THE CAUCHY PROBLEM FOR THE DIFFUSIVE-VLASOV-ENSKOG EQUATIONS

by

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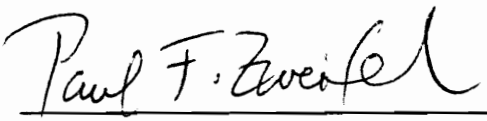

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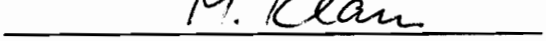

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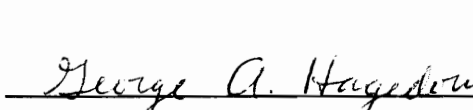

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ABSTRACT

In order to better describe dense gases, a smooth attractive tail arising from a Coulomb-type potential is added to the hard core repulsion of the Enskog equation, along with a velocity diffusion. By choosing the diffusing term of Fokker-Planck type with or without dynamical friction forces, the Cauchy problem for the Diffusive-Vlasov-Poisson-Enskog equation (DVE) and the Cauchy problem for the Fokker-Planck-Vlasov-Poisson-Enskog equation (FPVE) are addressed.

Chapters II - V focus on global existence of *renormalized solutions* of (DVE). The main tool used here is a sequential stability theorem, which, based on the fact that the operator $L_\lambda = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla_{\vec{r}} - \lambda \Delta_{\vec{v}}$ acts like a hypoelliptic operator from $L^1((0,T) \times R^3 \times R^3))$ to $L^1((0,T) \times R^3 \times R^3) \oplus L^1((0,T) \times R^3_{\vec{r}}; L^2(R^3_{\vec{v}}))$, concludes that any weakly compact set of solutions of (DVE) is strongly compact and the limits are renormalized solutions. The existence of global-in-time solutions to the renormalized equation (DVE) is proved for arbitrary L^1 initial conditions with finite

mass, energy and entropy. In Chapter VI, these results are extended to the equation (FPVE).

The last part of the paper, Chapter VII, introduces the concept of $\mathcal{B}$-type mild solutions for non-linear evolution equations in general Banach spaces. The existence and uniqueness of this kind of solution, locally and globally, is investigated for such equations even with unbounded discontinuous nonlinear terms. The theory is applied finally to address the global existence of mild solutions of the Fokker-Planck-Vlasov equation, the equation (DVE) and the equation (FPVE) with special geometrical factors.

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CONTENTS

Chapter I. Introduction

The aim of statistical mechanics is to explain and predict the properties of macroscopic matter from the properties of its microscopic constituents. The subject may naturally be divided into an *equilibrium* and a *nonequilibrium* part. Historically, nonequilibrium statistical mechanics has taken two directions: *kinetic theory of gases* and *Brownian motion theory*. The subject studied in this paper, roughly speaking, belongs to the first direction.

§1.1 Boltzmann and Enskog Equations

In this section we shall derive the equations which we wish to study along the line of their historic development. As is well known, the basic concept in kinetic theory is the (one-particle) *density function (distribution function)*, which, denoted by $f(\vec{r}, \vec{v}, t)$, is defined, according to Boltzmann, in such a way that

$$f(\vec{r}, \vec{v}, t)d\vec{r}d\vec{v}$$

is the *probable* number of the molecules (or other kinds of particles studied) that lie at time t within an element $d\vec{r}$ around point $\vec{r}$ and have a velocity in the element $d\vec{v}$ around $\vec{v}$. Consider a gas in which each molecule of mass m is subject to an external force $m\vec{F}$, which may be a function of $\vec{r}$ and t, but not of $\vec{v}$. Between the time t and $t + dt$ the velocity $\vec{v}$ of any molecule that does not collide with another will change, according to the Newton's second law, to $\vec{v} + \vec{F}dt$, and its position vector $\vec{r}$ will change to $\vec{r} + \vec{v}dt$; the number in this set is

$$f(\vec{r} + \vec{v}dt, \vec{v} + \vec{F}dt, t + dt)d\vec{r}d\vec{v}.$$

Since there are collisions happening among molecules, the number of molecules in the second set generally deffers from that in the first set. The number of net gain (or loss) of molecules to the second set is approximately proportional to $d\vec{r}d\vec{v}dt$, and is denoted by $(\partial f/\partial t)_{coll}d\vec{r}d\vec{v}dt$. A first order approximation gives

$$\{f(\vec{r}+\vec{v}dt, \vec{v}+\vec{F}dt, t+dt) - f(\vec{r},\vec{v},t)\}d\vec{r}d\vec{v} = (\frac{\partial f}{\partial t})_{coll}d\vec{r}d\vec{v}dt.$$

Dividing both sides by $d\vec{r}d\vec{v}dt$, and letting $dt \to 0$, we get the following form of the *Liouville Equation*:

$$\frac{\partial}{\partial t}f(\vec{r},\vec{v},t) + \vec{v}\cdot\nabla_{\vec{r}}f(\vec{r},\vec{v},t) + \vec{F}(\vec{r},t)\cdot\nabla_{\vec{v}}f = (\frac{\partial f}{\partial t})_{coll}. \tag{1.1}$$

The particular choice of physical model determines the different forms of the collision terms in equation (1.1), and accordingly, different equations. Two well-known assumptions in the dilute gases theory state as follows[12],[42]:

Assumption I. *In the low-density limit, we can limit ourselves to binary collisions and consider them as instantaneous and local in space.*

Assumption II. ("Molecular-chaos assumption" or "Stosszahlansatz"). *The number of pairs of molecules in the element $d\vec{r}$ with respective velocities in the range $(\vec{v}, \vec{v}+dv)$ and $(\vec{v}_1, \vec{v}_1+d\vec{v}_1)$, which are able to participate in a collision is given by*

$$f(\vec{r},\vec{v},t)d\vec{v}d\vec{r}f(\vec{r},\vec{v}_1,t)d\vec{r}d\vec{v}_1.$$

From assumption I, the collision term $(\partial f/\partial t)_{coll}$ can be decomposed into two terms: $(\frac{\partial f}{\partial t})_{coll} = C'' - C'$, where $C'd\vec{r}d\vec{v}dt$ is the number of *binary collisions* in the time interval dt of molecules lying in the range $(\vec{r}, \vec{r}+d\vec{r}; \vec{v}, \vec{v}+d\vec{v})$ and deflected to any other velocity $\vec{v}'$, and $C''d\vec{r}d\vec{v}dt$ is the number of *binary collisions* in the time interval

dt of molecules lying in the element $(\vec{r}, \vec{r}+d\vec{r})$ with arbitrary initial velocity $\vec{v}'$ and ending up after the collision with a velocity in the given range $(\vec{v}. \vec{v}+d\vec{v})$. Using the molecular-chaos assumption, we can express the *gain-loss* term explicitly in terms of the *(differential) scattering cross section* $\sigma(\vec{\Omega}, \vec{g})$, where, roughly speaking, $\sigma(\vec{\Omega}, \vec{g})d\vec{\Omega}$ indicates the probability of a molecule, having the initial velocity $\vec{g}$ and sent to the fixed potential, that is deflected into a solid angle $d\vec{\Omega}$ around the polar angles $\vec{\Omega}$ of the final velocity $\vec{g}'$. If two molecules, with before-collision velocities of $\vec{v}$ and $\vec{v}_1$, emerge after collision with velocities $\vec{v}'$ and $\vec{v}_1'$. Then energy and momentum conservation demand:

$$\vec{v}_1 + \vec{v} = \vec{v}_1' + \vec{v}', \qquad v^2 + v_1^2 = v'^2 + v_1'^2. \tag{1.2}$$

In this case, $\vec{g} = \vec{v}_1 - \vec{v}$ and $\vec{g}' = \vec{v}_1' - \vec{v}'$ are the relative velocities before and after the collision. Integrating over all (possible) deflection angles $\vec{\Omega}$ and all velocities $\vec{v}_1$ as well as using the relationship

$$\frac{\partial(\vec{v}', \vec{v}_1')}{\partial(\vec{v}, \vec{v}_1)} = 1,$$

we derive the celebrated *Boltzmann Equation* in the dilute gas theory:

$$\begin{aligned} &\frac{\partial}{\partial t} f(\vec{r}, \vec{v}, t) + \vec{v} \cdot \nabla_{\vec{r}} f(\vec{r}, \vec{v}, t) + \vec{F}(\vec{r}, t) \cdot \nabla_{\vec{v}} f = \left(\frac{\partial f}{\partial t}\right)_{coll} = \\ &= \int d\vec{v}_1 \int d\vec{\Omega}\, \sigma(\vec{\Omega}, |\vec{g}|)|\vec{g}|[f(\vec{r}, \vec{v}', t) f(\vec{r}, \vec{v}_1', t) - f(\vec{r}, \vec{v}, t) f(\vec{r}, \vec{v}_1, t)]. \end{aligned} \tag{BB}$$

Despite its pre-eminent role in nonequilibrium statistical mechanics, the Boltzmann equation is known to be valid only in the dilute-gas regime, indeed yielding transport coefficients of an ideal fluid. Enskog, first in 1921, attempted to rectify this situation by introducing a Boltzmann-like collision process with hard core interaction representing particles with non-zero diameter. The Enskog equation, in several

modified and *revised* versions introduced in the 1970's in order to obtain correct hydrodynamics, describes a non-ideal fluid with transport coefficients within 10% of those of realistic numerical models up to one-half close packing density.

Unlike the Boltzmann theory, which describes the behavior of a dilute gas, the Enskog equation deals with a dense gas consisting of hard spheres. Interestingly enough, in the derivation of the equation, multiple collisions are neglected completely and the dynamics is described by two-body events, just as in the Boltzmann theory. Since the gas is dense, Enskog modifies the Boltzmann theory in the following way:

(i) Modifying the frequency of these binary events by invoking geometrical effects;

(ii) Taking into account the *collision transfer*; i.e., in a collision, the momentum $\Delta\vec{p} = m(\vec{v}_1' - \vec{v}_1) = -m(\vec{v}' - \vec{v})$ exchanged between the molecules is suddenly transferred from the center of one molecule to the center of the other.

Since the molecule is a sphere with diameter a, a collision does not take place *at a given point.* Rather, we have to replace

$$f(\vec{r},\vec{v},t)f(\vec{r},\vec{v}_1,t) \Longrightarrow f(\vec{r},\vec{v},t)f(\vec{r}-a\vec{\epsilon},\vec{v}_1,t),$$

where $\vec{\epsilon}$ is the unit vector along the line joining the centers of the molecules. We use Y^E to indicate the geometrical factor, which depends only on the density at the point of contact, $Y^E = Y^E(n(\vec{r} - \frac{1}{2}a\vec{\epsilon},t))$. From classical scattering theory, the scattering cross section for a hard sphere potential is $\sigma(\vec{\epsilon},\vec{g}) = a^2(\frac{\vec{\epsilon}\cdot\vec{g}}{g})\Theta(\vec{\epsilon}\cdot\vec{g})$. Therefore from (1.1) and (1.2) we can surmise the *Standard Enskog Equation*:

$$\frac{\partial}{\partial t}f + \vec{v}\cdot\nabla_{\vec{r}}f + \vec{F}\cdot\nabla_{\vec{v}}f = C_E(f,f) =$$
$$= a^2 \iint\limits_{R^3\times S^2_+} ((\vec{v}-\vec{v}_1)\cdot\vec{\epsilon})\Theta((\vec{v}-\vec{v}_1)\cdot\vec{\epsilon})\{Y^E(n(\vec{r}+\frac{a\vec{\epsilon}}{2},t))f_1(\vec{r},\vec{v}',t)f(\vec{r}+a\vec{\epsilon},\vec{v}_1',t)$$
$$- Y^E(n(\vec{r}-\frac{a\vec{\epsilon}}{2},t))f(\vec{r},\vec{v},t)f(\vec{r},\vec{v}_1,t)\}\, d\vec{v}_1 d^2\vec{\epsilon}. \qquad \text{(SEE)}$$

Unfortunately, this Standard Enskog Equation does not yield correct hydrodynamics. In the next section, we will indicate a derivation of a revised Enskog equation via BBGKY hierarchy. which corrects this deficiency.

§1.2 BBGKY Hierarchy and Vlasov-Enskog Equation

A limitation of the Enskog equation, unlike the Boltzmann equation, is that it incorporates only hard-sphere molecular interactions. In order to account for more realistic potentials, several extensions of the Enskog equation have been proposed in the statistical mechanics community, especially by de Sobrino, Grmela, Davis, Rice, Sengers, Stell, van Beijeren and co-workers (see, for example, [8], [21], [24], [25], [28], [29], [31], [32], [46], [47]).

One strategy toward improving the Enskog theory is based on the addition of an intermolecular potential tail to the *hard-core* repulsion. In this direction the addition of a square well potential to the repulsive hard core has been studied by Grmela, Davis, Rice, Sengers, and by Karkheck, van Beijeren, de Schepper, Stell, Liu, Greenberg, Polewczak, and others (cf. [24], [31], [46], [33]), obtaining a kinetic equation with multiple Enskog-like collision terms. Another approach is to add a smooth attractive tail to the hard core. This direction is first studied by Luis de Sobrino for the nonequilibrium problem of a van der Waals gas.

Consider a system consisting of N particles with Hamiltonian

$$H_N = H_N^0 + V_N, \tag{1.3a}$$

$$H_N^0 = \sum_{i=1}^{N} \frac{mv_i^2}{2}, \qquad V_N = \sum_{j>i=1}^{N} V(r_{ij}), \tag{1.3b}$$

where $(\vec{r}_1, \cdots, \vec{r}_N, \vec{v}_1, \cdots, \vec{v}_N) \equiv (\mathfrak{r}, \mathfrak{v})$ denotes the positions and velocities of the particles, H_N^0 is the total kinetic energy of the system, and V_N represents the total

potential energy, which is the sum of all distinct pair interactions $V(r_{ij})$ with $r_{ij} = |\vec{r}_i - \vec{r}_j|$. Define the *N-particles distribution function* $\rho_N(\mathfrak{r}, \mathfrak{v}, t)$ in the same manner as in the one-particle distribution function. It is well-known that the distribution function $\rho_N(\mathfrak{r}, \mathfrak{v}, t)$ satisfies the *N-particles Liouville equation*:

$$\frac{\partial}{\partial t}\rho_N(\mathfrak{r}, \mathfrak{v}, t) = \{H_N, \rho_N\}, \tag{1.4}$$

where we have used the *Poisson bracket* $\{ * , * \}$ defined as:

$$\{\hat{A}, \hat{B}\} = \frac{1}{m}\sum_{a=1}^{N}(\nabla_{\vec{r}_a}\hat{A} \cdot \nabla_{\vec{v}_a}\hat{B} - \nabla_{\vec{v}_a}\hat{A} \cdot \nabla_{\vec{r}_a}\hat{B}),$$

for any two functions $\hat{A}$ and $\hat{B}$ of $(\mathfrak{r}, \mathfrak{v})$. If we use

$$\theta_{ij} = \frac{1}{m}\nabla_{\vec{r}_i}V(r_{ij}) \cdot (\nabla_{\vec{v}_i} - \nabla_{\vec{v}_j}), \tag{1.5a}$$

then the Liouville equation (1.4) can be rewritten as

$$\frac{\partial}{\partial t}\rho_N(\mathfrak{r}, \mathfrak{v}, t) = -\sum_{i=1}^{N}\vec{v}_i \cdot \nabla_{\vec{r}_i}\rho_N + \sum_{b>i=1}^{N}\theta_{ij}\rho_N. \tag{1.5}$$

Introducing the *(reduced) specific distribution function* ρ_l and the *(reduced) generic distribution function* f_l by

$$\rho_l(\vec{r}_1, \cdots, \vec{r}_l, \vec{v}_1, \cdots, \vec{v}_l, t) = \int d\vec{r}_{l+1} \cdots d\vec{r}_N d\vec{v}_{l+1} \cdots d\vec{v}_N \rho_N(\mathfrak{r}, \mathfrak{v}, t), \tag{1.6a}$$

$$f_l(\vec{r}_1, \cdots, \vec{r}_l, \vec{v}_1, \cdots, \vec{v}_l, t) = \frac{N!}{(N-l)!}\rho_l(\vec{r}_1, \cdots, \vec{r}_l, \vec{v}_1, \cdots, \vec{v}_l, t), \tag{1.6b}$$

and integrating (1.5) over $\vec{r}_{l+1}, \cdots, \vec{r}_N, \vec{v}_{l+1}, \cdots, \vec{v}_N$, we find

$$\frac{\partial}{\partial t}f_l(\vec{r}_1, \cdots, \vec{r}_l, \vec{v}_1, \cdots, \vec{v}_l, t) =$$
$$= -\sum_{i=1}^{N}\vec{v}_i \cdot \nabla_{\vec{r}_i}f_l(\vec{r}_1, \cdots, \vec{r}_l, \vec{v}_1, \cdots, \vec{v}_l, t) + \sum_{j>i=1}^{l}\theta_{ij}f_l(\vec{r}_1, \cdots, \vec{r}_l, \vec{v}_1, \cdots, \vec{v}_l, t)$$

$$+\sum_{i=1}^{l}\int d\vec{r}_{l+1}d\vec{v}_{l+1}\theta_{i,l+1}f_{l+1}(\vec{r}_1,\cdots,\vec{r}_{l+1},\vec{v}_1,\cdots,\vec{v}_{l+1},t), \tag{1.7}$$

which, when we take $l = 1, 2, \cdots, N$, forms the well-known BBGKY hierarchy. In particular, for $l = 1$, we have

$$\begin{aligned}&\frac{\partial}{\partial t}f_1(\vec{r}_1,\vec{v}_1,t) + \vec{v}_1\cdot\nabla_{\vec{r}_1}f_1(\vec{r}_1,\vec{v}_1,t)\\ &= \int d\vec{r}_2 d\vec{v}_2\,\frac{1}{m}\nabla_{\vec{r}_1}V(r_{12})\cdot\nabla_{\vec{v}_1}f_2(\vec{r}_1,\vec{r}_2,\vec{v}_1,\vec{v}_2,t).\end{aligned} \tag{1.8}$$

Let us point out that equation (1.8) is asymptotically exact when we assume that the system is formed by a very large number, N, of identical particles and the boundary effects are negligible.

Define the correlation function

$$g_2(\vec{r}_1,\vec{r}_2,\vec{v}_1,\vec{v}_2,t) = \frac{f_2(\vec{r}_1,\vec{r}_2,\vec{v}_1,\vec{v}_2,t)}{f(\vec{r}_1,\vec{v}_1,t)f(\vec{r}_2,\vec{v}_2,t)}. \tag{1.9}$$

Then (1.8) can be rewritten as

$$\begin{aligned}&\frac{\partial}{\partial t}f(\vec{r}_1,\vec{v}_1,t) + \vec{v}_1\cdot\nabla_{\vec{r}_1}f(\vec{r}_1,\vec{v}_1,t)\\ &= \int d\vec{r}_2 d\vec{v}_2\,\frac{1}{m}\nabla_{\vec{r}_1}V(r_{12})\cdot\nabla_{\vec{v}_1}[g_2(\vec{r}_1,\vec{r}_2,\vec{v}_1,\vec{v}_2,t)f(\vec{r}_1,\vec{v}_1,t)]f(\vec{r}_2,\vec{v}_2,t).\end{aligned} \tag{1.8'}$$

For hard-sphere collisions, the potential $V(r)$ takes the form

$$V^H(r) = \begin{cases}\infty & r \le a,\\ 0 & r > a.\end{cases}$$

At the one particle level, one can assume that the correlation function does not depend on the velocity, i.e.,

$$f_2(\vec{r}_1,\vec{r}_2,\vec{v}_1,\vec{v}_2,t) = g_2(\vec{r}_1,\vec{r}_2|n(t))f(\vec{r}_1,\vec{v}_1,t)f(\vec{r}_2,\vec{v}_2,t), \tag{1.10}$$

where $n(t) = n(\vec{r},t) \equiv \int d\vec{v}\, f(\vec{r},\vec{v},t)$. Expanding the solution of (1.8′) $f(\vec{r},\vec{v},t+\tau)$ in terms of powers of $n(\vec{r},t)$ and letting $\tau \to 0$,[29] or using the *pseudo-Liouville equation*,[42] we get the following *revised Enskog equation*[47]:

$$[\frac{\partial}{\partial t} + \vec{v}\cdot\nabla_{\vec{r}}]f(\vec{r},\vec{v},t) = C_E(f,f)(\vec{r},\vec{v},t),$$
$$C_E(f,f)(\vec{r},\vec{v},t) = a^2 \iint_{R^3\times S^2} [g_2(\vec{r},\vec{r}+a\vec{\epsilon}|n(t))f(\vec{r},\vec{v}',t)f(\vec{r}+a\vec{\epsilon},\vec{v}_1',t) -$$
$$g_2(\vec{r},\vec{r}-a\vec{\epsilon}|n(t))f(\vec{r},\vec{v},t)f(\vec{r}-a\vec{\epsilon},\vec{v}_1,t)]\Theta(\vec{\epsilon}\cdot(\vec{v}-\vec{v}_1)) < \vec{\epsilon},\vec{v}-\vec{v}_1 > d\vec{\epsilon}d\vec{v}_1, \tag{REE}$$

where

$$\vec{v}' = \vec{v} - \vec{\epsilon} < \vec{\epsilon},\vec{v}-\vec{v}_1 >, \quad \vec{v}_1' = v_1 + \vec{\epsilon} < \vec{\epsilon},\vec{v}-\vec{v}_1 >,$$

and $\vec{\epsilon}$ is the unit vector from the center of the target particle to the center of colliding particle. Since the correlation function g_2 depends only on the local density at $\vec{r}$ and $\vec{r}\pm a\epsilon$, using traditional notations we can write $g_2(\vec{r},\vec{r}\pm a\vec{\epsilon}|n(t)) = Y(n(\vec{r},t), n(\vec{r}\pm a\vec{\epsilon}))$.

On the other hand, if the particles interact through a repulsive hard-core potential of diameter a, as well as a weak attractive pair potential $\phi^{tail}(r_{ij})$,

$$V(r) = \begin{cases} \infty & r \le a, \\ \phi^{tail}(r) & r > a, \end{cases}$$

then, in terms of the two-particle distribution function $f_2(\vec{r},\vec{v},\vec{r}',\vec{v}',t)$, from (1.8) we derive the following equation[23]

$$[\frac{\partial}{\partial t} + \vec{v}\cdot\nabla]f(\vec{r},\vec{v},t) - \frac{1}{m}\int d\vec{v}_1 \int_{|\vec{r}_1-\vec{r}|>a} d\vec{r}_1\, \nabla_{\vec{r}}\phi^{tail}|\vec{r}_1-\vec{r}|\cdot\nabla_{\vec{v}} f_2(\vec{r},\vec{v},\vec{r}_1,\vec{v}_1,t)$$
$$= \int d\vec{v}_1 \int_{S_2^+} d\vec{\epsilon}\vec{\epsilon}\cdot(\vec{v}_1-\vec{v})a^2[f_2(\vec{r},\vec{r}+a\vec{\epsilon},\vec{v},\vec{v}_1,t) - f_2(\vec{r},\vec{r}-a\vec{\epsilon},\vec{v},\vec{v}_1,t)], \tag{1.11}$$

where $S_2^+ = S_2^+(\vec{v},\vec{v}_1) = \{\vec{\epsilon}\in S_2 | \vec{\epsilon}\cdot(\vec{v}_1-\vec{v}) > 0\}$. The weakness of the attractive potential tail ϕ^{tail} permits us to neglect the correlations in the integral on the left-hand side of (1.11) and to extend the integration within the sphere $|\vec{r}_1-\vec{r}| \le a$,

since the contribution of the region where the correlations due to the hard core are important will be very small. Denoting the extended tail potential by $\mathcal{V}(\vec{r})$, in this case, the third term of equation (1.11) can thus be written

$$-\frac{1}{m}\nabla_{\vec{v}} f(\vec{r},\vec{v},t)\cdot\nabla_{\vec{r}}\int d\vec{r}_1\,\mathcal{V}(|\vec{r}_1-\vec{r}|)n(\vec{r}_1.t). \tag{1.12}$$

Replacing the two-particle distribution function in (1.1) by one-particle distribution, as was carried out in (REE), leads to the *Vlasov-Enskog* equations[29]:

$$[\frac{\partial}{\partial t}+\vec{v}\cdot\nabla_{\vec{r}}]f(\vec{x},t) = -\vec{E}\cdot\nabla_{\vec{v}} f(\vec{x},t)+C_E(f.f). \tag{VE1}$$

$$\vec{E} = \int d\vec{r}_2\,\nabla_{\vec{r}}\mathcal{V}(\vec{r}_{12})n(\vec{r}_2,t) = \int d\vec{r}_2\,\hat{\vec{r}}_{12}\mathcal{V}'(\vec{r}_{12})n(\vec{r}_2,t) \tag{VE2}$$

Equations similar to (VE1)-(VE2) were first derived by de Sorbrino,[32] but with the Enskog collision terms replaced by a perturbation of the Boltzmann collision. Grmela et al.[24] studied the solution of the linearized version of the Vlasov-Enskog equations. The Vlasov-Enskog equation studied in our paper was first derived by George Stell et al.[28][29] by using the method of maximization of entropy, and taking the Kac-limit for the tail potential, $\Phi_{12}^{tail} = \lim_{\gamma\to 0}\gamma^3 v(\gamma r)$. In their work, they also obtain some useful properties of the Vlasov-Enskog equation, such as an H-theorem.

§1.3 Fokker-Planck Terms

Let us mention some problem arising when trying to derive the Vlasov-Enskog equation from the exact classical dynamics. First, in the above derivation of the equation from the BBGKY hierarchy, the velocity correlations are completely neglected. One method of taking into account the velocity correlation effects is the addition of the Fokker-Planck terms.[26] Moreover, in the derivation above, we assume that the tail-potential has no influence on the binary collision. If we account for this

influence in some approximate way (small-angle scattering), an additional term of Fokker-Planck type should also appear. Therefore, we will consider in this paper the so-called Fokker-Planck-Vlasov-Enskog, or Diffusive-Vlasov-Enskog, equation.

In general, the Fokker-Planck term, or Kolmogorov forward process, as it is called by some authors, has different forms. The most simple, but very important, case is to treat the term just as the Laplacian $\Delta_{\vec{v}} f$ of the distribution function f about the velocity $\vec{v}$. Another very popular treatment is to write that process as $\Delta_{\vec{v}} f + \frac{\theta}{2}\mathrm{div}_{\vec{v}}(\vec{v} f)$. From the point of view of physics, $\Delta_{\vec{v}}$ denote the thermal background interaction, and $\mathrm{div}_{\vec{v}}(\vec{v} f)$, the dynamical friction forces. In other words, in the first case, the Fokker-Planck term takes only the thermal background interaction into account, but without the dynamical friction forces. The second situation takes care of both thermal background interaction and the dynamical friction forces. Although both cases are frequently referred to as Fokker-Planck terms in the literature, we will call the first case a diffusive term and the second one a Fokker-Planck term. In conclusion, in order to better describe the behavior of dense gases, we get two kinds of systems: the first one, called the Diffusive-Vlasov-Enskog equation in this paper, is given by

$$[\frac{\partial}{\partial t} + \vec{v} \cdot \nabla_{\vec{r}}] f(\vec{r}, \vec{v}, t) + \vec{E}(\vec{r}, t) \cdot \nabla_{\vec{v}} f(\vec{r}, \vec{v}, t) - \lambda \Delta_{\vec{v}} f = C_E(f, f), \qquad \text{(DVE)}$$

and the second one, called the Fokker-Planck-Vlasov-Enskog equation, is given by

$$[\frac{\partial}{\partial t} + \vec{v} \cdot \nabla_{\vec{r}}] f(\vec{r}, \vec{v}, t) + \vec{E}(\vec{r}, t) \cdot \nabla_{\vec{v}} f(\vec{r}, \vec{v}, t) - \eta \mathrm{div}_{\vec{v}}(\vec{v} f + \frac{\theta}{2} \nabla_{\vec{v}} f) = C_E(f, f), \quad \text{(FPVE)}$$

where $\vec{E}(\vec{r}, t)$ is assumed to satisfy the equation

$$\vec{E}(\vec{r}, t) = \int_{R^D} d\vec{r}_1 \, \nabla_{\vec{r}} \mathcal{V}(|\vec{r} - \vec{r}_1|) \, n(\vec{r}_1). \qquad \text{(VE)}$$

Here, R^D denotes real D-dimensional position or momentum space, θ, η and λ are positive numbers, and $C_E(f,f)$, is the Enskog collision term. Let us point out that, although it seems that the treatment of the (DVE) is simpler than that of the (FPVE), the conclusions about the solution of the two systems, from the point view of either mathematics or physics, can not be inferred from each other.

§1.4 Outline of Results

In this paper we will study the Cauchy problem for the equations (DVE)-(VE) as well as (FPVE)-(VE) with given initial condition

$$\lim_{t\to+0} f(\vec{r},\vec{v},t) = f_0(\vec{r},\vec{v}). \tag{IC}$$

in the Banach space L^1. We will use the revised Enskog collision term with the form of

$$C_E(f,f)(\vec{r},\vec{v},t) = a^2 \iint_{R^3\times S^2_+} [Y(n(\vec{r},t), n(\vec{r}-a\vec{\epsilon},t))f(\vec{r},\vec{v}',t)f(\vec{r}-a\vec{\epsilon},\vec{v}_1',t) - Y(n(\vec{r},t), n(\vec{r}+a\vec{\epsilon},t))f(\vec{r},\vec{v},t)f(\vec{r}+a\vec{\epsilon},\vec{v}_1,t)] < \vec{\epsilon}.\vec{v}-\vec{v}_1 > d\vec{\epsilon}d\vec{v}_1, \tag{ET}$$

$$\vec{v}' = \vec{v} - \vec{\epsilon} < \vec{\epsilon}, \vec{v}-\vec{v}_1 >, \quad \vec{v}_1' = v_1 + \vec{\epsilon} < \vec{\epsilon}, \vec{v}-\vec{v}_1 >,$$

and $n(\vec{r},t) = \int d\vec{v}\, f(\vec{r},\vec{v},t)$. The geometric factor $Y(\vec{r}_1,\vec{r}_2)$ in principle is a functional of f and should be determined by the Mayer cluster expansion.[37] The tail potential $\mathcal{V}$ is chosen as the *Coulomb potential*, $\mathcal{V}(\vec{r}) = -\alpha\Gamma(D/2)(2(2-D))^{-1}\pi^{-D/2}|\vec{r}|^{2-D}$, where $D > 2$ is the dimension of the space. The field term is assume to satisfy the following Poisson equation:

$$\mathrm{div}_{\vec{r}}\vec{E}(\vec{r},t) = -\alpha n(\vec{r},t). \tag{VP}$$

The paper is naturally divided into three parts. The first part. including Chapter II – Chapter V, deals with the so-called *renormalized solution* of the equations (defined below) with only the thermal background interaction considered in the diffusive term, i.e. the system (DVE)-(VP)-(IC). The second part, Chapter VI, consider the renormalized solution of the equation in which the diffusive term includes both thermal interaction and dynamical friction forces, that is, the (FPVE)-(VP)-(IC) equation. In the last chapter both (DVE)-(VP)-(IC) and (FPVE)-(VP)-(IC) are considered, for the *mild solutions* and *$\mathcal{B}$-type mild solutions* rather than renormalized solutions.

Now we introduce the main results of the first two parts of this paper and sketch the processes of the proof. A nonnegative function f of $C([0,\infty), L^1(R^3_{\vec{r}} \times R^3_{\vec{v}}))$ is called a renormalized solution of (DVE)-(VP)-(IC) if the composite function $g_\delta(f) = \frac{1}{\delta}\log(1+\delta f)$ satisfies the equation

$$\frac{\partial}{\partial t} g_\delta + \vec{v}\cdot\nabla_{\vec{r}} g_\delta - \lambda\Delta_{\vec{v}} g_\delta = \frac{1}{1+\delta f} C_E(f,f) - \vec{E}\cdot\frac{\partial}{\partial \vec{v}} g_\delta + \lambda\delta|\nabla_{\vec{v}} g_\delta|^2, \qquad \text{(RDVE)}$$

and f satisfies (VP)-(IC) in the sense of distributions. Similarly, f is called the renormalized solution of (FPVE)-(VP)-(IC) if g_δ satisfies

$$\begin{aligned} &\frac{\partial}{\partial t} g_\delta + \vec{v}\cdot\nabla_{\vec{r}} g_\delta + \vec{E}(\vec{r},t)\cdot\nabla_{\vec{r}} g_\delta - \eta \mathrm{div}_{\vec{v}}(g_\delta \vec{v} + \frac{\theta}{2}\nabla_{\vec{v}} g_\delta) + g_\delta = \\ &= \frac{1}{1+\delta g} C_E(f,f) + \delta\frac{\theta\eta}{2}|\nabla_{\vec{v}} g_\delta|^2 + N\frac{f}{1+\delta f}, \end{aligned} \qquad \text{(RME)}$$

and f satisfies (VP)-(IC) in the distribution sense. These equations are obtained from the original kinetic equations by replacing the unknown distribution f with $g_\delta(f)$.

The main result of first two parts is:

MAIN THEOREM. *a) Assume that $Y(\sigma,\tau) = Y(\tau,\sigma)$ is a jointly continuous function satisfying the boundedness condition $\sigma Y(\sigma,\tau) \le M_Y < \infty$, the initial value $f_0(\vec{r},\vec{v}) \ge$*

0 *satisfies the boundedness condition*

$$\iint_{R^3 \times R^3} d\vec{r}\, d\vec{v} f_0(\vec{r},\vec{v})(1 + |\vec{r}|^2 + |\vec{v}|^2 + |\log f_0|) \leq C < \infty,$$

and $\vec{E}_0(\vec{r}) = \nabla_{\vec{r}}(\frac{1}{|\vec{r}|} * n_0)(\vec{r}) = \nabla_{\vec{r}}(\frac{1}{|\vec{r}|} * \int f_0(\vec{r},\vec{v})\, d\vec{v})(\vec{r})$ *satisfies*

$$\int_{R^3} |\vec{E}_0(\vec{r})|^2\, d\vec{r} \leq C < \infty.$$

Then, there exists $f \in C([0,\infty); L^1(R^3 \times R^3))$ *which is a renormalized solution of (DVE)-(VP)-(IC).*

b) Under the same conditions as in a), the equations (FPVE)-(VT)-(IC) have renormalized solutions.

For the equation (DVE), the proof of the theorem is based upon the following ideas. In Chapter II, we give some useful estimates, including conservation of mass, a bound for the total kinetic and the field energy as well as a bound for the entropy, provided the system has classical solutions. From a mathematics point of view, these bounds indicate the weakly pre-compactness of the set of solutions. Chapter III addresses sequential stability results. From the boundedness arguments, approximate solutions can be obtained, which form a weakly pre-compact set in the Banach space $L^1((0,T) \times R^3 \times R^3)$. The sequential stabilities theorem concludes that the approximate solutions set is pre-compact, and limits are renormalized solutions of the system. This kind of idea is not new, for example, similar results are obtained by DiPerna and Lions for the Fokker-Planck-Boltzmann equation.[17] In fact, our approach to the proof is very similar to theirs. The increased difficulty here is in part due to the fact that we have to show the pre-compactness of $\{g_\delta^n\}$ with $L_\lambda g_\delta^n = h_1^n + h_2^n$, where

$$L_\lambda = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla_{\vec{r}} - \lambda \Delta_{\vec{v}},$$

$\{h_1^n\}$ is weakly pre-compact is $L^1((0,T)\times R^3\times R^3)$, but $\{h_2^n\}$ is only weakly pre-compact in $L^1([0,T]\times R^3_r; L^2(R^3_v))$ (see, e.g., Lemma 3.4). In other words, L_λ is similar to hypoelliptic operator from $L^1((0,T)\times R^3\times R^3)$ to $L^1((0,T)\times R^3\times R^3)\oplus L^1([0,T]\times R^3_r; L^2(R^3_v))$. From the compactness argument and some detailed measure and distribution theory considerations, we deduce that g_δ, a weak limit of g_δ^n, solves the (RDVE). Chapter IV and Chapter V provide the existence (and uniqueness) argument for the solution of the approximate equations and the existence of the renormalized solution of (DVE)-(VP)-(IC). The technique used for the existence of approximate solutions is the contraction mapping method, and the fact that the operator consisting of the Laplacian and the Vlasov term generates a positive semigroup in some functional Banach spaces.

The idea for demonstrating the solution for the (FPVE)-(VP)-(IC) is similar, but with some technical modification. For this reason, the second part of the paper is written in an abbreviated fashion. We do not present the proof of the theorems in great detail. Instead, we outline key facts differing from the first part, which set the new problem into the frame work of the first part.

The third part of this paper is basically to consider the following abstract nonlinear evolution equation

$$\begin{cases} \frac{d}{dt}f(t) - \mathcal{A}f(t) = \mathcal{J}(f(t)), \\ f(t=0) = f_0, \end{cases} \tag{ABS}$$

in some Banach space $\mathcal{X}$, where $\mathcal{A}:\mathcal{D}(\mathcal{A})\to\mathcal{X}$ is a closed linear operator which generates a c_0-semigroup $\mathbf{U}(t)$, and $\mathcal{J}(\cdot)$ is an unbounded discontinuous nonlinear operator from $\mathcal{D}(\mathcal{J})(\supset\mathcal{D}(\mathcal{A}))$ to $\mathcal{X}$. For a given closed linear operator $\mathcal{B}$, we call an $\mathcal{X}$-valued function $f(t)$ as a *$\mathcal{B}$-type mild solution* of the equation (ABS) if $f(t)$ is a

mild solution of (ABS), i.e., the solution of the *integral equation*

$$f(t) = \mathbf{U}(t)f_0 + \int_0^t \mathbf{U}(t-s)\mathcal{J}(f(s))\,ds,$$

and both $f(t)$ and $\mathcal{B}f(t)$ are continuous in t. We give some conditions for $\mathcal{A}$, $\mathcal{B}$, $\mathcal{J}$ under which the equation (ABS) has a unique $\mathcal{B}$-type mild solution locally and globally. Using the theory obtained, we show that the Fokker-Planck-Vlasov equation and the Diffusive-Vlasov equation have globally a unique mild solution $f(\vec{r}, \vec{v}, t)$ such that $\frac{\partial}{\partial v_i} f(\vec{r}, \vec{v}, t)$, $i = 1, 2, 3$, is a continuous $L^1(R^3 \times R^3) \cap C(R^3_{\vec{r}}; L^1(R^3_{\vec{v}}))$-function of t. Furthermore, we give some examples of the geometrical factor Y with which the Fokker-Planck-Vlasov-Enskog equation and Diffusive-Vlasov-Enskog equation have locally and globally a $\nabla_{\vec{v}}$-type mild solutions.

§1.5 A Short Review

In the remainder of this section, we review some works on related subjects. The first local in time existence theorem of the Enskog equation was obtained by Lachowicz.[30] A global in time existence theorem was obtained by Toscani and Bellomo[45] in the case of a perturbation of the vacuum. Polewczak[39] showed the solution obtained in [30] is actually a classical solution if the initial datum is smooth. Cercignani[11] obtained global in time solutions for small initial data in L^1 and $Y = 1$. All of the above results deal with the standard Enskog equation, but with easy modifications can be extended to the revised Enskog equation. Furthermore, those results refer either to small initial data or to local in time existence. For large initial data, Cercignani[10] obtained global in time L^1-solutions in the case of one-spatial dimension and $Y = 1$. Arkeryd[2] considered the two-spatial-dimension case using a weak compactness argument in L^1, however with the range of integration with respect to

$\vec{\epsilon}$ extended to the whole sphere S^2, together with the assumption that $Y \equiv 1$, i.e., the *Enskog-Boltzmann* equation. Moreover, Arkeryd[3] obtained global existence for $Y \equiv 1$ under the assumption that the initial value was differentiable in $\vec{r}$ in the L^1 sense. and Arkeryd and Cercignani[4] gave a proof of global existence with arbitrary L^1 data in the case of periodic boundary conditions for $Y \equiv 1$. For the Vlasov-Enskog equation, Grmela[24] first studied existence for the linearized Enskog-Vlasov equation as well as bifurcation of the equilibrium stationary solution to the nonlinear Enskog-Vlasov equation. The Enskog-like term in that paper is actually an improved Boltzmann collision operator. Global existence of a renormalized solution of the Fokker-Planck-Boltzmann equation was first studied by DiPerna and Lions.[15] Existence of global weak solutions for Vlasov-Poisson-Fokker-Planck systems was studied by Victory et al.[48][49] and for the Maxwell-Vlasov system by Dressler, DiPerna, Lions etc.[15][16][19] The reader is referred the references there for further review on related subjects.

Chapter II. Conserved Quantities and Bounds

In this chapter, we obtain some useful identities and inequalities, including bounds for mass, energy, spatial momentum as well as entropy. These bounds are obtained based upon the assumption that f is a classical solution of equation (DVE)-(VP) if no additional description is provided.

§2.1 Conservation of Mass and Energy Bounds

First, we assume a symmetry condition on the geometric function

$$Y(\sigma,\tau) = Y(\tau,\sigma). \tag{2.1}$$

Then we have[37],[38]

$$\begin{aligned}
&\iint_{R^3\times R^3} \phi(\vec{r},\vec{v})\, C_E(f,f)\, d\vec{v}d\vec{r} = \\
&= \frac{a^2}{2} \iiiint_{R^3\times R^3\times R^3\times S^2_+} [\phi(\vec{r},\vec{v}') + \phi(\vec{r}+a\vec{\epsilon},\vec{v}_1') - \phi(\vec{r},\vec{v}) - \phi(\vec{r}+a\vec{\epsilon},\vec{v}_1)] \\
&\qquad \times f(\vec{r},\vec{v},t) f(\vec{r}+a\vec{\epsilon},\vec{v}_1,t) Y(\vec{r},\vec{r}+a\vec{\epsilon}) < \epsilon, \vec{v}-\vec{v}_1 > d\vec{\epsilon}d\vec{v}_1 d\vec{v}d\vec{r},
\end{aligned} \tag{2.2}$$

which gives

$$\begin{aligned}
&\frac{d}{dt} \iint_{R^3\times R^3} f(\vec{r},\vec{v},t)\, d\vec{v}d\vec{r} = \\
&= \lambda \iint_{R^3\times R^3} \Delta_{\vec{v}} f\, d\vec{v}d\vec{r} - \iint_{R^3\times R^3} \vec{E}\cdot\nabla_{\vec{v}} f\, d\vec{v}d\vec{r} + \iint_{R^3\times R^3} C_E(f,f)\, d\vec{v}d\vec{r}
\end{aligned}$$

$$=\lambda \iint\limits_{R^3\times R^3} \mathrm{div}_{\vec{v}}(\nabla_{\vec{v}} f) - \iint\limits_{R^3\times R^3} \mathrm{div}_{\vec{v}} f\vec{E}\, d\vec{v}d\vec{r} = 0,$$

or

$$\iint\limits_{R^3\times R^3} f(\vec{r},\vec{v},t)\, d\vec{v}d\vec{r} = \iint\limits_{R^3\times R^3} f(\vec{r},\vec{v},0)\, d\vec{v}d\vec{r} = M_1 \quad \forall t \geq 0,$$

(conservation of mass.) (2.3)

Next, we want to get some information about the energy. Multiplying both sides of (DVE) by v^2 and integrated over $R^3 \times R^3$, we have formally

$$\iint\limits_{R^3\times R^3} [v^2 \frac{\partial f(\vec{x},t)}{\partial t} + v^2\vec{v}\cdot\nabla f(\vec{x},t)]\, d\vec{v}d\vec{r} + \iint\limits_{R^3\times R^3} v^2\vec{E}\cdot\frac{\partial}{\partial\vec{v}} f(\vec{x},t)\, d\vec{v}d\vec{r} =$$

$$=\lambda \iint\limits_{R^3\times R^3} v^2\Delta_{\vec{v}} f\, d\vec{v}d\vec{r} + \iint\limits_{R^3\times R^3} C_E(f,f)\, d\vec{v}d\vec{r}. \tag{2.4}$$

Since $\vec{E}$ is a function of $\vec{r}$ and t only,

$$\mathrm{div}_{\vec{v}}((v^2 f))\vec{E} = (\mathrm{div}_{\vec{v}}\vec{E})v^2 f + \vec{E}\cdot\nabla_{\vec{v}} v^2 f = 2\vec{E}\cdot\vec{v} f + v^2\vec{E}\cdot\nabla_{\vec{v}} f,$$

or

$$v^2\vec{E}\cdot\nabla_{\vec{v}} f = \mathrm{div}_{\vec{v}}(v^2 f\vec{E}) - 2\vec{E}\cdot\vec{v} f.$$

Also, by (2.2), the last term on the RHS of (2.4) is 0, and

$$\iint\limits_{R^3\times R^3} v^2\Delta_{\vec{v}} f\, d\vec{v}d\vec{r} = \iint\limits_{R^3\times R^3} \mathrm{div}_{\vec{v}}(v^2\nabla_{\vec{v}} f)\, d\vec{v}d\vec{r} - 2\iint\limits_{R^3\times R^3} \vec{v}\cdot\nabla_{\vec{v}} f\, d\vec{v}d\vec{r} =$$

$$= -2\iint\limits_{R^3\times R^3} \mathrm{div}_{\vec{v}} f\vec{v}\, d\vec{v}d\vec{r} + 2\iint\limits_{R^3\times R^3} f(\vec{r},\vec{v},t)\, d\vec{v}d\vec{r} = 2\iint\limits_{R^3\times R^3} f(\vec{r},\vec{v},t=0)\, d\vec{v}d\vec{r}.$$

Therefore, at least formally, we have

$$\frac{d}{dt}\iint\limits_{R^3\times R^3} v^2 f(...)\, d\vec{v}d\vec{r} = 2\lambda \iint\limits_{R^3\times R^3} f(\vec{r},\vec{v},t=0)\, d\vec{v}d\vec{r} + 2\int_{R^3} d\vec{r}\vec{E}\cdot\vec{j}, \tag{2.5}$$

where

$$\vec{j}(\vec{r},t) = \int \vec{v} f(\vec{r},\vec{v},t)\, d\vec{v}. \tag{2.5a}$$

Next, let us consider $\frac{d}{dt}\int_{R^3} E^2\, d\vec{r}$. Note that, at least formally, $\int_{R^3} \vec{E}\cdot\frac{\partial f}{\partial \vec{v}}\, d\vec{v} = \int_{R^3} \mathrm{div}_{\vec{v}} f\vec{E}\, d\vec{v} = 0$, $\int_{R^3} \Delta_{\vec{v}} f\, d\vec{v} = \int_{R^3} \mathrm{div}_{\vec{v}}\nabla_{\vec{v}} f\, d\vec{v} = 0$, and from (DVE)-(VP) we have

$$\frac{\partial}{\partial t} n(\vec{r},t) = -\int d\vec{v}\, \vec{v}\cdot\nabla_{\vec{r}} f + \int_{R^3} C_E(f.f)(\vec{r},\vec{v},t)\, d\vec{v},$$

$$\begin{aligned}
\frac{\partial}{\partial t}\vec{E} &= \int_{R^3} \nabla_{\vec{r}_2-\vec{r}}\mathcal{V}(\vec{r}_2-\vec{r})\, d\vec{r}_2 \int_{R^3} C_E(f,f)(\vec{r}_2,\vec{v},t)\, d\vec{v} \\
&\quad - \int_{R^3} \nabla_{\vec{r}_2-\vec{r}}\mathcal{V}(\vec{r}_2-\vec{r})\, d\vec{r}_2 \int_{R^3} \vec{v}\cdot\frac{\partial f}{\partial \vec{r}}\, d\vec{v} \\
&= \int_{R^3} \nabla_{\vec{r}_2-\vec{r}}\mathcal{V}(\vec{r}_2-\vec{r})\, d\vec{r}_2 \int_{R^3} C_E(f,f)(\vec{r}_2,\vec{v},t)\, d\vec{v} - \iint\limits_{R^3\times R^3} \nabla_{\vec{r}_2-\vec{r}}\mathcal{V}(\vec{r}_2-\vec{r}) div_{\vec{r}_2} f\vec{v} \\
&= \int_{R^3} \nabla_{\vec{r}_2-\vec{r}}\mathcal{V}(\vec{r}_2-\vec{r})\, d\vec{r}_2 \int_{R^3} C_E(f,f)(\vec{r}_2,\vec{v},t)\, d\vec{v} - \iint\limits_{R^3\times R^3} d\vec{r}_2 d\vec{v}\, \nabla_{\vec{r}}(\nabla_{\vec{r}}\mathcal{V}\cdot f\vec{v}) + \\
&\quad + \iint\limits_{R^3\times R^3} d\vec{r}_2 d\vec{v}\, \Delta_{\vec{r}}\mathcal{V}(\vec{r}_2-\vec{r}) f\vec{v},
\end{aligned} \tag{2.6}$$

where we have used

$$\nabla_{\vec{r}}(\nabla_{\vec{r}}\mathcal{V}\cdot f\vec{v}) = \Delta_{\vec{r}}\mathcal{V} f\vec{v} + \nabla_{\vec{r}}\mathcal{V}\mathrm{div}_{\vec{r}}(f\vec{v}).$$

In (2.2), let $\phi(\vec{r}_2,\vec{v}) = \phi(\vec{r}_2) = \nabla\mathcal{V}(\vec{r}_2-\vec{r})$. We assert that the first term on the right hand side of (2.6) is 0. Also, at least formally with some restriction, the second term on the RHS of (2.6) is equal to zero. Therefore, we have

$$\frac{d}{dt}\int_{R^3} d\vec{r}\,|\vec{E}|^2 = 2\int_{R^3} d\vec{r}\,\vec{E}\cdot\frac{d\vec{E}}{dt} = 2\int_{R^3} d\vec{r}\,\vec{E}\cdot\iint\limits_{R^3\times R^3} d\vec{r}_2 d\vec{v}\,\Delta\mathcal{V}(\vec{r}_2-\vec{r}) f\vec{v} \tag{2.7}$$

Now, if $\mathcal{V}$ satisfies (LPE), then

$$\frac{d}{dt}\int_{R^3} d\vec{r}\,|\vec{E}|^2 = 2\int_{R^3} d\vec{r}\,\vec{E}\cdot\iint\limits_{R^3\times R^3} d\vec{r}_2 d\vec{v}\, f\vec{v}(-\alpha)\delta(\vec{r}_{12}) =$$

$$= -2\alpha\int_{R^3} d\vec{r}\,\vec{E}\cdot\int_{R^3}\vec{v}f(\vec{r},\vec{v},t)\,d\vec{v} = -2\alpha\int_{R^3} d\vec{r}\,\vec{E}\cdot\vec{j}. \tag{2.8}$$

Thus, (2.5) and (2.8) imply

$$\frac{d}{dt}(\alpha\iint\limits_{R^3\times R^3} v^2 f(\vec{r},\vec{v},t)\,d\vec{v}d\vec{r} + \int_{R^3} d\vec{r}\,|\vec{E}|^2) = const. = 2\lambda\alpha M_1. \tag{2.9}$$

With no loss of generality we assume $\alpha = 1$, which, in turn, implies that

$$\iint\limits_{R^3\times R^3} v^2 f(\vec{r},\vec{v},t)\,d\vec{v}d\vec{r} + \int_{R^3} d\vec{r}\,|\vec{E}(\vec{r},t)|^2 = M_2 + 2\lambda M_1 t, \quad \forall t \geq 0, \tag{2.10}$$

where

$$M_2 = \iint\limits_{R^3\times R^3} v^2 f(\vec{r},\vec{v},t=0)\,d\vec{v}d\vec{r} + \int_{R^3} d\vec{r}\,|\vec{E}(\vec{r},t=0)|^2.$$

Now let us estimate the term $\iint r^2 f\,d\vec{v}d\vec{r}$. Introduce the *Liapunov* function $\mathbf{E}(t)$ by

$$\mathbf{E}(t) = \iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}(\vec{r}-t\vec{v})^2 f(\vec{r},\vec{v},t). \tag{2.11}$$

Since

$$(\frac{d}{dt} + \vec{E}(\vec{r}+t\vec{v},t)\cdot\nabla_{\vec{v}} - \lambda\Delta_{\vec{v}})f(\vec{r}+t\vec{v},\vec{v},t) = C_E(f,f)(\vec{r}+t\vec{v},\vec{v},t),$$

$$(\frac{d}{dt} + \vec{E}(\vec{r}+t\vec{v},t)\cdot\nabla_{\vec{v}} - \lambda\Delta_{\vec{v}})r^2 f(\vec{r}+t\vec{v},\vec{v},t) = r^2 C_E(f,f)(\vec{r}+t\vec{v},\vec{v},t),$$

$$\frac{d}{dt}\mathbf{E}(t) = \iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}(\vec{r}-t\vec{v})^2 f(\vec{r},\vec{v},t) =$$

$$= -\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}\vec{E}(\vec{r},t)\cdot\nabla_{\vec{v}}(\vec{r}-t\vec{v})^2 f(\vec{r},\vec{v},t) + \lambda\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}\Delta_{\vec{v}} f(\vec{r},\vec{v},t)(\vec{r}-t\vec{v})^2 +$$

$$+ \iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}(\vec{r}-t\vec{v})^2 C_E(f,f)(\vec{r},\vec{v},t)$$

$$= \iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}(\vec{r}-t\vec{v})^2 C_E(f,f)(\vec{r},\vec{v},t) = -2t \iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}\vec{r}\cdot\vec{v}C_E(f,f)(\vec{r},\vec{v},t) =$$

$$= -ta^3 \iiiint\limits_{R^3\times R^3\times R^3\times S^2_+} d\vec{\epsilon}d\vec{v}_1 d\vec{v}d\vec{r} < \vec{\epsilon}, \vec{v}-\vec{v}_1 >^2 f(\vec{r},\vec{v},t)f(\vec{r}+a\vec{\epsilon},\vec{v}_1,t)Y(n(\vec{r}),n(\vec{r}+a\vec{\epsilon}))$$

$$\leq 0, \tag{2.11'}$$

we have

$$\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}(\vec{r}-t\vec{v})^2 f(\vec{r},\vec{v},t) = \mathbf{E}(t) \leq \mathbf{E}(0) = \iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}r^2 f_0(\vec{r},\vec{v}) \equiv M_3. \tag{2.12}$$

By the *Cauchy-Schwarz* inequality and (2.10),

$$\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}r^2 f \leq (t\sqrt{M_2+2\lambda M_1 t}+\sqrt{M_3})^2. \tag{2.13}$$

§2.2 A Bound for Entropy

Multiplying both sides of (DVE) by $\log f$ and integrating over $R^3\times R^3$,

$$\log f\,\frac{\partial f}{\partial t} + \log f\,\vec{v}\cdot\frac{\partial f}{\partial \vec{r}} + \log f\,\vec{E}\cdot\frac{\partial f}{\partial \vec{v}} = \log f\,C_E(f,f) + \lambda \log f \Delta_{\vec{v}} f,$$

$$\{\frac{\partial}{\partial t} + \vec{v}\cdot\nabla_{\vec{r}} + \vec{E}\cdot\nabla_{\vec{v}} - \lambda\Delta_{\vec{v}}\} f\log f = (1+\log f)C_E(f,f) - \lambda f^{-1}|\nabla_{\vec{v}} f|^2,$$

$$\frac{d}{dt}\iint\limits_{R^3\times R^3} f\log f\,d\vec{v}d\vec{r} =$$

$$= \iint\limits_{R^3\times R^3} \mathrm{div}_{\vec{v}}(\vec{E}f\log f + \lambda\nabla_{\vec{v}} f\log f)\,d\vec{v}d\vec{r} + \iint\limits_{R^3\times R^3} (\log f C_E(f,f) - \lambda f^{-1}|\nabla_{\vec{v}} f|^2)\,d\vec{v}d\vec{r}$$

$$= \iint\limits_{R^3\times R^3} \log f C_E(f,f)\,d\vec{v}d\vec{r} - 4\lambda \iint_{R^3\times R^3} |\nabla_{\vec{v}}\sqrt{f}|^2\,d\vec{v}d\vec{r}. \tag{2.14}$$

Define

$$\Gamma(t) = \iint\limits_{R^3 \times R^3} f(\vec{r}, \vec{v}, t) \log f(\vec{r}, \vec{v}, t)\, d\vec{v} d\vec{r} - \int_0^t \mathbf{I}(s)\, ds + \int_0^t \mathbf{J}(s)\, ds, \tag{2.15}$$

where

$$\begin{aligned}\mathbf{I}(t) = \frac{1}{2} \iiiint_{R^3 \times R^3 \times R^3 \times S^2_+} & d\vec{\epsilon} d\vec{v}_1 d\vec{v} d\vec{r} f(\vec{r}, \vec{v}, t)[f(\vec{r} - a\vec{\epsilon}, \vec{v}_1, t) Y(n(\vec{r}), n(\vec{r} - a\vec{\epsilon})) \\ & - f(\vec{r} + a\vec{\epsilon}, \vec{v}_1, t) Y(n(\vec{r}), n(\vec{r} + a\vec{\epsilon}))] < \vec{\epsilon}, \vec{v} - \vec{v}_1 > \\ = \mathbf{I}^+(t) - \mathbf{I}^-(t), & \end{aligned} \tag{2.16a}$$

$$\mathbf{J}(t) = 4\lambda \iint_{R^3 \times R^3} |\nabla_{\vec{v}} \sqrt{f(\vec{r}, \vec{v}, t)}|^2\, d\vec{v} d\vec{r}. \tag{2.16b}$$

Letting $y = f(\vec{r}, \vec{v}, t) f(\vec{r} + a\vec{\epsilon}, \vec{v}_1, t)$, $z = f(\vec{r}, \vec{v}', t) f(\vec{r} + a\vec{\epsilon}, \vec{v}_1', t)$, using the inequalities $y(\log y - \log z) \geq (y - z)$, and integrating both sides of (DVE) over $R^3 \times R^3$, we have

$$\frac{d}{dt}\Gamma(t) \leq 0. \tag{2.17}$$

Furthermore,

$$\frac{d}{dt} \iint_{R^3 \times R^3} f \log f\, d\vec{v} d\vec{r} = \frac{d\Gamma(t)}{dt} + \mathbf{I}(t) - \mathbf{J}(t) \leq \mathbf{I}^+(t). \tag{2.18}$$

Now we assume that the scattering function $Y(\tau, \sigma)$ is continuous in τ, σ satisfying

$$\sup_{\sigma, \tau} \tau Y(\tau, \sigma) \leq M_Y < \infty. \tag{2.19}$$

Then

$$\begin{aligned}&\mathbf{I}^+(t) = \\ &= \frac{a^2}{2} \iiiint\limits_{R^3 \times R^3 \times R^3 \times S^2_+} f(\vec{r} - a\vec{\epsilon}, \vec{v}_1, t) f(\vec{r}, \vec{v}, t) Y(n(\vec{r}), n(\vec{r} - a\vec{\epsilon})) < \vec{\epsilon}, \vec{v} - \vec{v}_1 > d\vec{\epsilon} d\vec{v}_1 d\vec{v} d\vec{r}\end{aligned}$$

$$\leq \frac{a^2}{2} \iiiint\limits_{R^3\times R^3\times R^3\times S^2_+} f(\vec{r}-a\vec{\epsilon},\vec{v}_1,t)f(\vec{r},\vec{v},t)Y(n(\vec{r}),n(\vec{r}-a\vec{\epsilon}))(|\vec{v}|+|\vec{v}_1|)\,d\vec{\epsilon}d\vec{v}_1d\vec{v}d\vec{r}$$

$$\leq \frac{a^2}{2} \iiiint\limits_{R^3\times R^3\times R^3\times S^2_+} n(\vec{r}-a\vec{\epsilon},t)f(\vec{r},\vec{v},t)Y(n(\vec{r}-a\vec{\epsilon}),n(\vec{r}))\frac{1}{2}(1+v^2)\,d\vec{\epsilon}d\vec{v}_1d\vec{v}d\vec{r}$$

$$+\frac{a^2}{2} \iiiint\limits_{R^3\times R^3\times R^3\times S^2_+} n(\vec{r},t)f(\vec{r}-a\vec{\epsilon},\vec{v}_1,t)Y(n(\vec{r}),n(\vec{r}-a\vec{\epsilon}))\frac{1}{2}(1+v_1^2)\,d\vec{\epsilon}d\vec{v}_1d\vec{v}d\vec{r}$$

$$\leq 2\pi a^2 M_Y \iint\limits_{R^3\times R^3} R^3 f(\vec{r},\vec{v},t)(1+v^2)\,d\vec{v}d\vec{r} \qquad (\text{ from (2.3) and (2.10) })$$

$$\leq 2\pi a^2 M_Y(M_2+2\lambda M_1 t), \tag{2.20a}$$

and similarly,

$$\mathbf{I}^-(t) \leq 2\pi a^2 M_Y(M_2+2\lambda M_1 t). \tag{2.20b}$$

In addition, suppose that

$$M_4 = \iint\limits_{R^3\times R^3} f(\vec{r},\vec{v},t=0)|\log f(\vec{r},\vec{v},t=0)|\,d\vec{v}d\vec{r} < \infty. \tag{2.21}$$

Then (2.18) and (2.20) imply that there exists a constant $M_5 = M_5(M_1, M_2, M_4, M_Y, T)$ such that

$$\iint\limits_{R^3\times R^3} f(\vec{r},\vec{v},t)\log f(\vec{r},\vec{v},t)\,d\vec{v}d\vec{r} \leq M_5, \quad 0\leq t\leq T. \tag{2.22'}$$

Combining (2.3), (2.9), (2.12) and (2.22') and using the argument as in [17]-[20] we have

$$\iint\limits_{R^3\times R^3} f(\vec{r},\vec{v},t)|\log f(\vec{r},\vec{v},t)|\,d\vec{v}d\vec{r} \leq M_6(M_1,M_2,M_3,M_4,M_Y,T) < \infty \tag{2.22}$$

for $0\leq t\leq T$. In fact, suppose $\iint f\log f\,d\vec{v}d\vec{r}\leq R$, $\iint(1+|\vec{v}|^2+|\vec{r}|^2)f\,d\vec{v}d\vec{r}\leq R$. then

$$\iint f|\log f|\,d\vec{v}d\vec{r} = \iint f\log f\,d\vec{v}d\vec{r} + 2\iint \chi_{(f\leq 1)}f\log(\frac{1}{f})\,d\vec{v}d\vec{r} =$$

$$= R + 2\iint \chi_{(1>f>\exp(-(|\vec{r}|^2+|\vec{v}|^2))} f \log(\frac{1}{f})\, d\vec{v}d\vec{r} +$$

$$+ 2\iint \chi_{(f<\exp(-(|\vec{r}|^2+|\vec{v}|^2))} f \log(\frac{1}{f})\, d\vec{r}d\vec{v}$$

$$\leq 3R + 2\iint \chi_{(f<\exp(-(|\vec{r}|^2+|\vec{v}|^2))} f \log(\frac{1}{f})\, d\vec{r}d\vec{v}$$

$$(\because \exists C_0 = \text{const. such that } t\log\frac{1}{t} \leq C_0\sqrt{t}, \quad \forall 0 < t < 1)$$

$$\leq 3R + 2C_0 \iint \chi_{(f<\exp(-(|\vec{r}|^2+|\vec{v}|^2))} \sqrt{f}\, d\vec{v}d\vec{r} \leq 3R + 2C_0(2\pi)^3,$$

which proves the inequality (2.22).

Finally, integrating both sides of (2.14) on $(0,t)$ one gets

$$g(t) - g(0) = \int_0^t ds \iint_{R^3\times R^3} \log f C_E(f,f)\, d\vec{v}d\vec{r} - 4\lambda \int_0^t \iint_{R^3\times R^3} |\nabla_{\vec{v}}\sqrt{f}|^2\, d\vec{v}d\vec{r} =$$

$$= \iiiiint_{[0,t]\times R^3\times R^3\times R^3\times S^2_+} h(f)\, d\vec{\epsilon}d\vec{v}_1 d\vec{v}d\vec{r}ds - 4\lambda \iiint_{[0,t]\times R^3\times R^3} |\nabla_{\vec{v}}\sqrt{f}|^2\, d\vec{v}d\vec{r}ds,$$

where

$$g(t) = \iint_{R^3\times R^3} f\log f(\vec{r},\vec{v},t)\, d\vec{v}d\vec{r}, \tag{2.23a}$$

$$h(f) = \frac{a^2}{2}\log\frac{f(\vec{r},\vec{v}',s)\, f(\vec{r}+a\vec{\epsilon},\vec{v}_1',s)}{f(\vec{r},\vec{v},s)\, f(\vec{r}+a\vec{\epsilon},\vec{v}_1,s)} Y(n(\vec{r}),n(\vec{r}+a\vec{\epsilon})) < \vec{\epsilon},\vec{v}-\vec{v}_1 > . \tag{2.23b}$$

Letting $h^+(f) = max\{h(f),0\}$, $h^-(f) = max\{-h(f),0\}$, and using the fact $z(\log z - \log y) \geq z - y$ yields

$$\iiiiint_{[0,t]\times R^3\times R^3\times R^3\times S^2_+} h^+(f)\, d\vec{\epsilon}d\vec{v}_1 d\vec{v}d\vec{r}ds \leq \int_0^t \mathbf{I}^+(s)\, ds.$$

Note that $h = h^+ - h^-$, and from (2.3), (2.20a) and (2.22) we get that there exists a constant $M_7 = M_7(M_1, M_2, M_3, M_4, M_Y, T)$ such that

$$\iiiiint_{[0,t]\times R^3\times R^3\times R^3\times S^2_+} |h(f)|\, d\vec{\epsilon}d\vec{v}_1 d\vec{v}d\vec{r}dt + 4\lambda \iiint_{[0,T]\times R^3\times R^3} |\nabla_{\vec{v}}\sqrt{f}|^2\, d\vec{v}d\vec{r}dt \leq M_7. \tag{2.24}$$

Chapter III. Sequential Stability and the Proof

This chapter will address a key result of the first part of the paper, i.e., the sequential stability theorem. We first introduce the definition of a renormalized solution of the equation. Then we investigate some properties of the *transport-diffusion* operator. Finally, we give and prove the sequential stability results (Theorem 3.1).

§3.1 Sequential Stability

Consider a sequence f_n of nonnegative solutions of (DVE). We assume that $f_n \in W^{2,\infty}(R^n \times R^n \times [0,\infty))$, $f_n \to 0$ as $(\vec{r},\vec{v}) \to \infty$ uniformly in $t \in [0,T]$ for all $T < \infty$, $E^n \in W^{2,\infty}(R^3 \times [0,\infty))$, $E^n \to 0$ as $r \to \infty$, and there exists a constant C_T independent of n such that

$$\iint_{R^3 \times R^3} f_n(\vec{r},\vec{v},t)(1+|\vec{r}|^2+|\vec{v}|^2+|\log f_n|)\,d\vec{v}d\vec{r} \le C_T, \tag{3.1}$$

$$\int_{R^3} |E_n(\vec{r},t)|^2 d\vec{r} \le C_T, \tag{3.2}$$

$$\int_0^T dt \iint_{R^3 \times R^3} \{|\nabla_{\vec{v}}\sqrt{f_n}|^2 + |\log f_n C_E(f_n,f_n)|\}\,d\vec{v}d\vec{r} \le C_T. \tag{3.3}$$

Also note that assumptions (3.2) and (3.3) imply that for all $R < \infty$ there is a constant $C = C(T,R)$ such that

$$\int_{B_R} d\vec{v} \int_0^T dt \int_{R^3} d\vec{r} |\nabla_{\vec{v}}\sqrt{f_n} \cdot \vec{E}_n| \le C(T,R), \tag{3.4a}$$

and

$$\int_0^T dt \int d\vec{r} \Big(\int |\nabla_{\vec{v}}\sqrt{f_n} \cdot \vec{E}_n|^2\, d\vec{v} \Big)^{1/2} < C_T. \tag{3.4b}$$

In view of the preceding chapter, these bounds are automatically satisfied provided the basic physical identities (2.2), (2.3), (2.9) and (2.14) are justified and provided (3.1) and (3.2) hold at $t = 0$. The justification of these and related identities becomes necessary only when we address the question of the existence of a solution of (DVE) and analyze sequences of approximate solutions. For the moment we shall assume for simplicity that (3.1)-(3.4) hold. Because of (3.1) and (3.2) we may assume by passing to a subsequence that f_n converges weakly in $L^1(R^n \times R^n \times [0,T])$ to some f for all T. and $\vec{E}_n$ weakly in $L^2(R^n \times [0,T])$ to $\vec{E}$.

In order to deal with the term $C_E(f,f)$, DiPerna and Lions in their treatment of a Fokker-Planck-Boltzmann equation[17] introduce a new formulation which consisted of renormalization by a suitable non-linear transformation of the dependent variable f. Suppose f is a smooth nonnegative solution of (DVE)-(VP)-(IC). Then $g_\delta = \beta_\delta(f) \equiv \frac{1}{\delta}\log(1+\delta f)$ solves the following renormalized version of (DVE),

$$\frac{\partial}{\partial t}g_\delta + \vec{v}\cdot\nabla_{\vec{r}}g_\delta - \lambda\Delta_{\vec{v}}g_\delta = \frac{1}{1+\delta f}C_E(f,f) - \vec{E}\cdot\frac{\partial}{\partial\vec{v}}g_\delta + \lambda\delta|\nabla_{\vec{v}}g_\delta|^2, \qquad \text{(RDVE)}$$

$$\operatorname{div}_{\vec{r}}\vec{E} = -n(\vec{r},t), \qquad \text{(VP)}$$

which motivates the following definition.

DEFINITION. *A nonnegative element f of $C([0,\infty), L^1(R^3_{\vec{r}} \times R^3_{\vec{v}}))$ is a renormalized solution of (DVE)-(VP)-(IC) if for any $\delta > 0$, the composite function $g_\delta = \beta_\delta(f)$ satisfies (RDVE) and f satisfies (VP)-(IC) in the sense of distributions, where $\beta_\delta(t) = \frac{1}{\delta}\log(1+\delta t)$.*

We state now the sequential stability theorem for the Diffusive Vlasov-Enskog system. This result is a key ingredient in the construction of an existence theorem for (DVE)-(VE).

THEOREM 3.1. *Under the assumptions (3.1)-(3.2)-(3.3), $1 \le p < \infty$, $T > 0$, then the sequence f_n converges in $L^p(0,T;L^1(R^3_{\vec{r}} \times R^3_{\vec{v}}))$ to a renormalized solution f which satisfies (3.1)-(3.2) for a.e. $t \in (0,T)$ and (3.3)-(3.4). Furthermore, for every $\delta > 0$, the renormalized interaction terms satisfy*

$$\begin{cases} C_E^-(f,f)(1+\delta f)^{-1}|_{\vec{v}\in B_R} \in C([0,\infty);L^1(R^3_{\vec{r}} \times B_R)), & \forall R < \infty \\ C_E^+(f,f)(1+\delta f)^{-1}|_{\vec{v}\in B_R} \in L^1([0,\infty) \times R^3_{\vec{r}} \times B_R), & \forall R, T < \infty \end{cases}$$

and $g_\delta|_{\vec{v}\in B_R} \in L^2((0,T) \times R^3_{\vec{r}}; H^1(B_R)) \quad (\forall R, T < \infty)$.

§3.2 Properties of the Diffusive Transport Operator

To prove the theorem, we need some information about the partial diffusive transport operator:

$$L_\lambda = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla_{\vec{r}} - \lambda \Delta_{\vec{v}}. \tag{3.5}$$

LEMMA 3.1[17]. *The operator L_λ has a fundamental solution (Green's function) $p(t,\vec{r},\vec{v};\vec{r}',\vec{v}')$ satisfying*

(i)

$$\sup_{t\in(0,T],\,(\vec{r}'\vec{v}')\in R^3\times R^3} \iint p(t,\vec{r},\vec{v};\vec{r}',\vec{v}')\,d\vec{v}d\vec{r} \le C(T), \quad \forall T < \infty. \tag{3.6}$$

(ii)

$$p(t,\vec{r},\vec{v};\vec{r}',\vec{v}') \le C(T,h) \text{ if } t \in [h,T], (\vec{r},\vec{v}), (\vec{r}',\vec{v}') \in R^3 \times R^3, \quad \forall 0 < h < T < \infty, \tag{3.7}$$

(iii)

$$\iint\limits_{|(\vec{r}',\vec{v}')|\le M, t\in[h,T]} \iint\limits_{|(\vec{r},\vec{v})|\ge R} p(t,\vec{r},\vec{v};\vec{r}',\vec{v}')\,d\vec{v}d\vec{r} \to 0 \quad as \quad R \to \infty,$$

$$\forall M < \infty, \quad \forall 0 < h < T < \infty. \tag{3.8}$$

Moreover, the Green's function $p(t,\vec{r},\vec{v};\vec{r}',\vec{v}')$ *is given by*

$$\chi_{(0,\infty)}(t)\frac{3\sqrt{3}}{8}(\frac{1}{\lambda\pi t^2})^3\exp[-\frac{1}{\lambda t}(\frac{1}{4}(\vec{v}-\vec{v}')^2+\frac{1}{4}(\vec{v}+\vec{v}')^2-\frac{3(\vec{r}-\vec{r}')\cdot(\vec{v}+\vec{v}')}{t}+\frac{3(\vec{r}-\vec{r}')^2}{t^2})]. \tag{3.9}$$

PROOF: Let us first show briefly here how to deduce fundamental solutions of the operator L_λ. Since the coefficients of the homogeneous equation depend on the variable $\vec{v}$, we can not obtain the general solution of the equation by using the convolution of the right hand side of the equation with the *Green's function.* Instead, we seek *fundamental solutions* of the equation

$$(\frac{\partial}{\partial t}+\vec{v}\cdot\nabla_{\vec{r}}-\lambda\Delta_{\vec{v}})G(\vec{r},\vec{v},t;\vec{u})=\delta(\vec{r},\vec{v}-\vec{u},t), \tag{3.10}$$

where $\vec{u}$ is a (*3-d*) parameter. Taking the Fourier transform of (3.10) with respect to $\vec{r}$, denoting by $\vec{\alpha}$ the transformed variable and $\tilde{G}$ the transformed function, then

$$(\frac{\partial}{\partial t}+\vec{v}\cdot i\vec{\alpha}-\lambda\Delta_{\vec{v}})\tilde{G}(\vec{\alpha},\vec{v},t;\vec{u})=\delta(\vec{v}-\vec{u},t).$$

Fourier transforming with respect to $\vec{v}$, writing $\vec{\beta}$ and $\hat{G}$ as the transformed variable and the function, respectively, we have

$$(\frac{\partial}{\partial t}-\vec{\alpha}\cdot\nabla_{\vec{\beta}}+\lambda\beta^2)\hat{G}(\vec{\alpha},\vec{\beta},t;\vec{u})=e^{-i\vec{u}\cdot\vec{\beta}}\delta(t).$$

Using the method of characteristic for the PDE, we have

$$\frac{d}{ds}\hat{G}(\vec{\alpha},\vec{\beta}-s\vec{\alpha},s)+\lambda(\vec{\beta}-s\vec{\alpha})^2\hat{G}(\vec{\alpha},\vec{\beta}-s\vec{\alpha},s)=e^{-i\vec{u}\cdot(\vec{\beta}-s\vec{\alpha})}\delta(t).$$

$$\hat{G}(\vec{\alpha},\vec{\beta},t;\vec{u})=\begin{cases} e^{-i\vec{u}\cdot(\vec{\beta}+t\vec{\alpha})}e^{-(t\beta^2+t^2\vec{\alpha}\cdot\vec{\beta}+\frac{t^3}{3}\alpha^2)}, & t>0 \\ 0. & t<0 \end{cases}$$

Therefore, by taking the inverse Fourier transformation about $\vec{\beta}$ and $\vec{\alpha}$, respectively, we have for $t>0$,

$$\tilde{G}(\vec{\alpha},\vec{v},t;\vec{u})=e^{-\lambda\frac{t^3}{3}\alpha^2}e^{-it\vec{u}\cdot\vec{\alpha}}\int_{R^3}e^{i\beta\cdot(\vec{v}-\vec{u})}e^{-zl(t\beta^2+t^2\vec{\alpha}\cdot\vec{\beta})}$$

$$= e^{-\frac{i}{2}t(\vec{v}+\vec{u})\cdot\vec{\alpha}} e^{-\lambda\frac{t^3}{12}\alpha^2} \int e^{-i\vec{\beta}\cdot(\vec{v}-\vec{u})} e^{-\lambda t\beta^2} d\vec{\beta}$$

$$= (\frac{1}{4\lambda t\pi})^{3/2} e^{-\frac{i}{2}t(\vec{v}+\vec{u})\cdot\vec{\alpha}} e^{-\lambda\frac{t^3}{12}\alpha^2} e^{-\frac{(\vec{v}-\vec{u})^2}{4\lambda t\pi}}.$$

and

$$G(\vec{r},\vec{v},t;\vec{u}) = (\frac{1}{4\lambda t\pi})^{3/2} e^{-\frac{(\vec{v}-\vec{u})^2}{4\lambda t\pi}} \mathcal{F}^{-1}(e^{-\frac{i}{2}t(\vec{v}+\vec{u})\cdot\vec{\alpha}} e^{-\lambda\frac{t^3}{12}\alpha^2})$$

$$= \frac{3\sqrt{3}}{8}(\frac{1}{\lambda\pi t^2})^3 \exp\{-\frac{1}{\lambda t}[\frac{1}{4}(\vec{v}-\vec{u})^2 + \frac{3}{4}(\vec{v}+\vec{u})^2 - \frac{3}{t}\vec{r}\cdot(\vec{v}+\vec{u}) + \frac{3}{t^2}r^2]\}. \tag{3.9'}$$

Noting the coefficient of the operator L_λ is independent of the variables $\vec{r}$ and t. it is easy to show from (3.9′) and the classical theory of PDE that (3.9) is the fundamental solution of the operator L_λ.

Now, integrating (3.9) over $(\vec{r},\vec{v})$, then for each given $(\vec{q},\vec{u},t)$, we get

$$\iint_{R^3\times R^3} d\vec{r}\,d\vec{v}\,p(t,\vec{r},\vec{v};\vec{q},\vec{u})$$

$$= \frac{3\sqrt{3}}{8}(\frac{1}{\lambda\pi t^2})^3 \iint_{R^3\times R^3} d\vec{r}\,d\vec{v}\,\exp(-\frac{1}{\lambda t}\frac{1}{4}(\vec{v}-\vec{u})^2)\times$$

$$\exp(-\frac{1}{\lambda t}(\frac{3}{4}(\vec{v}+\vec{u}) - \frac{3(\vec{r}-\vec{q})\cdot(\vec{v}+\vec{u})}{t} + \frac{3(\vec{r}-\vec{q})^2}{t^2}))$$

$$= \frac{3\sqrt{3}}{8}(\frac{1}{\lambda\pi})^3 \frac{1}{t^6} \int \exp(-\frac{1}{4\lambda t}v^2)\,d\vec{v} \int \exp(-\frac{3r^2}{\lambda t^3})\,d\vec{r}$$

$$= \frac{3\sqrt{3}}{8}(\frac{1}{\lambda\pi})^3 \frac{1}{t^6}(4\lambda t\pi)^{3/2}(\frac{\lambda t^3\pi}{3})^{3/2} = 1, \tag{3.11}$$

which proves (3.6). (3.7) is obvious from (3.9), and (3.8) from the identity (3.11) and Lebesgue's Dominated Convergence Theorem. ■

Now let h^n be a bounded sequence in $L^1((0,T)\times R^3\times R^3)$ satisfying

$$\sup_n \int_0^T \iint_{|(\vec{r},\vec{v})|\geq R} |h^n|\,d\vec{r}d\vec{v}dt \to 0 \quad \text{as} \quad R\to\infty, \tag{3.12}$$

and g_0^n be a bounded sequence in $L^1(R^3 \times R^3)$ satisfying

$$\sup_n \iint_{|(\vec{r},\vec{v})|\geq R} |g_0^n|\, d\vec{r}d\vec{v} \to 0 \quad \text{as} \quad R \to \infty. \tag{3.13}$$

We denote by g^n the solution of $L_\lambda g^n = h^n$ in $(0,T) \times R^3 \times R^3$ with $g^n|_{t=0} = g_0^n$. Then, for each given $t \in [0,\infty)$, define the operator $U : L^1(R^3 \times R^3) \to L^1(R^3 \times R^3)$ by

$$(Uh)(\vec{r},\vec{v},t) = \iint_{R^3\times R^3} p(t,\vec{r}.\,\vec{v};\vec{q},\vec{u})h(\vec{q},\vec{u},t)\, d\vec{q}d\vec{u}.$$

Using the definition of the fundamental solution, we write g^n as

$$\begin{aligned} g^n(\vec{r},\vec{v},t) &= U(t)g_0 + \int_0^t U(t-s)h^n(s)\, ds = \\ &= \int_0^t \iint_{R^3\times R^3} h^n(s,\vec{q},d\vec{u})p(t-s,\vec{r},\vec{v};\vec{q},\vec{u})\, d\vec{q}d\vec{u}ds + \iint_{R^3\times R^3} g_0^n(\vec{q},\vec{v}_1 y)p(t,\vec{r},\vec{v},\vec{q},\vec{v})\, d\vec{q}d\vec{u} \end{aligned} \tag{3.14}$$

for $t \in (0,T)$, $\vec{r},\vec{u} \in (R^3 \times R^3$.

LEMMA 3.2. *The sequence g^n is pre-compact in $L^1((0,T) \times R^3 \times R^3)$.*

PROOF.: From the Dunford–Pettis property,[44] we only need to show

(a) for Borel sets A in $(0,T) \times R^3 times R^3$,

$$\sup_n \int_A |g^n(t,\vec{r},\vec{v})|\, d\vec{r}d\vec{v}dt \to 0 \quad \text{as} \quad meas(A) \to 0,$$

(b) for any $T > 0$,

$$\sup_n \int_0^T \iint_{|(\vec{r},\vec{v})|\geq R} d\vec{r}d\vec{v}dt|g^n(t,\vec{r},\vec{v})| \to 0 \quad \text{as} \quad R \to \infty,$$

(c) $\exists\{g_n'\} \subset \{g^n\}$ and g_n' converges a.e. on $(0,T) \times R^3 \times R^3$.

To prove (a), let us consider

$$\int_A |g^n|\, d\vec{r}d\vec{v}dt \leq \int_0^T \iint_{R^3\times R^3} \chi_A(t,\vec{r},\vec{v})dtd\vec{r}d\vec{v} \int_0^t \iint_{R^3\times R^3} |h^n(s,\vec{q},\vec{u})|p(t-s,\vec{r},\vec{v};\vec{q},\vec{u})d\vec{q}d\vec{v}ds$$

$$+\int_0^T \iint_{R^3\times R^3} \chi_A(t,\vec{r},\vec{v})dtd\vec{r}d\vec{v} \iint_{R^3\times R^3} |g_0^n(\vec{q},\vec{u})|p(t,\vec{r},\vec{v};\vec{q},\vec{u})\,d\vec{q}d\vec{u},$$

and thus (a) holds as soon as we have

$$\sup_{s\in[0,T],(\vec{q},\vec{u})\in R^3\times R^3} q_A(s.\vec{q},\vec{u}) \to 0, \quad \text{as} \quad meas|A| \to 0,$$

where $q_A(s.\vec{q},\vec{u}) = \int_s^T dt \iint_{R^3\times R^3} d\vec{r}d\vec{v}\chi_A(t,\vec{r},\vec{v})p(t-s,\vec{r},\vec{v};\vec{q},\vec{u})$. But this is obvious from (3.9′) and the absolute continuity of Lebesgue integral with respect to the measure.

Next,

$$\int_0^T \iint_{|(\vec{r},\vec{v})|\geq R} dtd\vec{r}d\vec{v}\ |g^n(t.\vec{r},\vec{v})| \leq$$

$$\int_0^T \iint_{R^3\times R^3} \chi_{|(\vec{r},\vec{v})|\geq R}(t,\vec{r},\vec{v})dtd\vec{r}d\vec{v} \int_0^t \iint_{R^3\times R^3} |h^n(s,\vec{q},\vec{u})|p(t-s,\vec{r},\vec{v},\vec{q},\vec{u})\,d(\vec{q},\vec{u})ds$$

$$+\int_0^T \iint_{R^3\times R^3} \chi_{|(\vec{r},\vec{v})|\geq R}(t,\vec{r},\vec{v})dtd\vec{r}d\vec{v} \int_0^t \iint_{R^3\times R^3} |g_0^n(\vec{q},\vec{u})|p(t,\vec{r},\vec{v},\vec{q},\vec{u})\,d\vec{q}d\vec{u}ds$$

$$=\int_0^T \iint_{R^3\times R^3} \chi_{|(\vec{r},\vec{v})|\geq R}(t,\vec{r}.\vec{v})dtd\vec{r}d\vec{v} \int_0^t \iint_{R^3\times R^3} |h^n(s,\vec{q},\vec{u})|\chi_{(|(\vec{q},\vec{u})|\leq M)}p(t-s,\cdots)\,d\vec{q}d\vec{u}ds$$

$$+\int_0^T \iint_{R^3\times R^3} \chi_{|(\vec{r},\vec{v})|\geq R}(t.\vec{r},\vec{v})dtd\vec{r}d\vec{v} \int_0^t \iint_{R^3\times R^3} |h^n(s,\vec{q},\vec{u})|\chi_{(|(\vec{q},\vec{u})|\geq M)}p(t-s,\cdots)\,d\vec{q}d\vec{u}ds$$

$$+\int_0^T \iint_{R^3\times R^3} \chi_{|(\vec{r},\vec{v})|\geq R}(t.\vec{r}.\vec{v})dtd\vec{r}d\vec{v} \int_0^t \iint_{R^3\times R^3} |g_0^n(\vec{q},\vec{u})|\chi_{(|(\vec{q},\vec{u})|\leq M)}p(t,\cdots)\,d\vec{q}d\vec{u}ds$$

$$+\int_0^T \iint_{R^3\times R^3} \chi_{|(\vec{r},\vec{v})|\geq R}(t.\vec{r}.\vec{v})dtd\vec{r}d\vec{v} \int_0^t \iint_{R^3\times R^3} |g_0^n(\vec{q},\vec{u})|\chi_{(|\vec{q},\vec{u}|\geq M)}p(t,\cdots)\,d\vec{q}d\vec{u}ds$$

$$=\int_0^T ds \iint_{R^3\times R^3} d\vec{q}d\vec{u} \int_0^{T-s} dt' \iint_{R^3\times R^3} d\vec{r}d\vec{v}\chi_{|(\vec{r},\vec{v})|\geq R}(\vec{r},\vec{v})\chi_{|(\vec{q},\vec{u})|\leq M}h^n(s,\vec{q},\vec{u})p(t',\cdots)+\cdots.$$

Decomposing $\int_0^{T-s} dt'$ into $\int_0^{\delta\wedge(T-s)} dt' + \int_\delta^{T-s} dt'$, and using Lemma 3.1 and the

condition (3.12)-(3.13), we have

$$\int_0^T \iint_{|(\vec{r},\vec{v})|\geq R} dtd\vec{r}d\vec{v}\ |g^n(t,\vec{r},\vec{v})| \leq C\delta + C \sup_{|(\vec{q},\vec{u})|\leq M, t\in(\delta,T)} \iint_{|(\vec{r},\vec{v})|\geq R} p(t,\vec{r},\vec{v},\vec{q},\vec{u})\, d\vec{r}d\vec{v} +$$

$$+ C\int_0^T underset|(\vec{q},\vec{u})| \geq M \rightarrow \iint dtd\vec{q}d\vec{u}|h^n| + C\int_{|(\vec{q},\vec{u})|\geq M} |g_0^n|d\vec{q}d\vec{u},$$

and (b) is proved.

(c) is proved by the following facts. First, without loss of generality, we may assume that h^n converges strongly to some bounded measure μ on $[0,T]\times R^3\times R^3$ and g_0^n to some bounded measure λ on $R^3\times R^3$. It follows from (3.6)-(3.7) that

$$g_0^n(\vec{q},\vec{u})p(t,\vec{r},\vec{v},\vec{q},\vec{u})\, d\vec{q},\vec{u} \rightarrow_n \iint_{R^3\times R^3} p(t,\vec{r},\vec{v},\vec{q},\vec{u})\, d\lambda(\vec{q},\vec{u}).$$

Therefore we can assume $g_0^n \equiv 0$ without loss of generality.

Let $\delta > 0$ and $\phi_\delta \in C^\infty(R)$, $0\leq\phi_\delta\leq 1$, $\phi_\delta \equiv 0$ if $s\leq\delta$, $\phi_\delta\equiv 1$ if $s\geq 2\delta$. Set

$$g_\delta^n(t,\vec{r},\vec{v}) = \int_0^T \iint_{R^3\times R^3} dsd\vec{q}d\vec{u}h^n(s,\vec{q},\vec{u})\phi_\delta(t-s)p(t-s,\vec{r},\vec{v},\vec{q},\vec{u}).$$

From (3.6)-(3.8) we have

$$g_\delta^n(t,\vec{r},\vec{v}) \rightarrow_n \iint \phi_\delta(t-s)p(t-s,\vec{r},\vec{v},\vec{q},\vec{u})\, d\mu(s,\vec{q},\vec{u}),$$

$$\int_0^T dt \iint_{R^3\times R^3} d\vec{r}d\vec{v}|g_\delta^n - g^n|$$

$$\leq \int_0^T dt \iint_{R^3\times R^3} d\vec{r}d\vec{v}\int_0^t ds \iint_{R^3\times R^3} d\vec{q}d\vec{u}|h^n(s,\vec{q},\vec{u})|(1-\phi_\delta(t-s))p(t-s,\vec{r},\vec{v},(\vec{q},\vec{u}))$$

$$\leq \int_0^T dt \iint_{R^3\times R^3} d\vec{r}d\vec{v}\{\int_{(t-s\delta)^+}^t \iint_{R^3\times R^3} dsd\vec{q}d\vec{u}|h^n(s,\vec{q},\vec{u})|p(t-s,\vec{r},\vec{v},\vec{q},\vec{u})\}$$

$$\leq 2C_T\delta \int_0^T \iint_{R^3\times R^3} |h^n(s,\vec{r},\vec{v})|\, dsd\vec{q}d\vec{u} \leq C\delta,$$

which proves (c). ■

The next lemma is quite important for our argument.

LEMMA 3.3. *In addition to (3.6)-(3.8) in Lemma 3.1, the fundamental solution of* L_λ *satisfies also*

(iv) for any Borel set A *and* $\forall T < \infty$,

$$\sup_{\vec{r}' \in R^3} \int_A d\vec{r} d\vec{v} ds \Big(\int p^2(t, \vec{r}, \vec{v}; \vec{r}', \vec{v}')\, d\vec{v}' \Big)^{1/2} \le C(T, meas(A)) \to 0$$

as $meas(A) \to 0$;

(v) for all $M < \infty$ *and for all* $0 < h < T < \infty$,

$$\sup_{|\vec{r}', \vec{v}'| \le M, t \in [h,T]} \int d\vec{r} \Big(\int_{|\vec{r}|+|\vec{v}| \ge R} p^2(t, \vec{r}, \vec{v}; \vec{r}', \vec{v}')\, d\vec{v} \Big)^{1/2} d\vec{r} \to 0 \text{ as } R \to \infty.$$

PROOF: Since

$$\begin{aligned}
&\Big(\int_{R^3} p^2(t, \vec{r}, \vec{v}; \vec{r}', \vec{v}') d\vec{v} \Big)^{1/2} \\
=&\frac{C_1}{t^6} \Big(\int_{R^3} \exp\{ -\frac{1}{\lambda t} (\frac{1}{2} (\vec{v} - \vec{v}')^2 + \frac{3}{2} (\vec{v} + \vec{v}')^2 - \frac{6}{t} (\vec{v} + \vec{v}') \cdot (\vec{r} - \vec{r}') + \frac{6}{t^2} (\vec{r} - \vec{r}')^2) \} d\vec{v} \Big)^{\frac{1}{2}} \\
=&\frac{C_1}{t^6} \exp\Big(\frac{3}{4\lambda t^3} (\vec{r} - \vec{r}' - t\vec{v}')^2 \Big) \Big(\int_{R^3} \exp\Big(-\frac{2}{\lambda t} \Big(\vec{v} + \frac{1}{2} \Big(\vec{v}' - \frac{3(\vec{r} - \vec{r}')}{t} \Big) \Big) \Big) d\vec{v} \Big)^{1/2} \\
=&\frac{C_1}{t^6} \Big(\frac{\lambda t \pi}{2} \Big)^{3/4} \exp\Big(\frac{3}{4\lambda t^3} (\vec{r} - \vec{r}' - t\vec{v}')^2 \Big), \\
&\int_{R^3} \frac{1}{t^5} \frac{1}{t^{1/4}} \exp\Big(-\frac{3(\vec{r} - \vec{r}' - t\vec{v}')^2}{4\lambda t^3} \Big)\, d\vec{r} = C \frac{1}{t^{3/4}},
\end{aligned}$$

and

$$\int_0^T \frac{1}{t^{3/4}}\, dt = C(T) < \infty, \qquad \forall T < \infty,$$

(iv) follows from the absolute continuity of Lebesgue integral with respect to the measure, and (v) from Lebesgue's Dominated Convergence Theorem. ■

Now, let $\tilde{h}^n$ be a bounded sequence in $L^1((0,T] \times R^3_{\vec{r}}; L^2(R^3_{\vec{v}}))$ satisfying

$$\sup_n \int_0^T dt \int d\vec{r} \Big(\int_{|\vec{r}|+|\vec{v}| \ge R} |\tilde{h}^n|^2(\vec{r}, \vec{v}, t)\, d\vec{v} \Big)^{1/2} \to 0 \quad \text{as } R \to \infty \tag{3.15}$$

Consider

$$\tilde{g}^n(\vec{r},\vec{v},t) = \int_0^t ds \int_{R^3} d\vec{r}' \int d\vec{v}' \tilde{h}^n(s,\vec{r}',\vec{v}') p(t-s,\vec{r},\vec{v},\vec{r}',\vec{v}'), \tag{3.14'}$$

Then from Lemma 3.3 and the boundedness of $\tilde{h}^n$ we have, for each Borel set $A \subset (0,T) \times R^3 \times R^3$,

$$\begin{aligned}
&\int_A |\tilde{g}_n(\vec{r},\vec{v},t)|\, d\vec{v}d\vec{r}dt \leq \\
&\leq \int_0^T dt \int_{R^3} d\vec{r} \int_{R^3} d\vec{v} \chi_A(\vec{r},\vec{v},t) \int_0^t ds \int_{R^3} d\vec{r}' \int d\vec{v}' |\tilde{h}^n|(s,\vec{r}',\vec{v}') p(t-s,\vec{r},\vec{v},\vec{r}',\vec{v}') \\
&\leq \int_0^T dt \int_{R^3} d\vec{r} \int_{R^3} d\vec{v} \chi_A(\vec{r},\vec{v},t) [(\int_0^t ds \int_{R^3} d\vec{r}' (\int d\vec{v}' |\tilde{h}^n|^2(s,\vec{r}',\vec{v}'))^{1/2} \\
&\qquad \times (\int d\vec{v}' p^2(t-s,\vec{r},\vec{v},\vec{r}',\vec{v}'))^{1/2}] \\
&\leq C(T, meas(A)) \to 0 \text{ as } meas(A) \to 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^T dt \int_{|\vec{r}|+|\vec{v}|\geq R} d\vec{r}d\vec{v} |\tilde{g}_n(\vec{r},\vec{v},t)| \leq \\
&\leq \int dt \int d\vec{r}d\vec{v} \chi_{(|\vec{r}|+|\vec{v}|\geq R)} \int_0^t ds \int d\vec{r}' (\int |\tilde{h}^n(s,\vec{r}',\vec{v}')|^2 \chi_{(|\vec{r}'|+|\vec{v}'|\geq M)} d\vec{v}')^{1/2} \\
&\qquad \times (\int p^2(t-s,\vec{r},\vec{v},\vec{r}',\vec{v}') \chi_{(|\vec{r}'|+|\vec{v}'|\leq M)} d\vec{v}')^{1/2} \\
&\quad + \int dt \int d\vec{r}d\vec{v} \chi_{(|\vec{r}|+|\vec{v}|\geq R)} \int_0^t ds \int d\vec{r}' (\int |\tilde{h}^n(s,\vec{r}',\vec{v}')|^2 \chi_{(|\vec{r}'|+|\vec{v}'|\leq M)} d\vec{v}')^{1/2} \\
&\qquad \times (\int p^2(t-s,\vec{r},\vec{v},\vec{r}',\vec{v}') \chi_{(|\vec{r}'|+|\vec{v}'|\leq M)} d\vec{v}')^{1/2} \\
&\to 0 \qquad \text{as} \quad R \to \infty.
\end{aligned}$$

Therefore, using the same argument as in the proof of Lemma 3.2 we have the following lemma.

LEMMA 3.4. *Suppose the sequences* $\{h^n\}$, $\{\tilde{h}^n\}$ *and* $\{g_0^n\}$ *satisfy (3.12), (3.15) and (3.13) respectively. Then the set of solutions* $\{g^n\}$ *of the equations*

$$L_\lambda g^n = h^n + \tilde{h}^n \quad in \quad (0,T) \times R^3 \times R^3,$$

$$g^n|_{t=0} = g_0^n,$$

is pre-compact in $L^1((0,T) \times R^3 \times R^3)$.

§3.3 Proof of the Sequential Stability Theorem

Now let us start the

PROOF OF THEOREM 3.1.: We will divide this proof into four steps. In the first step. we show that $\{C_E(f_n, f_n)\}$ forms a bounded set in $L^1([0,T] \times R^3 \times R^3)$. Then using assumption (3.3) and Lemma 3.4, we prove in the second step that $\{g_\delta^n\}$ is pre-compact in $L^1([0,T] \times R^3 \times R^3)$. Supposing, passing into a subsequence if necessary, that $g_\delta^n = \beta_\delta(f_n) \to \beta_\delta(f) = g_\delta$, we will in step 3 demonstrate that $C_E(f_n, f_n)(1+\delta f_n)^{-1} \to C_E(f,f)(1+\delta f)^{-1}$ in L^1. The fourth step will address how to pass to the limit in the (RDVE) such that f is the renormalized solution of the equations.

Step 1 We first remark that $C_E^-(f_n, f_n)(1+\delta f_n)^{-1}$ is bounded in $L^\infty(0,T; L^1(R^3_{\vec{r}} \times R^3_{\vec{v}}))$. In fact, letting $Y_n(n(\vec{r}), n(\vec{r}+a\vec{\epsilon})) = Y(n_n(\vec{r},t), n_n(\vec{r}+a\vec{\epsilon},t))$. then

$$\begin{aligned} C_E^-(f_n,f_n)(1+\delta f_n)^{-1} &= \\ &= \frac{f_n}{1+\delta f_n} a^2 \iint_{R^3 \times S^2_+} f(\vec{r}+a\vec{\epsilon}, \vec{v}_1, t) Y_n(n(\vec{r}), n(\vec{r}+a\vec{\epsilon})) < \vec{\epsilon}, \vec{v} - \vec{v}_1 > \, d\vec{\epsilon} d\vec{v}_1 \\ &\le \frac{f_n}{1+\delta f_n} a^2 \iint_{R^3 \times S^2_+} f(\vec{r}+a\vec{\epsilon}, \vec{v}_1, t) Y_n(n(\vec{r}), n(\vec{r}+a\vec{\epsilon})) \frac{1}{2}(2+v^2+v_1^2)\, d\vec{\epsilon} d\vec{v}_1. \end{aligned}$$

Therefore,

$$
\begin{aligned}
&\|C_E^-(f_n,f_n)(1+\delta f_n)^{-1}\|_{L^1(R^3\times R^3)} \le \\
&\le \frac{a^2}{2}\iiiint \frac{f_n(\vec r,\vec v,t)}{1+f_n(\vec r,\vec v,t)}(2+v^2+v_1^2)f(\vec r+a\vec\epsilon,\vec v_1,t)Y_n(n(\vec r),n(\vec r+a\vec\epsilon))\,d\vec\epsilon d\vec v_1 d\vec v d\vec r \\
&\le \frac{a^2}{2}\iint d\vec r d\vec\epsilon\{\int (1+v_1^2)f(\vec r+a\vec\epsilon,\vec v_1,t)n(\vec r,t)Y_n(n(\vec r),n(\vec r+a\vec\epsilon))\,d\vec v_1 + \\
&\quad + \int \frac{f_n(\vec r,\vec v,t)}{1+f_n(\vec r,\vec v,t)}(1+v^2)f(\vec r+a\vec\epsilon,\vec v_1,t)n(\vec r+a\vec\epsilon,t)Y_n(n(\vec r),n(\vec r+a\vec\epsilon))\,d\vec v\} \\
&\le 2\pi a^2 M_Y\{\iint d\vec r d\vec v_1(1+v_1^2)f(\vec r,\vec v_1,t)+\iint d\vec r d\vec v(1+v^2)f(\vec r,\vec v,t)\} \\
&\le C(T), \qquad (3.16)
\end{aligned}
$$

i.e., $\{C_E^-(f_n,f_n)(1+\delta f_n)^{-1}\}$ is bounded in $L^\infty(0,T;L^1(R^3_{\vec r}\times R^3_{\vec v}))$.

Now let us estimate the norm of $C_E^+(f_n,f_n)$. First, one has,[37][38] for each $M>0$,

$$
\begin{aligned}
C_E^+(f_n,f_n) \le & Ma^2\iint_{R^3\times S_+^2} Y_n(n(\vec r),n(\vec r-a\vec\epsilon))f_n(\vec r,\vec v,t)f_n(\vec r-a\vec\epsilon,\vec v_1,t) \\
& \times <\vec\epsilon,\vec v-\vec v_1>\, d\vec\epsilon d\vec v_1+\frac{1}{\log M}\alpha(f_n,f_n), \qquad (3.17)
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha(f,f) = & a^2\iint_{R^3\times S_+^2} Y(n(\vec r),n(\vec r-a\vec\epsilon))f(\vec r,\vec v',t)f(\vec r-a\vec\epsilon,\vec v_1',t)\times \\
& \times |\log\frac{f(\vec r,\vec v',s)\,f(\vec r+a\vec\epsilon,\vec v_1',s)}{f(\vec r,\vec v,s)\,f(\vec r+a\vec\epsilon,\vec v_1,s)}| <\vec\epsilon,\vec v-\vec v_1>\, d\vec\epsilon d\vec v_1.
\end{aligned}
$$

Then, define $h(f)$ by (2.23b). Using equality (2.23a), inequality (2.24) and the fact that

$$
\iiint_{[0,T]\times R^3\times R^3}\alpha(f,f)\,d\vec v d\vec r ds = \iiiint\limits_{[0,t]\times R^3\times R^3\times R^3\times S_+^2} 2[h^+(f)+h^-(f)]\,d\vec\epsilon d\vec v_1 d\vec v d\vec r ds,
$$

it can be concluded that

$$\sup_n \|\alpha(f_n, f_n)\|_{L^1([0,T]\times R^3\times R^3)} \le C(T) < \infty,$$

and

$$\sup_n \|C_E^+(f_n, f_n)\|_{L^1} \le C'(T) < \infty. \tag{3.18}$$

Step 2 We want to get the pre-compactness of $\{g_\delta^n\}$. In step 1 we showed that $\{L_\lambda g_\delta^n\} = \{h_\delta^n + \vec{E}\cdot\nabla_{\vec{v}} g_\delta^n\}$ with $\{h_\delta^n\}$ bounded in $L^1((0,T)\times R^3\times R^3)$, and $\{\vec{E}_n\cdot\nabla_{\vec{v}} g_\delta^n\}$ bounded in $L^1((0,T)\times R^3_{\vec{r}}, L^2(R^3_{\vec{v}}))$ by the assumption. In order to get the pre-compactness of $\{g_\delta^n\}$ by using Lemma 3.4. we consider the problem in the following way. For any cut-off function $\phi(\vec{r},\vec{v})$ in $\mathcal{D}(R^3\times R^3)$, with $supp(\phi)\subset\{B_R\times B_R\}$ for some $R<\infty$, we observe that

$$L_\lambda(\phi g_\delta^n) = \phi(L_\lambda g_\delta^n) + g_\delta^n \vec{v}\cdot\frac{\partial}{\partial\vec{r}}\phi - \lambda(g_\delta^n\Delta_{\vec{v}}\phi + 2\nabla_{\vec{v}}\phi\cdot\nabla_{\vec{v}} g_\delta^n) = h_\delta^n.$$

Then $L_\lambda(\phi g_\delta^n)$ is bounded in $L^1((0,T)\times R^3\times R^3)\oplus L^1([0,T]\times R^3_r; L^2(R^3_{\vec{v}}))$ and Lemma 3.4 shows that $\{\phi g_\delta^n\}$ is pre-compact.

Next, choose $\phi_m\in\mathcal{D}(R^3\times R^3)$ such that $supp(\phi_m)\in B_{m+1}\times B_{m+1}$, $0\le\phi_m\le 1$, and $\phi_m|_{B_m\times B_m} = 1$. The above argument implies that $\{\phi_m g_\delta^n\}$ is pre-compact in $L^1((0,T)\times R^3\times R^3)$ for each positive integer m. Using the diagonal method we can choose a subsequence of $\{g_\delta^n\}$, denoted still by $\{g_\delta^n\}$ for simplicity, such that there is a function g, $g_\delta^n\to g$ in $L^1((0,T)\times B_m\times B_m)$ for each m. Then classical measure theory shows that $g_\delta^n|_{(0,T)\times B_m\times B_m}\to g|_{(0,T)\times B_m\times B_m}$ in the topology of convergence in measure. Fatou's lemma and the boundedness condition (3.1) imply

$$\int_{(0,T)\times B_m\times B_m} g d\vec{r}d\vec{v}dt \le \liminf \int_{(0,T)\times B_m\times B_m} g_\delta^n d\vec{r}d\vec{v}dt \le C(T) < \infty \tag{3.19}$$

for $C(T)$ independent of m, and

$$\int_{(0,T)\times R^3\times R^3} |g - g_\delta^n| d\vec{r}d\vec{v}dt =$$

$$\int_{(0,T)\times B_m\times B_m} |g-g_\delta^n|d\vec{r}d\vec{v}dt + \int_{(0,T)\times B_m\times B_m-(0,T)\times B_m\times B_m} |g-g_\delta^n|d\vec{r}d\vec{v}dt \to 0.$$

In other words, $g \in L^1((0,T)\times R^3\times R^3)$ and $g_\delta^n \to g$ in $L^1((0,T)\times R^3\times R^3)$. Therefore, g_δ^n converges also to g in measure. That is, $meas(|g_\delta^n - g| > \epsilon) \to_n 0$ for any $\epsilon > 0$. Since $f^n = \frac{1}{\delta}[\exp(\delta g_\delta^n) - 1]$, $f^n \to_n f = \frac{1}{\delta}[\exp(\delta f) - 1]$, $f^n \to f$ in measure for any set with finite measure. Next, we recall $f^n \to f$ weakly in $L^1((0,T) \times R^3 \times R^3)$. Combining this information with the convergence in measure, we conclude that $f^n \to f$ in $L^1((0,T)\times R^3\times R^3)$ by using Schur's theorem (see, for instance, [50]). Therefore, using the $L^\infty(0,T;L^1(R^3 \times R^3))$ boundedness of f^n implied by (3.1), it follows that f^n converges in $L^p(0,T;L^1(R^3 \times R^3))$ to f for all $1 \le p < \infty$ and for all $T < \infty$. Hence, g_δ^n converges in $L^p(0,T;L^1(R^3 \times R^3))$ to $g_\delta = \beta_\delta(f)$ for all $1 \le p < \infty$ and for all $T < \infty$.

Step 3 Now we want to show that $C_E(f^n,f^n)(1+\delta f^n)^{-1}$ converges in L^1 to $C_E(f,f)(1+\delta f)^{-1}$, and that $C_E^{\pm}(f,f)(1+\delta f)^{-1} \in L^1$. In fact, we shall show that

$$C_E^-(f,f)(1+\delta f)^{-1} \in L^\infty(0,T;L^1(R^3 \times R^3)),$$

$$C_E^+(f,f)(1+\delta f)^{-1} \in L^1((0,T) \times R^3 \times R^3),$$

and that

$$C_E^-(f^n,f^n)(1+\delta f^n)^{-1} \to_n C_E^-(f,f)(1+\delta f)^{-1} \quad \text{in} \quad L^p(0,T;L^1), \quad \forall p < \infty,\ T < \infty, \tag{3.20a}$$

$$C_E^+(f^n,f^n)(1+\delta f^n)^{-1} \to_n C_E^+(f,f)(1+\delta f)^{-1} \quad \text{in} \quad L^1 \qquad \forall T < \infty \tag{3.20b}$$

The convergence (3.20a) is easy to prove since

$$C_E^-(f^n,f^n)(1+\delta f^n)^{-1} = f^n(1+\delta f^n)^{-1}L(f^n), \tag{3.21}$$

$$L(f^n) = \iint_{R^3 \times S^2_+} f^n(\vec{r} + a\vec{\epsilon}, \vec{v}_1, t) Y_n(n(\vec{r}), n(\vec{r} + a\vec{\epsilon})) < \vec{\epsilon}, \vec{v} - \vec{v}_1 > d\vec{\epsilon} d\vec{v}_1. \quad (3.22)$$

The argument used in step 2 and the continuity of Y ensure that $Y(n_n(\vec{r}), n_n(\vec{r} + a\vec{\epsilon}))$ converges in measure to $Y(n(\vec{r}), n(\vec{r} + a\vec{\epsilon}))$ and the argument in step 1 shows that we can use the boundedness condition (2.19) and Lebesgue's theorem to get (3.20a).

For (3.20b), it is sufficient to show $I^n \equiv C_E^+(f^n, f^n)(1 + \delta f^n)^{-1}$ converges in measure to $I \equiv C_E^+(f, f)(1+\delta f)^{-1}$ on every set with finite measure. In fact, supposing local convergence in measure, we observe that the sequence I^n is bounded in L^1 and satisfies (*cf.* (3.17))

$$0 \le I^n \le Mh^n + \frac{1}{\log K} e^n,$$

where $h^n \ge 0$ converges in L^1 while $e^n \ge 0$ remain bounded in L^1 (e.g., see the proof of lemma 3.5). Therefore I^n is weakly pre-compact in L^1. The strong convergence of I^n follows from Schur's theorem.[50]

To prove the convergence in measure of I^n to I, we notice that $(1 + \delta f^n)^{-1}$ converges in measure to $(1 + \delta f)^{-1}$. Therefore what we need is to show $C_E^+(f^n, f^n)$ converges is measure on each set with finite measure to $C_E^+(f, f)$. This is easy if one notices that our bounded condition (2.19) makes the Y *regular* in the sense of Polewczak (see [37,p469]), and we omit the detailed proof.

Step 4 Now we consider how to pass to the limit in (RDVE). Comparing the right side of the renormalized equation with the kinetic equation (cf. [37], [38]), we should consider the weak convergence of both $|\nabla_{\vec{v}} g^n_\delta|^2$ and $\vec{E} \cdot \nabla_{\vec{v}} g^n_\delta$. Since $\nabla_{\vec{v}} g^n_\delta$ converges weakly in L^2 to $\nabla_{\vec{v}} g_\delta$, we have

LEMMA 3.5. *There exists a bounded nonnegative measure* $\mu^{(1)}$ *on* $(0, T) \times R^3 \times R^3$

such that

$$|\nabla_{\vec{v}} g_\delta^n|^2 \to_n |\nabla_{\vec{v}} g_\delta|^2 + \mu^{(1)} \quad in \quad \mathcal{D}'((0,T) \times R^3 \times R^3). \tag{3.23}$$

In fact, for each n, define the measure μ_n on $(0,T) \times R^3 \times R^3$ by

$$d\mu_n(\vec{r}, \vec{v}, t) = |\nabla_{\vec{v}} g_\delta^n|^2 d\vec{r} d\vec{v} dt.$$

Then the weak convergence of $\nabla_{\vec{v}} g_\delta^n$ in L^2 and the Uniform Boundedness Theorem imply that $\{\mu_n\} \subset \mathcal{BV}(X)$, the space of bounded variation measures on $X \equiv (0,T) \times R^3 \times R^3$, and μ_n is a bounded sequence in the space. X is obviously a locally compact Hausdorff space. Using the Riesz-Kakutani theorem,[44,p.151-153],[41,p.138-142] $\mathcal{BV}(X)$ is congruent (*isometrically isomorphic*) to the conjugate $C_0^*(X)$ of the space $C_0(X)$. Now using the Alaoglu's Theorem,[44,p.174] $\{\mu_n\}$ is w^* pre-compact in $\mathcal{BV}(X)$. Since $C_0(X)$ is separable, there exists $\mu^{(1)} \in \mathcal{BV}(X)$ and $\{\mu_{n'}\} \subset \{\mu_n\}$ such that

$$\int_X f d\mu_{n'} \to_{n'} \int_X f d\nu + \int_X f d\mu^{(1)}$$

for each $f \in C_0(X)$, where the measure ν is given by $d\nu \equiv |f|^2 d\vec{r} d\vec{v} dt$, which in turn implies that

$$|\nabla_{\vec{v}} g_\delta^{n'}|^2 \to |\nabla_{\vec{v}} g_\delta|^2 + \mu^{(1)} \quad \text{in} \quad \mathcal{D}'((0,T) \times R^3 \times R^3).$$

Furthermore, it follows from Fatou's lemma that μ is a nonnegative measure. Then (3.23) is easily obtained by using the weak convergence of ∇g_δ^n.

Using the same argument, we conclude, passing to a subsequence if necessary, that there exists a locally bounded (may change sign) measure $\mu^{(2)}$ such that

$$-\vec{E}^n \cdot \nabla_{\vec{v}} g_\delta^n \to_n -\vec{E} \cdot \nabla_{\vec{v}} g_\delta + \mu^{(2)} \quad \text{in} \quad \mathcal{D}'((0,T) \times R^3 \times R^3)$$

and $\int_0^T \int_{R^3_{\vec{r}}} (\int_{R^3_{\vec{v}}} |\mu_{(2)}|^2)^{1/2}$ bounded.

Now we may pass to the limit in (RDVE) in the sense of distributions and deduce, using the above convergence, that $g_\delta = \beta_\delta(f)$ solves

$$\frac{\partial}{\partial t} g_\delta + \vec{v} \cdot \nabla_{\vec{r}} g_\delta - \lambda \Delta_{\vec{v}} g_\delta = \lambda \delta |\nabla_{\vec{v}} g_\delta|^2 - \vec{E} \cdot \nabla_{\vec{v}} g_\delta + \mu \quad \text{in} \quad \mathcal{D}'((0,T) \times R^3 \times R^3), \tag{3.24}$$

$$\operatorname{div}_{\vec{r}} \vec{E} = -n(\vec{r}, t) \quad \text{in} \quad \mathcal{D}'((0,T) \times R^3). \tag{3.25}$$

where $\mu = \lambda \mu^{(1)} - \mu^{(2)}$. Of course, f and $\vec{E}$ satisfy (3.1)-(3.4).

To get (3.25) from the fact that $\operatorname{div}_{\vec{r}} \vec{E}_n = -n_n(\vec{r}, t)$ is easy. In fact, since $\vec{E}^n$ is weakly pre-compact in L^2, we have, passing to a subsequence if necessary,

$$\lim_{n \to \infty} \vec{E}^n(\vec{r}, t) \to^w \vec{E}(\vec{r}, t) \quad \text{in} \quad L^2,$$

and for all $\psi \in \mathcal{D}((0,T) \times R^3)$,

$$\lim_{n \to \infty} \iint \psi \nabla_{\vec{r}} \cdot \vec{E}^n \, d\vec{r} \, dt = -\lim \iint \vec{E}^n \cdot \nabla_{\vec{r}} \psi \;\to\; -\iint \vec{E} \cdot \nabla_{\vec{r}} \psi = \iint \psi \nabla_{\vec{r}} \cdot \vec{E},$$

i.e.,

$$\nabla_{\vec{r}} \cdot \vec{E}^n \quad \to \quad \nabla_{\vec{r}} \cdot \vec{E} \quad \text{in} \quad \mathcal{D}'((0,T) \times R^3).$$

On the other hand, from the weak pre-compactness of f^n (which comes from the bounded condition (3.1)[37],[38]) we have

$$n(f^n) = \int f^n \, d\vec{v} \;\to_n\; \int_{R^3} f \, d\vec{v} = n(f) \quad \text{in} \quad \mathcal{D}'((0,T) \times R^3).$$

Then (3.25) follows from the uniqueness of the limit of distributions.[22]

Now we show that the measure μ on the right hand side of (3.24) vanishes. The formal argument in Chapter II, if it could be justified, hints that μ should be zero. We get the conclusion by considering a modified version of the formal proof.

Let $\Phi_{\theta R}(t) = \exp \theta(t \wedge R) \equiv \exp[\theta \min\{t, R\}]$, $\Psi_{\theta,R}(t) = \int_0^t \Phi_{\theta R}(s) \, ds$. Multiply the equation (RDVE) by $\Phi_{\theta R}(g_\delta)$, with $0 < \theta < \delta$, $R > 0$ fixed. Multiplying by $\phi(\vec{r}, t) \in \mathcal{D}'((0,T) \times R^3)$ we have

$$\phi(\vec{r},t)\Phi_{\theta R}(g_\delta^n)\frac{\partial}{\partial t}g_\delta^n + \phi(\vec{r},t)\Phi_{\theta R}(g_\delta^n)\vec{v}\cdot\nabla_{\vec{r}}g_\delta^n - \phi(\vec{r},t)\Phi_{\theta R}(g_\delta^n)\frac{1}{1+\delta f^n}C_E$$
$$-\lambda\phi(\vec{r},t)\Phi_{\theta R}\Delta_{\vec{v}}g_\delta^n - \lambda\delta\phi(\vec{r},t)\Phi_{\theta R}|\nabla_{\vec{v}}g_\delta^n|^2(f^n,f^n) + \phi(\vec{r},t)\Phi_{\theta R}(g_\delta^n)\vec{E}^n\cdot\nabla_{\vec{v}}g_\delta^n = 0. \tag{3.26}$$

Since

$$\phi(\vec{r},t)\Phi_{\theta R}(g_\delta^n)\frac{\partial}{\partial t}g_\delta^n = \frac{\partial}{\partial t}(\phi(\vec{r},t)\Psi_{\theta R}(g_\delta^n)) - \Psi_{\theta R}(g_\delta^n)\frac{\partial}{\partial t}\phi(\vec{r},t),$$

$$\phi(\vec{r},t)\Phi_{\partial R}(g_\delta^n)\vec{v}\cdot\nabla_{\vec{r}}g_\delta^n = \mathrm{div}_{\vec{r}}(\phi(\vec{r},t)\Psi_{\theta R}(g_\delta^n)\vec{v}) - \Psi_{\theta R}(g_\delta^n)\vec{v}\cdot\nabla_{\vec{r}}\phi(\vec{r},t),$$

$$\phi(\vec{r},t)\Phi_{\theta R}(g_\delta^n)\Delta_{\vec{v}}g_\delta^n = \phi(\vec{r},t)\Phi_{\theta R}(g_\delta^n)\mathrm{div}_{\vec{v}}\cdot\nabla_{\vec{v}}g_\delta^n =$$
$$= \mathrm{div}_{\vec{v}}\cdot(\phi(\vec{r},t)\Phi_{\theta R}(g_\delta^n)\nabla_{\vec{v}}g_\delta^n) - \theta\phi(\vec{r},t)\Phi_{\theta R}(g_\delta^n)\chi_{(g_\delta^n\le R)}|\nabla_{\vec{v}}g_\delta^n|^2,$$

and

$$\vec{E}^n\cdot\nabla_{\vec{v}}g_\delta^n\Phi_{\theta R}(g_\delta^n)\phi(\vec{r},t) = \Phi_{\theta R}\vec{E}^n\cdot\nabla_{\vec{v}}\Psi_{\theta R}(g_\delta^n) = \mathrm{div}_{\vec{v}}(\phi(\vec{r},t)\Psi_{\theta R}(g_\delta^n)\vec{E}^n).$$

Substituting all of these equalities into (3.26), we get

$$\frac{\partial}{\partial t}(\phi(\vec{r},t)\Psi_{\theta R}(g_\delta^n)) - \Psi_{\theta R}(g_\delta^n)\frac{\partial}{\partial t}\phi(\vec{r},t) + \mathrm{div}_{\vec{r}}(\phi(\vec{r},t)\Psi_{\theta R}(g_\delta^n)\vec{v})$$
$$-\Psi_{\theta R}(g_\delta^n)\vec{v}\cdot\nabla_{\vec{r}}\phi(\vec{r},t) - \lambda\mathrm{div}_{\vec{v}}(\phi(\vec{r},t)\Phi_{\theta R}(g_\delta^n)\nabla_{\vec{v}}g_\delta^n)$$
$$+\lambda\theta\phi(\vec{r},t)\Phi_{\theta R}(g_\delta^n)\chi_{(g_\delta^n\le R)}|\nabla_{\vec{v}}g_\delta^n|^2 - \lambda\delta\phi(\vec{r},t)\Phi_{\theta R}(g_\delta^n)|\nabla_{\vec{v}}g_\delta^n|^2$$
$$-\phi(\vec{r},t)\Phi_{\theta R}\frac{1}{1+\delta f^n}C_E(f^n,f^n) + \mathrm{div}_{\vec{v}}(\phi(\vec{r},t)\Psi_{\theta R}(g_\delta^n)\vec{E}^n) = 0 \tag{3.27}$$

Integrating (3.27) over $(0,T)\times R^3\times R^3$, at least formally we get

$$-\int_0^T dt\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}\{(\frac{\partial}{\partial t}\phi(\vec{r},t) + \vec{v}\cdot\nabla_{\vec{r}}\phi(\vec{r},t))\Psi_{\theta R}(g_\delta^n)$$
$$+\phi(\vec{r},t)\frac{1}{1+\delta f^n}C_E(f^n,f^n)\Phi_{\theta R}(g_\delta^n)\}$$

$$=\lambda \int_0^T dt\, \phi(\vec{r},t) \iint\limits_{R^3\times R^3} d\vec{r}\, d\vec{v}\, [\delta|\nabla_{\vec{v}} g_\delta^n|^2 - \theta|\nabla_{\vec{v}} g_\delta^n|^2 \chi_{(g_\delta^n \le R)}]\Phi_{\theta R}(g_\delta^n). \tag{3.28}$$

The correctness of this integration by parts will be shown later.

Now observe that the right-hand side of (3.28) is bounded by

$$\lambda \sup|\phi(\vec{r},t)|\{(\delta-\theta)\int_0^T dt \iint\limits_{R^3\times R^3} d\vec{r}\, d\vec{v}\, |\nabla_{\vec{v}} g_\delta^n|^2 \Phi_{\theta R}(g_\delta^n) +$$

$$+ \theta e^{\theta R} \int_0^T dt \iint\limits_{R^3\times R^3} d\vec{r}\, d\vec{v}\, |\nabla_{\vec{v}} g_\delta^n|^2 \Phi_{\theta R}(g_\delta^n)\chi_{(g_\delta^n > R)}\}.$$

Since $\nabla_{\vec{v}} g_\delta^n = \nabla_{\vec{v}}(\frac{1}{\delta}\log(1+\delta f^n) = (1+\delta f^n)^{-1}$ and $\Phi_{\theta R}(g_\delta^n) = \exp(\theta(\frac{1}{\delta}\log(1+\delta f^n) \wedge R) \le (1+\delta f^n)^{\frac{\theta}{\delta}}$, we can bound the first integral by

$$\int_0^T dt \iint\limits_{R^3\times R^3} d\vec{r}\, d\vec{v}\, |\nabla_{\vec{v}} f^n|^2 (1+\delta f^n)^{-2}(1+\delta f^n)^{\frac{\theta}{\delta}} \le \frac{1}{\delta}\int_0^T dt \iint\limits_{R^3\times R^3} d\vec{r}\, d\vec{v}\, |\nabla_{\vec{v}} f^n|^2 (f^n)^{-1}.$$

Also, since $(1+\delta f^n)|_{g_\delta^n \ge R} \ge \Phi_{\theta R}(g_\delta^n)|_{g_\delta^n \ge R} = e^{\theta R}$, $(1+\delta f^n)^{-1}|_{g_\delta^n \ge R} \le e^{-\delta R}|_{g_\delta^n \ge R}$, $(1+\delta f^n)|_{g_\delta^n \ge R} \ge \Phi_{\theta R}(g_\delta^n)|_{g_\delta^n \ge R} = e^{\theta R}$, and $(1+\delta f^n)^{-1}|_{g_\delta^n \ge R} \le e^{-\delta R}|_{g_\delta^n \ge R}$, we bound the second integral by

$$\int_0^T dt \iint\limits_{R^3\times R^3} d\vec{r}\, d\vec{v}\, |\nabla_{\vec{v}} f^n|^2 (1+\delta f^n)^{-1} e^{-\delta R} \le e^{-\delta R}\frac{1}{\delta}\int_0^T dt \iint\limits_{R^3\times R^3} d\vec{r}\, d\vec{v}\, |\nabla_{\vec{v}} f^n|^2 (f^n)^{-1}.$$

In conclusion, using (3.3) we deduce

$$\Big| \int_0^T dt \iint\limits_{R^3\times R^3} d\vec{r}\, d\vec{v} (\frac{\partial}{\partial t}\phi(\vec{r},t) + \vec{v}\cdot\nabla_{\vec{r}}\phi(\vec{r},t))\Psi_{\theta R}(g_\delta^n)$$

$$+ \int_0^T dt\, \phi(\vec{r},t) \iint\limits_{R^3\times R^3} d\vec{r}\, d\vec{v}\, \frac{1}{1+\delta f^n} C_E(f^n,f^n)\Phi_{\theta R}(g_\delta^n)\Big|$$

$$\le C \sup|\phi(\vec{r},t)|[(\delta-\theta) + e^{(\theta-\delta)R}]$$

for some constant $C \ge 0$ independent of n, θ, R. Passing to the limit, we deduce

$$\Big| \int_0^T dt \iint\limits_{R^3\times R^3} d\vec{r}\, d\vec{v} (\frac{\partial}{\partial t}\phi(\vec{r},t) + \vec{v}\cdot\nabla_{\vec{r}}\phi(\vec{r},t))\Psi_{\theta R}(g_\delta)$$

$$+\int_0^T dt\,\phi(\vec{r},t)\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}\,\frac{1}{1+\delta f}C_E(f,f)\Phi_{\theta R}(g_\delta)|$$

$$\leq C\sup|\phi(\vec{r},t)|[(\delta-\theta)+e^{(\theta-\delta)R}] \tag{3.29}$$

Next, multiplying equation (3.24) by $\phi(\vec{r},t)\Phi_{\theta R}(g_\delta)$ and integrating over $(0,T)\times R^3\times R^3$, we deduce exactly as before

$$-\int_0^T dt\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}(\frac{\partial}{\partial t}\phi(\vec{r},t)+\vec{v}\cdot\nabla_{\vec{r}}\phi(\vec{r},t))\Psi_{\theta R}(g_\delta)$$

$$-\int_0^T dt\,\phi(\vec{r},t)\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}\,\frac{1}{1+\delta f}C_E(f,f)\Phi_{\theta R}(g_\delta)$$

$$=\lambda\int_0^T dt\,\phi(\vec{r},t)\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}\,[\delta|\nabla_{\vec{v}}g_\delta|^2-\theta|\nabla_{\vec{v}}g_\delta|^2\chi_{(g_\delta\leq R)}]\Phi_{\theta R}(g_\delta)$$

$$+\iiint_{(0,T)\times R^3\times R^3}\phi(\vec{r},t)\Phi_{\theta R}(g_\delta)\,d\mu((\vec{r},\vec{v},t))$$

and

$$|\int_0^T dt\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}(\frac{\partial}{\partial t}\phi(\vec{r},t)+\vec{v}\cdot\nabla_{\vec{r}}\phi(\vec{r},t))\Psi_{\theta R}(g_\delta)$$

$$+\int_0^T dt\,\phi(\vec{r},t)\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}\,\frac{1}{1+\delta f}C_E(f,f)\Phi_{\theta R}(g_\delta)|$$

$$\geq C_1\sup|\phi(\vec{r},t)|[(\delta-\theta)+e^{(\theta-\delta)R}]+|\iiint_{(0,T)\times R^3\times R^3}o(\vec{r},t)\Phi_{\theta R}(g_\delta)\,d\mu| \tag{3.30}$$

for some (nonpositive) constant C_1 independent of θ and R. Combining the inequality with (3.29) we deduce

$$\left|\iiint_{(0,T)\times R^3\times R^3}\phi(\vec{r},t)\Phi_{\theta R}(g_\delta)\,d\mu((\vec{r},\vec{v},t))\right|\leq C(\sup|\phi(\vec{r},t)|)[(\delta-\theta)+e^{(\theta-\delta)R}] \tag{3.31'}$$

for every $\phi(\vec{r},t)\in\mathcal{D}((0,T)\times R^3)$. We get, by letting $R\to\infty$ and then $\theta\to\delta$, that

$$\left|\iiint_{(0,T)\times R^3\times R^3}\phi(\vec{r},t)\Phi_{\theta R}(g_\delta)\,d\mu((\vec{r},\vec{v},t))\right|=0.\qquad\forall\phi(\vec{r},t)\in\mathcal{D}((0,T)\times R^3) \tag{3.31}$$

Since $\Phi_{\theta R}(g_\delta) \geq 1$, we conclude by standard distribution theory that the measure $\mu((\vec{r},\vec{v},t))$ vanishes if we can prove further $\forall \hat{\psi} \in \mathcal{D}(R^3_{\vec{v}})$,

$$\iiint_{(0,T)\times R^3\times R^3} \hat{\psi}\Phi_{\theta R}(g_\delta)\, d\mu((\vec{r},\vec{v},t)) = 0, \qquad \forall \hat{\psi} \in \mathcal{D}(R^3_{\vec{v}}). \tag{3.32}$$

We use the same trick as before to prove (3.32). Since

$$\begin{aligned}
\hat{\psi}\Phi_{\theta R}(g^n_\delta)\frac{\partial}{\partial t}g^n_\delta &= \frac{\partial}{\partial t}(\Psi_{\theta R}(g^n_\delta)\hat{\psi}),\\
\hat{\psi}\Phi_{\theta R}(g^n_\delta)\vec{v}\cdot\nabla_{\vec{r}}g^n_\delta &= \mathrm{div}_{\vec{r}}(\Psi_{\theta R}(g^n_\delta)\hat{\psi}\vec{v}),\\
\hat{\psi}\Phi_{\theta R}(g^n_\delta)\Delta_{\vec{v}}g^n_\delta &= \Delta_{\vec{v}}(\hat{\psi}\Psi_{\theta R}(g^n_\delta)) - 2\Phi_{\theta R}(g^n_\delta)\nabla_{\vec{v}}g^n_\delta\cdot\nabla_{\vec{v}}\hat{\psi}\\
&\quad - \Psi_{\theta R}(g^n_\delta)\Delta_{\vec{v}}\hat{\psi} - \theta\hat{\psi}\Phi_{\theta R}(g^n_\delta)\chi_{(g^n_\delta\leq R)}|\nabla_{\vec{v}}g^n_\delta|^2,
\end{aligned}$$

and

$$\hat{\psi}\Phi_{\theta R}(g^n_\delta)\vec{E}^n\cdot\nabla_{\vec{v}}g^n_\delta = \mathrm{div}_{\vec{v}}(\hat{\psi}\Psi_{\theta R}(g^n_\delta)\vec{E}^n) - \vec{E}^n\cdot\Psi_{\theta R}(g^n_\delta)\nabla_{\vec{v}}\hat{\psi},$$

multiplying (RDVE) by $\hat{\psi} \in \mathcal{D}(R^3_{\vec{v}})$ and $\Phi_{\theta R}(g^n_\delta)$, integrating over $(0,T)\times R^3\times R^3$, we deduce

$$\begin{aligned}
&2\lambda \iiint_{(0,T)\times R^3\times R^3} dt\, d\vec{r}\, d\vec{v}\, \Phi_{\theta R}(g^n_\delta)\nabla_{\vec{v}}g^n_\delta\cdot\nabla_{\vec{v}}\hat{\psi} + \lambda \iiint_{(0,T)\times R^3\times R^3} dt\, d\vec{r}\, d\vec{v}\Psi_{\theta R}\Delta\hat{\psi}\\
&\quad - \iiint_{(0,T)\times R^3\times R^3} dt\, d\vec{r}\, d\vec{v}\{\hat{\psi}\Phi_{\theta R}(g^n_\delta)\frac{1}{1+\delta f^n}C_E(f^n,f^n) + \Psi_{\theta R}(g^n_\delta)\vec{E}^n\cdot\nabla_{\vec{v}}\hat{\psi}\}\\
&= \lambda \iiint_{(0,T)\times R^3\times R^3} dt\, d\vec{r}\, d\vec{v}\hat{\psi}[\delta\Phi_{\theta R}(g^n_\delta)|\nabla_{\vec{v}}g^n_\delta|^2 - \theta\Phi_{\theta R}(g^n_\delta)\chi_{(g^n_\delta\leq R)}|\nabla_{\vec{v}}g^n_\delta|^2].
\end{aligned} \tag{3.28$'$}$$

Noticing the similarity between (3.28) and (3.28′) and remembering the weak precompactness of $\{\vec{E}^n(\vec{r},t)\}$ in $L^2((0,T)\times R^3_{\vec{r}})$, we deduce (3.32) by using the same argument as above. Therefore, μ vanishes in $\mathcal{D}'((0,T)\times R^3\times R^3)$.

Finally, we have to justify the nonlinear multiplication by $\Phi_{\theta R}$ and the resulting integration by parts, (3.27), (3.28), and so on. This can be checked by the convolution

regularization. In fact, let ρ_ϵ denote a regularizing kernel ρ in $((\vec{r},\vec{v},t))$,

$$\rho_\epsilon = \frac{1}{\epsilon^7}\rho(\frac{\cdot}{\epsilon}), \qquad \rho \in \mathcal{D}(R^7),\ \rho \geq 0, \qquad \int \rho = 1.$$

We check directly that

$$g_\delta^\epsilon = \rho_\epsilon * g_\delta = \iiint g_\delta(\vec{r}-\vec{r}',\vec{v}-\vec{v}',t-t')\rho_\epsilon(\vec{r}',\vec{v}',t')dt'd\vec{v}'d\vec{r}'$$

satisfies

$$\begin{aligned}&\frac{\partial}{\partial t}g_\delta^\epsilon + \vec{v}\cdot\nabla_{\vec{r}}g_\delta^\epsilon - \lambda\Delta_{\vec{v}}g_\delta^\epsilon + \vec{E}\cdot\nabla_{\vec{v}}g_\delta^\epsilon = \\ =&\{\frac{1}{1+\delta f}C_E(f,f) + \lambda\delta|\nabla_{\vec{v}}g_\delta^\epsilon|^2\} * \rho_\epsilon + \mu * \rho_\epsilon + r_{1\epsilon} + r_{2\epsilon} \qquad (3.33)\end{aligned}$$

in $(\alpha_\epsilon, T)\times R^3\times R^3$, where $\alpha_\epsilon \in (0,T)$, $\alpha_\epsilon \to 0$ as $\epsilon \to 0$, $r_{1\epsilon} = \vec{v}\cdot\nabla_{\vec{r}}g_\delta^\epsilon - (\vec{v}\cdot\nabla_{\vec{r}}g_\delta^\epsilon)*\rho_\epsilon$, $r_{2\epsilon} = \vec{E}\cdot\nabla_{\vec{v}}g_\delta^\epsilon - (\vec{E}\cdot\nabla_{\vec{v}}g_\delta^\epsilon)*\rho_\epsilon$.

Notice that

$$\begin{aligned}r_{1\epsilon} &= \vec{v}\cdot(g_\delta^\epsilon * \nabla_{\vec{r}}\rho_\epsilon) - (\vec{v}\cdot\nabla_{\vec{r}}g_\delta^\epsilon)*\rho_\epsilon = \\ &= \int dt'\,d\vec{v}'\,d\vec{r}'\,(\vec{v}'-\vec{v})g_\delta(\vec{r}',\vec{v}',t')\cdot\nabla_{\vec{r}}\rho_\epsilon(\vec{r}-\vec{r}',\vec{v}-\vec{v}',t-t') = g_\delta * \mathbf{K}_\epsilon^{(1)},\end{aligned}$$

with $\mathbf{K}_\epsilon^{(1)} = \epsilon^{-7}\mathbf{K}^{(1)}(\cdot/\epsilon), \mathbf{K}^{(1)} = -\vec{v}\cdot\nabla_{\vec{r}}((\vec{r},\vec{v},t))$. Hence

$$r_{1\epsilon} \to_\epsilon (\int \mathbf{K}^{(1)})g_\delta = 0 \quad \text{in } L^1.$$

Similarly, $r_{2\epsilon} = g_\delta * \mathbf{K}_\epsilon^{(2)}$, with $\mathbf{K}_\epsilon^{(2)} = \epsilon^{-7}\mathbf{K}^{(2)}(\cdot/\epsilon)$, $\mathbf{K}^{(2)} = -\vec{E}(\vec{r},t)\cdot\nabla_{\vec{v}}\rho((\vec{r},\vec{v},t))$. Hence

$$r_\epsilon = r_{1\epsilon} + r_{2\epsilon} \to_\epsilon (\int(\mathbf{K}^{(1)} + \mathbf{K}^{(2)}))g_\delta \equiv 0 \quad \text{in } L^1 \oplus L^1([0,T]\times R_r^3; L^2(R_{\vec{v}}^3)).$$

Then for $\phi(\vec{r},t) \in \mathcal{D}((0,T)\times R^3)$, $\hat{\psi} \in \mathcal{D}(R^3)$, $\|\phi\|_{C^\infty} = 1$, multiplying equation (3.33) by $\phi\Phi_{\theta R}(g_\delta^\epsilon)$ and $\hat{\psi}\Phi_{\theta R}(g_\delta^\epsilon)$ respectively, observing $\Phi_{\theta R}(g_\delta^\epsilon) \geq 1$, we deduce that

$$-\int_0^T dt \iint_{R^3\times R^3} d\vec{r}\,d\vec{v}\{(\frac{\partial}{\partial t}\phi(\vec{r},t) + \vec{v}\cdot\nabla_{\vec{r}}\phi(\vec{r},t))\Psi_{\theta R}(g_\delta^\epsilon) +$$

$$+ \phi(\vec{r},t)\,\Phi_{\theta R}(g_\delta^\epsilon)\{\frac{1}{1+\delta f}C_E(f,f)\} * \rho_\epsilon\}$$

$$\geq \lambda \int \phi(\vec{r},t)[\delta\Phi_{\theta R}(g_\delta^\epsilon)|\nabla_{\vec{v}} g_\delta^\epsilon|^2 * \rho_\epsilon - |\nabla_{\vec{v}} g_\delta^\epsilon|^2 \nabla_{\vec{v}}\Phi_{\theta R}(g_\delta^\epsilon)]$$

$$+ \int \phi(\vec{r},t)\Phi_{\theta R}(g_\delta^\epsilon)\mu * \rho_\epsilon - e^{\theta R}(\sup|\phi(\vec{r},t)|)[\|r_{1\epsilon}\|_{L^1} + \|r_{2\epsilon}\|_{L^1([0,T]\times R^3_r; L^2(R^3_{\vec{v}}))}] \tag{3.34a}$$

and

$$2\lambda \int \Phi_{\theta R}(g_\delta^\epsilon)\nabla_{\vec{v}} g_\delta^\epsilon \cdot \nabla_{\vec{v}}\hat{\psi} + \lambda \int \Psi_{\theta R}\Delta\hat{\psi} - \int \hat{\psi}\Phi_{\theta R}(g_\delta^\epsilon)\{\frac{1}{1+\delta f}C_E(f,f)\} * \rho_\epsilon$$

$$- \int \Psi_{\theta R}(g_\delta^\epsilon)\vec{E}\cdot\nabla_{\vec{v}}\hat{\psi}$$

$$\geq \int \phi(\vec{r},t)\Phi_{\theta R}(g_\delta^\epsilon)\mu * \rho_\epsilon - e^{\theta R}(\sup|\phi(\vec{r},t)|)[\|r_{1\epsilon}\|_{L^1} + \|r_{2\epsilon}\|_{L^1([0,T]\times R^3_r; L^2(R^3_{\vec{v}}))}]$$

$$+ \lambda \int \phi(\vec{r},t)[\delta\Phi_{\theta R}(g_\delta^\epsilon)|\nabla_{\vec{v}} g_\delta^\epsilon|^2 * \rho_\epsilon - |\nabla_{\vec{v}} g_\delta^\epsilon|^2\nabla_{\vec{v}}\Phi_{\theta R}(g_\delta^\epsilon)]. \tag{3.34b}$$

Let $\epsilon \to 0$. Then we may argue that the left sides of both (3.34a) and (3.34b) are bounded below and conclude that μ vanishes by the argument as before, provided that we have shown the following inequalities:

$$|\nabla_{\vec{v}} g_\delta|^2 \leq \frac{4}{\delta}|\nabla_{\vec{v}} f^{1/2}|^2 \quad \text{a.e.}, \quad |\nabla_{\vec{v}}(g_\delta \wedge R)|^2 \leq \frac{4}{\delta}e^{-\delta R}|\nabla_{\vec{v}} f^{1/2}|^2 \quad \text{a.e.}$$

Formally, these bounds are obvious if we remember that $g_\delta = \beta_\delta(f) = \frac{1}{\delta}\log(1+\delta f)$ and $f \geq 0$. Using a simple approximation argument, say, taking $f_{n,\epsilon} = \rho_\epsilon * (f + 1/n\exp[-n(r^2+v^2)])$ and observing that f_n satisfies the above inequalities and $f_n \to f$, we have justified these inequalities.

Now we have proved that g_δ solves the (RDVE) and that $(1+\delta f^n)^{-1}C_E(f^n,f^n)$ converges to $(1+\delta f)^{-1}C_E(f,f)$ in $L^1((0,T)\times R^3\times R^3)$ for all $T<\infty$. The next step is to show that both $\mu^{(1)}$ and $\mu^{(2)}$ vanish. As the matter of fact, we know that $0 = \mu = \lambda\mu^{(1)} - \mu^{(2)}$ and $\mu^{(1)} = \delta^{-1}\mu^{(2)} \geq 0$. Integrating (RDVE) over $(0,T)\times R^3\times R^3$

allows us. at least formally (and rigorously by a similar proof as above), to deduce that

$$\int (\frac{1}{1+\delta f^n} C_E(f^n, f^n) - \frac{1}{1+\delta f} C_E(f,f)) + \lambda\delta \int (|\nabla_{\vec{v}} g_\delta^n|^2 - |\nabla_{\vec{v}} g_\delta|^2) = 0.$$

Letting $n \to \infty$, we may see that

$$\int |\nabla_{\vec{v}} g_\delta^n|^2 \to \int |\nabla_{\vec{v}} g_\delta|^2, \tag{3.35}$$

and $\mu^{(2)} = \lambda\mu(1) = 0$. Also

$$\int_0^T dt \iint_{R^3_{\vec{r}} \times B_R(\vec{v})} \vec{E}^n \cdot \nabla_{\vec{v}} g_\delta^n \, d\vec{v} d\vec{r} \to \int_0^T dt \iint_{R^3_{\vec{r}} \times B_R(\vec{v})} \vec{E} \cdot \nabla_{\vec{v}} g_\delta \, d\vec{v} d\vec{r}, \quad \forall R < \infty \tag{3.36'}$$

and

$$\int_0^T dt \int_{R^3} d\vec{r} (\int_{R^3} |\vec{E}^n \cdot \nabla_{\vec{v}} g_\delta^n|^2 \, d\vec{v})^{1/2} \to \int_0^T dt \int_{R^3} d\vec{r} (\int_{R^3} |\vec{E} \cdot \nabla_{\vec{v}} g_\delta|^2 \, d\vec{v})^{1/2}. \tag{3.36}$$

This completes the proof of the theorem. ■

COROLLARY. *Using the above notation,*

$$g_\delta \to f \quad in \quad L^\infty((0,T); L^1(R^3 \times R^3)) \quad as \quad \delta \to 0_+.$$

In fact, from the standard results on weak convergence, (3.35) implies $\nabla_{\vec{v}} g_\delta^n$ converges to $\nabla_{\vec{v}} g_\delta$ in $L^2((0,T) \times R^3 \times R^3)$ and $|\nabla_{\vec{v}} g_\delta^n|^2$ converges to $|\nabla_{\vec{v}} g_\delta|^2$ in $L^1((0,T) \times R^3 \times R^3)$. Also, $\vec{E}^n \cdot \nabla_{\vec{v}} g_\delta^n$ converges in $L^1((0,T) \times R^3 \times B_R(\vec{v}))$, $R < \infty$ and in $L^1((0,T) \times R^3; L^2(R^3(\vec{v})))$ to $\vec{E} \cdot \nabla_{\vec{v}} g_\delta$. In particular, $L_\lambda g_\delta^n$ converges in $L^1((0,T) \times R^3 \times R^3) \oplus L^1((0,T) \times R^3_{\vec{r}}; L^2(R^3_{\vec{v}}))$ to $L_\lambda g_\delta$ in the sense of Lemma 3.4. The standard results of Lemma 3.4 show that $g_\delta \in C([0,T]; L^1(R^3 \times R^3))$ for all $\delta > 0$.

From this fact, it is straightforward to deduce $f \in C([0,T];L^1(R^3 \times R^3))$: one just needs to observe that for a.e. $t \in [0,T]$,

$$\iint_{R^3 \times R^3} d\vec{r}\, d\vec{v}\, |f((\vec{r},\vec{v},t)) - g_\delta((\vec{r},\vec{v},t))| \leq$$

$$\leq \epsilon_\delta \iint_{R^3 \times R^3} d\vec{r}\, d\vec{v}\, f((\vec{r},\vec{v},t)) + 2 \iint_{R^3 \times R^3} d\vec{r}\, d\vec{v} f((\vec{r},\vec{v},t))\chi_{(f \geq R)},$$

where $R > 0$, and $\epsilon_\delta = \sup_{[0,R]} |1 - \beta_\delta(s)s^{-1}| \to 0$ as $\delta \to 0_+$. Because of (3.1), this implies that $g_\delta \to f$ in $L^\infty((0,T);L^1(R^3 \times R^3))$ as $\delta \to 0_+$.

COROLLARY. *In theorem 3.1, if we assume further that* $\|f_n|_{t=0} - f_0\|_{L^1(R^3 \times R^3)} \to 0$ *for some* $f_0 \in L^1$, *then the renormalized solution satisfies also the initial condition* $f|_{t=0} = f_0$. *In other words,* f *is a renormalized solution of (DVE)-(VP)-(IC).*

Chapter IV. Existence Of Approximate Solutions

In this chapter, we construct positive solutions for the approximate equations. The strategy of the construction is as follows. First, we find the semigroup $\mathbf{U}_{\mathcal{A}}(t)$ generated by the diffusive operator $\mathcal{A}$: $\mathcal{A}f = -\vec{v}\cdot\nabla_{\vec{r}}f + \lambda\Delta_{\vec{v}}$, and the evolution system $\mathbf{U}_{\mathcal{B}}(t,s)$ determined by the time-dependent generators $\mathcal{B}(t)$: $\mathcal{B}(t)f(\vec{r},\vec{v},t) = -\vec{E}\cdot\nabla_{\vec{v}}f - L(\vec{r},\vec{v},t)f$. Then we show that the linear evolution transport operators $\mathcal{C}(t) = \mathcal{A}+\mathcal{B}(t)$ determine a positive evolution system $\mathbf{U}(t,s)$ by using Kato's theorem and the Trotter product formula. Observing that the approximate Enskog collision terms $\tilde{C}_E^{\pm}$ are Lipschitz functions, we conclude that the approximate equations have positive solutions for each n, by a contraction type fixed point argument. Finally, at the end of this chapter, we estimate some bounds related to the solutions of the equations.

§4.1 Semigroups Generated By The Diffusive Operator

Let $X = L^p(R^3\times R^3)$ $(1\le p<\infty)$ or $C_\infty(R^3\times R^3)$. Define the diffusive transport operator $\mathcal{A}:\mathcal{D}(\mathcal{A})(\subset X)\to X$ as the closure of the operator

$$\mathcal{A}f(\vec{r},\vec{v}) = -\vec{v}\cdot\nabla_{\vec{r}}f + \lambda\Delta_{\vec{v}}f = \mathcal{A}_\infty f + \mathcal{A}_\epsilon f \qquad \text{(A)}$$

on $W^{p,\infty}(R^3\times R^3)$ (for $1\le p<\infty$) and $\mathcal{S}(R^3\times R^3)$ (for $C_\infty(R^3\times R^3)$), respectively. Since $\mathcal{A}_\infty$ is an infinitesimal generator of a c_0-semigroup $\mathbf{U}(t)$,[37][38] $\mathcal{A}_\epsilon(=\lambda\Delta_{\vec{v}})$ is dissipative both in L^p and C_∞ (see [35] or [43, §X.8]), and $\mathcal{D}(\mathcal{A}_2)\subset\mathcal{D}(\mathcal{A}_1)$, the

operator $\mathcal{A}$ generates a c_0-semigroup $\mathbf{U}_{\mathcal{A}}(t)$ in X. Using the relationship of the c_0-semigroup and its infinitesimal generator and the fundamental solution of the operator L_λ given in (3.9), we deduce that the semigroup generated by $\mathcal{A}$ is given by

$$\mathbf{U}_{\mathcal{A}}(t)f = \iint\limits_{R^3\times R^3} \exp[-\frac{1}{\lambda t}(\frac{1}{4}(\vec{v}-\vec{u})^2 + \frac{3}{4}(\vec{v}+\vec{u})^2 - \frac{3(\vec{r}-\vec{q})\cdot(\vec{v}+\vec{u})}{t} + \frac{3(\vec{r}-\vec{q})^2}{t^2})]$$
$$\times \frac{3\sqrt{3}}{8}(\frac{1}{\lambda\pi t^2})^3 f(\vec{q},\vec{u})\, d\vec{q}\, d\vec{u}, \qquad t>0. \tag{4.1}$$

Using the fact, given by (3.11), that $\iint p(t,\vec{r},\vec{v};\vec{q},\vec{u})\, d\vec{r}d\vec{v} = 1$ for each $(\vec{q},\vec{u},t) \in R^3\times R^3\times R_+$, (4.1) and the Fubini Theorem, we have

$$\begin{aligned}\|\mathbf{U}_{\mathcal{A}}(t)f\|_{L^1(R^3\times R^3)} &= \iint d\vec{r}d\vec{v} \iint d\vec{q}d\vec{u}\, p(t,\vec{r},\vec{v};\vec{q},\vec{u}) f(\vec{q},\vec{u}) \\ &= \iint f(\vec{q},\vec{u}) d\vec{q}d\vec{u} \iint h(t,\vec{r},\vec{v};\vec{q},\vec{u})\, d\vec{r}d\vec{v} \\ &= \iint f(\vec{q},\vec{u}) d\vec{q}d\vec{u},\end{aligned}$$

and conclude that $\mathbf{U}_{\mathcal{A}}(t)$ is a contraction semigroup in L^1. The Young's inequality [43] shows that $\|\mathbf{U}_{\mathcal{A}}(t)\|_\infty \le 1$. Then classical L^p-interpolation theory (e.g., Riesz-Thorin theorem) implies $\mathbf{U}_{\mathcal{A}}(t)$ is a contraction semigroup on L^p $(1\le p<\infty)$ and C_∞. That $\mathbf{U}_{\mathcal{A}}(t)$ given by (4.1) does define a semigroup can also be justified directly by using the following integration:

$$\begin{aligned}1 = {} & \frac{3\sqrt{3}}{8}(\frac{(t_1+t_2)^2}{\lambda\pi t_1^2 t_2^2})^3 \iint d\vec{r}''d\vec{v}'' \times \\ & \times \exp[-\frac{1}{4\lambda t_2}(v-v'')^2 - \frac{1}{4\lambda t_1}(v'-v'')^2 + \frac{1}{4\lambda(t_1+t_2)}(v'-v)^2] \times \\ & \times \exp[-\frac{3}{4\lambda t_2}(v+v'')^2 - \frac{3}{4\lambda t_1}(v'+v'')^2 + \frac{1}{4\lambda(t_1+t_2)}(v'+v)^2] \times \\ & \times \exp[\frac{3}{\lambda t_2^2}(\vec{r}-\vec{r}'')\cdot(\vec{v}+\vec{v}'') + \frac{3}{\lambda t_1^2}(\vec{r}'-\vec{r}'')\cdot(\vec{v}'+\vec{v}'') - \frac{3}{\lambda(t_1+t_2)^2}(\vec{r}'-\vec{r})\cdot(\vec{v}'+\vec{v})] \\ & \times \exp[-\frac{3}{\lambda t_2^3}(\vec{r}-\vec{r}'')^2 - \frac{3}{\lambda t_1^3}(\vec{r}'-\vec{r}'')^2 + \frac{3}{\lambda(t_1+t_2)^3}(\vec{r}'-\vec{r})^2],\end{aligned}$$

which can be checked by elementary calculation.

Let $\mathcal{T}_+$ be the cone of non-negative functions in X. As seen from (4.1), we also have

$$\mathbf{U}_{\mathcal{A}}(t)\mathcal{T}_+ \subseteqq \mathcal{T}_+. \tag{4.2}$$

§4.2 The Vlasov-Enskog Operator

Suppose $\vec{E}(\vec{r},t)$ is a (bounded) continuous function on $R^3 \times [0,T]$ and $L(\vec{r},\vec{v},t)$ a non-negative measurable function on $R^3 \times R^3 \times [0,T]$. Define the time-dependent Vlasov-Enskog operator $\mathcal{B}(t) : \mathcal{D}(\mathcal{B}(t))(\subset X) \to X$ by

$$\mathcal{B}(t)f(\vec{r},\vec{v},t) = -\vec{E}(\vec{r},t)\cdot\nabla_{\vec{v}}f(\vec{r},\vec{v},t) - L(\vec{r},\vec{v},t)f(\vec{r},\vec{v},t), \tag{B}$$

$$\mathcal{D}(\mathcal{B}(t)) = \{f \in X;\quad \lim_{v\to\infty} f(\vec{r},\vec{v},t) = 0,\ |\nabla_{\vec{v}}f| \in X\}.$$

It is not difficult to prove that, for each given $s \in [0,\infty)$, the operator $\mathcal{B}(s)$ satisfies the *Hille-Phillips-Yosida* condition and generates a c_0-semigroup $\mathbf{U}_{s,\mathcal{B}}(t)$. In particular, $\{\mathcal{B}(t)\}$ itself satisfies the Kato's conditions[50 p.430-438] [43] [35] for generating a two-parameter contractive *evolution operator*[13] (*evolution system,*[35] *integration of evolution equation,*[50] *propagator*[43]) $\mathbf{U}_{\mathcal{B}}(t,s)$. That is, there exists a two-parameter family of bounded operators $\mathbf{U}_{\mathcal{B}}(t,s)$ ($0 \le s \le t < \infty$) on X, such that

(1) $\mathbf{U}_{\mathcal{B}}(s,s) = 1$, $\quad \mathbf{U}_{\mathcal{B}}(t,r)\mathbf{U}_{\mathcal{B}}(r,s) = \mathbf{U}_{\mathcal{B}}(t,s)$ for $0 \le s \le r \le t < \infty$;

(2) $(t,s) \to \mathbf{U}_{\mathcal{B}}(t,s)$ is strongly continuous for $0 \le s \le t < \infty$;

(3) $\forall f_0 \in \mathcal{D}(\mathcal{B}(0))$, the equation

$$\frac{df(t)}{dt} = \mathcal{B}(t)f(t), \quad f(0) = f_0,\ f(t) \in \mathcal{D}(\mathcal{B}(t)) \tag{4.3}$$

is solved by $f(t) = \mathbf{U}_{\mathcal{B}}(t,0)f_0$ which satisfies the estimate $\|f(t)\| \le \|f_0\|$.

In fact, checking directly. $\mathbf{U}_{\mathcal{B}}(t,s)$ is given by

$$f(\vec{r},\vec{v},t) = (\mathbf{U}_{\mathcal{B}}(t,s)g)(\vec{r},\vec{v},t) =$$
$$= \exp\{-\int_s^t L(\vec{r},\vec{v}-\int_{t_1}^t \vec{E}(\vec{r},\tau)d\tau,t_1)dt_1\}g(\vec{r},\vec{v}-\int_s^t \vec{E}(\vec{r},\tau)d\tau,s). \qquad g \in X \tag{4.4}$$

and $\|f(t)\|_{L^1} \leq \|g\|_{L^1}$, $\|f(t)\|_{C_\infty} \leq \|g\|_{C_\infty}$. Using the interpolation theory again we conclude that $\mathbf{U}_{\mathcal{B}}(t,s)$ is a contractive evolution operator in L^p, $1 \leq p < \infty$. Note that (4.4) also implies

$$\mathbf{U}_{\mathcal{B}}(t,s)\mathcal{T}_+ \subseteq \mathcal{T}_+, \qquad 0 \leq s \leq t < \infty. \tag{4.5}$$

§4.3 Solutions of The Linear Homogeneous Equation

Let X be a Banach space, $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to X$ and $\mathcal{B}(t) : \mathcal{D}(\mathcal{B}(t)) \to X$, $(t \in [0,T])$, be dissipative operators. Suppose $\mathcal{D}(A) \subset D(B(t))$ for each t. Define the linear evolution operator $\mathcal{C}(t) : D(C(t)) (\supseteq D(A) \cap D(B(t)) = D(A) = D(C(0)))) \to X$ as the closure of the addition of $\mathcal{A}$ and $\mathcal{B}(t)$: $C(t) = \overline{A + B(t)}$. The $\mathcal{C}(t)$ are linear closed operators with $\mathcal{D}(C(t))$ independent of t and dense in X. Assume that

(H1) For every $\lambda \geq 0$ and t, $0 \leq t \leq T$, the resolvent $(\lambda I - \mathcal{C}(t))^{-1}$ exists as an operator belonging to $L(X,X)$ such that

$$\|(\lambda I - \mathcal{C}(t))^{-1}\| \leq \lambda^{-1} \quad \text{for} \quad \lambda > 0: \tag{4.6}$$

(H2) For each $x \in X$, $(t-s)^{-1}(\mathcal{C}(t)\mathcal{C}(s)^{-1} - I)x$ is bounded and uniformly strongly continuous in t and s, $t \neq s$, and $s - \lim_{k\to\infty} k\mathcal{C}(t)\mathcal{C}(t-1/k)x = \mathcal{K}(t)x$ exists uniformly in $t(\in [0,T])$ so that $\mathcal{K}(t) \in L(X,X)$ is strongly continuous in t.

With above assumptions, we have the following

KATO'S THEOREM. [50,p.432] *For any positive integer k and $0 \le s \le t \le T$, define the operator $\mathbf{U}_k(t,s) \in L(X,X)$ by*

$$\begin{cases} \mathbf{U}_k(t,s) = \exp((t-s)\mathcal{C}(\frac{i-1}{k}T)) & \text{for } \frac{i-1}{k}T \le s \le t \le \frac{i}{k}T \ (1 \le i \le k), \\ \mathbf{U}_k(t,r) = \mathbf{U}_k(t,s)\mathbf{U}_k(s,r) & \text{for } 0 \le r \le s \le t \le T. \end{cases} \tag{4.7}$$

Then, for every $x \in X$ and $0 \le s \le t \le T$, $s - \lim_{k\to\infty} \mathbf{U}_k(t,s)x = \mathbf{U}_{\mathcal{C}}(t,s)x$ exists uniformly in t and s. Moreover, for $y \in \mathcal{D}(\mathcal{C}(0))$, the Cauchy problem

$$\frac{dx(t)}{dt} = \mathcal{C}(t)x(t), \ x(0) = y \quad \text{for} \quad x(t) \in \mathcal{D}(\mathcal{C}(t)), \ 0 \le t \le T, \tag{C'}$$

is solved by $x(t) = \mathbf{U}_{\mathcal{C}}(t,0)y$ which satisfies the estimate $\|x(t)\| \le \|y(t)\|$.

The following theorem tell us how to "compute" $\mathbf{U}_{\mathcal{C}}$ in terms of $\mathbf{U}_{\mathcal{A}}$ and $\mathbf{U}_{\mathcal{B}}$.

THE TROTTER PRODUCT FORMULA. [43,p.245] *Let $\mathcal{A}$ and $\mathcal{B}$ be the generators of contraction semigroups in X. Suppose that the closure of $(\mathcal{A}+\mathcal{B})|_{\mathcal{D}(\mathcal{A})\cap\mathcal{D}(\mathcal{B})}$ generates a contraction semigroup on X. Then for all $\phi \in X$,*

$$e^{-t\mathcal{C}}\phi = e^{-t\overline{(\mathcal{A}+\mathcal{B})}}\phi = \lim_{n\to\infty}(e^{-t\mathcal{A}/n}e^{-t\mathcal{B}/n})^n\phi. \tag{4.8}$$

Substituting (4.8) into (4.7) for each k, and using an argument similar to that in the proof of Kato's theorem, we have

THEOREM 4.1. [13] *Under the assumptions of (H1) and (H2), then the evolution operators $\mathbf{U}_{\mathcal{C}}(t,s)$, generated by $\mathcal{C}(t)$, are given by*

$$\mathbf{U}(t,s;\vec{E},L) = \mathbf{U}_{\mathcal{C}}(t,s) = \tag{C}$$
$$= s\text{-}\lim_{\substack{m\to\infty \\ \delta(m)\to 0}} \lim_{n\to\infty} \left([\mathbf{U}_{\mathcal{A}}(\frac{t_m - t_{m-1}}{n})\mathbf{U}_{\mathcal{B}}(\frac{t_m}{n}, \frac{t_{m-1}}{n})]^n \times \cdots \times [\mathbf{U}_{\mathcal{A}}(\frac{t_1 - t_0}{n})\mathbf{U}_{\mathcal{B}}(\frac{t_1}{n}, \frac{t_0}{n})]^n \right)$$

and $\mathbf{U}_{\mathcal{C}}(t,s)$ solves the evolution equation (C'). In addition, if $\mathbf{U}_{\mathcal{A}}\mathcal{T}_+ \subset \mathcal{T}_+$ and $\mathbf{U}_{\mathcal{B}}(t,s)\mathcal{T}_+ \subset \mathcal{T}_+$, then $\mathbf{U}_{\mathcal{C}}(t,s)\mathcal{T}_+ \subset \mathcal{T}_+$.

Now let us consider the operators $\mathcal{A}$ given by (A) and $\mathcal{B}(t)$ by (B). We have

THEOREM 4.2. *Suppose that $\vec{E}(\vec{r},t)$ is continuous in $\vec{r}$ and $L(\vec{r},\vec{v},t)$ is non-negative measurable on $(0,T)\times R^3\times R^3$. Then there exists a system of evolution operators $\mathbf{U}_{\mathcal{C}}(t,s)$ such that for each $f_0\in\mathcal{D}(A)$ and $s\in(0,T)$, the homogeneous evolution equation*

$$\begin{cases} \frac{\partial}{\partial t}f(\vec{r},\vec{v},t)+\vec{v}\cdot\nabla_{\vec{r}}f-\lambda\Delta_{\vec{v}}f+\vec{E}\cdot\nabla_{\vec{v}}f+Lf=0, \quad 0\le s<t\le T \\ \lim_{t\to s^+}\|f(t)-f_0\|_X=0 \end{cases} \tag{HE}$$

has a unique solution $f(t)=\mathbf{U}_{\mathcal{C}}(t,0)f_0$ with $\|f(t)\|\le\|f_0\|$. Furthermore, $f(\vec{r},\vec{v},t)\ge 0$ if $f_0\ge 0$. If $L(\vec{r},\vec{v},t)$ is assumed bounded but not necessarily positive, the above is still valid, except that the estimation $\|f(t)\|\le\|f_0\|$ may fail.

We remark that Theorem 4.2 is not a direct consequence of the Kato's theorem. If $\inf L(\vec{r},\vec{v},t)=0$, we may apply the theorem to the solution of

$$\frac{d}{dt}g(t)-(\mathcal{A}+\mathcal{B}(t)+I)g(t)=0, \quad \lim_{t\to 0^+}f(t)=f_0$$

and deduce that $\|g(t)\|_X\le e^{-t}\|f_0\|_X$. Then we justify that $f(t)=e^t g(t)$ is the solution of (HE) with $\|f(t)\|\le\|f_0\|$. ■

§4.4 The Solution of Approximate Equations

Let us consider the solution of the following non-linear equation

$$\begin{cases} \frac{\partial}{\partial t}f(\vec{r},\vec{v},t)+\vec{v}\cdot\nabla_{\vec{v}}f-\lambda\Delta_{\vec{v}}f+\vec{E}(\vec{r},t)\cdot\nabla_{\vec{v}}f=\tilde{C}_E(f,f) \\ \lim_{t\to 0+}f(\vec{r},\vec{v},t)=f_0(\vec{r},\vec{v}). \end{cases} \tag{AE1}$$

where

$$\vec{E}(\vec{r},t)=\frac{1}{4\pi}\int d\vec{r}'\nabla_{\vec{r}}\frac{1}{|\vec{r}-\vec{r}'|}\int d\vec{v}f(\vec{r}',\vec{v},t), \tag{AE2}$$

$\tilde{C}_E(f,f)(\vec{r},\vec{v},t)$

$$= \iint_{R^3 \times S^2_+} [\mathcal{Y}(n(\vec{r},t), n(\vec{r}-a\vec{\epsilon},t))\eta_B(\vec{v},\vec{v}_1)f(\vec{r},\vec{v}',t)f(\vec{r}-a\vec{\epsilon},\vec{v}_1',t) -$$
$$\mathcal{Y}(n(\vec{r},t), n(\vec{r}+a\vec{\epsilon},t))\eta_B(\vec{v},\vec{v}_1)f(\vec{r},\vec{v},t)f(\vec{r}+a\vec{\epsilon},\vec{v}_1,t)] < \vec{\epsilon}, \vec{v}-\vec{v}_1 > d\vec{\epsilon}d\vec{v}_1$$
$$= \tilde{C}_E^+ - \tilde{C}_E^-, \qquad \text{(AE3)}$$

$B = \{(\vec{v},\vec{v}_1);\ |\vec{v}|^2+|\vec{v}_1|^2 \le k\}$ for some positive constant k and $n(r,t) = \int_B f(\vec{r},\vec{v},t)d\vec{v}$. We assume

(A1) $f_0 \in C_0^\infty(R^3 \times R^3) \cap L^1_+(R^3 \times R^3)$;

(A2) there exists a bounded function $\tilde{Y}$ satisfying $\tilde{Y}(\sigma,\tau) = \tilde{Y}(\tau,\sigma)$, $\sup_{\sigma,\tau} \tau\tilde{Y}(\tau,\sigma) \le M_Y < \infty$, and $\mathcal{Y}(n(\vec{r}_1), n(\vec{r}_2)) = (1+\epsilon n(\vec{r}_1))^{-1}(1+\epsilon n(\vec{r}_2))^{-1}\tilde{Y}(n(\vec{r}_1), n(\vec{r}_2))$ for some $\epsilon > 0$.

(A3) $\tilde{Y}$ satisfies the *Lipschitz condition*: $|\tilde{Y}(\sigma_1,\tau_1)-\tilde{Y}(\sigma_2,\tau_2)| \le C(|\sigma_1-\sigma_2|+|\tau_1-\tau_2|)$ for some constant C independent of σ and τ;

(A4) $\eta_B(\vec{v},\vec{v}_1) = \eta_B(\vec{v}_1,\vec{v}) = \eta_B(\vec{v}_1',\vec{v}')$ is a smooth nonnegative function of $(\vec{v},\vec{v}_1)$. $\eta \le 1$, and supported in B.

It seems that the assumptions (A1) and (A3) are artificial. We will, however, show how to alter the initial value f_0 and the geometric factor Y to satisfy those assumptions for the purpose of constructing approximate solutions in the proof of global existence (Theorem 5.1).

THEOREM 4.3. *Under the assumptions (A1)-(A4), the equation (AE1)-(AE3) has a unique non-negative solution which belongs to* $C([0,T]; L^p(R^3_{\vec{r}}; L^1(R^3_{\vec{v}})))$, $1 \le p \le \infty$. *for each* $T \in (0,\infty)$.

We will need the following easy lemma.[15]

LEMMA 4.4. *Let* $d \ge 3$ *and* $\mathcal{V}(\vec{r}) = 1/|\vec{r}|^{d-2}$. *Then for all* $\phi \in L^1(R^d) \cap L^\infty(R^d)$, *the*

convolution $\mathcal{V} * \phi$ is differentiable and

$$\nabla_{\vec{r}}(\mathcal{V} * \phi) = (\nabla_{\vec{r}}\mathcal{V}) * \phi.$$

Furthermore $\mathcal{V} * \phi$ and its gradient $\nabla_{\vec{r}}(\mathcal{V} * \phi)$ are bounded (i.e., $\mathcal{V} * \phi \in H^{1,\infty}(R^d)$),

$$|\nabla_{\vec{r}}\mathcal{V} * \phi(\vec{r})| \le \|\nabla_{\vec{r}}\mathcal{V}\|_{L^1(B_R)}\|\phi\|_\infty + \|\nabla_{\vec{r}}\mathcal{V}\|_{L^\infty(B_R^c)}\|\phi\|_1,$$

where $B_R = \{|\vec{r}| \le R\}$.

Before we prove the theorem, we also need to introduce the following spaces: the space $X = C(R^3_{\vec{r}}; L^1(R^3_{\vec{v}}))$ with the norm $\|f\|_X = \sup_{\vec{r}} \int |f(\vec{r},\vec{v})|\, d\vec{v}$, the space $M = X \cap L^1(R^3 \times R^3)$ with the norm $\|f\|_M = \max\{\|f\|_{L^1}, \|f\|_X\}$, and the space $M_{[0,T]} = C(0,T;M)$ with the norm $\|\hat{f}\|_{[0,T]} = \sup_{0\le t\le T}\|\hat{f}(t)\|_M$. The following lemma shows some important properties of the operator $\mathbf{U}(t,s;\vec{E},L)$.

LEMMA 4.5. *If $L(\vec{r},\vec{v},t)$ is a non-negative measurable function, then $\mathbf{U}(t,s;\vec{E},L)$ defined by (C) is a contractive operator for $t \ge s \ge 0$ in both X and M.*

In fact, for any $t \ge s \ge 0$, we have

$$\begin{aligned}
&\|\mathbf{U}_{\mathcal{A}}(t)f\|_X = \\
&= \sup_{\vec{r}} \int d\vec{v} \Big| \iint\limits_{R^3\times R^3} \exp[-\frac{1}{\lambda t}(\frac{1}{4}(\vec{v}-\vec{u})^2 + \frac{3}{4}(\vec{v}+\vec{u})^2 - \frac{3(\vec{r}-\vec{q})\cdot(\vec{v}+\vec{u})}{t} + \frac{3(\vec{r}-\vec{q})^2}{t^2})] \\
&\qquad \times \frac{3\sqrt{3}}{8}(\frac{1}{\lambda\pi t^2})^3 f(\vec{q},\vec{u})\, d\vec{q}d\vec{u}\Big| \\
&\le \sup_{\vec{r}} \int d\vec{v} \frac{3\sqrt{3}}{8}(\frac{1}{\lambda\pi t^2})^3 \exp\big(-\frac{1}{4\lambda t}(\vec{v}-\vec{u})^2\big) \times \\
&\qquad \times \iint\limits_{R^3\times R^3} d\vec{u}\, d\vec{q}\, |f(\vec{q},\vec{u})| \exp\big(-\frac{3}{4\lambda t^3}(\vec{r}-\vec{q}-t(\vec{v}+\vec{u}))^2\big) \\
&\le \int_{R^3} d\vec{V}\, \frac{3\sqrt{3}}{8}(\frac{1}{\lambda\pi t^2})^3 \exp(-\frac{1}{\lambda t}V^2) \int_{R^3} d\vec{r}\big(\sup_{\vec{q}} \int |f(\vec{q},\vec{v})|\, d\vec{v}\big) \exp(-\frac{3}{4\lambda t^3}r^2)
\end{aligned}$$

$= \|f\|_X.$

and

$$\|\mathbf{U}_{\mathcal{B}}(t,s)g\|_X \leq \sup_{\vec{r}} \int d\vec{v}\, |g(\vec{r},\vec{v} - \int_s^t \vec{E}(\vec{r},\tau)d\tau, s)| = \|g(s)\|_X.$$

It follows from the product formula (4.8) that $\mathbf{U}(t,s;\vec{E},L)$, $0 \leq s \leq t$, is contractive, since $\mathbf{U}_{\mathcal{A}}(t)$ and $\mathbf{U}_{\mathcal{B}}(t,s)$ are. The conclusion in M follows from the fact that $\mathbf{U}(t,s;\vec{E},L)$ is contractive in both $L^1(R^3 \times R^3)$ and X.

PROOF OF THEOREM 4.3: The theorem is proved in three steps. First we demonstrate a local existence result in a special space. We then finish the proof by extending that result to general spaces and global time.

Let us first consider the following integral equation:

$$f(t) = \mathbf{U}_{\mathcal{A}}(t)f_0 + \int_0^t \mathbf{U}_{\mathcal{A}}(t-s)[C_E(f,f)(s) - \vec{E}\cdot\nabla_{\vec{v}}f(s)]\,ds. \tag{4.9}$$

Using the explicit expression for $\mathbf{U}_{\mathcal{A}}$ and C_E, we rewrite (4.9) explicitly in terms of $(\vec{r},\vec{v},t)$ as

$$\begin{aligned}
&f(\vec{r},\vec{v},t) = \\
&= \frac{3\sqrt{3}}{8}\int_0^t \iint_{R^3\times R^3} d\vec{u}\,d\vec{q}\,(\frac{1}{\lambda\pi(t-s)^2})^3 \\
&\qquad \times \exp[-\frac{1}{\lambda(t-s)}(\frac{1}{4}(\vec{v}-\vec{u})^2 + \frac{3}{4}(\vec{v}+\vec{u})^2 - \frac{3(\vec{r}-\vec{q})\cdot(\vec{v}+\vec{u})}{(t-s)} + \frac{3(\vec{r}-\vec{q})^2}{(t-s)^2})] \\
&\qquad \times [\tilde{C}_E(f,f)(\vec{q},\vec{u},s) - \vec{E}(\vec{q},s)\cdot\nabla_{\vec{u}}f(\vec{q},\vec{u},s)] \\
&\quad + \iint \frac{3\sqrt{3}}{8} d\vec{u}\,d\vec{q}\,(\frac{1}{\lambda\pi t^2})^3 f_0(\vec{q},\vec{u}) \\
&\qquad \times \exp[-\frac{1}{\lambda t}(\frac{1}{4}(\vec{v}-\vec{u})^2 + \frac{3}{4}(\vec{v}+\vec{u})^2 - \frac{3(\vec{r}-\vec{q})\cdot(\vec{v}+\vec{u})}{t} + \frac{3(\vec{r}-\vec{q})^2}{t^2})] \\
&= \int_0^t ds \iint_{R^3\times R^3} d\vec{u}\,d\vec{q}\,\frac{3\sqrt{3}}{8}(\frac{1}{\lambda\pi(t-s)^2})^3 \times
\end{aligned}$$

$$\times \exp[-\frac{1}{\lambda(t-s)}(\frac{1}{4}(\vec{v}-\vec{u})^2+\frac{3}{4}(\vec{v}+\vec{u})^2-\frac{3(\vec{r}-\vec{q})\cdot(\vec{v}+\vec{u})}{(t-s)}+\frac{3(\vec{r}-\vec{q})^2}{(t-s)^2})]\times$$

$$\times \iint\limits_{R^3\times S^2_+} <\vec{\epsilon},\vec{u}-\vec{u}_1>[\mathcal{Y}(n(\vec{q},s),n(\vec{q}-a\vec{\epsilon}))\eta_B(\vec{v},\vec{v}_1)f(\vec{q},\vec{u}',s)f(\vec{q}-a\vec{\epsilon},\vec{u}_1',s)$$

$$-\mathcal{Y}(n(\vec{q},s),n(\vec{q}+a\vec{\epsilon}))\eta_B(\vec{v},\vec{v}_1)f(\vec{q},\vec{u},s)f(\vec{q}+a\vec{\epsilon},\vec{u}_1,s)]\,d\vec{\epsilon}d\vec{u}_1$$

$$-\int_0^t ds\iint\limits_{R^3\times R^3} d\vec{u}\,d\vec{q}\,\frac{3\sqrt{3}}{8}(\frac{1}{\lambda\pi(t-s)^2})^3\vec{E}(\vec{q},s)\cdot\nabla_{\vec{u}}f(\vec{q},\vec{u},s)\times$$

$$\times \exp[-\frac{1}{\lambda(t-s)}(\frac{1}{4}(\vec{v}-\vec{u})^2+\frac{3}{4}(\vec{v}+\vec{u})^2-\frac{3(\vec{r}-\vec{q})\cdot(\vec{v}+\vec{u})}{(t-s)}+\frac{3(\vec{r}-\vec{q})^2}{(t-s)^2})]$$

$$+\iint d\vec{u}\,d\vec{q}\,\frac{3\sqrt{3}}{8}(\frac{1}{\lambda\pi t^2})^3\exp[-\frac{1}{\lambda t}(\cdots)]f_0(\vec{q},\vec{u})$$

$$=I_1-I_2+I_3. \tag{4.10}$$

Now we want to show each term in (4.10) satisfies a Lipschitz condition. Since I_3 is independent of f, we need only to consider I_1 and I_2. For the first term, we have

$$\|\tilde{C}_E(\psi,\psi)-\tilde{C}_E(\phi,\phi)\|_{L^1}\le$$

$$\le \iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}\{|\int_{R^3\times S^2_+} <\vec{\epsilon},\vec{v}-\vec{v}_1>[\mathcal{Y}(n_\psi(\vec{r}),n_\psi(\vec{r}-a\vec{\epsilon}))\psi(\vec{r},\vec{v}')\psi(\vec{r}-a\vec{\epsilon},\vec{v}_1')$$

$$-\mathcal{Y}(n_\phi(\vec{r}),n_\phi(\vec{r}-a\vec{\epsilon}))\phi(\vec{r},\vec{v}')\phi(\vec{r}-a\vec{\epsilon},\vec{v}_1')]\eta_B(\vec{v}',\vec{v}_1')\,d\vec{v}_1d\vec{\epsilon}|$$

$$+|\int_{R^3\times S^2_+} <\vec{\epsilon},\vec{v}-\vec{v}_1>[\mathcal{Y}(n_\psi(\vec{r}),n_\psi(\vec{r}+a\vec{\epsilon}))\psi(\vec{r},\vec{v})\psi(\vec{r}+a\vec{\epsilon},\vec{v}_1)$$

$$-\mathcal{Y}(n_\phi(\vec{r}),n_\phi(\vec{r}+a\vec{\epsilon}))\phi(\vec{r},\vec{v})\phi(\vec{r}+a\vec{\epsilon},\vec{v}_1)]\eta_B(\vec{v},\vec{v}_1)\,d\vec{v}_1d\vec{\epsilon}|\}$$

$$\le 4k\iiiint\limits_{R^3\times R^3\times R^3\times S^2_+} d\vec{\epsilon}d\vec{v}_1d\vec{v}d\vec{r}\,|\mathcal{Y}(n_\psi(\vec{r}),n_\psi(\vec{r}+a\vec{\epsilon}))\psi(\vec{r},\vec{v})\psi(\vec{r}+a\vec{\epsilon},\vec{v}_1)$$

$$-\mathcal{Y}(n_\phi(\vec{r}),n_\phi(\vec{r}+a\vec{\epsilon}))\phi(\vec{r},\vec{v})\phi(\vec{r}+a\vec{\epsilon},\vec{v}_1)|\le$$

$$\le 4k\{\iiiint\limits_{R^3\times R^3\times R^3\times S^2_+} d\vec{\epsilon}d\vec{v}_1d\vec{v}d\vec{r}\,|\frac{\tilde{Y}(\hat{\psi},\hat{\psi}')}{(1+\hat{\psi})(1+\hat{\psi}')}\psi\psi'-\frac{\tilde{Y}(\hat{\psi},\hat{\psi}')}{(1+\hat{\phi})(1+\hat{\phi}')}\psi\psi'|+$$

$$+\iiiint\limits_{R^3\times R^3\times R^3\times S^2_+} d\vec{\epsilon}d\vec{v}_1d\vec{v}d\vec{r}\,|\frac{\tilde{Y}(\hat{\psi},\hat{\psi}')}{(1+\hat{\phi})(1+\hat{\phi}')}\psi\psi'-\frac{\tilde{Y}(\hat{\psi},\hat{\psi}')}{(1+\hat{\phi})(1+\hat{\phi}')}\phi\phi'|+$$

$$+ \iiiint\limits_{R^3\times R^3\times R^3\times S^2_+} d\vec{\epsilon}d\vec{v}_1 d\vec{v}d\vec{r}\,|\frac{\tilde{Y}(\hat{\psi},\hat{\psi}')}{(1+\hat{\psi})(1+\hat{\psi}')}\psi\psi' - \frac{\tilde{Y}(\hat{\phi},\hat{\phi}')}{(1+\hat{\phi})(1+\hat{\phi}')}\phi\phi'|\} \equiv$$

$$\equiv 4k(P_1+P_2+P_3),$$

where, for the sake of simplicity, we have written $\psi = \psi(\vec{r},\vec{v})$, $\psi' = \psi'(\vec{r}+a\vec{\epsilon},\vec{v}_1)$, $\hat{\psi} = n_\psi$, etc., and set $\epsilon = 1$. Now,

$$P_1 = \iiiint\limits_{R^3\times R^3\times R^3\times S^2_+} d\vec{\epsilon}d\vec{v}_1 d\vec{v}d\vec{r}\,|\tilde{Y}(\hat{\psi},\hat{\psi}')\frac{v\psi'}{(1+\hat{\psi})(1+\hat{\psi}')(1+\hat{\phi})(1+\hat{\phi}')}\times$$
$$\times[(\hat{\phi}-\hat{\psi})+(\hat{\phi}'-\hat{\psi}')+\hat{\phi}(\hat{\phi}'-\hat{\psi}')+\hat{\psi}'(\hat{\phi}+\hat{v})]|.$$

Since

$$\iiiint\limits_{R^3\times R^3\times R^3\times S^2_+} d\vec{\epsilon}d\vec{v}_1 d\vec{v}d\vec{r}\,\tilde{Y}(\hat{\psi},\hat{\psi}')\frac{\psi\psi'}{(1+\hat{\psi})(1+\hat{\psi}')(1+\hat{\phi})(1+\hat{\phi}')}\hat{\phi}|\hat{\phi}'-\hat{\psi}'| \le$$

$$\le \int d\vec{r}\int d\vec{\epsilon}\,(\int d\vec{v}\,\frac{\hat{\phi}}{1+\hat{\phi}}\tilde{Y}(\hat{\psi},\hat{\psi}')v)(\int d\vec{v}_1\,\frac{\psi'}{1+\hat{\psi}'}|\hat{\phi}'-\hat{\psi}'|) \le$$

$$\le \int d\vec{r}\int d\vec{\epsilon}\tilde{Y}(\hat{\psi},\hat{\psi}')\hat{\psi}\frac{\hat{\psi}'}{1+\hat{\psi}'}|\hat{\phi}'-\hat{\psi}'| \le$$

$$\le M_Y\int d\vec{r}\int d\vec{\epsilon}|\hat{\phi}'-\hat{\psi}'| \le$$

$$\le 4\pi M_Y\|\phi'-\psi'\|_{L^1(R^3\times R^3)},$$

we have $P_1 \le C_1\|\phi-\psi\|$. Use the assumptions (A2) and (A3) and similar decomposition, we have a similar estimation for P_2 and P_3. That is.

$$\|\tilde{C}_E(v,\psi)-\tilde{C}_E(\phi,\phi)\|_{L^1} \le C\|\psi-\phi\|_{L^1}, \tag{4.11a}$$

with C independent of ϕ and ψ. In other words, $\tilde{C}_E(v,\psi)$ is a global Lipschitz operator. Using a similar argument we have

$$\|\tilde{C}_E(\psi,\psi)-\tilde{C}_E(\phi,\phi)\|_X \le C\|\psi-\phi\|_X. \tag{4.11b}$$

We conclude that I_1 in (4.10) is globally Lipschitz continuous in M.

Next, we rewrite I_2 as

$$
\begin{aligned}
&I_2(\vec{r},\vec{v},t) = \\
&\int_0^t ds \iint\limits_{R^3\times R^3} \frac{3\sqrt{3}}{8}\Big(\frac{1}{\lambda\pi(t-s)^2}\Big)^3 f(\vec{q},\vec{u},s)\vec{E}(\vec{q},s)\cdot\Big[\frac{3(\vec{r}-\vec{q})}{\lambda(t-s)^2}+\frac{(\vec{v}-\vec{u})}{2\lambda(t-s)}-\frac{3(\vec{v}+\vec{u})}{2\lambda(t-s)}\Big] \\
&\times\exp\Big[-\frac{1}{\lambda(t-s)}\Big(\frac{1}{4}(\vec{v}-\vec{u})^2+\frac{3}{4}(\vec{v}+\vec{u})^2-\frac{3(\vec{r}-\vec{q})\cdot(\vec{v}+\vec{u})}{(t-s)}+\frac{3(\vec{r}-\vec{q})^2}{(t-s)^2}\Big)\Big]\,d\vec{q}d\vec{u} \qquad (4.12)
\end{aligned}
$$

Let $\vec{k}(\vec{r},\vec{q};\vec{v},\vec{u};t-s)$ denote the integral kernel of the integral operator defined by (4.12). We have

$$
\begin{aligned}
&\int_0^t ds \iint\limits_{R^3\times R^3} d\vec{u}\,d\vec{q}\,|\vec{k}(\vec{r},\vec{q};\vec{v},\vec{u};t-s)| = \\
&=\int_0^t ds \iint\limits_{R^3\times R^3} d\vec{u}\,d\vec{q}\,\frac{3\sqrt{3}}{8}\Big(\frac{1}{\lambda\pi(t-s)^2}\Big)^3\Big|\frac{3(\vec{r}-\vec{q})}{\lambda(t-s)^2}+\frac{(\vec{v}-\vec{u})}{2\lambda(t-s)}-\frac{(\vec{v}+\vec{u})}{2\lambda(t-s)}\Big| \\
&\times\exp\Big[-\frac{1}{\lambda(t-s)}\Big(\frac{1}{4}(\vec{v}-\vec{u})^2+\frac{3}{4}(\vec{v}+\vec{u})^2-\frac{3(\vec{r}-\vec{q})\cdot(\vec{v}+\vec{u})}{(t-s)}+\frac{3(\vec{r}-\vec{q})^2}{(t-s)^2}\Big)\Big] \\
&=\frac{3\sqrt{3}}{8}\int_0^t d\tau \iint\limits_{R^3\times R^3} d\vec{u}\,d\vec{q}\left(\frac{1}{\lambda\pi\tau^2}\right)^3\Big|\frac{3}{\lambda\tau^2}(\vec{q}-\tau\vec{u})+\frac{\vec{u}-2\vec{v}}{2\lambda\tau}\Big| \\
&\times\exp\left(-\frac{1}{\lambda\tau}\Big(\frac{1}{4}(\vec{u}-2\vec{v})^2+\frac{3}{\tau^2}(\vec{q}-\tau\vec{u}/2)^2\Big)\right) \\
&=\frac{3\sqrt{3}}{8}\int_0^t d\tau \iint d\vec{Q}d\vec{U}\left(\frac{1}{\lambda\pi\tau^2}\right)^3\Big|\frac{3}{\lambda\tau^2}\vec{Q}+\frac{1}{2\lambda\tau}\vec{U}\Big|\times \\
&\times\exp\left(-\frac{1}{\lambda\tau}\Big(\frac{1}{4}U^2+\frac{3}{\tau^2}Q^2\Big)\right) \\
&=\frac{3\sqrt{3}}{8}\int_0^t d\tau \iint d\vec{Q}d\vec{U}\left(\frac{1}{\lambda\pi\tau^2}\right)^3\frac{3}{\lambda\tau^2}|\vec{Q}|\exp\left(-\frac{1}{\lambda\tau}\Big(\frac{1}{4}U^2+\frac{3}{\tau^2}Q^2\Big)\right) \\
&\quad+\frac{3\sqrt{3}}{8}\int_0^t d\tau \iint d\vec{Q}d\vec{U}\left(\frac{1}{\lambda\pi\tau^2}\right)^3\frac{1}{2\lambda\tau}|\vec{U}|\exp\left(-\frac{1}{\lambda\tau}\Big(\frac{1}{4}U^2+\frac{3}{\tau^2}Q^2\Big)\right) \\
&=\frac{3\sqrt{3}}{8}\int_0^t d\tau \int\frac{2}{3\lambda^2\pi^2\tau^2}e^{-\frac{u^2}{4\lambda\tau}}\,d\vec{u}+\frac{3\sqrt{3}}{8}\int_0^t d\tau \int\frac{1}{6}\lambda^2\pi^2\tau^5 e^{-\frac{3q^2}{\lambda\tau^3}}\,d\vec{q}
\end{aligned}
$$

$$= \frac{\sqrt{3}+1}{2}\sqrt{\frac{t}{\lambda\pi}}.$$

On the other hand, from Lemma 4.4 and the definition of the norm $\|\cdot\|_M$, we can find a constant $C_2\,(=2\max\{\|\nabla_{\vec{r}}\mathcal{V}\|_{L^1(B_1)},\ \|\nabla_{\vec{r}}\mathcal{V}\|_{L^1(B_1^c)}\})$ such that

$$\|\vec{E}_\phi\|_{L^\infty} \le C_2\|\phi\|_M,$$

where $\vec{E}_\phi$ and ϕ are related by (AE2). Now

$$\begin{aligned}
&\|I_2(\phi) - I_2(\psi)\|_{L^1} \le \\
&\le \frac{8}{\sqrt{\lambda\pi}}t^{1/2}\{\int_0^t ds \iint\limits_{R^3\times R^3} d\vec{u}\,d\vec{q}\,|\vec{E}_\psi(\vec{q},s)||\psi(\vec{q},\vec{u},s) - \phi((\vec{q},\vec{u},s))| + \\
&\qquad + \int_0^t ds \iint\limits_{R^3\times R^3} d\vec{u}\,d\vec{q}\,|\vec{E}_\psi(\vec{q},s) - \vec{E}_\phi(\vec{q},s)||\phi((\vec{q},\vec{u},s))|\} \\
&\le \frac{8}{\sqrt{\lambda\pi}}t^{1/2}\{\|\vec{E}_\psi\|_{L^\infty(R^3)}\|\psi-\phi\|_L^1 + \|\vec{E}_\psi - \vec{E}_\phi\|_{L^\infty}\|\phi\|_{L^1}\} \le \\
&\le \frac{8}{\sqrt{\lambda\pi}}C_2t^{1/2}(\|\phi\|_M + \|\psi\|_M)\|\psi-\phi\|_M,
\end{aligned}$$

and similarly

$$\|I_2(\psi) - I_2(\phi)\|_X \le \frac{8}{\sqrt{\lambda\pi}}C_2t^{1/2}(\|\phi\|_M + \|\psi\|_M)\|\psi-\phi\|_M.$$

Therefore,

$$\|I_2(\psi) - I_2(\phi)\|_{[0,t]} \le \frac{8}{\sqrt{\lambda\pi}}C_2t^{1/2}(\|\phi\|_{[0,t]} + \|\psi\|_{[0,t]})\|\psi-\phi\|_{[0,t]}. \tag{4.13}$$

In other word, $I_2(f,t)$ is local Lipschitz continuous in the space $M_{[0,t]}$ with the Lipschitz constant depending only on the norm $\|f\|_{[0,t]}$.

Now we consider the norm of the solution of the approximate equation. Consider the following sequence of functions:

$$f^{(0)} = 0,$$

$$L^{(i)}(\vec{r},\vec{v},t) = \tilde{C}_E^-(f^{(i)},f^{(i)}) =$$

$$= \iint\limits_{R^3 \times S_+^2} \mathcal{Y}(n^{(i)}(\vec{r},t), n^{(i)}(\vec{r}+a\vec{\epsilon},t))\chi_B(\vec{v},\vec{v}_1) f^{(i)}(\vec{r}+a\vec{\epsilon},\vec{v}_1,t) < \vec{\epsilon}, \vec{v}-\vec{v}_1 > d\vec{\epsilon}d\vec{v}_1, \tag{4.14a}$$

$$\vec{E}^{(i)}(\vec{r},t) = \iint \nabla_{\vec{r}}\left(\frac{1}{|\vec{r}-\vec{r}_1|}\right) f^{(i)}(\vec{r}_1,\vec{v},t)\, d\vec{r}_1 d\vec{v}, \tag{4.14b}$$

$$f^{(i+1)} = \mathbf{U}(t,0;\vec{E}^{(i)},L^{(i)})f_0 + \int_0^t \mathbf{U}(t,s;\vec{E}^{(i)},L^{(i)})\tilde{C}_E^+(f^{(i)},f^{(i)})(s)\, ds, \tag{4.14c}$$

where $\tilde{C}_E^+(f^{(i)},f^{(i)})$ is given by (AE3).

It is not difficult to show that the sequence defined above converges to some function f, the solution of the equation, for small time t. Let us estimate the norm of functions of this sequence. Looking at the structure of the operator $\tilde{C}_E^+(f,f)$ and using the assumption (A2) about $\mathcal{Y}$, it is easy to show that $\tilde{C}_E^+(f,f) \in M$ if f is, and

$$\tilde{C}_E^+(f^{(i)},f^{(i)})(s) =$$

$$= \iint d\vec{\epsilon}d\vec{v}_1 f^{(i)}(\vec{r},\vec{v}',s) f^{(i)}(\vec{r}+a\vec{\epsilon},\vec{v}_1',s)\mathcal{Y}(n(\vec{r},s),n(\vec{r}-a\vec{\epsilon},s)) < \vec{v}-\vec{v}_1,\vec{\epsilon} >$$

$$\leq k \iint d\vec{\epsilon}d\vec{v}_1 f^{(i)}(\vec{r},\vec{v},s) f^{(i)}(\vec{r}+a\vec{\epsilon},\vec{v}_1,s)\mathcal{Y}(n(\vec{r},s),n(\vec{r}+a\vec{\epsilon}))$$

$$\leq 4k\pi M_Y f^{(i)}(\vec{r},\vec{v},s),$$

$$\|\tilde{C}_E^+(f,f)\|_M \leq C_1\|f\|_M,$$

with the constant C_1 independent of f. Therefore, using the Lemma 4.5 we conclude from (4.14c) that

$$\|f^{(i+1)}\|_M \leq \|f_0\|_M + tC_1\|f^{(i)}\|_M.$$

Letting

$$B_0 = \|f_0\|_M, \qquad T_1 = \frac{1}{2C_1},$$

we have easily

$$\|f^{(i)}\|_{[0,T_1]} \leq 2B_0, \qquad i = 1,2,\cdots.$$

Furthermore, following from (4.14c) and the fact that $f_0 \in L^1_+$, the solution f of the integral equation (4.9), if it exists, is a nonnegative function.

Now let us go back to (4.10),

$$f = I_1 - I_2 + I_3.$$

Since $\|I_1(\phi,t)-I_1(\psi,t)\|_{[0,t]} \le C_1' t\|\phi-\psi\|_{[0,t]}$, $\|I_2(\phi,t)-I_2(\psi,t)\|_{[0,t]} \le C_2' t^{1/2}(\|\phi\|_{[0,t]}+\|\psi\|_{[0,t]})\|\phi-\psi\|_{[0,t]} \le 4b_0 C_2' t^{1/2}\|\phi-\psi\|_{[0,t]}$ for $t \in [0,T_1]$, and I_3 independent of ψ or ϕ, we conclude by the contractive mapping theorem that (4.10) has a unique solution $f(t) \ge 0$ in $M_{[0,T']}$ for some $T' \in [0,T_1]$.

Next we want to extend the solution given above to any given $M_{[0,T]}$. In view of the proof of local existence, it is enough to show that if $f(t)$ is a solution of the equation on an arbitrary interval [0, T], then we can find a constant $B > 0$ such that

$$\|f\|_{[0,t]} \le B \qquad \forall t \in [0,T]. \tag{4.15}$$

This conclusion is based on the fact that the choice of the 'existence interval' $[0.T']$ in the first step depends only on the norm $\|f_0\|_{[0.t]}$ of the initial value. Now we prove 4.15. If $f(\vec{r},\vec{v},t) \ge 0$ is a solution of (AE1)-(AE2)-(AE3), integrating both sides of the equation over $R^3 \times R^3$, we have

$$\|f(t)\|_{L^1} = \iint d\vec{r}d\vec{v}\, f(\vec{r},\vec{v},t) = \iint d\vec{r}d\vec{v}\, f_0(\vec{r},\vec{v}) = B_1 \equiv \|f_0\|_{L^1}.$$

On the other hand, using hypothesis (A2) and (A4), it follows from (4.14c) that $\|f^{(i+1)}\|_M \le \|f_0\|_M + 4k\pi M_Y t/\epsilon \le B_0 + 4k\pi M_Y T/\epsilon$. Letting $B = \max\{B_1, B_0 + 4k\pi M_Y T/\epsilon\}$, we assert that the integral equation (4.9) has a unique solution in $C(0,T; M \cap L^+)$ for any $T \ge 0$.

Finally we finish the proof of the theorem by the following two remarks: 1) We have shown that the solution $f(t)$ of the integral equation belongs to $L^1(R^3_{\vec{r}}; L^1(R^3_{\vec{v}})) \cap$

$C(R^3_{\vec{r}}; L^1(R^3_{\vec{v}}))$ for each t. It then follows from interpolation theory that $f(t) \in L^p(R^3_{\vec{r}}; L^1(R^3_{\vec{v}}))$ for any $1 \le p \le \infty$; 2) to make $f(t)$, the solution of "the integral equation" (4.9), be the solution of "the differential equation" (AE1)-(AE3), it is sufficient to show that[34] $f(t) \in C(0,T;\mathcal{D}(\mathcal{A}))$. which, in our case, follows from the estimation about the derivatives of $f(\vec{r},\vec{v},t)$ stated at the end of this chapter (*cf.* (4.20)). This completes the proof of Theorem 4.3. ■

In the reminder of this chapter we give estimations of some quantities related to the solutions of the approximate equations. First we estimate the *'k-th moment'* of the system. For $k = 0, 2$, we have already (using the same arguments as in Chapter II) that $\iint f(t)\,d\vec{v}d\vec{r} = \iint f_0\,d\vec{v}d\vec{r} < \infty$ and $\sup\limits_{0\le t\le T} \iint f(t)(|\vec{r}|^2 + |\vec{v}|^2)\,d\vec{v}d\vec{r} \le C(t) < \infty$, and (by the Cauchy-Schwarz inequality) $\sup\limits_{0\le t\le T} \iint f(t)(|\vec{r}| + |\vec{v}|)\,d\vec{v}d\vec{r} \le C(t) < \infty$. Now multiplying equation (AE1) by $|\vec{v}|^k + |\vec{r}|^k$. integrating over $R^3 \times R^3$, and using the facts that

$$|\vec{r}|^k\vec{v}\cdot\nabla_{\vec{r}}f(\vec{r},\vec{v},t) = \mathrm{div}_{\vec{r}}(|\vec{r}|^k f\vec{v}) - k|\vec{r}|^{k-2}\vec{r}\cdot\vec{v}$$

and

$$|\vec{v}|^k\Delta_{\vec{v}}f = \mathrm{div}_{\vec{v}}(|\vec{v}|^k\nabla_{\vec{v}}f) - k\mathrm{div}_{\vec{v}}(f|\vec{v}|^{k-2}\vec{v}) + k(k-1)f(\vec{r},\vec{v},t)|\vec{v}|^{k-2}.$$

we have

$$\begin{aligned}
&\frac{d}{dt}\iint f(\vec{r},\vec{v},t)(|\vec{r}|^k + |\vec{v}|^k)\,d\vec{r}d\vec{v} - k\iint f(\vec{r},\vec{v},t)|\vec{r}|^{k-2}\vec{r}\cdot\vec{v}\,d\vec{v}d\vec{r}\\
&- k\iint |\vec{v}|^{k-2}\vec{v}\cdot\vec{E}(\vec{r},t)f\,d\vec{v}d\vec{r} - \lambda k(k-1)\iint f(\vec{r},\vec{v},t)|\vec{v}|^{k-2}\\
&= \iint |\vec{v}|^k\tilde{C}_E(f,f)\,d\vec{v}d\vec{r}.
\end{aligned}$$

From Lemma 4.4 we know $\vec{E}(\vec{r},t)$ is bounded. and therefore

$$\frac{d}{dt}\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}f(\vec{r},\vec{v},t)(|\vec{r}|^k + |\vec{v}|^k)\,d\vec{r}d\vec{v}$$

$$\leq C \iint\limits_{R^3 \times R^3} d\vec{r}\, d\vec{v}(|\vec{r}|^{k-1}|\vec{v}| + |\vec{v}|^{k-1} + |\vec{v}|^{k-2}) + \iint\limits_{R^3 \times R^3} d\vec{r}\, d\vec{v}|\tilde{C}_E(f,f)||\vec{v}|^k.$$

Since $\mathcal{Y}\eta_B(\vec{v}, \vec{v}_1)$ is compactly supported in $\vec{v}$, one can easily bound the second term by $C \iint\limits_{R^3 \times R^3} d\vec{r}\, d\vec{v} f(1 + |\vec{v}|^k)$. It follows inductively from the Grönwall's inequality that

$$\sup_{0 \leq t \leq T} \iint\limits_{R^3 \times R^3} d\vec{r}\, d\vec{v} f(\vec{r}, \vec{v}, t)(1 + |\vec{r}|^k + |\vec{v}|^k)\, d\vec{r} d\vec{v} < \infty \quad \text{for all} \quad k \geq 1. \tag{4.16}$$

Now, we establish an estimate on the derivatives of f in $L^\infty(0, T; L^1 \cap L^\infty(R^3 \times R^3))$. We begin with $\nabla_{\vec{r}} f$ and $\nabla_{\vec{v}} f$. Set $\vec{F} = \nabla_{\vec{r}} f = (\frac{\partial}{\partial r_1} f, \frac{\partial}{\partial r_2} f, \frac{\partial}{\partial r_3} f), g = |\nabla_{\vec{r}} f| = \left((\frac{\partial}{\partial r_1} f)^2 + (\frac{\partial}{\partial r_2} f)^2 + (\frac{\partial}{\partial r_3} f)^2\right)^{1/2}$. Note that

$$\begin{aligned}
\frac{\partial}{\partial t} g &= \frac{1}{g} \vec{F} \cdot \frac{\partial}{\partial t} \vec{F} = \frac{dg}{d\vec{F}} \cdot \nabla_{\vec{r}} (\frac{\partial}{\partial t} f), \\
\vec{v} \cdot \nabla_{\vec{r}} g &= \frac{dg}{d\vec{F}} \cdot v_1 \frac{\partial}{\partial r_1} \vec{F} + \frac{dg}{d\vec{F}} \cdot v_2 \frac{\partial}{\partial r_2} \vec{F} + \frac{dg}{d\vec{F}} \cdot v_3 \frac{\partial}{\partial r_3} \vec{F} = \\
&= \frac{dg}{d\vec{F}} \cdot (\vec{v} \cdot \nabla_{\vec{r}}) \vec{F} = \frac{dg}{d\vec{F}} \cdot \nabla_{\vec{r}} (\vec{v} \cdot \nabla_{\vec{r}} f), \\
\vec{E} \cdot \nabla_{\vec{v}} g &= \frac{dg}{d\vec{F}} \cdot ((\vec{E} \cdot \nabla_{\vec{v}}) \vec{E}) \\
&= \frac{dg}{d\vec{F}} \cdot \nabla_{\vec{r}} (\vec{E} \cdot \nabla_{\vec{v}} f) - \frac{dg}{d\vec{F}} \cdot ((\nabla_{\vec{r}} \otimes \vec{E}) \nabla_{\vec{v}} f),
\end{aligned}$$

where $\otimes$ indicates the vector tensor product, and

$$\begin{aligned}
\Delta_{\vec{v}} g =& \nabla_{\vec{v}} \cdot \left(\frac{dg}{d\vec{F}} \cdot \frac{\partial}{\partial v_1} \vec{F}, \quad \frac{dg}{d\vec{F}} \cdot \frac{\partial}{\partial v_2} \vec{F}, \quad \frac{dg}{d\vec{F}} \cdot \frac{\partial}{\partial v_3} \vec{F} \right)^T = \\
=& -\frac{1}{g^3} [(\vec{F} \cdot \frac{\partial}{\partial v_1} \vec{F})^2 + (\vec{F} \cdot \frac{\partial}{\partial v_2} \vec{F})^2 + (\vec{F} \cdot \frac{\partial}{\partial v_3} \vec{F})^2] \\
&+ \frac{1}{g} [(\frac{\partial}{\partial v_1} \vec{F})^2 + (\frac{\partial}{\partial v_2} \vec{F})^2 + (\frac{\partial}{\partial v_3} \vec{F})^2] + [\frac{dg}{d\vec{F}} \cdot \Delta_{\vec{v}} \vec{F}] \\
=& \frac{1}{g} [(\frac{dg}{d\vec{F}} \times \frac{\partial}{\partial v_1} \vec{F})^2 + (\frac{dg}{d\vec{F}} \times \frac{\partial}{\partial v_2} \vec{F})^2 + (\frac{dg}{d\vec{F}} \times \frac{\partial}{\partial v_3} \vec{F})^2] + [\frac{dg}{d\vec{F}} \cdot \Delta_{\vec{v}} \vec{F}].
\end{aligned}$$

Therefore,

$$\frac{\partial}{\partial t} g + \vec{v} \cdot \nabla_{\vec{r}} g + \vec{E} \cdot \nabla_{\vec{v}} g - \lambda \Delta_{\vec{v}} g =$$

$$= \frac{dg}{d\vec{F}} \cdot \nabla_{\vec{r}}[\frac{\partial}{\partial t}f + \vec{v} \cdot \nabla_{\vec{r}}f + \vec{E} \cdot \nabla_{\vec{v}}f - \lambda\Delta_{\vec{v}}f] - \frac{dg}{d\vec{F}} \cdot [(\nabla_{\vec{r}} \otimes \vec{E})\nabla_{\vec{v}}f]$$
$$- \lambda\frac{1}{g}[(\frac{dg}{d\vec{F}} \times \frac{\partial}{\partial v_1}\vec{F})^2 + (\frac{dg}{d\vec{F}} \times \frac{\partial}{\partial v_2}\vec{F})^2 + (\frac{dg}{d\vec{F}} \times \frac{\partial}{\partial v_3}\vec{F})^2]$$
$$= \frac{dg}{d\vec{F}} \cdot \nabla_{\vec{r}}\tilde{C}_E(f.f) - \frac{dg}{d\vec{F}} \cdot [(\nabla_{\vec{r}} \otimes \vec{E})\nabla_{\vec{v}}f]$$
$$- \lambda\frac{1}{g}[(\frac{dg}{d\vec{F}} \times \frac{\partial}{\partial v_1}\vec{F})^2 + (\frac{dg}{d\vec{F}} \times \frac{\partial}{\partial v_2}\vec{F})^2 + (\frac{dg}{d\vec{F}} \times \frac{\partial}{\partial v_3}\vec{F})^2].$$

Taking integral over $R^3 \times R^3$, we get

$$\frac{d}{dt}\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}|\nabla_{\vec{r}}f|d\vec{r}d\vec{v}$$
$$= \iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}\frac{dg}{d\vec{F}} \cdot \nabla_{\vec{r}}\tilde{C}_E(f,f) - \iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}\frac{dg}{d\vec{F}} \cdot [(\nabla_{\vec{r}} \otimes \vec{E})\nabla_{\vec{v}}f]$$
$$- \frac{\lambda}{g}\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}[(\frac{dg}{d\vec{F}} \times \frac{\partial}{\partial v_1}\vec{F})^2 + (\frac{dg}{d\vec{F}} \times \frac{\partial}{\partial v_2}\vec{F})^2 + (\frac{dg}{d\vec{F}} \times \frac{\partial}{\partial v_3}\vec{F})^2]. \tag{4.17}$$

Similarly, letting $\vec{H} = (\frac{\partial}{\partial v_1}f, \frac{\partial}{\partial v_2}f, \frac{\partial}{\partial v_3}f,)$, $h = |\vec{H}|$, $\hat{H} = \frac{\vec{H}}{h}$, we have

$$\frac{\partial}{\partial t}h = \hat{H} \cdot \frac{\partial}{\partial t}\vec{H} = \hat{H} \cdot \nabla_{\vec{v}}(\frac{\partial}{\partial t}f),$$
$$\vec{v} \cdot \nabla_{\vec{r}}h = \vec{v} \cdot \begin{pmatrix} \hat{H} \cdot \nabla_{\vec{v}}(\frac{\partial}{\partial r_1}f) \\ \hat{H} \cdot \nabla_{\vec{v}}(\frac{\partial}{\partial r_2}f) \\ \hat{H} \cdot \nabla_{\vec{v}}(\frac{\partial}{\partial r_3}f) \end{pmatrix} = \vec{v} \cdot [(\hat{H} \cdot \nabla_{\vec{v}})\nabla_{\vec{r}}f]$$
$$= \hat{H} \cdot \nabla_{\vec{v}}(\vec{v} \cdot \nabla_{\vec{r}}f) - \hat{H} \cdot \nabla_{\vec{r}}f,$$
$$\vec{E} \cdot \nabla_{\vec{v}}h = \vec{E} \cdot [(\hat{H} \cdot \nabla_{\vec{v}})\nabla_{\vec{v}}f] = \hat{H} \cdot \nabla_{\vec{v}}(\vec{E} \cdot \nabla_{\vec{v}}f),$$
$$\Delta_{\vec{v}}h = \nabla_{\vec{v}} \cdot \nabla_{\vec{v}}h = \nabla_{\vec{v}} \cdot (\hat{H} \cdot \nabla_{\vec{v}})f$$
$$= \hat{H} \cdot \Delta_{\vec{v}}\vec{H} + \frac{1}{h}[(\hat{H} \times \frac{\partial}{\partial v_1}\vec{H})^2 + (\hat{H} \times \frac{\partial}{\partial v_2}\vec{H})^2 + (\hat{H} \times \frac{\partial}{\partial v_3}\vec{H})^2]$$
$$= \hat{H} \cdot \nabla_{\vec{v}}(\Delta_{\vec{v}}f) + \frac{1}{h}[(\hat{H} \times \frac{\partial}{\partial v_1}\vec{H})^2 + (\hat{H} \times \frac{\partial}{\partial v_2}\vec{H})^2 + (\hat{H} \times \frac{\partial}{\partial v_3}\vec{H})^2],$$

and

$$\frac{\partial}{\partial t}h + \vec{v} \cdot \nabla_{\vec{r}}h + \vec{E} \cdot \nabla_{\vec{v}}h - \lambda\Delta_{\vec{v}}h =$$

$$= \hat{H} \cdot \nabla_{\vec{v}} \tilde{C}_E(f,f) - \hat{H} \cdot \nabla_{\vec{r}} f - \lambda \frac{1}{h}[(\hat{H} \times \frac{\partial}{\partial v_1}\vec{H})^2 + (\hat{H} \times \frac{\partial}{\partial v_2}\vec{H})^2 + (\hat{H} \times \frac{\partial}{\partial v_3}\vec{H})^2].$$

Therefore,

$$\frac{d}{dt} \iint\limits_{R^3 \times R^3} d\vec{r}\, d\vec{v}\; h = -\lambda \iint\limits_{R^3 \times R^3} d\vec{r}\, d\vec{v} \frac{1}{h}[(\hat{H} \times \frac{\partial}{\partial v_1}\vec{H})^2 + (\hat{H} \times \frac{\partial}{\partial v_2}\vec{H})^2 + (\hat{H} \times \frac{\partial}{\partial v_3}\vec{H})^2]$$
$$+ \iint\limits_{R^3 \times R^3} d\vec{r}\, d\vec{v} \hat{H} \cdot \nabla_{\vec{v}} \tilde{C}_E(f,f) - \iint\limits_{R^3 \times R^3} d\vec{r}\, d\vec{v} \hat{H} \cdot \nabla_{\vec{r}} f, \tag{4.18}$$

and

$$\frac{d}{dt} \iint\limits_{R^3 \times R^3} d\vec{r}\, d\vec{v}(g+f) \leq \iint\limits_{R^3 \times R^3} d\vec{r}\, d\vec{v} \left(\frac{\vec{F}}{g} \cdot \nabla_{\vec{r}} + \hat{H} \cdot \nabla_{\vec{v}} \right) \tilde{C}_E(f,f)$$
$$- \iint\limits_{R^3 \times R^3} d\vec{r}\, d\vec{v} \left(\frac{\vec{F}}{g} \cdot [(\nabla_{\vec{r}} \otimes \vec{E}) \nabla_{\vec{v}} f] + \hat{H} \cdot \nabla_{\vec{r}} f \right). \tag{4.19}$$

Similar to the estimate in [17], there is a constant $C(\epsilon)$ such that

$$\iint\limits_{R^3 \times R^3} d\vec{r}\, d\vec{v} \left(\frac{\vec{F}}{g} \cdot \nabla_{\vec{r}} + \hat{H} \cdot \nabla_{\vec{v}} \right) \tilde{C}_E(f,f) \leq C(\epsilon) \iint\limits_{R^3 \times R^3} d\vec{r}\, d\vec{v}(g+f).$$

Since $n(\vec{r},t) \in L^1 \cap L^\infty$, it follows [16, p.174] that each element in $\nabla_{\vec{r}} \otimes \vec{E}$ belongs to L^∞. In addition, $|\hat{H}| = |\vec{F}|/g = 1$, and we get a similar estimate for the second integral on the RHS of (4.19). Hence,

$$\frac{d}{dt} \iint\limits_{R^3 \times R^3} d\vec{r}\, d\vec{v}(|\nabla_{\vec{r}} f| + |\nabla_{\vec{v}} f|) \leq C \iint\limits_{R^3 \times R^3} d\vec{r}\, d\vec{v}(|\nabla_{\vec{r}} f| + |\nabla_{\vec{v}} f|), \tag{4.20}$$

which implies that both $|\nabla_{\vec{r}} f|$ and $|\nabla_{\vec{v}} f|$ are bounded (in L^1). In addition, using the estimation[15][16]

$$\|\nabla_{\vec{r}} \vec{E}\|_{L^\infty} \leq C(1 + \|n\|_1 + \|n\|_\infty [1 + \log(1 + \|\nabla_{\vec{r}} n\|_\infty)])$$

and a similar argument we can bound the second order and other derivatives.

Chapter V Global Existence

In this chapter we state and prove the main result of the first part of this paper, i.e., global existence of the renormalized solution.

THEOREM 5.1. *Assume that $Y(\sigma,\tau) = Y(\tau,\sigma)$ is a jointly continuous function satisfying the symmetry condition (2.1) and the boundedness condition (2.19), the initial value $f_0(\vec{r},\vec{v}) \geq 0$ satisfies the bounded condition*

$$\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v} f_0(\vec{r},\vec{v})(1+|\vec{r}|^2+|\vec{v}|^2+|\log f_0|) \leq C < \infty,$$

and $\vec{E}_0(\vec{r}) = \nabla_{\vec{r}}(\frac{1}{|\vec{r}|} * n_0)(\vec{r}) = \nabla_{\vec{r}}(\frac{1}{|\vec{r}|} * \int f_0(\vec{r},\vec{v})\,d\vec{v})(\vec{r})$ satisfies

$$\int_{R^3} |\vec{E}_0(\vec{r})|^2\,d\vec{r} \leq C < \infty.$$

Then, there exist $f \in C([0,\infty);\, L^1(R^3\times R^3))$ which satisfies $f|_{t=0} = f_0$, (3.1), (3.2), (3.3) and such that, for all $\delta > 0$, $g_\delta = \beta_\delta(f)$ satisfies (RDVE)-(VP)-(IC) in the sense of distributions and $g_\delta|_{(0,T)\times R^3\times B_R} \in L^2([0,T]\times R^3,\, H^1(B_R))$ $(\forall R, T < \infty)$. In particular f is a renormalized solution of (DVE)-(VP).

PROOF: The theorem is proved by combining use of Theorem 3.1 and Theorem 4.3. First, truncating f_0 and regularizing the truncated function by convolution, we get a sequence $f_0^n \in \mathcal{D}(R^3\times R^3)$ such that $f_0 \geq 0$ and

$$\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}|f_0 - f_0^n|(1+|\vec{r}|^2+|\vec{v}|^2) \to_n 0, \tag{5.1}$$

$$\iint_{R^3\times R^3} d\vec{r}\,d\vec{v} f_0^n |\log f_0^n| \le C, \tag{5.2}$$

for some constant $C \ge 0$ independent of n, and such that

$$\int d\vec{r}\,|\vec{E}_0^n - \vec{E}_0|^2 \to_n 0. \tag{5.3}$$

Next, in equation (AE3) of Chapter IV, let $\epsilon = 1/n$ (in assumption (A2)) and let $\eta_B(\vec{v},\vec{v}_1) = \eta_n(\vec{v},\vec{v}_1) \in \mathcal{D}(R^3 \times R^3)$ such that $0 \le \eta_n \le 1$, $supp\,\eta_n \subset B_{n+1}$, and $\eta_n|_{B_n} = 1$. Furthermore, since $Y(\tau,\sigma)$ is bounded and continuous, we can choose $\tilde{Y}_n = \tilde{Y}_n(\sigma,\tau) \in C^\infty(R \times R)$ such that $\tilde{Y}_n$ satisfies the *symmetry condition* (A2) and the *Lipschitz condition* (A3) in the last chapter (with the Lipschitz constant $C = C(n)$ depending on n only) such that

$$\lim_n \sup_{\sigma,\tau} |Y(\sigma,\tau) - \tilde{Y}_n(\sigma,\tau)| = 0. \tag{5.4}$$

Define the approximate geometric factors $\mathcal{Y}_n(\sigma,\tau) = (1+1/n\sigma)^{-1}(1+1/n\tau)^{-1}\,\tilde{Y}_n(\sigma,\tau)$ $\chi_{\{|\vec{r}|\le n\}}$. Consider the solution of the following system of equations:

$$L_\lambda f^n + \vec{E}_n(\vec{r},t)\cdot\nabla_{\vec{v}} f^n = \tilde{C}_E^n(f^n,f^n), \quad f^n|_{t=0} = f_0^n, \tag{APE}$$

with $\vec{E}_n$ satisfying (AE2) in the last chapter. As shown in the last chapter, for each n, there exists uniquely a non-negative solution $f^n(t) \in L^1 \cap L^\infty(R^3 \times R^3)$ of the equation (APE). Furthermore, the conditions (3.1)-(3.4a-b) are automatically satisfied for f^n and $\tilde{C}_E^n(f^n,f^n)$, provided the identities and inequalities (2.3), (2.10), (2.13) and (2.15) are justified for the f^n and $\vec{E}_n$ given above. (2.3). (2.10) and (2.13) can be checked without difficulty by the regularity and decay of f^n and $\vec{E}_n$ stated at the end of the last chapter. In view of the choice of f_0^n, (2.13) may be justified by the lower bound method used in [17].

Finally, despite the fact that f^n is not a solution of (DVE)-(VP), regarding the statement of Lemma 3.4 and the proof of Theorem 3.1, we see that Theorem 3.1 and its proof still apply to this sequence of solutions of approximate equations and yield, passing to a subsequence if necessary, convergence in $C([0,T];L^1(R^3,R^3))$ $(\forall T<\infty)$ to some f satisfying all the properties listed in Theorem 5.1.

In fact, since f^n is the solution of the approximate equation, $g_\delta(f^n)$ automatically satisfies the approximate renormalized equation

$$\frac{\partial}{\partial t}g_\delta^n+\vec{v}\cdot\nabla_{\vec{r}}g_\delta^n-\lambda\Delta_{\vec{v}}g_\delta^n=\frac{1}{1+\delta f^n}\tilde{C}_E^n(f^n,f^n)-\vec{E}^n\cdot\nabla_{\vec{v}}g_\delta^n+\lambda\delta|\nabla_{\vec{v}}g_\delta^n|^2. \qquad (5.5)$$

Comparing this equation with the renormalized equation (RDVE), the only difference is the collision term. We may complete the proof by following remarks. First, *Step 1* in the proof of Theorem 3.1, i.e., the boundedness of $\{C_E^{\pm}(f^n,f^n)(1+\delta f^n)^{-1}$ in L^1, depends only on the bounds in (3.1) and M_Y, which is obviously correct in our case. Second, *Step 2* and *Step 4* in that proof depend only on the linear operator L_λ and the form of the additional terms $\vec{E}\cdot\nabla_{\vec{v}}g_\delta$ and $|\nabla_{\vec{v}}g_\delta|^2$ in the renormalized equation, and not on the collision term $C_E(f^n,f^n)$. finally, passing to the limit $C_E(f^n,f^n)(1+\delta f^n)^{-1}\rightarrow_n C_E(f,f)(1+\delta f)^{-1}$ in *Step 3* is mainly based upon the bounds (3.1)-(3.4a-b) and the convergence in measure, passing to a subsequence if necessary, of the integrands in (3.20a)-(3.20b), which, in our case, can be proved in the same manner by using (5.4). This completes the proof. ■

Chapter VI. The Fokker-Planck-Vlasov-Enskog Equation

In this chapter we consider the renormalized solution of the following Fokker-Planck-Vlasov-Enskog equation:

$$[\frac{\partial}{\partial t}+\vec{v}\cdot\nabla_{\vec{r}}]f(\vec{r},\vec{v},t)+\vec{E}(\vec{r},t)\cdot\nabla_{\vec{v}}f(\vec{r},\vec{v},t)-\eta\mathrm{div}_{\vec{v}}(\vec{v}f+\frac{\theta}{2}\nabla_{\vec{v}}f)=C_E(f,f),\ \text{(FPVE)}$$

where, as before, $\vec{E}(\vec{r},t)$ is the *field function* and satisfies the equation

$$\vec{E}(\vec{r},t)=-\alpha\int_{R^N}d\vec{r}_1\,\nabla_{\vec{r}}\Big(\frac{1}{|\vec{r}-\vec{r}_1|^{N-2}}\Big)\int_{R^N}f(\vec{r}_1,\vec{v},t)\,d\vec{v},\qquad\text{(VE)}$$

$\alpha>0$ is a constant, R^N denotes real N-dimensional position or momentum space. The terms $\eta\mathrm{div}_{\vec{v}}(\vec{v}f)$ and $\eta\theta\mathrm{div}_{\vec{v}}(\nabla_{\vec{v}}f)/2$ denote the dynamical friction forces and thermal background interaction, respectively: The last term $C_E(f,f)$ is the Enskog collision term.

§6.1 Conserved Quantities and Bounds

The system we consider is (FPVE)-(VE), defined above. Suppose that $Y(\sigma,\tau)=Y(\tau,\sigma)$. Then we still have equation (2.1), i.e.,

$$\begin{aligned}&\iint\phi(\vec{r},\vec{v})C_E(f,f)(\vec{r},\vec{v},t)\,d\vec{v}d\vec{r}\\&=\frac{a^2}{2}\iiiint[\phi(\vec{r},\vec{r}')+\phi(\vec{r}+a\vec{\epsilon},\vec{v}_1')-\phi(\vec{r},\vec{v})-\phi(\vec{r}+\vec{v}_1)]\times\\&\qquad\times f(\vec{r},\vec{v},t)f(\vec{r}+a\vec{\epsilon},\vec{v}_1,t)Y(\vec{r},\vec{r}+a\vec{\epsilon})<\vec{\epsilon},\vec{v}-\vec{v}_1>\,d\vec{\epsilon}d\vec{v}_1d\vec{v}d\vec{r}\end{aligned}$$

for any given $\phi=\phi(\vec{r},\vec{v})$. Let $\phi=1$. Then

$$\iint d\vec{r}d\vec{v}\{\frac{\partial}{\partial t}f+\vec{v}\cdot\nabla_{\vec{r}}f+\vec{E}(\vec{r},t)\cdot\nabla_{\vec{v}}f(\vec{r},\vec{v},t)-\eta\mathrm{div}_{\vec{v}}(\vec{v}f+\frac{\theta}{2}\nabla_{\vec{v}}f)\}=0,$$

$$\frac{d}{dt}\iint d\vec{r}d\vec{v}\, f(\vec{r}+\vec{v},\vec{v},t) =$$
$$= \iint d\vec{r}d\vec{v}\,\mathrm{div}_{\vec{v}}\{f(\vec{r}+t\vec{v},\vec{v},t)\vec{E}(\vec{r}+t\vec{v},t) - \eta(f\vec{v}+\frac{\theta}{2}\nabla_{\vec{v}}f(\vec{r}+t\vec{v},\vec{v},t))\} = 0,$$

or

$$\iint f(\vec{r},\vec{v},t)d\vec{r}d\vec{v} = \iint f_0(\vec{r},\vec{v})d\vec{r}d\vec{v}, \qquad (\text{ Mass Conservation }). \tag{6.1}$$

Next we consider the energy bound. Since

$$\frac{\partial}{\partial t}v^2 f + \mathrm{div}_{\vec{r}}(v^2 f\vec{v}) + v^2\mathrm{div}_{\vec{v}}(f\vec{E}) - \eta v^2\mathrm{div}_{\vec{v}}(f\vec{v}+\frac{\theta}{2}\nabla_{\vec{v}}f) = v^2 C_E(f,f),$$
$$\frac{\partial}{\partial t}\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}v^2 f + \iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}v^2\mathrm{div}_{\vec{v}}(v\vec{E}) - \eta\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}v^2\mathrm{div}_{\vec{v}}(f\vec{v}+\nabla_{\vec{v}}f)$$
$$= \iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}v^2 C_E(f,f) = 0,$$

using

$$v^2\mathrm{div}_{\vec{v}}(f\vec{E}) = \mathrm{div}_{\vec{v}}(v^2 f\vec{E}) - 2\vec{E}\cdot f\vec{v},$$
$$v^2\mathrm{div}_{\vec{v}}(f\vec{v}) = \mathrm{div}_{\vec{v}}(v^2 f\vec{v}) - 2v^2 f,$$
$$v^2\mathrm{div}_{\vec{v}}(\nabla_{\vec{v}}f) = \mathrm{div}_{\vec{v}}(v^2\nabla_{\vec{v}}f) - 2\mathrm{div}_{\vec{v}}(f\vec{v}) + 2Nf,$$

we have

$$\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}\{\vec{E}(\vec{r}+t\vec{v},t)\cdot v^2\nabla_{\vec{v}}f(\vec{r}+t\vec{v},\vec{v},t) - \eta v^2\mathrm{div}_{\vec{v}}(f\vec{v}+\frac{\theta}{2}\nabla_{\vec{v}}f)\} =$$
$$= -2\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}\vec{E}\cdot f\vec{v} + 2\eta\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}v^2 f - N\theta\eta\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}f$$
$$= -2\int_{R^3} d\vec{r}\,\vec{E}\cdot\vec{j} + 2\eta\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}v^2 f - N\theta\eta\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}f_0,$$

where

$$\vec{j} = \int_3 \vec{v}f(\vec{r},\vec{v},t)d\vec{v},$$

and $N = 3$ is the spatial dimension, and

$$\frac{d}{dt}\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v} f(\vec{r},\vec{v},t)v^2 = 2\int_{R^3} d\vec{r}\,\vec{E}\cdot\vec{j} + N\theta\eta \iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v} f_0 - 2\eta \iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v} f v^2. \tag{6.2}$$

On the other hands, let $n(\vec{r},t) = \int d\vec{v}\, f(\vec{r},\vec{v},t)$. Then $n(\vec{r},t)$ satisfies the following equation:

$$\frac{\partial}{\partial t} n(\vec{r},t) + \mathrm{div}_{\vec{r}}(\vec{j}(\vec{r},t)) = \int C_E(f,f) d\vec{v}.$$

But

$$\vec{E}(\vec{r},t) = -\alpha \int d\vec{r}_1\, \nabla_{\vec{r}} \frac{1}{|\vec{r}-\vec{r}_1|} \int f(\vec{r}_1,\vec{v},t)\, d\vec{v},$$

and therefore

$$\begin{aligned} &\frac{\partial}{\partial t}\vec{E}(\vec{r},t) = \\ &= -\alpha \int d\vec{r}_1\, \nabla_{\vec{r}}\left(\frac{1}{|\vec{r}-\vec{r}_1|}\right)[\mathrm{div}_{\vec{r}_1}(\vec{j}(\vec{r}_1,t)) + \int d\vec{v}\, C_E(f,f)(\vec{r}_1,\vec{v},t)] \\ &= -4\pi\alpha\vec{j}(\vec{r},t), \end{aligned}$$

and

$$\frac{d}{dt}\int_{R^3} d\vec{r}\,|\vec{E}(\vec{r},t)|^2 = -8\pi\alpha \int d\vec{r}\,\vec{j}\cdot\vec{E}(\vec{r},t). \tag{6.3}$$

From (6.2) and (6.3) we get

$$\frac{d}{dt}\{\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v} v^2 f(\vec{r},\vec{v},t) + \frac{1}{4\pi\alpha}\int_{R^3} d\vec{r}\,|\vec{E}(\vec{r},t)|^2\} = -2\eta \iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v} f v^2 + N\theta\eta M_1, \tag{6.4}$$

where $M_1 = \iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v} f_0(\vec{r},\vec{v})$. Obviously, we have

$$\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v} v^2 f(\vec{r},\vec{v},t) + \frac{1}{4\pi\alpha}\int_R 3\, d\vec{r}\,|\vec{E}(\vec{r},t)|^2 \le M_2 + N\theta\eta M_1 t, \tag{6.5}$$

where, $M_2 = \iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v} v^2 f_0(\vec{r},\vec{v}) + \frac{1}{4\pi\alpha}\int_{R^3} d\vec{r}\,|\vec{E}(\vec{r},t=0)|^2$.

Since

$$\frac{d}{dt}\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}r^2 f = \frac{d}{dt}\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}(\vec{r}+t\vec{v})^2 f(\vec{r}+t\vec{v},\vec{v},t) =$$

$$= \iint (\hat{r})\cdot\vec{v}f(\hat{r},\vec{v},t)\,d\hat{r}d\vec{v} + \iint (\hat{r})^2(\frac{\partial}{\partial t}f(\hat{r},\vec{v},t) + \vec{v}\cdot\nabla_{\hat{r}}f(\hat{r},\vec{v},t))d\hat{r}d\vec{v}$$

$$= \iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}\vec{r}\cdot\vec{v}f(\vec{r},\vec{v},t) + \iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}\mathrm{div}_{\vec{v}}\{r^2(-f\vec{E}(\vec{r},t) + f\vec{v} + \nabla_{\vec{v}}f)\} +$$

$$+ \iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}r^2 C_E(f,f)(\vec{r},\vec{v},t)$$

$$= \iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}\vec{r}\cdot\vec{v}f(\vec{r},\vec{v},t),$$

we have

$$\frac{d}{dt}\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}r^2 f(\vec{r},\vec{v},t) \le 2\Big(\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}v^2 f(\vec{r},\vec{v},t)\Big)^{1/2}\Big(\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}r^2 f(\vec{r},\vec{v},t)\Big)^{1/2},$$

and

$$\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}r^2 f(\vec{r},\vec{v},t) \le \frac{1}{M_1^2}(M_2 + N\theta\eta M_1 t)^3 + 2M_3, \tag{6.6}$$

where $M_3 = \iint f_0(\vec{r},\vec{v})r^2 d\vec{r}d\vec{v}$.

Finally, let us consider the *entropy*. Formally we have,

$$\{\frac{\partial}{\partial t} + \vec{v}\cdot\nabla_{\vec{r}} + \vec{E}\cdot\nabla_{\vec{v}}\}f\log f =$$

$$= (1+\log f)C_E(f,f) + \eta(1+\log f)\mathrm{div}_{\vec{v}}(f\vec{v} + \frac{\theta}{2}\nabla_{\vec{v}}f),$$

$$(1+\log f)\mathrm{div}_{\vec{v}}(f\vec{v}) = \mathrm{div}_{\vec{v}}(f\log f\vec{v}) + Nf,$$

$$(1+\log f)\mathrm{div}_{\vec{v}}(\nabla_{\vec{v}}f) = \mathrm{div}_{\vec{v}}(\nabla_{\vec{v}}f\log f) - f^{-1}|\nabla_{\vec{v}}f|^2,$$

and therefore

$$\frac{d}{dt}\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}f\log f = \iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}\log f C_E(f,f) - 2\eta\theta\iint\limits_{R^3\times R^3} d\vec{r}\,d\vec{v}|\nabla_{\vec{v}}\sqrt{f}|^2 + N\eta M_1.$$

Define

$$\Gamma(t) = \iint\limits_{R^3 \times R^3} f(\vec{r}, \vec{v}, t) \log f(\vec{r}, \vec{v}, t)\, d\vec{v} d\vec{r} - \int_0^t \mathbf{I}(s)\, ds + \int_0^t \mathbf{J}(s)\, ds, \tag{6.7}$$

where $\mathbf{I}(t)$ and $\mathbf{J}(t)$ are given by (2.16a) and (2.16b), respectively. Using the fact that $y(\log y - \log z) \geq (y - z)$ for $y, z > 0$, and integrating both sides of equation (6.7) over $R^3 \times R^3$, we obtain, as in chapter II,

$$\frac{d}{dt}\Gamma(t) \leq N\eta M_1, \tag{6.8}$$

and

$$\frac{d}{dt} \iint\limits_{R^3 \times R^3} d\vec{r}\, d\vec{v} f \log f = \frac{\Gamma(t)}{dt} + \mathbf{I}(t) - \mathbf{J}(t) \leq I^+ + \eta N M_0. \tag{6.9}$$

Now assume the geometric function $Y(\tau, \sigma)$ is continuous in τ, σ satisfying (2.19). Then we have as before that

$$I^+(t) \leq 2\pi a^2 M_Y (M_2 + N\theta\eta M_1 t),$$

$$I_-(t) \leq 2\pi a^2 M_Y (M_2 + N\theta\eta M_1 t).$$

In addition,

$$\iint\limits_{R^3 \times R^3} d\vec{r}\, d\vec{v} f(\vec{r}, \vec{v}, t) |\log f(\vec{r}, \vec{v}, t)| \leq M_5, \quad 0 \leq t \leq T, \tag{6.10}$$

where $M_5 = M_5(M_1, M_2, M_3, M_4, M_Y, T)$ is a constant depending on M_1, M_2, M_3, M_4, M_Y, T, and

$$M_4 = \iint\limits_{R^3 \times R^3} d\vec{r}\, d\vec{v} f_0(\vec{r}, \vec{v}) |\log f_0(\vec{r}, \vec{v})| < \infty.$$

Finally, as in Chapter II, we can find a constant $M_6 = M_6(M_1, M_2, M_3, M_4, M_Y, T) < \infty$ such that

$$\int_0^T dt \iint_{R^3 \times R^3} \{|\nabla_{\vec{v}} \sqrt{f}|^2 + |\log f C_E(f, f)|\}\, d\vec{v} d\vec{r} \leq M_6.$$

§6.2 The Green's Function

Now let us consider the *fundamental solution* or the *Green's function* of the *field free* Vlasov-Fokker-Planck equation. Consider the following equation:

$$\frac{\partial}{\partial t}p + \vec{v}\cdot\nabla_{\vec{r}}p - \eta\nabla_{\vec{v}}(\vec{v}p + \frac{\theta}{2}\nabla_{\vec{v}}p) = \delta(\vec{r},\vec{v}-\vec{u},t),$$

where θ and $\vec{u}$ are parameters.

Taking a Fourier transformation with respect to $\vec{r}$, and denoting the transformed variable as $\vec{\alpha}$,

$$\frac{\partial}{\partial t}\hat{p} + \vec{v}\cdot i\vec{\alpha}\hat{p} - \eta\mathrm{div}_{\vec{v}}(\hat{p}\vec{v} + \frac{\theta}{2}\nabla_{\vec{v}}\hat{p}) = \delta(\vec{v}-\vec{u},t).$$

Fourier transforming with respect to $\vec{v}$, with transfored variable $\vec{\beta}$,

$$\frac{\partial}{\partial t}\tilde{p} - \vec{\alpha}\cdot\nabla_{\vec{\beta}}\tilde{p} - \eta - i\eta\vec{\beta}\cdot[i\nabla_{\vec{\beta}}\tilde{p} + \frac{\theta}{2}i\vec{\beta}\tilde{p}] = \delta(t)e^{-i\vec{u}\cdot\vec{\beta}},$$

$$\frac{\partial}{\partial t}\tilde{p} - \vec{\alpha}\cdot\nabla_{\vec{\beta}}\tilde{p} + \eta\vec{\beta}\cdot\nabla_{\vec{\beta}}\tilde{p} + \frac{\theta}{2}\eta\beta^2\tilde{p} = \delta(t)e^{-i\vec{\beta}\cdot\vec{u}}.$$

$$\frac{d}{dt}\tilde{p}(\vec{\alpha},\vec{\beta}-t\vec{\alpha}+t\eta\vec{\beta},t)e^{\int_0^t\frac{\theta}{2}\eta(\vec{\beta}-s\vec{\alpha}-s\eta\vec{\beta})^2ds} = \delta(t)e^{\int_0^t\frac{\theta}{2}\eta(\vec{\beta}-s\vec{\alpha}-s\eta\vec{\beta})^2ds}e^{i(\vec{\beta}-t\vec{\alpha}+t\eta\vec{\beta})\cdot\vec{u}}.$$

The solution of this equation is, for $t>0$,

$$\begin{aligned}
&\tilde{p}(\vec{\alpha},(1+t\eta)\vec{\beta}-t\vec{\alpha},t) = e^{-\frac{\theta}{2}\eta\int_0^t((1+\eta\tau)\vec{\beta}-\tau\vec{\alpha})^2d\tau}e^{-i\vec{\beta}\cdot\vec{u}}\\
&= \exp\{-\frac{\theta}{2}\eta[(t+\eta t^2+\frac{\eta^2}{3}t^3)\beta^2 - (t^2+\frac{2}{3}\eta t^3)\vec{\alpha}\cdot\vec{\beta} + \frac{1}{3}\alpha^2t^3]\}e^{-i\vec{\beta}\cdot\vec{u}}\\
&= \exp\{-\frac{\theta}{2}\eta t[(1+\eta t+\frac{1}{3}t^2\eta^2)\beta^2 - (t+\frac{2}{3}\eta t^2)\vec{\alpha}\cdot\vec{\beta} + \frac{1}{3}\alpha^2t^2]\}e^{-i\vec{\beta}\cdot\vec{u}}.
\end{aligned}$$

Let $\vec{B} = (1+t\eta)\vec{\beta} - t\vec{\alpha}$, $\vec{\beta} = \frac{\vec{B}+t\vec{\alpha}}{1+t\eta}$, then

$$(1+\eta t+\frac{1}{3}t^2\eta^2)\beta^2 - (t+\frac{2}{3}\eta t^2)\vec{\alpha}\cdot\vec{\beta} + \frac{t^2}{3}\alpha^2$$

$$= \frac{1+\eta t+\frac{1}{3}t^2\eta^2}{(1+t\eta)^2}(\vec{B}^2+2t\vec{\alpha}\cdot\vec{B}+t^2\alpha^2)-\frac{t+\frac{2}{3}\eta t^2}{1+t\eta}(\vec{\alpha}\cdot\vec{B}+t\alpha^2)+\frac{1}{3}t^2\alpha^2$$
$$= \frac{1}{(1+t\eta)^2}[(1+\eta t+\frac{1}{3}t^2\eta^2)\vec{B}^2+(t+\frac{2}{3}\eta t^2)\vec{\alpha}\cdot\vec{B}+\frac{1}{3}t^2\alpha^2].$$

Therefore, for $t>0$.

$$\tilde{p}(\vec{\alpha},\vec{\beta},t;\vec{u})e^{i\frac{\vec{\beta}+t\vec{\alpha}}{1+t\eta}\cdot\vec{u}}$$
$$=\exp\{-\frac{\theta}{2}\frac{\eta t}{(1+t\eta)^2}[(1+\eta t+\frac{1}{3}t^2\eta^2)\beta^2+(t+\frac{2}{3}\eta t^2)\vec{\alpha}\cdot 1+\eta t+\frac{1}{3}t^2\eta^2+\frac{t^2}{3}\alpha^2]\}=$$
$$=\exp\{-\frac{\theta\eta(1+\eta t+\frac{1}{3}t^2\eta^2)t}{2(1+t\eta)^2}$$
$$[(\vec{\beta}+\frac{1}{2}\frac{t+\frac{2}{3}\eta t^2}{1+\eta t+\frac{1}{3}t^2\eta^2}\vec{\alpha})^2+(\frac{t^2}{3(1+\eta t+\frac{1}{3}t^2\eta^2)}-\frac{1}{4}(\frac{t+\frac{2}{3}\eta t^2}{1+t\eta})^2)\alpha^2]\}$$
$$=\exp\{-\frac{\theta}{2}\frac{\eta t}{(1+t\eta)^2}(1+\eta t+\frac{1}{3}t^2\eta^2)(\vec{\beta}+\frac{1}{2}\frac{t+\frac{2}{3}\eta t^2}{1+\eta t+\frac{1}{3}t^2\eta^2}\vec{\alpha})^2\}$$
$$\times\exp\{-\frac{\theta}{2}\frac{\eta t^3}{12(1+\eta t+\frac{1}{3}t^2\eta^2)}\alpha^2\},$$

and

$$\tilde{p}(\vec{\alpha},\vec{\beta},t;\vec{u})=$$
$$=\begin{cases}\exp\{-\frac{\theta}{2}\frac{\eta t}{(1+t\eta)^2}(1+\eta t+\frac{1}{3}t^2\eta^2)[(\vec{\beta}+\frac{1}{2}\frac{t+\frac{2}{3}\eta t^2}{1+\eta t+\frac{1}{3}t^2\eta^2}\vec{\alpha})^2+\frac{t^2(1+t\eta)^2}{12}\alpha^2]\}, & t>0\\ 0. \qquad t<0\end{cases} \tag{A1}$$

Taking an inverse Fourier transform for $\vec{\beta}$ and for $\vec{\alpha}$, then for $t>0$.

$$p(\vec{r},\vec{v},t;\vec{u})=(\frac{(1+t\eta)^2}{2\theta\eta t(1+\eta t+\frac{1}{3}t^2\eta^2)\pi})^{3/2}\exp\{-\frac{(1+t\eta)^2}{2\theta\eta t(1+\eta t+\frac{1}{3}t^2\eta^2)}(\vec{v}-\frac{\vec{u}}{1+t\eta})^2\}\times$$
$$\mathcal{F}_{\vec{\alpha}}^{-1}\left(\exp\left(-\frac{1}{2(1+\eta t+\frac{1}{3}t^2\eta^2)}[\frac{\theta\eta t^3}{12}\alpha^2-i\vec{\alpha}\cdot((t+\frac{2}{3}\eta t^2)\vec{v}+\frac{t+\frac{4}{3}\eta t^2+\frac{2}{3}\eta^2t^3}{1+t\eta}\vec{u})\right)\right)$$
$$=(\frac{\sqrt{3}(1+t\eta)}{\pi\eta\theta t^2})^3\exp\left(-\frac{(1+t\eta)}{2\theta\eta t(1+\eta t+\frac{1}{3}t^2\eta^2)}(\vec{v}-\frac{\vec{u}}{1+t\eta})^2\right)\times$$
$$\times\exp\{-\frac{6(1+\eta t+\frac{1}{3}t^2\eta^2)}{\theta\eta t^3}r^2+\frac{6}{\theta\eta t^2}\vec{r}\cdot((1+\frac{2}{3}\eta t)\vec{v}+\frac{1+\frac{4}{3}\eta t+\frac{2}{3}\eta^2t^2}{1+t\eta}\vec{u})\}\times$$

$$\times \exp\{-\frac{3}{2\theta\eta t(1+\eta t+\frac{1}{3}t^2\eta^2)}((1+\frac{2}{3}\eta t)\vec{v}+\frac{1+\frac{4}{3}\eta t+\frac{2}{3}\eta^2t^2}{1+t\eta}\vec{u})^2\}. \tag{6.11}$$

Consider the linear operator

$$L_{\eta,\theta}(\vec{v})f=\frac{\partial}{\partial t}f+\vec{v}\cdot\nabla_{\vec{r}}f-\eta\nabla_{\vec{v}}(\vec{v}f+\frac{\theta}{2}\nabla_{\vec{v}}f).$$

Its Green's function is given by

$$G(\vec{r},\vec{v},t;\vec{q},\vec{u},s)=p(\vec{r}-\vec{q},\vec{v},t-s;\vec{u}). \tag{6.12}$$

That is, the solutions of the nonhomogeneous equation

$$\begin{cases}\frac{\partial}{\partial t}f(\vec{r},\vec{v},t)+\vec{v}\cdot\nabla_{\vec{r}}f-\eta\mathrm{div}_{\vec{v}}(\vec{v}f+\frac{\theta}{2}\nabla_{\vec{v}}f)=h(\vec{r},\vec{v},t),\\ \lim_{t\to+0}f(\vec{r},\vec{v},t)=f_0(\vec{r},\vec{v}),\end{cases}$$

are given by

$$\begin{aligned}f(\vec{r},\vec{v},t)=&\iint\limits_{R^3\times R^3}G(\vec{r},\vec{v},t;\vec{q},\vec{u},0)f_0(\vec{q},\vec{u})d\vec{q}d\vec{u}+\\&+\int_0^t ds\iint\limits_{R^3\times R^3}G(\vec{r},\vec{v},t;\vec{q},\vec{u},s)h(\vec{q},\vec{u},s)d\vec{q}d\vec{u}\end{aligned} \tag{6.13}$$

The following identities and inequalities will be used later. First, for all $(\vec{q},\vec{u})\in R^3\times R^3$ and $t>0$,

$$\begin{aligned}&\iint\limits_{R^3\times R^3}G(\vec{r},\vec{v},t;\vec{q},\vec{u},0)\,d\vec{r}d\vec{v}=\\&=(\frac{\sqrt{3}(1+t\eta)}{\pi\eta\theta t^2})^3\iint\limits_{R^3\times R^3}d\vec{v}d\vec{r}\exp\left(-\frac{(1+t\eta)}{2\theta\eta t(1+\eta t+\frac{1}{3}t^2\eta^2)}(\vec{v}-\frac{\vec{u}}{1+t\eta})^2\right)\times\\&\times\exp\left(-\frac{6(1+\eta t+\frac{1}{3}t^2\eta^2)}{\theta\eta t^3}(\vec{r}-(\frac{t+\frac{2}{3}\eta t^2}{2(1+\eta t+\frac{1}{3}t^2\eta^2)}\vec{v}+\frac{t+\frac{4}{3}\eta t^2+\frac{2}{3}\eta^2t^3}{2(1+\eta t+\frac{1}{3}t^2\eta^2)(1+t\eta)}))\right)\\&=1.\end{aligned} \tag{6.14}$$

Let

$$\vec{V} = \vec{v} - \Big(\frac{\vec{u}}{1+t\eta} + \frac{6(1+\eta t+\frac{1}{3}t^2\eta^2)(t+\frac{2}{3}\eta t^2)}{t^2(7+10\eta t+\frac{11}{3}\eta^2t^2)}\Big(2\vec{r} - \frac{2t+2\eta t^2+2\eta^2t^2/3}{2(1+t\eta)(1+\eta t+\frac{1}{3}t^2\eta^2)}\vec{u}\Big)\Big),$$

$$\vec{R} = \vec{r} - \frac{2t+2\eta t^2+2\eta^2t^2/3}{2(1+t\eta)(1+\eta t+\frac{1}{3}t^2\eta^2)}\vec{u},$$

then for every $(\vec{q},\vec{u}) \in R^3 \times R^3$,

$$\begin{aligned}
&\frac{(1+t\eta)}{2\theta\eta t(1+\eta t+\frac{1}{3}t^2\eta^2)}(\vec{v} - \frac{\vec{u}}{1+t\eta})^2 + \frac{6}{\theta\eta t^2}\vec{r}\cdot((1+\frac{2}{3}\eta t)\vec{v} + \frac{1+\frac{4}{3}\eta t+\frac{2}{3}\eta^2t^2}{1+t\eta}\vec{u}) + \\
&+ \frac{6(1+\eta t+\frac{1}{3}t^2\eta^2)}{\theta\eta t^3}r^2 + \frac{3}{2\theta\eta t(1+\eta t+\frac{1}{3}t^2\eta^2)}((1+\frac{2}{3}\eta t)\vec{v} + \frac{1+\frac{4}{3}\eta t+\frac{2}{3}\eta^2t^2}{1+t\eta}\vec{u})^2 \\
=&\frac{12(1+\eta t+\frac{1}{3}t^2\eta^2)(1+t\eta)^2}{\theta\eta t^3(7+10\eta t+\frac{11}{3}\eta^2t^2)}R^2 + \frac{7+10\eta t+\frac{11}{3}\eta^2t^2}{2\theta\eta t(1+\eta t+\frac{1}{3}t^2\eta^2)}V^2,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{R^3}\Big(\int_{R^3} G^2(\vec{r},\vec{v},t;\vec{q},\vec{u},0)\,d\vec{v}\Big)^{1/2} d\vec{r} \\
=&\int_{R^3}\Big(\int_{R^3} p^2(\vec{r}-\vec{q},\vec{v},t;\vec{u})\,d\vec{v}\Big)^{1/2} d\vec{r} = \int_{R^3}\Big(\int_{R^3} p^2(\vec{r},\vec{v},t;\vec{u})\,d\vec{v}\Big)^{1/2} d\vec{r} \\
=&\Big(\frac{\sqrt{3}(1+t\eta)}{\pi\eta\theta t^2}\Big)^3 \int_{R^3} d\vec{R}\,\exp\{-\frac{12(1+\eta t+\frac{1}{3}t^2\eta^2)(1+t\eta)^2}{\theta\eta t^3(7+10\eta t+\frac{11}{3}\eta^2t^2)}R^2\}\times \\
&\qquad\times\Big(\int_{R^3} d\vec{v}\,\exp\{-\frac{7+10\eta t+\frac{11}{3}\eta^2t^2}{\theta\eta t(1+\eta t+\frac{1}{3}t^2\eta^2)}V^2\}\Big)^{1/2} \\
=&\Big(\frac{7+10\eta t+\frac{11}{3}\eta^2t^2}{16\pi\eta\theta t(1+\eta t+\frac{1}{3}t^2\eta^2)}\Big)^{3/4} < \frac{1}{8}\Big(\frac{1}{11\pi\eta\theta t}\Big)^{3/4}.
\end{aligned} \tag{6.15}$$

Similarly, we also have

$$\iint_{R^3\times R^3} G(\vec{r},\vec{v},t;\vec{q},\vec{u},s)\,d\vec{q}d\vec{u} = (1+\eta(t-s))^3. \tag{6.16}$$

§6.3 Sequential Stability

Now we consider the solution of the following equations:

$$\begin{cases} \frac{\partial}{\partial t}g^n(\vec{r},\vec{v},t) + \vec{v}\cdot\nabla_{\vec{r}}g^n - \eta\mathrm{div}_{\vec{v}}(\vec{v}g^n + \frac{\theta}{2}\nabla_{\vec{v}}g^n) = h^n(\vec{r},\vec{v},t), \\ \lim_{t\to+0} g^n(\vec{r},\vec{v},t) = g_0^n(\vec{r},\vec{v}), \end{cases} \tag{IHS}$$

From (3.3), the solutions of equations (IHS) are given by

$$g^n(\vec{r},\vec{v},t) = \iint\limits_{R^3\times R^3} G(\vec{r},\vec{v},t;\vec{q},\vec{u},0)g_0^n(\vec{q},\vec{u})d\vec{q}d\vec{u} + \\ + \int_0^t ds \iint\limits_{R^3\times R^3} G(\vec{r},\vec{v},t;\vec{q},\vec{u},s)h^n(\vec{q},\vec{u},s)d\vec{q}d\vec{u}$$

Using (6.16), (6.17) and the argument in the proof of Lemma 3.5, we have

LEMMA 6.1. *Suppose that* $\{g_0^n\}$ *and* $\{h^n\}$ *satisfy the following hypothesis:*

(1) $\{g_0^n\}$ *are bounded in* $L^1(R^3\times R^3)$, *and*

$$\sup_n \iint\limits_{|(\vec{r},\vec{v})|\geq R} g_0^n(\vec{r},\vec{v})\,d\vec{r}d\vec{v} \to 0 \qquad as \qquad R\to\infty;$$

(2) $h^n = \hat{h}^n + \tilde{h}^n$, $\{\hat{h}^n\}$ *is bounded in* $L^1((0,T)\times R^3\times R^3)$, $\{\tilde{h}^n\}$ *is bounded in* $L^1((0,T)\times R^3_{\vec{r}}; L^2(R^3_{\vec{v}}))$, *and*

$$\begin{cases} \sup_n \int_0^T \iint_{|(\vec{r},\vec{v})|\geq R} |\hat{h}^n|\,d\vec{r}d\vec{v}dt \to 0 \quad as \quad R\to\infty, \\ \sup_n \int_0^T dt \int d\vec{r}\big(\int_{|\vec{r}|+|\vec{v}|\geq R} |\tilde{h}^n|^2(\vec{r},\vec{v},t)\,d\vec{v}\big)^{1/2} \to 0 \quad as\ R\to\infty. \end{cases}$$

Then the set of solutions $\{g^n(\vec{r},\vec{v},t)\|$ *of the equations* (IHS) *is compact in* $L^1((0,T)\times R^3\times R^3)$.

Now we can consider the sequential stability result for the equation (FPVE). Suppose that $f(\vec{r},\vec{v},t)$ is a nonnegative solution of the equation (FPVE). Then, the function g_δ of f, $g_\delta = \beta_\delta(f) \equiv \frac{1}{\delta}\log(1+\delta f)$ solves

$$\frac{\partial}{\partial t}g_\delta + \vec{v}\cdot\nabla_{\vec{r}}g_\delta - \eta\,\mathrm{div}_{\vec{v}}(g_\delta\vec{v} + \frac{\theta}{2}\nabla_{\vec{v}}g_\delta) = \\ = \frac{1}{1+\delta g}C_E(f,f) + \delta\frac{\theta\eta}{2}|\nabla_{\vec{v}}g_\delta|^2 + N\frac{\eta f}{1+\delta f} - \vec{E}(\vec{r},t)\cdot\nabla_{\vec{r}}g_\delta. \qquad \text{(RME)}$$

This motivates the following definition.

DEFINITION 6.2. *A nonnegative element f of $C([0,\infty), L^1(R^3 \times R^3))$ is a renormalized solution of* (FPVE)-(VE) *if the composite function $g_\delta = \beta_\delta(f)$ satisfies* (RME)-(VE) *in the sense of the distributions, where $\beta_\delta(t) = \frac{1}{\delta}\log(1+\delta t)$.*

THEOREM 6.3. *Assume that $f^n \in W^{2,\infty}(R^3 \times R^3 \times [0,\infty))$ is a sequence of nonnegative solutions of* (DVE)-(VE), *$f^n \to 0$ as $|(\vec{r},\vec{v})| \to \infty$ uniformly in $t \in [0,T]$ for all $T < \infty$. Assume that there is a constant C_T independent of n such that*

$$\iint_{R^3\times R^3} f^n(\vec{r},\vec{v},t)(1+|\vec{r}|^2+|\vec{v}|^2+|\log f^n|)\,d\vec{v}d\vec{r} \le C_T, \tag{6.17a}$$

$$\int_{R^3} |E_n(\vec{r},t)|^2 d\vec{r} \le C_T, \tag{6.17b}$$

$$\int_0^T dt \iint_{R^3\times R^3} \{|\nabla_{\vec{v}}\sqrt{f^n}|^2 + |\log f^n C_E(f^n,f^n)|\}\,d\vec{v}d\vec{r} \le C_T. \tag{6.17c}$$

Then the sequence f^n converges in $L^p(0,T;L^1(R^3\times R^3))$, $1 \le p < \infty$, $0 < T < \infty$. to a renormalized solution f which satisfies (6.17a)-(6.17b) *for a.e. $t \in (0,T)$ and* (6.17c). *Furthermore. for each $\delta > 0$, the renormalized interaction terms satisfy*

$$\begin{cases} C_E^-(f,f)(1+\delta f)^{-1}|_{\vec{v}\in B_R} \in C([0,\infty); L^1(R^3_{\vec{r}}\times B_R)), & \forall R < \infty \\ C_E^+(f,f)(1+\delta f)^{-1}|_{\vec{v}\in B_R} \in L^1([0,\infty)\times R^3_{\vec{r}}\times B_R), & \forall R, T < \infty \end{cases}$$

and $g_\delta|_{\vec{v}\in B_R} \in L^2((0,T)\times R^3_{\vec{r}}; H^1(B_R))$ $(\forall R, T < \infty)$.

We point out that Theorem 6.3 can be proved in the same manner as the proof of Theorem 3.1 with little modification, noting in particular that the results of Lemma 3.4 are given by Lemma 6.1.

§6.4 Global Existence

As we did in chapter IV and V, we first consider the solution of the following non-linear approximate equation:

$$\begin{cases} \frac{\partial}{\partial t} f(\vec{r},\vec{v},t) + \vec{v}\cdot\nabla_{\vec{r}} f - \eta(\vec{v} f + \frac{\theta}{2}\nabla_{\vec{v}} f) + \vec{E}(\vec{r},t)\cdot\nabla_{\vec{v}} f = \tilde{C}_E(f,f) \\ \lim_{t\to 0+} f(\vec{r},\vec{v},t) = f_0(\vec{r},\vec{v}). \end{cases} \tag{AE1}$$

where

$$\vec{E}(\vec{r},t) = \frac{1}{4\pi}\int d\vec{r}'\nabla_{\vec{r}}\frac{1}{|\vec{r}-\vec{r}'|}\int d\vec{v}f(\vec{r}',\vec{v},t), \tag{AE2}$$

$$\begin{aligned}
&\tilde{C}_E(f,f)(\vec{r},\vec{v},t)\\
&= \iint_{R^3\times S^2_+}[\mathcal{Y}(n(\vec{r},t),n(\vec{r}-a\vec{\epsilon},t))\eta_B(\vec{v},\vec{v}_1)f(\vec{r},\vec{v}',t)f(\vec{r}-a\vec{\epsilon},\vec{v}_1',t)\\
&\quad -\mathcal{Y}(n(\vec{r},t),n(\vec{r}+a\vec{\epsilon},t))\eta_B(\vec{v},\vec{v}_1)f(\vec{r},\vec{v},t)f(\vec{r}+a\vec{\epsilon},\vec{v}_1,t)] < \vec{\epsilon},\vec{v}-\vec{v}_1 > d\vec{\epsilon}d\vec{v}_1\\
&= \tilde{C}_E^+ - \tilde{C}_E^-.
\end{aligned} \tag{AE3}$$

$B = \{(\vec{v},\vec{v}_1);\ |\vec{v}|^2+|\vec{v}_1|^2 \le k\}$ for some positive constant k and $n(r,t) = \int_B f(\vec{r},\vec{v},t)d\vec{v}$. We assume that $\mathcal{Y}$ and f_0 satisfy the assumptions (A1)-(A4) in Section 4.4. Then by considering the following form of the integral equation

$$\begin{aligned}
f(\vec{r},\vec{v},t) = &\iint_{R^3\times R^3} G(\vec{r},\vec{v},t;\vec{q},\vec{u},0)f_0(\vec{q},\vec{u})d\vec{q}d\vec{u}+\\
&+\int_0^t ds\iint_{R^3\times R^3} G(\vec{r},\vec{v},t;\vec{q},\vec{u},s)[\tilde{C}_E(f,f)(\vec{q},\vec{u},s)+\vec{E}(\vec{q},s)\cdot\nabla_{\vec{u}}f(\vec{q},\vec{u},s)d\vec{q}d\vec{u},
\end{aligned}$$

and using the lemma 6.1, we can follow the proof of Theorem 4.3 and conclude

THEOREM 6.4. *Under the assumptions* (A1)-(A4), *the equation* (AE1)-(AE3) *has a unique non-negative solution which belongs to* $C([0,T];L^p(R^3_{\vec{r}};L^1(R^3_{\vec{v}})))$, $1\le p\le\infty$, *for each* $T\in(0,\infty)$.

Finally, truncating f_0 and regularizing the truncated function by convolution, we get a sequence $f_0^n\in\mathcal{D}(R^3\times R^3)$ such that $f_0\ge 0$ and

$$\iint_{R^3\times R^3} d\vec{r}\,d\vec{v}|f_0-f_0^n|(1+|\vec{r}|^2+|\vec{v}|^2)\to_n 0, \tag{6.18a}$$

$$\iint_{R^3\times R^3} d\vec{r}\,d\vec{v}f_0^n|\log f_0^n|\le C, \tag{6.18b}$$

for some constant $C \geq 0$ independent of n, and such that

$$\int d\vec{r}\,|\vec{E}_0^n - \vec{E}_0|^2 \to_n 0. \tag{6.18c}$$

In equation (AE3) let $\epsilon = 1/n$ (in the assumption (A2)) and let $\eta_B(\vec{v}, \vec{v}_1) = \eta_n(\vec{v}, \vec{v}_1) \in \mathcal{D}(R^3 \times R^3)$ such that $0 \leq \eta_n \leq 1$, $supp\,\eta_n \subset B_{n+1}$, and $\eta_n|_{B_n} = 1$. Since $Y(\tau, \sigma)$ is bounded and continuous, we can choose $\tilde{Y}_n = \tilde{Y}_n(\sigma, \tau) \in C^\infty(R \times R)$ such that $\tilde{Y}_n$ satisfies the *symmetry condition* (A2) and the *Lipschitz condition* (A3) in the last chapter (with the Lipschitz constant $C = C(n)$ depending on n only) and such that

$$\lim_n \sup_{\sigma,\tau} |Y(\sigma, \tau) - \tilde{Y}_n(\sigma, \tau)| = 0. \tag{6.19}$$

Define the approximate geometric factors $\mathcal{Y}_n(\sigma, \tau) = (1+1/n\sigma)^{-1}(1+1/n\tau)^{-1}\tilde{Y}_n(\sigma, \tau)$ $\chi_{\{|\vec{r}| \leq n\}}$. Consider the solution of the following system of equations:

$$L_\lambda f^n + \vec{E}_n(\vec{r}, t) \cdot \nabla_{\vec{v}} f^n = \tilde{C}_E^n(f^n, f^n), \quad f^n|_{t=0} = f_0^n, \tag{APE}$$

with $\vec{E}_n$ satisfying (AE2). Then, according theorem 6.4, there exists a unique non-negative solution $f^n(t) \in L^1 \cap L^\infty(R^3 \times R^3)$ of the equation (APE), which satisfies the bounded condition in Theorem 6.3. Using Theorem 6.3, we have

THEOREM 6.5. *Assume that $Y(\sigma, \tau) = Y(\tau, \sigma)$ is a jointly continuous function satisfying the symmetry condition and the boundedness condition (2.13), the initial value $f_0(\vec{r}, \vec{v}) \geq 0$ satisfies the bounded condition*

$$\iint_{R^3 \times R^3} d\vec{r}\,d\vec{v} f_0(\vec{r}, \vec{v})(1 + |\vec{r}|^2 + |\vec{v}|^2 + |\log f_0|) \leq C < \infty,$$

and $\vec{E}_0(\vec{r}) = \nabla_{\vec{r}}(\frac{1}{|\vec{r}|} * n_0)(\vec{r}) = \nabla_{\vec{r}}(\frac{1}{|\vec{r}|} * \int f_0(\vec{r}, \vec{v})\,d\vec{v})(\vec{r})$ satisfies

$$\int_{R^3} |\vec{E}_0(\vec{r})|^2\,d\vec{r} \leq C < \infty.$$

Then, there exist $f \in C([0,\infty);\ L^1(R^3 \times R^3))$ *which satisfies* $f|_{t=0} = f_0$. *(3.9a)-(3.9b)-(3.9c) and such that, for all* $\delta > 0$, $g_\delta = \beta_\delta(f)$ *satisfies (RDVE)-(VP)-(IC) in the sense of distributions and* $g_\delta|_{(0,T)\times R^3 \times B_R} \in L^2([0,T] \times R^3,\ H^1(B_R))$ $(\forall R, T < \infty)$. *In particular* f *is a renormalized solution of (DVE)-(VP).*

Chapter VII Mild and $\mathcal{B}$-Type Mild Solution

In this chapter we study the existence of weak solution and semi-stronh solution for some special models. We first give a general theory for the existence of solution for semi-linear evolution equations. We then apply this theory to some kinetic equations.

§7.1 Solution of Non-linear Evolution Equations

Let $\mathcal{X}$ be a Banach space, $\mathcal{A} : \mathcal{D}(\mathcal{A})(\subset \mathcal{X}) \to \mathcal{X}$ be a densely defined closed linear operator which generates a c_0-semigroup $\mathbf{U}(t)$. Let $\mathcal{J}(f) : \mathcal{D}(\mathcal{A})(\supset \mathcal{D}(\mathcal{A})) \to \mathcal{X}$ be a non-linear operator. Consider the following non-linear evolution equation:

$$\begin{cases} \frac{d}{dt}f(t) - \mathcal{A}f(t) = \mathcal{J}(f(t)) \\ f(t=0) = f_0. \end{cases} \tag{7.1}$$

A $\mathcal{X}$-valued function $f(t)$ is called a *strong solution* of equation (7.1) if $f(t) \in \mathcal{D}(\mathcal{A})$ for each $t > 0$, strongly differentiable about t, and satisfies the equation in the strong topological sense. $f(t)$ is called a *mild solution* of equation (7.1) if $f(t)$ satisfies the integral equation

$$f(t) = \mathbf{U}f_0 + \int_0^t \mathbf{U}(t-s)\mathcal{J}(f(s))\,ds. \tag{7.2}$$

Suppose that $\mathcal{B} : \mathcal{D}(\mathcal{B}) \to \mathcal{X}$ is another densely defined closed linear operator satisfying $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{B}) \subset \mathcal{X}$. We shall say that $f(t)$ is a *$\mathcal{B}$-type mild solution* of equation (7.1) if $f(t) \in \mathcal{D}(\mathcal{B})$, satisfies (7.2), and both $f(t)$ and $\mathcal{B}f(t)$ are continuous. Obviously a strong solution or a $\mathcal{B}$-type mild solution is a mild solution, but the reverse may not be true.

In the following, we want to find some conditions under which equation (7.1) has a $\mathcal{B}$-type solution. In order to do so, we need to restrict both $\mathcal{A}$ and $\mathcal{J}$. Since $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{B}) \subset \mathcal{X}$ and $\mathcal{B}$ is a closed operator, for any $\phi \in \mathcal{X}$, $\mathbf{U}(t)\phi \in \mathcal{D}(\mathcal{A})(\subset \mathcal{D}(\mathcal{B}))$ as long as $t > 0$, that is, for any $t > 0$, $\mathcal{B}\mathbf{U}(t)$ is a closed operator defined on $\mathcal{X}$. By the closed graph theorem, $\mathcal{B}\mathbf{U}(t)$ is a bounded operator. The c_0-semigroup $\mathbf{U}(t)$ will be called *$\mathcal{B}$-regular* if there exists a continuous function $\sigma(t)$, $0 < t < \infty$, such that

(R1) $\quad \|\mathcal{B}\mathbf{U}(t)\| \leq \sigma(t), \ \ t > 0;$

(R2) $\quad \|\mathbf{U}(t)\mathcal{B}\phi\| \leq \sigma(t)\|\phi\|, \ \text{for } \phi \in \mathcal{D}(\mathcal{B}) \text{ and } t > 0;$

(R3) $\quad \int_0^T \sigma(t)\, dt = p(T) < \infty \text{ for any } T > 0.$

The closed operator $\mathcal{A}$ is called a *$\mathcal{B}$-regular operator* if the semigroup $\mathbf{U}(t)$ generated by $\mathcal{A}$ is $\mathcal{B}$-regular. For the nonlinear operator $\mathcal{J}(\cdot)$, we will say that $\mathcal{J}(\cdot)$ is *$\mathcal{B}$-bounded continuous* if $\mathcal{D}(\mathcal{B}) \subset \mathcal{D}(\mathcal{J})$ and $\mathcal{J}$ satisfies the following conditions:

(B1) $\quad \|\mathcal{J}(\phi)\| \leq C(\|\phi\|, \|\mathcal{B}\phi\|)(\|\mathcal{B}\phi\| + \|\phi\|);$

(B2) $\quad \|\mathcal{J}(\phi) - \mathcal{J}(\psi)\| \leq C(\|\phi\|, \|\psi\|, \|\mathcal{B}\phi\|, \|\mathcal{B}\psi\|)(\|\mathcal{B}(\phi - \psi)\| + \|\phi - \psi\|)$

for all $\phi, \psi \in \mathcal{X}$, where each constant C is a monotone increasing (everywhere finite) function of the norm indicated.

THEOREM 7.1. (Local Existence) *Suppose $\mathcal{A}$ is $\mathcal{B}$-regular and $\mathcal{J}$ is $\mathcal{B}$-bounded continuous. Then, for each $\phi_0 \in \mathcal{D}(\mathcal{A})$, there is a $T > 0$ so that equation (7.1) has a unique $\mathcal{B}$-type mild solution for $t \in (0, T)$. For each $a, b > 0$, T can be chosen uniformly for all ϕ_0 in the set $\{\phi \mid \|\phi\| \leq a, \|\mathcal{B}\phi\| \leq b\}$.*

PROOF: We first assume that $\sigma(t)$ is decreasing and that $\mathbf{U}(t)$ is contractive. Let X_T be the set of $\mathcal{D}(\mathcal{B})$-valued functions on (0, T) for which $\phi(t)$ and $\mathcal{B}\phi(t)$ are continuous and

$$\|\phi(\cdot)\|_T \equiv \sup_{t\in(0,T)} \|\phi(t)\| + \sup_{t\in(0,T)} \|\mathcal{B}\phi(t)\| < \infty.$$

Since $\mathcal{B}$ is closed, X_T with the norm $\|\phi(\cdot)\|_T$ is a Banach space. For fixed $\epsilon > 0$, let $\phi_0 \in \mathcal{D}(\mathcal{B})$ be given and let X_{T,ϵ,ϕ_0} consist of those X_T with $\phi(0) = \phi_0$ and $\|\phi(\cdot) - \mathbf{U}(t)\phi_0\|_T \leq \epsilon$. We will show that the map

$$(\mathbf{S}\phi)(t) = \mathbf{U}(t)\phi_0 + \int_0^t U(t-s)\mathcal{J}(\phi(s))\,ds \tag{7.3}$$

is a contraction on X_{T,ϵ,ϕ_0} if T is small enough. We denote by C_ϵ any of the constants in the conditions (B1)-(B2) with arguments $\|\phi\|, \|\psi\| = \|\phi_0\| + \epsilon$ and $\|\mathcal{B}\phi\|, \|\mathcal{B}\psi\| = \|\mathcal{B}\phi_0\| + \epsilon$. Suppose that $\phi(\cdot) \in X_{T,\epsilon,\phi_0}$, if $s \in (0,t)$ and $s + h \in (0,t)$. Then

$$\begin{aligned}
&\|\mathbf{U}(t-(s+h))\mathcal{J}(\phi(s+h)) - \mathbf{U}(t-s)\mathcal{J}(\phi(s))\| \\
&\leq \|\mathbf{U}(t-s-h)(\mathcal{J}(\phi(s+h)) - \mathcal{J}(\phi(s)))\| + \|(\mathbf{U}(t-s-h) - \mathbf{U}(t-s))\mathcal{J}(\phi(s))\| \\
&\leq C_\epsilon(\|\mathcal{B}\phi(s+h) - \mathcal{B}\phi(s)\| + \|\phi(t+h) - \phi(t)\|) + \|(\mathbf{U}(h) - I)\mathbf{U}(t-s)\mathcal{J}(\phi(s))\|.
\end{aligned}$$

For fixed t and s, let $\psi(s) = \mathbf{U}(t-s)\mathcal{J}(\phi(s))$. Then $\|\psi(s)\| \leq \|\mathcal{J}(\phi(s))\| \leq C_\epsilon(\|\mathcal{B}\| + \|\phi_0\| + 2\epsilon)$. Therefore, the last term above tends to zero as $h \to 0$ since $U(t)$ is c_0-semigroup, and the first term on the right hand side of the above estimation converges to zero by the assumption. Hence, the integrand on the right hand side of (7.3) is continuous for $s \in (0,t)$. Similarly, for any $b \in (0,t)$, if $s \in (0,b)$ and $s + h \in (0,b)$,

$$\begin{aligned}
&\|\mathcal{B}\mathbf{U}(t-(s+h))\mathcal{J}(\phi(s+h)) - \mathcal{B}\mathbf{U}(t-s)\mathcal{J}(\phi(s))\| \leq \\
&\leq \|\mathcal{B}\mathbf{U}(t-(s+h))(\mathcal{J}(\phi(s+h)) - \mathcal{J}(\phi(s)))\| \\
&\quad + \|\mathcal{B}(\mathbf{U}(t-(s+h)) - \mathbf{U}(t-s))\mathcal{J}(\phi(s))\| \\
&\leq C_\epsilon\sigma(t-b)(\|\mathcal{B}\phi(s+h) - \mathcal{B}zf(s)\| + \|\phi(t+h) - \phi(t)\|) \\
&\quad + \sigma(t-b)\|(\mathbf{U}(h) - I)\mathcal{J}(\phi(s))\|.
\end{aligned}$$

A similar proof shows that $\mathcal{B}\mathbf{U}(t-s)\mathcal{J}(\phi(s))$ is continuous for $s \in (0,t)$. Therefore, the right hand side of (7.3) can be defined as a generalized Riemann integral. For

any $b \in (0,t)$ given, rewrite (7.3) as

$$(\mathbf{S}\phi)(t) = \mathbf{U}(t)\phi_0 + \int_0^b \mathbf{U}(t-s)\mathcal{J}(\phi(s))\,ds + \int_b^t \mathbf{U}(t-s)\mathcal{J}(\phi(s))\,ds, \tag{7.3a}$$

and let

$$\eta_n(t) \equiv \sum_{1\le m\le[n\frac{b}{t}]} \frac{1}{n}\mathbf{U}(t-\frac{m}{n}t)\mathcal{J}(\phi(\frac{m}{n}t)), \tag{7.4a}$$

$$\eta_b(t) \equiv \int_0^b \mathbf{U}(t-s)\mathcal{J}(\phi(s))\,ds, \tag{7.4b}$$

$$\eta(t) \equiv \int_0^t \mathbf{U}(t-s)\mathcal{J}(\phi(s))\,ds. \tag{7.4c}$$

Then $\eta_n(t) \to \eta_b(t)$ as $n \to \infty$. Furthermore,

$$\lim_{b\to t}\|\eta(t)-\eta_b(t)\| \le \lim_{(t-b)\to 0}\int_0^{t-b} \|\mathbf{U}(s)\mathcal{J}(\psi(t-s))\|\,ds$$

$$\le \lim_{(t-b)\to 0}\int_0^{t-b} \sigma(s)\,ds\; C_\epsilon(\|\phi_0\| + \|\mathcal{B}\phi_0\| + 2\epsilon) = 0. \tag{7.4d}$$

Now, each $\eta_n(t) \in \mathcal{D}(\mathcal{A})(\subset \mathcal{D}(\mathcal{B}))$, so

$$\mathcal{B}\eta_n(t) = \sum_{1\le m\le[nb/t]} \mathcal{B}\frac{1}{n}\mathbf{U}(t-\frac{m}{n}t)\mathcal{J}(\phi(\frac{m}{n}t))$$

$$\to \int_0^b \mathcal{B}U(t-s)\mathcal{J}(\phi(s))\,ds = \mathcal{B}\eta_b(t) \tag{7.5a}$$

and

$$\lim_{b\to t}\|\mathcal{B}\eta_b(t) - \mathcal{B}\eta(t)\| = \lim_{b\to t}\|\mathcal{B}\int_b^t \mathbf{U}(t-s)\mathcal{J}(\phi(s))\,ds\|$$

$$\le \lim_{b\to t}\int_b^t \sigma(t-s)C_\epsilon(\|\mathcal{B}\phi_0\| + \|\phi_0\| + 2\epsilon)\,ds \to 0. \tag{7.5b}$$

Therefore, $\eta(t) \in \mathcal{D}(\mathcal{B})$, and

$$\|\mathcal{B}\eta(t+h) - \mathcal{B}\eta(t)\| \le$$

$$\leq \| \int_t^{t+h} \mathcal{B}\mathbf{U}(t+h-s)\mathcal{J}(\phi(s))\,ds\| + \| \int_0^t \mathcal{B}\mathbf{U}(t-s)(\mathbf{U}(h)-I)\mathcal{J}(\phi(s))\,ds\|$$
$$\leq C_\epsilon \|\phi\|_T \int_t^{t+h} \sigma(t+h-s)\,ds \; + \; \int_0^t \|\mathcal{B}\mathbf{U}(t-s)(\mathbf{U}(h)-I)\mathcal{J}(\phi(s))\|\,ds. \tag{7.6}$$

The first integration tends to zero as $h \to 0$. The integrand in the second term converges to zero as $h \to 0$ for each s, and by the hypotheses on $\mathcal{J}$, the integrand

$$\|\mathcal{B}\mathbf{U}(t-s)(\mathbf{U}(h)-I)\mathcal{J}(\phi(s))\| \leq 2C_\epsilon\sigma(t-s)(\|\mathcal{B}\phi_0\| + \|\phi_0\| + 2\epsilon),$$

and $\int_0^t \sigma(t-s)\,ds = p(t) < \infty$. Thus, by Dominated Convergence Theorem, the right-hand side of (7.6) converges to zero as $h \to 0$. That is, $\mathcal{A}\eta(t)$ is continuous, and similarly, $\eta(t)$ is continuous. Moreover, exactly the same kind of estimation as above shows that for any $\phi(\cdot)$, $\psi(\cdot) \in X_{T,\epsilon,\phi_0}$, we have

$$\|(\mathbf{S}\phi)(t) - \mathbf{U}(t)\phi_0\| \leq C_\epsilon T(\sup_{t\in(0,T)} \|\phi(t)\| + \sup_{t\in(0,T)} \|\mathcal{B}\phi(t)\|),$$
$$\|(\mathcal{B}\mathbf{S}\phi)(t) - \mathcal{B}\mathbf{U}(t)\phi_0\| \leq C_\epsilon p(T)(\sup_{t\in(0,T)} \|\mathcal{B}\phi(t)\| + \sup_{t\in(0,T)} \|\phi(t)\|),$$
$$\|(\mathbf{S}\phi)(t) - (\mathbf{S}\psi)(t)\| \leq C_\epsilon T(\sup_{t\in(0,T)} \|\phi(t)-\psi(t)\| + \sup_{t\in(0,T)} \|\mathcal{B}\phi(t) - \mathcal{B}\psi(t)\|),$$
$$\|(\mathcal{B}\mathbf{S}\phi)(t) - (\mathcal{B}\mathbf{S}\psi)(t)\| \leq C_\epsilon p(T)(\sup_{t\in(0,T)} \|\mathcal{B}\phi(t) - \mathcal{B}\psi(t)\| + \sup_{t\in(0,T)} \|\phi(t)-\psi(t)\|.$$

Thus, since $\lim_{T\to 0}\ p(T) = 0$, for T small enough, $\mathbf{S}$ is a contraction on X_{T,ϵ,ϕ_0}. By the Schauder fixed point theorem, $\mathbf{S}$ has a unique fixed point $\phi(\cdot)$ in X_{T,ϵ,ϕ_0}, which satisfies (7.2). Comparing the definition of a $\mathcal{B}$-type solution and the space X_{T,ϵ,ϕ_0}, we see that the fixed point $\phi(\cdot)$ is actually a $\mathcal{B}$-type solution of (7.1).

We complete the proof of this theorem with the following two remarks: First, the assumption that $\sigma(t)$ is decreasing and $\mathbf{U}(t)$ is a contraction at the beginning of this proof is just for convenience, but not necessary. Second, what was used in the above estimations was the norm of ϕ and $\mathcal{B}\phi$. Hence T can be chosen uniformly

whenever the initial data ϕ_0 is in the set $\{\phi|\ \|\phi\| \le a, \|\mathcal{B}\| \le b\}$. This completes the proof. ■

From the proof above, it is evident that the existence result can be continued globally.

COROLLARY 7.2. *Suppose $\mathcal{A}$ itself is an $\mathcal{A}$-regular operator, and $\mathcal{J}()$ is $\mathcal{A}$-bounded continuous. Then for any $\phi_0 \in \mathcal{D}(\mathcal{A})$, there is a $T = T(\|\phi_0\|, \|\mathcal{A}\phi_0\|) > 0$ so that equation (7.1) has uniquely a strong solution.*

PROOF: First, by Theorem 7.1, equation (7.1) has an $\mathcal{A}$-type mild solution $\phi(\cdot)$. We want to show the $\mathcal{A}$-type mild solution is actually the strong solution of equation (7.1). First, suppose that $\tilde{\phi}(t)$ is a continuous $\mathcal{D}(\mathcal{A})$-valued solution of (7.1) on the interval $(0, \tilde{T})$ with $\tilde{\phi}(0) = \phi_0$. From the differential equation, $\mathcal{A}\tilde{\phi}(t)$ is continuous, so $\tilde{\phi}(t) \in X_{T,\epsilon,\phi_0}$ for t in some interval $(0, T_0)$, where, X_{T,ϵ,ϕ_0} is the space used in the proof of Theorem 7.1. Since $\tilde{\phi}$ obeys (7.2), $\phi(t) = \tilde{\phi}(t)$ for $t < T_0$. Let T_1 be the supremum of such T_0, since X_{T,ϵ,ϕ_0} is closed and $\tilde{\phi}(T_1) \in X_{T,\epsilon,\phi_0}$. Now if $T_1 < T$, then since $\phi(T_1) = \tilde{\phi}(T_1)$, the same argument as above shows that $\tilde{\phi}(t) = \phi(t)$ for some small interval of T_1, which contradicts the maximality of T_1. Thus $T_1 \ge T$, and $\tilde{\phi}(t) = \phi(t)$ for $t \in (0, T)$. That is, any strong solution of (7.1) on (0,T) equals the $\mathcal{A}$-type mild solution ϕ.

Next, we want to show the differentiability of $\phi(t)$. We write

$$\frac{\phi(t+h) - \phi(t)}{h} = (\frac{\mathbf{U}(h) - I}{h})\mathbf{U}(t)\phi_0 + \frac{1}{h}\int_t^{t+h} \mathbf{U}(t)\mathcal{J}(\phi(s))\, ds + \int_0^t (\frac{\mathbf{U}(t) - I}{h})\mathbf{U}(t-s)\mathcal{J}(\phi(s))\, ds. \tag{7.7}$$

Since $\phi_0 \in \mathcal{D}(\mathcal{A})$, the first term converges to $\mathcal{A}\mathbf{U}(t)\phi_0$ as $h \to 0$. The integrand in the second term, as shown in the proof of Theorem 7.1, is continuous, and its norm is dominated by $C_\epsilon(\|\phi_0\| + \|\mathcal{A}\phi_0\| + 2\epsilon)$. Therefore, the second term converges to

$\mathcal{J}(\phi(t))$. The integrand in the last term converges to $\mathcal{A}\mathbf{U}(t-s)\mathcal{J}(\phi(s))$ for each s and its norm is dominated by a integrable function $C_\epsilon(\|\phi_0\|+\|\mathcal{A}\phi_0\|+2\epsilon)\sigma(t-s)$, so, by Dominated Convergence Theorem, the last term converges to $\int_0^t \mathcal{A}\mathbf{U}(t-s)\phi(s)\,ds = \mathcal{A}\int_0^t \mathbf{U}(t-s)\phi(s)\,ds$. In conclusion, $\phi(t)$ is strongly differentiable for $t \in (0,T)$ and satisfies (7.1), i.e., a strong solution. This completes the proof. ■

Remark: It is not quite clear under what conditions an unbounded operator $\mathcal{A}$ itself is $\mathcal{A}$-regular. In fact, the remark after the proof of the next lemma shows that, for self-adjoint operators $\mathcal{A}$ in *Hilbert space, $\mathcal{A}$ is $\mathcal{A}$-regular is nearly equivalent to that $\mathcal{A}$ is bounded.*

COROLLARY 7.3. *Suppose that $\mathcal{X}$ is Hilbert space and $\mathcal{A}$ is self-adjoint with spectrum contained in $(-\infty,\alpha_0]$ for some finite α_0. If there is a ρ, $0 \le \rho < 1$, such that $\mathcal{J}(\cdot)$ is $|\mathcal{A}|^\rho$-bounded continuous, then equation (7.1) has uniquely an $|\mathcal{A}|^\rho$-type mild solution in $(0, T(\|\phi_0\|, \||\mathcal{A}|^\rho\phi_0\|))$ for any $\phi_0 \in \mathcal{D}(|\mathcal{A}|^\rho)$.*

PROOF: By the theorem, we only need to show that $\mathcal{A}$ is $|\mathcal{A}|^\rho$-regular. Since $\mathcal{A}$ is self-adjoint, $\mathcal{A}$ generates a spectral family $\{\mathcal{E}_\mu : -\infty < \mu < \infty\}$, and the operator $|\mathcal{A}|^\rho\mathbf{U}(t)$ can be represented as

$$|\mathcal{A}|^\rho\mathbf{U}(t) = \int_{-\infty}^{\infty} |\mu|^\rho e^{t\mu}\, d\mathcal{E}_\mu \tag{7.8}$$

in the strong topology sense. Formal computation shows that

$$\||\mathcal{A}|^\rho\mathbf{U}(t)\| \le \max\{|\alpha_0|^\rho e^{t\alpha_0}, \frac{\rho^\rho}{t^\rho}e^{-1}\}, \tag{7.8a}$$

and the right hand side above, if $0 \le \rho < 1$, is finite integrable on any finite interval, i.e., $\mathcal{A}$ is $|\mathcal{A}|^\rho$-regular for $0 \le \rho < 1$. ■

Remark: If the spectrum $\sigma(\alpha) \supset (-\infty,\alpha_1]$ for some constant α_1, then for $t \le |\alpha_1|$, it can be shown that (7.8a) takes equality in this case. Since $\int_0^T 1/t\,dt = \infty$, $\mathcal{A}$ can

not be $\mathcal{A}$-regular. Nevertheless, the equation (7.1) can have uniquely a strong solution if $\mathcal{J}(\cdot)$ satisfies some conditions. The reader is referred to the last chapter of [43] for some examples.

§7.2. The Fokker-Planck-Vlasov Equation

This part will consider the solution of the *Diffusive-Vlasov* equation and the *Fokker-Planck-Vlasov* equation. Since the arguments are almost the same, we only consider the Diffusive-Vlasov equation:

$$[\frac{\partial}{\partial t} + \vec{v}\cdot\nabla_{\vec{r}}]f(\vec{r},\vec{v},t) + \vec{E}(\vec{r},t)\cdot\nabla_{\vec{v}}f(\vec{r},\vec{v},t) - \lambda\Delta_{\vec{v}}f = 0, \tag{DV}$$

$$\vec{E}(\vec{r},t) = -\frac{1}{4\pi}\int d\vec{r}_1\nabla_{\vec{r}}(\frac{1}{|\vec{r}-\vec{r}_1|})\int f(\vec{r}_1,\vec{v},t)\,d\vec{v}, \tag{PV}$$

$$\lim_{t\to+0} f(\vec{r},\vec{v},t) = f_0(\vec{r},\vec{v}) \tag{IC}$$

in the space $M = X\cap L^1(R^3\times R^3) \equiv C(R^3_{\vec{r}}; L^1(R^3_{\vec{v}})\cap L^1(R^3\times R^3)$ defined in Chapter IV. Define operator $\mathcal{A}:\ \mathcal{D}(\mathcal{A})\to X$ and $\mathcal{J}(\cdot):\ \mathcal{D}(\mathcal{A})\to X$ by

$$\mathcal{A}\phi(\vec{r},\vec{v}) = -\vec{v}\cdot\nabla_{\vec{r}}\phi(\vec{r},\vec{v}) + \lambda\Delta_{\vec{v}}\phi(\vec{r},\vec{v}), \tag{7.8a}$$

$$\mathcal{J}(\phi)(\vec{r},\vec{v}) = -\frac{1}{4\pi}(\int\frac{\vec{r}-\vec{r}_1}{|\vec{r}-\vec{r}_1|^3}\,d\vec{r}_1\int\phi(\vec{r}_1,\vec{v}_1)\,d\vec{v}_1)\cdot\nabla_{\vec{v}}\phi(\vec{r},\vec{v}) \tag{7.8b}$$

for $\phi\in\mathcal{D}(\mathcal{A})$, where the domain of the operator $\mathcal{A}$ is given by

$$\mathcal{D}(\mathcal{A}) = \left\{\phi\in X:\ \left\{\begin{array}{l}\phi(\vec{r},\vec{v}) \text{ is abs. cont., } \lim\limits_{|\vec{r}|+|\vec{v}|\to\infty}\phi(\vec{r},\vec{v}) = 0\\ \vec{v}\cdot\nabla_{\vec{r}}\phi(\vec{r},\vec{v})\in X,\ |\nabla_{\vec{v}}\phi|\in X,\ \lim\limits_{|\vec{r}|+|\vec{v}|\to\infty}|\nabla_{\vec{v}}\phi| = 0\\ |\nabla_{\vec{v}}\phi(\vec{r},\vec{v})| \text{ is abs. cont., } \Delta_{\vec{v}}\in X\end{array}\right.\right\} \tag{7.8c}$$

We call that the function $f(\vec{r},\vec{v},t)$ is a $\nabla_{\vec{v}}$-type mild solution (of (DV)-(PV)-(IC)) if f is the solution of the integral equation

$$f(\vec{r},\vec{v},t) = \iint p(t,\vec{r},\vec{v};\vec{q},\vec{u})f_0(\vec{u},\vec{q})\,d\vec{u}d\vec{q}$$

$$-\int_0^t \iint p(t-s,\vec{r},\vec{v};\vec{q},\vec{u})\vec{E}(\vec{q},s)\cdot\nabla_{\vec{u}}f(\vec{q},\vec{u},s)\,d\vec{q}d\vec{u} \tag{7.9}$$

with $\vec{E}$ satisfying (PV), and $\frac{\partial}{\partial v_i}f(\vec{r},\vec{v},t)$ is a continuous X-valued function of t for $i=1,2,3$. Define a *vector* operator $\vec{\mathcal{B}}:\mathcal{D}(\vec{\mathcal{B}})\to X\times X\times X$ by

$$\vec{\mathcal{B}}o(\vec{r},\vec{v})=\nabla_{\vec{v}}o(\vec{r},\vec{v}), \tag{10}$$

where

$$\mathcal{D}(\vec{\mathcal{B}})=\{\phi\in X:\ \lim_{|\vec{r}|+|\vec{v}|\to 0}\phi(\vec{r},\vec{v})=0,\ |\nabla_{\vec{v}}\phi|\in X\}.$$

Let $X^3=X\times X\times X$ with the norm $\|\vec{\phi}\|_{X^3}=(\|\phi_1\|_X^2+\|\phi_2\|_X^2+\|\phi_3\|_X^3)^{1/2}$. Then X^3 is a Banach space, and for any $t>0$, by an argument similar to the proof of Theorem 4.3 (cf. (4.12)), we know that

$$\begin{aligned}
&\mathcal{B}\mathbf{U}(t)\phi(\vec{r},\vec{v})=\\
&=\iint_{R^3\times R^3} d\vec{q}d\vec{u}\,\frac{3\sqrt{3}}{8}\Big(\frac{1}{\lambda\pi t^2}\Big)^3 o(\vec{q},\vec{u})\Big[\frac{3(\vec{r}-\vec{q})}{\lambda t^2}-\frac{(\vec{v}-\vec{u})}{2\lambda t}-\frac{3(\vec{v}+\vec{u})}{2\lambda t}\Big]\times\\
&\quad\times\exp\Big[-\frac{1}{\lambda t}\Big(\frac{1}{4}(\vec{v}-\vec{u})^2+\frac{3}{4}(\vec{v}+\vec{u})^2-\frac{3(\vec{r}-\vec{q})\cdot(\vec{v}+\vec{u})}{t}+\frac{3(\vec{r}-\vec{q})^2}{t^2}\Big)\Big]\\
&=\iint_{R^3\times R^3} d\vec{q}d\vec{u}\,\vec{K}(\vec{r},\vec{q};\vec{v},\vec{u};t)o(\vec{q},\vec{u}),
\end{aligned}\tag{7.11}$$

and

$$\|\mathcal{B}\mathbf{U}(t)\|_{X\to X^3}\le\frac{\sqrt{3}+1}{2}\sqrt{\frac{t}{\lambda\pi}}. \tag{7.12}$$

Next, by Lemma 4.4 in Chapter 4, we know that $\vec{E}\in C(R^3_{\vec{r}})\times C(R^3_{\vec{r}})\times C(R^3_{\vec{r}})$ if $\phi\in X$, and $\|\vec{E}\|_{L^\infty}\le C\|\phi\|_X$ for some constant C. Therefore, $\mathcal{J}(\cdot)$ is $\vec{\mathcal{B}}$-bounded continuous in the following sense:

$$\|\mathcal{J}(\phi)\|_X\le C\|\phi\|_X\|\vec{\mathcal{B}}o\|_{X^3},$$

$$\|\mathcal{J}(\phi)-\mathcal{J}(\psi)\|_X\le C(\|o\|_X,\|\vec{\mathcal{B}}\psi\|)(\|\phi-\psi\|_X+\|\vec{\mathcal{B}}\phi-\vec{\mathcal{B}}\psi\|_{X^3}),$$

for all $\phi,\ \psi\in\mathcal{D}(\mathcal{B})$. Thus, using Theorem 7.1 on each component of $\vec{\mathcal{B}}\phi$, we have

THEOREM 7.4. (Local Existence) *For each $f_0 \in \mathcal{D}(\vec{\mathcal{B}})$, there exists a T such that the Diffusive-Vlasov equation (DV)-(PV)-(IC) has a unique mild solution $f(t)$ for $0 \le t < T$. Furthermore, $\frac{\partial}{\partial v_i} f(t)$ is a continuous X-valued function in the interval. The conclusion is also true for the Fokker-Planck-Vlasov Equation.*

THEOREM 7.5. (Global Existence) *For any $T > 0$, the equation (DV)-(PV)-(IC) has uniquely an X-valued mild solution $f(t)$ for $0 \le t \le T$. Furthermore, the X-valued function $\frac{\partial}{\partial v_i}$, $(i = 1,\ 2,\ 3,)$ is continuous for $0 \le t \le T$ if $f_0 \in \mathcal{D}(\mathcal{B})$.*

PROOF: The following proof is essentially based on the proof of Theorem 4.2. Consider the solution of the integral equation (7.2) with operator $\mathcal{A}$ defined by (7.8a)-(7.8c) and operator $\mathcal{J}(\cdot)$ given by (7.8b). We can rewrite the integral equation as

$$
\begin{aligned}
&f(\vec{r},\vec{v},t) = \mathbf{U}(t)f_0 + \\
&+ \int_0^t ds \iint\limits_{R^3\times R^3} \frac{3\sqrt{3}}{8}\Big(\frac{1}{\lambda\pi(t-s)^2}\Big)^3 f(\vec{q},\vec{u},s)\Big[\frac{3(\vec{r}-\vec{q})}{\lambda(t-s)^2} - \frac{(\vec{v}-\vec{u})}{2\lambda(t-s)} - \frac{3(\vec{v}+\vec{u})}{2\lambda(t-s)}\Big]\cdot \vec{E}(\vec{q},\vec{u}) \\
&\quad \times \exp\Big[-\frac{1}{\lambda(t-s)}\Big(\frac{1}{4}(\vec{v}-\vec{u})^2 + \frac{3}{4}(\vec{v}+\vec{u})^2 - \frac{3(\vec{r}-\vec{q})\cdot(\vec{v}+\vec{u})}{(t-s)} + \frac{3(\vec{r}-\vec{q})^2}{(t-s)^2}\Big)\Big]\, d\vec{q}d\vec{u}.
\end{aligned}
$$

By using Lemma 4.4 and the expression for $\mathbf{U}(t)\mathcal{J}(f)$ above, we can show that, in addition to satisfying (R1)-(R2)-(R3) and (B1)-(B2) relating to the operator $\mathcal{B}$, the operators $\mathcal{A}$ and $\mathcal{J}$ also satisfy

(A1) $\|\mathbf{U}(t)\mathcal{J}(\phi)\| \le C(\|\phi\|)\sigma(t)\|\phi\|$;

(A2) $\|\mathbf{U}(t)\mathcal{J}(\phi) - \mathbf{U}(t)\mathcal{J}(\psi)\| \le C(\|\phi\|,\|\psi\|)\sigma(t)\|\phi-\psi\|$;

(A3) $\int_0^t \sigma(\tau)\, d\tau = p(t) < \infty$.

The same proof as in Theorem 7.1 shows that, under condition (A1)-(A2)-(A3), the equation (7.1) has uniquely a local (mild) solution, and can be extended globally if one can bound that the norm $\|f(t)\|$ of the solution $f(t)$ is upper bounded on any (finite) existence interval (0, T). Suppose that the equation has a mild solution $f(t)$

for $0 \le t \le T_1$. Let

$$\vec{E}_1(\vec{r},t) = -\frac{1}{4\pi}\int d\vec{r}_1\, \nabla_{\vec{r}}(\frac{1}{|\vec{r}-\vec{r}_1|})\int f(\vec{r}_1,\vec{v}.t)\, d\vec{v}, \tag{7.13}$$

which at least is well-defined on $(0,T_1)$. Consider the solution of

$$\begin{cases} \frac{\partial}{\partial t}f(\vec{r},\vec{v},t) = -\vec{v}\cdot\nabla_{\vec{r}}f + \lambda\Delta_{\vec{v}}f - \vec{E}_1\cdot\nabla_{\vec{v}}f, \\ \lim\limits_{t\to 0}\|f(t)-f_0\|_X = 0. \end{cases} \tag{7.14}$$

Since $f(\vec{r},\vec{v},t)$ is also the solution of the integral equation (7.9), it can be shown that $\int f(\vec{r},\vec{v},t)\, d\vec{v}$ is continuous in $\vec{r}$, and therefore $\vec{E}_1(\vec{r},t)$ is continuous in $\vec{r}$. For each given $t \in (0,T)$, the operator $\mathcal{E}_1(t)$: $(\mathcal{E}_1(t)f)(\vec{r},\vec{v},t) = -\vec{E}_1(\vec{r},t)\cdot\nabla_{\vec{v}}f(\vec{r},\vec{v})$ generates a contraction semigroup. By Theorem 4.2, the operator $\mathcal{A}+\mathcal{E}_1(t)$ defined via the right hand side of (7.14) generates a contractive (two parameters) system of evolution operators $\mathbf{U}_{\mathcal{C}}(t,s)$ for $0 \le s \le t \le T$ and the equation (7.14) has uniquely a solution $\tilde{f}(\vec{r},\vec{v},t)$ with $\|\tilde{f}\|_X \le \|f_0\|_X$. On the other hand, since $\vec{E}_1$ is given by (7.13) via the mild solution (DV)-(PV)-(IC), it can be shown that $f(\vec{r},\vec{v},t) = \tilde{f}(\vec{r},\vec{v},t)$ and $\|f(\vec{r},\vec{v},t)\|_X \le \|f_0\|_X$. That is, the Diffusive-Vlasov equation has globally a unique mild solution. The second half of the theorem can be proved by using the explicit expression of the integral equation and the existence of the mild solution. This completes the proof. ■

§7.3 The Solution of (DVE) and (FPVE)

Now we use the theory in §7.1 and §7.2 to get existence of a unique $\nabla_{\vec{v}}$-type mild solution of the Diffusive-Vlasov-Enskog equation (DVE) and a unique $\nabla_{\vec{v}}$-type mild solution of Fokker-Planck-Vlasov-Enskog Equation (FPVE). From Theorem 7.1 and Theorem 7.4 we have

THEOREM 7.5. *Suppose that in the Enskog collision term, the geometrical factor* $Y = Y(\vec{r},\vec{v},\vec{r}\pm a\vec{\epsilon},\vec{v}_1|f)$ *is so chosen such that the Enskog collision operator* C_E

is a bounded continuous operator in X. Then, for any $f_0 \in \mathcal{D}(\mathcal{B})$, there is a $T = T(\|f_0\|, \|cbf_0\|)$ so that the equation (DVE)-(PV)-(IC) (or the equation (FPVE)-(PV)-(IC)) has uniquely a mild solution $f(t)$ for $0 \leq t \leq T$ with continuous X-valued derivative $\frac{\partial}{\partial v_i}$, $(i = 1, 2, 3)$.

In the following we give some examples of the existence of mild solutions as well as the existence of $\nabla_{\vec{v}}$-type mild solution.

EXAMPLE 1: (The cut-off model) Suppose that $Y = \tilde{Y}(n(\vec{r}, t), n(\vec{r} + a\vec{\epsilon}, t))\eta_B(\vec{v}, \vec{v}_1)$ with $\tilde{Y}$ satisfying (A2)-(A3)-(A4) given in Chapter IV. Then we know that the collision operator C_E is bounded continuous in this case. By Theorem 7.5, the equation has uniquely a $\nabla_{\vec{v}}$-type solution for small time t. In fact, Theorem 4.3 has shown the global existence of the strong solution in this case.

EXAMPLE 2: (The decreasing velocity model) In this case, we assume that $Y = Y(\vec{r}, \vec{v}, \vec{r}_1, \vec{v}_1)$ is a continuous function of its arguments, and bounded by a decreasing velocity function: $Y(\vec{r}, \vec{v}; \vec{r}_1, \vec{v}_1) \leq M \frac{1}{1+|\vec{v}-\vec{v}_1|^\gamma}$ with constant M and positive $\gamma > 0$. By the definition of the space X, we know that C_E is bounded continuous , satisfying (B1) and (B2) in this case. According to Theorem 7.5, the (DVE) or (FPVE) has uniquely a $\nabla_{\vec{v}}$-type solution for small t. In fact, one can show that the equation has uniquely a global mild solution in this case. The proof is almost the same as Theorem 7.4 and we omit the details.

EXAMPLE 3: (Maxwell Distribution Model) In this case, we assume that the geometrical factor Y is bounded by the Maxwell distribution: $Y(\vec{r}, \vec{v}, \vec{r}_1, \vec{v}_1) \leq M \exp\{-\gamma|\vec{v} - \vec{v}_1|^2\}$ with constants $M, \gamma > 0$. It can be show that in this case Y also satisfies the velocity decreasing model. Therefore, the equation has a unique local $\nabla_{\vec{v}}$-type mild solution, and a unique global mild solution.

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12. Peng Lei, and W. Greenberg , The Weak Solution for the Diffusive-Fokker-Planck Enskog Equation, *submitted for publication.*

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