THE FINITE ELEMENT ANALYSIS OF LAMINATED COMPOSITES

by

Fu Tien Lin

Thesis submitted to the Graduate Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY

in

Engineering Mechanics

APPROVED:

Daniel Frederick
Daniel Frederick, Chairman

Richard M. Barker

George A. Gray

Robert A. Heller

Chin-Bing Ling

December, 1971

Blacksburg, Virginia
ACKNOWLEDGEMENTS

This investigation was supported by the Department of Defense, Project THEMIS, Contract Number DAA F07-69-C-0449 with Watervliet Arsenal, Watervliet, New York.

The author expresses his sincere appreciation to Dr. Daniel Frederick, Director of Project THEMIS and Chairman of his committee for providing guidance and encouragement throughout the investigation.

Special thanks must go to Dr. Richard M. Parker for his help and advice in this investigation. Thanks are also extended to members of his graduate committee for the assistance they have provided during his study at Virginia Polytechnic Institute and State University.

Also, the patience and support of the author's wife, Chii Mei, is gratefully acknowledged.
## TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>LIST OF FIGURES</td>
<td>v</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>vi</td>
</tr>
<tr>
<td>LIST OF SYMBOLS</td>
<td>vii</td>
</tr>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. REVIEW OF LITERATURE</td>
<td>5</td>
</tr>
<tr>
<td>III. ANALYTICAL METHOD</td>
<td>10</td>
</tr>
<tr>
<td>A. Development of the Governing Equations for a Lamina</td>
<td>10</td>
</tr>
<tr>
<td>Assumptions and Definitions</td>
<td>10</td>
</tr>
<tr>
<td>Hellinger-Reissner Variational Principle</td>
<td>13</td>
</tr>
<tr>
<td>Equations of Motion</td>
<td>17</td>
</tr>
<tr>
<td>Stress Resultant-Displacement Relationships</td>
<td>18</td>
</tr>
<tr>
<td>Basic Equations for the Bending of Rectangular Plates</td>
<td>18</td>
</tr>
<tr>
<td>Governing Equations for the Extension of Plates</td>
<td>20</td>
</tr>
<tr>
<td>B. Cylindrical Bending of a Two-Fly Laminate</td>
<td>22</td>
</tr>
<tr>
<td>Examples and Discussion</td>
<td>27</td>
</tr>
<tr>
<td>Fourier Series Representation for a Uniform Load</td>
<td>29</td>
</tr>
<tr>
<td>IV. FINITE ELEMENTS IN TWO DIMENSIONS</td>
<td>36</td>
</tr>
<tr>
<td>A. 16-DOF Element Formulation</td>
<td>36</td>
</tr>
<tr>
<td>Displacement Functions</td>
<td>38</td>
</tr>
<tr>
<td>Derivation of Element Stiffness Matrix</td>
<td>39</td>
</tr>
<tr>
<td>Consistent Element Joint Loads</td>
<td>41</td>
</tr>
<tr>
<td>Trace of the Element Stiffness Matrix</td>
<td>42</td>
</tr>
<tr>
<td>Assembly and Analysis</td>
<td>43</td>
</tr>
<tr>
<td>B. 24-DOF Element Formulation</td>
<td>44</td>
</tr>
<tr>
<td>Shape Functions</td>
<td>44</td>
</tr>
<tr>
<td>Coordinate Transformation</td>
<td>47</td>
</tr>
<tr>
<td>Element Stiffness Matrix</td>
<td>49</td>
</tr>
<tr>
<td>Trace of the Element Stiffness Matrix</td>
<td>49</td>
</tr>
<tr>
<td>Consistent Element Joint Loads</td>
<td>50</td>
</tr>
<tr>
<td>Stress Analysis</td>
<td>52</td>
</tr>
<tr>
<td>Chapter</td>
<td>Page</td>
</tr>
<tr>
<td>---------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>C. Applications and Discussion</td>
<td>52</td>
</tr>
<tr>
<td>Aluminum-Steel Laminate in Cylindrical Bending</td>
<td>52</td>
</tr>
<tr>
<td>2-Ply Laminate</td>
<td>56</td>
</tr>
<tr>
<td>3-Ply Laminate</td>
<td>58</td>
</tr>
<tr>
<td>V. FINITE ELEMENTS IN THREE DIMENSIONS</td>
<td>61</td>
</tr>
<tr>
<td>Shape Function for 72-DOF Element</td>
<td>61</td>
</tr>
<tr>
<td>Coordinate Transformation</td>
<td>63</td>
</tr>
<tr>
<td>The Elasticity Matrix</td>
<td>65</td>
</tr>
<tr>
<td>Element Stiffness Matrix</td>
<td>67</td>
</tr>
<tr>
<td>Trace of the Stiffness Matrix</td>
<td>69</td>
</tr>
<tr>
<td>Consistent Element Joint Loads</td>
<td>69</td>
</tr>
<tr>
<td>Application and Discussion</td>
<td>73</td>
</tr>
<tr>
<td>VI. CONCLUSIONS</td>
<td>78</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>80</td>
</tr>
<tr>
<td>APPENDICES</td>
<td>83</td>
</tr>
<tr>
<td>A. The elements of selected matrices used in Chapter IV</td>
<td>83</td>
</tr>
<tr>
<td>B. The elements of selected matrices used in Chapter V</td>
<td>85</td>
</tr>
<tr>
<td>C. 16-DOF Program Listing</td>
<td>91</td>
</tr>
<tr>
<td>D. 24-DOF Program Listing</td>
<td>115</td>
</tr>
<tr>
<td>E. 72-DOF Program Listing</td>
<td>139</td>
</tr>
<tr>
<td>VITA</td>
<td>170</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Laminate, Lamina and Element of the Lamina</td>
<td>11</td>
</tr>
<tr>
<td>2. Aluminum-Steel Laminate in Cylindrical Bending</td>
<td>23</td>
</tr>
<tr>
<td>3. Displacement and Stress Distributions for Aluminum-Steel Laminate, $h_1 = .1''$, $h_2 = .5''$, $s = 10$</td>
<td>30</td>
</tr>
<tr>
<td>4. Fourier Series Expansion for Uniform Load $\bar{q} = 1$</td>
<td>31</td>
</tr>
<tr>
<td>5. 16-DOF Element, Distributed Force and Trace</td>
<td>37</td>
</tr>
<tr>
<td>6. 24-DOF Element, $4 \times 4$ Gauss Rule and Trace</td>
<td>45</td>
</tr>
<tr>
<td>7. Consistent Element Joint Loads for Cubic Curve Line</td>
<td>51</td>
</tr>
<tr>
<td>8. Stress Distributions for Aluminum-Steel Laminate in Cylindrical Bending by Sinusoidal Load, $s = 10$</td>
<td>53</td>
</tr>
<tr>
<td>9. Stress Distributions for Aluminum-Steel Laminate in Cylindrical Bending by Uniform Load, $s = 10$</td>
<td>54</td>
</tr>
<tr>
<td>10. Stress Distributions for 2-Ply Laminate in Cylindrical Bending by Sinusoidal Load, $s = 4$</td>
<td>57</td>
</tr>
<tr>
<td>11. Displacement and Stress Distributions for 3-Ply Laminate in Cylindrical Bending by Sinusoidal Load, $s = 4$</td>
<td>59</td>
</tr>
<tr>
<td>12. 72-DOF Element, Parent Elements and Trace of Right Prism</td>
<td>62</td>
</tr>
<tr>
<td>13. Consistent Element Joint Loads for Distributed Surface Load</td>
<td>71</td>
</tr>
<tr>
<td>14. Symmetric 3-Ply Square Laminate and Idealization</td>
<td>74</td>
</tr>
<tr>
<td>15. Stress and Displacement Distributions in Symmetric 3-Ply Square Laminate</td>
<td>77</td>
</tr>
</tbody>
</table>
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. Displacements and Stresses for Aluminum-Steel Laminate with Difference s</td>
<td>28</td>
</tr>
<tr>
<td>II. Displacement and Stress Distributions for Aluminum-Steel Laminate by Fourier Series for a Uniform Load $\bar{q} = 1 \text{ k/&quot;}, s = 10 $</td>
<td>32</td>
</tr>
<tr>
<td>III. Displacements and Stresses along the interface for Aluminum-Steel Laminate by Fourier Series for a Uniform Load $\bar{q} = 1 \text{ k/&quot;}, s = 10 $</td>
<td>34</td>
</tr>
<tr>
<td>IV. Consistent Joint Loads for the Cubic-Cubic Plane</td>
<td>72</td>
</tr>
<tr>
<td>V. Comparison of Three-Dimensional Plate Solutions</td>
<td>76</td>
</tr>
</tbody>
</table>
LIST OF SYMBOLS

1, 2, 3  Lamina principal directions
A, B, C  Element side lengths in x, y, z directions
$a_{ij}$  Direction cosines for angle between lamina and global axes
D  Flexural rigidity
E  Modulus of elasticity for isotropic material
G  Shear modulus for isotropic material
h  Plate thickness
$H_i$  Weighing coefficients for Gaussian points
K  Kinetic energy
l  Length of a side of an element
m, n  $\cos \theta, \sin \theta$ respectively where $\theta$ is the angle between lamina and global axes
$M_x, M_y, M_{xy}$  Bending stress resultants
N  Total number of layers in a laminate
$P_x, P_y, P_{xy}$  In-plane stress resultants
$P_i$  Intensity of distributed load
q  Distributed load
$q\overline{q}$  Maximum intensity of sinusoidal distributed load
$S_1$  Boundary surface on which stress prescribed
$S_2$  Boundary surface on which displacement prescribed
$S_x^\pm, S_y^\pm, S_z^\pm$  $\tau_{xz} \pm \tau_{xz}, \tau_{yz} \pm \tau_{yz}, \sigma_z \pm \sigma_z$, respectively
$V_x, V_y$  Shear stress resultants
\( \bar{u}, \bar{v}, \bar{w} \)  
Displacement functions at the midplane in \( x, y, z \) direction

\( u, v, w \)  
Displacements of a point in \( x, y, z \) directions

\( x, y, z \)  
Global cartesian coordinates

\( w \)  
Mixed strain energy density function

\( c_i \)  
Constants in displacement functions

\( \alpha, \beta, \gamma \)  
Linear term of \( u, v, w \) expansion

\( \xi, \eta, \zeta \)  
Local curvilinear coordinates

\( \sigma_x, \sigma_y, \sigma_z \)  
Normal stress components with respect to global system

\( \tau_{xy}, \tau_{xz}, \tau_{yz} \)  
Shear stress components with respect to global system

\( \epsilon_x, \epsilon_y, \epsilon_z \)  
Normal strain components with respect to global system

\( \gamma_{xy}, \gamma_{xz}, \gamma_{yz} \)  
Shear strain components with respect to global system

\( \sigma^T_x, \sigma^T_y, \sigma^T_z \)  
Stresses at the top face of lamina

\( \tau^B_{xz}, \tau^B_{yz}, \sigma^B_z \)  
Stresses at the bottom face of lamina

\( \sigma_{ij} \)  
Stresses in constitutive relations for an anisotropic material

\( \varepsilon_{ij} \)  
Strains in constitutive relations for an anisotropic material

\( v \)  
Poisson's ratio for isotropic material

\( v_{ij} \)  
Poisson's ratio relating normal strain in \( j \)-direction due to uniaxial normal stress in \( i \)-direction

\( \rho \)  
Mass density per unit volume

\( [A] \)  
Matrix relating nodal displacements to constant

\( [\hat{E}] \)  
Matrix relating strains to nodal displacements

\( [C] \)  
Matrix representing differentiation of \( \{w\} \) with respect to \( \xi, \eta, \zeta \)

\( [D] \)  
Elasticity matrix
{f} Element nodal force vector
{F} Global force vector
[J] Jacobian Matrix
|J| Determinant of [J]
[k] Element stiffness matrix
[K] Global stiffness matrix
[M] Matrix relating displacement matrix \{u\} to constant matrix \{a\}
{N} Shape functions
{P} Consistent element joint loads vector
{u_i} Element displacement vector
{u} Global displacement vector
[XYZ] Matrix representing element node global coordinates
{σ} Vector of stress
{ε} Vector of strain
{a} Matrix representing constants in displacement functions
I. INTRODUCTION

A composite material consists of high strength, continuous or discontinuous filaments embedded in matrix material. Filament materials which have been used widely are glass, steel, boron and graphite; and matrix materials in use are aluminum, epoxies, epoxy phenolics. When a composite is subjected to external loads the matrix transfers stresses to the embedded high strength fibers. With such a configuration of filaments and matrix, a high strength-to-weight ratio is achieved. Common applications of composites are for aircraft, space vehicle and deep sea structures.

There are various kinds of constituent materials and many options in the fiber arrangements of a lamina, and in lamina orientations of a composite. Of the different combinations for a composite, one may design a composite such that the laminae are oriented in an optimum manner with respect to some prescribed design criteria.

An individual lamina, or basic ply, of a multilayered composite is considered to be a homogeneous and orthotropic material. But the layered laminate, which is formed by bonding arbitrary oriented laminae in the form of laminated composite plates and shells, in a complex, non-homogeneous, anisotropic structure.

Classical laminated plate theories (CPT) have been used to analyze composite structures and some boundary value problems have been formulated and solved. Owing to certain assumptions of CPT, such as the classical small deformation assumption, the results for deformation and stress
distributions have not been in good agreement with the exact elasticity solutions. Although some recent exact solutions are available, they are restricted to certain special types of boundary value problems.

It is the purpose of this dissertation to present a truly three dimensional finite element analysis for laminated composites which removes the undesirable restrictions existing in the CPT and some refined theories. The basic restriction, which is one straight line normal rotation through the plate thickness, is removed by taking a linear variation of displacements through the element thickness and using one or more elements for each layer of the laminate. Thus a complete three-dimensional analysis of the composite is developed which includes the thickness-stretching deformation as well as extension deformation, and transverse shear strains. Most of the problems solved by the CPT, refined theories and exact elasticity theory have been for rectangular plates with simply-supported edges. In this study generality is achieved by using a curved isoparametric element with cubic displacements expansion in plane which enables the element to fit boundaries of arbitrary shape.

Prior to presenting the two- and three-dimensional finite element analyses of laminated composites, an analytical method for composite with isotropic lamina is presented. A linear expansion for displacements is assumed in each lamina. By applying the Hellinger-Reissner variational principle, governing equations for bending and extension are derived. Because of the complexity of the governing equations and difficulty of achieving exact solutions, the equations for the three-dimensional problem are specialized to the two-dimensional case of cylindrical bending.
Solutions are obtained in this case for sinusoidal and uniform loadings. A Fourier series solution is obtained for the latter.

Two elements are developed for the finite element analysis of two-dimensional laminated composites. Because the plate bending is usually associated with cubic displacement functions, and the individual layers of laminated plates are relatively thin compared with the other dimensions of plate, a 16 degree of freedom (DOF) element is presented first. A cubic expansion of displacements in the plane direction and a linear variation of displacements through the thickness of the element are used. In order to evaluate the performance of the 16-DOF element, a 24-DOF element is developed for which a cubic expansion of displacements in the thickness direction is employed. In the 16-DOF element, rectangular cartesian coordinates are used and the element stiffness matrix is formulated directly and explicitly for a rectangular element. In the 24-DOF element, a local curvilinear coordinates are used to describe the nodal displacements of the element and its geometry, and a numerical integration procedure is applied to formulate the element stiffness matrix. Results of the finite element analyses are compared with the solutions by an analytical method and, wherever possible, with an exact elasticity solution.

A curved, isoparametric, 72-DOF element is developed for the three-dimensional finite element analysis of the laminated composites. The displacement expansion is the three-dimensional analog of the 16-DOF element, that is, a cubic expansion of the displacement in the element plane and a linear variation of displacement through the thickness of the
element are made. A numerical integration procedure is employed to formulate the element stiffness matrix. Results are compared with the exact elasticity solution.

It is believed that the refined, curved, isoparametric element to be presented herein offers an improvement over the simple, linear triangular and quadrilateral elements and that the finite element method facilitates the investigation of the arbitrary shaped layered composites. Furthermore the applicability of the solutions for the laminated composites can be expanded to more general fields.
II. REVIEW OF LITERATURE

The first report on composite plates laminated with thin orthotropic layers or plies was published by Smith [8] in 1953. Reissner and Stavsky [10] were first to recognize the coupling phenomenon between in-plane stretching and transverse bending for nonsymmetrical laminated plates. This phenomenon does not occur in the theory of homogeneous plates and had been overlooked by Smith. The coupling effect was presented by considering two orthotropic layers of equal thickness laminated in such a way that the axes of elastic symmetry are located at an angle of +θ with the plate axes in one layer and an angle of -θ in the other layer. Results obtained by considering coupling were quite different from those obtained without coupling.

In 1962, a general small-deflection theory for the elastostatic extension and flexure of thin laminated anisotropic shells and plates was formulated by Dong, Pister and Taylor [11]. The multilayered composites were composed of arbitrary numbers of bonded layers, each of different thickness, orientation, and/or anisotropic elastic properties. The Kirchhoff-Love assumptions were retained and the governing equations were specialized for the cylindrical shells with orthotropic laminae of equal thickness in various lamina orientation by using the Donnell theory. Although transverse shear deformations were neglected in the theory, the equilibrium equations were used to derive the interlaminar shear stresses.

Whitney and Leissa [12] solved a number of problems relating to

---

8 Numbers in brackets [ ] refer to references given in the Bibliography.
the bending, vibration, and stability of coupled laminates by including the influence of bending-extensional coupling in unsymmetrical laminates. Later, Ashton [13] solved the special class of simply-supported laminated plates by "reduced stiffness matrix" concept to uncouple the governing equations. His approximate solutions were acceptable in comparison with the solutions in reference [12].

The previously described development for the laminated composites was clearly outlined and discussed in references [1] and [16] and was named the classical laminated plate theory (CPT). Later, some researchers realized the limitations of CPT and developed refined theories such as shear deformation theory (SDT) [6] and exact elasticity solutions [2,4].

Stavsky [14] was the first to introduce the shear deformation into laminated plate theory. Ambartsumyan [14] developed a rather cumbersome approach to define transverse shear stresses that satisfied the required continuity conditions at interfaces. Three boundary conditions per edge were specified, but the analysis was restricted to symmetric laminates in which the orthotropic axes of each layer coincided with the plate axes. Whitney [3] extended Ambartsumyan's approach to solve certain specific boundary valued problems for more general material properties and geometries. The most general linear theory for laminates was developed by Yang, Norris and Stavsky [15], and solved the frequency equations for the propagation of harmonic waves in a two-layer isotropic plate of infinite extent. Whitney and Pagano [6] investigated the bending theory of Yang, Norris and Stavsky [15]. Good agreement was observed in numerical results for plate bending as compared to CPT but poor
agreement was found in comparison with the results of Pagano [2]. Pagano presented exact solutions for composite laminates in cylindrical bending [2] and for rectangular bidirectional composites and sandwich plates with simply supported boundary conditions for the static bending [4]. Results for 2-ply and symmetric 3-ply laminates were compared with those of CPT to give insight into the assumptions required for the formulation of more general laminated plate theories. Recently, Pagano [5] extended the investigation to consider the influence of shear coupling for the angle-ply laminates. The general range of validity of CPT were offered.

The finite element method (FEM) was developed originally as a concept of structural analysis based on matrix formulation using electronic computers [18, 19]. Since 1956, there has been a concurrent and rapid development of electronic computers, matrix techniques, and FEM. The FEM can be used to deal with structural problems of complex and irregular geometric shapes, complex in loading-patterns and nonlinear non-homogeneous and anisotropic properties of materials. The FEM deals with the solutions of the element analysis and of the total system analysis. The element analysis involves: (1) the selection of a function that uniquely describes the displacements within the elements in terms of the nodal point displacements \( \{u\} \), (2) the derivation of corresponding stresses, and (3) the derivation of consistent element joint loads \( \{f\} \) from the distributed boundary stresses. The element analysis yields a relationship between nodal point forces and nodal point displacements, \( \{f\} = [k]\{u\} \), where \([k]\) is the element stiffness matrix. Selecting the most efficient type of element for various purposes is a
major task in FEM. Once the element analysis is completed, the system analysis may be formulated in such a way that it is completely unaffected by the type of element used.

The most recent FEM for laminated anisotropic plates was published by Pryor [22]. A rectangular 28-DOF element which includes extension, bending, and transverse shear deformation states at the midplane was employed for the analysis of rectangular anisotropic laminate plates. The properties of the individual layers were integrated through the thickness of the plate. The results were in agreement quantitatively with the solutions of Whitney [3] but disagreed with the solutions of Pagano [2] because the normal to the midplane was severely distorted for sandwich plates with a large difference in material properties between layers.

An increase of available parameters associated with an element usually leads to improved accuracy of solution for a given number of parameters representing the whole assembly. Thus it is possible to use fewer elements for the solution. Based with this concept, Ergatoudis, Irons, and Zienkiewicz [7] introduced curved, isoparametric, quadrilateral elements and these were subsequently expanded to three-dimensional isoparametric elements by Zienkiewicz et al. [21]. Clough [20] compared the directly formed, refined hexahedron element with the one assembled by tetrahedron elements and concluded that the former was superior to the latter, with regard to their structural efficiency and to their general applicability. By using advanced isoparametric cubic elements, the number of unknown nodal displacements and therefore the
size of the coefficient matrix of the simultaneous equations for the structure becomes considerably smaller than in the standard procedure.
III. ANALYTICAL METHOD

A. Development of the Governing Equations for a Lamina

The governing equations for an individual lamina of the laminate are derived by applying the Hellinger-Reissner variational principle [9]. Each lamina is assumed to be a linear elastic, homogeneous, isotropic rectangular layer of constant thickness with normal and shear stresses applied at the top and bottom faces, and all deflections are assumed to be small.

Assumptions and Definitions

An element of a ply of the laminate is shown in Figure 1 oriented with respect to rectangular cartesian coordinates. The displacements are assumed to be expanded in the following form:

\[ u(x, y, z, t) = \bar{u}(x, y, t) + \alpha(x, y, t)z \]
\[ v(x, y, z, t) = \bar{v}(x, y, t) + \beta(x, y, t)z \quad (3.1) \]
\[ w(x, y, z, t) = \bar{w}(x, y, t) + \gamma(x, y, t)z \]

where \( \bar{u}, \bar{v} \) and \( \bar{w} \) are displacements of the middle surface and \( \alpha \) and \( \beta \) are rotations about the middle surface in the \( x \) and \( y \) directions and \( \gamma \) is the normal strain in a direction orthogonal to the middle surface. The in-plane, bending moment, and twisting moment stress resultants are defined as

\[ (P_x, P_y, P_{xy}) = \int_{-h/2}^{h/2} \sigma_x, \sigma_y, \tau_{xy} \, dz \]

*Equations are indicated by numbers in parenthesis.
Figure 1. Laminate, Lamina and an Element of the Lamina.
\[
(M_x, M_y, M_{xy}) = \frac{jh}{2} (\sigma, \sigma, \tau_{xy}) dz
\] (3.2)

\[
(V_x, V_y) = \frac{jh}{2} (\tau_{xz}, \tau_{yz}) dz
\]

Using the generalized Hooke's law and the equilibrium equations of elasticity without the body force terms, the stress components may be derived in the following form:

\[
(\sigma_x, \sigma_y, \tau_{xy}) = \frac{1}{h} (P_x, P_y, P_{xy}) + \frac{j}{h} z (M_x, M_y, M_{xy})
\]

\[
\tau_{xz} = \frac{S_z}{2} + \frac{S_x}{xh} + (V_x - \frac{h}{2} S_x)[1 - \left(\frac{2z}{h}\right)^2] \frac{3}{2h}
\] (3.3)

\[
\tau_{yz} = \frac{S_z}{2} + \frac{S_x}{yh} + (V_y - \frac{h}{2} S_y)[1 - \left(\frac{2z}{h}\right)^2] \frac{3}{2h}
\]

\[
\sigma_x = \frac{S_z}{2} + \left[\frac{S_x}{h} + \frac{1}{2} \left(\frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y}\right) + \rho h \nu \frac{\partial^2}{\partial y^2} \right] [1 - \left(\frac{2z}{h}\right)^2] \frac{z}{2h} + \left[\frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y}\right] \frac{z}{2h}
\]

where

\[
S_z = \frac{\tau_x}{xh} \pm \frac{B}{x^2}, \quad \sigma_z = \frac{\sigma_x}{z} \pm \frac{B}{z^2} \quad \text{and} \quad \sigma_x = \frac{T}{z} \pm \frac{\rho}{z^2},
\]

\[
\tau_{xz} = \tau_{xy} (x, y, \frac{h}{2}, t), \quad \tau_{yz} = \tau_{xz} (x, y, \frac{h}{2}, t), \text{etc.; } \rho \text{ is the mass density per unit volume.}
\]
**Hellinger-Reissner Variational Principle**

The equations of motion and the stress-displacement relations of elasticity satisfying prescribed boundary conditions can be derived by employing the following variational equation:

\[
\delta \int_{t_0}^{t_1} (W - k) \, dt = 0
\]

where

\[
W = \int_R \left( \sigma_x x_x + \sigma_y y_y + \sigma_z z_z + \tau_{xy} \gamma_{xy} + \tau_{xz} \gamma_{xz} + \tau_{yz} \gamma_{yz} \
- \frac{1}{2E} \left( \sigma_x^2 + \sigma_y^2 + \sigma_z^2 + 2\mu (\sigma_x \sigma_y + \sigma_x \sigma_z + \sigma_y \sigma_z) + 2(1+\nu) \right) \right) \, dv
\]

\[
- \int_{S_1} \left( p_x' \dot{u} + p_y' \dot{v} + p_z' \dot{w} \right) \, ds
\]

\[
K = \frac{\mu}{2} \left( \dot{u}^2 + \dot{v}^2 + \dot{w}^2 \right)
\]

and

\[
p_x' = \sigma_x' \gamma_x + \tau_{xy}' \gamma_{xy} + \tau_{xz}' \gamma_{xz}
\]

\[
p_y' = \tau_{xy}' \gamma_y + \sigma_y' \gamma_{xy} + \tau_{yz}' \gamma_{yz}
\]

\[
p_z' = \tau_{xz}' \gamma_z + \sigma_z' \gamma_{xz} + \tau_{yz}' \gamma_{yz}
\]

are the components of the stress vector on the boundary surface \( S_1 \) where the stresses are prescribed, and \( l, m \) and \( n \) are direction cosines of the
outer unit normal vector. The body forces are neglected in the derivation. That part of the boundary surface \( S_2 \) where the displacements are prescribed does not enter into the equation since the variation of the known displacements vanishes there. The boundary conditions on either displacements or stress resultants arise from this principle also. For plate bending and extension, the surface integral term in the equation (3.5) becomes

\[
- \int_{S_{1E}} (p'x u + p'y v + p'z w) dS_{1E} + \{ \int_{S_{1B}} \sigma_{w}^{B}(x, y, -\frac{h}{2}) \\
+ \tau_{xz}^{B} u(x, y, -\frac{h}{2}) + \tau_{yz}^{B} u(x, y, -\frac{h}{2}) \} \\
- \int_{S_{1T}} \tau_{w}^{T}(x, y, \frac{h}{2}) + \tau_{xz}^{T} u(x, y, \frac{h}{2}) + \tau_{yz}^{T} v(x, y, \frac{h}{2}) \} \ dx \ dy
\]

where \( S_{1E} \) is that part of the edge surface on which the stresses are prescribed; and \( S_{1T} \) and \( S_{1B} \) are those portions of the top and bottom surfaces, respectively, where the stresses are prescribed. Substituting the stress components from equation (3.3) and strains from equation (3.1) into equation (3.4) and employing standard techniques from the calculus of variationals, the following equation is obtained.

\[
\int_{S_{1E}} \left\{ \left[ -\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y} - S_{x} + \rho h u \right] \delta u + \left[ -\frac{\partial ^{2} F}{\partial y} - \frac{\partial F}{\partial y} - S_{y} + \rho h v \right] \delta v \\
+ \left[ -\frac{\partial ^{2} F}{\partial x} - \frac{\partial ^{2} F}{\partial y} - S_{z} + \rho h w \right] \delta w \right\} \ dx \ dy
\]
\[
\begin{align*}
+ [ - \frac{\partial M}{\partial x} - \frac{\partial M}{\partial y} + \gamma - \frac{h}{2} S^+ + \frac{\rho h}{12} \alpha ] \delta \alpha \\
+ [ - \frac{\partial M}{\partial x} - \frac{\partial M}{\partial y} + \gamma - \frac{h}{2} S^+ + \frac{\rho h}{12} \beta ] \delta \beta + [ - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{P_{xy}}{\gamma} ] \delta P_{xy} \\
+ \left[ \frac{\partial u}{\partial x} - \frac{P}{E_h} + \frac{P}{E_h} + \frac{v}{2E} S^+ + \frac{v h}{12 E} (s_{xx,x} + s_{yy,y}) - \frac{v h}{1} \gamma \right] \delta P_x \\
+ \left[ \frac{\partial v}{\partial y} - \frac{P}{E_h} + \frac{P}{E_h} + \frac{v}{2E} S^+ + \frac{v h}{12 E} (s_{xx,x} + s_{yy,y}) - \frac{v h}{1} \gamma \right] \delta P_y \\
+ \left[ \frac{\partial a}{\partial x} - \frac{12}{E_h^3} M_x + \frac{12 u}{E_h^3} M_y + \frac{6 u}{E_h^3} M_z + \frac{v}{10 E} (s_{xx,x} + s_{yy,y}) - \frac{v}{5E} \rho \alpha \right] \delta M_x \\
+ \left[ \frac{\partial b}{\partial y} - \frac{12}{E_h^3} M_y + \frac{12 u}{E_h^3} M_x + \frac{6 u}{E_h^3} M_z + \frac{v}{10 E} (s_{xx,x} + s_{yy,y}) \right] \delta M_y \\
- \frac{v}{5E} \rho \beta \right] \delta M_y + \left[ \frac{\partial w}{\partial x} - \frac{6v}{5E} + \frac{v h}{10 E} \right] \delta v_x + \left[ \frac{\partial v}{\partial y} - \frac{6v}{5E} + \frac{v h}{10 E} \right] \delta v_x \\
+ \left[ \beta + \frac{\partial w}{\partial y} - \frac{6v}{5E} + \frac{v h}{10 E} \right] \delta v_x + \frac{1}{2E} \left[ \frac{3}{2} \rho h \frac{2}{10 E} (s_{xx,x} + s_{yy,y}) \right] \\
- \frac{\rho h^3}{10 E} \gamma + \frac{5}{6} \left( \delta v_x + \delta v_y \right) \delta v_x + \frac{1}{2E} \left[ \frac{\rho h}{12} S^+ + \frac{\rho h}{10} (s_{xx,x} + s_{yy,y}) \right] \\
- \frac{\rho h^3}{60} \gamma - \frac{v h^2}{6} \left( \delta v_x + \delta v_y \right) - \frac{\rho h^3}{12} \gamma \delta v_x + \left[ \frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} \right] \delta v_x
\end{align*}
\]
\[\begin{align*}
+ \left[ \left( \frac{P_y - P'_y}{y} \right) \delta \omega + \left( \frac{M_y - M'_y}{y} \right) \delta \beta \right] m + \left[ \left( \frac{V_x - V'_x}{x} \right) \delta \omega + \frac{h^2}{12} \left( \frac{S^-}{y} - \frac{S'^-}{y} \right) \delta \gamma \right] l \\
+ \left[ \left( V_y - V'_y \right) \delta \omega + \frac{h^2}{12} \left( S^- - S'^- \right) \delta \gamma \right] m \cdot \frac{dc}{dt}
\end{align*}\]

\[\begin{align*}
+ \int_{t_0}^{t_1} \left\{ \left[ P_x \delta \omega + M_x \delta \alpha \right] l + \left[ P_y \delta \omega + M_y \delta \alpha \right] m + \left[ P_{xy} \delta \nu + M_{xy} \delta \beta \right] l \\
+ \left[ P_y \delta \nu + M_y \delta \beta \right] m \right. \\
+ \left[ \frac{h^2}{12} \left( S^- + \delta \gamma \right) l + \left[ V_x \delta \omega + \frac{h^2}{12} S^- \delta \gamma \right] m \right. \\
\left. \right\} \cdot \frac{dc}{dt}
\end{align*}\]

\[\begin{align*}
+ \int_S \left\{ \frac{1}{2E} \left[ \frac{3}{70} \rho \rho \frac{h^2}{12} \varepsilon_z + \frac{h^2}{210} \left( S^+ + S^- \right) \right] - \frac{\rho^2}{105} \frac{2h^3}{w} - \frac{2}{5} \mu \left( M_x + M_y \right) \right. \\
\left. \cdot \frac{dc}{dt} \right\}
\end{align*}\]

\[\begin{align*}
- \int_S \left\{ \frac{1}{2E} \left[ \frac{3}{70} \rho \rho \frac{h^2}{12} \varepsilon_z + \frac{h^2}{210} \left( S^+ + S^- \right) \right] - \frac{\rho^2}{105} \frac{2h^3}{w} - \frac{2}{5} \mu \left( M_x + M_y \right) \right. \\
\left. \cdot \frac{dc}{dt} \right\}
\end{align*}\]

\[\begin{align*}
+ \int_S \left\{ \frac{1}{2E} \left[ \frac{h^3}{12} \varepsilon_z + \frac{h^4}{60} \left( S^+ + S^- \right) \right] - \frac{\rho^2}{60} \frac{2h^5}{w} - \frac{2}{5} \mu \left( P_x + P_y \right) \right. \\
\left. \cdot \frac{dc}{dt} \right\}
\end{align*}\]

\[\begin{align*}
- \frac{\rho h^3}{12} \gamma \delta \gamma \right\} \cdot \frac{dc}{dt}
\end{align*}\]

\[\begin{align*}
- \int_S \left\{ \frac{1}{2E} \left[ \frac{h^3}{12} \varepsilon_z + \frac{h^4}{60} \left( S^+ + S^- \right) \right] - \frac{\rho^2}{60} \frac{2h^5}{w} - \frac{2}{5} \mu \left( P_x + P_y \right) \right. \\
\left. \cdot \frac{dc}{dt} \right\}
\end{align*}\]
The line integral involving $C_1$ is taken over boundary where the stresses are prescribed, and the one involving $C_2$ is taken over the remaining portion of the boundary where the displacements are prescribed. In equation (3.8), the coefficients of each variations must vanish individually.

Equations of Motion

From the first five brackets of equation (3.8) arise five equations of motion:

\[
\begin{align*}
\frac{\partial P}{\partial x} + \frac{\partial P_{xy}}{\partial y} + S_x &= \frac{\partial}{\partial x} (\rho h u) \\
\frac{\partial P_{xy}}{\partial x} + \frac{\partial P}{\partial y} + S_y &= \frac{\partial}{\partial y} (\rho h v) \\
\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + S_z &= \frac{\partial}{\partial z} (\rho h w) \\
\frac{\partial M}{\partial x} + \frac{\partial M_{xy}}{\partial y} - V_x + \frac{h}{2} S_x &= \frac{1}{12} \rho h \frac{\partial}{\partial x} \rho h \frac{\partial}{\partial y} \\
\frac{\partial M_{xy}}{\partial x} + \frac{\partial M}{\partial y} - V_y + \frac{h}{2} S_y &= \frac{1}{12} \rho h \frac{\partial}{\partial x} \rho h \frac{\partial}{\partial y}
\end{align*}
\]
Stress Resultant-Displacement Relationships

Setting the coefficients of the variation of the stress resultants in equation (3.8) yields

\[
\frac{\partial u}{\partial x} = \frac{P x - u P y}{E h} - \frac{v}{2E} S_z - \frac{u h}{12E} (S_x^{+} - S_y^{-}) + \frac{v}{12E} \rho h^2 \gamma
\]

\[
\frac{\partial v}{\partial y} = \frac{P y - u P x}{E h} - \frac{v}{2E} S_z - \frac{u h}{12E} (S_x^{+} - S_y^{-}) + \frac{v}{12E} \rho h^2 \gamma
\]

\[
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{P x y}{G h}
\]

\[
\frac{\partial \alpha}{\partial x} = \frac{12}{E h^3} (M_x - u M_y) - \frac{6v}{5E h} S_z - \frac{u}{10E} (S_x^{+} + S_y^{-}) + \frac{v}{5E} \rho \omega
\]

\[
\frac{\partial \beta}{\partial y} = \frac{12}{E h^3} (M_y - u M_x) - \frac{6v}{5E h} S_z - \frac{u}{10E} (S_x^{+} + S_y^{-}) + \frac{v}{5E} \rho \omega
\]

\[
\frac{\partial \gamma}{\partial x} + \frac{\partial \omega}{\partial y} = \frac{12}{G h^3} M_{xy}
\]

\[
\alpha + \frac{\partial \omega}{\partial x} = \frac{6}{5G h} V x - \frac{S_x^{+}}{10G}
\]

\[
\beta + \frac{\partial \omega}{\partial y} = \frac{6}{5G h} V y - \frac{S_y^{+}}{10G}
\]

\[
\gamma = \frac{1}{2E} S_z + \frac{h}{10E} (S_x^{+} + S_y^{-}) - \frac{u}{E h} (P_x + P_y) - \frac{1}{10E} \rho h^2 \gamma
\]

Basic Equations for the Bending of Rectangular Plates

Solving equations (3.9) and (3.10), yields six equations for six
unknowns, \( \overline{w}, V_x, V_y, M_x, M_{xy} \) and \( M_y \). They are

\[\nabla^4 w = S_z^+ - k \nabla^2 (S_z^+) - \frac{k}{12} \left( \frac{\partial}{\partial x} \nabla^2 S_x^+ + \frac{\partial}{\partial y} \nabla^2 S_y^+ \right) - \frac{\partial}{\partial w} S_z^+ + \frac{3 \partial}{\partial x} \nabla^2 w + \frac{3 \partial}{\partial y} \nabla^2 w - \frac{17-6u}{60(1-u)} \phi h^3 \nabla^2 \overline{w} + \frac{\phi h^3}{120 \epsilon} (S_x^+ + S_y^+) \nabla^2 w + \frac{\phi h^2}{10 \epsilon} (S_z^+ - \phi h \overline{w}) \]

\[+ \frac{h}{2} (S_x^+ + S_y^+) \]

\[ \frac{h^2}{10} \nabla^2 V_x - V_x = \frac{\phi h^2}{10 \epsilon} \nabla^2 V_x + D \frac{\partial}{\partial x} \nabla^2 \overline{w} - \frac{h}{2} S_x^+ + \frac{h^2}{10 (1-u)} S_{z,x}^+ \]

\[+ \frac{k}{12} h S_{x,xx}^+ + \frac{h}{120} S_{x,yy}^+ + \frac{h^3}{120} \left( 1 - \frac{1}{1-u} \right) S_{x,xy}^+ - \frac{11 \phi h^3}{60 (1-u)} \overline{w} \]

\[- \frac{\phi h^3}{120 \epsilon} S_x^+ \]

\[ \frac{h^2}{10} \nabla^2 V_y - V_y = \frac{\phi h^2}{10 \epsilon} \nabla^2 V_y + D \frac{\partial}{\partial y} \nabla^2 \overline{w} - \frac{h}{2} S_y^+ + \frac{h^2}{10 (1-u)} S_{z,y}^+ \]

\[+ \frac{k}{12} h S_{y,yy}^+ + \frac{h}{120} S_{y,xx}^+ + \frac{h^3}{120} \left( 1 - \frac{1}{1-u} \right) S_{y,xy}^+ - \frac{11 \phi h^3}{60 (1-u)} \overline{w} \]

\[- \frac{\phi h^3}{120 \epsilon} S_y^+ \]

\[ M_x = D \frac{6}{5} \frac{1-u}{C h} V_x - (\overline{w}_{,xx} + \overline{w}_{,yy}) - \frac{3v}{5Ch} S_x^+ - \frac{2-u}{20 \epsilon} S_x^+ - \frac{u}{20 \epsilon} S_y^+ \]

\[+ \frac{11}{60} \frac{v}{1-u} \phi h \overline{w} \]
\[ M_{xy} = \frac{Gh^3}{12} [\frac{6}{5Gh} (V_{x,y} + V_{y,x}) - 2 \frac{w_{x,y}}{w_{x,y}} - \frac{1}{10G} (S_{x,y}^+ + S_{y,x}^+)] \]

\[ M_y = D \left[ \frac{6}{5Gh} V_{y,y} - (\overline{w}_{yy} + \overline{w}_{xx}) - \frac{3u}{5Gh} \varepsilon_x^z - \frac{2-v}{20G} S_{y,y}^+ \right. \]

\[ \left. - \frac{u}{20G} S_{x,x}^+ + \frac{11}{60} \frac{u}{1-u} \rho h \frac{\ddot{w}}{w} \right] \]

where \( D = \frac{E}{12} \frac{h^3}{(1-v)^2} \) and \( k = \frac{h^2}{10} \frac{2-v}{1-u} \). The boundary conditions for this problem are also obtained from the variational equation (3.9). On a straight edge of constant \( x \), these are

\[ V_x = V'_x \quad \text{or} \quad \overline{w} = \overline{w}' \]

\[ M_x = M'_x \quad \text{or} \quad \alpha = \alpha' \]

\[ M_{xy} = M'_{xy} \quad \text{or} \quad \beta = \beta' \]

Whereas on a straight edge of constant \( y \), the boundary conditions are

\[ V_y = V'_y \quad \text{or} \quad \overline{w} = \overline{w}' \]

\[ M_y = M'_y \quad \text{or} \quad \beta = \beta' \]

\[ M_{xy} = M'_{xy} \quad \text{or} \quad \alpha = \alpha' \]

Thus three boundary conditions must be satisfied on an edge instead of two as in the classical theory.

**Governing Equations for the Extension of Plates**

From the first three equations of (3.10) and the first two equations of (3.9), proper manipulation gives two equations on \( \overline{u} \) and \( \overline{v} \) and three equations for the loads in terms of the displacements \( \overline{u} \) and \( \overline{v} \). These are
\[
C \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} - (1 + \nu) \frac{\partial^2 \Gamma}{\partial x^2} \right) + 4\Gamma \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} \right) + S^+ = \rho \dot{u}
\]

\[
C \left( \frac{\partial^2 v}{\partial y^2} + \nu \frac{\partial^2 u}{\partial x \partial y} - (1 + \nu) \frac{\partial^2 \Gamma}{\partial y^2} \right) + 4\Gamma \left( \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y} \right) + S^- = \rho \dot{v}
\]

\[
P_x = C \left[ \frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} \right] - (1 + \nu) \Gamma
\]

\[
P_y = C \left[ \frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x} \right] - (1 + \nu) \Gamma
\]

\[
P_{xy} = 4\Gamma \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right]
\]

where \( C = \frac{Eh}{1-u^2} \) and \( \Gamma = \frac{u}{2E} S^+ - \frac{uh}{12E} (S^-_{x,x} + S^-_{y,y}) - \frac{u}{12E} \rho h^2 \gamma \).

The boundary conditions for this problem, for rectangular plates, are:

a) on a straight edge of constant \( x \)

\[
P_x = P'_x \quad \text{or} \quad \dot{u} = \dot{u}'
\]

\[
P_{xy} = P'_x \quad \text{or} \quad \dot{v} = \dot{v}'
\]

b) on a straight edge of constant \( y \)

\[
P_{xy} = P'_y \quad \text{or} \quad \dot{u} = \dot{u}'
\]

\[
P_y = P'_y \quad \text{or} \quad \dot{v} = \dot{v}'
\]

The equation for \( \gamma \) is shown in the last equation of (3.19).
From the last integral of equation (3.8), twelve initial conditions arise. These may be initial displacements and initial velocities. For the bending problem, there will be six conditions: two on \( \bar{w} \), two on \( \alpha \) and two on \( \beta \). There are also six conditions for the extension problem: two on \( \bar{u} \), two on \( \overline{\nu} \) and two on \( \gamma \).

B. Cylindrical Bending of a Two-ply Laminate

The derived governing equations for three-dimensional problem are specialized to solve the two-dimensional problem. A two-ply laminate is shown in Figure 2. A distributed load \( q(x) \) is applied at the top face. Recognizing that almost any loading function can be expressed in the form of Fourier series, a sinusoidal loading is used for \( q(x) \). The laminate is simply supported at both ends and is in a state of plane strain with respect to the \( xz \) plane.

The governing equations of a single ply of the laminate subjected to static bending can be shown to be:

\[
\frac{d^4 w}{dx^4} = \frac{S_z}{D} + \frac{h}{2D} \frac{d \alpha}{dx} - \frac{k}{D} \frac{d^3 \gamma}{dx^3} - \frac{kh}{12D} \frac{d \gamma}{dx} \]

\[
\frac{d^2 u}{dx^2} = -\frac{1 - \nu}{2Gc} \frac{S_z}{x} - \frac{\nu}{6G} \frac{d \gamma}{dx} - \frac{h\nu}{2Gc} \frac{d \gamma}{dx} \]

\[
\alpha = -\frac{d w}{dx} + \frac{h^2}{5(1 - \nu)} \frac{d^3 w}{dx^3} + \frac{1}{2Gc} \frac{S_z}{x} - \frac{6k}{5Gc} \frac{d \gamma}{dx} - \frac{k}{10G} \frac{d^2 \gamma}{dx^2} \]
Figure 2. Aluminum-Steel Laminate in Cylindrical Bending.
\[ \gamma = -\frac{u}{1-v} \frac{d\bar{u}}{dx} + \frac{1}{4G} \frac{1-2v}{1-v} S_z \frac{d S^-}{dx} + \frac{3-3v-5v^2}{30E(1-v)} h \frac{d S^-}{dx} \]  

\[ V_x = -D \frac{d^2 \bar{w}}{dx^3} + \frac{h}{2} S_x^+ - \kappa \frac{d S^-}{dx} - \frac{kh}{12} \frac{d^2 S^+}{dx^2} \]  

\[ M_x = -D \frac{d^2 \bar{w}}{dx^3} - \kappa S_z^- - \frac{kh}{12} \frac{d S^+}{dx} \]  

\[ P_x = \frac{2Gh}{1-v} \frac{d\bar{u}}{dx} + \frac{h}{2} \frac{v}{1-v} S_z^+ + \frac{h^2}{12} \frac{v}{1-v} \frac{d S^-}{dx} \]  

All unknowns are expressed in terms of applied load \( q(x) \) at the top face and the unknown interface stresses \( \sigma \) and \( z \). The interface stresses are solved by the continuity conditions of displacement at the interface, these are

\[ \bar{u}_1 - \frac{h_1}{2} \alpha_1 = \bar{u}_2 + \frac{h_2}{2} \alpha_2 \]  

\[ \bar{w}_1 - \frac{h_1}{2} \gamma_1 = \bar{w}_2 + \frac{h_2}{2} \gamma_2 \]  

Recognizing that the boundary conditions of a simply-supported laminate are \( \bar{w}, M_x \), and \( P_x \) equal to zero and that the leading function \( q(x) \) is in the form of a single sinusoidal function, the following forms of the solutions of \( \tau \) and \( \sigma \) are used

\[ \tau = c_1 \cos \frac{\pi x}{\ell} + c_3 (1 - \frac{2x}{\ell}) \]  

\[ \sigma = c_2 \sin \frac{\pi x}{\ell} + c_4 \]
Writing the equations (3.11) for each ply and applying the continuity conditions (3.12), it can be shown that $C_3$ and $C_4$ are zero and $C_1$ and $C_2$ are given by solution of the following simultaneous equations

$$
C_1\left[\frac{n^3}{D_1} - \frac{h_2}{D_2}\right] + \frac{n}{10}\left(\frac{1-3v_1}{G_1} - \frac{1-3v_2}{G_2}\right) + \frac{1}{12n}\frac{h_1}{1-v_1}\left(\frac{v_1}{G_1} + \frac{3-3v_1-5v_1^2}{5E_1}\right)
$$

$$
- \frac{1}{12n}\frac{h_2^2}{1-v_2}\left(\frac{v_2^2}{G_2} + \frac{3-3v_2-5v_2^2}{5E_2}\right)
$$

$$
+ C_2\left[\left(\frac{1}{D_1} + \frac{1}{D_2}\right)n^4 + \left(\frac{k_1}{D_1} + \frac{k_2}{D_2}\right)n^2 + \frac{1-v_1}{8G_1}h_1 + \frac{1-v_2}{8G_2}h_2\right] = \bar{q}\left[\frac{n}{D_1} + \frac{k_2n}{D_2} - \frac{1-v_2}{8G_2}h_2\right]
$$

$$
C_1\left[\frac{\nu_1}{15}h_1 + \frac{h_2v_2}{G_2}\right] - 2n^2\left(\frac{1-v_1}{G_1h_1} + \frac{1-v_2}{G_2h_2}\right)
$$

$$
+ C_2\left[\frac{n}{20}\left(\frac{v_1}{G_1} - \frac{v_2}{G_2}\right) - \frac{n^3}{2}\left(\frac{h_1}{D_1} - \frac{h_2}{D_2}\right)\right] = \bar{q}\left[\frac{n}{2}\frac{h_2}{D_2} - \frac{n}{20}\frac{v_2}{G_2}\right]
$$

(3.14)

where $n = \frac{1}{2}$. Once $C_1$ and $C_2$ are obtained, the displacements and stress resultants are solved by the following equations

$$
\bar{w}_1 = -\frac{n}{D_1}\left[C_2n^3 + C_1h_1\frac{n^2}{2} + C_2k_1n + C_1\frac{k_1n^2}{12}\right] \sin \frac{\pi x}{L}
$$

$$
\bar{w}_2 = \frac{n}{D_2}\left[(C_2 - \bar{q})n^3 - C_1h_2\frac{n^2}{2} + (C_2 - \bar{q})k_2n - C_1\frac{k_2n^2}{12}\right] \sin \frac{\pi x}{L}
$$

$$
\gamma_1 = \frac{1}{G_1}\left[C_1\frac{v_1}{12n} - \frac{1-v_1}{4}C_2 - \frac{h_1}{6(1-v_1)}\left(\frac{v_2}{4} + \frac{G_1}{F_1}\frac{3-3v_1-5v_1^2}{5}\right)\frac{C_1}{n}\right] \sin \frac{\pi x}{L}
$$
\[ y_2 = \frac{1}{G_2} \left[ \frac{v_2}{2h_2} \right] n + \frac{1-u_2}{4} (c_2 + \overline{q}) - \frac{h_2}{6(1-u_2)} \left( \frac{u_2}{4} + \frac{G_2}{E_2} \frac{3-3u_2-5u_2^2}{5} \right) \]

\[ \frac{c_1}{n} \] \sin \frac{\pi x}{\lambda} \]

\[ \overline{u}_1 = -\frac{1}{G_1} \left[ \frac{1-u_1}{2h_1} \right] n^2 - c_2 \frac{u_1}{4} n - \frac{h_1 u_1}{24} c_1] \cos \frac{\pi x}{\lambda} \]

\[ \overline{u}_2 = \frac{1}{G_2} \left[ \frac{1-u_2}{2h_2} \right] n^2 + (c_2 + \overline{q}) n - \frac{h_2 u_2}{24} c_1 \] \cos \frac{\pi x}{\lambda} \]

\[ a_1 = \left[ \frac{c_2}{D_1} n^3 + c_1 \frac{h_1}{2D_1} \right] n^2 - c_2 \frac{3v_1}{5G_1 h_1} n - \frac{u_1}{20G_1} c_1 \] \cos \frac{\pi x}{\lambda} \]

\[ a_2 = \left[ (c_2 - \overline{c}) \frac{3}{D_2} + c_1 \frac{h_2}{2D_2} \right] n^2 + (c_2 - \overline{q}) \frac{3v_2}{5G_2 h_2} n - \frac{u_2}{20G_2} c_1 \] \cos \frac{\pi x}{\lambda} \]

\[ (3.15) \]

\[ v_{x_1} = -c_2 n \cos \frac{\pi x}{\lambda} \]

\[ v_{x_2} = (c_2 - \overline{q}) n \cos \frac{\pi x}{\lambda} \]

\[ M_{x_1} = -n(c_2 n + c_1 \frac{h_1}{2}) \sin \frac{\pi x}{\lambda} \]

\[ M_{x_2} = n [(c_2 - \overline{q}) n - c_1 \frac{h_2}{2}] \sin \frac{\pi x}{\lambda} \]

\[ p_{x_1} = c_1 n \sin \frac{\pi x}{\lambda} \]

\[ p_{x_2} = -c_1 n \sin \frac{\pi x}{\lambda} \]
The displacement distributions and stress distributions for a section are obtained by substituting equations (3.15) into (3.1) and (3.3), respectively.

**Examples and Discussion**

Solution for the laminate as shown in Figure 2 are shown in Table I for various s, which is the span-to-thickness ratio of the laminate. Three cases are shown for comparison. Solution A is calculated by the elementary beam theory in which the transformed section of the two materials is used to calculate displacements and stresses. Solution B is obtained in this study. In order to observe the significant effect of the displacement function γ, solution C is obtained by dropping γ in the expansion of vertical displacement w. The normalized maximum vertical deflections are shown for the three cases. The solution B is about 5.6% and the solution C is about 4.3% larger than the solution A for s = 4. This indicates that both shear deformation and normal straining should be considered in which s is less than 10. The normalized horizontal rotations about y-axis θ₁ and θ₂ are shown for solutions B and C. In the beam theory, α is a constant through the thickness. It is shown in the table that α is not the same for each layer, specially when s is less than 10. The solutions of the maximum normal stress σₓ at each layer are almost the same in three cases. The maximum shearing stress are at the neutral axis for the solution A and calculated at the interface for the solution B and C. In this example, the neutral axis is located near the interface. The shearing stresses at the three points of the cross-section are the same for both solutions
<table>
<thead>
<tr>
<th>s</th>
<th>Normalized Max. w</th>
<th>Normalized $\alpha_1$</th>
<th>Normalized $\alpha_2$</th>
<th>Max. $\sigma_{x1}$ psi</th>
<th>Max. $\sigma_{x2}$ psi</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>A</td>
<td>B</td>
<td>C</td>
<td>B</td>
</tr>
<tr>
<td>4</td>
<td>1.00</td>
<td>1.056</td>
<td>1.043</td>
<td>1.217</td>
<td>1.209</td>
</tr>
<tr>
<td>10</td>
<td>1.00</td>
<td>.932</td>
<td>.929</td>
<td>1.034</td>
<td>1.032</td>
</tr>
<tr>
<td>20</td>
<td>1.00</td>
<td>.913</td>
<td>.912</td>
<td>1.008</td>
<td>1.008</td>
</tr>
<tr>
<td>30</td>
<td>1.00</td>
<td>.910</td>
<td>.909</td>
<td>1.004</td>
<td>1.003</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>s</th>
<th>Max. $\tau$ psi</th>
<th>$\tau(z_1 = 0)$ psi</th>
<th>$\tau(z_2 = 0)$ psi</th>
<th>$\tau_z$ @ interface</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>A</td>
<td>B,C</td>
<td>A</td>
</tr>
<tr>
<td>4</td>
<td>1.91</td>
<td>1.92</td>
<td>1.35</td>
<td>1.38</td>
</tr>
<tr>
<td>10</td>
<td>4.77</td>
<td>4.75</td>
<td>3.37</td>
<td>3.35</td>
</tr>
<tr>
<td>20</td>
<td>9.53</td>
<td>9.49</td>
<td>6.73</td>
<td>6.67</td>
</tr>
<tr>
<td>30</td>
<td>14.30</td>
<td>14.23</td>
<td>10.10</td>
<td>9.99</td>
</tr>
</tbody>
</table>

A: Elementary Solution
B: Solution for $w = \bar{w} + \gamma z$
C: Solution for $w = \bar{w}$
B and C. This is also for the normal stress $c_n$ at the interface.

In Figure 3, an example is shown where the neutral axis is not near the interface. Comparison with the solution of the elementary beam theory shows excellent agreement for the maximum normal stresses and shearing stress for $s = 10$. Thus it is natural to conclude that the theory developed with linear terms in the expansion of the displacement functions is good for the isotropic, homogeneous ply in the laminate.

**Fourier Series Representation for a Uniform Load**

The Fourier series representation for a uniform load $\bar{q}$ is the following

$$q(x) = \frac{4q}{\pi} \left[ \sin \frac{nx}{l} + \frac{1}{3} \sin \frac{3nx}{l} + \frac{1}{5} \sin \frac{5nx}{l} + \ldots \right]$$

or

$$q(x) = \frac{4q}{\pi} \sum_{n=1,3}^{2} \frac{1}{n} \sin \frac{n\pi x}{l} \quad (3.16)$$

The partial loads represented by the first three terms are shown in Figure 4(a), 4(b) and 4(c). The sum of the first and second term is shown in Figure 4(d), and the sum of the first three terms is represented in Figure 4(e). The sum of all the terms of the series gives the straight line $\bar{q} = 1$. In general, only the first few terms of this series are needed to achieve sufficient accuracy.

The laminate as shown in Figure 2 is solved by two Fourier series representations of the uniform load $\bar{q} = 1$ for $s = 10$. Solution for stresses and displacement distributions are shown in Table II. Three cases are shown for comparison. Solution A is obtained by using only the first three partial loads. Solution B is obtained by using the
Figure 3. Displacement and Stress Distributions for Aluminum-Steel Laminate, \( h_1 = .1" \), \( h_2 = .5" \), \( s = 10 \)
Figure 4. Fourier Series Expansion for Uniform Load $\overline{q} = 1$. 
TABLE II. STRESS AND DISPLACEMENT DISTRIBUTIONS FOR ALUMINUM-STEEL LAMINATE BY FOURIER SERIES REPRESENTATION FOR A UNIFORM LOAD \( \bar{q} = 1 \text{k/"}, h_1 = .1", h_2 = .2", s = 10 \).

<table>
<thead>
<tr>
<th>( \sigma_x(\ell/2) ) ksi</th>
<th>( \tau_{xz}(\ell) ) ksi</th>
<th>( \sigma_z(\ell/2) ) ksi</th>
</tr>
</thead>
<tbody>
<tr>
<td>A  B  C</td>
<td>A  B  C</td>
<td>A  B</td>
</tr>
<tr>
<td>-59.50 -59.42 -59.10</td>
<td>0  0  0</td>
<td>1.103 1.063</td>
</tr>
<tr>
<td>3.10 3.27 3.42</td>
<td>1.038 .999</td>
<td>A: 3 terms Fourier series</td>
</tr>
<tr>
<td>3.43 3.44 3.11</td>
<td>.865 .832</td>
<td>B: 5 terms Fourier series</td>
</tr>
<tr>
<td>5.41 5.56 5.81</td>
<td>C: Elementary Solution</td>
<td></td>
</tr>
<tr>
<td>6.67 6.86 7.15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.98 7.19 7.47</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.94 5.09 5.29</td>
<td></td>
<td></td>
</tr>
<tr>
<td>101.92 101.72 102.65</td>
<td>0  0  0</td>
<td>0  0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( u(\ell) \times 10^{-2} \text{ in.} )</th>
<th>( w(\ell/2) \times 10^{-1} \text{ in.} )</th>
<th>( V_x(\ell) ) k/in</th>
<th>( M_x(\ell/2) ) kip</th>
</tr>
</thead>
<tbody>
<tr>
<td>A  B</td>
<td>A  B  C</td>
<td>A#  B#  C</td>
<td>A##  B##  C</td>
</tr>
<tr>
<td>-.5199</td>
<td>-.5199</td>
<td>.2692 .2692</td>
<td>1.40 1.44 1.5</td>
</tr>
<tr>
<td>.0318</td>
<td>.0319</td>
<td>.2716 .2715 .2916</td>
<td>1.127 1.126 1.125</td>
</tr>
<tr>
<td>.3181</td>
<td>.3183</td>
<td>.2710 .2709</td>
<td>* ( V_x = V_{x1} + V_{x2} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( * )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( \ast )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( ** )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( *** )</td>
</tr>
</tbody>
</table>

\( * \) \( \text{V}_x = \text{V}_{x1} + \text{V}_{x2} \)

\( ** \text{M}_x + \text{M}_{x1} + \text{M}_{x2} + P_{x1}z_1 + P_{x2}z_2 \)
first five terms partial loads. Solution C is calculated by the elementary beam theory by using the transformed section. The stress \( \sigma_x \) is shown for mid span at the top face, interface and bottom face. Two values are shown at the interface, since the elasticity moduli are different. Solution for \( \sigma_x \) is good compared with the elementary solution even only three terms are used. The shear stress resultant for the solution B is better than the solution A compared with the solution C. Therefore the distribution of \( \tau_{xz} \) is the same. This indicates that in order to obtain a better solution for \( \tau_{xz} \), more terms in the series be used. The solutions for displacement \( u \) and \( w \) are good for the solution A. The maximum value for \( w \) is occurred at the interface. The normal stress \( \sigma_z \) for the solution B is better than the solution A at the midspan, while it is zero at the supported edges by the Fourier analysis.

The displacements and stresses along the interface for this problem are shown in Table III for solutions A and B. The displacements \( u \) and \( w \) are the same for solutions A and B along the \( x \)-direction. The stresses \( \tau_{xz} \), \( \sigma_z \) and \( \varphi_x \) from the solution B is better than the solution A. The stress \( \sigma_x \), \( M_x \) and \( P_x \) are good already from the solution A. Thus, more terms are needed in the solution of shearing stress and normal stress \( \sigma_z \) by a Fourier series representation of the uniform load than the normal stress \( \sigma_x \) and displacements \( u \) and \( w \).

In general, for a \( N \)-ply laminate, there are \( (N - 1) \) sets of equations (3.14) obtained from the displacement continuity conditions at the \( (N - 1) \) interfaces. The solutions of the interface stresses
<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>l/12</th>
<th>l/6</th>
<th>l/4</th>
<th>l/3</th>
<th>5l/12</th>
<th>l/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>u</td>
<td>A</td>
<td>.318</td>
<td>.307</td>
<td>.273</td>
<td>.221</td>
<td>.155</td>
<td>.080</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>.318</td>
<td>.307</td>
<td>.273</td>
<td>.221</td>
<td>.155</td>
<td>.080</td>
</tr>
<tr>
<td>v</td>
<td>A</td>
<td>0</td>
<td>.072</td>
<td>.138</td>
<td>.194</td>
<td>.236</td>
<td>.263</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>0</td>
<td>.072</td>
<td>.138</td>
<td>.194</td>
<td>.236</td>
<td>.263</td>
</tr>
<tr>
<td>x</td>
<td>10^-3</td>
<td>St. 0</td>
<td>2.352</td>
<td>5.489</td>
<td>7.537</td>
<td>9.030</td>
<td>9.937</td>
</tr>
<tr>
<td></td>
<td>A</td>
<td>Al. 0</td>
<td>.074</td>
<td>-1.732</td>
<td>-2.490</td>
<td>-3.038</td>
<td>-3.337</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>St. 0</td>
<td>2.941</td>
<td>5.420</td>
<td>7.547</td>
<td>9.050</td>
<td>9.916</td>
</tr>
<tr>
<td></td>
<td>A</td>
<td>Al. 0</td>
<td>.075</td>
<td>-1.754</td>
<td>-2.499</td>
<td>-3.018</td>
<td>-3.332</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>6.980</td>
<td>6.389</td>
<td>5.022</td>
<td>3.613</td>
<td>2.470</td>
<td>1.326</td>
</tr>
<tr>
<td></td>
<td>7.192</td>
<td>6.297</td>
<td>4.988</td>
<td>3.770</td>
<td>2.454</td>
<td>1.257</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>A</td>
<td>0</td>
<td>.286</td>
<td>-.383</td>
<td>-.322</td>
<td>-.274</td>
<td>-.314</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>0</td>
<td>.378</td>
<td>-.597</td>
<td>-.311</td>
<td>-.328</td>
<td>-.298</td>
</tr>
<tr>
<td>x</td>
<td>1.399</td>
<td>1.283</td>
<td>1.011</td>
<td>.730</td>
<td>.497</td>
<td>.266</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>1.439</td>
<td>1.266</td>
<td>.989</td>
<td>.758</td>
<td>.495</td>
<td>.253</td>
</tr>
<tr>
<td></td>
<td>A</td>
<td>0</td>
<td>.340</td>
<td>.628</td>
<td>.845</td>
<td>.997</td>
<td>1.093</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>0</td>
<td>.324</td>
<td>.625</td>
<td>.844</td>
<td>1.000</td>
<td>1.093</td>
</tr>
<tr>
<td>p</td>
<td>A</td>
<td>0</td>
<td>1.694</td>
<td>3.129</td>
<td>4.204</td>
<td>4.961</td>
<td>5.439</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>0</td>
<td>1.718</td>
<td>3.111</td>
<td>4.197</td>
<td>4.977</td>
<td>5.438</td>
</tr>
</tbody>
</table>
are solved once the properties of each lamina and the dimensions of the laminate are known. With the displacement and stress equations, the stress and displacement for any point in a laminate are obtained. The formulation described in sections III-A and III-B can be expanded to solve the general anisotropic laminate. Since the formulation is lengthy and restricted to the special case of boundary shape of composites, the numerical analysis, the finite element method, will be used to solve the linear elastic, nonhomogeneous, anisotropic laminated composites with boundaries of arbitrary shape.
IV. FINITE ELEMENTS IN TWO DIMENSIONS

An analysis of a laminated plate by discrete elements connected at a finite number of nodal points can be performed by using a direct stiffness or displacement formulation. Solutions to the fundamental equation of the linear theory for laminated composites, which was described in Chapter III, will be obtained using this approach. In this chapter, a 16-DOF element and a 24-DOF element will be developed for the two-dimensional analysis of laminated composites with various loading and boundary conditions.

A. 16-DOF Element Formulation

For the analysis of arbitrary laminated anisotropic plates, a rectangular plate element shown in Figure 5 with 2-DOF at each nodal point will be used. This element will perform the deformation with a cubic expansion in the x-direction and with a linear variation in the z-direction. The deformation pattern for the displacement corresponds to the linear theory described in Chapter III. The solution accuracy of the displacement formulation depends on the ability of the assumed functions to accurately model the deformation modes of the composite. In selecting this element, a cubic deformation pattern for the displacement $w$ is assumed and for most loading conditions this will give the correct solution and there is no need to have many divisions in the x-direction.
Figure 5. Sixteen DOF Element, Distributed Force and Trace.
Displacement Functions

The assumed displacement functions associated with the degrees of freedom shown in Figure 5 are

\[ u = a_1 + a_2x + a_3z + a_4x^2 + a_5xz + a_6x^3 + a_7x^2z + a_8x^3z \]  
\[ w = a_9 + a_{10}x + a_{11}z + a_{12}x^2 + a_{13}xz + a_{14}x^3 + a_{15}x^2z + a_{16}x^3z \]  
(4.1)

or in matrix form

\[ \{u\} = [M]\{a\} \]  
(4.2)

where \([M]\) is a function of \(x\) and \(z\). The sixteen constants \(\{a\}\) are evaluated by solving the two sets of eight simultaneous equations which result when the nodal coordinates and the corresponding nodal displacements are inserted into equations (4.1). Expressing \(\{a\}\) in terms of the nodal displacements \(\{u_i\}\) which are now the unknowns of the problem, if the inverse of \([A]\) exists,

\[ \{u_i\} = [A]\{a\} \]  
(4.3)

\[ \{a\} = [A]^{-1}\{u_i\} \]  
(4.4)

where \([A]\) is the matrix relating the nodal displacements and the constants \(\{a\}\). If \([A]^{-1}\) does not exist, then different displacement
functions must be chosen. The displacements for any point in the element are obtained by substituting \( \{u\} \) in equation (4.4) into equation (4.3), thus

\[
\{u\} = [M] [A]^{-1} \{u_i\} = [N] \{u_i\}
\]

(4.5)

where \([N]\) are shape functions such that \(N_i = 1\) at node \(i\) and is equal to zero at all other nodes. It defines the deformation of any point in terms of the nodal displacements.

**Derivation of Element Stiffness Matrix**

With the displacement functions formulated, the strains are calculated from equations (4.5) and expressed in terms of the nodal displacements. This is done by appropriate differentiation of \([M]\), which yields

\[
\{\varepsilon\} = \begin{bmatrix}
\varepsilon_x \\
\varepsilon_z \\
\gamma_{xz}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial w}{\partial z} \\
\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}
\end{bmatrix} = [Q][A]^{-1}\{u_i\} = [B]\{u_i\}
\]

(4.6)

where \([Q]\) is the differentiation of \([M]\) and \([B]\) is the differentiation of \([N]\). The stresses are related to the strains by the following equation
\[
\{\sigma\} = \begin{bmatrix}
\sigma_x \\
\sigma_z \\
\tau_{xz}
\end{bmatrix} = [D]\{\varepsilon\} \quad (4.7)
\]

where \([D]\) is the elasticity matrix. For the transversely isotropic composite material \([23]\) in a state of plane stress

\[
[D] = \begin{bmatrix}
\frac{E_{11}}{1 - \nu_{13} \nu_{31}} & \frac{E_{11} \nu_{31}}{1 - \nu_{13} \nu_{31}} & 0 \\
\frac{E_{33}}{1 - \nu_{13} \nu_{31}} & 0 & 0 \\
\text{Symmetry} & G_{13}
\end{bmatrix} \quad (4.8)
\]

and in a state of plane strain

\[
[D] = \begin{bmatrix}
\frac{E_{11} (1 - \nu_{23} \nu_{32})}{F} & \frac{E_{11} \left(\nu_{31} + \nu_{21} \nu_{32}\right)}{F} & 0 \\
\frac{E_{33} (1 - \nu_{12} \nu_{21})}{F} & 0 & 0 \\
\text{Symmetry} & G_{13}
\end{bmatrix} \quad (4.9)
\]

where \(F = (1 - \nu_{12} \nu_{21} - \nu_{23} \nu_{32} - \nu_{13} \nu_{31} - \nu_{12} \nu_{23} \nu_{31} - \nu_{21} \nu_{13} \nu_{22})\); and 1, 2 and 3 refer to the principal directions of the lamina, and \(\nu_{13}\) is the Poisson ratio measuring normal strain in the 3-direction due to uniaxial
normal stress in the l-direction. In this study the axes of elastic symmetry of the various layers are parallel to the structure axes. The inclusion of anisotropic properties for composite material applications is readily accomplished at the element level by inserting elastic constants associated with the principal directions of the composite.

Thus, with the displacement functions in equation (4.5) and the elasticity matrix in equation (4.8) or (4.9), the element stiffness matrix is formulated straightforwardly by the following equation

$$[k] = \int_v [B]^T [D] [B] dv$$  \hspace{1cm} (4.10)

This element stiffness matrix relates the nodal displacements \( \{u_i\} \) directly to the nodal forces \( \{f_i\} \) by

$$\{f_i\} = [k] \{u_i\}$$  \hspace{1cm} (4.11)

As shown, the element stiffness matrix can be determined by a series of systematic steps in matrix algebra based on assumed displacement functions and known material properties. The more realistic the displacement function, the closer the stiffness will be to the true value.

**Consistent Element Joint Loads**

The nodal forces due to the distributed force as shown in Figure 5 are calculated by the following equation

$$\oint_s \delta u ds = P_1 \delta u_1 + P_2 \delta u_2 + P_3 \delta u_3 + P_4 \delta u_4$$  \hspace{1cm} (4.12)
where \( \overline{p} \) is the load intensity; \( \overline{p}_1, \overline{p}_2, \overline{p}_3 \) and \( \overline{p}_4 \) are equivalent joint loads; and \( u_1, u_2, u_3 \) and \( u_4 \) are joint displacements. With Figure 5, it can be shown that

\[
\overline{p} = \left[ 1 - \frac{11}{2} \left( \frac{x}{L} \right)^2 - \frac{9}{2} \left( \frac{x}{L} \right)^3 \right] \overline{p}_1 + \left[ 9 \left( \frac{x}{L} \right) - \frac{45}{2} \left( \frac{x}{L} \right)^2 \right] \overline{p}_2 \\
+ \frac{27}{2} \left( \frac{x}{L} \right)^3 \overline{p}_3 + \left[ -9 \left( \frac{x}{L} \right) + 18 \left( \frac{x}{L} \right)^3 \right] \overline{p}_4 + \left[ \left( \frac{x}{L} \right) - \frac{9}{2} \left( \frac{x}{L} \right)^2 \right] \overline{p}_4
\]  

(4.13)

where \( \overline{p}_1, \overline{p}_2, \overline{p}_3 \) and \( \overline{p}_4 \) are load intensity at node 1, 2, 3 and 4, respectively. Similarly, \( u \) can be expressed in terms of \( u_1, u_2, u_3 \) and \( u_4 \) with the same form. Substituting equation (4.13) into the left side of equation (4.12) and integrating the consistent joint loads coefficients become

\[
\begin{bmatrix}
\overline{p}_1 \\
\overline{p}_2 \\
\overline{p}_3 \\
\overline{p}_4
\end{bmatrix} = \frac{L}{1680} \begin{bmatrix}
128 & 99 & -36 & 19 \\
99 & 648 & -81 & -36 \\
-36 & -81 & 648 & 99 \\
13 & -36 & 99 & 128
\end{bmatrix} \begin{bmatrix}
\overline{p}_1 \\
\overline{p}_2 \\
\overline{p}_3 \\
\overline{p}_4
\end{bmatrix} = [P][\overline{p}] 
\]

(4.14)

For a uniformly distributed load, the coefficients are shown in the top of Figure 5(b).

**Trace of the Element Stiffness Matrix**

The trace of the element stiffness matrix, which is the sum of the diagonal terms in the matrix, provides a convenient measure of the
quality of the element [20]. The best element of a group of elements having the same geometry and degrees of freedom is, in general, the one with the lowest trace. The trace of each 16-DOF rectangular element with various aspect ratios of A and C but with constant area, equal to AC, is shown in Figure 5. Evidently, the best one has an aspect ratio located between 3 and 4, but an element with aspect ratios from 1 to 10 will perform very well. This provides a guide for selecting an idealization of the structure by recommending that relatively thin elements should be used.

**Assembly and Analysis**

Beginning with the idealization of the structure and the assumed displacement functions, followed by the formulation of element stiffness matrix and joint loads, the last step involved in every finite-element analysis is the summation of equation (4.11) over all the elements in the structure. This result can be written as follows

\[
\{F\} = [k] \{u\} \tag{4.15}
\]

where \(\{F\}\) is the load vector due to the external forces including concentrated and distributed loads; \([k]\) is the structure stiffness matrix, and \(\{u\}\) is the unknown nodal displacement vector. The nodal displacements can be determined by solving the system of simultaneous equations (4.15).

With the displacements known, the stresses can be calculated by equations (4.7) and (4.6), which give
\{\sigma\} = [D][B]\{u, i\}

(4.16)

The stresses are calculated at four points along the mid-line of the element rather than along the external faces because the formulation gives a linear stress variation in the z-direction and an average stress value is appropriate.

B. 24-DOF Element Formulation

In this section, an improved two-dimensional element stiffness matrix will be developed by a numerical integration procedure and the solutions will be compared with the results of the 16-DOF element described in Section III-A. In most cases, the thickness of each lamina in the composite is thin and the assumption of linear variation in thickness direction such as in the 16-DOF element may yield good results. In order to evaluate the performance of the 16-DOF element (linear transverse displacement), a 24-DOF element (cubic transverse displacement) is introduced. A curved, isoparametric cubic element [7] with 24-DOF is shown in Figure 6. With cubic displacement expansions in both directions the element can model deformations with greater accuracy and can fit boundaries of arbitrary shape.

Shape Functions

The local curvilinear coordinates \(\xi-\zeta\) is introduced as shown in Figure 6 for each element. These coordinates are so determined as to give \(\xi = 1\) on the curve \((1, 9, 10, 5)\), \(\xi = -1\) on the curve \((4, 11, 12, 8)\), \(\zeta = 1\) on the curve \((1, 2, 3, 4)\) and \(\zeta = -1\) on the curve \((5, 6, 7, 0)\).
Figure 6. Twenty-four DOF Element, 4 x 4 Gauss Rule and Trace.
The relationships between the Cartesian and local coordinates are

\[ x = \alpha_1 + \alpha_2 \xi + \alpha_3 \zeta + \alpha_4 \xi^2 + \alpha_5 \xi \zeta + \alpha_6 \xi^3 + \alpha_7 \zeta^2 + \alpha_8 \xi^2 \zeta 
+ \alpha_9 \xi \zeta^2 + \alpha_{10} \xi^3 \zeta + \alpha_{11} \xi^2 \zeta^3 + \alpha_{12} \xi \zeta^3 \]  \hspace{1cm} (4.17)

\[ z = \text{same form} \]

or in matrix form

\[ \{x\} = [M]\{\alpha\} \]  \hspace{1cm} (4.18)

Substituting the corresponding nodal coordinates in the two systems for each node results in

\[ \{x_i\} = [A]\{\alpha_i\} \]  \hspace{1cm} (4.19)

Solving the above equation yields

\[ \{\alpha\} = [A]^{-1}\{x_i\} \]  \hspace{1cm} (4.20)

Equations (4.19) and (4.20) are similar to equations (4.3) and (4.4).

Substituting equation (4.20) into equation (4.18) gives

\[ x = N_1 x_1 + N_2 x_2 + \ldots + N_{12} x_{12} = \{N\}^T \{x_i\} \]  \hspace{1cm} (4.21)

\[ z = N_1 z_1 + N_2 z_2 + \ldots + N_{12} z_{12} = \{N\}^T \{z_i\} \]
where $N_i$ are shape functions and are functions of $\xi$ and $\zeta$ [7]. They have a value of unity at the point in question and zero elsewhere. It is a most convenient method to establish the coordinate transformation by using these shape functions. The twelve shape functions are shown in Appendix A for the 24-DOF element.

The displacement functions can also be defined by $N_i$ such that

$$u(\xi,\zeta) = N_1 u_1 + N_2 u_2 + \ldots + N_{12} u_{12} = \{N\}^T \{u_1\}$$

$$v(\xi,\zeta) = N_1 v_1 + N_2 v_2 + \ldots + N_{12} v_{12} = \{N\}^T \{v_1\}$$

(4.22)

Since the shape functions $N_i$ defining geometry (4.21) and functions (4.22) are the same, the elements are called isoparametric [6].

**Coordinate Transformation**

As shown in equations (4.6) and (4.10), the differentiation and the integration are taken with respect to global coordinates $x-z$, but the $N_i$ and integrand are functions of $\xi$ and $\zeta$. It is necessary to express the global derivatives in terms of local derivatives and to carry out the integration over the element surface in terms of the local coordinates with an appropriate change of limits of integration. For derivatives of $N_i$, we have

$$\begin{bmatrix}
\frac{\partial N_i}{\partial \xi} \\
\frac{\partial N_i}{\partial \zeta}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial \xi} & \frac{\partial}{\partial \zeta}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial N_i}{\partial x} \\
\frac{\partial N_i}{\partial \xi}
\end{bmatrix}
= \left[ JJ \right]
\begin{bmatrix}
\frac{\partial N_i}{\partial x} \\
\frac{\partial N_i}{\partial \xi}
\end{bmatrix}
$$

(4.23)
where \([J]\) is the Jacobian Matrix. The global derivatives become

\[
\begin{bmatrix}
\frac{\partial N_i}{\partial \xi} \\
\frac{\partial N_i}{\partial \eta} \\
\frac{\partial N_i}{\partial \zeta}
\end{bmatrix} = [J]^{-1}
\begin{bmatrix}
\frac{\partial N_i}{\partial \xi} \\
\frac{\partial N_i}{\partial \eta} \\
\frac{\partial N_i}{\partial \zeta}
\end{bmatrix}
\quad (4.24)
\]

where \([J]^{-1}\) is the inverse of \([J]\).

For isoparametric formulation, \([J]\) is

\[
[J] =
\begin{bmatrix}
\frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \cdots & \frac{\partial N_{12}}{\partial \xi} \\
\frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \cdots & \frac{\partial N_{12}}{\partial \eta} \\
\frac{\partial N_1}{\partial \zeta} & \frac{\partial N_2}{\partial \zeta} & \cdots & \frac{\partial N_{12}}{\partial \zeta}
\end{bmatrix}
\begin{bmatrix}
x_1 & z_1 \\
x_2 & z_2 \\
\vdots \\
x_{12} & z_{12}
\end{bmatrix} = [C][XZ] \quad (4.25)
\]

where \([C]\) is the matrix for derivatives of \(N_i\) with respect to \(\xi\) and \(\eta\), shown in Appendix A, and \([XZ]\) is the element global coordinate matrix.

The transformation of variables in the surface integration is

\[
dx\,d\eta = |J| \, d\xi \, d\zeta \quad (4.26)
\]

where \(|J|\) is the determinant of \([J]\). The region with respect to the integration can be written as

\[
\int_{-1}^{1} \int_{-1}^{1} C(\xi, \eta) \, d\xi \, d\eta 
\quad (4.27)
\]
where $G(\xi, \xi')$ stands for $[B]^T[D][B]|\mathcal{J}$; $[\mathcal{E}]$ is shown in Appendix A and $[D]$ is the same elasticity matrix as given in Section A.

**Element Stiffness Matrix**

The formulation of the element stiffness matrix, equation (4.10), can be rewritten as

$$[k] = \int [B]^T[D][B] dv = \int_{-1}^{1} \int_{-1}^{1} G(\xi, \xi') d\xi' d\xi$$

(4.20)

Due to the $[J]^{-1}$ matrix, in which polynomials occur in the denominator, the above integration is carried out by numerical procedures.

The location of the sixteen Gaussian points $(a_i, c_j)$ and the corresponding weighting coefficients $H_i$ for the $4 \times 4$ Gauss Rule is shown in Figure 6. Equation (4.23) then becomes

$$[k] = \sum_{i=1}^{4} \sum_{j=1}^{4} H_i H_j G(a_i, c_j)$$

(4.29)

**Trace of the Stiffness Matrix**

The trace of the 24-DOF element stiffness matrix is shown in Figure 6 for various aspect ratios $A/C$ with constant element area. This comparison is for rectangular elements. As expected, the lowest trace is for the element which is square since the element is cubic in both directions. The curve is symmetric with respect to the lowest value of the trace.
Consistent Element Joint loads

A cubic curve and its curvilinear and cartesian coordinates with shape functions are shown in Figure 7. The following relationships exist

\[ u = N_i u_i, \quad p = N_i p_i, \quad x = N_i x_i \]  \hspace{1cm} (4.30)

The element consistent joint loads equation (4.12) can be written as

\[ \{\delta u\}^T \{p\} = \{\delta u\}^T \int_{-1}^{1} \{N\}{(C)}^T \{x\}{(N)}^T \{p\} \, d\xi \]

or

\[ \{P\} = \int_{-1}^{1} \{N\}{(C)}^T \{x\}{(N)}^T \{p\} \, d\xi \]  \hspace{1cm} (4.31)

where \( C_i = N_i, \xi \). Consider the element \( P_{11} \), it will be

\[ P_{11} = \begin{bmatrix} -3 & 537 & -3 & 139 \\ 1 & 1120 & 16 & 3360 \end{bmatrix} \{x\} \]

Setting \( \{x\}^T = [0 \frac{L}{3} \frac{2L}{3} L] \), \( P_{11} \) becomes

\[ P_{11} = \frac{537}{1120} \cdot \frac{L}{3} - \frac{3}{16} \cdot \frac{2L}{3} + \frac{139}{3360} \cdot L = \frac{8L}{105} \]

This is equal to \( P_{11} \) in equation (4.14). Similarly, other elements of \( P_{ij} \) can be obtained from equation (4.31).
Figure 7. Consistent Element Joint Loads for Cubic Curve Line.
Stress Analysis

Following the standard procedure to solve the unknown nodal displacements, the stresses are calculated by equation (4.16) at twelve nodal points on the edges of the element. This time the local coordinates $\xi$ and $\zeta$ are used.

C. Applications and Discussion

Three problems are analyzed by computer programs for the 16-DOF and the 24-DOF elements. The framework of the programs was taken from an existing program for rectangular elements with 8-DOF [24], in which the element stiffness matrix was formulated directly like the one for 16-DOF in this study; and from a general quadrilateral program with 8-DOF [25], in which numerical integration was used to formulate the element stiffness matrix like the one for 24-DOF in this study. For analyzing the transversely isotropic laminated composite, the element elasticity [D] subroutine was also modified.

Aluminum-Steel Laminate in Cylindrical Bending

This example was solved in Chapter III by the analytical method, in which the linear expansion of displacement functions are assumed, exactly for the sinusoidal load and approximately for the uniform load using a Fourier series expansion. The idealization of FEM analysis for 16-DOF and 24-DOF rectangular elements is shown in Figure 8 for the sinusoidal load and in Figure 9 for the uniform load. The uniform mesh of $3 \times 4$ for 24-DOF element and $12 \times 4$ for 16-DOF element is employed. Each of the 24-DOF elements are sliced equally into four layers. Since
Figure 8. Stress Distributions for Aluminum-Steel Laminate in Cylindrical Bending by Sinusoidal Load, s = 10.
Figure 9. Stress Distributions for Aluminum-Steel Laminate in Cylindrical Bending by Uniform Load, $z = 10$. 

Strain Energy

\begin{itemize}
  \item $x$ - 16 DOF  12 x 4 mesh \quad 0.01294
  \item $\circ$ - 24 DOF  3 x 4 mesh \quad 0.01296
  \item $\square$ - Fourier Series $n = 5$ \quad 0.01289
  \item $\triangle$ - Fourier Series $n = 9$ \quad 0.01289
\end{itemize}
it is a symmetric problem, only the half span of plate is considered. The aspect ratio $A/C$ of each element is 3.75 for the 24-DOF element and 15 for the 16-DOF element. These are within the ideal range of the trace as shown in Figures 5 and 6. The boundary conditions are zero horizontal displacements at the center of the span and zero vertical displacements at the support end. Since the solution for displacements are in excellent agreement with the analytical solution, only the comparison of stress distributions are shown. Note that the stresses calculated at the midpoint of the thickness for each element in 16-DOF, where the displacements vary linearly in the thickness direction, are fitted closely on the curves of the analytical solution. But in the 24-DOF element, stresses are evaluated at the nodal points along the edges, $\sigma_x$ and $\tau_{xz}$ are in good agreement with the analytical solution and $\sigma_z$ is not. $\sigma_z$ can be improved by using a finer $3 \times 8$ mesh.

The strain energy for the three cases are also shown in the Figures. Since no initial strains or initial stresses exist in this example, by the principle of energy conservation, the strain energy will be equal to the work done by the external loads which increase uniformly from zero. Thus the strain energy for the analytical method, $U = \frac{qV}{2}$.

For the finite element method [FEM], $U = \{u\}^T[k]\{u\}/2$, as shown by Zienkiewicz [19]. The larger the strain energy, the better the approximate solution. Generally use of a finer mesh will increase the strain energy. From a $3 \times 4$ mesh for the 24-DOF element and a $12 \times 4$ mesh for the 16-DOF element, it appears that one 24-DOF element is better than four 16-DOF elements for the laminate with isotropic layers. Results
from the Fourier series for the uniform load are also compared in Figure 9. It shows that $\sigma_x$ is in good agreement with the FEM solution, and $\sigma_z$ and $\tau_{xz}$ are approaching the FEM solution with increasing values of $n$. In general, solutions by Fourier series are satisfactory with $n = 9$ (5 terms) for this unbalanced, unsymmetric laminates.

**Two-Ply Laminate [2]**

In order to compare the FEM results to the corresponding exact elasticity solutions, a bidirectional (coupled) laminate with the fibers oriented perpendicular to the support ends in the bottom layer and parallel to the supports in the top layer subjected to cylindrical bending is studied. The idealizations are shown in Figure 10, where the layers are shown as being of equal thickness. Results for stresses are plotted in the same figure. The elastic constants for this composite material, graphite/epoxy, are [2]

$$
E_L = 25 \times 10^6 \text{ psi}, \quad E_T = 10^6 \text{ psi} \\
G_{LT} = .5 \times 10^6 \text{ psi}, \quad G_{TT} = .2 \times 10^6 \text{ psi} \\
u_{LT} = v_{TT} = .25, \quad v_{TL} = .01 (= \frac{E_L}{E_T} v_{LT})
$$

where $L$ signifies the fiber direction, $T$ the transverse direction, and $v_{LT}$ is the Poisson ratio measuring strain in the transverse direction under uniaxial normal stress in the $L$ direction.

As shown in Figure 10, the uniform mesh of $2 \times 4$ for the 24-DOF element is used along with the $12 \times 4$ mesh for the 16-DOF element. The aspect ratio of $A/C$ is 1 for the $2 \times 4$ mesh and 6 for the $12 \times 4$ mesh.
Elasticity Solutions Ref. [2] Strain Energy

\[ x \quad 16\text{-DOF} \quad 12 \times 4 \text{ mesh} \quad 0.2180 \]

\[ \circ \quad 24\text{-DOF} \quad 2 \times 4 \text{ mesh} \quad 0.2193 \]

Figure 10. Stress Distributions for 2-Ply Laminate in Cylindrical Bending by Sinusoidal Load, \( s = 4 \).
They are within the ideal range of the trace. While some difficulties occurred in the study by Pryor [22], results of both finite element analyses are in excellent agreement with those of the elasticity solution for both stresses and displacements. This indicates that the solution of the 16-DOF element is as good as the 24-DOF element even for a thick laminate, \( s = 4 \).

3-Ply Laminate [2]

This example was also solved exactly by Pagano [2]. A symmetric 3-ply laminate with layers of equal thickness, the L direction coinciding with \( x \) in the outer layers, while \( T \) is parallel to \( x \) in the central layer, is shown in Figure 11. The material properties are the same as in the previous example. The plate is in cylindrical bending under sinusoidal load and \( S = 4 \). An equidistant mesh, \( 3 \times 4 \) for the 24-DOF element and \( 16 \times 4 \) for the 16-DOF element, is used. Since Pryor assumed that normals remained straight after deformation, the displacement \( u \) and stresses were not accurate because the normals are severely distorted [22]. Results of both finite element analyses for displacement and stress solutions are in excellent agreement with the exact elasticity solutions and compared in Figure 11. It is concluded that the performance of 16-DOF element for two-dimensional finite element analysis of laminated composites is very good even though more elements are needed to achieve the same results with a 24-DOF element. Since the layers in a laminate are thin and the linear variation of displacements for the 16-DOF element is an excellent representation in the direction of the layer thickness, as shown by the above examples, the three dimensional analog of the 16-DOF
Elasticity Solutions Ref. [2] Strain Energy

16-DOF 18 x 4 mesh  \( \cdot1390 \)
24-DOF 3 x 4 mesh  \( \cdot1393 \)

\( \tau_{xz}(0) \text{ ksi} \)
\( \sigma_z(\frac{L}{2}) \text{ ksi} \)
\( \sigma_x(\frac{L}{2}) \text{ ksi} \)

**Figure 11.** Displacement and Stress Distributions for 3-Ply Laminate in Cylindrical Bending by Sinusoidal Load, \( z = \xi \).
element will be formulated for the analysis of three dimensional laminated composites in the next chapter.
V. FINITE ELEMENTS IN THREE DIMENSIONS

Recognizing the well-known fact that for isotropic plates the normals to the midplane remain practically straight after deformation and in general the thickness of each layer for a laminate is thin compared to the other dimensions, the three-dimensional analog of the 16-DOF element for two-dimensional elements, a 72-DOF element, is developed for the three-dimensional analysis of laminated composites. An element similar to this one had been described by Ahmad, et al. [26] and was applied to isotropic shells and plates by condensing all of the degrees of freedom to the middle surface of the element. In so doing, the strain in the Z-direction was lost and assumed to be negligible. In this investigation, the degrees of freedom are retained at the corners of the element so that the strain in the Z-direction remains. Also with this element, the interconnections between layers of the laminate can be made at their interfaces and a better description of the important transverse shear stresses can be obtained through the thickness of the composite.

Shape Functions for 72-DOF Element

A curved isoparametric cubic element is shown in Figure 12. The external faces of the element are curved, while the sections across the thickness are generated by straight lines. Let $\xi$ and $\eta$ be two curvilinear coordinates in the midsurface of the element and $\zeta$ a linear coordinate in the thickness direction. $\xi$, $\eta$ and $\zeta$ vary between -1 and 1 on the respective faces of the element. Thus the general form of the relationship between the Cartesian coordinates and the local curvilinear
Figure 12. Seventy-two DOF Element, Parent Elements and Trace of Pigtight Prism.
coordinates is
\[ x = N_1 x_1 + N_2 x_2 + \ldots + N_{24} x_{24} = \{N\}^T \{x_i\} \]
\[ y = N_1 y_1 + N_2 y_2 + \ldots + N_{24} y_{24} = \{N\}^T \{y_i\} \]  \hspace{1cm} (5.1)
\[ z = N_1 z_1 + N_2 z_2 + \ldots + N_{24} z_{24} = \{N\}^T \{z_i\} \]

where \( N_i \) are functions of \( \xi, \eta \) and \( \zeta \) and named isoparametric shape functions [7]. They define a value of unity at the node in question and zero at all other nodes. These 24 shape functions are shown in Appendix B for the 72-DOF element.

The displacement functions can also be expressed in terms of \( N_i \) such that
\[ u = \{N\}^T \{u_i\} \]
\[ v = \{N\}^T \{v_i\} \]  \hspace{1cm} (5.2)
\[ w = \{N\}^T \{w_i\} \]

Using the isoparametric shape function is a most convenient way to both define the geometry and to establish the coordinate transformation.

**Coordinate Transformation**

As in equation (4.23) for the two-dimensional element, the corresponding equation for the three-dimensional element is
\[
\begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\
\frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta}
\end{bmatrix} = [J] = [J]^{-1}
\]

(5.3)

where \([J]\) is the Jacobian Matrix. Then the global derivative becomes

\[
\begin{bmatrix}
\frac{\partial x}{\partial \xi} \\
\frac{\partial y}{\partial \eta} \\
\frac{\partial z}{\partial \zeta}
\end{bmatrix} = [J]^{-1}
\]

(5.4)

where \([J]^{-1}\) is the inverse of \([J]\), which in isoparametric formulation is

\[
[J] = \begin{bmatrix}
\frac{\partial N_1}{\partial \xi} & \cdots & \frac{\partial N_{24}}{\partial \xi} \\
\frac{\partial N_1}{\partial \eta} & \cdots & \frac{\partial N_{24}}{\partial \eta} \\
\frac{\partial N_1}{\partial \zeta} & \cdots & \frac{\partial N_{24}}{\partial \zeta}
\end{bmatrix}
\]

\[
\begin{bmatrix}
x_1 & y_1 & z_1 \\
\vdots & \ddots & \vdots \\
x_{24} & y_{24} & z_{24}
\end{bmatrix} = [C][XYZ]
\]

(5.5)

where \([C]\) is the matrix for derivatives of \(N_i\) with respect to \(\xi\), \(\eta\) and \(\zeta\), which is shown in Appendix B, and \([XYZ]\) is the global coordinates for the element nodes.

Knowing the Jacobian \([J]\), the volume integration becomes

\[
d\xi d\eta d\zeta = |J| d\xi d\eta d\zeta
\]

(5.6)
where $|J|$ is the determinant of $[J]$.

The Elasticity Matrix

Consider each lamina or layer of the composite behaving as a homogeneous orthotropic material. Nine independent elastic constants are required to describe this material. For the principal axes of elastic symmetry (1, 2, 3) which coincide with the reference axes ($x, y, z$), the constitutive relations for a typical layer of a composite are

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\tau_{12} \\
\tau_{13} \\
\tau_{23}
\end{bmatrix} =
\begin{bmatrix}
D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\
D_{21} & D_{22} & D_{23} & 0 & 0 & 0 \\
D_{31} & D_{32} & D_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & D_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & D_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & D_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\gamma_{12} \\
\gamma_{13} \\
\gamma_{23}
\end{bmatrix}
\]  

(5.7)

where $D_{11} = \frac{1 - v_{23} v_{32}}{F} E_{11}$, $D_{22} = \frac{1 - v_{13} v_{32}}{F} E_{22}$, $D_{33} = \frac{1 - v_{12} v_{31}}{F} E_{33}$,

$D_{12} = \frac{v_{12} + v_{13} v_{32}}{F} E_{22}$, $D_{13} = \frac{v_{12} + v_{13} v_{23}}{F} E_{33}$,

$D_{23} = \frac{v_{13} + v_{12} v_{31}}{F} E_{33}$, $D_{44} = G_{12}$, $D_{55} = G_{13}$, $D_{66} = G_{23}$

and $F = 1 - v_{12} v_{21} - v_{13} v_{31} - v_{23} v_{32} - v_{12} v_{23} v_{31} - v_{21} v_{13} v_{32}$.

For an arbitrary orientation of a lamina, the principal axes will
not coincide with the reference axes of the laminate. The transformation laws of Cartesian tensors [27] of order 2 are

\[ \sigma'_{ij} = a_{ik} a_{jl} \sigma_{kl} \]  \hspace{1cm} (5.8)

\[ \varepsilon'_{ij} = a_{ik} a_{jl} \varepsilon_{kl} \]

where \( \sigma_{kl} \) and \( \varepsilon_{kl} \) are stresses and strains referred to the principal axes and \( \sigma'_{ij} \) and \( \varepsilon'_{ij} \) are stresses and strains referred to the reference or laminate axes; and \( a_{ik} \) are direction cosines of the reference axes with respect to the principal axes. In this study, the following transformation matrix is used

\[ a_{ij} = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix} \]  \hspace{1cm} (5.9)

Performing the operation in equations (5.8) and letting
\[ m = \cos \theta, \ n = \sin \theta, \] results in
\[
\begin{bmatrix}
    \sigma_x \\
    \sigma_y \\
    \sigma_z \\
    \tau_{xy} \\
    \tau_{xz} \\
    \tau_{yz}
\end{bmatrix}
= \begin{bmatrix}
    m^2 & n^2 & 0 & -2mn & 0 & 0 \\
    n^2 & m^2 & 0 & 2mn & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 \\
    mn & -mn & 0 & m^2-n^2 & 0 & 0 \\
    0 & 0 & 0 & 0 & m & -n \\
    0 & 0 & 0 & 0 & n & m
\end{bmatrix}
\begin{bmatrix}
    \epsilon_1 \\
    \epsilon_2 \\
    \epsilon_3 \\
    \gamma_{12} \\
    \gamma_{13} \\
    \gamma_{23}
\end{bmatrix}
= [T]
\begin{bmatrix}
    \epsilon_1 \\
    \epsilon_2 \\
    \epsilon_3 \\
    \gamma_{12} \\
    \gamma_{13} \\
    \gamma_{23}
\end{bmatrix}
(5.10)
\]

and \( \{\epsilon_x\} = [T]\{\epsilon_i\} \)  

(5.11)

where \( x \) refers to the reference axes and \( i \) refers to the principal axes.

Equation (5.11) yields

\( \{\epsilon_i\} = [T]^{-1}\{\epsilon_x\} \)  

(5.12)

where \([T]^{-1}\) is obtained by replacing \( n \) with \( -n \) in \([T]\). Substituting equation (5.12) into equation (5.7) and then into equation (5.10) gives

\( \{\sigma_x\} = [T][D][T]^{-1}\{\epsilon_x\} \)

(5.13)

Thus the elasticity matrix \([D]\) is modified by premultiplying with \([T]\) and postmultiplying with \([T]^{-1}\) for the dominate where the principal axes of its lamina are not coincident with the reference axes.

**Element Stiffness Matrix**

The strains and displacements relations in a three-dimensional continuum are
\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{xy} \\
\gamma_{xz} \\
\gamma_{yz}
\end{bmatrix}
= [B] \begin{bmatrix} u_1 \end{bmatrix}
\]  
(5.14)

where \([B]\) is shown in Appendix B. With \([B]\) given in equation (5.14), \([D]\) in equation (5.13) and \([J]\) in equation (5.5), the stiffness matrix becomes

\[
[k] = \int_B [B]^T [D][B] dv = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} [B]^T [D][B] d\xi d\eta d\zeta
\]

\[
= \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} G(\xi, \eta, \zeta) d\xi d\eta d\zeta
\]  
(5.15)

Due to polynomials in the denominator of \(G\), a numerical integration procedure is performed to evaluate \([k]\). The 32 Gaussian points \((a_i, b_j, c_k)\) and the corresponding weighting coefficients \(w_i\) for the \(4 \times 4 \times 2\) Gauss Rule are shown in Figure 6 for the \((a_1, b_1)\) and \(\zeta = .57735, \quad H = 1.0\) in \(\zeta\)-direction. Equation (5.15) becomes

\[
[k] = \sum_{i=1}^{4} \sum_{j=1}^{4} \sum_{k=1}^{2} H_i H_j H_k G(a_i, b_j, c_k)
\]  
(5.16)
Trace of the Stiffness Matrix

The trace of the 72-DOF element stiffness matrix is shown in Figure 12 for various aspect ratio A/C in each aspect ratio of A/B with constant element volume. This comparison is for right prisms. The lowest trace is for an element whose projection in the x-y plane is a square, A=B, and A=3C. This is expected from the results of 16-DOF and 24-DOF elements. In the 16-DOF element, the lowest value of the trace is for the element with A/C of 3 and in the 24-DOF element A/B = 1. These volumes will serve as a guide for the idealization of a continuum and for keeping the proportions of the element within an ideal range whenever possible.

Consistent Element Joint Loads

The equation for calculating consistent element joint loads due to distributed surface loads is, from equation (4.12),

$$\int_A \delta u \, dxdy = P_1 \delta u_1 + P_2 \delta u_2 + \ldots + P_i \delta u_i \quad (5.17)$$

where \( u = N_i u_i \), \( \overrightarrow{p} = N_i \overrightarrow{p}_i \), \( x = N_i x_i \), \( y = N_i y_i \) and \( P_i \) are consistent joint loads. Using the isoparametric shape functions \( N_i \) and performing the coordinate transformation for \( dxdy \), equation (5.17) becomes

$$\{\delta u\}^T \{P\} = \{\delta u\}^T \int_{-1}^{1} \int_{-1}^{1} \{N\}\{N\}^T \{\overrightarrow{p}\}\{|J|\} \, d\xi d\eta$$

or

$$\{\gamma\} = \int_{-1}^{1} \int_{-1}^{1} \{N\}\{N\}^T \{\overrightarrow{p}\}\{|J|\} \, d\xi d\eta \quad (5.18)$$
where \( |J| \) is the determinant of the Jacobian Matrix \([J]\) which is equal to

\[
[J] = \begin{bmatrix}
\frac{\partial N_1}{\partial \xi} & \cdots & \frac{\partial N_8}{\partial \xi} \\
\frac{\partial N_1}{\partial \eta} & \cdots & \frac{\partial N_8}{\partial \eta}
\end{bmatrix}
\begin{bmatrix}
x_1 & y_1 \\
\vdots & \vdots \\
x_8 & y_8
\end{bmatrix} = [C][XY]
\]

for the cubic-linear surface and equal to equation (4.25) for the cubic-cubic surface. The \( N_i \) and their derivatives with respect to \( \xi \) and \( \eta \), which are like those of the 16-DOF element in Section IV-A, are shown in Appendix B.

In the case of the parallelogram as shown in Figure 13, equation (5.18) becomes, for the distributed load in the cubic-linear plane,

\[
\begin{bmatrix}
P_1 \\
P_2 \\
P_3 \\
P_4 \\
P_5 \\
P_6 \\
P_7 \\
P_8
\end{bmatrix} = Q = \begin{bmatrix}
256 & 193 & -72 & 38 & 123 & 99 & -36 & 19 \\
1236 & -162 & -72 & 93 & 648 & -81 & -36 \\
1296 & 198 & -38 & 91 & 648 & 99 \\
256 & 19 & -36 & 99 & 128 \\
256 & 198 & -72 & 38 \\
1296 & -162 & -72 \\
1296 & 198 & 1296 & 198 \\
256 & 10080
\end{bmatrix}
\begin{bmatrix}
P_1 \\
P_2 \\
P_3 \\
P_4 \\
P_5 \\
P_6 \\
P_7 \\
P_8
\end{bmatrix}
\]

(5.19)

For the distributed load in the cubic-cubic plane equation (5.18) is expanded as Table IV. Simplified further, the consistent joint loads for the uniformly distributed load case are shown in Figure 13.
Figure 13. Consistent Element Joint Loads for Distributed Surface Load.
TABLE IV. CONSISTENT JOINT LOADS FOR THE CUBIC-CUBIC PLANE

\[ \begin{bmatrix}
  P_1 \\
  P_2 \\
  P_3 \\
  P_4 \\
  P_5 \\
  P_6 \\
  P_7 \\
  P_8 \\
  P_9 \\
  P_{10} \\
  P_{11} \\
  P_{12}
\end{bmatrix} =
\begin{bmatrix}
  12360 & -1620 & -2421 & -3546 & 6480 & -810 & -1494 & 2072 & 2268 & 2268 & 567 \\
  12360 & -4224 & -1494 & -810 & 6480 & -3546 & 2268 & 567 & 9072 & 2268 & 567 \\
  12360 & -1620 & -2421 & 2268 & 9072 & 567 & 2268 & 2268 & 9072 \\
  12360 & -4224 & 6480 & 567 & 2268 & 2268 & 9072 \\
  4126 & -1494 & -3546 & -2421 & -4224 \\
  12360 & -1620 & 6480 & -510 \\
  12960 & -810 & 6480 \\
  12960 & 12960
\end{bmatrix} \]

Symmetry, \( Q = \frac{AB}{100E00} \)
After the element stiffness matrix and joint loads are established, the unknown nodal displacements are solved by the standard procedure. Once the element nodal displacements are known, the stresses at any point within the element are calculated by equation (4.16).

Application and Discussion

The computer program for the 24-DOF elements in the two-dimensional problem is modified and expanded to include the 72-DOF elements in the three-dimensional problem. Because of the large increase in bandwidth brought about by the use of this advanced element in the three-dimensional problem and due to the maximum storage capacity of the computer, one division in x-y plane is used to analyze a three-ply laminated plate as shown in Figure 14. In order to observe the performance of this element and keep the aspect ratios within the best range of the trace for the element and also to be able to compare with the results in Pagano [4], one element for each layer is used for the finite-element analysis. The plate is a symmetric square laminate with equal thickness layers loaded by a sinusoidal load, the L direction which is in the longitudinal direction of the fibers, coincides with x in the outer layers, while T the transverse direction is parallel to x in the central layer. Since it is symmetric about the x and y axes, only one quarter of the plate needs to be considered. The idealization for a quarter of the plate and the material properties are also shown in Figure 14.

The boundary conditions for this problem are

At $x = \pm \frac{a}{2}$ \hspace{1cm} $w = v = 0$

At $y = \pm \frac{d}{2}$ \hspace{1cm} $w = u = 0$
\[ q = \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{a} \right) \]

\[ E_L = 25 \times 10^6 \text{ psi} \quad C_{LT} = .5 \times 10^6 \text{ psi} \quad v_{LT} = v_{TT} = .25 \]

\[ E_T = 10^6 \text{ psi} \quad C_{TT} = .2 \times 10^6 \text{ psi} \quad v_{TL} = \frac{E_T}{E_L} \quad v_{LT} = .01 \]

Figure 13. Symmetric 3-Ply Square Laminate and Idealization. \( s = 4 \).
Since the plate is under symmetric loading, the horizontal displacement $u$ at the plane of $x = 0$ and $v$ at the plane of $y = 0$ are equal to zero. The low span-to-depth ratio, $s$ equal to 4, is studied in order to compare with the available solutions.

Results of normalized stresses are compared with the exact elasticity solution and the classical laminated plate theory (CPT) solution for maximum values and are given in Table V. When the maximum values of $\bar{\sigma}_{xz}$ do not occur at $\bar{z} = 0$, two values are shown in the respective columns. The upper value gives the one at $\bar{z} = 0$, while the lower number represents the maximum value with the corresponding $\bar{z}$ being given in parentheses.

Note that the maximum values for $\bar{\sigma}_{y}$ occur in the central layer. The normalized stress and displacement distributions are shown in Figure 15. Even with one element for each layer, the results of the FEM analysis is in good agreement with the exact elasticity solutions and shows an improvement over the CPT solutions. One exception is the transverse shear stress $\bar{\tau}_{xz}$ which gives a maximum value in the central layer rather than in the outer layers. This distribution can be improved by placing additional elements in the thickness direction as was done in the two-dimensional analysis. The significant advantage of the higher order elements, like the 72-DOF element, over the lower elements is that with fewer elements a more exact solution can be obtained.
<table>
<thead>
<tr>
<th>Point</th>
<th>$\sigma_x$</th>
<th>$\sigma_y$</th>
<th>$\tau_{xz}$</th>
<th>$\tau_{yz}$</th>
<th>$\tau_{xy}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0, $\frac{a}{2}$)</td>
<td>0.801</td>
<td>0.534</td>
<td>-0.256</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0, 0, $\frac{1}{2}$)</td>
<td>-0.755</td>
<td>0.556</td>
<td>-0.282 (0.27)</td>
<td>-0.2172</td>
<td>-0.0511</td>
</tr>
<tr>
<td>(a/2, 0, 0)</td>
<td>0.753</td>
<td>0.536</td>
<td>-0.254</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0, a/2, 0)</td>
<td>-0.713</td>
<td>0.567</td>
<td>-0.257 (0.17)</td>
<td>-0.1953</td>
<td>-0.0459</td>
</tr>
<tr>
<td>(a/2, a/2, $\frac{1}{2}$)</td>
<td>±0.539</td>
<td>±0.180</td>
<td>-0.395</td>
<td>-0.0823</td>
<td>±0.0213</td>
</tr>
</tbody>
</table>

**Table V. Comparison of Three-Dimensional Plate Solutions**
Figure 15. Stress and Displacement Distributions in Symmetric 3-Ply Square Laminate.

\( s = \mu \).
VI. CONCLUSIONS

A solution of the problem of a laminated plate with isotropic layers has been developed by an analytical method based on the assumption of a linear variation of the displacement functions through the thickness of each layer. Thus, a complete analysis of the composite was developed which includes the thickness-stretching deformation as well as extension, bending and transverse shear. The governing equations were specialized for the static cylindrical bending of a plate with two isotropic layers of dissimilar material. The continuity conditions at the interfaces were satisfied for both stress and displacement. Results of stresses and displacements in each layer were in agreement with those of the elementary theory.

Two finite element analyses have been developed for applications to isotropic and anisotropic two-dimensional laminated plate problems. In a 15-DOF element a linear variation of displacement through the thickness of the element was assumed, while a cubic variation was used for a refined 24-DOF element. These two elements were applied to solve the laminate with isotropic layers previously solved by the analytical method. Results were in excellent agreement. The two elements were also applied to the two-dimensional analysis of anisotropic laminated composite plate, and showed excellent agreement compared with the exact elasticity solution. Results of two finite element analysis have shown that the element with linear variation of displacement through the thickness of the element matches the element with cubic variation, although more elements are required in the thickness direction.
A true three-dimensional finite element analysis has been developed for applications to nonhomogeneous anisotropic laminated plate problems. A curved isoparametric element with cubic displacement expansion in the plate plane and linear variation of displacement through the thickness of the element can be applied to solve arbitrary shape boundary and severe normal warping problems. Results of the finite element analysis for a 3-ply laminate were in good agreement with the exact elasticity solution, even when only one element is used for each layer.
BIBLIOGRAPHY


APPENDIX A

The elements of selected matrices used in Chapter IV are given as follows:

Cubic-Cubic Shape Functions \( N_i \)

\[
\begin{align*}
N_1 &= 1/32(1-\xi)(1-\zeta)[9(\xi^2+\zeta^2)-10] \\
N_2 &= 9/32(1-3\xi)(1-\zeta)(1-\xi^2) \\
N_3 &= 9/32(1+3\xi)(1-\zeta)(1-\xi^2) \\
N_4 &= 1/32(1+\xi)(1-\zeta)[9(\xi^2+\zeta^2)-10] \\
N_5 &= 1/32(1-\xi)(1+\zeta)[9(\xi^2+\zeta^2)-10] \\
N_6 &= 9/32(1-3\xi)(1+\zeta)(1-\zeta^2) \\
N_7 &= 9/32(1+3\xi)(1+\zeta)(1-\xi^2) \\
N_8 &= 1/32(1+\xi)(1+\zeta)[9(\xi^2+\zeta^2)-10] \\
N_9 &= 9/32(1-\xi)(1-3\zeta)(1-\zeta^2) \\
N_{10} &= 9/32(1-\xi)(1+3\xi)(1-\zeta^2) \\
N_{11} &= 9/32(1+\xi)(1-3\xi)(1-\zeta^2) \\
N_{12} &= 9/32(1+\xi)(1+3\xi)(1-\zeta^2)
\end{align*}
\]

\( N_{i*\xi} \)

\[
\begin{align*}
C_{11} &= 1/32(1-\xi)[9(2\xi-3\xi^2-\zeta^2)+10] \\
C_{12} &= 9/32(1-\xi)(9\xi^2-2\xi-3) \\
C_{13} &= 9/32(1-\xi)(3-9\xi^2-2\xi) \\
C_{14} &= 1/32(1-\xi)[9(2\xi+3\xi^2+\zeta^2)-10] \\
C_{15} &= 1/32(1+\xi)[9(2\xi-3\xi^2-\zeta^2)+10] \\
C_{16} &= 9/32(1+\xi)(3-9\xi^2-2\xi-3) \\
C_{17} &= 9/32(1+\xi)(3-9\xi^2-2\xi) \\
C_{18} &= 1/32(1+\xi)[9(2\xi+3\xi^2+\zeta^2)-10] \\
C_{19} &= 9/32(1-3\xi)(\zeta^2-1) \\
C_{1,10} &= 9/32(1+3\xi)(\zeta^2-1) \\
C_{1,11} &= 9/32(1-3\xi)(1-\zeta^2) \\
C_{1,12} &= 9/32(1+3\xi)(1-\zeta^2)
\end{align*}
\]

\( N_{i*\zeta} \)

\[
\begin{align*}
C_{21} &= 1/32(1-\xi)[9(2\xi-3\xi^2-\zeta^2)+10] \\
C_{22} &= 9/32(1-3\xi)(\xi^2-1) \\
C_{23} &= 9/32(1+3\xi)(\xi^2-1) \\
C_{24} &= 1/32(1+\xi)[9(2\xi-3\xi^2-\zeta^2)+10] \\
C_{25} &= 1/32(1-\xi)[9(2\xi+3\xi^2+\zeta^2)-10] \\
C_{26} &= 9/32(1-3\xi)(1-\zeta^2) \\
C_{27} &= 9/32(1+3\xi)(1-\zeta^2) \\
C_{28} &= 1/32(1+\xi)[9(2\xi+3\xi^2+\zeta^2)-10] \\
C_{29} &= 9/32(1-\xi)(3-9\xi^2-2\xi-3) \\
C_{2,10} &= 9/32(1-\xi)(3-9\xi^2-2\xi) \\
C_{2,11} &= 9/32(1+\xi)(9\xi^2-2\xi-3) \\
C_{2,12} &= 9/32(1+\xi)(3-9\xi^2-2\xi)
\end{align*}
\]

83
All elements of $B_{ij}$ are zero, except the following:

$$B_{1i} = \frac{\partial N_j}{\partial x} \left[ J_{12}C_{1j} - J_{12}C_{2j} \right]; \ j = 1, 2, \ldots 12, \ i = 2j-1.$$  

$$B_{2i} = \frac{\partial N_j}{\partial z} \left[ J_{12}C_{1j} - J_{21}C_{1j} \right]; \ j = 1, 2, \ldots 12, \ i = 2j.$$  

$$B_{3i} = B_{1j}; \ i = 2, 4, \ldots 24, \ j = i - 1.$$  

$$B_{3i} = B_{2j}; \ i = 1, 3, \ldots 23, \ j = i + 1.$$
The elements of selected matrices in Chapter V are given as follows:

Cubic-Cubic-Linear Shape Functions $N_i$

\[ N_1 = \frac{1}{6} (1-\xi)(1-\eta)(1-\zeta)[9(\xi^2 + \eta^2) - 10]\]

\[ N_2 = \frac{9}{6} \xi(1-\eta)(1-\zeta)(1-\xi^2)\]

\[ N_3 = \frac{9}{6} \xi(1+3\xi)(1-\eta)(1-\zeta)(1-\xi^2)\]

\[ N_4 = \frac{1}{6} \xi(1+\xi)(1-\eta)(1-\zeta)[9(\xi^2 + \eta^2) - 10]\]

\[ N_5 = \frac{1}{6} \xi(1-\xi)(1-\eta)(1+\xi)[9(\xi^2 + \eta^2) - 10]\]

\[ N_6 = \frac{9}{6} \xi(1-3\xi)(1-\eta)(1+\xi)(1-\xi^2)\]

\[ N_7 = \frac{9}{6} \xi(1+3\xi)(1-\eta)(1+\xi)(1-\xi^2)\]

\[ N_8 = \frac{1}{6} \xi(1+\xi)(1-\eta)(1+\xi)[9(\xi^2 + \eta^2) - 10]\]

\[ N_9 = \frac{9}{6} \xi(1-\xi)(1-3\eta)(1-\xi)(1-\eta^2)\]

\[ N_{10} = \frac{9}{6} \xi(1-\xi)(1+3\eta)(1-\xi)(1-\eta^2)\]

\[ N_{11} = \frac{9}{6} \xi(1+\xi)(1-3\eta)(1-\xi)(1-\eta^2)\]

\[ N_{12} = \frac{9}{6} \xi(1+\xi)(1+3\eta)(1-\xi)(1-\eta^2)\]

\[ N_{13} = \frac{9}{6} \xi(1-\xi)(1-3\eta)(1+\xi)(1-\eta^2)\]

\[ N_{14} = \frac{9}{6} \xi(1-\xi)(1+3\eta)(1+\xi)(1-\eta^2)\]

\[ N_{15} = \frac{9}{16} (1+\xi)(1-3\eta)(1+\xi)(1-\eta^2)\]

\[ N_{16} = \frac{9}{16} (1+\xi)(1+3\eta)(1+\xi)(1-\eta^2)\]

\[ N_{17} = \frac{1}{64} (1-\xi)(1+\eta)(1-\zeta)[9(\xi^2 + \eta^2) - 10]\]

\[ N_{18} = \frac{9}{64} (1-3\xi)(1+\eta)(1-\zeta)(1-\xi^2)\]

\[ N_{19} = \frac{9}{64} (1+3\xi)(1+\eta)(1-\zeta)(1-\xi^2)\]

\[ N_{20} = \frac{1}{64} (1+\xi)(1+\eta)(1-\zeta)[9(\xi^2 + \eta^2) - 10]\]
\[ N_{21} = \frac{1}{64}(1-\xi)(1+\eta)(1+\zeta)[g(\xi^2+\eta^2)-10] \]
\[ N_{22} = \frac{9}{64}(1-3\xi)(1+\eta)(1-\xi) \]
\[ N_{23} = \frac{9}{64}(1+3\xi)(1+\eta)(1-\zeta) \]
\[ N_{24} = \frac{1}{64}(1+\xi)(1+\eta)(1+\zeta)[g(\xi^2+\eta^2)-10] \]

\[ N_{1,4} \]
\[ C_{11} = \frac{1}{64}(1-\eta)(1-\zeta)[10+9(2\xi-3\xi^2-\eta^2)] \]
\[ C_{12} = \frac{9}{64}(1-\eta)(1-\zeta)(9\xi^2-2\xi-3) \]
\[ C_{13} = \frac{9}{64}(1-\eta)(1-\zeta)(3-2\xi-9\xi^2) \]
\[ C_{14} = \frac{1}{64}(1-\eta)(1-\zeta)[9(3\xi^2+\eta^2+2\xi)-10] \]
\[ C_{15} = \frac{1}{64}(1-\eta)(1+\zeta)[10+9(2\xi-3\xi^2-\eta^2)] \]
\[ C_{16} = \frac{9}{64}(1-\eta)(1+\zeta)(9\xi^2-2\xi-3) \]
\[ C_{17} = \frac{9}{64}(1-\eta)(1+\zeta)(3-2\xi-9\xi^2) \]
\[ C_{18} = \frac{1}{64}(1-\eta)(1+\zeta)[9(3\xi^2+\eta^2+2\xi)-10] \]
\[ C_{19} = \frac{9}{64}(1-3\eta)(1-\zeta)(\eta^2-1) \]
\[ C_{1,10} = \frac{9}{64}(1+3\eta)(1-\zeta)(\eta^2-1) \]
\[ C_{1,11} = \frac{9}{64}(1-3\eta)(1-\zeta)(1-\eta^2) \]
\[ C_{1,12} = \frac{9}{64}(1+3\eta)(1-\zeta)(1-\eta^2) \]
\[ C_{1,13} = \frac{9}{64}(1-3\eta)(1+\zeta)(\eta^2-1) \]
\[ C_{1,14} = \frac{9}{64}(1+3\eta)(1+\zeta)(\eta^2-1) \]
\[ C_{1,15} = \frac{9}{64}(1-3\eta)(1+\zeta)(1-\eta^2) \]
\[ C_{1,16} = \frac{9}{64}(1+3\eta)(1+\zeta)(1-\eta^2) \]
\[ C_{1,17} = \frac{1}{64}(1+\eta)(1-\zeta)[10+9(2\xi-3\xi^2-\eta^2)] \]
\[ C_{1,18} = \frac{9}{64}(1+\eta)(1-\zeta)(9\xi^2-2\xi-3) \]
\[ C_{1,19} = \frac{9}{64}(1+\eta)(1-\zeta)(3-2\xi-9\xi^2) \]
\[ C_{1,20} = \frac{1}{64}(1+\eta)(1-\eta)[3(3\xi^2+n^2+2\xi)-10] \]
\[ C_{1,21} = \frac{1}{64}(1+\eta)(1+\xi)[10+3(2\xi-3\xi^2-n^2)] \]
\[ C_{1,22} = \frac{9}{64}(1+\eta)(1+\xi)(9\xi^2-2\xi-3) \]
\[ C_{1,23} = \frac{9}{64}(1+\eta)(1+\xi)(3-2\xi-9\xi^2) \]
\[ C_{1,24} = \frac{1}{64}(1+\eta)(1+\xi)[9(3\xi^2+n^2+2\xi)-10] \]

\[ N_{1,n} \]
\[ C_{21} = \frac{1}{64}(1-\xi)(1-\xi)[10+9(2n-\xi^2-3n^2)] \]
\[ C_{22} = \frac{9}{64}(1-3\xi)(1-\xi)(\xi^2-1) \]
\[ C_{23} = \frac{9}{64}(1+3\xi)(1-\xi)(\xi^2-1) \]
\[ C_{24} = \frac{1}{64}(1+\xi)(1-\xi)[10+9(2n-\xi^2-3n^2)] \]
\[ C_{25} = \frac{1}{64}(1-\xi)(1+\xi)[10+9(2n-\xi^2-3n^2)] \]
\[ C_{26} = \frac{9}{64}(1-3\xi)(1+\xi)(\xi^2-1) \]
\[ C_{27} = \frac{9}{64}(1+3\xi)(1+\xi)(\xi^2-1) \]
\[ C_{28} = \frac{1}{64}(1+\xi)(1+\xi)[10+9(2n-\xi^2-3n^2)] \]
\[ C_{29} = \frac{9}{64}(1-\xi)(1-\xi)(9n^2-2n-3) \]
\[ C_{210} = \frac{9}{64}(1-\xi)(1-\xi)(3-2n-9n^2) \]
\[ C_{2,11} = \frac{9}{64}(1+\xi)(1-\xi)(9n^2-2n-3) \]
\[ C_{2,12} = \frac{9}{64}(1+\xi)(1-\xi)(3-2n-9n^2) \]
\[ C_{2,13} = \frac{9}{64}(1-\xi)(1+\xi)(9n^2-2n-3) \]
\[ C_{2,14} = \frac{9}{64}(1-\xi)(1+\xi)(3-2n-9n^2) \]
\[ C_{2,15} = \frac{9}{64}(1+\xi)(1+\xi)(9n^2-2n-3) \]
\[ C_{2,16} = \frac{9}{64}(1+\xi)(1+\xi)(9-2n-9n^2) \]
\[ C_{2,17} = \frac{1}{64}(1-\xi)(1-\xi)[9(3n^2+\xi^2+2n)-10] \]
\[ C_{2,18} = \frac{9}{64}(1-3\xi)(1-\xi)(1-\xi^2) \]
\[ C_{2,13} = \frac{9}{64}(1+3\xi)(1-\xi)(1-\xi^2) \]
\[ C_{2,20} = \frac{1}{64}(1+\xi)(1-\xi)[9(3n^2+\xi^2+2n)-10] \]
\[ C_{2,21} = \frac{1}{64}(1-\xi)(1+\xi)[9(3n^2+\xi^2+2n)-10] \]
\[ C_{2,22} = \frac{9}{64}(1-3\xi)(1+\xi)(1-\xi^2) \]
\[ C_{2,23} = \frac{9}{64}(1+3\xi)(1+\xi)(1-\xi^2) \]
\[ C_{2,24} = \frac{1}{64}(1+\xi)(1+\xi)[9(3n^2+\xi^2+2n)-10] \]

\[ N_{i,\xi} \]
\[ C_{31} = \frac{1}{64}(1-\xi)(1-\eta)[10-9(\xi^2+n^2)] \]
\[ C_{32} = \frac{9}{64}(1-3\xi)(1-\eta)(\xi^2-1) \]
\[ C_{33} = \frac{9}{64}(1+3\xi)(1-\eta)(\xi^2-1) \]
\[ C_{34} = \frac{1}{64}(1+\xi)(1-\eta)[10-9(\xi^2+n^2)] \]
\[ C_{35} = \frac{1}{64}(1-\xi)(1-\eta)[9(\xi^2+n^2)-10] \]
\[ C_{36} = \frac{9}{64}(1-3\xi)(1-\eta)(1-\xi^2) \]
\[ C_{37} = \frac{9}{64}(1+3\xi)(1-\eta)(1-\xi^2) \]
\[ C_{38} = \frac{1}{64}(1+\xi)(1-\eta)[9(\xi^2+n^2)-10] \]
\[ C_{39} = \frac{9}{64}(1-\xi)(1-3\eta)(n^2-1) \]
\[ C_{3,10} = \frac{9}{64}(1-\xi)(1+3\eta)(n^2-1) \]
\[ C_{3,11} = \frac{9}{64}(1+\xi)(1-3\eta)(n^2-1) \]
\[ C_{3,12} = \frac{9}{64}(1+\xi)(1+3\eta)(n^2-1) \]
\[ C_{3,13} = \frac{9}{64}(1-\xi)(1-3\eta)(1-n^2) \]
\[ C_{3,14} = \frac{9}{64}(1-\xi)(1+3\eta)(1-n^2) \]
\[ C_{3,15} = \frac{9}{64}(1+\xi)(1-3\eta)(1-n^2) \]
\[ C_{3,16} = \frac{9}{64}(1+\xi)(1+3\eta)(1-n^2) \]
\[ C_{3,17} = \frac{1}{64}(1-\xi)(1+\eta)[10-9(\xi^2+n^2)] \]
\[ C_{3,16} = \frac{9}{64}(1-3\xi)(1+n)(\xi^2-1) \]
\[ C_{3,19} = \frac{9}{64}(1+3\xi)(1+n)(\xi^2-1) \]
\[ C_{3,26} = \frac{1}{64}(1+\xi)(1+n)[10-9(\xi^2+n^2)] \]
\[ C_{3,21} = \frac{1}{64}(1-\xi)(1+n)[9(\xi^2+n^2)-10] \]
\[ C_{3,22} = \frac{9}{64}(1-3\xi)(1+n)(1-\xi^2) \]
\[ C_{3,23} = \frac{9}{64}(1+3\xi)(1+n)(1-\xi^2) \]
\[ C_{3,24} = \frac{1}{64}(1+\xi)(1+n)[9(\xi^2+n^2)-10] \]

All elements of \( B_{ij} \) are zero, except the following:

\[ B_{1i} = \frac{\partial N_i}{\partial x} \left[ J(1,k) C(k,j) \right]; \quad k = 1, 2, 3; \quad j = 1, 2, \ldots 2^4, \quad i = 3j-2. \]

\[ B_{2i} = \frac{\partial N_i}{\partial y} \left[ J(2,k) C(k,j) \right]; \quad k = 1, 2, 3; \quad j = 1, 2, \ldots 2^4, \quad i = 3j-1. \]

\[ B_{3i} = \frac{\partial N_i}{\partial z} \left[ J(3,k) C(k,j) \right]; \quad k = 1, 2, 3; \quad j = 1, 2, \ldots 2^4, \quad i = 3j. \]

\[ B_{4i} = B_{1j}; \quad i = 2, 5, 8, \ldots 71, \quad j = i-1. \]

\[ B_{5i} = B_{2j}; \quad i = 1, 4, 7, \ldots 70, \quad i = j+1. \]

\[ B_{6i} = B_{1j}; \quad i = 3, 6, 9, \ldots 72, \quad j = i-2. \]

\[ B_{6i} = B_{3j}; \quad i = 1, 4, 7, \ldots 70, \quad j = i+2. \]

\[ B_{6i} = B_{2j}; \quad i = 3, 6, 9, \ldots 72, \quad j = i-2. \]

\[ B_{6i} = B_{3j}; \quad i = 2, 5, 8, \ldots 71, \quad j = i-1. \]

where

\[ J(1,1) = \frac{[J(2,2)J(3,3)-J(2,3)J(3,2)]}{1J1} \]

\[ J(1,2) = \frac{[J(1,3)J(3,2)-J(1,2)J(3,3)]}{1J1} \]
\[ JJ(1,3) = \frac{J(1,2)J(2,3) - J(1,3)J(2,2)}{1JJ} \]
\[ JJ(2,1) = \frac{J(2,3)J(3,1) - J(2,1)J(3,3)}{1JJ} \]
\[ JJ(2,2) = \frac{J(1,1)J(3,3) - J(1,3)J(3,1)}{1JJ} \]
\[ JJ(2,3) = \frac{J(1,3)J(2,1) - J(1,1)J(2,3)}{1JJ} \]
\[ JJ(3,1) = \frac{J(2,1)J(3,2) - J(2,2)J(3,1)}{1JJ} \]
\[ JJ(3,2) = \frac{J(1,2)J(3,1) - J(1,1)J(3,2)}{1JJ} \]
\[ JJ(3,3) = \frac{J(1,1)J(2,2) - J(1,2)J(2,1)}{1JJ} \]

**Cubic-Linear Shape Functions** \( N_i \)

\[ N_1 = \frac{1}{32}(1-\xi)(1-\eta)(3\xi^2-1) \]
\[ N_2 = \frac{9}{32}(1-3\xi)(1-\eta)(1-\xi^2) \]
\[ N_3 = \frac{9}{32}(1+3\xi)(1-\eta)(1-\xi^2) \]
\[ N_4 = \frac{1}{32}(1+\xi)(1-\eta)(3\xi^2-1) \]
\[ N_5 = \frac{1}{32}(1-\xi)(1+\eta)(3\xi^2-1) \]
\[ N_6 = \frac{9}{32}(1-3\xi)(1+\eta)(1-\xi^2) \]
\[ N_7 = \frac{9}{32}(1+3\xi)(1+\eta)(1-\xi^2) \]
\[ N_8 = \frac{1}{32}(1+\xi)(1+\eta)(3\xi^2-1) \]

**\( N_{i,\xi} \)**

\[ C_{11} = \frac{1}{32}(1-\eta)[1+9\xi(2-3\xi)] \]
\[ C_{12} = \frac{9}{32}(1-\eta)(9\xi^2-2\xi-3) \]
\[ C_{13} = \frac{9}{32}(1-\eta)(3-2\xi-9\xi^2) \]
\[ C_{14} = \frac{1}{32}(1-\eta)[3\xi(2+3\xi)-1] \]
\[ C_{15} = \frac{1}{32}(1+\eta)[1+9\xi(2-3\xi)] \]
\[ C_{16} = \frac{9}{32}(1+\eta)(9\xi^2-2\xi-3) \]
\[ C_{17} = \frac{9}{32}(1+\eta)(3-2\xi-9\xi^2) \]
\[ C_{18} = \frac{1}{32}(1+\eta)[9\xi(2+3\xi)-1] \]

**\( N_{i,\eta} \)**

\[ C_{21} = \frac{1}{32}(1-\xi)(1-9\xi^2) \]
\[ C_{22} = \frac{9}{32}(1-3\xi)(\xi^2-1) \]
\[ C_{23} = \frac{9}{32}(1+3\xi)(\xi^2-1) \]
\[ C_{24} = \frac{1}{32}(1+\xi)(1-9\xi^2) \]
\[ C_{25} = \frac{1}{32}(1-\xi)(9\xi^2-1) \]
\[ C_{26} = \frac{9}{32}(1-3\xi)(1-\xi^2) \]
\[ C_{27} = \frac{9}{32}(1+3\xi)(1-\xi^2) \]
\[ C_{28} = \frac{1}{32}(1+\xi)(9\xi^2-1) \]
APPENDIX C

16-DOT Program Listing
C FINITE ELEMENT ANALYSIS OF 2-DIMENSIONAL LAMINATED COMPOSITES
C
C
C    Z    0----0----0----0----0
C    I    I
C    I    I
C    I    0----0----0----0----0
C    +---- x 1 2 3 4
C
C FEM 16-DOF (2 AT EACH NODE) PROGRAM FOR RECTANGULAR ELEMENTS
C
C FILED REAL*8 (A-Z)
C INTEGER CODE
C COMMON / HEAD / ICRO, LIST, HED(20), IPAGE, LINE
C COMMON / DATUM / QDDE(1520), X(1520), Z(1520), UX(1520), UZ(1520),
1      AS(500), CLS(500), E(8,4),
2      IX(500,5), NUMNP, NUMEL, NUMMAT, MTYPE, IBW, NE, NPS
C COMMON / SOLN / A(164,82), B(164), MNUBLK, IJK, NB, ND, ND2
C DEFINE FILE 1 (660, 5000, L, IJK)
C DEFINE FILE 2 (660, 5000, L, N)
C ICRO = 5
C LIST = 6
C
C CHECK FOR START OF NEW PROBLEM OR TERMINATION OF PROGRAM
C 50 READ (ICRO, 1000, END=5000) HED
C 51 IPAGE = 1
C CALL DOCUMT
C
C FORM STIFFNESS MATRIX
C CALL STIFF
C
C CALCULATE AND PRINT OUT DISPLACEMENTS
C CALL SYMOSL (NE, IBW)
C CALL TITLE
C WRITE (LIST, 2100)
C NBLK = 0
C N = 0
C 60 NBLK = NBLK + 1
L = 0.
70 N = N+1
L = L+1
I = MD + (2*N-((NBLK-1)*NU))
J = I-1
IF ( LINE .LT. 50 ) GO TO 80
CALL TITLE
WRITE (LIST,2100)
80 WRITE (LIST,2200) N,X(N),Y(N),A(J,NBLK),A(I,NBLK)
LINE = LINE +1
IF ( N.NE. NUMAP ) GO TO 90
IF (L.NE.) 70,50,60
90 CONTINUE
C CALCULATE AND PRINT OUT STRESSES
CALL STRESS
C START NEXT PROBLEM
GO TO 50
1000 FORMAT (20A4)
2100 FORMAT (1H4,9X,4HNODE,7X,1HX,11X,1HZ,14X,7HX-015PL,13X,7HZ-015PL)
2200 FORMAT (9X,15.2F12.5,2020.8)
5000 STOP
END
SUBROUTINE DOCNT
C SUBROUTINE FOR DOCUMENTATION OF PROBLEM
C( I/O ALL THE PROBLEM CHARACTERISTICS )
IMPLICIT REAL*8(A-G,O-Z)
INTEGER CODE
COMMON / NLOCAL / IORD,LIST,HEB(20),IPAGE,LINE
COMMON / DATATM / CODE(1520),X(1520),Z(1520),UX(1520),VX(1520),
1 AS(520),CS(520),E(8,4),
2 IX(520),NUMRE,NUMEL,NUMMAT,NTYPE,ISW,NE,NPS
C READ AND WRITE CONTROL INFORMATION AND MATERIAL PROPERTIES
READ (ICRD,1603) NUMNP,NUMEL,NUMMAT,NS
NC = 2*NUMNP
CALL TITLE
WRITE (LIST,2000) NUMNP,NUMEL,NUMMAT,NC
IF (NS .EQ. 1) WRITE(LIST,2007)
IF (NS .EQ. 1) WRITE(LIST,2006)
DO 50 N=1,NUMMAT
READ (ICRD,1001) MTYPE,(E(J,MTYPE),J=1,8)
WRITE(LIST,2010)
IF (LINE .LT. 50) GO TO 40
CALL TITLE
WRITE (LIST,2010)
40 LINE = LINE + 1
50 WRITE (LIST,2011) MTYPE, (E(I,MTYPE),I=1,8)
CALL TITLE
WRITE (LIST,2004)
L = 0
60 READ (ICRD,1602) N, CODE(N), X(N), Z(N), UX(N), UZ(N)
IF (N .EQ. 1) 65, 69, 65
CONTINUE
ZX=N-L
DX = (X(N)-X(L))/ZX
69 CONTINUE
NL=L+1
70 L=L+1
IF (N-L .EQ. 1) 100, 69, 65
80 CODE(L) = 0
X(L) = X(L-1) + DX
Z(L) = Z(L-1)
UX(L) = 0.0
UZ(L) = 0.0
GO TO 70
90 DO 97 K=1, N
IF (IINE .LT. 50) GO TO 91
CALL TITLE
WRITE (LIST,2004)
91 LINE = LINE + 1
92 WRITE (LIST,2002) K,CODE(K),x(K),Z(K),Ux(K),UZ(K)
IF (NUMNL .EQ. N) 160,110,60
100 WRITE (LIST,2009) N
STOP
110 CONTINUE
C I/O ELEMENT PROPERTIES
CALL TITLE
WRITE (LIST,2001)
N=0
130 READ (ICRD,1303) N,(IX(M,I), I=1,5)
IF (M .EQ. NUMNL) 140,140,132
132 WRITE (LIST,2010) M
STOP
140 N=N+1
IF(M>N) 170,170,150
150 DO 151 I=1,4
151 IX(N,I) = IX(N-1,I) + 3
IX(N,5) = IX(N-1,5)
170 CONTINUE
I = IX(N+1)
L = IX(N+4)
AS(N) = DABS(X(L) - X(I))
CS(N) = DABS(Z(L) - Z(I))
IF (AS(N)) 160,160,160
160 WRITE (LIST,2018) N
STOP
169 CONTINUE
AC = CS(N)/AS(N)=2.00
IF (LINE .LT. 50) GO TO 171
CALL TITLE
WRITE (LIST,2001)
171 LINE = LINE +1
WRITE (LIST,2003) N,(1X(N,I),I=1,5),AC,AS(R),CS(N)
IF (N - N) 180,180,140
180 IF (NUMEL - N) 190,190,130
190 CONTINUE
C DETERMINE THE HALF-BAND-WIDTH FOR THIS PROBLEM  ( MAX. BEING 32)
J=0
DO 340 K=1,NUMEL
DO 340 I=1,5
K = I + 1
DO 340 L=K,4
KK =LAMBDA ((1X(N,I)-1X(N,L))
IF (KK-J) 325,325,320
320 J=KK
325 CONTINUE
340 CONTINUE
C ISW = HALF-BAND-WIDTH OF PROBLEM IN QUESTION
ISW = 2*J + 2
1001 FORMAT (15,8F8.0)
1002 FORMAT (215,4F10.2,215)
1003 FORMAT (615)
2001 FORMAT (1HC / 10X,2HNUMBER OF NOAL POINTS...........15/
1 1X,2HNUMBER OF ELEMENTS............15/
2 1X,2HNUMBER OF DIFF. MATERIALS......15/
3 1X,2HNUMBER OF TOTAL OF DOF........15)
2001 FORMAT (1HC,8X,5HELEL. NO. I J K L MATERIAL 35
1/A, 11X,1HA,12X,1HA)
Cmai 2 FORMAT (7X,211C,6F17.4,2020.1,2,17)
2023 FORMAT (11X,13,2X,16,16,3F13.3)
2024 Format (1HC,8X,17HNDUAL POINT TYPE,9X,9FX=COORD.,9X,6HZ=COORD.,
1 6X,14FX(LOAD/DISPL.),6X,14HZ(LOAD/DISPL.))
2026 FORMAT (12X,13,4X,13,4)
2027 FORMAT (12X,'PLANE STRESS PROBLEM')
2028 FORMAT ('PLANE STRAIN PROBLEM')
2039 FORMAT (26HNDUAL POINT CARD ERROR N=,I5)
210 FORMAT (26HNDUAL coord. error =,I5)
211 FORMAT (26HNDUAL coord. error - SIDE OF ELEMENT,10,2X,SHEQUALS 3)
219 FORMAT (18HNDUAL ELEMENT ERROR M=,I5)
219 FORMAT (18HNDUAL ELEMENT ERROR M=,I5)
RETURN
END
SUBROUTINE TITLE
C TITLE SUBROUTINE
COMMON / HEAD / ICRD,LIST,RED(20),I PAGE,L INE
WRITE (LIST,150) LINE
WRITE (LIST,160) RED
I PAGE = I PAGE + 1
LINE = 0
150 FORMAT (1HC,8X,'FEM-16 0DF RECTANGULAR ELEMENTS WITH THREE POINTS'
1 ' ', 'IN X-DIRECTION',LINE',10X,'PAGE',I5)
160 FORMAT (1HC,12X,2044)
RETURN
END
SUBROUTINE STIFF
C FORM THE TOTAL STIFFNESS MATRIX IN BLOCKS
IMPLICIT REAL*8 (A-H,O-Z)
INTEGER CDOE
COMMON / HEAD / ICORD,LIST,RED(20),I PAGE,L INE
COMMON / DATU, CODE(1520), X(1520), Z(1520), UX(1520), UZ(1520),
       A(500), C(500), B(8,4),
       IX(500,5), NUMN, NUMEL, NUMAT, MTYPE, IDN, NE, NPS
COMMON / SOLN / A(164,82), B(164), MNULK, IJK, NY, NO, NQ2
COMMON / SWALK / B(16,16), C(16,16), D(16,16), S(16,16), S(4)
DIMENSION LM(2)

C INITIALIZATION OF MATRICES
N5 = 41
ND = 2*NB
NQ2 = 2*ND
NALK = 0
DO 10 N = 1, ND
  5(N) = 0.000
  DO 10 M = 1, ND
  10 A(N,M) = 0.000

C CALCULATE MAX. NO. OF BLOCKS (MNULK)
MNULK = NE/ND + 1
WRITE (6,1000) MNULK

C FORM BIG K IN BLOCKS
   60 NLK = NLK + 1
   NH = N5*(NLK + 1)
   NM = NH - N8
   NL = NH - N9 + 1
   IS = 2*NL - 2

C FORM ELEMENT STIFFNESS MATRIX
   DO 210 N = 1, NUMEL
     IF (IX(N,5) .LT. 1) GO TO 210

C DETERMINING IF THIS ELEMENT AFFECTS THIS BLOCK OF BIG K
   DO 210 I = 1, 4
     IF (IX(N,1) .GT. NL .AND. IX(N,1) .LT. NA) GO TO 90
   CONTINUE
   GO TO 210
90 IF (N .EQ. 1) GO TO 100
   QA = DABS(D5(N)/AR - 1.0D0)
   QC = DABS(D5(N)/CR - 1.0D0)
   IF (QA .LT. 0.50-9 .AND. QC .LT. 0.50-9) GO TO 110
100 CALL CINVER(N)
110 MR = D5(N)
   CR = D5(N)
   IF (N .EQ. 1) GO TO 120
   IF (IX(N,5) .EQ. MR) GO TO 130
120 CALL ELAS(N)
   IF (N .EQ. 1) GO TO 140
130 IF (QA .LT. 0.50-9 .AND. QC .LT. 0.50-9 .AND. IX(N,5).EQ.MR) GO TO 150
140 CALL OTOR(N)
   MR = IX(N,5)
C  MULTIPLY TRANSP. OF CINVER * OTOR = CINVER
C  Q = (QTOR) * CINVER
C  S = CINVER * (QTOR*C)
   DO 145 I=1,16
      DO 145 J=1,16
         G(I,J) = 0.00
      DO 145 K=1,16
         145 S(I,J) = G(I,J) + R(I,K) * C(K,J)
   DO 146 I=1,16
      DO 146 J=1,16
         S(I,J) = S(I,J) + R(I,K) * C(K,J)
   DO 146 K=2,16
      146 S(I,J) = S(I,J) + C(K,1) + Q(K,J)
      WRITE (2,*M) ((S(I,J),I=1,16),J=1,16)
150 CONTINUE
C ADD ELEMENT STIFFNESS TO TOTAL STIFFNESS
   LM(1) = 2*IX(N,1) - 2
   LM(2) = 2*IX(N,2) - 2
Ix(N,5) = -Ix(N,5)
DO 240 I=1,2
DO 240 K=1,9
II =LM(I)+K,1-IBS
KK = K + B*(I - 1)
DO 200 J=1,2
DO 200 L=1,8
JJ = LM(J)+L-II+1-IBS
LL = L + B*(J - 1)
IF (JJ) 200,202,170
170 CONTINUE
IF (LM(JJ) 180,190,190
190 WRITE (LIST,200) N,1ER,JJ
STOP
200 CONTINUE
210 CONTINUE
C ADD CONCENTRATED FORCES AT NODES
DO 250 K=1,NUMNP
K = 2*K-1-IBS
ICODE = CODE(N)+1
IF (K,LT,1,GT,NC) GO TO 250
GO TO (251,252,253,254),ICODE
251 A(K) = A(K)+UZ(N)
252 A(K-1) = A(K-1)+UX(N)
GO TO 250
255 CONTINUE
C APPLY DISPLACEMENT B.C.'S
DO 400 K=1,NUMNP
ICODE = CODE(N)+1
N = 2*K-1-IBS
IF ( N.LT. 1 .OR. N.GT. N02 ) GO TO 420

U = UX(N)
GO TO (415,315,320,330,1690)
315 CALL MODIFY (G,J,16W,N02)
GO TO 410
320 CALL MODIFY (N,U,16W,N02)
330 N = N + 1
U = VJ(M)
CALL MODIFY (G,J,16W,N02)
410 CONTINUE

C WRITE BLOCK OF EQU'S ON DISK AND SHIFT UP LOWER BLOCK
IJK = (MRLK=11) - 1
WRITE (1,1JK) ((A(I,J),I=1,NC),J=1,IBW,(B(X),X=1,NO))
GO 420 N = 1,ND
K = K + ND
B(N) = B(K)
B(K) = 0.000
GO 420 M = 1,ND
A(M,N) = A(K,M)
420 A(K,M) = 0.000

C CHECK FOR LAST BLOCK
IF ( NM .LE. NUMNP ) GO TO 60
GO 450 I = 1,NUMEL
450 IX(I,5) = -1*IX(I,5)
1000 FORMAT (1H7,9X,17HALF BAND WIDTH =,15,15H, NB. BLOCKS =,15)
2000 FORMAT (1X,46H BAND WIDTH EXCEEDS THAT CALCULATED FOR PROBLEM,15)
RETURN
END

SUBROUTINE ELAS(N)
C CALCULATE ELASTIC STIFFNESS MATRIX, DEPENDING ON MATERIAL PROPERTY
IMPLICIT REAL8 (A-G,Z-L)
INTHEX CODE
COMMON / DATUM /  CODE(1520),X(1520),Z(1520),UX(1520),UZ(1520),
1    AS(500),CS(500),E(8,4),
2    IX(500+2),NUMP,NUMEL,NUMMAT,MTYPE,IRX,NE,NPS
COMMON / SMLK / R(16,16),C(16,16),Q(16,16),S(16,16),D(4)
L = IX(N,L)
E11 = E(1,L)
E22 = E(2,L)
F33 = E(3,L)
XNU12 = E(4,L)
XNU13 = E(5,L)
XNU23 = E(6,L)
G13 = E(7,L)
TH = E(8,L)
XNU21 = XNU12*E22/E11
XNU31 = XNU13*E33/E11
XNU32 = XNU23*E33/E22
IF(NPS = 1) go to 61

C PLANE STRESS CASE
60 FACT = E11/ ((1.0 - XNU13*XNU31) *TH
D(1) = FACT
D(2) = FACT * E33 / E11
D(3) = FACT * XNU31
D(4) = G13 * TH
GO TO 62

C PLANE STRAIN CASE
61 FACT = ((1.0 - XNU13*XNU21) * (1.0 - XNU23*XNU32))
1    = (XNU13 + XNU12*XNU23)* (XNU21 + XNU23*XNU32)
D(1) = E11*(1.0 - XNU23*XNU32)*TH/FACT
D(2) = F33*(1.0 - XNU12*XNU21)*TH/FACT
D(3) = E33*(XNU13 + XNU12*XNU23)*TH/FACT
D(4) = G13 * TH
62 CONTINUE
RETURN
END

SUBROUTINE UTCOL(N)
C FORM MATRIX R = (CGTDQ)Vol, INTEG. OF GTDQ * RT* Vol FOR 16 DOF ELEMENT
IMPLICIT REAL*8 (A-G,0-9)
INTEGER CODE
COMMON / DATUM / CODE(1520),X(1520),Z(1520),UX(1520),UZ(1520),
1 AS(530),CS(530),E(S,4),
2 IX(560,5),NUMNP,NUMEL,NORMAT,MTYPE,LOW,NE,APS
COMMON / SMALL / R(16,16),C(16,16),D(16,16),S(16,16),D(4)
DO 10 I=1,16
C0:10 J=I,16
10 R(I,J) = 0.00
AA = AS(N)/3.00
CC = CS(N)
AC = A+CC
R(2,2) = 3.00*AC*D(1)
R(2,4) = 9.00*AC*A*A*C(1)
R(2,5) = 1.00*AC*C*C*D(1)
R(2,6) = 27.00*A*A*2*AC*D(1)
R(2,7) = 4.50*AC*2*C(1)
R(2,8) = 10.50*AC*2*A*A*D(1)
R(2,11) = 3.00*AC*D(3)
R(2,13) = 4.50*A*A*AC*C(3)
R(2,15) = 9.00*A*A*AC*2*AC*C(3)
R(2,16) = 27.00*A*A*3*AC*C(3)
DO 11 I=5,8
11 R(3,1) = R(2,1)*D(3)/D(4)
DO 12 I=2,3
12 R(I+3) = R(I+1)*D(4)/D(I)
R(4,4) = 9.00*A*A*2*3*AC*C(1)
R(4,7) = R(2,7)
$R(7,15) = AC \ast (2 \ast 15) + R(3,14) \ast 730$

$R(7,16) = CC \ast (4 \ast 3) + R(7,14) / 2 \ast 0$

$R(8,8) = 2 \ast 0 \ast AC \ast R(16,7) + R(7,14) \ast AA \ast 2 \ast 16 \ast D / 7 \ast 0$

$R(8,10) = R(7,12) / 2 \ast 0$

$R(8,11) = R(2,9) \ast 9(3) / 0(1)$

$R(9,12) = 2 \ast 0 \ast 8(7,14) / 3 \ast 0$

$R(8,13) = CC \ast (0 \ast 13) + R(7,12) / 2 \ast 0 \ast D / 2 \ast 0$

$R(9,14) = R(0,10) \ast D(4) / D(3)$

$R(8,15) = CC \ast (R(4,12) + R(6,15)) / 2 \ast 0$

$R(8,16) = CC \ast (R(3,14) + R(6,16)) / 2 \ast 0$

DO 13 J=2, I+2

13 IF J+1, I+2 = R(J,J) \* D(4) / D(1)

DO 14 I=3, K

14 IF I \* D(3) = R(K, I) \* D(4)

R(13,15) = AC \ast (0 \ast 13) \* AA \ast 2 \* 0(2) + CC \ast 2 \* D(4)

R(13,14) = R(3,14)

R(15,16) = R(3,15) \* 3 \* 0 \* D(2) / D(3) + R(3,15) \* AC \ast 2 \* 0

R(15,16) = R(0,15) \* 3 \* 0 \* D(2) / D(3) + R(3,15) \* AC \ast 2 \* 0

R(14,16) = R(0,14) \* 3 \* 0 \* D(2) / D(4) + R(12,16)

R(15,16) = R(7,15) \* AC \ast AA \ast 3 \* 0 \* D(2) / D(3) + 14 \* 0 \* AC \ast 3 \* AA \ast 2 \* 0(4)

DO 20 I=1, 15

J = I + 1

DO 20 K = J, 16

20 R(K, I) = R(I, K)

RETURN

END

SUBROUTINE CVRSE (N)

C FORM MATRIX C FOR INVERSE OF C

IMPLICIT REAL*4 (A-G, 0-Z)

INTEGER CODE
COMMON / DATUM / CODE(1520),X(1520),Z(1520),UX(1520),UZ(1520),
1 AS(320),CS(320),E18(4),
2 IX(50,5),NUMF,NUMK,NUMT,MTYPE,IBW,NE,NPS
COMMON / SMALL / K(16,16),C(16,16),U(16,16),S(36,16),J(4)
AA = 3.96/AS(N1)
BB = AA*92
CC = 1.00/CS(N1)
DE = AA*53
ES = BB*CC
FF = DD*CC
DO 10 I=1,16
DO 10 J=1,16
10 C(I,J) = 0.00
C(1,1) = 1.00
C(2,1) = -1.56*AA/6.64
C(2,2) = 3.06*AA
C(2,3) = -1.56*AA
C(2,7) = AA/3.60
C(3,1) = CC
C(3,9) = CC
C(4,1) = BB
C(4,3) = -2.50*BB
C(4,5) = 2.50*BB
C(4,7) = -BB/2.06
DO 16 I=1,7
16 C(5,1) = -C(2,1)*CC
DO 11 I=1,7
11 C(5,1+8) = -C(5,1)
C(6,1) = CC/6.50
C(6,3) = 0.07/2.65
C(6,5) = -C(6,1)
C(6,7) = -C(6,1)
DO 12 I=1,7
12 C(7,I) = -C(4,I)*CC
DO 13 I=1,7
13 C(7,I+3) = -C(7,I)
DO 14 I=1,7
14 C(8,I) = -C(6,I)*CC
DO 15 I=1,7
15 C(8,I+3) = -C(8,I)
DO 20 I=1,8
DO 20 J=1,18
20 C(I+8,J+1) = C(I,J)
RETURN
END

SUBROUTINE MODIFY (N,U,IBW,ML)
C MODIFY TOTAL STIFFNESS MATRIX FOR GIVEN DISPL. B. C.
IMPLICIT REAL*8 (A-H,U)
COMMON / SOLN / A(164,82),B(164),BNUSLK,IJK,NE,NQ,N02,NN,NN,NN
DO 250 M=2,18W
K= N- K +1
IF (K) 239,239,239
239 S(K) = (K)* A(K, M)* U
A(K, M) = DCC
235 K= N+ M-1
IF(ML - K) 250,240,240
240 P(K) = R(K) - A(IN, M)* U
A(IN, M) = DCC
250 CONTINUE
A(IN, 1) = 1.0CC
A(N) = U
RETURN
END

SUBROUTINE SYMSTL (NN,IBW)
C SOLVE THE SYSTEM OF BANDED, BLOCKED SIMULTANEOUS EQUATIONS BY AUSTAN K
IMPLICIT REAL*4 (A-G, I-Z)
COMMON / SOLN / A(164,62), B(164), ML(164), IJK, NS, NA, ND2
NS = NO
NL = NS+1
NH = NS+44
ML = 0
IJK = 0
GO TO 150
C REDUCE EQU'S BY BLOCKS; AND THEN SHIFT BLOCK OF EQU'S
150 NLK = NLK + 1
DO 125 N = 1, NN
NM = NS + N
N1 = J(NM)
A(NM) = 0.000
A(N1) = 0.000
GO TO 125
A(NM, N) = 0.000
C READ IN NEXT BLOCK OF EQU'S
IF (NLK .LE. MNUBLK ) GO TO 200
IJK = (NLK*11) +1
125 READ (11, IJK) ((A(I,J), I = NL, NM), J = 1, IHW), (B(K), K = NL, NH)
IF (NLK .LE. MNUBLK ) GO TO 100
C REDUCE BLOCK OF EQU'S
200 DO 300 N = 1, NN
IF ( A(N,1) .EQ. 0.000 ) GO TO 300
ML = B(N)/ A(N,1)
GO TO 275
L = 2, IHW
IF ( A(N,L) .EQ. 0.000 ) GO TO 275
C = A(N,L)/ A(N,1)
I = N + L - 1
J = 0
GO TO 250 K=L, I=N
J=J+1
IF (A(I,J) .EQ. 1.0EC0 .AND. B(I) .EQ. 0.0EC0 ) GO TO 260
250 A(I,J) = A(I,J) - C*PA(N,K)
     B(I) = B(I) - B(K,L) - B(N)
260 A(N,L) = C
270 CONTINUE
300 CONTINUE
C WRITE BLOCK OF EQN'S BACK ON DISK
   IF ( NBLK .EQ. MNUBLK ) GO TO 400
   IJK = ( NBLK+11 ) - 13
   WRITE (1,10) ((A(I,J), I=1,NN), J=1,NW), (B(K), K=1,NN)
300 GO TO 100
C BACK SUBSTITUTION; SOLVED UNKNOWN DISPLACEMENTS ARE STORED IN
C THE FIRST MNUBLK COLUMNS OF THE 2ND BLOCK OF MATRIX A
400 DO 45 K=1,NN
   N=NN+1-K
   GO TO 425 K=K+1
425 B(N) = B(N) - A(N,K)*PA(L)
   NM = N + NN
   B(NM) = B(N)
45 , A(NM,NUBLK) = B(N)
   NUBLK = NUBLK-1
   IF ( NUBLK < MNUBLK ) GO TO 300
   IJK = ( NUBLK+11 ) - 13
   READ (1,10) ((A(I,J), I=1,NN), J=1,NW), (B(K), K=1,NN)
   GO TO 300
400 CONTINUE
C subroutine stress
C CALCULATE STRESSES AT 4 NODES ALONG MID-LINE OF EACH ELEMENT
IMPLICIT REAL*8 (A-G, I-Z)

INTEGER CODE

COMMON / HEAD / ICRD, LIST, HED(20), IPAGE, LINE
COMMON / DATUM / CODE(1520), X(1520), Z(1520), UX(1520), UZ(1520),
1 AS(500), CS(500), E(8, 4),
2 IX(560, 5), NUMNF, NUMEL, NUMAT, MTPC, 104, NE, NEF
COMMON / SOLN / A(160, 16), R(160), MNTRLK, IJK, NR, ND, NO2
COMMON / SMALLK / R(16, 16), C(16, 16), Q(16, 16), S(16, 16), D(4)
DIMENSION LM(2), LB(2, 16), U(16), SIG(17), I(12, 16)

CALL TITLE
WRITE (LIST, 2000)

C CALCULATE Y = D4OF5-CNINF, 6 SETS OF 3X16 FOR EACH NODE OF THE ELEMENT
STEER = 0.0 DC
DO 94 K = 1, NUMEL
   IF (N .EQ. 1) GO TO 129
   GA = DARS(A(N)/AR - 1.00)
   WC = DARS(CS(N))/CR - 1.00
   IF (GA .LT. 0.50-9 A AND. WC .LT. 0.50-9) GO TO 133
94 CONTINUE

129 CALL CNINF(N)
   IF (N .EQ. 1) GO TO 140
130 IF (IX(N, 5) .NE. MR) GO TO 142
144 CALL ELAS(N)
   IF (N .EQ. 1) GO TO 143
142 IF (GA .LT. 0.50-9 A AND. WC .LT. 0.50-9 A AND. IX(N, 5) .EQ. MR) GO TO 145
145 CALL ELSTKS(K, L, T)

C PLACE PROPER LOCAL DISPLACEMENTS IN U FROM A
LM(1) = 2*IX(N, 1) - 2
LM(2) = 2*IX(N, 1) - 2
DO 20 J = 1, 2
20 DO K = 1, 8
   I = LM(J) + K
   U(I) = U(I) - GA
   U(I) = U(I) - WC
   U(I) = U(I) - GA
   U(I) = U(I) - WC
   U(I) = U(I) - GA
   U(I) = U(I) - WC
   U(I) = U(I) - GA
   U(I) = U(I) - WC
JJ = K + 3*J - 1
NLK = (I - 1)/NI + 1
II = ND + 1 - (ND*(NLK-1))
U(JJ) = A(I, NI)
20 CONTINUE
C SOLVE FOR STRESSES
DO 60 K=1, N
DO 30 I=1, 7
50 SIG(I)= 0.0
DO 35 I=1, 3
L = I + 3*(K - 1)
DO 35 J=1, 16
SIG(I)=SIG(I) + T(L, J)*U(J)
35 CONTINUE
C CALCULATE PRINCIPAL STRESSES
AA = (SIG(1)+SIG(2))/ 2
BB = (SIG(1)-SIG(2))/ 2
CC = 0.5*SQRT( (AA**2 + SIG(3)**2)**2 )
SIG(4)=AA+CC
SIG(5)=AA-CC
IF (SIG(1) = SIG(2)) 41, 37, 36
38 SIG(6)= (COTAN(SIG(3)/ BB ))* 28.64788750
GO TO 45
37 IF (SIG(2)) 41, 35, 36
38 SIG(6)= 45.00
GO TO 45
39 SIG(6)= 0.00
GO TO 45
40 SIG(6)= -45.00
GO TO 45
41 SIG(6)= (COTAN(SIG(3)/ BB ))* 28.64788750
1F (SIG(3)) 44, 43, 42
$42 \text{ SIG}(6) = 90.88 + \text{ SIG}(6)$

GO TO 45

$42 \text{ SIG}(6) = 90.88$

GO TO 45

$44 \text{ SIG}(6) = -90.88 + \text{ SIG}(6)$

CONTINUE

$\text{ SIG}(7) = 0.55 \times \text{ CABS} \left( \text{ SIG}(4) - \text{ SIG}(5) \right)$

IF (LINE = LT. 35) GO TO 50

CALL TITLE

WRITE (LIST, 2000)

CONTINUE

IF (K .EQ. 1) GO TO 55

WRITE (LIST, 2001) K, (SIG(I), I=1,7)

GO TO 56

WRITE (LIST, 2002) N, K, (SIG(I), I=1,7)

CONTINUE

LINE = LINE + 1

CONTINUE

IF (N .EQ. 1) GO TO 70


READ (2,N) ((S(I,J), I=1,16), J=1,16)

STEGEL = 0.00

MR = IC(N,S)

AK = AS(N)

CR = CS(N)

GO TO 200 II=1,16

STIFFU = 0.00

GO TO 100 JJ=1,16

STIFFU = STIFFU + S(II,JJ)*U(JJ)

STEGEL = STEGEL + STIFFU*U(II)/2.00

WRITE (LIST, 2009) STEGEL

STEGST = STEGST + STEGEL
WRITE (LIST, 2004) STRESS
2004 FORMAT (UHE, GX, SHEAR, GX, GMODE, GX, GY STRESS, GX, GZ STRESS,
15X, GX, GX X STRESS, GX, GMAX STRESS, GX, GMIN STRESS, GX, GHANGLE,
25X, GX, GMAX SHEAR)
2001 FORMAT (15X, 111, 4X, 5014, 4, F9.2, 014.4)
2002 FORMAT (1HE, 9X, 15, 111, 4X, 5014, 4, F9.2, 014.4)
2003 FORMAT (1DX, 'EL STRAIN ENERGY = ', E15.8)
2004 FORMAT (1DX, 'SI STRAIN ENERGY = ', E15.8)
RETURN
END
SUBROUTINE ELSTRES(N, DB, T)
C FORM ELEMENT NODE STRESS MATRIX T(12,16)
IMPLICIT REAL*8 (A-G, I-Z)
INTEGER CODE
COMMON / DATUM / CODE(1520), X(1520), Z(1520), UX(1520), UZ(1520),
1 AS(500), CS(500), E(8,4),
2 IX(500), BI, NUMNP, NUMEL, NUMMAT, MTYPE, 15W, NE, NPS
COMMON / SMALL / R(16,16), G(16,16), W(16,16), S(16,16), D(4)
DIMENSION L(2), DB(3,16), U(16), STG(7), T(12,16)
DO 20 K = 1, 4
XX = 0.00
ZZ = CS(N)/2.00
GO TO (21,22,23,24), K
22 XX = AS(N)/3.00
GO TO 21
23 XX = AS(N)+2.00/3.00
GO TO 21
24 XX = AS(N)
21 CONTINUE
DO 30 I = 1, 3
DO 30 J = 1, 16
30 DB(I,J) = 0.00
DB(1,2) = D(1)
DB(1,4) = 2.000*XX*Z+C(1)
DB(1,5) = 2.Z+C(1)
DB(1,6) = 3.000*XX*Z+C(1)
DB(1,7) = 2.000*XX*Z*C(1)
DB(1,8) = 3.000*XX*Z+C(1)
DB(1,11) = D(3)
DB(1,13) = XX+C(3)
DB(1,15) = XX*Z+C(3)
DB(1,16) = XX*X+C(3)
DB(2,11) = C(2)
DB(2,13) = XX+C(2)
DB(2,15) = XX*Z+C(2)
DB(2,16) = XX*X+C(2)
GO 40 I=2,3
40 DB(2,1) = DB(1,1)*D(3)/D(1)
DC 50 I=3,5
50 DB(3,1) = DB(2,I+8)*C(4)/D(2)
CC 60 I=2,3
60 DB(3,1+8) = DB(1,1)*D(4)/D(1)
C MULTIPLY DB*C FOR EACH OF EIGHT NODES
GO 100 I=1,3
L = 3*K + I - 3
GO 100 J=1,16
T(L,J) = C
GO 100 M=1,16
T(L,J) = T(L,J) + DB(I,M)*C(M,J)
100 CONTINUE
200 CONTINUE
RETURN
END
APPENDIX D

24-DOF Program Listing
FINITE ELEMENT ANALYSIS OF 2-DIMENSIONAL LAMINATED COMPOSITES

C FEM 24-DOF (2 @ EACH NODE) PROGRAM FOR GENERAL QUADRILATERAL
IMPLICIT REAL*8 (A-H,O-Z)
COMMON / READ / ICRO, LIST, HED(10), IPAGE, LINE
COMMON / DATA / X(500), Z(500), UX(500), UZ(500), AS(100), CS(100),
I CODE(500), IX(100), NP, NUMN, NUME, NUMMAT, ISW, NE, NPS
COMMON / SOLN / A(189,94), B(189), MNODE, IJK, NP, ND, ND2
DEFINE FILE 1 (939, 5000, L, IJK)
DEFINE FILE 2 (959, 5000, L, N)
ICRD = 8
LIST = 6
CHECK FOR START OF NEW PROBLEM OR TERMINATION OF PROGRAM
50 READ (ICRD, 1000, HED)
IPAGE = 1
C STORE PROBLEM INFORMATION
CALL DOCUNT
C FORM STIFFNESS MATRIX
CALL STIFF
C CALCULATE AND PRINT OUT DISPLACEMENTS
CALL SYMSON (NL, IBW)
CALL TITLE
WRITE (LIST, 2100)
NPLK = 0
M = 0
A0 = NPLK = NPLK+1
L = 0
70 N = N+1
   L = [L+1]
   I = NO + (2*N-((N*NLK-1)*NO))
   IF (L .LT. 50) GO TO 85
   CALL TITLE
   WRITE (LIST,2100)
35 WRITE (LIST,2200) N,X(N),Z(N),A(I-1,NL),A(I,NL)
   LINE = LINE + 1
   IF (N .LT. NUMNP) GO TO 80
   IF (L .GT. 70) 70,60,60
80 CONTINUE
C CALCULATE AND PRINT OUT STRESSES
   CALL STRESS
C START NEXT PROBLEM
   GO TO 50
100 FORMAT (100A1)
2100 FORMAT (14I4,9X,FNODE,7X,1H,NODE,11X,1HZ,14X,7HX-DISPL,13X,7HZ-DISPL)
2200 FORMAT (5X,15,2F12.3,2D20.8)
5000 STOP
END
SUBROUTINE DOCUMT
C SUBROUTINE FOR DOCUMENTATION OF PROBLEM
C INPUT AND OUTPUT ALL THE PROBLEM CHARACTERISTICS
C IMPLICIT REAL*8 (A-H,O-Z)
C COMMON / HEAD / ICRO, LIST, MOD(1L), IPAGE, LINE
C COMMON / DATUM / X(500),Z(500),UX(500),UZ(500),AS(500),CS(500),
C I IC(8,10),ICODE(500),IX(100,13),NUMNP,NUMEL,NUMMAT,IBN,NE,NPS
C READ AND PRINT CONTROL INFORMATION AND MATERIAL PROPERTIES
C READ (ICRO,1001) NUMNP,NUMFL,NUMMAT,NPS,NSYM
C NUM = 2*NUMP
   CALL TITLE
WRITE (LIST,2001) NUMNP,NUMEL,NUMMAT,NE
IF (NPS .EQ. 0) WRITE(LIST,2011)
IF (NPS .EQ. 1) WRITE(LIST,2012)
DO 50 M=1,NUMMAT
READ (ICRD,1002) MTYPE,(E(J,MTYPE),J=1,8)
WRITE(LIST,2002)
IF (LINE .LT. 65) GO TO 40
CALL TITLE
WRITE(LIST,2002)
LINE = LINE + 1
50 WRITE (LIST,2003) MTYPE, (E(I,MTYPE),I=1,8)
C I/O NODAL POINT DATA
CALL TITLE
WRITE (LIST,2004)
L = 0
60 READ (ICRD,1003) N,ICODE(N),X(N),Z(N),UX(N),UZ(N)
IF (N .LT. 1) 65,69,65
CONTINUE
70 X(N-L) = X(N) - X(L)
80 CONTINUE
90 CONTINUE
100 N=L+1
110 IF (N .LT. 100) 120,90,80
120 IF(NSYM .EQ. 0) ICODE(L) = 0
IF(NSYM .EQ. 1) ICODE(L) = ICODE(L-1)
X(L) = X(L-1) + OX
Z(L) = Z(L-1) + OZ
UX(L) = 0.0
UZ(L) = 0.0
GO TO 70
90 GO TO 92 K=NL,N
   IF (LINE .LT. 50) GO TO 91
   CALL TITLE
   WRITE (LIST,2004)
91   LINE = LINE + 1
92   WRITE (LIST,2005) K,1000E(K),X(K),Z(K),UX(K),UZ(K)
   IF (NUMNP .NE. N) 180,113,60
100  WRITE (LIST,2006) N
   STOP
C I/O ELEMENT PROPERTIES
110  CALL TITLE
   WRITE (LIST,2007)
   N=0
120  READ (ICRP,1001) M,(IX(M,I),I=1,13)
   IF (M .NE. NUMEL) 140,140,132
130  WRITE (LIST,2003) M
   STOP
140  N=N+1
   IF(N-M) 140,160,150
150  GO TO 151 I=1,8
151  IX(N,1) = IX(N-1,I) + 3
   GO TO 152 I=9,12
152  IX(N,1) = IX(N-1,I) + 2
   IX(N,13) = IX(N-1,13)
160  I = IX(N,1)
   L = IX(N,3)
   AS(N) = 0ABS(X(L) - X(I))
   CS(N) = 0ABS(Z(L) - Z(I))
   IF (AS(N)) 168,167,168
167  AC = CS(N)
   GO TO 169
168  AC = CS(N)/AS(N)
169 IF (LINE .LT. 50 ) GO TO 170
    CALL TITLE
    WRITE (LIST,2007)
170 LINE = LINE +1
    WRITE (LIST,2010) N,(IX(N,I),I=1,13),AC,AS(N),CS(N)
    IF (N - N) 180,180,140
140 IF (NUMEL-N) 190,190,130
C DETERMINE THE HALF-BAND WIDTH FOR PROBLEMS, MAX. 94 (8 DIVISIONS, 8X34)
130 J=0
   DO 340 N=1,NUMEL
      DO 340 I=1,11
           K = I + 1
           DO 340 L=K,12
           KK = FABS (IX(N,I)-IX(N,L))
           IF (KK-J) 240,340,320
320 J=KK
   340 CONTINUE
C IBW = HALF-BAND-WIDTH OF PROBLEM IN QUESTION
IBW = 2*J + 2
RETURN
1801 FORMAT (1415)
1802 FORMAT (15,6F8.0)
1803 FORMAT (215,4F10.2,215)
2001 FORMAT (IH,10X,28HNUMBER OF NOdal POINTS.........15/
   1 10X,28HNUMBER OF ELEMENTS............15/
   2 10X,28HNUMBER OF DIFF. MATERIALS......15/
   3 10X,28HNUMBER OF TOTAL OF DOf........15)
2002 FORMAT (1HC,5X,9HMATERIAL NO.,5X,3HE11,10X,3HE22,10X,3HE33,9X,4HNU12,  
1 9X,4HNU33,9X,4HNU32,10X,4HE13,7X,9THICKNESS)
2003 FORMAT (12X,13,3X,8D13.5)
2004 FORMAT (1HC,5X,17HNOdal POINT TYPE,5X,12X-COORDINATE,5X,12HZ-C  
10ORDINATE,5X,15DX-(LOAD/CISPL.),5X,15HZ-(LOAD/CISPL.))
2005 FORMAT (7X,211D,2F17.6,2D20.9)
2006 FORMAT (26HMONOCL. POINT CARD ERROR M=#)
2008 FORMAT (16HMONOCL. ERROR: M=#)
2009 FORMAT (37HMONOCL. ERROR: M=#)
2010 FORMAT (11X,14,3X,12I5,18,F13.3,2F16.3)
2011 FORMAT (//10X,'PLANE STRESS: PROBLEM')
2012 FORMAT (//10X,'PLANE STRAIN PROBLEM')

END

SUBROUTINE TITLE

C SUBROUTINE FOR TITLE
IMPLICIT REAL*8 (A-H,O-Z)
COMMON / HEAD / ICRD,LIST,HED(1C),IPAGE,LIN
WRITE (LIST,100) IPAGE
WRITE (LIST,101) HED
IPAGE= IPAGE+1
LIN = 0
RETURN

100 FORMAT (11H,'FEM 24-DOF GENERAL QUADRILATERAL CURVE ELEMENTS', 1X,LIN, 1X,'PAGE',13)

101 FORMAT (11H,'TOTAL STIFFNESS MATRIX')

C SUBROUTINE FOR FORMULATION OF TOTAL STIFFNESS MATRIX
IMPLICIT REAL*8 (A-H,O-Z)
COMMON / HEAD / ICRD,LIST,HED(1C),IPAGE,LIN
COMMON / DATUM / X(500),Z(500),UX(500),UZ(500),AS(100),CS(100),
& E(1:1),ICODE(500),IX(100),13,NUMNP,NUMEL,NUMMAT,ISW,NE,NPS
COMMON / SOLN / A(126,94),D(128),MNURLK,JK,NE,ND
COMMON / SMALL / S(24,24),BC(36,24),ETG(24,3),BA(3,24),C(2,12),
& XZ(19,2),D(3,3),D(2,2),D(2,2)
NB = 47
ND = 2*NB
ND2 = 2*ND
NOLK=9

C INITIALIZATION OF MATRICES
DO 10 N=1,ND2
R(N)= 0.00
DO 10 M=1,ND
10 A(N,M) = 0.00

C CALCULATE MAX. NO. OF BLOCKS, MNUBLK
MNUBLK = ME/NO + 1
WRITE (LIST,2000) IBW,MNUBLK

C FORM THE TOTAL STIFFNESS MATRIX IN BLOCKS
60 NBLK=NBLK+1
NH=NP*(NBLK+1)
NM= NH-NB
NL=NM-NB+1
IBS = 2*NJ-2

C FORM ELEMENT STIFFNESS MATRIX BY NUMERICAL INTEGRATION
DO 210 N=1,NUMEL
IF (IX(N,13) .LT. 1) GO TO 210
C DETERMINE IF THIS ELEMENT AFFECTS THIS BLOCK OF TOTAL STIFFNESS MATRIX
DO 80 I=1,12
IF (IX(N,1) .GE. NL .AND. IX(N,1) .LE. NM) GO TO 90
CONTINUE
GO TO 210
80 IF (M .EQ. 1) GO TO 100
QA = DABS(AS(N)/MR - 1.00)
QC = DABS(CS(N)/MR - 1.00)
IF (IX(N,13) .EQ. MR) GO TO 110
CALL FLAS(N)
IF (N .EQ. 1) GO TO 120
100 CALL FLSM(N)
110 CALL FLAS(N)
120 IF (N .EQ. 1) GO TO 120
110 IF(QA .LT. .5D-9 .AND. QC .LT. .5D-9 .AND. IX(N,13).EQ.MR) GO TO 130
120 CALL ELSTIF(N)
   MR = IX(N,13)
130 AR = AS(N)
   CR = CS(N)
   IF (AR .EQ. 0.0) AR = CS(N)
   IF (CR .EQ. 0.0) CR = AS(N)
   IX(N,13) = -IX(N,13)
C ADD ELEMENT STIFFNESS TO TOTAL STIFFNESS
DO 200 I=1,12
   DO 200 K=1,2
      II = 2*IX(N,K) - 2 + K - 185
      KK = K + 2*(I - 1)
   DO 200 J=1,12
      DO 203 L=1,2
         JJ = 2*IX(N,J) - 2 + L - II + 1 - 185
         LL = L + 2*(J - 1)
      DO 203 J=1,12
      200 CONTINUE
   201 CONTINUE
C ADD CONCENTRATED FORCES AT NODES
DO 250 N=1,NUMNP
   K = 2*N - 185
   TICODE = TICODE(N) + 1
   IF (K .LT. 1 .OR. K .GT. ND) GO TO 250
   GO TO (251,252,253,250), TICODE
   251 P(K) = P(K) + UZ(N)
   252 R(K-1) = R(K-1) + UX(N)
   253 R(K-1) = R(K-1) + UX(N)
GO TO 250
250 B(K) = A(K) + UZ(M)
250 CONTINUE
C APPLY DISPLACEMENT 3. C. S
DO 400 M = 1, NUMNP
    ICODE = ICODE(M) + 1
    N = 2*M - 1 - IS
    IF ( N .LT. 1 .OR. N .GT. ND2 ) GO TO 400
    U = UX(M)
    GO TO (410, 115, 132, 130), ICODE
315 CALL MODIFY (N, U, IBW, ND2)
    GO TO 400
330 CALL MODIFY (N, U, IBW, ND2)
359 N = N + 1
    U = UZ(M)
    CALL MODIFY (N, U, IBW, ND2)
400 CONTINUE
C WRITE BLOCK OF EQUATIONS ON DISK AND SHIFT UP LOWER BLOCK
   IJK = (NPLR+15) - 14
   WRITE (14, IJK) ((A(I,J), I=1, ND), J=1, IBW), (B(K), K=1, ND)
DO 420 N = 1, ND
    K = N + 1
    S(N) = S(K)
    S(K) = S(N)
    DO 420 M = 1, ND
    A(N,M) = A(K,M)
420 A(K,M) = 0.00
C CHECK FOR LAST BLOCK
IF ( NM .LE. NUMNP ) GO TO 60
DO 450 I = 1, NUMH
450 IX(I,13) = -IX(I,13)
RETURN
2000 FORMAT (1HC, 9X, 17H HALF RANK WIDTH =, 13, 15H, NO. BLOCKS =, 13)
2004 FORMAT (15X, 46H RANK WIDTH EXCEEDS THAT CALCULATED FOR PROBLEM 915)
END

SUBROUTINE ELASIN

C CALCULATE ELASTIC STIFFNESS MATRIX, DEPENDING ON MATERIAL PROPERTIES

IMPLICIT REAL*8 (A-H, O-Z)

COMMON / DATUM / X(500), Z(500), UX(500), UZ(500), AS(100), CS(100),
1 E(8, 10), ICODE(500), IX(100, 13), NUMNR, NUMFL, NUMMAT, 18M, NE, NPS

COMMON / SMALL / S(24, 24), RDH(36, 24), STD(24, 3), CA(3, 24), C(2, 12),
1 XL(12, 2), D(3, 3), OJ(2, 2), OJ1(2, 2)

DO 10 I = 1, 3
DO 10 J = 1, 3
10 D(I, J) = 0.00
L = IX(N, 13)
E11 = E(1, L)
E22 = E(2, L)
E33 = E(3, L)
XNU12 = E(4, L)
XNU13 = E(5, L)
XNU23 = E(6, L)
G13 = E(7, L)
TH = E(8, L)
XNU21 = XNU12*E22/E11
XNU31 = XNU13*E33/E11
XNU32 = XNU23*E33/E22
IF (NPS = 1) G0, 61, 61

C PLANE STRESS CASE

50 FACT = E11/((1.00 - XNU13*XNU31) * TH)
D(1, 1) = FACT
D(2, 1) = FACT*E32/E11
D(1, 2) = FACT*XNU31
D(2, 2) = D(1, 2)
C FORM ELEMENT STIFFNESS MATRIX \( (2 \times 2)^4 \) = HEAT TRANSFER BY GAUSSIAN

C GAUSSIAN FORMULAS FOR EACH GAUSS POINT OF 4x4 RULE

C 3 x 3 x 3 x 3 X 1

C CONTINUE

C END
W(4) = W(1)
DO 15 I=1, 4
DO 15 J=1, 4
M = J + 4*(I - 1)
15 K(M) = W(I) + W(J)
C FORM NORMAL POINT COORD. MATRIX XZ FOR J(2X2) = C(2X12)*XZ(12X2)
DO 20 I=1, 12
L = I*IN(I)
XZ(I, 1) = X(L)
20 XZ(I, 2) = X(L)
DO 25 K=1, 24
DO 25 L=1, 24
25 S(K, L) = C * 0.6
C CAL. JACOBY, J = G.P. I=1-4, AND FORM INV. OF IT FOR COORD. TRANSFORMAT.
DO 100 I=1, 4
DO 100 J=1, 4
M = 1 + 4*(J - 1)
C(I, J) = (1.0 - ZTA(I))* (81.00*XSJ(J) + 27.00*XSJ(J)**2 - 9.00*ZTA(I)**2
1 1 + 1.00)
C(I, 2) = (1.0 - ZTA(I))* (81.00*XSJ(J) + 27.00*XSJ(J)**2 - 9.00*ZTA(I)**2
1 1 + 1.00)
C(I, 3) = (1.0 - ZTA(I))* (27.00*XSJ(J) + 9.00*XSJ(J)**2 - 21.00*ZTA(I)**2
1 1 - 6.00)
C(I, 4) = (1.0 - ZTA(I))* (18.00*XSJ(J) + 27.00*XSJ(J)**2 + 9.00*ZTA(I)**2
1 1 - 6.00)
C(I, 5) = (1.0 - ZTA(I))* (18.00*XSJ(J) + 27.00*XSJ(J)**2 - 9.00*ZTA(I)**2
1 1 + 1.00)
C(I, 6) = (1.0 - ZTA(I))* (81.00*XSJ(J) + 27.00*XSJ(J)**2 - 9.00*ZTA(I)**2
1 1 + 1.00)
C(I, 7) = (1.0 - ZTA(I))* (27.00*XSJ(J) + 9.00*XSJ(J)**2 - 21.00*ZTA(I)**2
1 1 - 6.00)
C(I, 8) = (1.0 - ZTA(I))* (18.00*XSJ(J) + 27.00*XSJ(J)**2 + 9.00*ZTA(I)**2
1 1 - 6.00)
C(I, 9) = (1.0 - 3.00*ZTA(I))* (9.00*ZTA(I)**2 - 21.00*ZTA(I)**2
1 1 + 1.00)
C(I, 10) = (1.0 - 3.00*ZTA(I))* (9.00*ZTA(I)**2 - 21.00*ZTA(I)**2
1 1 + 1.00)
C(I, 11) = -C(I, 9)
\( C(1, 1, 2) = -C(1, 1, 0) \)
\( C(2, 1, 1) = (-1.00 \times SSI(J) \times (18.00 \times ZTA(1) - 27.00 \times ZTA(1)) \times 2 - 9.00 \times SSI(J) \times 2 \)
\( + 12.00) \)
\( C(2, 2, 1) = (1.00 \times 3.00 \times SSI(J)) \times (9.00 \times SSI(J)) \times 2 - 9.00) \)
\( C(2, 3) = (1.00 \times 3.00 \times SSI(J)) \times (9.00 \times SSI(J)) \times 2 - 9.00) \)
\( C(2, 4) = (1.00 \times SSI(J)) \times (18.00 \times ZTA(1) - 27.00 \times ZTA(1)) \times 2 - 9.00 \times SSI(J) \times 2 \)
\( + 10.00) \)
\( C(2, 5) = (1.00 \times SSI(J)) \times (18.00 \times ZTA(1) + 27.00 \times ZTA(1)) \times 2 + 9.00 \times SSI(J) \times 2 \)
\( + 10.00) \)
\( C(2, 6) = -C(2, 2) \)
\( C(2, 7) = -C(2, 3) \)
\( C(2, 8) = (1.00 \times SSI(J)) \times (18.00 \times ZTA(1) + 27.00 \times ZTA(1)) \times 2 + 9.00 \times SSI(J) \times 2 \)
\( + 10.00) \)
\( C(2, 9) = (1.00 \times SSI(J)) \times (9.00 \times ZTA(1)) \times 2 - 18.00 \times ZTA(1) - 27.00 \)
\( C(2, 10) = (1.00 \times SSI(J)) \times (27.00 \times 81.00 \times ZTA(1)) \times 2 - 18.00 \times ZTA(1) \)
\( C(2, 11) = (1.00 \times SSI(J)) \times (81.00 \times ZTA(1)) \times 2 - 18.00 \times ZTA(1) - 27.00 \)
\( C(2, 12) = (1.00 \times SSI(J)) \times (27.00 \times 81.00 \times ZTA(1)) \times 2 - 18.00 \times ZTA(1) \)

DO 30 I1 = 1, 2
DO 30 K1 = 1, 2
DO 30 JJ = 1, 12
DO 30 KK = 1, 2
DO 30 JJ = 1, 12
DE TJ = C(J1, J1, KK) + C(J1, J1, KK) \times XZ(JJ, KK) / 32.00
DE TJ = C(J1, J1) \times C(J1, J2) - D(J1, 2) \times D(J1, 2)
D(J1, 1, 1) = D(J1, 2, 1) / DE TJ
D(J1, 1, 2) = -D(J1, 2, 1) / DE TJ
D(J1, 2, 1) = -D(J1, 1, 2) / DE TJ
D(J1, 2, 2) = D(J1, 1, 1) / DE TJ

C FORM MATRIX \( A(3Y24) \), WHERE \( (B) = (BA) \)
DO 40 KK = 1, 3
DO 40 K1 = 1, 24
M(K1, K) = 0.00
DO 50 K1 = 1, 3, 2

L = (K - 1)/2 + 1
S4(1, K) = 0.0J(1,1)*C(1,L) + 0.0J(1,2)*C(2,L)) / 32.0
50 S4(2, K) = 0.0J(2,1)*C(2,L) + 0.0J(2,2)*C(2,L)) / 32.0
DO 60 K = 2, 24, 2
L = (K - 2)/2 + 1
PA(2, K) = 0.0J(2,1)*C(1,L) + 0.0J(2,2)*C(2,L)) / 32.0
60 S4(3, K) = 0.0J(1,1)*C(1,L) + 0.0J(1,2)*C(2,L)) / 32.0
DO 80 K = 1, 24
DO 80 L = 1, 3
BTD(K, L) = 0.0
80 DO KK = 1, 3
BTB(K, L) = BTB(K, L) + S4(KK, K) * B(KK, L)
DO 90 K = 1, 24
DO 90 L = 1, 24
BBD(K, L) = 0.0
90 DO KK = 1, 3
90 BBD(K, L) = BBD(K, L) + BTB(KK, K) * S4(KK, L)
DO 100 K = 1, 24
DO 100 L = 1, 24
BBD(K, L) = BBD(K, L) + H(M) * DETJ * BBD(K, L)
WRITE (2, N) ((S(I, J), I=1, 24), J=1, 24)
RETURN
END
SUBROUTINE MODIFY (N, U, UBW, ML)
C MODIFY TOTAL STIFFNESS MATRIX FOR GIVEN DISPLACEMENT S.C.'S
IMPLICIT REAL*8 (A, B, U)
COMMON / SCLN / A(188, 34), B(188), MNUBLK, [JK, NP, NO, ND]
DO 200 M = 1, NW
K = N - M + 1
IF (K) 235, 235, 230
235 S(K) = S(K) - A(K, M) * U
A(K, M) = 0.0
230 RETURN
END
235  N=N+1
      IF(ML -K)  250,240,240
240  H(K)= H(K) - A(N,N)*U
      A(N,N) = U
250  CONTINUE
      A(N,1) = 1.0CC
      C(N) = 0
      RETURN
      END

SUBROUTINE SYMSOL (NN,IPK)
C SOLVE THE SYSTEM OF CANDF, BLOCKED SIMULTANEOUS EQUATIONS BY GAUSS EL
IMPLICIT REAL*8 (A-H,O-Z)
COMMON / SOLN / A(188,74),B(188),MNULK,IK,JL,KL,NL,NN
      NL = NN+1
      NH = NN+NN
      MNULK = 0
      IJK = 1
      GO TO 150
C REDUCE EQUATIONS BY BLOCKS; AND THEN SHIFT BLOCK OF EQUATIONS
100  MNULK = MNULK + 1
      DO 125 M=1,NM
      NM = NN+NM
      S(N) = P(NM)
      S(NM) = 0.0DO
      DO 125 M=1,18W
      A(N,M) = A(NM,M)
125  A(NM,M) = 0.0DO
C READ IN NEXT BLOCK OF EQUATIONS
      IF (MNULK .EQ. MNULK - 1) GO TO 200
      IJK = (MNULK+15) + 1
150  READ (1,IJK) ((A(i,J),I=NL,NH),J=1,18W),(3(K),K=NL,NH)
      IF (MNULK .EQ. 0) GO TO 150
C REDUCE BLOCK OF EQUATIONS
200  DO 300 N=1,NN
   IF ( A(N,1) .EQ. 0.000 ) GO TO 300
   B(N) = B(N) / A(N,1)
   DO 275 L=2,IBW
   IF ( A(N,L) .EQ. 0.000 ) GO TO 275
   C = A(N,L) / A(N,1)
   I=N+L-1
   J=0
   DO 255 K=J,IBW
      J=J+1
      IF ( A(I,J) .EQ. 1.000 .AND. B(I) .EQ. 0.000 ) GO TO 260
250  A(I,J) = A(I,J) - C*A(N,K)
      B(I) = B(I) - A(N,L)*B(N)
260  A(N,L) = C
275  CONTINUE
300  CONTINUE
C WRITE BLOCK OF EQUATIONS BACK ON DISK
   IF ( NLK .EQ. NNBLK ) GO TO 400
   IJK = ( NLK+15 ) / 14
   WRITE ( 11,11 ) (( A(I,J), I=1,NN), J=2,IBW), ( P(K), K=1,NN)
   GO TO 100
C BACK SUBSTITUTION: SOLVED UNKNOWN DISPLACEMENTS ARE STORED IN
C THE FIRST NLK columns of the 2ND BLOCK of matrix A
400  DD 450 K=1,NN
    N=NN+1-M
    DO 425 K=2,IBW
       L=N+K-1
420  B(N) = B(N) - A(N,K)*B(L)
    NM = N + NN
    P(NM) = B(N)
450  A(NM,NLK) = P(N)
NBLK = NBLK-1
IF (NBLK, EQ, 0) GO TO 500
IJK = (NBLK*15) - 14
READ (10, IJK) ((A(I,J), I=1,NN), J=2,ISW), (S(K), K=1,NN)
GO TO 400
500 RETURN
END

SUBROUTINE STRESS
C CALCULATE STRESSES SIG(3) AT 12 NODAL POINTS OF EACH ELEMENT
IMPLICIT REAL*8 (A-H,O-Z)
COMMON / HEAD / ICRO, LIST, NER(10), IMAGE, LINE
COMMON / DATUM / X(5,20), Z(5,20), UX(5,20), UZ(5,20), AS(100), CS(100),
1 SX(8,10), SXME(80), IX(100,13), NUMVR, NUMEL, NUMMAT, IBW, NE, NPS,
COMMON / SELM / A(188,94), B(188), MNDDLK, IJK, NE, ND, N2D
COMMON / SMALLK / S(24,24), 6DO(36,24), 6DO(24,3), 6A(3,24), C(2,12),
1 X7(12,2), D(3,3), CJ(2,2), DJI(2,2)
DIMENSION B(24), SIG(7)

CALL TITLE
WRITE (LIST, 2001)
STEPST = 0.0D0
DO 300 N=1,NUMEL
IF (N, EQ, 1) GO TO 6
QAR = BARS(AS(N)/AR - 1.0D0)
QC = BARS(CS(N)/CR - 1.0D0)
IF (IX(N,13), EQ, 999) GO TO 7
6 CALL ELAS(N)
IF (N, EQ, 1) GO TO 8
7 IF (GA , LT , .50-9 , AND , QC , LT , .50-9 , AND , IX(N,13), EQ, NR) GO TO 9
8 CALL ELSTRES(N)
9 CONTINUE
C PLACE PROPER NODAL DISPLACEMENTS IN U FROM A FOR EACH ELEMENT
GO TO 27 J=1,12
DG 30 K=1,2
I = 2*IX(N,J) - 2 + K
JJ = K + 2*(J - 1)
NLK = (I - 1)/MD + 1
II = ND + I - (NE*(NLK - 1))
ZJ U(JJ) = A(I1,NLK)
C SOLVE FOR STRESSES
DG 53 K=1,12
DG 40 I=1,7
30 SIG(I) = 0.O
DG 35 I=1,3
L = I + 3*(K - 1)
DG 35 J=1,24
35 SIG(I) = SIG(I) + 982(L,J)*U(J)
C CALCULATE PRINCIPAL STRESSES
AA = (SIG(1)+SIG(2))* 0.500
BB = (SIG(1)-SIG(2))* 0.500
CC = 0.500*(AA**2 + SIG(3)**2)
SIG(4) = AA+CC
SIG(5) = AA-CC
IF (SIG(1) - SIG(2)) 41,37,36
36 SIG(6) = (DATAN(SIG(3)/ BB))* 28.6478837500
GO TO 45
37 IF (SIG(3)) 40,39,38
38 SIG(6) = 45.00
GO TO 45
39 SIG(6) = 0.00
GO TO 45
40 SIG(6) = -45.00
GO TO 45
41 SIG(6) = (DATAN(SIG(3)/ BB))* 28.6478837500
IF (SIG(3)) 4+,43,42
42 SIG(6) = .9000 + SIG(6)
   GO TO 45
43 SIG(6) = .9000
   GO TO 45
44 SIG(6) = -.9000 + SIG(6)
45 CONTINUE
   SIG(7) = .9000 + AABS(SIG(4) - SIG(5))
   IF (LINE.LT.48) GO TO 46
   CALL TITLE
   WRITE (LIST,2001)
46 CONTINUE
   IF (K.EQ.1) GO TO 47
   WRITE (LIST,2001) IX(N,K), (SIG(I),I=1,7)
   GO TO 48
47 WRITE (LIST,2002) N, IX(N,K), (SIG(I),I=1,7)
48 CONTINUE
   LINE = LINE + 1
49 CONTINUE
   IF (N .LE. 1) GO TO 60
   IF (DA.LT. .50-9 .AND. QC .LT. .50-9 .AND. IX(N,13).EQ.MR160) GO TO 70
60 READ (*) N, (S(I,J), I=1,24), J=1,24
70 STEGEL = 0.00
   MK = IX(N,13)
   AR = AS(N)
   CA = CS(N)
   IF (AR .GT. 0.00) AR = CS(N)
   IF (CR .GT. 0.00) CR = AS(N)
   DO 200 I=1,24
      STIFFU = 1.00
      DO 100 JJ=1,24
         STIFFU = STIFFU + S(I,JJ)*U(JJ)
      100 CONTINUE
      STEGEL = STEGEL + STIFFU*H(I)/2.00
   200 CONTINUE
WRITE (LIST, 2003) STEGEL
2003 STEGELS = STEGEL + STEGEL
WRITE (LIST, 2004) STEGEL
RETURN
2004 FORMAT (1HC, 8X, SHELL NO., 6X, 4HMDOE, 9X, 8HX-STRESS, 6X, 8HZ-STRESS,
          15X, 9HXZ-STRESS, 5X, 10HMAX-STRESS, 4X, 10HMIN-STRESS, 4X, 5HANGLE,
          25X, 9HMAX-SHEAR)
2001 FORMAT (15X, 11, 4X, 5014, 4, 9, 2, 514, 4)
2002 FORMAT (1HC, 9X, 15, 11, 4X, 5014, 4, 9, 2, 514, 4)
2003 FORMAT (1HC, 9X, EL, STRAIN ENERGY = 1, 015, 8)
2004 FORMAT (1HC, 1ST. STRAIN ENERGY = 1, 015, 8)
END
SUBROUTINE ELSTRE(N)
C FORM ELEMENT NODAL STRESSES MATRIX B0(N, 72X24)
IMPLICIT REALS (A-H, O-Z)
COMMON / DATUM / X(500), Z(500), UY(500), OZ(500), AG(100), CS(100),
1 E(2, 1), ICODE(500), IX(100, 13), NUMNP, NUMEL, NUMMAT, IBW, NE, NPS
COMMON / SMALK / S(24, 24), BDF(3A, 24), BT(24, 24), RA(3, 24), C(2, 12),
1 XZ(12, 2), N(3, 3), OJ(2, 2), OJ(2, 2)
C FORM NODAL POINT COORD. MATRIX XI FOR J = C * XI = C * XI
DO 21 L = 1, 12
21 CONTINUE
C CALCULATE JACOBIAN J AT EACH NODAL POINT
DO 10 K = 1, 12
10 CONTINUE
GO TO 1
1 XSIJI = -1.00
ZTAJI = -1.00
GO TO 15
2 XSIJI = -1.00/3.00
GO TO 15
3  XSI2 = 1.00 / 3.00
   GO TO 15
4  XSI2 = 1.00
   GO TO 15
5  ZTAA = 1.00
   XSI2 = -1.00
   GO TO 15
6  XSI2 = -1.00 / 3.00
   GO TO 15
7  XSI2 = 1.00 / 3.00
   GO TO 15
8  XSI2 = 1.00
   GO TO 15
9  XSI2 = -1.00
   ZTAA = -1.00 / 3.00
   GO TO 15
10  ZTAA = 1.00 / 3.00
    GO TO 15
11  XSI2 = 1.00
    ZTAA = -1.00 / 3.00
    GO TO 15
12  ZTAA = 1.00 / 3.00
15  CONTINUE
   C1 (1,1) = (1.00 - ZTAA) * (18.00 * XSI2 - 27.00 * XSI1 ** 2 - 9.00 * ZTAA ** 2 + 10.00)
   C1 (1,2) = (1.00 - ZTAA) * (81.00 * XSI1 ** 2 - 18.00 * XSI1 - 27.00)
   C1 (1,3) = (1.00 - ZTAA) * (27.00 - 81.00 * XSI1 + 27.00 * XSI2)
   C1 (1,4) = (1.00 - ZTAA) * (18.00 * XSI1 + 27.00 * XSI2 * XSI1 ** 2 + 9.00 * ZTAA ** 2 - 10.00)
   C1 (1,5) = (1.00 + ZTAA) * (18.00 * XSI1 - 27.00 * XSI1 ** 2 - 9.00 * ZTAA ** 2 + 10.00)
   C1 (1,6) = (1.00 + ZTAA) * (91.00 * XSI1 ** 2 - 18.00 * XSI1 - 27.00)
   C1 (1,7) = (1.00 + ZTAA) * (27.00 - 81.00 * XSI1 + 27.00 * XSI2)
   C1 (1,8) = (1.00 + ZTAA) * (18.00 * XSI1 + 27.00 * XSI2 * XSI1 ** 2 + 9.00 * ZTAA ** 2 - 10.00)
   C1 (1,9) = (1.00 - 3.00 * ZTAA) * (4.00 * ZTAA ** 2 - 9.00)
C(I, J) = (I. DD + 3. DD*ZI*A) * (9. DD*ZI*A**2 - 9. DD)
C(I, J) = -C(1, 9)
C(I, J) = -C(1, 10)
C(2, J) = (I. DD - XSII) * (9. DD*XSII**2 - 9. DD)
C(2, J) = (I. DD - XSII) * (9. DD*XSII**2 - 9. DD)
C(2, J) = -C(2, 2)
C(2, J) = -C(2, 3)
C(2, J) = -C(2, 4)
C(2, J) = -C(2, 5)

00 30 IT=1. 2
00 30 KK=1. 2
00 30 JJ=1. 12
30 DJ(II, KK) = DJ(II, KK) + C(II, JJ) * X(II, JJ, KK) / 32. 00

C FORM INVERSE OF J, DJ(2X2) FOR COORDINATE TRANSFORMATION
DETi = DJ(1, 1) * D(2, 1) - DJ(1, 2) * D(2, 1) - DJ(1, 2) * D(2, 2)
DJ(1, 1) = DJ(1, 1) / DETJ
DJ(1, 2) = DJ(1, 2) / DETJ
DJ(1, 1) = DJ(1, 1) / DETJ
DJ(1, 2) = DJ(1, 2) / DETJ

C CALCULATE MA = O*3, 12 SETS OF 3X24 FOR EACH NODE OF THE ELEMENT
00 40 I=1. 3
00 40 L=1. 24
40 8A(I, L) = 0. 00
00 50 m=1. 23, 2
L = (M - 1)/2 + 1
BA(1, M) = (DJI(1, 1)*C(1, L) + DJI(1, 2)*C(2, L))/32.0D
50 BA(2, M) = (DJI(2, 1)*C(1, L) + DJI(2, 2)*C(2, L))/32.0D
DO 60 M=2, 24, 2
   L = (M - 2)/2 + 1
   BA(2, M) = (DJI(2, 1)*C(1, L) + DJI(2, 2)*C(2, L))/32.0D
60 DO 80 M=1, 3
   J = 3*K + M - 3
   DO 80 L=1, 24
      CBR(J, L) = 0.0D
   DO 80 K=1, 3
   DOJ(J, L) = CBR(J, L) + D(M, I)*BA(I, L)
100 CONTINUE
RETURN
END
APPENDIX E

72-DOF Program Listing
C FINITE ELEMENT ANALYSIS OF 3-DIMENSIONAL LAMINATED COMPOSITES
C FEM 72-DOF AT EACH NODE PROGRAM FOR GENERAL HEXAHEDRON

IMPLICIT REAL*(8) (A-H,O-Z)

COMMON / REAL / FEO(10), ICRO, LIST, IPAGE, LINE
COMMON / DATUM / X(100), Y(100), Z(100), UX(100), UY(100), UZ(100),
1 E(10), AS(10), BS(10), CS(10),
2 ICODE(100), IX(10,25), NUMNP, NUMEL, NUMMAT, IEB, NE

COMMON / SOLN / A(144,72), B(144), MNUSL, IJK, NS, NO, NO2

DEFINE FILE 1 (999,5000,L,IJK)
DEFINE FILE 2 (999,5000,L,N)
ICRD = 6
LIST= 6

C CHECK FOR START OF NEW PROBLEM OR TERMINATION OF PROGRAM
GO TO 100 (ICRD,100,END=9999) FEO
IPAGE = 1
CALL DOCUM

C FORM STIFFNESS MATRIX
CALL STIFF

C CALCULATE AND PRINT OUT DISPLACEMENTS
CALL SYMPSL (NS,1BW)
CALL TITLE
WRITE (LIST,2100)
NBLK = 0
N = 0
60 NBLK = NBLK+1
L = 0
70 N = N+1
L = L+1
I = NO + (3*N-((NBLK-1)*NO))
IF ( LINE .EQ. 50 ) GO TO 60
CALL TITLE
WRITE (LIST,2100)
8: WRITE (LIST,2) N,X(N),Y(N),Z(N),A(I-2,NBLK),A(I-1,NBLK),A(I,NBLK)
   LINE = LINE +1
   IF (N .EQ. NUMNP) GO TO 90
   IF (L-NR) 70,60,60
90 CONTINUE
C CALCULATE AND PRINT OUT STRESSES SIG(A)
C CALL STRESS
C START NEXT PROBLEM
GO TO 50
500 STOP
Z FORMAT (1X,N12,9X,3F12.3,20X,3)
900 FORMAT (1F8)
200 FORMAT (1HC,9X,4FNCDE,7X,1HX,11X,1HY,11X,1HZ,14X,7HX,DISPL,13X,
1 7HY,DISPL,13X,7HZ,DISPL)
END
SUBROUTINE DOCUMT
IMPLICIT HEAL*8 (A-H,O-Z)
C SUBROUTINE FOR DOCUMENTATION OF PROBLEM, I/O ALL CHARACTERISTICS
COMMON / HEAD / HEAD(10),ICRD,LIST,IPAGE,LIN
COMMON / DATUMS / X(100),Y(100),Z(100),UX(100),UY(100),UZ(100),
1 E(100),P(100),PS(100),CS(100),
2 ICODE(100),IX(10,25),NUMNP,NUMEL,NUMMAT,IBW,NE
C READ AND WRITE CONTROL INFORMATION AND MATERIAL PROPERTIES
READ (ICRD,100) NUMNP,NUMEL,NUMMAT
NE= 3*NUMNP
CALL TITLE
WRITE (LIST,2000) NUMNP,NUMEL,NUMMAT
DO 50 M=1,NUMMAT
READ (ICRD,10001) MTYPE,(E(I,MTYPE),I=1,10)
WRITE (LIST,20001)
IF (LINE .LT. 90) GO TO 40
CALL TITLE
01 WRITE (LIST, 2001)
02 LINE = LINE + 1
03 WRITE (LIST, 2002) MTYPE, (G(I, MTYPE), I=1, 16)
04 C I/O NODE POINT DATA
05 CALL TITLE
06 WRITE (LIST, 2003)
07 L = 0
08 READ (ICPD, 1002) N, IMODE(N), X(N), Y(N), Z(N), UX(N), UY(N), UZ(N)
09 IF (N-1) 10, 65, 65
10 XY = N - 1
11 DX = (X(N) - X(L))/XY
12 DY = (Y(N) - Y(L))/XY
13 CONTINUE
14 NL=NL+1
15 L=L+1
16 IF (N-L) 100, 90, 90
17 IMODE(L) = IMODE(L-1)
18 X(L) = X(L-1) + DX
19 Y(L) = Y(L-1) + DY
20 Z(L) = Z(L-1)
21 UX(L) = 0.00
22 UY(L) = 0.00
23 UZ(L) = 0.00
24 GO TO 70
25 DO 42 K=NL, N
26 IF (LINE .LT. 50) GO TO 91
27 CALL TITLE
28 WRITE (LIST, 2003)
29 LINE = LINE + 1
30 WRITE (LIST, 2004) K, IMODE(K), X(K), Y(K), Z(K), UX(K), UY(K), UZ(K)
31 IF (NUMNP = N) 100, 110, 60
32 WRITE (LIST, 2011) N
STOP
110 CONTINUE
C I/O - CLUSTER PROPERTIES
CALL TITLE
WRITE (LIST, 2005)
N = 0
120 READ (ICKD, 1003) N, (IX(M, I), I = 1, 25)
IF (IN - NUMEL) 140, 140, 130
130 WRITE (LIST, 2012) N
STOP
140 N = N + 1
IF (M - N) 150, 150, 150
150 DB 151 I = 1, 8
151 IX(M, I) = IX(N-1, I) + 1
DB 152 I = 1, 10
152 IX(M, I) = IX(N-1, I) + 2
DB 153 I = 1, 13
153 IX(M, I) = IX(N-1, I) + 3
IX(M, 25) = IX(N-1, 25)
160 CONTINUE
I = IX(N, 1)
L = IX(N, 24)
AS(N) = MAX(SX(L) - X(I))
BS(N) = MAX(SY(L) - Y(I))
CS(N) = MAX(SZ(L) - Z(I))
IF (AS(N) .EQ. .999 999 999 999) .OR. BS(N) .EQ. .999 999 999 999 .OR. CS(N) .EQ. .999 999 999 999) GO TO 168
GO TO 160
168 WRITE (LIST, 2013) N, AS(N), BS(N), CS(N)
STOP
169 CONTINUE
MC= CS(N)/AS(N)*2.00
MC= CS(N)/BS(N)*2.00
IF (LINE .LT. 50) GO TO 170
CALL TITLE
WRITE (LIST,205)
170 LINE = LINE + 1
WRITE (LIST,206) N, (IX(N,I), I=1,25), ACC, DEC, AS(N), BS(N), CS(N)
IF (N .NE. NUMEL) 160, 180, 140
180 IF (NUMEL = N) 140, 190, 120
190 CONTINUE
C DETERMINING THE HALF-BAND-WIDTH FOR THIS PROBLEM
J = 0
GO TO 340 N = 1, NUMEL
GO TO 340 I = 1, 25
K = I + 1
GO TO 340 L = N, 24
KK = JABS (IX(N,I) - IX(N,L))
IF (KK .LT. 1) 825, 325, 320
320 J = KK
325 CONTINUE
340 CONTINUE
C IBW = HALF-BAND-WIDTH OF PROBLEM IN QUESTION
IBW = 3.4J + 3
RETURN
1001 FORMAT (915)
1002 FORMAT (15,3F8.0,3F5.1,3F8.0,F4.0)
1003 FORMAT (1615,1615)
2001 FORMAT (9X,28X8HNUMBER OF KOGAL POINTS.............,13/
1 10X,28XNUMBER OF ELEMENTS..................,13/
2 10X,28XNUMBER OF DIFF. MATERIALS...........,13)
2002 FORMAT (1H5,9X,9HFORMAT L NO., 4X,3H511, 8X,3H522,8X,3H633,7X,4HNU12, 1
7X,4HNU13,7X,4HNU23,8X,3H612, 8X,3H513,8X,3H523,7X,5HANGLE)
2003 FORMAT (1X,13,9X,1E11.4)
2003 FORMAT (16I1, 17HANGULAR POINT TYPE, 5X, 12FX-CMOCORDINATE, 5X, 12FY-CMORDI
1NATE, 5X, 12HZ-CMORDINATE, 5X, 16FX-LOAD/DISP., 5X, 15FY-LOAD/DISP.
2L., 9X, 13PZ-LOAD/CISPL.)
2004 FORMAT (18, 11I1, 31F17.9, 5D20.9)
2005 FORMAT (14H1, 'ELE. I J K L M N O P Q R S T',
1 2K', 'I J J K K L L W W NN QQ PP QQ FR SS TT M A L. ',
2 2X, '30/5 A B C D E')
2006 FORMAT (26, I4, 2X, 5F5.1)
2011 FORMAT (26HANGULAR POINT CARD ERROR N=, I5)
2012 FORMAT (18H-ELEMENT ERROR M=, J5)
2013 FORMAT (37HANGULAR COORD. ERROR = SIDE OF ELEMENT, I3, 5X, 3F12.4)
END
SUBROUTINE TITLE
IMPLICIT REAL*8 (A-H, O-Z)
COMMON / HANG / HANG, LIST, IPAGE, LINE
WRITE (LIST, 100) IPAGE
WRITE (LIST, 100) IPAGE
IPAGE = IPAGE + 1
LINE = 0
RETURN
100 FORMAT (14H, 'FEM 72-DF GENERAL HEXAHEDRONS, LIA', 13X, 'PAGE', 13)
101 FORMAT (14H, 'CASE')
END
SUBROUTINE STIFF
C FORM THE TOTAL STIFFNESS MATRIX IN BLOCKS
IMPLICIT REAL*8 (A-H, O-Z)
COMMON / DATA / X(100), Y(100), Z(100), UX(100), UY(100), UZ(100),
1 E(10, 1), AS(10), B(10), CS(10),
2 ICODE(100), IX(10, 25), NUMNP, NUMEL, NUMMAT, IHW, NE
COMMON / SCLEN / A(144, 72), B(144), MINUS, JJK, N3, N3, ND2
COMMON / ESMALL / S(72, 72), B4A(72, 72), BSTC(72, 6), 6A(6, 72), 0(3, 24),
1 XYZ(94, 3), U(6, 6), B(3, 3), B(3, 3)
C INITIALIZATION OF MATRICES
WA = 24
NL = 3*NM
KD2 = 2*NL
NBLK = 0
DO 10 N = 1, NM2
3(N) = 0.00
GO TO 10
10 A(N,N) = 0.00
C CALCULATE MAX. NO. OF BLOCKS (MNUBLK), AND FORM BIG K IN BLOCKS
MNUBLK = NF/NL + 1
WRITE (6,1000) 15, MNUBLK
10 NBLK = NBLK + 1
NH = NM - (NBLK + 1)
NG = NH - N2
NL = NM - NH + 1
BS = 3*NL - 3
C FORM ELEMENT STIFFNESS MATRIX BY GAUSSIAN QUADRATURE FORMULAE
DO 210 NR = 1, NUMEL
IF (IX(NM,25) .LT. 1) GO TO 210
C DETERMINE IF THIS ELEMENT AFFECTS THIS BLOCK OF BIG K
DO 200 I = 1, 2
IF (IX(NM,1) .GE. NL .AND. IX(NM,1) .LT. NG) GO TO 95
200 CONTINUE
GO TO 210
95 IF (NM .EQ. 1) GO TO 100
IF (IX(NM,25) .EQ. MK) GO TO 110
100 CALL CALAS(NM)
110 IF (NM .EQ. 1) GO TO 120
CA = CALAS(NM)/AR = 1.00
CB = CALAS1BS(NM)/ER = 1.00
CG = CALAS(CS(NM))/GR = 1.00
IF (CA .LT. 0.5D-5 .AND. CR .LT. 0.5D-5 .AND. GC .LT. 0.5D-5
     .AND. IX(NM,25) .EQ. MK) GO TO 150
120 CALL ELSTIF(NM)
   AR = AS(NM)
   BR = KS(NM)
   CR = CS(NM)
   IF (AR .LE. 1.00) AR = BS(NM)
   IF (BR .LE. 1.00) BR = CS(NM)
   IF (CR .LE. 1.00) CR = AS(NM)
   MK = IX(NM,25)
150 CONTINUE
   IX(NM,25) = -1*IX(NM,25)
C ADD ELEMENT STIFFNESS TO TOTAL STIFFNESS
DO 200 I=1,24
DO 200 K=1,3
   II = 3*IX(NK,1) - 3 + K - IBS
   KK = K + 3*(I - 1)
   DO 200 J=1,24
   DO 200 L=1,3
      JJ = 3*IX(NJ,J) - 3 + L - II + 1 - IBS
      LL = L + 3*(J - 1)
      IF (JJ) 200,200,170
200 CONTINUE
IF (IBK-JJ) 180,190,190
140 WRITE (LIST,2000) NM
STOP
190 A(IK,JJ) = A(IK,JJ) + S(KK,LL)
200 CONTINUE
210 CONTINUE
C ADD CONCENTRATED FORCES AT NODES
DO 250 K=1,NUMNP
   K = 3*NM - IBS
IF ( K \lt \text{.LT.} 1 \text{.OR.} K \gt \text{.GT.} \text{.NR}) \text{GO TO} 260
LDO = LDO0 + 1
\text{GO TO} (241, 242, 243, 244, 245, 246, 247, 250), \text{LDO}
241 A(K-2) = A(K-2) + UX(N)
242 A(K-1) = A(K-1) + UY(N)
243 A(K) = A(K) + UZ(N)
\text{GO TO} 250
244 B(K-1) = B(K-1) + UY(N)
\text{GO TO} 250
245 B(K) = B(K) + UZ(N)
\text{GO TO} 247
246 B(K-1) = B(K-1) + UY(N)
247 B(K-2) = B(K-2) + UX(N)
\text{CONTINUE}
\text{APPLY DISPLACEMENT S.C.'S}
\text{DO 400} \text{M=1,NUMNP}
\text{LDO} = \text{LDO0} + 1
390 \text{M} = 3 + \text{M} - 2 + 168
\text{IF} (\text{N \lt \text{.LT.} 1 \text{.OR.} N \gt \text{.GT.} \text{.NC2}}) \text{GO TO} 400
398 \text{UX} = \text{UX(N)}
\text{GO TO} (400, 402, 403, 404, 405, 406, 407, 408), \text{LDO}
402 \text{CALL MODIFY (N, U, I8W, NE)}
\text{GO TO} 400
403 \text{CALL MODIFY (N, U, I8W, NE)}
405 \text{N} = \text{N} + 1
397 \text{U} = \text{UY(K)}
\text{CALL MODIFY (N, U, I8W, NE)}
\text{GO TO} 400
404 \text{CALL MODIFY (N, U, I8W, NE)}
406 \text{N} = \text{N} + 2
396 \text{UZ} = \text{UZ(N)}
\text{CALL MODIFY (N, U, I8W, NE)}
GO TO 460
460 CALL MODIFY (N,U,15W,NE)
407 N = N + 1
U = UY(K)
CALL MODIFY (N,U,15W,NE)
N = N + 1
U = UZ(K)
CALL MODIFY (N,U,15W,NE)
420 CONTINUE

C WRITE BLOCK OF CDC'S ON DISK AND SHIFT UP LOWER BLOCK
IJK = (KBLX=9) - 8
WRITE (11,1J) ((A(I,J),I=1,NL),J=1,15W),((I,J),J=1,15W),((K),K=1,NO)
DO 420 N=1,NO
K=N+NO
I(1)=N
A(K)=C:0C
DO 420 M=1,NL
A(N,M)=A(K,M)
420 A(K,M)=C:0C

C CHECK FOR LAST BLOCK
IF (NG .LT. NUMAP) GO TO 60
DO 450 I=1,NUMEL
450 IX(1,25) = -IZ(1,25)
RETURN

1000 FORMAT (11H0,17F7.0,17H0,15W,NO. BLOCKS =,15)
2000 FORMAT (10X,17F7.0,17H0,NO. BLOCKS EXCEEDS THAT CALCULATED FOR PROB. N=,15)
END

SUBROUTINE ELAS(NM)
C FORM ELASTIC STIFFNESS MATRIX U, DEPENDS ON MAT.L PROPERTY & TYPE
IMPLICIT REALS (A-H,O-Z)
COMMON / ETON / X(10),Y(10),Z(10),UX(100),UY(100),UZ(100),
L,E(10),10),CS(10),BS(10),OS(10),CS(10),
2 ICODE(100), IX(10, 25), NUNNP, NUMEL, NUMMAT, INNP, NE
C OMMON / SMARK / S(72, 72), L(72, 72), PTSD(72, 6), A(6, 72), C(3, 24),
1 X(24, 3), O(6, 6), DJ(3, 31), DJI(3, 3)
DIMENSION T(6, 6), TD(6, 6)
CO 10 I = 1, 6
CO 10 J = 1, 6
T(I, J) = 1.00
TD(I, J) = 1.00
10 C(I, J) = 1.00
L = IX(NM, 25)
XNU21 = E(4, L) * E(2, L) / E(1, L)
XNU31 = E(4, L) * E(3, L) / E(1, L)
XNU32 = E(6, L) * E(3, L) / E(2, L)
FACT = 1.00 - E(4, L) * (XNU21 + E(6, L) * XNU31) - E(5, L) * (XNU31 + XNU32 * XNU21) -
1
E(6, L) * XNU32
U(I, 1) = E(I, L) * (1.00 - E(6, L) * XNU32) / FACT
U(1, I) = E(1, L) * (1.00 - E(5, L) * XNU31) / FACT
U(2, I) = E(2, L) * (1.00 - E(5, L) * XNU31) / FACT
U(3, I) = E(3, L) * (1.00 - E(5, L) * XNU21) / FACT
U(4, I) = E(4, L) / FACT
U(5, I) = E(5, L) / FACT
U(6, I) = E(6, L) / FACT
THETA = E(I, L) * 3.14159265359793260 / 180.00
C(THETA) = COS(THETA)
IF(THETA LT .5E-14) GO TO 50
T(1, 1) = .3COS(THETA) * 2
T(1, 2) = .5SIN(THETA) * 2
T(4,1) = C*COS(THETA)*D*SIN(THETA)
T(1,4) = -2*C*T(4,1)
T(2,1) = T(1,2)
T(2,2) = T(1,1)
T(2,4) = -T(1,4)
T(3,3) = 1*C
T(4,2) = -T(4,1)
T(4,4) = T(1,1) - T(1,2)
T(5,5) = C*COS(THETA)
T(5,6) = D*SIN(THETA)
T(6,5) = -T(5,6)
T(6,6) = T(5,5)
DO 20 I=1,6
DO 20 J=1,6
20 T(I,J) = T(I,J) + T(I,K)*D(K,J)
T(1,4) = -T(1,4)
T(2,4) = -T(2,4)
T(4,1) = -T(4,1)
T(4,2) = -T(4,2)
T(5,6) = -T(5,6)
T(6,5) = -T(6,5)
DO 30 I=1,6
DO 30 J=1,6
DO 30 K=1,6
30 D(I,J) = D(I,J) + T(I,K)*T(K,J)
50 CONTINUE
RETURN
END
SUBROUTINE ELSTIF(NM)
C FORM ELEMENT STIFFNESS MATRIX BY NUMERICAL INTEGRATION WITH GAUSS QUAD
C METHOD FORMULA: S(72X72) = 16*(9T+45)*D(I,J) AT EACH G.P., 4X4X2 RULE
IMPLICIT REAL*8 (A-H,O-Z)
COMMON / CATUR / X(100),Y(100),Z(100),UX(100),UY(100),UZ(100),
1  I(10),J(10),K(10),L(10),M(10),N(10),O(10),
2  ICCOE(100),1X(100),Z(100),NUMNF,NUMEL,NUMMAT,100,NE
COMMON / SMAXK / S(T2,72),B(E(72,72),T(72,6),P(9,72),C(9,24),
1  XYZ(24,3),0(6,6),J(3,3),LJ(3,3)
DIMENSION H(32),XSI(4),ETA(4),ZTA(2),W(4),WW(2)
C define GAUSS points for general hexahedron in Num. Int. with 4x4x2 rule
XSI(4) =-.16113631159405300
XSI(3) =-.33998124355485600
XSI(2) =-.51735626618962600
XSI(1) =-.66524615486254600
ZTA(2) =-.51735626618962600
ZTA(1) =-.66524615486254600
DO 10 1=1,4
10 ETA(1) = XSI(I)
C form weighting coefficients H(32) at gauss point
W(1) =.34735444513745400
W(2) =.65264555482546000
W(3) = W(2)
W(4) = W(1)
WW(I) = 1.0
WW(2) = 1.0
DO 15 I=1,4
DO 15 J=1,4
DO 15 K=1,2
M = K + 2*(J - 1) + 8*(I - 1)
15 H(M) = W(I)*W(J)*W(K)
C form nodal pt. coord. matrix XYZ and C for J(3x3) = J(3x24)*XYZ(24x3)
DO 25 I=1,24
L = I*(M,1)
XYZ(I,1) = X(L)
XYA(1,2) = Y(1)
XYA(1,3) = Z(L)
DC 25 N=1,72
CO 25 L=1,72
25 S(N,L) = N:0
DC 100 K=1,4
CO 100 J=1,4
CO 100 I=1,2
M = 1 + 2*(J - 1) + 8*(K - 1)
C(1,1) = (1.00 - ETA(J))*1.00 - ZTA(I)))*(13.00 + 19.00*XS1(K) - 1
     27.00*XS1(K)**2 - 9.00*ETA(J)**2)
C(1,2) = (1.00 - ETA(J))*1.00 - ZTA(I)))*(31.00*XS1(K)**2
     -18.00*XS1(K) - 27.00)
C(1,3) = (1.00 - ETA(J))*1.00 - ZTA(I)))*(27.00 - 18.00*XS1(K)
     -18.00*XS1(K)**2)
C(1,4) = (1.00 - ETA(J))*1.00 - ZTA(I)))*(27.00*XS1(K)**2 + 9.00*ETA(J)**2 + 18.00*XS1(K) - 11.00)
C(1,5) = (1.00 - ETA(J))*1.00 - ZTA(I)))*(15.00 + 18.00*XS1(K)
     -27.00*XS1(K)**2 - 9.00*ETA(J)**2)
C(1,6) = (1.00 - ETA(J))*1.00 - ZTA(I)))*(15.00 + 18.00*XS1(K)
     -18.00*XS1(K)**2)
C(1,7) = (1.00 - ETA(J))*1.00 - ZTA(I)))*(15.00 - 18.00*XS1(K)
     -18.00*XS1(K)**2)
C(1,8) = (1.00 - ETA(J))*1.00 - ZTA(I)))*(15.00 - 18.00*XS1(K)
     -18.00*XS1(K)**2 + 9.00*ETA(J)**2 + 18.00*XS1(K) - 11.00)
C(1,9) = (1.00 - ETA(J))*1.00 - ZTA(I)))*(9.00*ETA(J)**2 - 9.00)
C(1,10) = (1.00 + 3.00*ETA(J))*1.00 - ZTA(I))):(9.00*ETA(J)**2 - 9.00)
C(1,11) = -C(1,5)
C(1,12) = -C(1,10)
C(1,13) = (1.00 + 3.00*ETA(J))*1.00 - ZTA(I))):(9.00*ETA(J)**2 - 9.00)
C(1,14) = (1.00 + 3.00*ETA(J))*1.00 - ZTA(I))):(9.00*ETA(J)**2 - 9.00)
C(1,15) = -C(1,13)
\[ C(1,16) = -C(1,14) \]
\[ C(1,17) = (1.00 + ETA(J)) * (1.00 - ZTA(I)) * (10.00 + 18.00 * XSI(K) - 1) \]
\[ 27.00 * XSI(K) ** 2 - 9.00 * ETA(J) ** 2 \]
\[ C(1,18) = (1.00 + ETA(J)) * (1.00 - ZTA(I)) * (81.00 * XSI(K)) ** 2 \]
\[ -13.00 * XSI(K) - 27.00 \]
\[ C(1,19) = (1.00 + ETA(J)) * (1.00 - ZTA(I)) * (27.00 - 18.00 * XSI(K) - 1) \]
\[ 81.00 * XSI(K) ** 2 \]
\[ C(1,20) = (1.00 + ETA(J)) * (1.00 - ZTA(I)) * (27.00 * XSI(K) ** 2 + 1) \]
\[ 9.00 * ETA(J) ** 2 + 18.00 * XSI(K) - 18.00 \]
\[ C(1,21) = (1.00 + ETA(J)) * (1.00 + ZTA(I)) * (10.00 + 18.00 * XSI(K) - 1) \]
\[ 27.00 * XSI(K) ** 2 - 9.00 * ETA(J) ** 2 \]
\[ C(1,22) = (1.00 + ETA(J)) * (1.00 + ZTA(I)) * (10.00 * XSI(K)) ** 2 \]
\[ -18.00 * XSI(K) - 27.00 \]
\[ C(1,23) = (1.00 + ETA(J)) * (1.00 + ZTA(I)) * (27.00 - 18.00 * XSI(K) - 1) \]
\[ 81.00 * XSI(K) ** 2 \]
\[ C(1,24) = (1.00 + ETA(J)) * (1.00 + ZTA(I)) * (27.00 * XSI(K) ** 2 + 1) \]
\[ 9.00 * ETA(J) ** 2 + 18.00 * XSI(K) - 18.00 \]
\[ C(2,1) = (1.00 - XSI(K)) * (1.00 - ZTA(I)) * (10.00 + 18.00 * ETA(J)) \]
\[ 9.00 * ETA(J) ** 2 - 27.00 * ETA(J) ** 2 \]
\[ C(2,2) = (1.00 - 3.00 * XSI(K)) * (1.00 - ZTA(I)) * (9.00 * XSI(K)) ** 2 - 9.00 \]
\[ C(2,3) = (1.00 + 3.00 * XSI(K)) * (1.00 - ZTA(I)) * (9.00 * XSI(K)) ** 2 - 9.00 \]
\[ C(2,4) = (1.00 + XSI(K)) * (1.00 - ZTA(I)) * (10.00 + 13.00 * ETA(J)) \]
\[ -9.00 * ETA(J) ** 2 - 27.00 * ETA(J) ** 2 \]
\[ C(2,5) = (1.00 - XSI(K)) * (1.00 + ZTA(I)) * (10.00 + 18.00 * ETA(J)) \]
\[ 9.00 * ETA(J) ** 2 - 27.00 * ETA(J) ** 2 \]
\[ C(2,6) = (1.00 - 2.00 * XSI(K)) * (1.00 + ZTA(I)) * (9.00 * XSI(K)) ** 2 - 9.00 \]
\[ C(2,7) = (1.00 + 2.00 * XSI(K)) * (1.00 + ZTA(I)) * (9.00 * XSI(K)) ** 2 - 9.00 \]
\[ C(2,8) = (1.00 + XSI(K)) * (1.00 + ZTA(I)) * (10.00 + 13.00 * ETA(J)) \]
\[ -9.00 * ETA(J) ** 2 - 27.00 * ETA(J) ** 2 \]
\[ C(2,9) = (1.00 - XSI(K)) * (1.00 - ZTA(I)) * (27.00 - 18.00 * ETA(J) - 1) \]
\[ 18.00 * ETA(J) - 27.00 \]
\[ C(2,10) = (1.00 - XSI(K)) * (1.00 - ZTA(I)) * (27.00 - 18.00 * ETA(J) - 1) \]
\[ 18.00 * ETA(J) - 27.00 \]
C(2,11) = (1.0 + XSI(K))*1.0*ETAI(J)*2
C(2,12) = (1.0 + XSI(K))*1.0*ETAI(J)*2
C(2,13) = (1.0 - XSI(K))*1.0*ETAI(J)*2
C(2,14) = (1.0 - XSI(K))*1.0*ETAI(J)*2
C(2,15) = (1.0 + XSI(K))*1.0*ETAI(J)*2
C(2,16) = (1.0 + XSI(K))*1.0*ETAI(J)*2
C(2,17) = (1.0 - XSI(K))*1.0*ETAI(J)*2
C(2,18) = -C(2,2)
C(2,19) = -C(2,2)
C(2,20) = (1.0 + XSI(K))*1.0*ETAI(J)*2
C(2,21) = (1.0 - XSI(K))*1.0*ETAI(J)*2
C(2,22) = (1.0 - 3.0*XSI(K))*1.0*ETAI(J)*2
C(2,23) = (1.0 + 3.0*XSI(K))*1.0*ETAI(J)*2
C(2,24) = (1.0 + XSI(K))*1.0*ETAI(J)*2
C(3,1) = (1.0 - XSI(K))*1.0*ETAI(J)*2
C(3,2) = (1.0 + XSI(K))*1.0*ETAI(J)*2
C(3,3) = (1.0 + XSI(K))*1.0*ETAI(J)*2
C(3,4) = (1.0 + XSI(K))*1.0*ETAI(J)*2

DC 26 N=1,4
26 C(3, N+4) = -C(3, N)
C(3, 8) = (1.00 - XSI(K)) * (1.00 - XSI(J)) * (5.00 * ETA(J) * 2 - 9.00)
C(3, 11) = (1.00 + 3.00 * ETA(J)) * (1.00 - XSI(K)) * (5.00 * ETA(J) * 2 - 9.00)
C(3, 14) = (1.00 - 3.00 * ETA(J)) * (1.00 + XSI(K)) * (9.00 * ETA(J) * 2 - 9.00)
C(3, 12) = (1.00 + 3.00 * ETA(J)) * (1.00 + XSI(K)) * (9.00 * ETA(J) * 2 - 9.00)
DO 27 N = 9, 12
27 C(3, N+4) = -C(3, N)
C(3, 17) = (1.00 - XSI(K)) * (1.00 + ETA(J)) * (10.00 - 9.00 * XSI(K) * 2)
1        -9.00 * ETA(J) * 2
C(3, 18) = (1.00 - 3.00 * XSI(K)) * (1.00 + ETA(J)) * (9.00 * XSI(K) * 2 - 9.00)
C(3, 19) = (1.00 + 3.00 * XSI(K)) * (1.00 + ETA(J)) * (9.00 * XSI(K) * 2 - 9.00)
C(3, 20) = (1.00 + XSI(K)) * (1.00 + ETA(J)) * (10.00 - 9.00 * XSI(K) * 2)
1        -9.00 * ETA(J) * 2
DO 28 N = 17, 20
28 C(3, N+4) = -C(3, N)
C CALCULATE JACOBIAN MATRIX
J(3X3) = C(3X24) * XYZ(24X3) AT 32 PT.
DO 30 II = 1, 3
DO 30 KK = 1, 3
DJ(II, KK) = C(30)
DO 30 JJ = 1, 24
30 DJ(II, KK) = DJ(II, KK) + C(II, JJ) * XYZ(JJ, KK) / 64.0
C FORM INVERSE J MATRIX DJ(3X3) FOR COORDINATE TRANSFORMATION
DETJ = DJ(1, 1) * (DJ(2, 2) * DJ(3, 3) - DJ(2, 3) * DJ(3, 2)) - DJ(2, 3) * DJ(3, 2))
1 + DJ(1, 2) * (DJ(2, 3) * DJ(3, 1) - DJ(2, 1) * DJ(3, 3))
in
2 + DJ(1, 3) * (DJ(2, 3) * DJ(3, 2) - DJ(2, 2) * DJ(3, 3))
DJ(1, 1) = (DJ(2, 2) * DJ(3, 3) - DJ(2, 3) * DJ(3, 2)) / DETJ
DJ(1, 2) = (DJ(3, 3) * DJ(1, 1) - DJ(1, 3) * DJ(2, 1)) / DETJ
DJ(1, 3) = (DJ(1, 2) * DJ(2, 3) - DJ(1, 3) * DJ(2, 2)) / DETJ
DJ(2, 1) = (DJ(2, 3) * DJ(3, 1) - DJ(2, 1) * DJ(3, 3)) / DETJ
DJ(2, 2) = (DJ(3, 3) * DJ(1, 1) - DJ(1, 3) * DJ(2, 1)) / DETJ
DJ(2, 3) = (DJ(1, 2) * DJ(2, 3) - DJ(1, 3) * DJ(2, 2)) / DETJ
DJ(3, 1) = (DJ(2, 3) * DJ(3, 2) - DJ(2, 2) * DJ(3, 3)) / DETJ
DJ(3, 2) = (DJ(3, 3) * DJ(1, 1) - DJ(1, 3) * DJ(2, 1)) / DETJ
DJ(3, 3) = (DJ(1, 2) * DJ(2, 2) - DJ(1, 3) * DJ(2, 2)) / DETJ
J(3, 1) = (DJ(2, 1) * DJ(3, 1) - DJ(2, 2) * DJ(3, 1)) / DETJ
J(3, 2) = (DJ(2, 2) * DJ(3, 2) - DJ(2, 3) * DJ(3, 2)) / DETJ
J(3, 3) = (DJ(2, 3) * DJ(3, 3) - DJ(2, 3) * DJ(3, 3)) / DETJ
DJI(3,7) = (DJI(3,2) * DJI(1,2) - DJI(3,2) * DJI(1,1)) / DETJ
DJI(3,3) = (DJI(3,1) * DJI(2,2) - DJI(3,1) * DJI(2,1)) / DETJ

C FORM MATRIX B(6X72), WHERE (B) = (BA)
DO 40 N=1,6
DO 40 L=1,72
40 BA(N,L) = 0.00
DO 50 N=1,72,3
L = (N - 1)/2 + 1
BA(1,N) = (DJI(1,1) * C(1,L) + DJI(1,2) * C(2,L) + DJI(1,3) * C(3,L)) / 64.00
BA(4,N) = (DJI(2,1) * C(1,L) + DJI(2,2) * C(2,L) + DJI(2,3) * C(3,L)) / 64.00
BA(5,N) = (DJI(3,1) * C(1,L) + DJI(3,2) * C(2,L) + DJI(3,3) * C(3,L)) / 64.00
DO 60 N=2,71,3
L = (N - 2)/2 + 1
BA(2,N) = (DJI(2,1) * C(1,L) + DJI(2,2) * C(2,L) + DJI(2,3) * C(3,L)) / 64.00
BA(4,N) = (DJI(1,1) * C(1,L) + DJI(1,2) * C(2,L) + DJI(1,3) * C(3,L)) / 64.00
BA(6,N) = (DJI(3,1) * C(1,L) + DJI(3,2) * C(2,L) + DJI(3,3) * C(3,L)) / 64.00
DO 70 N=3,72,3
L = (N - 3)/2 + 1
BA(3,N) = (DJI(3,1) * C(1,L) + DJI(3,2) * C(2,L) + DJI(3,3) * C(3,L)) / 64.00
BA(5,N) = (DJI(1,1) * C(1,L) + DJI(1,2) * C(2,L) + DJI(1,3) * C(3,L)) / 64.00
BA(7,N) = (DJI(2,1) * C(1,L) + DJI(2,2) * C(2,L) + DJI(2,3) * C(3,L)) / 64.00
DO 80 N=1,72
DO 80 L=1,6
BLT(N,L) = 0.00
DO 80 NN=1,6
80 continued

9.3 BTM(N,L) = BTM(N,L) + BA(NN,N) * (NN,L)
DO 90 N=1,72
DO 90 L=1,72
90 continued

9.3 BCT(N,L) = BCT(N,L) + BTM(N,NN) * AA(NN,L)
DO 100 A=1,72
100 continued
DO 120 L=1,72
120 S(N,L) = S(N,L) + H(M) * DE1J * PCL(N,L)
N = (NM=10) + S
WRITE (2*N) ((S(I,J), I=1,72), J=1,72)
RETURN
END

SUBROUTINE MODIFY (N,U,IEW,ML)
C MODIFY TOTAL STIFFNESS MATRIX FOR GIVEN DISPL. B. C.
IMPLICIT REAL*8 (A,B,U)
COMMON / SOLN / A(144,72), B(144), ANUJALK, IJK, NB, NC, ND2
DO 250 M=2,18
K = N - M +1
IF (K) 225, 235, 230
230 Y(K) = Y(K) - A(K,M) * U
A(K,M) = 0.00
235 K = N - M +1
IF (ML - K) 240, 250, 240
240 Y(K) = Y(K) - A(N,K) * U
A(N,K) = 0.00
250 CONTINUE
A(N,1) = 1.00
S(N) = U
RETURN
END

SUBROUTINE SYMSOL (NB,IPK)
C SOLVE THE SYSTEM OF BANDED, BLOCKED SIMULTANEOUS EQUATIONS BY GAUSSIAN ELIMINATION.
IMPLICIT REAL*8 (A-H, U-Z)
COMMON / SOLN / A(144,72), B(144), ANUJALK, IJK, NB, NC, ND2
NL = NM+1
NH = KM+1
NALK = C
IJK=1
GO TO 150
C REDUCE EQU'S BY BLOCKS; AND THEN SHIFT BLOCK OF EQU'S
100 NLK = NLK + 1
DO 125 K = 1, NN
NM = NM + N
S(N) = B(NM)
B(NM) = $.00
DO 125 M = 1, IBW
A(N, M) = A(NM, M)
125 A(NM, M) = $.00
C READ IN NEXT BLOCK OF EQU'S
IF (NLK .EQ. MNUBLK) GO TO 200
IJK = (NLK + S) + 1
150 READ (I, IJK) ((A(I, J), I = NL, NH), J = 1, IBW), (I, K), K = NL, NH
IF (NLK .EQ. C) GO TO 100
C REDUCE BLOCK OF EQU'S
200 DO 300 M = 1, NN
IF (A(N, M) .EQ. C .AN. C .DO ) GO TO 300
B(N) = S(M) / A(N, M)
DO 275 L = 2, IBW
IF (A(N, L) .EQ. C .AN. C .DO ) GO TO 275
C = A(N, L) / A(N, 1)
I = N + L - 1
J = 0
DO 260 K = L, IBW
J = J + 1
IF (A(I, J) .EQ. 1 .AN. A(I, J) .EQ. C .AN. A(I, J) .EQ. C .do ) GO TO 260
250 A(I, J) = A(I, J) - C * A(N, K)
B(I) = B(I) - A(N, L) * B(N)
260 A(N, L) = C
275 CONTINUE
300 CONTINUE
C WRITE BLOCK OF EQU'S BACK ON DISK
IF (NBLK .EQ. 0) GO TO 460
IJK = (NBLK+5) - 8
WRITE (11, IJK) ((A(I,J), I=1,NN), J=2,IMW), (B(K), K=1,NN)
GO TO 100
C BACK SUBSTITUTION; SOLVED UNKNOWN DISPLACEMENTS ARE STORED IN
C THE FIRST NBLK repeat OF THE 2ND BLOCK OF MATRIX A
460 DO 470 I=1,NN
M=NN+1-I
DO 465 K=2,IMW
L=M+K-1
465 B(M) = B(K) = A(N,K)*B(N)
NM = N + NN
B(NM) = B(N)
470 A(NM,NBLK) = C(N)
NBLK = NBLK-1
IF (NBLK .EQ. C) GO TO 500
IJK = (NBLK+5) - 8
READ (11, IJK) ((A(I,J), I=1,NN), J=2,IMW), (B(K), K=1,NN)
GO TO 460
500 RETURN
END

SUBROUTINE STRESS
C CALCULATE STRESSES AT 24 NORMAL POINTS OF EACH ELEMENT
IMPLICIT REAL*8 (A-H,O-Z)
COMMON / HEAD / FDG(10), ICRE, LIST, IPAGE, LINE
COMMON / DATUM / X(100), Y(100), Z(100), UX(100), UY(100), UZ(100)
1 E(10,10), AS(10), BS(10), CS(10),
2 ICRE(100), IX(10,26), NUMNR, NUMNL, NUMMAT, IMW, NE
COMMON / SOLN / A(144,72), B(144), KMULK, IJK, NBJ, NC, NNO2
COMMON / SPALK / S(72,72), ELE(72,72), STD(72,6), PA(6,72), C(3,24),
1 XYZ(24,5), B(8,6), DJ(5,3), UJ(5,3)
DIMENSION U(72), SIG(6)

C SOLVE FOR STRESSES SIG(6) AT EACH NODAL POINTS OF EACH ELEMENT

CALL TITLE
WRITE (LIST, 2000)
STEIGST = 0.0
du = 3.0
nm = 1, numel
if(n* .eq. 1) go to 110
ca = dabs(as/nm)/ar - 1.64
cb = dabs(bs/nm)/br - 1.64
cg = dabs(cs/nm)/cr - 1.64
if(i(x(nm, 25) .eq. mr)) go to 120
110: CALL ELAS(nm)
120: continue

C PLACE PROPER NODAL DISPLACEMENTS IN U FROM A

00 130 j = 1, 24
00 130 k = 1, 3
01 2 = 3*nj + 1, j = 1 + k
01 jj = k + 3*(j - 1)
02 n p l k = (i - 1)/nd + 1
02 11 = nd + 1 - (nj*nblk - 1)
03 30 u(jj) = a(ii, nplk)

C FORM NODAL PT. COORDS. MATRIX XYZ FOR J(2X3) = C(3X24)*XYZ(24X3)

04 140 i = 1, 24
04 l = x(i, i)
04 xz(i, 1) = x(l)
04 xz(i, 2) = y(l)
04 xz(i, 3) = z(l)

C CALCULATE PA = U**2, 24 SETS OF (6X72) FOR EACH NODE OF THE ELEMENT

05 200 k = 1, 24
05 150 i = 1, 6
05 150 sig(i) = -0.0

go to (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,
1  $X_{III} = -1.00$
   $ET_{AA} = -1.00$
   $ZF_{AA} = -1.00$
   GO TO 25
2  $X_{III} = -1.00/3.00$
   GO TO 25
3  $X_{III} = 1.00/3.00$
   GO TO 25
4  $X_{III} = 1.00$
   GO TO 25
5  $Z_{TAA} = 1.00$
   $X_{III} = -1.00$
   GO TO 25
6  $X_{III} = -1.00/3.00$
   GO TO 25
7  $X_{III} = 1.00/3.00$
   GO TO 25
8  $X_{III} = 1.00$
   GO TO 25
9  $Z_{TAA} = -1.00$
   $X_{III} = -1.00$
   $LT_{AA} = -1.00/3.00$
   GO TO 25
10 $ET_{AA} = 1.00/3.00$
    GO TO 25
11 $X_{III} = 1.00$
   $LT_{AA} = -1.00/3.00$
   GO TO 25
12 $ET_{AA} = 1.00/3.00$
    GO TO 25
13 $Z_{TAA} = 1.00$
$XSII = -1.00$
$ETAA = -1.00 / 3.00$
GO TO 25
14 $ETAA = 1.00 / 3.00$
GO TO 25
15 $XSII = 1.00$
$ETAA = -1.00 / 3.00$
GO TO 25
16 $ETAA = 1.00 / 3.00$
GO TO 25
17 $ETAA = 1.00$
$ETAA = -1.00$
$XSII = -1.00$
GO TO 25
18 $XSII = -1.00 / 3.00$
GO TO 25
19 $XSII = 1.00 / 3.00$
GO TO 25
20 $XSII = 1.00$
GO TO 25
21 $ETAA = 1.00$
$XSII = -1.00$
GO TO 25
22 $XSII = -1.00 / 3.00$
GO TO 25
23 $XSII = 1.00 / 3.00$
GO TO 25
24 $XSII = 1.00$
25 CONTINUE
$C(1,1) = (1.00 - ETAA) * (1.00 - ETAA) * (10.00 + 10.00 * XSII - 27.00 * XSII * 1.2 - 1.00 * ETAA * 1.2)
C(1,2) = (1.00 - ETAA) * (1.00 - ETAA) * (61.00 * XSII * 1.2 - 19.00 * XSII - 27.00)$
\[
C(1, 5) = (1.0C - \text{ETA}) \times (1.0C - \text{ZTA}) \times (27.0C - 18.0C) \times \text{SII} - 1.0C \times \text{SII} \times 2 \\
C(1, 4) = (1.0C - \text{ETA}) \times (1.0C - \text{ZTA}) \times (27.0C) \times \text{SII} \times 2 + 9.0C \times \text{ETA} \times 2 \\
1.0C \times \text{SII} - 18.0C \\
C(1, 3) = (1.0C - \text{ETA}) \times (1.0C - \text{ZTA}) \times (18.0C) \times \text{SII} - 27.0C \times \text{SII} \times 2 \\
1.0C \times \text{ETA} \\
C(1, 2) = (1.0C - \text{ETA}) \times (1.0C - \text{ZTA}) \times (27.0C - 18.0C) \times \text{SII} - 81.0C \times \text{SII} \times 2 \\
1.0C \times \text{ETA} \\
C(1, 1) = (1.0C - \text{ETA}) \times (1.0C - \text{ZTA}) \times (27.0C) \times \text{SII} \times 2 + 9.0C \times \text{ETA} \times 2 \\
1.0C \times \text{SII} - 18.0C \\
C(1, 9) = (1.0C - \text{ETA}) \times (1.0C - \text{ZTA}) \times (9.0C) \times \text{ETA} \times 2 - 9.0C \\
C(1, 8) = (1.0C - \text{ETA}) \times (1.0C - \text{ZTA}) \times (9.0C) \times \text{ETA} \times 2 - 9.0C \\
C(1, 11) = -C(1, 9) \\
C(1, 12) = -C(1, 10) \\
C(1, 13) = (1.0C - \text{ETA}) \times (1.0C - \text{ZTA}) \times (9.0C) \times \text{ETA} \times 2 - 9.0C \\
C(1, 14) = (1.0C - \text{ETA}) \times (1.0C - \text{ZTA}) \times (9.0C) \times \text{ETA} \times 2 - 9.0C \\
C(1, 15) = -C(1, 13) \\
C(1, 16) = -C(1, 14) \\
C(1, 17) = (1.0C - \text{ETA}) \times (1.0C - \text{ZTA}) \times (9.0C) \times \text{ETA} \times 2 - 9.0C \times \text{ETA} \times 2 \\
C(1, 18) = (1.0C - \text{ETA}) \times (1.0C - \text{ZTA}) \times (81.0C) \times \text{SII} \times 2 - 18.0C \times \text{SII} - 27.0C \\
C(1, 19) = (1.0C - \text{ETA}) \times (1.0C - \text{ZTA}) \times (27.0C - 18.0C) \times \text{SII} - 9.0C \times \text{SII} \times 2 \\
C(1, 20) = (1.0C - \text{ETA}) \times (1.0C - \text{ZTA}) \times (27.0C) \times \text{SII} \times 2 + 9.0C \times \text{ETA} \times 2 \\
1.0C \times \text{SII} - 18.0C \\
C(1, 21) = (1.0C - \text{ETA}) \times (1.0C - \text{ZTA}) \times (18.0C) \times \text{SII} - 27.0C \times \text{SII} \times 2 \\
1.0C \times \text{ETA} \\
C(1, 22) = (1.0C - \text{ETA}) \times (1.0C - \text{ZTA}) \times (81.0C) \times \text{SII} \times 2 - 18.0C \times \text{SII} - 27.0C \\
C(1, 23) = (1.0C - \text{ETA}) \times (1.0C - \text{ZTA}) \times (27.0C - 18.0C) \times \text{SII} - 81.0C \times \text{SII} \times 2 \\
C(1, 24) = (1.0C - \text{ETA}) \times (1.0C - \text{ZTA}) \times (27.0C) \times \text{SII} \times 2 + 9.0C \times \text{ETA} \times 2 \\
1.0C \times \text{SII} - 18.0C \\
C(2, 1) = (1.0C - \text{XSI}) \times (1.0C - \text{ZTA}) \times (9.0C) \times \text{SII} \times 2 - 9.0C \\
C(2, 2) = (1.0C - \text{XSI}) \times (1.0C - \text{ZTA}) \times (9.0C) \times \text{SII} \times 2 - 9.0C \\
C(2, 3) = (1.0C - \text{XSI}) \times (1.0C - \text{ZTA}) \times (9.0C) \times \text{SII} \times 2 - 9.0C \\
C(2, 4) = (1.0C - \text{XSI}) \times (1.0C - \text{ZTA}) \times (9.0C) \times \text{SII} \times 2 - 9.0C \\
C(2, 5) = (1.0C - \text{XSI}) \times (1.0C - \text{ZTA}) \times (9.0C) \times \text{SII} \times 2 - 9.0C \\
C(2, 6) = (1.0C - \text{XSI}) \times (1.0C - \text{ZTA}) \times (9.0C) \times \text{SII} \times 2 - 9.0C \\
C(2, 7) = (1.0C - \text{XSI}) \times (1.0C - \text{ZTA}) \times (9.0C) \times \text{SII} \times 2 - 9.0C \\
C(2, 8) = (1.0C - \text{XSI}) \times (1.0C - \text{ZTA}) \times (9.0C) \times \text{SII} \times 2 - 9.0C \\
C(2, 9) = (1.0C - \text{XSI}) \times (1.0C - \text{ZTA}) \times (9.0C) \times \text{SII} \times 2 - 9.0C \\
C(2, 10) = (1.0C - \text{XSI}) \times (1.0C - \text{ZTA}) \times (9.0C) \times \text{SII} \times 2 - 9.0C \\
C(2, 11) = -C(2, 9) \\
C(2, 12) = -C(2, 10) \\
C(2, 13) = (1.0C - \text{XSI}) \times (1.0C - \text{ZTA}) \times (9.0C) \times \text{SII} \times 2 - 9.0C \\
C(2, 14) = (1.0C - \text{XSI}) \times (1.0C - \text{ZTA}) \times (9.0C) \times \text{SII} \times 2 - 9.0C \\
C(2, 15) = -C(2, 13) \\
C(2, 16) = -C(2, 14) \\
C(2, 17) = (1.0C - \text{XSI}) \times (1.0C - \text{ZTA}) \times (9.0C) \times \text{SII} \times 2 - 9.0C \times \text{SII} \\
C(2, 18) = (1.0C - \text{XSI}) \times (1.0C - \text{ZTA}) \times (9.0C) \times \text{SII} \times 2 - 9.0C \\
C(2, 19) = (1.0C - \text{XSI}) \times (1.0C - \text{ZTA}) \times (9.0C) \times \text{SII} \times 2 - 9.0C \\
C(2, 20) = (1.0C - \text{XSI}) \times (1.0C - \text{ZTA}) \times (9.0C) \times \text{SII} \times 2 - 9.0C \\
C(2, 21) = (1.0C - \text{XSI}) \times (1.0C - \text{ZTA}) \times (9.0C) \times \text{SII} \times 2 - 9.0C \\
C(2, 22) = (1.0C - \text{XSI}) \times (1.0C - \text{ZTA}) \times (9.0C) \times \text{SII} \times 2 - 9.0C \\
C(2, 23) = (1.0C - \text{XSI}) \times (1.0C - \text{ZTA}) \times (9.0C) \times \text{SII} \times 2 - 9.0C \\
C(2, 24) = (1.0C - \text{XSI}) \times (1.0C - \text{ZTA}) \times (9.0C) \times \text{SII} \times 2 - 9.0C
\[ C(3,4) = (1, DC + XSII) * (1, DO + ETA) * (10, DC - 9, DC + XSII) * (1 - 2, DG + ETA) \]

\[ \text{DO 26 N}=1, 4 \]

\[ C(3, N+4) = -C(3, N) \]

\[ C(2, 5) = (1, DC - 3, DG + ETA) * (1, DC - XSII) * (9, DC + ETA) * (2 - 9, DG) \]

\[ C(3, 11) = (1, DC + 3, DG + ETA) * (1, DC - XSII) * (9, DG + ETA) * (2 - 9, DG) \]

\[ C(2, 12) = (1, DC - 3, DG + ETA) * (1, DC + XSII) * (1 - 2, DG) * (9, DG + ETA) * (2 - 9, DG) \]

\[ \text{DO 27 N}=9, 2 \]

\[ C(3, N+4) = -C(3, N) \]

\[ C(3, 17) = (1, DC - XSII) * (1, DC + ETA) * (10, DC - 9, DC + XSII) * (1 - 2, DG + ETA) \]

\[ C(3, 18) = (1, DC - 3, DG + ETA) * (1, DC - XSII) * (9, DG + ETA) * (2 - 9, DG) \]

\[ C(3, 19) = (1, DC + 3, DG + ETA) * (1, DC - XSII) * (9, DG + ETA) * (2 - 9, DG) \]

\[ C(3, 20) = (1, DC + XSII) * (1, DC + ETA) * (10, DC - 9, DG + ETA) * (2 - 9, DG) \]

\[ \text{DO 28 N}=17, 2 \]

\[ C(3, N+4) = -C(3, N) \]

C. Calculate Jacobian matrix \( J(3x3) = C(3x4) \times XYZ(24x3) \) at nodal pt.

\[ \text{DO 29 H}=1, 3 \]

\[ \text{DO 30 KK}=1, 3 \]

\[ \text{OJ(II, KK)} = D\cdot DC \]

\[ \text{DO 30 JJ}=1, 24 \]

\[ \text{OJ(II, KK)} = OJ(II, KK) + OJ(II, JJ) \times XYZ(JJ, KK) / c^4 \cdot DC \]

C. Form inverse of matrix \( OJ(3x3) \)

\[ \text{DETJ} = \text{OJ}(1, 1) \times \text{OJ}(2, 2) \times \text{OJ}(3, 3) - \text{OJ}(2, 3) \times \text{OJ}(3, 2) \]

\[ 1 + \text{OJ}(1, 2) \times \text{OJ}(2, 3) \times \text{OJ}(3, 1) - \text{OJ}(2, 1) \times \text{OJ}(3, 3) \]

\[ 2 + \text{OJ}(1, 3) \times \text{OJ}(2, 3) \times \text{OJ}(3, 1) - \text{OJ}(2, 3) \times \text{OJ}(3, 1) \]

\[ \text{OJ}(1, 1) = \text{OJ}(2, 2) \times \text{OJ}(3, 3) - \text{OJ}(2, 3) \times \text{OJ}(3, 2) / \text{DETJ} \]

\[ \text{OJ}(1, 2) = \text{OJ}(3, 2) \times \text{OJ}(1, 3) - \text{OJ}(3, 3) \times \text{OJ}(1, 2) / \text{DETJ} \]

\[ \text{OJ}(1, 3) = \text{OJ}(2, 3) \times \text{OJ}(1, 2) - \text{OJ}(2, 2) \times \text{OJ}(1, 3) / \text{DETJ} \]

\[ \text{OJ}(2, 1) = \text{OJ}(1, 1) \times \text{OJ}(3, 3) - \text{OJ}(1, 3) \times \text{OJ}(3, 1) / \text{DETJ} \]

\[ \text{OJ}(2, 2) = \text{OJ}(3, 3) \times \text{OJ}(2, 1) - \text{OJ}(3, 1) \times \text{OJ}(2, 1) / \text{DETJ} \]

\[ \text{OJ}(2, 3) = \text{OJ}(1, 3) \times \text{OJ}(2, 1) - \text{OJ}(1, 1) \times \text{OJ}(2, 3) / \text{DETJ} \]

\[ \text{OJ}(3, 1) = \text{OJ}(3, 1) \times \text{OJ}(1, 2) - \text{OJ}(1, 2) \times \text{OJ}(3, 1) / \text{DETJ} \]
DO JI(3,2) = (DI(3,1)*DI(1,2) - DI(3,2)*CI(1,1)) / G2TJ
DI(2,3) = (DI(1,1)*CI(2,2) - DI(1,2)*CI(2,1)) / G2TJ

C FORM MATRIX $A(6\times72)$, WHERE $A = (3A)$

DC 4C I=1,6
dc 4o L=1,72
4c BA(i,l) = 0.0e
DO 50 N=1,72,3
l = (N-1)/3 + 1
50 BA(i,l) = (DI(1,1)*CI(1,l) + DI(1,2)*CI(2,l) + DI(1,3)*CI(3,l)) / 64.0e
BA(i,NN) = (DI(2,1)*CI(1,l) + DI(2,2)*CI(2,l) + DI(2,3)*CI(3,l)) / 64.0e
BA(i,NNN) = (DI(3,1)*CI(1,l) + DI(3,2)*CI(2,l) + DI(3,3)*CI(3,l)) / 64.0e
DO 70 N=2,72,3
l = (N-1)/3 + 1
70 BA(i,l) = (DI(2,1)*CI(1,l) + DI(2,2)*CI(2,l) + DI(2,3)*CI(3,l)) / 64.0e
BA(i,NN) = (DI(3,1)*CI(1,l) + DI(3,2)*CI(2,l) + DI(3,3)*CI(3,l)) / 64.0e
BA(i,NNN) = (DI(1,1)*CI(1,l) + DI(1,2)*CI(2,l) + DI(1,3)*CI(3,l)) / 64.0e
DO 80 N=1,6
dc 80 L=1,72
dc 80 NN=1,6
dc 80 NN=1,6
80 BA(N,l) = RA(N,l) + D(N,NN)*RA(NN,l)
dc 90 I=1,6
dc 90 J=1,72
90 SIC(I) = SIC(I) + ABA(I,J)*DI(J)
IF L(L,LT,48) GO TO 94
CALL TITLE
WRITE (LIST,2000)
54 CONTINUE
   IF (K.EQ.1) GO TO 95
   WRITE (LIST,2601) IX(NM,K), (SIG(I), I=1,6)
   GO TO 100
95  WRITE (LIST,2602) NM, IX(NM,K), (SIG(I), I=1,6)
100 CONTINUE
   LINE = LINE + 1
200 CONTINUE
   N = (106NM) - 9
   IF (NM .LE. 1) GO TO 250
   IF (CA .LT. .5E-9 .AND. QB .LT. .5E-9 .AND. CC .LT. .5D-9
     1 .AND. IX(NM,25) .EQ. MK) GO TO 260
250 READ (2'N') ((S(I,J), I=1,72), J=1,72)
260 STEGEL = 0.00
      MK = IX(NM,25)
      AR = AS(NM)
      BR = BS(NM)
      CR = CS(NM)
      IF (AR .LE. 0.0) AR = BS(NM)
      IF (BR .LE. 0.0) BR = CS(NM)
      IF (CR .LE. 0.0) CR = AS(NM)
      DO 280 JJ=1,72
         STIFFU = 0.00
         DO 277 TI=1,72
            STIFFU = STIFFU + S(TI,JJ)*U(JJ)
            DO 277 JJ=1,72
            STIFFU = STIFFU + S(TI,JJ)*U(JJ) / 2.10
            WRITE (LIST,2603) STEGEL
   300 STEGST = STEGST + STEGEL
   WRITE (LIST,2604) STEGST
   RETURN
260 FORMAT (II,6X,PEL,6X,6X,PHA0DE,6X,GHX-STRESS,6X,HEY-STRESS,
       16X,GHY-STRESS,6X,GHY-STRESS,6X,GHY-STRESS,6X,PHYZ-STRESS)
2001 FORMAT (1X,111,4X,6D14.4)
2002 FORMAT (1HC,9X,15,111,4X,6D14.4)
2003 FORMAT (10X,'EL. STRAIN ENERGY = ',1D15.8)
2004 FORMAT (10X,'ST. STRAIN ENERGY = ',1D15.8)
END
VITA

The author was born on September 1, 1934, in Taichung, Taiwan, China. He was graduated from the Civil Engineering Department of the National Taiwan University in July 1958. After two years of reserve officer's training in the Chinese Army, he worked for the Taiwan Provincial Water Conservancy Bureau as an engineer from July 1960 to November 1964. In December 1964 he came to the United States of America.

In March 1965, he began graduate work in the Civil Engineering Department at Virginia Polytechnic Institute and in March 1966 completed the requirements for the M.S. degree. From March 1966 to December 1967 he worked in a consulting firm, Edward and Bjorth, in New York City. Since January 1968, he has been in pursuit of the Doctor of Philosophy degree in the Engineering Mechanics Department at Virginia Polytechnic Institute and State University.

Fu Tien Lin
THE FINITE ELEMENT ANALYSIS OF LAMINATED COMPOSITES

by

Fu Tien Lin

(ABSTRACT)

Laminated plates with homogeneous isotropic layers in bending and extension were investigated by an analytical method. A linear variation through the thickness of the lamina was assumed for displacements. The governing equations were derived by the Hellinger-Reissner Variational Principle. Two isotropic layered plate problems were solved for specialized cylindrical bending by imposing continuity on stresses and displacements at the interface. Solutions were in excellent agreement when compared with the elementary theory.

Two finite elements were developed for the finite element analysis of two-dimensional laminated plates. A linear variation of displacement through the thickness of element was assumed for one element and a cubic variation for the other. Solutions from both finite element analyses are in excellent agreement with the analytical solution for isotropic layers and exact elasticity solution for transversely isotropic layers. Solutions using the element with linear variation displacement with an increased number of elements through the thickness match the solutions obtained by using the element with cubic variation.

A 72-degree-of-freedom element was developed for the three-dimensional analysis of nonhomogeneous, anisotropic laminated plates.
A linear variation of displacement through the thickness of element and cubic in plate plane was assumed. Results are in good agreement with the exact elasticity solutions even when one element is used for each layer in a 3-ply laminated plate.