FUNCTIONAL ANALYTIC TREATMENT OF LINEAR TRANSPORT EQUATIONS IN KINETIC THEORY AND NEUTRON TRANSPORT THEORY

by

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Chapter 1

INTRODUCTION

In the text to follow the conservative neutron transport equation (both one-speed and multi-group) and the temperature-density equation of Kinetic Theory will be studied for time-independent, one dimensional problems. The general form of these equations is,

$$\frac{\partial\psi(x,\mu)}{\partial x} + K^{-1}\psi(x,\mu) = q_o(x,\mu). \tag{1.1}$$

In the case of the neutron transport equation ψ represents the neutron distribution function and $2\Pi\psi(x,\mu)\, dxd\mu$ is the number of neutrons at position x, $x \in \mathbb{R}$, traveling in the direction specified by the polar angle $\cos^{-1}(\mu)$ where $\cos^{-1}(\mu)$ is the angle between the x axis and the velocity vector of the neutrons $(\mu\varepsilon[-1,1])$. q_o is a source function and K^{-1} is the neutron transport operator. Since we are dealing with the conservative neutron transport equation we are considering neutron transport in conserving media. As a result of this condition K^{-1} is non-invertible. In the case of the temperature-density equation $\mathbb{R}$ is a two component vector function depending on the spatial coordinate x, $x \in \mathbb{R}$, and the molecular velocity μ, $\mu \in \mathbb{R}$. ψ is a perturbation from an equilibrium distribution of a rarefied gas; one component of ψ is the perturbation of the density and the other the perturbation of the temperature. In this case q_o

is zero and K^{-1} is the temperature-density operator. An analogous equation for the perturbation in the transverse velocities of a gas near equilibrium exists where ψ is a simple function rather than a vector function. This "transverse velocity" equation will not be treated in detail.

Although the operator K^{-1} differs significantly for the neutron transport case as opposed to the temperature-density case, the similarities in the spectra of the two operators allow them to be treated using the same general technique. In fact, the linear Vlasov equation of plasma physics has been studied by applying the same method.[1]

The standard method of treating equations of the form of Eq(1.1) was introduced by Case[2] in dealing with the non-conservative, one-speed, one-dimensional, neutron transport equation. The conservative case was worked out later by Shure and Natelson.[3] The reason that Case's original approach could not be applied directly to the conservative case is basically due to the non-invertibility of the transport operator for this case. The same problem exists in the case of the temperature-density equation and the transverse velocity equation. The simpler transverse velocity equation was first treated by Cercignani [4] and the coupled temperature-density equation was treated by Kriese et al.[5]; in both cases the authors used the singular eigenmodes approach. In all of these papers the techniques applied are considered to be heuristic owing to their treatment of the continuous spectrum (for a critique of the singular eigenmodes approach see the first chapter of Hangelbroeck's thesis[6]). As pointed out by Greenberg and Zweifel[7] the singular eigenmodes approach in transport theory is analogous to Dirac's treatment of quantum mechanics[8] which ignores the fact that the continuous spectrum actually possesses no eigenvectors. In the case of quantum mechanics Dirac's work was justified by von Neuman's proof of the spectral theorem.[9] The justification of Case's work was to come later.

In 1973 a paper was introduced which employed what is now known as the Larsen-Habetler resolvent integration technique[10] for treating linear transport equations. Larsen and Habetler applied this method to the one-speed, one-dimensional, time-independent neutron transport equation and duplicated the results of Case's earlier paper for the non-conservative case while avoiding the mathematical irregularities committed by Case in his treatment of the continuous spectrum. Simultaneous to this work Hangelbroeck[6] developed a method for treating the one-speed neutron transport equation in a Hilbert space setting for the non-conservative case. Lekkerkerker[11] extended Hangelbroeck's results to the conservative case by restricting the domain of the transport operator to a subspace on which it is invertible, treating the reduced transport operator by Hangelbroeck's technique and then later extending the results to the full domain. Greenberg and Zweifel[12] applied this procedure to the same equation but were able to simplify notation and achieve more general results by applying the Larsen-Habetler technique to the reduced transport operator. The Larsen-Habetler method was used by Bowden, Sancaktar and Zweifel to obtain the eigenfunction expansions appropriate to full-range[13] and half-range[14] problems in multi-group transport theory. Later Bowden, Greenberg, and Zweifel[15] attacked the conservative multi-group case by treating the reduced multi-group transport operator with this technique; however, this procedure is troubled by notational complexities.

The rigorous means of dealing with the conservative neutron transport equation referred to above are not applicable to the temperature-density equation or to the transverse velocity equation because the domains of the operators corresponding to these cases cannot be decomposed, as in the conservative neutron transport equation, into a finite dimensional subspace and a subspace on which the transport operator is invertible. However, a modified version of the Larsen-Habetler technique can be used to attack these equations. In Chapter 2 it will be shown how this technique can be applied to the conservative neutron transport equation, both one-group and multi-group, yielding the proper results while avoiding the notational complexities of previous methods. In Chapter 4 this technique will be applied to the temperature-density equation which, until now, has only been treated by heuristic arguments. This approach has been shown to be applicable to the transverse velocity equation by Bowden and Garbanati.[16] The results presented in Chapter 4 and those obtained by Bowden and Garbanati have not only placed the Kinetic Theory equations on a rigorous footing but have also extended the results obtained by earlier heuristic arguments. The earlier work forced one to search for solutions to transport problems in the space of Hölder continuous functions, a space which is not complete under any reasonable norm. There are two objections to this constraint. From a purely technical standpoint one would rather work in the framework of a Banach space rather than an incomplete space. Secondly, and most important, the constraint of Hölder continuity has no physical basis. Both of these problems are solved by the approach presented here. The results obtained for the temperature-density equation are valid in a physically reasonable Banach space which is introduced in Chapter 4. In addition, an integral equation for the surface density is presented for the Kinetic Theory equations. When a mixing of diffuse and specular reflection of molecules is allowed at a boundary, the boundary conditions assume a particularly difficult form. Until now the method of handling this problem involved solving integral equations for the expansion coefficients. The technique we have applied leads quite naturally to an integral equation for the surface density.

Since the basis of this work depends on the Larsen-Habetler resolvent integration technique, the basic ideas of this method will be sketched. But first, a few general transport theory references are in order. In neutron transport theory the standard reference is Case and Zweifel[17], in Kinetic Theory Cercignani's[18,19] books are recommended and Chandrasekhar's[20] book on radiative transfer may be helpful as many of the methods used in radiative transfer are also applicable in neutron transport theory and Kinetic Theory.

The basic method of solution employed by Case[2] for subcritical problems assumes that a separation of variables exists of the form

$$(1.2)\qquad \psi(x,\mu) = \sum_{i=1}^{N} a_i p_i(x) q_i(\mu) + \int_{-1}^{+1} A(\nu)\phi_\nu(\mu)\theta_\nu(x)d\nu,$$

where a_i, $1 \leq i \leq N$, and $A(\nu)$ are expansion coefficients; $p_i(x)$ and $\theta_\nu(x)$ are eigenvectors of $\frac{\partial}{\partial x}$; $q_i(\mu)$ and $\phi_\nu(\mu)$ are eigenvectors of K^{-1}. Suppose that the following relations hold,

$$(1.3a)\qquad K^{-1}q_i(\mu) = \frac{1}{\nu_i}\, q_i(\mu) \;,$$

$$(1.3b)\qquad K^{-1}\phi_\nu(\mu) = \frac{1}{\nu}\phi_\nu(\mu).$$

Since $p_i(x)$ and $\theta_\nu(x)$ given by

$$(1.4a)\qquad p_i(x) = e^{-x/\nu_i} \;,$$

$$(1.4b)\qquad \theta_\nu(x) = e^{-x/\nu} \;,$$

are eigenvectors of $\frac{\partial}{\partial x}$ with eigenvalues of $-\frac{1}{\nu_i}$, $-\frac{1}{\nu}$ then $\psi(x,\mu)$ given by Eqs. (1.2), (1.3) and (1.4) is a solution of Eq. (1.1) for $q_o = 0$, i.e.,

$$(1.5)\qquad \left(\frac{\partial}{\partial x} + K^{-1}\right)\left[\sum_{i=1}^{N} a_i e^{-x/\nu_i} q_i(\mu) + \int_{-1}^{+1} A(\nu)\phi_\nu(\mu) e^{-x/\nu} d\nu\right] = 0 .$$

The expansion coefficients a_i and $A(\nu)$ can be determined by the boundary conditions at $x = 0$,

$$(1.6) \qquad \psi(0,\mu) = \sum_{i=1}^{N} a_i q_i(\mu) + \int_{-1}^{+1} A(\nu)\phi_\nu(\mu)d\nu.$$

Thus, we are led to seek an expansion of $\psi(0,\mu)$ in terms of the eigenfunctions of K^{-1}. The eigenfunctions corresponding to the discrete spectrum, $q_i(\mu)$, are readily obtained. The contribution due to the generalized eigenvectors, $\phi_\nu(\mu)$, which correspond to the continuous spectrum of K^{-1}, must be handled more delicately.

K^{-1} is an unbounded operator but possesses a bounded inverse. In order that the vast literature on bounded operators could be employed Larsen and Habetler chose to study the bounded operator K. In the case of the discrete spectrum, note that if η is an eigenvector of K with eigenvalue λ, then η is also an eigenvector of K^{-1} with eigenvalue $1/\lambda$. In the case of generalized eigenvectors and eigenvalues one can apply the spectral mapping theorem to obtain the same result. Thus, finding an expansion in terms of eigenfunctions of the more tractable operator K gives us the expansion we require in terms of eigenfunctions of K^{-1}. By employing the following identity for bounded operators,

$$(1.7) \qquad \frac{1}{2\Pi} \int_\Gamma (zI-K)^{-1}\psi(\mu)dz = \psi(\mu),$$

where Γ is a contour enclosing the spectrum of K, Larsen and Habetler obtained the desired eigenfunction expansion in an elegant fashion while avoiding the difficulties which were previously encountered in dealing with the continuous spectrum.

In the case of conservative neutron transport this technique is not directly applicable. As for the Kinetic Theory equations, this technique is again not directly applicable and furthermore, the method of dealing with a reduced transport operator is also unworkable. As mentioned earlier the reason for the difficulties encountered stems from the non-invertibility of K^{-1}. The solution to this problem has been suggested by Larsen. If one defines the operator

$$(1.8)\qquad S^{-1} = K^{-1} - z_o I$$

where z_o is a complex number not contained in the spectrum of K^{-1}, then S^{-1} is an unbounded but invertible operator. Its bounded inverse, S, can be treated by the standard Larsen-Habetler method and an expansion of $\psi(0,\mu)$ in terms of its eigenfunctions can be obtained. Again, applying the spectral mapping theorem, we see that if η is an eigenvector of S with eigenvalue λ then η is also an eigenvector of K^{-1} but with eigenvalue $(1/\lambda+z_o)$. A similar result is seen to hold for the continuous spectrum. Then, the eigenfunction expansion desired is obtained by studying the well-behaved, bounded operator S rather than the ill-behaved operator K^{-1}.

In the chapters to follow these sketchy ideas will be elaborated on at great length. In Chapter 2 the technique mentioned above will be applied to the one-group and multi-group conservative neutron transport equations. The full-range and half-range expansions will be obtained. These results will then be used to solve the Milne problem. Many of the details of these calculations parallel the details of previous approaches (c.f. Ref. 12-15) and thus these calculations will not be reproduced. The main objectives of this chapter are to demonstrate the technique and to note its obvious notational advantages over previous methods.

Chapter 3 contains a development of the primary equation of interest in this work, the temperature-density equation, from basic principles.

In Chapter 4 the modified Larsen-Habetler approach is applied to the temperature-density equation and the full-range and half-range expansions are obtained. These results are initially obtained for a restricted class of functions and then the results are extended to the full Banach space. In order that these expansions can be used to solve transport problems a functional calculus is developed for the operator S.

Using the results of the previous chapters the solution to the temperature-jump problem, for complete accommodation, is presented in Chapter 5. Also in Chapter 5, the temperature-jump problem with arbitrary accommodation at the boundary is studied. For this case the boundary conditions of both the temperature-density equation and the transverse velocity equation are particularly difficult to deal with. We will present integral equations for the surface densities in both cases. Once the surface densities are known these problems can be solved by the same technique that was applied to the temperature-jump problem with complete accommodation. In previous work the standard technique involved solving rather unwieldly integral equations for the expansion coefficients. In the case of the temperature-density equation the existence of solutions to these integral equations has not been proven. We will show that the integral equations we obtain have solutions for all but, at most, a finite number of values of the accommodation coefficient.

Chapter 2

CONSERVATIVE NEUTRON TRANSPORT

Section 1: The One-Speed, Full-Range Expansion

The homogeneous, one-speed neutron transport equation for a conservative medium can be written in the form,

$$\text{(1.1a)} \qquad \mu\frac{\partial\psi(x,\mu)}{\partial x} + K^{-1}\psi(x,\mu) = 0.$$

K^{-1} is an operator which acts only on the μ dependence of ψ and is given by

$$\text{(1.1b)} \qquad (K^{-1}f)(\mu) = \frac{1}{\mu}[f(\mu)-\frac{1}{2}\int_{-1}^{+1} f(s)ds].$$

The domain of K^{-1} is usually taken to be the space of Hölder continuous functions with index α defined on $[-1,1]$. We will designate this space by $Hö_\alpha([-1,1])$. Here

$$\text{(1.2a)} \qquad Hö_\alpha(X) = \{f: ||f||_\alpha<\infty\},$$

where $||\cdot||_\alpha$ is given by

$$\text{(1.2b)} \qquad ||f||_\alpha = \sup_{\substack{x,y\in X \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

With this domain K^{-1} is a non-invertible operator, in fact, (1.3a) $K^{-1}(1) = 0,$

(1.3b) $K^{-1}(\mu) = 1.$

The vectors $e_o(\mu) = 1$ and $e_1(\mu) = \mu$ span the zero-root linear manifold of K^{-1}. One easily verifies this assertion. Using Eq. (1.1b) and solving

(1.4) $(K^{-1}f_o)(\mu) = 0$

for f_o we obtain

(1.5) $f_o(\mu) \; \alpha \; \rho_o(\mu),$

where $\rho_i(\mu)$ is a polynomial in μ of degree i. Similarly, solving

(1.6a) $K^{-2}f_1 = 0,$

we have

(1.6b) $K^{-1}(K^{-1}f_1) = 0.$

Using Eqs. (1.4) and (1.5) this implies that

(1.6c) $(K^{-1}f_1)(\mu) \; \alpha \; \rho_o(\mu).$

Hence we obtain the result

(1.7) $f_1(\mu) \; \alpha \; \rho_1(\mu).$

Assume that

(1.8a) $(K^{-n}f_n)(\mu) = 0, \; n \geq 3,$ and

(1.8b) $(K^{-n+1}f_n)(\mu) \neq 0.$

Then

(1.9a) $K^{-2}(K^{-n+2}f_n)(\mu) = 0,$

and thus, by Eqs. (1.6a) and (1.7) we have

(1.9b) $K^{-1}(K^{-n+3}f_n)(\mu) \; \alpha \; \rho_1(\mu).$

Solving the above for $K^{-n+3}f_n$ we find that

(1.9c) $(K^{-n+3}f_n)(\mu) \; \alpha \; \rho_2(\mu).$

A simple calculation shows that this result contradicts Eqs. (1.8a) and (1.8b). Hence

(1.10) $(K^{-n}f)(\mu) = 0$

implies that $f(\mu)$ is a linear combination of the vectors $e_o(\mu) = 1$ and $e_1(\mu) = \mu$. In previous work the span of $\{1,\mu\}$ is the finite-dimensional subspace which is substracted from the domain of K^{-1} in order that the reduced transport operator [11,12] be invertible.

It is well known that the spectrum of $K^{-1}, \sigma(K^{-1})$, is restricted to the real axis. Thus, if we define S^{-1} by,

(1.11) $S^{-1} = K^{-1} - iI,$

then S^{-1} has a bounded inverse on $H\ddot{o}_\alpha([-1,1])$ and S^{-1} can be dealt with by applying the Larsen-Habetler resolvent integration technique.

S, the inverse of S^{-1}, may be computed by solving,

$$(1.12) \qquad S^{-1}f = \eta$$

for f in terms of η. Of course, $f,\eta\varepsilon H\ddot{o}_{\alpha}([-1,1])$. Using Eqs. (1.1b) and (1.11) we have

$$(1.13a) \qquad \frac{1}{\mu}[f(\mu)-\frac{1}{2}\int_{-1}^{+1}f(s)ds]-if(\mu) = \eta(\mu),$$

or,

$$(1.13b) \qquad f(\mu) = \frac{\mu}{1-i\mu}\,\eta(\mu) + \frac{1}{2(1-i\mu)}\int_{-1}^{+1}f(s)ds.$$

Integrating both sides of Eq. (1.13b) with respect to μ and rearranging we obtain the following expression for the integral on the r.h.s. of Eq. (1.13b),

$$(1.14a) \qquad \int_{-1}^{+1}f(s)ds = \Lambda^{-1}(-i)\int_{-1}^{+1}\frac{\mu}{1-i\mu}\,\eta(\mu)d\mu,$$

where $i = \sqrt{-1}$ and we have introduced the definition

$$(1.14b) \qquad \Lambda(z) = 1 - \frac{z}{2}\int_{-1}^{+1}\frac{ds}{z-s}.$$

Inserting the l.h.s. of Eq. (1.14a) into Eq. (1.13b) we have f in terms of η and thus, using Eq. (1.12), we conclude that the action of S on η is given by

$$(1.15) \qquad (S\eta)(\mu) = \frac{\mu}{1-i\mu}\,\eta(\mu) + \frac{(1-i\mu)^{-1}}{2\Lambda(-i)}\int_{-1}^{+1}\frac{s\eta(s)ds}{1-is}\,.$$

S is a bounded operator on the space $L_p([-1,1],|\mu|^p d\mu)$ given by,

$$(1.16)\quad L_p(X,g(s)ds) = \{f:\int_X |f(s)|^p g(s)ds<\infty\}.$$

We will restrict the domain of S to the smaller space, H_p, defined by

$$(1.17)\quad H_p = \{f:\mu f(\mu)\varepsilon H\eth_\alpha([-1,1])\},$$

in order that the resolvent of S may be studied. The results we obtain on this restricted domain can be extended to the full domain of S.[12,21]

The resolvent of S, $(zI-S)^{-1}$, is computed in the same way that S is obtained from S^{-1}. The result is

$$(1.18)\quad (zI-S)^{-1}f(\mu) = \frac{(1+iz)^{-1}}{t^{-1}(z)-\mu}\{(1-i\mu)f(\mu) + \frac{(1+iz)^{-1}}{2\Lambda(t^{-1}(z))}\int_{-1}^{+1}\frac{sf(s)ds}{t^{-1}(z)-s}\},$$

where we have employed Eq. (1.14b) and the following definitions:

$$(1.19a)\quad t(z) = \frac{z}{1-iz}$$

and

$$(1.19b)\quad t^{-1}(z) = \frac{z}{1+iz}\ .$$

Note that t^{-1} is the inverse of the operator t rather than the reciprocal of the function t(z), i.e., $(t^{-1}t)(z) = z$.

In order to employ the identity given by Eq. (1.1.7) we must determine the spectrum of the operator S. If we view the r.h.s. of Eq. (1.18) as a function of z then the spectrum of S will be those points at which this function fails to be analytic. We examine the leading multiplicative term,

$$(1.20)\qquad \frac{(1+iz)^{-1}}{t^{-1}(z)-\mu} = \frac{1}{z-(1+iz)\mu} \,,$$

and note that it does not give rise to a pole at $z = i$. However, this term does give rise to a first order pole at $z = t(\mu)$.

From Eq. (1.14b) it is clear that $\Lambda(z)$ is non-zero and analytic for z in the finite plane,[17] provided $z \notin [-1,1]$. For z large $\Lambda(z)$ can be expressed in the form,

$$(1.21)\qquad \Lambda(z) = 1 - \frac{1}{2}\int_{-1}^{+1} ds\left(1+ \frac{s}{z} + \left(\frac{s}{z}\right)^2 + \ . \ . \ . \right) \xrightarrow[|z|\to\infty]{} -\frac{1}{3}\frac{1}{z^2} \,.$$

Since $|t^{-1}(z)|\to\infty$ as $z\to i$, the factor of $\Lambda^{-1}(t^{-1}(z))$ in Eq. (1.18) gives rise to a second order pole at $z = i$.

Both $\Lambda^{-1}(t^{-1}(z))$ and the term

$$(1.22)\qquad \int_{-1}^{+1} \frac{sf(s)ds}{t^{-1}(z)-s} = g(z)$$

are discontinuous across $\{z : t^{-1}(z)\varepsilon[-1,1]\}$, since the Plemelj formulas applied to Eq. (1.22) yield,

$$(1.23a)\qquad g^{\pm}(\mu) = P\int_{-1}^{+1} \frac{sf(s)ds}{t^{-1}(\mu)-s} \mp t^{-1}(\mu)f(t^{-1}(\mu)),\ t^{-1}(\mu)\varepsilon[-1,1],$$

where,

$$(1.23b)\qquad g^{\pm}(\mu) = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \int_{-1}^{+1} \frac{sf(s)ds}{t^{-1}(\mu)\pm i\varepsilon-s} \,.$$

A similar result is obtained when the Plemelj formulas are applied to Eq. (1.14b). Thus, $\Lambda^{-1}(t^{-1}(z))$ and the integral term introduce a branch cut, $\{z:t^{-1}(z)\varepsilon[-1,1]\}$.

The complete spectrum is accounted for by the semi-circle, $\{z:z = \frac{1}{2}(i+\exp(i\theta)), -\Pi\leq\theta\leq 0\}$, and the point, $z=i$. We enclose the spectrum of S in two disjoint contours, Γ and Γ_i, with Γ surrounding the semi-circle and Γ_i surrounding the point $z=i$. Employing the identity (1.1.7) we calculate the contribution due to the semi-circle,

$$(1.24)\qquad \frac{1}{2\Pi i}\int_{\Gamma}(zI-S)^{-1}f(\mu)dz = \frac{1}{2\Pi i}\int_{\Gamma'}\frac{1}{z'-\mu}\left(f(\mu) + \frac{\Lambda^{-1}(z')}{2}\int_{-1}^{+1}\frac{sf(s)ds}{z'-s}\right)dz'.$$

Here Γ' is a contour surrounding $[-1,1]$ and the change of variable $z' = t^{-1}(z)$ has been applied. The r.h.s. of Eq. (1.24) is precisely the result of Ref. 12 for the branch cut integration. Thus, we are led directly to the standard formula,[10,12]

$$(1.25a)\qquad \frac{1}{2\Pi i}\int_{\Gamma}(zI-S)^{-1}f(\mu)dz = \int_{-1}^{+1}A(\nu)\Phi_{\nu}(\mu)d\nu,$$

with

$$(1.25b)\qquad A(\nu) = \frac{1}{N(\nu)}\int_{-1}^{+1}sf(s)\Phi_{\nu}(s)ds,$$

$$(1.25c)\qquad \Phi_{\nu}(\mu) = \frac{\nu}{2}\, P\,\frac{1}{\nu-\mu} + \frac{1}{2}[\Lambda^{+}(\nu)+\Lambda^{-}(\nu)]\delta(\nu-\mu),$$

and,

$$(1.25d)\qquad N(\nu) = \nu\Lambda^{+}(\nu)\Lambda^{-}(\nu).$$

Consider the integration about Γ_i. Note that the analytic behaviour of the first term on the r.h.s. of Eq. (1.18), in a neighborhood of z=i, is given by Eq. (1.20) and thus is analytic there. Consequently, the contribution due to the integration about Γ_i will be given entirely by the second term on the r.h.s. of Eq. (1.18). This term is given by

$$(1.26) \qquad \frac{1}{z-(1+iz)\mu}\left[\frac{\Lambda^{-1}(t^{-1}(z))}{2}\int_{-1}^{+1}\frac{sf(s)ds}{z-(1+iz)s}\right] = \frac{p(z)}{q(z)},$$

where p(z) is analytic in a neighborhood of i and is given by,

$$(1.27) \qquad p(z) = \frac{1}{2}[z-(1+iz)\mu]^{-1}\int_{-1}^{+1}\frac{sf(s)ds}{z-(1+iz)s},$$

and q(z) is given by,

$$(1.28) \qquad q(z) = \Lambda(t^{-1}(z)).$$

q(z) vanishes at z=i as well as its first derivative giving rise to a second order pole. The integration of the resolvent about Γ_i is obtained by applying the standard residue formula for a second order pole,

$$(1.29) \qquad \frac{1}{2\pi i}\oint\frac{p(z)}{q(z)}\,dz = \frac{2}{3(q''(z_0))^2}\,(3p'(z_0)q''(z_0)-p(z_0)q'''(z_0)),$$

and evaluating for z_0=i. The result is

$$(1.30) \qquad \frac{1}{2\pi i}\int_{\Gamma_i}(zI-S)^{-1}f(\mu)\,dz = \frac{3}{2}\left[\mu\int_{-1}^{+1}sf(s)ds + \int_{-1}^{+1}s^2f(s)ds\right].$$

Combining the results of these integrations, Eqs.(1.25) and (1.30), we obtain Eq.(10) of Ref. 12, i.e., the Case full-range expansion formula for c=1,

$$(1.31) \qquad f(\mu) = \tfrac{1}{2}a_0 - \tfrac{1}{2}a_1\mu + \int_{-1}^{+1} A(\nu)\Phi_\nu(\mu)d\nu,$$

where the expansion coefficients, a_i, are defined by,

$$(1.32) \qquad a_i = 3\int_{-1}^{+1} (-\mu)^{2-i} f(\mu)d\mu.$$

Section 2: The Half-Range Expansion

In this section we will obtain the half-range expansion making use of the results from Section 1. In the previous section we obtained the eigenfunction expansion,

$$(2.1) \qquad \psi(0,\mu) = \tfrac{1}{2}a_0 - \tfrac{1}{2}a_1\mu + \int_{-1}^{+1} A(\nu)\Phi_\nu(\mu)d\nu,$$

where the expansion coefficients are given by Eqs.(1.25b) and (1.32), substituting $\psi(0,\mu)$ for $f(\mu)$. $\psi(x,\mu)$ given by

$$(2.2) \qquad \psi(x,\mu) = \tfrac{1}{2}a_0 + \tfrac{1}{2}a_1\ (x-\mu) + \int_{-1}^{+1} A(\nu)\Phi_\nu(\mu)e^{-x/\nu}d\nu$$

is a solution to Eq.(1.1a) provided $\psi(0,\mu)$ is specified for $\mu\varepsilon[-1,1]$. For half-range problems $\psi(0,\mu)$ is specified only for $\mu\varepsilon[0,1]$ and the additional boundary condition,

$$(2.3) \qquad \lim_{x\to\infty} \psi(x,\mu) \to \text{constant}$$

is usually given. In order to satisfy boundary condition (2.3), $\psi(x,\mu)$ given by Eq. (2.2) must have

(2.4a) $\quad a_1 = 0$

and

(2.4b) $\quad A(\nu) = 0 \ , \ \nu\varepsilon[-1,0).$

Thus, for half-range problems, we seek an expansion of the form,

(2.5) $$\psi(0,\mu) = \tfrac{1}{2}a_0+\int_0^1 A(\nu)\phi_\nu(\mu)d\nu.$$

Functions satisfying Eq. (2.5) are clearly a subset of the functions expandable by the full-range expansion, Eq. (2.1).

It is convenient for the following analysis to introduce the operator $P^+:L_p([-1,1],|s|^p ds) \rightarrow L_p([0,1],|s|^p ds)$ defined by,

(2.6) $\quad (P^+f)(\mu) = f(\mu) \ , \ \mu\geq 0.$

Define the operator $E:L_p([0,1],|s|^p ds) \rightarrow L_p([-1,1],|s|^p ds)$ by

(2.7a) $\quad P^+EP^+\psi = P^+\psi,$

(2.7b) $$\int_{-1}^{+1} s(EP^+\psi)(s)ds = 0,$$

and,

(2.7c) $\quad (zI-S)^{-1}(EP^+\psi)(\mu)$ analytic in z for Re z<0.

If an E exists which satisfies the above conditions and if $\psi(0,\mu)$

is given by

$$(2.8)\qquad \psi(0,\mu) = (EP^{+}\psi)(0,\mu),$$

then $\psi(x,\mu)$ given by Eq. (2.2) will satisfy boundary condition (2.4a) due to Eq. (2.7b) and boundary condition (2.4b) due to Eq. (2.7c). Furthermore, if ψ is specified for $\Psi\varepsilon[0,1]$, which is the usual boundary condition for half-range problems, then ψ is given on the full-range by Eq. (2.8). In other words, specifying $\psi(0,\mu)$ for $\mu\varepsilon[0,1]$ with the added condition (2.3) is equivalent to specifying $\psi(0,\mu)$ on $[-1,1]$.

The operator E is given, as in Ref. 12, by

$$(2.9)\qquad (EP^{+}f)(\mu) = \begin{cases} f(\mu) \ , \ \mu\geq 0, \\ \dfrac{3}{2}\dfrac{1}{X(\mu)}\displaystyle\int_0^1 \dfrac{sf(s)\,ds}{X(-s)(s-\mu)} \ , \ \mu<0. \end{cases}$$

Here $X(z)$ provides the Wiener-Hopf factorization of $\Lambda(z)$[12]:

$$(2.10)\qquad X(z)X(-z) = 3\Lambda(z),$$

where $X(z)$ is analytic in $\mathbb{C}\setminus[0,1]$ and vanishes as z^{-1} as $|z|\to\infty$.

Not only does the existence of E imply that the boundary conditions (2.4a) and (2.4b) can be satisfied, the stronger statement

$$(2.11)\qquad \{a_1 = 0,\ A(\nu) = 0,\ \nu\varepsilon[-1,0)\} \rightleftarrows \psi(0,\mu) = (EP^{+}\psi)(0,\mu)$$

can be made. The proof of (2.11) is left for Appendix III.

According to Eq. (2.11) the expansion appropriate to half-range problems will be given by the full-range expansion of $EP^{+}\psi$.

In order to express this result in the usual form, Eq. (2.5), some rearrangement is necessary. The expansion coefficients can be obtained by applying Eqs. (1.25b) and (1.32) directly to $EP^{+}\psi$ or by using the identity (1.1.7) on $EP^{+}\psi$. We choose the latter method. Substituting Eq. (2.9) into Eq. (1.18) we obtain,

$$(2.12a) \quad [(zI-S)^{-1}EP^{+}\psi](\mu) = \frac{(1+iz)^{-1}}{t^{-1}(z)-\mu}\{(1-i\mu)\psi(\mu)+\frac{(1+iz)^{-1}}{2\Lambda(t^{-1}(z))} \times \int_{-1}^{+1}\frac{s(EP^{+}\psi)(s)}{t^{-1}(z)-s}ds \ , \ \mu>0,$$

and,

$$(2.12b) \quad [(zI-S)^{-1}EP^{+}\psi](\mu) = \frac{(1+iz)^{-1}}{t^{-1}(z)-\mu} \ \frac{(1-i\mu)}{X(\mu)}\frac{3}{2}\int_{0}^{1}\frac{s\psi(s)}{X(-s)(s-\mu)}ds + \frac{(1+iz)^{-1}}{2\Lambda(t^{-1}(z))}\int_{-1}^{+1}\frac{s(EP^{+}\psi)(s)}{t^{-1}(z)-s}ds\}, \ \mu<0.$$

Let us consider the term

$$(2.13) \quad \int_{1}^{0} s\frac{(EP^{+}\psi)}{t^{-1}(z)-s}(s)ds = \int_{-1}^{0}ds\int_{0}^{1}dt\ \frac{sX^{-1}(s)}{t^{-1}(z)-s}\ \frac{3}{2}\frac{t\psi(t)}{X(-t)(t-s)}.$$

Using a partial fraction decomposition of the denominator on the r.h.s. of Eq. (2.13), making the change of variable $s \to -s$, and interchanging limits of integration we arrive at the result,

$$(2.14) \quad \int_{-1}^{0}\frac{s(EP^{+}\psi)}{t^{-1}(z-s)}(s)ds = \int_{0}^{1}dt\int_{0}^{1}ds\ \frac{3}{2}\left[\frac{s}{X(-s)(s+t)} - \frac{s}{X(-s)(s+t^{-1}(z))}\right] \times \frac{t\Psi(t)}{X(-t)} .$$

To proceed we need the equation from Ref. 12,

$$(2.15)\qquad X(z) = \int_0^1 \frac{sX^-(s)}{2\Lambda^-(s)(s-z)}\,ds.$$

Using Eq. (2.10) in (2.15) we obtain,

$$(2.16)\qquad X(\mu) = \frac{3}{2}\int_0^1 \frac{s}{X(-s)(s-\mu)}\,ds.$$

Substituting the above into Eq. (2.14) and multiplying by $(2\Lambda(t^{-1}(z)))^{-1}$ we have

$$(2.17)\qquad \frac{1}{2\Lambda(t^{-1}(z))}\int_{-1}^{0} \frac{s(EP^+\psi)(s)}{t^{-1}(z)-s}\,ds = \frac{3}{2}\,\frac{1}{X(t^{-1}(z))}\int_0^1 \frac{s\psi(s)}{X(-s)(t^{-1}(z)-s)}\,ds$$

$$-\frac{1}{2\Lambda(t^{-1}(z))}\int_0^1 \frac{s\psi(s)}{t^{-1}(z)-s}\,ds,$$

and since EP^+ is the identity on $[0,1]$ we conclude that

$$(2.18)\qquad \frac{1}{2\Lambda(t^{-1}(z))}\int_{-1}^{+1} \frac{s(EP^+\psi)(s)}{t^{-1}(z)-s}\,ds = \frac{3}{2}\,\frac{1}{X(t^{-1}(z))}\int_0^1 \frac{s\psi(s)}{X(-s)(t^{-1}(z)-s)}\,ds.$$

With this result Eq. (2.12) can be cast in the form,

$$(2.19a)\qquad [(zI-S)^{-1}EP^+\psi](\mu) = \frac{(1+iz)^{-1}}{t^{-1}(z)-\mu}\{(1-i\mu)\psi(\mu) + \frac{(1+iz)^{-1}}{X(t^{-1}(z))}$$

$$\times\ \frac{3}{2}\int_0^1 \frac{s\psi(s)}{X(-s)(t^{-1}(z)-s)}\,ds\},\ \mu>0,$$

and

$$(2.19b)\qquad [(zI-S)^{-1}EP^{+}\psi](\mu) = \frac{3}{2}\,\frac{(1+iz)^{-1}}{t^{-1}(z)-\mu}\,\Big\{\frac{(1-i\mu)}{X(\mu)}\int_0^1 \frac{s\psi(s)}{X(-s)(s-\mu)}\,ds$$

$$+\frac{(1+iz)^{-1}}{X(t^{-1}(z))}\int_0^1 \frac{s\psi(s)}{X(-s)(t^{-1}(z)-s)}\,ds\ ,\quad \mu<0.$$

Eq. (2.19) can be used to quickly verify that E satisfies Eq. (2.7c). To see this, note that t^{-1} maps the left half complex plane into itself and is analytic except for a simple pole at $z=i$. Thus $X(t^{-1}(z))$ is analytic for Re $z<0$. Moreover, for $\mu>0$ and Re $z<0$, $|(1+iz)(t^{-1}(z)-\mu)|^{-1}$ is finite. Therefore, from Eq. (2.19a), we have that $(zI-S)^{-1}EP^{+}\psi$ is analytic in z for Re $z<0$ and $\mu>0$. To see that $(zI-S)^{-1}EP^{+}\psi$ is analytic for Re $z<0$ when $\mu<0$, we need only check that $z=t(\mu)$ is not a singularity of $(zI-S)^{-1}EP^{+}\psi$. This is done by recalling from Eq. (1.19) that $t^{-1}(t(\mu)) = \mu$. Thus, $(zI-S)^{-1}EP^{+}\psi$ is analytic for Re $z<0$. At $z=i$ we note from Eq. (2.19) that $(zI-S)^{-1}EP^{+}\psi$ has a simple pole induced by the zero of $X(t^{-1}(z))$.

Integrating $(zI-S)^{-1}EP^{+}\psi$ on z along a contour containing the point i and the semicircle, $\{z: z = \frac{1}{2}(i+\exp(i\theta)),\ -\Pi/2<\theta<0\}$, yields the Case half-range eigenfunction expansion,

$$(2.20a)\qquad \psi(\mu) = \tfrac{1}{2}a_0 + \int_0^1 A(\nu)\Phi_\nu(\mu)\,d\nu,$$

where,

$$(2.20b)\qquad a_0 = 2\int_0^1 \frac{s\psi(s)}{X(-s)}\,ds \div \int_0^1 \frac{s\,ds}{X(-s)}\ ,$$

and,

$$(2.20c) \qquad A(\nu) = \frac{X(-\nu)}{N(\nu)} \int_0^1 \frac{s\psi(s)\Phi_\nu(s)}{X(-s)}\,ds.$$

Section 3: The Multigroup Full-Range Expansion

We write the multi-group neutron transport equation in the form,

$$(3.1) \qquad \frac{\partial\psi(x,\mu)}{\partial x} + K^{-1}\psi(x,\mu) = \frac{q(x,\mu)}{\mu}, \mu \neq 0,$$

with,

$$(3.2) \qquad K^{-1}\psi(x,\mu) = \frac{1}{\mu}[\Sigma\psi(x,\mu) - C\int_{-1}^{+1} \psi(x,s)ds], \mu \neq 0.$$

Here ψ is an N-component vector where the i-th component represents the neutron angular densities in the i-th group, Σ is the diagonal cross section matrix, and C the group-group transfer matrix. The appropriate space to seek a solution is

$$X_p^N = \bigotimes_{i=1}^{N} X_p$$

and X_p is the Banach space mentioned in Section 1. As in the one-speed case, the computations are done in a dense subspace of Hölder continuous functions, and the results can be extended to X_p^N by continuity.[22]

We have the dispersion function

$$(3.3) \qquad \Lambda(z) = (\Sigma - 2C)C^{-1}\Sigma - \int_{-1}^{+1} sD(z,s)ds,$$

where,

$$(3.4)\qquad D(z,\mu) = (zI-\mu\Sigma^{-1})^{-1}.$$

As we are dealing with the case of conservative neutron transport we have

$$(3.5)\qquad \det(\Sigma-2C) = 0.$$

In this case K^{-1} given by Eq. (3.2) is not invertible on its range. Thus, defining S as before, i.e., $S = (K^{-1}-iI)^{-1}$, we find

$$(3.6a)\qquad S\eta(\mu) = B(\mu)(\mu\eta(\mu) + \Sigma(C^{-1} - \int_{-1}^{+1} B(s)ds)^{-1} \int_{-1}^{+1} sB(s)\eta(s)ds),$$

where,

$$(3.6b)\qquad B(\mu) = (\Sigma-i\mu I)^{-1}.$$

We have assumed that $z = i$ is in the resolvent set of K^{-1}. If not, any other point could be chosen, assuming the spectrum of K^{-1} does not consist of the entire complex plane. Furthermore, we have assumed that $\det \Lambda(z)$ vanishes as $1/z$ as $|z|\to\infty$.

It is convenient to define,

$$(3.7)\qquad F(z,\mu) = (zI-\mu B(\mu))^{-1}.$$

Then a direct computation gives

$$(3.8a)\qquad (zI-S)^{-1}\psi(\mu) = F(z,\mu)(\psi(\mu) + B(\mu)R^{-1}(z)[C^{-1}- \int_{-1}^{+1} B(s)ds]^{-1}$$

$$\times \int_{-1}^{+1} tB(t)F(z,t)\psi(t)dt).$$

Here we have defined

$$(3.8b)\qquad R(z) = I-[C^{-1}-\int_{-1}^{+1}B(s)ds]^{-1}\int_{-1}^{+1}tB^2(t)F(z,t)dt.$$

R is related to Λ by

$$(3.9)\qquad R(z) = [C^{-1}-\int_{-1}^{+1}B(s)ds]^{-1}\Sigma^{-1}\Lambda(t^{-1}(z))\Sigma^{-1}.$$

To see this we note that

$$(3.10a)\qquad \int_{-1}^{+1}sB^2(s)F(z,s)ds = \int_{-1}^{+1}\frac{s}{1+iz}\,\Sigma^{-1}(\frac{z}{1+iz}\,I-s\Sigma^{-1})^{-1}(\Sigma-isI)^{-1}ds,$$

and thus,

$$(3.10b)\qquad (C^{-1}-\int_{-1}^{+1}B(s)ds)R(z) = C^{-1}-\int_{-1}^{+1}\Sigma^{-1}(I-is\Sigma^{-1})^{-1}ds-\int_{-1}^{+1}\frac{s\Sigma^{-1}}{1+iz}$$

$$\times\,(\frac{z}{1+iz}\,I-s\Sigma^{-1})^{-1}(\Sigma^{-1}-isI)^{-1}ds,$$

which yields,

$$(3.10c)\qquad \Sigma(C^{-1}-\int_{-1}^{+1}B(s)ds)R(z)\Sigma^{-1} = \Sigma C^{-1}\Sigma-\Sigma\int_{-1}^{+1}[\frac{z}{1+iz}\,I-s\Sigma^{-1}$$

$$+\frac{s\Sigma^{-1}}{1+iz}](\frac{z}{1+iz}\,I-s\Sigma^{-1})(\Sigma-isI)^{-1}ds\Sigma$$

$$= \Sigma C^{-1}\Sigma-\int_{-1}^{+1}\Sigma\frac{z}{1+iz}(\frac{z}{1+iz}\,I-s\Sigma^{-1})^{-1}ds$$

$$= \Sigma C^{-1}\Sigma-2\Sigma-\int_{-1}^{+1}(\frac{z}{1+iz}\,\Sigma-\frac{z\Sigma}{1+iz}+sI)(\frac{z}{1+iz}\,I-s\Sigma^{-1})^{-1}ds$$

$$= (\Sigma-2C)C^{-1}\Sigma - \int_{-1}^{+1} sD(t^{-1}(z),s)ds = \Lambda(t^{-1}(z)),$$

and hence Eq. (3.9) is verified. Since det $\Lambda(z)$ has a double zero at infinity, it follows that det $R(z)$ will have a double zero at $t(\infty) = i$. The continuous spectrum of K transorms into the semicircle, $\{z : z = \frac{1}{2}(i + \exp(i\theta)), -\Pi \leq \theta \leq 0\}$, and the additional eigenvalues of K (zeroes of $\Lambda(z)$) transform by $\nu_i \to t(\nu_i)$.

The eigenfunction expansion is again obtained by integrating the resolvent around the spectrum. The integration around the continuous spectrum can be transformed into the identical form found in Ref. 15 (or see the result for the subcritical situation which is also identical)[13] by the change of variable $z' = t^{-1}(z)$. Similarly the integration about the isolated point eigenvalues, ν_i, can, by the same change of variable, be transformed into the expansion met in Refs. 15 and 17. Only the contribution from the double pole at i remains to be evaluated. Again the appropriate residue for a second order pole must be used.

Denoting the integral by I_1 we have from Eqs. (3.8a) and (3.9),

$$(3.11) \qquad I_1 = \frac{1}{2\Pi i} \int_{\Gamma_i} (F(z,\mu)B(\mu) \frac{\Sigma\Lambda_c^T(t^{-1}(z))}{\det\Lambda(z)} \Sigma \int_{-1}^{+1} sB(s)F(z,s)\psi(s)ds)dz.$$

Using the diagonal expansion of $\det\Lambda(z)$ (see Appendix I), we find

$$(3.12) \qquad \det \Lambda(z) = \det(\Sigma C^{-1}\Sigma - 2\Sigma) - \frac{2}{3z^2} \operatorname{Tr}(\Sigma^{-1}\Lambda_c(\infty)) + O(\frac{1}{z^4}),$$

where $\Lambda_c(z)$ is the cofactor matrix of $\Lambda(z)$. By definition of the critical multi-group problem, the first term on the r.h.s. of Eq. (3.12) is zero and the remaining term gives us,

$$(3.13)\qquad I_1 = \frac{1}{2\Pi i}\int_{\Gamma_i} (F(z,\mu)B(\mu)\Sigma\Lambda_c^T(t^{-1}(z))\Sigma\int_{-1}^{+1} sB(s)F(z,s)\psi(s)ds)$$

$$\times\ (-\ \frac{2}{3}(\frac{1+iz}{z})^2 Tr(\Sigma^{-1}\Lambda_c(\infty)) + O((\frac{1+iz}{z})^4))^{-1}dz.$$

We note the following equalities,

$$(3.14a)\qquad F(i,\mu)B(\mu) = B(\mu)F(i,\mu) = -i\Sigma^{-1},$$

$$(3.14b)\qquad \frac{d}{dz}F(z,\mu)\Big|_{z=i} = -F^2(i,\mu),$$

and,

$$(3.14c)\qquad F(i,\mu) = -(iI+\mu\Sigma^{-1}).$$

Using Eq. (3.14a) one finds,

$$(3.15)\qquad F(i,\mu)B(\mu)\Sigma\Lambda_c^T\Sigma\int_{-1}^{+1} sB(s)F(i,s)\psi(s)ds = -\Lambda_c^T\int_{-1}^{+1} s\psi(s)ds.$$

Using the above and simple residue theory for a second order pole we obtain

$$(3.16)\qquad I_1 = [\frac{2}{3}Tr(\Sigma^{-1}\Lambda_c^T(\infty))]^{-1}[-2i\Lambda_c^T(\infty)\int_{-1}^{+1} s\psi(s)ds - F(i,\mu)\Lambda_c^T(\infty)$$

$$\times \int_{-1}^{+1} s\psi(s)ds - \Lambda_c^T(\infty)\int_{-1}^{+1} F(i,s)s\psi(s)ds] = [\frac{2}{3}T_r(\Sigma^{-1}\Lambda_c^T(\infty))]^{-1}$$

$$\times\ [-2i\Lambda_c^T(\infty)\int_{-1}^{+1} s\psi(s)ds + i\Lambda_c^T(\infty)\int_{-1}^{+1} s\psi(s)ds + \mu\Sigma^{-1}\Lambda_c^T(\infty)$$

$$\times \int_{-1}^{+1} s\psi(s)ds + i\Lambda_c^T(\infty)\int_{-1}^{+1} s\psi(s)ds + \Lambda_c^T(\infty)\Sigma^{-1}\int_{-1}^{+1} s^2\psi(s)ds]$$

$$= [\tfrac{2}{3}\mathrm{Tr}(\Sigma^{-1}\Lambda_c^T(\infty))]^{-1}[\Lambda_c^T(\infty)\Sigma^{-1}\int_{-1}^{+1} s^2\psi(s)ds + \mu\Sigma^{-1}\Lambda_c^T(\infty)\int_{-1}^{+1} s\psi(s)ds].$$

Using the results of Ref. 15 for the integrations about the continuous spectrum and the isolated point eigenvalues and Eq. (3.16) for the integration about the double pole we have,

$$(3.17)\quad \psi(\mu) = \sum_{i=1}^{2n} \psi_{\nu_i} + \psi_\Gamma + [\tfrac{2}{3}\mathrm{Tr}(\Sigma^{-1}\Lambda_c(\infty))]^{-1}\,[\Lambda_c^T(\infty)\Sigma^{-1}\int_{-1}^{+1} s^2\psi(s)ds + \mu\Sigma^{-1}\Lambda_c^T(\infty)\int_{-1}^{+1} s\psi(s)ds].$$

The last term on the r.h.s. of Eq. (3.17) can be expressed in the manner of Ref. 15 by noting (see Appendix I) that

$$(3.18)\quad \Lambda_c^T(\infty) = \tfrac{2}{3}\mathrm{Tr}(\Sigma^{-1}\Lambda_c(\infty))\xi\hat{\xi}^T\Sigma,$$

and thus,

$$(3.18b)\quad \Lambda_c^T(\infty)\Sigma^{-1}\psi = \tfrac{2}{3}\mathrm{Tr}(\Sigma^{-1}\Lambda_c(\infty))[\psi,\hat{\xi}]\xi,$$

and,

$$(3.18c)\quad \Sigma^{-1}\Lambda_c^T(\infty)\psi = \tfrac{2}{3}\mathrm{Tr}(\Sigma^{-1}\Lambda_c(\infty))[\psi,\Sigma\hat{\xi}]\Sigma^{-1}\xi,$$

where ψ is a column vector and we have defined the inner product $[\cdot,\cdot]$ by,

$$(3.18d)\quad [a,b] = \sum_{i=1}^{N} a_i b_i .$$

Combining the above with Eq. (3.17) we obtain

$$(3.19)\qquad \psi(\mu) = \sum_{i=1}^{2n} \psi_{\nu_i} + \psi_\Gamma + \int_{-1}^{+1} ds\; s^2[\psi(s),\hat{\xi}]\xi + \int_{-1}^{+1} ds\; s[\psi(s),\Sigma\hat{\xi}]\mu\Sigma^{-1}\xi .$$

The first term on the r.h.s. is the contribution from the finite eigenvalues of K. This, along with ψ_Γ, is identical to the sub-critical result obtained in Ref. 13. Only the contribution from the eigenvalue at infinity is essentially different in the critical case.

Section 4: The Multigroup Half-Range Expansion

For the half-space expansion, again an "albedo operator" E must be introduced. This operator has precisely the same properties as in the one-speed case presented in Section 2. The appropriate E is

$$(4.1)\qquad (E\psi)_i(\sigma_i\mu) = \begin{cases} -[X^{-1}(\mu)\int_0^1 s(\mu-s)^{-1}Y^{-1}(-s)\Sigma^2\psi_\Sigma(s)ds]_i, & -1\le\sigma_i\mu\le 0, \\ \psi_i(\Sigma_i\mu), & \mu>0, \end{cases}$$

where X and Y provide the Wiener-Hopf factorization of Λ,[23]

$$(4.2)\qquad Y(-z)X(z) = \Lambda(z).$$

We now compute

$$(4.3)\qquad \frac{1}{2\pi i}\int (zI-S)^{-1}E\psi(\mu)dz$$

about the spectrum of S obtaining

$$(4.4) \quad \sum_{i=1}^{n} \psi_{\Gamma_i} + \psi_{\Gamma} + \frac{1}{2\Pi i} \int_{\Gamma_i} (zI-S)^{-1} E\psi(\mu)dz.$$

Here ψ_{Γ} and ψ_{Γ_i} are defined in Ref. 14 and are computed in analogous fashion. The remaining integral is of the same form as I_1 of Eq. (3.11) except with ψ replaced by $E\psi$. Using the results of Section 3 one finds,

$$(4.5) \quad \frac{1}{2\Pi i} \int_{\Gamma_i} (zI-S)^{-1} E\psi(\mu)dz = \int_{-1}^{+1} ds\ s^2[E\psi(s),\hat{\xi}]\xi + \int_{-1}^{+1} ds\ s$$

$$\times\ [E\psi(s),\Sigma\hat{\xi}]\mu\Sigma^{-1}\xi.$$

The second integral on the r.h.s. of Eq. (4.5) is calculated in Ref. 15 and found to be zero. Thus, we write the half-range expansion in the form,

$$(4.6) \quad \psi(\mu) = \sum_{i=1}^{n} \psi_{\Gamma_i} + \psi_{\Gamma} + \int_{-1}^{+1} ds\ s^2[E\psi(s),\hat{\xi}]\xi.$$

Section 5: Extension to X_p and the Development of the Functional Calculus for S.

In the following sections an outline of the extension to X_p and the development of the functional calculus for S is presented for the one-speed case. The procedures followed are identical in the multi-group case.[15] Since a discussion of the multi-group case would add no new insights it will be omitted.

S is a bounded operator with domain $Hö_\alpha([-1,1])$. As $Hö_\alpha([-1,1])$ is a dense subset of the Banach space X_p, the domain of S can be extended to X_p by continuity.[21] Define the operators $T:X_p \to X_p$ and $\tilde{T}:X_p \to X_p$ by

$$(5.1) \quad (TA)(\mu) = \int_{-1}^{+1} A(\nu)\Phi_\nu(\mu)d\nu,$$

and,

$$(5.2) \quad (\tilde{T}\psi)(\nu) = \frac{1}{N(\nu)} \int_{-1}^{+1} s\psi(s)\Phi_\nu(s)ds,$$

where $\Phi_\nu(\mu)$ and $N(\nu)$ are as defined in Eqs. (1.25c) and (1.25d). Define the linear functionals $\rho_i:X_p \to \mathbb{R}$ by

$$(5.3) \quad \rho_i(\psi) = 3\int_{-1}^{+1}(-s)^{2-i}\psi(s)ds.$$

Then the expansion, Eq. (1.31), can be cast in the form

$$(5.4) \quad \psi(\mu) = \frac{1}{2}\rho_0(\psi) - \frac{1}{2}\rho_1(\psi)\mu + (TA)(\mu),$$

where the expansion coefficient, A, is given by,

$$(5.15) \quad A(\nu) = (\tilde{T}\psi)(\nu).$$

One can show that the operators, ρ_i, T, $\tilde{T}$ are bounded on X_p.[21] Furthermore, if we define X'_p by

$$(5.6) \quad X'_p = \{f\varepsilon X_p : \rho_i(f) = 0,\ i\varepsilon\{0,1\}\}$$

then it can be shown that for every $A \varepsilon X_p$ there exists a corresponding

$\psi \varepsilon X'_p$ such that

(5.7) $\quad A = \tilde{T}\psi .$

And for every $\psi \varepsilon X'_p$ there exists a corresponding $A \; \varepsilon \; X_p$ such that

(5.8) $\quad \psi = TA \, .$

Using these facts one concludes that the expansion (1.31) is valid for $\psi \; \varepsilon \; X_p$.[21]

In order to solve transport problems it is necessary to develop a functional calculus for S. Following Ref. 21 we define the operator P(w) (corresponding to E(w) in Ref. 21) by,

(5.9) $\quad P(w)f(\mu) = \int_{-1}^{w} A(\nu)\phi_\nu(\mu)d\nu ,$

where $A(\nu)$ is the expansion coefficient, or Case transform, of f. $\Phi_\nu(\mu)$ is the Case continuum eigenfunction. One can show for $f \; \varepsilon \; X'_p$,[21]

(5.10) $\quad P(w_1)P(w_2)f = P(w_3)f, \; w_3 = \inf\{w_1, w_2\},$

(5.11) $\quad P(1) = I,$

and

(5.12) $\quad P(-1) = 0,$

where I denotes the identity operator. One can also prove that P(w) is a continuous function of w in the strong operator topology and that the following hold,

$$(5.13)\qquad Sf = \int_{-1}^{+1} t(w)dP(w)f,\ f\varepsilon X'_p,$$

where t is given by Eq. (1.19a), and

$$(5.14)\qquad SP(w) = P(w)S.$$

The facts cited above imply that P(w) is the spectral family of projection operators for S.[24]

Employing the results of Ref. 21 we also have that,

$$(5.15)\qquad (Sf)(\mu) = \int_{-1}^{+1} t(w)d[P(w)f](\mu).$$

If we define P_0 and P_1 by

$$(5.16)\qquad (P_0 f)(\mu) = \frac{1}{2}\rho_0(f),$$

and,

$$(5.17)\qquad (P_1 f)(\mu) = -\frac{1}{2}\rho_1(f)\mu,$$

then any $\psi \varepsilon X_p$ can be written as

$$(5.18)\qquad \psi(\mu) = [P_0+P_1+\int_{-1}^{+1} dP(w)]\psi.$$

Furthermore, if F is a rational function and $\psi' = (I-P_0-P_1)\psi$ then

$$(5.19)\qquad F(S)\psi' = \int_{-1}^{+1} F(t(w))dP(w)\psi'.$$

We are interested in the action of K^{-1} on ψ which can be written as

$$(5.20)\qquad K^{-1}\psi = K^{-1}(I-P_0-P_1)\psi + K^{-1}(P_0+P_1)\psi.$$

But,

$$(5.21)\qquad K^{-1} = S^{-1} + iI = \frac{1}{t^{-1}(S)} ,$$

with a slight abuse of notation, where t^{-1} is defined as in Eq. (1.12b). Employing Eqs. (5.21) and (5.19) we have

$$(5.22)\qquad K^{-1}(I-P_0-P_1)\psi = \int_{-1}^{+1} \frac{1}{t^{-1}(t(w))} \, dP(w)(I-P_0-P_1)\psi,$$

i.e.,

$$(5.23)\qquad K^{-1}(I-P_0-P_1)\psi = \int_{-1}^{+1} \frac{1}{w} \, dP(w)(I-P_o-P_1)\psi.$$

The action of $K^{-1}(P_0+P_1)$ on ψ is easily calculated and the action of $K^{-1}(I-P_0-P_1)$ is given by Eq. (5.23), which now puts us in a position to solve transport problems.

Section 6: The Milne Problem

The Milne problem involves the determination of the neutron distribution in a source-free half-space with zero incident flux.[17] A solution to the one-speed, homogeneous transport equation of the form,

$$(6.1)\qquad \Psi_M(x,\mu) = \frac{1}{2}a_0 + \frac{1}{2}(x-\mu) + \int_0^{+1} A(\nu)\phi_\nu(\mu)e^{-x/\nu}d\nu,$$

is sought which obeys the boundary condition,

$$(6.2)\qquad \Psi_M(0,\mu) = 0, \ \mu > 0.$$

Let

$$(6.3) \qquad \psi(\mu) = \frac{1}{2} a_0 + \int_0^{+1} A(\nu)\Phi_\nu(\mu)d\nu.$$

By Eq. (2.11) ψ must obey

$$(6.4) \qquad \psi = EP^+\psi.$$

The boundary condition, Eq. (6.2), implies

$$(6.5) \qquad P^+\psi - \frac{1}{2}P^+\mu = 0.$$

Applying E to Eq. (6.5) and using Eq. (6.4) we obtain,

$$(6.6) \qquad \psi = \frac{1}{2} EP^+\mu,$$

and thus the coefficients, a_0 and $A(\nu)$, are obtained by applying the half-range expansion to $\frac{1}{2}\mu$. The result is,

$$(6.7a) \qquad a_0 = \int_0^1 s\gamma(s)ds \Big/ \int_0^1 \gamma(s)ds,$$

and

$$(6.7b) \qquad A(\nu) = \frac{1}{2} \frac{\nu}{\gamma(\nu)N(\nu)} \int_0^1 s\gamma(s)\Phi_\nu(s)ds.$$

By construction the function obtained satisfies the boundary conditions. To show that it also satisfies the transport equation it is necessary to apply Eq. (5.23) of the previous section.

Chapter 3

INTRODUCTION TO THE TEMPERATURE DENSITY EQUATION

Section 1: Derivation of the Boltzmann Equation

The state of a gas containing N molecules enclosed in a finite volume, V, can be represented by a probability density, $P_N(x_1, x_2, \ldots x_N; \xi_1, \xi_2, \ldots \xi_N; t)$ where $P_N\, dx_1, dx_2, \ldots dx_N\, d\xi_1\, d\xi_2 \ldots d\xi_N$ is the probability of finding molecule n in the volume element dx_n centered at the position x_n with a velocity in the volume element $d\xi_n$ centered at the velocity ξ_n at time t with n ranging from 1 to N. It will be assumed that the molecules do not interact with the boundary of V except when they strike it. The only restriction on the type of interaction at the boundary is that the net flux of molecules through the boundary is zero, i.e.,

$$(1.1) \qquad \int_S \int_{-\infty}^{+\infty} \xi_i \cdot da_i P_N = 0,$$

where S is the boundary of V and da_i is the area element of the i-th coordinate. Note that Maxwell's boundary condition with arbitrary accommodation satisfies this restriction. It will also be assumed that the molecules have no internal degrees of freedom so that they can be considered as essentially point masses. In this case $P_N(X;\Xi;t) = 0$ for $x_i \notin V$, $1 \leq i \leq N$, because no molecules can escape V (here X represents the 3N dimensional space $x_1 \times x_2 \ldots \times x_N$ and Ξ the corresponding velocity space $\xi_1 \times \xi_2 \ldots \times \xi_N$).

Furthermore,

$$(1.2) \qquad \int_V \int_\Xi P_N(X;\Xi;t)dXd\Xi = 1,$$

since this expresses the certainty of finding all N molecules in the volume V.

The one-particle distribution function $P_N^1(x_1;\xi_1;t)$ is given by

$$(1.3) \qquad P_N^1(x_1;\xi_1;t) = \int_{X-x_1} \int_{\Xi-\xi_1} P_N(X;\Xi;t)d(X-x_1)d(\Xi-\xi_1),$$

i.e., all coordinates except x_1 and ξ_1 are integrated over. Consequently, $P_N(x_1;\xi_1;t)dx_1d\xi_1$ expresses the probability that molecule 1 is located in the volume element dx_1 centered at x_1 with a velocity in the velocity element $d\xi_1$ centered at ξ_1 at time t, independent of the positions and the velocities of the other N-1 molecules. Similarly, the two-particle distribution function, $P_N^2(x_1,x_2;\xi_1,\xi_2;t)$, is given by

$$(1.4) \qquad P_N^2(x_1,x_2;\xi_1,\xi_2;t) = \int_{X-x_1-x_2} \int_{\Xi-\xi_1-\xi_2} P_N(X;\Xi;t)d(X-x_1-x_2) \times d(\Xi-\xi_1-\xi_2).$$

If N is very large the molecules will be constantly undergoing collisions with each other. In addition, if we assume that the molecular interaction has a finite range, σ, and if it is assumed that the molecules are in thermal equilibrium, then the velocities and positions of the molecules will be uncorrelated.[18] Thus, the probability of finding molecule 1 at x_1 with velocity ξ_1 and simultaneously finding molecule 2 at position x_2 with velocity ξ_2 will simply be the product of the probability that molecule 1 has coordinates (x_1,ξ_1) independent of the other molecules and the probability that molecule 2 has coordinates (x_2,ξ_2) independent of the other molecules, i.e.,

$$(1.5)\qquad P_N^2(x_1,x_2;\xi_1,\xi_2;t) = P_N^1(x_1;\xi_1;t)P_N^2(x_2;\xi_2;t).$$

For a gas isolated from external forces and confined to the volume, V, the state of the gas will be constant in time and thus the total time derivative of $P_N(X;\Xi;t)$ must be zero. This requirement yields the Liouville equation,

$$(1.6)\qquad \frac{\partial P_N}{\partial t} + \sum_{i=1}^{N} \xi_i \cdot \nabla_i P_N + \sum_{i=1}^{N} F_i \cdot \nabla_{\xi_i} P_N = 0,$$

where ∇_i is the gradient with respect to the i-th spatial coordinates, ∇_{ξ_i} is the gradient with respect to the i-th velocity coordinates and F_i is the force on the i-th molecule. It is assumed that the intermolecular force is repulsive, of finite range, σ, and is directed along the line connecting the interacting molecules. In this case we have,

$$(1.7a)\qquad F_i = \sum_{\substack{j=1\\ j\neq i}}^{N} F(|x_i-x_j|)\,\frac{x_i-x_j}{|x_i-x_j|},$$

where,

$$(1.7b)\qquad F(s) = 0,\ s > \sigma.$$

If we restrict our attention to rarefied gases it is reasonable to assume that trinary collisions and higher order collisions may be neglected, i.e., assume that only binary collisions occur. Let V_{ij} denote the volume $\{x: |x_i - x| < \frac{1}{2}\sigma \text{ and } |x_j - x| < \frac{1}{2}\sigma\}$ and let S_{ij} be the surface of V_{ij}. In the region, $R = V - \underset{ij}{U} V_{ij}$, the Liouville equation reduces to,

$$(1.8) \quad \frac{\partial P_N}{\partial t} + \sum_{i=1}^{N} \xi_i \cdot \nabla_i P_N = 0,$$

since there are no intermolecular forces in this region. Integrating (1.8) over all spacial coordinates except x_1 and over all velocity coordinates except ξ_1, one obtains

$$(1.9a) \quad \frac{\partial P_N^1}{\partial t} + \int_{X-x_1} \int_{\Xi-\xi_1} \xi_1 \cdot \nabla_1 P_N d(X-x_1) d(\Xi-\xi_1)$$

$$+ \sum_{j=2}^{N} \int_{X-x_1} \int_{\Xi-\xi_1} \xi_i \cdot \nabla_i P_N d(X-x_1) d(\Xi-\xi_1) = 0.$$

Thus,

$$(1.9b) \quad \frac{\partial P_N^1(x_1;\xi_1;t)}{\partial t} + \xi_1 \cdot \nabla_1 P_N^1(x_1;\xi_1;t) - (N-1) \int_{S_{12}} \int (\xi_1 \cdot \hat{n})$$

$$\times P_N^2(x_1,x_2;\xi_1,\xi_2;t) da(x_2) d\xi_2 + (N-1) \int_{S_{12}} \int (\xi_2 \cdot \hat{n}) P_N^2(x_1,x_2;\xi_1,\xi_2;t)$$

$$\times da(x_2) d\xi_2 = 0,$$

where the velocity integration is over all velocity space and the surface integrals are over the sphere of radius σ centered at x_1 with inward normal $\hat{\eta}$. The first surface integral in Eq. (1.9b) is due to the dependence of the range of integration in Eq. (1.9a) on the x_1 coordinates and the second surface integral is due to the application of Gause's theorem. Additional surface integrals arise from the dependence of the integration range on the coordinates with subscripts greater than 1 but these cancel each other. The integration over the surface enclosing V is zero by Eq. (1.1).

Equation (1.9b) can be rewritten as,

$$(1.10)\quad \frac{\partial P_N^1(x;\xi;t)}{\partial t} + \xi \cdot \nabla P_N^1(x;\xi;t) = (N-1) \int_{\Xi-\xi_1} \int_{|x-x_1|=\sigma}$$

$$\times\ P_N^2(x,x_1;\xi,\xi_1;t)(\xi-\xi_1) \cdot \hat{\eta}\, da(x_1)d\xi_1 .$$

If we consider molecules such that $(\xi-\xi_1) \cdot \hat{\eta} > 0$, these molecules are just entering into a collision and, if N is very large, we may assume that they are uncorrelated, just as in the case of a dense gas. This is the molecular chaos assumption. Consequently,

$$(1.11)\quad P_\infty^2(x,x_1;\xi,\xi_1;t) = P_\infty^1(x;\xi;t)P_\infty^1(x_1;\xi_1;t), (\xi-\xi_1) \cdot \hat{\eta} > 0.$$

This assumption cannot be made for molecules which have just interacted because their trajectories are correlated by the collision. However, inside the sphere of radius σ we know that,

$$(1.12a)\quad \xi_1 = \frac{dx_1}{dt}, \ \xi = \frac{dx}{dt},$$

(1.12b) $F_1 = \frac{d\xi_1}{dt}$, $F = \frac{d\xi}{dt}$,

and thus,

(1.12c) $\frac{\partial P_N^2}{\partial t} + \frac{dx_1}{dt} \cdot \nabla_1 P_N^2 + \frac{dx}{dt} \cdot \nabla P_N^2 + F_1 \cdot \nabla_{\xi_1} P_N^2 + F \cdot \nabla_{\xi} P_N^2 = 0,$

i.e.,

(1.12d) $\frac{dP_N^2}{dt} (x,x_1;\xi,\xi_1;t) = 0.$

Thus, if the duration of the collision is taken to be τ we have,

(1.13a) $P_N^2(x,x_1;\xi,\xi_1;t) = P_N^2(x^*,x_1^*;\xi',\xi_1';t-\tau),$

where,

(1.13b) $(\xi-\xi_1) \cdot \hat{n} < 0$ and $(\xi'-\xi'_1) \cdot \hat{n} > 0.$

Here the coordinates x^*,x_1^*,ξ',ξ'_1 are the coordinates of the molecules before the collision and x,x_1,ξ,ξ_1 the coordinates after the collision. Since the molecules are uncorrelated before the collision we may write,

(1.14) $P_\infty^2(x,x_1;\xi,\xi_1;t) = P_\infty^1(x^*;\xi';t-\tau)P_\infty^1(x_1{}^*;\xi_1';t-\tau).$

Since the duration of a collision is very short and the region of interaction is of the same order of magnitude as σ, a very small quantity, Eq. (1.14) reduces to,

$$(1.15)\qquad P_N^2(x,x_1;\xi,\xi_1;t) = P_N^1(x;\xi';t)P_N^1(x_1;\xi_1';t),$$

where ξ', ξ_1' are related to ξ, ξ_1 by energy and momentum conservation. Thus, for N very large, we have,

$$(1.16)\qquad \frac{\partial P_N^1(x;\xi;t)}{\partial t} + \xi \cdot \nabla P_N^1(x;\xi;t) = N \int_{|x-x_1|=\sigma} \int_{(\xi-\xi_1)\cdot\hat{\eta}>0}$$

$$\times \ [P_N^1(x;\xi;t)P_N^1(x;\xi_1;t) - P_N^1(x;\xi';t)P_N^1(x;\xi'_1;t)]$$

$$\times \ |(\xi-\xi_1)\cdot\hat{\eta}|da(x_1)d\xi_1.$$

Define f by $f(x,\xi,t) = NP_N^1(x;\xi;t)$. Then the equation for f is,

$$(1.17a)\qquad \frac{\partial f(\xi)}{\partial t} + \xi \cdot \nabla f(\xi) = \int\int[f(\xi)f(\xi_1) - f(\xi')f(\xi'_1)]$$

$$\times \ V da(x_1)d\xi_1 \ ,$$

where,

$$(1.17b)\qquad V = |(\xi-\xi_1) \cdot \hat{\eta}|,$$

and the limits of integration are understood to be the same as in Eq. (1.16).

Equation (1.17) is the Boltzmann equation for f, the particle distribution function. Various moments of f give the macroscopic physical properties of the gas. The number density of the gas is given by, (1.18a) $\quad n(x) = \int f(x,\xi)d\xi,$

the fluid velocity by,

(1.18b) $\quad v(x) = \int f(x,\xi)\xi d\xi/n(x),$

and the temperature of the gas by,

(1.18c) $\quad T(x) = \frac{1}{3}\,\frac{m}{kn(x)} \int (\xi - v(x))f(x,\xi)d\xi,$

here m is the mass of a molecule and k is Boltzmann's constant.

Before moving on a few of the properties of Eq. (1.17) should be discussed; for a detailed discussion see Refs. 18 and 19. Eq. (1.17a) is usually written in the form,

(1.19) $\quad \frac{\partial f}{\partial t} + \xi \cdot \nabla f = Q(f,f),$

where $Q(f,f)$ is the bilinear functional defined by,

(1.20) $\quad Q(f,g) = \frac{1}{2} \int\int\{f(\xi)g(\xi_1) + g(\xi)f(\xi_1) - f(\xi')g(\xi'_1)$

$$- f(\xi'_1)g(\xi')\}\, V\, da(x_1)d\xi_1.$$

It can be shown that,

(1.21) $\quad \int Q(f,f)\phi(\xi)d\xi = 0,$

if and only if ϕ has the property,

(1.22) $\quad \phi(\xi) + \phi(\xi_1) = \phi(\xi') + \phi(\xi'_1),$

i.e., if ϕ is a conserved quantity. Functions satisfying (1.21) are called collision invariants because the average effect of collisions leaves them unchanged. The collision invariant functions are the set spanned by the five functions: $1, \xi_1, \xi_2, \xi_3, \xi^2$. The function, 1, corresponds to conservation of particles, ξ_1, ξ_2 and ξ_3 correspond to conservation of momentum, and ξ^2 corresponds to conservation of energy. Functions of the form,

$$(1.23) \quad q(\xi) = \exp(a+b\cdot\xi+c\xi^2),$$

where a and c are constants and b is a constant vector have the property,

$$(1.24) \quad Q(q(\xi),q(\xi)) = 0.$$

Finally, if one defines the quantities,

$$(1.25) \quad \hat{H} = \int f \ln(f)d\xi,$$

$$(1.26) \quad H_i = \int \xi_i f \ln(f)d\xi,$$

and,

$$(1.27) \quad H = \int_V \hat{H}dx,$$

then one can prove Boltzmann's H theorem using the fact that,

$$(1.28) \quad \int \ln(f)Q(f,f)d\xi \leq 0.$$

The H theorem states that the quantity H always decreases with time except when f is Maxwellian and then H remains constant.

Section 2: Derivation of the Temperature Density Equation and the Transverse Velocity Equation.

Due to the complicated nature of the collision operator, Q, model equations are studied in which the collision integral is replaced by a more tractable expression. We will study one of the simplest of these model equations, the BGK model, in which the collision operator is replaced by J(f) where,

(2.1) $J(f) = \nu(\hat{f}(x;\xi;t) - f(x;\xi;t))$.

Here the constant parameter, ν, is the collision frequency and $\hat{f}(x;\xi;t)$ is a Maxwellian satisfying,

(2.2) $\int(\hat{f}(x;\xi;t) - f(x;\xi;t)U(\xi)d\xi = 0$,

where,

(2.3) $$U(\xi) = \begin{bmatrix} 1 \\ \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi^2 \end{bmatrix}.$$

An $\hat{f}$ satisfying (2.2) will give, at any point in space and any instant in time, the same density, fluid velocity and temperature as f. This model term, J(f), has the advantage of relative simplicity as well as retaining many of the important properties of the collision operator, Q(f,f). It can be shown[18] that J has the same collision invariants as Q and that Eq. (1.28) remains valid with Q replaced by

J and thus, solutions to the model equation will obey an H theorem.

We will consider time-independent problems with plane symmetry, allowing Eq. (1.19) to be cast in the form,

$$(2.4)\qquad \xi \cdot \nabla f(x;\xi) = \nu(\hat{f}(x;\xi)-f(x;\xi)),$$

where $Q(f,f)$ has been replaced by $J(f)$. Equation (2.4) may appear simple. However, the requirement that $\hat{f}$ satisfy (2.2) introduces complications. To further simplify the equation we will assume that to zeroeth order the state of the gas can be described by a Maxwellian, $f_0(\xi)$, and to first order f is given by,

$$(2.5)\qquad f(x;\xi) = f_0(\xi)(1+h(x;\xi)),$$

so that terms of order $(f_0h)^2$ may be neglected.

Let

$$(2.6)\qquad f_0(\xi) = n_0\left(\frac{m}{2\Pi kT_0}\right)^{3/2} \exp\left(-\frac{m}{2kT_0}(\xi-V_0)^2\right),$$

where n_0 is the density, T_0 the temperature and V_0 the fluid velocity corresponding to $f_0(\xi)$. Also define the inner product,[18]

$$(2.7)\qquad (p,q) = \int f_0(\xi)p(\xi)q(\xi)d\xi.$$

Define the dimensionless velocity, C, by

$$(2.8)\qquad C = \sqrt{\frac{m}{2kT_0}}\,(\xi-V_0).$$

Then,

$$(2.9)\qquad f_0 d\xi = n_0\Pi^{-3/2}\exp(-C^2)dC.$$

Note that the basis vectors,

(2.10a) $e_0 = n_0^{-1/2}$,

(2.10b) $e_i = (\frac{2}{n_0})^{1/2} C_i,\ 1 \le i \le 3$,

(2.10c) $e_4 = (\frac{2}{3n_0})^{1/2}(C^2 - \frac{3}{2})$,

satisfy

(2.11) $(e_i, e_j) = \delta_{ij}$.

Equation (2.2),

$$\int (\hat{f}(x;\xi) - f(x;\xi))U(\xi)d\xi = 0,$$

is satisfied by $\hat{f}$ given by,

(2.12) $\hat{f}(x;\xi) = f_0(\xi) \sum_{i=1}^{4} (1+h(x;\xi), e_i)e_i$.

One easily verifies that $\hat{f}$ given by (2.12) gives the same density, fluid velocity and temperature as f, to first order, by using the orthogonality of the e_i. For example, the density corresponding to $\hat{f}$ is given by,

(2.13) $\hat{f}(x;\xi)d\xi = \int\{n_0^{1/2} e_0 f_0(\xi) \sum_{i=1}^{4} (1+h(x;\xi), e_i)e_i\}d\xi$,

$$= \sum_{i=1}^{4} (1+h(x;\xi), e_i) n_0^{1/2}(e_0, e_i) = n_0^{1/2}(1+h(x;\xi), e_0)$$

$$= \int f_0(\xi)(1+h(x;\xi))d\xi.$$

But the last term on the r.h.s. is the density given by f, to first order.

It is easily shown that with J(f) given by Eqs. (2.1) and (2.12), the collision invariants remain invariant, i.e., that

$$(2.14) \qquad \int J(f)\phi(\xi)d\xi = 0,$$

if $\phi(\xi)$ is a collision invariant function. We have,

$$(2.15) \qquad \int J(f)\phi(\xi)d\xi = \int \nu\{f_0(\xi) \sum_{i=1}^{4}(1+h(x;\xi),e_i)e_i - f(\xi)\}\phi(\xi)d\xi.$$

To first order we obtain,

$$(2.16) \qquad \int J(f)\phi(\xi)d\xi = \nu\{\sum_{i=1}^{4}(1+h,e_i)(e_i,\phi)-(1+h,\phi)\},$$

and thus, by Eq. (2.16), Eq. (2.14) is satisfied for the collision invariant functions, because the collision invariants span the same subspace as the e_i.

Finally, to show that the solutions will obey an H theorem one must prove that,

$$(2.17) \qquad \int \ln(f)J(f)d\xi \leq 0.$$

To first order we have,

$$(2.18) \qquad \int \ln(f)J(f)d\xi = \int \ln(f_0(1+h))f_0 h^{\perp} d\xi,$$

where $h^{\perp}$ is given by,

$$(2.19) \qquad h^{\perp} = \sum_{i=1}^{4}(h,e_i)e_i - h.$$

Equation (2.18) can be expanded as follows,

$$(2.20) \qquad \int \ln(f)J(f)d\xi = (\ln(f_0),h^{\perp}) + (\ln(1+h),h^{\perp}),$$

where the inner product given by Eq. (2.7) has been used. Since f_0 is a Maxwellian, $\ln(f_0)$ is a linear combination of the collision invariants which are orthogoanl to $h^{\perp}$. Taking this into account and expanding $\ln(1+h)$ to first order one has,

$$(2.21) \qquad \int \ln(f)J(f)d\xi = (h,h^{\perp}) = (h^{\perp},h^{\perp}).$$

But the term on the far r.h.s. of Eq. (2.21) is zero to first order and consequently Eq. (2.17) is obtained.

If we consider only problems with plane symmetry, upon substituting Eq. (2.12) into Eq. (2.4) and simplifying we obtain the linearized BGK equation,

$$(2.22) \qquad \xi_1 \frac{\partial h(x',\xi)}{\partial x'} = \nu\{\sum_{i=1}^{4}(h(x',\xi),e_i)e_i - h(x',\xi)\}.$$

For time-independent problems with plane symmetry, if V_{01} is the component of the fluid velocity in the x' direction, then $V_{01} = 0$. If not, any spatial variations of the distribution function along the x' axis would propagate along the x' axis contradicting the hypothesis of time-independence.

Let

$$(2.23a) \qquad x = \nu\left(\frac{m}{2kT_0}\right)^{1/2}x',$$

then,

(2.23b) $\frac{\partial}{\partial x'} = \nu(\frac{m}{2kT_0})^{1/2} \frac{\partial}{\partial x}$,

and Eq. (2.22) can be written as,

(2.24) $C_1 \frac{\partial h}{\partial x} = \sum_{i=1}^{4}(h,e_i)e_i - h,$

where C is given by Eq. (2.8). Multiplying Eq. (2.24) by f_0 and integrating over velocity space we see that,

(2.25) $\frac{\partial}{\partial x}(C_1,h) = 0.$

Define

(2.26a) $\overline{h} = h - (e_1,h)e_1,$

then,

(2.26b) $(\overline{h},e_1) = 0.$

Furthermore, using (2.25), we see that $\overline{h}$ satisfies (2.24) and thus, w.l.o.g., we may assume,

(2.27) $(h,e_1) = 0.$

We introduce a second inner product and basis vectors,[18]

(2.28) $(p,q)_2 = \int pq \exp(-C_2^2-C_3^2)dC_2dC_3,$

(2.29a) $\phi_0 = \Pi^{-1/2},$

(2.29b) $\phi_1 = \Pi^{-1/2}(C_2^2+C_3^2-1)$,

(2.29c) $\phi_i = (\frac{2}{\Pi})^{1/2}C_i$, $i\varepsilon\{2,3\}$,

and observe,

(2.30) $(\phi_i,\phi_j)_2 = \delta_{ij}$,

(2.31a) $(\phi_0,e_i)_2 = (\frac{\Pi}{n_0})^{1/2}\{\delta_{i0}+ \sqrt{2}C_1 + \sqrt{\frac{2}{3}}\,(C_1{}^2- \frac{1}{2})\delta_{i4}\}$,

(2.31b) $(\phi_1,e_i)_2 = (\frac{2\Pi}{3n_0})^{1/2}\delta_{i4}$,

(2.31c) $(\phi_2,e_i)_2 = (\frac{\Pi}{n_0})^{1/2}\delta_{i2}$,

(2.31d) $(\phi_3,e_i)_2 = (\frac{\Pi}{n_0})^{1/2}\delta_{i3}$.

Let $\mu \equiv C_1$ and take the (2) inner product of Eq. (2.24) with ϕ_0 and ϕ_1 to obtain,

(2.32) $\mu\,\frac{\partial}{\partial x}\,(h,\phi_0)_2 = (h,e_0)(e_0,\phi_0)_2 + (h,e_4)(e_4,\phi_0)_2 - (h,\phi_0)_2$,

and

(2.33) $\mu\,\frac{\partial}{\partial x}\,(h,\phi_1)_2 = (h,e_0)(e_0,\phi_1)_2 + (h,e_4)(e_4,\phi_1)_2 - (h,\phi_1)_2$.

Note that,

(2.34a) $(h,e_0) = \frac{\sqrt{n_0}}{\Pi}\int_{-\infty}^{+\infty} \exp(-\mu^2)(h,\phi_0)_2 d\mu$,

(2.34b) $(h,e_4) = \sqrt{\frac{2}{3}}\,\frac{\sqrt{n_0}}{\Pi}\int_{-\infty}^{+\infty}\exp(-\mu^2)\{(h,\phi_1)_2 + (\mu^2 - \frac{1}{2})(h,\phi_0)_2\}$.

If we define,

$$(2.35a) \qquad \psi_1(x,\mu) = (h,\phi_0)_2,$$

$$(2.35b) \qquad \psi_2(x,\mu) = (h,\phi_1)_2,$$

and

$$(2.35c) \qquad \Psi(x,\mu) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} ,$$

then, using Eqs. (2.35), (2.33) and (2.32), we obtain the following equation for Ψ,

$$(2.36) \quad \mu \frac{\partial\Psi(x,\mu)}{\partial x} + \Psi(x,\mu) = Q(\mu) \int_{-\infty}^{+\infty} Q^T(s)\Psi(x,s) \exp(-s^2)ds,$$

where Q^T is the transpose of Q and

$$(2.37) \quad Q(\mu) = \Pi^{-1/4} \begin{pmatrix} \sqrt{\frac{2}{3}}(\mu^2 - \frac{1}{2}) & 1 \\ \sqrt{\frac{2}{3}} & 0 \end{pmatrix} .$$

Equation (2.36) is the temperature-density equation and contains all information necessary to determine the perturbations in the temperature and density.

Note that,

$$(2.38) \quad \int (Q^T\Psi)_2 \exp(C_1{}^2)dC_1 = \Pi^{-3/4} \int h \exp(-C^2)dC,$$

and using (2.9) we see that,

$$(2.39) \quad \int (Q^T\Psi)_2 \exp(-C_1{}^2)dC_1 = \frac{\Pi^{3/4}}{n_0} \Delta n,$$

where,

(2.40) $n(x) = n_0 + \Delta n(x).$

Similarly the perturbation in the temperature, ΔT, can be found by considering,

(2.41) $$\int (Q^T\Psi)_1 \exp(-C_1{}^2)dC_1 = \sqrt{\frac{2}{3}}\,\Pi^{-3/4} \int \exp(-C^2)(C^2 - \tfrac{3}{2})h\,dC.$$

ΔT is given by,

(2.42) $$\Delta T = T_0 \Pi^{-3/2}\, \frac{2}{3} \int (C^2 - \tfrac{3}{2})h \exp(-C^2)dC,$$

and thus,

(2.43) $$\int (Q^T\Psi)_1 \exp(-C_1{}^2)dC_1 = \frac{\Pi^{3/4}}{T_0} \sqrt{\frac{3}{2}}\, \Delta T.$$

Thus the perturbations in the temperature and density are given by the components of $Q^T\Psi$ integrated over the velocity variable with the appropriate weight function.

Now take the (2) inner product of Eq. (2.24) with ϕ_i, $i \varepsilon \{2,3\}$, to obtain,

(2.44) $$\mu \frac{\partial}{\partial x}(h,\phi_i)_2 = \sum_{j=1}^{4} (h,e_j)(e_j,\phi_i)_2 - (h,\phi_i)_2$$

$$= (h,e_i)\left(\frac{\Pi}{n_0}\right)^{1/2} - (h,\phi_i)_2 = \frac{1}{\sqrt{\Pi}} \int_{-\infty}^{+\infty} (h,\phi_i)\exp(-\mu^2)d\mu - (h,\phi_i)_2.$$

Thus, if ψ is a solution to (2.44) with $i \varepsilon \{2,3\}$ then ψ must satisfy,

(2.45) $$\mu \frac{\partial \psi(x,\mu)}{\partial x} + \psi(x,\mu) = \frac{1}{\sqrt{\Pi}} \int_{-\infty}^{+\infty} \psi(x,s)\exp(-s^2)ds,$$

the transverse velocity equation. The solutions to Eq. (2.45) will give the perturbations in the components of the fluid velocity in the transverse directions.

The perturbation of the fluid velocity in the x direction is obtained by multiplying Eq. (2.24) by f_0 and integrating over velocity space to obtain Eq. (2.25).

Thus, the solutions to Eqs. (2.37), (2.45) and (2.25) yield the macroscopic quantities of interest; the fluid velocity, the temperature and the density of the gas.

Chapter 4

THE TEMPERATURE-DENSITY EXPANSION

Section 1: Full-Range

The equation of interest, the temperature-density equation, is given by Eq. (3.2.36),

$$(1.1)\quad \mu\,\frac{\partial\Psi(x,\mu)}{\partial x} + \Psi(x,\mu) = Q(\mu)\int_{-\infty}^{+\infty} Q^T(s)\Psi(x,s)\exp(-s^2)ds,$$

with,

$$(1.2)\quad Q(\mu) = \pi^{-1/4}\begin{pmatrix} \sqrt{\frac{2}{3}}\,(\mu^2-\frac{1}{2}) & 1 \\ \sqrt{\frac{2}{3}} & 0 \end{pmatrix} .$$

Multiplying Eq. (1.1) by $Q^T(\mu)\mu^{-1}$ and defining $\Xi(x,\mu)$ by,

$$(1.3)\quad \Xi(x,\mu) = Q^T(\mu)\Psi(x,\mu),$$

leads to the following equation for $\Xi(x,\mu)$:

$$(1.4)\quad \frac{\partial\Xi(x,\mu)}{\partial x} + K^{-1}\Xi(x,\mu) = 0,$$

where we have defined K^{-1} by,

$$(1.5)\quad K^{-1}\Xi(x,\mu) = \frac{1}{\mu}\,[\Xi(x,\mu)-F(\mu)\int_{-\infty}^{+\infty}\Xi(x,s)\exp(-s^2)ds],$$

and F is given by,

$$(1.6)\quad F(\mu) = Q^T(\mu)Q(\mu).$$

Recalling Eqs. (3.2.39) and (3.2.43) we see that,

$$(1.7)\qquad \int_{-\infty}^{+\infty} \Xi_1(x,\mu)\exp(-\mu^2)d\mu \;\alpha\; \Delta T(x),$$

i.e., the integral of Ξ_1 over μ with the weight function, $\exp(-\mu^2)$, is proportional to the perturbation in the temperature and,

$$(1.8)\qquad \int_{-\infty}^{+\infty} \Xi_2(x,\mu)\exp(-\mu^2)d\mu \;\alpha\; \Delta n(x),$$

i.e., the integral of Ξ_2 over μ with the weight function, $\exp(-\mu^2)$, is proportional to the density. Consequently, it is physically reasonable to seek solutions to Eq. (1.4) which are differentiable w.r.t. x and p-integrable w.r.t. μ with the weight function $\exp(-\mu^2)$ for each x. Define the I(p,n) norm by,

$$(1.9a)\qquad ||f||_{I(p,n)} = \left(\sum_{i=1}^{n} \int_{-\infty}^{+\infty} |f_i(\mu)|^p d\sigma(\mu)\right)^{1/p},$$

with,

$$(1.9b)\qquad d\sigma(\mu) = \exp(-\mu^2)d\mu.$$

Define the Banach space, $X_p^n(\mathbb{R})$ by,

$$(1.10)\qquad X_p^n(\mathbb{R}) = \{f: ||f||_{I(p,n)} < \infty\}.$$

We seek solutions to Eq. (1.4), $\Xi_x(\mu)$, such that $\Xi_x(\mu) \;\varepsilon\; X_p^2(\mathbb{R})$, p>1. For convenience the dependence of Ξ on x will be dropped and the identification,

$$(1.11)\qquad ||\cdot||_{I(p)} \equiv ||\cdot||_{I(p,2)}\ ,$$ will be made.

It is clear from Eqs. (1.4), (1.5), (2.1.1a), (2.1.1b), (2.3.1) and (2.3.2) that the one-speed and multigroup neutron transport equations are very similar to the temperature-density equation. Furthermore, the transport operator and the temperature-density operator have similar properties, i.e., in the case of conservative neutron transport, the transport operator is non-invertible as is the temperature-density operator.

The zero-root linear manifold of K^{-1} is spanned by,

(1.12) $\{F(\mu)\nu_0,\ F(\mu)\nu_1,\ \mu F(\mu)\nu_0,\ \mu F(\mu)\nu_1\}$

where,

(1.13a) $$\nu_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and

(1.13b) $$\nu_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

To see this we first note the form of K^{-1} from Eq. (1.5),

$$K^{-1}f = \frac{1}{\mu}[f(\mu)-F(\mu)\int_{-\infty}^{+\infty} f(s)d\sigma(s)].$$

Suppose that f_1 satisfies,

(1.14) $(K^{-1}f_1)(\mu) = 0,$

then,

(1.15) $f_1(\mu)\ \alpha\ F(\mu)\nu,\quad \nu \in \mathbb{R}^2.$

Similarly if f_2 satisfies,

$$(1.16) \quad (K^{-2}f_2)(\mu) = 0,$$

then,

$$(1.17) \quad K^{-1}(K^{-1}f_2)(\mu) = 0,$$

and hence, by (1.14) and (1.15),

$$(1.18) \quad (K^{-1}f_2)(\mu) \; \alpha \; F(\mu)\upsilon, \; \upsilon \in \mathbb{R}.$$

But Eq. (1.18) implies that,

$$(1.19) \quad f_2(\mu) \; \alpha \; F(\mu)\upsilon_1(\mu),$$

where υ_1 is of the form,

$$(1.20) \quad \upsilon_1(\mu) = \mu k_1 + k_0, \; k_1, k_0 \in \mathbb{R}^2.$$

Assume that,

$$(1.21) \quad (K^{-n}f_n)(\mu) = 0, \quad n \geq 3$$

and that

$$(1.22) \quad (K^{-n+1}f_n)(\mu) \neq 0.$$

Then, by Eqs. (1.18), (1.19) we have,

$$(1.23) \quad (K^{-n+2}f_n)(\mu) \; \alpha \; F(\mu)\upsilon_1(\mu),$$

and thus,

(1.24) $(K^{-n+3}f_n)(\mu) \;\alpha\; F(\mu)\upsilon_2(\mu)$,

where,

(1.25) $\upsilon_2(\mu) = \mu^2 k_2 + \mu k_1 + k_0, \quad k_i \in \mathbb{R}^2.$

Operating on both sides of Eq. (1.24) with K^{-2} and employing Eqs. (1.19) and (1.16) we obtain,

(1.26) $K^{-n+1}f_n \;\alpha\; \mu^2 F(\mu)k_2, \quad k_2 \in \mathbb{R}^2.$

Thus, by Eq. (1.21), we must have

(1.27) $K^{-1}(\mu^2 F(\mu)k_2) = 0,$

but,

(1.28) $K^{-1}[\mu^2 F(\mu)k_2] \neq 0.$

Thus we are led to a contradiction by the original assumptions, Eqs. (1.21) and (1.22), and hence the entire zero-root linear manifold is spanned by functions of the form of f_1 and f_2 of Eqs. (1.15) and (1.19).

We approach the temperature density equation in the same way as the conservative neutron transport equation and define the operator, S^{-1}, by,

(1.29) $S^{-1} = K^{-1} - iI,$

from which we obtain,

$$(1.30)\qquad (S\Xi)(\mu) = t^{-1}(\mu)\Xi(\mu) + \frac{F(\mu)}{(1-i\mu)}\,\Lambda^{-1}(-i)\int_{-\infty}^{+\infty} t^{-1}(s)\Xi(s)d\sigma(s).$$

Here we have defined,

$$(1.31a)\qquad \Lambda(z) = I + z\int_{-\infty}^{+\infty} \frac{F(s)}{s-z}\,d\sigma(s),$$

$$(1.31b)\qquad t(z) = \frac{z}{1+iz},$$

$$(1.31c)\qquad t^{-1}(z) = \frac{z}{1+iz},$$

and,

$$(1.31d)\qquad (tt^{-1})(z) = z.$$

In order to apply the Plemelj formulas to integrals which will appear later in the text, such as Eq. (1.52), we restrict the class of functions to be considered to those functions in $X_p^2(\mathbb{R})$ obeying a Hölder condition. In particular we will work in the space $H_p{}^2(\mathbb{R})$ where,

$$(1.32)\qquad H_p(\mathbb{R}) = \{f:\ sf(s)\exp(-s^2/p)\in L_p(\mathbb{R},ds)\cap H\ddot{o}_\alpha(\mathbb{R}),p>1\}.$$

Here,

$$(1.33)\qquad L_p(Y,f(y)dy) = \ g:(\int_Y|g(y)|^p f(y)dy)^{1/p}<\infty\},$$

$$(1.34)\qquad H\ddot{o}_\alpha(\mathbb{R}) = \{f:||f||_\alpha<\infty\},$$

and

$$(1.35)\qquad ||f||_\alpha = \sup_{\substack{x_1\neq x_2\\ x_1,x_2\in\mathbb{R}}} \frac{|f(x_1)-(f(x_2)|}{|x_1-x_2|^\alpha} \quad .$$

Note that the Hölder norm, $||\cdot||_\alpha$, is actually only a semi-norm. In section 2 it will be shown that $H_p^{\,2}(\mathbb{R})$ is a dense subset of $X_p^{\,2}(\mathbb{R})$ and the results of this section will be extended to the entire Banach space, $X_p^{\,2}(\mathbb{R})$.

We wish to apply the identity,

$$(1.36)\qquad \int_{\Gamma'} (zI-S)^{-1}\Xi(\mu)dz = \Xi(\mu),$$

in order to obtain the full-range expansion (here Γ' is a contour surrounding the spectrum of S). To apply this identity we must first establish that S is a bounded operator. From Eqs. (1.30), (1.31) and the definition of $||\cdot||_{I(p)}$ we have,

$$(1.37)\qquad ||S\Xi||_{I(p)} \le C_1||\Xi||_{I(p)} + C_2||\Xi||_{I(p)} \ ,$$

where,

$$(1.38a)\qquad C_1 = \sup_{\mu\in\mathbb{R}} |t^{-1}(\mu)|<\infty \ ,$$

$$(1.38b)\qquad C_2 = \sup_{i,j\in\{1,2\}} \left|\left(\int_{-\infty}^{+\infty} t^{-1}(s)F(s)d\sigma(s)\Lambda^{-1}(-i)\right)_{ij}\right| \sup_{\mu\in\mathbb{R}} \left|\frac{1}{1-i\mu}\right| <\infty \ ,$$

and hence S is bounded on $X_p^{\,2}(\mathbb{R})$.

To compute the resolvent operator we solve the following for g,

$$(1.39)\qquad (zI-S)f(\mu) = g(\mu),$$

i.e.,

$$(1.40)\qquad (z-t^{-1}(\mu))f(\mu) - \frac{F(\mu)}{1-i\mu}\Lambda^{-1}(-i)\int_{-\infty}^{+\infty} t^{-1}(s)f(s)d\sigma(s) = g(\mu).$$

Solving for f we find that,

$$(1.41)\qquad (zI-S)^{-1}g(\mu) = (z-t^{-1}(\mu))^{-1}\{g(\mu) + \frac{F(\mu)}{1-i\mu}(\Lambda(-i)+ \frac{1}{1+iz}\int_{-\infty}^{+\infty}\frac{s}{s-t(z)}$$

$$\times \frac{F(s)}{1-is}d\sigma(s))^{-1}\frac{1}{1+iz}\int_{-\infty}^{+\infty}\frac{sg(s)}{t(z)-s}d\sigma(s)\}.$$

Define the operator T_1 by,

$$(1.42)\qquad T_1 = \Lambda(-i)+\frac{1}{1+iz}\int_{-\infty}^{+\infty}\frac{s}{s-t(z)}\frac{F(s)}{1-is}d\sigma(s).$$

Then we have,

$$(1.43)\qquad (zI-S)^{-1}g(\mu) = (z-t^{-1}(\mu))^{-1}\{g(\mu) + \frac{F(\mu)}{1-i\mu}T_1^{-1}\frac{1}{1+iz}\int_{-\infty}^{+\infty}\frac{sg(s)}{t(z)-s}d\sigma(s)\}.$$

Now we wish to put T_1 into a more familiar form. To this end we note that,

$$(1.44)\qquad \frac{s}{1-is} = i - \frac{i}{1-is}.$$

Using the above and Eq. (1.31a) one finds,

$$(1.45)\qquad T_1 = \Lambda(-i) + \frac{i}{z}(\Lambda(t(z))-I) - \frac{i}{1+iz}\int_{-\infty}^{+\infty}\frac{F(s)}{(s-t(z))(1-is)}d\sigma(s).$$

Using the partial fraction decomposition,

$$(1.46)\qquad \frac{1}{s-t(z)}\,\frac{1}{1-is} = \frac{1+iz}{s-t(z)} + \frac{i(1+iz)}{1-is}\ ,$$

and Eq. (1.31a) we have,

$$(1.47)\quad T_1 = \Lambda(-i) + \frac{i}{z}\,(\Lambda(t(z))-I) - \frac{i}{t(z)}\,(\Lambda(t(z))-I) - (\Lambda(-i)-I)$$
$$= \Lambda(t(z)).$$

Thus, substituting Eq. (1.47) into Eq. (1.43) and noting that,

$$(1.48)\qquad (z-t^{-1}(\mu))^{-1} = \frac{1-i\mu}{1+iz}\,\frac{1}{t(z)-\mu}\ ,$$

we have,

$$(1.49)\qquad (zI-S)^{-1}\Xi(\mu) = \frac{(1+iz)^{-1}}{t(z)-\mu}\,\{(1-i\mu)\Xi(\mu) + \frac{F(\mu)}{1+iz}\,\Lambda^{-1}(t(z))$$
$$\times \int_{-\infty}^{+\infty} \frac{s\Xi(s)}{t(z)-s}\,d\sigma(s)\},$$

which is a convenient form of the resolvent operator.

To determine the spectrum of S, $\sigma(S)$, we examine the points where $(zI-S)^{-1}\Xi$ fails to be analytic in z. $\Lambda^{-1}(t(z))$ is analytic in the complex z plane except along the curve, $C = \{z: z = \frac{1}{2}(i+\exp(i\theta)),\ 0 \le \theta \le 2\Pi\}$, and det $\Lambda(t(z))$ has a double zero at z=i where,[5]

$$(1.50)\quad \lim_{z\to i} \Lambda^{-1}(t(z))_{ij} = -\,(t(z))^2 B_{ij}/(1+O[(t(z))^2]),$$

and,

$$(1.51)\qquad B = \frac{12}{5}\begin{pmatrix} \frac{1}{2} & -\sqrt{\frac{1}{6}} \\ -\sqrt{\frac{1}{6}} & \frac{7}{6} \end{pmatrix}\ .$$

This double zero accounts for the only eigenvalues of S. The spectrum of S consists of the points $\{z=\mu/(1-i\mu),\ \mu\in\mathbb{R}\}$. Therefore the entire spectrum of S consists of the circle C.

Let C be enclosed by the contour $\Gamma' = \bigcup_{j=1}^{2} \{\Gamma_j \cup \Gamma_{\mu j} \cup \Gamma_{ij}\}$ where the contours are as in Figure 1. If Γ' is collapsed around the spectrum of the resolvent operator then $\Gamma_\mu = \bigcup_{j=1}^{2} \Gamma_{\mu j}$ is a circular contour about $t^{-1}(\mu)$ with the points where the contour intersects C deleted. $\Gamma_i = \bigcup_{j=1}^{2} \Gamma_{ij}$ is a circle about z=i with two points deleted and $\Gamma = \bigcup_{j=1}^{2} \Gamma_j$ consists of two concentric circles enclosing the spectrum with the arcs contained in the circles, Γ_μ and Γ_i, deleted.

In evaluating the contour integral on the l.h.s. of Eq. (1.36) it is convenient to make the following designations,

$$(1.52)\qquad N(z) = \int_{-\infty}^{+\infty} \frac{s\Xi(s)}{z-s}\, d\sigma(s),$$

and,

$$(1.53)\qquad M^{\pm}(\mu) = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} M(\mu\pm i\varepsilon) = (\Lambda^{\pm}(\mu))^{-1}N^{\pm}(\mu),\ \mu\in\mathbb{R}.$$

Let $z' = t(z)$ and consider the integral about Γ_μ of $(zI-S)^{-1}\Xi$. Using Eq. (1.49) we obtain,

$$(1.54)\qquad \frac{1}{2\Pi i}\int_{\Gamma_\mu} (zI-S)^{-1}\Xi(\mu)dz = \frac{1}{2\Pi i}\int_{t(\Gamma_\mu)} \frac{(1-iz')^{-1}}{z'-\mu}\{(1-i\mu)\Xi(\mu)$$

$$+ F(\mu)(1-iz')\Lambda^{-1}(z')N(z')\}dz' = \Xi(\mu) + F(\mu)\,\frac{1}{2}\,\{M^{+}(\mu)+M^{-}(\mu)\}.$$

Employing the same change of variable and integrating over Γ we have,

$$(1.55)\qquad \frac{1}{2\Pi i}\int_{\Gamma}(zI-S)^{-1}\Xi(\mu)dz = \frac{1}{2\Pi i}F(\mu)P\int_{-\infty}^{+\infty}\frac{M^-(s)-M^+(s)}{s-\mu}\,ds,$$

where P denotes the Cauchy principal value. Applying the Plemelj formulas to Eq. (1.52) we find,

$$(1.56a)\qquad N^-(\mu) - N^+(\mu) = 2\Pi i\mu\Xi(\mu)\exp(-\mu^2),\ \mu \in \mathbb{R}.$$

Using Eq. (1.53) this can be cast in the form,

$$(1.56b)\qquad 2\Pi i\mu\Xi(\mu)\exp(-\mu^2) = \frac{1}{2}(\Lambda^-(\mu)-\Lambda^+(\mu))(M^-(\mu)+M^+(\mu))$$

$$+\frac{1}{2}(\Lambda^-(\mu)+\Lambda^+(\mu))(M^-(\mu)-M^+(\mu)).$$

Noting that,

$$(1.56c)\qquad \Lambda^-(\mu) - \Lambda^+(\mu) = -2\Pi i\mu F(\mu)\exp(-\mu^2),$$

by applying the Plemelj formulas to Eq. (1.31a) we arrive at the result,

$$(1.56d)\qquad \Xi(\mu) = -\frac{1}{2}F(\mu)(M^-(\mu)+M^+(\mu)) + \frac{\exp(\mu^2)}{2\Pi i\mu}\frac{1}{2}(\Lambda^-(\mu)+\Lambda^+(\mu))(M^-(\mu)-M^+(\mu)).$$

Employing the result of Eqs. (1.54) and (1.55) we have,

$$(1.57)\qquad \frac{1}{2\Pi i}\int_{\Gamma\cup\Gamma_\mu}(zI-S)^{-1}\Xi(\mu)dz = \frac{\exp(\mu^2)}{\mu}\frac{1}{2}(\Lambda^-(\mu)+\Lambda^+(\mu))\frac{1}{2\Pi i}(M^-(\mu)-M^+(\mu))$$

$$+\frac{1}{2\Pi i}F(\mu)\,P\int_{-\infty}^{+\infty}\frac{M^-(s)-M^+(s)}{s-\mu}\,ds.$$

We are left with the integral over Γ_i. Returning to Eq. (1.49) we note that,

$$(1.58)\qquad \lim_{z\to i} \frac{(1+iz)^{-1}}{t(z)-\mu} = -i,$$

and hence the first term on the r.h.s. of Eq. (1.49) is analytic in a neighborhood of z=i. Thus we have

$$(1.59)\qquad \frac{1}{2\pi i}\int_{\Gamma_i} (zI-S)^{-1}\Xi(\mu)dz = \frac{1}{2\pi i}\int_{\Gamma_i} \frac{F(\mu)}{z-\mu(1+iz)} \quad \frac{\Lambda^{-1}(t(z))N(t(z))}{1+iz}\, dz.$$

In a neighborhood of z=i, $N(t(z))(1+iz)^{-1}$ is analytic as is $(z-\mu(1+iz))^{-1}$. The behaviour of $\Lambda^{-1}(t(z))$, z=i, is given by Eq. (1.50). Using elementary means the residue can be evaluated to yield

$$(1.60)\qquad \frac{1}{2\pi i}\int_{\Gamma_i} (zI-S)^{-1}\Xi(\mu)dz = F(\mu)B\{\int_{-\infty}^{+\infty} s^2\Xi(s)d\sigma(s) +\mu \int_{-\infty}^{+\infty} s\Xi(s)d\sigma(s)\}.$$

Define ρ_i by

$$(1.61)\qquad \rho_i(\Xi) = B\int_{-\infty}^{+\infty} s^{2-i}\Xi(s)d\sigma(s).$$

Combining Eqs. (1.60), (1.57) and using Eq. (1.61) we have,

$$(1.62)\qquad \frac{1}{2\pi i}\int_{\Gamma'} (zI-S)^{-1}\Xi(\mu)dz = \lambda(\mu)A(\mu) + F(\mu)P\int_{-\infty}^{+\infty} \frac{sA(s)}{s-\mu}\, d\sigma(s)$$

$$+ F(\mu)(\rho_0(\Xi)+\mu\rho_1(\Xi)),$$

where we have made use of the definitions,

$$(1.63a)\qquad \lambda(\mu) = \tfrac{1}{2}(\Lambda^+(\mu)+\Lambda^-(\mu)),$$

and,

$$(1.63b) \quad A(\mu) = \frac{M^-(\mu)-M^+(\mu)}{2\Pi i\mu} \exp(\mu^2).$$

Note that,

$$(1.64) \quad M^-(\mu) - M^+(\mu) = (\Lambda^-(\mu))^{-1}N^-(\mu) - (\Lambda^+(\mu))^{-1}N^+(\mu)$$

$$= \frac{1}{2}((\Lambda^-(\mu))^{-1} - (\Lambda^+(\mu))^{-1})(N^-(\mu)+N^+(\mu))$$

$$+ \frac{1}{2}((\Lambda^-(\mu))^{-1} + (\Lambda^+(\mu))^{-1})(N^-(\mu) - N^+(\mu)),$$

where we have made use of the definitions, Eqs. (1.52) and (1.53). Define

$$(1.65a) \quad \lambda_-^-(\mu) = \frac{1}{2\Pi i}\{(\Lambda^-(\mu))^{-1} - (\Lambda^+(\mu))^{-1}\},$$

$$(1.65b) \quad \lambda_-^+(\mu) = \frac{1}{2}\{(\Lambda^-(\mu))^{-1} + (\Lambda^+(\mu))^{-1}\}.$$

Making use of these definitions and applying the Plemelj formulas one obtains from Eqs. (1.63b) and (1.64),

$$(1.66) \quad A(\mu) = \frac{\exp(\mu^2)}{\mu} \lambda_-^-(\mu) P \int_{-\infty}^{+\infty} \frac{s\Xi(s)}{\mu-s} d\sigma(s) + \lambda_-^-(\mu)\Xi(\mu).$$

The results of this section are summarized in the following theorem.

Theorem 1. Every function $\Xi \varepsilon H_p^2(\mathbb{R})$ can be expanded in the following eigenfunction expansion,

$$\Xi(\mu) = F(\mu)(\rho_0(\Xi)+\mu\rho_1(\Xi)) + \int_{-\infty}^{+\infty} \phi_\nu(\mu)A(\nu)d\sigma(\nu),$$

where,

$$\phi_\nu(\mu) = \lambda(\nu)\exp(\nu^2)\delta(\nu-\mu) + F(\mu)P\frac{\nu}{\nu-\mu},$$

and,

$$A(\nu) = \lambda_-^+(\nu)\Xi(\nu) + \frac{\exp(\nu^2)}{\nu}\,\lambda_-^-(\nu)P\int_{-\infty}^{+\infty} \frac{s\Xi(s)}{\nu-s}\,d\sigma(s).$$

Section 2: Extension of the Full-Range Expansion to $X_p^2(\mathbb{R})$

The results of Theorem 1 are valid for functions contained in $H_p^2(\mathbb{R})$. However this restriction is without a physical basis. Furthermore, $H_p^2(\mathbb{R})$ is not a Banach space, a fact which will interfere with the development of a functional calculus for S. Here the results of section 1 will be extended to $X_p^2(\mathbb{R})$, a more physically reasonable space and a space which is more easily dealt with mathematically.

The first matter at hand is to show that $H_p^2(\mathbb{R})$ is a dense subset of $X_p^2(\mathbb{R})$. Recall definition (1.32),

$$H_p(\mathbb{R}) = \{f:(sf(s)\exp(-s^2/p)) \in L_p(\mathbb{R},ds) \cap \ddot{Ho}_\alpha(\mathbb{R})\}.$$

We prove the following lemma.

<u>Lemma 2</u>. $H_p(\mathbb{R})$ is dense in $X_p(\mathbb{R})$.

<u>Proof</u>: It is well known that if $(sg(s)) \in L_p(\mathbb{R},ds)$ then there exists a sequence $\{sg_i\} \in \ddot{Ho}_\alpha(\mathbb{R})$ such that $\{sg_i\} \to sg$ in the L_p norm.

If $f \in X_p(\mathbb{R})$ then $(sf(s)\exp(-s^2/p)) \in L_p(\mathbb{R},ds)$ and thus there exists a sequence $\{sg_i\} \in \ddot{Ho}_\alpha(\mathbb{R})$ such that $\{sg_i\} \to sf(s)\exp(-s^2/p)$ in L_p norm.

Let $y_i(s) = g_i(s)\exp(+s^2/p)$, then $\{y_i\} \in H_p(\mathbb{R})$ and $\{sy_i(s)\exp(-s^2/p)\} \to sf(s)\exp(-s^2/p)$ in L_p norm, i.e.,

$$||f-y_i||_{I(p,1)} \to 0 \text{ for } i \to \infty.$$

Thus $H_p(\mathbb{R})$ is dense in $X_p(\mathbb{R})$ in the $I(p,1)$ norm.

By lemma 2 $H_p(\mathbb{R})$ is dense in $X_p(\mathbb{R})$ and thus we conclude that $H_p^2(\mathbb{R})$ is dense in $X_p^2(\mathbb{R})$ in the $I(p)$ norm.

It will be convenient to define the following operators, T and $\tilde{T}$, such that,

$$(2.1a)\qquad (TA)(\mu) = \lambda(\mu)A(\mu) + F(\mu)\ P\int_{-\infty}^{+\infty} \frac{sA(s)}{s-\mu}\, d\sigma(s),$$

$$(2.1b)\qquad (\tilde{T}\Xi)(\mu) = \lambda_-^+(\mu)\Xi(\mu) + \lambda_-^-(\mu)\ \frac{\exp(\mu^2)}{\mu}\ P\int_{-\infty}^{+\infty} \frac{s\Xi(s)}{\mu-s}\, d\sigma(s).$$

The expansion coefficient, $A(\nu)$, is given by,

$$(2.2)\qquad A(\nu) = (\tilde{T}\Xi)(\nu).$$

For $\Xi \in X_p^2(\mathbb{R})$ we will show that $A \in \tilde{X}_p^2(\mathbb{R})$ where,

$$(2.3a)\qquad \tilde{X}_p^n(\mathbb{R}) = \{f : ||f||_{II(p,n)} < \infty\},$$

and

$$(2.3b)\qquad ||f||_{II(p,n)} = \Big(\sum_{i=1}^{n} \int_{-\infty}^{+\infty} \frac{|sf_i(s)|^p}{|s^2+1|^p}\, d\sigma(s)\Big)^{1/p}.$$

Again we will make the identification,

$$||\cdot||_{II(p,2)} \equiv ||\cdot||_{II(p)},$$

in order to simplify notation.

We will prove that $\tilde{T}$ is a bounded operator from $X_p^2(\mathbb{R})$ to $\tilde{X}_p^2(\mathbb{R})$. Using Eq. (2.1b) we observe

$$(2.4)\qquad ||\tilde{T}\Xi||_{II(p)} \leq ||\lambda_-^+\Xi||_{II(p)} + ||\lambda_-^-(\mu)\frac{\exp(\mu^2)}{\mu} P\int_{-\infty}^{+\infty}\frac{s\Xi(s)}{\mu-s}\, d\sigma(s)\,||_{II(p)}.$$

$\lambda_-^+(\mu)$ and $\lambda_-^-(\mu)$ are bounded in the finite plane and their behavior for large μ is discussed in Appendix II. Here we quote the results of Appendix II on the limiting behavior of these functions for large μ;

$$(2.5a)\qquad \sup_{i,j\varepsilon\{1,2\}} |\lambda_-^+(\mu)_{ij}| \to \mu^2$$

and

$$(2.5b)\qquad \sup_{i,j\varepsilon\{1,2\}} |\lambda_-^+(\mu)_{ij}| \to |\mu^9\exp(-\mu^2)|.$$

Using Eq. (2.5a) one finds that the first term on the r.h.s. of Eq. (2.4) is bounded by

$$(2.6)\qquad ||\lambda_-^+\Xi||_{II(p)} \leq \sup_{\mu\varepsilon\mathbb{R}} k_1\left|\frac{\mu^2}{\mu^2+1}\right|^p ||\Xi||_{I(p)}$$

where k_1 is a constant. Using Eq. (2.5b) we find that the second term on the r.h.s. of Eq. (2.4) is bounded by,

$$(2.7)\qquad ||\lambda_-^-(\mu)\frac{\exp(\mu^2)}{\mu} P\int_{-\infty}^{+\infty}\frac{s\Xi(s)}{\mu-s}\, d\sigma(s)||_{II(p)} \leq k_2\sup_{\mu\varepsilon\mathbb{R}}\left|\frac{\mu^9}{\mu^2+1}\right|^p$$

$$\times\ \exp(-\mu^2)\ ||f||_p,$$

where k_2 is a constant and f is given by

$$(2.8)\qquad f = P\int_{-\infty}^{+\infty}\frac{s\Xi(s)}{\mu-s}\, d\sigma(s).$$

Here it is necessary to introduce the following theorem.

Theorem 2.[27] If $f \varepsilon L_p(\mathbb{R}, ds)$, $p > 1$, and if g is given by,

$$g(\mu) = P \int_{-\infty}^{+\infty} \frac{f(s)}{s-\mu}\, ds,$$

then $g \varepsilon L_p(\mathbb{R}, ds)$ and,

$$||g||_p \leq M_p ||f||_p$$

where M_p is a constant depending only on p.

Using Theorem 2 and Eq. (2.8) we obtain

$$(2.9) \qquad ||f||_p \leq M_p ||\Xi||_{I(p)}$$

and hence

$$(2.10) \qquad ||\lambda_-^-(\mu) \frac{\exp(\mu^2)}{\mu} P \int_{-\infty}^{+\infty} \frac{s\Xi(s)}{\mu - s}\, d\sigma(s)||_{II(p)} \leq k_3 ||\Xi||_{I(p)}$$

where K_3 is a constant depending only on p. Combining Eqs.(2.10), (2.6) and (2.4) we obtain

$$(2.11) \qquad ||\tilde{T}\Xi||_{II(p)} \leq \hat{M} ||\Xi||_{I(p)}$$

where M is a constant depending only on p. Thus $\tilde{T}$ is a bounded operator from $X_p^2(\mathbb{R})$ to $\tilde{X}_p^2(\mathbb{R})$. Hence for every $\Xi \varepsilon X_p^2(\mathbb{R})$ there exists an $A \varepsilon \tilde{X}_p^2(\mathbb{R})$ where A is given by Eq. (2.2).

Turning our attention to T, given by Eq. (2.1a), we note that the limiting behavior of $\lambda(\mu)$, for large μ, is given by

$$(2.12)\qquad \sup_{i,j\varepsilon\{1,2\}} |\lambda(\mu)_{ij}| \to \mu^{-2}.$$

Again the details are given in Appendix II. Using Eq. (2.12) we find,

$$(2.13)\qquad ||\lambda A||_{I(p)} \leq k_1 ||A||_{II(p)} ,$$

where k_1 is a constant. The principal value integral in Eq. (2.1a) can be bounded by applying the same reasoning as was used to bound the principal value term in Eq. (2.1b) with the result,

$$(2.14)\qquad ||F(\mu)P\int_{-\infty}^{+\infty}\frac{sA(s)}{s-\mu}\,d\sigma(s)||_{I(p)} \leq k_2||A||_{II(p)},$$

for some constant, k_2, depending on p. Using these results we find,

$$(2.15)\qquad ||TA||_{I(p)} \leq M||A||_{II(p)},$$

and thus T is a bounded operator from $\tilde{X}_p^2(\mathbb{R})$ to $X_p^2(\mathbb{R})$.

Finally we must prove that the operator, T', given by

$$(2.16)\qquad T'\Xi = F(\mu)[\rho_0(\Xi)+\mu\rho_1(\Xi)]$$

is a bounded operator from $H_p^2(\mathbb{R})$ to $X_p^2(\mathbb{R})$. Clearly T' is bounded if

$$(2.17)\qquad \sup_{i\varepsilon\{0,1\}}\ \sup_{j\varepsilon\{1,2\}} |[\rho_i(\Xi)]_j| \leq ||\Xi_j||_{I(p,1)}.$$

We have,

$$(2.18)\qquad |[\rho_i(\Xi)]_j| \leq \int_{-\infty}^{+\infty}|s^{2-i}\Xi_j(s)\exp(-s^2)|ds.$$

Applying Hölder's inequality[24] we obtain,

$$(2.19)\qquad |[\rho_i(\Xi)]_j| \leq \Big(\int_{-\infty}^{+\infty}|s\Xi_j(s)|^p d\sigma(s)\Big)^{1/p}\Big(\int_{-\infty}^{+\infty}|s^{1-i}\exp[-(1-\tfrac{1}{p})s^2]|^q ds\Big)^{1/q}$$

$$= \Big(\int_{-\infty}^{+\infty}|s^{1-i}\exp[-(1-\tfrac{1}{p})s^2]|^q ds\Big)^{1/q}||\Xi_j||_{I(p,1)}.$$

Hence there exist constants, k_i, $i \varepsilon \{0,1\}$, such that,

$$(2.20)\qquad ||F(\mu)\rho_0(\Xi)||_{I(p)} \leq k_0||\Xi||_{I(p)},$$

and,

$$(2.21)\qquad ||\mu F(\mu)\rho_1(\Xi)||_{I(p)} \leq k_1||\Xi||_{I(p)}.$$

T' defined by Eq. (2.16) is a bounded operator from $X_p^2(\mathbb{R})$ to $X_p^2(\mathbb{R})$. Since T', T, $\tilde{T}$ are all bounded operators and since $H_p^2(\mathbb{R})$ is a dense subset of $X_p^2(\mathbb{R})$ we obtain the desired extension which we state as a theorem.

<u>Theorem 3</u>. The domain of S may be extended to $X_p^2(\mathbb{R})$, $p > 1$, and the identity,

$$\Xi(\mu) = F(\mu)[\rho_0(\Xi)+\mu\rho_1(\Xi)] + \int_{-\infty}^{+\infty}\Phi_\nu(\mu)A(\nu)d\sigma(\nu),$$

with,

$$\Phi_\nu(\mu) = \lambda(\nu)\exp(\nu^2)\delta(\nu-\mu) + F(\mu)P\,\frac{\nu}{\nu-\mu}\ ,$$

and,

$$A(\nu) = \lambda_-^+(\nu)\Xi(\nu) + \lambda_-^-(\nu)\frac{\exp(\nu^2)}{\nu}\ P\int_{-\infty}^{+\infty}\frac{s\Xi(s)}{\nu-s}\,d\sigma(s),$$

holds for each $\Xi \ \varepsilon\ X_p^2(\mathbb{R})$.

Section 3: The Half-Range Expansion

The half-range expansion is generally of more physical interest than the full-range expansion and can be obtained by employing the methods of the previous sections. In section 2 the full-range expansion,

$$(3.1) \qquad \Xi(0,\mu) = F(\mu)[\rho_0(\Xi_0)+\mu\rho_1(\Xi_0)] + \int_{-\infty}^{+\infty} \phi_\nu(\mu)(\tilde{T}\Xi_0)(\nu)d\sigma(\nu),$$

was obtained where,

$$(3.2) \qquad \Xi_0(\mu) = \Xi(0,\mu).$$

$\Xi(x,\mu)$ given by,

$$(3.3) \qquad \Xi(x,\mu)= F(\mu)[\rho_0(\Xi_0)+(\mu-x)\rho_1(\Xi_0)] + \int_{-\infty}^{+\infty} \phi_\nu(\mu)(\tilde{T}\Xi_0)(\nu)e^{-x/\nu}d\sigma(\nu),$$

is a solution to the temperature-density equation, (1.4), provided that $\Xi_0(\mu)$ is given for $\mu \in \mathbb{R}$. For half-range problems $\Xi_0(\mu)$ is given only for $\mu \in \mathbb{R}^+$. But, as in the case of neutron transport, an additional boundary condition must be met, i.e.,

$$(3.4) \qquad \lim_{x\to\infty} \Xi(x,\mu) \quad \text{constant.}$$

In order to satisfy (3.4) we see from Eq. (3.3) that

$$(3.5a) \qquad (\tilde{T}\Xi_0)(\nu) = 0, \ \nu < 0,$$

and

$$(3.5b) \qquad \rho_1(\Xi_0) = 0.$$

Thus, for half-range problems, we seek an expansion of the form,

$$(3.6) \qquad \Xi(0,\mu) = F(\mu)v + \int_{-\infty}^{+\infty} \Phi_\nu(\mu)A(\nu)d\sigma(\nu),$$

where $v \in \mathbb{R}^2$. Functions of the form given by Eq. (3.6) are clearly a subset of the functions expandable by the full-range expansion, (3.1).

Note that boundary conditions (3.5a) and (3.5b) and expansions (3.1) and (3.6) are similar to the equations encountered in the case of neutron transport in section 2 of Chapter 2. We proceed as in the neutron transport case and define the operator $P^+:X_p^2(\mathbb{R}) \to X_p^2(\mathbb{R}^+)$ by,

$$(3.7) \qquad (P^+f)(\mu) = f(\mu), \ \mu \geq 0.$$

Define $E:X_p^2(\mathbb{R}^+) \to X_p^2(\mathbb{R})$ by,

$$(3.8a) \qquad P^+EP^+\Xi = P^+\Xi, \ \Xi \in X_p^2(\mathbb{R}),$$

$$(3.8b) \qquad \rho_1(EP^+\Xi) = 0,$$

and,

$$(3.8c) \qquad (zI-S)^{-1}(EP^+\Xi)(\mu) \text{ analytic in } z \text{ for } \mathrm{Re}\, z < 0.$$

If an E exists which satisfies the above conditions and if $\Xi(0,\mu)$ is given by

$$(3.9) \qquad \Xi(0,\mu) = EP^+\Xi(0,\mu),$$

then $\Xi(x,\mu)$ given by Eq. (3.3) will satisfy boundary condition (3.5b) by (3.8b) and boundary condition (3.5a) by (3.8c). If $\Xi(0,\mu)$ is given for $\mu \in \mathbb{R}^+$, the usual half-range boundary condition, then $\Xi(0,\mu)$ is given by Eq. (3.9) for $\mu \in \mathbb{R}$. In other words specifying $\Xi(0,\mu)$ for $\mu \in \mathbb{R}^+$ with the additional requirement (3.4) is equivalent to specifying $\Xi(0,\mu)$ for $\mu \in \mathbb{R}$.

We observe that E given by

$$(3.10) \qquad (EP^+\Xi)(\mu) = \begin{cases} \Xi(\mu), & \mu \geq 0, \\ F(\mu)H(-\mu)\int_0^\infty \frac{sH^T(s)\Xi(s)}{s-\mu}\,d\sigma(s), & \mu < 0, \end{cases}$$

satisfies properties (3.8a), (3.8b) and (3.8c). In Eq. (3.10) we have used,

$$(3.11) \qquad \Lambda(z) = H^T(z))^{-1}H^{-1}(z),$$

where $H(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}^-$ and is given by,[5]

$$(3.12) \qquad H(z) = I + zH(z)\int_0^\infty \frac{H^T(s)F(s)}{s+z}\,d\sigma(s).$$

Since H is of such importance to the half-range development we list a few of its properties:[5]

$$(3.13a) \qquad H(\mu) = I + \mu H(\mu)\int_0^\infty \frac{H^T(s)F(s)}{s+\mu}\,d\sigma(s), \quad \mu \in \mathbb{R}^+,$$

$$(3.13b) \qquad \int_0^\infty F(s)H(s)\,d\sigma(s) = I,$$

and

$$(3.13c) \qquad H^T(\mu)\lambda(\mu) = I+\mu P\int_0^\infty \frac{H^T(s)F(s)}{s-\mu}\,d\sigma(s), \quad \mu \in \mathbb{R}^+.$$

Since the half-range expansion we desire is given by the full-range expansion of $EP^+\Xi$ we use the results of Theorem 3 to obtain,

$$\text{(3.14)} \qquad \Xi(\mu) = F(\mu)[\rho_0(EP^+\Xi)+\mu\rho_1(EP^+\Xi)] + \int_{-\infty}^{+\infty}\phi_\nu(\mu)A(\nu)d\sigma(\nu),$$

where,

$$\text{(3.15)} \qquad A(\nu) = \lambda_-^+(\nu)(EP^+\Xi)(\nu) + \lambda_-^-(\nu)\frac{\exp(\nu^2)}{\nu}\, P\int_{-\infty}^{+\infty}\frac{s(EP^+\Xi)(s)}{\nu-s}\, d\sigma(s),$$

and $\Xi \in X_p^2(\mathbb{R})$. Define the operator, $\rho_0{}^+$, by,

$$\text{(3.16)} \qquad \rho_0{}^+(\Xi) = \rho_0(EP^+\Xi).$$

Using Eqs. (3.13a) and (3.13b) we find

$$\text{(3.17)} \qquad \rho_0{}^+(\Xi) = BH_1\int_0^\infty sH^T(s)\Xi(s)d\sigma(s),$$

where,

$$\text{(3.18)} \qquad H_i = \int_0^\infty s^i F(s)H(s)d\sigma(s).$$

Combining Eqs. (3.14), (3.16) and using (3.8b) the half-range expansion takes the form,

$$\text{(3.19)} \qquad \Xi(\mu) = F(\mu)\rho_0{}^+(\Xi) + \int_{-\infty}^{+\infty}\phi_\nu(\mu)A(\nu)d\sigma(\nu).$$

Let us define $\underset{\sim}{M}$ and $\underset{\sim}{N}$ by,

$$\text{(3.20a)} \qquad \underset{\sim}{M}(z) = \Lambda^{-1}(z)\underset{\sim}{N}(z),$$

and,

$$(3.20b)\qquad \underset{\sim}{N}(z) = \int_{-\infty}^{+\infty} \frac{s(EP^{+}\Xi)(s)}{z-s}\, d\sigma(s).$$

Then, using the results of section 1, we have,

$$(3.21)\qquad \int_{-\infty}^{+\infty} \Phi_{\nu}(\mu)A(\nu)d\sigma(\nu) = \frac{\lambda(\mu)}{2\pi i}\,\frac{\exp(\mu^2)}{\mu}\,(\underset{\sim}{M}^{+}(\mu) - \underset{\sim}{M}^{-}(\mu))$$

$$+ F(\mu)\,\frac{1}{2\pi i}\,P\int_{-\infty}^{+\infty} \frac{\underset{\sim}{M}^{-}(s)-\underset{\sim}{M}^{+}(s)}{s-\mu}\, ds.$$

By Eq. (1.55) we have,

$$(3.22)\qquad \frac{1}{2\pi i}\int_{\Gamma}(zI-S)^{-1}(EP^{+}\Xi)(\mu)dz = \frac{1}{2\pi i}\,F(\mu)P\int_{-\infty}^{+\infty} \frac{\underset{\sim}{M}^{-}(s)-\underset{\sim}{M}^{+}(s)}{s-\mu}\, ds.$$

In fact, we can decompose this contour integral into two integrals. Let Γ^{-} be the part of Γ for which Re $z < 0$ and Γ^{+} be the contribution from the remainder of Γ. Then,

$$(3.23a)\qquad \frac{1}{2\pi i}\int_{\Gamma^{-}}(zI-S)^{-1}(EP^{+}\Xi)(\mu)dz = \frac{1}{2\pi i}\,F(\mu)P\int_{-\infty}^{0} \frac{\underset{\sim}{M}^{-}(s)-\underset{\sim}{M}^{+}(s)}{s-\mu}\, ds,$$

and,

$$(3.23b)\qquad \frac{1}{2\pi i}\int_{\Gamma^{+}}(zI-S)^{-1}(EP^{+}\Xi)(\mu)dz = \frac{1}{2\pi i}\,F(\mu)P\int_{0}^{\infty} \frac{\underset{\sim}{M}^{-}(s)-\underset{\sim}{M}^{+}(s)}{s-\mu}\, ds.$$

According to Eq. (3.8c) the integrand of the contour integral on the l.h.s. of Eq. (3.23a) is analytic for z in the region of integration and thus the integral is zero yielding,

$$(3.24)\qquad \frac{1}{2\pi i}\,F(\mu)P\int_{-\infty}^{0} \frac{\underset{\sim}{M}^{-}(s)-\underset{\sim}{M}^{+}(s)}{s-\mu}\, ds = 0.$$

From the above and Eq. (3.21) we have,

$$(3.25)\qquad \int_{-\infty}^{+\infty} \Phi_\nu(\mu)A(\nu)d\sigma(\nu) = \frac{\lambda(\mu)}{2\Pi i}\frac{\exp(\mu^2)}{\mu}(\underset{\sim}{M}^+(\mu)-\underset{\sim}{M}^-(\mu))$$

$$+ F(\mu)\frac{1}{2\Pi i} P\int_0^\infty \frac{\underset{\sim}{M}^-(s)-\underset{\sim}{M}^+(s)}{s-\mu}ds.$$

Defining A^+ by,

$$(3.26)\qquad A^+(\nu) = \frac{\exp(\nu^2)}{\nu}\frac{\underset{\sim}{M}^-(\nu)-\underset{\sim}{M}^+(\nu)}{2\Pi i},$$

and inserting into Eqs. (3.25) and (3.19) we obtain the half-range expansion,

$$(3.27)\qquad \Xi(\mu) = F(\mu)\rho_0{}^+(\Xi) + \int_0^\infty \Phi_\nu(\mu)A^+(\nu)d\sigma(\nu).$$

Using the decomposition of $\underset{\sim}{M}^- - \underset{\sim}{M}^+$,

$$(3.28)\qquad \underset{\sim}{M}^- - \underset{\sim}{M}^+ = \lambda_-^-\frac{1}{2}(\underset{\sim}{N}^+ + \underset{\sim}{N}^-) + \lambda_-^-\frac{1}{2\Pi i}(\underset{\sim}{N}^- - \underset{\sim}{N}^+),$$

we have,

$$(3.29)\qquad A^+(\nu) = \lambda_-^+(\nu)\Xi(\nu) + \lambda_-^-(\nu)\frac{\exp(\nu^2)}{\nu} P\int_{-\infty}^{+\infty}\frac{s(EP^+\Xi)(s)}{\nu-s}d\sigma(s).$$

In order to obtain a more explicit expression for A^+ we consider the term,

$$(3.30)\qquad P\int_{-\infty}^{+\infty}\frac{s(EP^+\Xi)(s)}{\nu-s}d\sigma(s) = P\int_0^\infty \frac{s\Xi(s)}{\nu-s}d\sigma(s) - \int_0^\infty \frac{s(EP^+\Xi)(-s)}{\nu+s}d\sigma(s)$$

$$= P\int_0^\infty \frac{s\Xi(s)}{\nu-s}d\sigma(s) - \int_0^\infty d\sigma(s)\int_0^\infty d\sigma(t)\,\frac{sF(s)H(s)}{\nu+s}\,\frac{tH(t)\Xi(t)}{t+s}.$$

Making use of the partial fraction decomposition,

$$(3.31) \qquad \frac{st}{(\nu+s)(t+s)} = \frac{t}{t-\nu}\left(\frac{t}{t+s} - \frac{\nu}{\nu+s}\right),$$

changing the orders of integration and applying the transpose of Eq. (3.13a) one finds

$$(3.32) \qquad P\int_{-\infty}^{+\infty} \frac{s(EP^{+}\Xi)(s)}{\nu-s}\, d\sigma(s) = H^{T^{-1}}(\nu)\; P\int_{0}^{\infty} \frac{sH^{T}(s)\Xi(s)}{\nu-s}\, d\sigma(s).$$

Substituting Eq. (3.32) into (3.2a) we have,

$$(3.33) \qquad A^{+}(\nu) = \lambda_{-}^{+}(\nu)\Xi(\nu) + \lambda_{-}^{-}(\nu)H^{T^{-1}}(\nu)\,\frac{\exp(\nu^2)}{\nu}\; P\int_{0}^{\infty} \frac{sH^{T}(s)\Xi(s)}{\nu-s}\, d\sigma(s).$$

The results of this section are summarized in the following theorem.

Theorem 4. Each $\Xi \in X_p^2(\mathbb{R}^+)$ can be expanded in the following eigenfunction expansion,

$$\Xi(\mu) = F(\mu)\rho_0^{+}(\Xi) + \int_0^{\infty} \Phi_\nu(\mu)A^{+}(\nu)d\sigma(\nu),$$

where,

$$A^{+}(\nu) = \lambda_{-}^{+}(\nu)\Xi(\nu) + \lambda_{-}^{-}(\nu)H^{T^{-1}}(\nu)\,\frac{\exp(\nu^2)}{\nu}\; P\int_{0}^{\infty} \frac{sH^{T}(s)\Xi(s)}{\nu-s}\, d\sigma(s),$$

and,

$$\Phi_\nu(\mu) = \lambda(\mu)\exp(\mu^2)\delta(\nu-\mu) + F(\mu)P\,\frac{\nu}{\nu-\mu}\,.$$

Section 4: Development of the Functional Calculus for S.

In previous sections we have obtained expansions for functions in $X_p^2(\mathbb{R})$ and $X_p^2(\mathbb{R}^+)$ in terms of the eigenfunctions of the operator, S, where S is related to K^{-1} by,

$$(4.1) \qquad K^{-1} = S^{-1} + iI.$$

If λ is an element of the point spectrum of S with eigenvector η then a simple calculation shows that η is also an eigenvector of K^{-1} with eigenvalue $(\lambda^{-1}+i)$, i.e.,

$$(4.2) \qquad K^{-1}\eta = \frac{1+i\lambda}{\lambda}\,\eta.$$

A similar relation holds when λ is an element of the continuous spectrum of S, a fact which will be proven in the remainder of this section.

In order to find how the eigenvectors of S and K^{-1} are related it is convenient to subtract off the finite dimensional subspace corresponding to the discrete spectrum and deal with the rest of the Banach space separately. Let

$$(4.3) \qquad Y_p^2(\mathbb{R}) = \{f\varepsilon X_p^2(\mathbb{R}):\rho_i(f) = 0,\ i\varepsilon\{0,1\}\}.$$

Clearly any function $g \ \varepsilon\ X_p^2(\mathbb{R})$ can be decomposed by

$$(4.4) \qquad g(\mu) = F(\mu)[\rho_0(g)+\mu\rho_1(g)] + q(\mu),$$

where $q \ \varepsilon\ Y_p^2(\mathbb{R})$.

Define $P(w):Y_p^2(\mathbb{R}) \to Y_p^2(\mathbb{R})$ by,

$$(4.5a) \qquad P(w)f(\mu) = \int_{-\infty}^{w} \Phi_\nu(\mu)(\tilde{T}f)(\nu)d\sigma(\nu),$$

that is,

$$(4.5b) \qquad P(w)f(\mu) = \begin{cases} \lambda(\mu)A(\mu)+F(\mu)P\int_{-\infty}^{w} \frac{\nu A(\nu)}{\nu-\mu}\,d\sigma(\nu), & \mu \leq w, \\ F(\mu)\int_{-\infty}^{w} \frac{\nu A(\nu)}{\nu-\mu}\,d\sigma(\nu), & \mu > w, \end{cases}$$

where A is defined by

$$(4.5c) \qquad A(\nu) = (\tilde{T}f)(\nu).$$

The family of operators, $P(w)$ for $w \in \mathbb{R}$, forms a part of the spectral family of projections for the operator S and is essential to the development of this section. In order to prove that these operators are a part of the spectral family the following lemmas are introduced.

Lemma 3. $P(w)$ is a continuous function of w in the strong operator topology, i.e.,

$$(4.6) \qquad \lim_{\varepsilon\to 0^+} ||P(w+\varepsilon)f-P(w)f||_{I(p)} = 0, \quad f \in Y_p^2(\mathbb{R}), \; w \in \mathbb{R}.$$

Proof: Define f_1 and f_2 such that

$$(4.7) \qquad P(w+\varepsilon)f-P(w)f = f_1 + f_2,$$

where,

$$(4.8) \qquad f_1(\mu) = \begin{cases} \lambda(\mu)(\tilde{T}f)(\mu), & \mu \in (w,w+\varepsilon), \\ 0, & \mu \notin (w,w+\varepsilon), \end{cases}$$

and

$$(4.9)\qquad f_2(\mu) = F(\mu)P\int_{w}^{w+\varepsilon} \frac{\nu(\tilde{T}f)(\nu)}{\nu-\mu}\, d\sigma(\nu).$$

Clearly the norm of f_1 can be made as small as desired and the contribution due to f_2 is given by,

$$(4.10)\qquad ||f_2||_{I(p)} \le \left(\sum_{i=1}^{2} M_1 \int_{-\infty}^{+\infty} ds \left|P\int_{w}^{w+\varepsilon} \frac{\nu A_i(\nu)}{\nu-s}\, d\sigma(\nu)\right|^p\right)^{1/p}$$

$$\le \left(\sum_{i=1}^{2} M_2 \int_{w}^{w+\varepsilon} |\nu A_i(\nu)\exp(-\nu^2)|^p d\nu\right)^{1/p},$$

where M_1 and M_2 are constants determined by simple estimates and the application of Theorem 2. By the above we have

$$(4.11)\qquad \lim_{\varepsilon\to 0^+} \{||f_1||_{I(p)} + ||f_2||_{I(p)}\} = 0,$$

and thus, by Eq. (4.7), $P(w)$ is continuous in the strong operator topology.

<u>Lemma 4</u>. For $f \in Y_p^2(\mathbb{R})$:

(i) $P(w_1)P(w_2)f = P(\lambda)f$ where $\lambda = \inf\{w_1, w_2\}$;

(ii) $Sf = \int_{-\infty}^{+\infty} t^{-1}(\nu)\Phi_\nu(\mu)(\tilde{T}f)(\nu)d\sigma(\nu)$;

(iii) $SP(w)f = P(w)Sf$.

<u>Proof</u>: (i) $P(w)f = \int_{-\infty}^{w} \Phi_\nu(\mu)(\tilde{T}f)(\nu)d\sigma(\nu) = g(\mu)$, $g \in Y_p^2(\mathbb{R})$. Then

$$g(\mu) = \int_{-\infty}^{+\infty} \Phi_\nu(\mu)(\tilde{T}g)(\nu)d\sigma(\nu),$$

where,

$$(\tilde{T}g)(\mu) = \begin{cases} (\tilde{T}f)(\mu), & \mu \leq w, \\ 0 & , \mu > w. \end{cases}$$

Thus,

$$P(w_2)P(w_1)f = P(w_2)g = \int_{-\infty}^{w_2} \Phi_\nu(\mu)(\tilde{T}g)(\nu)d\sigma(\nu) = \int_{-\infty}^{\lambda} \Phi_\nu(\mu)(\tilde{T}f)(\nu)d\sigma(\nu)$$

$$= P(\lambda)f, \ \lambda = \inf\{w_1, w_2\}.$$

$$\text{(ii)} \quad Sf = \int_{\Gamma'} (zI-S)^{-1}Sfdz = \int_{\Gamma'} (S-zI+zI)(zI-S)^{-1}fdz$$

$$= \int_{\Gamma'} z(zI-S)^{-1}fdz = \int_{-\infty}^{+\infty} t^{-1}(\nu)\Phi_\nu(\mu)(\tilde{T}f)(\nu)d\sigma(\nu).$$

$$\text{(iii)} \quad P(w)Sf = \int_{-\infty}^{w} t^{-1}(\nu)\Phi_\nu(\mu)(\tilde{T}f)(\nu)d\sigma(\nu).$$

$$SP(w)f = S\int_{-\infty}^{w} \Phi_\nu(\mu)(\tilde{T}f)(\nu)d\sigma(\nu) = Sg, \ g \in Y_p^2(\mathbb{R}).$$

Expressing the expansion coefficient of g, $\tilde{T}g$, in terms of the expansion coefficient of f as in the proof of part (i) we obtain,

$$SP(w)f = \int_{-\infty}^{w} t^{-1}(\nu)\Phi_\nu(\mu)(\tilde{T}f)(\nu)d\sigma(\nu),$$

and hence,

$$P(w)S = Sp(w).$$

<u>Lemma 5</u>. $Sf(\mu) = \int_{-\infty}^{+\infty} t^{-1}(w)d[P(w)f](\mu)$, $f \in Y_p^2(\mathbb{R})$, where the integral is defined in the weak sense.

Proof: Define U(w) such that, $U(w) = \int_{-\infty}^{+\infty} g^T(\mu)P(w)f(\mu)d\sigma(\mu)$,

where $g^T(\mu)$ is an element of the dual of $Y_p^2(\mathbb{R})$. Now,

$$U(w) = \int_{-\infty}^{w} g^T(w)(\tilde{T}f)(\mu)d\sigma(\mu) + \int_{-\infty}^{w} g^T(\mu)F(\mu)P\int_{-\infty}^{+\infty} \frac{\nu(\tilde{T}f)(\nu)}{\nu-\mu} d\sigma(\nu)d\sigma(\mu)$$

$$= \int_{-\infty}^{w} g^T(\mu)\lambda(\mu)(\tilde{T}f)(\mu)d\sigma(\mu) + \int_{-\infty}^{w} P\int_{-\infty}^{+\infty} \frac{g^T(\nu)F(\nu)}{\mu-\nu} \mu(\tilde{T}f)(\mu)d\sigma(\nu)d\sigma(\mu),$$

and thus,

$$U'(w) = [g^T(w)\lambda(w)+P\int_{-\infty}^{+\infty} g^T(\nu)F(\nu) \frac{w}{w-\nu} d\sigma(\nu)]e^{-w^2}(\tilde{T}f)(w).$$

Now,

$$\int_{-\infty}^{+\infty} t^{-1}(w)U'(w)dw = \int_{-\infty}^{+\infty} d\sigma(w)t^{-1}(w)[g^T(w)\lambda(w)+P\int_{-\infty}^{+\infty} d\sigma(\nu)g^T(\nu)F(\nu) \frac{w}{w-\nu}]$$

$$\times (\tilde{T}f)(w) = \int_{-\infty}^{+\infty} d\sigma(w)g^T(w)(Sf)(w).$$

Using the above and the definition of U we have

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} t^{-1}(w)g^T(\mu)d[P(w)f](\mu)d\sigma(\mu) = \int_{-\infty}^{+\infty} g^T(w)(Sf)(w)d\sigma(w),$$

and thus,

$$(Sf)(\mu) = \int_{-\infty}^{+\infty} t^{-1}(w)d[P(w)f](\mu),$$

which yields the result,

$$S = \int_{-\infty}^{+\infty} t^{-1}(w)d[P(w)].$$

Let us write $f \in X_p^2(\mathbb{R})$ in the form,

(4.12) $$f = f_1 + f_2 ,$$

where

(4.13a) $$f_1(\mu) = f(\mu) - F(\mu)[\rho_0(f)+\mu\rho_1(f)],$$

(4.13b) $$f_2(\mu) = F(\mu)[\rho_0(f)+\mu\rho_1(f)].$$

From Eq. (4.3) we see that $f_1 \in Y_p^2(\mathbb{R})$. By Theorem 3 and Eq. (4.5a) we have,

(4.14a) $$\lim_{w\to-\infty} P(w)f_1 = 0,$$

(4.14b) $$\lim_{w\to\infty} P(w)f_1 = f_1.$$

These facts and the preceding lemmas lead to Theorem 5.

Theorem 5. (i) The family of projections, $P(w)$, is a generalized resolution of the identity on $Y_p^2(\mathbb{R})$;

(ii) for $f \in Y_p^2(\mathbb{R})$,

$$(K^{-1}f)(\mu) = \int_{-\infty}^{+\infty} \frac{1}{\nu}\Phi_\nu(\mu)(\tilde{T}f)(\nu)d\sigma(\nu);$$

(iii) $K^{-1}F(\mu) = 0$ and $K^{-1}[\mu F(\mu)] = F(\mu)$.

Proof: (i) The proof follows from lemmas 3 and 4 and the fact that $P(\infty) = I$ on $Y_p^2(\mathbb{R})$.

(ii) By Lemma 5 we have,

$$(Sf)(\mu) = \int_{-\infty}^{+\infty} t^{-1}(w)\frac{d}{dw}[P(w)f(\mu)]dw.$$

Consider,

$$S^{-1}f - \int_{-\infty}^{+\infty} \frac{1}{t^{-1}(w)} \frac{d}{dw}[P(w)f]dw = S^{-1}[f- \int_{-\infty}^{+\infty} \frac{1}{t^{-1}(w)} \frac{d}{dw}[P(w)Sf]dw],$$

where we have used part (iii) of lemma 4 to obtain the r.h.s. of the equation. Using lemma 5 we have,

$$S^{-1}f - \int_{-\infty}^{+\infty} \frac{1}{t^{-1}(w)} d[P(w)f] = S^{-1}[f- \int_{-\infty}^{+\infty} \frac{1}{t^{-1}(w)} \frac{d}{dw} \{P(w) \int_{-\infty}^{+\infty} t^{-1}(\nu)$$

$$\times \frac{d}{d\nu}[P(\nu)f]d\nu\}dw].$$

By part (i) of lemma 4 the r.h.s. of the above is,

$$S^{-1}[f-\int_{-\infty}^{+\infty} \frac{1}{t^{-1}(w)} \frac{d}{dw} \{\int_{-\infty}^{w} t^{-1}(\nu)\frac{d}{d\nu} [P(\nu)f]d\nu\}dw] =$$

$$S^{-1}[f- \int_{-\infty}^{+\infty} \frac{1}{t^{-1}(w)} \{t^{-1}(w)\frac{d}{dw} [P(w)f]\}dw] = S^{-1}[f- \int_{-\infty}^{+\infty} d[P(w)f]].$$

Thus we have

$$S^{-1}f = \int_{-\infty}^{+\infty}(\frac{1}{w} - i)d[P(w)f] = \int_{-\infty}^{+\infty} \frac{1}{w}d[P(w)f] - if,$$

where we have used definition (1.31c) to write

$$\frac{1}{w} - i = \frac{1}{t^{-1}(w)} .$$

By Eq. (1.29) we have

$$S^{-1} = K^{-1} - iI$$

and thus we obtain

$$K^{-1} = \int_{-\infty}^{+\infty} \frac{1}{w} d[P(w)f],$$

or, using (4.5a),

$$K^{-1}f = \int_{-\infty}^{+\infty} \frac{1}{\nu}\Phi_\nu(\mu)(\tilde{T}f)(\nu)d\sigma(\nu).$$

(iii) The proof follows by a direct calculation.

For $f \in X_p^2(\mathbb{R}^+)$ one can express Ef in the form,

(4.15) $$Ef = Ef_1 + Ef_2$$

where,

(4.16a) $$f_1(\mu) = f(\mu) - F(\mu)\rho_0(Ef)$$

and

(4.16b) $$f_2(\mu) = F(\mu)\rho_0(Ef).$$

Then $Ef_1 \in Y_p^2(\mathbb{R})$ and the previous results can be used. Note that

(4.17) $$P(w)(Ef_1)(\mu) = \frac{1}{2\pi i}\int_{\Gamma'} (zI-S)^{-1}(Ef_1)(\mu)dz - \int_w^\infty \Phi_\nu(\mu)(\tilde{T}Ef_1)(\nu)d\sigma(\nu).$$

Using the above and the properties of E we have,

(4.18) $$P(w)(Ef_1)(\mu) = \int_0^{w'} \Phi_\nu(\mu)(\tilde{T}Ef_1)(\nu)d\sigma(\nu),$$

where

(4.19) $w' = \sup\{0,w\}$,

i.e.,

(4.20) $P(w)Ef_1 = 0, \; w \leq 0.$

Now we are in position to solve Eq. (1.4).

Section 5: Solutions to Boundary Value Problems in Kinetic Theory

We seek solutions to the system,

(5.1a) $$(\frac{\partial}{\partial x} + K^{-1})\Xi(x,\mu) = 0$$

(5.1b) $$\Xi(0,\mu) = \Xi_0(\mu),$$

where K^{-1} is given by Eq. (1.5) and $\Xi_0(\mu) \varepsilon X_p^2(\mathbb{R})$ is given. The solution is given by

(5.2) $$\Xi(x,\mu) = F(\mu)[\rho_0(\Xi_0)+(\mu-x)\rho_1(\Xi_0)] + \int_{-\infty}^{+\infty} \Phi_\nu(\mu)(\tilde{T}\Xi_0)(\nu)e^{-x/\nu}d\sigma(\nu).$$

Clearly (5.2) satisfies the boundary condition (5.1b) since the r.h.s. of Eq. (5.2) reduces to the full-range expansion of Ξ_0 when $x = 0$. To show that $\Xi(x,\mu)$ given by (5.2) satisfies (5.1a) we first note that

(5.3) $$(\frac{\partial}{\partial x} + K^{-1})F(\mu)[\rho_0(\Xi_0)+(\mu-x)\rho_1(\Xi_0)] = 0,$$

by part (iii) of Theorem 5. And thus we must show that,

(5.4) $$(\frac{\partial}{\partial x} + K^{-1})\int_{-\infty}^{+\infty} \Phi_\nu(\mu)(\tilde{T}\Xi_0)(\nu)e^{-x/\nu}d\sigma(\nu) = 0.$$

Let Ξ_x be defined by

$$(5.5)\quad \Xi_x(\mu) = \int_{-\infty}^{+\infty} \Phi_\nu(\mu)(\tilde{T}\Xi_0)(\nu)e^{-x/\nu}d\sigma(\nu),$$

then

$$(5.6)\quad \Xi_x(\mu) = \int_{-\infty}^{+\infty} \Phi_\nu(\mu)(\tilde{T}\Xi_x)(\nu)d\sigma(\nu),$$

where

$$(5.7)\quad (\tilde{T}\Xi_x)(\nu) = (\tilde{T}\Xi_0)(\nu)e^{-x/\nu}.$$

Assuming $\Xi_x \in X_p^2(\mathbb{R})$ we then have, using part (ii) of Theorem 5,

$$(5.8)\quad (K^{-1}\Xi_x)(\mu) = \int_{-\infty}^{+\infty}\frac{1}{\nu}\Phi_\nu(\mu)(\tilde{T}\Xi_x)(\nu)d\sigma(\nu),$$

and hence,

$$(5.9)\quad K^{-1}\int_{-\infty}^{+\infty} \Phi_\nu(\mu)(\tilde{T}\Xi_0)(\nu)e^{-x/\nu}d\sigma(\nu) = \int_{-\infty}^{+\infty} \frac{1}{\nu}\Phi_\nu(\mu)(\tilde{T}\Xi_0)(\nu)e^{-x/\nu}d\sigma(\nu).$$

And thus Eq. (5.4) is verified.

For half-range problems $\Xi^+(\mu) \in X_p^2(\mathbb{R}^+)$ is given and

$$(5.10)\quad \Xi(0,\mu) = \Xi^+(\mu),\ \mu \geq 0.$$

The additional constraint,

$$(5.11)\quad \lim_{x\to\infty} \Xi(x,\mu) \to \text{constant},$$

is also applied. In this case the solution is of the form,

$$(5.12)\quad \Xi(x,\mu) = F(\mu)\rho_0{}^+(\Xi^+) + \int_0^{+\infty} \Phi_\nu(\mu)A^+(\nu)e^{-x/\nu}d\sigma(\nu),$$

where A^+ is the half-range expansion coefficient of Ξ^+ as in Theorem 4. As in the full-range case the r.h.s. of Eq. (5.12) reduces to the half-range expansion of Ξ^+ when $x = 0$. Thus boundary condition (5.10) is satisfied. Since $x \in \mathbb{R}^+$, $\Xi(x,\mu)$ given by (5.12) satisfies (5.11). Verification that $\Xi(x,\mu)$ is a solution to (5.1a) is obtained in the same way as the full-range case.

Chapter 5

THE TEMPERATURE-JUMP PROBLEM AND RELATED PROBLEMS IN KINETIC THEORY

Section 1: Introduction to the Temperature-Jump Problem

The Temperature-Jump problem is defined by a half-space of gas bounded by a wall at a uniform temperature, T_0. It is found that the temperature of the gas near the wall differs from the temperature of the wall. This is due to a layer of gas called the transition region or Knudsen layer. For a discussion of this phenomenon see Ref. 26. The thickness of the transition region is only a few mean free paths and the temperature gradient just beyond this region is constant. Figure 2 illustrates the qualitative behavior of the temperature near the wall. The temperature jump is defined to be the apparent temperature of the gas at the wall, extrapolated from the linear portion of the temperature curve just beyond the transition region, and T_0, the wall temperature. Thus if we define τ by

$$(1.1) \qquad \tau(x) = T(x) - T'_0(x),$$

where $T(x)$ is the temperature of the gas and $T'_0(x)$ is given by,

$$(1.2) \qquad T'_0(x) = T_0(1+\kappa' x),$$

where $\kappa' T_0$ is the asymptotic temperature gradient, then the temperature jump is $\tau(\infty)$. Of course, we must stretch the local coordinate in the

Knudsen layer in order to consider it a half-space.

The method of solution proceeds as follows. Denote by $f'_0(0,\xi)$ the Maxwellian corresponding to the wall temperature and expand the distribution function about $f'_0(x,\xi)$ given by,

$$(1.3) \quad f'_0(x;\xi) = n'_0\left(\frac{m}{2\pi kT'_0}\right)^{3/2}\exp\left(-\frac{m}{2kT'_0}(\xi-\upsilon_0)^2\right),$$

where,

$$(1.4) \quad n'_0 = n_o(1+\kappa' x)^{-1}.$$

Note that the pressure, $n'_0T'_0$, is constant.

To first order the distribution function, $f'(x;\xi)$, will be given by,

$$(1.5) \quad f'(x;\xi) = f'_0(x;\xi)(1+h(x;\xi)).$$

Note that $f'_0(x;\xi)$ can be expressed, to first order, as,

$$(1.6) \quad f'_0(x;\xi) = f_0(\xi)[1+\kappa' x(c^2-5/2)+h(x;\xi)],$$

where c is given by (3.2.8) and $f_0(\xi)$ is given by (3.2.6).

Substituting (1.6) into (1.5) yields,

$$(1.7) \quad f'(x;\xi) = f_0(\xi)[1+\kappa' x(c^2-5/2)+h(x;\xi)].$$

If $h'(x,\xi)$ is taken to be the deviation of $f'(x,\xi)$ from $f_0(\xi)$, i.e.,

$$(1.8) \quad h'(x,\xi) = \kappa' x(c^2-5/2)+h(x,\xi),$$

then it follows from (3.2.22) that

$$(1.9)\qquad \xi_1 \frac{\partial h'(x';\xi)}{\partial x'} = \nu[\sum_{i=0}^{4} (h'(x';\xi),e_i)e_i - h'(x';\xi)].$$

Proceeding as in section 2 of Chapter 3 the above can be transformed into,

$$(1.10)\qquad \mu \frac{\partial h}{\partial x} + \kappa\mu(c^2-5/2) = \sum_{i=0}^{4} (h,e_i)e_i - h,$$

with,

$$(1.11a)\qquad (h,e_1) = 0,$$

and

$$(1.11b)\qquad \kappa' = \frac{m}{2kT_0} \nu\kappa.$$

Note that,

$$(1.12)\qquad (\mu(c^2-5/2),e_i) = 0,\ 0 \le i \le 4,$$

and thus, using (1.10), we see that h" defined by,

$$(1.13)\qquad h''(x;\xi) = h(x;\xi) + \kappa\mu(c^2-5/2),$$

satisfies (3.2.24) and (3.2.25).

Let

$$(1.14a)\qquad \psi_1(x,\mu) = (h'',\phi_0)_2 ,$$

$$(1.14b)\qquad \psi_2(x,\mu) = (h'',\phi_1)_2 ,$$

and

$$(1.14c)\qquad \psi(x,\mu) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} .$$

Then Ψ will satisfy (3.2.36).

If $f_0(\xi)$ is the Maxwellian corresponding to the wall temperature then the general boundary condition on the distribution function, $f'(x,\xi)$, is,

$$f'(0;\xi_1,\xi_2,\xi_3) = \alpha f_0(\xi) + (1-\alpha)f'_0(0;-\xi_1,\xi_2,\xi_3),\ \xi_1 > 0. \tag{1.15}$$

Here $\alpha \in [0,1]$ is the accommodation coefficient. If $\alpha = 1$ then the distribution function is equal to the wall Maxwellian and the molecules perfectly accommodate to the wall. If $\alpha = 0$ then the molecules are specularly reflected from the wall. Values of α between 0 and 1 correspond to mixtures of perfect accommodation and specular reflection.

Substituting Eq. (1.5) into (1.15) we obtain the boundary condition on h,

$$f_0(\xi)[1+h(0;\xi_1,\xi_2,\xi_3)] = \alpha f_0(\xi) + (1-\alpha)f_0(\xi)[1+h(0;-\xi_1,\xi_2,\xi_3)],\quad \xi_1 > 0, \tag{1.16a}$$

or,

$$h(0;\xi_1,\xi_2,\xi_3) = (1-\alpha)h(0;-\xi_1,\xi_2,\xi_3),\ \xi_1 > 0. \tag{1.16b}$$

Assuming that the molecules perfectly accommodate to the wall, $\alpha = 1$, we must have,

$$h(0;\xi) = 0,\ \xi_1 > 0, \tag{1.17a}$$

and thus,

(1.17b) $\quad h''(0;\xi) = \kappa\mu(c^2-5/2),\ \mu > 0.$

Recalling Eq. (3.2.35) this boundary condition, in terms of $\Psi(0,\mu)$, is,

(1.18) $$\Psi(0,\mu) = \kappa\Pi^{3/4}\mu Q(\mu)\begin{pmatrix}\sqrt{2/3}\\ -1\end{pmatrix},\ \mu > 0.$$

Thus we wish to find solutions to Eq. (3.2.36), i.e.,

(1.19) $$\mu\frac{\partial\Psi(x,\mu)}{\partial x} + \Psi(x,\mu) = Q(\mu)\int_{-\infty}^{+\infty} Q^T(s)\Psi(x,s)d\sigma(s),$$

subject to the boundary condition given by Eq. (1.18). From this we can determine the temperature jump, $\tau(\infty)$, where $\tau(x)$ is given by Eq. (1.1).

Section 2: Solution to the Temperature-Jump Problem with Complete Accommodation.

It is convenient to define $\Xi(x,\mu)$ as in Eq. (4.1.3). Equation (1.19) with boundary condition (1.18) expressed in terms of $\Xi(x,\mu)$ are as follows:

(2.1) $$\Xi(0,\mu) = \mu F(\mu)v,\ \mu > 0,$$

where,

(2.2) $$v = k\Pi^{3/4}\begin{pmatrix}\sqrt{3/2}\\ -1\end{pmatrix},$$

and

(2.3) $$\frac{\partial\Xi(x,\mu)}{\partial x} + K^{-1}\Xi(x,\mu) = 0.$$

Furthermore, the additional boundary condition is needed,

$$\lim_{x\to\infty} \Xi(x,\mu) \to f(\mu). \tag{2.4}$$

The boundary condition, (2.4), is necessary because we are calculating a perturbation from an initial distribution whose temperature depends linearly on x. Thus the perturbed distribution, which is the product of the initial distribution and 1 + h, where h is the perturbation, will have a linear dependence on x, for large x, if h is independent of x for large x. We know that the temperature dependence in the Chapman-Enskog region, i.e. $x \to \infty$, is linear and thus to match this region with the Knudsen layer we must apply the boundary condition, (2.4).

Let P^+ be defined as in Eq. (4.3.7). Boundary condition (2.1) can be written as,

$$P^+\Xi(0,\mu) = \mu F(\mu)v. \tag{2.6}$$

The solution to Eq. (2.3) is of the form,

$$\Xi(x,\mu) = F(\mu)\rho_0{}^+(\Xi_0) + \int_0^\infty \Phi_\nu(\mu)A^+(\nu)e^{-x/\nu}d\sigma(\nu), \tag{2.7}$$

according to (4.5.12). A^+ is given by Theorem 4,

$$A^+(\nu) = \lambda_-^+(\nu)\Xi_0(\nu) + \lambda_-^-(\nu)H^{T^{-1}}(\nu)\,\frac{\exp(\nu^2)}{\nu}\,P\int_0^\infty \frac{sH^T(s)\Xi_0(s)}{\nu-s}\,d\sigma(s), \tag{2.8}$$

and

$$\Xi_0(\mu) = \mu F(\mu)v. \tag{2.9}$$

$\Xi(x,\mu)$ given by (2.7) satisfies boundary condition (2.1) since the r.h.s. of Eq. (2.7) reduces to the half-range expansion of $\mu F(\mu)v$ when $x = 0$. It satisfies boundary condition (2.4) since

$$\lim_{x\to\infty} \Xi(x,\mu) = F(\mu)\rho_0^{\ +}(\mu F(\mu)v). \tag{2.10}$$

Uniqueness is guaranteed by Ref. 28.

Using the definitions, (4.3.17) and (4.3.18), we have,

$$\rho_0^{\ +}(\mu F(\mu)v) = BH_1H_2^{\ T}v. \tag{2.11}$$

It can be shown that,

$$B = (H_1H_1^{\ T})^{-1} , \tag{2.12}$$

and

$$H_2^{\ T} = H_1^{\ T} , \tag{2.13}$$

and thus,

$$\rho_0^{\ +}(\mu F(\mu)v) = v. \tag{2.14}$$

By Eqs. (2.10) and (2.14) we have

$$\lim_{x\to\infty} \Xi(x,\mu) = F(\mu)v. \tag{2.15}$$

According to Eq. (1.8) the perturbation of the distribution is h' given by

$$h' = \kappa x(c^2-5/2) + h, \tag{2.16}$$

where h is given by Eq. (1.13),

$$(2.17) \qquad h = h'' - \kappa\mu(c^2-5/2).$$

Designate the perturbation to the temperature due to h by ΔT, due to h' by $\Delta T'$ and due to h" by $\Delta T''$. $\Delta T''$ can be written as,

$$(2.18) \qquad \Delta T''(x) = T_0 \Pi^{-3/4} \sqrt{2/3} \int_{-\infty}^{+\infty} \Xi_1(x,\mu)d\sigma(\mu),$$

using Eq. (3.2.43) and the definition of Ξ. ΔT is given by,

$$(2.19) \qquad \Delta T = T_0 \Pi^{-3/2} \frac{2}{3} \int (c^2-3/2)[h''-\kappa\mu(c^2-5/2)]d\sigma(c) = \Delta T'',$$

where Eqs. (3.2.42) and (2.18) have been used. Thus, the perturbation to T_0, $\Delta T'$, will be given by

$$(2.20) \qquad \Delta T'(x) = \Delta T(x) + \kappa x T_0 \Pi^{-3/2} \frac{2}{3} \int (c^2-5/2)(c^2-3/2)d\sigma(c)$$

$$= T_0 \Pi^{-3/4} \sqrt{2/3} \int_{-\infty}^{+\infty} \Xi_1(x,\mu)d\sigma(\mu) + T_0 \kappa x.$$

Consequently the temperature jump is given by,

$$(2.21) \qquad \tau(\infty) = T_0 \kappa .$$

Section 3: Treatment of the Temperature-Jump Problem with Arbitrary Accommodation.

For the case of arbitrary accommodation the boundary condition on h is given by Eq. (1.16b). Using Eq. (1.13) this translates into the following boundary condition on h";

$$(3.1)\qquad h''(0;\xi_1,\xi_2,\xi_3) = (1-\alpha)h''(0;-\xi_1,\xi_2,\xi_3) + (2-\alpha)\kappa\mu(c^2-5/2),\ \xi_1 > 0.$$

In terms of Ξ we use Eqs. (3.2.35), (4.1.3), (2.2) and the above to write,

$$(3.2)\qquad \Xi(0,\mu) = (1-\alpha)\Xi(0,-\mu) + (2-\alpha)\mu F(\mu)v,\ \mu > 0.$$

Define $S:X_p^2(\mathbb{R}) \to X_p^2(\mathbb{R})$ by

$$(3.3)\qquad (Sf)(\mu) = f(-\mu),$$

and let

$$(3.4)\qquad \Xi_0(\mu) = \Xi(0,\mu).$$

Then boundary condition (3.2) can be written in the form,

$$(3.5)\qquad (P^+\Xi_0)(\mu) = (1-\alpha)(P^+S\Xi_0)(\mu) + (2-\alpha)\mu F(\mu)v.$$

For half-range problems it has been shown that the additional boundary condition given by Eq. (2.4) can be satisfied if

$$(3.6)\qquad \Xi_0(\mu) = (EP^+\Xi_0)(\mu).$$

In fact, this additional boundary condition is satisfied if and only if (3.6) holds. The proof of this is identical to the proof in Appendix III for the neutron transport equation except that Ref. 28 is needed for uniqueness.

Substituting (3.6) into (3.5) we obtain,

$$(3.7)\qquad (P^+\Xi_0)(\mu) = (2-\alpha)\mu F(\mu)v + (1-\alpha)[P^+SE][P^+\Xi_0](\mu),$$

an integral equation for the surface density, $(P^{+}\Xi_0)(\mu)$.

It should be noted that the original boundary condition, (3.5), is difficult to use in half-space problems since $P^{+}\Xi_0$, which is usually a completely known function, is given in terms of a known function and the unknown function, $P^{+}S\Xi_0$. The approach used to overcome this difficulty in other problems with this boundary condition[29] has been to expand both sides of Eq. (3.5) using the full-range expansion and obtain rather complicated integral equations for the expansion coefficients. No papers have appeared in the literature to guarantee that these integral equations possess solutions or that the numerical techniques applied to them will converge. With our approach the surface density is obtained directly and in a comparatively simple form. We also claim that solutions to (3.7) exist for at least all but finitely many values of $\alpha \in [0,1]$.

To prove that Eq. (3.7) has a solution we need the following theorem which we state without proof

Theorem 6.[30] Let $\langle M,\tau\rangle$ be a measure space and $H = L^2(M,d\tau)$. Then a bounded linear operator, C, from H to H is Hilbert-Schmidt if and only if there is a function

$$K \in L^2(M\times M,\ d\tau \times d\tau)$$

with

$$(Cf)(x) = \int K(x,y)f(y)d\tau(y).$$

Moreover,

$$||C||^2 = \int |K(x,y)|^2 d\tau(x)d\tau(y).$$

With this theorem we are able to prove

Theorem 7. The operator, $C: X_2^2(\mathbb{R}^+) \to X_2^2(\mathbb{R}^+)$, defined by

$$(Cf)(\mu) = (P^+ SEf)(\mu)$$

is compact.

Proof: Using Eq. (4.3.10) the action of C on $f \in X_2^2(\mathbb{R}^+)$ is given by,

$$(Cf)(\mu) = \int_0^\infty \frac{F(\mu)H(\mu)sH^T(s)}{s+\mu} f(s)d\sigma(s).$$

Define the τ measure by

$$d\tau(s) = s^2 \exp(-s^2)ds,$$

then,

$$||f||_{I(2)} = \int_0^\infty |sf(s)|^2 d\sigma(s),$$

or,

$$||f||_{I(2)} = \int_0^\infty |f(s)|^2 d\tau(s).$$

Thus the action of C on f is also given by

$$(Cf)(\mu) = \int_0^\infty \frac{F(\mu)H(\mu)H^T(s)}{s(s+\mu)} f(s)d\tau(s).$$

Define the matrix norm, $||\cdot||_M$, by

$$||G||_M = \sup_{j\varepsilon\{1,2\}} \sum_{i=1}^{2} |G_{ij}|$$

where G_{ij} is the ij-th element of the matrix G. Using Theorem 6 we can show that C is compact if

$$\int_0^\infty \int_0^\infty \left|\left|\frac{F(\mu)H(\mu)H^T(s)}{s(s+\mu)}\right|\right|_M^2 d\tau(\mu)d\tau(s)$$

$$= \int_0^\infty \int_0^\infty ||F(\mu)H(\mu)H^T(s)||_M^2 \frac{\mu^2}{(s+\mu)^2} d\sigma(\mu)d\sigma(s) < \infty.$$

Since $[\mu/(\mu+s)]^2$ is well behaved and since F and H are bounded by polynomials the above condition is satisfied. As C is a compact operator from $X_2^2(\mathbb{R}^+)$ to $X_2^2(\mathbb{R}^+)$ its spectrum consists of at most a countable number of points. The only possible accumulation point in its spectrum is zero which corresponds to $\alpha = \infty$. Thus there are at most finitely many values of $\alpha \in [0,1]$ for which the operator,

$$(3.8) \qquad C' = \frac{1}{2-\alpha}[I-(1-\alpha)P^+SE],$$

is non-invertible. And hence the solution to Eq. (3.7) is given by

$$(3.9) \qquad (P^+\Xi_0)(\mu) = (C')^{-1}\mu F(\mu)v,$$

for all but at most a finite number of values of $\alpha \in [0,1]$.

We have been unable to find a satisfactory upper bound on $||C||$, however, we know that

$$(3.10) \qquad ||C|| \geq 1,$$

since,

$$(3.11) \qquad CF(\mu)v_1 = F(\mu)v_1, \quad v_1 \in \mathbb{R}^2.$$

To see that $F(\mu)v_1$ is an eigenvector of C we substitute $F(\mu)v_1$ for f into the definition of C given in Theorem 7,

$$(3.12)\quad CF(\mu)v_1 = F(\mu)H(\mu) \int_0^\infty H^T(s)F(s)\,\frac{s}{s+\mu}\,d\sigma(s)v_1$$

$$= F(\mu)H(\mu) \int_0^\infty H^T(s)F(s)\left(1-\frac{\mu}{s+\mu}\right)d\sigma(s)v_1 .$$

Using the properties of H given in Eqs. (4.3.13a) and (4.3.13b) we obtain (3.11). Thus, if there is a solution to the temperature-jump problem with $\alpha = 0$ it is not unique. This leads us to conjecture that the Neumann-Liouville series converges for $0 < \alpha \leq 1$.

Proceeding with the solution of the problem one would solve the integral equation for the surface density numerically. Once the surface density is obtained the method of solution proceeds exactly as in the case of complete accommodation. This, however, is beyond the scope of this work.

Section 4: Treatment of the Transverse Velocity Equation with Arbitrary Accommodation

The form of the transverse velocity equation is similar to that of the temperature-density equation although simpler. The transverse velocity equation is given by Eq. (3.2.45), i.e.,

$$(4.1)\quad \mu\,\frac{\partial\psi(x,\mu)}{\partial x} + \psi(x,\mu) = \frac{1}{\sqrt{\Pi}} \int_{-\infty}^{+\infty} \psi(x,s)d\sigma(s).$$

The appropriate space to search for solutions to Eq. (4.1) is $X_p^1(\mathbb{R})$, $p > 1$ (see Ref. 16). Since the elements of Ψ and ψ are appropriate inner products of the perturbation, h, with certain basis vectors (see Eqs. (3.2.35a), (3.2.35b), (3.2.44) and (3.2.45)) the boundary condition for arbitrary accommodation takes the same form in the transverse velocity case as in the temperature-density case, i.e.,

(4.2) $(P^+\psi)(\mu) = (1-\alpha)(P^+S\psi)(\mu) + (2-\alpha)k\mu.$

Here $P^+ : X^1_p(\mathbb{R}) \to X^1_p(\mathbb{R}^+)$ and $S : X^1_p(\mathbb{R}) \to X^1_p(\mathbb{R})$ are defined by,

(4.3) $(P^+\psi)(\mu) = \psi(\mu), \ \mu > 0,$

(4.4) $(S\psi)(\mu) = \psi(-\mu),$

and k is a constant. Boundary condition (4.2) can be derived in analogy with boundary condition (3.5) or consult Ref. 31. Equation (4.1) along with boundary condition (4.2) and the additional requirement,

(4.5) $\lim_{x\to\infty} \psi(x,\mu) \to f(\mu),$

define Kramer's problem (or "slip-flow" problem).[31]

As in the case of the temperature-density equation, Eq. (4.5) is satisfied if and only if,

(4.6) $\psi(0,\mu) = (\hat{E}P^+\psi)(\mu).$

Here $\hat{E}$ is given by,[16]

(4.8) $$(\hat{E}f)(\mu) = \begin{cases} f(\mu), \ \mu > 0 \\ \dfrac{1}{X(\mu)} \displaystyle\int_0^\infty \frac{\gamma(s)}{s-\mu} f(s)ds, \ \mu < 0, \end{cases}$$

where X and γ are as in Ref. 16. Thus, boundary conditions (4.2) and (4.5) can be combined as,

$$(4.9) \quad (P^{+}\psi)(\mu) = (1-\alpha)[P^{+}S\hat{E}][P^{+}\psi](\mu) + (2-\alpha)k\mu,$$

an integral equation for the surface density, $P^{+}\psi$. Cercignani[31] has dealt with Kramer's problem with arbitrary accommodation but with a different approach. Rather than deriving an integral equation for the surface density his approach was to expand both sides of Eq. (4.2) using the full-range expansion of ψ and employ boundary condition (4.5) to obtain an integral equation for the expansion coefficients. He then proved that the integral equation could be solved by a Neumann-Liouville series for $0 < \alpha \leq 1$. The integral equation that we have obtained, Eq. (4.9), can also be shown to have a convergent Neumann-Liouville series, however, we shall omit the proof.

The solution to Kramer's problem is obtained by first calculating the surface density using Eq. (4.9) and numerical procedures. Once the surface density is obtained the solution follows by applying the methods of Ref. 16.

Appendix I

Section 1: The Diagonal Expansion of the Determinant of the Multi-Group Dispersion Matrix

If A is an $N \times N$ matrix denote by A_{ij} the $(N-1) \times (N-1)$ matrix formed by deleting the i-th row and the j-th column of A. It will be convenient to label the rows and columns of A_{ij} with the same indices as the corresponding rows and columns of A. Thus the matrix, A_{ij}, will have its i-th row and j-th column missing. Keeping in mind this convention we define $(A_{ij})_{k\ell}$ to be the matrix formed by deleting the i-th and k-th rows of A and the j-th and ℓ-th columns of A. The element in the i-th row and j-th column of A will be given by a_{ij}. If the matrix A is perturbed by adding a factor, ε_n, to the element, a_{i_n, j_n}, the resulting matrix will be called $A(\varepsilon_n)$ and a matrix with many such perturbations will be labeled $A(\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n)$.

It is trivially verified that

$$(1.1) \qquad \det(A(\varepsilon_1)) = \det(A) + (-1)^{i_1+j_1}\varepsilon_1 \det(A_{i_1 j_1}).$$

We wish to show by induction that,

$$(1.2) \qquad \det(A(\varepsilon_1,\varepsilon_2,\ldots\varepsilon_n)) = \det\,(A) + \sum_{k=1}^{n} (-1)^{i_k j_k}\varepsilon_k \det(A_{i_k j_k}),$$

to first order in ε. Assume that the ε_i are all of the same order of magnitude, ε, and that $\det(A(\varepsilon_1,\varepsilon_2,\ldots\varepsilon_n))$ is given, to first order, by Eq. (1.2). Then,

(1.3) $$\det(A(\varepsilon_1,\varepsilon_2,\ldots\varepsilon_{n+1})) = \sum_{\ell=1}^{N} (-1)^{i_{n+1}\ell} a'_{i_{n+1}\ell} \det(A(\varepsilon_1,\varepsilon_2,\ldots\varepsilon_{n+1})_{i_{n+1}\ell}),$$

where $a'_{i_{n+1}\ell}$ is the (i_{n+1},ℓ)-th element of $A(\varepsilon_1,\varepsilon_2,\ldots\varepsilon_{n+1})$ and the r.h.s. of Eq. (1.3) is simply the cofactor expansion of $A(\varepsilon_1,\varepsilon_2,\ldots\varepsilon_{n+1})$. Let a''_{ij} be the (i,j)-th element of $A(\varepsilon_1,\varepsilon_2,\ldots\varepsilon_n)$. Then,

(1.4) $$\det(A(\varepsilon_1,\varepsilon_2,\ldots\varepsilon_{n+1})) = \varepsilon_{n+1}(-1)^{i_{n+1}j_{n+1}}\det(A(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)_{i_{n+1}j_{n+1}})$$
$$+ \sum_{\ell=1}^{N} a''_{i_{n+1}\ell}(-1)^{i_{n+1}\ell}\det(A(\varepsilon_1,\varepsilon_2,\ldots\varepsilon_n)_{i_{n+1}\ell}).$$

Here we have used the fact that

(1.5) $$A(\varepsilon_1,\varepsilon_2,\ldots\varepsilon_{n+1})_{i_{n+1}\ell} = A(\varepsilon_1,\varepsilon_2,\ldots\varepsilon_n)_{i_{n+1}\ell}.$$

The second term on the r.h.s. of Eq. (1.4) is simply $\det(A(\varepsilon_1,\varepsilon_2,\ldots\varepsilon_n))$ and thus, using the assumption, Eq. (1.2), we have

(1.6) $$\det(A(\varepsilon_1,\varepsilon_2,\ldots\varepsilon_{n+1})) = (-1)^{i_{n+1}j_{n+1}}\varepsilon_{n+1}\det(A_{i_{n+1}j_{n+1}})$$
$$+ \det(A) + \sum_{k=1}^{n} (-1)^{i_kj_k}\varepsilon_k\det(A_{i_kj_k}),$$

where we have used,

(1.7) $\quad \varepsilon_{n+1}\det(A(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)_{i_{n+1}j_{n+1}}) = \varepsilon_{n+1}\det(A_{i_{n+1}j_{n+1}}),$

to first order in ε. This yields the result,

(1.8) $\quad \det(A(\varepsilon_1,\varepsilon_2,\ldots\varepsilon_{n+1})) = \det(A)$

$$+\sum_{k=1}^{n+1}(-1)^{i_k+j_k}\varepsilon_k \det(A_{i_k j_k}).$$

Hence Eq. (1.2) is verified.

Suppose that the perturbation matrix is B, the elements of B being of order ε. Then, using Eq. (1.8), we have,

(1.9) $\quad \det(A+B) = \det(A) + \sum_{ij}(-1)^{i+j}b_{ij}\det(A_{ij}),$

where b_{ij} is the (i,j)-th element of B. By definition

(1.10) $\quad (-1)^{i+j}\det(A_{ij}) = c_{ji},$

where c_{ij} is the (i,j)-th element of A_c^T. Substituting Eq. (1.10) into Eq. (1.9) we have,

(1.11) $\quad \det(A+B) = \det(A) + \mathrm{Tr}(BA_c^T),$

to first order in ε.

$\Lambda(z)$ is given by,

(1.12) $\quad \Lambda(z) = \Sigma C^{-1}\Sigma - 2\Sigma - \int_{-1}^{+1}\mu(zI-\mu\Sigma^{-1})^{-1}d\mu.$

Define,

(1.13) $$G(z) = \int_{-1}^{+1} \mu(zI-\mu\Sigma^{-1})^{-1}d\mu.$$

Then the elements of G will be given by,

(1.14) $$g_{ij}(z) = 0, \; i \neq j,$$

and

(1.15) $$g_{ii}(z) = \int_{-1}^{+1} \frac{\mu}{z-\mu/\sigma_{ii}} d\mu,$$

where the non-zero elements of Σ are given by σ_{ii}. For large z we have,

(1.16) $$g_{ii}(z) = \frac{1}{z} \int_{-1}^{+1} \mu(\mu+ \frac{\mu^2}{\sigma_{ii}z} + \ldots)d\mu = \frac{2}{3\sigma_{ii}} \frac{1}{z^2} + O(z^{-4}).$$

To order z^{-2} we have,

(1.17) $$G(z) = \frac{2}{3} z^{-2}\Sigma^{-1},$$

and, substituting into Eq. (1.12), we have,

(1.18) $$\Lambda(z) = \Sigma C^{-1}\Sigma-2\Sigma- \frac{2}{3} z^{-2}\Sigma^{-1} + O(z^{-4}).$$

Using the expansion, Eq. (1.11), with

(1.19a) $$A = \Sigma C^{-1}\Sigma-2\Sigma$$

and

(1.19b) $$B = -\frac{2}{3} z^{-2}\Sigma^{-1},$$

we have,

(1.20) $\det(\Lambda(z)) = -\frac{2}{3} z^{-2}\mathrm{Tr}(\Sigma^{-1}\Lambda_c^T(\infty)) + O(z^{-4})$,

where we have used,

(1.21) $\det(\Sigma C^{-1}\Sigma - 2\Sigma) = 0$,

for the critical case, and

(1.22) $\Lambda_c^T(\infty) = (\Sigma C^{-1}\Sigma - 2\Sigma)_c^T$.

Note that Eq. (1.20) is equivalent to

(1.23) $\det(\Lambda(z)) = -\frac{2}{3} z^{-2}\mathrm{Tr}(\Sigma^{-1}\Lambda_c(\infty)) + O(z^{-4})$,

since Σ is diagonal. Thus the two expressions may be used interchangeably.

Section 2: Direct Product Representation of $\Lambda_c^T(\infty)$.

Following Ref. 15 we introduce the null vectors ξ and $\hat{\xi}$ where,

(2.1) $\Lambda(\infty)\xi = 0$

and

(2.2) $\Lambda^T(\infty)\hat{\xi} = 0$.

Equations (2.1) and (2.2) imply that,

(2.3) $\Lambda_c^T(\infty) = \begin{pmatrix} a_1\xi & a_2\xi & \cdots & a_N\xi \end{pmatrix}$

and

$$(2.4)\qquad \Lambda_c(\infty) = \begin{pmatrix} b_1\Sigma\hat{\xi} & b_2\Sigma\hat{\xi} & \cdots & b_N\Sigma\hat{\xi} \end{pmatrix} ,$$

i.e., the transpose of the cofactor matrices have columns proportional to the null vectors. Here the proportionality constants are a_i and b_i with $1 \le i \le N$. Taking the transpose of Eq. (2.4) and equating to Eq. (2.3) we have,

$$(2.5)\qquad \begin{pmatrix} a_1\xi & a_2\xi & \cdots & a_N\xi \end{pmatrix} = \begin{pmatrix} b_1(\Sigma\hat{\xi})^T \\ b_2(\Sigma\hat{\xi})^T \\ \vdots \\ b_N(\Sigma\hat{\xi})^T \end{pmatrix} ,$$

i.e.,

$$(2.6)\qquad a_i\xi = (\Sigma\hat{\xi})_i b,$$

or

$$(2.7)\qquad \xi = (\Sigma\hat{\xi})_i \frac{b}{a_i} .$$

The above implies that a is proportional to $\Sigma\hat{\xi}$ and thus, using (2.3), we have,

$$(2.8)\qquad \Lambda_c^T(\infty) \propto \xi(\Sigma\hat{\xi})^T.$$

Let the constant of proportionality be k,

$$(2.9)\qquad \Lambda_c^T(\infty) = k\xi(\Sigma\hat{\xi})^T,$$

and let the normalization of ξ and $\hat{\xi}$ be given by, (2.10) $\quad \hat{\xi}^T \xi = 3/2.$

Then k is determined by,

(2.11) $\quad \mathrm{Tr}(\Lambda_c^T(\infty)\Sigma^{-1}) = \frac{3}{2}\, k.$

The final result is,

(2.12) $\quad \Lambda_c^T(\infty) = \frac{2}{3}\, \mathrm{Tr}(\Lambda_c^T(\infty)\Sigma^{-1})\xi(\Sigma\hat{\xi})^T,$

or,

(2.13) $\quad \Lambda_c(\infty) = \frac{2}{3}\, \mathrm{Tr}(\Sigma^{-1}\Lambda_c(\infty))\Sigma\hat{\xi}\xi^T.$

Appendix II

Denote by Λ the dispersion matrix for the temperature-density equation. The solutions of the temperature-density equation depend on a knowledge of the limiting behavior of $\Lambda(\mu)$ for large μ. The dispersion matrix is given by Eq. (4.1.31a), i.e.,

$$(1) \qquad \Lambda(z) = I + z \int_{-\infty}^{+\infty} \frac{F(s)}{s-z}\, d\sigma(s).$$

By a minor rearrangement Eq. (1) can be written as,

$$(2) \qquad \Lambda(z) = I + \int_{-\infty}^{+\infty} \frac{(z-s+s)}{s-z} F(s)d\sigma(s)$$

$$= I - \int_{-\infty}^{+\infty} F(s)d\sigma(s) + \int_{-\infty}^{+\infty} \frac{s}{s-z} F(s)d\sigma(s).$$

Using,

$$(3) \qquad \int_{-\infty}^{+\infty} F(s)d\sigma(s) = I,$$

we have,

$$(4) \qquad \Lambda(z) = \int_{-\infty}^{+\infty} \frac{s}{s-z} F(s)d\sigma(s).$$

Making use of the Plemelj formulas we obtain,

$$(5) \qquad \Lambda^{\pm}(\mu) = P\int_{-\infty}^{+\infty} \frac{s}{s-\mu} F(s)d\sigma(s) \pm i\Pi\mu F(\mu)\exp(-\mu^2).$$

Define C and D by,

(6a) $$C = P\int_{-\infty}^{+\infty} \frac{s}{s-\mu} F(s)d\sigma(s)$$

and

(6b) $$D = i\Pi\mu F(\mu)\exp(-\mu^2).$$

Using Eq. (I.1.11) we have, for large μ,

(7) $$\det(\Lambda^{\pm}(\mu)) = \det(C\pm D) = \det(C) \pm \mathrm{Tr}(DC_c^T) + O(\mu^n \exp(-2\mu^2)),$$

where n is a positive integer. Since we are dealing with 2 × 2 matrices we have,

(8) $$(\Lambda^{\pm})^{-1} = \frac{\det(C)}{\det(\Lambda^{\pm})} C^{-1} \pm \frac{\det(D)}{\det(\Lambda^{\pm})} D^{-1},$$

and thus,

(9) $$(\Lambda^{+})^{-1}-(\Lambda^{-})^{-1} = C^{-1}\det(C)\left(\frac{1}{\det(\Lambda^{+})} - \frac{1}{\det(\Lambda^{-})}\right)$$

$$+ D^{-1}\det(D)\left(\frac{1}{\det(\Lambda^{+})} + \frac{1}{\det(\Lambda^{-})}\right)$$

and

(10) $$(\Lambda^{+})^{-1} + (\Lambda^{-})^{-1} = C^{-1}\det(C)\left(\frac{1}{\det(\Lambda^{+})} + \frac{1}{\det(\Lambda^{-})}\right)$$

$$+ D^{-1}\det(D)\left(\frac{1}{\det(\Lambda^{+})} - \frac{1}{\det(\Lambda^{-})}\right).$$

Expressing $\det(\Lambda^{\pm}(\mu))$ as the r.h.s. of Eq. (7) we obtain, (11a) $$\frac{1}{\det(\Lambda^{+})} - \frac{1}{\det(\Lambda^{-})} = -2\,\frac{\mathrm{Tr}(DC^{-1})}{\det(C)} + O(\mu^{n}\exp(-2\mu^{2})),$$

and

(11b) $$\frac{1}{\det(\Lambda^{+})} + \frac{1}{\det(\Lambda^{-})} = \frac{2}{\det(C)} + O(\mu^{n'}\exp(-2\mu^{2})),$$

where n and n' are integer constants. Substituting Eqs. (11a) and (11b) into Eqs. (9) and (10) we find,

(12a) $$(\Lambda^{+})^{-1}(\mu) - (\Lambda^{-})^{-1}(\mu) \to \mu^{9}\exp(-\mu^{2}),$$

for large μ, and

(12b) $$(\Lambda^{+})^{-1}(\mu) + (\Lambda^{-})^{-1}(\mu) \to \mu^{2},$$

for large μ. In the above we have made use of the following,

(13a) $$\sup_{i,j\varepsilon\{1,2\}} d_{ij} \to \mu^{5}\exp(-\mu^{2}),$$

(13b) $$\det(D) \to \mu^{2}\exp(-2\mu^{2}),$$

(13c) $$\sup_{i,j\varepsilon\{1,2\}} d_{ij}^{-1} \to \mu^{3},$$

(13d) $$\sup_{i,j\varepsilon\{1,2\}} c_{ij}^{-1} \to \mu^{2}$$

and

(13e) $$\det(C) \to \mu^{-4},$$

where d_{ij} is the (i,j)-th element of D, d_{ij}^{-1} is the (i,j)-th element of D^{-1}, and c_{ij}^{-1} is the (i,j)-th element of C^{-1}.

From Eq. (5) we easily obtain

$$(14) \quad (\Lambda^{+})^{-1}(\mu) + (\Lambda^{-})^{-1}(\mu) \to \mu^{-2}.$$

The results are summarized in the following lemma.

Lemma 6. The limiting behavior of λ_{-}^{+}, λ_{-}^{-} and λ, for large μ, is given by:

$$(15a) \quad \sup_{i,j} (\lambda_{-}^{+}(\mu))_{ij} \to \mu^{2},$$

$$(15b) \quad \sup_{i,j} (\lambda_{-}^{-}(\mu))_{ij} \to \mu^{9}\exp(-\mu^{2}),$$

and

$$(15c) \quad \sup_{i,j} (\lambda(\mu))_{ij} \to \mu^{-2}.$$

Appendix III

The solution to the homogeneous neutron transport equation, for the conservative case, is unique provided $\Psi(0,\mu)$ is specified for $\mu\varepsilon[-1,1]$. Such problems are known as full-range problems. The half-range case is when $\Psi(0,\mu)$ is specified for $\mu\varepsilon[0,1]$ and the additional requirement,

$$(1) \qquad \lim_{x\to\infty} \Psi(x,\mu) \to L(\mu),$$

is specified. Since the solution is given by (see Chapter 2, Section 6),

$$(2) \qquad \Psi(x,\mu) = \frac{1}{2}\, a_0 + \frac{1}{2}\, a_1\, (x-\mu) + \int_{-1}^{+1} A(\nu)\Phi_\nu(\mu)e^{-x/\nu}d\nu,$$

Eq. (1) implies that

$$(3a) \qquad a_1 = 0$$

and

$$(3b) \qquad A(\nu) = 0,\ \nu\varepsilon[-1,0].$$

Specifying $\Psi(0,\mu)$ on $[0,1]$ with the additional requirement given by Eq. (1) is equivalent to specifying $\Psi(0,\mu)$ for $\mu\varepsilon[-1,1]$. To establish this we prove

Theorem 8. Let $\psi\ \varepsilon\ X_p(\mathbb{R})$ then ψ is expandable in the form,

$$\psi(\mu) = \frac{1}{2}\, a_0 - \frac{1}{2}\, a_1\mu + \int_{-\infty}^{+\infty} A(\nu)\Phi_\nu(\mu)d\nu.$$

The conditions

$$a_1 = 0$$

and

$$A(\nu) = 0, \ \nu\varepsilon[-1,0]$$

are satisfied iff

$$\psi = EP^+\psi$$

where the operators E and P^+ are defined in Chapter 2.

Proof: (i) By construction of E

$$\psi = EP^+\psi \Rightarrow a_1 = 0, \ A(\nu) = 0, \ \nu\varepsilon[-1,0].$$

(ii) Assume that

$$\{a_1 = 0, \ A(\nu) = 0, \ \nu\varepsilon[-1,0]\} \not\Rightarrow \psi = EP^+\psi.$$

Then there exists a $\psi_1 \ \varepsilon \ X_p(\mathbb{R})$ such that

(4) $\psi_1 \neq EP^+\psi_1$,

and

(5a) $a_1(\psi_1) = 0$,

(5b) $(\tilde{T}\psi_1)(\nu) = 0, \quad \nu\varepsilon[-1,0)$,

where,

$$(6a)\qquad a_i(\psi) = -3 \int_{-1}^{+1} s^{2-i}\psi(s)ds,$$

$$(6b)\qquad (\tilde{T}\psi)(\nu) = \frac{1}{N(\nu)} \int_{-1}^{+1} s\psi(s)\Phi_\nu(s)ds = A(\nu).$$

Thus Ψ_1 given by

$$(7)\qquad \Psi_1(x,\mu) = a_0(\psi_1) + \int_0^\infty \Phi_\nu(\mu)(\tilde{T}\psi_1)(\nu)e^{-x/\nu}d\nu$$

is a solution to the system,

$$(8a)\qquad \Psi(0,\mu) = \psi_1(\mu),\ \mu > 0,$$

$$(8b)\qquad \lim_{x\to\infty} \Psi(x,\mu) \to \text{bdd},$$

$$(8c)\qquad \frac{\partial\Psi}{\partial x} + K^{-1}\Psi = 0,$$

where K^{-1} is the transport operator. But Ψ_2 given by,

$$(9)\qquad \Psi_2(x,\mu) = a_0(EP^+\psi_1) + \int_0^\infty \Phi_\nu(\mu)(\tilde{T}EP^+\psi_1)(\nu)e^{-x/\nu}d\nu,$$

is also a solution to this system, where we have employed the half-range expansion. But the system given by Eqs. (8a), (8b) and (8c) is known to have a unique[17] solution and thus,

$$(10)\qquad \Psi_1 = \Psi_2 \Longrightarrow \psi_1 = EP^+\psi_1.$$

This contradicts the original assumption and hence the proof.

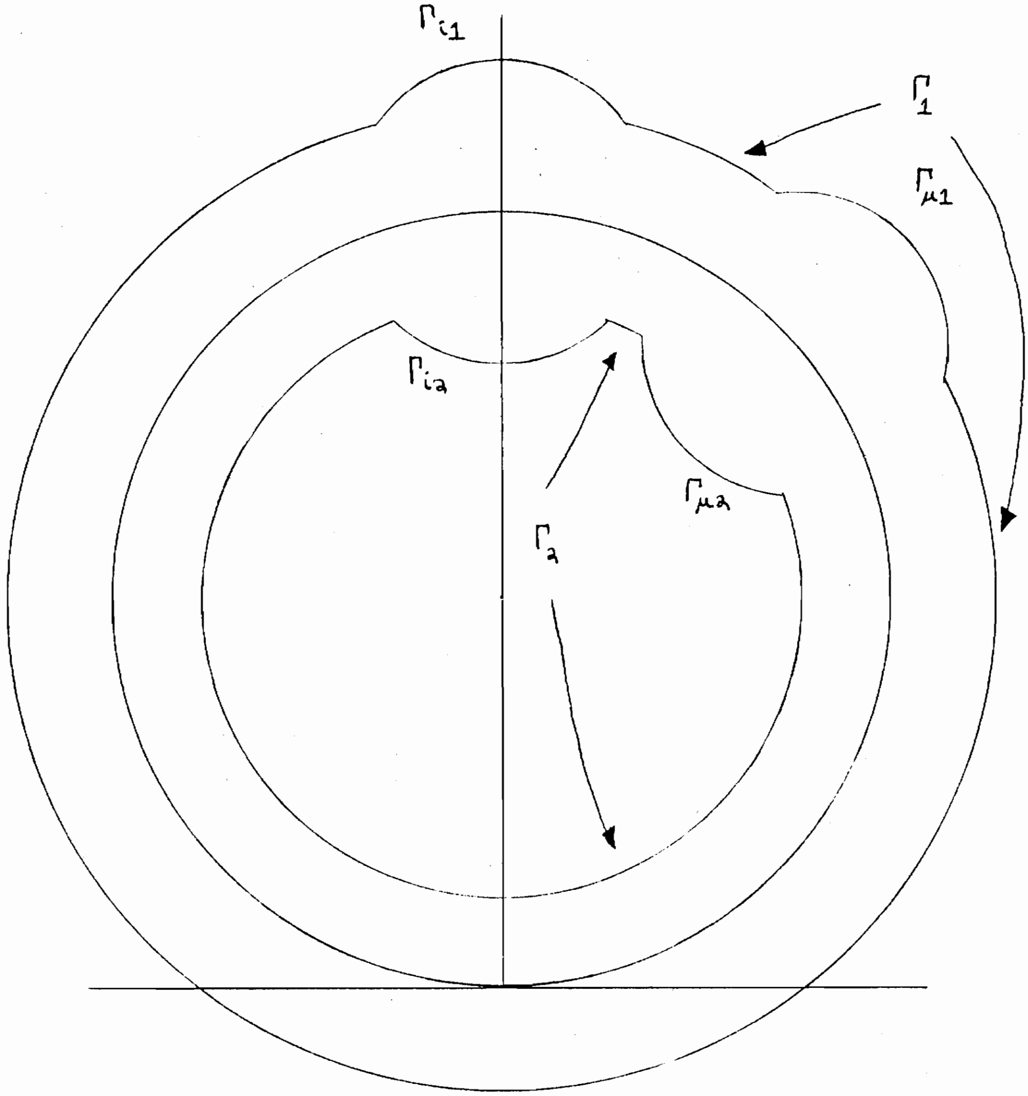

Figure 1: Contours Used in Evaluating the Contour Integral of the Resolvent of S.

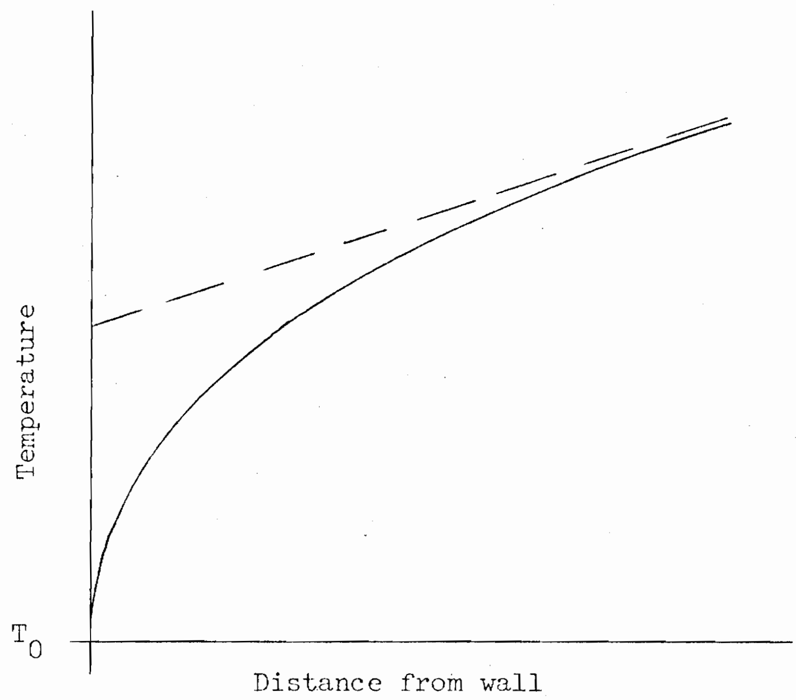

Figure 2: Temperature Profile of a Gas Very Near a Wall at Constant Temperature.

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William L. Cameron

FUNCTIONAL ANALYTIC TREATMENT OF LINEAR TRANSPORT EQUATIONS IN KINETIC THEORY AND NEUTRON TRANSPORT THEORY

by

William Lyle Cameron

(ABSTRACT)

The temperature-density equation of Kinetic Theory and the conservative neutron transport equation are studied. In both cases a modified version of the Larsen-Habetler resolvent integration technique is applied to obtain full-range and half-range expansions. For the neutron transport equation the method applied is seen to have notational advantages over previous approaches. In the case of the temperature-density equation this development extends previous results by enlarging the class of expandable functions and has the added advantage of rigor and simplicity. As a natural extension of the Kinetic Theory results, an integral equation for the surface density is derived for half-space problems involving the boundary condition of arbitrary accommodation.