Syzygy Decompositions and Projective Resolutions

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(ABSTRACT)

We give a projective resolution of a finite dimensional K-algebra Λ over its enveloping algebra $\Lambda^e = \Lambda^{op} \otimes_K \Lambda$. The description of this resolution is related to decompositions of the first syzygy module of Λ as a Λ^e module, denoted $\Omega^1_{\Lambda^e}(\Lambda)$. Resolutions of right Λ modules M_Λ may be obtained by tensoring M over Λ with this bimodule resoution. We describe how to obtain such a resolution when M is simple or when M is given in the form of a projective presentation. Computations of $Ext^n_\Lambda(S_v, S_w)$ for certain classes of algebras Λ are made using these resolutions, and applied to obtain results on global dimension.

For Stacy Lynn

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Chapter 1

Introduction

The central result of this thesis is the construction of a projective resolution of a finite dimensional k-algebra Λ over its enveloping algebra $\Lambda^e = \Lambda^{op} \otimes_k \Lambda$. Throughout it is assumed that Λ is a quotient of a path algebra, that is, that $\Lambda = k\Gamma/I$ where Γ is a finite directed graph and I is an ideal contained in J, the ideal generated by the set of all arrows in Γ. Recall that the path algebra $k\Gamma$ is defined to be the algebra with k-basis the set of all finite directed paths in Γ, (where we consider a vertex to be a path of length zero), and multiplication is defined by concatenation of paths, if possible, or zero if the concatenation would not be a directed path in Γ. Any finite dimensional algebra over an algebraically closed field is Morita equivalent to such a quotient of a path algebra.

The resolution mentioned above is constructed by repeatedly tensoring a canonical short exact sequence with the first syzygy module of the algebra over its enveloping algebra, denoted $\Omega^1_{\Lambda^e}(\Lambda)$. Decompositions of this syzygy module play an important role in the description of the modules in this projective resolution. These decompositions are determined by the structure of the reduced Gröbner basis for the ideal used in forming the quotient of the path algebra. Certain rewriting rules influenced by the structure of this reduced Gröbner basis are central to the structure of the maps between the projective modules in the resolution. As such the reader must be familiar with the basic results and terminology of non-commutative Gröbner basis theory.

Projective resolutions of modules have played a central role in ring and module theory since the introduction of homological techniques to algebra in the 1950s. One may view an algebra as a module over its enveloping algebra, and compute a projective resolution of the algebra in this sense. This resolution is intimately tied to the representation theory of the algebra and to the homological properties of the module category over the algebra. We will attempt to partially illustrate this with two examples of applications of such an enveloping algebra resolution. The first is in computing the Hochschild cohomology groups $HH^n(\Lambda)$ of the algebra, which are defined to be $HH^n(\Lambda) = Ext^n_{\Lambda^e}(\Lambda, \Lambda)$. Happel [11] gives a nice treatment of Hochschild cohomology and the use of enveloping algebra resolutions in its computation. These invariants are not only important to ring theory (global dimension of the ring), but have applications in other areas of mathematics, such as algebraic topology (simplicial homology), and algebraic geometry (infinitessimal automorphisms and deformations). The second application of enveloping algebra resolutions is that they provide a means of constructing functorial projective resolutions of one sided Λ-modules, which can be used to investigate homological properties of the category $mod(\Lambda)$, which is the category of finite dimensional modules over the ring Λ. (Among other things one is interested in the projective dimensions of Λ-modules M_Λ - the length of the minimal Λ resolution of M - and in the derived functors $Ext^n_\Lambda(M_\Lambda, \_)$ and $Tor^\Lambda_n(M_\Lambda, \_)$.) It is in this second area of application that this thesis will examine implications of the bimodule resolution given in the central result.

There are several known projective resolutions of Λ as a Λ^e-module. One of the earliest such examples given was the bar resolution (see for example [6]). Since we are dealing with artin algebras, one can define the minimal resolution, where minimal here means that the image of each of the maps $P^n \to P^{n-1}$ is contained in the radical of P^{n-1}. This resolution, which in a precise sense is the 'smallest' possible resolution, should be the easiest for computations, and moreover the minimal resolution is unique and is an obviously interesting invariant of Λ. However as one might imagine, it is rather difficult to compute. Happel gives a description of the projective modules in this resolution in [11], but not a description of the maps. Bardzell [4] described the maps in the minimal resolution in the case that Λ is a monomial algebra, that is, $\Lambda = k\Gamma/I$ where I is generated by a set of paths in Γ. In the case that Λ is not monomial, a resolution, not necessarily minimal, is given in [5]. Concerning resolutions of modules, techniques are given in [1] [9], and [8]. With the exception of the bar resolution, all of these examples are directed toward finding a minimal, or at least as small as possible, resolution. The resolution given in this thesis makes a departure from this track in that the resolution here is clearly nowhere near the minimal resolution. Rather than strive for minimality, the resolution here arises in a natural way, and the modules and maps can be described somewhat naturally from the structure of a minimal Gröbner basis for the ideal I.

We will use the enveloping algebra resolution to compute a one-sided resolution (i.e.. a Λ-resolution) of simple modules S_v. This resolution can be used to compute $Ext^n_\Lambda(S_v, M)$ for Λ-modules M. See [13] for example, for a thorough treatment of Ext. In the case that $M = S_w$, another simple module, descriptions of these Ext groups are crucial to understanding the cohomology algebra of Λ. This is the algebra $\coprod_i Ext^i_\Lambda(\Lambda/\mathbf{r},\Lambda/\mathbf{r})$ - $\mathbf{r}$ is the Jacobson radical of Λ - endowed with the vector space addition and the Yoneda product [10]. It is known that the dimension of the top of the nth projective module in the minimal enveloping algebra resolution of Λ is the sum of the dimensions of the modules $Ext^n_\Lambda(S_v, S_w)$, as v and w range over all vertices in Γ [11]. From this it follows that the non-existence of an N such that $Ext^n_\Lambda(S_v, S_w) = 0$ for all $n \geq N$ will guarantee infinite right global dimension of Λ [11]. The right global dimension of an algebra Λ is the supremum of the projective dimensions of all right Λ-modules. In the case that Λ is monomial it is known how to compute $Ext^n_\Lambda(S_v, S_w)$, and thus how to determine infinite global dimension or finite global dimension [10]. We will give some computations of these Ext groups for some non-monomial algebras, assuring infinite global dimension in these cases.

If we desire a projective resolution of a right Λ-module M_Λ which is given in terms of a projective presentation, it is necessary to use other techniques to obtain the resolution, since it is not clear how one tensors M with the bimodule resolution when one doesn't know an explicit k-basis for M. We give a method which may be used to calculate the resolution one would obtain by tensoring M with our bimodule resolution of Λ in the case that M is given in terms of a presentation. It turns out that this is an iterative process, starting with $P^1 \to P^0$, and computing first $P^2 \to P^1$, then $P^3 \to P^2$, and so on, rather than finding the resolution in one step as one can do in the simple case (or any other case in which one has an explicit k-basis for M). But if one is running through an iterative process it is possible to minimize the resolution at each step and compute the minimal projective resolution of M instead of the much larger resolution which would have been obtained had we tensored M with the bimodule resolution. It is possible, however, to start the iteration at any step, and so starting with M given as a presentation $P^1 \to P^0$ one could compute $P^{n+1} \to P^n \to P^{n-1}$ without first computing each step less than $n-1$, and this information might be used in computing $Ext^n_\Lambda(M, N)$ and $Tor^\Lambda_n(M, N)$. It is perhaps this application that is of most interest, in that other iterative processes for computing projective resolutions exist (see for example [9]).

Chapter 2

Background and Notation

We begin with the necessary background material which will be used in this thesis. Let Γ be a finite directed graph (quiver). The vertex set of Γ will be denoted Γ_0, and the arrow set Γ_1. We let B be the set of all finite directed paths in Γ. (Here a vertex will denote a path of length 0). The path algebra $K\Gamma$ is the K-algebra with basis B and multiplication given by $b_1 \cdot b_2 = b_1 b_2$ if $b_1 b_2$ is a path in Γ, or $b_1 \cdot b_2 = 0$ otherwise. We will be studying quotients $\Lambda = K\Gamma/I$ of path algebras, where I is assumed to be contained in J^2 - where J is the ideal in $k\Gamma$ generated by the arrows. It is also assumed that $J^n \subset I$ for some n, which of course guarantees us that Λ will be finite dimensional. An ideal I in $K\Gamma$ which satisfies the property $J^n \subset I \subset J^2$ is called admissible. See [3] for a more detailed discussion of path algebras.

The Jacobson radical of Λ (denoted $\mathbf{r}$) is the two sided ideal J/I. The top of a module $Top(M)$ is defined to be $M/M\mathbf{r}$. Thus we have that the top of Λ (as either a left or right module over itself) is equal to $\coprod_{v\in\Gamma_0} S_v$, that is, there are $\mid \Gamma_0 \mid$ non-isomorphic simple modules, one corresponding to each vertex. Each simple S_v is one dimensional with basis element e_v, and the module structure is given by $e_v \cdot v = e_v$ and $e_v \cdot b = 0$ for all $b \in B$ with $b \neq v$. It is also clear that v is an idempotent in Λ (where for simplicity of notation we are now considering elements b of B to be representatives of their equivalence class in $K\Gamma/I$), so $v\Lambda$ will be a projective right Λ-module, since $v\Lambda \oplus (1-v)\Lambda = \Lambda$. The map $v\Lambda \to S_v$ given by $v \mapsto e_v$ is easily seen to be the projective cover of S_v. The set $\{v\Lambda : v \in \Gamma_0\}$ is a complete set of non-isomorphic indecomposable finitely generated projective right Λ-modules. See [3] for a discussion. Thus all finitely generated projective Λ-modules are direct sums of the $v\Lambda$.

We will use the notation and theory of non-commutative Gröbner Bases to study quotients of path algebras. The theory hinges on the existence of an admissible order $<$ on the basis B. By admissible we mean that $<$ is a well order, if $b = b_1 b_2$ then $b > b_1, b_2$, and if $b_1 < b_2$ then $xb_1y < xb_2y$ whenever both products are non-zero. An example of such an order is the length-lexicographic order. Here we say that the length of a basis element b (denoted $len(b)$) is the number of arrows in b as a path in Γ, and we define $b_1 < b_2$ if $len(b_1) < len(b_2)$. We order the vertices arbitrarily, and the arrows arbitrarily, and then say $b_1 < b_2$ if $len(b_1) = len(b_2)$ and b_1 comes before b_2 in the "dictionary," considering a path to be a word in the arrows and using the order on the arrows. We now fix an admissible order on B. For an arbitrary element x of $k\Gamma$, $x = \sum k_i b_i$ we say the largest b_i in this sum with non-zero coefficient is the *tip* of x, denoted $tip(x)$. A subset $\mathcal{G}$ of I is a minimal Gröbner basis for I if for each $i \in I$, there is $g \in \mathcal{G}$ such that $tip(g)$ is a subpath of $tip(i)$, and if $tip(g)$ is not a subpath of $tip(g')$ for $g' \neq g$ in $\mathcal{G}$. It is clear that we can divide B into two disjoint subsets, those paths b which are divisible by $tip(i)$ for some $i \in I$, denoted $Tip(I)$, and those that do not, denoted $Nontip(I)$. It is well known that $k\Gamma \cong I \oplus \mathrm{span}_K(Nontip(I))$, so we may identify $Nontip(I)$ with a K-basis for $\Lambda = K\Gamma/I$.

We denote by Λ^e the K-algebra $\Lambda^{op} \otimes_K \Lambda$. Λ-Λ-bimodules ${}_\Lambda M_\Lambda$ correspond to right Λ^e-modules, where the multiplication is given by $M \cdot a \otimes_K b = aMb$. The non-isomorphic indecomposable projective Λ^e-modules are the modules $v \otimes_K w\Lambda^e$. Recall that if $\{v_i\}$ is a basis for the vector space V and if $\{w_i\}$ is a basis for the vector space W, then a basis for

$V \otimes_K W$ is the set $\{v_i \otimes_K w_j\}_{(i,j)}$. Thus we see that $m \otimes_K n$ with $m, n \in Nontip(I)$ forms a K-basis for Λ^e.

We end this section with some notational conventions that will be followed throughout the rest of this paper. Without a subscript, the symbol $\otimes$ will always refer to a tensor over the field K, $\otimes_K$. The length of a path will be denoted $len(p)$. The subset of $Nontip(I)$ consisting of those paths n with $len(n) \geq 1$ will be denoted $\mathcal{N}_{\geq 1}$. We will often need to consider the first and last arrows of a path (of positive length) separately from the rest, so we let α_p denote the first arrow of a path p and β_p denote the last arrow. As long as we restrict ourselves to paths of positive length α_p and β_p will always be nontrivial. The remaining portions of the path, which may be only vertices if $len(p) = 1$, will be denoted p^- and p^+ respectively, so we have that

$$p = \alpha_p \cdot p^- = p^+ \cdot \beta_p.$$

We will also need to speak specifically of subpaths which are either the first or last part of a path p. We say that a path q is a *prefix* of the path p if there is r in Γ with $p = q \cdot r$. If $len(r) \geq 1$ we say that q is a *proper prefix* of p. Similarly we say that a path q is a *suffix* of the path p if there is r in Γ with $p = r \cdot q$. If $len(r) \geq 1$ we say that q is a *proper suffix* of p. If one views paths as words with the arrows in Γ_1 serving as the letters this terminology is obvious.

Chapter 3

Syzygy Decompositions

We begin with the minimal projective cover of $\Lambda = k\Gamma/I$ as a right Λ^e-module, given by the projective bimodule $P^0 = \coprod_{v\in\Gamma_0} v \otimes v\Lambda^e$, with the map $d_0 : P^0 \to \Lambda$ given by $v \otimes v \mapsto v$. It is a well known fact that the first syzygy module $\Omega^1_{\Lambda^e}(\Lambda)$, which is the kernel of this map, is generated as a bimodule by $\{g_a = a \otimes t(a) - o(a) \otimes a : a \in \Gamma_1\}$. For the sake of completeness, we indicate a proof of this result here:

Lemma 3.1 *As a Λ^e-module $\Omega^1_{\Lambda^e}(\Lambda)$ is generated (minimally) by $\{g_a = a \otimes t(a) - o(a) \otimes a : a \in \Gamma_1\}$.*

Proof. It is clear that each of the g_a lies in the kernel of the map $P^0 \to \Lambda$. If $w = \sum p_i \otimes q_i$ (each p_i and q_i a path) is an element of $\Omega^1_{\Lambda^e}(\Lambda) = ker(d_0)$, we show w is in the sub-bimodule of P^0 spanned by the g_a. Write $p_i = a_{i,1}a_{i,2}\cdots a_{i,m_i}$, with each $a_{i,j} \in \Gamma_1$. Then

$$w - a_{1,1}\cdots a_{1,m_1-1}(a_{1,m_1} \otimes t(a_{1,m_1}) - o(a_{1,m_1}) \otimes a_{1,m_1})q_1$$

yields

$$w_1 = a_{1,1} \cdots a_{1,m_1-1} \otimes a_{1,m_1} q_1 + \sum_{i \neq 1} p_i \otimes q_i$$

where we may need to write $a_{1,m_1} q_1$ as a sum of Nontips if it is not in $Nontip(I)$. The thing to notice is that the path on the left hand side of the tensor in the $i = 1$ term (and in any rewritings) is shorter by one arrow. Again we subtract:

$$w_2 = w_1 - a_{1,1} \cdots a_{1,m_1-2}(a_{1,m_1-1} \otimes t(a_{1,m_1-1}) - o(a_{1,m_1-1}) \otimes a_{1,m_1-1}) a_{1,m_1} q_1$$

yielding

$$w_2 = a_{1,1} \cdots a_{1,m_1-2} \otimes a_{1,m_1-1} a_{1,m_1} q_1 + \sum_{i \neq 1} p_i \otimes q_i$$

where we may again need to rewrite, but again the first term has a path on the left hand side of the tensor which is shorter by one arrow. We continue this process, first along the first term until it has the form $v \otimes p_1'$ where v is a vertex and p_1' is a path, and then repeat this along each of the subsequent terms.

Note that each time we are subtracting an element of the kernel, so the difference remains in the kernel of d_0. We note that we now have something of the form $w' = \sum v_j \otimes p_j$, which is mapped to $\sum_j p_j$ in Λ. However, since $w' \in ker(d_0)$ we know $\sum_j p_j = 0$. It follows then that $w' = 0$. We then have the equation:

$$w - g = w' = 0$$

where g is in the submodule spanned by the g_a elements, and hence we have $w = g$. It is now established that $\{g_a\}_{a \in \Gamma_1}$ is a generating set for $\Omega^1_{\Lambda^e}(\Lambda)$. It remains to show that this generating set is minimal.

We suppose that for some arrow b, $g_b = \sum_{a \neq b} p g_a q$. Then we have:

$$\sum p(a \otimes t(a) - o(a) \otimes a)q = b \otimes t(b) - o(b) \otimes b$$

.

Then we have $\sum(pa \otimes t(a)q - p \otimes aq) - b \otimes t(b) = -o(b) \otimes b$. This is impossible, since $x \otimes y$ forms a K-basis for P^0, where x and y are nontips, and no term on the left hand side is of the form vertex $\otimes$ path. The minimality of our generating set is now established.□

It is clear then that P^1, the projective cover of $\Omega^1_{\Lambda^e}(\Lambda)$ must have the form $\coprod_{a \in \Gamma_1} o(a) \otimes t(a)\Lambda^e$, with $o(a) \otimes t(a)$ mapping to g_a.

BIMODULE DECOMPOSITIONS

We use the above minimal generating set to give a bimodule direct decompositition of the first syzygy into submodules that may not necessarily be an indecomposable decomposition. Then we give conditions on I and Γ which guarantee that the decompostion is indecomposable.

Let $\Lambda = K\Gamma/I$, where $I = <\rho_1, \rho_2, \ldots, \rho_m>$. We describe an algorithm for obtaining a direct decompostion of $\Omega^1_{\Lambda^e}(\Lambda)$ as a Λ-Λ-bimodule. Define the graph $G_{\Gamma,I}$ with one vertex v_a for each $a \in \Gamma_1$ and a directed edge $v_a \xrightarrow{ab} v_b$ if ab is a path in Γ. We also include a vertex v_{ρ_i} in G for each relation ρ_i. If $\rho_i = \sum_j k_j p_j$, where each p_j is a path in Γ, then we include an arrow $v_{\rho_i} \xrightarrow{p_j} v_{\alpha_{p_j}}$. Let $C_1, \cdots C_m$ be the connected components of $G_{\Gamma,I}$ when we forget about the orientation of the edges, and let B_k be the submodule of $\Omega^1_{\Lambda^e}(\Lambda)$ generated by $\{g_a = a \otimes t(a) - o(a) \otimes a : v_a \in C_k\}$ for each k. We will show that $\oplus B_k$ is a direct sum decomposition of $\Omega^1_{\Lambda^e}(\Lambda)$.

We will first make a few remarks about this decomposition, or more precisely, about the graph $G_{\Gamma,I}$. If I is monomial, then one may omit the vertices and arrows in $G_{\Gamma,I}$ which correspond to relations and still obtain the same decomposition. Also we have that whenever $v_a \in C_i$ ($g_a = a \otimes t(a) - o(a) \otimes a$ is an element of A_i), then $v_b \in C_i$ ($g_b = b \otimes t(b) - o(b) \otimes b \in A_i$) whenever there is a sequence of paths $(p_i)_{i=1}^n$ with $a \in p_1$, and $b \in p_n$ and $p_i \cup p_{i+1}$ equal to a path of length at least one. Furthermore, if there is some relation $\rho_\ell = \sum p_{\ell_j}$ with $a \in p_{\ell_s}$ and $b \in p_{\ell_t}$. then it should be clear that v_a and v_b are in the same C_i (g_a and g_b are in the same B_i).

Proposition 3.2 *As a $\Lambda - \Lambda$ bimodule, and hence a Λ^e-module, $\Omega^1_{\Lambda^e}(\Lambda) = \coprod B_i$.*

Proof. While it is clear that $\sum B_i = \Omega^1_{\Lambda^e}(\Lambda)$ we must show that $B_i \bigcap \cup_{j \neq i} B_j = \emptyset$. We note again that the components of G, the C_i's are determined by the existence of a sequence of paths overlapping in at least an arrow. That is, two G vertices v_a and v_b are in the same C_i if and only if there are paths $p_1, p_2, \ldots p_n$ in Γ with a in p_1, $b \in p_n$, and with p_i overlapping with p_{i+1} in at least an arrow. Now B_i is generated by elements of the form $a \otimes t(a) - o(a) \otimes a$ with $v_a \in C_i$ and we note that any nonzero mulitplication $(a \otimes t(a) - o(a) \otimes a) \cdot (p^o \otimes p)$ will necessitate that $p^o a p$ be a path in Γ, hence none of the G vertices associated with any arrow in $p^o a p$ will be in any component of G other than C_i, the same component as v_a. A complication arises if $p^o a$ or ap is equal to $tip(\rho_\ell)$ for some relation ρ_ℓ. In this case the elements of Λ corresponding to rewriting $p^o a$ or ap respectively as sums of nontips are merely nontip terms in ρ_ℓ. However each of the paths occuring as nontip terms in ρ_ℓ are by definition also in B_i. (Here again we mean a path p in Γ is "in" B_i if the generator g_a associated with each arrow a in p is in B_i.) Thus no multiplication of a generator of B_i by an element of the enveloping algebra Λ^e will produce a result outside of B_i itself, and we have our result that $\oplus B_i$ is a decomposition of $\Omega^1_{\Lambda^e}(\Lambda)$.$\square$

While we have given a direct bimodule decomposition of the first syzygy, we cannot say whether it is an indecomposable decomposition. In fact we note that if Λ is hereditary (that is, if submodules of projective modules are projective) it follows that $\Omega^1_{\Lambda^e}(\Lambda)$ is projective as a Λ-Λ-bimodule. From this it follows that g_a generates a Λ^e-projective inside of P^0, which must be isomorphic to $o(a) \otimes t(a)\Lambda^e$. There must be no intersection between this submodule of $\Omega^1_{\Lambda^e}(\Lambda)$ and the projective submodule generated by g_b for b with $o(b) \neq o(a)$ and $t(b) \neq t(a)$ (such arrows are termed *parallell*). So there are at least as many indecomposable submodules in the indecomposable decomposition of $\Omega^1_{\Lambda^e}(\Lambda)$ as there are non-parallell arrows in Γ.

To guarantee indecomposability we need to impose certain conditions on both the graph Γ and the ideal I. While this indecomposability result is included here for completeness at this time, it is not necessary for any subsequent results. We start with the following definition:

Definition $I =< \rho_1, \rho_2, \cdots, \rho_n >$ *saturates* Γ if I is monomial and if each arrow $a \in \Gamma_1$ with the property that a is contained in some path of length 2 also has the property that $a \in \rho_i$ for some i, and if whenever $\rho_i = x_1 x_2 \cdots x_j$ is contained in some path $a\rho_i b$ then the paths ax_1 and $x_j b$ are contained in some relations ρ_p and ρ_q respectively.

Example: J^n saturates any Γ for $n \geq 2$.

Example: $1 \rightarrow^a 2 \rightarrow^b 3 \rightarrow^c 4 \rightarrow^d 5 \rightarrow^e 6 \leftarrow^f 7$ with $I =< abcd, de >$, I saturates Γ.

Example: Same quiver, let $I =< abc, de >$, I does not saturate Γ.

Proposition 3.3 *Let $\Lambda = K\Gamma/I$ where I saturates Γ and Γ has neither oriented cycles nor multiple arrows between the same pair of vertices. Define the graph G as above with one vertex v_a for each $a \in \Gamma_1$ and an edge $\vec{e_{ab}}$ if ab is a path in Γ. Again let C_i be the connected components of G, and define the B_i's again to be the submodules of $\Omega^1_{\Lambda^e}(\Lambda)$ generated by*

$\{g_a = a \otimes t(a) - o(a) \otimes a : v_a \in C_i\}$ *for each* i *respectively. Then* $\coprod B_i$ *is an indecomposable decomposition of* $\Omega^1_{\Lambda^e}(\Lambda)$.

Proof. That the first syzygy is a direct sum of the B_i's is already proven above 3.2. B_i) what is new in this result is that the decomposition is indecomposable. We show that B_i is indecomposable. We consider $End_{\Lambda^e}(B_i)$ and prove that it is isomorphic to K. Let $\phi \in End_{\Lambda^e}(B_i)$. We first show that if B_i is generated by $\{g_a = a \otimes t(a) - o(a) \otimes a : v_a \in C_i\}$ then $\phi(g_a) = k_a g_a$ where $k_a \in K$. Indeed $g_a \in B_i(o(a) \otimes t(a))$, so $\phi(g_a) = \phi(g_a(o(a) \otimes t(a)))$ $= \phi(g_a) \cdot (o(a) \otimes t(a))$, so $\phi(g_a) \in B_i(o(a) \otimes t(a))$. But

$$\begin{aligned}
\phi(g_a) &= \sum_{v_b \in C_i} g_b \cdot (\lambda_b^o \otimes \lambda_b) \\
&= \sum_{v_b \in C_i} \lambda_b^o b \otimes t(b)\lambda_b - \lambda_b^o o(b) \otimes b\lambda_b \\
&= \sum_{v_b \in C_i} o(a)\lambda_b^o b \otimes t(b)\lambda_b t(a) - o(a)\lambda_b^o o(b) \otimes b\lambda_b t(a) \\
&= \qquad o(a)\lambda_a^o a \otimes t(a)\lambda_a t(a) - o(a)\lambda_a^o o(a) \otimes a\lambda_a t(a) \\
&\quad + \sum_{b \neq a} o(a)\lambda_b^o b \otimes t(b)\lambda_b t(a) - o(a)\lambda_b^o o(b) \otimes b\lambda_b t(a)
\end{aligned}$$

Now if λ_b^o and $\lambda_b \neq 0$, $\lambda_b^o \in o(a)B_i o(b)$, and there is a path in Γ from a to b, but $\lambda_b \in t(b)B_i t(a)$, and as above there must be a path in Γ from b to a, so then Γ has an oriented cycle, a contradiction, thus $\lambda_b^o \otimes \lambda_b = 0$ and $\phi(g_a) = o(a)\lambda_a^o a \otimes t(a)\lambda_a t(a) - o(a)\lambda_a^o o(a) \otimes a\lambda_a t(a)$, so $\lambda_a^o = ko(a)$ and $\lambda_a = k't(a)$, and we have $\phi(g_a) = k_a g_a$.

Now that we have that the image of any generator of B_i under ϕ is a scalar multiple of that generator, we show that all generators are sent to the same scalar multiple. Let $aa_1a_2 \ldots a_s = \rho_i$ be a relation in I and let $\phi(g_a) = k_a g_a$. We note that

$g_a \cdot o(a) \otimes a_1 a_2 \dots a_s = a \otimes a_1 a_2 \dots a_s - o(a) \otimes r_i = a \otimes a_1 a_2 \dots a_s$. Also

$g_{a_1} \cdot a \otimes a_2 a_3 \dots a_s = aa_1 \otimes a_2 a_3 \dots a_s - a \otimes a_1 a_2 \dots a_s$. Furthermore,

$g_{a_j} \cdot aa_1 a_2 \dots a_{j-1} \otimes a_{j+1} \dots a_s =$

$aa_1 a_2 \dots a_j \otimes a_{j+1} \dots a_s - aa_1 a_2 \dots a_{j-1} \otimes a_j \dots a_s$.

Now $\phi(aa_1 a_2 \dots a_{s-1} \otimes a_s - a \otimes a_1 a_2 \dots a_s)$=

$$\begin{aligned}
&- \phi(g_{a_s}) \cdot aa_1 a_2 \dots a_{s-1} \otimes t(a_s) - \phi(g_a) \cdot o(a) \otimes a_1 a_2 \dots a_s = \\
&k_{a_s}\ g_{a_s} aa_1 a_2 \dots a_{s-1} \otimes t(a_s) - k_a\ g_a o(a) \otimes a_1 a_2 \dots a_s = \\
&k_{a_s}\ (aa_1 a_2 \dots a_{s-1} \otimes a_s) - k_a\ (a \otimes a_1 a_2 \dots a_s). \qquad (3.1)
\end{aligned}$$

But we also have

$$\phi(aa_1a_2\dots a_{s-1}\otimes a_s - a\otimes a_1a_2\dots a_s) =$$

$$\phi(aa_1a_2\dots a_{s-1}\otimes a_s - aa_1a_2\dots a_{s-2}\otimes a_{s-1}a_s +$$

$$aa_1a_2\dots a_{s-2}\otimes a_{s-1}a_s - aa_1a_2\dots a_{s-3}\otimes a_{s-2}a_{s-1}a_s +$$

$$aa_1a_2\dots a_{s-3}\otimes a_{s-2}a_{s-1}a_s - aa_1a_2\dots a_{s-4}\otimes a_{s-3}a_{s-2}a_{s-1}a_s +$$

$$\dots +$$

$$aa_1\otimes a_2a_3\dots a_s - a\otimes a_1a_2\dots a_s) =$$

$$\phi(g_{a_{s-1}}aa_1a_2\dots a_{s-2}\otimes a_s +$$

$$g_{a_{s-2}}aa_1a_2\dots a_{s-3}\otimes a_{s-1}a_s +$$

$$g_{a_{s-3}}aa_1a_2\dots a_{s-4}\otimes a_{s-2}a_{s-1}a_s +$$

$$\dots +$$

$$g_{a_1}a\otimes a_2a_3\dots a_s) =$$

$$k_{a_{s-1}}\ g_{a_{s-1}}aa_1a_2\dots a_{s-2}\otimes a_s +$$

$$k_{a_{s-2}}\ g_{a_{s-2}}aa_1a_2\dots a_{s-3}\otimes as-1a_s +$$

$$k_{a_{s-3}}\ g_{a_{s-3}}aa_1a_2\dots a_{s-4}\otimes a_{s-2}a_{s-1}a_s +$$

$$\dots +$$

$$k_{a_1}\ g_{a_1}a\otimes a_1a_2\dots a_s =$$

$$k_{a_{s-1}}\ aa_1a_2\dots a_{s-1}\otimes a_s - k_{a_{s-1}}\ aa_1a_2\dots a_{s-2}\otimes a_{s-1}a_s +$$
$$k_{a_{s-2}}\ aa_1a_2\dots a_{s-2}\otimes a_{s-1}a_s - k_{a_{s-2}}\ aa_1a_2\dots a_{s-3}\otimes a_{s-2}a_{s-1}a_s +$$
$$k_{a_{s-3}}\ aa_1a_2\dots a_{s-3}\otimes a_{s-2}a_{s-1}a_s - k_{a_{s-3}}\ aa_1a_2\dots a_{s-4}\otimes a_{s-3}\dots a_s +$$
$$\dots +$$
$$k_{a_1}\ aa_1\otimes a_2a_3\dots a_s - k_{a_1}\ a\otimes a_1a_2\dots a_s). \tag{3.2}$$

Now equating the above (3.1) and (3.2) we get:

$$[k_{a_{s-1}} - k_{a_s}]\ (aa_1a_2\dots a_{s-1}\otimes a_s) +$$
$$[k_{a_{s-2}} - k_{a_{s-1}}]\ (aa_1a_2\dots a_{s-2}\otimes a_{s-1}a_s) +$$
$$[k_{a_{s-3}} - k_{a_{s-2}}]\ (aa_1a_2\dots a_{s-3}\otimes a_{s-2}as-1a_s) +$$
$$\dots +$$
$$[k_a - k_{a_1}]\ (a\otimes a_1a_2\dots a_s) = 0,$$

whence $k_{a_{s-1}} = k_{a_s}$, $k_{a_{s-2}} = k_{a_{s-1}}$, $k_{a_{s-3}} = k_{a_{s-2}}$, $\dots$, $k_{a_2} = k_{a_1}$, and $k_a = k_{a_1}$. We therefore have the result that for any two arrows x and y in some relation, $k_x = k_y$.

Now we use the fact that I saturates Γ to link any two generators via a sequence of relations, and obtain an isomorphism between $End_{\Lambda^e}(B_i)$ and K for each B_i in the bimodule decomposition of $\Omega^1_{\Lambda^e}(\Lambda)$. Let g_a and g_b be generators of B_i, and let $\phi \in End_{\Lambda^e}(\Lambda)$. Since v_a and v_b are in the same component C_i of G, a is contained in some path p in Γ and b is contained in some path q in Γ where p and q overlap on some arrow x. Now since I saturates Γ there is a relation ρ_a in I such that a occurs in ρ_a and a relation ρ_x in I such that x occurs in ρ_x. Furthermore we are guaranteed, since I saturates Γ, a 'sequence' of relations from ρ_a to ρ_x, or vice versa, since each arrow in Γ contained in a path of length at least two lies in some relation, and each of the relations in this sequence must overlap in an arrow with the next. Using the above result that each generator of B_i associated with an arrow in a particular relation is sent by ϕ to the same K multiple of itself, we see that if $\phi(g_a) = k_a g_a$ and $\phi(g_x) = k_x g_x$, we have that $k_a = k_x$. In a similar manner we will have $k_b = k_x$ from which it follows that $k_a = k_b$. This proves that for any generator g associated with a vertex in G we will have $\phi(g) = k_a \cdot g$, and hence for any $b \in B_i, \phi(b) = k_a \cdot b$. So we see now that ϕ is completely determined by k_a, and we have an obvious isomorphism between $End_{\Lambda^e}(B_i)$ and K. Thus we have that B_i is indecomposable for all I, and that the decomposition given in the first proposition,

$$\Omega^1_{\Lambda^e}(\Lambda) = \coprod B_i$$

is an indecomposable decomposition when I and Γ satisfy the conditions that I saturates Γ and Γ has neither oriented cycles nor multiple arrows between vertices.□

In a later chapter we will be describing projective resolutions which involve tensoring $\Omega^1_{\Lambda^e}(\Lambda)$ with itself. We would like to have an understanding of how the bimodules B_i in the decomposition of $\Omega^1_{\Lambda^e}(\Lambda)$ behave when tensoring $\Omega^1_{\Lambda^e}(\Lambda)$ with itself. A description of this lies in the following proposition.

Proposition 3.4 *Let $\Omega^1_{\Lambda^e}(\Lambda) \cong \coprod B_i$ be the direct sum decomposition of $\Omega^1_{\Lambda^e}(\Lambda)$ given by computing the graph $G_{\Gamma,I}$ as above. If $i \neq j$ then $B_i \otimes_\Lambda B_j = 0$.*

Proof. Recall that when tensoring over Λ we may pull elements of Λ across the tensor as if they were scalars. Thus $p \otimes_\Lambda q$ will be zero unless $t(p) = o(q)$. Let pg_aq be in B_i and rg_bs be in B_j with $i \neq j$ and p, q, r, and s paths in Γ. The assumption $i \neq j$ guarantees that in the graph $G_{\Gamma,I}$ the vertices v_a and v_b lie in different connected components. This meant that there is no path including both a and b in Γ, and therefore we are guaranteed that qr

is not a path in Γ, that is, $q \cdot r = 0$. Therefore if pa, aq, rb, and bs are all in $\mathcal{N}_{\geq 1}$ we see that $pg_a q \otimes_\Lambda rg_b s = 0$.

We are now left with the case that some of the four terms above must be rewritten. Recall that $pg_a q = pa \otimes q - p \otimes aq$. At worst we might have both pa and aq not in $Nontip(I)$, and we would need to rewrite $pg_a q = \sum k_m(p_m a_m \otimes q) - \sum k_n(p \otimes a_n q_n)$. [Here p_m and q_n are paths, not equal to p or q, and a_m and a_n are arrows which may or may not be equal to a]. If we need to rewrite one of rb or bs we do so, but we notice that when tensoring $pg_a q \otimes_\Lambda rg_b s$ we still are looking at sums of things of the form $\sigma_1 \otimes \sigma_2 \otimes_\Lambda \sigma_3 \otimes \sigma_4$ where σ_i is a path in $\mathcal{N}_{\geq 1}$. This tensor will be zero unless $t(\sigma_2) = o(\sigma_3)$. If this were the case then the last arrow of σ_2, β_{σ_2}, and the first arrow of σ_3, α_{σ_3} would be such that $v_{\beta_{\sigma_2}}$ and $v_{\alpha_{\sigma_3}}$ are in the same component of $G_{\Gamma,I}$. Because of the definition of $G_{\Gamma,I}$ we know that $v_{\beta_{\sigma_2}}$ and v_a are in the same component of $G_{\Gamma,I}$, and that $v_{\alpha_{\sigma_3}}$ and v_b are in the same component of $G_{\Gamma,I}$, and hence v_a and v_b are in the same component of $G_{\Gamma,I}$. This is impossible, since by assumption v_a and v_b must be in different components of $G_{\Gamma,I}$ since g_a is a generator of B_i and g_b is a generator of B_j with $i \neq j$.

We are therefore left with the result that if $i \neq j$ then $B_i \otimes_\Lambda B_j = 0$.$\square$

ONE SIDED DECOMPOSITIONS

We desire a decomposition of $\Omega^1_{\Lambda^e}(\Lambda)$ as a left or right Λ-module. We begin with the following obvious lemma:

Lemma 3.5 *If we consider the set of all elements of $\Omega^1_{\Lambda^e}(\Lambda)$ with the form $a_i \otimes p - o(a_i) \otimes a_i p$, that is, $g_{a_i} \cdot 1 \otimes p$, where p is a $Nontip(I)$ path with appropriate origin, we have a generating set for $\Omega^1_{\Lambda^e}(\Lambda)$ as a left Λ-module.*

Proof. Indeed, any element of $\Omega^1_{\Lambda^e}(\Lambda)$ may be realized as a bimodule sum

$$\sum(qa_i \otimes p - q \otimes a_i p),$$

which is easily seen to be a sum of the form

$$\sum q \cdot (a_i \otimes p - o(a_i) \otimes a_i p).\square$$

Since this will be a left generating set, we refer to the left generator $a \otimes p - o(a) \otimes p$ as ℓ_{ap}. While this set is a generating set, we desire a smaller set. In particular we note that if I is generated by a set of relations $\{\rho_1, \rho_2, \ldots, \rho_t\}$ which constitute a minimal Gröbner basis for I, whenever $tip(\rho_i) \mid ap$ the corresponding left generator ℓ_{ap} is superfluous. This result is contained in the following lemma:

Lemma 3.6 *If $tip(\rho) \mid ap$ for some ρ then ℓ_{ap} is in the left Λ submodule of $\Omega^1_{\Lambda^e}(\Lambda)$ generated by the set of all elements of the form $a_k \otimes p - o(a_k) \otimes a_k p$ where $a_k p \in Nontip(i)$.*

Proof. We prove the case that $ap = tip(\rho)$, with the other cases being obvious extensions of this one. So we let $(\rho) = ab_1 \cdots b_m + \sum_i k_i(b_{i,1} \cdots b_{i,m_i})$, where $tip(\rho) = ab_1 \cdots b_m$ and $p = b_1 \cdots b_m$. We begin with the following observation:

$$\begin{aligned}
ab_1 \cdots b_m \otimes t(b_m) - o(a) \otimes ab_1 \cdots b_m = \\
& ab_1 \cdots b_{m-1} \cdot (b_m \otimes t(b_m) - o(b_m) \otimes b_m) + \\
& ab_1 \cdots b_{m-2} \cdot (b_{m-1} \otimes b_m - o(b_{m-1}) \otimes b_{m-1} b_m) + \\
& \vdots \\
& a \cdot (b_1 \otimes b_2 \cdots b_m - o(b_1) \otimes b_1 \cdots b_m) + \\
& (a \otimes b_1 \cdots b_m - o(a) \otimes ab_1 \cdots b_m)
\end{aligned}$$

which we rewrite as follows:

$$\sum_{j=0}^{m-1} ab_1 \cdots b_j \cdot (\ell_{b_{j+1} \cdots b_m}) + \ell_{ap}.$$

Similarly we observe that for each of the nontip summands of ρ:

$$b_{i,1} \cdots b_{i,m_i} \otimes t(b_{i,m_i}) - o(b_{i,1}) \otimes b_{i,1} \cdots b_{i,m_i}) = \sum_{j=1}^{m_i-1} b_{i,1} \cdots b_{i,j} \cdot (\ell_{b_{i,j+1} \cdots b_{i,m_i}}).$$

So we see that:

$$\sum_{j=0}^{m-1} ab_1 \cdots b_j \cdot (\ell_{b_{j+1} \cdots b_m}) + \ell_{ap} = \\ - \sum_i k_i \sum_{j=0}^{m-1} b_{i,1} \cdots b_{i,j} \cdot (\ell_{b_{i,j+1} \cdots b_{i,m_i}}).$$

Solving for ℓ_{ap} gives the following formula:

$$\ell_{ap} = -\sum_{j=0}^{m-1} ab_1 \cdots b_j \cdot (\ell_{b_{j+1} \cdots b_m}) - \sum_i k_i \sum_{j=0}^{m-1} b_{i,1} \cdots b_{i,j} \cdot (\ell_{b_{i,j+1} \cdots b_{i,m_i}})$$

where every term on the right hand side is a left multiple of an element ℓ_{bq} where $bq \in Nontip(I)$.$\square$

This result has the following immediate corollary:

Proposition 3.7 *As a left Λ-module $\Omega^1_{\Lambda^e}(\Lambda)$ is generated by $\{\ell_{ap} : ap \in Nontip(I)\}$ when we have fixed a minimal Gröbner basis for I.*$\square$

This leads us to the following decomposition of $\Omega^1_{\Lambda^e}(\Lambda)$ as a left Λ-module:

Proposition 3.8 *As a left* Λ*-module* $\Omega^1_{\Lambda^e}(\Lambda)$ *is isomorphic to:*

$$\coprod_{a\in\Gamma_1}\coprod_{dim_K(a\Lambda)}\Lambda o(a).$$

Proof. We note that anything of the form $\sigma\cdot\ell_{ap}$ is in $\mathbf{r}\Omega^1_{\Lambda^e}(\Lambda)$ for σ any path in Γ with positive length. So the only possible elements of $Top(\Omega^1_{\Lambda^e}(\Lambda))$ are the ℓ_{ap}s. We claim that these elements do indeed comprise the Top, since if

$$\sum k_i\ell_{ap_i}=\sum k_j\sigma_j\ell_{bq_j}$$

where the σ_j are paths of positive length, then we have that

$$\sum k_ia_i\otimes p_i-k_io(a_i)\otimes a_ip_i=\sum k_j\sigma_jb_j\otimes q_j-k_j\sigma_j\otimes b_jq_j$$

and hence

$$\sum k_io(a_i)\otimes a_ip_i=\sum k_ia_i\otimes p_i-\sum k_j\sigma_jb_j\otimes q_j-k_j\sigma_j\otimes b_jq_j$$

where the elements on the right hand side of the equation all have a path of length at least one on the left hand side of the tensor, and as such are distinct from the elements on the right hand side of the equation. This of course is impossible, since *nontip* $\otimes$ *nontip* forms a k-basis for P^0. It is evident therefore that $Top(\Omega^1_{\Lambda^e}(\Lambda))$ is in one to one correspondence with the set of all ℓ_{ap}, and the result follows.$\square$

We remark that the obvious analogues to the preceding two propositions and the two lemmas before them also hold, that is:

Lemma 3.9 *If we consider the set of all elements of the form $pa \otimes t(a) - p \otimes a$ where p is any path with appropriate terminus, we have a generating set for $\Omega^1_{\Lambda^e}(\Lambda)$ as a right Λ-module.*$\square$

In an analogous manner, the generating elements $pa \otimes t(a) - p \otimes a$ of $\Omega^1_{\Lambda^e}(\Lambda)$ as a right module are denoted by r_{pa}.

Lemma 3.10 *If $tip(\rho) \mid pa$ for some ρ in a minimal Gröbner basis for I then r_{pa} is in the right submodule of $\Omega^1_{\Lambda^e}(\Lambda)$ generated by the set of all elements of the form $qb \otimes t(b) - q \otimes b$ where $qb \in Nontip(I)$.*$\square$

Proposition 3.11 *As a right Λ-module, $\Omega^1_{\Lambda^e}(\Lambda)$ is generated by $\{r_{pa} : pa \in Nontip(I)\}$ for some fixed minimal Gröbner basis for I.*$\square$

Proposition 3.12 *As a right Λ-module $\Omega^1_{\Lambda^e}(\Lambda)$ is isomorphic to:*

$$\coprod_{a \in \Gamma_1} \coprod_{dim_k(\Lambda a)} t(a)\Lambda$$

$\square$

We also remark here that the left and right decompositions of $\Omega^1_{\Lambda^e}(\Lambda)$ can be recast as follows:

$$\coprod_{n \in \mathcal{N}_{\geq 1}} \Lambda o(n) \qquad \coprod_{n \in \mathcal{N}_{\geq 1}} t(n)\Lambda$$

where we recall $\mathcal{N}_{\geq 1}$ denotes the subset of $Nontip(I)$ consisting of all elements of length greater than or equal to one. This description will prove useful in the future.

We also record here for future use the rewriting rules obtained in the proof of the second lemma for writing ℓ_{ap} where ap is not in $Nontip(I)$ in terms of the minimal generators. Let ρ denote the relation $a_1 \cdots a_m + \sum_i k_i b_{i,1} \cdots b_{i,m_i}$ with $tip(\rho) = a_1 \cdots a_m$. Then we have the following:

$$\begin{aligned}\ell_{a_1 \cdots a_m} = & \\ & -\sum_{j=1}^{m-1} a_1 \cdots a_j \cdot \ell_{a_{j+1} \cdots a_m} \\ & -\sum_i k_i \sum_{j=1}^{m_i - 1} b_{i,1} \cdots b_{i,j-1} \cdot \ell_{b_{i,j} \cdots b_{i,m_i}}\end{aligned}$$

$$\begin{aligned}\ell_{x_1 \cdots x_t a_1 \cdots a_m} = & \\ & -\sum_i k_i \ell_{x_1 \cdots x_t b_{i,1} \cdots b_{i,m_i}}\end{aligned}$$

$$\begin{aligned}\ell_{a_1 \cdots a_m x_1 \cdots x_t} = & \\ & -\sum_{j=1}^{m-1} a_1 \cdots a_j \cdot \ell_{a_{j+1} \cdots a_m x_1 \cdots x_t} \\ & -\sum_i k_i \sum_{j=1}^{m_i - 1} b_{i,1} \cdots b_{i,j-1} \cdot \ell_{b_{i,j} \cdots b_{i,m_i} x_1 \cdots x_t}\end{aligned}$$

Chapter 4

Enveloping Algebra Resolution

Now we will give an application of the previous decompositions in describing a projective resolution of Λ as a right module over its enveloping algebra $\Lambda^e = \Lambda^{op} \otimes \Lambda$. Recall that we have $Top(\Omega^1_{\Lambda^e}(\Lambda))$ generated as a left Λ-module by $\{\ell_{ap} = a \otimes p - o(a) \otimes ap : ap \in \mathcal{N}_{\geq 1}\}$ for some fixed minimal Gröbner basis for I. Our object will be to describe a Λ^e projective resolution of Λ by repeatedly tensoring (over Λ) a canonical short exact sequence with $\Omega^1_{\Lambda^e}(\Lambda)$. This description will be in the form of a description of the projective Λ^e-modules as a direct sum of modules of the form $(v \otimes w)\Lambda^e$, with v and w vertices in Γ, and a description of the maps between the projectives in terms of where each $(v \otimes w)$ is to be sent. We will begin with the following proposition:

Proposition 4.1 *Let ${}_\Lambda M_\Lambda$ be a Λ-Λ-bimodule, which is projective as a left Λ-module. Then $M \otimes_\Lambda (\Lambda v \otimes_K w\Lambda)$ is a projective Λ^e-module.*

Proof We note that $M \otimes_\Lambda (\Lambda v \otimes_K w\Lambda)$ is clearly isomorphic to $Mv \otimes_K w\Lambda$. Since M is projective as a left Λ-module, there is ${}_\Lambda N$ so that $M \oplus N$ is free as a left Λ-module. We note that since v is an idempotent, $Mv \oplus M(1-v) \cong M$, whence $Mv \oplus (M(1-v) \oplus N)$ is free, and Mv is still projective as a left Λ-module. Thus $Mv \cong \coprod \Lambda u$ where u is a vertex in Λ and hence $M \otimes_\Lambda (\Lambda v \otimes_K w\Lambda) \cong Mv \otimes_K w\Lambda \cong \coprod(\Lambda u \otimes_K w\Lambda)$, which is clearly a Λ^e projective module.□

We now have the following immediate corollary:

Corollary 4.2 *If P is a finitely generated projective Λ^e-module, and M is a $\Lambda - \Lambda$ bimodule which is left projective, then $M \otimes_\Lambda P$ is a projective Λ^e-module.*□

We now note that we have the following short exact sequence:

$$0 \to \Omega^1_{\Lambda^e}(\Lambda) \to P^0 \to \Lambda \to 0$$

with $P^0 \cong \coprod_{v \in \Gamma_0}(v \otimes v)\Lambda^e$, and the map from P^0 to Λ given by $v \otimes v \mapsto v$. We can tensor this over Λ with $\Omega^1_{\Lambda^e}(\Lambda)$, and obtain the following short exact sequence (exactness follows from the fact that $\Omega^1_{\Lambda^e}(\Lambda)$ is projective in $mod(\Lambda)$)

$$0 \to \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \to \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0 \to \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda \Lambda \to 0$$

which we can tensor again with $\Omega^1_{\Lambda^e}(\Lambda)$ to obtain:

$$0 \to \otimes^3_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \to \otimes^2_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0 \to \otimes^2_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda \Lambda \to 0$$

and continue in this manner to obtain:

$$0 \to \otimes^{n+1}_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \to \otimes^n_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0 \to \otimes^n_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda \Lambda \to 0.$$

Note that in each case we see that the middle module is projective as a Λ^e-module by the previous proposition, and that the modules $\otimes^n_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda \Lambda$ and $\otimes^n_\Lambda \Omega^1_{\Lambda^e}(\Lambda)$ are clearly isomorphic. Thus we can assemble these short exact sequences in the following manner:

$$
\begin{tikzcd}
0 \arrow[r] & \Omega^1_{\Lambda^e}(\Lambda) \arrow[r] & P^0 \arrow[r] & \Lambda \arrow[r] & 0 \\
0 \arrow[r] & \otimes^2_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \arrow[r] & \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0 \arrow[r] & \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda \Lambda \arrow[r] \arrow[ull, bend right] & 0 \\
0 \arrow[r] & \otimes^3_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \arrow[r] & \otimes^2_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0 \arrow[r] & \otimes^2_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda \Lambda \arrow[r] \arrow[ull, bend right] & 0 \\
 & & \cdots & & \\
0 \arrow[r] & \otimes^{n+1}_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \arrow[r] & \otimes^n_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0 \arrow[r] & \otimes^n_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda \Lambda \arrow[r] & 0
\end{tikzcd}
$$

and we see that if we define P^n to be the middle term in the nth sequence, $\otimes^n_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0$, by following the series of the next three maps we have a projective resolution of Λ as a Λ^e-module. We are now ready to describe the projective modules in this resolution as direct sums of indecomposable projective Λ^e-modules $(v \otimes w)\Lambda^e$. We need the following result:

Proposition 4.3

$$\Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda (v \otimes w)\Lambda^e \cong \coprod_{\{p \in \mathcal{N}_{\geq 1} : t(p) = v\}} o(p) \otimes w \quad \Lambda^e.$$

Proof. We know from the previous proposition that $\Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda (v \otimes w)\Lambda^e$ will be a projective Λ^e modlule, so we look for its top. A generic element of the tensor module will be a sum of elements of the form $\lambda_{a1} g_a \lambda_{a2} \otimes_\Lambda \lambda_{a3}(v \otimes w)\lambda_{a4}$. Clearly this will be in the top if and only if $\lambda_{a1} = o(a)$ and $\lambda_{a4} = w$. So we are left with the top of $\Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda (v \otimes w)\Lambda^e$ consisting of elements of the form $g_a\lambda_{a2}\otimes_\Lambda\lambda_{a3}(v\otimes w) = g_a\lambda_{a2}\lambda_{a3}\otimes_\Lambda(v\otimes w) = \ell_{a\lambda_{a2}\lambda_{a3}}\otimes_\Lambda(v\otimes w)$. We recall that if $a\lambda_{a2}\lambda_{a3}$ is not a nontip, we may rewrite it in some form $\ell_{a\lambda_{a2}\lambda_{a3}} = \sum \ell_{bq}$ where each bq is a nontip, and that otherwise there is no interdependence among the ℓ_{ap}s with ap a non-tip (except via right multiplication, which will never happen here), and so we have a description of the top of $\Omega^1_{\Lambda^e}(\Lambda)\otimes_\Lambda (v\otimes w)\Lambda^e$ as one $o(a)\otimes w$ for each minimal ℓ_{ap} with $t(p) = v$. The isomorphism now immediately follows since there are exactly $\mid \{p \in \mathcal{N}_{\geq 1} : t(p) = v\} \mid$ such ℓ_{ap}s.$\square$

As an immediate corollary to this we have that

$$\Omega^1_{\Lambda^e}(\Lambda)\otimes_\Lambda P^0 \cong \coprod_{p\in\mathcal{N}_{\geq 1}} o(p)\otimes t(p) \quad \Lambda^e.$$

and we remark that it is evident from the proof of Proposition 4.3 above that the element $o(p)\otimes t(p)$ of the top of $\Omega^1_{\Lambda^e}(\Lambda)\otimes_\Lambda P^0$ arises from the minimal left generator $\ell_{\alpha_p p^-}$ of $\Omega^1_{\Lambda^e}(\Lambda)$ tensored with the element $t(p)\otimes t(p)$ of $top(P^0)$.

We also see that $\otimes^2_\Lambda\Omega^1_{\Lambda^e}(\Lambda)\otimes_\Lambda P^0 \cong \Omega^1_{\Lambda^e}(\Lambda)\otimes_\Lambda \coprod_{p\in\mathcal{N}_{\geq 1}} o(p)\otimes t(p) \quad \Lambda^e$, which gives us that

$$\otimes^2_\Lambda\Omega^1_{\Lambda^e}(\Lambda)\otimes_\Lambda P^0 \cong \coprod_{\{p_1,p_2\in\mathcal{N}_{\geq 1}: t(p_1)=o(p_2)\}} o(p_1)\otimes t(p_2) \quad \Lambda^e.$$

where the element of the top $o(p_1)\otimes t(p_2)$ corresponds to $\ell_{\alpha_{p_1}p_1^-}\otimes_\Lambda \ell_{\alpha_{p_2}p_2^-}\otimes_\Lambda t(p_2)\otimes t(p_2)$.

We see then that inductively we have the following:

Proposition 4.4 *As a projective Λ^e-module,*

$$\otimes^n_\Lambda\Omega^1_{\Lambda^e}(\Lambda)\otimes_\Lambda P^0 \cong \coprod_{\{p_1,\dots,p_n\in\mathcal{N}_{\geq 1}: t(p_i)=o(p_{i+1})\}} o(p_1)\otimes t(p_n) \quad \Lambda^e.\square$$

where $o(p_1) \otimes t(p_n)$ corresponds to:

$$\ell_{\alpha_{p_1} p_1^-} \otimes_\Lambda \ell_{\alpha_{p_2} p_2^-} \otimes_\Lambda \ldots \otimes_\Lambda \ell_{\alpha_{p_n} p_n^-} \otimes_\Lambda t(p_n) \otimes t(p_n).$$

In order to simplify notation we make the following definition:

$$Seq(n) = \{(p_1, \ldots, p_n) \in (\mathcal{N}_{\geq 1})^n : t(p_i) = o(p_{i+1})\}.$$

We see that this enables us to recast our isomorphisms above to obtain:

$$\otimes_\Lambda^n \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0 \cong \coprod_{(p_1,\ldots,p_n)\in Seq(n)} o(p_1) \otimes t(p_n),$$

where we point out again that $o(p_1) \otimes t(p_n)$ corresponds to:

$$\ell_{\alpha_{p_1} p_1^-} \otimes_\Lambda \ell_{\alpha_{p_2} p_2^-} \otimes_\Lambda \ldots \otimes_\Lambda \ell_{\alpha_{p_n} p_n^-} \otimes_\Lambda t(p_n) \otimes t(p_n).$$

Now we turn our attention to the maps in the resolution. Recall that we have $P^0 \cong \coprod_{\Gamma_0} v \otimes v$ and $P^n \cong \otimes_\Lambda^n \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0$. If one refers back to the diagram of short exact sequences given above it is clear that the maps in the resolution arise from the following commutative diagram:

$$\begin{array}{lcl} \otimes_\Lambda^n \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0 & \xrightarrow{d_n} & \otimes_\Lambda^{n-1} \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0 \\ \downarrow & & \uparrow \\ \otimes_\Lambda^n \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda \Lambda & \rightarrow & \otimes_\Lambda^n \Omega^1_{\Lambda^e}(\Lambda) \end{array}$$

where the map d_n will result from following the left, bottom, and right edges of the square. Thus the map will be the identity on $n-1$ copies of $\Omega^1_{\Lambda^e}(\Lambda)$ and arise from the composition of the maps between P^0 and Λ and the inclusion from $\Omega^1_{\Lambda^e}(\Lambda)$ into P^0 (ignoring the isomorphism on the bottom). We wish to compute this map when we think of P^n as $\coprod_{Seq(n)} o(p_1) \otimes t(p_n)$. In terms of the generators of the top, $(p_1, \dots, p_n)$ in $Seq(n)$ corresponds to $\ell_{\alpha_{p_1} p_1^-} \otimes_\Lambda \dots \ell_{\alpha_{p_n} p_n^-} \otimes_\Lambda t(p_n) \otimes t(p_n)$, so the map down the left hand side of the square takes $o(p_1) \otimes t(p_n)$ to $\ell_{\alpha_{p_1} p_1^-} \otimes_\Lambda \dots \ell_{\alpha_{p_n} p_n^-} \otimes_\Lambda t(p_n)$. The isomorphism on the bottom of the square effectively drops the $t(p_n)$, so we are left with following $\ell_{\alpha_{p_1} p_1^-} \otimes_\Lambda \dots \ell_{\alpha_{p_n} p_n^-}$ up the right hand side of the square. This amounts to viewing $\ell_{\alpha_{p_n} p_n^-} = \alpha_{p_n} \otimes p_n^- - o(p_n) \otimes \alpha_{p_n} p_n^-$ as a sum of two basic tensors in P_0. We note furthermore that in our description of $\otimes_\Lambda^{n-1} \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0$ we pull all paths to the left of the last $\otimes_\Lambda$ leaving only something of the form $v \otimes v$ on the right, so our element $\ell_{\alpha_{p_1} p_1^-} \otimes_\Lambda \dots \ell_{\alpha_{p_n} p_n^-}$ maps to

$$\ell_{\alpha_{p_1} p_1^-} \otimes_\Lambda \dots \ell_{\alpha_{p_{n-1}} p_{n-1}^- \alpha_{p_n}} \otimes_\Lambda t(\alpha_{p_n}) \otimes t(\alpha_{p_n}) \cdot p_n^- - \ell_{\alpha_{p_1} p_1^-} \otimes_\Lambda \dots \ell_{\alpha_{p_{n-1}} p_{n-1}^-} \otimes_\Lambda t(p_{n-1}) \otimes t(p_{n-1}) \cdot p_n.$$

In this case we have that $o(p_1) \otimes t(p_n)$ associated with $(p_1, \dots, p_n) \in Seq(n)$ maps to $o(p_1) \otimes t(\alpha_{p_n}) \cdot p_n^- - o(p_1) \otimes t(p_{n-1}) \cdot p_n$, where the corresponding elements of $Seq(n)$ are $(p_1, \dots, p_{n-2}, p_{n-1}\alpha_{p_n})$ and $(p_1, \dots, p_{n-2}, p_{n-1})$. One problem is the fact that $Seq(n)$ is defined to be sequences of $Nontip(I)$ paths, and it is quite possible that $p_{n-1}\alpha_{p_n}$ is no longer a nontip. In this case we must employ the rewriting rules given at the end of the section on one sided decompositions of $\Omega^1_{\Lambda^e}(\Lambda)$ to rewrite $\ell_{p_{n-1}\alpha_{p_n}}$ as a sum $\sum_i \lambda_i \cdot \ell_{b_i q_i}$ of left multiples of minimal left generators of $\Omega^1_{\Lambda^e}(\Lambda)$, $\ell_{b_i q_i}$ where $b_i q_i \in Nontip(I)$. Following our convention we move each of the λ_i to the left hand side of the tensor $\otimes_\Lambda$, and have now that $d_n(o(p_1) \otimes t(p_n)$ maps to $\sum_i o(p_1) \otimes t(b_i q_i) p_n^- - o(p_1) \otimes t(p_{n-1}) \cdot p_n$, where the idempotent $o(p_1) \otimes t(b_i q_i)$ corresponds to $(p_1, \dots, p_{n-2}\lambda_i, b_i q_i)$ in $Seq(n-1)$. Naturally it may be the case that some of the $p_{n-2}\lambda_i$ are not nontip paths in $\mathcal{N}_{\geq 1}$, and so these will also necessarily be rewritten in computing $d_n o(p_1) \otimes t(p_n)$, and this process will continue until either there need be no more rewritings, or until we have rewritten in the first position $\ell_{p_1\sigma} = \sum \lambda_j \ell_{b_j q_j}$, and the image of $o(p_1) \otimes t(p_n)$ will contain summands of the form $\lambda_j o(b_j q_j) \otimes t(p_{n-1}) p_n^-$, and no more rewritings are possible.

The above discussion is recorded in the following theorem:

Theorem 4.5 *A projective resolution of* Λ *as a right* Λ^e*-module is given by*

$$\cdots P^n \xrightarrow{d_n} P^{n-1} \xrightarrow{d_{n-1}} \cdots \to P^1 \xrightarrow{d_1} P^0 \to \Lambda \to 0$$

where

$$P^0 \cong \coprod_{v \in \Gamma_0} v \otimes v \ \Lambda^e \qquad P^n \cong \coprod_{(p_1, \ldots, p_n) \in Seq(n)} o(p_1) \otimes t(p_n) \ \Lambda^e$$

for $n > 0$, $v \otimes v \overset{d_0}{\mapsto} v$ *and for* $n > 0$, d_n *is given by:*

$$\underset{(p_1, \ldots, p_n)}{o(p_1) \otimes t(p_n)} \mapsto \underset{(p_1, \ldots, p_{n-1}\alpha_{p_n})}{o(p_1) \otimes t(p_{n-1}\alpha_{p_n}) \cdot p_n^-} \quad \underset{(p_1, \ldots, p_{n-1})}{-o(p_1) \otimes t(p_{n-1}) \cdot p_n}$$

unless $p_{n-1}\alpha_{p_n} \notin Nontip(I)$*, in which case we rewrite:*

$$\ell_{\alpha_{p_{n-1}} p_{n-1}^- \alpha_{p_n}} = \sum_{i=1}^{m_{n-1}} \lambda_{n-1,i} \ell_{a_i q_i}^{n-1}$$

to obtain:

$$\underset{(p_1, \ldots, p_n)}{o(p_1) \otimes t(p_n)} \mapsto \underset{(p_1, \ldots, p_{n-2}\lambda_{n-1,i}, a_i q_i)}{\textstyle\sum_{i=1}^{m} o(p_1) \otimes t(q_i) \cdot p_n^-} \quad \underset{(p_1, \ldots, p_{n-1})}{-o(p_1) \otimes t(p_{n-1}) \cdot p_n}$$

unless $p_{n-2}\lambda_{n-1,i} \notin Nontip(I)$*, in which case we rewrite again:*

$$\ell_{p_{n-2}\lambda_{n-1,i}} = \sum_{j=1}^{m_{n-2}} \lambda_{n-2,j} \ell_{a_j q_j}^{n-2}$$

to obtain:

$$
\begin{array}{cccc}
o(p_1)\otimes t(p_n) & \mapsto & \sum_{i,j} o(p_1)\otimes t(q_i)\cdot p_n^- & -o(p_1)\otimes t(p_{n-1})\cdot p_n \\
(p_1,\dots,p_n) & & (p_1,\dots,p_{n-3}\lambda_{n-2,j},a_jq_j,a_iq_i) & (p_1,\dots,p_{n-1})
\end{array}
$$

and so on, until we need rewrite no further.□

From this it should be clear that if Γ contains no oriented cycles then $k\Gamma/I$ is of finite global dimension, since at some point one runs out of possible sequences of paths of lengths at least one, and hence the Λ^e resolution of Λ stops. (This is a well known fact). It should be pointed out that it is also well known that the converse is not true, that is, there are algebras of finite global dimension whose quiver does contain an oriented cycle. An example is given by any algebra Λ whose quiver is an oriented cycle of length n and whose one relation for the ideal I is a path of length n. Such an algebra is clearly monomial, and results of [10] guarantee finite global dimension.

Chapter 5

Resolutions of Simple Modules and $Ext^n_\Lambda(S_v, S_w)$

As a first application of this bimodule resolution, we will compute projective resolutions of simple Λ-modules S_v and use them to obtain information about $Ext^n_\Lambda(S_v, S_w)$. We remark that by using one point extension quivers, the ability to construct projective resolutions of simple modules makes it possible to compute projective resolutions of an arbitrary Λ-module M (see [9]).

The resolution of the simple Λ-module S_v associated with the vertex v in Γ_0 will be obtained by by tensoring S_v over Λ with the above bimodule resolution. We note that the tensor $S_v \otimes_\Lambda \lambda_1 w_1 \otimes w_2 \lambda_2$ will be zero unless $\lambda_1 = w_1 = v$, and hence the projectives in our resolution of S_v will arise from precisely those $(p_1, \ldots, p_n)$ in $Seq(n)$ such that $o(p_1) = v$. We define $Seq(n, v)$ to be the subset of $Seq(n)$ such that $o(p_1) = v$ and arrive at the conclusion that the projective resolution will be given by the following:

Proposition 5.1 *The projective resolution of S_v obtained by tensoring S_v over Λ with our bimodule projective resolution of Λ is given by:*

$$Q^0 = v\Lambda \quad Q^n = \coprod_{(p_1,\dots,p_n)\in Seq(n,v)} t(p_n)\Lambda$$

with the maps given as before, d_0 being the usual projective cover of S_v and $d_n : Q^n \to Q^{n-1}$ given by

$$\begin{array}{ccccc} t(p_n) & \mapsto & t(p_{n-1}\alpha_{p_n}) \cdot p_n^- & - & t(p_{n-1}) \cdot p_n \\ (p_1, \dots, p_n) & & (p_1, \dots, p_{n-1}\alpha_{p_n}) & & (p_1, \dots, p_{n-1}) \end{array}$$

unless $p_{n-1}\alpha_{p_n}$ is not a nontip, in which case we rewrite

$$\ell_{\alpha_{p_{n-1}} p_{n-1}^- \alpha_{p_n}} = \sum_{i=1}^{m} \lambda_i \ell_{a_i p_i}$$

and obtain:

$$\begin{array}{ccccc} t(p_n) & \mapsto & \sum_{i=1}^{m} t(a_i p_i) \cdot p_n^- & - & t(p_{n-1}) \cdot p_n \\ (p_1, \dots, p_n) & & (p_1, \dots, p_{n-2}\lambda_i, a_i p_i) & & (p_1, \dots, p_{n-1}) \end{array}$$

unless $p_{n-2}\lambda_i \notin Nontip(I)$ and we rewrite another step back, continuing as far as necessary to obtain an element of $Seq(n-1, v)$.$\square$

We will now make the convention that when we are referring to the element $t(p_n)$ of $Top(Q^n)$ associated with $(p_1, \dots, p_n)$ in $Seq(n, v)$ we will simply write $(p_1, \dots, p_n)$. We now wish to consider $Ext^n_\Lambda(S_v, S_w)$, and so we will need to apply the functor $Hom_\Lambda(\_, S_w)$ to the above projective resolution. We make the following observations. First, if $u\Lambda$ is an indecomposable projective Λ-module, $Hom_\Lambda(u\Lambda, S_w) = 0$ unless $u = w$, in which case $Hom_\Lambda(u\Lambda, S_w) \cong K$, that is $w \mapsto ke_w$, where e_w is the basis element of the one-dimensional vector space S_w. Second, since we are dealing with finite sums, $Hom(Q^n, S_w) \cong \coprod_{Seq(n,v)} Hom(t(p_n), S_w)$. We say a sequence $(p_1, \dots, p_n) \in Seq(n, v)$ *fits* vertex w if $t(p_n) =$

w, and we define $Fit(n, v, w)$ to be the subset of $Seq(n, v)$ consisting of all sequences that fit vertex w. Using this notation we obtain the following:

Proposition 5.2 *If Q^n is the nth projective in the above projective resolution of the simple S_v,*

$$Hom(Q^n, S_w) \cong \coprod_{Fit(n,v,w)} K \square$$

We will again identify the basis element $(0, 0, \dots, 0, 1, 0, \dots, 0)$ of $Hom(Q^n, S_v)$ with the 1 in the position corresponding to $(p_1, \dots, p_n)$ in $Fit(n, v, w)$ with the element $(p_1, \dots, p_n)$ itself. Hence we are now identifying $Seq(n, v)$ with $Top(Q^n)$, and $Fit(n, v, w)$ with a canonical basis for $Hom(Q^n, S_w)$.

Of course we now know a projective resolution of $\Lambda/\mathbf{r} \cong \coprod_{v \in \Gamma_0} S_v$, we merely take the direct sum of all the resolutions of the simples. Here we have that the tops of the projectives have basis $Seq(n)$, and if we apply $Hom(\_, \Lambda/\mathbf{r})$ to this resolution we have a basis for the Hom set also equal to $Seq(n)$. In what follows we will be considering $Ext^n(S_v, S_w)$. One can either pretend that we are computing this directly by resolving S_v and applying $Hom(\_, S_w)$ or that we have really resolved $\Lambda/\mathbf{r}$ and are applying $Hom(\_, S_w)$ or $Hom(\_, \Lambda/\mathbf{r})$ and picking out the appropriate elements. That is, we can consider $Ext^n(S_v, S_w)$ directly or we may consider $Ext^n(\Lambda/\mathbf{r}, \Lambda/\mathbf{r})$, of which $Ext^n(S_v, S_w)$ is a direct summand.

For the moment we will assume that we have resolved S_v and will be computing information about $Ext^n(S_v, S_w)$ directly. We note that by applying the Hom functor to our projective resolution, we obtain the following complex:

$$\dots Hom(Q^{n+1}, S_w) \overset{d^*_{n+1}}{\leftarrow} Hom(Q^n, S_w) \overset{d^*_n}{\leftarrow} Hom(Q^{n-1}, S_w) \leftarrow \dots$$

and if we wish to compute $Ext^n_\Lambda(S_v, S_w)$ we must take the homology of this complex, computing both $Ker(d^*_{n+1})$ and $Im(d^*_n)$. To do this we must figure out what the maps d^*_i do to elements of $Fit(n, v, w)$. Recall that if $f \in Hom(Q^{n-1}, S_w)$, then $d^*_n(f) = f \circ d_n$. So we take an element of $Hom(Q^{n-1}, S_w)$, $(p_1, \ldots, p_{n-1})$, and apply it, (it is a map now, taking $t(p_{n-1})$ to $1 \cdot e_w$ and all other elements of $\mathrm{Top}(Q^{n-1})$ to zero), to $d_n((q_1, \ldots, q_n))$ for $(q_1, \ldots, q_n)$ in $Seq(n, v)$. Hence if we really want to understand d^*_n we will need a thorough understanding of how d_n acts on elements of $Seq(n, v)$. We begin with the observation that if $(q_1, \ldots, q_n) \in Seq(n, v)$ with $len(q_n) \geq 2$, then q^-_n is a path of positive length, and hence d_n takes it to $(q_1, \ldots, q_{n-1}\alpha_{q_n}) \cdot q^-_n - (q_1, \ldots, q_{n-1}) \cdot q_n$, both terms of which will result in zero when mapped into S_w, as any element of S_w times a non-zero length path will be zero. Thus we may restrict ourselves to considering those elements of $Seq(n, v)$ such that the length of the last path is one. We also note that if $(q_1, \ldots, q_n)$ is an element of $Seq(n, v)$ but $t(q_n) \neq w$ then no matter what $d_n(q_1, \ldots, q_n)$ is in Q^{n-1}, all elements of $Fit(n-1, v, w)$ will take it to zero, since $t(d_n(q_1, \ldots, q_n)) = t(q_n)$. So we really need to analyze the behavior of d_n only on elements of $Fit(n, v, w) \subset Seq(n, v)$ with $len(q_n) = 1$, as $d^*_n(Fit(n-1, v))$ will be contained in the subspace of $Hom(Q^n, S_w)$ spanned by these elements. Finally we note that if $(q_1, \ldots, q_n) \in Fit(n, v, w)$ with $len(q_n) = 1$ we have $d_n((q_1, \ldots, q_n)) = (q_1, \ldots, q_{n-1}q_n) - (q_1, \ldots, q_{n-1}) \cdot q_n$. When we apply any element of $Hom(Q^{n-1}, S_w)$ we see that $(q_1, \ldots, q_{n-1}) \cdot q_n$ is mapped to zero. Therefore we really need only consider the first summand of the image of any element of $Fit(n, v, w)$, since the second is always maped to zero by any element of $Hom(Q^{n-1}, S_w)$. We define the map $\tilde{d}_n : Q^n \to Q^{n-1}$ to pick out only this first summand (and any elements of Q^{n-1} which arise from rewriting it), and note that it is enough to consider $\tilde{d}_n$ when computing $Ext(S_v, S_w)$.

Suppose that we list all basis elements of $Fit(n, v, w)$ with $len(q_n) = 1$, calling them $x_1, x_2, \ldots, x_s$. We list all basis elements of $Fit(n-1, v, w)$, regardless of the length of the last path, calling them $y_1, y_2, \ldots, y_t$. We will construct a matrix for $\tilde{d}_n$. So we apply $\tilde{d}_n$ to each of the x_j, noting that the result will now be merely a K-linear combination of the y_is, $\tilde{d}_n(x_j) = \sum_{i=1}^{t} k_{ij} y_i$, and obtain an s by t matrix D_n, where the ij entry is k_{ij}. So the image of x_j under the abbreviated chain map $\tilde{d}_n$ will be recorded in the jth column of D_n. Recall that $\{y_1, \ldots, y_t\}$ is a K-basis for $Hom(Q^{n-1}, S_w)$, and if we wish to compute $\tilde{d}_n^*(y_i)(x_j)$ we compute $y_i \circ \tilde{d}_n(x_j)$. But this is $y_i(\sum_{\ell=1}^{t} k_{\ell j} y_\ell) = k_{ij} e_w$, where e_w again is the basis element of S_w. We note that if we now consider x_j to be the basis element of $Hom(Q^n, S_w)$ taking the element $t(x_j)$ of Q^n to e_w and all other elements of Q^n not K-multiples of this element to zero, we have the image of y_i under $\tilde{d}_n^*$ recorded in the ith row of D_n in terms of the partial basis $\{x_1, \ldots, x_s\}$ for $Hom(Q^n, S_w)$. We note now that if we row reduce the matrix D_n, we will obtain, via the non-zero rows, a basis for $Im(\tilde{d}_n^*)$, (and the zero rows will correspond to a basis for $Ker(\tilde{d}_n^*)$). It will be in precisely this manner that we will obtain information about $Im(\tilde{d}_n^*)$ in $Ext^n(S_v, S_w) = Ker(\tilde{d}_{n+1}^*)/Im(\tilde{d}_n^*)$.

Before we begin though, we need to know something about $Ker(\tilde{d}_{n+1}^*)$, and the manner in which we obtain this information will not be in the form of the matrix D_{n+1} but rather in terms of "liftings" of elements of $Hom(Q^n, S_w)$ to $Hom(Q^{n+1}, S_w)$. We make this notion more precise with the following definition.

Definition. We say a basis element y_i in $Fit(n, v, w)$ *lifts* to a basis element $(p_1, \ldots, p_n, p_{n+1})$ of $Fit(n+1, v, w)$ if $\tilde{d}_{n+1}^*(y_i)(p_1, \ldots, p_n) \neq 0$ in S_w. Clearly this is equivalent to y_i occurring as a term of $\tilde{d}_{n+1}((p_1, \ldots, p_{n+1}))$.

As we have already noted, if $y_i = (p_1, \ldots, p_n)$ with $len(p_n) \geq 2$, we write $p_n = p_n^+ \cdot \beta_{p_n}$ where $len(\beta_{p_n}) = 1$ and we see that $(p_1, \ldots, p_n)$ will lift to $(p_1, \ldots, p_{n-1}, p_n^+, \beta_{p_n})$ in $Fit(n+1, v, w)$. We are guaranteed here that $len(p_n^+) \geq 1$, so this is indeed an element of $Fit(n+1, v, w)$. This lifting is easily seen by considering $\tilde{d}_{n+1}((p_1, \ldots, p_{n-1}, p_n^+, \beta_{p_n})) = (p_1, \ldots, p_{n-1}, p_n^+ \cdot \beta_{p_n}) = (p_1, \ldots, p_{n-1}, p_n)$, and hence $\tilde{d}_{n+1}^*(y_i)$ will act in a non-zero way on $(p_1, \ldots, p_{n-1}, p_n^+, \beta_{p_n})$.

(This of course implies that $d^*_{n+1}(y_i)$ will act in a non-zero manner as well.) Of course we see that if $(p_1, \ldots, p_n) \in Fit(n, v, w)$ with $len(p_n) = 1$ we will not be able to lift in this manner. However, there is another way in which elements of $Fit(n, v, w)$ may lift. Let ρ be a relation in a Gröbner basis for I, with $tip(\rho) = a_1 \cdots a_m$, and $\rho = tip(\rho) + \sum_{i=1}^{r} k_i \sum_{j=1}^{m_i} b_{i,1} \cdots b_{i,m_i}$, where the a_i and $b_{i,j}$ are arrows in Γ. We consider the action of $\tilde{d}_{n+1}$ on an element of the form $(p_1, \ldots, p_{n-1}, a_1 \cdots a_{m-1}, a_m)$ in $Fit(n+1, v, w)$. $\tilde{d}_{n+1}$ takes this element to $(p_1, \ldots, p_{n-1}, tip(\rho))$, which of course must be rewritten. The reader is asked to recall the rewriting formulas for the ℓ_{ap} elements for which $ap \notin Nontip(I)$ given at the end of the chapter on left and right syzygy decompositions. From these rules we see that our rewriting takes the form:

$$
\begin{aligned}
(p_1, \ldots, p_{n-1}, a_1 \cdots a_m) = \\
&- \sum_{j=1}^{m-1} (p_1, \ldots, p_{n-1} a_1 \cdots a_j, a_{j+1} \cdots a_m) \\
&- \sum_{i=1}^{r} k_i \sum_{j=0}^{m_i - 1} (p_1, \ldots, p_{n-1} b_{i,1} \cdots b_{i,j}, b_{i,j+1} \cdots b_{i,m_i}).
\end{aligned}
$$

Of course it is entirely possible that one or more of the terms now in the $n-1$ position is no longer in $Nontip(I)$, and will then necessarily be rewritten again. The point of this example is that we are able to obtain a lifting of an element $(p_1, \ldots, p_n)$ of $Fit(n, v)$ when $p_{n-1}p_n$ contains a term of ρ as a proper suffix. It is also quite possible that if $p_{n-1}a_1 \cdots a_j$ or $p_{n-1}b_{i,1} \cdots b_{i,j}$ were not in $Nontip(I)$ and the element in the $n-1$ position was rewritten, that $(p_1, \ldots, p_n)$ might occur as a term of that rewriting, with there necessarily now being a pair of relations overlapping.

In order to get a complete understanding of the possible liftings we will require a consideration of the rewritings which can arise under the map $\tilde{d}_{n+1}$. In the example above, the path $p_{n-1}p_n$ contained a term of ρ as a proper suffix. We now consider the case that there are two relations $\rho_1 = a_1 \cdots a_m + \sum_{i=1}^{r_1} k_i b_{i,1} \cdots b_{i,m_i}$ and $\rho_2\ c_1 \cdots c_{m'} + \sum_{i=1}^{r_2} k_i' d_{i,1} \cdots d_{i,m_i'}$, such that $tip(\rho_2) = c_1 \cdots c_{m'}$ overlaps with a term of ρ_1, that is either we have:

$$\begin{array}{llll} c_1 \cdots c_j & c_{j+1} \cdots c_{m'} & & \\ & a_1 \cdots a_{m'-j} & a_{m'-j+1} \cdots a_m & \end{array}$$

or

$$\begin{array}{llll} c_1 \cdots c_j & c_{j+1} \cdots c_{m'} & & \\ & b_{i,1} \cdots b_{i,m'-j} & b_{i,m'-j+1} \cdots b_{i,m_i}. & \end{array}$$

If we now consider the action of $\tilde{d}_{n+1}$ on $(p_1, \ldots, p_{n-2}, c_1 \cdots c_j, a_1 \cdots a_{m-1}, a_m)$ we have this element being mapped to $(p_1, \ldots, p_{n-2}, c_1 \cdots c_j, tip(\rho_1))$, which we rewrite as follows:

$$-\sum_{i=1}^{m-1}(p_1, \ldots, p_{n-2}, c_1 \cdots c_j a_1 \cdots a_i, a_{i+1} \cdots a_m)$$

$$-\sum_{i=1}^{r_1} -k_i \sum_{h=0}^{m_i-1}(p_1, \ldots, p_{n-2}, c_1 cdots c_j b_{i,1} \cdots b_{i,h}, b_{i,h+1} \cdots b_{i,m_i})$$

and in the case that the overlapping relations have overlapping tips (the first case in the diagram), we will have a term $(p_1, \ldots, p_{n-2}, tip(\rho_2), a_{m'-j+1} \cdots a_m)$, or in the case that $tip(\rho_2)$ overlapps with a nontip summand of ρ_1 (the second case in the diagram), we will have a term $(p_1, \ldots, p_{n-2}, tip(\rho_2), b_{i,m'-j+1} \cdots b_{i,m_i})$. In both of these cases we rewrite and end up splitting terms of ρ_2 between the $n-2$ and $n-1$ position.

We see then that when we are dealing with rewriting twice (that is, there are two overlapping relations), the path in the p_n position must be the part of ρ_1 which 'hangs off' of the end of $tip(\rho_2)$. Also the path $p_{n-2}p_{n-1}$ must contain a term of ρ_2 as a proper suffix.

To consider one further case before stating the general result, we suppose that there are three relations with tips overlapping, $\rho_1 = a_1 a_2 \ldots a_m + \sum_i k_i b_{i,1} \ldots b_{i,m_i}$, $\rho_2 = c_1 c_2 \ldots c_m' + \sum_i k_i' d_{i,1} \ldots d_{i,m_i'}$, and $\rho_3 = e_1 e_2 \ldots e_m'' + \sum_i k_i'' f_{i,1} \ldots f_{i,m_i''}$. There are several possibilities for the way in which the tips of these relations overlap with the other relations, we will list each of these ways and then list the liftings possible under such configurations, all of which may be checked by routine calculation:

To begin with we could have the following configuration:

$$\begin{array}{lll} a_1 \cdots a_j & a_{j+1} \cdots a_m & \\ & c_1 \cdots c_{m-j} & c_{m-j+1} \cdots c_m \end{array}$$

and

$$\begin{array}{lll} c_1 \cdots c_{j'} & c_{j'+1} \cdots c_{m'} & \\ & e_1 \cdots e_{m'-j'} & e_{m'-j'+1} \cdots e_{m''}. \end{array}$$

This is the case where the tip of the middle relation overlaps with the tip of the other two. In this case we have the following:

$$(p_1, \ldots, p_{n-4}, Xa_1 \cdots a_s, a_{s+1} \cdots a_m, c_{m-j+1} \cdots c_m, e_{m'-j+1} \cdots e_{m''})$$

and

$$(p_1, \ldots, p_{n-4}, Xb_{i,1} \cdots b_{i,s}, b_{i,s+1} \cdots b_{i,m_i}, c_{m-j+1} \cdots c_m, e_{m'-j+1} \cdots e_{m''})$$

lifting to $(p_1, \ldots, p_{n-4}, X, a_1 \cdots a_j, c_1 \cdots c_{j'}, e_1 \cdots e_{m''-1}, e_{m''})$.

We might have the tip of the first relation overlapping with a nontip summand of the second, and the tip of the second overlapping with the tip of the third:

$$\begin{array}{lll} a_1 \cdots a_j & a_{j+1} \cdots a_m & \\ & d_{i,1} \cdots d_{i,m-j} & d_{i,m-j+1} \cdots d_{i,m'_i} \end{array}$$

and

$$\begin{array}{lll} c_1 \cdots c_{j'} & c_{j'+1} \cdots c_{m'} & \\ & e_1 \cdots e_{m'-j'} & e_{m'-j'+1} \cdots e_{m''}. \end{array}$$

Here we have the following liftings:

$$(p_1, \ldots, p_{n-4}, Xa_1 \cdots a_s, a_{s+1} \cdots a_m, d_{i,m-j+1} \cdots d_{i,m_i'}, e_{m'-j'+1} \cdots e_{m''})$$

and

$$(p_1, \ldots, p_{n-4}, Xb_{i,1} \cdots b_{i,s}, b_{i,s+1} \cdots b_{i,m_i}, d_{i,m-j+1} \cdots d_{i,m_i'}, e_{m'-j'+1} \cdots e_{m''})$$

lifting to $(p_1, \ldots, p_{n-4}, X, a_1 \cdots a_j, c_1 \cdots c_{j'}, e_1 \cdots e_{m''-1}, e_{m''})$.

A third possible configuration of tip overlaps is when the tip of ρ_1 overlaps with $tip(\rho_2)$, and $tip(\rho_2)$ overlaps with a nontip summand of ρ_3:

$$\begin{array}{llll} a_1 \cdots a_j & a_{j+1} \cdots a_m & & \\ & c_1 \cdots c_{m-j} & & c_{m-j+1} \cdots c_{m'} \end{array}$$

and

$$\begin{array}{lll} c_1 \cdots c_{j'} & c_{j'+1} \cdots c_{m'} & \\ & f_{i,1} \cdots f_{i,m'-j'} & f_{i,m'-j'+1} \cdots f_{i,m_i''}. \end{array}$$

In this case we have the following elements:

$$(p_1, \ldots, p_{n-4}, Xa_1 \cdots a_s, a_{s+1} \cdots a_m, c_{m-j+1} \cdots c_{m'}, f_{i,m'-j'+1} \cdots f_{i,m_i''})$$

and

$$(p_1, \ldots, p_{n-4}, Xb_{i,1} \cdots b_{i,s}, b_{i,s+1} \cdots b_{i,m_i}, c_{m-j+1} \cdots c_{m'}, f_{i,m'-j'+1} \cdots f_{i,m_i''})$$

lifting to $(p_1, \cdots, p_{n-4}, X, a_1 \cdots a_j, c_1 \cdots c_{j'}, e_1 \cdots e_{m''-1}, e_{m''})$.

Finally we could have $tip(\rho_1)$ overlapping with a nontip summand of ρ_2, and $tip(\rho_2)$ overlapping with a nontip summand of ρ_3:

$$\begin{array}{llll} a_1 \cdots a_j & a_{j+1} \cdots a_m & & \\ & d_{i,1} \cdots d_{i,m-j} & d_{i,m-j+1} \cdots d_{i,m'_i} & \end{array}$$

and

$$\begin{array}{llll} c_1 \cdots c_{j'} & c_{j'+1} \cdots c_{m'} & & \\ & f_{i,1} \cdots f_{i,m'-j'} & f_{i,m'-j'+1} \cdots f_{i,m''_i}. & \end{array}$$

In this last case we have the following liftings:

$$(p_1, \cdots, p_{n-4}, Xa_1 \cdots a_s, a_{s+1} \cdots a_m, d_{i,m-j+1} \cdots d_{i,m'_i}, f_{i,m'-j+1} \cdots f_{i,m''_i})$$

and

$$(p_1, \cdots, p_{n-4}, Xb_{i,1} \cdots b_{i,s}, b_{i,s+1} \cdots b_{i,m_i}, d_{i,m-j+1} \cdots d_{i,m'_i}, f_{i,m'-j+1} \cdots f_{i,m''_i})$$

lifting to $(p_1, \cdots, p_{n-4}, X, a_1 \cdots a_j, c_1 \cdots c_{j'}, e_1 \cdots e_{m''-1}, e_{m''})$.

After computing all of the previous examples, it should be clear the the following lemmas describe how elements of $Seq(n, v)$ lift using the relations. Before stating them we will need the following definition:

Definition. If $a_1 \cdots a_n$ is the tip of some relation ρ in a reduced Gröbner Basis for I, and if $p = b_1 \cdots b_m$ is a path in Γ, where $tip(\rho)$ and p overlap as follows:

$$\begin{array}{lll} a_1 a_2 \cdots a_j & a_{j+1} \cdots a_n & \\ & b_1 \cdots b_{n-j} & b_{n-j+1} \cdots b_m \end{array}$$

we say the path $b_{n-j+1} \cdots b_m$ is the *tail of the tip overlap of* ρ *and* p, and the path $a_1 \cdots a_j$ is the *head of the tip overlap of* ρ *and* p. We do allow the case that $b_{n-j+1} \cdots b_m$ is merely a vertex, that is, p is a suffix of $tip(\rho)$. In this case we define the tail of the tip overlap to be the vertex $t(\rho)$. Clearly it is also possible for $tip(\rho)$ to overlap with any other (not a suffix) sub-path of itself, however these cases will be of no interest to us and we do not consider them.

We can now state the lemmas describing liftings:

Lemma 5.3 *If there are t relations $\rho_1, \dots \rho_t$ such that $tip(\rho_i)$ overlaps with a term of ρ_{i+1}, and σ_{i+1} is the tail of the tip overlap of ρ_i and the term of ρ_{i+1} with which it overlaps, then*

$$(\dots, p_1, p_2, \sigma_2, \sigma_3, \sigma_4, \dots, \sigma_t)$$

where a term of ρ_1 is a proper suffix of p_1p_2, lifts to

$$(\dots, p, \tau_1, \tau_2, \dots, \tau_{t-1}, tip(\rho_t)^+, \beta_{tip(\rho_t)}),$$

where τ_i is the head of the tip overlap of ρ_i and the term of ρ_{i+1} with which it overlaps, and p is the prefix of the term of ρ_1 in the path p_1p_2. In this lemma we assume all of the overlaps are non-trivial, that is, the tail of each overlap is a path of positive length.

Proof. Merely a computation. Notice that under the abbreviated mapping $\tilde{d}_{n+1}$ we have $(\dots, p_1, \tau_1, \tau_2, \dots, \tau_{t-1}, tip(\rho_t)^+, \beta_{tip(\rho_t)})$ mapping to $(\dots, p_1, \tau_1, \tau_2, \dots, \tau_{t-1}, tip(\rho_t))$, which will rewrite to $(\dots, p_1, \tau_1, \tau_2, \dots, \tau_{t-2}, tip(\rho_{t-1}), \sigma_t)$, which in turn rewrites to
$(\dots, p_1, \tau_1, \tau_2, \dots, \tau_{t-3}, tip(\rho_{t-2}), \sigma_{t-1}, \sigma_t)$, and so on until we obtain:
$(\dots, p_1, tip(\rho_1), \sigma_2, \sigma_3, \dots, \sigma_t)$, which will then rewrite to:
$(\dots, p_1, tip(\rho_1), \sigma_2, \sigma_3, \dots, \sigma_t)$ which will leave us with a term of ρ_1 being split between the positions now occupied by p_1 and $tip(\rho_1)$, or the special case of a nontip term of ρ_1 remaining entirely in the position now ocupied by $tip(\rho_1)$. In either case we now have rewritten to an element of the form $(\dots, p_1, p_2, \sigma_2, \sigma_3, \dots, \sigma_t)$ where a term of ρ_1 is a proper suffix of p_1p_2.$\square$

It is not clear from the above lemma what happens if $tip(\rho_i)$ contains a nontip term of ρ_{i+1} as a (necessarily proper) suffix. In this case σ_i will be a vertex, and hence $(\ldots, \sigma_i, \ldots)$ will not be an element of $Seq(n)$ since we required that each path have positive length. In essence, σ_i does not appear in the element to be lifted, and one concatenates τ_i and τ_{i+1} in the element to which is is lifted, as the next lemma describes:

Lemma 5.4 *Again we consider t relations $\rho_1, \ldots, \rho_t$ such that $tip(\rho_i)$ overlaps with a term of ρ_{i+1}. We again denote the tail of the tip overlap of ρ_i and the term of ρ_{i+1} by σ_{i+1}, and we denote the head of the tip overlap of ρ_i and the term of ρ_{i+1} by τ_i. We assume here that for some j, $1 < j < t$ we have $len(\sigma_j) = 0$. Again, if $p_1 p_2$ contains a term of ρ_1 as a proper tail then:*

$$(\ldots, p_1, p_2, \sigma_2, \ldots, \sigma_{j-1}, \sigma_{j+1}, \ldots, \sigma_t)$$

lifts to:

$$(\ldots, p, \tau_1, \tau_2, \ldots, \tau_{j-1}\tau_j, \tau_{j+1}, \ldots, \tau_{t-1}, tip(\rho_t)^+, \beta_{tip(\rho_t)}).$$

Proof. Again just a computation. Applying the map $\tilde{d}_{n+1}$ to the lifting above we obtain:

$$(\ldots, p, \tau_1, \tau_2, \ldots, \tau_{j-1}\tau_j, \tau_{j+1}, \ldots, \tau_{t-1}, tip(\rho_t))$$

which rewrites to:

$$(\ldots, p, \tau_1, \tau_2, \ldots, \tau_{j-1}\tau_j, \tau_{j+1}, \ldots, tip(\rho_{t-1}), \sigma_t))$$

and we continue this process as before until we reach the step:

$$(\dots, p, \tau_1, \tau_2, \dots, \tau_{j-1}tip(\rho_j), \sigma_{j+1}, \dots, \sigma_{t-1}), \sigma_t)).$$

At this point we are rewriting the element $\ell_{\tau_{j-1}tip(\rho_j)}$ of $\Omega^1_{\Lambda^e}(\Lambda)$, and rewritings take the form $\sum_i k_i \ell_{\tau_{j-1}n_i}$ where n_i are nontip terms of ρ_j. One of these nontip terms will complete $tip(\rho_{j-1})$ when concatenated with τ_{j-1}, so we now have:

$$(\dots, p, \tau_1, \tau_2, \dots, tip(\rho_{j-1}), \sigma_{j+1}, \dots, \sigma_{t-1}), \sigma_t))$$

and the rewriting continues in the same manner as in the previous lemma.□

It should be a straightforward extension of the above lemma to describe lifitings where there are two or more σ_i with length equal to zero, and where one of these length zero tails occurs in the tth position. So far we have described certain elements of $Seq(n)$ and how they lift to $Seq(n+1)$, but we do not know that we have a complete description of all the possible liftings. The next lemma will guarantee this.

Lemma 5.5 *If $(p_1, \dots, p_n) \in Fit(n, v, w)$ with $len(p_n) \geq 2$, then $(p_1, \dots, p_n)$ lifts to the element $(p_1, \dots, p_{n-1}, p_n^+, \beta_{p_n})$ in $Fit(n+1, v, w)$. Furthermore, aside from this case, liftings like those given in the above lemmas (with t overlapping relations, $p_n = \sigma_t, p_{n-1} = \sigma_{t-1}, \dots p_{n-t+2} = \sigma_2$, and $p_{n-t}p_{n-t+1}$ has a proper suffix a term of ρ_1), are the only possible liftings , if we also include the exceptional cases that $t \geq n$ and we lift the element $(p_2, \sigma_2, \dots, \sigma_t)$ where p_2 is a nontip term of ρ_1 to $(\tau_1, \tau_2, \dots, \tau_{t-1}, tip(\rho_t)^+, \beta_{tip(\rho_t)})$ and the cases where $len(\sigma_i) = 0$ and we adjust the $Fit(n, v, w)$ elements and the liftings accordingly.*

Proof. The first lifting is obvious. All of the rewritings in the proof of the above lemmas do give possible liftings, and the exceptional case when $t \geq n$ which is mentioned in the statement of this lemma and should be an obvious extension of the proof of the preceding lemmas. It now remains to show that these are the only possible liftings. So we consider an element $(p_1, \ldots, p_n)$ of $Fit(n, v, w)$. Let $(q_1, \ldots, q_{n+1}) \in Fit(n+1, v, w)$ with $len(q_{n+1}) = 1$ be a lifting. We will show that $(p_1, \ldots, p_n)$ has the form $p_n = \sigma_t, p_{n-1} = \sigma_{t-1}$, etc., and that $(q_1, \ldots q_{n+1})$ has the form $q_{n+1} = \beta_{\rho_t}, q_n = \rho_t^+, q_{n-1} = \tau_{t-1}$, etc. We begin with $(q_1, \ldots, q_{n+1})$. $\tilde{d}_{n+1}$ maps this to $(q_1, \ldots, q_{n-1}, q_n q_{n+1})$. If $len(p_n) \geq 2$ then it is possible that $q_n q_{n+1} = p_n$, and we must have $p_i = q_i$ for i ranging from 1 to $n-1$. This is the obvious lifting of an element of $Fit(n, v)$ with $len(p_n) \geq 2$. If it is not the case that $q_n q_{n+1} = p_n$, then we must have $q_n q_{n+1}$ not an element of $\mathcal{N}_{\geq 1}$, and we must rewrite. Since $q_n \in \mathcal{N}_{\geq 1}$ we conclude that by adding the last arrow q_{n+1} we have introduced a tip. There are two possibilities as to how this might happen. We have that $q_n q_{n+1} = X \cdot tip(\rho)$, and the two cases are that $len(X) = 0$ or $len(X) \geq 1$. If $len(X) \geq 1$ we rewrite

$$\ell_{Xtip(\rho)} = \sum_i k_i \ell_{Xm_i}$$

where $\rho = tip(\rho) - \sum_i k_i m_i$, and we now have

$$(q_1, \ldots, q_{n-1}, Xm_i)$$

or if $len(X) = 0$ we rewrite to obtain

$$(q_1, \ldots, q_{n-1}a, b)$$

and we now compare to see if Xm_i or b is equal to p_n and q_{n-1} or $q_{n-1}a$ is equal to p_{n-1}, in which case we have a lifting. Otherwise, we need another rewriting. In the first case we have σ_t equal to a vertex, and continue rewriting in the nth position, or in the second case we necessarily have $q_{n-1}a$ not in $\mathcal{N}_{\geq 1}$, and so b, which must be p_n, is a tail of a tip overlap. The rewritings, if necessary, continue in this manner, but we point out that each rewriting requires a tip overlap and leaves either $(p_1, \ldots, p_n)$, or a tail of a tip overlap and another rewriting. The result of the lemma follows from these observations.$\square$

At this point what we really have described is all basis elements of $Hom(Q^n, S_w)$, ie. elements of $Fit(n, v, w)$ which lie in $Ker(d^*_{n+1})$.

Proposition 5.6 $\eta = (p_1, \ldots, p_n) \in Fit(n, v, w)$ *is an element of* $Ker(d^*_{n+1})$ *if and only if* η *is not of the form of the elements above which lift to elements of* $Fit(n+1, v, w)$.

Proof It is clear that if η does not lift, then $\eta \in Ker(d^*_{n+1})$ since for each basis element x of $Top(Q^{n+1})$ we have that η does not occur as a term of $d_{n+1}(x)$ (if it did η would have lifted to x), and so $d^*_{n+1}(\eta)$ is zero on all of $Top(Q^{n+1})$ (and hence all of Q^{n+1}.) The above lemma describes completely the basis elements which lift, and the result now follows.$\square$

In order to make some homological computations we will now make some assumptions about the quiver Γ and the ideal I. Let $\rho_1, \rho_2, \ldots, \rho_{n-1}$ be overlapping monomial relations (not necessarily distinct) in a reduced Gröbner basis G for I. We assume that ρ_i does not overlap with ρ_{i+2}. We denote the head of the tip overlap of ρ_i and ρ_{i+1} by τ_i, and we denote the tail of the tip overlap of ρ_i and ρ_{i+1} by σ_{i+1}. Define the operation $*$ on paths p and q to be

$$p * q = (p \setminus q) \cdot (p \cap q) \cdot (q \setminus p),$$

which in essence removes the overlap, and let P denote the path $\rho_1 * \rho_2 * \rho_3 \ldots * \rho_{n-1}$. We assume that $\rho_i * \rho_{i+1}$ does not contains as a subpath any term of any relation in G except ρ_i and ρ_{i+1}. Among other things this avoids the complications which arise from tip overlaps with tails being vertices. Note that we are not assuming a monomial ideal I, merely monomial relations along P and no nontip terms of other relations being subpaths of $\rho_i * \rho_{i+1}$.

Proposition 5.7 *Under the above assumptions we have that*

$$Ext^n_\Lambda(o(P), t(P)) \neq 0$$

if $len(\sigma_{n-1}) \geq 2$ or if $\sigma_i = \rho_i^-$ for all i, that is, if ρ_{i-1} and ρ_i overlap in only an arrow.

Proof. It will be clear from the previous discussion on liftings that the assumptions guarantee us that the element

$$\eta = (\tau_1, \tau_2, \dots, \tau_{n-2}, \rho^+_{n-1}, \beta_{\rho_{n-1}})$$

is in $Ker(d^*_{n+1})$. To see this we note that $len(\beta_{\rho_{n-1}}) = 1$ and $\rho^+_{n-1} \cdot \beta_{\rho_{n-1}}$ does not contain a term of any relation as a proper tail by assumption. If $\beta_{\rho_{n-1}}$ is indeed the tail of a tip overlap (which must happen for η to lift, then we must have either ρ^+_{n-1} the tail of a tip overlap or $\tau_{n-2}\rho^+_{n-1}$ containing a term of some relation as a proper tail. The second case cannot happen since $\tau_{n-2}\rho^+_{n-1}$ is a prefix of $\rho_{n-2} * \rho_{n-1}$, which by assumption does not contain terms of other relations. So if η were to lift we would need ρ^+_{n-1} to be the tail of a tip overlap as well as $\beta_{\rho_{n-1}}$ being a tail of a tip overlap. At this point the lifting of η will require either $\tau_i\tau_{i+1}$ containing a term of some relation as a proper tail, which our assumptions rule out, or τ_1 being a nontip term in some relation, which our assumptions again rule out.

We now compute $\tilde{d}_n(\eta)$. This takes the form:

$$(\tau_1, \tau_2, \ldots, \tau_{n-2}, \rho_{n-1})$$

which must be rewritten to give elements of the form

$$(\tau_1, \tau_2, \ldots, \tau_{n-2} sub_{n-1}, sub'_{n-1})$$

where $sub_{n-1} sub'_{n-1}$ is equal to ρ_{n-1}. At some point we will rewrite to an element of the form:

$$(\tau_1, \tau_2, \ldots, \rho_{n-2}, \sigma_{n-1})$$

which rewrites to elements of the form

$$(\tau_1, \tau_2, \ldots, \tau_{n-3} sub_{n-2}, sub'_{n-2}, \sigma_{n-1})$$

until we reach

$$(\tau_1, \tau_2, \ldots, \rho_{n-3}, \sigma_{n-2}, \sigma_{n-1})$$

and so on until we are rewriting ρ_1, all rewritings of which go to zero, and we have the final elements of the form

$$(\tau_1 sub_2, sub'_2, \sigma_3, \ldots, \sigma_{n-2}, \sigma_{n-1}).$$

We note that sub'_i is always a longer path than σ_i.

It is evident that if ρ_i and ρ_{i+1} overlap in only one arrow for all i, that is, there are no sub_i in the sense of the above computations of $\tilde{d}_n(\eta)$, we have $\tilde{d}_n(\eta) = 0$, in which case it is clear that η considered as an element of $Hom(Q^n, S_w)$ is not in $Im(d^*_{n-1})$, and hence is a nonzero element of $Ext^n_\Lambda(S_{o(\rho_1)}, S_{t(\rho_{n-1})})$. It is however, as we see from the above computations, too much to expect that $d_n(\eta) = 0$ most of the time. This does not necessarily imply that $\eta \in Im(d^*_n)$. We recall our matrix D_n and note that we have the following:

$$\begin{array}{cc} & \eta \\ x_1 & k_1 \\ x_2 & k_2 \\ \vdots & \\ x_t & k_t \end{array}$$

where k_i is nonzero in k and each x_i is one of the elements

$$(\tau_1, \tau_2, \ldots, \tau_j sub_{j+1}, sub'_{j+1}, \sigma_{j+1}, \ldots \sigma_{n-1}).$$

If $len(\sigma_{n-1}) \geq 2$ each of the x_i lifts in the obvious way to y_i which equals

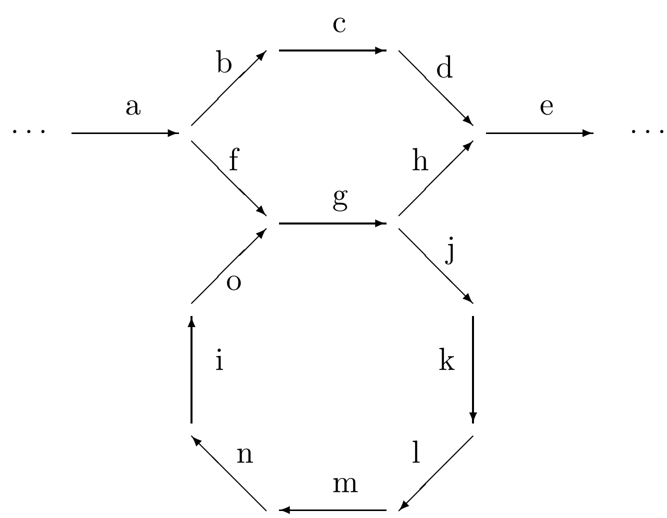
$$(\tau_1, \tau_2, \ldots, \tau_j sub_{j+1}, sub'_{j+1}, \sigma_{j+2}, \ldots \sigma^+_{n-1}, \beta_{\sigma_{n-1}}).$$

We claim that except for this and the lifting of x_i to η, there are no other liftings of x_i. We are assured that $\sigma_i\sigma_{i+1}$ does not contain a term of any relation as a proper suffix since $\sigma_i\sigma_{i+1}$ is a subpath of $\rho_i * \rho_{i+1}$. $sub'_{j+1}\sigma_{j+2}$ is a subpath of $\rho_{j+1} * \rho_{j+2}$, and as such contains no term of any relation as a proper suffix. $\tau_j sub_{j+1} sub'_{j+1}$ contains ρ_j as a proper suffix, which gives us the lifting to η in the first place, but no other terms of any other relations as proper suffixes since $\tau_j sub_{j+1} sub'_{j+1}$ is equal to $\rho_j * \rho_{j+1}$. $\tau_{j-1}\tau_j sub_{j+1}$ is a subpath of $\rho_{j-1} * \rho_j$, and so contains no terms of any other relations as proper suffixes. Also $\tau_i\tau_{i+1}$ is a subpath of $\rho_i * \rho_{i+1}$ and as such contains no terms of any relations as a proper suffix, and it is clear that τ_1 cannot be a term in any relation. This establishes that the x_i elements lift to nothing other than η and y_i (assuming $len(\sigma_{n-1}) \geq 2$). Our matrix D_n then contains a block of the following form:

$$\begin{array}{cccccc} & y_1 & y_2 & \dots & y_t & \eta \\ x_1 & 1 & 0 & \dots & 0 & k_1 \\ x_2 & 0 & 1 & \dots & 0 & k_2 \\ \vdots & & & & & \\ x_t & 0 & 0 & \dots & 1 & k_t \end{array}$$

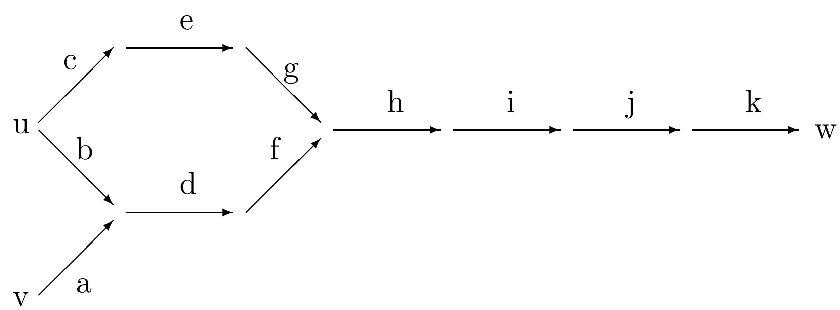
from which we deduce that η is a non-zero element of $Ext^n_\Lambda(o(P), t(P))$, since it canot lie in $Im(d_n^*)$.$\square$

An immediate corollary to this is the infinite global dimension of algebras for which there are overlapping monomial relations $\rho_1, \dots \rho_t$ along some cycle such that ρ_t overlaps with ρ_1 and the relations in $\mathcal{G}$ satisfy the assumptions of the above proposition. Again we point out that we do not require that I be monomial, merely that the relations ρ_i are monomial, and that no terms of the other relations occur as subpaths of $\rho_i * \rho_{i+1}$. An example of this is given by the following:



We let the relations be: $bcd - fgh$, igj, jkl, lmn, and nio, along with anything at all on the outside of the .... Letting ρ_i be successive relations around the octagon, we will have P being powers of the path around the octagon. The overlap of these monomial relations is one arrow, so this falls into the easy case of proving η not in $Im(d_n^*)$. The relations clearly satisfy the assumptions we made to prove the non-vanishing of Ext. Notice that we even have a non-monomial relation intersecting the monomial relations around P, we just don't have a *term* of the relation contained in $\rho_i * \rho_{i+1}$.

Another example of an algebra which exhibits this type of behavior is:



where the ideal I is generated by the relations $\{adf, dfhi, hijk, ceg - bdf, gh\}$. A Gröbner basis for I under the length lexicographic order is given by $\mathcal{G} = \{adf, dfhi, hijk, ceg - bdf, gh, bdfh\}$. Our results guarantee us, via the overlapping relations adf, $dfhi$, and $hijk$ that $Ext^4_\Lambda(S_v, S_w) \neq 0$. They do not guarantee the case of the overlapping relations $bdfh, dfhi$, and $hijk$, since bdf, a term in $ceg - bdf$, is a sub-path of $bdfh * dfhi$. So our techniques do not guarantee the non-vanishing of $Ext^4_\Lambda(S_u, S_w)$. It should be pointed out that by using minimal resolutions it is possible to compute $dim_K Ext^4_\Lambda(S_u, S_w) = 1$ in this case. This algebra is clearly of finite global dimension.

We also note that the vanishing of $Ext^n_\Lambda(S_v, S_w)$ is of interest. It is known that the number of times the indecomposable projective $v \otimes w\Lambda^e$ occurs in the nth projective in the minimal Λ^e resolution of Λ is equal to the k-dimension of $Ext^n_\Lambda(S_v, S_w)$. It is clear from the description of the modules in the resolution of S_v that if there is no oriented path from v to w in Γ then $w\Lambda$ will never occur as a summand of Q^n. Furthermore if there are no paths of length greater than m between v and w we see that $w\Lambda$ cannot occur as a summand of Q^i for $i > m$ in the projective resolution of S_v.

Chapter 6

Comparison With Minimal Resolutions

BIMODULE RESOLUTION

We described a projective resolution of Λ as a right module over its enveloping algebra Λ^e by repeatedly tensoring the short exact sequence:

$$0 \to \Omega^1_{\Lambda^e}(\Lambda) \to P^0 \to \Lambda \to 0$$

with the bimodule $\Omega^1_{\Lambda^e}(\Lambda)$. An interesting invariant of Λ is the minimal projective resolution of Λ as a Λ^e-module. We would like to know how our resolution differs from the minimal resolution.

We will denote by Q^n the nth projective in the minimal resolution of Λ. P^n will denote $\otimes^n_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0$, the nth projective in the resolution of Λ given in this thesis. The nth syzygy of Λ, denoted $\Omega^n_{\Lambda^e}(\Lambda)$ is the image of the map $Q^n \to Q^{n-1}$. Recall that $\otimes^n_\Lambda \Omega^1_{\Lambda^e}(\Lambda)$ is the image of the map $P^n \to P^{n-1}$.

To begin our comparison let us consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \to & \Omega^1_{\Lambda^e}(\Lambda)\otimes_\Lambda \Omega^1_{\Lambda^e}(\Lambda) & \to & \Omega^1_{\Lambda^e}(\Lambda)\otimes_\Lambda P^0 & \to & \Omega^1_{\Lambda^e}(\Lambda)\otimes_\Lambda \Lambda & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^2_{\Lambda^e}(\Lambda) & \to & Q^1 & \to & \Omega^1_{\Lambda^e}(\Lambda) & \to & 0
\end{array}$$

where the rightmost vertical map is clearly an isomorphism, and the middle vertical map must be onto since Q^1 is the minimal projective cover of $\Omega^1_{\Lambda^e}(\Lambda)$. The snake lemma (see [13] for example) will force the leftmost vertical map to be surjective, and the kernels of the left and middle vertical maps to be identical according to the following diagram:

$$\begin{array}{ccccccccc}
 & & K^2 & & K^2 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^1_{\Lambda^e}(\Lambda)\otimes_\Lambda \Omega^1_{\Lambda^e}(\Lambda) & \to & P^1 & \to & \Omega^1_{\Lambda^e}(\Lambda)\otimes_\Lambda \Lambda & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^2_{\Lambda^e}(\Lambda) & \to & Q^1 & \to & \Omega^1_{\Lambda^e}(\Lambda) & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The middle column is clearly split, and hence we have that $P^1 \cong Q^1 \oplus K^2$, or to say it a different way, we have $K^2 \cong P^1/Q^1$. A diagram chase guarantees that the left column is split, so

$$\otimes^2_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \cong \Omega^2_{\Lambda^e}(\Lambda) \oplus P^1/Q^1.$$

Similarly, we can now consider the following commutative exact diagram:

$$\begin{array}{ccccccccc}
0 & \to & K^3 & \to & K^3 & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \otimes^3_\Lambda \Omega^1_{\Lambda^e}(\Lambda) & \to & P^2 & \to & \begin{array}{c}\otimes^2_\Lambda \Omega^1_{\Lambda^e}(\Lambda)\\ || \\ \Omega^2_{\Lambda^e}(\Lambda)\oplus K^2\end{array} & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^3_{\Lambda^e}(\Lambda) & \to & Q^2\oplus K^2 & \to & \Omega^2_{\Lambda^e}(\Lambda)\oplus K^2 & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & 0 & \to & 0 & &
\end{array}$$

and in a manner similar to the last time we find $\otimes^3_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \cong \Omega^3_{\Lambda^e}(\Lambda) \oplus K^3$ and $K^3 \cong P^2/(Q^2 \oplus K^2)$. Inductively we have the following diagram:

$$\begin{array}{ccccccccc}
0 & \to & K^n & \to & K^n & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \otimes^n_\Lambda \Omega^1_{\Lambda^e}(\Lambda) & \to & P^{n-1} & \to & \begin{array}{c}\otimes^{n-1}_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \\ || \\ \Omega^{n-1}_{\Lambda^e}(\Lambda) \oplus K^{n-1}\end{array} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^n_{\Lambda^e}(\Lambda) & \to & Q^{n-1} \oplus K^{n-1} & \to & \Omega^{n-1}_{\Lambda^e}(\Lambda) \oplus K^{n-1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & 0 & \to & 0 & &
\end{array}$$

and we'll have that

$$\otimes^n_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \cong \Omega^n_{\Lambda^e}(\Lambda) \oplus K^n$$

Now we will describe the modules K^n. We saw that

$$K^2 \cong P^1/Q^1.$$

The next diagram gave us that

$$K^3 \cong P^2/(Q^2 \oplus K^2)$$

which, if we substitute for K^2 gives us

$$K^3 \quad \cong \quad \frac{P^2}{Q^2 \oplus \dfrac{P^1}{Q^1}}.$$

Inductively we will have:

$$K^n \cong \cfrac{P^n}{Q^n \oplus \cfrac{P^{n-1}}{\begin{array}{c}\vdots \\ Q^{n-1} \oplus \cdots \oplus \cfrac{P^2}{Q^2 \oplus \cfrac{P^1}{Q^1}}\end{array}}}.$$

ONE SIDED MODULE RESOLUTIONS

In the last section we obtained projective resolutions of simple right Λ-modules S_v by tensoring S_v with our Λ^e resolution of Λ. These resolutions were used to investigate homological properties of simple modules. It is evident however that these resolutions were not the minimal resolutions of the simple modules. It should be clear that the minimal resolution of a right Λ-module M_Λ is an interesting invariant of its own right. We note that the process used in the last section of tensoring (over Λ) a right Λ-module M with a Λ^e projective resolution of Λ to obtain a Λ projective resolution of M had nothing to do with the fact that we were considering the case that M was simple. It is clearly possible to obtain a Λ projective resolution of any right Λ-module M_Λ by tensoring M over Λ with the bimodule resolution. While these resolutions can also be used to investigate homological properties of general right modules, we will be interested in this section in comparing the projective resolution of a Λ-module M_Λ obtained in this way with the minimal Λ projective resolution of M. These results will parallel the above computations in the bimodule case.

We are assured of a minimal projective resolution of M:

$$\cdots Q^n \to Q^{n-1} \cdots Q^1 \to Q^0 \to M.$$

The minimal ith syzygy of M will be denoted W^i. Following the notation of previous sections, we assume that we have a bimodule projective resolution of Λ as follows:

$$\cdots P^n \to P^{n-1} \cdots P^1 \to P^0 \to \Lambda$$

where $P^0 = \coprod_{v\in\Gamma_0} v \otimes v\Lambda^e$ and $P^n \cong \otimes^n_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P_0$. The nth kernel in this resolution was $\otimes^n_\Lambda \Omega^1_{\Lambda^e}(\Lambda)$. We recall that each of the bimodule projectives P^0, and each of the kernels $\otimes^n_\Lambda \Omega^1_{\Lambda^e}(\Lambda)$ are projective as right Λ-modules.

A short exact sequence $\delta = 0 \to A \to B \to C \to 0$ is called *pure exact* if $M \otimes_\Lambda \delta$ remains exact for all right Λ-modules M. Clearly if C is projective in $mod(\Lambda)$ then δ will be pure exact, since applying the functor $M \otimes_\Lambda \_$ to δ will result in the following:

$$Tor^1_\Lambda(M, C) \to M \otimes_\Lambda A \to M \otimes_\Lambda B \to M \otimes_\Lambda C \to 0$$

and $Tor^1_\Lambda(M, C)$ is zero since C is projective. We have therefore established the following lemma.

Lemma 6.1 *The short exact sequences*

$$0 \to \otimes^{n+1}_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \to \otimes^n_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0 \to \otimes^n_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \to 0$$

arising from our bimodule resolution of Λ *are all pure exact.*$\square$

We note now that we have the following commutative exact diagram:

$$\begin{array}{ccccccccc}
0 & \to & M \otimes_\Lambda \Omega^1_{\Lambda^e}(\Lambda) & \to & M \otimes_\Lambda P^0 & \to & M \otimes_\Lambda \Lambda & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & W^1 & \to & Q^0 & \to & M & \to & 0
\end{array}$$

where the rightmost vertical map is an isomorphism and the middle vertical map is surjective since Q^0 is the projective cover of M. The snake lemma now assures us that we have the following commutative exact diagram:

$$\begin{array}{ccccccccc}
0 & \to & K^1 & \to & K^1 & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M \otimes_\Lambda \Omega^1_{\Lambda^e}(\Lambda) & \to & M \otimes_\Lambda P^0 & \to & M \otimes_\Lambda \Lambda & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & W^1 & \to & Q^0 & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & 0 & \to & 0 & &
\end{array}$$

and it is evident that the kernels of the first two vertical maps will be isomorphic, and we denote this module by K^1. It is clear, since Q^0 is projective, that $M \otimes_\Lambda P^0 \cong Q^0 \oplus K^1$, and hence we see that $K^1 \cong (M \otimes_\Lambda P^0/Q^0)$. It is a simple diagram chase to determine that K^1, when viewed as a submodule of $M \otimes_\Lambda P^0$, is in the kernel of the map from $M \otimes_\Lambda P^0 \to M \otimes_\Lambda \Lambda$, and hence K^1 is in the image of $M \otimes_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \to M \otimes_\Lambda P^0$. It is obvious therefore that the sequence

$$0 \to K^1 \to M \otimes_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \to W^1 \to 0$$

is split exact, since we have a 'back' map from the middle term to the first term. It is therefore clear that $M \otimes_\Lambda \Omega^1_{\Lambda^e}(\Lambda) \cong W^1 \oplus K^1$, and we have described the first syzygy of our resolution obtained by tensoring M with the bimodule resolution as the minimal first syzygy of M plus a projective module.

We extend this process by considering the following diagram:

$$\begin{array}{ccccccccc}
0 & \to & K^2 & \to & K^2 & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & & M \otimes_\Lambda \Omega^1_{\Lambda^e}(\Lambda) & & \\
0 & \to & M \otimes_\Lambda \otimes^2_\Lambda \Omega^1_{\Lambda^e}(\Lambda) & \to & M \otimes_\Lambda P^1 & \to & || & \to & 0 \\
 & & & & & & W^1 \oplus K^1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & W^2 & \to & Q^1 \oplus K^1 & \to & W^1 \oplus K^1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & 0 & \to & 0 & &
\end{array}$$

which arises in the same manner as the first diagram. It is easy to see that we have the analogous result here that $W^2 \oplus K^2 \cong M \otimes_\Lambda \otimes^2_\Lambda \Omega^1_{\Lambda^e}(\Lambda)$. Inductively we see that we will have the following diagram:

$$\begin{array}{ccccccccc}
0 & \to & K^n & \to & K^n & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & & M \otimes_\Lambda \otimes^{n-1}_\Lambda \Omega^1_{\Lambda^e}(\Lambda) & & \\
0 & \to & M \otimes_\Lambda \otimes^n_\Lambda \Omega^1_{\Lambda^e}(\Lambda) & \to & M \otimes_\Lambda P^{n-1} & \to & || & \to & 0 \\
 & & & & & & W^{n-1} \oplus K^{n-1} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & W^n & \to & Q^{n-1} \oplus K^{n-1} & \to & W^{n-1} \oplus K^{n-1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & 0 & \to & 0 & &
\end{array}$$

which will give us that $W^n \oplus K^n \cong M \otimes_\Lambda \otimes^n_\Lambda \Omega^1_{\Lambda^e}(\Lambda)$.

It will now be our goal to determine what the K^n modules look like. We recall that we have $K^1 \cong (M \otimes_\Lambda P^0/Q^0)$. From the above diagram we see that

$$K^2 \cong (M \otimes_\Lambda P^1)/(Q^1 \oplus K^1)$$

which is isomorphic to

$$K^2 \cong \cfrac{M \otimes_\Lambda P^1}{Q^1 \oplus \cfrac{M \otimes_\Lambda P^0}{Q^0}}.$$

With $n = 3$ in the above diagram we would have that

$$K^3 \cong \cfrac{M \otimes_\Lambda P^2}{Q^2 \oplus \cfrac{M \otimes_\Lambda P^1}{Q^1 \oplus \cfrac{M \otimes_\Lambda P^0}{Q^0}}}.$$

Inductively then we have:

$$K^n \cong \cfrac{M \otimes_\Lambda P^{n-1}}{Q^{n-1} \oplus \cfrac{M \otimes_\Lambda P^{n-2}}{\begin{array}{c}\vdots \\ Q^{n-2} \oplus \cdots \oplus \cfrac{M \otimes_\Lambda P^1}{Q^1 \oplus \cfrac{M \otimes_\Lambda P^0}{Q^0}}\end{array}}}.$$

A few words now about some decompositions. Recall that we gave a decomposition $\Omega^1_{\Lambda^e}(\Lambda) \cong \coprod B_i$ in Proposition 3.2 and that we established that $B_i \otimes_\Lambda B_j = 0$ for $i \neq j$ (Proposition 3.4.) As a result of this we see that $\otimes^n_\Lambda \Omega^1_{\Lambda^e}(\Lambda)$ is isomorphic to $\coprod \otimes^n_\Lambda B_i$. Suppose that B_i is such that $\otimes^m_\Lambda B_i = 0$. Then for any module with $M \otimes_\Lambda B_j = 0$ for $j \neq i$ we will have that $pd_\Lambda(M) \leq m$ where $pd_\Lambda(M)$ is the projective dimension of M. Furthermore for any module M we have that $M \otimes_\Lambda \Omega^1_{\Lambda^e}(\Lambda)$ is isomorphic to $\coprod M \otimes_\Lambda B_i$. The resolution of M in this sense 'spreads out' into the direct sum of the resolutions of $M \otimes_\Lambda B_i$. That is, $M \otimes_\Lambda \otimes^2_\Lambda B_i$, $M \otimes_\Lambda \otimes^3_\Lambda B_i$, etc. become the syzygies of the B_i summand of the resolution. Any B_i with the nilpotency property described above will then guarantee that the 'part' of M which has infinite global dimension must come from $M \otimes_\Lambda B_j$ for $j \neq i$. In fact any B_i

which has the property that $\otimes_{\Lambda}^{m} B_i$ is projective in $mod(\Lambda^e)$ will likewise contribute nothing to the infinite projective dimension of M since we have already shown $M \otimes_{\Lambda} P$ is projective in $mod(\Lambda)$ when P is projective as a Λ-Λ-bimodule. It is easy to see that the existence of an oriented cycle in $G_{\Gamma,I}$ is sufficient to guarantee $\otimes_{\Lambda}^{m} B_i$ does not vanish when B_i is the bimodule associated to the component of $G_{\Gamma,I}$ containing the cycle. However the eventual projectivity of $\otimes_{\Lambda}^{m} B_i$ is obviously more subtle. Conditions which either guarantee or prohibit such behavior would be very interesting to see.

Chapter 7

Resolutions of Modules Given by Presentations

In this section we use the decompositions of $\Omega^1_{\Lambda^e}(\Lambda)$ to give a new method of computing a projective resolution of an arbitrary Λ-module M given in the form of a projective presentation. Unlike other methods of computing resolutions, this one does not require the computation of a Gröbner basis at each step in the resolution, but rather relies solely on the Gröbner basis for the ideal I in the path algebra $K\Gamma$. As was mentioned in the introduction, this is an iterative process, and hence is subject to minimization at each step, hence allowing the construction of the minimal projective resolution of M. It may be interesting, if not in any way useful, to note that if one does not bother to minimize at each step the resolution of M obtained by this process would be exactly the resolution obtained by tensoring M over Λ with the enveloping algebra resolution of Λ given previously. It is certainly interesting to note that it is possible to begin this resolution at any step, that is, to begin by computing $Q^{n+1} \to Q^n \to Q^{n-1}$ without computing any prior projectives or maps (although one cannot compute the minimal resolution at these steps without first computing all previous projectives and the maps.)

We begin by considering a right Λ-module M, which is given to us in the form of a projective presentation

$$Q_1 \xrightarrow{f} Q_0 \to M \to 0.$$

The data we assume we know is a matrix for f in the following form: let $Q_1 \cong \coprod_J w_j\Lambda$ and let $Q_0 \cong \coprod_I v_i\Lambda$, then the matrix for f will be of dimension $\mid I \mid \times \mid J \mid$ and each entry f_{ij} will be an element of $v_i\Lambda w_j$. In the case that one knows Γ and (f_{ij}) one can easily determine the indecomposable projective summands of Q_1 and Q_0. We will construct the first three terms of a deleted projective resolution for M, the first term of which will be Q_0, that is, we will construct the following:

$$Q_2'' \xrightarrow{f_2} Q_1' \xrightarrow{f_1} Q_0$$

where M is the cokernel of f_1. One might then take $Q_2'' \to Q_1'$ as a presentation of $\Omega^1_\Lambda(M)$ and use this as the input to repeat this process, obtaining a three term deleted projective resolution $Q_3'' \to Q_2' \to Q_1' \to \Omega^1_\Lambda(M)$ of $\Omega^1_\Lambda(M)$, the first term of which is Q_1', and continue in this manner to produce a resolution of M as follows:

$$\begin{array}{ccccccc} & & Q_3'' & & Q_2'' & & Q_1 \\ & & \downarrow & & \downarrow & & \downarrow \\ \ldots & \to & Q_2' & \to & Q_1' & \to & Q_0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \Omega^2_\Lambda(M) \oplus proj. & & \Omega^1_\Lambda(M) & & M \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

where the middle row $\ldots \to Q_2' \to Q_1' \to Q_0$ is the desired resolution of M.

In order to do this we will need to consider the minimal projective presentation of Λ as a right Λ^e-module, which is given below:

$$P_1 \cong \coprod_{a \in \Gamma_1} o(a) \otimes t(a)\Lambda^e \xrightarrow{\delta} P_0 \cong \coprod_{v \in \Gamma_0} v \otimes v\Lambda^e \to \Lambda \to 0$$

where $\delta(o(a) \otimes t(a)) = a \otimes t(a) - o(a) \otimes a$. We will need the following well known result, a proof of which we will indicate here for the sake of completeness:

Proposition 7.1 *Let M be a finitely generated projective right Λ-module, and let P be a finitely generated projective Λ^e-module. Then $M \otimes_\Lambda P$ is projective as a right Λ-module.*

Proof. Since $M \otimes_\Lambda (\coprod A_i) \cong \coprod (M \otimes_\Lambda A_i)$ it suffices to show that $M \otimes_\Lambda \Lambda^e$ is projective as a right Λ-module. $M \otimes_\Lambda \Lambda^e = M \otimes_\Lambda \Lambda^{op} \otimes \Lambda \cong M \otimes \Lambda \cong \Lambda^{dim_K(M)}$, which is clearly projective. $\square$

Since the functors $\_ \otimes_\Lambda P_1$ and $\_ \otimes_\Lambda P_0$ are right exact, we may apply them to the projective presentation of M, obtaining the following:

$$Q_1 \otimes_\Lambda P_1 \xrightarrow{f \otimes id_1} Q_0 \otimes_\Lambda P_1 \to M \otimes_\Lambda P_1 \to 0$$

$$Q_1 \otimes_\Lambda P_0 \xrightarrow{f \otimes id_0} Q_0 \otimes_\Lambda P_0 \to M \otimes_\Lambda P_0 \to 0$$

where by the previous proposition, $M \otimes_\Lambda P_1$ and $M \otimes_\Lambda P_0$ are projective. Therefore the epimorphisms $Q_0 \otimes_\Lambda P_1 \to M \otimes_\Lambda P_1$ and $Q_0 \otimes_\Lambda P_0 \to M \otimes_\Lambda P_0$ split, and we have the following split exact sequences:

$$0 \to Im(f \otimes id_1) \to Q_0 \otimes_\Lambda P_1 \to M \otimes_\Lambda P_1 \to 0 \tag{7.1}$$

$$0 \to Im(f \otimes id_0) \to Q_0 \otimes_\Lambda P_0 \to M \otimes_\Lambda P_0 \to 0 \tag{7.2}$$

We note that since we know f as a matrix (f_{ij}), with each $f_{ij} \in v_i\Lambda w_j$, it is easy to determine the indecomposable direct summands of Q_1 and Q_0, and then one may compute the following modules: $Q_1 \otimes_\Lambda P_1$, $Q_1 \otimes_\Lambda P_0$, $Q_0 \otimes_\Lambda P_1$, and $Q_0 \otimes_\Lambda P_0$. In fact we have the following:

Proposition 7.2 *If* $Q = \coprod_I v_i\Lambda$*,* $P_1 = \coprod_{\Gamma_1} o(a) \otimes t(a)\Lambda^e$*, and* $P_0 = \coprod_{\Gamma_0} v \otimes v\Lambda^e$*, then as right* Λ*-modules,*

$$Q \otimes_\Lambda P_1 \cong \coprod_I \coprod_{a\in\Gamma_1} \coprod_{dim_K v_i\Lambda o(a)} t(a)\Lambda$$

$$Q \otimes_\Lambda P_0 \cong \coprod_I \coprod_{v\in\Gamma_0} \coprod_{dim_K v_i\Lambda v} v\Lambda.$$

Proof. Note that the previous proposition guarantees that $Q\otimes_\Lambda P_1$ and $Q\otimes_\Lambda P_0$ are projective as right Λ-modules. We describe an isomorphism from the top of the modules on the right hand side of the $\cong$ to the top of the modules on the left. First we describe the top of the modules on the left. It is clear that $v_i\Lambda \otimes_\Lambda \Lambda o(a) \otimes t(a)\Lambda$ is isomorphic to $v_i\Lambda o(a) \otimes t(a)\Lambda$ and that $v_i\Lambda \otimes_\Lambda \Lambda v \otimes v\Lambda$ is isomorphic to $v_i\Lambda v \otimes v\Lambda$. As we are now tensoring over K, we may obtain any element of $v_i\Lambda o(a)$ or of $v_i\Lambda v$ from basis elements of $v_i\Lambda o(a)$ and $v_i\Lambda v$. Thus for each i, there will be $dim_K(v_i\Lambda o(a))$ copies of $t(a)\Lambda$ for the summand $v_i\Lambda \otimes_\Lambda \Lambda o(a) \otimes t(a)$ of $Q \otimes_\Lambda P_1$, and $dim_K(v_i\Lambda v)$ copies of $v\Lambda$ for the summand $v_i\Lambda \otimes_\Lambda \Lambda v \otimes v\Lambda$ of $Q \otimes_\Lambda P_0$. The isomorphisms between the tops of the projective modules are now clear, and the result immediately follows.$\square$

Now we are prepared to compute $Im(f \otimes id)$ in both of the above split exact sequences (1) and (2). From now on, we will represent an element of the form $v_1\lambda \otimes_\Lambda v_2 \otimes v_3$ by $v_1\lambda\ v_2\ \otimes\ v_3$. Let $w_j\ \lambda\ o(a) \otimes\ t(a)$ be an element of Top$(Q_1 \otimes_\Lambda P_1)$. We wish to find the image of this element under $f \otimes\ id_1$. If we consider how this element is labeled, we see that it arises from $w_j\ \lambda\ \otimes_\Lambda\ o(a) \otimes t(a)$, and now applying $f \otimes\ id$, keeping in mind that $f(w_j)\ = \sum_I\ f_{ij}$, we find that $f \otimes id_1(w_j\ \lambda\ o(a)\ \otimes t(a)) = \sum_I\ f_{ij}\lambda\ o(a)\ \otimes\ t(a)$. Similarly, if

$w_j\lambda v \otimes v$ is in $\mathrm{Top}(Q_1 \otimes_\Lambda P_0)$, we find that $f \otimes id_0(w_j\lambda v \otimes v) = \sum_I f_{ij}\lambda v \otimes v$. We see then that $f \otimes id_1(\mathrm{Top}(Q_1 \otimes_\Lambda P_1)) \in \mathrm{Top}(Q_0 \otimes_\Lambda P_1)$ and that $f \otimes id_0(\mathrm{Top}(Q_1 \otimes_\Lambda P_0)) \in \mathrm{Top}(Q_0 \otimes_\Lambda P_0)$. First we will compute $f \otimes id_0(\mathrm{Top}(Q_1 \otimes_\Lambda P_0))$, and to do this we consider a matrix of dimesion $dim_K(\mathrm{Top}(Q_1 \otimes_\Lambda P_0)) \times dim_K(\mathrm{Top}(Q_0 \otimes_\Lambda P_0))$, where the column corresponding to $w_j\lambda v \otimes v$ will have entries equal to zero in every row except those corresponding to a term of $f_{ij}\lambda v \otimes v$ for some i, and in these rows we have the coefficient of that term in f_{ij}. Note that what we have defined here is a matrix corresponding to the vector space map between $\mathrm{Top}(Q_1 \otimes_\Lambda P_0)$ and $\mathrm{Top}(Q_0 \otimes_\Lambda P_0)$ induced by $f \otimes id_0$. We column reduce this matrix, and the columns now correspond to a new K-basis for $\mathrm{Top}(Q_0 \otimes_\Lambda \ P_0)$ such that those basis elements corresponding to non-zero columns will map one to one onto a K-basis for $f \otimes \ id_0(Q_1 \otimes_\Lambda \ P_0)$. The K-basis for the image is of course obtained by reading down each column, and we obtain a basis $\{x_1, x_2, \ldots, x_s\}$ for $f \otimes id_0 \ (Q_1 \otimes_\Lambda \ P_0)$ where each x_i is a linear combination of elements of our previous basis $\{v_i\lambda v \otimes v\}$ for $\mathrm{Top}(Q_0 \otimes_\Lambda \ P_0)$. Thus the inclusion $f \otimes \ id_0 \ (Q_1 \otimes_\Lambda \ P_0) \to \mathrm{Top}(Q_0 \otimes_\Lambda \ P_0)$ is obvious.

The construction of $f \otimes id_1(Q_1 \otimes_\Lambda P_1)$ inside of $Q_0 \otimes_\Lambda P_1$ takes exactly the same form, first we construct a matrix of dimension $dim_K(\mathrm{Top}(Q_1 \otimes_\Lambda P_1)) \times dim_K(\mathrm{Top}(Q_0 \otimes_\Lambda P_1))$ with each column corresponding to the image of a basis element $w_j\lambda o(a) \otimes t(a)$ of $Q_1 \otimes_\Lambda P_1$ in terms of the basis $\{v_i\lambda o(a) \otimes t(a)\}$ of $Q_0 \otimes_\Lambda P_1$. Column reducing this matrix we obtain a basis $\{y_1, y_2, \ldots y_t\}$ of the non-zero columns for $Im(f \otimes id_1)$, with each y_i a linear combination of the $v_i\lambda o(a) \otimes t(a)$s. Again the inclusion $f \otimes id_1(Q_1 \otimes_\Lambda P_1) \to Q_0 \otimes_\Lambda P_1$ is obvious.

At this point we should mention something about the split exact sequences (7.1) and (7.2). We note that there is a natural short exact sequence $0 \to \Omega^1_\Lambda(M) \to Q_0 \to M \to 0$ in $mod(\Lambda)$. If we apply the functors $\_ \otimes_\Lambda P_1$ and $\_ \otimes_\Lambda P_0$ to this short exact sequence, the image will remain exact, and we will have the following:

$$0 \to \Omega^1_\Lambda(M) \otimes_\Lambda P_1 \to Q_0 \otimes_\Lambda P_1 \to M \otimes_\Lambda P_1 \to 0 \tag{7.3}$$

$$0 \to \Omega^1_\Lambda(M) \otimes_\Lambda P_0 \to Q_0 \otimes_\Lambda P_0 \to M \otimes_\Lambda P_0 \to 0 \tag{7.4}$$

which split since $M \otimes_\Lambda P_1$ and $M \otimes_\Lambda P_0$ are projective. Since the last two modules of the split exact sequences (7.1) and (7.3) are the same, and the last two modules of (7.2) and (7.4) are the same, we see that $Im(f \otimes_\Lambda id_1) \cong \Omega^1_\Lambda(M) \otimes_\Lambda P_1$ and $Im(f \otimes_\Lambda id_0) \cong \Omega^1_\Lambda(M) \otimes_\Lambda P_0$. Putting all of this together we see that we have the following commutative exact diagram, with the first and the second column split:

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\Omega^1_\Lambda(M) \otimes_\Lambda P_1 & \to & \Omega^1_\Lambda(M) \otimes_\Lambda P_0 & \to & \Omega^1_\Lambda(M) \otimes_\Lambda \Lambda & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
Q_0 \otimes_\Lambda P_1 & \stackrel{id_{Q_0} \otimes \delta}{\to} & Q_0 \otimes_\Lambda P_0 & \to & Q_0 \otimes_\Lambda \Lambda & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
M \otimes_\Lambda P_1 & \to & M \otimes_\Lambda P_0 & \to & M \otimes_\Lambda \Lambda & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

We note that by Proposition 7.1 $\Omega^1_\Lambda(M) \otimes_\Lambda P_1$ and $\Omega^1_\Lambda(M) \otimes_\Lambda P_0$ are projective in $mod(\Lambda)$, and hence the top row is a projective presentaion of $\Omega^1_\Lambda(M)$. Furthermore, if we consider $\Omega^1_\Lambda(M) \otimes_\Lambda P_1$ and $\Omega^1_\Lambda(M) \otimes_\Lambda P_0$ as submodules of $Q_0 \otimes_\Lambda P_1$ and $Q_0 \otimes_\Lambda P_0$ respectively, we see that we really have $\Omega^1_\Lambda(M) \otimes_\Lambda P_1 \to \Omega^1_\Lambda(M) \otimes_\Lambda P_0 \to Q_0 \to M \to 0$, the first three projectives in a projective resolution of M. We have already described $\Omega^1_\Lambda(M) \otimes_\Lambda P_1$ and $\Omega^1_\Lambda(M) \otimes_\Lambda P_0$ as $Im(f \otimes id_1)$ and $Im(f \otimes id_0)$ respectively, and we have a description of their tops. We now wish to describe the map between them induced by the map $id_{Q_0} \otimes \delta$. Recall that $\delta(o(a) \otimes t(a)) = a \otimes t(a) - o(a) \otimes a$. Recall also that $\{v_i \lambda o(a) \otimes t(a)\}$ forms a basis for Top$(Q_0 \otimes_\Lambda P_1)$, and that $\{v_i \lambda v \otimes v\}$ forms a basis for Top$(Q_0 \otimes_\Lambda P_0)$. We now see that $id_{Q_0} \otimes \delta(v_i \lambda o(a) \otimes t(a)) = v_i \lambda a t(a) \otimes t(a) - v_i \lambda o(a) \otimes o(a) \cdot a$. If we identify $\Omega^1_\Lambda(M) \otimes_\Lambda P_1$

and $\Omega^1_\Lambda(M) \otimes_\Lambda P_0$, with $Im(f \otimes id_0)$ and $Im(f \otimes id_1)$ respectively, we can now construct the map $\Omega^1_\Lambda(M) \otimes_\Lambda P_1 \to Q_0 \otimes_\Lambda P_0$ in the commutative diagram above. Recall that we have previously included $Im(f \otimes id_0) \cong \Omega^1_\Lambda(M) \otimes_\Lambda P_0$ into $Q_0 \otimes_\Lambda P_0$, and we know that restricting $id_{Q_0} \otimes \delta$ to $\Omega^1_\Lambda(M) \otimes_\Lambda P_1$ will result in an image inside $\Omega^1_\Lambda(M) \otimes_\Lambda P_0$, but the image is in terms of our basis for $Q_0 \otimes_\Lambda P_0$, and not in terms of our basis for $\Omega^1_\Lambda(M) \otimes_\Lambda P_0$.

We must observe that $id_{Q_0} \otimes \delta$ takes an element of $\text{Top}(\Omega^1_\Lambda(M) \otimes P_1)$ to an element of $\text{Top}(\Omega^1_\Lambda(M) \otimes P_0) \cdot (\Gamma_0 \cup \Gamma_1)$. A basis for this subspace of $\Omega^1_\Lambda(M) \otimes P_0$ is just $\{x_1, x_2, \ldots, x_s\} \cup \{x_i \cdot a : a \in \Gamma_1, x_i \cdot a \neq 0\}_{i=1}^s$. Recall that we have a basis $\{v_i \lambda v \otimes v\}$ for $\text{Top}(Q_0 \otimes_\Lambda P_0)$, and that the x_i may all be written in terms of this basis. Furthermore we note that $id_{Q_0} \otimes \delta$ takes an element of $\text{Top}(\Omega^1_\Lambda(M) \otimes P_1)$ to an element of $\text{Top}(Q_0 \otimes_\Lambda P_0) \cdot (\Gamma_0 \cup \Gamma_1)$. We see that $\{v_i \lambda v \otimes v\} \cup \{v_i \lambda v \otimes v \cdot a : a \in \Gamma_1, v \cdot a \neq 0\}$ is a basis of this subspace, and note that any element of $\{x_1, x_2, \ldots, x_s\} \cup \{x_i \cdot a : a \in \Gamma_1, x_i \cdot a \neq 0\}_{i=1}^s$ may be written as a linear combination of elements of $\{v_i \lambda v \otimes v\} \cup \{v_i \lambda v \otimes v \cdot a : a \in \Gamma_1, v \cdot a \neq 0\}$ if we know how to write each x_i in terms of the $v_i \lambda v \otimes v$, which we do know how to do.

Thus we are left with the following problem, given an element z of $\text{Top}(\Omega^1_\Lambda(M) \otimes_\Lambda P_0) \cdot (\Gamma_0 \cup \Gamma_1)$ written in terms of our basis $\{v_i \lambda v \otimes v\} \cup \{v_i \lambda v \otimes v \cdot a : a \in \Gamma_1, v \cdot a \neq 0\}$ for $\text{Top}(Q_0 \otimes_\Lambda P_0) \cdot (\Gamma_0 \cup \Gamma_1)$, how do we write this element in terms of the basis $\{x_1, x_2, \ldots, x_s\} \cup \{x_i \cdot a : a \in \Gamma_1, x_i \cdot a \neq 0\}_{i=1}^s$ for $\text{Top}(\Omega^1_\Lambda(M) \otimes_\Lambda P_0) \cdot (\Gamma_0 \cup \Gamma_1)$.

For ease of notation we will recast our problem in the following way: Given a K vector space V with basis $\{v_1, v_2, \ldots v_n\}$ and a subspace W with basis $\{w_1, w_2, \ldots w_m\}$ where we know how to write $w_j = \sum_{i=1}^n v_i k_{ij}$, and given an element z of W which is written $z = \sum_{i=1}^n v_i \ell_i$, how do we write $z = \sum_{j=1}^m w_j k'_j$? We see that this is the same problem we have above, with $V = \text{Top}(Q_0 \otimes_\Lambda P_0) \cdot (\Gamma_0 \cup \Gamma_1)$, $W = \text{Top}(\Omega^1_\Lambda(M) \otimes_\Lambda P_0) \cdot (\Gamma_0 \cup \Gamma_1)$, the bases for V and W corresponding as above, and our element z is $id_{Q_0} \otimes \delta \circ incl$ of an element of $\text{Top}(\Omega^1_\Lambda(M) \otimes_\Lambda P_1)$. We now give the solution to our problem in the simplified notation.

Since we know that $z \in W$, we know that there is some way to write it $z = \sum_{j=1}^{m} w_j k'_j$, and we have, since $w_j = \sum_{i=1}^{n} v_i k_{ij}$, that $z = \sum_{j=1}^{m} (\sum_{i=1}^{n} v_i k_{ij}) k'_j$. We switch the order of the sums, obtaining $z = \sum_{i=1}^{n} v_i \sum_{j=1}^{m} k_{ij} k'_j$. But we know that $z = \sum_{i=1}^{n} v_i \ell_i$, and so we see that for each i, $\ell_i = \sum_{j=1}^{m} k_{ij} k'_j$, and treating the k'_j as indeterminants, we have a system of n equations in m unknowns, which we may hopefully solve for the k'_j. Of course, if there are too many dependencies among the equations, we will not be able to do so, but we note that if there were two solutions for the system, $\{k'_j\}_{j=1}^{m}$ and $\{k''_j\}_{j=1}^{m}$, we have that $z = \sum_{j=1}^{m} w_j k'_j = \sum_{j=1}^{m} w_j k''_j$, and since the w_j are a basis for W, we have for all j that $k'_j = k''_j$. Thus there really will only be one solution for our system, and we may obtain the k'_j algorithmically.

Recall that we are working toward the following resolution of M:

$$\Omega^1_\Lambda(M) \otimes_\Lambda P_1 \to \Omega^1_\Lambda(M) \otimes_\Lambda P_0 \to Q_0 \to M \to 0.$$

At this point we know $\Omega^1_\Lambda(M) \otimes_\Lambda P_1$ as a right Λ-module, $\Omega^1_\Lambda(M) \otimes_\Lambda P_0$ as a right Λ-module, and the map $\mathrm{Top}(\Omega^1_\Lambda(M) \otimes_\Lambda P_1) \to \Omega^1_\Lambda(M) \otimes_\Lambda P_0$. We now wish to find the map from $\mathrm{Top}(\Omega^1_\Lambda(M) \otimes_\Lambda P_0)$ to Q_0. But this is easy, recalling that we have a basis $\{x_1, \ldots x_s\}$ for $\mathrm{Top}(\Omega^1_\Lambda(M) \otimes_\Lambda P_0)$ with the additional information of how to write x_i as a linear combination of basis elements $\{v_i \lambda v \otimes v\}$ for $\mathrm{Top}(Q_0 \otimes_\Lambda P_0)$, and hence the inclusion map is obvious, we need only compute the image of $v_i \lambda v \otimes v$ in $Q_0 \otimes_\Lambda \Lambda$ and use the obvious isomorphism to Q_0. But this is $v_i \lambda \otimes_\Lambda v$ which maps to $v_i \lambda$ under the isomorphism. Finally composing, we see how to construct the map $\Omega^1_\Lambda(M) \otimes_\Lambda P_0 \to Q_0$, which completes the resolution. We will illustrate with two examples:

Example 1

Consider the following graph:

$$\Gamma : \quad \begin{tikzcd} (1) \arrow[dr, "a"] & & (4) \arrow[dr, "e"] & \\ & (3) \arrow[ur, "c"] \arrow[dr, "d"'] & & (6) \\ (2) \arrow[ur, "b"] & & (5) \arrow[ur, "f"'] & \end{tikzcd}$$

Let I be the ideal $< ac, bd, df - ce >$. If we use length lex ordering we have that the set $\{ac, bce, bd, df - ce\}$ is a Gröbner basis for I. We will resolve S_2, the simple module associated with the vertex (2). A presentation for this module is given by: $(3)\Lambda \xrightarrow{F} (2)\Lambda$ with F given by the matrix (b). We will now give bases for $\mathrm{Top}(Q_i \otimes_\Lambda P_i)$:

$Q_1 \otimes_\Lambda P_1$

$$(3) \otimes_\Lambda (3) \overset{c}{\otimes} (4), (3) \otimes_\Lambda (3) \overset{d}{\otimes} (4), (3)c \otimes_\Lambda (4) \overset{e}{\otimes} (6), (3)d \otimes_\Lambda (5) \overset{f}{\otimes} (6)$$

$Q_0 \otimes_\Lambda P_1$

$$(2) \otimes_\Lambda (2) \overset{b}{\otimes} (3), (2)b \otimes_\Lambda (3) \overset{c}{\otimes} (4), (2)b \otimes_\Lambda (3) \overset{d}{\otimes} (5), (2)bc \otimes_\Lambda (4) \overset{e}{\otimes} (6)$$

$Q_1 \otimes_\Lambda P_0$

$$(3) \otimes_\Lambda (3) \otimes (3), (3)c \otimes_\Lambda (4) \otimes (4), (3)d \otimes_\Lambda (5) \otimes (5), (3)ce \otimes_\Lambda (6) \otimes (6)$$

$Q_0 \otimes_\Lambda P_0$

$$(2) \otimes_\Lambda (2) \otimes (2), (2)b \otimes_\Lambda (3) \otimes (3), (2)bc \otimes_\Lambda (4) \otimes (4)$$

where for ease in keeping track, any tensor of the form $o(a) \otimes t(a)$ will be denoted $o(a) \overset{a}{\otimes} t(a)$. We will now denote anything of the form $v\lambda \otimes_\Lambda w \otimes u$ by $v\lambda w \otimes u$. We now wish to calculate the image of $F \otimes id_1 : Q_1 \otimes_\Lambda P_1 \to Q_0 \otimes_\Lambda P_1$. To do this we have the following matrix:

	$(3)(3) \overset{c}{\otimes} (4)$	$(3)(3) \overset{d}{\otimes} (4)$	$(3)c(4) \overset{e}{\otimes} (6)$	$(3)d(5) \overset{f}{\otimes} (6)$
$(2)(2) \overset{b}{\otimes} (3)$	0	0	0	0
$(2)b(3) \overset{c}{\otimes} (4)$	1	0	0	0
$(2)b(3) \overset{d}{\otimes} (5)$	0	1	0	0
$(2)bc(4) \overset{e}{\otimes} (6)$	0	0	1	0

where again for the sake of clarity we have included the basis elements as headings for the rows and columns corresponding to them. We see that this matrix is already column redued for us, and hence a basis for $F \otimes id_1(\mathrm{Top}(Q_1 \otimes_\Lambda P_1)) \cong \mathrm{Top}(\Omega^1_\Lambda(S_2) \otimes P_1)$ is given by $\{(2)b(3) \overset{c}{\otimes} (4), (2)b(3) \overset{d}{\otimes} (5), (2)bc(4) \overset{e}{\otimes} (6)\}$.

In order to calculate the image of $F \otimes id_0 : Q_1 \otimes_\Lambda P_0 \to Q_0 \otimes_\Lambda P_0$ we have the following matrix:

	$(3)(3) \otimes (3)$	$(3)c(4) \otimes (4)$	$(3)d(5) \otimes (5)$	$(3)ce(6) \otimes (6)$
$(2)(2) \otimes (2)$	0	0	0	0
$(2)b(3) \otimes (3)$	1	0	0	0
$(2)bc(4) \otimes (4)$	0	1	0	0

where we have again included the basis elements for clarity. We see again that this matrix too is alreay column reduced, so a basis for $\mathrm{Top}(Im(F \otimes id_0)) \cong \mathrm{Top}(\Omega^1_\Lambda(S_2) \otimes_\Lambda P_0)$ will be: $\{(2)b(3) \otimes (3), (2)bc(4) \otimes (4)\}$.

The map $id_{Q_0} \otimes \delta : Q_1 \otimes_\Lambda P_0 \to Q_0 \otimes_\Lambda P_0$ is given by the following matrix:

	$(2)(2) \overset{b}{\otimes} (3)$	$(2)b(3) \overset{c}{\otimes} (4)$	$(2)b(3) \overset{d}{\otimes} (5)$	$(2)bc(4) \overset{e}{\otimes} (6)$
$(2)(2) \otimes (2)$	$-b$	0	0	0
$(2)b(3) \otimes (3)$	1	$-c$	$-d$	0
$(2)bc(4) \otimes (4)$	0	1	0	$-e$

and now we may write the map from $\Omega^1_\Lambda(S_2) \otimes_\Lambda P_1$ to $\Omega^1_\Lambda(S_2) \otimes_\Lambda P_0$ as follows:

$$
\begin{array}{lccc}
 & (2)b(3)\overset{c}{\otimes}(4) & (2)b(3)\overset{d}{\otimes}(5) & (2)bc(4)\overset{e}{\otimes}(6) \\
(2)b(3)\otimes(3) & -c & -d & 0 \\
(2)bc(4)\otimes(4) & 1 & 0 & -e
\end{array}.
$$

Finally we have that the map from $\mathrm{Top}(\Omega^1_\Lambda(S_2)) \otimes P_0$ to Q_0 is given by $(2)b(3) \otimes (3) \mapsto b$ and $(2)bc(4) \otimes (4) \mapsto bc$. We put this all together to produce the following resolution:

$$
\begin{array}{ccccccccccccccc}
(4)\Lambda & \coprod & (5)\Lambda & \coprod & (6)\Lambda & \to & (3)\Lambda & \coprod & (4)\Lambda & \to & (2)\Lambda & \to & S_2 & \to & 0 \\
((4) & , & 0 & , & 0\ ) & \to & (-c & , & (4)) & & & & & & \\
(\ 0 & , & (5) & , & 0\ ) & \to & (-d & , & 0\ ) & & & & & & \\
(\ 0 & , & 0 & , & (6)) & \to & (\ 0 & , & -e) & & & & & & \\
 & & & & & & ((3) & , & 0\ ) & \to & (b) & & & & \\
 & & & & & & (\ 0 & , & (4)) & \to & (bc) & & & &
\end{array}
$$

Example 2

Consider the following graph Γ:

$$
\begin{tikzcd}
(1) \arrow[r, shift left, "a"] & (2) \arrow[l, shift left, "b"] \arrow[r, "c"] & (3) \arrow[loop right, "d"]
\end{tikzcd}
$$

and let I be the ideal generated by all paths of length four. Let $\Lambda = K\Gamma/I$ and we will resolve the right Λ-module M given by the following matrix:

$$
\begin{pmatrix} ba & cd - c \\ 0 & dd \end{pmatrix}
$$

We see that in this case Q_1 and Q_0 are both isomorphic to $(2)\Lambda \coprod (3)\Lambda$. We give bases for the tops of the modules $Q_i \otimes_\Lambda P_1$ and $Q_i \otimes_\Lambda P_0$. Again for brevity we abbreviate elements of the form $v\lambda \otimes_\Lambda w \otimes u$ by $v\lambda w \otimes u$.

$\mathrm{Top}(Q_i \otimes_\Lambda P_1)$:

$(2)\ (2) \overset{b}{\otimes}\ (1) = x_1$ $\qquad$ $(2)\ (2) \overset{c}{\otimes}\ (3) = x_2$

$(2)b(1) \overset{a}{\otimes}\ (2) = x_3$ $\qquad$ $(2)c(3) \overset{d}{\otimes}\ (3) = x_4$

$(2)ba(2) \overset{b}{\otimes}\ (1) = x_5$ $\qquad$ $(2)ba(2) \overset{c}{\otimes}\ (3) = x_6$

$(2)cd(3) \overset{d}{\otimes}\ (3) = x_7$ $\qquad$ $(2)bab(1) \overset{a}{\otimes}\ (2) = x_8$

$(2)bac(3) \overset{d}{\otimes}\ (3) = x_9$ $\qquad$ $(2)cdd(3) \overset{d}{\otimes}\ (3) = x_{10}$

$(3)\ (3) \overset{d}{\otimes}\ (3) = x_{11}$ $\qquad$ $(3)d(3) \overset{d}{\otimes}\ (3) = x_{12}$

$(3)dd(3) \overset{d}{\otimes}\ (3) = x_{13}$ $\qquad$ $(3)ddd(3) \overset{d}{\otimes}\ (3) = x_{14}$

$\mathrm{Top}(Q_i \otimes_\Lambda P_0)$:

$(2)(2) \otimes (2) = v_1$ $\qquad$ $(2)b(1) \otimes (1) = v_2$ $\qquad$ $(2)c(3) \otimes (3) = v_3$

$(2)ba(2) \otimes (2) = v_4$ $\qquad$ $(2)cd(3) \otimes (3) = v_5$ $\qquad$ $(2)bab(1) \otimes (1) = v_6$

$(2)bac(3) \otimes (3) = v_7$ $\qquad$ $(2)cdd(3) \otimes (3) = v_8$ $\qquad$ $(3)(3) \otimes (3) = v_9$

$(3)d(3) \otimes (3) = v_{10}$ $\qquad$ $(3)dd(3) \otimes (3) = v_{11}$ $\qquad$ $(3)ddd(3) \otimes (3) = v_{12}$

Now we compute a basis for $f \otimes id_1 \mathrm{Top}(Q_1 \otimes_\Lambda P_1)$. We obtain the following matrix:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}
x_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_4	0	0	0	0	0	0	0	0	0	0	-1	0	0	0
x_5	1	0	0	0	0	0	0	0	0	0	0	0	0	0
x_6	0	1	0	0	0	0	0	0	0	0	0	0	0	0
x_7	0	0	0	0	0	0	0	0	0	0	1	-1	0	0
x_8	0	0	1	0	0	0	0	0	0	0	0	0	0	0
x_9	0	0	0	1	0	0	0	0	0	0	0	0	0	0
x_{10}	0	0	0	0	0	0	0	0	0	0	0	1	-1	0
x_{11}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_{12}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_{13}	0	0	0	0	0	0	0	0	0	0	1	0	0	0
x_{14}	0	0	0	0	0	0	0	0	0	0	0	1	0	0

representing the map on the above bases, which column reduces to the following:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	$x_{11}+x_{12}+x_{13}$	$x_{12}+x_{13}$	x_{13}	x_{14}
x_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_4	0	0	0	0	0	0	0	0	0	0	-1	0	0	0
x_5	1	0	0	0	0	0	0	0	0	0	0	0	0	0
x_6	0	1	0	0	0	0	0	0	0	0	0	0	0	0
x_7	0	0	0	0	0	0	0	0	0	0	0	-1	0	0
x_8	0	0	1	0	0	0	0	0	0	0	0	0	0	0
x_9	0	0	0	1	0	0	0	0	0	0	0	0	0	0
x_{10}	0	0	0	0	0	0	0	0	0	0	0	0	-1	0
x_{11}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_{12}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_{13}	0	0	0	0	0	0	0	0	0	0	1	0	0	0
x_{14}	0	0	0	0	0	0	0	0	0	0	1	1	0	0

which gives us the following basis for $f \otimes id_1(\mathrm{Top}(Q_1 \otimes_\Lambda P_1)) \cong \mathrm{Top}(\Omega^1_\Lambda(M) \otimes P_1)$:

$$\{x_5, x_6, x_8, x_9, x_4 - x_{13} - x_{14}, x_7 - x_{14}, x_{10}\},$$

which is $\{(2)ba(2) \overset{b}{\otimes} (1), (2)ba(2) \overset{c}{\otimes} (3), (2)bab(1) \overset{a}{\otimes} (2), (2)bac(3) \overset{d}{\otimes} (3), (2)c(3) \overset{d}{\otimes} (3) - (3)dd(3) \overset{d}{\otimes} (3) - (3)ddd(3) \overset{d}{\otimes} (3), (2)cd(3) \overset{d}{\otimes} (3) - (3)ddd(3) \overset{d}{\otimes} (3), (2)cdd(3) \overset{d}{\otimes} (3) = x_{10}\}$.

We compute the basis for $\mathrm{Top}(\Omega^1_\Lambda(M) \otimes P_0$ in the same way, first obtaining this matrix:

$$
\begin{array}{ccccccccccccc}
 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} & v_{12} \\
v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
v_4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
v_6 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_7 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
v_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
v_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}
$$

which column reduces to:

$$
\begin{array}{ccccccccccccc}
 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & -\;v_9+v_{10}+v_{11} & -\;v_{10}+v_{11} & v_{11} & v_{12} \\
v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
v_4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
v_6 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_7 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
v_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
v_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0
\end{array}
$$

which gives us the following basis for $\mathrm{Top}(\Omega^1_\Lambda(M) \otimes_\Lambda P_0$: $\{v_4, v_6, v_7, v_3 - v_{11} - v_{12}, v_5 - v_{12}, v_8\}$ $= \{(2)ba(2) \otimes (2), (2)bab(2) \otimes (2), (2)bac(3) \otimes (3), (2)c(3) \otimes (3) - (3)dd(3) \otimes (3) - (3)ddd(3) \otimes (3), (2)cd(3) \otimes (3) - (3)ddd(3) \otimes (3), (2)cdd(3) \otimes (3) = v_8\}$. We can now construct the map from $\mathrm{Top}(\Omega^1_\Lambda(M)\otimes_\Lambda P_1)\mathrm{Top}Q_0\otimes_\Lambda P_0$. It is given by the following matrix:

$$\begin{array}{cccccccc}
 & x_5 & x_6 & x_8 & x_9 & x_4-x_{13}-x_{14} & x_7-x_{14} & x_{10} \\
v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_3 & 0 & 0 & 0 & 0 & -d & 0 & 0 \\
v_4 & -b & -c & 0 & 0 & 0 & 0 & 0 \\
v_5 & 0 & 0 & 0 & 0 & (3) & -d & 0 \\
v_6 & (1) & 0 & -a & 0 & 0 & 0 & 0 \\
v_7 & 0 & (3) & 0 & -d & 0 & 0 & 0 \\
v_8 & 0 & 0 & 0 & 0 & 0 & (3) & -d \\
v_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_{11} & 0 & 0 & 0 & 0 & d & 0 & 0 \\
v_{12} & 0 & 0 & 0 & 0 & -(3)+d & d & 0
\end{array}$$

Now we are ready to compute these elements in terms of our basis for $\Omega^1_\Lambda(M)\otimes_\Lambda P_0$. We recall that the following is a basis for $\mathrm{Top}(\Omega^1_\Lambda(M)\otimes_\Lambda P_0)$: $\{v_4, v_6, v_7, v_3-v_{11}-v_{12}, v_5-v_{12}, v_8\}$. It is then obvious that the image of x_5 will be $v_6 - v_4\cdot b$, the image of x_6 will be $v_7 - v_4\cdot c$, the image of x_8 will be $v_6\cdot -a$, the image of x_9 will be $v_7\cdot -d$, and the image of x_{10} will be $v_8\cdot -d$. However the images of the remaining two basis elements in the domain are less clear. For this we will need to use the method explained in the discussion above. So we take the following obvious bases of $\mathrm{Top}(Q_0\otimes_\Lambda P_0)\cdot(\Gamma_0\cup\Gamma_1)$ and $\mathrm{Top}(\Omega^1_\Lambda(M)\otimes_\Lambda P_0)\cdot(\Gamma_0\cup\Gamma_1)$, $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_1b, v_1c, v_2a, v_3d, v_4b, v_4c, v_5d, v_6a,$ $v_7d, v_8d, v_9d, v_{10}d, v_{11}d, v_{12}d\}$ and $\{v_4, v_6, v_7, v_3-v_{11}-v_{12}, v_5-v_{12}, v_8, v_4b, v_4c,$ $v_6a, v_7d, (v_3-v_{11}-v_{12})d, (v_5-v_{12})d, v_8d\}$ respectively. We see how to write the image of $x_4-x_{13}-x_{14}$ in terms of the v_i and $v_i\cdot\alpha$s, we have it equal to $v_3d\cdot -1 + v_5\cdot 1 + v_{11}d\cdot 1 + v_{12}\cdot -1 + v_{12}d\cdot 1$. If we also consider this image as a linear combination of elements in the other basis, we have it equal to $v_4\cdot k'_1 + v_6\cdot k'_2 + v_7\cdot k'_3 + (v_3-v_{11}-v_{12})\cdot k'_4 + (v_5-v_{12})\cdot k'_5 + v_8\cdot k'_6 + v_4b\cdot k'_{1b} + v_4c\cdot k'_{1c} + v_6a\cdot k'_{2a} + v_7d\cdot k'_{3d} + (v_3-v_{11}-v_{12})d\cdot k'_{4d} + (v_5-v_{12})d\cdot k'_{5d} + v_8d\cdot k'_{6d}$, we reorder the sum to obtain $v_3\cdot k'_4 + v_4\cdot k'_1 + v_5\cdot k'_5 + v_6\cdot k'_2 + v_7\cdot k'_3 + v_8\cdot k'_6 + v_{11}\cdot -k'_4 + v_{12}\cdot(-k'_4 - k'_5) + v_3d\cdot k'_{4d} + v_4b\cdot k'_{1b} + v_4c\cdot k'_{1c} + v_5d\cdot k'_{5d} + v_6a\cdot k'_{2a} + v_7d\cdot k'_{3d} + v_8d\cdot k'_{6d} + v_{11}d\cdot -k'_{4d} + v_{12}d\cdot(-k'_{4d} - k'_{5d})$

and we obtain the following equations:

$$
\begin{array}{ll}
k'_4 = 0 & k'_1 = 0 k'_5 = 0 \\
k'_2 = 0 & k'_3 = 0 k'_6 = 0 \\
-k'_4 = 0 & -k'_4 - k'_5 = -1 k'_5 = 1 \\
k'_{4d} = -1 & k'_{1b} = 0 k'_{1c} = 0 \\
k'_{5d} = 0 & k'_{2a} = 0 k'_{3d} = 0 \\
k'_{6d} = 0 & -k'_{4d} = 1 - k'_{4d} - k'_{5d} = 1
\end{array}
$$

so we see that $k'_5 = 1, k'_4 = 0, k'_{4d} = -1, and k'_{5d} = 0$, so we have the image of $x_4 - x_{13} - x_{14}$ will be equal to $(v_5 - v_{12}) \cdot 1 + (v_3 - v_{11} - v_{12}) \cdot d$ which is $(2)cd(3) \otimes (3) + ((2)c(3) \otimes (3) - (3)dd(3) \otimes (3) - (3)ddd(3) \otimes (3)) \cdot -d$. Similarly one computes that the image of $x_7 - x_{14}$ is $(v_5 - v_1 2) \cdot -d + v_8$. Thus we have computed the following matrix for the map between $\mathrm{Top}(\Omega^1_\Lambda(M) \otimes_\Lambda P_1)$ and $\Omega^1_\Lambda(M) \otimes_\Lambda P_0$ to be:

	x_5	x_6	x_8	x_9	$x_4 - x_{13} - x_{14}$	$x_7 - x_{14}$	x_{10}
v_4	$-b$	$-c$	0	0	0	0	0
v_6	(2)	0	$-a$	0	0	0	0
v_7	0	(3)	0	$-d$	0	0	0
$v_3 - v_{11} - v_{12}$	0	0	0	0	$-d$	0	0
$v_5 - v_{12}$	0	0	0	0	(3)	$-d$	0
v_8	0	0	0	0	0	(3)	$-d$

Now we must compute the map between $\mathrm{Top}(Omega^1_\Lambda(M) \otimes_\Lambda P_0)$ and Q_0. The map between $\mathrm{Top}(Q_0 \otimes_\Lambda P_0)$ and Q_0 is given by:

$$v_1 \mapsto (2) \qquad v_2 \mapsto (2)b \qquad v_3 \mapsto (2)c$$

$$v_4 \mapsto (2)ba \qquad v_5 \mapsto (2)cd \qquad v_6 \mapsto (2)bab$$

$$v_7 \mapsto (2)bac \qquad v_8 \mapsto (3) \qquad v_{10} \mapsto (3)d$$

$$v_{11} \mapsto (3)dd \qquad v_{12} \mapsto (3)ddd$$

so we have the following map between $\mathrm{Top}(\Omega^1_\Lambda(M) \otimes_\Lambda P_0)$ and Q_0:

$$v_4 \mapsto (2)ba \qquad v_6 \mapsto (2)bab \qquad v_7 \mapsto (2)bac$$

$$(v_3 - v_{11} - v_{12}) \mapsto (2)c - (3)dd - (3)ddd \qquad (v_5 - v_{12}) \mapsto (2)cd - (3)ddd$$

$$v_8 \mapsto (3)$$

and we put all of this together to obtain the following resolution:

$$(1)\Lambda \coprod (2)\Lambda \coprod \coprod_5 (3)\Lambda \xrightarrow{d_2} (1)\Lambda \coprod (2)\Lambda \coprod \coprod_4 (3)\Lambda \xrightarrow{d_1} (2)\Lambda \coprod (3)\Lambda \to M \to 0$$

where the matrices d_2 and d_1 are given below:

$$\begin{array}{r|ccccccc} & x_5 & x_8 & x_6 & x_9 & \begin{array}{c} x_4 \\ -x_{13} \\ -x_{14} \end{array} & \begin{array}{c} x_7 \\ -x_{14} \end{array} & x_{10} \\ \hline v_6 & (1) & -a & 0 & 0 & 0 & 0 & 0 \\ v_4 & -b & 0 & -c & 0 & 0 & 0 & 0 \\ v_7 & 0 & 0 & (3) & -d & 0 & 0 & 0 \\ v_3 - v_{11} - v_{12} & 0 & 0 & 0 & 0 & -d & 0 & 0 \\ v_5 - v_{12} & 0 & 0 & 0 & 0 & (3) & -d & 0 \\ v_8 & 0 & 0 & 0 & 0 & 0 & (3) & -d \end{array}$$

$$\begin{array}{c|cccccc} & v_6 & v_4 & v_7 & v_3 - v_{11} - v_{12} & v_5 - v_{12} & v_8 \\ \hline (2) & bab & ba & bac & c & cd & 0 \\ (3) & 0 & 0 & 0 & -dd - ddd & -ddd & (3) \end{array}$$

Now we point out how one could begin this process at any step in the resolution. We again begin with M given in the form of a presentation

$$Q^1 \xrightarrow{f} Q^0$$

where $M \cong \mathrm{Coker}(f)$. Suppose one is interested in computing the $n+1$st, nth and $n-1$st projectives in a resolution of M, along with the necessary maps between them. If we had a projective presentation of $\Omega_\Lambda^{n-1}(M)$ (or $\Omega_\Lambda^{n-1}(M) \oplus P$ for some projective module P) we could use the above techniques to compute the desired part of the projective resolution.

To do this, we compute $Q^1 \otimes_\Lambda \otimes_\Lambda^{n-1} \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P_0$, $Q^1 \otimes_\Lambda \otimes_\Lambda^{n-2} \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0$, $Q^0 \otimes_\Lambda \otimes_\Lambda^{n-1} \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0$, and $Q^0 \otimes_\Lambda \otimes_\Lambda^{n-2} \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0$. We note that we have the following picture:

$$\begin{array}{ccc}
Q^1 \otimes_\Lambda \otimes_\Lambda^{n-1} \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0 & \rightarrow & Q^1 \otimes_\Lambda \otimes_\Lambda^{n-2} \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0 \\
\downarrow & & \downarrow \\
Q^0 \otimes_\Lambda \otimes_\Lambda^{n-1} \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0 & \rightarrow & Q^0 \otimes_\Lambda \otimes_\Lambda^{n-2} \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0 \\
\downarrow & & \downarrow \\
M \otimes_\Lambda \otimes_\Lambda^{n-1} \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0 & \rightarrow & M \otimes_\Lambda \otimes_\Lambda^{n-2} \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0 \\
\downarrow & & \downarrow \\
0 & 0 & 0.
\end{array}$$

The modules in the bottom row are projective, and hence the epimorphisms split, with the kernels of the bottom vertical maps being equal to the images of the top vertical maps, $\Omega^1_\Lambda(M) \otimes_\Lambda \otimes_\Lambda^{n-1} \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0$ and $\Omega^1_\Lambda(M) \otimes_\Lambda \otimes_\Lambda^{n-2} \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0$ respectively.

If we recall that $\otimes_\Lambda^j \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0 \cong \coprod_{(p_1,\cdots,p_j) \in Seq(j)} o(p_1) \otimes t(p_j)$ it is an easy extension of previous results to obtain the following:

Lemma 7.3 *If $Q = \coprod_I v_i \Lambda$ is a projective Λ-module, and $P^j = \otimes_\Lambda^j \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0$ then*

$$Q \otimes_\Lambda P^j \cong \coprod_I \coprod_{(p_1,\dots,p_j)\in Seq(j)} \coprod_{v_i\Lambda o(p_1)} t(p_j)\Lambda\Box$$

In this way we can compute the modules $Q^1 \otimes_\Lambda \otimes_\Lambda^{n-1} \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0$, $Q^1 \otimes_\Lambda \otimes_\Lambda^{n-2} \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0$, $Q^0 \otimes_\Lambda \otimes_\Lambda^{n-1} \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0$, and $Q^0 \otimes_\Lambda \otimes_\Lambda^{n-2} \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0$. Computing the vertical maps between them is done in exactly the same manner as was done in the previous resolution example. One takes a basis element of $Top(Q^1 \otimes_\Lambda -)$ and applies $f \otimes_\Lambda id$ to it, to obtain an element of $Top(Q^0 \otimes_\Lambda -)$. A matrix is obtained, column reduced to produce a basis for the image, and we obtain bases for $\Omega^1_\Lambda(M) \otimes_\Lambda -$. One computes the map between these two modules in the same way as in the above resolution example, and in this way obtains a projective presentation:

$$\begin{array}{c}
\Omega^1_\Lambda(M) \otimes_\Lambda \otimes_\Lambda^{n-1} \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0 \\
\downarrow \\
\Omega^1_\Lambda(M) \otimes_\Lambda \otimes_\Lambda^{n-2} \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda P^0 \\
\downarrow \\
\Omega^1_\Lambda(M) \otimes_\Lambda \otimes_\Lambda^{n-2} \Omega^1_{\Lambda^e}(\Lambda) \otimes_\Lambda \Lambda \\
\downarrow \\
0
\end{array}$$

of a module which is isomorphic to $\Omega^{n-1}_\Lambda(M) \oplus P$ where P is projective. This presentation is then input into the method of computing a projective resolution of a module given in the form of a presentation to obtain a projective resolution.

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Vita

Nathan A. Smith was born on November 16, 1971. He grew up in Burtonsville, MD and graduated high school in 1989, following which he enrolled in the Virginia Polytechnic Institute and State University to pursue a degree in Horticulture. During his sophomore year he declared a double major in Mathematics, and in May 1994 Nathan graduated from Virginia Tech, receiving B.S. degrees in both Horticulture and Mathematics. In August of that year he began graduate study in Mathematics at Virginia Tech, and in September of that year he married Stacy Lynn Mehringer. Assuming all goes well Nathan will receive the doctoral degree in Mathematics in May, 1999 and will begin his career as Assistant Professor of Mathematics at the University of Texas at Tyler. Nathan and Stacy are expecting their first child in September, 1999.