OSCILLATIONS AND WAVES IN ANISOTROPIC PLASMAS

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Chapter 1

Introduction

Section 1 Derivation of Equations

A mathematically rigorous study of the Vlasov-Maxwell equations describing the deviation, f_1, of the one particle electron distrubution function, f, from its equilibrium value, f_0, is undertaken in the text to follow. This study will have particular bearing on questions of plasma stability, as we shall see in the next section. The physical situation we visualize is a one dimensional model of a neutral plasma where the electrons move subject to external and self consistent electric and magnetic fields according to the Vlasov-Maxwell equations[1,2] given by

$$\text{(1.1a)} \qquad \frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f + \frac{e}{m}(\vec{E} + \frac{1}{c} \vec{v} \times \vec{B}) \cdot \vec{\nabla}_{\vec{v}} f = 0 \ ,$$

$$\text{(1.1b)} \qquad \vec{\nabla} \cdot \vec{E} = 4\pi e(\int f \, d^3v - N_I),$$

$$\text{(1.1c)} \qquad \vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} + \frac{4\pi e}{c} \int \vec{v} \, f \, d^3v,$$

$$\text{(1.1d)} \qquad \vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t},$$

$$\text{(1.1e)} \qquad \vec{\nabla} \cdot \vec{B} = 0,$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields respectively, e is the electron charge, m is the electron mass and N_I is the density of the relatively immobile ion background.

These equations are linearized by writing

$$(1.2) \qquad f(z,\vec{v},t) = n_0 f_0(\vec{v}) + f_1(z,\vec{v},t)$$

where n_0 is the electron density and is equal to the density of the relativily immobile ion background, N_I (this is the neutral plasma criterion), and z is the spatial coordinate. f_1 is to be regarded as a "small" quantity. We will also assume that the steady state current, $n_0\, e \int \vec{v}\, f_0(\vec{v}) d^3v$, is small enough to be ignored in Eq. (1.1c). This assumption is certainly valid for Maxwellian equilibrium distribution functions or any equilibrium distribution that is an even function of the velocity, since then the steady state current vanishes identically. We shall consider plasma equilibrium distribution functions that are essentially Maxwellian but have been perturbed so as to have two relative maxima, and the steady state current will be viewed as maintained by external charges and currents not described by f. Thus the charge and current distributions of interest to us will be due principally to f_1, as in Ref. 2.

We point out here that previous analyses of the longitudinal and transverse mode plasma initial value problem[3-12] have also assumed non-Maxwellian equilibria. However, earlier studies of the longitudinal mode plasma boundary value problem[2,13] have assumed only Maxwellian equilibria. Hence, this is the first study where more general equilibria are considered for longitudinal plasma waves.

With the aforementioned conditions and Eq. (1.2), Eqs. (1.1) become the linearized Vlasov-Maxwell set of equations

$$\frac{\partial f_1}{\partial t} + u \frac{\partial f_1}{\partial z} + \frac{n_0 e}{m}\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right) \cdot \vec{\nabla}_{\vec{v}} f_0 = 0 , \tag{1.3a}$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi e \int f_1 \, d^3v , \tag{1.3b}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi e}{c} \int \vec{v} f_1 \, d^3v , \tag{1.3c}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} , \tag{1.3d}$$

$$\vec{\nabla} \cdot \vec{B} = 0 , \tag{1.3e}$$

where $u = v_z$.

Since our model is one dimensional, we shall consider the plasma to be axially symmetric about the z-direction. Thus, we will integrate over the velocity components v_x and v_y and introduce the reduced distribution functions

$$f(z,u,t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(z,\vec{v},t) \, dv_x \, dv_y , \tag{1.4a}$$

$$j_x(z,u,t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v_x \, f_1(z,\vec{v},t) \, dv_x \, dv_y , \tag{1.4b}$$

$$j_y(z,u,t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v_y \, f_1(z,\vec{v},t) \, dv_x \, dv_y . \tag{1.4c}$$

Multiplying the linearized Vlasov equation (1.3a) by 1, v_x and v_y respectively, and integrating over v_x and v_y, one obtains equations for each of these functions. Each equation may be combined with Maxwell's equations for the relevant field components, and this leads to one set of equations for longitudinal modes and two sets of equations for transverse modes. They are

$$(1.5a)\qquad \frac{\partial f}{\partial t} + u\,\frac{\partial f}{\partial z} + \frac{n_0 e}{m}\, E\, F'(u) = 0 \quad ,$$

$$(1.5b)\qquad \frac{\partial E}{\partial z} = 4\pi e \int_{-\infty}^{\infty} f(z,u,t)\, du \quad ,$$

$$(1.5c)\qquad \frac{\partial E}{\partial t} = -\, 4\pi e \int_{-\infty}^{\infty} u\, f(z,u,t)\, du$$

for the longitudinal modes, where

$$(1.6)\qquad F(u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_0(\vec{v})\, dv_x\, dv_y \quad ,$$

and E is the z-component of the electric field; and

$$(1.7)\qquad \frac{\partial \Psi_\pm}{\partial t} + T\,\frac{\partial \Psi_\pm}{\partial z} + A_\pm\, \Psi_\pm = 0 \quad ,$$

for the transverse modes[11], where

$$(1.8a)\qquad \Psi_\pm = \begin{bmatrix} j_x \pm i\, j_y \\ E_x \pm i\, E_y \\ \pm(B_x \pm i\, B_y) \end{bmatrix} \quad ,$$

$$(1.8b)\qquad T = \begin{bmatrix} u & 0 & 0 \\ 0 & 0 & -ic \\ 0 & ic & 0 \end{bmatrix}$$

$$(1.8c)\qquad A_{\pm} = \begin{bmatrix} \pm i\omega_c & -\dfrac{n_0 e}{m} F(u) & \dfrac{n_0 e}{imc} G(u) \\ 4\pi e \int_{-\infty}^{\infty} du & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$(1.8d)\qquad G(u) = u\,F(u) + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tfrac{1}{2}(v_x^2 + v_y^2)\, f_0(\vec{v})\, dv_x\, dv_y \;,$$

and $\omega_c = eB_0/(mc)$ is the cyclotron frequency when an external magnetic field $\vec{B}_0 = B_0\hat{z}$ is present. Note, Ψ_+ represents right circularly polarized solutions and Ψ_- represents left circularly polarized solutions.

We wish to point out that in the Vlasov theory of plasmas, the microfields produced by plasma particles are replaced by the average fields that the particles produce at a given space point. It is these average fields that are assumed to obey Maxwell's equations and hence our development of the Vlasov-Maxwell set of equations in this section. The Vlasov equation, based on this model, correctly describes the behavior of plasmas for time intervals much less than the binary collision times.

The question of stability in longitudinal and transverse plasma oscillations, specifically whether an initial displacement from equilibrium will grown or damp in time, is apparently mainly of theoretical interest. At present, experimental measurements of the electron density in plasmas are sometimes not sufficiently reproducable. The problem here is partly due to strong turbulent effects (e.g.

binary collisions) for which the linearized Vlasov model is not appropriate.

Similar difficulties also plague the question of stability in longitudinal and transverse plasma waves, specifically how boundary conditions affect an initial displacement from equilibrium as a function of the spatial coordinate. However for the slab problem, which is a model of the plasma capacitor, it has been possible to obtain qualitative agreement for longitudinal plasma waves and oscillations.[13]

Section 2 Methods Employed

Equations (1.5) and (1.7) represent the starting point for our analysis of plasma stability. Many previous analyses of plasma stability, for example those found in standard texts,[1] involve only studies of the plasma dispersion function $D(\omega,k)$ which is obtained by Fourier transforming Eqs. (1.5) and (1.7) in both the time and space coordinates with Fourier transform variables ω (fixed frequency) and k (fixed wave number) respectively. If D has zeros with nonvanishing imaginary part (hereafter referred to as "complex zeros") the plasma is, ipso facto, linearly unstable. This is easily seen because the zeros of D occur in complex conjugate pairs, so that if $\omega_0(k_0)$ is a zero of D then both $\omega_0 = \omega_{0r} \pm i\,\omega_{0i}$ are zeros of D. Thus, of the two terms corresponding to these zeros, $\exp\left[i(\omega_0 t+k_0 z)\right]$ and $\exp\left[i(\bar{\omega}_0 t+k_0 z)\right]$, one will be exponentially damped and the other will represent a growing unstable mode.

The actual magnitude of these stable and unstable modes is not a factor in determining stability. This is true of course only for linear stability. But the question of nonlinear stability, specifically whether nonlinear effects can stabilize a linearly unstable plasma or destabilize a linearly stable plasma is of vital importance. One approach, as has been followed by Simon and Rosenbluth,[14] is to use a perturbative expansion about the linear solution as is done in quantum mechanics. For this procedure, a complete solution of the linear equation is necessary, and is in part a practical motivation for our study.

In order to study the magnitude of the moments of f_1, Eqs. (1.4), we take only one Fourier transform of Eqs. (1.5) and (1.7) and obtain equations of the form

$$\frac{\partial \phi}{\partial t} + i\, k\, K_k \phi = 0 \quad , \tag{1.9a}$$

$$\frac{\partial \phi}{\partial z} + i\, \omega\, K_\omega \phi = 0 \quad , \tag{1.9b}$$

where ϕ is either the Fourier transform of f or $\Psi_\pm$. This defines four problems;

(i) longitudinal plasma oscillations [Eq. (1.9a) with ϕ as the Fourier space transform of f],

(ii) transverse plasma oscillations [Eq. (1.9a) with ϕ as the Fourier space transform of $\Psi_\pm$].

(iii) longitudinal plasma waves [Eq. (1.9b) with ϕ as the Fourier time transform of f],

(iv) transverse plasma waves [Eq. (1.9b) with ϕ as the Fourier time transform of $\Psi_\pm$].

In Chapter 2 we shall study problem (i), in Chapter 3 we study problem (ii) and in Chapter 4 we study problem (iii), reserving problem (iv) for a later date. In each of the problems we study, $K_{k,\omega}$ is the relevant transport operator, whose specific form will be given later.

There is a considerable body of literature which deals with longitudinal plasma oscillations. In particular some studies, notably by N. G. van Kampen[3] and K. M. Case,[4,5] have expressed the solution to the longitudinal plasma initial value problem as a sum of exponentials (van Kampen-Case discrete modes) plus an integral over the continuum eigenfunctions (van Kampen-Case singular eigenmodes).

The van Kampen-Case discrete modes depend on the zeros of the fixed wave number plasma dispersion function, Λ_k, which is a real function depending on a complex variable. Like $D(\omega,k)$, the zeros of Λ_k occur in complex conjugate pairs. It is possible for Λ_k to have real simple zeros or, if one thinks of a complex conjugate pair coalescing on the real axis for a critical value of the electron density, n_0, then Λ_k may have real second order zeros. Although Case stated[4] that he considers all eigenvalues to be simple "for simplicity only," he did not describe how to deal with this important case. Furthermore, the expansion coefficients of the stable plasma modes corresponding just to simple real zeros of Λ_k had been obtained incorrectly by the singular eigenmode approach.[15]

There are other serious mathematical difficulties in the usual development of the singular eigenmode approach. This point has been stressed extensively in the context of neutron transport theory[16,17] and these objections carry over into the plasma case essentially unchanged. In 1973, Larsen and Habetler[16] presented a method of analysis which overcomes these difficulties. This approach has come to be called the resolvent integration technique.

The general method of solution is to apply the operator identity

$$(1.10) \qquad \phi = -\frac{1}{2\pi i} \oint_{\Gamma} (K_{k,\omega}-\rho)^{-1}\phi \, d\rho$$

where the contour Γ is a closed contour that surrounds the spectrum of $K_{k,\omega}$. In each of the cases we study, the operator $K_{k,\omega}$ is unbounded with spectrum extending to $\pm\infty$, so that an attempt to carry out the contour integration directly could lead to serious difficulties. Bareiss[18] encountered this problem in dealing with the neutron transport equation, and resolved it by constructing a sequence of approximating bounded operators. We can avoid this complication by following a suggestion of Larsen,[19] to deal not with $K_{k,\omega}$ but with the resolvent of $K_{k,\omega}$ evaluated at a particular point and to apply the resolvent integration technique to the resolvent.

For illustrational purposes, we shall follow this resolvent - resolvent procedure in solving the longitudinal mode plasma initial value problem in Chapter 2. As a result of this calculation we are able to present for the first time the correct expansion coefficients for stable plasma oscillations corresponding not only to simple real zeros of Λ_k but also to second order real zeros of Λ_k. (Subsequent to the publication of Ref. 15, Siewert [20] and Case [21] developed a procedure for treating stable oscillations correctly by the singular eigenfunction method. However, the mathematical objections to the singular eigenfunction approach persist.)

We have found[22] in studying the transverse mode plasma initial value problem that it is unnecessary to calculate the second resolvent. In Chapter 3, we are able to obtain estimates on the norm of the resolvent and appeal to well known semigroup theorems[23,24,25] for the existence and uniqueness of solutions. Again, as in the case of longitudinal plasma oscillations, we are able to present the expansion coefficients for stable transverse oscillations corresponding to simple and second order real zeros of the appropriate plasma dispersion function.

The use of semigroup theory is particularly well suited for initial value problems (c.f. Refs. 23, 24 and 25). In order to obtain an existence and uniqueness theorem for the solutions to the longitudinal mode plasma boundary value problem of a plasma half space with $z > 0$, we will proceed in Sec. 1 of Chapter 4 as in our earlier paper[26] by studying equations (1.5a) and (1.5c) as a two component system. The boundary condition for f is the usual linear combination of specular and diffuse reflection

$$(1.11) \qquad f(0,u) = \alpha\, f(0,-u) + (1-\alpha)\,\frac{g(u)}{u}\int_{-\infty}^{0} |s|\, f(0,s)\, ds \;;\; u > 0 \;,$$

and is incorporated directly into the "free" semigroup for the two component system of equations. Here, α is the accommodation coefficient where $\alpha = 1$ corresponds to purely specular reflection and $\alpha = 0$ corresponds to purely diffuse reflection. The function g is obtained by also requiring F to satisfy the boundary condition (1.11) and is given by

$$(1.12) \qquad g(u) = uF(u) - \alpha uF(-u) \;/\; (1-\alpha)\left(\int_{-\infty}^{0}|s|F(s)\, ds\right) \;.$$

Our treatment of the longitudinal mode plasma boundary value problem is not merely a generalization of the problem to plasma equilibria that are not necessarily even functions of the velocity, which requires careful tratment of the fixed frequency plasma dispersion function, Λ_ω,[27] but it is also unique in the sense that we begin our study with Ampere's Law, Eq. (1.5c), rather than Gauss' Law, Eq. (1.5b).[28] In previous studies[2,18] the use of Gauss' Law led to a two component system of equations. This is an unnecessary and confusing mathematical complication because the transport operator then appears to have an eigenfunction with corresponding zero eigenvalue. Fortunately, this so-called "zero eigenmode" was identified with an applied electric field, E_A, which we shall also consider in Chapter 4. However, in view of Ampere's Law, the "zero eigenmode" is more appropriately regarded as a response to the applied electric field rather than a true physical mode.

We shall see in Chapter 4 that by using Ampere's Law and Fourier transforming in time we are able to eliminate the self consistent electric field and study a one component equation. This much simpler equation will then be treated with half range techniques similar to those developed by Larsen and Habetler[11] for the neutron transport problem.

Returning to Eq. (1.10), we now discuss the general method for evaluating the closed contour integral term. The integrand, $(K_{k,\omega}-\rho)^{-1}\phi$, can be expressed as a quotient of functions T and Λ, analytic in the cut plane $\mathbb{C}\setminus(-\infty,\infty)$

$$(1.13) \qquad (K_{k,\omega}-\rho)^{-1}\phi = T(\rho)/\Lambda(\rho) \quad .$$

Thus, the integrand will be analytic everywhere in the cut plane $\mathbb{C}\setminus(-\infty,\infty)$ except at the zeros of Λ. Since the closed contour Γ surrounds the singularities of the integrand, we find from residue theory[29] that

$$(1.14)\quad \frac{1}{2\pi i}\oint_\Gamma (K_{k,\omega}-\rho)^{-1}\phi(u)\,d\rho = \frac{1}{2\pi i}\int_{-\infty}^{\infty} P\left[T^-(s)/\Lambda^-(s) - T^+(s)/\Lambda^+(s)\right] ds$$

$$+ \tfrac{1}{2}\{\mathrm{Res}\left[T^-(s)/\Lambda^-(s),u\right] + \mathrm{Res}\left[T^+(s)/\Lambda^+(s),u\right]\}$$

$$+ \sum_k \mathrm{Res}\left[T(\rho)/\Lambda(\rho),\nu_k\right]$$

$$+ \sum_\ell \tfrac{1}{2}\{\mathrm{Res}\left[T^-(s)/\Lambda^-(s),\nu_{0\ell}\right]+\left[\mathrm{Res}\quad T^+(s)/\ ^+(s),\nu_{0\ell}\right]\}$$

where the integral on the right hand side is to be regarded as a Cauchy principle value integral, and the sums over k and ℓ are due to complex and real zeros of Λ respectively. The Cauchy principle value integral is combined with the residue terms at u to yield an integral over the "continuum eigenmodes". The residue terms are all of the form $\mathrm{Res}\left[T(\rho)/\Lambda(\rho),\nu\right]$, and may be evaluated by using the standard formula[29]

$$(1.15)\qquad \mathrm{Res}\left[\frac{T(\rho)}{\Lambda(\rho)},\nu\right] = \lim_{\rho\to\nu}(m-1)!\ \frac{d^{m-1}}{d\rho^{m-1}}\left[(\rho-\nu)^m\frac{T(\rho)}{(\rho)}\right]$$

where m is the order of the zero, ν, of Λ. If ν is a simple zero of $\Lambda(m=1)$ then Eq. (1.15) becomes

$$(1.16)\qquad \mathrm{Res}\left[\frac{T(\rho)}{\Lambda(\rho)},\nu\right] = \frac{T(\nu)}{\Lambda'(\rho)}\quad .$$

However, if ν is a second order zero of $\Lambda(m=2)$ then Eq. (1.15) becomes[30]

$$(1.17) \qquad \mathrm{Res}\left[\frac{T(\rho)}{\Lambda(\rho)},\nu\right] = \frac{2}{3}\left[\frac{3T'(\nu)}{\Lambda''(\nu)} - \frac{T(\nu)\Lambda'''(\nu)}{[\Lambda''(\nu)]^2}\right] .$$

Equations (1.16) and (1.17) are instrumental in carrying out the analysis of problems (i), (ii) and (iii). Fortunately, we are able to simplify these terms even further, as we shall see in the chapters ahead.

In the three plasma problems that we study, the Larsen and Habetler resolvent integration technique is a central theme and our application of it to longitudinal plasma oscillations is primarily responsible for our new results regarding the expansion coefficients for stable longitudinal plasma modes. The generalization of the plasma equilibrium distribution to a non-even function in the case of longitudinal plasma waves leads to a restudy of the corresponding plasma dispersion function with the new result that more stable and unstable plasma modes are possible than had previously been considered. Finally, our use of semigroup theory in proving existence and uniqueness theorems for solutions to the transverse plasma oscillation and longitudinal plasma wave problems and our application of the resolvent - resolvent technique (hence working with a bounded operator) to the longitudinal plasma oscillations problem has placed the study of the Vlasov-Maxwell plasma model on a much more rigorous mathematical foundation.[31]

Chapter 2

Longitudinal Oscillations

Section 1 Computation of the Resolvent

As discussed in Chapter 1, we study here the Fourier space transform of Eqs. (1.5a) and (1.5b). These are

$$(2.1a) \qquad \frac{\partial f_k}{\partial t} + i\, k\, u\, f_k + \frac{n_0 e}{m}\, E_k\, F'(u) = 0$$

$$(2.16) \qquad E_k = \frac{4\pi e}{ik} \int_{-\infty}^{\infty} f_k(u,t)\, du$$

where the fixed wave number k is the Fourier transform variable and

$$(2.3) \qquad f_k(u,t) = \int_{-\infty}^{\infty} e^{-ikz}\, f(z,u,t)\, dz\ .$$

The Fourier transform of the electric field, E_k, is easily eliminated from Eqs. (2.1) and we may write

$$(2.4) \qquad \frac{\partial f}{\partial t} + i\, k\, K\, f = 0,$$

where we have omitted writing the explicit dependence of f on k, and the operator K is given by

$$(2.5a) \qquad (Kf)(u,t) = u\, f(u,t) + \eta(u) \int_{-\infty}^{\infty} f(s,t)\, ds\ ,$$

$$(2.5b) \qquad \eta(u) = -\,(\omega_p/k)^2\, F'(u)\ ,$$

and $\omega_p = (4\pi n_0 e^2/m)^{\frac{1}{2}}$ is the plasma frequency.

The resolvent of K is obtained by writing $(K-\rho)g = f$, and solving for g in terms of f. Since this procedure is used again in Chapters 3 and 4, we shall present the details of this type of calculation here. We have from Eq. (2.5a)

$$f(u) = (u-\rho)g(u)+\eta(u)\int_{-\infty}^{\infty} g(s)\, ds \ , \tag{2.6a}$$

so that

$$g(u) = f(u)(u-\rho)^{-1} - \eta(u)(u-\rho)^{-1}\int_{-\infty}^{\infty} g(s)\, ds \tag{2.6b}$$

and integrating both sides of this last equation yields

$$\int_{-\infty}^{\infty} g(s)\, ds \left[1+\int_{-\infty}^{\infty} \frac{\eta(u)}{u-\rho}\, du\right] = \int_{-\infty}^{\infty} \frac{f(u)}{u-\rho}\, du \ . \tag{2.6c}$$

Equation (2.6b) gives the resolvent of K provided we can express $\int_{-\infty}^{\infty} g(u)\, du$ in terms of f. This is achieved through Eq. (2.6c) when the function in brackets is different from zero.

The function in brackets is called the plasma dispersion function, Λ, and plays an important role in the following analysis

$$\Lambda(\rho) = 1 + \int_{-\infty}^{\infty} \frac{\eta(u)}{u-\rho}\, du \tag{2.7}$$

(again, we omit the explicit dependence of Λ on k). Λ is an analytic function of ρ on the cut plane $\mathbb{C}/\mathbb{R}$ with continuous boundary values $\Lambda^{\pm}$ evaluated on the branch cut $\mathbb{R}$ from above and below respectively, and given by

$$(2.8a) \qquad \Lambda^{\pm}(u) = \lambda(u) \pm i\pi\eta(u) \quad ,$$

$$(2.8b) \qquad \lambda(u) = 1 + \int_{-\infty}^{\infty} P \frac{\eta(s)}{s-u} ds \quad ,$$

where P indicates that the Cauchy principal value is to be taken. Combining Eqs. (2.6b), (2.6c) and (2.7), we may write the resolvent of K as

$$(2.9) \qquad (K-\rho)^{-1}f(u) = \frac{f(u)}{u-\rho} + \frac{\eta(u)}{u-\rho} \frac{1}{\Lambda(\rho)} \int_{-\infty}^{\infty} \frac{f(s)}{s-\rho} ds \quad .$$

The spectrum of K, $\sigma(K)$, is easily determined by viewing this expression of the resolvent of K as an operator-valued analytic function of the complex variable ρ. The singularities of $(K-\rho)^{-1}$ occur when $\rho = u$ or $\Lambda(\rho) = 0$. Since u is permitted to have any real value, we conclude that the continuous spectrum of K, $C\sigma(K)$, consists of the entire real line, while the point spectrum of K, $P\sigma(K)$, consists of the zeros of Λ ($\sigma(K) = C\sigma(K) \bigcup P\sigma(K)$).

As mentioned in Chapter 1, difficulties could arise in integrating the resolvent of K about its continuous spectrum. To avoid this, we consider the resolvent of K evaluated at a point not in the spectrum of K (i.e. at a point in the resolvent set of K,$\rho(K)$). We choose $\rho = i$ arbitrarily and define

$$(2.10) \qquad S \equiv (K-i)^{-1} \quad .$$

The operator S is a Caley transform of the operator K. The Caley transform function is given by $U(\rho) = (\rho-i)^{-1}$, so that $S = U(K)$. The spectrum of S is related to the spectrum of K by the spectral mapping theorem, $\sigma(S) = U[\sigma(K)]$. It is easily seen that for any real number u, we have

$$(2.11)\qquad |U(u) - \tfrac{1}{2}i| = \tfrac{1}{2} \quad .$$

Thus, the continuous spectrum of K maps into a circle of radius ½, centered at the point ½i. The point spectrum of K, which consists of complex conjugate pairs, is mapped into pairs of points, conjugate in the sense of linear fractional transformations.[29]

The resolvent of S is obtained by writing $(S-\rho)g = f$ and solving for g in terms of f. The procedure is identical to that given earlier for calculating the resolvent of K. We state the result

$$(2.12a)\qquad (S-\rho)^{-1}f(u) = \frac{f(u)}{(u-i)^{-1}-\rho} - \frac{\eta(u)}{(u-i)}\,\frac{1}{(u-i)^{-1}-\rho}\,\frac{1}{\Omega(\rho)} \times \int_{-\infty}^{\infty} \frac{f(s)}{\rho(s-i)-1}\,ds$$

$$(2.12b)\qquad \Omega(\rho) = \Lambda(\rho^{-1}+i) \quad .$$

Furthermore, the spectrum of S can now be expressed as

$$(2.13a)\qquad \sigma(S) = C\sigma(S) \bigcup P\sigma(S) \; ,$$

where the continuous spectrum consists of the circle

(2.13b) $$C\sigma(S) = \{\rho : |\rho - \tfrac{1}{2}i| = \tfrac{1}{2}\} \quad ,$$

while the point spectrum occurs at the zeros of Ω

(2.13c) $$P\sigma(S) = \{\rho_i : \Omega(\rho_i) = 0,\ i = 1, 2, \ldots, n\} \ .$$

For use in later computations we will require the boundary values of

(2.14) $$M(\rho) \equiv \frac{1}{\Omega(\rho)} \int_{-\infty}^{\infty} \frac{f(s)}{\rho(s-i)-1}\, ds \quad .$$

We define $M^{\pm}$ as the limiting values of M as we approach $C\sigma(S)$ from outside and inside the circle, respectively. These can be obtained from the Plemelj formulas[32] after the change of integration variable $t \to (s-i)^{-1}$:

(2.15) $$M^{\pm}\left[(u-i)^{-1}\right]\Omega^{\pm}\left[(u-i)^{-1}\right] = \pm\, i\pi f(u)(u-i) + \int_{-\infty}^{\infty} P\frac{f(s)}{(u-i)^{-1}(s-i)-1}\, ds \quad .$$

Using this result and recalling (2.8) and (2.12b) we obtain, after some algebra

(2.16) $$f(u) - \frac{\eta(u)}{u-i}\, \tfrac{1}{2}\{M\left[(u-i)^{-1}\right] + M^{-}\left[(u-i)^{-1}\right]\}$$
$$= \frac{1}{2\pi i}\, \frac{\lambda(u)}{u-i}\, \{M^{+}\left[(u-i)^{-1}\right] - M^{-}\left[(u-i)^{-1}\right]\} \ .$$

In section 3 we integrate the resolvent, Eq. (2.12a), around the spectrum of S and obtain the eigenfunction expansion for the operator S. Equation (2.16) plays an important role in the analysis.

Section 2 Analysis of Λ

The fact that the zeros of Λ occur in complex conjugate pairs has been mentioned in Chapter 1 and in the previous section. The symbolic statement of this is $\overline{\Lambda(\rho)} = \Lambda(\bar{\rho})$. Thus, in determining the number of zeros of Λ, we need only consider the zeros of Λ in the upper half plane.

We shall utilize the argument principle which states that the number of zeros minus the number of poles of an analytic function in a given region is equal to the change in the argument of the function in question as the boundary is traversed in the counterclockwise sense, divided by 2π. We shall determine the change in the argument of Λ by drawing the so-called Nyquist diagram for a particular contour. The Nyquist diagram is a graph in the Λ plane of the image under Λ of the contour enclosing the region of interest. The number of times this graph encircles the origin gives the number of zeros of Λ in the region (note Λ has no poles in the upper half plane).

The contour we consider traverses from $-\infty$ to $+\infty$ just above the real axis, and closes in a semicircular arc in the upper half plane (see Fig. 2.1). Along the curved portion of the contour, Λ assumes its limiting value at ∞. In particular,

$$\Lambda(\rho) \to 1 \text{ as } \rho \to \infty \quad .$$

Thus, the curved portion of the contour does not contribute to the change in the argument of Λ.

Along the real axis, we must consider Λ^+, which we write as

(2.17) $$\Lambda^{+}(u) = 1 - \left(\frac{\omega_p}{k}\right)^2 \int_{-\infty}^{\infty} P \frac{F'(s)}{s-u}\, ds - i\pi\left(\frac{\omega_p}{k}\right)^2 F'(u) \ .$$

This equation follows from (2.5b), (2.8a) and (2.8b).

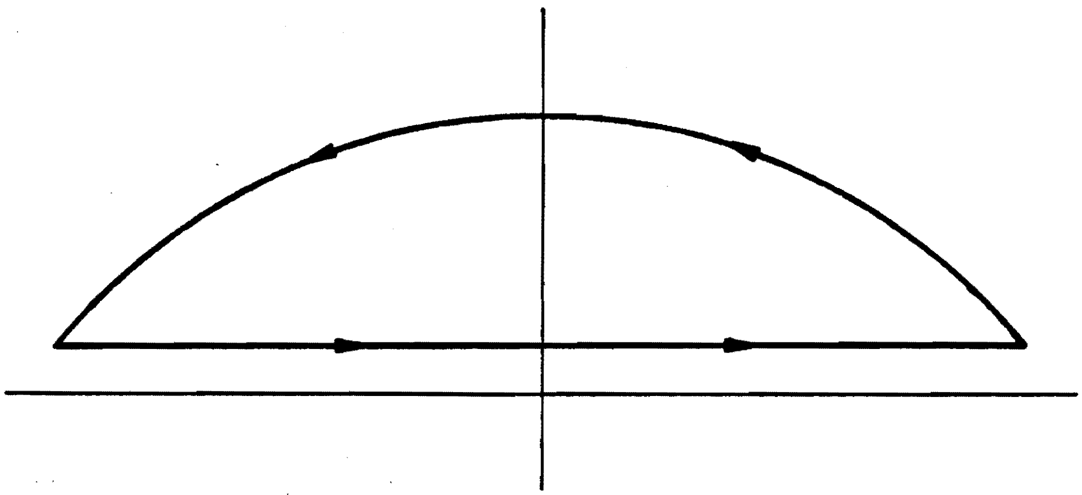

Fig. 2.1 The Contour Enclosing the Upper Half Plane

As we proceed from $-\infty$ to $+\infty$ just above the real axis, the imaginary part of Λ^+ starts out negative, changes sign three times, then approaches 0 from positive values. We denote the values of the zeros of F' by u_0, u_1 and u_2. These are also the points at which the imaginary part of Λ^+ vanishes. We define

(2.18) $$M = \Lambda(u_0)\Lambda(u_1)\Lambda(u_2) \ .$$

In Figs. (2.2) and (2.3) we show representative cases for Nyquist diagrams with $M < 0$ and $M > 0$ respectively. From Fig. 2.2 we see that Λ has a zero in the upper half plane when $M < 0$. From Fig. 2.3 we see that Λ has no zeros in the upper half plane when $M > 0$. Finally, when $M = 0$ then Λ clearly has a real zero at one or more of the zeros of F'.

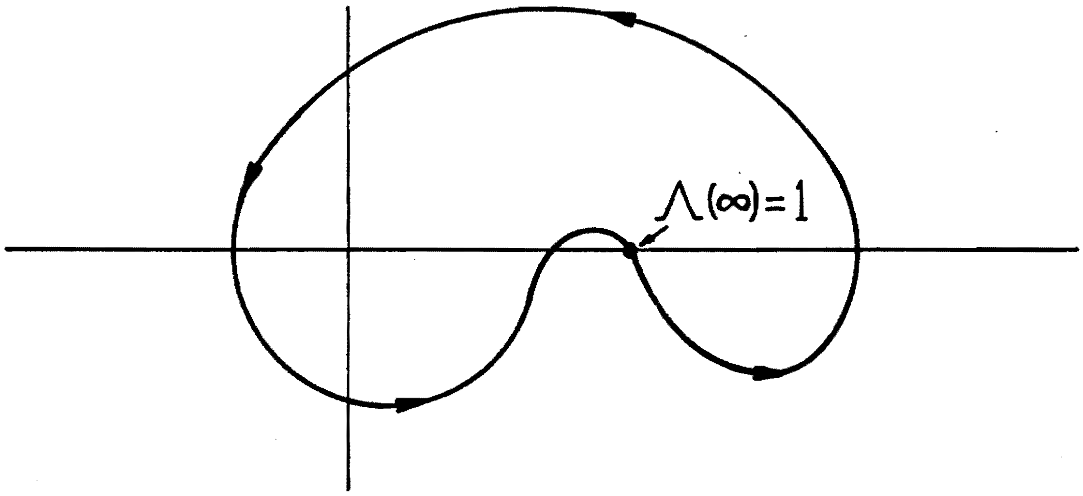

Fig. 2.2 A Nyquist Diagram for the case $M < 0$

Although the complex zeros of Λ here will be simple, the real zeros of Λ may be either simple or second order as discussed in Sec. 2 of Chapter 1. We shall denote the complex zeros of Λ by ν_k while the real zeros of Λ will be denoted by $\nu_{0\ell}$. However, from Eq. (2.12b) we see that $(\nu_k - i)^{-1}$ and $(\nu_{0\ell} - i)^{-1}$ are then the zeros of Ω and hence spectral points of the operator S.

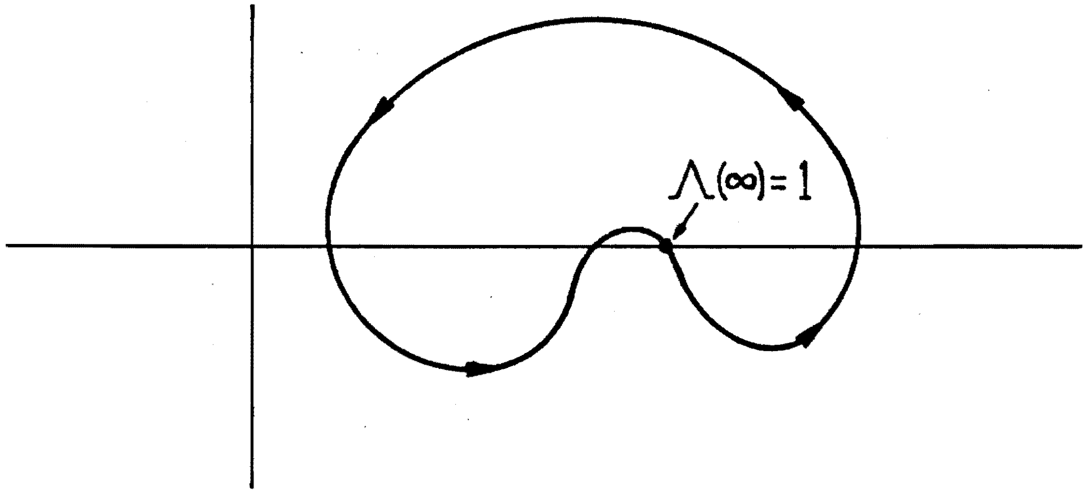

Fig. 2.3 A Nyquist Diagram for

the case $M > 0$

Section 3 Integration of the Second Resolvent

We apply the identity

$$(2.19) \qquad -\frac{1}{2\pi i}\oint_{\Gamma} (S-\rho)^{-1}\, h(u)\,d\rho \;=\; h(u)$$

to Eq. (2.12a). The contour Γ is shown in Fig. 2.4.

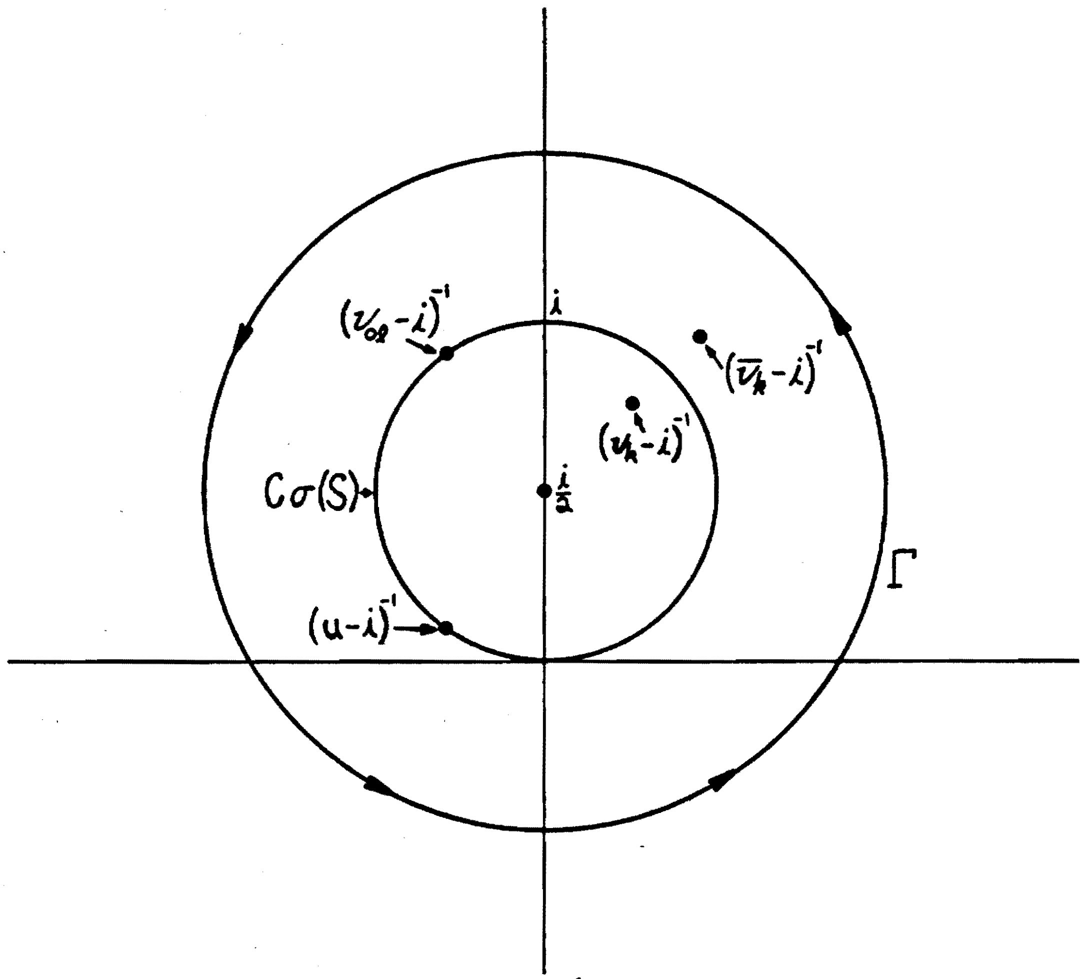

Fig. 2.4 The Contour Γ and $\sigma(S)$

In order to apply the theory of residues, we shall add the contour

Γ_- (see Fig. 2.5) to the integration in Eq. (2.19). Since Γ_- does not enclose any singularities of $(S-\rho)^{-1}$, this has the effect of adding zero to the left hand side of Eq. (2.19). We also deform the original contour Γ into Γ_+ as shown in Fig. 2.5.

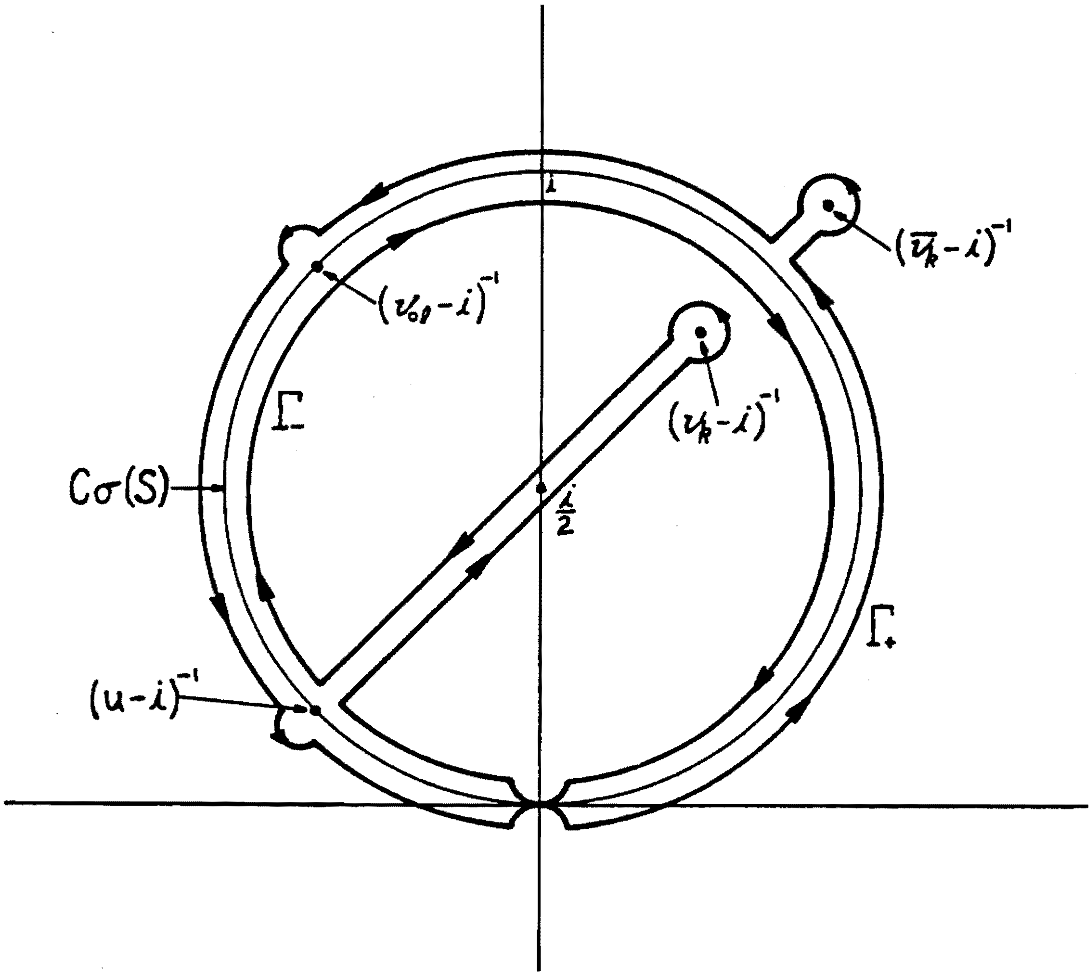

Fig. 2.5 The Contours Γ_+ and Γ_-

We further deform the contours Γ_+ and Γ_- so that Γ_+ becomes $\hat{\Gamma}_+ \bigcup \Gamma_{\bar{v}_k} \bigcup \Gamma_{v_{0\ell+}} \bigcup \Gamma_{u+}$ and Γ_- becomes $\hat{\Gamma}_- \bigcup \Gamma_{v_k} \bigcup \Gamma_{v_{0\ell-}} \bigcup \Gamma_{u-}$. These contours are shown in Fig. 2.6.

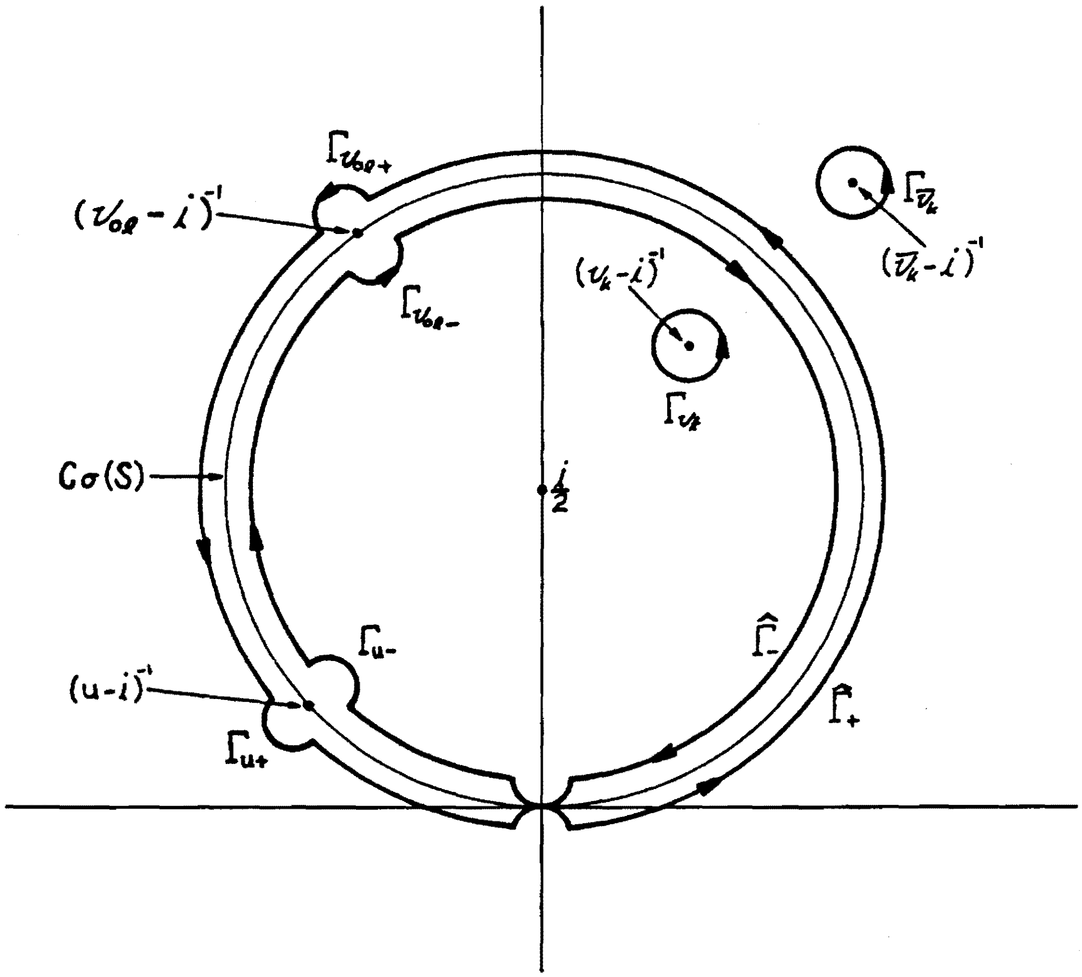

Fig. 2.6 The Detailed Composition

of Contours

Having made these changes in the contour, Eq. (2.19) becomes

$$(2.20) \qquad -\frac{1}{2\pi i}\oint_{\hat{\Gamma}} (S-\rho)^{-1} h(u)\, d\rho = h(u)$$

where

(2.21) $$\hat{\hat{\Gamma}} = \hat{\Gamma}_+ \bigcup \hat{\Gamma}_- \bigcup \Gamma_{u+} \bigcup \Gamma_{u-} \bigcup \Gamma_{\nu_k} \bigcup \Gamma_{\bar{\nu}_k} \bigcup \Gamma_{\nu_{0\ell+}} \bigcup \Gamma_{\nu_{0\ell-}} .$$

We have only indicated a single pair of complex zeros $(\nu_k, \bar{\nu}_k)$ since for the the type of equilibrium distribution function, F, we are considering, the analysis of Λ in Sec. 2 shows that we have at most one such pair. However, for even more general equilibria, there may be several such pairs and we sould have to sum over the index k. In the expressions to follow we shall include such a sum which is viewed as taken over all conjugate pairs of complex eigenvalues.

Similarly, we have only indicated one real eigenvalue, $\nu_{0\ell}$. For cases where there is more than one real eigenvalue, we include a sum over the index ℓ. With these ideas in mind, Eq. (2.20) becomes

(2.22) $$h(u) = \frac{-1}{2\pi i} \left(\int_{\hat{\Gamma}_+} + \int_{\hat{\Gamma}_-} + \int_{\Gamma_{u+}} + \int_{\Gamma_{u-}} + \sum_{\ell} \int_{\Gamma_{\nu_{0\ell\pm}}} + \sum_{k} \int_{\Gamma_{\nu_k}} \right)$$

$$(S-\rho)^{-1} h(u)\, d\rho$$

The integrals over the Γ_{ν_k} are simple exercises in residue theory. The results are the following:

(a) Simple pole; $\Lambda(\nu_k) = 0$, $\Lambda'(\nu_k) \neq 0$; $\mathrm{Im}\nu_k \neq 0$.

(2.23) $$-\frac{1}{2\pi i} \int_{\Gamma_{\nu_k}} (S-\rho)^{-1} h(u)\, d\rho$$

$$= \frac{\eta(u)}{u-\nu_k} \int h(s)(s-\nu_k)^{-1} ds \left[\int \eta(s)(s-\nu_k)^{-2}\, ds \right]^{-1}$$

$$= \phi_{\nu_k}(u) \int \bar{\phi}_{\nu_k}(s)h(s)\ ds \left[\Lambda'(\nu_k) \right]^{-1}$$

$$= A_{\nu_k}\ \phi_{\nu_k}(u)$$

where

(2.24a) $$\phi_{\nu_k}(u) = -\eta(u)/(u-\nu_k)$$

(2.24b) $$\tilde{\phi}_{\nu_k}(u) = -1/(u-\nu_k)$$

(2.24c) $$A_{\nu_k} = \int \tilde{\phi}_{\nu_k}(s)h(s)\ ds \left[\Lambda'(\nu_k) \right]^{-1}$$

This is, of course, the classical Case result.[4]

(b) Double pole; $\Lambda(\nu_k) = \Lambda'(\nu_k) = 0$; $\Lambda''(\nu_k) \neq 0$; $\mathrm{Im}\nu_k \neq 0$.

(2.25) $$-\frac{1}{2\pi i} \int_{\Gamma_{\nu_k}} (S-\rho)^{-1}h(u)\ d\rho = \eta(u)(u-\nu_k)^{-2}\ 2\int h(s)(s-\nu_k)^{-1}$$

$$\times\ ds \left[\Lambda''(\nu_k)\right]^{-1} + \frac{\eta(u)}{(u-\nu_k)} \{2\int h(s)(s-\nu_k)^{-2}\ ds \left[\Lambda''(\nu_k) \right]^{-1}$$

$$- \tfrac{2}{3} \Lambda'''(\nu_k) \int h(s)(s-\nu_k)^{-1}\ ds \left[\Lambda''(\nu_k) \right]^{-2}\} .$$

$$= A^{(1)}_{\nu_k}\ \phi^{(1)}_{\nu_k}(u) + A^{(2)}_{\nu_k}\ \phi^{(2)}_{\nu_k}(u)$$

where

(2.26a) $$\phi^{(1)}_{\nu_k}(u) = -\eta(u)/(u-\nu_k) \quad ,$$

$$\text{(2.26b)}\qquad \phi^{(2)}_{\nu_k}(u) = -\eta(u)/(u-\nu_k)^2 \quad ,$$

$$\text{(2.26c)}\qquad A^{(1)}_{\nu_k} = -\{2 \int h(s)(s-\nu_k)^{-2}\, ds \left[\Lambda''(\nu_k)\right]^{-1}$$

$$- \tfrac{2}{3}\Lambda'''(\nu_k) \int h(s)(s-\nu_k)^{-1}\, ds \left[\Lambda''(\nu_k)\right]^{-2}\} \; ,$$

$$\text{(2.26d)}\qquad A^{(2)}_{\nu_k} = -2 \int h(s)(s-\nu_k)^{-1}\, ds \left[\Lambda''(\nu_k)\right]^{-1}$$

We recognize that the first term is proportional to the usual Case discrete eigenvector $\phi_{\nu_k}(\nu)$. The second term is proportional to the generalized eigenvector, since

$$\text{(2.27)}\qquad (K-\nu_k)\,\frac{\eta(u)}{(u-\nu_k)^2} = \frac{\eta(u)}{u-\nu_k} \; .$$

In constructing solutions to the initial value problem, special care must be used in dealing with the generalized eigenvector (see Sec. 4).

The integrals over the $\Gamma_{\nu_{0\ell\pm}}$ are done by applying the residue theorem separately to $\Gamma_{\nu_{0\ell+}}$ and $\Gamma_{\nu_{0\ell-}}$. We obtain

(a) $\nu_{0\ell}$ a simple pole;

$$\text{(2.28)}\qquad -\frac{1}{2\pi i}\int_{\Gamma_{\nu_{0\ell+}}\cup\Gamma_{\nu_{0\ell-}}} (S-\rho)^{-1} h(u)\, d\rho$$

$$= \frac{\eta(u)}{2(u-\nu_k)}\{\left[\int P\, h(s)(s-\nu_k)^{-1}\, ds + i\pi h(\nu_k)\right]\left[\Lambda''(\nu_k)\right]^{-1}$$

$$+\left[\int Ph(s)(s-\nu_k)^{-1}\,ds - i\pi h(\nu_k)\right]\left[\Lambda'^-(\nu_k)\right]^{-1}\} \quad .$$

$$= A_{\nu_{0\ell}}\,\phi_{\nu_{0\ell}}(u)$$

where

$$(2.29a) \qquad \phi_{\nu_{0\ell}}(u) = -\eta(u)/(u-\nu_{0\ell}) \ ,$$

$$(2.29b) \qquad A_{\nu_{0\ell}} = -\mathrm{Re}\{\left[\int Ph(s)(s-\nu_{0\ell})^{-1}\,ds + i\pi h(\nu_{0\ell})\right]\left[\Lambda'^+(\nu_{0\ell})\right]^{-1}\}.$$

This coefficient of $\phi_{\nu_{0\ell}}$ differs from Case's result unless $\Lambda'^+(\nu_{0\ell}) = \Lambda'^-(\nu_{0\ell})$ which occurs if, and only if, $\eta'(\nu_{0\ell}) = 0$. However, there is no reason to assume, in general, that $\eta'(\nu_{0\ell}) = 0$.

As we discussed earlier, there is some physical reason to suspect that double poles may occur in the continuum.

(b) $\nu_{0\ell}$ a double pole:

$$(2.30) \qquad -\frac{1}{2\pi i}\int_{\Gamma_{\nu_{0\ell+}}\cup\,\Gamma_{\nu_{0\ell-}}}(S-\rho)^{-1}h(u)\,d\rho$$

$$= A^{(1)}_{\nu_{0\ell}}\,\phi_{\nu_{0\ell}}(u) + A^{(2)}_{\nu_{0\ell}}\,\phi^{(2)}_{\nu_{0\ell}}(u) \quad ,$$

where

$$(2.31a) \qquad \phi^{(1)}_{\nu_{0\ell}}(u) = -\eta(u)/(u-\nu_{0\ell}) \quad ,$$

(2.31b) $$\phi_{\nu_{0\ell}}^{(2)}(u) = -\eta(u)/(u-\nu_{0\ell})^2 \quad ,$$

(2.31c) $$A_{\nu_{0\ell}}^{(1)} = -\tfrac{1}{3}\{3\left[\int Ph(s)(s-\nu_{0\ell})^{-2}\, ds + i\, h'(\nu_{0\ell})\right]\left[\Lambda^{+''}(\nu_{0\ell})\right]^{-1}$$

$$-\left[\int Ph(s)(s-\nu_{0\ell})^{-1}\, ds + i\pi h(\nu_0)\right]\Lambda^{+'''}(\nu_0)\left[\Lambda^{+''}(\nu_{0\ell})\right]^{-2}$$

$$+3\left[\int Ph(s)(s-\nu_{0\ell})^{-2}\, ds - i\pi h'(\nu_0)\right]\left[\Lambda^{-''}(\nu_{0\ell})\right]^{-1}$$

$$-\int Ph(s)(s-\nu_{0\ell})^{-1}\, ds + i\pi h(\nu_{0\ell})\Lambda^{-'''}(\nu_{0\ell})\left[\Lambda^{-''}(\nu_{0\ell})\right]^{-2}$$

and

(2.31d) $$A_{\nu_{0\ell}}^{(2)} = \left[\int Ph(s)(s-\nu_{0\ell})^{-1}\, ds + i\pi h(\nu_{0\ell})\right]\left[\Lambda^{+''}(\nu_{0\ell})\right]^{-1}$$

$$+\left[\int Ph(s)(s-\nu_{0\ell})^{-1}\, ds - i\pi h(\nu_{0\ell})\right]\left[\Lambda^{-''}(\nu_{0\ell})\right]^{-1} \quad .$$

Again, see Sec. 4 for the time dependence of these modes.

The integration around $C\sigma(S)$ is slightly more complicated to carry out, but the procedure is completely analogous to that of Ref. 16. Denoting

$$\Gamma_4 = \hat{\Gamma}_+ \bigcup \hat{\Gamma}_- \bigcup \Gamma_u \bigcup \Gamma_{u-}$$

we have, using Eq. (2.9)

$$(2.32)\qquad -\frac{1}{2\pi i}\int_{\Gamma_4}(S-\rho)^{-1}h(u)\,d\rho = h(u) - \frac{\eta(u)}{u-i}\,\frac{1}{2\pi i}\int_{\Gamma_4}\frac{M(\rho)}{\rho-(u-i)^{-1}}d\rho\ .$$

The second term can be broken up into its contributions from $\Gamma_{u\pm}$ and $\Gamma_{\pm}$. We immediately obtain

$$(2.33)\qquad -\frac{1}{2\pi i}\int_{\Gamma_4}(S-\rho)^{-}H(u)\,d\rho$$

$$= h(u) - \frac{\eta(u)}{u-i}\,\tfrac{1}{2}\{M^{+}\left[(\nu - i)^{-1}\right] + M^{-}\left[(\nu - i)^{-1}\right]\}$$

$$+ \frac{\eta(u)}{u-i}\,\frac{1}{2\pi i}\int_{-\infty}^{\infty} P\,\frac{M^{+}\left[(s-i)^{-1}\right] - M^{-}\left[(s-i)^{-1}\right]}{\left[(s-i)^{-1} - (u-i)^{-1}\right]}\,\frac{ds}{(s-i)^{2}}\ .$$

Thus, using Eq. (2.16) we have

$$(2.34)\qquad -\frac{1}{2\pi i}\int_{\Gamma_4}(S-\rho)^{-1}h(u)\,d\rho$$

$$= \frac{1}{2\pi i}\,\lambda(u)\,\frac{M^{+}\left[(u-i)^{-}\right] - M^{-}\left[(u-i)^{-}\right]}{(u-i)}$$

$$- \frac{1}{2\pi i}\int P\,\frac{\eta(u)}{u-s}\,\frac{M^{+}\left[(s-i)^{-}\right] - M^{-}\left[(s-i)^{-}\right]}{s-i}\,ds\ .$$

This can be written as a Case continuum eigenmode expansion by setting

$$(2.35)\qquad A(\nu) = \frac{M^{+}\left[(\nu - i)^{-1}\right] - M^{-}\left[(\nu - i)^{-1}\right]}{2\pi i(\nu - i)}$$

A straighforward computation yields

(2.36) $$A(\nu) = \frac{\eta(\nu)}{\Lambda^+(\nu)\Lambda^-(\nu)} \int_{-\infty}^{\infty} \tilde{\phi}_\nu(s)h(s)\, ds$$

with

(2.37) $$\tilde{\phi}_\nu(s) = P\, \frac{-1}{s-\nu} + \frac{\lambda(s)}{\eta(s)}\, \delta(s-\nu) \;.$$

We define

(2.37) $$\phi_\nu(u) \equiv P\, \frac{-\eta(u)}{u-\nu} + \lambda(u)\delta(u-\nu)$$

so that Eq. (2.34) becomes

(2.38) $$-\frac{1}{2\pi i} \int_{\Gamma_4} (S-\rho)^{-1}h(u)\, d\rho = \int_{-\infty}^{\infty} \phi_\nu(u)A(\nu)\, d\nu$$

This is in exact agreement with Ref. 4.

Combining the results of Eqs. (2.23), (2.25), (2.28), (2.30) and (2.38) in Eq. (2.22) we have

(2.39) $$h(u) = \int_{-\infty}^{\infty} \phi_\nu(u)A(\nu)\, d\nu + \sum_k A_{\nu_k} \phi_{\nu_k}(u)$$

$$+ \sum_k A_{\nu_k}^{(1)} \phi_{\nu_k}^{(1)}(u) + A_{\nu_k}^{(2)} \phi_{\nu_k}^{(2)}(u)$$

$$+ \sum_\ell A_{\nu_{0\ell}} \phi_{\nu_{0\ell}}(u)$$

$$+ \sum_{\ell} A^{(1)}_{\nu 0 \ell} \, \phi^{(1)}_{\nu 0 \ell}(u) + A^{(2)}_{\nu 0 \ell} \, \phi^{(2)}_{\nu 0 \ell}(u)$$

where the A's are given, by construction, in terms of h in Eqs.(2.24c), (2.26b), (2.26d), (2.29b), (2.31c), (2.32d) and (2.36).

We summarize the results of this section by stating

<u>Theorem 2.1</u>. For any function $h \in L_p(-\infty,\infty)$ which is Hölder continuous on every bounded interval of $\mathbb{R}$, there is a collection of expansion coefficients, A's, such that h can be written as in Eq. (2.39).

Section 4 Solution of the Initial Value Problem

In order to solve the initial value problem for Eq. (2.4), one would expand the initial data, f(u,0) by the expansion formula Eq. (2.39). Then,

$$(2.40) \qquad f(u,t) = \int_{-\infty}^{\infty} \exp(-ik\nu t)\ A(\nu)\ \phi_\nu(u)\ d\nu$$

$$+ \sum_k A_{\nu_k}\ \phi_{\nu_k}(u)\ \exp(-ik\nu_k t)$$

$$+ \sum_k A^{(1)}_{\nu_k}\phi^{(1)}_{\nu_k}(u)\exp(-ik\nu_k t)$$

$$+ A^{(2)}_{\nu_k}\left[\phi^{(2)}_{\nu_k}(u) - ikt\phi^{(1)}_{\nu_k}(u)\right]\exp(-ik\nu_k t)$$

$$+ \sum_\ell A_{\nu_{0\ell}}\phi_{\nu_{0\ell}}(u)\exp(-ik\nu_{0\ell} t)$$

$$+ \sum_\ell A^{(1)}_{\nu_{0\ell}}\phi^{(1)}_{\nu_{0\ell}}(u)\ \exp(-ik\nu_{0\ell} t)$$

$$+ A^{(2)}_{\nu_{0\ell}}\left[\ \phi^{(2)}_{\nu_{0\ell}}\ (u)\ -\ ikt\ \phi^{(1)}_{\nu_{0\ell}}\ (u)\right]\ \exp(-ik\nu_{0\ell} t)$$

As a consequence of Theorem 2.1, the coefficients A are given by choosing h(u) = f(u,0). Note that the coefficients A were given in Sec. 3 in terms of an arbitrary function which is now identified as the initial data.

The fact that Eq. (2.40) is a solution of Eq (2.4) follows from the following considerations

Theorem 2.2. The functions $\phi_\nu(u)$, $\phi_{\nu_k}(u)$ and $\phi_{\nu_{0\ell}}(u)$, are eigenfunctions of K with corresponding eigenvalues ν, ν_k and $\nu_{0\ell}$ respectively.

Proof.

$$K\,\phi_\nu(u) = u\,P\,\frac{-\eta(u)}{u-\nu} + u\lambda(u)\delta(u-\nu)$$

$$+\,\eta(u)\int_{-\infty}^{\infty} P\,\frac{-\eta(s)}{s-\nu}\,ds + \lambda(\nu)\eta(u)$$

$$= (u-\nu+\nu)\,P\,\frac{-\eta(u)}{u-\nu} + \nu\lambda(u)\delta(u-\nu)$$

$$-\,\eta(u)\int_{-\infty}^{\infty} P\,\frac{\eta(s)}{s-\nu}\,ds + \lambda(\nu)\eta(u)$$

$$= \nu\{P\,\frac{-\eta(u)}{u-\nu} + \lambda(u)\delta(u-\nu)\} + 0$$

$$= \nu\phi_\nu(u) \quad .$$

Direct calculation yields the results for ν_k and $\nu_{0\ell}$ also.

Corollary. It is easy to see that the terms $\phi_\nu(u)\exp(-ik\nu t)$, $\phi_{\nu_k}(u)\exp(-ik\nu_k t)$ and $\phi_{\nu_{0\ell}}(u)\exp(-ik\nu_{0\ell}t)$ are solutions of Eq.(2.4). The terms with $\phi^{(1)}_{\nu_k}(u)\exp(-ik\nu_k t)$ and $\phi^{(1)}_{\nu_{0\ell}}(u)\exp(-ik\nu_{0\ell}t)$ are also solutions of Eq (2.4) since $\phi^{(1)}_{\nu_k} = \phi_{\nu_k}$ and $\phi^{(1)}_{\nu_{0\ell}} = \phi_{\nu_{0\ell}}$.

We need to show,

Theorem 2.3. The second order eigenmodes

$$\psi_{\nu_k}(u,t) = \left[\phi^{(2)}_{\nu_k}(u) - ikt\phi^{(1)}_{\nu_k}(u)\right] \exp(-ik\nu_k t) \tag{2.41}$$

and

$$\psi_{\nu_{0\ell}}(u,t) = \left[\phi^{(2)}_{\nu_{0\ell}}(u) - ikt\phi^{(1)}_{\nu_{0\ell}}(u)\right] \exp(-ik\nu_{0\ell} t) \tag{2.42}$$

are solutions of Eq. (2.4).

<u>Proof</u>. For $\psi_{\nu_k}(u,t)$ we have

$$\frac{\partial\psi_{\nu_k}(u,t)}{\partial t} = \{(-ik\nu_k)\left[\phi^{(2)}_{\nu_k}(u) - ikt\ \phi^{(1)}_{\nu_k}(u)\right] -ik\ \phi^{(1)}_{\nu_k}(u)\}$$

$$\times \exp(-ik\nu_k t)$$

and

$$ikK\psi_{\nu_k}(u,t) = \left[ikK\phi^{(2)}_{\nu_k}(u) + k^2\nu_k t\phi^{(1)}_{\nu_k}(u)\right] \exp(-ik\nu_k t)$$

adding these we obtain

$$0 = (-k^2\nu_k t - ik)\phi^{(1)}_{\nu_k}(u) - ik\nu_k\phi^{(2)}_{\nu_k}(u)$$

$$+ ikK\ \phi^{(2)}_{\nu_k}(u) + k^2\nu_k\, t\, \phi^{(1)}_{\nu_k}(u)$$

This is easily seen to be true by using Eq. (2.27). The proof for $\psi_{\nu_{0\ell}}(u,t)$ is analogous.

From the Corolary to Theorem 2.2 and from Theorem 2.3, it follows directly that f(u,t) defined by Eq. (2.40) is a solution of Eq. (2.4).

It would be desirable to obtain uniqueness of the solution to Eq. (2.4) by appealing to the uniqueness of the spectral resolution, Eqs. (2.19) and (2.39). However, in order to do this, we would have to show that the bounded operator S is of scalar type,[33] a procedure requiring a very careful definition of the domain of the operator as in a similar study of the neutron transport problem.[34] We could avoid these difficulties by showing that the operator -ikK is the generator of a strongly continuous semigroup, and the existence of a unique solution would follow immediately from Ref. 23. Furthermore, the solution is given explicitly in terms of an integral of the resolvent of K, rather than an integral of the second resolvent. We shall not pursue this point any further for the longitudinal plasma oscillations problem since these techniques are used in the next chapter for the transverse plasma oscillations problem. However, we assert that such a procedure does indeed lead to an expression of the unique solution to Eq. (2.4) given by Eq. (2.40).

Chapter 3

Transverse Oscillations

Section 1 Computation of the Resolvent

We will now consider the Fourier space transform of Eq. (1.7) given by

$$(3.1) \qquad \frac{\partial \psi_{\pm}(u,t)}{\partial t} + (ikT + A_{\pm})\ \psi_{\pm}(u,t) = 0 \quad ,$$

where

$$(3.2) \qquad \psi_{\pm}(u,t) = \int_{-\infty}^{\infty} \exp(-ikz)\ \psi_{\pm}(z,u,t)\ dz \quad .$$

We shall consider the equation for right circularly polarized solutions, dropping the dual ± sign in the equations and retaining only the + sign where necessary. When we are done, the left circularly polarized solution can be obtained by an appropriate change of sign.

With this change in notation, Eq. (3.1) becomes

$$(3.3) \qquad \frac{\partial \psi}{\partial t} + ik\ \underset{\approx}{K}\ \psi = 0$$

where $\psi(u,t) = \psi_{+}(u,t)$, and the operator $\underset{\approx}{K}$ is given explicitly by

$$(3.4) \qquad \underset{\approx}{K} = \begin{bmatrix} u + u_c & \frac{-n_0 e}{ikm} F(u) & -\frac{n_0 e}{kmc} G(u) \\ \frac{4\pi e}{ik} \int_{-\infty}^{\infty} du & 0 & -ic \\ 0 & ic & 0 \end{bmatrix},$$

where $u_c = \omega_c/k$.

The calculation of the resolvent of K proceeds as in Sec. 1 of Chapter 2 by writing $(\underset{\approx}{K}-\rho)g = f$ and solving for g in terms of f. Note that here, f and g each have three components, that is

$$(3.5) \qquad f(u) = \begin{bmatrix} f_1(u) \\ f_2 \\ f_3 \end{bmatrix} \in L_1(-\infty,\infty) \times \mathbf{C} \times \mathbf{C} = X .$$

We present the results of the resolvent calculation, component by component as

$$(3.6) \qquad (\underset{\approx}{K} - \rho)^{-1} f = \begin{Bmatrix} \left[(\underset{\approx}{K}-\rho)^{-1} f\right]_1 \\ \left[(\underset{\approx}{K}-\rho)^{-1} f\right]_2 \\ \left[(\underset{\approx}{K}-\rho)^{-1} f\right]_3 \end{Bmatrix}$$

where

$$(3.7a) \qquad \left[(\underset{\approx}{K} - \rho)^{-1} f\right]_1 = \frac{f_1(u)}{u+u_c-\rho} - \left(\frac{\omega_p}{k}\right)^2 \frac{G(u)-\rho F(u)}{u+u_c-\rho}$$

$$\times \frac{1}{\Lambda(\rho)} \int_{-\infty}^{\infty} \frac{f_1(s)}{s+u_c-\rho}\, ds - \frac{n_0 e}{imk} \frac{G(u)-\rho F(u)}{u+u_c-\rho} \frac{f_2}{\Lambda-\rho}$$

$$+\frac{n_0 e}{mkc} H_1(u,\rho)\ f_3$$

(3.7b) $$\left[(\underset{\approx}{K} - \rho)^{-1} f\right]_2 = -\frac{\rho}{\Lambda(\rho)} \frac{4\pi e}{ik} \int_{-\infty}^{\infty} \frac{f_1(s)}{s+u_c-\rho}\, ds$$

$$+\frac{\rho}{\Lambda(\rho)}\, f_2 + H_2(\rho)\, f_3$$

(3.7c) $$\left[(\underset{\approx}{K} - \rho)^{-1} f\right]_3 = -\frac{ic}{\Lambda(\rho)} \frac{4\pi e}{ik} \int_{-\infty}^{\infty} \frac{f_1(s)}{s+u_c-\rho}\, ds$$

$$+\frac{ic}{\Lambda(\rho)}\, f_2 + H_3(\rho)\, f_3$$

and

(3.8) $$\Lambda(\rho) = c^2 - \rho^2 + \left(\frac{\omega_p}{k}\right)^2 \int_{-\infty}^{\infty} \frac{G(s)-\rho F(s)}{s+u_c-\rho}\, ds \quad ,$$

(3.9) $$H_1(u,\rho) = \frac{1}{\Lambda(\rho)} \left\{ \frac{\rho G(u) - c^2 F(u)}{u+u_c-\rho} + \frac{(\omega_p/k)^2}{u+u_c-\rho} \right.$$

$$\left. \times\left[G(u) \int_{-\infty}^{\infty} \frac{F(s)}{s+u_c-} ds - F(u) \int_{-\infty}^{\infty} \frac{G(s)}{s+u_c-\rho}\, ds\right]\right\},$$

(3.10) $$H_2(\rho) = \frac{1}{ic} + \frac{1}{ic\Lambda(\rho)}\left[\rho^2 + \left(\frac{\omega_p}{k}\right)^2 \int_{-\infty}^{\infty} \frac{\rho F(s)}{u+u_c-\rho}\right] ds \quad ,$$

(3.11) $$H_3(\rho) = \frac{1}{\Lambda(\rho)}\left[\rho + \left(\frac{\omega_p}{k}\right)^2 \int_{-\infty}^{\infty} \frac{\rho F(s)}{s+u_c-\rho}\right] ds \quad .$$

The spectrum of $\underset{\approx}{K}$ can be obtained from the singularities of $(\underset{\approx}{K}-\rho)^{-1}$. Clearly, the continuous spectrum of $\underset{\approx}{K}$, $C\sigma(\underset{\approx}{K})$, consists of $\mathbb{R}$ (the real line). The eigenvalues, or point spectrum $P\sigma(\underset{\approx}{K})$, occur where $\Lambda(\rho) = 0$ (note that the function Λ referred to in this Chapter is completely different from the one considered in Chapter 2).

Section 2 Analysis of Λ

As in the case of longitudinal plasma oscillations, the plasma dispersion function, Eq. (3.8), for transverse oscillations satisfies the property that $\overline{\Lambda(\rho)} = \Lambda(\bar{\rho})$ so that the zeros of Λ occur in complex conjugate pairs. Thus, we shall use the same techniques applied in Sec. 2 of Chapter 2. We consider the contour drawn in Fig. 2.1, in order to determine the number of zeros of Λ in the upper half plane.

Along the curved portion of the contour, we have from Eq. (3.8),

$$\Lambda(\rho) \approx -\rho^2 \quad , \quad |\rho| \text{ large.} \tag{3.12}$$

For the portion of the contour just above the real axis, we need the limiting value of Λ, Λ^+. We present both limiting values, although for the moment we need only Λ^+

$$\begin{aligned} \Lambda^{\pm}(u) &= \lim_{\epsilon\to 0+} \Lambda(u \pm i\epsilon) \\ &= \lambda(u) \pm i\pi \left(\frac{\omega_p}{k}\right)^2 q(u) \quad , \end{aligned} \tag{3.13}$$

where

$$\lambda(u) = c^2 - u^2 + \left(\frac{\omega_p}{k}\right)^2 \int_{-\infty}^{\infty} P\, \frac{G(s)-uF(s)}{s+u_c-u}\, ds \quad , \tag{3.14}$$

$$q(u) = \frac{d}{du} \iint \tfrac{1}{2}(v_x^2 + v_y^2)\, f_0(v_x^2 + v_y^2,\, u - u_c)\, dv_x\, dv_y \tag{3.15}$$

Since we regard the equilibrium distribution function as having two relative maxima, then q will have three zeros, in increasing order u_0, u_1 and u_2.

We may now consider the Nyquist diagram for Λ with respect to the contour enclosing the upper half plane drawn in Fig. 2.1. Since q has three zeros, the image contour must cross the axis three times, and we have along the real axis,

$$q(u) > 0 \qquad u < u_0 \qquad \text{or} \qquad u_1 < u < u_2$$

and $$q(u) < 0 \qquad u_0 < u < u_1 \quad \text{or} \qquad u_2 < u$$

so that if

1) $\lambda(u_0) < 0$, then Λ has one conjugate pair of zeros
2) $\lambda(u_0) > 0$ and
   a) $\lambda(u_1)\lambda(u_2) < 0$, then Λ has one conjugate pair of zeros
   b) $\lambda(u_1)\lambda(u_2) > 0$, then Λ has no zeros

As an example of these conclusions, we show the Nyquist diagram for case 2a) in Fig. 3.1. The other cases are fairly simple modifications of this diagram.

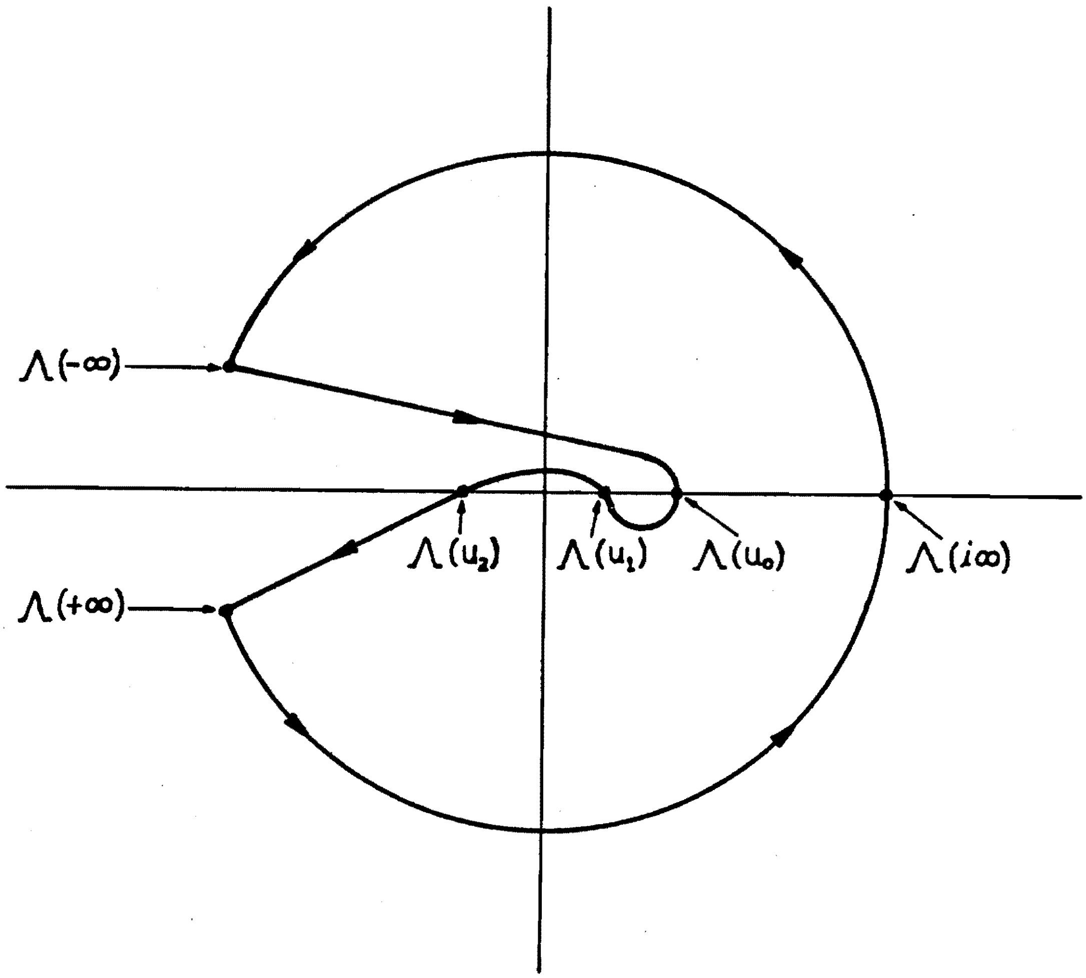

Fig. 3.1 A Nyquist Diagram

for Transverse Λ

Real zeros occur if $\Lambda^+(u) = 0$. Clearly, these can only occur at u_0, u_1 and u_2 where the imaginary part of Λ^+ vanishes. Thus the condition for $\Lambda^+(u_j)$ to vanish is that $\lambda(u_j) = 0$, $j = 0, 1, 2$ and thus we see that there may exist one, two or three real zeros. (A similar analysis for the case of the isotropic equilibrium distribution in Ref. 2 shows that Λ has no real zeros.)

Section 3 Integration of the Resolvent

We shall apply the identity

$$(3.16)\qquad h(u) = -\frac{1}{2\pi i}\oint_{\Gamma}(K - \rho)^{-1}h(u)\,d\rho$$

where the contour Γ encloses the spectrum of $\underset{\approx}{K}$, $\sigma(\underset{\approx}{K})$. We have

$$(3.17)\qquad \sigma(\underset{\approx}{K}) = C\sigma(\underset{\approx}{K}) \bigcup P\sigma(\underset{\approx}{K}) \quad ,$$

$$(3.18)\qquad C\sigma(\underset{\approx}{K}) = \mathbb{R} \quad ,$$

and

$$(3.19)\qquad P\sigma(\underset{\approx}{K}) = \{\nu : \Lambda(\nu) = 0\}$$

The contour Γ can be viewed as a deformed and modified Bromowich contour as in Ref. 22 given by

$$(3.20)\qquad \Gamma = \hat{\Gamma}_{+} \bigcup \hat{\Gamma}_{-} \bigcup \Gamma_{u+} \bigcup \Gamma_{u-} \bigcup \Gamma_{\bar{\nu}_k} \bigcup \Gamma_{\nu_k} \bigcup \Gamma_{\nu_{0\ell+}} \bigcup \Gamma_{\nu_{0\ell-}}$$

where the components of Γ are indicated in Fig. 3.2.

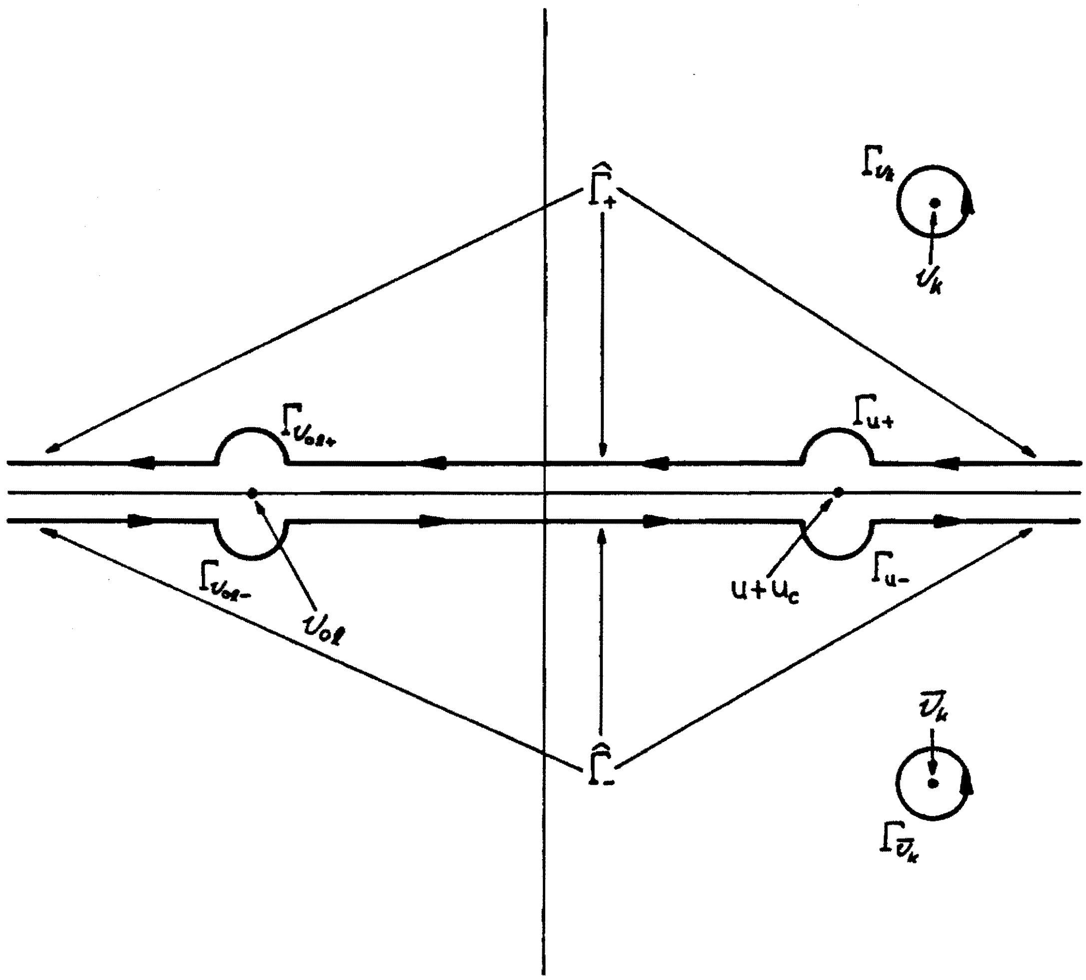

Fig. 3.2 The Components of Γ

Equation (3.16) becomes

$$h(u) = -\frac{1}{2\pi i}\left(\int_{\Gamma_4} + \sum_k \int_{\Gamma_{\nu_k}} + \sum_\ell \int_{\Gamma_{\nu_{0\ell\pm}}}\right)(\underset{\sim}{K} - \rho)^{-1} h(u) \quad d\rho \tag{3.21}$$

where

$$\Gamma_4 = \hat{\Gamma}_+ \bigcup \hat{\Gamma}_- \bigcup \Gamma_{u+} \bigcup \Gamma_{u-} \tag{3.22}$$

and the sum on the index k is viewed as running over the conjugate pairs of complex zeros of Λ while the sum on the index ℓ includes all the real zeros of Λ.

In order to evaluate the integrals in Eq. (3.21) using Eq. (3.6) we will need the limiting values of the functions

$$M(\rho) = \frac{1}{\Lambda(\rho)} \int_{-\infty}^{\infty} \frac{h_1(s)}{s+u_c-\rho}\, ds \tag{3.23}$$

$$N_1(\rho) = \frac{1}{\Lambda(\rho)} \int_{-\infty}^{\infty} \frac{F(s)}{s+u_c-\rho}\, ds \tag{3.24}$$

$$N_2(\rho) = \frac{1}{\Lambda(\rho)} \int_{\infty}^{\infty} \frac{G(s)}{s+u_c-\rho}\, ds \tag{3.25}$$

where we define $M^+(u)$, $N_1{}^+(u)$ and $N_2{}^+(u)$ as the limiting values of $M(\rho)$, $N_1(\rho)$ and $N_2(\rho)$ respectively as ρ approaches u from above the real axis while $M^-(u)$, $N_1{}^-(u)$ and $N_2{}^-(u)$ are defined as the limiting values of $M(\rho)$, $N_1(\rho)$ and $N_2(\rho)$ respectively as ρ approaches u from below the real axis. Using the Plemelj formulas [32] we obtain

$$M^\pm(u)\Lambda^\pm(u) = \int_{-\infty}^{\infty} P\, \frac{h_1(s)}{s+u_c-u}\, ds \pm i\,\pi\, F(u-u_c) \quad , \tag{3.26}$$

$$N_1^\pm(u)\Lambda^\pm(u) = \int_{-\infty}^{\infty} P\, \frac{F(s)}{s+u_c-u}\, ds \pm i\,\pi\, F(u-u_c) \quad , \tag{3.27}$$

and

$$N_2^{\pm}(u)\Lambda^{\pm}(u) = \int_{-\infty}^{\infty} P \frac{G(s)}{s+u_c-u} ds \pm i \pi G(u-u_c) \quad . \tag{3.28}$$

The application of residue theory in evaluating the integrals in Eq. (3.21) is carried out for each component. For the first component, the integral about the continuum, $C\sigma(\underset{\approx}{K})$, involves the use of Eq. (3.26), (3.27) and (3.28) as well as a change in the order of integration, where the integrand contains a product of principle value functions. The Poincaré-Bertrand formula[32] must be used in this change in the order of integration which results in a great simplification of the expressions. The second and third components are more easily integrated about the continuum since they involve only Eqs. (3.26) and (3.27) without any products of principle value functions. The results are

$$-\frac{1}{2\pi i} \int_{\Gamma_4} (\underset{\approx}{K}-\rho)^{-1} h(u)\, d\rho = \int_{-\infty}^{\infty} \phi_\nu(u) A(\nu)\, d\nu \quad , \tag{3.29}$$

where

$$\phi_\nu(u) = \begin{bmatrix} \left(\frac{\omega_p}{k}\right)^2 P \dfrac{\nu F(u)-G(u)}{\nu-u_c-u} - \lambda(\nu)\delta(\nu-u_c-u) \\ 4\pi e\nu/(ik) \\ 4\pi ec/k \end{bmatrix} \tag{3.30}$$

and

$$A(\nu) = -\frac{M^+(\nu)-M^-(\nu)}{2\pi i} + \frac{n_0 e}{ikm} \frac{g(\nu-u_c)-\nu F(\nu-u_c)}{\Lambda^+(\nu)\ \Lambda^-(\nu)}\, h_2 \tag{3.31}$$

$$+ \frac{n_0 e}{kmc} \frac{h_3}{\Lambda^+(\nu)\Lambda^-(\nu)} \int_{-\infty}^{\infty} \left(\frac{\omega_p}{k}\right)^2 P \frac{G(\nu-u_c)F(s-u_c)-G(s-u_c)F(\nu-u_c)}{s-\nu} ds$$

These are in complete agreement with results previously obtained in Ref. 12.

The integrals around the complex zeros, ν_k, of Λ (i.e. the isolated eigenvalues of $\underset{\approx}{K}$) are straightforward exercises in residue theory. We present here the results for simple ν_k, recalling that the calculation is done component by component

$$(3.32) \qquad -\frac{1}{2\pi i} \int_{\Gamma_{\nu_k}} (\underset{\approx}{K}-\rho)^{-1} h(u)\, d\rho = A_{\nu_k} \phi_{\nu_k}(u) \quad ,$$

where

$$(3.33) \qquad \phi_{\nu_k}(u) = \begin{bmatrix} \left(\frac{\omega_p}{k}\right)^2 \frac{\nu_k F(u)-G(u)}{\nu_k - u_c - u} \\ 4\pi e\nu_k/(ik) \\ 4\pi ec/k \end{bmatrix}$$

and

$$(3.34) \qquad A_{\nu_k} = \frac{1}{\Lambda'(\nu_k)} \{ \int_{-\infty}^{\infty} \frac{h_1(s-u_c)}{s-\nu_k} ds - \frac{ikh_2}{4\pi e}$$

$$+ \frac{kh_3}{4\pi ec} (\nu_k + \left(\frac{\omega_p}{k}\right)^2 \int_{-\infty}^{\infty} \frac{F(s-u_c)}{s-\nu_k} ds)\}$$

The integrals around the real zeros, $\nu_{0\ell}$, of Λ (i.e. the eigenvalues of $\underset{\approx}{K}$ embedded in the continuum) are done by applying the residue theorem to $\Gamma_{\nu_{0\ell+}}$ and $\Gamma_{\nu_{0\ell-}}$ separately and adding the result. If $\nu_{0\ell}$ is a simple zero of Λ we have

$$(3.35) \qquad -\frac{1}{2\pi i}\int_{\Gamma_{\nu_{0\ell}} \bigcup \Gamma_{\nu_{0\ell}}} (\underset{\approx}{K}-\rho)^{-1} h(u)\, d\rho = A_{\nu_{0\ell}}\, \phi_{\nu_{0\ell}}(u) \quad ,$$

where

$$(3.36) \qquad \phi_{\nu_{0\ell}}(u) = \begin{bmatrix} \left(\frac{\omega_p}{k}\right)^2 \dfrac{\nu_{0\ell}F(u)-G(u)}{\nu_{0\ell} - u_c - u} \\ \\ 4\pi e\nu_{0\ell}/(ik) \\ \\ 4\pi ec/k \end{bmatrix} ,$$

and

$$(3.37) \qquad A_{\nu_{0\ell}} = \left[\Lambda'^{+}(\nu_{0\ell})\Lambda'^{-}(\nu_{0\ell})\right]^{-1}\{\lambda'(\nu_{0\ell}) \int_{-\infty}^{\infty} P\, \frac{h_1(s-u_c)}{s-\nu_{0\ell}}\, ds$$

$$- \pi^2\left[\Lambda'^{-}(\nu_{0\ell})-\Lambda'^{+}(\nu_{0\ell})\right] h_1(\nu_{0\ell} - u_c)/(2\pi i)$$

$$- \frac{ik}{4\pi e}\lambda'(\nu_{0\ell})h_2 + \frac{k}{4\pi ec}\{\lambda'(\nu_{0\ell})\left[\nu_{0\ell} + \right.$$

$$\left(\frac{\omega_p}{k}\right)^2 \int_{-\infty}^{\infty} P\, \frac{F(s-u_c)}{s - \nu_{0\ell}}\, ds - \pi^2\left(\frac{\omega_p}{k}\right)^2\left[\Lambda'^{-}(\nu_{0\ell})-\Lambda'^{+}(\nu_{0\ell})\right]$$

$$\left. \times\, F(\nu_{0\ell} - u_c)/(2\pi i)\right]\}\, h_3\} \quad .$$

However, if $\nu_{0\ell}$ is a second order zero of Λ, we have

$$(3.38) \qquad -\frac{1}{2\pi i}\int_{\Gamma_{\nu_{0\ell+}}\bigcup\Gamma_{\nu_{0\ell-}}}(\underset{\approx}{K}-\rho)^{-1}h(u)\ d\rho = A^{(1)}_{\nu_{0\ell}}\phi^{(1)}_{\nu_{0\ell}}(u) + A^{(2)}_{\nu_{0\ell}}\phi^{(2)}_{\nu_{0\ell}}(u) \quad ,$$

where

$$(3.39) \qquad \phi^{(1)}_{\nu_{0\ell}}(u) = \phi_{\nu_{0\ell}}(u) \quad ,$$

$$(3.40) \qquad \phi^{(2)}_{\nu_{0\ell}}(u) = \begin{bmatrix} \left(\frac{\omega_p}{k}\right)^2 \dfrac{G(u)-(u+u_c)F(u)}{(\nu_{0\ell}-u_c-u)^2} \\ \\ 4\pi e/(ik) \\ \\ 0 \end{bmatrix} ,$$

$$(3.41) \qquad A^{(1)}_{\nu_{0\ell}} = 2\ \mathrm{Re}\ (\{3\left[\Lambda''^{+}(\nu_{0\ell})\right]^2\}^{-1}\{3\left[\Lambda''^{+}(\nu_{0\ell})\right.$$

$$\times\ \{\int_{-\infty}^{\infty} P\ \frac{h_1{*}s-u_c)}{(s-\nu_{0\ell})^2}\ ds + i\pi h_1(\nu_{0\ell}-u_c)\}\Big]$$

$$-\Lambda'''^{+}(\nu_{0\ell})\ \left[\int_{-\infty}^{\infty} P\ \frac{h_1(s-u_c)}{s-\nu_{0\ell}}\ ds + i\pi h_1(\nu_{0\ell}-u_c)\right]\})$$

$$-\frac{1}{4\pi e}\ h_2\ \ 2\mathrm{Re}\{\tfrac{1}{3}\Lambda'''^{+}(\nu_{0\ell})\left[\ \Lambda''^{+}(\nu_{0\ell})\right]^{-2}\}$$

$$-\frac{1}{4\pi ec}\ h_3\ \ 2\mathrm{Re}\ (\tfrac{i}{3}\left[\Lambda''^{+}(\nu_{0\ell})\right]^{-2}\{\Lambda'''^{+}(\nu_{0\ell})$$

$$\times\left[\left(\frac{\omega_p}{k}\right)^2 \int_{-\infty}^{\infty} P\ \frac{F(s-u_c)}{s-\nu_{0\ell}}\ ds + i\pi\left(\frac{\omega_p}{k}\right)^2\ F(\nu_{0\ell}-u_c)\right]\}$$

$$- \left[\Lambda''^{+}(\nu_{0\ell}) \right]^{-1} \left[1 + \left(\frac{\omega_p}{k} \right)^2 \int_{-\infty}^{\infty} P \, \frac{F(s-u_c)}{(s-\nu_{0\ell})^2} ds \right.$$

$$\left. + \, i\pi \left(\frac{\omega_p}{k} \right)^2 \quad F'(\nu_{0\ell}-u_c) \right]) \quad ,$$

and

$$A^{(2)}_{\nu_{0\ell}} = 2 \, \mathrm{Re}\{ \left[\Lambda''^{+}(\nu_{0\ell}) \right]^{-1} \left[\int_{-\infty}^{\infty} P \, \frac{h(s-u_c)}{s - \nu_{0\ell}} \, ds \right. \tag{3.42}$$

$$\left. + \, i\pi h_1 (\nu_{0\ell}-u_c) \right] \} - \frac{ikh_2}{4\pi e} \, 2\mathrm{Re}\{ \left[\Lambda''^{+}(\nu_{0\ell}) \right]^{-1}$$

$$+ \, \frac{kh_3}{4\pi ec} \, 2\mathrm{Re}\{ \left[\Lambda''^{+}(\nu_{0\ell}) \right]^{-1} \left[\nu_{0\ell} + \left(\frac{\omega_p}{k} \right)^2 \int_{-\infty}^{\infty} P \, \frac{F(s-u_c)}{s - \nu_{0\ell}} \, ds \right.$$

$$\left. + \, i \, \pi \left(\frac{\omega_p}{k} \right)^2 \, F(\nu_{0\ell} - u_c) \right] \} \, .$$

Combining the results of Eqs. (3.29), (3.32), (3.35) and (3.38) in Eq. (3.21), we have

$$h(u) = \int_{-\infty}^{\infty} \phi_\nu(u) A(\nu) \, d\nu \; + \; \sum_k A_{\nu_k} \phi_{\nu_k}(u) \tag{3.43}$$

$$+ \sum_\ell A_{\nu_{0\ell}} \phi_{\nu_{0\ell}}(u) \; + \; \sum_\ell A^{(1)}_{\nu_{0\ell}} \phi_{\nu_{0\ell}}(u) \; + \; A^{(2)}_{\nu_{0\ell}} \phi^{(2)}_{\nu_{0\ell}}(u)$$

This equation will be used in obtaining the solution to the initial value problem for transverse oscillations. We must first consider the meaning of such solutions since a rigorous derivation of the solution to Eq. (3.3) presents certain technical problems for unbounded operators such as $\underset{\sim}{K}$.

Section 4 Solution of the Initial Value Problem

A solution of the differential equation (3.3) is understood to be a differentiable function

$$f(u,\cdot) : \mathbb{R} \to X \tag{3.44}$$

with values in the Banach space

$$X = L_1(-\infty,\infty) \times \mathbb{C} \times \mathbb{C} \tag{3.45}$$

with norm

$$||f|| = \int_{-\infty}^{\infty} |f_1(u)| \, du + |f_2| + |f_3| \ . \tag{3.46}$$

Using a semigroup approach, a necessary and sufficient condition for Eq. (3.3) to have a unique solution is that the operator $-ik\underset{\approx}{K}$ be the generator of a strongly continuous semigroup.[23] In order to prove this, we decompose $-ik\underset{\approx}{K}$ as follows

$$-ik\underset{\approx}{K} = -(ik\underset{\approx}{K}_1 + ik\underset{\approx}{K}_2) \quad , \tag{3.47}$$

where

$$(3.48) \qquad ik\underset{\approx}{K}_1 = \begin{bmatrix} ik(u+u_c) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(3.49) \qquad ik\underset{\approx}{K}_2 = \begin{bmatrix} 0 & \frac{n_0 e}{m} F(u) & \frac{n_0 e}{m} G(u) \\ 4\pi e \int du & 0 & kc \\ 0 & -kc & 0 \end{bmatrix}$$

<u>Lemma 3.1</u>. The operator $ik\underset{\approx}{K}_2$ is bounded in $\mathcal{X}$.

<u>Proof</u>. By direct calculation, we have

$$\begin{aligned} ||ik\underset{\approx}{K}_2(f)|| &= \int_{-\infty}^{\infty} \left|\frac{n_0 e}{imc} f_3 \, G(u) - \frac{n_0 e}{m} f_2 \, F(u) \right| du \\ &\quad + \left| \int_{-\infty}^{\infty} f_1(u) \, du + kc \, f_3 \right| \\ &\quad + \left| - kc \, f_2 \right| \\ &\le \int_{-\infty}^{\infty} |f_1(u)| du + \left\{\frac{n_0 e}{m} \int_{\infty}^{\infty} |F(u)| du + |kc|\right\} f_2 \\ &\quad + \left\{\frac{n_0 e}{m} \int_{-\infty}^{\infty} |G(u)| du + |kc|\right\} |f_3| \end{aligned}$$

The function F, although a perturbed Maxwellian, is certainly integrable. Similarly, the function G is also integrable [see Eq. (1.8d)]. Let

$$B = \max \{1, \frac{n_0 e}{m} \int_{-\infty}^{\infty} |F(u)|du + |kc| ,$$

$$\frac{n_0 e}{m} \int_{-\infty}^{\infty} |G(u)|du + |kc| \} ,$$

then

$$||ik\underset{\approx}{K}_2(f)|| \le B \; ||f|| \quad ,$$

thus the lemma is proved.

Lemma 3.2. The operator $ik\underset{\approx}{K}_1$ is the generator of a strongly continuous semigroup.

Proof. By virtue of the Hille-Yosida-Phillips theorem,[25] $ik\underset{\approx}{K}_1$ will be the generator of such a semigroup if, and only if

(i) $(\gamma,\infty) \not\subset \sigma(ik\underset{\approx}{K}_1)$

and (ii) $||(\xi + ik\underset{\approx}{K}_1)^{-n}|| \le B'(\xi - \gamma)^{-n} \;\forall\; \xi > \gamma$

where B' and γ are positive constants.

In fact, (i) is satisfied immediately by virtue of the spectral mapping theorem

$$\sigma(ik\underset{\approx}{K}_1) = ik\sigma(\underset{\approx}{K}_1) = i\mathbb{R}$$

and clearly the line segment (γ,∞) is not contained in $i\mathbb{R}$ (the imaginary axis) for any positive γ.

The estimate (ii) is easily verified since

$$||(\xi + ik\underset{\approx}{K}_1)^{-n}|| = \sup_{f\in X}\{\int_{-\infty}^{\infty} |[\xi + i\ k\ (u + u_c)]^{-n} f_1(u)|du/||f||\}$$

$$\leq \xi^{-n} \sup_{f\in X}\{\int_{-\infty}^{\infty}|f_1(u)|du/||f||\}$$

$$\leq \xi^{-n}$$

$$\leq (\xi - \gamma)^{-n} \quad \forall\ \xi > \gamma$$

(note that the constant B' is chose to be 1).

In Lemma 3.1 and Lemma 3.2 we have shown that $-ik\underset{\approx}{K}$ is the generator of a strongly continuous semigroup, $-ik\underset{\approx}{K}_1$, that is perturbed by a bounded operator, $-ik\underset{\approx}{K}_2$. From a theorem in Kato (Ref. 24, p. 495) it follows immediately that $-ik\underset{\approx}{K}$ is indeed the generator of a strongly continuous semigroup. Thus, the solution of Eq. (3.3) may be uniquely represented (see Ref. 23, p. 31) by the Laplace transform

$$f(u,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} (ik\underset{\approx}{K} + s)^{-1} f(u,0)\ \exp(st)\ ds\ . \tag{3.50}$$

The contour in Eq (3.50) is a Bromowich contour where $\gamma > \max\{|\mathrm{Re}\,i\nu| : \nu\in P\sigma(\underset{\approx}{K})\}$ and is shown in Fig. 3.3.

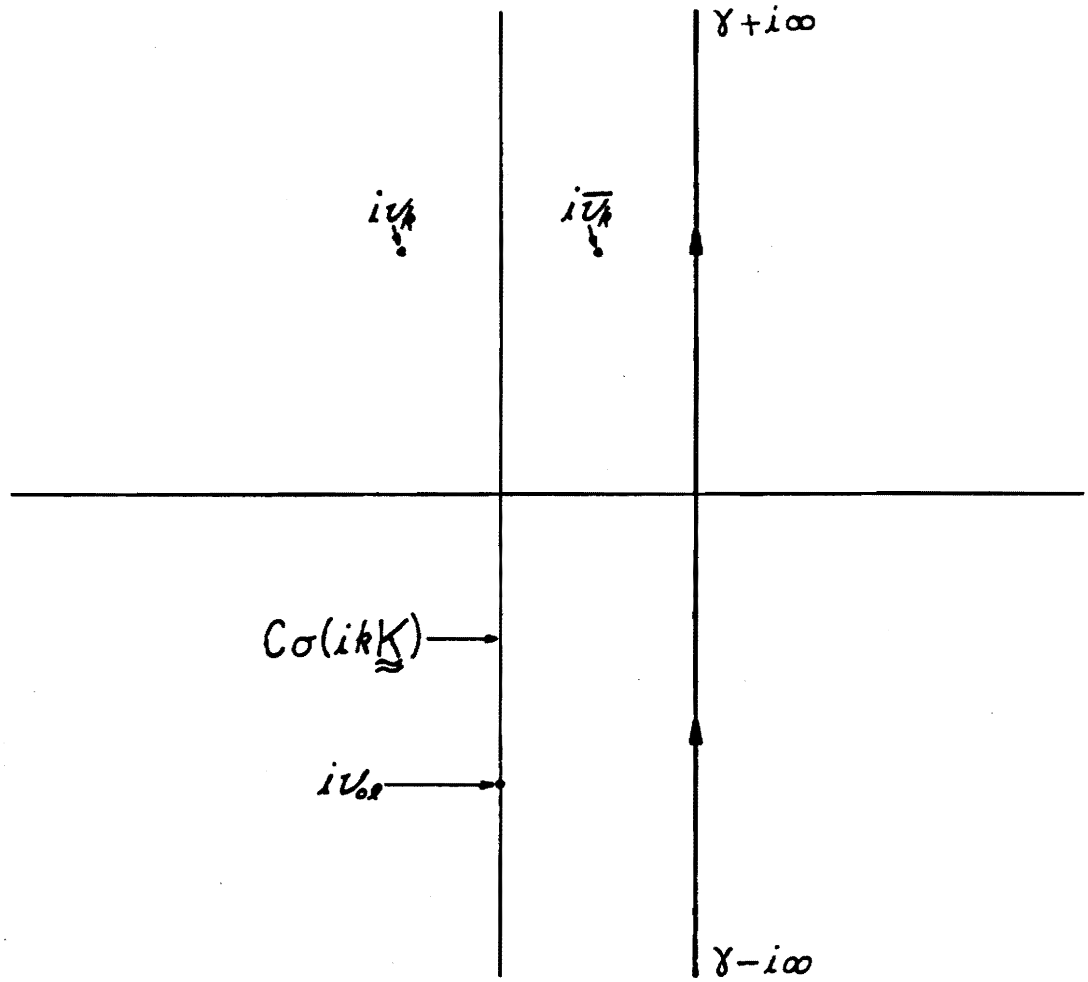

Fig. 3.3 The Bromowich Contour

We first deform the Bromowich contour of Fig. 3.3 and add contours enclosing the spectral points in the left half plane as is shown in Fig. 3.4. These added contours contribute nothing to the integral in Eq. 3.50) since one is directed in the clockwise sense and the other is directed in the counterclockwise sense and as a pair of contours they cancel one another.

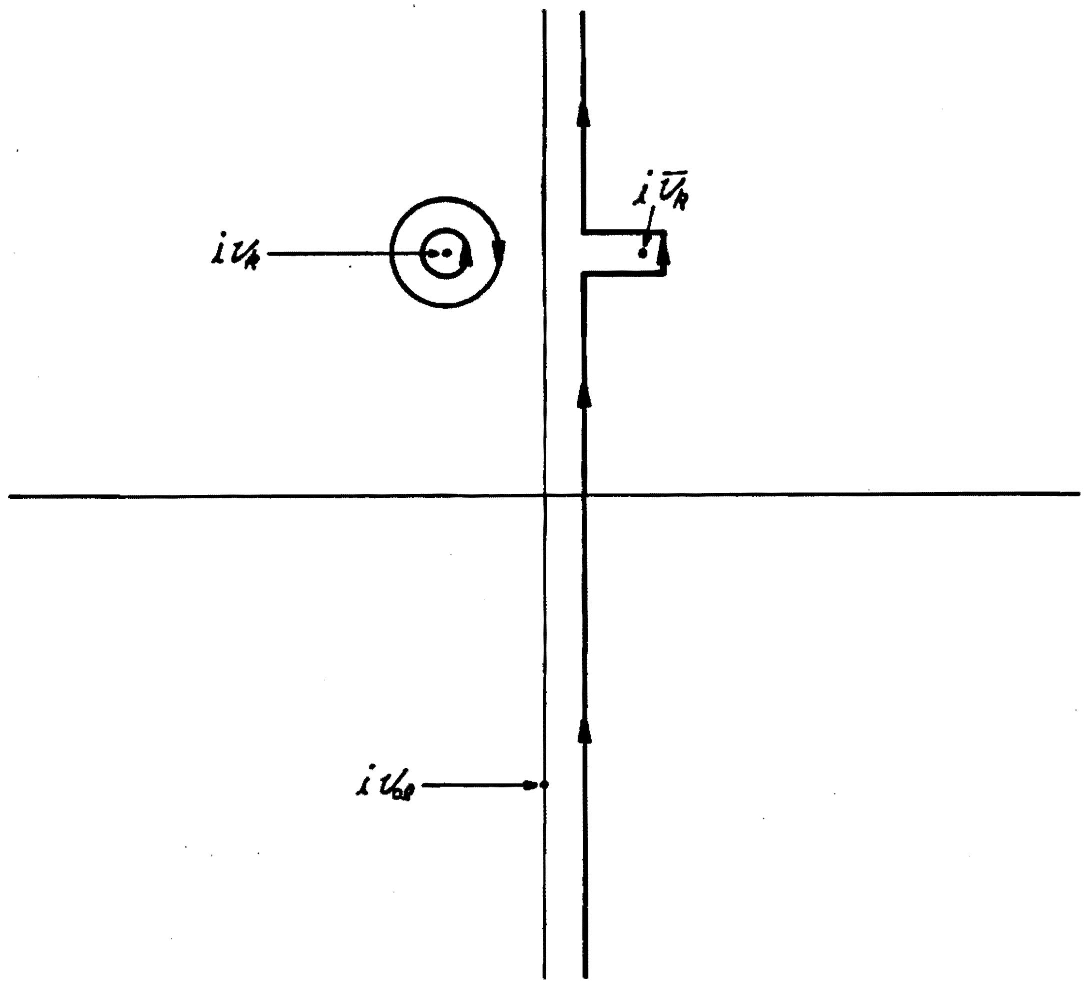

Fig. 3.4 The First Contour Deformation

In a second deformation shown in Fig. 3.5, the clockwise contour is expanded into a contour running parallel to the imaginary axis and closing in a large semicircular arc in the left half plane. We have also further deformed the portion of the contour lying in the right half plane.

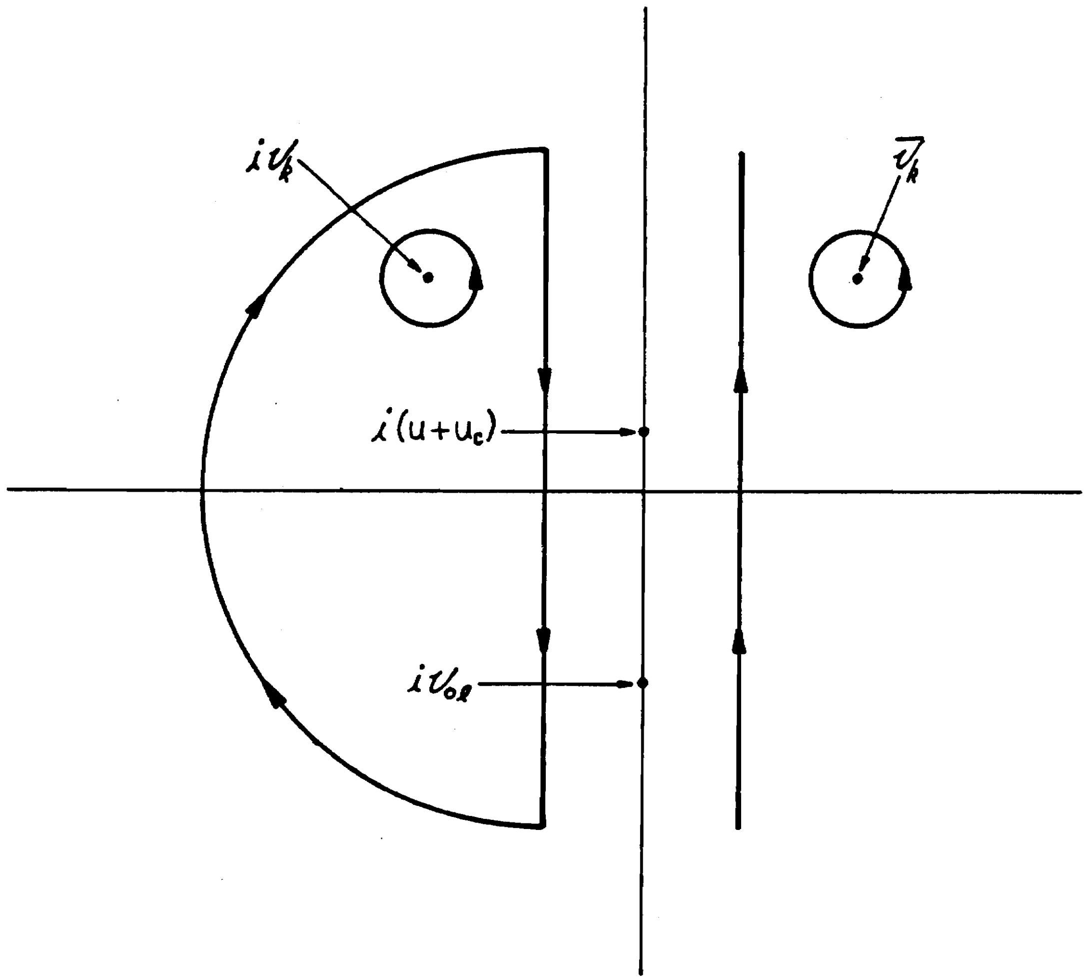

Fig 3.5 The Second Contour Deformation

The integrand in Eq. (3.50), $(ik\cdot\underset{\approx}{K} + s)^{-1}f(o,u)$, is easily seen to vanish on the curved portion of the contour in Fig. 3.5 since along this portion of the contour $s \to \infty$. Thus, there is no contribution to Eq. (3.50) from the curved portion of the contour and we omit it in Fig. 3.6. Figure 3.6 also includes semicircular arcs about the singularities of the integrand imbedded in the continuum.

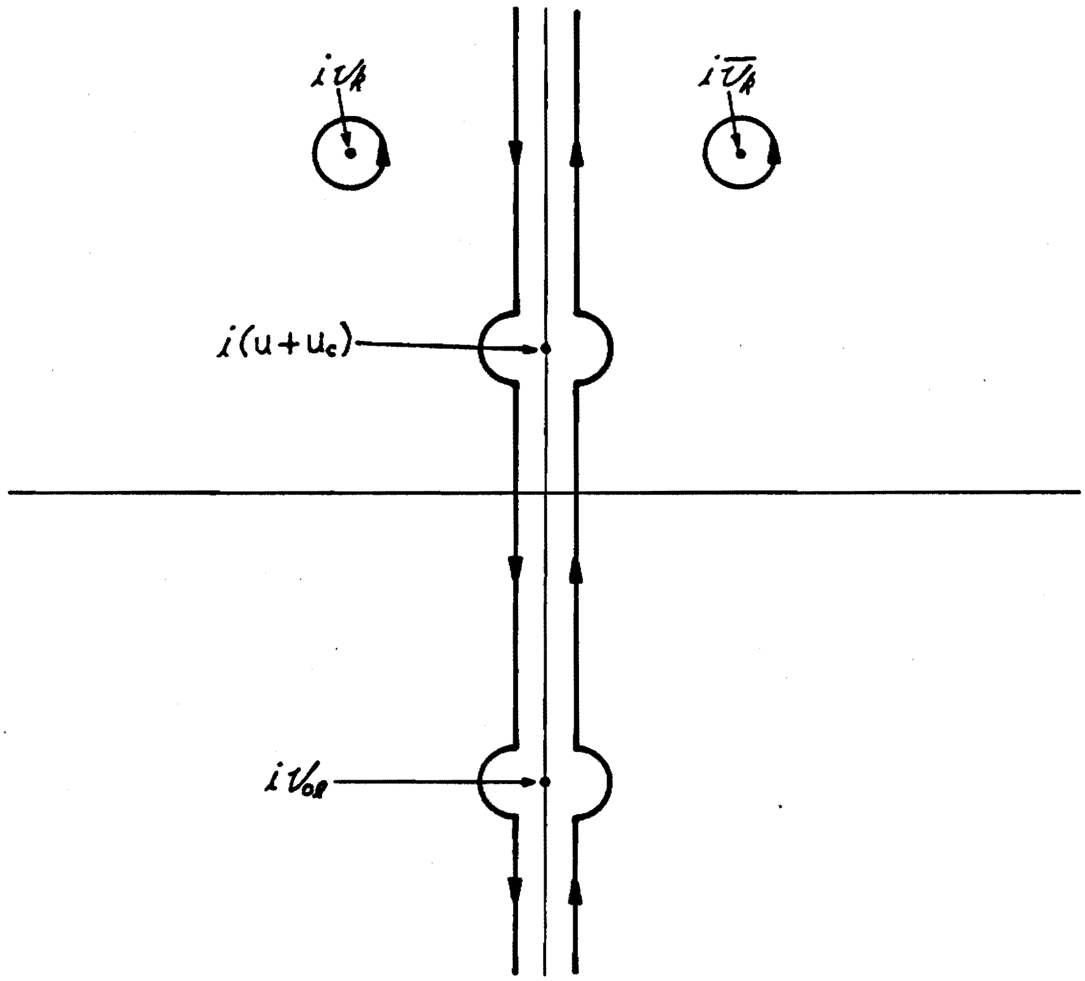

Fig. 3.6 Integration Contour Before Rotation

We now make a change of variable in Eq. (3.50)

$$s \rightarrow ik\rho$$

so that the integration contour of Fig. 3.6 becomes the contour Γ'

discussed in Sec. 3 and Eq. (3.50) becomes

$$(3.51) \qquad f(u,t) = -\frac{1}{2\pi i} \int_\Gamma (\underset{\approx}{K} - \rho)^{-1} f(u,0) \exp(-ik\rho t)\, d\rho \quad .$$

For t = 0, this is precisely the resolution of the identity formula, Eq. (3.16), that we expanded in Sec. 3.

By analogy with Chapter 2 or by carefully considering the effect of the exponential factor in Eq. (3.51) we arrive at a more useful form for the unique solution to Eq. (3.3)

$$\begin{aligned}
(3.52) \qquad f(u,t) = & \int_{-\infty}^{\infty} \phi_\nu(u)\, A(\nu) \exp(-ik\nu t) \\
& + \sum_k A_{\nu_k} \phi_{\nu_k}(u) \exp(-ik\nu_k t) \\
& + \sum_\ell A^{(1)}_{\nu_{0\ell}} \phi^{(1)}_{\nu_{0\ell}}(u) \exp(-ik\nu_{0\ell} t) \\
& \quad + A^{(2)}_{\nu_{0\ell}} \left[\phi^{(2)}_{\nu_{0\ell}}(u) - i\,k\,t\,\phi^{(1)}_{\nu_{0\ell}}(u)\right] \exp(-ik\nu_{0\ell} t)
\end{aligned}$$

where the coefficients, A, are given by the appropriate equation in Sec. 3 with h identified as the initial data

$$h(u) = f(u,0) \ .$$

Chapter 4

Longitudinal Waves

Section 1 Existence and Uniqueness

We now turn to the boundary value problem for longitudinal plasma waves in a semi-infinite medium ($z > 0$). Before carrying out the Fourier time transform of Eqs. (1.5a) and (1.5c), we shall first consider existence and uniqueness of solutions to these equations using the theory of perturbed semigroups[24] and incorporating the boundary condition, Eq. (1.10), directly into the "free" semigroup. After we have established existence and uniqueness for the coupled system of equations, we will then work towards a constructive solution to the transformed equations which will also include an inhomogeneous forcing term due to a small applied electric field, E_A.

The coupled equations, (1.5a) and (1.5c), may be written as a single matrix equation

$$\frac{\partial \psi}{\partial t} + T\,\psi + A\,\psi = 0 \quad , \tag{4.1}$$

where

$$\psi = \begin{bmatrix} f \\ E \end{bmatrix} \tag{4.2}$$

$$T = \begin{bmatrix} u\frac{\partial}{\partial z} & 0 \\ 0 & 0 \end{bmatrix} \tag{4.3}$$

and

$$(4.4) \qquad A = \begin{bmatrix} 0 & \frac{n_0 e}{m} F'(u) \\ 4\pi e \int_{-\infty}^{\infty} ds \; s & 0 \end{bmatrix} .$$

We shall also require

$$(4.5) \qquad \int_{-\infty}^{\infty} (1 + |u|)|F'(u)| \; du < \infty \quad ,$$

which is not a severe restriction since F is viewed as a perturbed Maxwellian so that F or any of its derivatives will still be integrable even when multiplied by a polynomial. At the boundary, $z = 0$, we impose the boundary condition, Eq. (1.11), where the condition

$$(4.6) \qquad \int_{0}^{\infty} g(u) \; du = 1 \quad ,$$

represents zero net current across the boundary. For technical reasons we must also require

$$(4.7) \qquad C = \int_{0}^{\infty} (1 + |u|) \; g(u) \; du < \infty \quad ,$$

which again, is not a severe restriction since, from Eq. (1.11), g inherits the Maxwellian behavior of F at $\pm \infty$.

We are going to treat Eq. (4.1) as an evolution equation in the Banach space

$$(4.8) \qquad X = \{f\in L_1(\mathbb{R}_+\times \mathbb{R}): \int_{-\infty}^{\infty}\int_{0}^{\infty}(1+|u|)|f|\ dz\ du < \infty\} \times L_1(\mathbb{R}_+),$$

with norm given by

$$(4.9) \qquad ||\psi|| = \int_{-\infty}^{\infty}\int_{0}^{\infty} (1 + |u|)|f|\ dz\ du + \int_{0}^{\infty} |E(z)|\ dz \quad .$$

We describe the free motion due to the operator T alone by incorporating the boundary condition (1.10) directly into the one-parameter family of operators

$$(4.10) \qquad W(t) = \begin{bmatrix} U(t) & 0 \\ 0 & I \end{bmatrix} ,$$

where

$$(4.11) \qquad \left[U(t)f\right](z,u) = \begin{cases} f(z-ut,u) \quad , & ut < z \\ \alpha f(ut-z,-u) & \\ +(1-\alpha)g(u)/u \int_{-\infty}^{\infty}|s|f(sz/u-st,s)\ ds, & ut > z \end{cases}$$

Here, $\{W(t):t \geq 0\}$ is the "free" semigroup. We now state

Lemma 4.1. $\{W(t):t \geq 0\}$ is a bounded strongly continuous semigroup on X, whose generator, T, is formally given by Eq. (4.3).

Proof. Clearly it is sufficient to show that $\{U(t:t)\geq 0\}$ is a bounded strongly continuous semigroup on the first component of X (which we denote by X_1). In order to show that U is a semigroup, we note that once the boundary condition has taken effect, the sign of the velocity changes and a second application of U will not involve the boundary condition since this would represent multiple reflections. Thus, it is easily seen that $U(t_1)U(t_2) = U(t_1 + t_2)$. Furthermore U is bounded since

$$||U(t)f||_{X_1} = \int\int_{ut<z}(1+|u|)|f(z-ut,u)|dzdu$$

$$+ \int\int_{ut>z} (1+|u|)|\alpha f(ut-z,-u)+(1-\alpha)g(u) \int_{-\infty}^{0}|s|f(sz/u-st,s)ds|dzdu$$

$$\leq \int_{-\infty}^{0}\int_{-ut}^{\infty} + \int_{0}^{\infty}\int_{0}^{\infty} (1+|u|)|f(z',u)|dz' du$$

$$+ \int_{-\infty}^{0}\int_{\infty}^{ut} (1+|u|)|f(z,u)|dzdu + (1-\alpha)C \int_{-\infty}^{0}\int_{\infty}^{-st} |f(z''',s)|dz'''ds$$

$$\leq C \int_{-\infty}^{\infty}\int_{0}^{\infty} (1+|u|)|f(z,u)|dzdu = C\,||f||_{X_1} \quad .$$

In order to prove strong continuity, it is now sufficient to show that $U(t)f \to f$ as $t \to 0$ for f in a dense subset of X_1. If $f \in C_c((0,\infty)\times \mathbb{R})$ where $C_c((0,\infty)\times \mathbb{R})$ is the set of continuous functions with compact support in $(0,\infty)\times\mathbb{R}$, then $U(t)f(z,u) = f(z-ut,u)$ for t sufficiently small. Thus,

$$||U(t)f-f||_{X_1} \leq C_1 \sup_{z,u} |f(z-ut,u)-f(z,u)| \to 0 \text{ as } t \to 0 \quad .$$

For $f \in C_c^1((0,\infty)\times \mathbb{R}) = \{f \in C_c((0,\infty)\times \mathbb{R})$: f is continuously differentiable$\}$ it is easy to show that

$$\lim_{t\to 0} t^{-1}\Big[U(t)f-f\Big] = -u \frac{\partial}{\partial z} f \ .$$

Thus, it follows that on a subset of X_1, the generator, T, of W is given by Eq. (4.3), completing the proof of Lemma 4.1. (It is not necessary for us to explicitly determine the domain of T.)

Lemma 4.2. T + A is the generator of a strongly continuous semigroup on X.

Proof. In view of a pertubation theorem (Ref. 24, p. 495) and Lemma 4.1, it is sufficient to show that A is bounded on X. The fact that A is bounded follows immediately from the definition of A and our choice of X.

Finally, we can state our desired result as a consequence of Lemma 4.2.

Theorem 4.1. For $\Psi_0 \in D(T)$, there is a unique solution, $\Psi(t)$, of Eq. (4.1) satisfying $\Psi(0) = \Psi_0$.

Although this result is more or less technical, it tells us that X is an appropriate space in which to look for solutions to Eq. (4.1). Furthermore, the construction of solutions to Eq. (4.1) by the eigenfunction techniques of Ref. 4 or the resolvent integration techniques of Chapters 2 and 3 is supported by this existence and uniqueness result.

Section 2 Computation of the Resolvent

We now Fourier transform Eqs. (1.5) with respect to the time variable and simplify by letting

$$f(z,u) = \int_{-\infty}^{\infty} \exp(i\omega t) f(z,ut)\, dt \tag{4.12}$$

$$E(z) = \int_{-\infty}^{\infty} \exp(i\omega t) E(z,t)\, dt \tag{4.13}$$

$$E_A = \int_{-\infty}^{\infty} \exp(i\omega t) E_A(t)\, dt \tag{4.14}$$

where the total electric field in Eq.(1.5a) is viewed as the sum of a self consistent electric field $E(z,t)$ and an externally applied field $E_A(t)$, so that we have

$$\frac{\partial f}{\partial z} + \frac{i\omega}{u} f + \frac{n_0 e}{m} \frac{F'(u)}{u} E = -\frac{n_0 e}{m} \frac{F'(u)}{u} E_A \tag{4.15a}$$

$$\frac{\partial E}{\partial z} = 4\pi e \int_{-\infty}^{\infty} f(z,u)\, du \tag{4.15b}$$

$$E = -\frac{4\pi e}{i\omega} \int_{-\infty}^{\infty} u f(z,u)\, du \tag{4.15c}$$

Note that the externally applied electric field, E_A, appears in the Vlasov equation (4.15a) as a forcing term. We view E_A as being maintained by external charges and currents not described by f.

As mentioned in Chapter 1, previous analyses[2,5,13] of this set of equations have taken Eqs. (4.15a) with Gauss' Law, Eq. (4.15b), and studied the resulting two component matrix equation. This was done primarily because it was desired to have an electrostatic model (note the magnetic fields have been neglected). However, we shall show that Eqs. (4.15a) and (4.15b) imply Eq. (4.15c) up to a constant; thus justifying the use of Ampere's Law from the outset. If we multiply Eq. (4.15a) by u and integrate with respect to u, we obtain

$$(4.16) \qquad \frac{\partial}{\partial z} \int_{-\infty}^{\infty} uf(z,u)\, du + i\omega \int_{-\infty}^{\infty} f(z,u)\, du = 0$$

where we have used the fact that $\int_{-\infty}^{\infty} F'(u)\, du = 0$. We now incorporate Eq. (4.15b) into Eq. (4.16), thus

$$(4.17) \qquad \frac{\partial}{\partial z} \{ \int_{-\infty}^{\infty} uf(z,u)\, du + \frac{i\omega}{4\pi e} E(z) \} = 0$$

so that

$$(4.18) \qquad E(z) = - \frac{4\pi e}{i\omega} \int_{-\infty}^{\infty} uf(z,u)\, du + \text{constant} \quad .$$

In view of Ampere's Law, Eq. (4.15c), the constant in Eq.(4.18) is zero. However, earlier studies attributed this constant to an externally applied electric field. This led to the notion that the relevant transport operator had a "zero eigenmode". We now see that this "zero eignemode" is superfluous. Furthermore, our argument also shows that the three equations (4.15) in two unknowns, E and f, forms a consistent set of equations. The same conclusions are also valid if one applies our method to Eq.(1.5).

We may now eliminate the electric field, E, from Eqs. (4.15a) and

and (4.15c) so that

$$\frac{\partial f}{\partial z} + i\,\omega\,K\,f = -\frac{n_0 e}{i\omega m}\,E_A\,\frac{F'(u)}{u} \tag{4.19}$$

where the fixed frequency longitudinal transport operator, K, is given by

$$(Kf)(z,u) = f(z,u)/u + \sigma^2\left[F'(u)/u\right]\int_{-\infty}^{\infty} s f(z,s)\,ds \tag{4.20}$$

and $\sigma = \omega_p/\omega$.

The resolvent of K is obtained by writing $(K-\rho)g=f$ and solving for g in terms of f. The procedure is identical to that given earlier in Section 1 of Chapter 2. The result is

$$(K-\rho)^{-1}h(u) = \frac{uh(u)}{1-\rho u} - \frac{\sigma^2 F'(u)}{1-\rho u}\,\frac{1}{\Lambda(1/\rho)}\int_{-\infty}^{\infty}\frac{s^2 h(s)}{1-\rho s}\,ds \tag{4.21}$$

where the fixed frequency longitudinal dispersion function Λ is given by

$$\Lambda(\rho) = 1 - \sigma^2\rho^2\int_{-\infty}^{\infty}\frac{F'(s)}{s-\rho}\,ds \quad . \tag{4.22}$$

Note that the Λ defined here is completely different from the one discussed in Chapters 2 and 3.

The spectrum of K is easily determined from Eq. (4.21) by viewing the resolvent as an operator valued analytic function of the complex variable ρ. We see immediately that singularities occur for $\rho = \frac{1}{u}$

and $\Lambda(1/\rho) = 0$. Thus, since u can be any real number we may write the spectrum of K as

$$(4.23) \qquad \sigma(K) = C\sigma(K) \cup P\sigma(K)$$

where

$$(4.24) \qquad C\sigma(K) = \mathbb{R}$$

and

$$(4.25) \qquad P\sigma(K) = \{\nu : \Lambda(\nu) = 0\} \quad .$$

For reasons which will become apparent later, we make the change of variable $\rho \to 1/\rho$ in Eq. (4.21) so that we have

$$(4.26) \qquad (K - \tfrac{1}{\rho})^{-1}h(u) = \frac{\rho u h(u)}{\rho - u} - \frac{\rho^2\sigma^2 F'(u)}{\rho - u}\,\frac{1}{\Lambda(\rho)} \int_{-\infty}^{\infty} \frac{s^2 h(s)}{\rho - s}\, ds \; .$$

Furthermore, we will need the limiting values of $\Lambda, \Lambda^{\pm}$, as the real line is approached from above and below respectively

$$(4.27) \qquad \Lambda^{\pm}(u) = \lim_{\epsilon \to 0+} \Lambda(u \pm i\epsilon)$$

$$= 1 - \sigma^2 u^2 \int_{-\infty}^{\infty} P\, \frac{F'(s)}{s - u}\, ds \mp i\pi\sigma^2 u^2 F'(u) \quad .$$

Section 3 Analysis of Λ

For the case of isotropic plasma (i.e. F is an even function), it is well known[2,5] that Λ has no zeros if $\sigma^2 < 1$; K then has no eigenvalues, and the plasma waves are dissipative. However, we are interested in anisotropic plasmas, for example "bump on tail" or "two stream" equilibria (see Ref. 1), for which eigenvalues may indeed exist. We first discuss the zeros of Λ and later construct the Wiener-Hopf factorization of Λ. The Wiener-Hopf factorization of Λ is needed in the construction of the resolvent of the half range transport operator in Sec. 4. For completeness, we shall also discuss the coupled non-linear integral equations for the Wiener-Hopf factors.

Suppose F has two relative maxima. We distinguish three cases for convenience;

Case 1: "bump on left tail", F(u) is an ordinary Maxwellian with a relative maximum for $u < 0$.

Case 2: "bump on right tail", F(u) is an ordinary Maxwellain with a relative maximum for $u > 0$.

Case 3: "two stream", F(u) is the sum of two ordinary Maxwellians, one of which having its maximum for $u > 0$ and the other having its maximum for $u < 0$.

In each case F' will vanish at three finite points. We call them $u_0 < u_1 < u_2$ (for Case 1, $u_2 \leq 0$; for Case 2, $u_0 \geq 0$; for Case 3 $u_0 < 0 < u_2$). Mathematically these three cases could be treated as one, but the above division helps clarify the physics. We need the following

Lemma 4.3. For $u > 0$, $\operatorname{Im} \Lambda(iu) < 0$ in Case 1 and $\operatorname{Im} \Lambda(iu) > 0$

in Case 2.

Proof. From Eq. (4.22) we have

(4.28) $$\Lambda(iu) = 1 + \sigma^2 u^2 \int_{-\infty}^{\infty} \frac{F'(u)}{s - iu} \frac{s+iu}{s+iu} ds$$

so that

(4.29) $$\operatorname{Im} \Lambda(iu) = \sigma^2 u^3 \int_{-\infty}^{\infty} \frac{F'(s)}{s^2+u^2} ds$$

For Case 1, the only contribution to the integral in Eq. (4.29) comes from the perturbing bump, call if F_1. Decompose F_1' into its even and odd parts, F'_{1e} and F'_{1o} respectively. Then only F'_{1e} contributes to the integral in Eq. (4.29). F'_{1e} will vanish at two points, say $\pm$ y, y > 0. Furthermore,

(4.30a) $$F'_{1e}(u) < 0 \text{ for } |u| < y$$

and

(4.30b) $$F'_{1e}(u) > 0 \text{ for } |u| > y \quad .$$

We observe that

(4.31) $$0 = -\frac{u^3\sigma^2}{y^2+u^2} \int_{-\infty}^{\infty} F'_{1e}(s) ds \quad .$$

By adding Eq (4.29) and Eq (4.31) we obtain

$$(4.32) \qquad \operatorname{Im}\Lambda(iu) = \sigma^2u^3 \int_{-\infty}^{\infty} F_{1e}'(s)\{\frac{1}{s^2+u^2} - \frac{1}{y^2+u^2}\} \; ds$$

$$= \sigma^2u^3 \int_{-\infty}^{\infty} \frac{(y^2-s^2)F_{1e}'(s)}{(s^2+u^2)(y^2+u^2)} \; ds$$

Thus, from the inequalities (4.30 a and b) it is easy to see that the integrand in Eq. (4.32) is always negative, completing the proof of the Lemma for Case 1. For Case 2, everything is the same with the exception of the inequalities (4.30a and b) which become

$$(4.33a) \qquad F_{1e}'(u) < 0 \quad \text{for} \quad |u| < y$$

and

$$(4.33b) \qquad F_{1e}'(u) > 0 \quad \text{for} \quad |u| > y \quad .$$

Thus, the integrand in Eq. (4.32 is always positive, completing the proof of the Lemma for Case 2.

We now let

$$(4.34) \qquad M = \Lambda(u_0)\Lambda(u_1)\Lambda(u_2) \quad .$$

We then have

<u>Theorem 4.2</u>

1. $1 - \sigma^2 > 0$

a) $M > 0$ then for Case 1, 2 and 3, Λ has no zeros

b) $M < 0$ then for Case 1, Λ has two zeros in the left half plane

for Case 2, Λ has two zeros in the right half plane

for Case 3, Λ has two zeros

2. $1 - \sigma^2 < 0$

a) $M > 0$ then for Case 1, Λ has two zeros in the right half plane

for Case 2, Λ has two zeros in the left half plane

for Case 3, Λ has two zeros

b) $M < 0$ then for Case 1 and 2, Λ has two zeros in both the left and right half planes

for Case 3, Λ has either no zeros or four zeros

Proof. We shall utilize the same technique as outlined in Sec. 2 of Chapter 2 relying on the argument principle and Nyquist diagrams for Λ. Again, from Eq. (4.22), we see that $\overline{\Lambda(\rho)} = \Lambda(\bar{\rho})$, thus the zeros of Λ occur in complex conjugate pairs and we need only consider the zeros of Λ in the upper half plane. We shall draw the Nyquist diagram for $\Lambda(\rho)$ as ρ proceeds from $-\infty$ to $+\infty$ just above the real axis and closes in a semicircular arc in the upper half plane (see Fig. 2.1). Along the semicircular arc,

$$\Lambda(\rho) \to 1 - \sigma^2 = \text{constant as } |\rho| \to \infty \quad ,$$

so that portion of the contour makes no change in the argument of Λ.

We show in Figs. 4.1a, 4.1b, 4.2a and 4.2b representative Nyquist diagrams corresponding to situations 1a, 1b, 2a and 2b respectively in Case 1. Note that from Eq. (4.27), the contour in the Λ plane crosses the real axis at $\pm\infty$, u_0, u_1, u_2, and 0. Also, since Λ is analytic in the upper half plane, we must insure that the bounded component of the Λ plane is to the left as one transverses the contour (otherwise the diagram obtained represents a function that has a pole in the upper half plane).

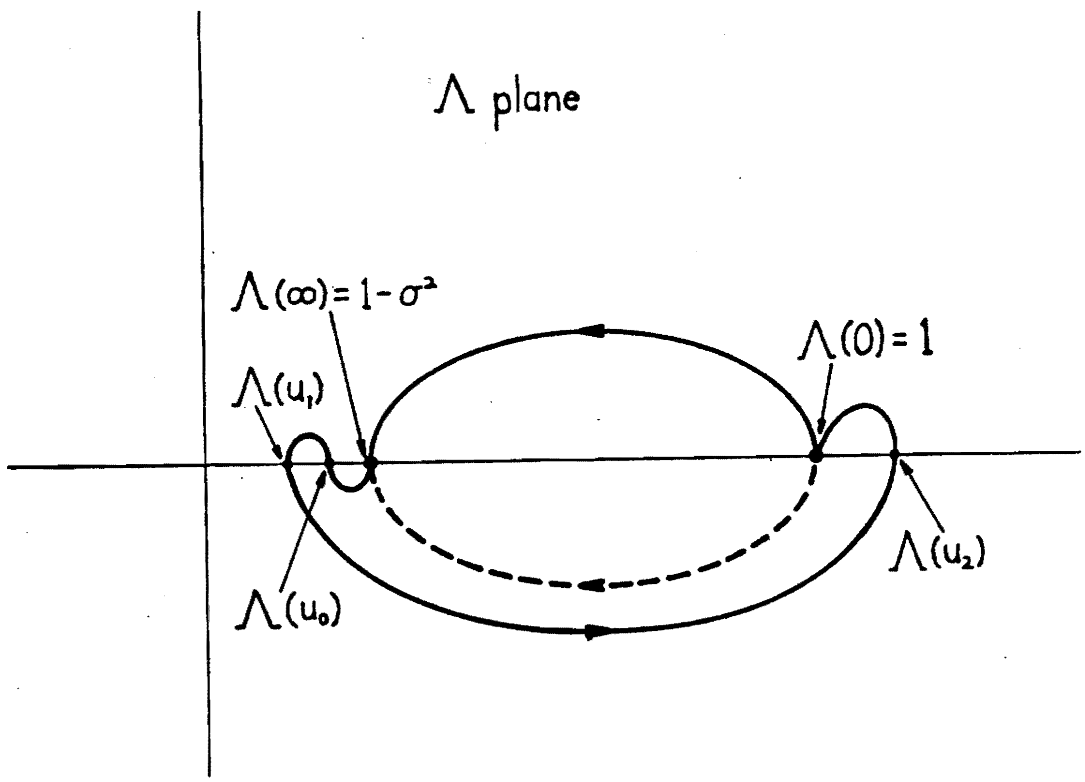

Fig.(4.1a) Nyquist Diagram for $1-\sigma^2 > 0$, $M > 0$, Case 1

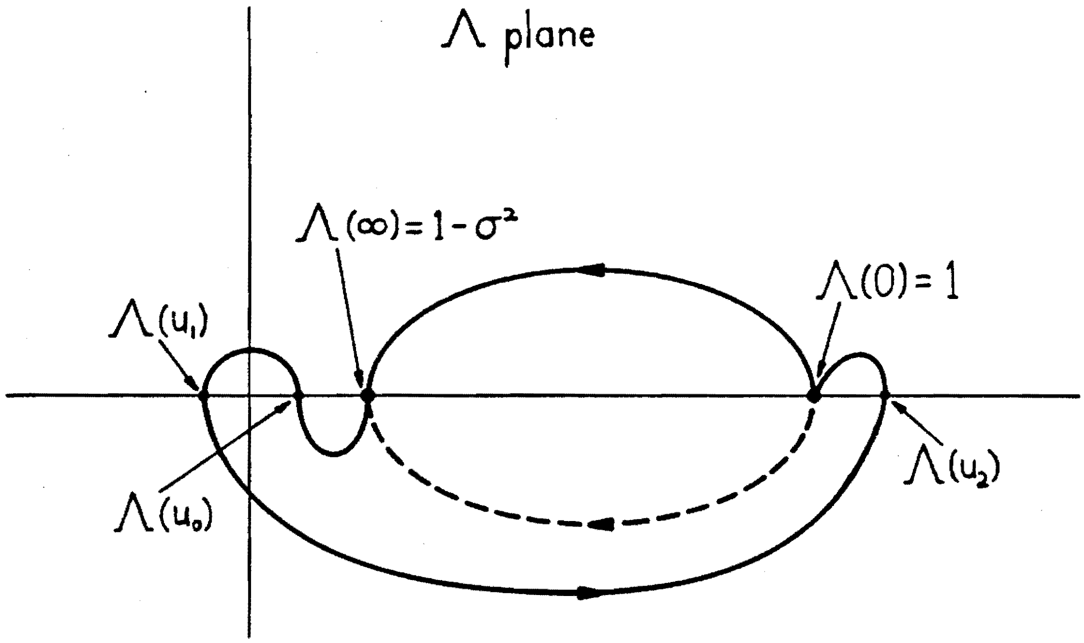

Fig. (4.1b) Nyquist Diagram for $1-\sigma^2 > 0$, $M < 0$, Case 1

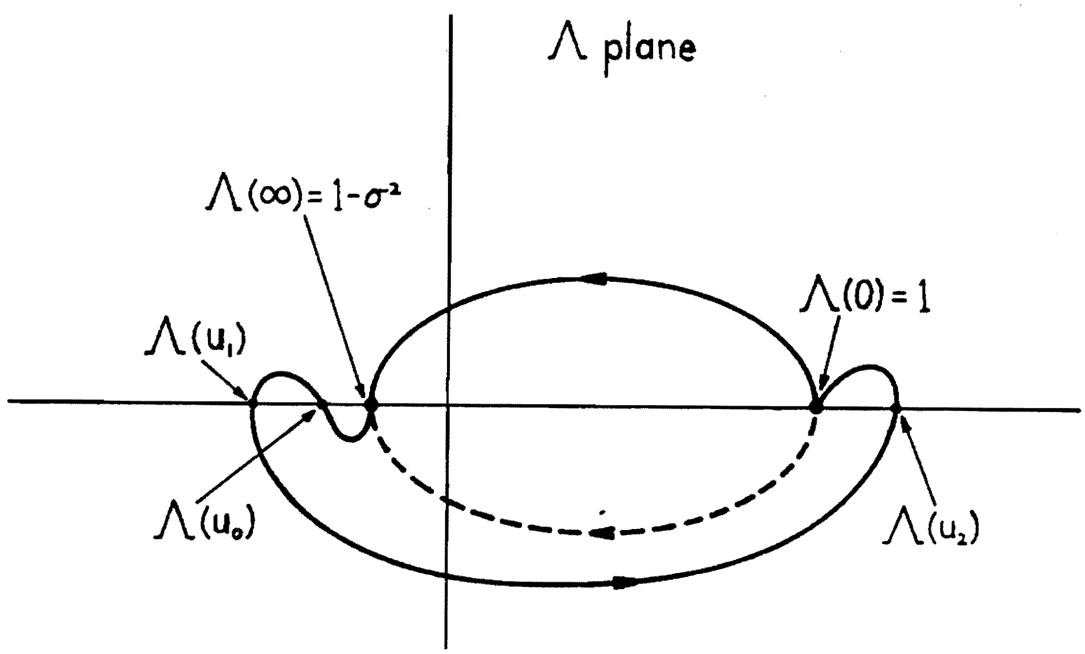

Fig. (4.2a) Nyquist Diagram for $1-\sigma^2 < 0$, $M > 0$, Case 1

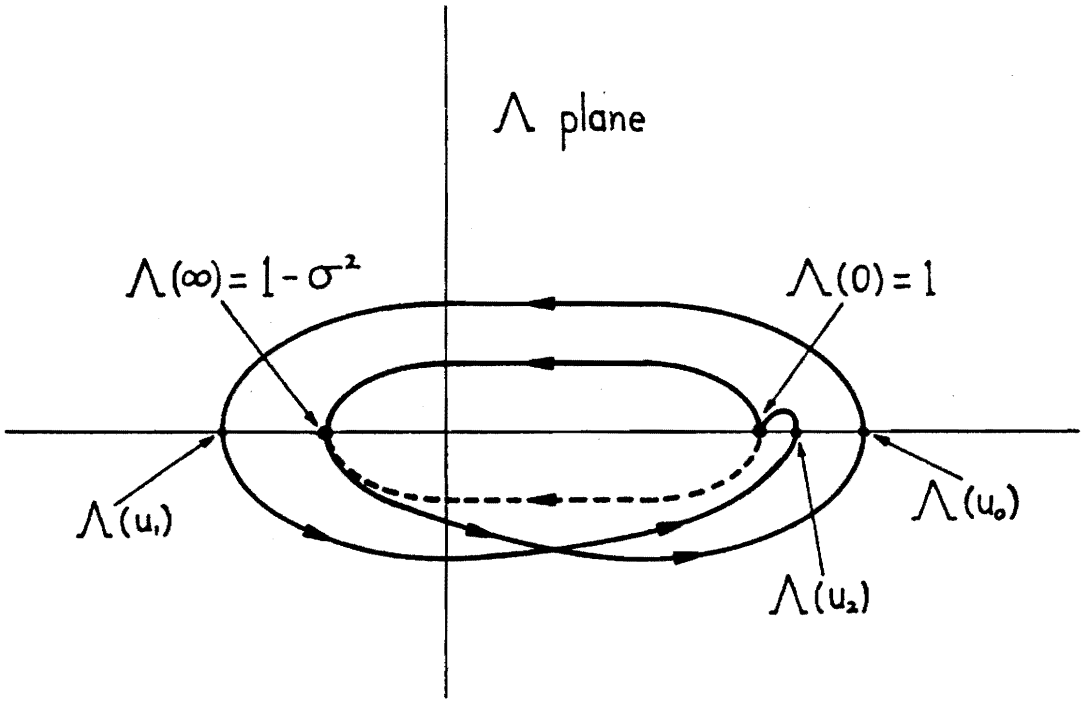

Fig. (4.2b) Nyquist Diagram for $1-\sigma^2 < 0$, $M < 0$, Case 1

To prove that the zeros are located in the stated half plane in each of the situations 1a, 1b, 2a and 2b for Case 1, consider Lemma 4.3 and follow the solid curve in the Λ plane as u advances from $-\infty$ to 0 just above the real axis and follow the dashed curve in the Λ plane as i u advances from 0 to ∞ upward along the imaginary axis. This closed contour encloses the origin of the Λ plane in Figs. 4.1b and 4.2b but not in Figs. 4.1a and 4.2a thus proving the assertions of the Theorem for Case 1. Cases 2 and 3 can be analized in the same fashion, completing the proof of the Theorem.

We note that real zeros can occur, but only at u_0, u_1 or u_2, where Im Λ vanishes, or at ∞ in the special case $\sigma^2 = 1$.

For an anisotropic plasma, the factorization of Refs. 2 and 5 is not applicable to the equilibrium distribution functions we consider. We require

$$\Lambda(\rho) = X(\rho)Y(-\rho) \tag{4.35}$$

with X and Y analytic for $\mathrm{Re}\rho<0$. Furthermore, if we let ν_r and ν_ℓ represent the zeros of Λ in the right and left half planes respectively, then we also require

$$X(\nu_r) = 0 \text{ for } \mathrm{Re}\ \nu_r>0 \ ,$$

$$Y(-\nu_\ell) = 0 \text{ for } \mathrm{Re}\ \nu_\ell<0$$

If Λ has no zeros, then the following functions are immediately seen to factor Λ

$$X_0(\rho) = (1-\sigma^2)^{\frac{1}{2}} \exp \frac{1}{2\pi i} \int_0^\infty \ln\frac{\Lambda^+(s)}{\Lambda^-(s)} \frac{ds}{s-\rho} \ , \tag{4.36a}$$

$$Y_0(\rho) = (1-\sigma^2)^{\frac{1}{2}} \exp\{-\frac{1}{2\pi i} \int_0^\infty \ln\frac{\Lambda^+(-s)}{\Lambda^-(-s)} \frac{ds}{s-\rho}\} \quad . \tag{4.36b}$$

To include the zeros of Λ, X_0 and Y_0 must be modified to

$$X_1(\rho) = (\rho-\nu_r)(\rho-\bar{\nu}_r)\ X_0(\rho) \quad , \tag{4.37a}$$

$$Y_1(-\rho) = (\rho-\nu_\ell)(\rho-\bar{\nu}_\ell)\ Y_0(-\rho) \quad . \tag{4.37b}$$

X_1 and Y_1 are still not adequate since $\Lambda(\rho) \to 1-\sigma^2$ as $\rho \to \infty$ and the product $X_1(\rho)Y_1(-\rho)$ diverges as ρ^n ($n = 2$ or 4). Thus, X_1 and Y_1 must be modified to

(4.38a) $$X(\rho) = X_1(\rho) / \rho^{\epsilon_1}$$

(4.38b) $$Y(-\rho) = Y_1(-\rho)/\rho^{\epsilon_2}$$

where $\epsilon_1(\epsilon_2)$ is either 2 or 0 depending on whether Λ does or does not have a zero in the right (left) half plane.

To verify that X and Y have no pole at $\rho = 0$, we must determine the behavior of $X_0(0)$ and $Y_0(0)$. Since $\rho \sim 0$, the largest contribution in Eqs. (4.36) comes from $s \sim 0$, so that we cut off the range of integration at, say $a > 0$. Then a simple calculation shows for $\rho \to 0$,

(4.39a) $$X_0(\rho) \sim \left(\frac{\rho-a}{\rho}\right)^{\theta_1(0)/\pi} \sim \rho^{-\theta_1(0)/\pi}$$

(4.39b) $$Y_0(\rho) \sim \left(\frac{\rho+a}{\rho}\right)^{-\theta_2(0)/\pi} \sim \rho^{\theta_2(0)/\pi}$$

where

(4.40a) $$\theta_1(s) = \tfrac{1}{2}\ln\left[\Lambda^+(s) / \Lambda^-(s)\right] , \quad s > 0 \quad ,$$

(4.40b) $$\theta_2(s) = \tfrac{1}{2}\ln\left[\Lambda^+(-s)/\Lambda^-(-s)\right] , \quad s > 0 \quad ,$$

We observe that

$$\theta_1(0) = \Delta_{(0,\infty)} \arg\Lambda^+$$

$$= \Delta_{(0,\infty)} \arg\Lambda^+ + \Delta_{(0,+i\infty)} \arg\Lambda \quad ,$$

where the second equality follows from Lemma 4.3. Similarly,

$$\theta_2(0) = \Delta_{(-\infty,0)} \arg\Lambda^+$$

$$= \Delta_{(-\infty,0)} \arg\Lambda^+ + \Delta_{(0,+i\infty)} \arg\Lambda \quad ,$$

where again we have used Lemma 4.3. Clearly, if the root occurs in the left half plane, $\theta_1(0) = 0$ and $\theta_2(0) = 2\pi$, whereas if there is a root in the right half plane, $\theta_1(0) = -2\pi$ and $\theta_2(0) = 0$. The results are summarized in Table I. Comparing these results with our definition of ϵ_1 and ϵ_2 we have

(4.41a) $$\epsilon_1 = -\theta_1(0)/\pi \quad ,$$

(4.41b) $$\epsilon_2 = \theta_2(0) / \pi \quad .$$

TABLE I
Values of $\theta_1(0)$ and $\theta_2(0)$

		Case 1	Case 2
1. $1-\sigma^2>0$			
	a) $M>0$	$\theta_1(0) = 0$ $\theta_2(0) = \pi$	$\theta_1(0) = 0$ $\theta_2(0) = 0$
	b) $M<0$	$\theta_1(0) = 0$ $\theta_2(0) = 2\pi$	$\theta_2(0) = -2\pi$ $\theta_2(0) = 0$
2. $1-\sigma^2<0$			
	a) $M>0$	$\theta_1(0) = -2\pi$ $\theta_2(0) = 0$	$\theta_1(0) = 0$ $\theta_2(0) = 2\pi$
	b) $M<0$	$\theta_1(0) = -2\pi$ $\theta_2(0) = 2\pi$	$\theta_1(0) = -2\pi$ $\theta_2(0) = 2\pi$

We see, incidentally, that the existence of a zero of Λ in the right or left half plane induces a double zero in X_0 or Y_0 respectively, so that the notation in Eqs. (4.37) is appropriate.

The result, Eqs. (4.41), directly implies the following

Theorem 4.3. The functions X and Y defined by Eqs. (4.37), (4.38) and (4.39) constitute a Wiener-Hopf factorization of Λ given by Eq. (4.35) and

$$X(\rho) \sim \text{constant as } \rho \to 0 \quad ,$$

$$Y(\rho) \sim \text{constant as } \rho \to 0 \quad .$$

Without loss of generality we assumme that $X(0) = Y(0) = 1$.

For computational purposes, the explicit representation of X and Y may not be so convenient as the iterative solution of coupled integral equations. These may easily be determined from Cauchy's theorem. In particular, from Eq. (4.35)

$$(4.42a) \qquad X^+(u) - X^-(u) = \frac{1}{Y(-u)}\left[\Lambda^+(u)-\Lambda^-(u)\right] , \quad u > 0 \quad ,$$

$$(4.42b) \qquad \left[Y(-u)\right]^+ - \left[Y(-u)\right]^- = \frac{1}{X(u)}\left[\Lambda^+(u)-\Lambda^-(u)\right] , \quad u < 0 \quad .$$

Using Eq. (4.27) and the behavior of X and Y at infinity, Cauchy's theorem yields

$$(4.43a) \qquad X(\rho) = (1-\sigma^2)^{\frac{1}{2}} - \int_0^\infty \frac{s^2\sigma^2 F'(s)}{Y(-s)(s-\rho)} \, ds$$

$$(4.43b) \qquad Y(-\rho) = (1-\sigma^2)^{\frac{1}{2}} - \int_{-\infty}^0 \frac{s^2\sigma^2 F'(s)}{X(s)(s-\rho)} \, ds$$

These equations can be solved iteratively for the values of $X(\rho)$ and $Y(-\rho)$. A more convenient iteration scheme is defined by taking the limit as $\rho \to 0$ in Eqs.(4.43). Then

$$(4.44a) \qquad 1 = (1-\sigma^2)^{\frac{1}{2}} - \int_0^\infty s\sigma^2 F'(s)/Y(-s) \, ds \quad ,$$

$$(4.44b) \qquad 1 = (1-\sigma^2)^{\frac{1}{2}} - \int_{-\infty}^0 s\sigma^2 F'(s)/X(s) \, ds \quad ,$$

and rewriting Eq. (4.43) as

$$X(-\rho) = 1 + \rho\sigma^2 \int_0^\infty \frac{sF'(s)}{Y(-s)(s+\rho)}\, ds \quad , \tag{4.45a}$$

$$Y(-\rho) = 1 - \rho\sigma^2 \int_0^\infty \frac{sF'(-s)}{X(-s)(s+\rho)}\, ds \quad . \tag{4.45b}$$

If we make the following change of dependent variable

$$U_1(\rho) = X^{-1}(-\rho)\rho\sigma^2 F'(-\rho) \quad , \tag{4.46a}$$

$$U_1(\rho) = -Y^{-1}(-\rho)\rho\sigma^2 F'(\rho) \quad , \tag{4.46b}$$

then Eq. (4.43) reduces to the bilinear matrix equation

$$\mathcal{U} = \mathcal{F} + \mathcal{A}(\mathcal{U},\mathcal{U}) \quad , \tag{4.47a}$$

$$\mathcal{U} = [U_1, U_2] \quad , \tag{4.47b}$$

$$\mathcal{F}(\rho) = \rho\sigma^2 [F'(-\rho), -F'(\rho)] \quad , \tag{4.47c}$$

$$\mathcal{A}(\mathcal{U},\mathcal{V})(\rho) = \left[-\rho \int_0^\infty V_1(\rho)U_2(s)\frac{ds}{s+\rho} \; , \; \rho \int_0^\infty U_1(s)V_2(\rho)\frac{ds}{s+\rho}\right] \quad . \tag{4.47d}$$

The convergence of the iteration scheme to Eq.(4.47a) has been studied previously.[35] If we define a Banach space with

$$||\mathcal{U}|| = \max_{i=1,2} \int_0^\infty |U_i(s)|\, ds \quad .$$

then it is shown by fixed point arguments that Eq.(4.47a) has a unique solution in the ball

$$S = \{\mathcal{U} : \| \mathcal{U} - \mathcal{F} \| < \tfrac{1}{2} \}$$

subject to the condition $\| \mathcal{F} \| < \frac{1}{2}$, and that an iteration scheme converges if the initial guess is chosen in S (note that if $\mathcal{U} \in S$ then $\| \mathcal{U} \| < 1$). We now show that the solution to Eq. (4.47a) lying in S is the "physical" solution. We observe that U_1 and U_2 obey

$$\text{(4.48a)} \qquad U_1(\rho) = F_1(\rho)/\left[1 - \int_0^\infty U_2(s)\frac{\rho}{s+\rho}\, ds\right]$$

$$\text{(4.48b)} \qquad U_2(\rho) = F_2(\rho)/\left[1 - \int_0^\infty U_1(s)\frac{\rho}{s+\rho}\, ds\right] \quad .$$

Consider Case 1 for situation 1b of Theorem 4.2. Then Y has zeros in the right half plane which implies that U_2 has poles in the left half plane. Thus U_2 must be analytic in the right half plane and U_1 analytic in $\mathbb{C}\backslash[-\infty,0]$. Since we are dealing with nonlinear integral equations which may have more than one solution, we must prove

Theorem 4.4. For Case 1, $1-\sigma^2 > 0$ and $M<0$, the solution to Eq. (4.47a), $\mathcal{U} = [U_1, U_2]$, in the ball S, is analytic in the right half plane.

Proof Writing $\rho=\alpha x+i\beta y$ we have

$$\left|\frac{\rho}{\rho+s}\right| = \left[\frac{\alpha^2+\beta^2}{(\alpha+s)^2+\beta^2}\right]^{\frac{1}{2}} < 1 \text{ for } 0 < s < \infty \text{ and } \alpha > 0 \quad ,$$

thus

$$\left| \int_0^\infty U_{1,2}(s)\frac{\rho}{\rho+s}\, ds\right| < \int_0^\infty |U_{1,2}(s)|\left|\frac{\rho}{\rho+s}\right| ds < 1 \quad .$$

The result follows from Eq. (4.48).

To get a feeling for the range of parameters for which an iteration scheme corresponding to Eq. (4.47a) converges, we have computed $\| F \|$ for a bump on tail distribution considered in Ref. 1 and given explicitly by

$$F(u) = (1-\beta)\left(\frac{m}{2\pi kT_1}\right)^{\frac{1}{2}}\exp\left(-\frac{mu^2}{2kT_1}\right) + \beta\left(\frac{m}{2\pi kT_2}\right)^{\frac{1}{2}}\exp\left[-\frac{m(u-V_0)^2}{2kT_2}\right], \tag{4.49}$$

A straightforward integration yields

$$\| F\| \leq \sigma^2 \left[\tfrac{1}{2}-\tfrac{1}{2}\beta\ E_2\left(\frac{mV_0^2}{2kT_2}\right)^{\frac{1}{2}} + 2\beta\left(\frac{mV_0^2}{2\pi kT_2}\right)^{\frac{1}{2}}\right] \tag{4.50}$$

where E_2 is the error function defined in Ref. 36. A more convenient, if less exact bound is

$$\| F\| \leq \sigma^2 \left[\tfrac{1}{2} + 2\beta\left(\frac{mV_0^2}{2\pi kT_2}\right)^{\frac{1}{2}}\right] , \tag{4.51}$$

We conclude that for certain values of σ^2, β, V_0 and T_2, an iteration scheme converges if the initial guess is chosen in S. For values outside this range, it is necessary to evaluate X and Y from the explicite definitions. We now develop these definitions into a form more useful for computation by a procedure similar to one used in Ref. 5, p. 130.

From Eqs. (4.36), (4.37) and (4.38) we have

$$\text{(4.52a)} \qquad X(\rho) = (\rho-\nu_r)(\rho-\bar{\nu}_r)\rho^{-\epsilon_1}(1-\sigma^2)^{\frac{1}{2}}\exp\Big[\frac{1}{\pi}\int_0^{\infty}\frac{\theta(s)}{s-\rho}\,ds\Big] \quad ,$$

$$\text{(4.52b)} \qquad Y(-\rho) = (\rho-\nu_\ell)(\rho-\bar{\nu}_\ell)\rho^{-\epsilon_2}(1-\sigma^2)^{\frac{1}{2}}\exp\Big[\frac{1}{\pi}\int_{-\infty}^{0}\frac{\theta(s)}{s-\rho}\,ds\Big] \quad ,$$

where

$$\text{(4.53)} \qquad \theta(s) = \arg\Lambda^+(s) = \tan^{-1}\Big[\frac{-\pi\sigma^2 s^2 F'(s)}{\lambda(s)}\Big] ,$$

and

$$\text{(4.54)} \qquad \lambda(s) = \tfrac{1}{2}\Big[\Lambda^+(s) + \Lambda^-(s)\Big] \ .$$

It is useful to write in Eq. (4.52)

$$\text{(4.55)} \qquad \int_a^{\infty}\frac{\theta(s)}{s-\rho}\,ds = \int_0^{\infty}\theta(s)\frac{d}{ds}\ln(s-\rho)\,ds \quad ,$$

and integrating by parts

$$\text{(4.56a)} \qquad \int_0^{\infty}\frac{\theta(s)}{s-\rho}ds = \epsilon_1\ln(-\rho) - \int_0^{\infty}\frac{d\theta}{ds}\ln\,(s-\rho)\,ds \quad ,$$

$$\text{(4.56b)} \qquad \int_{-\infty}^{0}\frac{\theta(s)}{s-\rho}ds = \epsilon_2\ln(-\rho) - \int_{-\infty}^{0}\frac{d\theta}{ds}\ln(s-\rho)\,ds \quad .$$

Here we have used Eq. (4.41) and the fact that $\theta(\infty) = 0$. Calculating $d\theta/ds$ from Eq. (4.53) and using Eqs. (4.52) and (4.56), we obtain after some algebra

$$X(\rho) = (\rho-\nu_r)(\rho-\bar{\nu}_r)(1-\sigma^2)^{\frac{1}{2}} \times \exp\{-\frac{1}{\pi}\int_0^\infty \mathrm{Im}\left[\frac{\Lambda^{+\prime}(s)}{\Lambda^+(s)}\right]\ln(s-\rho)\ ds\} \quad , \tag{4.57a}$$

$$Y(-\rho) = (\rho-\nu_\ell)(\rho-\bar{\nu}_\ell)(1-\sigma^2)^{\frac{1}{2}} \times \exp\{-\frac{1}{\pi}\int_{-\infty}^0 \mathrm{Im}\left[\frac{\Lambda^{+\prime}(s)}{\Lambda^+(s)}\right]\ln(s-\rho)\ ds\} \quad . \tag{4.58b}$$

Section 4 Resolvent of the Half Range Transport Operator

We shall construct an operator, K', whose resolution of the identity will yield the appropriate half range expansion. We define

$$(4.59) \qquad X_2' = \{h \in L_1(\mathbb{R}_+) : \int_0^\infty (1+|u|)|h(u)| \, du < \infty \text{ and } h \text{ is Hölder continuous on compact subsets of } \mathbb{R}_+\}$$

and

$$(4.60) \qquad X_2 = \{h \in L_1(\mathbb{R}) : \int_{-\infty}^\infty (1+|u|)|h(u)| \, du < \infty \text{ and } h \text{ is Hölder continuous on compact subsets of } \mathbb{R}\}$$

We construct an operator $E : X_2' \to X_2$ such that for $h \in X_2'$

$$(4.61) \qquad (Eh)(u) = h(u) \quad , \quad u > 0$$

and

$$(4.62) \qquad (K-\tfrac{1}{\rho})^{-1}(Eh)(u) \text{ is analytic in } \rho \text{ for } \operatorname{Re}\rho < 0.$$

The motivation for constructing such an operator is that integrals of $(K-\frac{1}{\rho})^{-1}(Eh)$ will be zero for $\operatorname{Re}\rho < 0$ and $\rho = \nu_\ell$ (where ν_ℓ denotes the zeros of Λ such that $\operatorname{Re}\rho < 0$). Thus, the full range expansion of Eh will contain no "negative" eigenfunctions and therefore (since Eh is an extension of h) must reduce to the half range expansion of h for $u > 0$. By analogy with Ref. 16 we will then have $K' = KE$.

Now assume that the operator E exists satisfying Eqs. (4.61) and (4.62). In order to derive E, we let

$$T(\rho) = \int_{-\infty}^{\infty} \frac{s^2 (Eh)(s)}{s - \rho} \, ds \tag{4.63}$$

and

$$G(\rho) = T(\rho)/\Lambda(\rho) \tag{4.64}$$

Then by the resolvent formula, Eq. (4.26),

$$(K - \tfrac{1}{\rho})^{-1} (Eh)(u) = (\rho - u)^{-1} \left[\rho u (Eh)(u) + \rho^2 \sigma^2 F'(u) G(\rho) \right] \quad . \tag{4.65}$$

This is analytic for $\mathrm{Re}\,\rho < 0$ if, and only if

$$T(\nu_\ell) = 0 \tag{4.66}$$

and

$$G^+(u) = G^-(u) = (Eh)(u) / \left[\sigma^2 F'(u) \right] , \quad u < 0 \tag{4.67}$$

where

$$G^{\pm}(u) = \lim_{\epsilon \to 0^+} G(u \pm i\epsilon) \quad . \tag{4.68}$$

The first part of Eq. (4.67) can be put into the form

$$(4.69) \qquad 0 = \frac{T^+(u)}{\Lambda^+(u)} - \frac{T^-(u)}{\Lambda^-(u)} \quad , \quad u < 0$$

At this point we utilize the results of Sec. 3 whereby the functions

$$(4.70) \qquad X(\rho) = (\rho-\nu_r)(\rho-\bar{\nu}_r)\rho^{-\epsilon_1}(1-\sigma^2)^{\frac{1}{2}}$$

$$\times \exp\left[\frac{1}{2\pi i}\int_0^\infty \ln\frac{\Lambda^+(s)}{\Lambda^-(s)}\frac{ds}{(s-\rho)}\right]$$

and

$$(4.71) \qquad Y(-\rho) = (\rho-\nu_\ell)(\rho-\bar{\nu}_\ell)\rho^{-\epsilon_2}(1-\sigma^2)^{\frac{1}{2}}$$

$$\times \exp\left[-\frac{1}{2\pi i}\int_0^\infty \ln\frac{\Lambda^+(-s)}{\Lambda^-(-s)}\frac{ds}{(s-\rho)}\right]$$

constitute a Wiener-Hopf factorization of Λ given by Eq. (4.35). The important point is that Y carries the zeros of Λ in the left half plane, so that in order to accomodate Eq. (4.66), Eq. (4.69) becomes

$$(4.72) \qquad 0 = \frac{T^+(u)}{[Y(-u)]^+} - \frac{T^-(u)}{[Y(-u)]^-} \quad , \quad u < 0$$

Now consider the function Q defined as

$$(4.73) \qquad Q(\rho) = \frac{T(\rho)}{Y(-\rho)} - \int_0^\infty \frac{s^2 h(s)}{Y(-s)}\frac{ds}{s-\rho} \quad .$$

Q is bounded as $\rho \to \pm\infty$ and 0 (Theorem 4.3). Also, utilizing Eqs.

(4.63) and (4.72), one can show that Q is continuous across the cut $(-\infty,\infty)$. Q is bounded near $\rho = \nu_\ell$ by virtue of Eq. (4.66). Finally, $Q(\rho) \to 0$ as $\rho \to \infty$. Therefore by Liouville's theorem, $Q \equiv 0$ and

$$(4.74) \qquad T(\rho) = Y(-\rho) \int_0^\infty \frac{s^2 h(s)}{Y(-s)} \frac{ds}{s-\rho}$$

We now determine (Eh)(u) for u < 0 by equating Eq. (4.63) and Eq. (4.74), and using the Plemlj formula[32] and Eq.(4.35). The result is

$$(4.75) \qquad (Eh)(u) = \frac{\sigma^2 F'(u)}{X(u)} \int_0^\infty \frac{s^2 h(s)}{Y(-s)} \frac{ds}{s-u}, \quad u < 0 \ .$$

Equations (4.75) and (4.62) define the operator E. We must however show that E satisfies the full equation (4.67). To do this, we divide Eq. (4.74) by $\Lambda(\rho)$ and use Eqs. (4.35) and (4.64) to obtain

$$(4.76) \qquad G(\rho) = \frac{1}{X(\rho)} \int_0^\infty \frac{s^2 h(s)}{Y(-s)} \frac{ds}{s-\rho}$$

Now that E has been determined, we can derive the resolvent of $K' = KE$. Combining Eqs.(4.61) and (4.65) we have for u > 0

$$(4.77) \qquad (K' - \tfrac{1}{\rho})^{-1} h(u) = (\rho-u)^{-1}\left[\rho u h(u) + \rho^2\sigma^2 F'(u) G(\rho)\right]$$

where $G(\rho)$ is given by Eq. (4.78).

For use in later computation we will require the boundary values of G, Eq.(4.68), along the cut $(0,\infty)$. These are given by

$$G^{\pm}(u)X^{\pm}(u) = \int_{0}^{\infty} P \frac{s^2h(s)}{Y(-s)} \frac{ds}{s-u} \pm i\pi u \; h(u)/Y(-u) \quad . \tag{4.78}$$

using this and recalling Eqs. (4.27) and (4.35), we obtain after some algebra

$$\tfrac{1}{2}\sigma^2F'(u)\left[G^{+}(u) + G^{-}(u)\right] = -h(u) + \lambda(u)\frac{1}{2\pi i}\left[\frac{G^{+}(u)-G^{-}(u)}{u^2}\right] \tag{4.79}$$

where λ is given by Eq. (4.54). In the next section we integrate the resolvent, Eq. (4.77), around the spectrum of K' and obtain the eigenfunction expansion for the operator K'. Equation (4.79) plays an important role in this analysis.

Section 5 Resolution of the Identity for the Half Range Operator

We apply the identity

$$h(u) = -\frac{1}{2\pi i}\oint (K'-\rho)^{-1}h(u)\, d\rho \tag{4.80}$$

where the contour of integration surrounds the spectrum of K'. First we change the integration variable $\rho \to 1/\rho$ so that Eq. (4.80) becomes

$$h(u) = \frac{1}{2\pi i}\oint (K'-\tfrac{1}{\rho})^{-1}h(u)\rho^{-2}\, d\rho \tag{4.81}$$

$$= \frac{1}{2\pi i}\oint \frac{uh(u)}{\rho(\rho-u)}\, d\rho + \sigma^2 F'(u)\,\frac{1}{2\pi i}\oint \frac{G(\rho)}{\rho - u}\, d\rho$$

where we have used Eq. (4.77) in the second equality.

In the first integral of Eq.(4.81) we note that there is apparently a pole at zero. To see that this is not the case, we change the integration variable in this term back from $1/\rho$ to ρ, obtaining

$$h(u) = \frac{1}{2\pi i}\int \frac{h(n)}{\rho-1/u}\, d\rho + \sigma^2 F'(u)\,\frac{1}{2\pi i}\oint_\Gamma \frac{G(\rho)}{\rho - u}\, d\rho \tag{4.82}$$

where the contour Γ is given by

$$\Gamma = \Gamma_+ \cup \Gamma_- \cup \Gamma_{u+} \cup \Gamma_{u-} \cup \Gamma_{\nu_r} \cup \Gamma_{\bar{\nu}_r} \cup \Gamma_{u_j+} \cup \Gamma_{u_j-} \tag{4.83}$$

with its components shown in Fig 4.3.

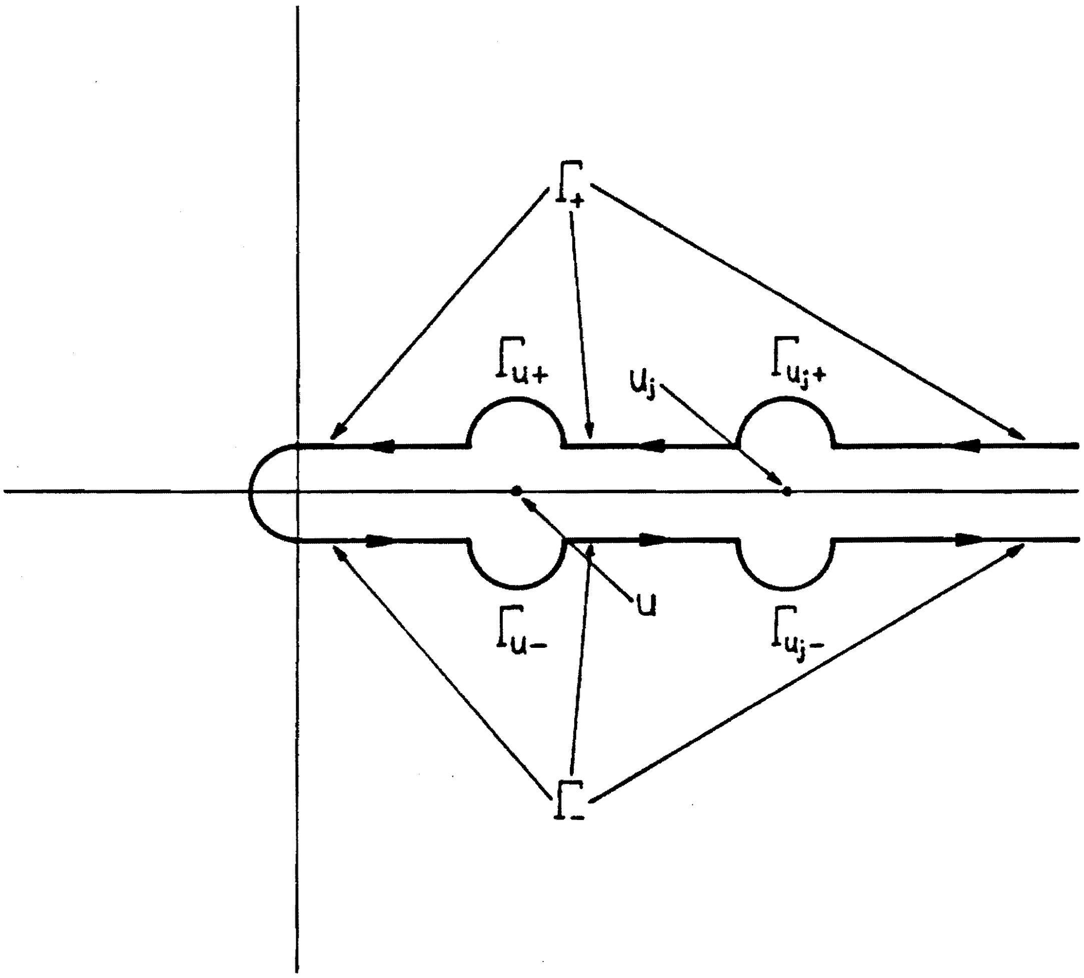

Fig. 4.3 The Components of the Contour Γ

Thus, Eq. (4.82) becomes

$$h(u) = h(u) + \tfrac{1}{2}\sigma^2 F'(u)\left[G^+(u) + G^-(u)\right] \tag{4.84}$$

$$+ \sigma^2 F'(u)\,\frac{1}{2\pi i}\int_0^\infty P\,\frac{G^+(\nu) - G^+(\nu)}{\nu - u}\,d\nu$$

$$+ \sigma^2 F'(u)\{(\nu_r-u)^{-1}\mathrm{Res}\left[G^+(\rho),\nu_r\right]$$

$$+(\bar{\nu}_r-u)^{-1}\mathrm{Res}\left[G(\rho),\bar{\nu}_r\right]\}$$

$$+ \sigma^2 F'(u)\{(u_j-u)^{-1}\tfrac{1}{2}\mathrm{Res}\left[G^+(u),u_j\right]$$

$$+(u_j-u)^{-1}\tfrac{1}{2}\mathrm{Res}\left[G^-(u),u_j\right]\}$$

We now apply Eq. (4.79) and evaluate the residues for simple complex eigenvalues, ν_r and $\bar{\nu}_r$, and for simple and second order real eigenvalues, $u_j > 0$. The result is

(4.85) $$h(u) = \int_{\infty}^{\infty} A(\nu)\phi_\nu(u)\, d\nu + \sum_{\nu=\nu_r\bar{\nu}_r} A_\nu \phi_\nu(u)$$

$$+ \sum_j A_{u_j}\phi_{u_j}(u)$$

$$+ \sum_j A^{(1)}_{u_j}\phi^{(1)}_{u_j}(u) + A^{(2)}_{u_j}\phi^{(2)}_{u_j}(u)$$

Where for continuum eigenvalues we have

(4.86) $$\phi_\nu(u) = P\,\frac{\sigma^2\nu^2F'(u)}{u-\nu} + \lambda(\nu)\delta(\nu-u)$$

(4.87) $$A(\nu) = \frac{1}{\Lambda^+(\nu)\Lambda^-(\nu)W(\nu)}\int_0^\infty \phi_\nu(u)W(u)h(u)\, du$$

(4.88) $$W(u) = \frac{u^2}{F'(u)Y(-u)}$$

and for simple complex eigenvalues we have

(4.89) $$\phi_{\nu_r}(u) = \frac{\sigma^2 \nu_r^2 F'(u)}{u - \nu_r}$$

(4.90) $$A_{\nu_r} = \frac{\int_o^\infty \phi_{\nu_r}(u) W(u) h(u) du}{\int_o^\infty \phi_{\nu_r}^2 (u) W(u) du}$$

and for simple real eigenvalues we have

(4.91) $$\phi_{u_j}(u) = \frac{\sigma^2 u_j^2 F'(u)}{u - u_j}$$

(4.92) $$A_{u_j} = \frac{1}{2u_j^2} \left[\frac{X^{+\prime}(u_j) + X^{-\prime}(u_j)}{X^{+\prime}(u_j) X^{-\prime}(u_j)} \right] \int_o^\infty P \frac{u^2 h(u)}{Y(-u)} \frac{du}{u-u_j}$$

$$+ \left[\frac{X^{-\prime}(u_j) - X^{+\prime}(u_j)}{X^{+\prime}(u_j) X^{-\prime}(u_j)} \right] \frac{i\pi h(u_j)}{2Y(-u_j)}$$

and for second order real eigenvalues we have

(4.93) $$\phi_{u_j}^{(1)}(u) = \phi_{u_j}(u)$$

(4.94) $$\phi_{u_j}^{(2)}(u) = \frac{\sigma^2 u_j^2 F'(u)}{(u-u_j)^2}$$

(4.95) $$A_{u_j}^{(1)} = - \tfrac{2}{3} \mathrm{Re}\{3 \left[u_j^{-2} \int_o^\infty P \frac{u^2 h(u)}{Y(-u)} \frac{du}{u-u_j} + \frac{2\pi i h(u_j)}{u_j Y(-u_j)} \right.$$

$$\left. + \frac{i\pi h'(u_j)}{Y(-u_j)} + \frac{i\pi h(u_j) Y'(-u_j)}{\left[Y(-u_j) \right]^2} \right] \left[X^{+\prime\prime}(u_j) \right]^{-2}$$

$$- \left[u_j^{-2} \int_0^\infty P \frac{u^2 h(u)}{Y(-u)} \frac{du}{u-u_j} + \frac{i\pi h(u_j)}{Y(-u_j)} \right] X^{+'''}(u_j) \left[X^{+''}(u_j) \right]^{-2} \}$$

(4.96) $$A_{u_j}^{(1)} = -\mathrm{Re}\{ \left[u_j^{-2} \int_0^\infty P \frac{u^2 h(u)}{Y(-u)} \frac{du}{u-u_j} + \frac{i\pi h(u_j)}{Y(-u_j)} \right] \left[X^{+''}(u_j) \right]^{-1} \} .$$

The spatial dependence of the second order modes, Eqs. (4.93) and (4.94), must be treated carefully (see Sec. 6). We now summarize the results of this section as

Theorem 4.5. Assume $\sigma \neq 1$ and let $h \in X_2'$. Then the half range expansion, Eq. (4.85), holds for $u > 0$ with coefficients A given by Eqs. (4.87), (4.90), (4.92), (4.95) and (4.96) in terms of h.

Section 6 Solution of the Boundary Value Problem

A solution of the differential eauation (4.19) is understood to be a differentiable function

$$f(\cdot,u) : \mathbb{R}_+ \to X_2 \tag{4.97}$$

with values in the Banach space X_2 defined in Eq. (4.60). The solution to the inhomogeneous equation (4.19) can be determined by direct calculation and is found to be

$$f_p(u) = \frac{\sigma^2}{1-\sigma^2}\frac{1}{4\pi e} E_A F'(u) \quad . \tag{4.98}$$

To this particular solution we must add the homogeneous solution which, by construction, is found using the half range expansion formula of Sec. 5, including the exponential factors of the form $\exp(-i\omega z/\rho)$. Thus, we may write the general solution of Eq. (4.19) as

$$\begin{aligned} f(z,u) = f_p(u) &+ \int_0^\infty A(\nu)\phi_\nu(u)\exp(-i\omega z/\nu)\, d\nu \\ &+ \sum_r A_{\nu_r}\phi_{\nu_r}(u) \exp(-i\omega z/\nu_r) \\ &+ \sum_j A_{u_j}\phi_{u_j}(u) \exp(-i\omega z/u_j) \\ &+ \sum_j \{A^{(1)}_{u_j}\phi^{(1)}_{u_j}(u) + A^{(2)}_{u_j}\left[\phi^{(2)}_{u_j}(u) - i\omega z\phi^{(1)}_{u_j}(u)\right]\} \\ &\qquad \times \exp(-i\omega z/u_j) \end{aligned} \tag{4.99}$$

where we have included all possible cases of complex eigenvalues and simple and second order real eigenvalues cumulatively in this equation. The coefficients, A, are given in terms of a function, h (see Theorem 4.5) which we must determine.

We now apply the boundary condition (1.10) which we rewrite here for convenience

$$f(0,u) = \alpha\, f(0,-u) + (1-\alpha) g(u)/u \times \int_{-\infty}^{0} |s| f(0,s)\, ds \,, \quad u > 0. \tag{4.100}$$

Setting z equal to zero in Eq.(4.99), using Eq.(4.100) and subtracting $f_p(u)$ we arrive at the expansion formula (4.85) on the right hand side and on the left hand side we have the function h given by

$$h(u) = \alpha f(0,-u) + (1-\alpha) g(u)/u \int_{-\infty}^{0} |s| f(0,s)\, ds - f_p(u), \quad u > 0. \tag{4.101}$$

Note that the solution, f, still appears in this equation.

In order to avoid merely representing the solution in the form of an unsolved integral equation, we must either regard the values, f(0,u) for u < 0, as given (this would be analogous to an incident current), or specialize to the case of purely diffuse reflection ($\alpha = 0$). In the latter situation, we recall Eq.(1.11) so that h may be expressed as

(4.102) $$h(u) = C\,F(u) - f_p(u)$$

where the constant C is the ratio of the perturbed current to the equilibrium current given by

(4.103) $$C = \int_{-\infty}^{0} |s| f(0,s)\,ds \Big/ \int_{-\infty}^{0} |s| F(s)\,ds \quad .$$

Note that according to the linearization hypothesis ($f \ll F$), C must be a small number. Thus, the solution to the boundary value problem for purely diffuse reflection is given by Eq.(4.99) where the coefficients, A, defined in Eqs.(4.87), (4.92), (4.95) and (4.96) are determined with h defined by Eq.(4.102).

Chapter 5

Conclusion and General Considerations

We have carried out the resolvent integration technique in its application to longitudinal plasma waves and oscillations and transverse plasma oscillations. It was seen that the operators treated have continuous spectra consisting of the real line as well as point spectra associated with the zeros of the appropriate plasma dispersion function. Out treatment of the dispersion function with a more general, non-even equilibrium distribution function, F (hence valid for anisotropic plasmas), has led to the possibility of more plasma modes than had previously been considered. The techniques we employed make it much easier to calculate the expansion coefficients for these plasma modes associated with both real and complex zeros of the appropriate plasma dispersion function. In fact, our computation of the expansion coefficients for plasma modes corresponding to simple real zeros of the longitudinal, fixed wave number plasma dispersion function replaces previous such calculations based on the van Kampen-Case generalized eigenfunction expansion technique which were found to be in error.

We have further presented, for the first time, expansion coefficients for plasma modes corresponding to second order real zeros of the plasma dispersion function. This is not merely an academic exercise since there is good physical reason to suspect that such modes exist. Our calculations may easily be extended to higher order real and complex zeros of the plasma dispersion function by careful consideration of the residue formula, Eq. (1.15).

In our analysis of the longitudinal plasma wave problem, the use of Ampere's Law has eliminated consideration of the "zero eigenmode" that arises when only Gauss' Law is considered. It is interesting to note that had Ampere's Law been used in the study of longitudinal plasma oscillations instead of Gauss' Law, the resulting matrix transport operator would also have had a superfluous "zero eigenmode". This symmetry is in part due to the consistency of the Vlasov-Maxwell equations.

More important, mathematically, is our use of semigroup theory in proving existence and uniqueness theorems which places the study of plasma oscillations and waves on a rigorous foundation. Since the Banach spaces employed in these theorems may not be the most appropriate for some physical considerations, a further mathematical study may lead to more general Banach spaces for which the existence and uniqueness theorems are still valid.

In each of the three problems we studied, the solution to the relevant transport equation is expressed as

$$\phi(u,t) = -\frac{1}{2\pi i}\oint (K_k-\rho)^{-1}\phi(u,0)\exp(-ik\rho t)\, d\rho \tag{5.1a}$$

or

$$\phi(z,u) = -\frac{1}{2\pi i}\oint (K_\omega-\rho)^{-1}\phi(0,u)\exp(-i\omega\rho z)\, d\rho \quad . \tag{5.1b}$$

The integrals in Eqs. (5.1a and b) define the appropriate semigroup operators which we write as

$$U_k(t)\phi = -\frac{1}{2\pi i}\oint (K_k-\rho)^{-1}\phi\exp(-ik\rho t)\,d\rho \tag{5.2a}$$

and

$$U_\omega(z)\phi = -\frac{1}{2\pi i}\oint (K_\omega-\rho)^{-1}\phi\exp(-i\omega\rho z)\,d\rho \quad . \tag{5.2b}$$

We shall drop the subscripts k and ω in the discussion of stability to follow.

The question of whether higher order effects (higher order in the linearization) combine in such a way so as to cancel the first order linear instabilities does not appear to be the most appropriate approach to plasma stability analysis. A more suitable approach would be to consider the equation

$$\phi(\tau) = U(\tau)\phi(0) + \int_0^\tau U(\tau-s)V(s)\phi\,ds \quad , \tag{5.3}$$

where the operator V arises from consideration of the appropriate non-linear integro-differential equation describing the plasma state. This nonlinear Volterra equation has been studied in general[37] and may prove to be the key in understanding nonlinear effects in plasmas. The main effort in this dissertation has been to understand completely the linear term, $U(\tau)\phi(0)$, which is a first step in any plasma stability theory.

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OSCILLATIONS AND WAVES IN ANISOTROPIC PLASMAS

by

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(ABSTRACT)

The linearized Vlasov-Maxwell equations describing anisotropic plasma oscillations and waves are studied using an operator theoretic approach. The model considered is one dimensional so that after velocity averages perpendicular to this direction have been taken, the equations can be naturally grouped into one set of equations for longitudinal modes and another set of equations for transverse modes.

The problems of longitudinal and transverse plasma oscillations are studied by Fourier transforming the equations in the space variable and analyzing the resulting operator equations using the theory of semigroups. Existence and uniqueness theorems are proved, and solutions are constructed by the resolvent integration technique. The solutions are put into the form of a generalized eigenfunction expansion with eigenmodes corresponding to zeros of the appropriate plasma dispersion function. The expansion coefficients for eigenmodes corresponding to simple and second order real zeros of the plasma dispersion function are also presented, and constitute some of the new results obtained by our analysis.

Existence and uniqueness of the solution to the longitudinal plasma wave boundary value problem is proved by writing the longitudinal equations in operator form and again using the theory of semi-

groups. The solution to the plasma wave boundary value problem is arrived at by a Fourier time transformation of the Vlasov equation coupled to Ampere's Law rather than Gauss' Law, and analyzing a scalar operator as opposed to the more complicated matrix operator that has previously been studied. Special care is used in constructing the half range transport operator whose resolution of the identity yields the solution in the form of a half range generalized eigenfunction expansion where again, new results are presented for the expansion coefficients for eigenfunctions corresponding to simple and second order real zeros of the fixed frequency longitudinal plasma dispersion function.

Since this study is concerned with anisotropic plasmas, a non-even plasma equilibrium distribution function is assumed with the direct result that more stable and unstable plasma modes corresponding to real and complex zeros of the plasma dispersion function are possible that has previously been considered. Also, for the longitudinal plasma wave problem, the Wiener-Hopf factorization of the fixed frequency longitudinal plasma dispersion function is presented and the coupled nonlinear integral equations for the Wiener-Hopf factors are studied. These Wiener-Hopf factors are required in the construction of the half range transport operator.