Control Charts with Missing Observations

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Abstract

Traditional control charts for process monitoring are based on taking samples from the process at regular time intervals. However, it is often possible in practice for observations, and even entire samples, to be missing. This dissertation investigates missing observations in Exponentially Weighted Moving Average (EWMA) and Multivariate EWMA (MEWMA) control charts. The standardized sample mean is used since this adjusts the sample mean for the fact that part of the sample may be missing. It also allows for constant control limits even though the sample size varies randomly. When complete samples are missing, the weights between samples should also be adjusted.

In the univariate case, three approaches for adjusting the weights of the EWMA control statistic are investigated: (1) ignoring missing samples; (2) adding the weights from previous consecutive missing samples to the current sample; and (3) increasing the weights of non-missing samples in proportion, so that the weights sum to one. Integral equation and Markov chain methods are developed to find and compare the statistical properties of these charts. The EI chart, which adjusts the weights by ignoring the missing samples, has the best overall performance.

The multivariate case in which information on some of the variables is missing is also examined using MEWMA charts. Two methods for adjusting the weights of the MEWMA control statistic are investigated and compared using simulation: (1) ignoring

all the data at a sampling point if the data for at least one variable is missing; and (2) using the previous EWMA value for any variable for which all the data are missing. Both of these methods are examined when the in-control covariance matrix is adjusted at each sampling point to account for missing observations, and when it is not adjusted. The MS control chart, which uses the previous value of the EWMA statistic for a variable if all of the data for that variable is missing at a sampling point, provides the best overall performance. The in-control covariance matrix needs to be adjusted at each sampling point, unless the variables are independent or only weakly correlated.

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Chapter 1

Introduction

Statistical process control (SPC) is a set of statistical methods used to monitor, control, and improve processes. One of the main tools of SPC is the control chart. Control charts are designed to continuously monitor a production process so that any deterioration in quality can be quickly detected and the cause of the deterioration can be identified and eliminated. Although originally developed for manufacturing processes, the application of control charts has been extended into many different areas, including finance, accounting, marketing, health care, and public health surveillance.

In any process there exists a certain amount of variability in the observed values of the characteristics or variables that determine the quality of the process output. Smaller variation usually corresponds to higher quality, so the closer the variable is to its specified target, the higher the quality. This process variation can be classified as resulting from common (non-assignable) causes and special (assignable) causes. Variation due to common causes can be treated as inherent to the process. However, special causes can alter the distribution of the quality variable, usually by changing the values of the distribution parameters. The purpose of a control chart is to detect special causes of variation so that they can be identified and eliminated, thus improving process quality.

A control chart is maintained by taking samples from the process at regular time intervals. A sample statistic, such as the sample mean, is plotted versus the sample time or sample number, and is then compared to the control limits. If the statistic falls within the control limits, then it is assumed that the variability is due only to common causes and that the process is in control. If the statistic falls outside of the control limits, it is called a signal. A signal means that there is strong evidence that a special cause is present, and immediate action should be taken to find and remove this special cause. However, a signal could also be a false alarm, which is a point that falls outside of the control limits when the process is actually in control. Therefore, the control limits must be carefully chosen in order to detect a change in the process quickly, while avoiding a high false alarm rate.

Shewhart control charts were developed by Walter A. Shewhart in the 1920s. They are the most commonly used in practice due to their simplicity and ease of interpretation, but they are relatively insensitive to small and moderate shifts in the process parameters. The exponentially weighted moving average (EWMA) control chart was introduced by Roberts (1959), and the cumulative sum (CUSUM) control chart was first proposed by Page (1954). Many studies have shown that EWMA and CUSUM charts have similar performance, and they are both alternatives to Shewhart charts that are much more effective for detecting small and moderate parameter changes.

Shewhart, EWMA, and CUSUM control charts are designed to monitor a single quality characteristic. However, it is common in practical applications for the quality of a process to be characterized by multiple variables. A multivariate extension of the EWMA chart, called the MEWMA control chart, was proposed by Lowry et. al (1992) to monitor several quality variables simultaneously. Multivariate CUSUM charts have been investigated by Woodall and Ncube (1985), Healy (1987), Crosier (1988), and Pignatiello and Runger (1990).

Traditional control charts for process monitoring, including EWMA and MEWMA charts, are based on taking samples from the process at regular time intervals.

However, it is often possible in practice for observations, and even entire samples to be missing. Suppose that someone simply forgets to record the measurements taken at a particular sampling time. A sample could be sent to the lab for analysis, but the results never received, or an item in the sample could be defective and unable be measured. The actual activities of the process being monitored may be performed by a group of people who work in different locations, such as in a financial services organization, and poor communication among them could result in missing observations. When monitoring disease counts over several regions, it is possible that data may not be reported for a particular region causing data to be missing.

If observations are missing, the sampling rate of the control chart is no longer fixed. The sample size may change from one sampling point to the next, and if entire samples are missing, then the sampling interval also varies. When control charts are used to monitor processes in practical applications, missing observations and samples would most likely simply be ignored, even though this violates the assumptions of the standard charts.

When using Shewhart control charts, it is not difficult to accommodate varying sample size by simply adjusting the control limits based on the value of n. For example, in some applications of the Shewhart p-chart to monitor the fraction of nonconforming items, the sample may be the number of units produced in each time period, which may vary somewhat from day to day. There are several approaches that can be used in this situation. The simplest one is to adjust the control limits for each individual sample based on the specific sample size. A second approach is to compute the control limits using the average sample size. This results in constant control limits, but may have poor performance if there is a large amount of variation in the sample sizes. Another approach to dealing with variable sample size is to use a standardized control chart where the points are plotted in standard deviation units. Conceptually, this chart may be more difficult to understand and interpret since the points are not plotted in the original units. Although Shewhart charts can be easily adjusted to accommodate varying sample sizes, it is not clear how to do this for more complicated charts, such as the EWMA chart.

Missing observations can be particularly damaging in the multivariate case. Consider the situation in which a traditional MEWMA chart is used to simultaneously monitor several quality variables. Suppose that at a particular sampling time the data for one variable is missing, but the data for the other variables are known. Then the standard control statistic cannot be computed, and so none of the data from that sampling time can be used. This means that all of the observations that were not missing must be thrown away. If this data could instead be used to help monitor the process, it is possible that process changes could be detected more quickly.

The objective of this dissertation is to investigate missing observations in control charts. I will focus on EWMA control charts in the univariate case, and on MEWMA charts in the multivariate case. Three approaches for adjusting the weights of the EWMA control statistic are considered. Methods are developed for evaluating the statistical properties of EWMA charts with missing observations. In the multivariate case, a modification of the MEWMA control statistic to account for missing observations is also examined. The goal is to determine methods to adjust the EWMA and MEWMA control charts to handle missing observations, while still using the data that are not missing to efficiently detect changes in the process.

Several types of control charts that vary the sampling rate are discussed in Chapter 2, and the need for investigation of control charts with observations missing at random is explained in more detail. Chapter 3 proposes several methods for adjusting the control statistic in EWMA and MEWMA control charts to account for missing observations. Markov chain and integral equation methods for evaluating the properties of EWMA control charts with missing observations are developed in Chapters 4 and 5. Chapters 6 and 7 evaluate the performance of the EWMA and MEWMA charts proposed. Conclusions and recommendations are discussed in Chapter 8.

Chapter 2

Review of Control Charts with Varying Sampling Rate

Control charts are an important tool used in process monitoring to detect changes that may adversely affect the quality of the process output. When applying a control chart, the standard practice is to sample using a fixed sampling rate (FSR). FSR control charts take samples of fixed size using fixed length sampling intervals. If observations are missing, the sampling rate of the control chart is no longer fixed. The sample size may change from one sampling point to the next, and if entire samples are missing, then the sampling interval also varies.

Recently, variable sampling rate (VSR) control charts which vary the sampling rate as a function of current and past sample results have been developed. For a given average false alarm rate and in-control average sampling rate, a VSR chart is able to detect small and moderate shifts in the process parameters more quickly than an FSR chart. One approach that can be used to deliberately vary the sampling rate is a variable sampling interval (VSI) chart, which varies the sampling interval as a function of the process data. A second approach is a variable sample size (VSS) chart, which varies the sample size as a function of the sample results. Shamma et al. (1991), Saccucci et al. (1992) and Reynolds (1995, 1996a, 1996b) investigated VSI EWMA charts, and Reynolds (1996b) presented some limited results for VSS EWMA charts. Reynolds and

Arnold (2001) provided results on the effect of EWMA charts with the VSS and VSI features separately and in combination. They found that using either the VSS or VSI feature in an EWMA chart substantially improves the chart's ability to detect shifts in the process mean. The VSI feature tends to provide more improvement in detection ability than the VSS feature, and the combination of the two features together provides the best performance.

In most applications of multivariate control charts, it is assumed that the same sample size is used for each variable. However, there are many practical applications in which it is appropriate to use unequal sample sizes. For example, some variables may be more likely than others to be affected by special causes or sampling constraints may prevent the use of the same sample size for each variable. Another situation, which was investigated by Kim and Reynolds (2005), occurs when one variable is more important to the quality of the process than the other variables. They found that when the correlation between the variables is small, increasing the sample size of the important variable improves the ability of the unequal sample size (USS) MEWMA control chart to detect shifts in this variable, but the ability to detect shifts in the other variables is decreased.

Although VSS and VSI control charts vary the sample size and sampling interval, they still assume that there are no observations missing at random. In addition, VSS and VSI charts vary the sampling rate as a function of the process data. The USS MEWMA control chart uses different sample sizes for the variables being monitored, but also assumes that there are no observations missing at random. However, the case investigated in this dissertation assumes that there is a probability p that an individual observation is missing, so the sampling rate varies randomly.

Furutani et al. (1988) proposed a method to apply multivariate control charts to multivariate biochemical data with missing observations. They suggested obtaining estimates of missing values by the method of least squares using all of the observed data. Since estimates of missing observations are obtained by using the data after all of it has been collected, this is a retrospective method. While it may be useful in Phase I monitoring, this research is concerned with the prospective monitoring of Phase II. Furutani et al.'s method cannot be applied during Phase II, since in this phase data are examined as they are observed.

Traditional control charts for which samples are taken at regular time intervals, including the EWMA and MEWMA charts, have been examined in detail by numerous authors. However, it is always assumed that no observations are missing. Additional references for the EWMA and MEWMA control charts are given in Chapter 3. VSR charts which vary the sampling rate as a function of the sample results have also been studied in recent years, but they are still based on the assumption that all the data is known. We were unable to find any investigation of the use of control charts to monitor a process which may have observations missing at random. The objective of this dissertation is to develop and evaluate different methods for modifying the EWMA and MEWMA control charts to adjust for missing observations.

Chapter 3

Control Charts

The first section in this chapter describes the process and notation that will be used for the univariate case. Next three methods are proposed for adjusting the EWMA control statistic to account for missing observations. The third section discusses the notation and assumptions for a process that is characterized by multiple quality variables. Then two approaches are proposed for modifying the MEWMA control statistic when observations may be missing at random. Measures of performance and methods for evaluation are discussed in the remainder of this chapter.

3.1 Notation and Assumptions for the Univariate Case

Consider monitoring a process with a single quality variable of interest. A process observation will be denoted by X, where the observations are assumed to be independent and follow a normal distribution with mean μ and standard deviation σ. The process is said to be in control when μ is at its in-control or target value μ_0, and σ is at its in-control value σ_0. In the standard case where it is assumed that no observations are missing, samples of size $n \geq 1$ are taken from the process at the sampling times t_1, t_2, t_3, …. In the case of missing observations, the times t_1, t_2, t_3, … will continue to be referred to as sampling times or sampling points even though a sample may not exist for some of them.

Process monitoring is assumed to start at time $t_0 = 0$, and sampling times t_{k-1} and t_k are d time units apart for $k = 1, 2, 3, \ldots$. Let $\mathbf{X}_k = (X_{k1}, X_{k2}, \ldots, X_{kn})^T$ represent the sample of n observations taken at time t_k. In the case of $n = 1$, $\mathbf{X}_k$ reduces to X_k, the individual observation sampled at time t_k.

Process monitoring is assumed to continue over a relatively long period of time, and the objective is to detect any special cause that changes μ from μ_0. Control chart performance in Phase II is considered here, and it is assumed that the in-control parameter values, μ_0 and σ_0, have been estimated accurately enough during Phase I that they can be treated as known quantities.

3.2 EWMA Control Charts

The EWMA control chart was originally proposed by Roberts (1959). Since then, many authors have evaluated the properties and performance of this chart, including Crowder (1987 and 1989), Lucas and Saccucci (1990), and Yashchin (1993). The EWMA control chart plots a weighted average of the current and past sample means, and is very efficient at detecting small to moderate shifts in the process parameters. When the objective is to detect both small and large shifts, it is often recommended that EWMA charts be used in combination with a Shewhart chart (see Lucas (1982), Klein (1996), Montgomery (2005)). Reynolds and Stoumbos (2001, 2004, 2005) have recently shown that a combination of the standard EWMA chart for μ and an EWMA chart of squared deviations from target is very effective for monitoring both μ and σ.

The standard EWMA control chart for monitoring μ is based on the control statistic

$$E_k^X = (1-\lambda)E_{k-1}^X + \lambda \overline{X}_k, \qquad k = 1, 2, \ldots, \tag{3.1}$$

where λ is a smoothing parameter satisfying $0 < \lambda \le 1$, $\overline{X}_k = \sum_{j=1}^{n} X_{kj} / n$ is the sample mean at time t_k, and the starting value is usually taken to be $E_0^X = \mu_0$. The value of λ determines the amount of weight given to the current sample mean relative to past sample means, and it can be used to tune the chart to be sensitive to specific shifts in the process parameter. In particular, using a small value of λ will make the EWMA chart sensitive to small shifts, and using a large value of λ will make the chart sensitive to large shifts. The chart signals at time t_k if E_k^X falls outside of the control limits

$$\mu_0 \pm h\sqrt{\lambda/(n(2-\lambda))}\sigma_0, \qquad (3.2)$$

where $\sqrt{\lambda/(n(2-\lambda))}\sigma_0$ is the asymptotic in-control standard deviation of E_k^X. The parameter h is usually chosen to yield a specified in-control average time to signal (ATS).

Consider the situation in which an EWMA control chart is applied to monitor a process where samples of size $n \ge 1$ are taken every d time units, but some of the observations may be missing. The standard EWMA control statistic E_k^X is a weighted average of the current and past sample means, but when there are missing observations, the sample size is not constant across samples. In addition, if an entire sample is missing, then the time between samples is no longer d time units. Statistical principles suggest that the weights in the EWMA control statistic should be adjusted to account for this. In the case of VSSVSI control charts, Reynolds and Arnold (2001) considered several EWMA control statistics that adjusted the weights based on the individual sample sizes. They recommended the control chart based on the standardized sample mean.

The EWMA control chart for monitoring μ based on the standardized sample mean has control statistic

$$E_k^Z = (1-\lambda)E_{k-1}^Z + \lambda Z_k, \qquad k = 1, 2, \ldots, \qquad (3.3)$$

where

$$Z_k = \frac{\left(\overline{X}_k - \mu_0\right)}{\sigma_0 / \sqrt{n}} \tag{3.4}$$

and there are no observations missing. The starting value is $E_0^Z = 0$, and the chart signals at time t_k if E_k^Z falls outside of the control limits

$$0 \pm h\sqrt{\lambda/(2-\lambda)}\sigma_0 . \tag{3.5}$$

The EWMA chart based on the control statistic E_k^Z will be referred to as the EZ chart.

Suppose that at time t_k there are m_k missing observations where $0 \le m_k < n$, so the sample size is $n - m_k \ge 1$. Then $\overline{X}_k$, the sample mean at time t_k, is the mean of the $n - m_k$ non-missing observations, and the standardized sample mean is

$$Z_k = \frac{\left(\overline{X}_k - \mu_0\right)}{\sigma_0 / \sqrt{n - m_k}} . \tag{3.6}$$

By using the standardized sample mean, the weights are adjusted by making the weight used for the current sample mean proportional to the square root of the sample size for this sample. When the process is in control, the distribution of Z_k is standard normal, and thus does not depend on the sample size. This means that the control limits are constant even though the sample size varies randomly. When $m_k < n$, using Z_k adjusts the sample mean $\overline{X}_k$ for the fact that part of the sample is missing. It is also possible that $m_k = n$, so that the entire sample at time t_k is missing. When complete samples are missing, the weights between samples should also be adjusted. Three different methods for adjusting the weights of the EWMA control statistics to account for observations missing at random are considered.

Suppose that at time t_k, $m_k < n$ and the previous i consecutive samples at times t_{k-1}, t_{k-2}, …, t_{k-i} are missing. The first EWMA statistic considered ignores the missing samples, as might typically be done in practice. This statistic is

$$E_k^I = (1-\lambda)E_{k-(i+1)}^I + \lambda Z_k , \qquad k = 1, 2, \ldots, \tag{3.7}$$

where the superscript "I" on E_k^I indicates that missing observations are ignored. The EWMA chart based on the control statistic E_k^I will be referred to as the EI chart.

The standard EWMA control statistic given by (3.3) can be written as

$$\begin{aligned} E_k^Z &= (1-\lambda)^k E_0^Z + \sum_{j=0}^{k-1}(1-\lambda)^j \lambda Z_{k-j} \\ &= (1-\lambda)^k E_0^Z + \sum_{j=i+1}^{k-1}(1-\lambda)^j \lambda Z_{k-j} + \sum_{j=1}^{i}(1-\lambda)^j \lambda Z_{k-j} + \lambda Z_k . \end{aligned} \tag{3.8}$$

Then the sum of the weights from the missing standardized sample means $Z_{k-1}, Z_{k-2}, \ldots, Z_{k-i}$ is $\sum_{j=1}^{i}(1-\lambda)^j \lambda$. Adding this to the weight of the current sample mean gives the second EWMA control statistic, which is

$$\begin{aligned} E_k^A &= (1-\lambda)^k E_0^A + \sum_{j=i+1}^{k-1}(1-\lambda)^j \lambda Z_{k-j} + \sum_{j=0}^{i}(1-\lambda)^j \lambda Z_k \\ &= (1-\lambda)^{i+1}\left[(1-\lambda)^{k-(i+1)} E_0^A + \sum_{j=i+1}^{k-1}(1-\lambda)^{j-(i+1)} \lambda Z_{k-j}\right] + \sum_{j=0}^{i}(1-\lambda)^j \lambda Z_k \\ &= (1-\lambda)^{i+1}\left[(1-\lambda)^{k-(i+1)} E_0^A + \sum_{j=0}^{k-(i+1)-1}(1-\lambda)^j \lambda Z_{k-(i+1)-j}\right] + \sum_{j=0}^{i}(1-\lambda)^j \lambda Z_k \\ &= (1-\lambda)^{i+1} E_{k-(i+1)}^A + \sum_{j=0}^{i}(1-\lambda)^j \lambda Z_k . \end{aligned} \tag{3.9}$$

The superscript "A" on E_k^A indicates that the weights are adjusted by adding the weights from the missing samples to the weight of the current sample. The EWMA chart based on the control statistic E_k^A will be referred to as the EA chart.

The sum of the weights of the standard EWMA statistic is one. When there are missing samples, this is no longer true. The third EWMA control statistic to be considered adjusts the weights so that they once again sum to one by increasing the weights of all the non-missing observations in proportion. This statistic, say E_k^P, is

$$E_k^P = \frac{(1-\lambda)^k}{1-\sum_{j=1}^{i}(1-\lambda)^j\lambda} E_0^P + \sum_{h=i+1}^{k-1}\frac{(1-\lambda)^h\lambda}{1-\sum_{j=1}^{i}(1-\lambda)^j\lambda} Z_{k-h} + \frac{\lambda}{1-\sum_{j=1}^{i}(1-\lambda)^j\lambda} Z_k$$

$$= \frac{(1-\lambda)^{i+1}}{1-\sum_{j=1}^{i}(1-\lambda)^j\lambda}\left[(1-\lambda)^{k-(i+1)} E_0^P + \sum_{h=i+1}^{k-1}(1-\lambda)^{h-(i+1)}\lambda Z_{k-h}\right] + \frac{\lambda}{1-\sum_{j=1}^{i}(1-\lambda)^j\lambda} Z_k$$

$$= \frac{(1-\lambda)^{i+1}}{1-\sum_{j=1}^{i}(1-\lambda)^j\lambda}\left[(1-\lambda)^{k-(i+1)} E_0^P + \sum_{h=0}^{k-(i+1)-1}(1-\lambda)^{h}\lambda Z_{k-(i+1)-h}\right] + \frac{\lambda}{1-\sum_{j=1}^{i}(1-\lambda)^j\lambda} Z_k$$

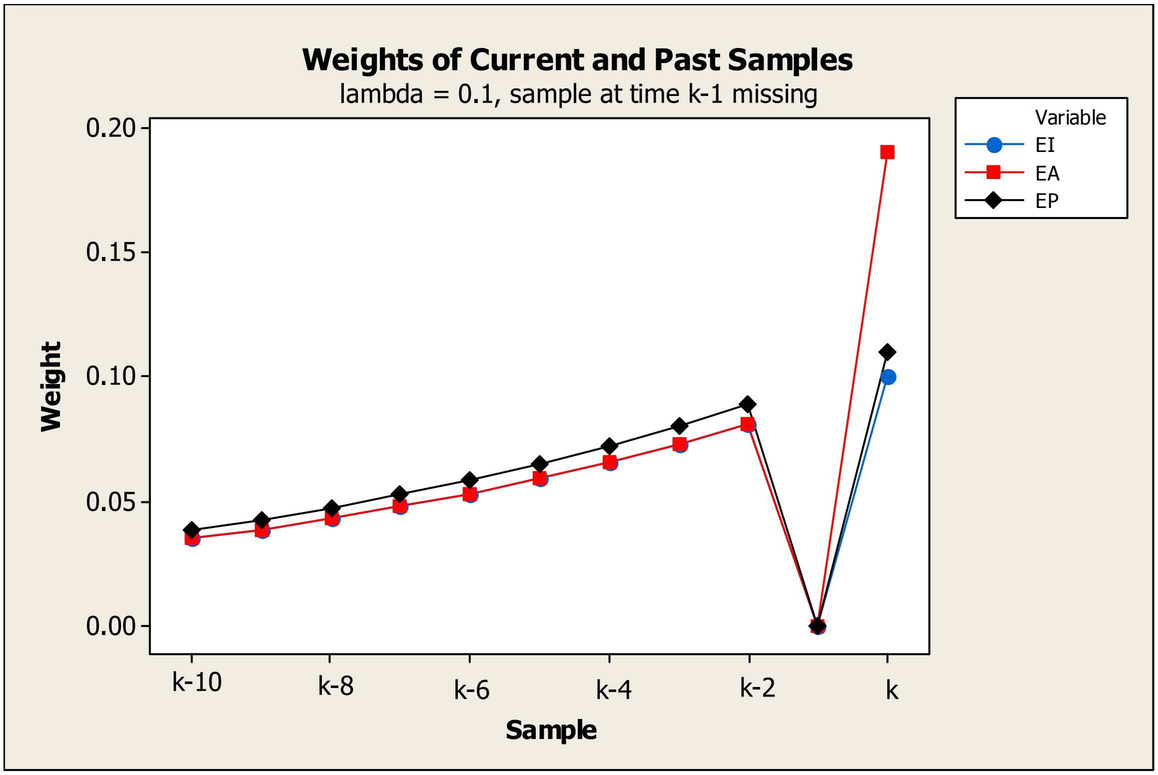
$$= \frac{(1-\lambda)^{i+1}}{1-\sum_{j=1}^{i}(1-\lambda)^j\lambda} E_{k-(i+1)}^P + \frac{\lambda}{1-\sum_{j=1}^{i}(1-\lambda)^j\lambda} Z_k\,, \tag{3.10}$$

where the superscript "P" on E_k^P indicates that the weights are adjusted by increasing them all in proportion. The EWMA control chart based on the statistic E_k^P will be referred to as the EP chart.

The form of the control limits that will be used for the EA, EI, and EP charts are the same as for the EZ chart and are given by (3.5); however, the value of h necessary to achieve specified in-control properties depends on the particular chart. Note that if there are no missing observations, then the EZ, EA, EI, and EP control charts are all equivalent.

The weights of the EI, EA, and EP control charts for the case when $\lambda = 0.1$ and the sample at time t_{k-1} is missing are shown in Figure 3.1. At the current time t_k, the weight on the current standardized sample mean Z_k is 0.1 for the EI chart, 0.19 for the EA chart, and 0.10989 for the EP chart. The EA chart gives almost twice as much weight to Z_k as the EI and EP charts. The EI and EA charts give the same amount of weight to the past non-missing samples; in fact, they give the same amount of weight to these samples as the EZ control chart does when no observations are missing. However, the EP chart increases the weight given to all previous non-missing sample means.

Figure 3.1. Comparison of Weights of EI, EA, and EP Control Charts when $\lambda = 0.1$ and the Sample at Time t_{k-1} is Missing.



3.3 Notation and Assumptions for the Multivariate Case

Consider monitoring a process with b variables that have a multivariate normal distribution with mean vector $\boldsymbol{\mu}$, covariance matrix $\boldsymbol{\Sigma}$, and vector of standard deviations $\boldsymbol{\sigma}$. Let $\boldsymbol{\mu}_0$, $\boldsymbol{\Sigma}_0$, and $\boldsymbol{\sigma}_0$ represent the in-control or target values for $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$, and $\boldsymbol{\sigma}$, respectively. It is assumed that the in-control parameter values are known or estimated with negligible error.

In the standard case which assumes that no observations are missing, a sample of $n \geq 1$ independent observation vectors will be taken from the process at each sampling point, where the sampling points are *d* time units apart. Let X_{kva} represent observation *a* for variable *v* at sampling time t_k, where $a = 1, 2, \ldots, n$, $v = 1, 2, \ldots, b$, and $k = 1, 2, \ldots$. Let the corresponding standardized observation be

$$Z_{kva} = \frac{X_{kva} - \mu_{ov}}{\sigma_{0v}} \tag{3.11}$$

where μ_{0v} is the *v*-th component of $\boldsymbol{\mu}_0$ and σ_{0v} is the *v*-th component of $\boldsymbol{\sigma}_0$. Let

$$\mathbf{Z}_{ka} = \left(Z_{k1a}, Z_{k2a}, \ldots, Z_{kba}\right)^T \qquad a = 1, 2, \ldots, n \tag{3.12}$$

be the vector of standardized observations for observation vector *a* at sampling point t_k. Let $\boldsymbol{\Sigma}_Z$ be the covariance matrix of $\mathbf{Z}_{ka}$, and let $\boldsymbol{\Sigma}_{Z0}$ be the in-control value of $\boldsymbol{\Sigma}_Z$. When the process is in control, then Z_{kva} follows the standard normal distribution, so $\boldsymbol{\Sigma}_{Z0}$ is also the in-control correlation matrix of the unstandardized observations. Let

$$\overline{X}_{kv} = \sum_{a=1}^{n} X_{kva} / n \tag{3.13}$$

be the sample mean for variable *v* at sampling time t_k, and so the standardized sample mean is

$$Z_{kv} = \frac{\overline{X}_{kv} - \mu_{0v}}{\sigma_{0v} / \sqrt{n}} \qquad v = 1, 2, \ldots, b. \tag{3.14}$$

3.4 MEWMA Control Charts

The MEWMA control chart proposed by Lowry et al. (1992) is a multivariate extension of the EWMA chart for monitoring several quality variables simultaneously. The properties and performance of this chart have been evaluated by many authors, including Kramer and Schmid (1997), Prabhu and Runger (1997) and Reynolds and Kim (2005). For the problem of monitoring both $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, Reynolds and Cho (2006) and Reynolds and Kim (2007) have recently investigated combinations of the standard

MEWMA chart for monitoring **μ** and MEWMA-type charts based on the squared deviations from target for monitoring **Σ**.

At sampling time t_k, let the EWMA statistic of standardized sample means for variable v be

$$E_{kv}^{Z} = (1-\lambda)E_{k-1,v}^{Z} + \lambda Z_{kv} \qquad v = 1, 2, \ldots, b \tag{3.15}$$

where the starting value is $E_{0v}^{Z} = 0$ and the superscript "Z" indicates that this statistic is based on Z_{kv}. The variance of E_{kv}^{Z} is

$$\begin{aligned}\operatorname{Var}\left(E_{kv}^{Z}\right) &= \operatorname{Var}\left((1-\lambda)^{k} E_{0v}^{Z} + \sum_{j=0}^{k-1}(1-\lambda)^{j}\lambda Z_{k-j,v}\right) \\ &= \sum_{j=0}^{k-1}(1-\lambda)^{2j}\lambda^{2}\operatorname{Var}\left(Z_{k-j,v}\right) \\ &= c_k \end{aligned} \tag{3.16}$$

where

$$c_k = \frac{\lambda\left[1-(1-\lambda)^{2k}\right]}{2-\lambda} \qquad k = 1, 2, \ldots. \tag{3.17}$$

The covariance of E_{kv}^{Z} and $E_{kv'}^{Z}$ for $v \neq v'$ is

$$\begin{aligned}\operatorname{Cov}\left(E_{kv}^{Z}, E_{kv'}^{Z}\right) &= \operatorname{Cov}\left((1-\lambda)^{k} E_{0v}^{Z} + \sum_{j=0}^{k-1}(1-\lambda)^{j}\lambda Z_{k-j,v}, (1-\lambda)^{k} E_{0v'}^{Z} + \sum_{j=0}^{k-1}(1-\lambda)^{j}\lambda Z_{k-j,v'}\right) \\ &= \sum_{j=0}^{k-1}(1-\lambda)^{2j}\lambda^{2}\operatorname{Cov}\left(Z_{k-j,v}, Z_{k-j,v'}\right) \\ &= c_k \rho_{vv'} \end{aligned} \tag{3.18}$$

where $\rho_{vv'}$ is the element in row v and column v' of $\mathbf{\Sigma}_{Z0}$. So $\rho_{vv'}$ is the correlation (and covariance) between Z_{kv} and $Z_{kv'}$, and also the correlation between X_{kva} and $X_{kv'a}$. The asymptotic variance and covariance is obtained by letting $k \rightarrow \infty$ in (3.17), which gives

$$c_{\infty} = \frac{\lambda}{2-\lambda}. \tag{3.19}$$

Thus, the in-control covariance matrix for $\left(E_{k1}^{Z}, E_{k2}^{Z}, \ldots, E_{kb}^{Z}\right)^{T}$ is $\mathbf{\Sigma}_{Zk0} = c_k \mathbf{\Sigma}_{Z0}$, and the asymptotic in-control covariance matrix is $c_{\infty}\mathbf{\Sigma}_{Z0}$.

The MEWMA control chart for monitoring $\boldsymbol{\mu}$ when all observations are known is based on $\left\{E_{kv}^{Z}\right\}$, and the control statistic is

$$M_k^Z = c_{\infty}^{-1}\left(E_{k1}^{Z}, E_{k2}^{Z}, \ldots, E_{kb}^{Z}\right)\mathbf{\Sigma}_{Z0}^{-1}\left(E_{k1}^{Z}, E_{k2}^{Z}, \ldots, E_{kb}^{Z}\right)^{T}. \tag{3.20}$$

This chart signals at time t_k if $M_k^Z >$ UCL. Using the exact covariance matrix would allow the MEWMA control chart to detect initial out-of-control conditions more quickly. However, it may be more likely that the process will stay in control for a while and then shift out of control, and so the asymptotic in-control covariance matrix is used here. The MEWMA chart based on the control statistic M_k^Z will be referred to as the MZ chart.

Consider the situation in which an MEWMA control chart is used to monitor a process where some of the observations may be missing. Suppose that at sampling point t_k, there are m_{kv} missing observations for variable v where $0 \le m_{kv} \le n$ for $v = 1, 2, \ldots, b$. Note that if $m_{kv} = n$, the entire sample for variable v is missing at t_k and the control statistic M_k^Z cannot be computed in the standard way for that sampling time. This means that any non-missing observations for the remaining variables cannot be used in the standard way.

Suppose that at time t_k, there are $m_{kv} < n$ missing observations for variable v, so the sample size for variable v is $n - m_{kv} \ge 1$ for $v = 1, 2, \ldots, b$. Let

$$I_{kva} = \begin{cases} 0 & \text{if observation } X_{kva} \text{ is missing} \\ 1 & \text{if observation } X_{kva} \text{ is not missing} \end{cases} \tag{3.21}$$

for $a = 1, 2, \ldots, n$, and so

$$\overline{X}_{kv} = \sum_{a=1}^{n} I_{kva} \frac{X_{kva}}{\left(n - m_{kv}\right)} \tag{3.22}$$

is the sample mean of the $n - m_{kv}$ non-missing observations for variable *v*. The standardized sample mean is

$$Z_{kv} = \frac{\overline{X}_{kv} - \mu_{0v}}{\sigma_{0v} \big/ \sqrt{n - m_{kv}}} \qquad v = 1, 2, \ldots, b. \tag{3.23}$$

Suppose that at time t_k, there are $m_{kv'} < n$ missing observations for variable v' where $v \neq v'$. Let

$$m_{kvv'} = n - \sum_{a=1}^{n} I_{kva} I_{kv'a} \tag{3.24}$$

be the number of observations at time t_k for which both variable *v* and v' are missing. Then covariance of Z_{kv} and $Z_{kv'}$ is

$$\begin{aligned}
\mathrm{Cov}\left(Z_{kv}, Z_{kv'}\right) &= \mathrm{Cov}\left(\frac{\overline{X}_{kv} - \mu_{0v}}{\sigma_{0v} \big/ \sqrt{n - m_{kv}}}, \frac{\overline{X}_{kv'} - \mu_{0v'}}{\sigma_{0v'} \big/ \sqrt{n - m_{kv'}}} \right) \\
&= \frac{\sqrt{\left(n - m_{kv}\right)\left(n - m_{kv'}\right)}}{\sigma_{0v}\sigma_{0v'}} \mathrm{Cov}\left(\overline{X}_{kv}, \overline{X}_{kv'}\right) \\
&= \frac{\sqrt{\left(n - m_{kv}\right)\left(n - m_{kv'}\right)}}{\sigma_{0v}\sigma_{0v'}} \mathrm{Cov}\left(\sum_{a=1}^{n} I_{kva} \frac{X_{kva}}{n - m_{kv}}, \sum_{a=1}^{n} I_{kv'a} \frac{X_{kv'a}}{n - m_{kv'}} \right) \\
&= \frac{\sqrt{\left(n - m_{kv}\right)\left(n - m_{kv'}\right)}}{\sigma_{0v}\sigma_{0v'}} \left(\frac{1}{\left(n - m_{kv}\right)\left(n - m_{kv'}\right)} \right) \sum_{a=1}^{n} I_{kva} I_{kv'a} \mathrm{Cov}\left(X_{kva}, X_{kv'a}\right) \\
&= \frac{1}{\sqrt{\left(n - m_{kv}\right)\left(n - m_{kv'}\right)}} \sum_{a=1}^{n} I_{kva} I_{kv'a} \rho_{vv'} \\
&= \frac{\left(n - m_{kvv'}\right)}{\sqrt{\left(n - m_{kv}\right)\left(n - m_{kv'}\right)}} \rho_{vv'} .
\end{aligned} \tag{3.25}$$

When $m_{kv} < n$, using Z_{kv} adjusts the sample mean for variable *v*, $\overline{X}_{kv}$, for the fact that part of the sample for variable *v* is missing. It is also possible that $m_{kv} = n$, so that the entire sample for variable *v* is missing at time t_k. When an entire sample for a variable is missing, the weights between samples of the EWMA statistic for that variable should also be adjusted. Two different MEWMA control statistics are considered to handle missing observations.

Suppose that at the current time t_k, $m_{kv} < n$ and the previous i consecutive sampling times t_{k-1}, t_{k-2}, …, t_{k-i} have at least one variable for which the entire sample is missing where $0 \le i < k$. One method for adjusting the MEWMA control statistic to account for missing observations ignores the data for all variables at a sampling point if the entire sample for at least one variable is missing. Then the EWMA statistic for variable v is

$$E_{kv}^{ZI} = (1-\lambda)E_{k-(i+1),v}^{ZI} + \lambda Z_{kv} \qquad v = 1, 2, \ldots, b \tag{3.26}$$

where $E_{0v}^{ZI} = 0$ and $0 \le m_{kv} < n$.

Let ζ_k be the total number of sampling points from the start of process monitoring at time t_0 to the current sampling point t_k where there is at least one variable for which the entire sample is missing, i.e. the total number of sampling points for which $m_{kv} = n$ for at least one v = 1, 2, …, b. Then there are a total of $k - \varsigma_k$ sampling points for which at least one observation is known for all b variables. Call the standardized sample means at these sampling points $Z_{(1),v}, Z_{(2),v}, \ldots, Z_{(k-\zeta_k),v}$ for v = 1, 2, …, b. Then E_{kv}^{ZI} in (3.26) can be rewritten as

$$E_{kv}^{ZI} = (1-\lambda)^{k-\varsigma_k} E_{0v}^{ZI} + \sum_{j=0}^{k-\varsigma_k-1} (1-\lambda)^j \lambda Z_{(k-\varsigma_k-j),v} \qquad v = 1, 2, \ldots, b \tag{3.27}$$

where $E_{0v}^{ZI} = 0$ and $0 \le m_{kv} < n$. The variance of E_{kv}^{ZI} is

$$\begin{aligned} \operatorname{Var}\left(E_{kv}^{ZI}\right) &= \operatorname{Var}\left((1-\lambda)^{k-\zeta_k} E_{0v}^{ZI} + \sum_{j=0}^{k-\varsigma_k-1} (1-\lambda)^j \lambda Z_{(k-\varsigma_k-j),v} \right) \\ &= \sum_{j=0}^{k-\varsigma-1} (1-\lambda)^{2j} \lambda^2 \operatorname{Var}\left(Z_{(k-\varsigma-j),v}\right) \\ &= c_{k-\varsigma_k} \, . \end{aligned} \tag{3.28}$$

Let p be the probability that an individual observation is missing, independent of other observations, where $0 \le p < 1$. Then the probability that of the entire sample of n observations being missing for at least one variable is $1 - p^{nb}$. For fixed k, ζ_k follows a

Binomial$\left(k, 1-p^{nb}\right)$ distribution. Let $\overline{\zeta}_k = \sum_{j=1}^{k} \frac{\zeta_j}{k}$, and by the Strong Law of Large Numbers, for every $\varepsilon > 0$

$$P\left(\lim_{k\to\infty}\left|\overline{\zeta}_k - k\left(1-p^{nb}\right)\right| < \varepsilon\right) = 1. \tag{3.29}$$

Then letting $k \to \infty$ in (3.28), the asymptotic variance is c_∞. Thus, this EWMA statistic has the same asymptotic variance as the standard EWMA statistic E_{kv}^{Z}, which assumes all observations are known. The covariance of E_{kv}^{ZI} and $E_{kv'}^{ZI}$ for $v \neq v'$ is

$$\begin{aligned} \mathrm{Cov}\left(E_{kv}^{ZI}, E_{kv'}^{ZI}\right) &= \mathrm{Cov}\left((1-\lambda)E_{k-(i+1),v}^{\mathrm{ZI}} + \lambda Z_{kv}, (1-\lambda)E_{k-(i+1),v'}^{ZI} + \lambda Z_{kv'}\right) \\ &= (1-\lambda)^2 Cov\left(E_{k-(i+1),v}^{ZI}, E_{k-(i+1),v'}^{ZI}\right) + \lambda^2 Cov\left(Z_{kv}, Z_{kv'}\right). \end{aligned} \tag{3.30}$$

When there are no observations missing, the in-control covariance matrix of Z_{ka} is $\mathbf{\Sigma}_{Z0}$. From (3.25), we can see that when there are observations missing, the in-control covariance matrix of Z_{ka} depends on the number of missing observations at time t_k. Then by (3.30), the in-control covariance of E_{kv}^{ZI} and $E_{kv'}^{ZI}$ also depends on the number of missing observations at time t_k, as well as the covariance of the EWMA statistics at time $t_{k-(i+1)}$. Let $\mathbf{\Sigma}_{Zk0}$ be the in-control covariance matrix of Z_{ka}, and let $\mathbf{\Sigma}_{Ik0}$ be the in-control covariance matrix for $\left(E_{k1}^{ZI}, E_{k2}^{ZI}, \ldots, E_{kb}^{ZI}\right)^T$. Then from (3.30), we see that $\mathbf{\Sigma}_{Ik0}$ can be obtained by

$$\mathbf{\Sigma}_{Ik0} = (1-\lambda)^2 \mathbf{\Sigma}_{I,k-1,0} + \lambda^2 \mathbf{\Sigma}_{Zk0} \qquad k = 1, 2, 3, \ldots. \tag{3.31}$$

Letting $\mathbf{\Sigma}_{I00} = \mathbf{\Sigma}_{Z0}$ gives the exact covariance matrix of $\left(E_{k1}^{ZI}, E_{k2}^{ZI}, \ldots, E_{kb}^{ZI}\right)^T$. However, just as for the MZ chart, it is likely that the process will stay in control for a while and then shift out-of-control. Thus, the asymptotic in-control covariance matrix of $\mathbf{Z}_{ka}$ will be used as the starting value, so $\mathbf{\Sigma}_{I00} = c_\infty \mathbf{\Sigma}_{Z0}$. Note that $\mathbf{\Sigma}_{Ik0}$ is an iterative process that depends only on the previous value $\mathbf{\Sigma}_{I,k-1,0}$ and the current sample at time t_k.

The control statistic for the MEWMA control chart based on $\{E_{kv}^{ZI}\}$ is

$$M_k^I = \left(E_{k1}^{ZI}, E_{k2}^{ZI}, \ldots, E_{kb}^{ZI}\right)\mathbf{\Sigma}_{Ik0}^{-1}\left(E_{k1}^{ZI}, E_{k2}^{ZI}, \ldots, E_{kb}^{ZI}\right)^T. \tag{3.32}$$

The superscript "I" indicates that missing samples are ignored, and the MEWMA chart based on the control statistic M_k^I will be referred to as the MI chart.

The MI chart does not use any of the data at a sampling point if all the data for one variable is missing. However, it seems that if we could use the data that are not missing, we might be able to better monitor the process and detect changes more quickly. One method that adjusts the MEWMA control statistic to account for missing observations while still using all of the data available is considered next. Suppose that at time t_k we have $m_{kv} = n$, so the entire sample is missing for variable v. Substitute the EWMA statistic for variable v from the previous sampling point for the missing EWMA statistic. This gives the EWMA statistic

$$E_{kv}^{ZS} = \begin{cases} E_{k-1,v}^{ZS} & \text{if } m_{kv} = n \\ (1-\lambda)E_{k-1,v}^{ZS} + \lambda Z_{kv} & \text{if } m_{kv} < n \end{cases} \qquad v = 1, 2, \ldots, b \tag{3.33}$$

where $E_{0v}^{ZS} = 0$.

Let ζ_{kv} be the total number of sampling points from the start of process monitoring at time t_0 to the current sampling point t_k where the entire sample is missing for variable v, i.e. the total number of sampling points for which $m_{kv} = n$. Then there are a total of $k - \varsigma_{kv}$ sampling points for which at least one observation is known for variable v and the sample mean for variable v can be computed. Call these $k - \varsigma_{kv}$ non-missing standardized sample means $Z_{(1),v}$, $Z_{(2),v}$, …, $Z_{(k-\zeta_{kv}),v}$. The EWMA statistic in (3.33) can be rewritten as

$$E_{kv}^{ZS} = (1-\lambda)^{k-\zeta_{kv}} E_{0v}^{ZS} + \sum_{j=0}^{k-\zeta_{kv}-1} (1-\lambda)^j \lambda Z_{(k-\zeta_{kv}-j),v} \qquad v = 1, 2, \ldots, b \text{ and } k = 1, 2, 3, \ldots. \tag{3.34}$$

Although E_{kv}^{ZI} in (3.27) and E_{kv}^{ZS} in (3.34) have similar formulas, note that E_{kv}^{ZI} does not exist unless $m_{kv} < n$ at sampling time t_k, while E_{kv}^{ZS} exists for all values of k = 0, 1, 2, ...

The variance of E_{kv}^{ZS} is

$$\begin{aligned}\mathrm{Var}\left(E_{kv}^{ZS}\right) &= \sum_{j=0}^{k-\zeta_{kv}-1} (1-\lambda)^{2j} \lambda^2 \mathrm{Var}\left(Z_{(k-\zeta_{kv}-j),v}\right) \\ &= c_{k-\varsigma_{kv}} \qquad v = 1, 2, \ldots, b. \end{aligned} \tag{3.35}$$

For fixed k and v, ζ_{kv} follows a $\mathrm{Binomial}\left(k, 1-p^n\right)$ distribution. Let $\overline{\zeta}_{kv} = \sum_{j=1}^{k} \frac{\zeta_{jv}}{k}$, and by the Strong Law of Large Numbers, for every $\varepsilon > 0$

$$P\left(\lim_{k\to\infty} \left| \overline{\zeta}_{kv} - k\left(1-p^n\right)\right| < \varepsilon\right) = 1. \tag{3.36}$$

Then letting $k \to \infty$ in (3.36), the asymptotic variance of E_{kv}^{ZS} is c_∞, which is the same as for E_{kv}^{Z} in the MZ chart and E_{kv}^{ZI} in the MI chart. The covariance of E_{kv}^{ZS} and $E_{kv'}^{ZS}$ for $v \neq v'$ is

$$\mathrm{Cov}\left(E_{kv}^{ZS}, E_{kv'}^{ZS}\right) =$$

$$\begin{cases} \mathrm{Cov}\left(E_{k-1,v}^{ZS}, E_{k-1,v'}^{ZS}\right) & \text{if } m_{kv} = m_{kv'} = n \\ (1-\lambda)\mathrm{Cov}\left(E_{k-1,v}^{ZS}, E_{k-1,v'}^{ZS}\right) & \text{if } m_{kv} = n \text{ and } m_{kv'} < n \text{ or if } m_{kv} < n \text{ and } m_{kv'} = n \\ (1-\lambda)^2 \mathrm{Cov}\left(E_{k-1,v}^{ZS}, E_{k-1,v'}^{ZS}\right) + \lambda^2 \mathrm{Cov}\left(Z_{kv}, Z_{kv'}\right) & \text{if } m_{kv} < n \text{ and } m_{kv'} < n. \end{cases} \tag{3.37}$$

Let $\mathbf{\Sigma}_{Sk0}$ be the in-control covariance matrix for $\left(E_{k1}^{ZS}, E_{k2}^{ZS}, \ldots, E_{kb}^{ZS}\right)^T$. Then $\mathbf{\Sigma}_{Sk0}$ can be obtained by

$$\mathbf{\Sigma}_{Sk0} = \begin{cases} \mathbf{\Sigma}_{S,k-1,0} & \text{if } m_{kv} = m_{kv'} = n \\ (1-\lambda)\mathbf{\Sigma}_{S,k-1,0} & \text{if } m_{kv} = n \text{ and } m_{kv'} < n \text{ or if } m_{kv} < n \text{ and } m_{kv'} = n \\ (1-\lambda)^2 \mathbf{\Sigma}_{S,k-1,0} + \lambda^2 \mathbf{\Sigma}_{Zk0} & \text{if } m_{kv} < n \text{ and } m_{kv'} < n. \end{cases}$$

$$k = 1, 2, 3, \ldots. \tag{3.38}$$

Letting $\mathbf{\Sigma}_{S00} = \mathbf{\Sigma}_{Z0}$ gives the exact covariance matrix of $\left(E_{k1}^{ZS}, E_{k2}^{ZS}, \ldots, E_{kb}^{ZS}\right)^T$. However, just as in the MI chart, the asymptotic in-control covariance matrix of $\mathbf{Z}_{ka}$ will be used as the starting value, so $\mathbf{\Sigma}_{S00} = c_\infty \mathbf{\Sigma}_{Z0}$. The in-control covariance matrix $\mathbf{\Sigma}_{Sk0}$ is also an iterative process that depends only on the previous value $\mathbf{\Sigma}_{S,k-1,0}$ and the current sample at time t_k.

The control statistic for the MEWMA control chart based on $\{E_{kv}^{ZS}\}$ is

$$M_k^S = \left(E_{k1}^{ZS}, E_{k2}^{ZS}, \ldots, E_{kb}^{ZS}\right)\mathbf{\Sigma}_{Sk0}^{-1}\left(E_{k1}^{ZS}, E_{k2}^{ZS}, \ldots, E_{kb}^{ZS}\right)^T. \qquad (3.39)$$

The superscript "S" indicates that the previous control statistic for a variable will be substituted for the current statistic when a sample is missing. This allows all of the non-missing data to be used to help continue monitoring the process. The MEWMA control chart based on the statistic M_k^S will be referred to as the MS chart. Both the MI and MS charts signal if the control statistic is greater than UCL.

Note that if there are no missing observations, then the MZ, MI, and MS control charts are all equivalent. In addition, if all the variables of interest are independent, so $\rho_{vv'} = 0$ for all $v \neq v'$, then the asymptotic value of the in-control covariance matrices $\mathbf{\Sigma}_{Ik0}$ and $\mathbf{\Sigma}_{Sk0}$ is $c_\infty \mathbf{\Sigma}_{Z0}$. This means that when $\rho = 0$, it is not necessary to adjust the covariance matrix for each sampling point even though observations may be missing. Thus, for the special case in which all of the variables are independent, the unadjusted MI chart with control statistic

$$M_k^{IU} = c_\infty^{-1}\left(E_{k1}^{ZI}, E_{k2}^{ZI}, \ldots, E_{kb}^{ZI}\right)\mathbf{\Sigma}_{Z0}^{-1}\left(E_{k1}^{ZI}, E_{k2}^{ZI}, \ldots, E_{kb}^{ZI}\right)^T \qquad (3.40)$$

and the unadjusted MS chart with control statistic

$$M_k^{SU} = c_\infty^{-1}\left(E_{k1}^{ZS}, E_{k2}^{ZS}, \ldots, E_{kb}^{ZS}\right)\mathbf{\Sigma}_{Z0}^{-1}\left(E_{k1}^{ZS}, E_{k2}^{ZS}, \ldots, E_{kb}^{ZS}\right)^T \qquad (3.41)$$

can be used.

3.5 Measures of Control Chart Performance

Having defined several approaches to dealing with missing observations, the next objective is to investigate the statistical performance of the proposed EWMA and MEWMA control charts in detecting changes in the process mean when observations are missing. The ability of control charts to detect process changes can be evaluated by considering the average time to signal (ATS), which is defined as the expected length of time from the start of process monitoring to the time that the chart signals. When the process is in control, a large ATS is desirable since this corresponds to a low false alarm rate. When a shift in the process occurs causing the process to go out of control, a small ATS is necessary in order to detect the change as quickly as possible.

The different control charts to be compared are set to have the same false alarm rate per unit time, which is equivalent to the requirement that the charts have the same in-control ATS. The false alarm rate used for the EWMA charts corresponds to an in-control ATS of 1481.6, which is the in-control ATS of the Shewhart $\overline{X}$ control chart with standard three-sigma control limits and samples of size n = 4 observations taken every $d = 4$ time units. This specified false alarm rate is obtained for each control chart by adjusting the value of h in the control limits to achieve the desired in-control ATS of 1481.6. The MEWMA control charts being compared have an in-control ATS of 800.0, which can be obtained by adjusting the UCL values for the various charts. These in-control values for the ATS have been used in some recent papers, including Reynolds and Cho (2006) and Reynolds and Stoumbos (2008).

The average number of samples to signal (ANSS) is defined as the expected number of samples from the start of process monitoring at time $t_0 = 0$ to the time that the chart signals. Similarly, the average number of observations to signal (ANOS) is defined to be the expected number of individual observations from t_0 until a signal is generated. When there are no observations missing, the time to signal is equal to the length of the sampling interval times the number of samples to signal, so ATS = d ANSS. The number of observations to signal is the sample size times the number of samples to signal, so

ANOS = n ANSS = (n/d) ATS. In the multivariate case, define the average number of measurements to signal (ANMS) as the expected number of measurements, i.e. individual observations, from the start of process monitoring at time t_0 until the chart signals. When there are no observations missing, ANMS = b ANOS = bn ANSS = (bn/d) ATS. However, these relationships change when observations or samples are missing. If observations may be missing at random, the number of non-missing observations per sample varies from sample to sample. In addition, since entire samples can be missing, the time between sample statistics plotted on the control chart varies randomly. Therefore, the ANSS, ANOS, and ANMS are also useful in evaluating the performance of the control charts considered here.

When there is a shift in a process parameter, the ATS is the appropriate measure of control chart performance only if the shift is assumed to occur when process monitoring starts. However, in reality, a shift may occur at some random time in the future after the start of process monitoring. In this case, the appropriate measure of detection time is the expected length of time from the time of the shift in the process parameter until a signal is generated.

EWMA and MEWMA statistics accumulate information over time, and the expected length of time from the shift until the chart signals depends on the value of the control statistic at the time the shift occurs. If the shift occurs a relatively long time after monitoring has started, this can be modeled by using the steady-state ATS (SSATS). The SSATS assumes that the control statistic has reached its steady-state or stationary distribution by the time at which the shift occurs. It also allows for the possibility that the shift in the parameter may occur within a time interval between two samples. The distribution of the time of the shift within this interval is assumed to be uniformly distributed over the interval. Similarly, we can define the steady-state ANSS (SSANSS) and the steady-state ANOS (SSANOS) to be the expected number of samples and the expected number of observations from the shift to the signal, respectively. In the multivariate case, the steady-state ANMS (SSANMS) is defined as the expected number of measurements from the time of the shift until the chart signals.

Consider the situation in which there is a probability p that each individual observation is missing, independently of other observations. Chapters 4 and 5 develop Markov chain and integral equation methods to evaluate the properties of the EWMA control charts with missing values in this case. Another situation of interest occurs when there is a probability p_1 that each individual observation is missing, and a probability p_2 that an entire sample is missing. Simulation with 100,000 runs was used to evaluate the performance of the EWMA charts in this second situation. The MEWMA control charts with missing observations were also evaluated using simulation for the case in which there is a probability p that an individual observation is missing. The steady-state values were obtained by simulating the process for 400 in-control time units and then introducing a shift in $\boldsymbol{\mu}$. The performance of the EWMA and MEWMA charts is evaluated in Chapters 6 and 7.

Chapter 4

Evaluating Properties of EWMA Control Charts Using the Markov Chain Method

This chapter develops a method to model the EWMA control chart with missing observations as a Markov process. The Markov chain method can then be used to evaluate the properties of these control charts. Formulas are derived to compute the ATS, ANSS, and ANOS of the EWMA chart with missing observations. Finally, expressions for the SSATS, SSANSS, and SSANOS are presented in the last three sections of this chapter.

4.1 Markov Chain Method

Consider an EWMA control chart applied to monitor a process where samples of size $n \geq 1$ are taken every d time units. Assume that each individual observation has probability p of being missing, independently of other observations, and let $q = 1 - p$. This could then result in part of a sample or all of a sample being missing. The probability of m observations missing in a sample, i.e. having a sample of size $n - m$ non-missing observations, is $\binom{n}{m} p^m q^{n-m}$, for $m = 0, 1, 2, \ldots, n$.

Suppose that the maximum number of consecutive samples missing is η. We must set a maximum limit on the number of consecutive samples missing in order to use either the Markov chain or integral equation methods to evaluate the properties of the EWMA control charts with missing observations. In practice this is a realistic constraint, since after several consecutive missing samples have occurred, the process should be evaluated to determine the cause. The missing samples could be the result of a change in the process and adjustments may need to be made. The probability of η consecutive missing samples is $p^{n\eta}$, so as $\eta \to \infty$, $p^{n\eta} \to 0$ and it becomes highly unlikely for many consecutive samples to be missing.

The EWMA control statistic is continuous when the process observations are assumed to follow a normal distribution. The approximate properties of the EWMA control chart can be obtained using a discretized version. Let C represent the in-control region, such that if the control statistic E_k falls in C at time t_k, then no signal is given and monitoring continues to time t_{k+1} when the next sample is taken, but if E_k falls outside of C, then the chart signals. Partition C into r intervals, I_1, I_2, …, I_r, and let y_j represent the midpoint of interval I_j for j = 1, 2, …, r. Let w be the width of each interval, so $w = y_{j+1} - y_j$ for all j = 1, 2, …, r-1. Suppose the sample at time t_k has $m < n$ missing observations, so at least one observation is known. Also suppose that the previous i consecutive samples from times t_{k-1}, t_{k-2}, …, t_{k-i} are missing. If it is known that E_{k-i-1} was in interval I_j, we will assume as an approximation that $E_{k-i-1} = y_j$. Then the probability the control statistic is in interval $I_{j'}$ at time t_k given the control statistic is in interval I_j at time t_{k-i-1} can be computed as

$$p_{im}(j'|j) = P(E_k \in I_{j'} | E_{k-i-1} = y_j \text{, sample at } t_k \text{ has } m \text{ missing observations,}$$
$$\text{samples at } t_{k-1}, t_{k-2}, \ldots, t_{k-i} \text{ are missing)}$$
$$= P\left(y_{j'} - \frac{w}{2} \le E_k \le y_{j'} + \frac{w}{2} \middle| E_{k-i-1} = y_j \text{, sample at } t_k \text{ has } m \text{ missing}\right.$$
$$\text{observations, samples at } t_{k-1}, t_{k-2}, \ldots, t_{k-i} \text{ are missing).} \qquad (4.1)$$

The probability in (4.1) can then be computed for the chosen control statistic E_k by using the standard normal distribution.

For example, if the EI chart is used with the control statistic given in (3.7), then (4.1) is

$$p_{im}\left(j'\middle|j\right) = P\left(y_{j'} - \frac{w}{2} \le E_k^I \le y_{j'} + \frac{w}{2} \middle| E_{k-i-1}^I = y_j \right)$$

$$= P\left(y_{j'} - \frac{w}{2} \le (1-\lambda)E_{k-i-1}^I + \lambda Z_k \le y_{j'} + \frac{w}{2} \middle| E_{k-i-1}^I = y_j \right)$$

$$= P\left(y_{j'} - \frac{w}{2} \le (1-\lambda)y_j + \lambda Z_k \le y_{j'} + \frac{w}{2} \right)$$

$$= P\left(\frac{y_{j'} - \frac{w}{2} - (1-\lambda)y_j}{\lambda} \le Z_k \le \frac{y_{j'} + \frac{w}{2} - (1-\lambda)y_j}{\lambda} \right) \tag{4.2}$$

Since process observations are assumed to follow a normal distribution with mean μ_0 and variance σ_0^2, then Z_k follows a standard normal distribution. The probability given by (4.2) can then be evaluated as

$$\Phi\left(\frac{y_{j'} + \frac{w}{2} - (1-\lambda)y_j}{\lambda} \right) - \Phi\left(\frac{y_{j'} - \frac{w}{2} - (1-\lambda)y_j}{\lambda} \right) \tag{4.3}$$

where $\Phi(z)$ is the standard normal c.d.f.

If the EA chart is used with the control statistic given in (3.9), then (4.1) is

$$p_{im}\left(j'\middle|j\right) = P\left(y_{j'} - \frac{w}{2} \le E_k^A \le y_{j'} + \frac{w}{2} \middle| E_{k-(i+1)}^A = y_j \right)$$

$$= P\left(y_{j'} - \frac{w}{2} \le (1-\lambda)^{i+1} E_{k-(i+1)}^A + \sum_{l=0}^{i} (1-\lambda)^l \lambda Z_k \le y_{j'} + \frac{w}{2} \middle| E_{k-(i+1)}^A = y_j \right)$$

$$= P\left(y_{j'} - \frac{w}{2} \le (1-\lambda)^{i+1} y_j + \sum_{l=0}^{i} (1-\lambda)^l \lambda Z_k \le y_{j'} + \frac{w}{2} \right)$$

$$= P\left(\frac{y_{j'} - \frac{w}{2} - (1-\lambda)^{i+1} y_j}{\sum_{l=0}^{i} (1-\lambda)^l \lambda} \le Z_k \le \frac{y_{j'} + \frac{w}{2} - (1-\lambda)^{i+1} y_j}{\sum_{l=0}^{i} (1-\lambda)^l \lambda} \right). \qquad (4.4)$$

The probability given by (4.4) can then be evaluated as

$$\Phi\left(\frac{y_{j'} + \frac{w}{2} - (1-\lambda)^{i+1} y_j}{\sum_{l=0}^{i} (1-\lambda)^l \lambda} \right) - \Phi\left(\frac{y_{j'} - \frac{w}{2} - (1-\lambda)^{i+1} y_j}{\sum_{l=0}^{i} (1-\lambda)^l \lambda} \right). \qquad (4.5)$$

If the EP chart is used with the control statistic given in (3.10), then (4.1) is

$$p_{im}(j'|j) = P\left(y_{j'} - \frac{w}{2} \le E_k^P \le y_{j'} + \frac{w}{2} \middle| E_{k-(i+1)}^P = y_j \right)$$

$$= P\left(y_{j'} - \frac{w}{2} \le \frac{(1-\lambda)^{i+1}}{1 - \sum_{l=1}^{i} (1-\lambda)^l \lambda} E_{k-(i+1)}^P + \frac{\lambda}{1 - \sum_{l=1}^{i} (1-\lambda)^l \lambda} Z_k \le y_{j'} + \frac{w}{2} \middle| E_{k-(i+1)}^P = y_j \right)$$

$$= P\left(y_{j'} - \frac{w}{2} \le \frac{(1-\lambda)^{i+1}}{1 - \sum_{l=1}^{i} (1-\lambda)^l \lambda} y_j + \frac{\lambda}{1 - \sum_{l=1}^{i} (1-\lambda)^l \lambda} Z_k \le y_{j'} + \frac{w}{2} \right)$$

$$= P\left(\left(y_{j'} - \frac{w}{2} \right) \left(\frac{1 - \sum_{l=1}^{i} (1-\lambda)^l}{\lambda} \right) - \frac{(1-\lambda)^{i+1} y_j}{\lambda} \le Z_k \right.$$

$$\left. \le \left(y_{j'} + \frac{w}{2} \right) \left(\frac{1 - \sum_{l=1}^{i} (1-\lambda)^l}{\lambda} \right) - \frac{(1-\lambda)^{i+1} y_j}{\lambda} \right). \qquad (4.6)$$

The probability given by (4.6) can then be evaluated as

$$\Phi\left(\left(y_{j'}+\frac{w}{2}\right)\left(\frac{1-\sum_{l=1}^{i}(1-\lambda)^{l}}{\lambda}\right)-\frac{(1-\lambda)^{i+1}y_j}{\lambda}\right)$$

$$-\Phi\left(\left(y_{j'}-\frac{w}{2}\right)\left(\frac{1-\sum_{l=1}^{i}(1-\lambda)^{l}}{\lambda}\right)-\frac{(1-\lambda)^{i+1}y_j}{\lambda}\right).$$

(4.7)

The transition matrix **Q** used in the Markov process can be partitioned into a number of submatrices. Let $\mathbf{Q}_{im}$ be the submatrix corresponding to the situation in which the previous i consecutive samples are missing and the current sample has $m < n$ observations missing. The submatrix $\mathbf{Q}_{im}$ depends on i and m due to the fact that adjustments may be made to the weighting of the observations in the EWMA control statistic when some past samples are missing and/or when some observations of the current sample are missing. There are r states corresponding to the partition of C into r subintervals, so $\mathbf{Q}_{im}$ has dimension $r \times r$ for each $i = 0, 1, 2, \ldots, \eta$ and $m = 0, 1, 2, \ldots, n\text{-}1$. Being in state j of $\mathbf{Q}_{im}$ at time t_k corresponds to the EWMA control statistic being in interval I_j at time t_k for $j = 1, 2, \ldots, r$ when the previous i consecutive samples are missing and the current sample has at least one non-missing observation. Then the element of $\mathbf{Q}_{im}$ in row j, column j' is $p_{im}(j'|j)$, the probability that the EWMA control statistic is in interval $I_{j'}$ at time t_k given that the control statistic is in interval I_j at time $t_{k\text{-}i\text{-}1}$, the samples at times t_{k-1}, $t_{k\text{-}2}, \ldots,$ t_{k-i} are missing, and the sample at time t_k has $m < n$ observations missing. Note that the submatrix $\mathbf{Q}_{im}$, as defined here, does not contain any terms for the probability of observations being missing.

4.2 Average Time to Signal

In this section, an expression for the ATS will be derived using the Markov chain method. Let A_{ij} be the expected time to a signal from a sampling point at which the previous i consecutive samples are missing, including the current sample, and the EWMA control statistic is in interval I_j. Let $\mathbf{A}_i = (A_{i1}, A_{i2}, ..., A_{ir})^T$ and let $\mathbf{d}$ be the $r \times 1$ vector with each component d.

Consider a sampling point at which the previous i consecutive samples are missing, including the current sample, for $i = 0, 1, 2, \ldots, \eta - 1$. The probability that m observations will be missing in the next sample is $\binom{n}{m} p^m q^{n-m}$ for $m = 0, 1, 2, \ldots, n$. If $m < n$, then the time to signal is d time units plus the time to signal from a sampling point at which at least one observation in the previous sample is known. If $m = n$, then the time to signal is d time units plus the time to signal from a sampling point at which the previous $i + 1$ consecutive samples are missing. Then

$$\begin{aligned}\mathbf{A}_i &= \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left(\mathbf{d} + \mathbf{Q}_{i,m}\mathbf{A}_0\right) + p^n \left(\mathbf{d} + \mathbf{A}_{i+1}\right) \\ &= \mathbf{d} + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{i,m}\mathbf{A}_0 + p^n \mathbf{A}_{i+1}. \qquad (4.8)\end{aligned}$$

Now consider a sampling point at which the previous η consecutive samples are missing, including the current sample, so at least one observation in the next sample must be known. The probability that m observations will be missing is $\frac{1}{1-p^n}\binom{n}{m} p^m q^{n-m}$ for $m =$ 0, 1, 2, …, $n - 1$. Then the time to signal is

$$\begin{aligned}\mathbf{A}_\eta &= \frac{1}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left(\mathbf{d} + \mathbf{Q}_{\eta,m}\mathbf{A}_0\right) \\ &= \mathbf{d} + \frac{1}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta,m}\mathbf{A}_0. \qquad (4.9)\end{aligned}$$

Therefore, the set $\{\mathbf{A}_i\}$ must satisfy the following equations.

$$
\begin{aligned}
\mathbf{A}_0 &= \mathbf{d} + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{0,m} \mathbf{A}_0 + p^n \mathbf{A}_1 \\
\mathbf{A}_1 &= \mathbf{d} + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{1,m} \mathbf{A}_0 + p^n \mathbf{A}_2 \\
\mathbf{A}_2 &= \mathbf{d} + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{2,m} \mathbf{A}_0 + p^n \mathbf{A}_3 \\
&\vdots \\
\mathbf{A}_{\eta-1} &= \mathbf{d} + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta-1,m} \mathbf{A}_0 + p^n \mathbf{A}_\eta \\
\mathbf{A}_\eta &= \mathbf{d} + \frac{1}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta,m} \mathbf{A}_0
\end{aligned}
\tag{4.10}
$$

Writing these equations in matrix form gives

$$
\mathbf{A} = \mathbf{d} + \mathbf{Q}\mathbf{A} \tag{4.11}
$$

where $\mathbf{A} = (\mathbf{A}_0, \mathbf{A}_0, \ldots \mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_\eta)^{\mathrm{T}}$ is a vector containing n copies of $\mathbf{A}_0$ and one copy of each of $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_\eta$, and $\mathbf{Q}$ is the $(r(n + \eta)) \times (r(n + \eta))$ transition matrix

$$
\mathbf{Q} = \begin{bmatrix}
q^n \mathbf{Q}_{0,0} & npq^{n-1}\mathbf{Q}_{0,1} & \cdots & np^{n-1}q\mathbf{Q}_{0,n-1} & p^n\mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
q^n \mathbf{Q}_{0,0} & npq^{n-1}\mathbf{Q}_{0,1} & \cdots & np^{n-1}q\mathbf{Q}_{0,n-1} & p^n\mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & & \\
q^n \mathbf{Q}_{0,0} & npq^{n-1}\mathbf{Q}_{0,1} & \cdots & np^{n-1}q\mathbf{Q}_{0,n-1} & p^n\mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
q^n \mathbf{Q}_{1,0} & npq^{n-1}\mathbf{Q}_{1,1} & \cdots & np^{n-1}q\mathbf{Q}_{0,n-1} & \mathbf{0} & p^n\mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\
q^n \mathbf{Q}_{2,0} & npq^{n-1}\mathbf{Q}_{2,1} & \cdots & np^{n-1}q\mathbf{Q}_{0,n-1} & \mathbf{0} & \mathbf{0} & p^n\mathbf{I} & & \mathbf{0} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \ddots & \\
q^n \mathbf{Q}_{\eta-1,0} & npq^{n-1}\mathbf{Q}_{\eta-1,1} & \cdots & np^{n-1}q\mathbf{Q}_{0,n-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & p^n\mathbf{I} \\
\frac{q^n \mathbf{Q}_{\eta,0}}{1-p^n} & \frac{npq^{n-1}\mathbf{Q}_{\eta,1}}{1-p^n} & \cdots & \frac{np^{n-1}q\mathbf{Q}_{\eta,n-1}}{1-p^n} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}
\end{bmatrix}
\tag{4.12}
$$

Solving equation (4.11) for $\mathbf{A}$ gives

$$
\mathbf{A} = (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{d} \tag{4.13}
$$

where $\mathbf{I}$ is the $(r(n + \eta)) \times (r(n + \eta))$ identity matrix.

However, note that the $(r(n + \eta)) \times (r(n + \eta))$ transition matrix $\mathbf{Q}$ has a fairly simple structure. In fact, it is possible to obtain a solution by working only with the submatrices $\mathbf{Q}_{im}$ of dimension $r \times r$, which may be especially useful when r, η and n are large. To obtain the solution using the $r \times r$ submatrices, the set of $\eta + 1$ equations given by (4.10) can be solved to obtain an expression for $\mathbf{A}_0$ by starting with the last one and working backwards.

$$\begin{aligned}
\mathbf{A}_{\eta-1} &= \mathbf{d} + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta-1,m} \mathbf{A}_0 + p^n \mathbf{A}_\eta \\
&= \mathbf{d} + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta-1,m} \mathbf{A}_0 + p^n \left[\mathbf{d} + \frac{1}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta,m} \mathbf{A}_0 \right] \\
&= (1 + p^n)\mathbf{d} + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta-1,m} \mathbf{A}_0 + \frac{p^n}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta,m} \mathbf{A}_0 \qquad (4.14)
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}_{\eta-2} &= \mathbf{d} + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta-2,m} \mathbf{A}_0 + p^n \mathbf{A}_{\eta-1} \\
&= \mathbf{d} + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta-2,m} \mathbf{A}_0 + p^n \left[(1 + p^n)\mathbf{d} + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta-1,m} \mathbf{A}_0 \right. \\
&\qquad \left. + \frac{p^n}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta,m} \mathbf{A}_0 \right] \\
&= (1 + p^n + p^{2n})\mathbf{d} + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta-2,m} \mathbf{A}_0 + p^n \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta-1,m} \mathbf{A}_0 \\
&\qquad + \frac{p^{2n}}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta,m} \mathbf{A}_0 \qquad (4.15)
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}_{\eta-3} &= \mathbf{d} + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta-3,m} \mathbf{A}_0 + p^n \mathbf{A}_{\eta-2} \\
&= \mathbf{d} + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta-3,m} \mathbf{A}_0 + p^n \left[(1 + p^n + p^{2n})\mathbf{d} + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta-2,m} \mathbf{A}_0 \right. \\
&\qquad \left. + p^n \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta-1,m} \mathbf{A}_0 + \frac{p^{2n}}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta,m} \mathbf{A}_0 \right]
\end{aligned}$$

$$= \left(1 + p^n + p^{2n} + p^{3n}\right)\mathbf{d} + \sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{\eta-3,m}\mathbf{A}_0 + p^n\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{\eta-2,m}\mathbf{A}_0$$

$$+ p^{2n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{\eta-1,m}\mathbf{A}_0 + \frac{p^{3n}}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{\eta,m}\mathbf{A}_0 \qquad (4.16)$$

Continuing in this manner, gives

$$\mathbf{A}_i = \left(\sum_{j=0}^{\eta-i} p^{jn}\right)\mathbf{d} + \sum_{j=0}^{\eta-i-1} p^{jn}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{i+j,m}\mathbf{A}_0 + \frac{p^{(\eta-i)n}}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{\eta,m}\mathbf{A}_0$$

$$= \left(\frac{1-p^{(\eta-i+1)n}}{1-p^n}\right)\mathbf{d} + \left[\sum_{j=0}^{\eta-i-1} p^{jn}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{i+j,m} + \frac{p^{(\eta-i)n}}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{\eta,m}\right]\mathbf{A}_0 \qquad (4.17)$$

for $i = 0, 1, \ldots, \eta$. Then

$$\mathbf{A}_0 = \left(\frac{1-p^{(\eta+1)n}}{1-p^n}\right)\mathbf{d} + \left[\sum_{j=0}^{\eta-1} p^{jn}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{j,m} + \frac{p^{\eta n}}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{\eta,m}\right]\mathbf{A}_0$$

(4.18)

and so the solution for $\mathbf{A}_0$ is

$$\mathbf{A}_0 = \left(\mathbf{I} - \left[\sum_{j=0}^{\eta-1} p^{jn}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{j,m} + \frac{p^{\eta n}}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{\eta,m}\right]\right)^{-1}\left(\frac{1-p^{(\eta+1)n}}{1-p^n}\right)\mathbf{d}\,.$$

(4.19)

Once the solution for $\mathbf{A}_0$ is obtained, the solutions for $\mathbf{A}_1$, $\mathbf{A}_2$, …, $\mathbf{A}_\eta$ can be obtained by substituting in equation (4.17), which gives

$$\mathbf{A}_i = \left(\frac{1-p^{(\eta-i+1)n}}{1-p^n}\right)\mathbf{d} + \left[\sum_{j=1}^{\eta-i} p^{(\eta-i-j)n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{\eta-j,m} + \frac{p^{(\eta-i)n}}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{\eta,m}\right]$$

$$* \left(\mathbf{I} - \left[\sum_{j=0}^{\eta-1} p^{jn}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{j,m} + \frac{p^{\eta n}}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{\eta,m}\right]\right)^{-1}\left(\frac{1-p^{(\eta+1)n}}{1-p^n}\right)\mathbf{d}.$$

(4.20)

The ATS can then be expressed as

ATS = (time from t_0 to t_1) + $\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}$ *(expected time from t_1 to signal

given sample at t_1 has m missing observations)

+ p^n *(expected time from t_1 to signal given sample at t_1 is missing)

$$= d + \sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\sum_{j=1}^{r} p_{om}\left(j|s\right)A_{0j} + p^n A_{1s} \tag{4.21}$$

where s is the starting state of the EWMA control statistic.

4.3 Average Number of Samples to Signal

An expression for the ANSS using the Markov chain method can be obtained in a manner similar to the derivation of the expression for the ATS in Section 4.2. Let N_{ij} be the number of samples to signal from a sampling point at which the previous i consecutive samples are missing, including the current sampling point, and the EWMA statistic is in interval I_j. Let $\mathbf{N}_i = \left(N_{i1}, N_{i2}, ..., N_{ir}\right)^T$ and let $\mathbf{1}$ be the $r \times 1$ vector with each component 1.

Consider a sampling point at which the previous i consecutive samples are missing, including the current sample, for $i = 0, 1, 2, \ldots, \eta - 1$. If $m < n$, then the number of samples to signal is one plus the number of samples to signal from a sampling point at which at least one observation in the previous sample is known. If $m = n$, then the number of samples to signal is the number of samples to signal from a sampling point at which the previous $i+1$ consecutive samples are missing. Then

$$\mathbf{N}_i = \sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left(\mathbf{1} + \mathbf{Q}_{i,m}\mathbf{N}_0\right) + p^n \mathbf{N}_{i+1} \tag{4.22}$$

Now consider a sampling point at which the previous η consecutive samples are missing, including the current sample, so at least one observation in the next sample must be known. Then the number of samples to signal is

$$\mathbf{N}_{\eta} = \frac{1}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left(\mathbf{1}+\mathbf{Q}_{\eta,m}\mathbf{N}_0\right)$$

$$= \mathbf{1} + \frac{1}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{\eta,m}\mathbf{N}_0\,. \tag{4.23}$$

Therefore, the set $\{\mathbf{N}_i\}$ must satisfy the following equations,

$$\begin{aligned}
\mathbf{N}_0 &= \sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left(\mathbf{1}+\mathbf{Q}_{0,m}\mathbf{N}_0\right) + p^n\mathbf{N}_1 \\
\mathbf{N}_1 &= \sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left(\mathbf{1}+\mathbf{Q}_{1,m}\mathbf{N}_0\right) + p^n\mathbf{N}_2 \\
\mathbf{N}_2 &= \sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left(\mathbf{1}+\mathbf{Q}_{2,m}\mathbf{N}_0\right) + p^n\mathbf{N}_3 \\
&\;\;\vdots \\
\mathbf{N}_{\eta-1} &= \sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left(\mathbf{1}+\mathbf{Q}_{\eta-1,m}\mathbf{N}_0\right) + p^n\mathbf{N}_{\eta} \\
\mathbf{N}_{\eta} &= \mathbf{1} + \frac{1}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{\eta,m}\mathbf{N}_0
\end{aligned} \tag{4.24}$$

Writing these equations in matrix form gives

$$\mathbf{N} = \mathbf{v} + \mathbf{QN} \tag{4.25}$$

where $\mathbf{N} = (\mathbf{N}_0, \mathbf{N}_0, \ldots, \mathbf{N}_0, \mathbf{N}_1, \mathbf{N}_2, \ldots, \mathbf{N}_{\eta})^{\mathrm{T}}$ is a vector containing n copies of $\mathbf{N}_0$ and one copy of each of $\mathbf{N}_1, \mathbf{N}_2, \ldots, \mathbf{N}_{\eta}$, $\mathbf{v}$ represents an $(r(n+\eta)) \times 1$ vector with the first $r(n+\eta-1)$ components $\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}$ and the last r components 1, and $\mathbf{Q}$ is the $(r(n+\eta)) \times (r(n+\eta))$ transition matrix given by (4.12). Solving equation (4.25) for $\mathbf{N}$ gives

$$\mathbf{N} = (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{v}\,. \tag{4.26}$$

Since the transition matrix $\mathbf{Q}$ has a fairly simple structure, it is possible to obtain a solution by working only with the submatricies $\mathbf{Q}_{im}$ in a manner similar to the method used in Section 4.2. To obtain the solution using the $r \times r$ submatrices, the set of $\eta + 1$

equations in (4.24) can be solved to obtain an expression for $\mathbf{N}_0$ by starting with the last one and working backwards.

$$\begin{aligned}
\mathbf{N}_{\eta-1} &= \sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left(\mathbf{1}+\mathbf{Q}_{\eta-1,m}\mathbf{N}_0\right)+p^n\mathbf{N}_\eta \\
&= \sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left(\mathbf{1}+\mathbf{Q}_{\eta-1,m}\mathbf{N}_0\right)+p^n\left[\mathbf{1}+\frac{1}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{\eta,m}\mathbf{N}_0\right] \\
&= \left(p^n+\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\right)\mathbf{1}+\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{\eta-1,m}\mathbf{N}_0 \\
&\qquad +\frac{p^n}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{\eta,m}\mathbf{N}_0 \\
&= \mathbf{1}+\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{\eta-1,m}\mathbf{N}_0+\frac{p^n}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{\eta,m}\mathbf{N}_0 \qquad (4.27)
\end{aligned}$$

$$\begin{aligned}
\mathbf{N}_{\eta-2} &= \sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left(\mathbf{1}+\mathbf{Q}_{\eta-2,m}\mathbf{N}_0\right)+p^n\mathbf{N}_{\eta-1} \\
&= \sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left(\mathbf{1}+\mathbf{Q}_{\eta-2,m}\mathbf{N}_0\right)+p^n\left[\mathbf{1}+\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{\eta-1,m}\mathbf{N}_0\right. \\
&\qquad \left.+\frac{p^n}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{\eta,m}\mathbf{N}_0\right] \\
&= \mathbf{1}+\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{\eta-2,m}\mathbf{N}_0+p^n\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{\eta-1,m}\mathbf{N}_0 \\
&\qquad +\frac{p^{2n}}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{\eta,m}\mathbf{N}_0 \qquad (4.28)
\end{aligned}$$

$$\begin{aligned}
\mathbf{N}_{\eta-3} &= \sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left(\mathbf{1}+\mathbf{Q}_{\eta-3,m}\mathbf{N}_0\right)+p^n\mathbf{N}_{\eta-2} \\
&= \sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left(\mathbf{1}+\mathbf{Q}_{\eta-3,m}\mathbf{N}_0\right)+p^n\left[\mathbf{1}+\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\mathbf{Q}_{\eta-2,m}\mathbf{N}_0\right.
\end{aligned}$$

$$+ p^n \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta-1,m} \mathbf{N}_0 + \frac{p^{2n}}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta,m} \mathbf{N}_0 \Bigg]$$

$$= \mathbf{1} + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta-3,m} \mathbf{N}_0 + p^n \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta-2,m} \mathbf{N}_0$$

$$+ p^{2n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta-1,m} \mathbf{N}_0 + \frac{p^{3n}}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta,m} \mathbf{N}_0 \qquad (4.29)$$

Continuing in this manner, gives

$$\mathbf{N}_i = \mathbf{1} + \left[\sum_{j=0}^{\eta-i-1} p^{jn} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{i+j,m} + \frac{p^{(\eta-i)n}}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta,m} \right] \mathbf{N}_0 \qquad (4.30)$$

for $i = 0, 1, \ldots, \eta$. Then

$$\mathbf{N}_0 = \mathbf{1} + \left[\sum_{j=0}^{\eta-1} p^{jn} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{j,m} + \frac{p^{\eta n}}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta,m} \right] \mathbf{N}_0 \qquad (4.31)$$

and so the solution for $\mathbf{N}_0$ is

$$\mathbf{N}_0 = \left(\mathbf{I} - \left[\sum_{j=0}^{\eta-1} p^{jn} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{j,m} + \frac{p^{\eta n}}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta,m} \right] \right)^{-1} \mathbf{1}. \qquad (4.32)$$

Once the solution for $\mathbf{N}_0$ is obtained, the solutions for $\mathbf{N}_1$, $\mathbf{N}_2$, …, $\mathbf{N}_\eta$ can be obtained by substituting it in equation (4.30), which gives

$$\mathbf{N}_i = \mathbf{1} + \left[\sum_{j=0}^{\eta-i-1} p^{jn} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{i+j,m} + \frac{p^{(\eta-i)n}}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta,m} \right]$$

$$* \left(\mathbf{I} - \left[\sum_{j=0}^{\eta-1} p^{jn} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{j,m} + \frac{p^{\eta n}}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta,m} \right] \right)^{-1} \mathbf{1}. \qquad (4.33)$$

Therefore, the ANSS can be expressed as

ANSS = $\sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m}$ *(number of samples from t_0 to t_1 + expected number of

samples from t_1 to signal given m observations missing in sample at t_1)

$+ p^n$ *(expected number of samples from t_1 to signal given sample at t_1 is missing)

$$= \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[1 + \sum_{j=1}^{r} p_{0m}(j|s) N_{0j} \right] + p^n N_{1s}$$

$$= 1 - p^n + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \sum_{j=1}^{r} p_{0m}(j|s) N_{0j} + p^n N_{1s} \qquad (4.34)$$

where s is the starting state of the EWMA control statistic.

4.4 Average Number of Observations to Signal

An expression for the ANOS can also be obtained using the Markov chain method in a manner similar to the derivation of the expressions for the ATS and ANSS. Let O_{ij} be the number of observations to signal from a sampling point at which the previous i consecutive samples are missing, including the current sample, and the EWMA statistic is in interval I_j, and let $\mathbf{O}_i = (O_{i1}, O_{i2}, ..., O_{ir})^T$.

First consider a sampling point at which the previous i consecutive samples are missing, including the current sample, for i = 0, 1, 2, …, η – 1. If $m < n$, then the number of observations to signal is $n - m$ plus the number of observations to signal from a sampling point at which at least one observation in the previous sample is known. If m = n, then the number of observations to signal is the number of observations to signal from a sampling point at which the previous i + 1 consecutive samples are missing. Then

$$\mathbf{O}_i = \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[(n-m)\mathbf{1} + \mathbf{Q}_{i,m}\mathbf{O}_0 \right] + p^n \mathbf{O}_{i+1}. \qquad (4.35)$$

Next consider a sampling point at which the previous η consecutive samples are missing, including the current sample, so at least one observation in the next sample must be known. Then the number of observations to signal is

$$\mathbf{O}_\eta = \frac{1}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[(n-m)\mathbf{1} + \mathbf{Q}_{\eta m}\mathbf{O}_0 \right]. \qquad (4.36)$$

Then the set $\{\mathbf{O}_i\}$ must satisfy the following equations.

$$\mathbf{O}_0 = \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[(n-m)\mathbf{1} + \mathbf{Q}_{0,m}\mathbf{O}_0\right] + p^n \mathbf{O}_1$$

$$\mathbf{O}_1 = \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[(n-m)\mathbf{1} + \mathbf{Q}_{1,m}\mathbf{O}_0\right] + p^n \mathbf{O}_2$$

$$\mathbf{O}_2 = \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[(n-m)\mathbf{1} + \mathbf{Q}_{2,m}\mathbf{O}_0\right] + p^n \mathbf{O}_3$$

$$\vdots \tag{4.37}$$

$$\mathbf{O}_{\eta-1} = \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[(n-m)\mathbf{1} + \mathbf{Q}_{\eta-1,m}\mathbf{O}_0\right] + p^n \mathbf{O}_\eta$$

$$\mathbf{O}_\eta = \frac{1}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[(n-m)\mathbf{1} + \mathbf{Q}_{\eta m}\mathbf{O}_0\right]$$

Writing these equations in matrix form gives

$$\mathbf{O} = \mathbf{w} + \mathbf{QO} \tag{4.38}$$

where $\mathbf{O} = (\mathbf{O}_0, \mathbf{O}_0, \ldots, \mathbf{O}_0, \mathbf{O}_1, \mathbf{O}_2, \ldots, \mathbf{O}_\eta)^{\mathrm{T}}$ is a vector containing n copies of $\mathbf{O}_0$ and one copy of each of $\mathbf{O}_1, \mathbf{O}_2, \ldots, \mathbf{O}_\eta$, $\mathbf{w}$ represents an $(r(n + \eta)) \times 1$ vector with the first $r(n+\eta-1)$ components $\sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m}(n-m)$ and the last r components $\frac{1}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m}(n-m)$, and $\mathbf{Q}$ is the $(r(n + \eta)) \times (r(n + \eta))$ transition matrix given by (4.12). Solving equation (4.38) for $\mathbf{O}$ gives

$$\mathbf{O} = (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{w}\,. \tag{4.39}$$

Due to the structure of the transition matrix **Q,** a solution can be obtained by working only with the submatricies $\mathbf{Q}_{im}$, in a manner similar to the method used in Sections 4.2 and 4.3. To obtain the solution using the $r \times r$ submatrices, the set of $\eta + 1$ equations in (4.37) can be solved to obtain an expression for $\mathbf{O}_0$ by starting with the last one and working backwards.

$$\mathbf{O}_{\eta-1} = \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[(n-m)\mathbf{1} + \mathbf{Q}_{\eta-1,m}\mathbf{O}_0\right] + p^n \mathbf{O}_\eta$$

$$= \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[(n-m)\mathbf{1} + \mathbf{Q}_{\eta-1,m}\mathbf{O}_0 \right] + \frac{p^n}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[(n-m)\mathbf{1} + \mathbf{Q}_{\eta m}\mathbf{O}_0 \right]$$

$$= \left(1 + \frac{p^n}{1-p^n}\right) \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} (n-m)\mathbf{1} + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta-1,m}\mathbf{O}_0$$

$$+ \frac{p^n}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta m}\mathbf{O}_0 \qquad (4.40)$$

$$\mathbf{O}_{\eta-2} = \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[(n-m)\mathbf{1} + \mathbf{Q}_{\eta-2,m}\mathbf{O}_0 \right] + p^n \mathbf{O}_{\eta-1}$$

$$= \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[(n-m)\mathbf{1} + \mathbf{Q}_{\eta-2,m}\mathbf{O}_0 \right] + p^n \left[\left(1 + \frac{p^n}{1-p^n}\right) \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} (n-m)\mathbf{1} \right.$$

$$\left. + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta-1,m}\mathbf{O}_0 + \frac{p^n}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta m}\mathbf{O}_0 \right]$$

$$= \left(1 + p^n + \frac{p^{2n}}{1-p^n}\right) \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} (n-m)\mathbf{1} + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta-2,m}\mathbf{O}_0$$

$$+ p^n \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta-1,m}\mathbf{O}_0 + \frac{p^{2n}}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta m}\mathbf{O}_0 \qquad (4.41)$$

$$\mathbf{O}_{\eta-3} = \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[(n-m)\mathbf{1} + \mathbf{Q}_{\eta-3,m}\mathbf{O}_0 \right] + p^n \mathbf{O}_{\eta-2}$$

$$= \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[(n-m)\mathbf{1} + \mathbf{Q}_{\eta-3,m}\mathbf{O}_0 \right] + p^n \left[\left(1 + p^n + \frac{p^{2n}}{1-p^n}\right) \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} (n-m)\mathbf{1} \right.$$

$$\left. + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta-2,m}\mathbf{O}_0 + p^n \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta-1,m}\mathbf{O}_0 + \frac{p^{2n}}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta m}\mathbf{O}_0 \right]$$

$$= \left(1 + p^n + p^{2n} + \frac{p^{3n}}{1-p^n}\right) \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} (n-m)\mathbf{1} + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta-3,m}\mathbf{O}_0$$

$$+ p^n \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta-2,m}\mathbf{O}_0 + p^{2n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta-1,m}\mathbf{O}_0 + \frac{p^{3n}}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta m}\mathbf{O}_0$$

$$(4.42)$$

Continuing in this manner, gives

$$\mathbf{O}_i = \left(\sum_{j=0}^{\eta-i-1} p^{jn} + \frac{p^{(\eta-i)n}}{1-p^n} \right) \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} (n-m) \mathbf{1} + \sum_{j=0}^{\eta-i-1} p^{jn} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{i+j,m} \mathbf{O}_0$$

$$+ \frac{p^{(\eta-i)n}}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta,m} \mathbf{O}_0$$

$$= \frac{1}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} (n-m) \mathbf{1} + \left[\sum_{j=0}^{\eta-i-1} p^{jn} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{i+j,m} \right.$$

$$\left. + \frac{p^{(\eta-i)n}}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta,m} \right] \mathbf{O}_0 \qquad (4.43)$$

for $i = 0, 1, \ldots, \eta$. Then

$$\mathbf{O}_0 = \frac{1}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} (n-m) \mathbf{1} + \left[\sum_{j=0}^{\eta-1} p^{jn} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{j,m} \right.$$

$$\left. + \frac{p^{\eta n}}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta,m} \right] \mathbf{O}_0 \qquad (4.44)$$

and so the solution for $\mathbf{O}_0$ is

$$\mathbf{O}_0 = \left(\mathbf{I} - \left[\sum_{j=0}^{\eta-1} p^{jn} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{j,m} + \frac{p^{\eta n}}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta,m} \right] \right)^{-1}$$

$$* \left(\frac{1}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} (n-m) \mathbf{1} \right). \qquad (4.45)$$

Once the solution for $\mathbf{O}_0$ is obtained, the solutions for $\mathbf{O}_1$, $\mathbf{O}_2$, …, $\mathbf{O}_\eta$ can be obtained by substituting in the equations above, which gives

$$\mathbf{O}_i = \frac{1}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} (n-m) \mathbf{1} + \left[\sum_{j=0}^{\eta-i-1} p^{jn} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{i+j,m} + \frac{p^{(\eta-i)n}}{1-p^n} \right.$$

$$\left. * \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta,m} \right] \left(\mathbf{I} - \left[\sum_{j=0}^{\eta-1} p^{jn} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{j,m} + \frac{p^{\eta n}}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \mathbf{Q}_{\eta,m} \right] \right)^{-1}$$

$$* \left(\frac{1}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} (n-m) \mathbf{1} \right). \qquad (4.46)$$

Thus, the ANOS can be expressed as

ANOS = $\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}$ *(number of observations from t_0 to t_1 + expected number of observations from t_1 to signal given m observations missing in sample at t_1) + p^n *(expected number of observations from t_1 to signal given sample at t_1 is missing)

$$= \sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left[(n-m)+\sum_{j=1}^{r} p_{0m}(j|s)O_{0j}\right]+p^n O_{1s} \tag{4.47}$$

where s is the starting state of the EWMA control statistic.

4.5 Steady-State Average Time to Signal

Let π_{ij} be the stationary distribution of the control statistic conditional on no false alarms when there are i consecutive missing samples and the control statistic is in interval I_j. Let

$$\boldsymbol{\pi}_i = (\pi_{i1}, \pi_{i2}, \ldots, \pi_{ir}) \tag{4.48}$$

and let

$$\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_0, \ldots, \boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \ldots, \boldsymbol{\pi}_\eta). \tag{4.49}$$

Then $\boldsymbol{\pi}$ is the normalized left eigenvector of $\mathbf{Q}$ corresponding to the largest eigenvalue. See Darroch and Senata (1965) and Reynolds (1995) for more details.

It is assumed that when the shift falls within a sampling interval, the position of the shift within the interval is uniformly distributed on the interval. This assumption implies that the expected position of the shift is halfway into the interval. Suppose that the shift in the process parameter falls between times t_k and t_{k+1}, so t_k is the last sampling point before the shift. Also suppose that after t_k, there are i consecutive samples missing, including the one at t_k, and the control statistic is in interval I_j. Then the expected time from the shift to the signal is $A_{ij} - \frac{d}{2}$. Using equation (4.13), the SSATS can then be expressed as

$$\begin{aligned} \text{SSATS} &= \sum_{i=0}^{\eta}\sum_{j=1}^{n}\pi_{ij}\left(A_{ij} - \frac{d}{2}\right) \\ &= \boldsymbol{\pi}\left(\mathbf{A} - \frac{\mathbf{d}}{2}\right) \\ &= \boldsymbol{\pi}\left((\mathbf{I}-\mathbf{Q})^{-1} - \frac{\mathbf{1}}{2}\right)\mathbf{d}. \end{aligned} \tag{4.50}$$

4.6 Steady-State Average Number of Samples to Signal

The SSANSS can also be obtained using the Markov chain method. Suppose that the shift in the process parameter falls between times t_k and t_{k+1}, so t_k is the last sampling point before the shift. Also suppose that after t_k, there are i consecutive samples missing, including the one at t_k, and the control statistic is in interval I_j. Then the expected number of samples from the shift to the signal is N_{ij}. Using equation (4.26), the SSANSS can be then expressed as

$$\begin{aligned} \text{SSANSS} &= \sum_{i=0}^{\eta}\sum_{j=1}^{n}\pi_{ij}N_{ij} \\ &= \boldsymbol{\pi}\mathbf{N} \\ &= \boldsymbol{\pi}(\mathbf{I}-\mathbf{Q})^{-1}\mathbf{v}. \end{aligned} \tag{4.51}$$

4.7 Steady-State Average Number of Observations to Signal

An expression for the SSANOS can also be obtained with the Markov chain method using arguments similar to those used for the SSATS and SSANSS. Suppose that the shift in the process parameter falls between times t_k and t_{k+1}, so t_k is the last sampling point before the shift. Also suppose that after t_k, there are i consecutive samples missing, including the one at t_k, and the control statistic is in interval I_j. Then the expected number of observations from the shift to the signal is O_{ij}. Using equation (4.39), the SSANOS can then be expressed as

$$\begin{aligned} \text{SSANOS} &= \sum_{i=0}^{\eta}\sum_{j=1}^{n}\pi_{ij}O_{ij} \\ &= \boldsymbol{\pi}\mathbf{O} \\ &= \boldsymbol{\pi}(\mathbf{I}-\mathbf{Q})^{-1}\mathbf{w}\,. \end{aligned} \tag{4.52}$$

Chapter 5

Evaluating Properties of EWMA Control Charts Using the Integral Equation Method

The EWMA control statistic is continuous when the process observations are assumed to follow a normal distribution, and so its properties can be obtained using the integral equation method. This chapter derives formulas for the ATS, ANSS, and ANOS of the EWMA control chart with missing observations. Expressions for the SSATS, SSANSS, and SSANOS are also presented.

5.1 Integral Equation Method

Let $f_{i,m}(y'|y) = f_{E_k|E_{k-i-1}}(y'|y)$ be the transition density from any point $y \in C$ to a point $y' \in C$ when the i previous consecutive samples at times t_{k-1}, t_{k-2}, …, t_{k-i} are missing, the sample at time t_k has $m < n$ observations missing, and C is the in-control region. The transition density can be found for a given control statistic by using the standard normal distribution.

If the EI chart is used with the control statistic given in (3.7), then the transition density is

$$f_{im}(y'|y) = \phi\left(\frac{y'-(1-\lambda)y}{\lambda}\right)\left(\frac{1}{\lambda}\right) \tag{5.1}$$

where $\phi(z)$ is the standard normal p.d.f. If the EA chart is used with the control statistic given in (3.9), then the transition density is

$$f_{im}(y'|y) = \phi\left(\frac{y'-(1-\lambda)^{i+1}y}{\sum_{l=0}^{i}(1-\lambda)^{l}\lambda}\right)\left(\frac{1}{\sum_{l=0}^{i}(1-\lambda)^{l}\lambda}\right). \tag{5.2}$$

If the EP chart is used with the control statistic given in (3.10), then the transition density is

$$f_{im}(y'|y) = \phi\left(\frac{1-\sum_{l=1}^{i}(1-\lambda)^{l}\lambda}{\lambda}y' - \frac{(1-\lambda)^{i+1}}{\lambda}y\right)\left(\frac{1-\sum_{l=1}^{i}(1-\lambda)^{l}\lambda}{\lambda}\right). \tag{5.3}$$

The ATS, ANSS, and ANOS of the control charts can be expressed using the integral of a function involving the transition density. Approximate numerical solutions to these integral equations can then be found using a quadrature approximation.

5.2 Average Time to Signal

In this section, an expression for the ATS of an EWMA control chart with missing observations will be derived. Let $A_i(y)$ be the time to a signal from a sampling point at which the previous i consecutive samples are missing, including the current sample, and the EWMA statistic corresponds to y. Consider a sampling point at which the previous i consecutive samples are missing, including the current sample, for i = 0, 1, 2, …, $\eta - 1$. If $m < n$, then the time to signal is d time units plus the time to signal from a sampling point at which at least one observation in the previous sample is known. If $m = n$, then the time to signal is d time units plus the time to signal from a sampling point at which the previous $i+1$ consecutive samples are missing, including the current sample. Then

$$A_i(y) = d + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \int_C A_0(y') f_{i,m}(y'|y) dy' + p^n A_{i+1}(y). \qquad (5.4)$$

Now consider a sampling point at which the previous η consecutive samples are missing, so at least one observation in the next sample must be known. Then the time to signal is

$$A_\eta(y) = d + \frac{1}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \int_C A_0(y') f_{\eta,m}(y'|y) dy'. \qquad (5.5)$$

This gives the following set of $\eta + 1$ equations,

$$\begin{aligned}
A_0(y) &= d + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \int_C A_0(y') f_{0,m}(y'|y) dy' + p^n A_1(y) \\
A_1(y) &= d + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \int_C A_0(y') f_{1,m}(y'|y) dy' + p^n A_2(y) \\
A_2(y) &= d + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \int_C A_0(y') f_{2,m}(y'|y) dy' + p^n A_3(y) \\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.6) \\
A_{\eta-1}(y) &= d + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \int_C A_0(y') f_{\eta-1,m}(y'|y) dy' + p^n A_\eta(y) \\
A_\eta(y) &= d + \frac{1}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \int_C A_0(y') f_{\eta,m}(y'|y) dy'
\end{aligned}$$

Thus, ATS can be expressed as

ATS = (time from t_0 to t_1) + $\sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m}$ *(expected time from t_1 to signal

given sample at t_1 has m missing observations)

$+ p^n$ *(expected time from t_1 to signal given sample at t_1 is missing)

$$= d + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \int_C A_0(y) f_{0,m}(y|s) dy + p^n A_1(s) \qquad (5.7)$$

where s is the starting state of the EWMA control statistic.

Numerical solutions to $A_0(y)$, $A_1(y)$, …, $A_\eta(y)$ can be obtained by approximating the integrals using a numerical quadrature approximation. Let y_1, y_2, …, y_r represent a set of r quadrature points in C, and let $a_1, a_2, \ldots, a_r$ be the corresponding quadrature weights. Then the numerical quadrature approximation to the integral of the function $A_i(y)$ is

$$\int_C A_i(y)dy \approx \sum_{j=1}^{r} a_j A_i(y_j). \tag{5.8}$$

Let $\tilde{A}_i(y)$ represent the approximation to $A_i(y)$ when the quadrature approximation is applied to the integrals in (5.6). If $\tilde{A}_i(y)$ is evaluated at $y = y_j$ for $j = 1, 2, \ldots, r$, then the following system of r equations is obtained for each $i = 0, 1, 2, \ldots, \eta - 1$.

$$\begin{aligned}
\tilde{A}_i(y_j) &= d + \sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\sum_{j=1}^{r} a_j f_{i,m}(y_j|y_1)\tilde{A}_0(y_j) + p^n \tilde{A}_{i+1}(y_1) \\
\tilde{A}_i(y_2) &= d + \sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\sum_{j=1}^{r} a_j f_{i,m}(y_j|y_2)\tilde{A}_0(y_j) + p^n \tilde{A}_{i+1}(y_2) \\
&\vdots \\
\tilde{A}_i(y_r) &= d + \sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\sum_{j=1}^{r} a_j f_{i,m}(y_j|y_r)\tilde{A}_0(y_j) + p^n \tilde{A}_{i+1}(y_r)
\end{aligned} \tag{5.9}$$

If $\tilde{A}_\eta(y)$ is evaluated at $y = y_j$ for $j = 1, 2, \ldots, r$, then the following system of r equations is obtained.

$$\begin{aligned}
\tilde{A}_\eta(y_1) &= d + \frac{1}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\sum_{j=1}^{r} a_j f_{\eta,m}(y_j|y_1)\tilde{A}_0(y_j) \\
\tilde{A}_\eta(y_2) &= d + \frac{1}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\sum_{j=1}^{r} a_j f_{\eta,m}(y_j|y_2)\tilde{A}_0(y_j) \\
&\vdots \\
\tilde{A}_\eta(y_r) &= d + \frac{1}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\sum_{j=1}^{r} a_j f_{\eta,m}(y_j|y_r)\tilde{A}_0(y_j)
\end{aligned} \tag{5.10}$$

Let $\tilde{\mathbf{A}}_i = \left(\tilde{A}_i(y_1), \tilde{A}_i(y_2), \ldots, \tilde{A}_i(y_r)\right)^T$. Then the set $\{\tilde{\mathbf{A}}_i\}$ must satisfy the following equations,

$$\tilde{\mathbf{A}}_0 = \mathbf{d} + \sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\tilde{\mathbf{Q}}_{0,m}\tilde{\mathbf{A}}_0 + p^n \tilde{\mathbf{A}}_1$$

$$
\begin{aligned}
\tilde{\mathbf{A}}_1 &= \mathbf{d} + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \tilde{\mathbf{Q}}_{1,m} \tilde{\mathbf{A}}_0 + p^n \tilde{\mathbf{A}}_2 \\
\tilde{\mathbf{A}}_2 &= \mathbf{d} + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \tilde{\mathbf{Q}}_{2,m} \tilde{\mathbf{A}}_0 + p^n \tilde{\mathbf{A}}_3 \\
&\vdots \\
\tilde{\mathbf{A}}_{\eta-1} &= \mathbf{d} + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \tilde{\mathbf{Q}}_{\eta-1,m} \tilde{\mathbf{A}}_0 + p^n \tilde{\mathbf{A}}_\eta \\
\tilde{\mathbf{A}}_\eta &= \mathbf{d} + \frac{1}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \tilde{\mathbf{Q}}_{\eta,m} \tilde{\mathbf{A}}_0
\end{aligned}
\tag{5.11}
$$

where $\tilde{\mathbf{Q}}_{im}$ is the $r \times r$ matrix

$$
\tilde{\mathbf{Q}}_{im} = \begin{bmatrix}
a_1 f_{i,m}(y_1|y_1) & a_2 f_{i,m}(y_2|y_1) & \cdots & a_r f_{i,m}(y_r|y_1) \\
a_1 f_{i,m}(y_1|y_2) & a_2 f_{i,m}(y_2|y_2) & \cdots & a_r f_{i,m}(y_r|y_2) \\
\vdots & \vdots & & \vdots \\
a_1 f_{i,m}(y_1|y_r) & a_2 f_{i,m}(y_2|y_r) & \cdots & a_r f_{i,m}(y_r|y_r)
\end{bmatrix}
\tag{5.12}
$$

for $i = 0, 1, 2, \ldots, \eta$ and $m = 0, 1, 2, \ldots, n-1$. Writing these equations in matrix form gives

$$
\tilde{\mathbf{A}} = \mathbf{d} + \tilde{\mathbf{Q}}\tilde{\mathbf{A}}, \tag{5.13}
$$

where $\tilde{\mathbf{A}} = \left(\tilde{\mathbf{A}}_0, \tilde{\mathbf{A}}_0, \ldots, \tilde{\mathbf{A}}_0, \tilde{\mathbf{A}}_1 \tilde{\mathbf{A}}_2, \ldots, \tilde{\mathbf{A}}_\eta\right)^T$ is an $(r(n+\eta)) \times 1$ vector containing n copies of $\tilde{\mathbf{A}}_0$ and one copy of each of $\tilde{\mathbf{A}}_1$, $\tilde{\mathbf{A}}_2$, …, $\tilde{\mathbf{A}}_\eta$ and $\tilde{\mathbf{Q}}$ is the $(r(n+\eta)) \times (r(n+\eta))$ matrix

$$
\tilde{\mathbf{Q}} = \begin{bmatrix}
q^n \tilde{\mathbf{Q}}_{0,0} & npq^{n-1}\tilde{\mathbf{Q}}_{0,1} & \cdots & np^{n-1}q\tilde{\mathbf{Q}}_{0,n-1} & p^n\mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
q^n \tilde{\mathbf{Q}}_{0,0} & npq^{n-1}\tilde{\mathbf{Q}}_{0,1} & \cdots & np^{n-1}q\tilde{\mathbf{Q}}_{0,n-1} & p^n\mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & & \\
q^n \tilde{\mathbf{Q}}_{0,0} & npq^{n-1}\tilde{\mathbf{Q}}_{0,1} & \cdots & np^{n-1}q\tilde{\mathbf{Q}}_{0,n-1} & p^n\mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
q^n \tilde{\mathbf{Q}}_{1,0} & npq^{n-1}\tilde{\mathbf{Q}}_{1,1} & \cdots & np^{n-1}q\tilde{\mathbf{Q}}_{0,n-1} & \mathbf{0} & p^n\mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\
q^n \tilde{\mathbf{Q}}_{2,0} & npq^{n-1}\tilde{\mathbf{Q}}_{2,1} & \cdots & np^{n-1}q\tilde{\mathbf{Q}}_{0,n-1} & \mathbf{0} & \mathbf{0} & p^n\mathbf{I} & & \mathbf{0} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \ddots & \\
q^n \tilde{\mathbf{Q}}_{\eta-1,0} & npq^{n-1}\tilde{\mathbf{Q}}_{\eta-1,1} & \cdots & np^{n-1}q\tilde{\mathbf{Q}}_{0,n-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & p^n\mathbf{I} \\
\dfrac{q^n \tilde{\mathbf{Q}}_{\eta,0}}{1-p^n} & \dfrac{npq^{n-1}\tilde{\mathbf{Q}}_{\eta,1}}{1-p^n} & \cdots & \dfrac{np^{n-1}q\tilde{\mathbf{Q}}_{\eta,n-1}}{1-p^n} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}
\end{bmatrix}
\tag{5.14}
$$

Solving equation (5.13) for $\tilde{\mathbf{A}}$ gives

$$\tilde{\mathbf{A}} = \left(\mathbf{I} - \tilde{\mathbf{Q}}\right)^{-1}\mathbf{d}\,. \tag{5.15}$$

Note that the $(r(n + \eta)) \times (r(n + \eta))$ matrix $\tilde{\mathbf{Q}}$ has a structure similar to the transition matrix $\mathbf{Q}$ in (4.12) used in the Markov chain method. A solution to the set of equations in (5.11) can be found using only the $r \times r$ submatricies $\tilde{\mathbf{Q}}_{im}$. From (4.19), the solution to $\tilde{\mathbf{A}}_0$ is

$$\tilde{\mathbf{A}}_0 = \left(\mathbf{I} - \left[\sum_{j=0}^{\eta-1} p^{jn} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \tilde{\mathbf{Q}}_{j,m} + \frac{p^{\eta n}}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \tilde{\mathbf{Q}}_{\eta,m}\right]\right)^{-1} \left(\frac{1-p^{(\eta+1)n}}{1-p^n}\right)\mathbf{d}\,. \tag{5.16}$$

From (4.20),

$$\tilde{\mathbf{A}}_i = \left(\frac{1-p^{(\eta-i+1)n}}{1-p^n}\right)\mathbf{d} + \left[\sum_{j=0}^{\eta-i-1} p^{jn} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \tilde{\mathbf{Q}}_{i+j,m} + \frac{p^{(\eta-i)n}}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \tilde{\mathbf{Q}}_{\eta,m}\right]$$
$$\left(\mathbf{I} - \left[\sum_{j=0}^{\eta-1} p^{jn} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \tilde{\mathbf{Q}}_{j,m} + \frac{p^{\eta n}}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \tilde{\mathbf{Q}}_{\eta,m}\right]\right)^{-1} \left(\frac{1-p^{(\eta+1)n}}{1-p^n}\right)\mathbf{d} \tag{5.17}$$

for $i = 1, 2, \ldots, \eta$. Applying the quadrature approximation to the integrals in (5.7) and substituting $\tilde{A}_i(y)$ for $A(y)$, an approximation to the ATS can be expressed as

$$\text{ATS} \approx d + \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \sum_{j=1}^{r} a_j \tilde{A}_0(y_j) f_{0,m}(y_j|s) + p^n \tilde{A}_1(s)\,. \tag{5.18}$$

5.3 Average Number of Samples to Signal

A model for the number of samples from a sampling point to the signal by the EWMA chart can also be obtained using the integral equation method. Let $N_i(y)$ be the number of samples to signal from a sampling point at which the previous i consecutive observations are missing, including the current sample, and the EWMA statistic corresponds to y.

Consider a sampling point at which the previous i consecutive observations are missing, including the current sample, for $i = 0, 1, 2, \ldots, \eta - 1$. If $m < n$, then the number of samples to signal is one plus the number of samples to signal from a sampling point at which at least one observation in the previous sample is known. If $m = n$, then the number of samples to signal is the number of samples to signal from a sampling point at which the previous $i + 1$ consecutive samples are missing. Then

$$N_i(y) = \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[1 + \int_C N_0(y') f_{i,m}(y'|y) dy' \right] + p^n N_{i+1}(y). \tag{5.19}$$

Now consider a sampling point at which the previous η consecutive samples are missing, including the current sample, so at least one observation in the next sample must be known. Then the number of samples to signal is

$$N_\eta(y) = \frac{1}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[1 + \int_C N_0(y') f_{\eta,m}(y'|y) dy' \right]. \tag{5.20}$$

This gives the following set of $(\eta + 1)$ equations.

$$\begin{aligned}
N_0(y) &= \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[1 + \int_C N_0(y') f_{0,m}(y'|y) dy' \right] + p^n N_1(y) \\
N_1(y) &= \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[1 + \int_C N_0(y') f_{1,m}(y'|y) dy' \right] + p^n N_2(y) \\
N_2(y) &= \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[1 + \int_C N_0(y') f_{2,m}(y'|y) dy' \right] + p^n N_3(y) \\
&\vdots \\
N_{\eta-1}(y) &= \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[1 + \int_C N_0(y') f_{\eta-1,m}(y'|y) dy' \right] + p^n N_\eta(y) \\
N_\eta(y) &= \frac{1}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[1 + \int_C N_0(y') f_{\eta,m}(y'|y) dy' \right]
\end{aligned} \tag{5.21}$$

The ANSS can then be expressed as

ANSS = $\sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m}$ *(number of samples from t_0 to t_1 + expected number of

samples from t_1 to signal given m observations missing in sample at t_1)
$+ p^n$ *(expected number of samples from t_1 to signal given sample at t_1 is missing)

$$= \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[1 + \int_C N_0(y) f_{0,m}(y|s) dy \right] + p^n N_1(s) \tag{5.22}$$

where s is the starting state of the EWMA control statistic.

Numerical solutions to $N_0(y)$, $N_1(y)$, …, $N_\eta(y)$ can be obtained by approximating the integrals using a numerical quadrature approximation. Let $\tilde{N}_i(y)$ represent the approximation to $N_i(y)$ when the quadrature approximation is applied to the integrals in (5.21). Then the following system of r equations is obtained for each i = 0, 1, 2, …, $\eta - 1$.

$$\begin{aligned}
\tilde{N}_i(y_1) &= \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[1 + \sum_{j=1}^{r} a_j f_{i,m}(y_j|y_1) \tilde{N}_0(y_j) \right] + p^n \tilde{N}_{i+1}(y_1) \\
\tilde{N}_i(y_2) &= \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[1 + \sum_{j=1}^{r} a_j f_{i,m}(y_j|y_2) \tilde{N}_0(y_j) \right] + p^n \tilde{N}_{i+1}(y_2) \\
&\vdots \\
\tilde{N}_i(y_r) &= \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[1 + \sum_{j=1}^{r} a_j f_{i,m}(y_j|y_1) \tilde{N}_0(y_r) \right] + p^n \tilde{N}_{i+1}(y_r)
\end{aligned} \tag{5.23}$$

If $\tilde{N}_\eta(y)$ is evaluated at $y = y_j$ for j = 1, 2, …, r, then the following system of r equations is obtained.

$$\begin{aligned}
\tilde{N}_\eta(y_1) &= \frac{1}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[1 + \sum_{j=1}^{r} a_j f_{\eta,m}(y_j|y_1) \tilde{N}_0(y_j) \right] \\
\tilde{N}_\eta(y_2) &= \frac{1}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[1 + \sum_{j=1}^{r} a_j f_{\eta,m}(y_j|y_2) \tilde{N}_0(y_j) \right] \\
&\vdots \\
\tilde{N}_\eta(y_r) &= \frac{1}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[1 + \sum_{j=1}^{r} a_j f_{\eta,m}(y_j|y_r) \tilde{N}_0(y_j) \right]
\end{aligned} \tag{5.24}$$

Let $\tilde{\mathbf{N}}_i = \left(\tilde{N}_i(y_1), \tilde{N}_i(y_2), \ldots, \tilde{N}_i(y_r)\right)^T$. Then the set $\{\tilde{\mathbf{N}}_i\}$ must satisfy the following equations.

$$
\begin{aligned}
\tilde{\mathbf{N}}_0 &= \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[\mathbf{1} + \tilde{\mathbf{Q}}_{0,m} \tilde{\mathbf{N}}_0\right] + p^n \tilde{\mathbf{N}}_1 \\
\tilde{\mathbf{N}}_1 &= \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[\mathbf{1} + \tilde{\mathbf{Q}}_{1,m} \tilde{\mathbf{N}}_0\right] + p^n \tilde{\mathbf{N}}_2 \\
\tilde{\mathbf{N}}_2 &= \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[\mathbf{1} + \tilde{\mathbf{Q}}_{2,m} \tilde{\mathbf{N}}_0\right] + p^n \tilde{\mathbf{N}}_3 \\
&\vdots \\
\tilde{\mathbf{N}}_{\eta-1} &= \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[\mathbf{1} + \tilde{\mathbf{Q}}_{\eta-1,m} \tilde{\mathbf{N}}_0\right] + p^n \tilde{\mathbf{N}}_\eta \\
\tilde{\mathbf{N}}_\eta &= \frac{1}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[\mathbf{1} + \tilde{\mathbf{Q}}_{\eta,m} \tilde{\mathbf{N}}_0\right]
\end{aligned}
\tag{5.25}
$$

Writing these equations in matrix form gives

$$\tilde{\mathbf{N}} = \mathbf{v} + \tilde{\mathbf{Q}}\tilde{\mathbf{N}}, \tag{5.26}$$

where $\tilde{\mathbf{N}} = \left(\tilde{\mathbf{N}}_0, \tilde{\mathbf{N}}_0, \ldots, \tilde{\mathbf{N}}_0, \tilde{\mathbf{N}}_1, \tilde{\mathbf{N}}_2, \ldots, \tilde{\mathbf{N}}_\eta\right)^T$ is a vector containing n copies of $\tilde{\mathbf{N}}_0$ and one copy of each of $\tilde{\mathbf{N}}_1$, $\tilde{\mathbf{N}}_2$, …, $\tilde{\mathbf{N}}_\eta$ and $\tilde{\mathbf{Q}}$ is the $(r(\eta + n)) \times (r(\eta + n))$ matrix in (5.14). Solving (5.26) for $\tilde{\mathbf{N}}$ gives

$$\tilde{\mathbf{N}} = \left(\mathbf{I} - \tilde{\mathbf{Q}}\right)^{-1} \mathbf{v}. \tag{5.27}$$

Once again, due to the simple structure of $\tilde{\mathbf{Q}}$, a solution to the set of equations in (5.25) can be found using only the $r \times r$ submatrices $\tilde{\mathbf{Q}}_{im}$. From (4.32), the solution to $\tilde{\mathbf{N}}_0$ is

$$\tilde{\mathbf{N}}_0 = \left(\mathbf{I} - \left[\sum_{j=0}^{\eta-1} p^{jn} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \tilde{\mathbf{Q}}_{j,m} + \frac{p^{\eta n}}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \tilde{\mathbf{Q}}_{\eta,m}\right]\right)^{-1} \mathbf{1} \tag{5.28}$$

and from (4.33), the solution to $\tilde{\mathbf{N}}_i$ is

$$\tilde{\mathbf{N}}_i = \mathbf{1} + \left[\sum_{j=0}^{\eta-i-1} p^{jn} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \tilde{\mathbf{Q}}_{i+j,m} + \frac{p^{(\eta-i)n}}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \tilde{\mathbf{Q}}_{\eta,m} \right]$$

$$* \left(\mathbf{I} - \left[\sum_{j=0}^{\eta-1} p^{jn} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \tilde{\mathbf{Q}}_{j,m} + \frac{p^{\eta n}}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \tilde{\mathbf{Q}}_{\eta,m} \right] \right)^{-1} \mathbf{1} \quad (5.29)$$

for i = 1, 2, …, η. Applying the quadrature approximation to the integrals in (5.22) and substituting $\tilde{N}_i(y)$ for $N(y)$, an approximation to the ANSS can be expressed as

$$\text{ANSS} \approx \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[1 + \sum_{j=1}^{r} a_j f_{0,m}(y_j|s) \tilde{N}_0(y_j) \right] + p^n \tilde{N}_1(s). \quad (5.30)$$

5.4 Average Number of Observations to Signal

An expression for the ANOS can also be obtained using the integral equation method in a manner similar to the derivation of the expressions for the ATS and ANSS in Sections 5.2 and 5.3. Let $O_i(y)$ be the number of observations to signal from a sampling point at which the previous i consecutive samples are missing, including the current sample, and the EWMA statistic corresponds to y.

First consider a sampling point at which the previous i consecutive samples are missing, including the current sample, for i = 0, 1, 2, …, η – 1. If $m < n$, then the number of observations to signal is $n - m$ plus the number of observations to signal from a sampling point at which at least one observation in the previous sample is known. If $m = n$, then the number of observations to signal is the number of observations to signal from a sampling point at which the previous i + 1 consecutive samples are missing. This gives

$$O_i(y) = \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[(n-m) + \int_C O_0(y') f_{i,m}(y'|y) dy' \right] + p^n O_{i+1}(y). \quad (5.31)$$

Next consider a sampling point at which the previous η consecutive samples are missing, including the current sample, so at least one observation in the next sample must be known. Then the number of observations to signal is

$$O_\eta(y)=\frac{1}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left[(n-m)+\int_C O_0(y')f_{\eta,m}(y'|y)dy'\right]. \qquad (5.32)$$

Thus, the set $\{O_i(y)\}$ must satisfy the following set of (η+1) equations.

$$O_0(y)=\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left[(n-m)+\int_C O_0(y')f_{0,m}(y'|y)dy'\right]+p^n O_1(y)$$

$$O_1(y)=\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left[(n-m)+\int_C O_0(y')f_{1,m}(y'|y)dy'\right]+p^n O_2(y)$$

$$O_2(y)=\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left[(n-m)+\int_C O_0(y')f_{2,m}(y'|y)dy'\right]+p^n O_3(y)$$

$$\vdots \qquad (5.33)$$

$$O_{\eta-1}(y)=\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left[(n-m)+\int_C O_0(y')f_{\eta-1,m}(y'|y)dy'\right]+p^n O_\eta(y)$$

$$O_\eta(y)=\frac{1}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left[(n-m)+\int_C O_0(y')f_{\eta,m}(y'|y)dy'\right]$$

The ANOS can then be expressed as

ANOS = $\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}$ *(number of observations from t_0 to t_1 + expected number of observations from t_1 to signal given m observations missing in sample at t_1) + p^n *(expected number of observations from t_1 to signal given sample at t_1 is missing)

$$=\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left[(n-m)+\int_C O_0(y)f_{0,m}(y|s)dy\right]+p^n O_1(s) \qquad (5.34)$$

where s is the starting state of the EWMA control statistic.

Numerical solutions to $O_0(y)$, $O_1(y)$, …, $O_\eta(y)$ can be obtained by approximating the integrals using a numerical quadrature approximation. Let $\tilde{O}_i(y)$ represent the approximation to $O_i(y)$ when the quadrature approximation is applied to the integrals in (5.33). Then the following system of r equations is obtained for each i = 0, 1, 2, …, $\eta - 1$.

$$\tilde{O}_i(y_1) = \sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left[(n-m) + \sum_{j=1}^{r} a_j f_{i,m}(y_j|y_1)\tilde{O}_0(y_j)\right] + p^n \tilde{O}_{i+1}(y_1)$$

$$\tilde{O}_i(y_2) = \sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left[(n-m) + \sum_{j=1}^{r} a_j f_{i,m}(y_j|y_2)\tilde{O}_0(y_j)\right] + p^n \tilde{O}_{i+1}(y_2)$$

$$\vdots \tag{5.35}$$

$$\tilde{O}_i(y_r) = \sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left[(n-m) + \sum_{j=1}^{r} a_j f_{i,m}(y_j|y_1)\tilde{O}_0(y_r)\right] + p^n \tilde{O}_{i+1}(y_r)$$

If $\tilde{O}_\eta(y)$ is evaluated at $y = y_j$ for j = 1, 2, …, r, then the following system of r equations is obtained.

$$\tilde{O}_\eta(y_1) = \frac{1}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left[(n-m) + \sum_{j=1}^{r} a_j f_{\eta,m}(y_j|y_1)\tilde{O}_0(y_j)\right]$$

$$\tilde{O}_\eta(y_2) = \frac{1}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left[(n-m) + \sum_{j=1}^{r} a_j f_{\eta,m}(y_j|y_2)\tilde{O}_0(y_j)\right]$$

$$\vdots \tag{5.36}$$

$$\tilde{O}_\eta(y_r) = \frac{1}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left[(n-m) + \sum_{j=1}^{r} a_j f_{\eta,m}(y_j|y_r)\tilde{O}_0(y_j)\right]$$

Let $\tilde{\mathbf{O}}_i = \left(\tilde{O}_i(y_1), \tilde{O}_i(y_2), \ldots, \tilde{O}_i(y_r)\right)^T$. Then the set $\{\tilde{\mathbf{O}}_i\}$ must satisfy the following equations.

$$\tilde{\mathbf{O}}_0 = \sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left[(n-m)\mathbf{1} + \tilde{\mathbf{Q}}_{0,m}\tilde{\mathbf{O}}_0\right] + p^n\tilde{\mathbf{O}}_1$$

$$\tilde{\mathbf{O}}_1 = \sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left[(n-m)\mathbf{1} + \tilde{\mathbf{Q}}_{1,m}\tilde{\mathbf{O}}_0\right] + p^n\tilde{\mathbf{O}}_2$$

$$\tilde{\mathbf{O}}_2 = \sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left[(n-m)\mathbf{1} + \tilde{\mathbf{Q}}_{2,m}\tilde{\mathbf{O}}_0\right] + p^n\tilde{\mathbf{O}}_3$$

$$\vdots \tag{5.37}$$

$$\tilde{\mathbf{O}}_{\eta-1} = \sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left[(n-m)\mathbf{1} + \tilde{\mathbf{Q}}_{\eta-1,m}\tilde{\mathbf{O}}_0\right] + p^n\tilde{\mathbf{O}}_\eta$$

$$\tilde{\mathbf{O}}_{\eta} = \frac{1}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\left[(n-m)\mathbf{1} + \tilde{\mathbf{Q}}_{\eta,m}\tilde{\mathbf{O}}_0\right]$$

Writing these equations in matrix form gives

$$\tilde{\mathbf{O}} = \mathbf{w} + \tilde{\mathbf{Q}}\tilde{\mathbf{O}}, \tag{5.38}$$

where $\tilde{\mathbf{O}} = \left(\tilde{\mathbf{O}}_0, \tilde{\mathbf{O}}_0, \ldots, \tilde{\mathbf{O}}_0, \tilde{\mathbf{O}}_1, \tilde{\mathbf{O}}_2, \ldots, \tilde{\mathbf{O}}_{\eta}\right)^T$ is a vector containing n copies of $\tilde{\mathbf{O}}_0$ and one copy of each of $\tilde{\mathbf{O}}_1$, $\tilde{\mathbf{O}}_2$, …, $\tilde{\mathbf{O}}_{\eta}$, $\mathbf{w}$ is the $(r(\eta + n)) \times 1$ vector with the first $(r(n+\eta-1))$ components $\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}(n-m)$ and the last r components $\frac{1}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}(n-m)$, and $\tilde{\mathbf{Q}}$ is the $(r(n + \eta)) \times (r(n + \eta))$ matrix in (5.14). Solving (5.38) for $\tilde{\mathbf{O}}$ gives

$$\tilde{\mathbf{O}} = \left(\mathbf{I} - \tilde{\mathbf{Q}}\right)^{-1}\mathbf{w}. \tag{5.39}$$

Once again, due to the simple structure of $\tilde{\mathbf{Q}}$, a solution to the set of equations in (5.37) can be found using only the $r \times r$ submatrices $\tilde{\mathbf{Q}}_{im}$. From (4.45), the solution to $\tilde{\mathbf{O}}_0$ is

$$\tilde{\mathbf{O}}_0 = \left(\mathbf{I} - \left[\sum_{j=0}^{\eta-1}p^{jn}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\tilde{\mathbf{Q}}_{j,m} + \frac{p^{\eta n}}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\tilde{\mathbf{Q}}_{\eta,m}\right]\right)^{-1} * \left(\frac{1}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}(n-m)\mathbf{1}\right) \tag{5.40}$$

and from (4.46), the solution to $\tilde{\mathbf{O}}_i$ is

$$\tilde{\mathbf{O}}_i = \frac{1}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}(n-m)\mathbf{1} + \left[\sum_{j=0}^{\eta-i-1}p^{jn}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\tilde{\mathbf{Q}}_{i+j,m} + \frac{p^{(\eta-i)n}}{1-p^n} * \sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\tilde{\mathbf{Q}}_{\eta,m}\right]\left(\mathbf{I} - \left[\sum_{j=0}^{\eta-1}p^{jn}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\tilde{\mathbf{Q}}_{j,m} + \frac{p^{\eta n}}{1-p^n}\sum_{m=0}^{n-1}\binom{n}{m}p^m q^{n-m}\tilde{\mathbf{Q}}_{\eta,m}\right]\right)^{-1}$$

$$* \left(\frac{1}{1-p^n} \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} (n-m) \mathbf{1} \right) \tag{5.41}$$

for i = 1, 2, …, η. Applying the quadrature approximation to the integrals in (5.34) and substituting $\tilde{O}_i(y)$ for $O(y)$, an approximation to the ANSS can be expressed as

$$\text{ANOS} \approx \sum_{m=0}^{n-1} \binom{n}{m} p^m q^{n-m} \left[(n-m) + \sum_{j=1}^{r} a_j f_{0,m}(y_j | s) \tilde{O}_0(y_j) \right] + p^n \tilde{O}_1(s). \tag{5.42}$$

5.5 Steady-State Average Time to Signal

Let $\pi_i(y)$ be the stationary distribution of the control statistic conditional on no false alarms for $y \in C$ when there are i consecutive missing samples. Let $\tilde{\pi}_i(y)$ be the approximation to $\pi_i(y)$ based on the quadrature approximation approach, let

$$\tilde{\boldsymbol{\pi}}_i = (\tilde{\pi}_i(y_1), \tilde{\pi}_i(y_2), \ldots, \tilde{\pi}_i(y_r)) \tag{5.43}$$

and let

$$\tilde{\boldsymbol{\pi}} = (\tilde{\boldsymbol{\pi}}_0, \tilde{\boldsymbol{\pi}}_0, \ldots, \tilde{\boldsymbol{\pi}}_0, \tilde{\boldsymbol{\pi}}_1, \tilde{\boldsymbol{\pi}}_2, \ldots, \tilde{\boldsymbol{\pi}}_\eta). \tag{5.44}$$

Then $\tilde{\boldsymbol{\pi}}$ is the normalized left eigenvector of $\tilde{\mathbf{Q}}$ corresponding to the largest eigenvalue. See Darroch and Seneta (1965) and Reynolds (1995) for more details. Let

$$\tilde{\boldsymbol{\alpha}}_i = (a_1 \tilde{\pi}_i(y_1), a_2 \tilde{\pi}_i(y_2), \ldots, a_r \tilde{\pi}_i(y_r)) \tag{5.45}$$

and let

$$\tilde{\boldsymbol{\alpha}} = (\tilde{\boldsymbol{\alpha}}_0, \tilde{\boldsymbol{\alpha}}_0, \ldots, \tilde{\boldsymbol{\alpha}}_0, \tilde{\boldsymbol{\alpha}}_1, \tilde{\boldsymbol{\alpha}}_2, \ldots, \tilde{\boldsymbol{\alpha}}_\eta). \tag{5.46}$$

It is assumed that when the shift falls within an interval, the position of the shift within the interval is uniformly distributed on the interval. This assumption implies that the expected position of the shift is halfway into the interval. Suppose that the shift in the process parameter falls between times t_k and t_{k+1}, so t_k is the last sampling point before the shift. Also suppose that after t_k, there are i consecutive missing samples, including the sample at t_k, and the control statistic corresponds to y. Then the expected time from the shift to the signal is $A_i(y)-\frac{d}{2}$. The SSATS can then be expressed as

$$\text{SSATS}=\sum_{i=0}^{\eta}\int_C \pi_i(y)\left(A_i(y)-\frac{d}{2}\right)dy\,. \tag{5.47}$$

Using the quadrature approximation to the integral in (5.47) and the approximations (5.15) and (5.44) gives the approximation

$$\begin{aligned}\text{SSATS} &\approx \sum_{i=0}^{\eta}\sum_{j=1}^{r} a_j \tilde{\pi}_i(y_j)\left(\tilde{A}_i(y_j)-\frac{d}{2}\right)\\ &= \tilde{\boldsymbol{\alpha}}\left(\tilde{\mathbf{A}}-\frac{\mathbf{d}}{2}\right)\\ &= \tilde{\boldsymbol{\alpha}}\left(\left(\mathbf{I}-\tilde{\mathbf{Q}}\right)^{-1}-\frac{\mathbf{1}}{2}\right)\mathbf{d}\,.\end{aligned} \tag{5.48}$$

5.6 Steady-State Average Number of Samples to Signal

The SSANSS can also be obtained using the integral equation method. Suppose that the shift in the process parameter falls between times t_k and t_{k+1}, so t_k is the last sampling point before the shift. Also suppose that after t_k, there are i consecutive missing samples, including the sample at t_k, and the control statistic corresponds to y. Then the expected number of samples from the shift to the signal is $N_i(y)$. Then the SSANSS can be expressed as

$$\text{SSANSS}=\sum_{i=0}^{\eta}\int_C \pi_i(y)N_i(y)dy\,. \tag{5.49}$$

Using the quadrature approximation to the integral in (5.49) and the approximations (5.27) and (5.44) gives the approximation

$$\begin{aligned}\text{SSANSS} &\approx \sum_{i=0}^{\eta}\sum_{j=1}^{r} a_j \tilde{\pi}_i(y_j)\tilde{N}_i(y_j)\\ &= \tilde{\boldsymbol{\alpha}}\tilde{\mathbf{N}}\end{aligned}$$

$$= \tilde{\boldsymbol{\alpha}}\left(\mathbf{I} - \tilde{\mathbf{Q}}\right)^{-1}\mathbf{v}\,. \tag{5.50}$$

5.7 Steady-State Average Number of Observations to Signal

An expression for the SSANOS can also be obtained using the integral equation method using arguments similar to those used for the SSATS and SSANSS. Suppose that the shift in the process parameter falls between times t_k and t_{k+1}, so t_k is the last sampling point before the shift. Also suppose that after t_k, there are i consecutive missing samples, including the sample at t_k, and the control statistic corresponds to y. Then the expected number of observations from the shift to the signal is $O_i(y)$. Then the SSANOS can be expressed as

$$\text{SSANOS} = \sum_{i=0}^{\eta}\int_C \pi_i(y)O_i(y)dy \tag{5.51}$$

Using the quadrature approximation to the integral in (5.51) and the approximations (5.39) and (5.44) gives the approximation

$$\begin{aligned}\text{SSANOS} &\approx \sum_{i=0}^{\eta}\sum_{j=1}^{r} a_j\tilde{\pi}_i(y_j)\tilde{O}_i(y_j) \\ &= \tilde{\boldsymbol{\alpha}}\tilde{\mathbf{O}} \\ &= \tilde{\boldsymbol{\alpha}}\left(\mathbf{I} - \tilde{\mathbf{Q}}\right)^{-1}\mathbf{w}\,.\end{aligned} \tag{5.52}$$

Chapter 6

Performance of EWMA Control Charts

In this chapter, the performance of different ways for modifying the EWMA control chart to adjust for missing observations is investigated. The three methods discussed in Section 3.2 will be examined: (1) ignoring missing samples; (2) adding the weights from previous consecutive missing sample means to the current sample mean; and (3) increasing the weights of non-missing sample means in proportion so that the weights sum to one. Performance of these methods will be evaluated for various combinations of the parameters λ, n, d, p, and η over a range of shifts in μ.

6.1 Control Chart Parameters

The sampling rate is set to be one observation per unit time so $n/d = 1.0$, and two sampling patterns are considered. One sampling pattern is based on taking samples of $n = 1$ every $d = 1$ time unit, and the other is based on samples of $n = 4$ every $d = 4$ time units. For convenience, the time unit is referred to as an hour.

When evaluating EWMA control charts, the value chosen for the smoothing parameter λ can have a significant effect on the conclusions reached. As discussed in Section 3.2, λ satisfies $0 < \lambda \le 1$, and determines the weight given to current observations relative to past observations. Using a small value of λ gives more weight to the older observations and makes the EWMA chart more sensitive to small shifts in the process mean. When the value of λ is larger, less weight is given to older observations, and the chart is more sensitive to large shifts in μ.

When investigating EWMA control charts with different sample sizes, λ must be chosen carefully to make the results comparable. When there are no missing observations, one approach which allows for a fair comparison was introduced by Reynolds and Stoumbos (2004). This approach chooses the value of λ when $n > 1$ so that the sum of the weights for a set of n individual observations is equal to the weight of one sample mean when samples of $n > 1$ are taken. Let λ_n represent the value of λ when the sample size is n. Then the EWMA statistic at time t_k if $n > 1$ is

$$E_k^Z = (1-\lambda_n)E_{k-1}^Z + \lambda_n Z_k , \tag{6.1}$$

so the weight of one sample mean is λ_n. Now consider taking samples of size $n = 1$, i.e. individual observations. The EWMA statistic at time t_k is

$$\begin{aligned} E_k^Z &= (1-\lambda_1)E_{k-1}^Z + \lambda_1 Z_k \\ &= (1-\lambda_1)^k E_0^Z + \sum_{j=0}^{k-1} (1-\lambda_1)^j \lambda_1 Z_{k-j} \\ &= (1-\lambda_1)^k E_0^Z + \sum_{j=n}^{k-1} (1-\lambda_1)^j \lambda_1 Z_{k-j} + \sum_{j=0}^{n-1} (1-\lambda_1)^j \lambda_1 Z_{k-j} , \end{aligned} \tag{6.2}$$

so the sum of the weights for a set of n individual observations is

$$\begin{aligned} \sum_{j=0}^{n-1} (1-\lambda_1)^j \lambda_1 &= \frac{\lambda_1\left(1-(1-\lambda_1)^n\right)}{1-(1-\lambda_1)} \\ &= 1-(1-\lambda_1)^n . \end{aligned} \tag{6.3}$$

Then the relationship between λ_1 and λ_n can be expressed as

$$\lambda_n = 1-(1-\lambda_1)^n \qquad \text{or} \qquad \lambda_1 = 1-(1-\lambda_n)^{1/n} . \tag{6.4}$$

Thus, equations (6.4) provide a method for determining λ_n from λ_1 and vice versa. For example, if $n = 4$ and $\lambda_4 = 0.1$, then the corresponding value of λ_1 for $n = 1$ is $\lambda_1 = 1-(1-\lambda_4)^{1/4} = 1-(1-0.1)^{1/4} = 0.026$.

By using this method, the individual observations in the control charts with different sample sizes have approximately the same weight, and the control charts with different sample sizes have approximately the same sensitivity to any given shift. In evaluating the EWMA control charts in this dissertation, $\lambda_1 = 0.026$ for $n = 1$ and the corresponding value $\lambda_4 = 0.1$ for $n = 4$, as well as $\lambda_1 = 0.11989$ for $n = 1$ and the corresponding value $\lambda_4 = 0.4$ for $n = 4$ are considered.

The performance of the charts investigated here is evaluated for $p = 0.01$, 0.1, 0.2, 0.5, and 0.9. A maximum of zero, one, or two consecutive missing samples is considered, i.e. $\eta = 0$, 1, or 2. The probability of η consecutive missing samples is $p^{n\eta}$, so as $\eta \to \infty$, $p^{n\eta} \to 0$. For example, if $n = 1$, the probability of three consecutive samples missing is $0.01^3 = 0.000001$ for $p = 0.01$, $0.1^3 = 0.001$ if $p = 0.1$, and $0.2^3 = 0.008$ if $p = 0.2$. Thus, for these values of p it is highly unlikely for many consecutive samples to be missing, and the results for $\eta > 2$ are very similar to the results for $\eta = 2$ shown here. The charts are also examined when $p = 0.5$ and 0.9 in order to see how they perform under more extreme conditions. Although these values lead to a higher probability of consecutive missing samples, they are unlikely to occur in practical applications.

Shifts in μ are expressed in terms of $\delta = |\mu - \mu_0| / \sigma_0$, and the shift sizes considered range from 0.0 to 10.0. First the control limits are set to have an in-control ATS of 1481.6, so that the EI, EA, and EP charts can be compared when they all have the same false alarm rate per unit time. Next, the performance of the charts is examined when the control limits are not adjusted, and instead the same limits are used for all three control charts.

6.2 Comparison of EWMA Control Charts when the Control Limits are Adjusted

Tables 6.1 through 6.50 present the SSATS, SSANSS, and SSANOS values for the three different EWMA control charts designed to detect changes in the process mean when there is a probability p that an individual observation is missing, independent of other observations. The control limits for all of these charts are set to have an in-control ATS of 1481.6, so they all have the same false alarm rate per unit time. Tables 6.1 – 6.10 give results for the case when $n = 1$, $d = 1$, $\lambda = 0.026$, and $p = 0.01$, 0.1, 0.2, 0.5, or 0.9. The results for the case when $n = 1$, $d = 1$, $\lambda = 0.11989$, and $p = 0.01$, 0.1, 0.2, 0.5, or 0.9 are shown in Tables 6.11 – 6.20. When $n = 1$, each sample consists of an individual observation, so if an observation is missing at time t_k then the entire sample is missing at t_k. Note that the SSANOS values are not shown for this case since the number of samples and the number of observations to signal are equivalent. The case when $n = 4$, $d = 4$, $\lambda = 0.1$, and $p = 0.01$, 0.1, 0.2, 0.5, or 0.9 is shown in Tables 6.21 – 6.35, and the results for $n = 4$, $d = 4$ and $\lambda = 0.4$ are presented in Tables 6.36 – 6.50.

For each table, the first column gives the different magnitudes of shifts considered in terms of δ, and the second column corresponds to the standard EZ chart with $\eta = 0$, so there are no missing observations. Recall that when no observations are missing, the EZ, EI, EA, and EP charts are all equivalent. The next three columns correspond to the EI, EA, and EP charts with a maximum of $\eta = 1$ consecutive missing sample, and the last three columns give the results for each of the three charts with $\eta = 2$.

Table 6.1. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 1$, $d = 1$, $\lambda = 0.026$, and $p = 0.01$.

	η = 0	η = 1			η = 2		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6
0.25	109.1	109.8	111.1	109.8	109.8	111.2	109.8
0.50	36.9	37.2	37.4	37.2	37.2	37.5	37.2
1.00	15.1	15.2	15.3	15.2	15.2	15.3	15.2
2.00	6.8	6.9	6.9	6.9	6.9	6.9	6.9
5.00	2.6	2.6	2.6	2.6	2.6	2.6	2.6
10.00	1.3	1.3	1.3	1.3	1.3	1.3	1.3
h	2.8334	2.8296	2.8607	2.8301	2.8296	2.8613	2.8300

Table 6.2. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 1$, $d = 1$, $\lambda = 0.026$, and $p = 0.01$.

	η = 0	η = 1			η = 2		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1466.9	1466.9	1466.9	1466.8	1466.8	1466.8
0.25	109.6	109.2	110.5	109.2	109.2	110.6	109.2
0.50	37.4	37.3	37.6	37,32	37.3	37.6	37.3
1.00	15.6	15.6	15.6	15.6	15.6	15.6	15.6
2.00	7.3	7.3	7.3	7.3	7.3	7.3	7.3
5.00	3.1	3.1	3.1	3.1	3.1	3.1	3.1
10.00	1.8	1.8	1.8	1.8	1.8	1.8	1.8
h	2.8334	2.8296	2.8607	2.8301	2.8296	2.8613	2.8300

Tables 6.1 and 6.11 show that there is very little difference in the SSATS values of the EI, EA, and EP charts with $n = 1$, $d = 1$, $\lambda = 0.026$ or 0.11989 and $p = 0.01$. In fact, when there is only a one percent chance that an observation is missing, the SSATS for all three charts is very similar to that of the standard EZ chart, which has no missing observations. Increasing the maximum number of consecutive missing samples from one to two has little to no effect on the performance of any of the charts.

The EI and EP charts result in the same SSATS values when $\lambda = 0.026$. For $\lambda = 0.11989$, the EI chart has the best overall performance, since it is slightly more efficient for small shifts in the process mean. In both cases, the EA chart has the poorest performance. For example, when $\lambda = 0.026$ and $\delta = 0.25$, the EI and EP charts are able to detect this change in μ in 109.8 hours, while it takes 111.1 hours for the EA chart to signal. If $\lambda = 0.11989$, the EI and EP charts signal after 228.8 and 229.1 hours, respectively, but the EA chart takes 236.9 hours to detect this change when $\eta = 1$. It is not surprising that the EA chart does not perform as well for small shifts, since this control statistic is adjusted to account for missing observations by increasing the weight on the current sample, which makes the chart less sensitive to smaller shifts in the parameter. Tables 6.2 and 6.12 show similar results for the SSANSS values; however, note that the in-control ANSS is approximately 1467 samples when $p = 0.01$ for the EI, EA, and EP charts for both values of λ, as compared to an in-control ANSS of 1481.6 when $p = 0$.

Table 6.3. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 1$, $d = 1$, $\lambda = 0.026$, and $p = 0.1$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6
0.25	109.1	116.0	127.0	116.1	116.7	131.0	116.8
0.50	36.9	39.8	41.7	39.8	40.1	42.7	40.1
1.00	15.1	16.4	16.7	16.4	16.5	17.0	16.5
2.00	6.8	7.4	7.5	7.4	7.5	7.6	7.5
5.00	2.6	2.8	2.8	2.8	2.9	2.8	2.9
10.00	1.3	1.4	1.4	1.4	1.4	1.4	1.4
h	2.8334	2.7967	3.0712	2.8013	2.7931	3.1165	2.7982

Table 6.4. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=1$, $d=1$, $\lambda=0.026$, and $p=0.1$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1346.9	1346.9	1346.9	1334.8	1334.8	1334.8
0.25	109.6	106.0	116.0	106.0	105.6	118.5	105.7
0.50	37.4	36.7	38.4	36.6	36.6	39.0	36.6
1.00	15.6	15.4	15.6	15.3	15.3	15.8	15.3
2.00	7.3	7.2	7.3	7.2	7.2	7.3	7.2
5.00	3.1	3.0	3.0	3.0	3.0	3.0	3.0
10.00	1.8	1.8	1.7	1.7	1.7	1.7	1.7
h	2.8334	2.7967	3.0712	2.8013	2.7931	3.1165	2.7982

The SSATS values for the EI, EA, and EP charts with $n=1$, $d=1$, $\lambda=0.026$ or 0.11989 and $p=0.1$ are shown in Tables 6.3 and 6.13. For large shifts in μ, all three methods have similar performance. The EI and EP charts are more efficient than the EA chart for detecting small to moderate shifts when $\lambda=0.026$. When $\lambda=0.11989$, the EI chart is the most efficient for small shifts. If $\eta=2$, $\lambda=0.11989$, and $\delta=0.5$, the EI chart is able to detect this change in 51.8 hours, while the EP chart takes 52.3 hours. The EA chart takes the longest to signal, with a SSATS of 68.9 hours. SSANSS values are shown in Tables 6.4 and 6.14. Although the in-control ATS for all EWMA charts is set to be 1481.6 hours, when there are missing observations the in-control ANSS decreases, as we would expect. For $\eta=2$, the in-control ANSS for the EI, EA, and EP charts is only 1334.8 samples.

Tables 6.5 and 6.15 show the results of increasing p to 0.20, so that each observation has a twenty percent chance of being missing. When $\lambda=0.026$, there is still no difference in the performance of the EI and EP charts. However, for $\lambda=0.11989$, the EI chart is still the most efficient since it detects small shifts in the process mean more quickly than either the EA or EP charts. For example, when $\delta=0.5$ and there is a maximum of two consecutive missing samples, the SSATS value of the EI chart is 55.5, while the EA and EP charts take 89.9 and 56.7 hours to signal, respectively. Increasing η significantly increases the SSATS of the EA chart, especially for small shifts in μ, but has only a small effect on the EI and EP charts. Tables 6.6 and 6.16 show the SSANSS values. For $\eta = 2$, the in-control ANSS for the EI, EA, and EP charts is only 1194.8 samples.

Table 6.5. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 1$, $d = 1$, $\lambda = 0.026$, and $p = 0.2$.

	η = 0	η = 1			η = 2		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6
0.25	109.1	122.7	141.1	122.8	125.4	153.8	125.5
0.50	36.9	42.6	45.6	42.6	43.8	48.9	43.8
1.00	15.1	17.6	17.9	17.6	18.2	18.9	18.1
2.00	6.8	8.0	8.0	8.0	8.3	8.4	8.3
5.00	2.6	3.1	3.0	3.1	3.2	3.1	3.2
10.00	1.3	1.5	1.5	1.5	1.6	1.6	1.6
h	2.8334	2.7626	3.2514	2.7718	2.7497	3.3907	2.7608

Table 6.6. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 1$, $d = 1$, $\lambda = 0.026$, and $p = 0.2$.

	η = 0	η = 1			η = 2		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1234.7	1234.7	1234.7	1194.8	1194.8	1194.8
0.25	109.6	102.7	118.1	102.8	101.5	124.5	101.6
0.50	37.4	36.0	38.4	36.0	35.7	39.8	35.7
1.00	15.6	15.1	15.4	15.1	15.1	15.7	15.0
2.00	7.3	7.1	7.1	7.1	7.1	7.2	7.1
5.00	3.1	3.0	2.9	3.0	3.0	2.9	3.0
10.00	1.8	1.7	1.7	1.7	1.7	1.7	1.7
h	2.8334	2.7626	3.2514	2.7718	2.7497	3.3907	2.7608

Table 6.7. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=1$, $d=1$, $\lambda=0.026$, and $p=0.5$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6
0.25	109.1	141.5	170.1	141.8	156.1	213.4	156.6
0.50	36.9	50.7	53.7	50.7	57.2	65.7	57.1
1.00	15.1	21.2	20.5	21.1	24.2	24.1	24.0
2.00	6.8	9.7	9.0	9.7	11.1	10.3	11.1
5.00	2.6	3.7	3.3	3.7	4.3	3.7	4.3
10.00	1.3	1.9	1.6	1.9	2.2	1.8	2.2
h	2.8334	2.6735	3.6085	2.6957	2.6102	4.0618	2.6435

Table 6.8. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=1$, $d=1$, $\lambda=0.026$, and $p=0.5$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	987.7	987.7	987.8	846.6	846.6	846.6
0.25	109.6	94.8	113.9	95.0	89.6	122.3	89.8
0.50	37.4	34.3	36.3	34.2	33.1	37.9	33.0
1.00	15.6	14.6	14.1	14.5	14.2	14.1	14.1
2.00	7.3	6.9	6.4	6.9	6.8	6.3	6.7
5.00	3.1	2.9	2.6	2.9	2.9	2.5	2.8
10.00	1.8	1.7	1.5	1.7	1.7	1.4	1.6
h	2.8334	2.6735	3.6085	2.6957	2.6102	4.0618	2.6435

Table 6.9. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 1$, $d = 1$, $\lambda = 0.026$, and $p = 0.9$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6
0.25	109.1	164.2	193.2	164.7	204.8	267.1	205.7
0.50	36.9	60.7	60.3	60.6	79.3	82.1	79.0
1.00	15.1	25.6	22.6	25.5	34.1	28.9	33.8
2.00	6.8	11.7	9.8	11.6	15.7	12.0	15.5
5.00	2.6	4.4	3.6	4.4	6.0	4.3	5.9
10.00	1.3	2.2	1.6	2.2	3.0	1.9	2.9
h	2.8334	2.5758	3.8799	2.6146	2.4225	4.635	2.4944

Table 6.10. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 1$, $d = 1$, $\lambda = 0.026$, and $p = 0.9$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	779.8	779.8	779.8	546.7	546.7	546.7
0.25	109.6	86.9	102.2	87.2	76.0	99.0	76.4
0.50	37.4	32.4	32.3	32.4	29.8	30.8	29.6
1.00	15.6	14.0	12.4	13.9	13.1	11.1	12.9
2.00	7.3	6.7	5.6	6.6	6.3	4.9	6.2
5.00	3.1	2.8	2.4	2.8	2.7	2.1	2.7
10.00	1.8	1.6	1.4	1.6	1.6	1.2	1.6
h	2.8334	2.5758	3.8799	2.6146	2.4225	4.635	2.4944

The SSATS values for the three charts with $\lambda = 0.026$ and 0.11989 when p is 0.5 and 0.9 are shown in Tables 6.7, 6.9, 6.17, and 6.19. The EI and EP charts once again have similar performance and are able to detect small changes in the process mean much more quickly than the EA chart. However, the EA chart is more efficient for moderate and large shifts in μ. If $p = 0.5$, $\lambda = 0.026$, $\eta = 2$, and $\delta = 2.0$, then the EA chart has a SSATS of 10.3, while the EI and EP charts take 11.1 hours to signal. Tables 6.8 and 6.18 show that if $p = 0.5$, although the in-control ATS is 1481.6, the in-control ANSS decreases to 987.7 samples if $\eta = 1$ and 846.6 samples if $\eta = 2$.

For small values of p, increasing the maximum number of consecutive missing samples from one to two has little effect on the EI and EP charts, but increases the SSATS values of the EA chart, especially for small shifts in μ. For example, when $p = 0.1$ and η is increased from 1 to 2, the SSATS of the EA chart with $\lambda = 0.11989$ increases from 63.7 to 68.9 hours for a shift of size $\delta = 0.5$. However, for larger values of p, including $p = 0.2$, 0.5 and 0.9, increasing η increases the time required to detect a change in μ for all three methods. If $p = 0.5$, $\lambda = 0.11989$, $\delta = 0.5$, and $\eta = 1$, the SSATS values for the EI, EA, and EP charts are 62.3, 92.6, and 64.4 hours, respectively. When η is increased to 2, these SSATS values increase to 68.6, 133.6 and 72.5 hours, respectively. Increasing the probability that an observation is missing increases the SSATS for the EI, EA, and EP control charts, as we would expect.

Table 6.11. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=1$, $d=1$, $\lambda=0.11989$, and $p=0.01$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6
0.25	228.1	228.8	236.9	229.1	228.9	237.3	229.1
0.50	48.5	48.8	50.3	48.9	48.8	50.4	48.9
1.00	11.9	12.0	12.1	12.0	12.0	12.1	12.0
2.00	4.2	4.3	4.3	4.3	4.3	4.3	4.3
5.00	1.4	1.4	1.4	1.4	1.4	1.4	1.4
10.00	0.6	0.6	0.6	0.6	0.6	0.6	0.6
h	3.2237	3.2206	3.2611	3.2231	3.2205	3.2622	3.2231

Table 6.12. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=1$, $d=1$, $\lambda=0.11989$, and $p=0.01$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1466.9	1466.9	1466.9	1466.8	1466.8	1466.8
0.25	228.6	227.1	235.1	227.3	227.1	235.4	227.3
0.50	49.0	48.8	50.3	48.9	48.8	50.4	48.9
1.00	12.4	12.3	12.5	12.3	12.3	12.5	12.3
2.00	4.7	4.7	4.7	4.7	4.7	4.7	4.7
5.00	1.9	1.9	1.9	1.9	1.9	1.9	1.9
10.00	1.1	1.1	1.1	1.1	1.1	1.1	1.1
h	3.2237	3.2206	3.2611	3.2231	3.2205	3.2622	3.2231

Table 6.13. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=1$, $d=1$, $\lambda=0.11989$, and $p=0.1$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6
0.25	228.1	235.7	295.4	238.2	236.5	320.1	239.4
0.50	48.5	51.5	63.7	51.9	51.8	68.9	52.3
1.00	11.9	12.8	14.2	12.9	13.0	14.8	13.0
2.00	4.2	4.6	4.8	4.6	4.7	4.9	4.7
5.00	1.4	1.6	1.6	1.6	1.6	1.6	1.6
10.00	0.6	0.6	0.7	0.6	0.7	0.7	0.7
h	3.2237	3.1934	3.5261	3.2183	3.1905	3.5991	3.2188

Table 6.14. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=1$, $d=1$, $\lambda=0.11989$, and $p=0.1$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1346.9	1346.9	1346.9	1334.8	1334.8	1334.8
0.25	228.6	214.8	269.0	217.0	213.5	288.8	216.1
0.50	49.0	47.3	58.4	47.7	47.1	62.5	47.6
1.00	12.4	12.1	13.4	12.2	12.1	13.8	12.1
2.00	4.7	4.7	4.8	4.6	4.7	4.9	4.6
5.00	1.9	1.9	1.9	1.9	1.9	1.9	1.9
10.00	1.1	1.1	1.1	1.0	1.0	1.1	1.0
h	3.2237	3.1934	3.5261	3.2183	3.1905	3.5991	3.2188

Table 6.15. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=1$, $d=1$, $\lambda=0.11989$, and $p=0.2$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6
0.25	228.1	243.0	332.5	247.8	245.8	393.7	252.2
0.50	48.5	54.3	74.3	55.2	55.5	89.9	56.7
1.00	11.9	13.8	15.9	13.8	14.2	18.0	14.3
2.00	4.2	5.0	5.2	5.0	5.2	5.6	5.1
5.00	1.4	1.7	1.7	1.7	1.8	1.8	1.8
10.00	0.6	0.7	0.8	0.7	0.8	0.9	0.8
h	3.2237	3.1655	3.7266	3.2141	3.1549	3.9263	3.2165

Table 6.16. In-Control ANOS and Out-of-Control SSANOS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=1$, $d=1$, $\lambda=0.11989$, and $p=0.2$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1234.7	1234.7	1234.7	1194.8	1194.8	1194.8
0.25	228.6	202.9	277.5	206.9	198.7	317.9	203.8
0.50	49.0	45.7	62.3	46.4	45.1	72.9	46.1
1.00	12.4	12.0	13.7	12.0	11.9	14.9	11.9
2.00	4.7	4.6	4.7	4.6	4.6	4.9	4.6
5.00	1.9	1.9	1.8	1.9	1.9	1.9	1.8
10.00	1.1	1.0	1.1	1.0	1.0	1.1	1.0
h	3.2237	3.1655	3.7266	3.2141	3.1549	3.9263	3.2165

Table 6.17. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=1$, $d=1$, $\lambda=0.11989$, and $p=0.5$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6
0.25	228.1	262.8	386.6	273.8	277.8	502.9	296.9
0.50	48.5	62.3	92.6	64.4	68.6	133.6	72.5
1.00	11.9	16.5	19.4	16.6	18.8	26.2	18.9
2.00	4.2	6.1	5.9	6.0	7.0	7.2	6.9
5.00	1.4	2.1	1.8	2.0	2.5	2.1	2.4
10.00	0.6	0.9	1.0	0.9	1.1	1.3	1.1
h	3.2237	3.0927	4.0851	3.2067	3.0413	4.6070	3.2235

Table 6.18. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=1$, $d=1$, $\lambda=0.11989$, and $p=0.5$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	987.7	987.7	987.7	846.6	846.6	846.6
0.25	228.6	175.7	258.2	183.0	159.1	287.8	170.0
0.50	49.0	42.0	62.2	43.4	39.6	76.7	41.8
1.00	12.4	11.5	13.4	11.5	11.1	15.4	11.2
2.00	4.7	4.5	4.4	4.4	4.4	4.5	4.3
5.00	1.9	1.9	1.6	1.8	1.8	1.6	1.7
10.00	1.1	1.0	1.1	1.0	1.0	1.1	1.0
h	3.2237	3.0927	4.0851	3.2067	3.0413	4.6070	3.2235

Table 6.19. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=1$, $d=1$, $\lambda=0.11989$, and $p=0.9$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6
0.25	228.1	285.9	421.6	303.6	325.2	565.5	361.2
0.50	48.5	72.0	105.8	75.5	89.6	164.5	97.5
1.00	11.9	19.9	22.1	19.9	26.4	33.4	26.5
2.00	4.2	7.4	6.5	7.2	10.0	8.6	9.5
5.00	1.4	2.5	1.9	2.4	3.5	2.2	3.2
10.00	0.6	1.0	1.0	1.0	1.5	1.5	1.5
h	3.2237	3.0134	3.3428	3.2041	2.8901	5.1223	3.2565

Table 6.20. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=1$, $d=1$, $\lambda=0.11989$, and $p=0.9$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	779.8	779.8	779.8	546.7	546.7	546.7
0.25	228.6	151.0	222.4	160.3	120.5	209.2	133.8
0.50	49.0	38.4	56.2	40.2	33.5	61.2	36.5
1.00	12.4	11.0	12.1	11.0	10.2	12.8	10.3
2.00	4.7	4.4	3.9	4.3	4.2	3.7	4.0
5.00	1.9	1.8	1.5	1.8	1.8	1.3	1.6
10.00	1.1	1.0	1.0	1.0	1.0	1.0	1.0
h	3.2237	3.0134	3.3428	3.2041	2.8901	5.1223	3.2565

Tables 6.21 and 6.36 show that there is no difference in the SSATS values of the EI, EA, and EP charts with $n = 4$, $d = 4$, $\lambda = 0.1$ or 0.4, and $p = 0.01$. In this case, the SSATS for all three charts is the same as for the standard EZ chart for moderate and large shifts in μ, and only slightly higher for small shifts. Increasing the maximum number of consecutive missing samples from one to two has no effect on the performance of any of the charts. When taking samples of size $n = 4$, the probability that an entire sample is missing is $p^n = 0.01^4$. Thus, almost every sample has at least one observation.

Table 6.21. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 4$, $d = 4$, $\lambda = 0.1$, and $p = 0.01$.

	η = 0	η = 1			η = 2		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6
0.25	108.5	109.5	109.5	109.5	109.5	109.5	109.5
0.50	36.2	36.5	36.5	36.5	36.5	36.5	36.5
1.00	14.5	14.6	14.6	14.6	14.6	14.6	14.6
2.00	6.5	6.5	6.5	6.5	6.5	6.5	6.5
5.00	2.1	2.1	2.1	2.1	2.1	2.1	2.1
10.00	2.0	2.0	2.0	2.0	2.0	2.0	2.0
h	2.7015	2.7015	2.7015	2.7015	2.7015	2.7015	2.7015

Table 6.22. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=4$, $d=4$, $\lambda=0.1$, and $p=0.01$.

	η = 0	η = 1			η = 2		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	370.4	370.4	370.4	370.4	370.4	370.4	370.4
0.25	27.6	27.9	27.9	27.9	27.9	27.9	27.9
0.50	9.6	9.6	9.6	9.6	9.6	9.6	9.6
1.00	4.1	4.2	4.2	4.2	4.2	4.2	4.2
2.00	2.1	2.1	2.1	2.1	2.1	2.1	2.1
5.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0
10.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0
h	2.7015	2.7015	2.7015	2.7015	2.7015	2.7015	2.7015

Table 6.23. In-Control ANOS and Out-of-Control SSANOS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=4$, $d=4$, $\lambda=0.1$, and $p=0.01$.

	η = 0	η = 1			η = 2		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1466.8	1466.8	1466.8	1466.8	1466.8	1466.8
0.25	110.5	110.4	110.4	110.4	110.4	110.4	110.4
0.50	38.2	38.1	38.1	38.1	38.1	38.1	38.1
1.00	16.5	16.5	16.5	16.5	16.5	16.5	16.5
2.00	8.5	8.4	8.4	8.4	8.5	8.4	8.4
5.00	4.1	4.1	4.1	4.1	4.1	4.1	4.1
10.00	4.0	4.0	4.0	4.0	4.0	4.0	4.0
h	2.7015	2.7015	2.7015	2.7015	2.7015	2.7015	2.7015

When p is increased to 0.1, there is still no difference in the SSATS values for the EI, EA, and EP charts, which can be seen in Tables 6.24 and 6.39. However, as we would expect, a larger probability that an observation is missing increases the time required to detect a change when compared to the EZ chart. For example, when $\lambda=0.1$ and $\delta=0.5$, the EI, EA, and EP charts signal in 39.3 hours for both $\eta=1$ and $\eta=2$, while the EZ chart is able to detect this change in only 36.2 hours. Since $n = 4$, increasing the maximum number of consecutive missing samples has no effect. Increasing the probability that an observation is missing increases the SSATS, especially for small shifts in μ. For $\lambda = 0.1$ and $\delta = 0.5$, the SSATS is 36.5 hours if $p = 0.01$, and 39.3 hours if $p = 0.1$.

Table 6.24. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 4$, $d = 4$, $\lambda = 0.1$, and $p = 0.1$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6
0.25	108.5	119.5	119.6	119.5	119.5	119.6	119.5
0.50	36.2	39.3	39.3	39.3	39.3	39.3	39.3
1.00	14.5	15.6	15.6	15.6	15.6	15.6	15.6
2.00	6.5	6.9	6.9	6.9	6.9	6.9	6.9
5.00	2.1	2.3	2.3	2.3	2.3	2.3	2.3
10.00	2.0	2.0	2.0	2.0	2.0	2.0	2.0
h	2.7015	2.7014	2.7017	2.7014	2.7014	2.7017	2.7014

Table 6.25. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 4$, $d = 4$, $\lambda = 0.1$, and $p = 0.1$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	370.4	370.4	370.4	370.4	370.4	370.4	370.4
0.25	27.6	30.4	30.4	30.4	30.4	30.4	30.4
0.50	9.6	10.3	10.3	10.3	10.3	10.3	10.3
1.00	4.1	4.4	4.4	4.4	4.4	4.4	4.4
2.00	2.1	2.2	2.2	2.2	2.2	2.2	2.2
5.00	1.0	1.1	1.1	1.1	1.1	1.1	1.1
10.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0
h	2.7015	2.7014	2.7017	2.7014	2.7014	2.7017	2.7014

Table 6.26. In-Control ANOS and Out-of-Control SSANOS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 4$, $d = 4$, $\lambda = 0.1$, and $p = 0.1$.

	η = 0	η = 1			η = 2		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1333.4	1333.4	1333.4	1333.4	1333.4	1333.4
0.25	110.5	109.4	109.4	109.4	109.4	109.4	109.4
0.50	38.2	37.2	37.2	37.2	37.2	37.2	37.2
1.00	16.5	15.8	15.8	15.8	15.8	15.8	15.8
2.00	8.5	8.0	8.0	8.0	8.0	8.0	8.0
5.00	4.1	3.9	3.9	3.9	3.9	3.9	3.9
10.00	4.0	3.6	3.6	3.6	3.6	3.6	3.6
h	2.7015	2.7014	2.7017	2.7014	2.7014	2.7017	2.7014

The SSATS values for $p = 0.2$ and 0.5 are shown in Tables 6.27, 6.30, 6.42, and 6.45. There is still no significant difference in the performance of the three methods for medium and large shifts in the process mean, but the EI chart is slightly more efficient for small shifts in μ. For $p = 0.2$, $\lambda = 0.4$, and $\delta = 0.25$, the SSATS for the EI chart is 288.5 hours, while the EA and EP charts have SSATS values of 290.1 and 288.9 hours, respectively.

When p is increased to 0.9, the EI chart is able to detect small shifts in μ much more quickly than the EA and EP charts for both $\lambda = 0.1$ and $\lambda = 0.4$. However, for moderate and large shifts in the process mean, the EA and EP charts are more efficient when $\lambda = 0.1$. For example, if $\eta = 1$ and $\delta = 2.0$, the SSATS for the EA chart is 18.9 hours, while it takes 20.7 and 20.3 hours for the EI and EP charts to signal, respectively. If $\delta = 10.0$, the SSATS of the EP chart is 3.6 hours, which is better than the SSATS values of 3.7 and 3.8 hours for the EI and EA charts. If $\lambda = 0.4$, the EI chart has the best performance for moderate shifts, but the EP chart is slightly better for large shifts in μ.

Table 6.27. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 4$, $d = 4$, $\lambda = 0.1$, and $p = 0.2$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6
0.25	108.5	133.3	133.6	133.3	133.3	133.6	133.3
0.50	36.2	43.2	43.2	43.2	43.2	43.2	43.2
1.00	14.5	16.9	16.9	16.9	16.9	16.9	16.9
2.00	6.5	7.5	7.5	7.5	7.5	7.5	7.5
5.00	2.1	2.5	2.5	2.5	2.5	2.5	2.5
10.00	2.0	2.0	2.0	2.0	2.0	2.0	2.0
h	2.7015	2.7008	2.7057	2.7011	2.7008	2.7057	2.7011

Table 6.28. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 4$, $d = 4$, $\lambda = 0.1$, and $p = 0.2$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	370.4	369.8	369.8	369.8	369.8	369.8	369.8
0.25	27.6	33.8	33.8	33.8	33.8	33.8	33.8
0.50	9.6	11.3	11.3	11.3	11.3	11.3	11.3
1.00	4.1	4.7	4.7	4.7	4.7	4.7	4.7
2.00	2.1	2.4	2.4	2.4	2.4	2.4	2.4
5.00	1.0	1.1	1.1	2.4	1.1	1.1	1.1
10.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0
h	2.7015	2.7008	2.7057	2.7011	2.7008	2.7057	2.7011

Table 6.29. In-Control ANOS and Out-of-Control SSANOS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 4$, $d = 4$, $\lambda = 0.1$, and $p = 0.2$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1185.3	1185.3	1185.3	1185.3	1185.3	1185.3
0.25	110.5	108.2	108.5	108.2	108.2	108.5	108.2
0.50	38.2	36.2	36.2	36.2	36.2	36.2	36.2
1.00	16.5	15.1	15.1	15.1	15.1	15.1	15.1
2.00	8.5	7.6	7.6	7.6	7.6	7.6	7.6
5.00	4.1	3.6	3.6	7.6	3.6	3.6	3.6
10.00	4.0	3.2	3.2	3.2	3.2	3.2	3.2
h	2.7015	2.7008	2.7057	2.7011	2.7008	2.7057	2.7011

Table 6.30. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 4$, $d = 4$, $\lambda = 0.1$, and $p = 0.5$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6
0.25	108.5	199.7	217.4	200.2	200.1	221.2	200.7
0.50	36.2	62.9	66.3	62.9	63.1	67.1	63.1
1.00	14.5	23.5	24.0	23.5	23.6	24.3	23.6
2.00	6.5	10.2	10.3	10.2	10.3	10.4	10.2
5.00	2.1	3.8	3.8	3.7	3.8	3.9	3.8
10.00	2.0	2.3	2.3	2.3	2.3	2.3	2.3
h	2.7015	2.6781	2.8493	2.6893	2.6767	2.8679	2.6888

Table 6.31. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 4$, $d = 4$, $\lambda = 0.1$, and $p = 0.5$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	370.4	348.6	348.6	348.6	347.3	347.3	347.3
0.25	27.6	47.5	51.6	47.6	47.4	52.3	47.5
0.50	9.6	15.3	16.1	15.3	15.3	16.2	15.3
1.00	4.1	6.0	6.1	6.0	6.0	6.2	6.0
2.00	2.1	2.9	2.9	2.9	2.9	2.9	2.9
5.00	1.0	1.4	1.4	1.4	1.4	1.4	1.4
10.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0
h	2.7015	2.6781	2.8493	2.6893	2.6767	2.8679	2.6888

Table 6.32. In-Control ANOS and Out-of-Control SSANOS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 4$, $d = 4$, $\lambda = 0.1$, and $p = 0.5$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	743.7	743.7	743.7	741.0	741.0	741.0
0.25	110.5	101.3	110.2	101.5	101.1	111.6	101.4
0.50	38.2	32.6	34.3	32.6	32.6	34.6	32.6
1.00	16.5	12.8	13.1	12.8	12.8	13.1	12.8
2.00	8.5	6.1	6.2	6.1	6.1	6.2	6.1
5.00	4.1	2.9	2.9	2.9	2.9	2.9	2.9
10.00	4.0	2.2	2.2	2.1	2.2	2.2	2.1
h	2.7015	2.6781	2.8493	2.6893	2.6767	2.8679	2.6888

Table 6.33. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 4$, $d = 4$, $\lambda = 0.1$, and $p = 0.9$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6
0.25	108.5	390.6	489.6	398.5	431.3	607.4	445.4
0.50	36.2	135.3	161.4	136.6	157.4	214.9	160.0
1.00	14.5	49.3	50.3	48.9	59.3	64.5	58.5
2.00	6.5	20.7	18.9	20.3	25.4	22.8	24.5
5.00	2.1	7.6	6.3	7.3	9.6	7.6	9.0
10.00	2.0	3.7	3.8	3.6	5.0	5.4	5.0
h	2.7015	2.5006	3.4990	2.6064	2.4035	4.0052	2.5837

Table 6.34. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 4$, $d = 4$, $\lambda = 0.1$, and $p = 0.9$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	370.4	223.7	223.7	223.7	177.5	177.5	177.5
0.25	27.6	390.6	74.4	60.6	52.1	73.2	53.8
0.50	9.6	20.9	24.8	21.1	19.3	26.2	19.6
1.00	4.1	7.9	8.1	7.8	7.5	8.1	7.4
2.00	2.1	3.6	3.3	3.5	3.5	3.1	3.3
5.00	1.0	1.6	1.4	1.6	1.6	1.3	1.5
10.00	1.0	1.0	1.0	1.0	1.0	1.1	1.0
h	2.7015	2.5006	3.4990	2.6064	2.4035	4.0052	2.5837

Table 6.35. In-Control ANOS and Out-of-Control SSANOS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 4$, $d = 4$, $\lambda = 0.1$, and $p = 0.9$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	260.1	260.1	260.1	206.5	206.5	206.5
0.25	110.5	69.1	86.5	70.5	60.6	85.1	62.6
0.50	38.2	24.3	28.9	24.5	22.4	30.4	22.8
1.00	16.5	9.2	9.4	9.1	8.7	9.5	8.6
2.00	8.5	4.2	3.8	4.1	4.0	3.7	3.9
5.00	4.1	1.9	1.6	1.8	1.8	1.5	1.7
10.00	4.0	1.2	1.2	1.2	1.2	1.2	1.2
h	2.7015	2.5006	3.4990	2.6064	2.4035	4.0052	2.5837

The SSANSS and SSANOS values show results similar to the SSATS. The in-control ANSS for the EI, EA, and EP charts is 370.4 samples – the same as for the EZ chart – for both values of λ when $p_1 = 0.01$ or 0.1. The in-control ANSS decreases to 369.8 for $p = 0.2$, 347.3 for $p = 0.5$ and 177.5 for $p = 0.9$ if $\eta = 2$. When $p = 0.01$, the in-control ANOS for all three charts decreases from 1481.6 observations to 1466.8 observations, and to 1333.4 observations when $p = 0.1$. The in-control ANOS is only 1185.3 for $p = 0.2$, 741.0 for $p = 0.5$, and 206.5 for $p = 0.9$. Although the control limits are set so that all the charts have an in-control ATS of 1481.6, the in-control ANSS and ANOS decreases significantly as p increases.

The EWMA charts with $n = 4$ are able to detect moderate changes in the process mean more quickly for smaller values of p, but the corresponding charts with $n = 1$ have smaller SSATS values for small and large shifts in μ. However, the difference in the performance of the two sampling patterns is small for smaller values of p. As p increases, the difference in the SSATS values of the two sampling schemes also increases. For $p = 0.5$ and $p = 0.9$, the EWMA charts with $n = 1$ have better performance for all magnitudes of shifts.

If the sampling scheme based on $n = 1$ is used, then the EI chart is recommended. When $n = 4$, the EI, EA, and EP charts are equivalent for small values of p. For $p = 0.5$ and $p = 0.9$, the EI chart has the best overall performance. However, for both sampling patterns, the EA and EP charts are able to detect some moderate and large shifts in the process mean more quickly than the EI chart if p is very large.

Table 6.36. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 4$, $d = 4$, $\lambda = 0.4$, and $p = 0.01$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6
0.25	230.3	232.7	232.7	232.7	232.7	232.7	232.7
0.50	48.3	48.8	48.8	48.8	48.8	48.8	48.8
1.00	11.2	11.3	11.3	11.3	11.3	11.3	11.3
2.00	3.7	3.7	3.7	3.7	3.7	3.7	3.7
5.00	2.0	2.0	2.0	2.0	2.0	2.0	2.0
10.00	2.0	2.0	2.0	2.0	2.0	2.0	2.0
h	2.9589	2.9589	2.9589	2.9589	2.9589	2.9589	2.9589

Table 6.37. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 4$, $d = 4$, $\lambda = 0.4$, and $p = 0.01$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	370.4	370.4	370.4	370.4	370.4	370.4	370.4
0.25	58.1	58.7	58.7	58.7	58.7	58.7	58.7
0.50	12.6	12.7	12.7	12.7	12.7	12.7	12.7
1.00	3.3	3.3	3.3	3.3	3.3	3.3	3.3
2.00	1.4	1.4	1.4	1.4	1.4	1.4	1.4
5.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0
10.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0
h	2.9589	2.9589	2.9589	2.9589	2.9589	2.9589	2.9589

Table 6.38. In-Control ANOS and Out-of-Control SSANOS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 4$, $d = 4$, $\lambda = 0.4$, and $p = 0.01$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1466.8	1466.8	1466.8	1466.8	1466.8	1466.8	1466.8
0.25	232.3	232.4	232.4	232.4	232.4	232.4	232.4
0.50	50.3	50.3	50.3	50.3	50.3	50.3	50.3
1.00	13.2	13.2	13.2	13.2	13.2	13.2	13.2
2.00	5.7	5.6	5.6	5.6	5.6	5.6	5.6
5.00	4.0	4.0	4.0	4.0	4.0	4.0	4.0
10.00	4.0	4.0	4.0	4.0	4.0	4.0	4.0
h	2.9589	2.9589	2.9589	2.9589	2.9589	2.9589	2.9589

Table 6.39. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=4$, $d=4$, $\lambda=0.4$, and $p=0.1$.

	$\eta=0$	$\eta=1$			$\eta=2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6
0.25	230.3	256.7	256.8	256.7	256.7	256.8	256.7
0.50	48.3	54.8	54.8	54.8	54.8	54.8	54.8
1.00	11.2	12.4	12.4	12.4	12.4	12.4	12.4
2.00	3.7	4.0	4.0	4.0	4.0	4.0	4.0
5.00	2.0	2.0	2.0	2.0	2.0	2.0	2.0
10.00	2.0	2.0	2.0	2.0	2.0	2.0	2.0
h	2.9589	2.9589	2.9592	2.9590	2.9589	2.9592	2.9590

Table 6.40. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=4$, $d=4$, $\lambda=0.4$, and $p=0.1$.

	$\eta=0$	$\eta=1$			$\eta=2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	370.4	370.4	370.4	370.4	370.4	370.4	370.4
0.25	58.1	64.7	64.7	64.7	64.7	64.7	64.7
0.50	12.6	14.2	14.2	14.2	14.2	14.2	14.2
1.00	3.3	3.6	3.6	3.6	3.6	3.6	3.6
2.00	1.4	1.5	1.5	1.5	1.5	1.5	1.5
5.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0
10.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0
h	2.9589	2.9589	2.9592	2.9590	2.9589	2.9592	2.9590

Table 6.41. In-Control ANOS and Out-of-Control SSANOS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=4$, $d=4$, $\lambda=0.4$, and $p=0.1$.

	$\eta=0$	$\eta=1$			$\eta=2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1466.8	1333.4	1333.4	1333.4	1333.4	1333.4	1333.4
0.25	232.3	232.8	232.9	232.8	234.8	232.9	232.8
0.50	50.3	51.1	51.2	51.1	51.1	51.2	51.1
1.00	13.2	13.0	13.0	13.0	13.0	13.0	13.0
2.00	5.7	5.4	5.4	5.4	5.4	5.4	5.4
5.00	4.0	3.6	3.6	3.6	3,6	3.6	3.6
10.00	4.0	3.6	3.6	3.6	3.6	3.6	3.6
h	2.9589	2.9589	2.9592	2.9590	2.9589	2.9592	2.9590

Table 6.42. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 4$, $d = 4$, $\lambda = 0.4$, and $p = 0.2$.

	η = 0	η = 1			η = 2		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6
0.25	230.3	288.5	290.1	288.9	288.5	290.1	288.9
0.50	48.3	63.2	63.5	63.3	63.2	63.5	63.3
1.00	11.2	14.0	14.0	14.0	14.0	14.0	14.0
2.00	3.7	4.5	4.5	4.5	4.5	4.5	4.5
5.00	2.0	2.0	2.0	2.0	2.0	2.0	2.0
10.00	2.0	2.0	2.0	2.0	2.0	2.0	2.0
h	2.9589	2.9584	2.9634	2.9602	2.9584	2.9635	2.9601

Table 6.43. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 4$, $d = 4$, $\lambda = 0.4$, and $p = 0.2$.

	η = 0	η = 1			η = 2		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	370.4	369.8	369.8	369.8	369.8	369.8	369.8
0.25	58.1	72.5	72.9	72.6	72.5	72.9	72.6
0.50	12.6	16.3	16.4	16.3	16.3	16.4	16.3
1.00	3.3	4.0	4.0	4.0	4.0	4.0	4.0
2.00	1.4	1.6	1.6	1.6	1.6	1.6	1.6
5.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0
10.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0
h	2.9589	2.9584	2.9634	2.9602	2.9584	2.9635	2.9601

Table 6.44. In-Control ANOS and Out-of-Control SSANOS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 4$, $d = 4$, $\lambda = 0.4$, and $p = 0.2$.

	η = 0	η = 1			η = 2		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1185.3	1185.3	1185.3	1185.3	1185.3	1185.3
0.25	232.3	232.4	233.7	232.7	232.4	233.7	232.7
0.50	50.3	52.2	52.4	52.2	52.2	52.4	52.2
1.00	13.2	12.8	12.8	12.8	12.8	12.8	12.8
2.00	5.7	5.2	5.2	5.2	5.2	5.2	5.2
5.00	4.0	3.2	3.2	3.2	3.2	3.2	3.2
10.00	4.0	3.2	3.2	3.2	3.2	3.2	3.2
h	2.9589	2.9584	2.9634	2.9602	2.9584	2.9635	2.9601

Table 6.45. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 4$, $d = 4$, $\lambda = 0.4$, and $p = 0.5$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6
0.25	230.3	423.8	495.6	442.3	424.2	505.2	446.6
0.50	48.3	106.3	126.6	111.5	106.5	129.4	112.7
1.00	11.2	22.4	24.6	22.9	22.5	24.9	23.1
2.00	3.7	6.7	6.9	6.7	6.8	7.0	6.7
5.00	2.0	2.4	2.4	2.4	2.4	2.5	2.4
10.00	2.0	2.2	2.2	2.2	2.3	2.3	2.3
h	2.9589	2.9393	3.1174	3.0024	2.9382	3.1320	3.0095

Table 6.46. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 4$, $d = 4$, $\lambda = 0.4$, and $p = 0.5$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	370.4	348.6	348.6	348.6	347.3	347.3	347.3
0.25	58.1	100.2	117.1	104.6	99.9	118.9	105.2
0.50	12.6	25.5	30.3	26.7	25.4	30.8	26.9
1.00	3.3	5.7	6.3	5.9	5.7	6.3	5.9
2.00	1.4	2.1	2.1	2.0	2.1	2.1	2.1
5.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0
10.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0
h	2.9589	2.9393	3.1174	3.0024	2.9382	3.1320	3.0095

Table 6.47. In-Control ANOS and Out-of-Control SSANOS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 4$, $d = 4$, $\lambda = 0.4$, and $p = 0.5$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	743.7	743.7	743.7	741.0	741.0	741.0
0.25	232.3	213.7	249.8	223.1	213.6	253.7	224.4
0.50	50.3	54.4	64.6	57.0	54.3	65.7	57.4
1.00	13.2	12.2	13.3	12.5	12.2	13.5	12.5
2.00	5.7	4.4	4.5	4.4	4.4	4.5	4.4
5.00	4.0	2.2	2.2	2.2	2.2	2.2	2.2
10.00	4.0	2.1	2.1	2.1	2.1	2.1	2.1
h	2.9589	2.9393	3.1174	3.0024	2.9382	3.1320	3.0095

Table 6.48. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=4$, $d=4$, $\lambda=0.4$, and $p=0.9$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6
0.25	230.3	695.1	886.3	788.4	724.4	1002.2	899.9
0.50	48.3	240.4	366.4	294.9	262.7	478.1	381.4
1.00	11.2	56.6	83.4	66.7	65.9	120.0	91.2
2.00	3.7	15.3	16.6	15.5	18.8	22.1	19.7
5.00	2.0	4.2	4.1	4.0	5.5	5.6	5.4
10.00	2.0	3.6	3.6	3.6	4.9	4.9	4.9
h	2.9589	2.7919	3.6873	3.2286	2.7121	4.0436	3.4904

Table 6.49. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=4$, $d=4$, $\lambda=0.4$, and $p=0.9$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	370.4	223.7	223.7	223.7	177.5	177.5	177.5
0.25	58.1	105.4	134.2	119.5	87.2	120.5	108.2
0.50	12.6	36.7	55.8	45.0	31.9	57.7	46.1
1.00	3.3	9.0	13.0	10.5	8.3	14.8	11.3
2.00	1.4	2.8	3.0	2.8	2.7	3.1	2.8
5.00	1.0	1.1	1.1	1.1	1.1	1.1	1.1
10.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0
h	2.9589	2.7919	3.6873	3.2286	2.7121	4.0436	3.4904

Table 6.50. In-Control ANOS and Out-of-Control SSANOS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=4$, $d=4$, $\lambda=0.4$, and $p=0.9$.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	260.1	260.1	260.1	206.5	206.5	206.5
0.25	232.3	122.6	156.1	139.0	101.4	140.2	125.9
0.50	50.3	42.7	64.9	52.3	37.1	67.1	53.6
1.00	13.2	10.5	15.2	12.2	9.7	17.2	13.2
2.00	5.7	3.2	3.5	3.3	3.1	3.6	3.2
5.00	4.0	1.3	1.3	1.2	1.3	1.3	1.2
10.00	4.0	1.2	1.2	1.2	1.2	1.2	1.2
h	2.9589	2.7919	3.6873	3.2286	2.7121	4.0436	3.4904

6.3 Comparison of the EWMA Control Charts when the Same Control Limits Are Used

Another issue of interest is the effect on the EZ control chart if observations are missing. The control limits of the EZ chart with no missing observations are set so that the in-control ATS is 1481.6. These limits are then used for the EI, EA, and EP charts. This allows us to investigate the effect of missing observations on the performance of the three charts. In addition, it is common in practical applications to use the standard control limits without adjusting them to obtain a specific in-control ATS. The resulting SSATS and SSANSS values are shown in Tables 6.51 through 6.62 for the case when $n = 1$, $d = 1$, $\lambda = 0.026$ or 0.11989, and $p = 0.01$ or 0.1.

From these tables, we can see that the in-control ATS for the EI chart increases by $p \cdot 100\%$ if $\eta = 1$ and by $(p + p^2) \cdot 100\%$ if $\eta = 2$. This results in a false alarm rate per unit time that is smaller than that of the EZ chart. The in-control ANSS for the EI chart remains the same – 1481.6 samples. The in-control ATS and ANSS for the EA chart are less than 1481.6, and they decrease as p increases. The in-control ATS of the EP chart is higher than 1481.6 hours for both values of λ and p, while the in-control ANSS is less than 1481.6. Although the in-control ATS of the EP chart is higher than that of the EZ chart (and the EA chart), it is always smaller than that of the EI chart. For example, if $\lambda = 0.11989$, $p = 0.1$, and $\eta = 1$, the in-control ATS is 1629.8, 673.8, and 1506.7 hours, for the EI, EA and EP charts respectively. This means that the false alarm rate per unit time for the EI chart is 10% lower than the EZ chart, and the false alarm rate for this EA chart is more than twice that of the EZ chart.

If $p = 0.01$, the SSATS values for the EI and EP charts for small shifts in μ are similar to the SSATS values for the EZ chart. The EA chart signals more quickly than the other charts; however, this is not unexpected since the EA chart has a much smaller in-control ATS. For moderate and large changes in the process mean, the EI, EA, and EP charts have similar performance to the EZ chart. If $p = 0.1$, the EA chart is the most efficient for detecting small shifts in μ, while the EI chart takes the longest to signal. The EA chart detects moderate shifts in approximately the same amount of time as the EZ chart, while the EI and EP charts have larger SSATS values. For large changes in the process mean, the EI, EA, and EP charts all signal in the same amount of time as the EZ chart. Thus, the time required to detect large shifts in μ is not affected by the missing observations or the differences in the false alarm rate. If the probability that an observation is missing is $p = 0.2$, the in-control ATS of the EI and EP charts is once again higher than 1481.6, while the in-control ATS of the EA chart decreases to 499.8 hours and 333.7 hours for $\lambda = 0.026$ and 0.11989, respectively. The EA chart is able to detect shifts in μ of all magnitudes in approximately the same amount of time as the EZ chart, while the EI chart takes the longest to signal.

Tables 6.21 – 6.26 and 6.36 – 6.41 show that when $n = 4$ and $p = 0.01$, the control limits that give an in-control ATS of 1481.6 hours for the EI, EA, and EP charts are the same as the limits for the EZ chart, and when $p = 0.1$, the control limits are almost equivalent. Missing observations have very little or no effect on the false alarm rate of the EI, EA, and EP charts. This is again due to the fact that the probability that an entire sample is missing is very small. From Tables 6.27 – 6.29 and 6.42 – 6.44, we can see that for $p = 0.2$, if the in-control ATS is 1481.6, the control limits for the EI and EP charts are very similar to those of the EZ chart. This indicates that even with a twenty percent chance of an observation being missing, using the same control limits for the EI and EP charts that would be used in the standard EZ chart with no missing observations still results in a similar false alarm rate. However, the limits for the EA chart are wider than those of the EZ chart, so for $p = 0.2$, using the same control limits for the EZ and EA charts would result in an EA chart with smaller in-control ATS, just as when using the sampling scheme based on $n = 1$.

Table 6.51. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=1$, $d=1$, $\lambda=0.026$, $p=0.01$, when the Same Control Limits Are Used for All Charts.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1496.4	1381.3	1494.6	1496.6	1379.3	1494.7
0.25	109.1	110.2	108.3	110.2	110.2	108.3	110.2
0.50	36.9	37.3	36.9	37.3	37.3	36.9	37.3
1.00	15.1	15.2	15.1	15.2	15.2	15.1	15.2
2.00	6.8	6.9	6.8	6.9	6.9	6.8	6.9
5.00	2.6	2.6	2.6	2.6	2.6	2.6	2.6
10.00	1.3	1.3	1.3	1.3	1.3	1.3	1.3
h	2.8334	2.8334	2.8334	2.8334	2.8334	2.8334	2.8334

Table 6.52. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=1$, $d=1$, $\lambda=0.026$, $p=0.01$, when the Same Control Limits Are Used for All Charts.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1481.6	1376.7	1479.8	1481.6	1365.5	1479.8
0.25	109.6	109.6	107.8	109.6	109.6	107.7	109.6
0.50	37.4	37.4	37.0	37.4	37.4	37.0	37.4
1.00	15.6	15.6	15.4	15.6	15.6	15.4	15.6
2.00	7.3	7.3	7.3	7.3	7.3	7.3	7.3
5.00	3.1	3.1	3.0	3.1	3.1	3.0	3.1
10.00	1.8	1.8	1.8	1.8	1.8	1.8	1.8
h	2.8334	2.8334	2.8334	2.8334	2.8334	2.8334	2.8334

Table 6.53. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=1$, $d=1$, $\lambda=0.026$, $p=0.1$, when the Same Control Limits Are Used for All Charts.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1629.8	861.2	1610.2	1644.6	790.7	1622.7
0.25	109.1	120.1	102.7	119.6	121.2	101.8	120.7
0.50	36.9	40.6	36.7	40.5	41.0	36.7	40.9
1.00	15.1	16.6	15.1	16.6	16.8	15.2	16.7
2.00	6.8	7.5	6.8	7.5	7.6	6.9	7.6
5.00	2.6	2.9	2.6	2.9	2.9	2.6	3.0
10.00	1.3	1.4	1.3	1.4	1.5	1.3	1.5
h	2.8334	2.8334	2.8334	2.8334	2.8334	2.8334	2.8334

Table 6.54. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=1$, $d=1$, $\lambda=0.026$, $p=0.1$ when the Same Control Limits Are Used for All Charts.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1481.6	782.9	1463.8	1481.6	712.4	1461.9
0.25	109.6	109.6	93.8	109.2	109.6	92.2	109.2
0.50	37.4	37.4	33.8	37.3	37.4	33.5	37.3
1.00	15.6	15.6	14.2	15.5	15.6	14.1	15.5
2.00	7.3	7.3	6.7	7.3	7.3	6.6	7.3
5.00	3.1	3.1	2.8	3.1	3.1	2.8	3.1
10.00	1.8	1.8	1.6	1.8	1.8	1.6	1.8
h	2.8334	2.8334	2.8334	2.8334	2.8334	2.8334	2.8334

Table 6.55. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=1$, $d=1$, $\lambda=0.026$, $p=0.2$ when the Same Control Limits Are Used for All Charts.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1777.9	617.6	1735.6	1837.2	499.8	1784.3
0.25	109.1	131.0	98.1	130.0	135.4	95.5	134.2
0.50	36.9	44.3	36.5	44.1	45.9	36.5	45.6
1.00	15.1	18.2	15.2	18.1	18.8	15.2	18.7
2.00	6.8	8.3	6.9	8.2	8.6	6.9	8.5
5.00	2.6	3.1	2.6	3.1	3.3	2.6	3.3
10.00	1.3	1.6	1.3	1.6	1.7	1.3	1.7
h	2.8334	2.8334	2.8334	2.8334	2.8334	2.8334	2.8334

Table 6.56. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=1$, $d=1$, $\lambda=0.026$, $p=0.2$ when the Same Control Limits Are Used for All Charts.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1481.6	514.6	1446.3	1481.6	403.0	1439.0
0.25	109.6	109.6	82.2	108.8	109.6	77.4	108.6
0.50	37.4	37.4	30.9	37.2	37.4	29.9	37.1
1.00	15.6	15.6	13.1	15.5	15.6	12.7	15.5
2.00	7.3	7.3	6.2	7.3	7.3	6.0	7.3
5.00	3.1	3.1	2.6	3.0	3.1	2.5	3.0
10.00	1.8	1.8	1.5	1.8	1.8	1.5	1.8
h	2.8334	2.8334	2.8334	2.8334	2.8334	2.8334	2.8334

Table 6.57. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=1$, $d=1$, $\lambda=0.11989$, $p=0.01$ when the Same Control Limits Are Used for All Charts.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1496.4	1320.9	1484.4	1496.6	1317.0	1484.4
0.25	228.1	230.3	219.4	229.4	230.4	219.3	229.4
0.50	48.5	49.0	48.0	48.9	49.0	48.0	48.9
1.00	11.9	12.0	11.9	12.0	12.0	11.9	12.0
2.00	4.2	4.3	4.2	4.3	4.3	4.2	4.3
5.00	1.4	1.4	1.4	1.4	1.4	1.4	1.4
10.00	0.6	0.6	0.6	0.6	0.6	0.6	0.6
h	3.2237	3.2237	3.2237	3.2237	3.2237	3.2237	3.2237

Table 6.58. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=1$, $d=1$, $\lambda=0.11989$, $p=0.01$ when the Same Control Limits Are Used for All Charts.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1481.6	1307.9	1469.7	1481.6	1303.8	1469.6
0.25	228.6	228.6	217.8	227.6	228.6	217.6	227.6
0.50	49.0	49.0	48.1	48.9	49.0	48.1	48.9
1.00	12.4	12.4	12.2	12.3	12.4	12.2	12.3
2.00	4.7	4.7	4.7	4.7	4.7	4.7	4.7
5.00	1.9	1.9	1.9	1.9	1.9	1.9	1.9
10.00	1.1	1.1	1.1	1.1	1.1	1.1	1.1
h	3.2237	3.2237	3.2237	3.2237	3.2237	3.2237	3.2237

Table 6.59. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 1$, $d = 1$, $\lambda = 0.11989$, $p = 0.1$ when the Same Control Limits Are Used for All Charts.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1629.8	673.8	1506.7	1644.6	590.4	1504.2
0.25	228.1	250.9	168.1	240.8	253.2	161.2	241.8
0.50	48.5	53.4	44.5	52.3	53.9	44.1	52.6
1.00	11.9	13.1	11.9	12.9	13.2	11.9	13.0
2.00	4.2	4.7	4.2	4.6	4.7	4.3	4.7
5.00	1.4	1.6	1.4	1.6	1.6	1.4	1.6
10.00	0.6	0.7	0.6	0.6	0.7	0.7	0.7
h	3.2237	3.2237	3.2237	3.2237	3.2237	3.2237	3.2237

Table 6.60. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n = 1$, $d = 1$, $\lambda = 0.11989$, $p = 0.1$ when the Same Control Limits Are Used for All Charts.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1481.6	612.5	1369.7	1481.6	531.9	1355.1
0.25	228.6	228.6	153.3	219.4	228.6	145.7	218.3
0.50	49.0	49.0	40.9	48.0	49.0	40.2	47.9
1.00	12.4	12.4	11.2	12.2	12.4	11.1	12.2
2.00	4.7	4.7	4.3	4.7	4.7	4.3	4.6
5.00	1.9	1.9	1.8	1.9	1.9	1.8	1.9
10.00	1.1	1.1	1.0	1.0	1.1	1.1	1.0
h	3.2237	3.2237	3.2237	3.2237	3.2237	3.2237	3.2237

Table 6.61. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=1$, $d=1$, $\lambda=0.11989$, $p=0.2$ when the Same Control Limits Are Used for All Charts.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1777.9	444.7	1525.7	1837.2	333.7	1514.6
0.25	228.1	273.7	138.5	252.6	282.9	123.8	255.9
0.50	48.5	58.3	41.8	55.8	60.3	40.6	57.2
1.00	11.9	14.3	11.9	13.9	14.8	11.9	14.3
2.00	4.2	5.1	4.3	5.0	5.3	4.3	5.2
5.00	1.4	1.8	1.4	1.7	1.9	1.5	1.8
10.00	0.6	0.7	0.7	0.7	0.8	0.8	0.8
h	3.2237	3.2237	3.2237	3.2237	3.2237	3.2237	3.2237

Table 6.62. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA, and EP Control Charts for Shifts in μ when $n=1$, $d=1$, $\lambda=0.11989$, $p=0.2$ when the Same Control Limits Are Used for All Charts.

	$\eta = 0$	$\eta = 1$			$\eta = 2$		
δ	EZ	EI	EA	EP	EI	EA	EP
0.00	1481.6	1481.6	370.6	1271.5	1481.6	269.1	1221.4
0.25	228.6	228.6	115.9	210.9	228.6	100.2	206.8
0.50	49.0	49.0	35.2	47.0	49.0	33.2	46.5
1.00	12.4	12.4	10.3	12.0	12.4	10.0	12.0
2.00	4.7	4.7	4.0	4.6	4.7	3.9	4.6
5.00	1.9	1.9	1.6	1.9	1.9	1.6	1.8
10.00	1.1	1.1	1.0	1.0	1.1	1.0	1.0
h	3.2237	3.2237	3.2237	3.2237	3.2237	3.2237	3.2237

6.4 Comparison of the EWMA Control Charts with p_1 and p_2

So far the case for which there is a probability p that an individual observation is missing has been investigated. This section now considers a second case that could occur in which there is a probability that an observation is missing and a probability that an entire sample is missing. For example, an observation could be missing because it was defective and unable to be measured, and an entire sample could be missing because it was sent to the lab for analysis and the results were never received.

Let p_1 be the probability that an individual observation is missing, independently of other observations, and let p_2 be the probability that an entire sample is missing, independently of other samples. These two events are also independent, so even if $p_2 = 0$, an entire sample could still be missing with probability p_1^n. Therefore, the probability of an entire sample being missing is $p_1^n + p_2 - p_1^n p_2$. Chapters 4 and 5 developed Markov chain and integral equation methods for evaluating the properties for the case when $p_2 = 0$, and Sections 6.2 and 6.3 evaluated the performance of the three methods for adjusting the EWMA control statistic in this case. In this section, the performance of the three EWMA statistics is evaluated for several combinations of the parameters λ, n, d, p_1, and p_2 over a range of shifts in μ. The results shown were found using simulation.

The sampling pattern considered here is based on taking samples of $n = 4$ every $d = 4$ hours. The sampling pattern based on samples of $n = 1$ every $d = 1$ is not considered. When $n = 1$, if an individual observation is missing then the entire sample is missing, and so this would be the same as the case of $n = 1$ considered in Chapter 6 with $p = p_1 + p_2 - p_1 p_2$. The values of λ considered are 0.1 and 0.4.

The performance of the three methods for adjusting the EWMA control charts when there are missing observations is investigated for $p_1 = 0.1$ and $p_2 = 0.1$, and

$p_1 = 0.1$ and $p_2 = 0.2$. In order to use the Markov chain and integral equation methods developed in Chapters 4 and 5, a maximum number of consecutive missing samples, denoted by η, was required. It was also shown in Sections 6.2 and 6.3 that increasing η had little effect when p was reasonably small, i.e. $p \le 0.2$. In this section, simulation was used to evaluate the EWMA control charts, so it is not necessary to specify a maximum number of consecutive missing samples and $\eta = \infty$ was used. The probability of i consecutive samples missing is $\left(p_1^n + p_2 - p_1^n p_2\right)^i$, so as $i \to \infty$, $\left(p_1^n + p_2 - p_1^n p_2\right)^i \to 0$. For example, the probability of three consecutive missing samples is $\left(0.1^4 + 0.1 - 0.1^4 0.1\right)^3 = 0.001003$ if $p_1 = 0.1$ and $p_2 = 0.1$, and $\left(0.1^4 + 0.2 - 0.1^4 0.2\right)^3$ = 0.00801 if $p_1 = 0.1$ and $p_2 = 0.2$. Thus, it is highly unlikely for many consecutive samples to be missing, and setting $\eta < \infty$ would have very little or no effect on the results. Shifts in μ are expressed in terms of $\delta = \left|\mu - \mu_0\right| / \sigma_0$, and the shift sizes considered range from 0.0 to 10.0.

Tables 6.63 through 6.68 present the SSATS, SSANSS, and SSANOS values for the three methods of adjusting the EWMA control chart designed to detect changes in the process mean when there is a probability p_1 that an observation is missing and probability p_2 that a sample is missing. The control limits for all of these charts are again set to have an in-control ATS of 1481.6. Tables 6.63 – 6.65 give results for the case of $p_1 = 0.1$ and $p_2 = 0.1$, and Tables 6.66 – 6.68 present the results for $p_1 = 0.1$ and $p_2 = 0.2$.

Table 6.63. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA and EP Control Charts for Shifts in μ when $n = 4$, $d = 4$, $p_1 = 0.1$, and $p_2 = 0.1$

	$\lambda = 0.1$				$\lambda = 0.4$			
	$p_1 = 0$ $p_2 = 0$	$p_1 = 0.1$ $p_2 = 0.1$			$p_1 = 0$ $p_2 = 0$	$p_1 = 0.1$ $p_2 = 0.1$		
δ	EZ	EI	EA	EP	EZ	EI	EA	EP
0.00	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6
0.25	108.1	127.2	147.5	127.6	230.7	264.4	359.5	293.1
0.50	36.1	42.7	46.4	42.6	48.5	58.2	77.2	63.8
1.00	14.5	17.2	18.1	17.2	11.2	13.7	15.6	14.2
2.00	6.5	7.8	8.2	7.8	3.7	4.6	5.1	4.7
5.00	2.2	2.8	3.0	2.8	2.0	2.4	2.4	2.4
10.00	2.0	2.4	2.4	2.4	2.0	2.4	2.4	2.4
h	2.7028	2.6600	2.9739	2.6802	2.9589	2.9246	3.2334	3.0455

Table 6.64. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA and EP Control Charts for Shifts in μ when $n = 4$, $d = 4$, $p_1 = 0.1$, and $p_2 = 0.1$

	$\lambda = 0.1$				$\lambda = 0.4$			
	$p_1 = 0$ $p_2 = 0$	$p_1 = 0.1$ $p_2 = 0.1$			$p_1 = 0$ $p_2 = 0$	$p_1 = 0.1$ $p_2 = 0.1$		
δ	EZ	EI	EA	EP	EZ	EI	EA	EP
0.00	370.4	333.3	333.3	333.3	370.4	333.3	333.3	333.3
0.25	27.5	29.1	33.6	29.2	58.2	59.9	81.3	66.4
0.50	9.5	10.1	10.9	10.0	12.6	13.5	17.8	14.8
1.00	4.1	4.3	4.5	4.3	3.3	3.5	4.0	3.6
2.00	2.1	2.2	2.3	2.2	1.4	1.5	1.6	1.5
5.00	1.0	1.1	1.1	1.1	1.0	1.0	1.0	1.0
10.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
h	2.7028	2.6600	2.9739	2.6802	2.9589	2.9246	3.2334	3.0455

Table 6.65. In-Control ANOS and Out-of-Control SSANOS Values for the EZ, EI, EA and EP Control Charts for Shifts in μ when $n = 4$, $d = 4$, $p_1 = 0.1$, and $p_2 = 0.1$

	λ = 0.1				λ = 0.4			
	p_1 = 0 p_2 = 0	p_1 = 0.1 p_2 = 0.1			p_1 = 0 p_2 = 0	p_1 = 0.1 p_2 = 0.1		
δ	EZ	EI	EA	EP	EZ	EI	EA	EP
0.00	1481.6	1200.2	1200.2	1200.2	1481.6	1200.2	1200.2	1200.2
0.25	110.1	104.7	121.1	105.0	232.7	215.8	292.8	239.0
0.50	38.1	36.2	39.2	36.2	50.5	48.7	64.1	53.3
1.00	16.5	15.6	16.3	15.5	13.2	12.7	14.3	13.1
2.00	8.5	7.9	8.2	7.9	5.7	5.4	5.7	5.5
5.00	4.2	3.9	4.1	3.9	4.0	3.6	3.6	3.6
10.00	4.0	3.6	3.6	3.6	4.0	3.6	3.6	3.6
h	2.7028	2.6600	2.9739	2.6802	2.9589	2.9246	3.2334	3.0455

From Table 6.63, we can see that for $\lambda = 0.1$, $p_1 = 0.1$ and $p_2 = 0.1$ the EI and EP charts have similar performance and are more efficient than the EA chart. If $\lambda = 0.1$ and $\delta = 1.0$, the SSATS for the EA chart is 18.1 hours, while the EI and EP charts both signal in approximately 17.2 hours. When $p_1 = 0.1$ and $p_2 = 0.2$, the EI and EP charts still have better performance than the EA chart, as shown in Table 6.66 . If $\delta = 1.0$, the EA chart has a SSATS value of 21.0, while the EI and EP charts have SSATS values of 19.2 and 19.1 hours, respectively. Increasing p_2 from 0.1 to 0.2 significantly increases the time it takes the charts to signal a change in μ. This is what we would expect, since the number of missing samples is increasing from ten to twenty percent.

Table 6.66. In-Control ATS and Out-of-Control SSATS Values for the EZ, EI, EA and EP Control Charts for Shifts in μ when $n=4$, $d=4$, $p_1=0.1$, and $p_2=0.2$

	λ = 0.1				λ = 0.4			
	p_1 = 0 p_2 = 0	p_1 = 0.1 p_2 = 0.2			p_1 = 0 p_2 = 0	p_1 = 0.1 p_2 = 0.2		
δ	EZ	EI	EA	EP	EZ	EI	EA	EP
0.00	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6	1481.6
0.25	108.1	137.0	183.6	138.4	230.7	273.7	452.8	342.4
0.50	36.1	47.1	55.5	47.2	48.5	62.7	103.3	76.8
1.00	14.5	19.2	21.0	19.1	11.2	15.3	19.4	16.6
2.00	6.5	8.9	9.6	8.8	3.7	5.4	6.2	5.7
5.00	2.2	3.4	3.9	3.4	2.0	3.0	3.0	3.0
10.00	2.0	3.0	3.0	3.0	2.0	3.0	3.0	3.0
h	2.7028	2.6135	3.2660	2.6608	2.9589	2.8853	3.4766	3.1558

Table 6.67. In-Control ANSS and Out-of-Control SSANSS Values for the EZ, EI, EA and EP Control Charts for Shifts in μ when $n=4$, $d=4$, $p_1=0.1$, and $p_2=0.2$

	λ = 0.1				λ = 0.4			
	p_1 = 0 p_2 = 0	p_1 = 0.1 p_2 = 0.2			p_1 = 0 p_2 = 0	p_1 = 0.1 p_2 = 0.2		
δ	EZ	EI	EA	EP	EZ	EI	EA	EP
0.00	370.4	296.3	296.3	296.1	370.4	296.3	296.3	296.3
0.25	27.5	27.8	37.1	28.1	58.2	55.1	90.9	68.9
0.50	9.5	9.8	11.5	9.8	12.6	12.9	21.0	15.8
1.00	4.1	4.2	4.6	4.2	3.3	3.5	4.3	3.7
2.00	2.1	2.2	2.3	2.2	1.4	1.5	1.6	1.5
5.00	1.0	1.1	1.2	1.1	1.0	1.0	1.0	1.0
10.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
h	2.7028	2.6135	3.2660	2.6608	2.9589	2.8853	3.4766	3.1558

Table 6.68. In-Control ANOS and Out-of-Control SSANOS Values for the EZ, EI, EA and EP Control Charts for Shifts in μ when $n = 4$, $d = 4$, $p_1 = 0.1$, and $p_2 = 0.2$

	λ = 0.1				λ = 0.4			
	p_1 = 0 p_2 = 0	p_1 = 0.1 p_2 = 0.2			p_1 = 0 p_2 = 0	p_1 = 0.1 p_2 = 0.2		
δ	EX	EI	EA	EP	EX	EI	EA	EP
0.00	1481.6	1066.8	1066.8	1066.8	1481.6	1066.8	1066.8	1066.8
0.25	110.1	100.1	133.6	101.1	232.7	198.5	327.4	248.0
0.50	38.1	35.3	41.4	35.4	50.5	46.6	75.8	56.7
1.00	16.5	15.2	16.6	15.2	13.2	12.4	15.4	13.4
2.00	8.5	7.8	8.4	7.8	5.7	5.3	5.9	5.5
5.00	4.2	3.9	4.3	3.9	4.0	3.6	3.6	3.6
10.00	4.0	3.6	3.6	3.6	4.0	3.6	3.6	3.6
h	2.7028	2.6135	3.2660	2.6608	2.9589	2.8853	3.4766	3.1558

If $\lambda = 0.4$, the EI chart is once again the most efficient, especially for small shifts in the process mean. For example, if $p_1 = 0.1$, $p_2 = 0.1$, and $\delta = 1.0$, the EI chart has a SSATS of 13.7, while the EP chart takes 14.2 hours to signal, and the EA chart takes 15.6 hours. If p_2 is increased to 0.2, the SSATS for all charts increases, but EI chart still has the best overall performance.

Both $\lambda = 0.1$ and $\lambda = 0.4$ have similar performance for large shifts in the process mean, regardless of the values of p_1 and p_2. For small shifts in μ, setting $\lambda = 0.1$ results in smaller SSATS values, and for moderate to large shifts $\lambda = 0.4$ allows the charts to detect a change more quickly. For both combinations of p_1 and p_2 and both values of λ, the EI chart has the best overall performance.

The SSANSS and SSANOS values show similar results similar to the SSATS. The in-control ATS is set to 1481.6 for all charts, which corresponds to an in-control ANSS of 370.4 and an in-control ANOS of 1481.6 for the EZ chart, which has $p_1 = p_2 = 0$. When $p_1 = 0.1$ and $p_2 = 0.1$, the in-control ANSS decreases to 333.3 and the in-control ANOS decreases to 1200.2 for both $\lambda = 0.1$ and $\lambda = 0.4$. Increasing p_2 to 0.2 decreases the in-control ANSS to 296.3 and the in-control ANOS to 1066.8 for both values of λ.

6.5 Conclusions

In this chapter, first the case in which there is a probability p that an individual observation is missing was investigated. Two sampling patterns were considered – one based on taking samples of $n = 1$ every $d = 1$ hour, and the other based on samples of $n = 4$ every $d = 4$ hours.

If $n = 1$, the EI and EP charts have similar performance, but the EI chart is slightly more efficient at detecting small shifts in the process mean. The EA chart has the worst overall performance since it takes the longest to signal that a change has occurred. If the sampling pattern based on $n = 4$ is used, the EI, EA, and EP charts are equivalent for small values of p. In fact, all three charts have SSATS values similar to the EZ chart, which has no observations missing. For large values of p, the EI chart again has the best overall performance. Increasing the probability that an observation is missing increases the SSATS for all three charts, as we would expect.

The EWMA control charts with $n = 4$ are able to detect moderate changes in the process mean more quickly for smaller values of p, but the corresponding charts with $n = 1$ are more efficient for small and large shifts in μ. However, the difference in the SSATS values of the two sampling schemes is small for smaller values of p. As p increases, the difference in the performance of the two sampling schemes also increases.

The performance of the EI, EA, and EP charts was also compared when the same control limits are used for all three charts. If $n = 1$, the in-control ATS of the EI chart increases by $\left(\sum_{i=1}^{\eta} p^i\right) \cdot 100\%$ as compared to the EZ chart with no missing observations.

The in-control ATS of the EP chart also increases as p increases, but the in-control ATS of the EA chart decreases as p increases. Thus, the false alarm rate per unit time for the EI and EP charts is lower than for the EZ chart, but it is higher for the EA chart. If $n = 4$, missing observations have very little or no effect on the in-control ATS and false alarm rate per unit time of the EI, EA, and EP charts.

The use of the standardized sample mean is recommended when observations are missing. By using the standardized sample mean, the weights are adjusted by making the weight used for the current sample mean proportional to the square root of the sample size for this sample. In addition, when the process is in control, Z_k follows a standard normal distribution, and thus does not depend on the sample size. This means that the control limits are constant even though the sample size varies randomly. Using Z_k adjusts the sample mean $\overline{X}_k$ for the fact that part of the sample is missing, but when complete samples are missing, the weights between samples should also be adjusted.

The EI chart, which adjusts the weights between missing samples by ignoring the missing samples, is recommended in practice when there is a probability p that an observation is missing. Recall from (3.7) that at time t_k, if $m_k < n$ and the previous i consecutive samples at times t_{k-1}, t_{k-2}, …, t_{k-i} are missing, the control statistic for the EI chart is

$$E_k^I = (1-\lambda)E_{k-(i+1)}^I + \lambda Z_k, \qquad k = 1, 2, \ldots.$$

If the standard control limits, which can be found using statistical software, are used, then the out-of-control SSATS values of the EI control chart are similar to those of the EZ chart. However, if $n = 1$, the in-control ATS of the EI chart increases by $\left(\sum_{i=1}^{\eta} p^i\right) \cdot 100\%$ as compared to the EZ chart with no missing observations. If $n = 4$, missing observations have very little effect on the in-control ATS and false alarm rate per unit time of the EI chart.

Chapter 7

Performance of MEWMA Control Charts

In this chapter, the performance of different ways for modifying the MEWMA control chart to adjust for missing observations is investigated. The two methods discussed in Section 3.4 will be examined: (1) ignoring all the data at a sampling point if the data for at least one variable is missing and adjusting the covariance matrix; and (2) using the previous EWMA value for any variable for which all the data are missing and adjusting the covariance matrix. Performance of these methods will be evaluated for various combinations of the parameters λ, n, d, p, and ρ over a range of shifts in $\boldsymbol{\mu}$. The results shown were found using simulation.

7.1 Control Chart Parameters

First, we consider the case in which there are $b = 4$ quality variables of interest. Two sampling patterns are considered, each with a sampling rate of one observation per unit time so $n/d = 1.0$. One sampling pattern is based on taking samples of $n = 1$ from each of the four variables every $d = 1$ hour, and the other is based on taking samples of $n = 4$ from each of the four variables every $d = 4$ hours. The values chosen for the smoothing parameter are $\lambda = 0.026$ and $\lambda = 0.11989$ for $n = 1$, and the corresponding values $\lambda = 0.1$ and $\lambda = 0.4$ for $n = 4$. Recall that these are the same combinations of n, d and λ that were investigated in the univariate case. First, the situation in which the variables are independent is examined, so $\rho = 0$. Then the case in which the variables of interest are highly correlated with $\rho = 0.9$ is investigated. Next, we consider the case in which there are $b = 2$ variables, and samples of size $n = 2$ are taken from each variable every $d = 2$ hours. The performance of the MI and MS control charts is then examined over a range of values of ρ from -0.9 to 0.9.

For the EWMA control charts examined in Chapter 6, the in-control ATS was set to 1481.6, which is the in-control ATS of the Shewhart $\overline{X}$ chart with standard three-sigma control limits and samples of size $n = 4$ observations taken every $d = 4$ hours. The control limits of the MEWMA charts investigated in this chapter are adjusted to give an in-control ATS of 800. This ATS value was chosen since it has been used in some recent papers, including Reynolds and Cho (2006) and Reynolds and Stoumbos (2008).

The performance of the MEWMA charts proposed is evaluated with a one percent or ten percent chance of an individual observation being mission, so $p = 0.01$ or 0.1. Simulation with 100,000 runs was used to evaluate the MEWMA control charts. Just as in Section 6.4, it is not necessary to specify a maximum number of consecutive samples, and $\eta = \infty$ was used. The probability of all n observations for one variable being missing is p^n, and the probability of an entire sample being missing, i.e. all n observations for all b variables being missing, is p^{nb}. Then the probability of i consecutive missing samples is p^{nbi}, so as $i \to \infty$, $p^{nbi} \to 0$. For example, if $b = 4$, and $n = 1$, the probability of two consecutive missing samples is $0.01^8 = 10^{-16}$ if $p = 0.01$ and $0.1^8 = 10^{-8}$ if $p = 0.1$. Thus, it is highly unlikely for many consecutive samples to be missing, and setting $\eta < \infty$ would have very little or no effect on the results.

Shifts in $\boldsymbol{\mu}$ are expressed in terms of the standardized mean shift vector, defined as $\mathbf{u} = (u_1, u_2, \ldots, u_b)^T$, where

$$u_v = (\mu_v - \mu_{0v})/\sigma_{0v}, \qquad v = 1, 2, \ldots, b. \tag{7.1}$$

The out-of-control properties of the MZ chart depend on $\mathbf{u}$ only through the noncentrality parameter

$$\delta = \sqrt{\mathbf{u}^T \mathbf{\Sigma}_{Z0}^{-1} \mathbf{u}}\ , \tag{7.2}$$

so shifts in $\boldsymbol{\mu}$ are expressed in terms of (7.2) and the shift sizes considered range from 0.0 to 10.0. A random shift direction was used since this provides an average over all shift directions and results in an average SSATS.

7.2 Comparison of MEWMA Control Charts when the Control Limits Are Adjusted

Tables 7.1 through 7.12 present the SSATS, SSANSS, and SSANMS values for the two different MEWMA control charts designed to detect changes in the process mean when there is a probability p that an observation is missing. The control limits for all of the MEWMA control charts are set to have an in-control ATS of 800.0, so they all have the same false alarm rate per unit time. Tables 7.1 – 7.6 give results for the case when $b = 4$, $n = 1$, $d = 1$, $\rho = 0$, $\lambda = 0.026$ or 0.11989, and $p = 0.01$ or 0.1. When $n = 1$, an individual observation is taken for each variable, so if an observation for variable v is missing at time t_k, then all the data for variable v is missing at t_k. Results for $b = 4$, $n = 4$, $d = 4$, $\rho = 0$, $\lambda = 0.1$ or 0.4, and $p = 0.01$ or 0.1 are shown in Tables 7.7 – 7.12. Tables 7.13 – 7.18 give the results for the case when $b = 4$, $n = 1$, $d = 1$, $\rho = 0.9$, $\lambda = 0.026$ or 0.11989, and $p = 0.01$ or 0.1, and results for $b = 4$, $n = 4$, $d = 4$, $\rho = 0.9$, $\lambda = 0.1$ or 0.4, and $p = 0.01$ or 0.1 are shown in Tables 7.19 – 7.24.

Table 7.1. In-Control ATS and Out-of-Control SSATS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $\rho = 0$, $n = 1$, $d = 1$, and $\lambda = 0.026$.

	p = 0	*p* = 0.01		*p* = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	800.0	800.0	800.0	800.0	800.0
0.25	131.1	133.6	131.5	164.2	138.4
0.50	44.1	45.3	44.5	60.3	47.7
1.00	17.7	18.3	17.8	25.3	19.3
2.00	8.0	8.3	8.0	11.6	8.7
5.00	3.0	3.1	3.0	4.5	3.3
10.00	1.5	1.5	1.5	2.4	1.6
UCL	13.6816	13.5574	13.6516	12.4516	13.4146

Table 7.2. In-Control ANSS and Out-of-Control SSANSS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $\rho = 0$, $n = 1$, $d = 1$, and $\lambda = 0.026$.

	p = 0	*p* = 0.01		*p* = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	800.0	768.5	800.0	524.9	799.9
0.25	131.6	128.9	132.0	108.0	138.9
0.50	44.6	44.0	45.0	39.9	48.2
1.00	18.2	18.1	18.3	16.9	19.8
2.00	8.5	8.4	8.5	8.0	9.2
5.00	3.5	3.5	3.5	3.3	3.8
10.00	2.0	2.0	2.0	1.9	2.1
UCL	13.6816	13.5574	13.6516	12.4516	13.4146

Table 7.3. In-Control ANMS and Out-of-Control SSANMS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $\rho = 0$, $n = 1$, $d = 1$, and $\lambda = 0.026$.

	p = 0	*p* = 0.01		*p* = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	3200.0	3073.9	3167.9	2099.5	2880.0
0.25	526.4	515.4	522.7	432.0	500.1
0.50	178.5	176.1	178.1	159.7	173.7
1.00	72.8	72.3	72.6	67.8	71.4
2.00	33.9	33.7	33.8	31.9	33.2
5.00	13.9	13.8	13.9	13.2	13.6
10.00	7.8	7.8	7.8	7.5	7.5
UCL	13.6816	13.5574	13.6516	12.4516	13.4146

For each table, the first column gives the different magnitudes of shifts considered in terms of δ, and the second column corresponds to the standard MZ chart with no missing observations. Recall that when $p = 0$, no observations are missing, and the MZ, MI, and MS charts are all equivalent. The next two columns correspond to the MI and MS charts with $p = 0.01$, and the last two columns give the results for the two MEWMA charts with $p = 0.1$.

Table 7.4. In-Control ATS and Out-of-Control SSATS Values for the MZ, MI, and MS Control Charts for Shifts in $\mathbf{\mu}$ when $b = 4$, $\rho = 0$, $n = 1$, $d = 1$, and $\lambda = 0.11989$.

	p = 0	p = 0.01		p = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	800.0	800.0	800.0	800.0	800.0
0.25	279.1	281.5	279.5	305.0	286.9
0.50	68.6	69.9	68.8	85.3	72.2
1.00	15.0	15.5	15.1	21.0	16.6
2.00	5.0	5.2	5.1	7.5	5.5
5.00	1.6	1.7	1.6	2.6	1.8
10.00	0.7	0.7	0.7	1.2	0.7
UCL	16.6490	16.5485	16.6272	15.5890	16.4574

Table 7.5. In-Control ANSS and Out-of-Control SSANSS Values for the MZ, MI, and MS Control Charts for Shifts in $\mathbf{\mu}$ when $b = 4$, $\rho = 0$, $n = 1$, $d = 1$, and $\lambda = 0.11989$.

	p = 0	p = 0.01		p = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	800.0	768.5	800.0	524.9	799.9
0.25	279.6	270.0	280.0	200.5	287.3
0.50	69.1	67.6	69.3	56.3	72.7
1.00	15.5	15.4	15.6	14.1	16.8
2.00	5.5	5.5	5.6	5.2	6.0
5.00	2.1	2.1	2.1	2.1	2.3
10.00	1.2	1.2	1.2	1.1	1.2
UCL	16.6490	16.5485	16.6272	15.5890	16.4574

Table 7.6. In-Control ANMS and Out-of-Control SSANMS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $\rho = 0$, $n = 1$, $d = 1$, and $\lambda = 0.11989$.

	p = 0	p = 0.01		p = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	3200.0	3074.0	3168.1	2099.6	2879.9
0.25	1118.3	1083.7	1108.9	801.9	1034.5
0.50	276.3	270.5	274.3	225.1	261.9
1.00	61.9	61.4	61.6	56.5	60.4
2.00	22.0	21.9	22.0	20.9	21.7
5.00	8.5	8.5	8.5	8.2	8.3
10.00	4.6	4.6	4.6	4.5	4.5
UCL	16.6490	16.5485	16.6272	15.5890	16.4574

Tables 7.1 and 7.4 show that for $\lambda = 0.026$ and $\lambda = 0.11989$ when the variables are independent, the MS chart is slightly more efficient than the MI chart when $p = 0.01$. For example, if $\lambda = 0.026$ and $\delta = 1.0$, the MI chart takes 18.3 hours to signal, while the MS chart detects the change in 17.8 hours. When $p = 0.1$, the MS chart is able to detect changes in the process mean much more quickly than the MI chart, but as p is increased, the SSATS values for both charts also increase, as we would expect. If $\lambda = 0.026$ and $\delta = 1.0$, the MI chart has a SSATS of 25.3 hours and the MS chart has a SSATS of only 19.3 hours.

Similar results are shown for the SSANSS and SSANMS values in Tables 7.2, 7.3, 7.5, and 7.6. If all n observations for at least one variable are missing at a sampling point, then none of the data at that sampling point is used in the MI chart. The probability of all n observations being missing for at least one variable is $1-(1-p^n)^b$, and so the in-control ANSS for the MI chart decreases by $1-(1-p^n)^b \cdot 100\%$. When $p = 0.1$ and $\lambda = 0.026$ or 0.11989, the in-control ANSS decreases from 800 samples to only 524.9 samples for the MI chart, and the in-control ANMS decreases from 3200 measurements to approximately 2100 measurements. However, for the MS chart, the in-control ANSS is 799.9 samples and the in-control ANMS is 2880 when $p = 0.1$. Since the MS chart is able to use all of the non-missing data, the in-control ANMS for this chart decreases by $p \cdot 100\%$.

Tables 7.7 and 7.10 show that for the case of $n = 4$ with $\lambda = 0.1$ and $\lambda = 0.4$, the MI and MS charts have similar performance when $p = 0.01$. When $p = 0.1$, the MS chart is slightly more efficient than the MI chart, especially for small shifts. The MI chart takes 77.8 hours to detect a shift of size $\delta = 0.5$ when $\lambda = 0.4$, but the MS chart signals in 77.6 hours. When $n = 4$, it is very unlikely that all the data for one variable is missing, so we would expect the two charts to have similar performance. For both $\lambda = 0.1$ and $\lambda = 0.4$, when $p = 0.01$ and $p = 0.1$ the in-control ANSS for the MI and MS charts is 200 samples – the same as the in-control ANSS of the MZ chart with no observations missing. However, the in-control ANMS decreases from 3200 measurements to 3168 measurements when $p = 0.01$ and to 2880 measurements when $p = 0.1$ for both the MI and MS charts.

The MEWMA charts with $n = 4$ are able to detect moderate changes in the process mean more quickly, but the corresponding charts with $n = 1$ are slightly more efficient for small and large shifts in $\boldsymbol{\mu}$ in some cases. For example, when $p = 0.01$, the MS chart with $n = 1$ and $\lambda = 0.11989$ is able to detect a shift of size $\delta = 0.5$ in 72.2 hours and a shift of $\delta = 2.0$ in 5.5 hours. The MS chart with $n = 4$ and $\lambda = 0.4$ has SSATS values of 77.6 and 4.9 hours for shifts of $\delta = 0.5$ and 2.0, respectively.

Table 7.7. In-Control ATS and Out-of-Control SSATS Values for the MZ, MI, and MS Control Charts for Shifts in $\mathbf{\mu}$ when $b = 4$, $\rho = 0$, $n = 4$, $d = 4$, and $\lambda = 0.1$.

	p = 0	*p* = 0.01		*p* = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	800.0	800.0	800.0	800.0	800.0
0.25	130.4	131.7	131.7	142.8	142.9
0.50	43.4	44.0	44.0	47.0	47.1
1.00	17.2	17.3	17.3	18.4	18.4
2.00	7.6	7.6	7.6	8.1	8.1
5.00	2.5	2.5	2.5	2.7	2.7
10.00	2.0	2.0	2.0	2.0	2.0
UCL	12.7143	12.7167	12.7167	12.7100	12.7108

Table 7.8. In-Control ANSS and Out-of-Control SSANSS Values for the MZ, MI, and MS Control Charts for Shifts in $\mathbf{\mu}$ when $b = 4$, $\rho = 0$, $n = 4$, $d = 4$, and $\lambda = 0.1$.

	p = 0	*p* = 0.01		*p* = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	200.0	200.0	200.0	199.9	200.0
0.25	33.1	33.4	33.4	36.2	36.2
0.50	11.3	11.5	11.5	12.2	12.3
1.00	4.8	4.8	4.8	5.1	5.1
2.00	2.4	2.4	2.4	2.5	2.5
5.00	1.1	1.1	1.1	1.2	1.2
10.00	1.0	1.0	1.0	1.0	1.0
UCL	12.7143	12.7167	12.7167	12.7100	12.7108

Table 7.9. In-Control ANMS and Out-of-Control SSANMS Values for the MZ, MI, and MS Control Charts for Shifts in $\mathbf{\mu}$ when $b = 4$, $\rho = 0$, $n = 4$, $d = 4$, and $\lambda = 0.1$.

	p = 0	*p* = 0.01		*p* = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	3200.0	3168.0	3168.0	2879.1	2880.0
0.25	529.7	529.6	529.6	520.9	521.5
0.50	181.4	182.0	182.0	176.2	176.7
1.00	76.6	76.3	76.3	73.4	73.3
2.00	38.3	38.2	38.2	36.3	36.3
5.00	17.9	17.8	17.8	16.9	16.9
10.00	16.0	15.8	15.8	14.4	14.4
UCL	12.7143	12.7167	12.7167	12.7100	12.7108

Table 7.10. In-Control ATS and Out-of-Control SSATS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $\rho = 0$, $n = 4$, $d = 4$, and $\lambda = 0.4$.

	p = 0	*p* = 0.01		*p* = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	800.0	800.0	800.0	800.0	800.0
0.25	280.4	283.7	283.7	306.1	304.7
0.50	68.2	69.4	69.4	77.8	77.6
1.00	14.3	14.4	14.4	16.0	16.0
2.00	4.5	4.5	4.5	4.9	4.9
5.00	2.0	2.0	2.0	2.0	2.0
10.00	2.0	2.0	2.0	2.0	2.0
UCL	14.5687	14.5700	14.5700	14.5709	14.572

Table 7.11. In-Control ANSS and Out-of-Control SSANSS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $\rho = 0$, $n = 4$, $d = 4$, and $\lambda = 0.4$.

	p = 0	*p* = 0.01		*p* = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	200.0	200.0	200.0	199.9	200.0
0.25	70.6	71.4	71.4	77.0	76.7
0.50	280.7	17.8	17.8	19.9	19.9
1.00	65.1	4.1	4.1	4.5	4.5
2.00	25.9	1.6	1.6	1.7	1.7
5.00	16.0	1.0	1.0	1.0	1.0
10.00	16.0	1.0	1.0	1.0	1.0
UCL	14.5687	14.5700	14.5700	14.5709	14.572

Table 7.12. In-Control ANMS and Out-of-Control SSANMS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $\rho = 0$, $n = 4$, $d = 4$, and $\lambda = 0.4$.

	p = 0	*p* = 0.01		*p* = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	3200.0	3167.9	3167.9	2879.0	2880.0
0.25	1129.5	1131.4	1131.4	1108.6	1104.2
0.50	280.7	282.6	282.6	287.2	286.5
1.00	65.1	65.0	65.0	64.7	64.7
2.00	25.9	25.9	25.9	24.8	24.8
5.00	16.0	15.8	15.8	14.4	14.4
10.00	16.0	15.8	15.8	14.4	14.4
UCL	14.5687	14.5700	14.5700	14.5709	14.572

The SSATS values for $n = 1$, $\lambda = 0.026$ and $\lambda = 0.11989$ when $\rho = 0.9$ are shown in Tables 7.13 and 7.16. For small shifts in the process mean, the MI chart is more efficient than the MS chart. However, the MS chart is able to detect moderate and large shifts in **μ** more quickly than the MI chart. If $\lambda = 0.026$ and $p = 0.1$, the MI chart has a SSATS of 60.1 hours for a shift of size $\delta = 0.5$, while the SSATS of the MS chart is 61.5 hours. However, when $\delta = 2.0$, the MI chart takes 11.6 hours to detect the change, while the MS chart signals in 9.0 hours.

Tables 7.19 and 7.21 show the results for the case of $n = 4$ and $\rho = 0.9$ with $\lambda = 0.1$ and 0.4. When $\lambda = 0.1$, the MS chart has better overall performance than the MI chart. However, when $\lambda = 0.4$, the MI chart is more efficient for small shifts, while the MS chart has smaller SSATS values for moderate and large shifts in the process mean.

When $\rho = 0.9$, the MEWMA charts with $n = 4$ are able to detect moderate changes in the process mean more quickly, but the corresponding charts with $n = 1$ are slightly more efficient for small and large shifts in **μ**. For example, when $p = 0.01$, the MS chart with $n = 1$ and $\lambda = 0.11989$ is able to detect a shift of size $\delta = 0.5$ in 71.9 hours and a shift of $\delta = 2.0$ in 5.1 hours. The chart with $n = 4$ and $\lambda = 0.4$ has SSATS values of 73.0 and 5.6 hours for shifts of $\delta = 0.5$ and 2.0, respectively.

Increasing the correlation from $\rho = 0$ to $\rho = 0.9$ when $n = 1$ has little effect on the performance of the MI and MS control charts for moderate and large shifts in the process mean. For small shifts in **μ**, increasing the correlation increases the SSATS for the MS chart. When the sampling pattern is based on taking samples of $n = 4$, increasing the correlation has little effect on either the MI or MS chart for large shifts. However, for moderate and large shifts in **μ**, increasing ρ increases the SSATS for both the MI and MS charts, especially when $p = 0.1$. If $n = 4$, $\lambda = 0.4$, $p = 0.1$, and $\rho = 0$, then the MI and MS charts detect a shift of size $\delta = 0.5$ in 77.8 and 77.6 hours, respectively. When $\rho = 0.9$, the SSATS values of the MI and MS charts are 117.6 and 120.3 hours, respectively.

Table 7.13. In-Control ATS and Out-of-Control SSATS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $\rho = 0.9$, $n = 1$, $d = 1$, and $\lambda = 0.026$.

	p = 0	*p* = 0.01		*p* = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	800.0	800.0	800.0	800.0	800.0
0.25	130.6	133.1	137.4	162.7	188.3
0.50	44.2	45.4	46.0	60.1	61.5
1.00	17.7	18.5	18.2	25.1	21.9
2.00	8.0	8.3	8.1	11.6	9.0
5.00	3.0	3.1	3.0	4.5	3.1
10.00	1.5	1.5	1.5	2.4	1.5
UCL	13.6816	13.5574	13.6250	12.4516	13.3266

Table 7.14. In-Control ANSS and Out-of-Control SSANSS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $\rho = 0.9$, $n = 1$, $d = 1$, and $\lambda = 0.026$.

	p = 0	*p* = 0.01		*p* = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	800.0	768.5	800.0	524.9	800.0
0.25	131.1	128.4	137.9	107.1	188.8
0.50	44.7	44.1	46.5	39.8	62.0
1.00	18.2	18.1	18.7	16.8	22.4
2.00	8.5	8.4	8.6	7.9	9.5
5.00	3.5	3.5	3.5	3.3	3.6
10.00	2.0	2.0	2.0	1.9	2.0
UCL	13.6816	13.5574	13.6250	12.4516	13.3266

Table 7.15. In-Control ANMS and Out-of-Control SSANMS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $\rho = 0.9$, $n = 1$, $d = 1$, and $\lambda = 0.026$.

	p = 0	*p* = 0.01		*p* = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	3200.0	3073.9	3168.1	2099.5	2880.2
0.25	524.5	513.4	545.9	428.3	679.7
0.50	178.8	176.4	184.0	159.0	223.2
1.00	72.8	72.3	73.9	67.2	80.8
2.00	33.9	33.7	33.9	31.7	34.1
5.00	13.9	13.8	13.8	13.2	13.1
10.00	7.8	7.8	7.8	7.5	7.2
UCL	13.6816	13.5574	13.6250	12.4516	13.3266

Table 7.16. In-Control ATS and Out-of-Control SSATS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $\rho = 0.9$, $n = 1$, $d = 1$, and $\lambda = 0.11989$.

	p = 0	p = 0.01		p = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	800.0	800.0	800.0	800.0	800.0
0.25	277.8	282.0	287.9	300.2	361.6
0.50	68.1	69.9	71.9	83.8	107.7
1.00	15.0	15.5	15.6	20.7	22.0
2.00	5.0	5.2	5.1	7.4	6.4
5.00	1.6	1.7	1.6	2.6	1.8
10.00	0.7	0.7	0.7	1.2	0.8
UCL	16.6490	16.5485	16.6034	15.5890	16.3082

Table 7.17. In-Control ANSS and Out-of-Control SSANSS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $\rho = 0.9$, $n = 1$, $d = 1$, and $\lambda = 0.11989$.

	p = 0	p = 0.01		p = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	800.0	768.5	800.0	524.9	800.0
0.25	278.3	271.4	288.4	197.4	362.1
0.50	68.6	67.6	72.4	55.3	108.2
1.00	15.5	15.3	16.1	13.9	22.5
2.00	2.1	5.5	5.6	5.2	6.9
5.00	2.1	2.1	2.1	2.0	2.3
10.00	1.2	1.2	1.2	1.1	1.3
UCL	16.6490	16.5485	16.6034	15.5890	16.3082

Table 7.18. In-Control ANMS and Out-of-Control SSANMS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $\rho = 0.9$, $n = 1$, $d = 1$, and $\lambda = 0.11989$.

	p = 0	p = 0.01		p = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	3200.0	3074.0	3168.1	2099.6	2880.0
0.25	1113.1	1085.4	1141.9	789.5	1303.7
0.50	274.4	270.4	286.8	221.2	389.5
1.00	62.0	61.3	63.9	55.7	81.2
2.00	22.0	21.9	22.3	30.7	24.9
5.00	8.5	8.5	8.5	8.1	8.4
10.00	4.6	4.6	4.6	4.5	4.6
UCL	16.6490	16.5485	16.6034	15.5890	16.3082

Table 7.19. In-Control ATS and Out-of-Control SSATS Values for the MZ, MI, and MS Control Charts for Shifts in $\mathbf{\mu}$ when $b = 4$, $\rho = 0.9$, $n = 4$, $d = 4$, and $\lambda = 0.1$.

	p = 0	p = 0.01		p = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	800.0	800.0	800.0	800.0	800.0
0.25	129.7	137.4	137.9	206.0	203.2
0.50	43.3	45.7	45.4	67.2	64.8
1.00	17.2	17.9	17.6	24.3	22.3
2.00	7.6	7.8	7.7	9.8	8.9
5.00	2.5	2.6	2.5	3.2	3.0
10.00	2.0	2.0	2.0	2.0	2.0
UCL	12.7143	12.7791	12.6986	13.0881	12.6587

Table 7.20. In-Control ANSS and Out-of-Control SSANSS Values for the MZ, MI, and MS Control Charts for Shifts in $\mathbf{\mu}$ when $b = 4$, $\rho = 0.9$, $n = 4$, $d = 4$, and $\lambda = 0.1$.

	p = 0	p = 0.01		p = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	200.0	200.0	200.0	199.9	200.0
0.25	32.9	34.8	35.0	52.0	51.3
0.50	11.3	11.9	11.9	17.3	16.7
1.00	4.8	5.0	4.9	6.6	6.1
2.00	2.4	2.4	2.4	2.9	2.7
5.00	1.1	1.1	1.1	1.3	1.2
10.00	1.0	1.0	1.0	1.0	1.0
UCL	12.7143	12.7791	12.6986	13.0881	12.6587

Table 7.21. In-Control ANMS and Out-of-Control SSANMS Values for the MZ, MI, and MS Control Charts for Shifts in $\mathbf{\mu}$ when $b = 4$, $\rho = 0.9$, $n = 4$, $d = 4$, and $\lambda = 0.1$.

	p = 0	p = 0.01		p = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	3200.0	3168.2	3167.9	2879.2	2880.0
0.25	526.8	552.0	553.9	748.6	738.6
0.50	181.3	188.9	187.9	249.1	240.4
1.00	76.6	78.6	77.6	94.7	87.5
2.00	38.4	38.8	38.4	42.4	39.1
5.00	17.9	18.1	18.0	18.8	18.0
10.00	16.0	15.8	15.8	14.5	14.4
UCL	12.7143	12.7791	12.6986	13.0881	12.6587

Table 7.22. In-Control ATS and Out-of-Control SSATS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $\rho = 0.9$, $n = 4$, $d = 4$, and $\lambda = 0.4$.

	p = 0	p = 0.01		p = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	800.0	800.0	800.0	800.0	800.0
0.25	279.9	294.1	292.1	390.2	394.2
0.50	68.3	73.0	73.0	117.6	120.3
1.00	14.3	15.1	15.0	23.5	23.4
2.00	4.5	4.7	4.6	6.7	6.3
5.00	2.0	2.0	2.0	2.2	2.1
10.00	2.0	2.0	2.0	2.0	2.0
UCL	14.5687	14.7255	14.5580	15.6959	14.5264

Table 7.23. In-Control ANSS and Out-of-Control SSANSS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $\rho = 0.9$, $n = 4$, $d = 4$, and $\lambda = 0.4$.

	p = 0	p = 0.01		p = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	200.0	200.0	200.0	199.9	200.0
0.25	70.5	74.0	73.5	98.0	99.1
0.50	17.6	18.8	18.8	29.9	30.6
1.00	4.1	4.3	4.3	6.4	6.3
2.00	1.6	1.7	1.7	2.2	2.1
5.00	1.0	1.0	1.0	1.0	1.0
10.00	1.0	1.0	1.0	1.0	1.0
UCL	14.5687	14.7255	14.5580	15.6959	14.5264

Table 7.24. In-Control ANMS and Out-of-Control SSANMS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $\rho = 0.9$, $n = 4$, $d = 4$, and $\lambda = 0.4$.

	p = 0	p = 0.01		p = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	3200.0	3168.1	3167.8	2879.1	2880.0
0.25	1127.5	1172.5	1164.5	1411.5	1426.3
0.50	281.3	297.1	297.0	430.4	440.3
1.00	65.3	67.6	67.5	91.9	91.3
2.00	26.0	26.5	26.3	31.3	29.7
5.00	16.0	15.9	15.9	15.1	14.8
10.00	16.0	15.8	15.8	14.4	14.4
UCL	14.5687	14.7255	14.5580	15.6959	14.5264

7.3 Comparison of the MEWMA Control Charts when the Same Control Limits Are Used

An additional issue of interest is the effect of missing observations on the standard MEWMA control chart. The control limits of the MZ chart which has no missing observations are set to provide an in-control ATS of 800. These limits are then used for the MI and MS charts. The resulting SSATS, SSANSS, and SSANMS values for the case of $b = 4$, $\rho = 0..9$, $n = 1$, $d = 1$, $\lambda = 0.026$ or 0.11989, and $p = 0.01$ or 0.1 are shown in Tables 7.25 through 7.30. Tables 7.31 – 7.36 present the results for $b = 4$, $\rho = 0.9$, $n = 4$, $d = 4$, $\lambda = 0.1$ or 0.4, and $p = 0.01$ or 0.1.

When $n = 1$, the in-control ATS for both the MI and MS control charts increases as p increases, which results in false alarm rates per unit time that are lower than for the MZ chart. For $\lambda = 0.026$ and 0.11989 with both $p = 0.01$ and 0.1, the in-control ATS for the MI chart is significantly higher than for the MS chart. For example, when $\lambda = 0.026$ and $p = 0.1$, the in-control ATS of the MI chart is 1221.3 hours, while it is only 905.8 hours for the MS chart. The in-control ANSS and ANMS for the MI chart are approximately the same as for the MZ chart for both $p = 0.01$ and 0.1. For the MS chart, both the in-control ANSS and ANMS increase as p increases.

Table 7.25. In-Control ATS and Out-of-Control SSATS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b=4$, $\rho=0.9$, $n=1$, $d=1$, and $\lambda=0.026$ when the Same Control Limits Are Used for All Charts.

	$p = 0$	$p = 0.01$		$p = 0.1$	
δ	MZ	MI	MS	MI	MS
0.00	800.0	833.7	817.0	1221.3	905.8
0.25	130.6	136.6	138.5	198.1	203.8
0.50	44.2	45.8	46.0	66.9	64.3
1.00	17.7	18.5	18.2	27.1	22.6
2.00	8.0	8.3	8.1	12.4	9.2
5.00	3.0	3.1	3.0	4.8	3.2
10.00	1.5	1.5	1.5	2.5	1.5
UCL	13.6816	13.6816	13.6816	13.6816	13.6816

Table 7.26. In-Control ANSS and Out-of-Control SSANSS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b=4$, $\rho=0.9$, $n=1$, $d=1$, and $\lambda=0.026$ when the Same Control Limits Are Used for All Charts.

	$p = 0$	$p = 0.01$		$p = 0.1$	
δ	MZ	MI	MS	MI	MS
0.00	800.0	800.8	817.0	801.2	905.7
0.25	131.1	131.7	139.0	130.3	204.3
0.50	44.7	44.5	46.5	44.2	64.8
1.00	18.2	18.2	18.7	18.1	23.1
2.00	8.5	8.5	8.6	8.4	9.7
5.00	3.5	3.5	3.5	3.5	3.7
10.00	2.0	2.0	2.0	2.0	2.0
UCL	13.6816	13.6816	13.6816	13.6816	13.6816

Table 7.27. In-Control ANMS and Out-of-Control SSANMS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $\rho = 0.9$, $n = 1$, $d = 1$, and $\lambda = 0.026$ when the Same Control Limits Are Used for All Charts.

	p = 0	*p* = 0.01		*p* = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	3200.0	3203.4	3235.2	3205.0	3260.9
0.25	524.5	526.8	550.5	521.1	735.5
0.50	178.8	178.0	184.3	176.9	233.2
1.00	72.8	72.8	73.9	72.5	83.2
2.00	33.9	33.9	34.0	33.7	34.8
5.00	13.9	13.9	13.9	13.9	13.3
10.00	7.8	7.8	7.8	7.8	7.3
UCL	13.6816	13.6816	13.6816	13.6816	13.6816

Table 7.28. In-Control ATS and Out-of-Control SSATS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $\rho = 0.9$, $n = 1$, $d = 1$, and $\lambda = 0.11989$ when the Same Control Limits Are Used for All Charts.

	p = 0	*p* = 0.01		*p* = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	800.0	830.7	815.7	1218.3	916.6
0.25	277.8	289.4	292.1	417.8	405.7
0.50	68.1	71.4	73.2	103.2	117.0
1.00	15.0	15.6	15.6	22.8	23.0
2.00	5.0	5.2	5.1	7.8	6.5
5.00	1.6	1.7	1.7	2.7	1.9
10.00	0.7	0.7	0.7	1.3	0.8
UCL	16.6490	16.6490	16.6490	16.6490	16.6490

Table 7.29. In-Control ANSS and Out-of-Control SSANSS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $\rho = 0.9$, $n = 1$, $d = 1$, and $\lambda = 0.11989$ when the Same Control Limits Are Used for All Charts.

	p = 0	p = 0.01		p = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	800.0	797.9	815.7	799.3	916.5
0.25	278.3	278.4	292.6	274.4	406.2
0.50	68.6	69.0	73.7	68.0	117.5
1.00	15.5	15.4	16.1	15.2	23.5
2.00	2.1	5.5	5.6	5.4	7.0
5.00	2.1	2.1	2.2	2.1	2.4
10.00	1.2	1.2	1.2	1.2	1.3
UCL	16.6490	16.6490	16.6490	16.6490	16.6490

Table 7.30. In-Control ANMS and Out-of-Control SSANMS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $\rho = 0.9$, $n = 1$, $d = 1$, and $\lambda = 0.11989$ when the Same Control Limits Are Used for All Charts.

	p = 0	p = 0.01		p = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	3200.0	3191.7	3230.1	3197.2	3299.7
0.25	1113.1	1113.8	1158.5	1097.8	1462.4
0.50	274.4	276.7	291.9	272.2	423.0
1.00	62.0	61.7	63.9	61.0	84.4
2.00	22.0	22.0	22.3	21.7	25.4
5.00	8.5	8.5	8.5	8.4	8.6
10.00	4.6	4.6	4.6	4.6	4.7
UCL	16.6490	16.6490	16.6490	16.6490	16.6490

When $n = 4$, the in-control ATS of the MI chart decreases as p increases, which results in a false alarm rate per unit time that is higher than that of the MZ chart. However, the in-control rate of the MS chart increases as p increases, resulting in a smaller false alarm rate per unit time than the MZ chart. For example, when $\lambda = 0.1$ and $p = 0.1$, the in-control ATS of the MI chart is 703.9 hours, while the in-control ATS of the MS chart is 815.8 hours. This means that the false alarm rate per unit time of the MI chart is 12.5% higher than for the MZ chart, but the false alarm rate per unit time of the MS chart it is less than 2% lower than for the MZ chart. The in-control ANSS of the MI chart decreases as p increases. For the MS chart, the in-control ANSS increases only slightly as p increases. The in-control ANMS for both charts decreases as p increases as we would expect, but is much less for the MI chart than for the MS chart.

Although the in-control ATS for the MS chart increases for both $n = 1$ and $n = 4$, this increase is much higher when $n = 1$. If $n = 1$, $\lambda = 0.11989$ and $p = 0.1$, the in-control ATS for the MS chart is 916.6 hours, but for the corresponding chart with $n = 4$, $\lambda = 0.4$, and $p = 0.1$, the in-control ATS is only 814.3 hours. For both sampling patterns, if the same control limits are used for both charts, the MI chart is able to detect small shifts in $\boldsymbol{\mu}$ more quickly. However, the MS chart is more efficient for most moderate and large shifts.

Table 7.31. In-Control ATS and Out-of-Control SSATS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $\rho = 0.9$, $n = 4$, $d = 4$, and $\lambda = 0.1$ when the Same Control Limits Are Used for All Charts.

	$p = 0$	$p = 0.01$		$p = 0.1$	
δ	MZ	MI	MS	MI	MS
0.00	800.0	780.7	804.8	703.9	815.8
0.25	129.7	136.4	137.5	189.9	207.4
0.50	43.3	45.4	45.6	64.6	65.7
1.00	17.2	17.8	17.6	23.6	22.3
2.00	7.6	7.8	7.7	9.5	8.8
5.00	2.5	2.6	2.5	3.2	3.0
10.00	2.0	2.0	2.0	2.0	2.0
UCL	12.7143	12.7143	12.7143	12.7143	12.7143

Table 7.32. In-Control ANSS and Out-of-Control SSANSS Values for the MZ, MI, and MS Control Charts for Shifts in **μ** when $b = 4$, $\rho = 0.9$, $n = 4$, $d = 4$, and $\lambda = 0.1$ when the Same Control Limits Are Used for All Charts.

	p = 0	*p* = 0.01		*p* = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	200.0	195.2	201.2	175.9	203.9
0.25	32.9	34.6	34.9	47.9	52.4
0.50	11.3	11.9	11.9	16.6	16.9
1.00	4.8	4.9	4.9	6.4	6.1
2.00	2.4	2.4	2.4	2.9	2.7
5.00	1.1	1.1	1.1	1.3	1.2
10.00	1.0	1.0	1.0	1.0	1.0
UCL	12.7143	12.7143	12.7143	12.7143	12.7143

Table 7.33. In-Control ANMS and Out-of-Control SSANMS Values for the MZ, MI, and MS Control Charts for Shifts in **μ** when $b = 4$, $\rho = 0.9$, $n = 4$, $d = 4$, and $\lambda = 0.1$ when the Same Control Limits Are Used for All Charts.

	p = 0	*p* = 0.01		*p* = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	3200.0	3091.6	3186.8	2533.1	2936.7
0.25	526.8	548.0	552.6	690.5	753.9
0.50	181.3	187.8	188.4	239.8	243.8
1.00	76.6	78.2	77.5	92.0	87.6
2.00	38.4	38.7	38.4	41.5	39.0
5.00	17.9	18.0	18.0	18.6	18.0
10.00	16.0	15.8	15.8	14.5	14.4
UCL	12.7143	12.7143	12.7143	12.7143	12.7143

Table 7.34. In-Control ATS and Out-of-Control SSATS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $\rho = 0.9$, $n = 4$, $d = 4$, and $\lambda = 0.4$ when the Same Control Limits Are Used for All Charts.

	p = 0	*p* = 0.01		*p* = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	800.0	750.5	803.2	526.3	814.3
0.25	279.9	277.8	293.2	270.8	400.5
0.50	68.3	70.7	72.8	91.4	122.1
1.00	14.3	14.9	15.1	20.7	23.5
2.00	4.5	4.6	4.6	6.2	6.3
5.00	2.0	2.0	2.0	2.2	2.1
10.00	2.0	2.0	2.0	2.0	2.0
UCL	14.5687	14.5687	14.5687	14.5687	14.5687

Table 7.35. In-Control ANSS and Out-of-Control SSANSS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $\rho = 0.9$, $n = 4$, $d = 4$, and $\lambda = 0.4$ when the Same Control Limits Are Used for All Charts.

	p = 0	*p* = 0.01		*p* = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	200.0	187.6	200.8	131.5	203.6
0.25	70.5	69.9	73.8	68.2	100.6
0.50	17.6	18.2	18.7	23.3	31.0
1.00	4.1	4.2	4.3	5.7	6.4
2.00	1.6	1.7	1.7	2.1	2.1
5.00	1.0	1.0	1.0	1.0	1.0
10.00	1.0	1.0	1.0	1.0	1.0
UCL	14.5687	14.5687	14.5687	14.5687	14.5687

Table 7.36. In-Control ANMS and Out-of-Control SSANMS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $\rho = 0.9$, $n = 4$, $d = 4$, and $\lambda = 0.4$ when the Same Control Limits Are Used for All Charts.

	p = 0	*p* = 0.01		*p* = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	3200.0	2972.1	3180.8	1893.9	2931.5
0.25	1127.5	1107.8	1168.9	981.9	1448.9
0.50	281.3	287.8	296.3	336.1	446.8
1.00	65.3	66.7	67.3	81.6	91.8
2.00	26.0	26.3	26.3	29.7	29.7
5.00	16.0	15.9	15.9	15.0	14.8
10.00	16.0	15.8	15.8	14.4	14.4
UCL	14.5687	14.5687	14.5687	14.5687	14.5687

7.4 Illustrative Example

Sections 7.2 and 7.3 have investigated the MI and MS charts when all pairs of the variables have the same correlation ρ. In this section, the performance of the two control charts will be examined by looking at an example of simulated data based on a problem from Duncan (1986). This problem was also used by Hawkins (1993), and deals with cotton spinning where the four variables being monitored are

X_1 = fiber fineness

X_2 = fiber length

X_3 = fiber strength

X_4 = skein strength.

The in-control correlation matrix is

$$\boldsymbol{\Sigma}_{Z0} = \begin{bmatrix} 1 & 0 & -0.16 & -0.39932 \\ 0 & 1 & 0 & 0.606 \\ -0.16 & 0 & 1 & 0.40688 \\ -0.39932 & 0.606 & 0.40688 & 1 \end{bmatrix}.$$

It seems that changes to the fiber fineness cause changes in the fiber strength since finer fiber has less strength, and this can be seen by the negative correlation between X_1 and X_3. However, fiber length is an inherent property of the cotton fibers and is not affected by changes in fiber fineness or strength. The skein strength is affected by changes to any of X_1, X_2, or X_3. Finer fiber decreases the skein strength, while longer and stronger fiber increases the skein strength.

Data was simulated using the in-control correlation matrix $\mathbf{\Sigma}_{Z0}$ when $n = 4$, $d = 4$, $\lambda = 0.1$, and $p = 0.01$ or 0.1. The results are shown in Tables 7.37 through 7.39. The control limits are set to give an in-control ATS of 800 for the MZ chart with no observations missing, and these control limits are then used for the MI and MS charts. For this example, we see that even though the control limits are not adjusted, the in-control ATS for both the MI and MS control charts are still approximately 800 hours when $p = 0.01$ and $p = 0.1$. The MI and MS charts have similar SSATS values, but the MS chart has better overall performance.

Table 7.37. In-Control ATS and Out-of-Control SSATS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $n = 4$, $d = 4$, and $\lambda = 0.1$.

	$p = 0$	$p = 0.01$		$p = 0.1$	
δ	MZ	MI	MS	MI	MS
0.00	800.0	798.8	800.3	795.3	801.5
0.25	130.6	132.7	133.2	159.9	159.6
0.50	43.3	44.2	44.2	52.5	51.9
1.00	17.1	17.4	17.3	20.0	19.5
2.00	7.6	7.7	7.7	8.5	8.4
5.00	2.5	2.5	2.5	2.9	2.8
10.00	2.0	2.0	2.0	2.0	2.0
UCL	12.7143	12.7143	12.7143	12.7143	12.7143

Table 7.38. In-Control ANSS and Out-of-Control SSANSS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $n = 4$, $d = 4$, and $\lambda = 0.1$.

	p = 0	*p* = 0.01		*p* = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	200.0	199.7	200.1	198.7	200.3
0.25	33.1	33.7	33.8	40.5	40.4
0.50	11.3	11.5	11.6	13.6	13.5
1.00	4.8	4.8	4.8	5.5	5.4
2.00	2.4	2.4	2.4	2.6	2.6
5.00	1.1	1.1	1.1	1.2	1.2
10.00	1.0	1.0	1.0	1.0	1.0
UCL	12.7143	12.7143	12.7143	12.7143	12.7143

Table 7.39. In-Control ANMS and Out-of-Control SSANMS Values for the MZ, MI, and MS Control Charts for Shifts in $\boldsymbol{\mu}$ when $b = 4$, $n = 4$, $d = 4$, and $\lambda = 0.1$.

	p = 0	*p* = 0.01		*p* = 0.1	
δ	MZ	MI	MS	MI	MS
0.00	3200.0	3163.3	3169.3	2862.0	2885.3
0.25	530.3	533.6	535.3	582.8	581.6
0.50	181.3	182.9	183.0	196.1	194.2
1.00	76.6	76.8	76.6	79.1	77.6
2.00	38.4	38.3	38.3	38.0	37.3
5.00	17.9	17.9	17.8	17.5	17.3
10.00	16.0	15.8	15.8	14.4	14.4
UCL	12.7143	12.7143	12.7143	12.7143	12.7143

7.5 Comparison of MEWMA Control Charts with Adjusted and Unadjusted Covariance Matrix

Recall from Chapter 3 the control statistic for the MI chart from equation (3.32) is

$$M_k^I = \left(E_{k1}^{ZI}, E_{k2}^{ZI}, \ldots, E_{kb}^{ZI}\right)\mathbf{\Sigma}_{Ik0}^{-1}\left(E_{k1}^{ZI}, E_{k2}^{ZI}, \ldots, E_{kb}^{ZI}\right)^T$$

and the control statistic for the MS chart from equation (3.39) is

$$M_k^S = \left(E_{k1}^{ZS}, E_{k2}^{ZS}, \ldots, E_{kb}^{ZS}\right)\mathbf{\Sigma}_{Sk0}^{-1}\left(E_{k1}^{ZS}, E_{k2}^{ZS}, \ldots, E_{kb}^{ZS}\right)^T.$$

For both of these charts, the in-control covariance matrices are adjusted at each sampling point in order to account for missing observations. For the special case in which all of the variables are independent, so $\rho = 0$, then the asymptotic value of the in-control covariance matrices $\mathbf{\Sigma}_{Ik0}$ and $\mathbf{\Sigma}_{Sk0}$ is $c_\infty \mathbf{\Sigma}_{Z0}$. This means that when $\rho = 0$, it is not necessary to adjust the in-control covariance matrix for each sampling point even though observations may be missing.

An issue of interest is how much of an effect adjusting the covariance matrix has on the performance of the MI and MS charts when the variables are correlated. Recall that for the MI and MS charts with the unadjusted covariance matrix, the asymptotic in-control covariance matrix is used at each sampling point. Thus, the control statistic for the unadjusted MI chart from equation (3.40) is

$$M_k^{IU} = c_\infty^{-1}\left(E_{k1}^{ZI}, E_{k2}^{ZI}, \ldots, E_{kb}^{ZI}\right)\mathbf{\Sigma}_{Z0}^{-1}\left(E_{k1}^{ZI}, E_{k2}^{ZI}, \ldots, E_{kb}^{ZI}\right)^T$$

and the control statistic for the unadjusted MS chart from equation (3.41) is

$$M_k^{SU} = c_\infty^{-1}\left(E_{k1}^{ZS}, E_{k2}^{ZS}, \ldots, E_{kb}^{ZS}\right)\mathbf{\Sigma}_{Z0}^{-1}\left(E_{k1}^{ZS}, E_{k2}^{ZS}, \ldots, E_{kb}^{ZS}\right)^T.$$

Let $b = 4$, $n = 4$, $d = 4$, $\rho = 0.9$, and $p = 0.1$, and suppose that the control limits are set to give an in-control ATS of 800 for the MZ chart. These control limits are then used for all of the charts. From Table 7.31, when $\lambda = 0.1$, the in-control ATS for the MI chart when the covariance matrix is adjusted to account for missing samples is 703.9 hours. If the covariance matrix is not adjusted, the in-control ATS decreases to only

129.7 hours. For the MS chart, the in-control ATS when the covariance matrix is adjusted is 815.8 hours, while it is only 128.6 hours if the covariance matrix is not adjusted. From Table 7.34, when $\lambda = 0.4$, the in-control ATS for the MI and MS charts are 526.3 and 814.3 hours, respectively, when the covariance matrix is adjusted for missing observations. However, when it is not adjusted, the in-control ATS decreases to approximately 62 hours for both charts.

Therefore, when the variables are highly positively correlated with $\rho = 0.9$, adjusting the covariance matrix to account for missing observations has a large impact on the performance of both the MI and MS control charts. If the covariance matrix is not adjusted at each sampling point, then there is a significant decrease in the in-control ATS, which corresponds to a significant increase in the false alarm rate per unit time. Next, the performance of the MI and MS charts with both the adjusted and unadjusted covariance matrices will be investigated over a range of correlation values.

Suppose there are $b = 2$ quality variables of interest and samples of size $n = 2$ are taken every $d = 2$ hours with $\lambda = 0.05$. There is a ten percent chance that each individual observation is missing, independently of other observations, so $p = 0.1$. Tables 7.46 through 7.48 present the in-control ATS, ANSS, and ANMS values for the MI and MS control charts over a range of correlation values from $\rho = -0.9$ to 0.9. For each table, the first column gives the different levels of correlation between the two variables, and the second column corresponds to the MZ chart with no missing observations. The next two columns correspond to the MI and MS charts when the covariance matrix is not adjusted with the control statistics. The last two columns correspond to the MI and MS charts when the covariance matrix is adjusted to account for missing observations. The control limits are set to give an in-control ATS of 800 hours for the MZ chart. These limits are then used for the MI and MS charts.

Table 7.46 shows that as the level of correlation between the variables increases, the in-control ATS of both the MI and MS control charts decreases if the covariance matrix is not adjusted. For highly correlated variables, the in-control ATS is significantly less than 800. For example, when $\rho = -0.75$, the in-control ATS is 503.5 and 481.6 hours for the MI and MS charts, respectively, if the covariance matrix is not adjusted to account for missing observations. However, if the covariance matrix is adjusted, then the in-control ATS is 806 hours for both the MI and MS charts. For weakly correlated variables, adjusting the covariance matrix has little effect on the performance of the two charts. When $\rho = 0.25$, the in-control ATS of the unadjusted MI and MS charts are 791.0 and 783.5 hours, respectively. The in-control ANSS and ANMS for both charts also decrease as the level of correlation increases if the covariance matrix is not adjusted when observations are missing.

Table 7.40. In-Control ATS Values for the MZ, MI, and MS Control Charts for when $b = 2$, $n = 2$, $d = 2$, $\lambda = 0.05$, and $p = 0.1$ with the Unadjusted and Adjusted Covariance Matrix.

		unadjusted		adjusted	
ρ	MZ	MI	MS	MI	MS
-0.90	800.0	245.4	226.7	784.5	817.1
-0.75	800.0	503.5	481.6	806.3	806.9
-0.50	800.0	708.4	695.1	813.5	807.3
-0.25	800.0	791.0	783.5	814.7	807.8
0.00	800.0	814.6	805.6	814.6	805.6
0.25	800.0	791.8	781.4	814.1	805.9
0.50	800.0	711.4	695.5	814.0	804.5
0.75	800.0	506.2	483.9	807.0	806.3
0.90	800.0	246.1	227.7	786.4	815.4
UCL	9.0287	9.0287	9.0287	9.0287	9.0287

Table 7.41. In-Control ANSS Values for the MZ, MI, and MS Control Charts for when $b = 2$, $n = 2$, $d = 2$, $\lambda = 0.05$, and $p = 0.1$ with the Unadjusted and Adjusted Covariance Matrix.

		unadjusted		adjusted	
ρ	MZ	MI	MS	MI	MS
-0.90	400.0	120.3	113.4	384.5	408.5
-0.75	400.0	246.7	240.8	395.1	403.4
-0.50	400.0	347.2	347.5	398.6	403.4
-0.25	400.0	287.6	391.7	399.2	403.6
0.00	400.0	399.1	402.7	399.2	402.7
0.25	400.0	388.0	390.7	398.9	402.9
0.50	400.0	348.6	347.7	398.9	402.2
0.75	400.0	248.1	241.9	395.5	403.1
0.90	400.0	120.6	113.8	385.4	407.7
UCL	9.0287	9.0287	9.0287	9.0287	9.0287

Table 7.42. In-Control ANMS Values for the MZ, MI, and MS Control Charts for when $b = 2$, $n = 2$, $d = 2$, $\lambda = 0.05$, and $p = 0.1$ with the Unadjusted and Adjusted Covariance Matrix.

		unadjusted		adjusted	
ρ	MZ	MI	MS	MI	MS
-0.90	1600.0	437.3	408.1	1398.1	1470.8
-0.75	1600.0	897.3	866.9	1436.9	1452.4
-0.50	1600.0	1262.4	1251.3	1449.6	1453.2
-0.25	1600.0	1409.5	1410.3	1451.9	1454.1
0.00	1600.0	1451.3	1450.0	1451.6	1450.0
0.25	1600.0	1411.1	1406.5	1450.7	1450.7
0.50	1600.0	1267.8	1252.0	1450.6	1448.0
0.75	1600.0	902.1	871.1	1438.0	1451.4
0.90	1600.0	438.6	409.9	1401.3	1467.7
UCL	9.0287	9.0287	9.0287	9.0287	9.0287

7.6 Conclusions

When $\rho = 0$, so the variables are independent, the MS chart has better performance than the MI chart for all magnitudes of shifts in **μ**. Increasing the probability that an observation is missing increases the SSATS for both charts, as we would expect. The MEWMA control charts with $n = 4$ are able to detect moderate changes in the process mean more quickly than the corresponding charts with $n = 1$, but the charts with $n = 1$ are more efficient for small and large shifts in **μ**. However, the difference in the SSATS values of the two sampling schemes is small for $p = 0.01$. When p increases to 0.1, the difference in the performance of the two sampling schemes also increases.

If the variables are highly correlated with $\rho = 0.9$, the MI chart has better performance for small shifts in **μ**, but the MS chart is more efficient for moderate and large shifts. Once again, the MEWMA charts with $n = 4$ have smaller SSATS values for moderate changes in the process mean, while the corresponding charts with $n = 1$ have are able to detect small and large shifts in **μ** more quickly.

When $n = 1$, increasing the correlation from 0 to 0.9 has little effect on the performance of the MI and MS control charts for moderate and large shifts in **μ**. For small shifts in the process mean, increasing the correlation increases the SSATS for the MS chart. When $n = 4$, increasing ρ has little effect on either the MI or MS chart for large shifts. However, for moderate and large shifts in **μ**, increasing the correlation increases the SSATS for both the MI and MS charts, especially when $p = 0.1$.

If the standard control limits are used for all the charts, the in-control ATS for both the MI and MS charts increases as p increases when $n = 1$, resulting in a lower false alarm rate per unit time than the MZ chart. The in-control ATS for the MI chart is always higher than for the MS chart. If $p = 0.1$, for both $\lambda = 0.026$ and $\lambda = 0.11989$, the in-control ATS is approximately 1220 and 905 hours for the MI and MS charts, respectively. When $n = 4$, the in-control ATS of the MI chart decreases as p increases, while the in-control ATS of the MS remains at approximately 800 hours.

As in the univariate case, the use of the standardized sample mean is recommended when observations are missing. By using the standardized sample mean, the weights are adjusted to account for missing observations. In addition, when the process is in control, Z_{kv} follows a standard normal distribution and does not depend on the sample size, so the control limits are constant even though the sample size varies randomly. Using Z_{kv} adjusts the sample mean $\overline{X}_{kv}$ for the fact that part of the sample is missing for variable v. The MI and MS control charts then use two different methods for adjusting the weights between samples when the entire sample for at least one variable is missing.

The MS chart, which uses the previous value of the EWMA statistic for variable v if the entire sample for variable v is missing, is recommended for use in practice. Recall from (3.33), for the MS control chart

$$E_{kv}^{ZS} = \begin{cases} E_{k-1,v}^{ZS} & \text{if } m_{kv} = n \\ (1-\lambda)E_{k-1,v}^{ZS} + \lambda Z_{kv} & \text{if } m_{kv} < n \end{cases} \qquad v = 1, 2, \ldots, b$$

where $E_{0v}^{ZS} = 0$. If the variables are independent or weakly correlated, then the in-control covariance matrix does not need to be adjusted at each sampling point, and instead the asymptotic in-control covariance matrix can be used. In this case, the control statistic for the unadjusted MS chart given by equation (3.41) is

$$M_k^{SU} = c_\infty^{-1}\left(E_{k1}^{ZS}, E_{k2}^{ZS}, \ldots, E_{kb}^{ZS}\right)\mathbf{\Sigma}_{Z0}^{-1}\left(E_{k1}^{ZS}, E_{k2}^{ZS}, \ldots, E_{kb}^{ZS}\right)^T.$$

However, if $|\rho| > 0.25$, the in-control covariance matrix needs to be adjusted at each sampling point. Then the in-control covariance matrix at time t_k, $\mathbf{\Sigma}_{Sk0}$, can be obtained by equation (3.38)

$$\mathbf{\Sigma}_{Sk0} = \begin{cases} \mathbf{\Sigma}_{S,k-1,0} & \text{if } m_{kv} = m_{kv'} = n \\ (1-\lambda)\mathbf{\Sigma}_{S,k-1,0} & \text{if } m_{kv} = n \text{ and } m_{kv'} < n \text{ or if } m_{kv} < n \text{ and } m_{kv'} = n \\ (1-\lambda)^2\mathbf{\Sigma}_{S,k-1,0} + \lambda^2\mathbf{\Sigma}_{Zk0} & \text{if } m_{kv} < n \text{ and } m_{kv'} < n. \end{cases}$$

for k = 1, 2, 3, …, where $\mathbf{\Sigma}_{S00} = c_{\infty}\mathbf{\Sigma}_{Z0}$. In this case, the control statistic for the MS chart given by equation (3.39) is

$$M_k^S = \left(E_{k1}^{ZS}, E_{k2}^{ZS}, \ldots, E_{kb}^{ZS}\right)\mathbf{\Sigma}_{Sk0}^{-1}\left(E_{k1}^{ZS}, E_{k2}^{ZS}, \ldots, E_{kb}^{ZS}\right)^T.$$

For both situations, the standard control limits, which can be found using statistical software, can be used.

Chapter 8

Conclusions and Recommendations

This dissertation investigated missing observations in EWMA and MEWMA control charts. The standardized sample mean is used when observations are missing since this adjusts the weights by making the weight used for the current sample mean proportional to the square root of the sample size for this sample. In addition, when the process is in control, the standardized sample mean follows a standard normal distribution, and thus does not depend on the sample size. This means that the control limits are constant even though the sample size varies randomly. Using the standardized sample means adjusts the sample mean for the fact that part of the sample is missing, but when complete samples are missing, the weights between samples should also be adjusted.

In the univariate case, three approaches for adjusting the weights of the EWMA control statistic were investigated: (1) ignoring missing samples; (2) adding the weights from previous consecutive missing sample means to the current sample mean; and (3) increasing the weights of non-missing sample means in proportion so that the weights sum to one. Two methods for adjusting the weights of the MEWMA control statistic were examined: (1) ignoring all the data at a sampling point if the data for at least one variable is missing; and (2) using the previous EWMA value for any variable for which all the data are missing. Both of these methods are examined when the in-control covariance matrix is adjusted at each sampling point to account for missing observations, and when it is not adjusted.

In the univariate case, the EI chart has the best overall performance for both $n = 1$ and $n = 4$ when there is a probability p that an individual observation is missing. The EWMA control charts with $n = 4$ are able to detect moderate shifts in μ more quickly, but the corresponding charts with $n = 1$ are more efficient for small and large shifts in the process mean. For smaller values of p, the difference in the SSATS values of the two sampling schemes is small. As p increases, the difference in the performance of the two sampling schemes also increases. The case in which there is a probability p_1 that an individual observation is missing and a probability p_2 that an entire sample is missing was also examined when $n = 4$. The EI chart once again has the best overall performance in this case.

The performance of the EI, EA, and EP charts was compared when the same control limits are used for all three charts. If $n = 1$, the in-control ATS of the EI chart increases by $\left(\sum_{i=1}^{\eta} p^i\right) \cdot 100\%$ as compared to the EZ chart. The in-control ATS of the EP chart also increases as p increases, but the in-control ATS of the EA chart decreases as p increases. Thus, the false alarm rate per unit time for the EI and EP charts is lower than for the EZ chart, but it is higher for the EA chart. If $n = 4$, missing observations have very little or no effect on the in-control ATS, and false alarm rate per unit time of the EI, EA, and EP charts.

The EI chart, which adjusts the weights between missing samples by ignoring the missing samples and has control statistic given by (3.7), is recommended when observations may be missing at random. If the standard control limits are used, then the out-of-control SSATS values of the EI control chart are similar to those of the EZ chart However, if $n = 1$, the in-control ATS of the EI chart increases by $\left(\sum_{i=1}^{\eta} p^i\right) \cdot 100\%$ as compared to the EZ chart with no missing observations. When $n = 4$, missing observations have little effect on the in-control ATS of the EI chart.

In the multivariate case, when the variables are independent, the MS chart has better performance than the MI chart for all magnitudes of shifts in $\boldsymbol{\mu}$ for both of the sampling patterns considered. The MEWMA control charts with $n = 4$ are more efficient for moderate shifts in $\boldsymbol{\mu}$, but the corresponding charts with $n = 1$ are able to detect small and large changes more quickly. The difference in the SSATS values of the two sampling schemes is small for $p = 0.01$, but as p increases, the difference in the performance of the two sampling schemes also increases.

When the variables are highly correlated with $\rho = 0.9$, the MI chart is more efficient for small shifts in $\boldsymbol{\mu}$, but the MS chart has better performance for moderate and large shifts. Once again, the MEWMA charts with $n = 4$ have smaller SSATS values for moderate changes in the process mean, while the corresponding charts with $n = 1$ have are able to detect small and large shifts in $\boldsymbol{\mu}$ more quickly.

When the correlation increases from 0 to 0.9 there is little effect on the performance of the MI and MS control charts for moderate and large shifts in $\boldsymbol{\mu}$ when $n = 1$. However, for small shifts in the process mean, increasing the correlation increases the SSATS for the MS chart. When $n = 4$, increasing ρ has little effect on either the MI or MS chart for large shifts. For moderate and large shifts in $\boldsymbol{\mu}$, increasing the correlation increases the SSATS for both the MI and MS charts, especially when $p = 0.1$.

If the standard control limits are used for all the charts, the in-control ATS for both the MI and MS charts increases as p increases when $n = 1$, resulting in a lower false alarm rate per unit time than the MZ chart. The in-control ATS for the MI chart is always higher than for the MS chart. When $n = 4$, the in-control ATS of the MI chart decreases as p increases, while the in-control ATS of the MS remains at approximately 800 hours.

Therefore, the MS control chart is recommended for use in practice. The MS chart uses the previous value of the EWMA statistic for variable v if the entire sample for variable v is missing, and this statistic is given in equation (3.33). If $|\rho| > 0.25$, the in-control covariance matrix needs to be adjusted at each sampling point using equation (3.38). In this case, the control statistic for the MS chart is given by equation (3.39). However, if the variables are independent or weakly correlated, then the in-control covariance matrix does not need to be adjusted at each sampling point. Instead, the asymptotic in-control covariance matrix can be used, and the control statistic for the unadjusted MS chart is given by equation (3.41). For both situations, the standard control limits can be used, which can be found using statistical software.

The case for which each individual observation has a probability p of being missing, independent of other observations, was investigated here. This assumes that each observation from each variable is equally likely to be missing, but this is not always true in practice. Observations and/or samples may not be independent of each other. If one observation or sample is missing, the next observation or sample may be more likely to also be missing.

When there are multiple quality variables of interest, there are many possible ways that observations may be missing. Some variables may be more likely to have missing observations than other variables. For example, certain variables may be more difficult to measure, causing more observations to be missing. In this case, each variable v could have a different probability p_v that an individual observation is missing.

Chapter 7 investigated the univariate case where there is a probability p_1 that each individual observation is missing, and a probability p_2 that the entire sample is missing. A similar situation could also occur in the multivariate case. There could be a probability p_1 that each individual observation is missing, and a probability p_2 that the entire set of n observation vectors for the b variables of interest is missing at a sampling point. Each variable v could have a different probability p_{1v} of individual observations being missing, and/or a different probability p_{2v} that the sample of n observations for variable v is missing at a sampling point.

I think that if there is a different probability p_{1v} or an individual observation being missing for each variable v and/or if there is a different probability p_{2v} that the sample of n observations for variable v is missing at a sampling point could have very interesting results. I would expect the direction of the shift to have a large impact on the SSATS. Suppose one variable has a much higher probability of observations or samples being missing. If there is a shift in the direction of this variable I think that the SSATS would be higher than if the shift occurred in the direction of another variable.

This dissertation investigated three methods for adjusting the EWMA control chart and two methods for adjusting the MEWMA control chart to account for missing observations. However, there are many other possible approaches to adjusting the weights of the control statistics that could be examined to determine if the performance of the charts could be improved further.

EWMA and MEWMA control charts for monitoring the process mean when observations are missing at random are considered here. Monitoring the variability of the process is also usually of interest. The standard EWMA and MEWMA charts for monitoring σ could be adjusted to account for missing observations. In addition, control chart combinations for monitoring μ and σ simultaneously also need to be investigated for both the univariate and multivariate cases when observations are missing.

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