D:S}$	$\rho_{EE'}\sigma_E\sigma_{E'}+n_1\rho_{DD'}\sigma_D\sigma_{D'}$
Progeny (P:D:S)	N-d	σ^2_E	$\rho_{EE'}\sigma_E\sigma_{E'}$
Total	N-1		

* The notation is as given by King and Henderson (1954).

**This notation follows the discussion by McBride (1958); the primed subscripts indicating differences in micro environments for the different traits. The E(MP) follows from a paper by Henderson (1953).

Tabulations of the estimated components with indicated degrees of freedom can be found in Tables 6.2, 6.3, 6.4, 6.5, 6.6, and 6.7. Each table is a 6 x 6 arrangement of these components, within a classification and sex. Tables 6.2 - 6.4 give those estimated from males, Tables 6.5 - 6.7 contain a tabulation obtained from the female stochastic parameters. These results were obtained from all possible analysis of variance and "analysis of cross products" among the six stochastic parameters for the male and female estimates. A total of 42 such analyses were thus performed.

A summary of the means, within sibship standard deviations and the corresponding coefficients of variation for these stochastic parameters are given in Table 6.8 for male and female progeny. These measurements indicate the differences between the stochastic parameters for the different sexes. Males were found to reach heavier asymptotic weights (β) and have a higher maximum growth rate (R), but little difference was found in their ages at half final juvenile weight (τ). Differences can be seen between the other stochastic parameters. The coefficients of variation are relatively small for both sexes, however those estimated from female progeny were larger (five out of six) than those from males' data.

Table 6.2. Estimates of male variance and covariance components for the Progeny (full-sib) classification, d.f. = 44.

stochastic			parameters		
γ	τ	β	α	ρ	R
8.017×10^{-5}	-4.934×10^{-3}	-2.030×10^{-1}	1.526×10^{-2}	-6.763×10^{-5}	4.628×10^{-2}
	5.756×10^{-1}	6.780×10^{-1}	6.398×10^{-1}	3.386×10^{-3}	1.382×10^{-1}
		4.264×10^{4}	1.822×10^{2}	1.481×10^{-1}	1.691×10^{3}
(symmetric)			8.441×10^{0}	-1.305×10^{-2}	1.838×10^{1}
				5.729×10^{-5}	-4.000×10^{-2}
					1.049×10^{2}

Table 6.3. Estimates of male variance and covariance components for the Dam classification, d.f. = 24.

stochastic			parameters		
γ	τ	β	α	ρ	R
9.992×10^{-5}	-4.934×10^{-3}	-5.169×10^{-1}	2.169×10^{-2}	-8.425×10^{-5}	4.931×10^{-2}
	3.387×10^{-1}	1.678×10^{1}	-7.621×10^{-1}	4.173×10^{-3}	-2.921×10^{0}
		1.952×10^{4}	-1.447×10^{2}	4.463×10^{-1}	4.342×10^{2}
(symmetric)			5.738×10^{0}	-1.823×10^{-2}	9.030×10^{0}
				7.094×10^{-5}	-4.121×10^{-2}
					5.287×10^{1}

Table 6.4. Estimates of male variance and covariance components for the Sire classification, d.f. = 9.

stochastic			parameters		
γ	τ	β	α	ρ	R
0*	$1.700\text{x}10^{-3}$	$1.109\text{x}10^{-1}$	$-2.237\text{x}10^{-3}$	$1.875\text{x}10^{-5}$	$-2.130\text{x}10^{-2}$
	0*	$1.212\text{x}10^{1}$	$2.768\text{x}10^{-1}$	$-1.437\text{x}10^{-3}$	$1.740.10^{0}$
		$1.148\text{x}10^{4}$	$-8.852\text{x}10^{0}$	$9.300\text{x}10^{-2}$	$3.984\text{x}10^{2}$
(symmetric)			$2.047\text{x}10^{-1}$	$1.872\text{x}10^{-3}$	$-2.261\text{x}10^{0}$
				0*	$1.795\text{x}10^{-2}$
					$8.058\text{x}10^{-1}$

*Variance component estimate was negative.

Table 6.5. Estimates of female variance and covariance components for the Progeny (full-sib) classification, d.f. = 60.

stochastic			parameters		
γ	τ	β	α	ρ	R
1.280×10^{-4}	-9.041×10^{-3}	-9.166×10^{-1}	1.426×10^{-2}	-1.097×10^{-4}	3.356×10^{-2}
	1.072×10^{0}	1.283×10^{2}	7.588×10^{-2}	7.740×10^{-3}	7.266×10^{-2}
		3.983×10^{4}	3.919×10^{1}	7.833×10^{-1}	1.046×10^{3}
(symmetric)			4.418×10^{0}	-1.226×10^{-2}	9.271×10^{0}
				9.408×10^{-5}	-2.888×10^{-2}
					5.897×10^{1}

Table 6.6. Estimates of female variance and covariance components for the Dam classification, d.f. = 28.

stochastic			parameters		
γ	τ	β	α	ρ	R
0*	-1.733×10^{-3}	-2.352×10^{-1}	6.454×10^{-3}	1.225×10^{-7}	-8.665×10^{-3}
	4.177×10^{-2}	9.936×10^{0}	4.915×10^{-1}	-1.454×10^{-3}	1.023×10^{0}
		1.244×10^{4}	-7.020×10^{1}	2.056×10^{-1}	3.258×10^{2}
(symmetric)			3.574×10^{0}	-5.468×10^{-3}	2.197×10^{-1}
				0*	7.543×10^{-3}
					7.348×10^{0}

*Variance component estimate was negative.

Table 6.7. Estimates of female variance and covariance components for the Sire classification, d.f. = 9.

stochastic			parameters		
γ	τ	β	α	ρ	R
5.359×10^{-6}	-2.604×10^{-4}	-2.892×10^{-2}	7.818×10^{-4}	-4.576×10^{-6}	1.716×10^{-3}
	0*	9.502×10^{0}	-1.075×10^{-1}	2.131×10^{-4}	2.914×10^{-1}
		1.035×10^{3}	1.878×10^{1}	2.386×10^{-2}	3.144×10^{1}
(symmetric)			0*	-6.862×10^{-4}	1.285×10^{0}
				3.907×10^{-6}	-1.491×10^{-3}
					2.315×10^{0}

*Variance component estimate was negative.

Table 6.8. Estimates of the mean, within sibship standard deviation and the corresponding coefficient of variation for male (N=78) and female (N=98) stochastic parameters.

stochastic parameter estimated	$\hat{\mu}$		$\hat{\sigma}_E$		C.V.	
	♂♂	♀♀	♂♂	♀♀	♂♂	♀♀
β	2776.4	2145.9	206.50	199.58	7.438	9.300
α	21.618	16.306	2.9054	2.1019	13.44	12.89
ρ	.84353	.85750	.00757	.00970	.8973	1.131
γ	.17025	.15379	.00895	.01131	5.259	7.356
τ	18.009	18.109	.75866	1.0355	4.213	5.718
R	117.97	82.210	10.242	7.6793	8.682	9.341

Heritability estimates were calculated for each stochastic parameter. The maternal and paternal half-sib and full-sib heritabilities were obtained according to the following formulae:

The paternal heritability estimate based on half-sib performance,

$$h_S^2 = \frac{4\hat{\sigma}_S^2}{(\hat{\sigma}_S^2 + \hat{\sigma}_D^2 + \hat{\sigma}_E^2)} \quad ;$$

the maternal heritability estimate based on half-sib performance

$$h_D^2 = \frac{4\hat{\sigma}_D^2}{(\hat{\sigma}_S^2 + \hat{\sigma}_D^2 + \hat{\sigma}_E^2)} \quad ;$$

and the full-sib heritability estimate,

$$h^2_{S+D} = \frac{2(\hat{\sigma}^2_S + \hat{\sigma}^2_D)}{(\hat{\sigma}^2_S + \hat{\sigma}^2_D + \hat{\sigma}^2_E)} \quad ;$$

were estimated for male and female progeny separately, using the component estimates given in Tables 6.2 through 6.7.

These heritabilities, for male and female progeny, are given in Table 6.9 for each stochastic parameter. The maternal heritability of some of these characters is obviously too high to be reflecting only additive genetic variance. Overall, the stochastic parameters associated with male progeny were under greater genetic influence than those for

Table 6.9. Estimates of h^2_S, h^2_D, and h^2_{S+D} for the six stochastic parameters for male and female progeny.

stochastic parameter	h^2_S		h^2_D		h^2_{S+D}	
	♂♂	♀♀	♂♂	♀♀	♂♂	♀♀
β	.6232	.0776	1.0600	.9336	.8416	.5056
α	.0568	.0000	1.5956	1.7940	.8262	.8970
ρ	.0000	.1592	2.2126	.0000	1.1064	.0796
γ	.0000	.1604	2.2192	.0000	1.1096	.0802
τ	.0000	.0000	1.4820	.1496	.7410	.0748
R	.0200	.1348	1.3336	.4280	.6768	.2814

females as is apparent from the h^2_{S+D} estimates. Estimates of h^2_D were generally greater for male progeny than for females; h^2_S estimates were small for both sexes, except for male progeny β.

There are several reasons why differences in σ^2_S and σ^2_D, in the numerators of h^2_S and h^2_D, can occur. It is well known (Lerner, 1958) that σ^2_S for female progeny includes 1/4 of the additive genetic variation 1/16 epistasis plus sex-linked effects. σ^2_D for female progeny also includes 1/4 of the additive genetic variation, 1/4 of the dominance less than 3/16 epistasis and maternal effects, but no sex-linked effects. Therefore the heritability estimates h^2_S and h^2_D for female progeny reflect these different sources of genetic variation. These estimates, then for the stochastic parameters β, α, τ, and R, indicate possible maternal and/or dominance effects since h^2_D were greater than h^2_S. ρ and γ for female progeny were lowly heritable and the reverse relationship among h^2_S and h^2_D was observed indicating the possibility of sex-linkage.

For male progeny the $\hat{\sigma}^2_D$ for the stochastic parameters were larger than the $\hat{\sigma}^2_S$, as is idicated by their h^2_D and h^2_S. This could indicate epistasis and/or dominance since σ^2_D contains less than 3/16 epistatic variance while σ^2_S contains only 1/16, σ^2_D contains 1/4 of the dominance and maternal effects while σ^2_S has none, and both contain 1/4 of the additive genetic variance. Each of these heritability estimates assume a random mating population. This was very likely violated since the males actually represented a sub-population with respect to a behavior trait. This may have resulted in some homogeneity among the males as regards growth. In addition, natural mortality among males, resulting from the dominance-subordinate relationships may have further delimited the population of males with respect to growth characteristics.

Therefore, although the stochastic parameters are apparently highly heritable, the estimates given in Table 6.9 may be biased. (It should be noted that only 10 sires were involved in the study, hence σ^2_S may be poorly estimated.)

The correlations ($\hat{\rho}_{EE'}$) among the stochastic parameters are given in Tables 6.10 and 6.11 for male and female progeny, respectively. γ and ρ are seen to be highly correlated for males and females. This probably is the result of their functional relationship, since

$$\rho = e^{-\gamma} \quad .$$

Maximum growth rate (R) and asymptotic juvenile weight (β) are highly correlated in both sexes (.7993 and .6827 for

Table 6.10. Correlations among the stochastic parameters for male progeny.

stochastic			parameters		
γ	τ	β	α	ρ	R
1	-.5976	-.1098	.5866	-.9979	.5048
	1	.4330	.2903	.5896	-.0176
		1	.3037	.0948	.7993
(symmetric)			1	-.5933	.6176
				1	-.5161
					1

Table 6.11. Correlations among the stochastic parameters for female progeny.

stochastic			parameters		
γ	τ	β	α	ρ	R
1	-.7717	-.4060	.5999	-.9999	.3863
	1	.6207	.0349	.7706	.0091
		1	.0934	.4046	.6827
(symmetric)			1	-.6012	.5743
				1	-.3878
					1

male and female progeny respectively). However, τ and R are nearly uncorrelated(.0176 and .0091 for males and females respectively), τ and β are moderately correlated (.4330 and .6207, males and females respectively).

Earlier it was suggested that writing the functional observation in terms of β_i, τ_i, and R_i had appeal because these stochastic parameters are easily interpreted. Since R and τ are nearly uncorrelated, considerable selection pressure presumably could be applied to them. However, R and β are correlated to the extent that it may be impossible to find functional observations in the population characterized by small R and large β or vice versa. Therefore, should a mating plan be advanced with selection pressure on τ and R, its probable effect on β can be evaluated through the R, β correlation. Thus if the mating plan for τ and R was arranged in a 2^2 factorial so that individuals with, say, high and low τ and high and low β (relative to the mean τ and β) were mated in all possible configurations, the genetic advance expected in β as a result of having selected R could be evaluated using an expression given by Siegel (1962a):

$$\frac{\Delta\hat{G}_{\beta}}{\Delta G_{R}} = \frac{h_{S+D,\beta}}{h_{S+D,R}} \; \frac{\hat{\sigma}_{E,\beta}}{\hat{\sigma}_{E,R}} \; r_{\beta R} \quad .$$

In the above

$\Delta\hat{G}_{\beta}$ = expected change in β,

ΔG_{R} = change in R,

$h_{S+D,\beta}$ and $h_{S+D,R}$ = average heritability (square root) estimates for β and R, respectively,

$\hat{\sigma}_{E,\beta}$ and $\hat{\sigma}_{E,R}$ = estimated standard deviations for β and R, respectively,

and

$r_{\beta R}$ = usual product moment correlation between β and R.

Use of this method assumes additive gene action and a constant genetic correlation. Using the above formulae $\Delta\hat{G}_{\beta}$ for male and female progeny are 63.04 grams and 87.46 grams respectively, based on the previously tabulated heritability correlation estimates.

Such an experiment as this would be useful in studying, more thoroughly, the inheritance of growth patterns and would make possible the estimation of the regression of offspring performance on parent, a superior estimate of the heritability of functional observations of growth.

6.3 Structural Relationships between Male and Female Stochastic Parameters

In order that selection procedures might be more appropriately applied, it is necessary to investigate the relationships existing between the stochastic parameters for males and females of similar genetic background. This necessity is due to the absence of any *a priori* information as to what kind of female functional observation might correspond most closely genetically to a given male functional observation. Since the stochastic parameters of the functional observation characterize the function, any relationships existing must be expressed through them. A proposal to investigate this relationship for the present data (in which parental stochastic parameters are not known) is given below.

Suppose that a set of structural equations exists which relate male offspring performance to that of their sires. In particular, let the son's performance be measured by a vector of stochastic parameters, say $\underline{Y}_i$ and his sire's corresponding stochastic parameter vector be $\underline{X}_1$. Let the structural relationships between the two vectors be B_{11}, then the set of structural equations can be written as

$$(6.1) \qquad \underline{Y}_1 = B_{11} \underline{X}_1 \quad .$$

A set relating the son's stochastic parameter vector to that of his dam's, $\underline{X}_2$, could be written as

$$\underline{Y}_1 = B_{12}\, \underline{X}_2 \quad . \tag{6.2}$$

Similarly, sets relating daughters' vectors of stochastic parameters to those of their sires' or dams' could be written as

$$\underline{Y}_2 = B_{21}\, \underline{X}_1 \tag{6.3}$$

$$\underline{Y}_2 = B_{22}\, \underline{X}_2 \quad . \tag{6.4}$$

The vector subscripts denote a male's stochastic parameters if 1, a female's if 2; the subscripts on B reflect the subscript of the predicted vector and variable vector in that order. Vectors of stochastic parameters denoted by $\underline{Y}$ are reserved for offspring and those written as $\underline{X}$ indicate parent vectors. Estimation of B_{11}, B_{12}, B_{21}, and B_{22} require that the parental stochastic parameters be known.

Two sets of structural equations are thus written which depend on the same parental vector. This fact can be symbolically used to define a new set of structural equations relating daughters and sons of the same parentage. (6.1) and (6.3) can be written as

(6.5) $$\underline{X}_1 = B_{11}^{-1}\,\underline{Y}_1$$

and

(6.6) $$\underline{X}_1 = B_{21}^{-1}\,\underline{Y}_2 \quad ,$$

respectively, which when equated give

$$B_{11}^{-1}\,\underline{Y}_1 = B_{21}^{-1}\,\underline{Y}_2$$

or

(6.7) $$\underline{Y}_2 = B_{21}B_{11}^{-1}\,\underline{Y}_1 \quad ,$$

if all B matrices are non-singular. Similarly the use of (6.2) and (6.4) give

$$B_{12}^{-1}\,\underline{Y}_1 = B_{22}^{-1}\,\underline{Y}_2$$

or

(6.8) $$\underline{Y}_2 = B_{22}B_{12}^{-1}\,\underline{Y}_1 \quad .$$

The sets of structural equations (6.7) and (6.8) are free of the parental vectors and relate son and daughter stochastic parameters. The set (6.7) was found by holding $\underline{X}_1$ constant and (6.8) by holding $\underline{X}_2$ constant. The resulting structural equations suggest using

(6.9) $$\underline{Y}_2 = \underline{A} + B\,\underline{Y}_1 \quad ,$$

that is, relating full-sibs for which the unknown dam and sire vectors would be equal. The vector $\underline{A}$ allows for possible intercepts and the matrix B for coefficients of the male stochastic parameters.

Each sibship involved in the biological study usually involved chicks of both sexes. However in some cases only male or only female chicks were obtained within a full-sibship at hatching. Of the surviving sibships there were thirty-one which had both male and female members. Within the jth sibships there were, say, n_{1j} males and n_{2j} females represented.

From the previous discussion of the average heritability of the stochastic parameters, it was seen that the performance of male offspring was more heritable than was that of the female progeny (see Table 6.10).

The structural equations (6.9) were estimated to study the structural relationship between male and female full-sibs according to the following procedure. Average values of the six stochastic parameters were found for males and females within each sibship. Then the set of stochastic parameter means for female full-sibs was weighted by n_{2j}, their number. These weighted means were then regressed on the son means. The structural equations were thus estimated using the model:

$$(6.10) \qquad \underline{\bar{Y}}_2 = A + B\, \underline{\bar{Y}}_1 + \underline{\bar{\varepsilon}} \quad ,$$

where $\underline{\bar{\varepsilon}}$ is a vector of additive random errors, assumed to be

$$NID(0, \sigma^2 \text{ diag. } \frac{1}{n_{2j}}) \quad ,$$

and $\underline{\bar{Y}}_1$ and $\underline{\bar{Y}}_2$ represent the vectors of means of full-sib male and female stochastic parameters. In particular, they were the respective male or female means of $\hat{\beta}$, $\hat{\alpha}$, $\hat{\rho}$, $\hat{\tau}$, $\hat{\gamma}$, and $\hat{R}$.

An ordinary multiple regression analysis was performed and variables eliminated which did not contribute significantly ($\alpha = .05$) to the sum of squares accounted for by the structural equations. Each female stochastic parameter was thus predicted using all six male means, then successive eliminations were made, until a reasonably optimal set was obtained. Among all six female parameters a set of three was selected which would uniquely define a functional observation. The resulting equations then, were:

$$\hat{R}_{♀} = 53.96772 + .22972\ \bar{R}_{♂}$$

(6.11)
$$\hat{\beta}_{♀} = 1226.625 + 7.63613\ \bar{R}_{♂}$$

$$\hat{\tau}_{♀} = -10.44014 + .10239\ \bar{\alpha}_{♂} + 31.9669\ \bar{\rho}_{♂} \quad .$$

Note that no requirement was made of this set to be invertable, i.e., it should only be used to predict female stochastic parameters.

These equations should have some merit in predicting what female is similar to a given male, for they are estimated on the basis of brother-sister relationships.

These equations suggest an experiment to evaluate the genetic compatability of females and males whose mating they predict. If these equations identify like genotypic compositions, the resulting functional observations of the progeny should be more uniform than would be expected from random mating.

A set of stochastic parameters for females was predicted using (6.11) for each male in the A-C sample of chickens. Then an actual bird was found among the females in the sample which had similar values for her particular stochastic parameters.

An example of such a predicted mating is given below. The predicted parameter set for a female was found from (6.11) to be

$$\beta = 2262$$

$$R = 85.13$$

$$\tau = 18.16 \quad ,$$

based on the stochastic parameters of a male with parameters

$\hat{\gamma} = .1689$ $\hat{\alpha} = 22.02$

$\hat{\tau} = 18.31$ $\hat{\rho} = .8446$

$\hat{\beta} = 3213$ $\hat{R} = 136.7$.

A female was then found among the A-C sample which had stochastic parameters

$$\hat{\beta} = 2247$$

$$\hat{R} = 85.10$$

$$\hat{\tau} = 18.17 \quad ,$$

which is seen to be close to the desired values.

CHAPTER VII

SUMMARY AND SUGGESTED FURTHER RESEARCH

7.1 Summary

A formulation is presented which describes a new method of handling responses which are characteristically a function relating measurable random variables to mathematical variables. The formulation considers each response a random function which is called a functional observation. This statistical formulation results in each response being summarized in a convenient functional form which can be used to derive other functional relationships which describe the response.

A particular formulation is derived to study the juvenile growth of chickens. The parametric family

$$\eta(t) = \frac{\beta}{1 + \alpha\rho^t}$$

is proposed as being most appropriate, after evaluating several alternative forms using actual growth data, to generate functional observations which represent the individual growth patterns. Each functional observation is characterized by an individual's set of β, α, and ρ, called stochastic parameters and designated β_i, α_i, and ρ_i. Thus a particular functional observation may be written as

$$Y_{it} = \frac{\beta_i}{1 + \alpha_i \rho_i^t} \quad ,$$

where Y_{it} represents the body weight of individual i at age t. The functional observations represent juvenile growth, a period when an individual chicken progressed from 20 to 140 days of age.

A biological experiment provided the actual growth data of the chickens. A sample (approximately 200 animals) of Athens-Canadian Random Bred chickens were used. Each bird was weighed every four days for a period of about five months. These data were used in this study to evaluate the appropriateness of various statistical formulations and to estimate the heritability of the stochastic parameters and through them the functional observations.

Several growth characteristics were derived from the functional observations to describe chicken growth. It was found that β (final juvenile weight), R (maximum growth rate), and τ (age at half final juvenile weight and age when R occurs), could be used to express the parametric family. Thus the alternative representation,

$$\eta(t) = \frac{\beta}{1 + e^{-\frac{4R}{\beta}(t-\tau)}} \quad ,$$

generates functional observations with meaningful biological interpretations.

The probability structure of a functional observation, a random function, is developed and discussed. The stochastic parameters are assumed to be elements of particular probability densities which with the form specified for η define the probability structure.

A particular probability structure is found by assuming different independent log-normal probability densities for α_i, ρ_i, and β_i. The resulting conditional probability density of Y given t could not be obtained in a closed form. A series to calculate the ordinates of $f(Y|t)$ was obtained,

$$f(Y|t) = \{\frac{e^{-\frac{1}{2\sigma_\beta^2}(\ell nY-\mu_\beta)^2-\frac{\mu_X}{2\sigma_X^2}}}{2\pi\ Y\sigma_X\sigma_\beta}\}$$

$$\cdot \{\sum_{i=0}^{2^n-1} \frac{1}{2^n}\, e^{(M-N)[A-C_1(M-N)]+N(B-C_2N)}\} \quad ,$$

where $M = \ell n(1 - i/2^n)$,

$N = \ell n(i/2^n)$,

$$A = [(\mu_\alpha + t\mu_\rho)/(\sigma_\alpha^2 + t^2\sigma_\rho^2)] - 1 = \frac{\mu_X}{\sigma_X^2} - 1 ,$$

$$B = \left(\frac{\ell nY - \mu_\beta}{\sigma_\beta^2}\right) - 2 ,$$

$$C_1 = 1/2\sigma_X^2 , \quad \text{and} \quad C_2 = 1/2\sigma_\beta^2 .$$

Several conditional probability densities were computed numerically and were found to be slightly positively skewed, with variance increasing from $t = 5$ to $t = 22.5$, then decreasing slightly. Their modal ordinates varied indirectly with the apparent variance, projecting approximately a general logistic in the (t,Y) plane. This probability structure is seen to be realistic in that it reflects characteristics commonly observed from actual growth data.

The inheritance of entire growth patterns was estimated using the stochastic parameters which characterize the functional observations. Heritability estimates were found for male and female progeny separately. Overall, the heritability of stochastic parameters associated with male progeny were larger than those for females. The individual heritability estimates were found to reflect some additive genetic variance but also dominance, and epistasis for male stochastic parameters, and maternal and/or dominance effects for female stochastic parameters. This may have been the result of a bias in sampling sires, giving a poor estimate of σ_S^2.

A set of structural equations are proposed which relate stochastic parameters of full-sibs of opposite sex. These equations were derived as a means of determining genetic similarities in male and female growth patterns. These equations may also be used to develop a mating scheme, by matching actual female stochastic parameters to those predicted by the equations, to possibly produce progeny with uniform growth patterns.

7.2 Suggestions for Further Research

This paper has laid the ground work through theoretical formulation and biological experimentation, of a new statistical method for evaluating a continuous functional response. The statistical formulation has been in terms of functional observations and their application to juvenile growth of chickens.

Other responses could be investigated in the general framework of functional observations. The specific problems associated with such a study would be similar to those discussed in this paper.

In particular, considerably more research could be undertaken to extend this study of juvenile growth of chickens. Perhaps a more appropriate parametric family could be advanced with functional observations characterizing the response for all values of t, determined without assuming serial independence in the noise. This necessarily presumes a study of the correlation structure of the noise. The probability structure of functional observations could be investigated much more thoroughly from many aspects.

Several genetic experiments have been proposed and could be performed, utilizing the sample of A-C chickens used in this study as parental stock. Besides estimating heritabilities, these experiments could be continued for several generations, with selection maintained for several distinct growth patterns. In this way it may be possible to develop lines with uniform growth patterns.

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BIBLIOGRAPHY

Abplanalp, H. and I. L. Kosin. 1952. Heritability of body measurements in turkeys. Poultry Science 31:781-791.

Aitchison, John and J.A.C. Brown. 1957. The Lognormal Distribution. Cambridge University Press.

Barott, H. G., J. C. Fritz, E. M. Pringle, and H. W. Titus. 1938. Heat production and gaseous metabolism of young male chickens. Journal of Nutrition 15:145-167.

Bray, D. F. and E. L. Iton. 1962. The effect of egg weight on strain differences in embryonic and post embryonic growth in the domestic fowl. British Poultry Science, Volume 3, Number 3, pp. 175-188.

Eisenhart, Churchill. 1947. The assumptions underlying the analysis of variance. Biometrics 3:1-21.

Federighi, Enrico T. 1959. Extended tables of the percentage points of Student's t-distribution. Journal of the American Statistical Association 54:683-688.

Feller, W. 1957. An Introduction to Probability Theory and Its Applications. Volume I. John Wiley and Sons.

Hald, A. 1952. Statistical Theory with Engineering Applications. John Wiley and Sons, Inc.

Henderson, C. R. 1953. Estimation of variance and covariance components. Biometrics 9:226-252.

Hess, Carl W. 1962. Randombred populations of the Southern Regional Poultry Breeding Project. World's Poultry Science Journal 18:147-152.

Hutt, F. B. and R. D. Crawford. 1960. On breeding chicks resistant to pullorum disease without exposure thereto. Canadian Journal of Genetics and Cytology 2:357-370.

Kavanau, J. Lee. 1960. A model of growth and growth control in mathematical terms, II. Compensatory organ growth in the adult. Proceedings of the National Academy of Science 46:1658-1673.

Kavanau, J. Lee. 1961. Predictions of the growth model for normal chicken growth. Science 134:1627-1628.

King, S. C. and C. R. Henderson. 1954. Variance components analysis in heritability studies. Poultry Science 33:147-154.

Lerner, I. Michael. 1958. The Genetic Basis of Selection. John Wiley and Sons, Inc.

McBride, G. 1958. The environment and animal breeding problems. Animal Breeding Abstracts, Volume 26, Number 4, pp. 349-358.

McDevitt, John H. 1956. Relationship between egg weight, initial chick weight, and growth rate of the chick for a period of eight weeks. Unpublished M.S. thesis. The Library, Virginia Polytechnic Institute, Blacksburg, Virginia.

Myers, M. H. 1960. Conditional distributions arising from variation of parameters in non-linear response functions. Unpublished thesis. The Library, Virginia Polytechnic Institute, Blacksburg, Virginia.

Nelder, J. A. 1961. A note on some growth patterns in a simple theoretical organism. Biometrics 17:220-228.

Nelder, J. A. 1962. An alternative form of a generalized logistic equation (Note). Biometrics 18:614-616.

Pearl, Raymond. 1928. Biology of Population Growth. Knopt, Inc.

Rao, C. Radhakrishna. 1958. Some statistical methods for comparison of growth curves. Biometrics 14:1-17.

Richards, F. J. 1959. A flexible growth function for empirical use. Journal of Experimental Botany 10:290-300.

Severo, N. C. and E. G. Olds. 1956. A comparison of tests on the mean of a logarithmico-normal distribution with known variance. Annals of Mathematical Statistics 27:670-686.

Siegel, Paul B. 1962. Selection for body weight at eight weeks of age. Poultry Science 41:954-962.

Siegel, Paul B. 1962a. A double selection experiment for body weight and breast angle at eight weeks of age in chickens. Genetics 47:1313-1319.

Stevens, W. L. 1951. Asymptotic regression. Biometrics 7:247-267.

Turner, Malcolm E., Robert J. Monroe, and Henry L. Lucas, Jr. 1961. Generalized asymptotic regression and non-linear path analysis. Biometrics 17:120-143.

Verhagen, A.M.W. 1960. Growth curves and their functional form. The Australian Journal of Statistics 2:122-127.

Weiss, Paul and J. Lee Kavanau. 1957. A model of growth and growth control in mathematical terms. The Journal of General Physiology 41:1-47.

Wishart, J. 1938. Growth rate determination in nutrition studies with the bacon pig, and their analysis. Biometrika 30:16-28.

Yoshida, Minoru and Hiroshi Morimoto. 1961. Normal growth curve of White Leghorn chicks. Bulletin of the National Institute of Agricultural Sciences (Japan), Series G, Number 20:167-175.

VITA

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After graduation from high school he attended Kansas State College at Manhattan, Kansas, for one year. During the next few months he operated a livestock farm with his father. In January,1954, he was drafted into the U. S. Army where he served on active duty until October, 1955. He was honorably discharged in June, 1962.

He returned to Kansas State University at Manhattan, Kansas, in October, 1955, where he obtained a B.S. in Agriculture in January, 1958. After a year in the Graduate School there, he was awarded a M.S. degree in January, 1959, having studied invertebrate toxicology and statistics.

In September, 1959, he joined the staff of the Department of Statistics at Kansas State University as an instructor. His principal duties involved teaching statistical subjects and consulting with resident staff associated with the Agricultural Experiment Station.

He was married to Janet D. Moyer in June, 1956. They have three children, Karen, Heidi, and Seth.

Mr. Krause is a member of the scholastic societies: Gamma Sigma Delta, Sigma Xi, and Pi Mu Epsilon. He is also a member of the Biometric Society, American Statistical Association, and The Central States Entomological Society.

Gary L. Krause

A P P E N D I C E S

APPENDIX A

Functional Observation:

$$Y_{it} = \frac{\beta_i}{1 + \left(\frac{A_i}{t}\right)^{R_i}} + \varepsilon_{it} \quad ,$$

depressing subscripts and writing

$$Q_t = 1 + \left(\frac{A}{t}\right)^R$$

and

$$\Psi = \frac{\beta}{Q_t} \quad ,$$

the normal equations which must be solved result from differentiating

$$L = \lambda\Sigma\left(y_t - \frac{\beta}{Q_t}\right)^2 + \lambda'$$

with respect to β, A, and R successively (λ and λ' do not involve A, β, and R.),

$$\underline{X} = \begin{bmatrix} \frac{\partial L}{\partial \beta} \\ \\ \frac{\partial L}{\partial R} \\ \\ \frac{\partial L}{\partial A} \end{bmatrix} = -2\lambda \begin{bmatrix} \sum (y_t - \Psi) \frac{1}{Q_t} \\ \\ -\sum (y_t - \Psi) \frac{\left(\frac{A}{t}\right)^R \ln\left(\frac{A}{t}\right)}{Q_t^2} \\ \\ -\, BRA^{R-1} \sum (y - \Psi) \frac{1}{t^R Q_t^2} \end{bmatrix} .$$

Solving $\underline{X}$ set equal to zero gives estimates b, r, and a.

The iteration matrix V is, writing q for Q_t which now involves estimates,

$$V = -2 \begin{bmatrix} \sum \frac{1}{q^2} & -\sum \frac{(\frac{a}{t})^r \ \ell n(\frac{a}{t})}{q^3} & -bra^{r-1} \sum \frac{1}{t^r q^3} \\ & \sum \frac{(\frac{a}{t})^{2r} \ell n(\frac{a}{t})^{2r}}{q^4} & bra^{r-1} \sum \frac{(\frac{a}{t})^r \ell n(\frac{a}{t})}{t^r q^4} \\ \text{(symmetric)} & & b^2 r^2 a^{2r-1} \sum \frac{1}{t^{2r} q^4} \end{bmatrix} .$$

The iteration equation is

$$\underline{\theta}_k = \underline{\theta}_{k-1} - V^{-1} \ \underline{X}(\underline{\theta}_k) \quad ,$$

which is evaluated repeatedly until

$$\underline{\theta}_k - \underline{\theta}_{k-1} \leq \begin{bmatrix} .0005 \\ .00000005 \\ .000005 \end{bmatrix} \quad ,$$

and the estimates result.

Execution of the FORTRAN digital computer program follows the usual procedure.

Data preparation follows the following format:

card columns

4 - 6 - numerical identification code,

7 - 10 - value of t (as an integer) as a 4-0 number,

11 - 20 - body weight as a 10-5 number.

Only complete sets of data may be used, that is, t must take values 1,2, ..., N where N is the age when the last weight was taken. Data cards must be ordered with respect to t.

One control card is necessary, prepared as follows:

card columns

4 - 6 - numerical identification (same as above),

7 - 10 - number of data cards, 4-0 number,

11 - 20 - starting value for r, 10-5 number,

21 - 30 - starting value for a, 10-5 number,

31 - 40 - starting value for b, 10-5 number,

41 - 50 - accuracy criteria for b, 10-8 number,

51 - 60 - accuracy criteria for r, 10-8 number,

61 - 70 - accuracy criteria for a, 10-8 number,

71 - 80 - blank.

Switch settings:

SW 1	ON	typewriter prints values of a, b, r at the end of each iteration
	OFF	no type out
SW 2	ON	punch out final inverse of V
	OFF	no punch out
SW 3, SW 4		not interrogated .

Deck sequence:

1. Object program
2. Suitable subroutine package
3. Control card
4. Data card
5. Further sets of control cards and data.

Output:

1 card, ID, a, b, r

1 card, X̲ evaluated for a, b, r

n cards, ID, y, predicted y, (w), y-w

if SW 2 ON

2 cards, elements of V^{-1} in the following order,

first card, inverse elements v^{11}, v^{12}, v^{13},

second card v^{22}, v^{23}, v^{33}.

FORTRAN SOURCE STATEMENTS

```
C     ESTIMATES STOCHASTIC PARAMETERS IN YIT=B/(1.+(A/T)**R)
      DIMENSION TN(100)
      DIMENSION W(100)
      DIMENSION Y(200)
 1000 READ 100,ID,N,R,A,B,CRIT1,CRIT2,CRIT3
  100 FORMAT(I6,I4,F10.5,F10.5,F10.5,F10.8,F10.8 ,F10.8 )
      DO 50 I=1,N
      READ 103,ID,J,Y(I)
  103 FORMAT(I6,I4,F10.5)
      IF(J-I+1) 2,50,2
    2 PRINT 101,I,J
      PAUSE
      GO TO 1000
   50 CONTINUE
  101 FORMAT(18H CARD OUT OF ORDER,I5,I5)
      COUNT=0
      DELR=0
      DELB=0
      DELA=0
      TN( 1)= 0.00000000000000
      TN( 2)=  .693147180559945
      TN( 3)= 1.098612288668100
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
TN( 4)= 1.386294361119890
TN( 5)= 1.609437912434100
TN( 6)= 1.791759469228050
TN( 7)= 1.945910149055310
TN( 8)= 2.079441541679830
TN( 9)= 2.197224577336210
TN(10)= 2.302585092994040
TN(11)= 2.397895272798370
TN(12)= 2.484906649788000
TN(13)= 2.564949357461530
TN(14)= 2.639057329615250
TN(15)= 2.708050201102210
TN(16)= 2.772588722239780
TN(17)= 2.833213344056210
TN(18)= 2.890371757896160
TN(19)= 2.944438979166440
TN(20)= 2.995732273553990
TN(21)= 3.044522437723420
TN(22)= 3.091042453358310
TN(23)= 3.135494215929140
TN(24)= 3.178053830347940
TN(25)= 3.218875824868200
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
      TN(26)= 3.258096538021480
      TN(27)= 3.295836866004320
      TN(28)= 3.332204510175200
      TN(29)= 3.367295829986470
      TN(30)= 3.401197381662150
      TN(31)= 3.433987204485140
      TN(32)= 3.465735902799720
      TN(33)= 3.496507561466480
      TN(34)= 3.526360524616160
      TN(35)= 3.555348061489410
      TN(36)= 3.583518938456110
      TN(37)= 3.610917912644220
      TN(38)= 3.637586159726380
      TN(39)= 3.663561646129640
      TN(40)= 3.688879454113930
      TN(41)= 3.713572066704300
      TN(42)= 3.737669618283360
      TN(43)= 3.761200115693560
      TN(44)= 3.784189633918260
      TN(45)= 3.806662489770310
      TN(46)= 3.828641396489090
      TN(47)= 3.850147601710050
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
TN(48)= 3.871201010907890
TN(49)= 3.891820298110620
TN(50)= 3.912023005428140
TN(51)= 3.931825632724320
TN(52)= 3.951243718581420
TN(53)= 3.970291913552120
TN(54)= 3.988984046564270
TN(55)= 4.007333185232470
TN(56)= 4.025351690735140
TN(57)= 4.043051267834550
TN(58)= 4.060443010546410
TN(59)= 4.077537443905710
TN(60)= 4.094344562222100
TN(61)= 4.110873864173310
TN(62)= 4.127134385045090
TN(63)= 4.143134726391530
TN(64)= 4.158883083359670
TN(65)= 4.174387269895630
TN(66)= 4.189654742026420
TN(67)= 4.204692619390960
TN(68)= 4.219507705176100
TN(69)= 4.234106504597250
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
      TN(70)= 4.248495242049350
      TN(71)= 4.262679877041310
      TN(72)= 4.276666119016050
      TN(73)= 4.290459441148390
      TN(74)= 4.304065093204160
      TN(75)= 4.317488113536310
      TN(76)= 4.330733340286330
      TN(77)= 4.343805421853680
      TN(78)= 4.356708826689590
      TN(79)= 4.369447852467020
      TN(80)= 4.382026634673880
      TN(81)= 4.394449154672430
      TN(82)= 4.406719247264250
      TN(83)= 4.418840607796590
      TN(84)= 4.430816798843310
      TN(85)= 4.442651256490310
      TN(86)= 4.454347296253500
      TN(87)= 4.465908118654580
      TN(88)= 4.477336814478200
      TN(89)= 4.488636369732130
      TN(90)= 4.499809670330260
      TN(91)= 4.510859506516850
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
      TN(92)= 4.521788577049040
      TN(93)= 4.532599493153250
      TN(94)= 4.543294782270000
      TN(95)= 4.553876891600540
      TN(96)= 4.564348191467830
      TN(97)= 4.574710978503380
      TN(98)= 4.584967478670570
      TN(99)= 4.595119850134580
    3 CONTINUE
      COUNT=COUNT+1.
      B=B+DELB
      R=R+DELR
      A=A+DELA
      A11=0
      A12=0
      A13=0
      A22=0
      A23=0
      A33=0
      PHI=0
      PSI=0
      THET=0
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
      DO 5 I=1,N
      X=I
      P5=LOGF(A)
      P=P5-TN(I)
      U=EXPF(R*P)
      Z=EXPF(R*TN(I))
      V=U*Z/A
      Q=1.+U
      Q2=Q*Q
      Q3=Q2*Q
      Q4=Q3*Q
      PHI=PHI+(Y(I)/Q-B/Q2)
      PSI=PSI-B*((Y(I)-B/Q)*(U*P)/Q2)
      THET=THET-B*((Y(I)-B/Q)*(R*V/(Z*Q2))   )
      A11=A11+1./Q2
      A13=A13-B*(R*V/(Z*Q3))
      A12=A12-B*(U*P/Q3)
      A33= A33+B*B*(R*R*V*V/(Z*Z*Q4))
      A23=A23+B*B*(R*V*U*P/(Z*Q4))
    5 A22=A22+B*B*(U*U*P*P/Q4)
      D1=A11*(A22*A33-A23*A23)
      D2=A12*(A12*A33-A13*A23)
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
      D3=A13*(A12*A23-A13*A22)
      DETA=D1-D2+D3
      AL11=A22*A33-A23*A23
      AL12=-1.*(A12*A33-A13*A23)
      AL13=A12*A23-A13*A22
      AL22=A11*A33-A13*A13
      AL23=-1.*(A11*A23-A13*A12)
      AL33=A11*A22-A12*A12
      C11=AL11/DETA
      C12=AL12/DETA
      C13=AL13/DETA
      C22=AL22/DETA
      C23=AL23/DETA
      C33=AL33/DETA
      DELB=C11*PHI+C12*PSI+C13*THET
      DELR=C12*PHI+C22*PSI+C23*THET
      DELA=C13*PHI+C23*PSI+C33*THET
      ZB=DELB
      ZR=DELR
      ZA=DELA
      IF(ZB)30,33,31
   30 ZB=ZB*(-1.)
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
   31 IF(ZB-CRIT1)33,33,41
   33 IF(ZR) 32,35,34
   32 ZR=-1.*ZR
   34 IF(ZR-CRIT2)35,35,41
   35 IF(ZA) 36,40,37
   36 ZA=-1.*ZA
   37 IF(ZA-CRIT3)40,40,41
   41 IF(SENSE SWITCH1) 42,3
   42 PRINT111,A,B,R
      PRINT 120,ID,COUNT
  120 FORMAT (I6,F5.1)
      GO TO 3
   40 PUNCH134,ID,A,B,R
      PUNCH111,PSI,PHI,THET
      U=N
      ST=0
      SST=0
      SSAR=0
      DO 6 I=1,N
      X=I
      ST=ST+Y(I)
      SST=SST+Y(I)*Y(I)
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
      W(I)=B/(1.+(A/X)**R)
      E=Y(I)-W(I)
      SSAR=SSAR+E*E
    6 PUNCH 121, ID,I,Y(I),I,W(I),E
      CTSS=SST-ST*ST/U
      SSDR=CTSS-SSAR
      PUNCH130,CTSS
      PUNCH131,SSDR
      PUNCH132,SSAR
      PC=SSDR/CTSS
      PUNCH133,PC
      IF(SENSE SWITCH2)45,1000
   45 PUNCH111,C11,C12,C13
      PUNCH111,C22,C23,C33
      GO TO 1000
  134 FORMAT(I6,2HA=F18.9,2HB=F18.9,2HR=F18.9)
  121 FORMAT(I6,4X,3HY( I4,2H)=F10.1,4X,3HW( I4,2H)=F15.6,4X,F10.5)
  130 FORMAT(20HCORRECTED TOTAL SS =F20.10)
  131 FORMAT( 15HSS DUE TO REG =F20.10    )
  132 FORMAT(14HSS AFTER REG =F20.10)
  133 FORMAT(26HPROPORTION ACCOUNTED FOR =F12.10)
  111 FORMAT(F20.10,F20.10,F20.10)
      END
```

APPENDIX B

Functional observation:

$$W_{it} = C_i + D_i\rho_i^t + \delta_{it} \quad ,$$

depressing the i subscripts the normal equations result from successively differentiating

$$L = \lambda\Sigma(w-C-D\rho^t)^2 + \lambda'$$

with respect to C, D, and ρ (λ and λ' do not involve C, D, and ρ.). The summations are over the N values of of t. The normal equations are

(b.1) $$\frac{\partial L}{\partial C} = -2\lambda\Sigma(w-C-D\rho^t)$$

(b.2) $$\frac{\partial L}{\partial D} = -2\lambda\Sigma(w-C-D\rho^t)\rho^t$$

(b.3) $$\frac{\partial L}{\partial \rho} = -2\lambda\Sigma(w-c-D\rho^t)Dt\rho^{t-1} \quad ,$$

which when set equal to zero give estimates c, d, and r.

From (b.1)

$$c = \Sigma w - d\Sigma r^t \quad ,$$

and substituting this result in (b.2) gives

$$d = \frac{\frac{1}{N}\Sigma w\Sigma r^t - \Sigma wr^t}{\frac{1}{N}(\Sigma r^t)^2 - \Sigma r^{2t}} \quad .$$

Equation (b.4) can be evaluated using both results, giving a function of r only

$$H(r) = 0 = (\Sigma wtr^{t-1} - \frac{1}{N}\Sigma w \Sigma tr^{t-1})$$

$$\times \left(\frac{\frac{1}{N}\Sigma w \Sigma r^{t} - \Sigma wr^{t}}{\frac{1}{N}(\Sigma r^{t})^2 - \Sigma r^{2t}}\right)$$

$$\times \left(\frac{1}{N}\Sigma r^{t} \Sigma tr^{t-1} - \Sigma tr^{2t-1}\right) \quad .$$

Thus c, d, and r are seen to be functions of r only.

The FORTRAN source statements given at the end of this appendix execute both passes described in the body of Section 4.

Execution of Pass I is attained by loading the object deck, appropriate subroutine package, Control Card I, and data deck prepared according to the instructions and conditions given on page 134. Control Card I is prepared as follows:

Card columns

1 - 3 - blank,

4 - 6 - numerical identification,

7 - 10 - number of data cards (N),

11 - 20 - blank,

21 - 30 - 0000010000,

Card columns

31 - 40 - 0000090000,

41 - 80 - blank.

The following switch settings must be made for Pass I:

SW 1 ON punch r and H(r) for each incrementation.

SW 2 ON execution of Pass I

Output Pass I:

9 cards, ID, r, H(r) .

Pass II is accomplished by entering Control Card I, data deck, and Control Card II. Control Card II is prepared according to the following instructions:

Card columns

1 - 10 - starting value of r as a 10-5 number,

11 - 20 - blank

21 - 30 - estimate of H'(r) with appropriate sign punched in cc 21,

31 - 40 - criteria to insure various numbers of decimal points in the estimate of r as a 10-8 number (usually 0000000500).

Switch settings for Pass II:

SW 1, SW 3, SW 4 not interrogated

SW 2 OFF

Output Pass II:

1 card, ID, r, H(r)

1 card, c, d, r

1 card, b, a, r (estimates for 4.9)

N cards, ID, y, $\hat{Y}$ (predicted), $y-\hat{Y}$

4 cards, self explanatory.

FORTRAN SOURCE STATEMENTS

```
C     ESTIMATES STOCHASTIC PARAMETERS IN 1./YIT=WIT=1./B+A/B*R**T
      DIMENSION Y(100)
      DIMENSION C(100)
 1000 READ 100,ID,IE,N,R,DELTA,CONST
  100 FORMAT (I3,I3,I4,F10.5,F10.5,F10.5)
      DO 3 I=1,N,1
      READ 103,ID,IE,J,C(I)
  103 FORMAT (I3,I3,I4,F10.5)
      IF(J-I+1)2,3,2
    2 PRINT 101,I,J
      PAUSE
      GO TO 1000
    3 CONTINUE
   11 DO 6 I=1,N
    6 Y(I)=1./C(I)
      IF (SENSE SWITCH 2) 35,12
   12 READ 106,R,DELTR,DER,SONST
   13 R=R-DELTR
      GO TO 36
   35 R=R+DELTA
   36 SUM=Y(1)
      X=1
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
      DO 4 I=2,N
      RI=I
      Q=RI/(RI-1.)
      X=R*Q*X
    4 SUM=SUM+X*Y(I)
      SUM2=0
      DO 5 I=1,N
    5 SUM2=SUM2+Y(I)
      SUM3=1.
      SUM4=R
      W=1.
      SUM8=R*R
      Z=R*R
      X=1
      DO 8 I=2,N
      RI=I
      G=RI/(RI-1.)
      X=X*G*R
      SUM3=SUM3+X
      W=W*R
      SUM4=SUM4+W*R
      Z=R*R*Z
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
    8 SUM8=SUM8+Z
      SUM9=0
      X=1.
      DO 9 I=1,N
      X=R*X
    9 SUM9=SUM9+Y(I)*X
      SUM10=R
      Z=R*R
      DO 10 I=2,N
      RI=I
      Z=R*R*Z
      Q=Z/R
   10 SUM10=SUM10+RI*Q
      SUM11=SUM4*SUM4
      RN=N
      SUM12=(SUM2*SUM4/RN-SUM9)/(SUM11/RN-SUM8)
      RN=N
      FR=SUM-(SUM2*SUM3)/RN+SUM12*((SUM4*SUM3)/RN-SUM10)
      IF (SENSE SWITCH 2) 38,37
   38 IF (SENSE SWITCH 1 ) 15, 30
   37 G=FR
      IF(G) 40,39,39
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
40 G=G*(-1.)
39 IF (G-SONST) 20,20,19
19 DELTR=FR/DER
   GO TO 13
20 PUNCH 102,ID,IE,R,FR
   A=(SUM2-SUM12*SUM4)/RN
   PUNCH 107,ID,IE,A,SUM12,R
   IF (SENSE SWITCH 3) 1000,50
50 U=0
   B=SUM12
   B=B/A
   A=1./A
   PUNCH 107,ID,IE,A,B,R
   DO 51 I=1,N
   T=A/(1.+B*R**I)
   E=C(I)-T
   F=E
   G=F*F
   U=U+G
51 PUNCH 108,ID,IE,C(I),T,E
   PUNCH 110,ID,IE,U
   V=0
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
      H=0
      DO 52 I=1,N
      T=N
      H=H+C(I)
   52 V=V+C(I)*C(I)
      SST=V-H*H/T
      SSR=SST-U
      RSQ=SSR/SST
      PUNCH 109,ID,IE,SST
  110 FORMAT(I3,I3,27H SUM OF SQUARES AFTER REG =F20.10)
      PUNCH 111,ID,IE,SSR
  111 FORMAT(I3,I3,28H SUM OF SQUARES DUE TO REG =F20.10)
      PUNCH 112,ID,IE,RSQ
  112 FORMAT(I3,I3,33H PROPORTION OF TOTAL SUM TO REG =F10.8)
      GO TO 1000
   30 PUNCH 104,SUM,SUM2,SUM3,SUM4
      PUNCH 105,SUM8,SUM9,SUM10,SUM12
   15 PUNCH 102,ID,IE,R,FR
      IF (R-CONST) 35,16,16
   16 PAUSE
      GO TO 1000
  101 FORMAT (18H CARD OUT OF ORDER,I5,I5)
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
102 FORMAT (I3,I3,3HR= F15.5,5X,6HF(R)= F15.5)
104 FORMAT (F15.10,2X,F15.10,2X,F15.10,2X,F15.10)
105 FORMAT (F15.10,2X,F15.10,2X,F15.10,2X,F15.10)
106 FORMAT (F10.5,F10.5,F10.5,F10.8)
107 FORMAT(I3,I3,3H A=F10.5,3H B=F15.7,3H R=F15.7)
108 FORMAT(I3,I3,3H W=F15.5,3H Y=F15.5,7H ERROR=F20.10)
109 FORMAT(I3,I3,10H TOTAL SS=F30.15)
    END
```

APPENDIX C

Functional observation:

$$Y_{it} = \frac{\beta_i Y_{i0}}{Y_{i0}+(\beta_i - Y_{i0})\rho_i^t} + \varepsilon_{it} \quad ,$$

suppressing subscripts and writing

$$Q_t = Y_0 + (\beta - Y_0)\rho^t \quad ,$$

the normal equations result when

$$L = \lambda\Sigma(y - \frac{\beta Y_0}{Q})^2 + \lambda'$$

is successively differentiated with respect to β and ρ (λ and λ' do not involve β or ρ.). The summations involved are over the values of t and y indicates a realization of Y.

The normal equations are

$$\underline{X} = \begin{bmatrix} \dfrac{\partial L}{\partial \beta} \\ \\ \dfrac{\partial L}{\partial \rho} \end{bmatrix} = -2\lambda \begin{bmatrix} -y_0^2 \sum (y_t - \dfrac{\beta y_0}{Q_t})(\dfrac{1-\rho^t}{Q_t^2}) \\ \\ \beta y_0(\beta - y_0) \sum (y_t - \dfrac{\beta y_0}{Q_t}) \dfrac{t\rho^{t-1}}{Q_t^2} \end{bmatrix} \quad ,$$

which when set equal to zero gives estimates b and r. Q_t will be written as q_t since it involves the estimates. The iterant matrix V in this case is

$$V = \begin{bmatrix} -y_0^4 \sum \frac{(1-r^t)^2}{q_t^4} & by_0^3(b-y_0) \sum \frac{tr^{t-1}(1-r^t)}{q_t^4} \\ \\ \text{(symmetric)} & -b^2y_0^2(b-y_0)^2 \sum \frac{t^2r^{2t-2}}{q_t^4} \end{bmatrix} .$$

The linear corrections are found according to the iteration equation

$$\begin{bmatrix} b_{k+1} \\ r_{k+1} \end{bmatrix} = \begin{bmatrix} b_k & - V_k^{-1} \underline{X}_k \\ r_k & \end{bmatrix} ,$$

where V^{-1} and $\underline{X}$ are evaluated for the k<u>th</u> values of b and r, iteration ceases when the correction is less than the criteria vector

$$\begin{bmatrix} .00000005 \\ .00000005 \end{bmatrix}$$

insuring seven significant digits in the resulting maximum likelihood estimates.

Operating instructions for the digital computer program, compiled from the FORTRAN source statements given later, are discussed below.

Data preparation is the same as described on page 134. The same conditions apply to the data as are given there.

Deck sequence is: Object program, appropriate subroutine package, Control Card I, and the data deck.

Control Card I is prepared in the following way:

Card columns

1 - 6	-	numerical identification,
7 - 10	-	number of data cards
11 - 20	-	starting value for r as a 10-5 number,
21 - 30	-	criteria to determine when iteration should cease as a 10-8 number (usually 00 00000005).

Switch settings:

SW 1	ON	typewriter prints value of b and r at the end of each iteration, with current iteration number
	OFF	no type out

SW 2, SW 3, SW 4 not interrogated.

Output:

1 card, ID, r, b, final number of iterations

N cards, ID, y, $\hat{Y}$, y-$\hat{Y}$

5 cards, self explanatory .

FORTRAN SOURCE STATEMENTS

```
C     ESTIMATES STOCHASTIC PARAMETERS IN YIT=B*YO/(1.+(B-YO)*R**T)
      DIMENSION W(100)
      DIMENSION Y(100)
 1000 READ 100,ID,N,R,CRIT
      DO 50I=1,N
      READ 103,ID,J,Y(I)
      IF(J-I+1)2,50,2
    2 PRINT 101,I,J
      PAUSE
      GO TO 1000
   50 CONTINUE
      JOUNT=0
      DR=0
      DB=0
      YO=Y(1)
      B=Y(N)
    3 B=B+DB
      R=R+DR
      C=B-YO
      D=B*YO
      SUM=0
      S=1.
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
      DO 4 I=2,N
      S=S*R
      TOP=(1.-S)*Y(I)
      Q=YO+C*S
      Q=Q*Q
    4 SUM=SUM+TOP/Q
      FIRST=YO*YO*SUM
      SUM1=0
      S=1.
      DO 5 I=2,N
      S=S*R
      TOP=1.-S
      Q=YO+C*S
      Q=Q*Q*Q
    5 SUM1=SUM1+TOP/Q
      TWO=B*YO*YO*YO*SUM1
      FRB=FIRST - TWO
      S=1.
      SUM2=Y(2)/((YO+C*R)*(YO+C*R))
      T=R
      SUM3=1./((YO+C*R)*(YO+C*R)*(YO+C*R))
      DO 6 I=3,N
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
      S=S*R
      T=T*R
      X=I
      TOP=Y(I)*(X-1.)*S
      Q=YO+C*T
      SUM2=SUM2+TOP/(Q*Q)
      TOP2=(X-1.)*S
    6 SUM3=SUM3+TOP2/(Q*Q*Q)
      THREE=D*C*SUM2
      FOUR=D*D*C*SUM3
      HRB=FOUR-THREE
      T=1.
      SUM5=0
      SUM6=0
      SUM7=0
      DO 10 I=2,N
      T=T*R
      Q=YO+C*T
      TOP=(1.-T)*T
      TOP2=(1.-T)*(1.-T)
      SUM5=SUM5+Y(I)*TOP/(Q*Q*Q)
      SUM6=SUM6+TOP/(Q*Q*Q*Q)
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
   10 SUM7=SUM7+TOP2/(Q*Q*Q*Q)
      FOURB=2.*YO*YO*SUM5
      FIVE=2.*YO*YO*D*SUM6
      SIX=YO*YO*YO*YO*SUM7
      Q= YO+C*R
      SUM6=(Q+2.*(1.-R)*C)*Y(2)/(Q*Q*Q)
      S=1
      T=R
      DO 12 I=3,N
      T=T*R
      S=S*R
      X=I
      Q=YO+C*T
      TOP=(X-1.)*S*(Q+2.*(1.-T)*C)*Y(I)
   12 SUM6=SUM6+TOP/(Q*Q*Q)
      SEVEN=YO*YO*SUM6
      Q=YO+C*R
      SUM7=(Q+2.*(1.-R)*C)/(Q*Q*Q*Q)
      S=1.
      T=R
      DO13 I=3,N
      S=S*R
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
      T=R*T
      Q=YO+C*T
      X=I
      TOP=(X-1.)*S*(Q+2.*(1.-T)*C)
   13 SUM7=SUM7+TOP/(Q*Q*Q*Q)
      EIGHT=B*YO*YO*YO*SUM7
      Q=YO+C*R
      SUM8=(1.-R)/(Q*Q*Q*Q)
      S=1.
      T=R
      DO 14 I=3,N
      S=S*R
      T=T*R
      Q=YO+C*T
      X=I
      TOP=(1.-T)*(X-1.)*S
   14 SUM8=SUM8+TOP/(Q*Q*Q*Q)
      XINE=B*YO*YO*YO*C*SUM8
      T=1.
      U=R
      V=1.
      TOP1=-1.*C*2.
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
      Q=YO+C*R
      SUM9=Y(2)*TOP1/(Q*Q*Q)
      SUM10=TOP1/(Q*Q*Q*Q)
      DO 15 I=3,N
      T=T*R
      P=T/R
      U=U*R
      Q=YO+C*U
      X=I
      V=V*R*R
      TOP=(X-1.)* ((Q*(X-2.)*P)-(2.*C*V*(X-1.)))
      SUM9=SUM9+Y(I)*TOP/(Q*Q*Q)
   15 SUM10=SUM10+TOP/(Q*Q*Q*Q)
      TEN=C*D*SUM9
      ELEVN=C*D*D*SUM10
      Q=YO+C*R
      SUM11=1./(Q*Q*Q*Q)
      T=R
      S=1
      DO 16 I=3,N
      S=S*R
      T=T*R
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
      Q=YO+C*T
      X=I
      TOP=(X-1.)*S
      TOP=TOP*TOP
   16 SUM11=SUM11+TOP/(Q*Q*Q*Q)
      TWELV=D*D*C*C*SUM11
      JOUNT=JOUNT+1
      A11=FIVE-FOURB-SIX
      A12=EIGHT+XINE-SEVEN
      A22=ELEVN-TEN-TWELV
      DEN=A11*A22-A12*A12
      DB=(-1.*FRB*A22+HRB*A12)/ DEN
      DR=(A11*(-1.)*HRB-A12*(-1.)*FRB)/DEN
      ZR=DR
      ZB=DB
      IF (ZB) 30,31,31
   30 ZB=ZB*(-1.)
   31 IF (ZB-CRIT) 33,33,24
   33 IF (ZR) 32,34,34
   32 ZR=ZR*(-1.)
   34 IF (ZR-CRIT) 22,22,24
   24 IF(SENSE SWITCH 1) 25,3
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
   25 PRINT 109,ID,R,B,JOUNT
      GO TO 3
   22 PUNCH 109,ID,R,B,JOUNT
      RSS=0
      SSS=0
      G=N
      TSS=0
      T=1.
      DO 18 I=1,N
      T=T*R
      Q=T/R
      W(I)=B*YO/(YO+C*Q)
      E=Y(I)-W(I)
      F=E*E
      TSS=TSS+F
      RSS=RSS+Y(I)*Y(I)
      SSS=SSS+Y(I)
   18 PUNCH 112,ID,Y(I),W(I),E
      CF=SSS*SSS/G
      TSQ=RSS-CF
      SSM=TSQ-TSS
      RAT=SSM/TSQ
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
      PUNCH 113,ID,DEN
      PUNCH 104,TSQ
      PUNCH105,TSS
      PUNCH106,SSM
      PUNCH 107,RAT
      GO TO 1000
  112 FORMAT(I6,5X,5HY(I)=F12.6,5X,5HW(I)=F12.6,5X,2HE=F12.6)
  100 FORMAT(I6,I4,F10.5,F10.8)
  101 FORMAT(18H CARD OUT OF ORDER,I5,I5)
  103 FORMAT(I6,I4,F10.5)
  109 FORMAT(3HID=I6,5X,2HR=F10.5,5X,2HB=F12.5,5X,6HCOUNT=I4)
  113 FORMAT(I6,2X,17HDETERMINANT OF A=F20.5)
  104 FORMAT(33H SUM OF SQUARES ABOUT MEAN OF Y =F20.5)
  105 FORMAT(5X,28H SUM OF SQUARES FROM MODEL =F20.5)
  106 FORMAT(3X,30H SUM OF SQUARES DUE TO MODEL =F20.5)
  107 FORMAT(45H PROPORTION OF TOTAL ACCOUNTED FOR BY MODEL =F10.8)
      END
```

APPENDIX D

Functional observation:

$$Y_{it} = \left(\frac{\beta_i}{1 + \alpha_i \rho_i^t}\right) + \varepsilon_{it} , \quad (t \geq 0) \quad .$$

Suppressing the i subscript, $\varepsilon_{it} \sim NID(0, \sigma^2/w_t)$, the normal equations result from successively differentiating

$$L = \lambda \Sigma w_t (y_t - \frac{\beta}{1+\alpha\rho^t})^2 + \lambda' \quad ,$$

where the w_t are weights applied to the deviations, λ and λ' do not involve β, α, or ρ, giving:

$$\underline{X} = \begin{bmatrix} \frac{\partial L}{\partial \beta} \\ \\ \frac{\partial L}{\partial \rho} \\ \\ \frac{\partial L}{\partial \alpha} \end{bmatrix} = -2\lambda \begin{bmatrix} -\sum w_t (y_t - \frac{\beta}{1+\alpha\rho^t})(\frac{1}{1+\alpha\rho^t}) \\ \\ \alpha\beta \sum w_t (y_t - \frac{\beta}{1+\alpha\rho^t})(\frac{t\rho^{t-1}}{(1+\alpha\rho^t)^2}) \\ \\ \beta \sum w_t (y_t - \frac{\beta}{1+\alpha\rho^t})(\frac{\rho^t}{(1+\alpha\rho^t)^2} \end{bmatrix} \quad .$$

Solving $\underline{X} = 0$ gives estimates, b, r, and a of the random variables.

The iteration matrix V is

$$V = \begin{bmatrix} \sum \left(\dfrac{w_t}{1+ar^t}\right)^2 & -ab\sum \dfrac{w_t^2 t r^{t-1}}{(1+ar^t)^3} & -b\sum \dfrac{w_t^2 r^t}{(1+ar^t)^3} \\ & a^2b^2\sum \dfrac{w_t^2 t^2 r^{2t-2}}{(1+ar^t)^4} & ab^2\sum \dfrac{w_t^2 t r^{2t-1}}{(1+ar^t)^4} \\ \text{(symmetric)} & & b^2\sum \dfrac{w_t^2 r^{2t}}{(1+ar^t)^4} \end{bmatrix} .$$

The iteration equation, as before, is

$$\begin{bmatrix} b_{k+1} \\ r_{k+1} \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} b_k \\ r_k \\ a_k \end{bmatrix} - \begin{bmatrix} V_k^{-1} \ \underline{X}_k \end{bmatrix} ,$$

where V^{-1} and $\underline{X}$ are evaluated for the k<u>th</u> values of b, r, and a. Iteration ceases when the correction is less than the vector

$$\begin{bmatrix} .0005 \\ .00000005 \\ .000005 \end{bmatrix} ,$$

giving the maximum likelihood estimates.

Operation instructions for the program described by the FORTRAN source statements given at the end of this Appendix follow.

Deck sequence is: Object deck, appropriate subroutine package, Control Card I, and data deck.

Control Card I is prepared as follows:

Card column

1 - 6	-	numerical identification,
7 - 10	-	number of data cards, N,
11 - 20	-	starting value for r as a 10-8 number,
21 - 30	-	starting value for a as a 10-6 number,
31 - 40	-	starting value for b as a 10-5 number,
41 - 50	-	criteria to insure ℓ significant digits in b (usually 0000050000),
51 - 60	-	criteria to insure ℓ significant digits in r (usually 0000000005),
61 - 70	-	criteria to insure ℓ significant digits in a (usually 0000000500).

Data is prepared according to:

Card column

1 - 6	-	numerical identification,
7 - 10	-	age (t) corresponding to the body weight to be entered. t must be integer valued, entered as a 4-0 number,
11 - 20	-	body weight as a 10-5 number,
21 - 60	-	blank,
61 - 70	-	w_t as a 10-5 number. If left <u>blank</u> this field will be interpreted as being 1.0.

Data cards are ordered with respect to card columns 7-10. The first value of t may be any value, less than 100-N. The values of t need not be equally spaced, but as stated previously must be integer valued.

Switch settings:

SW 1	ON	typewriter prints values of a, b, and r at the end of each iteration, also included are the ID and current number of iterations performed
	OFF	no type out
SW 2	ON	punch of current values of V, V^{-1}, and $\underline{X}$
	OFF	no punch out of above
SW 3	ON	permits operator to enter new values of a, b, and r via the typewriter. Format is 3, 10-5 numbers
	OFF	suppresses above
SW 4	ON	evaluate V and invert V for current values of a, b, and r in next iteration
	OFF	program uses previous value of V and V^{-1} in next iteration.

Output:

1 card, ID, a, b, r

1 card, $\underline{X}'$ evaluated for a, b, and r

N cards, ID, Y, $\hat{Y}$, Y-$\hat{Y}$, w_t

9 cards, self explanatory.

FORTRAN SOURCE STATEMENTS

```
C     ESTIMATES STOCHASTIC PARAMETERS IN YIT=B/(1.+A*R**T)
C     LOGISTIC MODEL,EQUAL OR UNEQUAL SPACINGS,WTS/NWTS
      DIMENSION X(100)
      DIMENSION V(100)
      DIMENSION U(100)
      DIMENSION Y(100)
      DIMENSION W(100)
      DIMENSION WT(100)
 1000 READ 100,ID,N,R,A,B,CRIT1,CRIT2,CRIT3
  100 FORMAT(I6,I4,F10.8,F10.6,F10.5,F10.8,F10.8,F10.8)
      DO50 I=1,N
      READ 103, ID,X(I),Y(I),WT(I)
      IF(WT(I))2,3,99
    3 WT(I)=1.
   99 IF (X(I)) 2,50,89
   89 IF (I-1) 1,50,1
    1 IF(X(I)-X(I-1))2,2,50
    2 PRINT 101,I,I
  101 FORMAT(18H CARD OUT OF ORDER,I6,I6)
      PAUSE
      GO TO 1000
   50 CONTINUE
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
      COUNT=0
    5 S=1.
      M=X(N)+1.
      DO 51  I=1,M
      S=S*R
      V(I)=S/R
   51 U(I)=S/R*R
      IF(COUNT) 140,11,57
   11 PHI=0
      PSI=0
      THET=0
      A11=0
      A12=0
      A13=0
      A22=0
      A23=0
      A33=0
      DO12  I=1,N
      K=X(I)+1.
      E2=1.+A*V(K)
      E3=E2*E2
      E4=E2*E3
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
      E5=E2*E4
      E1=(Y(I)-B/E2)*WT(I)
      PHI=PHI-E1*(1./E2)
      PSI=PSI+E1*(B*A*X(I)*U(K)/E3)
      THET=THET+E1*(B*V(K)/E3)
      A11=A11+WT(I)/E3
      A12=A12-WT(I)*(B*A*X(I)*U(K)/E4)
      A13=A13-WT(I)*(B*V(K)/E4)
      A22=A22+WT(I)*(B*B*A*A*X(I)*X(I)*U(K)*U(K)/E5)
      A23=A23+WT(I)*(B*B*A*X(I)*V(K)*U(K)/E5  )
   12 A33=A33+WT(I)*(B*B*V(K)*V(K)/E5  )
   60 D1=A11*(A22*A33-A23*A23)
      D2=A12*(A12*A33-A13*A23)
      D3=A13*(A12*A23-A13*A22)
      DETA=D1-D2+D3
      AL11=A22*A33-A23*A23
      AL12=-1.*(A12*A33-A13*A23)
      AL13=A12*A23-A13*A22
      AL22=A11*A33-A13*A13
      AL23=-1.*(A11*A23-A13*A12)
      AL33=A11*A22-A12*A12
      C11=AL11/DETA
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
      C12=AL12/DETA
      C13=AL13/DETA
      C22=AL22/DETA
      C23=AL23/DETA
      C33=AL33/DETA
      IF (COUNT) 61,61,87
   61 PHI=-PHI
      PSI=-PSI
      THET=-THET
      DELB=C11*PHI+C12*PSI+C13*THET
      DELR=C12*PHI+C22*PSI+C23*THET
      DELA=C13*PHI+C23*PSI+C33*THET
      ZB=DELB
      ZR=DELR
      ZA=DELA
      IF(ZB)30,33,31
   30 ZB=ZB*(-1.)
   31 IF(ZB-CRIT1)33,33,41
   33 IF(ZR) 32,35,34
   32 ZR=-1.*ZR
   34 IF(ZR-CRIT2)35,35,41
   35 IF(ZA) 36,40,37
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
 36 ZA=-1.*ZA
 37 IF(ZA-CRIT3)40,40,41
 41 IF (SENSE SWITCH 1) 42,19
 42 PRINT 111,A,B,R
    PRINT 120,ID,COUNT
    IF (SENSE SWITCH 2 ) 930,931
930 PUNCH 111,PSI,PHI,THET
    PUNCH 111,A11,A12,A13
    PUNCH 111,A22,A23,A33
    PUNCH 111,C11,C12,C13
    PUNCH 111,C22,C23,C33
931 IF (SENSE SWITCH 4 )  19,501
501 CONTINUE
    B1=B
    A1=A
    R1=R
    IF (SENSE SWITCH 3) 110,19
110 ACCEPT 109,A,B,R
    IF (R) 58,58,50
 58 R=R1
    B=B1
    A=A1
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
109 FORMAT(F10.5,F10.5,F10.5)
 19 R=R+DELR
    B=B+DELB
    A=A+DELA
    COUNT=COUNT+1.
    IF (SENSE SWITCH 4 ) 50,5
 57 PSI=0
    PHI=0
    THET=0
    DO 52 I=1,N
    K=X(I)+1.
    E2=1.+A*V(K)
    E3=E2*E2
    E4=E2*E3
    E5=E2*E4
    E1=(Y(I)-B/E2)*WT(I)
    PHI=PHI-E1*(1./E2)
    PSI=PSI+E1*(B*A*X(I)*U(K)/E3 )
 52 THET=THET+E1*(B*V(K)/E3)
    GO TO 61
 40 COUNT = COUNT+1.
    GO TO 60
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
   87 PUNCH134,ID,A,B,R
      PUNCH111,PSI,PHI,THET
      COUNT=-1.
      GO TO 5
  140 ST=0
      SST=0
      TN=N
      SSAR=0
      DO 93 I=1,N
      ST=ST+Y(I)
      J=X(I)+1.
      L=X(I)
      W(I)=B/(1.+A*V(J))
      E=Y(I)-W(I)
      SST=SST+Y(I)*Y(I)
      SSAR=SSAR+E*E
   93 PUNCH 121,ID,L,Y(I),L,W(I),E,WT(I)
      CTSS=SST-ST*ST/TN
      SSDR=CTSS-SSAR
      PUNCH130,CTSS
      PUNCH131,SSDR
      PUNCH132,SSAR
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
      PC=SSDR/CTSS
      PUNCH133,PC
      PUNCH 138
      PUNCH 301,C11,C12,C13
      PUNCH 302,C22,C23
      PUNCH 303,C33
  301 FORMAT (E18.12,2X,E18.12,2X,E18.12)
  302 FORMAT (20X,E18.12,2X,E18.12)
  303 FORMAT (40X,E18.12)
      GAM=-LOGF(R)
      ALPL =LOGF (A)
      TCRIT = ALPL/GAM
      PUNCH 137,ID,GAM,TCRIT,B
  137 FORMAT(I6,6HGAMMA=F12.10,14HAGE AT HALF B=F10.6,2HB=F14.8)
      VAR=SSAR/(TN-3.)
      RVAR=SQRTF(VAR)
      PUNCH 304,VAR,RVAR
  304 FORMAT(4HVAR=F20.10,5X,5HRVAR=F20.10)
      GO TO 1000
  134 FORMAT(I6,2HA=F18.9,2HB=F18.9,2HR=F18.9)
  121 FORMAT(I6,4X,2HY(I4,2H)=F7.1,4X,2HW(I4,2H)=F11.5,4X,F10.5,F7.2)
  130 FORMAT(20HCORRECTED TOTAL SS =F20.10)
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
  131 FORMAT( 15HSS DUE TO REG =F20.10    )
  132 FORMAT(14HSS AFTER REG =F20.10)
  133 FORMAT(26HPROPORTION ACCOUNTED FOR =F12.10)
  111 FORMAT(F20.10,F20.10,F20.10)
  103 FORMAT(I6,F4.0,F10.5,40X,F10.5)
  120 FORMAT (I6,F5.1)
  138 FORMAT(37H INVERSE OF INFORMATION MATRIX, BELOW)
      END
```

APPENDIX E

A list of FORTRAN source statements are given for a digital computer which evaluate ordinates of $f(Y|t)$. The expression

$$f(Y|t) = \frac{e^{-\frac{1}{2\sigma_\beta^2}(\ell nY-\mu_\beta)^2 - \frac{\mu_X^2}{2\sigma_X^2}}}{2\pi\ \sigma_X\sigma_\beta Y} \sum_{i=1}^{2^n-1} \frac{1}{2^n}\, e^{(X-Z)(A-C_1)(X-Z)+Z(B-C_2Z)}$$

is evaluated for given values of (or estimates of)

$$\begin{array}{cccc} \mu_\alpha & \mu_\beta & \mu_\rho & Y \\ \sigma_\alpha^2 & \sigma_\beta^2 & \sigma_\rho^2 & t \end{array} .$$

The program requires four control cards prepared as follows:

Control Card I

Card columns

1 - 12 - μ_a as a 12-9 number,

13 - 24 - μ_b as a 12-9 number,

25 - 36 - μ_r as a 12-9 number,

37 - 48 - σ_a^2 as a 12-9 number,

49 - 60 - σ_b^2 as a 12-9 number,

61 - 72 - σ_r^2 as a 12-9 number.

Control Card II

Card columns

1 - 10 - Test criteria on the exponent involved in the summation (usually 50). Used to prevent underflow. Entered as a 10-5 number.

Control Card III

Card columns

1 - 4 - n as a fixed point number (4-0).

Control Card IV

Card columns

1 - 5 - t as a 5-1 number,

6 - 13 - Y as an 8-1 number.

Deck sequence is: Object deck, appropriate subroutine package, Control Cards I, II, III, and IV.

Switch settings:

SW 1	ON	Omits punch out of partial sums.
	OFF	Partial sums are punched out.
SW 2	ON	Reads only additional Control Card III, utilizes previously read Control Card I and II.
	OFF	Reads Control Card I, Control Card II, and Control Card III.

Output:

1 card per ordinate containing the current values of N, T, Y, $f(Y|t)$.

FORTRAN SOURCE STATEMENTS

```
C       COMPUTES POINTS OF F(Y/T)
 1000 READ 100,AM,BM,GM,SA,SB,SG
  100 FORMAT(6F12.9)
      READ 300,ZT
  300 FORMAT(F10.5)
      READ 101,N
  101 FORMAT(I4)
   13 READ 102,T,Y
  102 FORMAT(F5.1,F8.2)
      SX=SA+T*T*SG
      SQ=SX*SB
      SRX=SQRTF(SQ)
      XM=AM-T*GM
      YL=LOGF(Y)
      FN=(YL-BM)*(YL-BM)/(2.*SB)+(XM*XM)/2.*SX)
       FN=(YL-BM)*(YL-BM)/(2.*SB)+(XM*XM)/(2.*SX)
      FY=1./(2.*3.1415927*SRX*Y)
      B=(YL-BM)/SB-2.
      A=XM/SX-1.
      C1=1./(2.*SX)
      C2=1./(2.*SB)
      L1=(2.**N)-1.
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
      L=L1+1
      SUM=0
      PUNCH 120,B,A,C1,C2,FN,FY
  120 FORMAT(5F10.5,F20.15)
      V1=L
      SUMT=0
      DO 5 I=1,L1
      V=I
      W=V/V1
      X=LOGF(1.-W)
      Z=LOGF(W)
      ZZ=(X-Z)*(A-C1*(X-Z))+Z*(B-(C2*Z)) -FN
      IF(ZZ+ZT) 5,5,4
    4 ZE=EXPF(ZZ)
      SUM=SUM+ZE
      IF(SUM-SUMT)2,2,3
    3 SUMT=SUM
      IF(SENSE SWITCH 2) 5,21
   21 PUNCH 201,I,W,SUM
    5 CONTINUE
      201 FORMAT(I4,F10.5,E14.0)
    2 TL=1./V1
```

FORTRAN SOURCE STATEMENTS (CONTINUED)

```
      YT=TL*SUM*FY
      PUNCH 200,N,T,Y,YT
  200 FORMAT(I4,F6.0,F10.2,F20.15)
      IF (SENSE SWITCH 1 ) 13,1000
      END
```

APPENDIX F

A set of stochastic parameters which define the functional observation for each individual in the sample of Athens-Canadian Randombred chickens identified by pedigree and sex.

Table F.1. Stochastic parameters for each individual by pedigree and sex.

Sire Number	Dam Number	Stochastic Parameters: Female Progeny $\hat{\tau}$	$\hat{\beta}$	$\hat{R}$	Male Progeny $\hat{\tau}$	$\hat{\beta}$	$\hat{R}$
10	0	17.44	2193	89.69	16.73	2759	120.70
					17.22	2881	124.96
					18.80	3063	133.92
					18.48	2880	118.00
					16.76	2859	129.51
10	1				18.63	2521	100.71
					17.95	2501	107.54
					18.38	2588	109.73
					18.61	2328	88.69
					18.76	2301	99.92
10	2	17.36	2008	76.20	18.20	2639	111.16
		17.05	2073	84.06			
		18.13	2286	86.56			
10	3	16.98	2055	88.36			
		18.52	2343	89.97			
10	4	18.86	1795	66.54	18.71	2271	92.42

Table F.1 (continued)

Sire Number	Dam Number	Female Progeny $\hat{\tau}$	Female Progeny $\hat{\beta}$	Female Progeny $\hat{R}$	Male Progeny $\hat{\tau}$	Male Progeny $\hat{\beta}$	Male Progeny $\hat{R}$
11	0	18.59	1884	77.85	18.30	2672	108.01
		17.70	2035	83.63	17.75	2461	105.45
		18.41	1996	81.23	17.85	2639	103.11
11	1	17.87	1952	71.05	17.75	2876	126.75
					17.83	2780	113.21
					17.14	2724	117.33
					18.55	3122	127.06
11	2	17.90	2207	84.30	18.22	2860	114.47
					18.30	2635	97.23
11	3				17.40	2497	115.73
12	1	16.66	1950	76.53	17.34	3065	145.43
		18.25	2012	73.33			
12	2	19.99	2472	90.78	18.72	3278	138.41
		17.36	1961	76.57	18.63	2999	110.13
12	3	16.94	2009	77.39	16.21	2517	114.46
		17.39	1800	63.00			
		16.25	1794	75.43			
		18.50	2311	83.54			
		16.74	2195	92.29			
12	4	18.84	1562	65.64			
		17.04	1842	77.45			
13	0	18.66	2036	70.29	16.56	2660	118.17
		16.46	1780	69.68	17.47	2599	106.23
		20.07	2395	84.90			
		16.64	1803	72.88			
		18.75	2410	83.26			
13	1	22.34	2736	83.24			

Table F.1 (continued)

Sire Number	Dam Number	Stochastic Parameters: Female Progeny $\hat{\tau}$	$\hat{\beta}$	$\hat{R}$	Male Progeny $\hat{\tau}$	$\hat{\beta}$	$\hat{R}$
13	2	18.66	2036	72.07	18.18	2622	108.09
		17.49	2084	83.04	18.14	2996	130.62
		17.96	2174	83.18			
		17.67	2323	93.67			
13	3	19.82	2484	84.26	18.12	2763	115.14
		18.27	2311	88.10	17.99	2886	123.52
					18.86	2931	120.46
					17.66	3178	131.72
					18.11	2961	127.10
14	1	16.45	1935	78.60	18.57	3258	139.19
					18.06	2847	117.22
14	2	18.05	2110	86.24			
		20.06	2346	89.67			
		17.88	2458	103.66			
14	3	18.63	2334	85.77	18.95	2972	132.03
		19.21	2386	87.92			
		19.43	2389	88.27			
14	4	17.15	2051	81.62	18.84	2782	115.31
		18.79	1997	71.54	16.59	2902	136.10
					18.20	2563	113.73
15	0	17.66	1999	66.91	17.75	3054	122.46
		16.65	1960	81.04	16.81	2608	113.64
		18.15	2262	81.77			
15	1				18.22	2585	113.22

Table F.1 (continued)

Sire Number	Dam Number	Stochastic Parameters: Female Progeny $\hat{\tau}$	$\hat{\beta}$	$\hat{R}$	Male Progeny $\hat{\tau}$	$\hat{\beta}$	$\hat{R}$
16	1	17.94	2140	79.23	16.70	2432	100.68
		16.34	1852	70.65	17.25	2841	118.46
		18.17	2247	85.10			
16	3	16.76	2278	89.41	19.63	2948	114.82
		18.32	2185	76.14	17.22	2665	111.93
16	4	18.94	2270	77.86	19.11	2446	95.33
		19.19	2193	81.41	16.53	2651	128.37
		17.63	2175	93.68	17.52	2801	122.05
17	0	18.64	2288	82.08	18.34	2768	105.18
		18.70	2300	85.33			
		17.43	2086	85.99			
17	1	18.65	2083	80.87	18.40	2620	108.27
		18.41	1947	70.91	18.37	2644	113.09
		17.42	1712	72.46			
17	2	17.40	2036	78.13	17.88	2872	130.81
		17.16	1747	69.96			
		17.12	2269	92.57			
		18.97	2422	93.36			
		18.52	2543	90.40			
17	3	19.76	2196	81.03			
		19.06	2088	79.91			
		16.87	1690	72.58			
		16.75	2101	91.02			
17	4	17.00	1976	78.74	18.83	2830	126.78
		19.24	2228	77.86	18.31	3213	135.66
		17.84	2003	71.75	19.03	2885	120.73

Table F.1 (continued)

Sire Number	Dam Number	Stochastic Parameters: Female Progeny $\hat{\tau}$	$\hat{\beta}$	$\hat{R}$	Male Progeny $\hat{\tau}$	$\hat{\beta}$	$\hat{R}$
18	0	19.03	2352	82.90			
		17.77	2127	91.67			
		16.81	2039	85.17			
18	1	18.41	1915	86.17	18.89	2831	121.66
					17.41	2440	113.82
18	2	17.33	1972	78.92	17.75	2574	106.17
					19.11	2997	125.57
					17.17	2478	104.38
					16.98	2394	103.60
18	3	19.58	2450	92.97	19.96	3187	137.51
		16.59	1977	90.89	17.31	2249	104.35
					17.93	2412	103.77
18	4	17.93	2353	92.94	16.72	2688	136.08
		18.79	2347	79.50	16.23	2344	123.99
		18.20	2104	83.52			
19	0	18.55	2554	92.96	18.23	3239	127.94
		18.28	2562	105.93	16.81	3102	149.12
		19.48	2448	81.21			
19	1	18.73	2178	76.33	17.47	3015	125.19
		18.25	2446	94.59	18.71	2872	128.73
		19.26	2410	89.10	17.78	2982	135.23
		17.11	2022	79.71			
19	2	19.26	2480	87.11	20.13	3374	120.45
		18.44	2332	96.19	20.08	3158	114.08
					20.08	3179	111.98
19	3	16.84	1960	82.51			
		19.52	1993	76.38			

Table F.1 (continued)

Sire Number	Dam Number	Stochastic Parameters: Female Progeny $\hat{\tau}$	$\hat{\beta}$	$\hat{R}$	Male Progeny $\hat{\tau}$	$\hat{\beta}$	$\hat{R}$
19	4	17.02	2058	76.96	16.98	2543	117.61
		18.27	2000	75.75			
		18.99	2575	95.01			
		18.43	2059	74.79			

ABSTRACT

A formulation is presented which describes a new method of handling responses which are characteristically a function relating measurable random variables to mathematical variables. The formulation considers each response a random function which is called a functional observation. This statistical formulation results in each response being summarized in a convenient functional form which can be used to derive other functional relationships which describe the response.

A particular formulation is derived to study the juvenile growth of chickens. The parametric family

$$\eta(t) = \frac{\beta}{1 + \alpha\rho^{t}}$$

is proposed as being most appropriate, after evaluating several alternative forms using actual growth data, to generate functional observations which represent the individual growth patterns. Each functional observation is characterized by an individual's set of β, α, and ρ, called stochastic parameters and designated β_i, α_i, and ρ_i. Thus a particular functional observation may be written as

$$Y_{it} = \frac{\beta_i}{1 + \alpha_i \rho_i^t} \quad ,$$

where Y_{it} represents the body weight of individual i at age t. The functional observations represent juvenile growth, a period when an individual chicken progressed from 20 to 140 days of age.

A biological experiment provided the actual growth data of the chickens. A sample (approximately 200 animals) of Athens-Canadian Random Bred chickens were used. Each bird was weighed every four days for a period of about five months. These data were used in this study to evaluate the appropriateness of various statistical formulations and to estimate the heritability of the stochastic parameters and through them the functional observations.

Several growth characteristics were derived from the functional observations to describe chicken growth. It was found that β (final juvenile weight), R (maximum growth rate), and τ (age at half final juvenile weight and age when R occurs), could be used to express the parametric family.

The probability structure of a functional observation, a random function, is developed and discussed. The stochastic parameters are assumed to be elements of particular probability densities which with the form specified for η define the probability structure.

A particular probability structure is found by assuming different independent log-normal probability densities for α_i, ρ_i, and β_i. The resulting conditional probability density of Y given t could not be obtained in a closed form. A series to calculate the ordinates of $f(Y|t)$ was obtained, and several conditional probability densities were computed numerically and were found to be slightly positively skewed, with variance increasing from $t = 5$ to $t = 22.5$, then decreasing slightly. Their modal ordinates varied indirectly with the apparent variance, projecting approximately a general logistic in the (t,Y) plane. This probability structure is seen to be realistic in that it reflects characteristics commonly observed from actual growth data.

The inheritance of entire growth patterns was estimated using the stochastic parameters which characterize the functional observations. Heritability estimates were found for male and female progeny separately and found to be quite large. Overall, the heritability of stochastic parameters associated with male progeny were larger than those for females.

A set of structural equations are proposed which relate stochastic parameters of full-sibs of opposite sex. These equations were derived as a means of determining genetic similarities in male and female growth patterns. These equations may also be used to develop a mating scheme, by matching actual female stochastic parameters to those predicted by the equations, to possibly produce progeny with uniform growth patterns.