Dynamic Compensators for A Nonlinear Conservation Law

by

Hamadi Marrekchi

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APPROVED:

John A. Burns, Chairman

Eugene M. Cliff

Janet S. Peterson

Terry L. Herdman

Robert L. Wheeler

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(ABSTRACT)

In this paper we consider the problem of designing dynamic compensators to control a class of nonlinear parabolic distributed parameter systems. We concentrate on a system with unbounded input and output operators governed by Burgers' equation. This equation provide a one dimensional model for certain convection–diffusion phenomena. A linearized model is used to compute a robust controller (MinMax), a LQG controller and a fixed-order-finite-dimensional control law (Optimal Projection) by minimizing various energy functionals. These control laws are then applied to the nonlinear model. Different approximation schemes are used to design suboptimal active feedback controllers. This approach provides important practical information. In particular, we show how functional gains can be used to locate new sensors.

Numerical results are given to illustrate the basic ideas and to compare the various controllers.

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TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

INTRODUCTION

1.1 Review of Literature

In recent years considerable attention has been devoted to the problem of using feedback to control fluid dynamic systems. This problem is complex and particularly difficult when one is faced with phenomena such as shocks. Moreover, these systems are governed by nonlinear partial differential equations so that the natural state of the system is infinite dimensional. If one assumes that "full state feedback" is necessary to design practical controllers, then one would conclude that feedback control of fluid dynamic systems is "not practical". However, it is well known that even in finite dimensional control systems one rarely has the ability to accurately sense all states, so that some form of dynamic compensation must be used.

This idea clearly extends to infinite dimensional problems and there is a growing literature on observers/compensators for distributed parameter systems. In this paper we consider a boundary control problem governed by Burgers' equation. We selected this problem because Burgers' equation is an infinite dimensional model that captures some phenomena (e.g., shocks) often observed in fluid flows and because it is simple enough to provide real insight into the problem. The goal is to show that it is possible to use modern control theory to produce practical dynamic compensators for boundary control of nonlinear partial differential equations of the type that occur naturally in fluid dynamics.

We shall present short summaries of three approaches (LQG, MinMax and the optimal projection method) and show how these approaches can be used in conjunction with standard numerical schemes to produce realizable controllers.

In the infinite-dimensional system theory literature, the attention was primarily focussed on the standard linear quadratic Gaussian (LQG) problem. In [13], Gibson and Adamian presented an approximation theory for the LQG optimal control problem for flexible structure whose distributed models have bounded input and output operators.

As noted above it is almost impossible to observe the whole state. Controls and sensors are limited to a few points or segments of the boundary, so it is necessary to construct an appropriate observer (estimator) of the state and design a feedback control law (called a compensator) based on the information available from the observed (estimated) state variable. Boundary control and observation often lead to unbounded input and output operators. Stabilization by dynamic feedback (or compensation) has been considered by Curtain [7], Fujii [12], and Nambu [25] for classes of parabolic as well as hyperbolic systems, including control and observation at the boundary. An advance was made by Pritchard and Salamon [27], when they introduced an interesting class of systems in order to solve the linear quadratic regulator problem for several infinite-dimensional systems with unbounded input and output operators. This class of systems is now commonly referred to as the

Pritchard–Salamon class. The reason for its popularity is that it is an extension of the "bounded class" including many "unbounded" examples, such that most of the system theoretic properties of the bounded class are retained. In particular, its structure is rich enough to solve the linear quadratic Gaussian problem and the H^∞-control problem.

State-space results for the H^∞–control problem for infinite–dimensional systems were obtained by Pritchard and Townley [28], Van Keulen et al. [19] and McMillan and Triggiani [23, 24].

Pritchard and Townley [28] were the first to consider the infinite–dimensional H^∞-control problem with state feedback. Van Keulen [19] and McMillan and Triggiani [23, 24] also restrict their work to the state feedback only. Paper [19] treats the bounded input/output case and in [23, 24] boundary control systems are considered.

The measurement–feedback (or compensator) problem is solved by Van Keulen [16, 17, 18] and Curtain [9] for several classes of infinite–dimensional systems. Van Keulen [17, 18] and Curtain [9] solved the problem for semigroup control systems with bounded input and output operators, whereas Van Keulen [16] solved the problem for the Pritchard–Salamon class using the procedures that were developed in [17, 18, 19].

All of these approaches produce stabilization schemes that either have the same finite order as that of a high-order approximate model, or alternatively, open-loop model reduction or closed-loop control reduction techniques are applied to achieve a lower-order compensator. An advance was made by Schumacher [29], when he gave a theory for designing finite-dimensional compensators for a large class of systems, including parabolic and delay systems. However, in his theory it was assumed that the control and observation operators are bounded. Curtain [6] presented an alternative compensator design which applied to the same class of systems, except that unbounded inputs and outputs were allowed.

The optimal projection method is one of many approaches to this problem. We shall make use of this method because a very nice theory has already been developed (for bounded input and output operators) and we are more interested in illustrating that recent results in distributed parameter control theory can be used to design practical feedback laws, than in discussing the "best" approach to the problem.

The possibility of applying this approach to distributed parameter systems was first suggested by Johnson [14] and Pearson [26]. The idea of fixing the order of the finite-dimensional compensator, while retaining the distributed parameter model was expanded and developed by Bernstein and Hyland in [3] and Rosen in [4]. The method extends the full order LQG case to an "optimal fixed-finite-order compensator" characterized by four equations; two modified Riccati equations and two modified Lyapunov equations, coupled by an oblique projection whose rank is precisely equal to the order of the compensator. Bernstein and Hyland assumed that the control and observation operators were bounded and hence boundary control and observations were not covered by their theory. Thus, one naturally is led to the problem of obtaining a fixed–finite–dimensional compensator for infinite–dimensional systems with unbounded input and output operators. We show that, in certain cases, the Bernstein/Hyland theory can be extended to unbounded control and observation.

1.2 Summary of the Main Results

The main results obtained in this thesis may be summarized as follows:

- Theorem 2.3.3 provides the basic theory for MinMax problems of Pritchard–Salamon class. This result is a generalization of the results given in [16, 23, 24]. In particular, the structure of the proof emerges from a combination of ideas in [16] and [10] where we show that under certain conditions the existence of a dynamic compensator that solves the MinMax problem is equivalent to the solvability of two coupled Riccati equations.

- Theorem 2.4.1 is an extension of Bernstein/Hyland theory to systems of Pritchard–Salamon class.

- We present a comprehensive computational study of all three approaches (LQG, MinMax, Optimal Projection). In particular, we apply these methods to the nonlinear Burgers' equation, examine the performance of each method and investigate the affects of various approximations to the optimal design.

1.3 Notation

If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed linear spaces, then $\mathcal{L}(X,Y)$ will denote the space of all bounded linear operators from X to Y. For any $A \in \mathcal{L}(X,Y)$, $\|A\|$ or $\|A\|_{\mathcal{L}(X,Y)}$ will denote the operator norm on the space $\mathcal{L}(X,Y)$. In the event that $X = Y$ we denote $\mathcal{L}(X,Y)$ by $\mathcal{L}(X)$. From time to time we will use $\|\cdot\|$ without any subindex for vector or operator norm. In all cases the appropriate index for $\|\cdot\|$ will be understood from the context. For a Hilbert space X, we denote the inner product on $X \times X$ by $< \cdot, \cdot >_X$. Given a linear operator A from X into itself, we denote its domain, spectrum, resolvent and adjoint by $\mathcal{D}(A)$, $\sigma(A)$, $\rho(A)$ and A^*, respectively. For real numbers a and b with $a < b$, $L^p(a, b; X)$, $1 < p < \infty$, will be the space of all Lebesgue measurable functions f from (a, b) to X such that $\|f\|_{L^p(a,b)} = \left(\int_a^b |f(x)|^p \, dx\right)^{\frac{1}{p}} < \infty$. The space $H^k(a, b)$ is the standard Sobolov space defined by $H^k(a, b) = \left\{f \in L^2(a, b) | f^{(j)} \in L^2(a, b);\ j = 0, 1, \cdots, k\right\}$.

Chapter 2

COMPENSATION: LQG, MINMAX AND OPTIMAL PROJECTION THEORIES

In this chapter, we present a summary of three approaches that will be used in designing a compensator for Burgers' equation with Neumann boundary control and Dirichlet boundary observation. Moreover, we will explore the possibility of obtaining a closed–loop system with an exponential decay rate independent of the viscosity parameter $\epsilon > 0$.

In Section 2.1 an abstract framework for a distributed parameter linear quadratic-Gaussian problem is given. The results in Section 2.1 will be applied to the Burgers' equation in Section 2.2. A short summary of the linear quadratic differential game theory will be presented in Section 2.3. Finally, a third approach based on the optimal projection method due to Bernstein and Hyland will be given in Section 2.4.

2.1 Distributed Parameter Linear Quadratic-Gaussian Problems

We consider the following abstract Cauchy problem

$$\dot{z}(t) = Az(t) + Bu(t) + H_1\eta(t) \qquad z(0) = z_o \in H \tag{2.1.1}$$

$$y(t) = Cz(t) + H_2\eta(t) \tag{2.1.2}$$

$$w(t) = E_1z(t) + E_2u(t) \tag{2.1.3}$$

where H is a Hilbert space, $u(\cdot) \in L^2(0\ T; U)$, $y(\cdot) \in L^2(0\ T; Y)$, and $w(\cdot) \in L^2(0\ T; \tilde{H})$ denote respectively, the input control, the measured output and the controlled output. In addition, assume that the state and measurements are corrupted by zero-mean Gaussian white noise $\eta(t)$ in the Hilbert space $\aleph$, $H_1 \in \mathcal{L}(\aleph,\ H)$ and $H_2 \in \mathcal{L}(\aleph,\ Y)$. Moreover, we assume that the disturbance and measurements are independent, i.e., $H_1\, H_2^* = 0$ and that $V_1 = H_1H_1^* \in \mathcal{L}(H)$ and $V_2 = H_2\, H_2^* \in \mathcal{L}(Y)$ is positive definite. Also, it is assumed that there is no cross weighting between the state and control input, i.e., $E_2^*E_1 = 0$ and that $R_1 = E_1^*E_1 \in \mathcal{L}(H)$, $R_2 = E_2^*E_2 \in \mathcal{L}(U)$ are positive definite and that $E_1 \in \mathcal{L}(H, \tilde{H})$ and $E_2 \in \mathcal{L}(U, \tilde{H})$.

We assume the operator A is the infinitesimal generator of analytic semigroup $S(t)$ on H, generally unstable, with exponential growth rate

$$w_0 = \lim_{t\to\infty} t^{-1} \log \|S(t)\|_{\mathcal{L}(H)} > 0, \tag{2.1.4}$$

so that

$$\|S(t)\|_{\mathcal{L}(H)} \le Me^{(w_0+\epsilon)t} \quad \text{for all} \quad \epsilon > 0,\ t \ge 0 \tag{2.1.5}$$

for some constant $M = M(w_0, \epsilon) \geq 1$. Throughout the remainder of this paper we let $\bar{A}$ denote the translation $\bar{A} = -A + wI$, where w is fixed and $w > w_0$, so that $\bar{A}$ has well-defined fractional powers $(\bar{A})^\mu$ on H and $-\bar{A}$ is the generator of a strongly continuous analytic contraction semigroup $\bar{S}(t)$ on H satisfying

$$\|\bar{S}(t)\|_{\mathcal{L}(H)} \leq \bar{M}e^{-\bar{w}t}, \quad t \geq 0 \qquad \bar{w} = w - w_0 > 0. \tag{2.1.6}$$

In order to allow for unbounded operators B and C, we assume that B $\in \mathcal{L}(U, Z)$ and C $\in \mathcal{L}(W, Y)$, where W and Z are also Hilbert spaces such that

$$\mathcal{D}(A) \subseteq W \hookrightarrow H \hookrightarrow Z \tag{2.1.7}$$

are continuous dense injections. More precisely, we assume that B^* is $[\bar{A}^*]^\gamma$-bounded, or equivalently,

(H-1-a) $$[\bar{A}]^{-\gamma}B \in \mathcal{L}(U, H) \qquad \text{for some} \quad 0 \leq \gamma < 1. \tag{2.1.8}$$

Similarly, for the operator C we assume that

(H-1-b) $$C[\bar{A}]^{-r} \in \mathcal{L}(H, Y) \qquad \text{for some} \quad 0 \leq r < 1. \tag{2.1.9}$$

Solutions of (2.1.1) are given by the variations of parameter formula. In particular, the solution $z(t)$ is given by

$$z(t) = S(t)z_o + \int_0^t S(t-s)Bu(s)ds, \quad 0 \leq t \leq T \tag{2.1.10}$$

and formally the output by

$$y(t) = CS(t)z_0 + C\int_0^t S(t-s)Bu(s)ds, \tag{2.1.11}$$

which will be discussed later in more details. Equation (2.1.11) is consistent in the sense that the restriction of $S(t)$ to W and its extension to Z defines an analytic semigroup on W and Z respectively, so we will use the one symbol for all three $S(t)$ and its generator, A. Also, we suppose that $B \in \mathcal{L}(U, Z)$ and $C \in \mathcal{L}(W, Y)$ induce a smooth controllability and observability map with respect to W and Z on $[0,\ T]$ for all finite T, respectively i.e.,

(H-2) For each $T > 0$, there exists a constant $b(T) > 0$ such that for $u \in L^2(0, T; U)$, $\int_0^T S(T-s)Bu(s)ds \in W$ and

$$\left\| \int_0^T S(T-s)Bu(s)ds \right\|_W \leq b(T)\|u(\cdot)\|_{L^2(0\ T;U)}. \tag{2.1.12}$$

(H-3) For each $T > 0$, there exists a constant $c(T) > 0$ such that for every $x \in W$

$$\int_0^T \|CS(t)x\|_Y dt \leq c(T)\|x\|_Z. \tag{2.1.13}$$

Remark 2.1.1 (i) We remark that (H-2) and (H-3) are dual conditions in the sense that B satisfies (H-2) condition with respect to $S(t)$ if and only if B^* satisfies (H-3) condition with respect to $S^*(t)$.

(ii) Hypothesis (H-2) implies that $z(\cdot)$ is a continuous function on $[0\ ,\ T]$ with values in W and the function $y(\cdot)$ defined by equation (2.1.2) is well defined and continuous on $[0, T]$ with values in Y. However, if $z_o \in Z$, then $z(\cdot)$ is a continuous function with values in Z. In this case the output function $y(\cdot)$ will make sense only if hypothesis (H-3) is satisfied. In particular for any $z_o \in Z$, we define $y(\cdot)$ in (2.1.2) as the continuous extension to Z of the mapping $z \in W \to y(\cdot) \in L^2(0,T;Y)$.

We consider the steady-state performance index

$$\begin{aligned} J(u) &= \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{E}\left[\|w(s)\|_{\tilde{H}}^2\right] ds \\ &= \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{E}\left[< R_1 z(s), z(s) > + < R_2 u(s), u(s) > \right] ds, \end{aligned} \tag{2.1.14}$$

where $R_2 \in \mathcal{L}(U)$ satisfies the inequality $< R_2 u, u >_U \geq d_1 \|u\|_U^2$ for some $d_1 > 0$ and for every $u \in U$ and $\mathbf{E}[\cdot]$ is the expectation function.

The linear quadratic-Gaussian problem is:

(LQG) To design an infinite-dimensional compensator of the form

$$\dot{z}_c(t) = A_c z_c(t) + B_c y(t) \qquad z_c(0) = z_{c_o} \tag{2.1.15}$$

$$u(t) = C_c z_c(t) \tag{2.1.16}$$

for which system (2.1.1)–(2.1.2) can be stabilized by the input control $u(\cdot) \in L^2(0,\infty;U)$ that minimizes the cost functional given by equation (2.1.14).

In order to obtain an admissible control u and an infinite-dimensional compensator of the form (2.1.15)–(2.1.16) such that $J(u) < \infty$ and that the closed-loop system in $\mathcal{H} = H \times H$ given by

$$\frac{\mathrm{d}}{\mathrm{dt}} \begin{bmatrix} z \\ z_c \end{bmatrix} = \mathcal{A} \begin{bmatrix} z \\ z_c \end{bmatrix} \quad \text{where} \quad \mathcal{A} = \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix} \tag{2.1.17}$$

is exponentially stable, we need the following hypotheses (H-4)-(H-5):

(H-4) Stabilizability Condition (S.C.)

There exists an operator $K \in \mathcal{L}(H, U)$ such that $A_K = A + BK$ generates an analytic semigroup $S_K(t) = e^{(A+BK)t}$ and $S_K(t)$ is exponentially stable on H, i.e.,

$$\|S_K(t)\|_{\mathcal{L}(H)} \leq M_K e^{-w_K t}, \qquad \text{for } \; w_K > 0. \tag{2.1.18}$$

(H-5) Detectability Condition (D.C.)

There exists an operator $G \in \mathcal{L}(Y, H)$ such that $A_G = A - GC$ generates an analytic semigroup $S_G(t) = e^{(A-GC)t}$ and $S_G(t)$ is exponentially stable on H, i.e.,

$$\|S_G(t)\|_{\mathcal{L}(H)} \leq M_G e^{-w_G t}, \qquad \text{for } \; w_G > 0. \tag{2.1.19}$$

In fact, if (H-1)-(H-5) hold, then there exists an infinite-dimensional compensator for system (2.1.1) in the form

$$\begin{aligned} \frac{d}{dt} z_c(t) &= (A + BK - GC) z_c(t) + Gy(t) \\ u(t) &= K z_c(t) \end{aligned} \tag{2.1.20}$$

which leads to a stable closed-loop system in H. Indeed, if we define $e = z - z_c$, then (2.1.1) and (2.1.20) can be written as

$$\frac{d}{dt} z(t) = (A + BK) z(t) - BKe(t) \tag{2.1.21}$$

$$\frac{d}{dt} e(t) = (A - GC) e(t) \tag{2.1.22}$$

Thus, hypotheses (H-4)-(H-5) imply the existence of a dynamic compensator and moreover the semigroup $\mathcal{S}(t)$ generated by $\mathcal{A}$ is exponentially stable in $\mathcal{H}$, i.e., there exist a constant

$\mathcal{M} \geq 1$ and a constant $w_c > 0$ such that

$$\|\mathcal{S}(t)\|_{\mathcal{L}(\mathcal{H})} \leq \mathcal{M}e^{-w_c t} \qquad t \geq 0 \tag{2.1.23}$$

where $w_c < \min(w_K, w_G)$. We will construct a stabilizing feedback and observer gain operators $K \in \mathcal{L}(H, U)$ and $G \in \mathcal{L}(Y, H)$ respectively, by solving the linear quadratic-Gaussian problem. In order to guarantee unique solvability, we assume the following stabilizability/detectability conditions

$$\textbf{(C-1)} \qquad (A, E_1) \text{ is detectable} \qquad (A, H_1) \text{ is stabilizable} \tag{2.1.24}$$

Next, we state the main theorem of this section (see [13] and [16]):

Theorem 2.1.2 Let (H-1)-(H-5) and (C-1) be satisfied. Then there is a unique optimal dynamic compensator $z_c(\cdot)$ for the linear quadratic-Gaussian (LQG) problem, satisfying equations (2.1.1) and (2.1.15) where the unique optimal control $u(\cdot) \in L^2(0\ T; U)$ is given by the feedback law

$$u(t) = -R_2^{-1}B^*Pz_c(t) = -Kz_c(t), \qquad t \geq 0 \tag{2.1.25}$$

and the unique optimal observer operator $G \in \mathcal{L}(H)$ is given by

$$G = QC^*V_2^{-1} \tag{2.1.26}$$

where P and $Q \in \mathcal{L}(H)$ are the unique nonnegative self-adjoint solutions of the following Algebraic Riccati Equations (ARE):

$$
\begin{aligned}
0 &= < Px, Ay >_H + < Ax, Py >_H + < R_1 x, y >_H - \\
&\quad < PBR_2^{-1}B^*Px, y >_H \qquad \forall (x,y) \in \mathcal{D}(A) \qquad (2.1.27) \\
0 &= < Qx, A^*y >_H + < A^*x, Qy >_H + < V_1 x, y >_H - \\
&\quad < QC^*V_2^{-1}CQx, y >_H \qquad \forall (x,y) \in \mathcal{D}(A^*) \qquad (2.1.28)
\end{aligned}
$$

Moreover, the closed-loop semigroup $\mathcal{S}(t) \in \mathcal{L}(\mathcal{H})$, generated by $\mathcal{A}$, decays exponentially.

Next, if we let $C_c = -K$ and $B_c = G$ given by equations (2.1.25) and (2.1.26) respectively, then we obtain the following well-posedness result for the connected system (2.1.1) and (2.1.15). This result and proof may be found in [6].

Proposition 2.1.3 Let (H-1)-(H-3) be satisfied, then for all z_o, $z_{c_o} \in W$ there exists a unique solution pair $z(t)$ and $z_c(t)$ of (2.1.1) and (2.1.15). This means that $z(t)$ is continuous in H and absolutely continuous in Z, that (2.1.1) is satisfied for almost every $t \geq 0$ where $u(t)$ is given by (2.1.16), and that $z_c(t) \in H$ is continuously differentiable and satisfies (2.1.15) where $y(t)$ is given by (2.1.2).

To apply the above result to a nonlinear problem such as Burgers' equation, we must linearize the problem. In particular, we start with a nonlinear system (with $f(0) = 0$)

$$
\begin{aligned}
\frac{d}{dt} z(t) &= Az(t) + f(z(t)) + Bu(t) \\
y(t) &= Cz(t) \qquad (2.1.29) \\
z(0) &= z_o
\end{aligned}
$$

and construct the linearized system

$$\begin{aligned}\frac{d}{dt}z(t) &= Az(t) + Bu(t)\\ y(t) &= Cz(t) \qquad (2.1.30)\\ z(0) &= z_o.\end{aligned}$$

We then construct a linear controller of the form (2.1.15)–(2.1.16) for (2.1.30). The system (2.1.15)–(2.1.16) is "extended" to the nonlinear compensator

$$\begin{aligned}\frac{d}{dt}z_c(t) &= A_c z_c(t) + f(z_c(t)) + B_c y(t)\\ u(t) &= C_c z_c(t)\end{aligned} \qquad (2.1.31)$$

yielding a non-linear controller (linear feedback with non-linear observer).

2.2 Applications to Burgers' Equation

We consider the following Burgers' equation, with Neumann boundary control and Dirichlet boundary observation on a finite interval $[0, \ell]$ given by

$$\frac{\partial}{\partial t}z(x,t) = \epsilon\frac{\partial^2}{\partial x^2}z(x,t) - z(x,t)\frac{\partial}{\partial x}z(x,t), \qquad x \in (0,\ell) \;\; t > 0 \qquad (2.2.1)$$

$$u_1(t) = -\frac{\partial}{\partial x}z(0,t), \qquad (2.2.2)$$

$$u_2(t) = \frac{\partial}{\partial x}z(\ell,t), \qquad (2.2.3)$$

$$z(x,0) = z_o(x), \qquad (2.2.4)$$

and we observe on the boundary

$$y(t) = \begin{bmatrix} y_1(t) \\ \\ y_2(t) \end{bmatrix} = \begin{bmatrix} z(0,t) \\ \\ z(\ell,t) \end{bmatrix}, \tag{2.2.5}$$

where $\epsilon = \frac{1}{\mathrm{Re}} > 0$, and Re is the Reynolds number. Initially, we will consider the linearized system

$$\frac{\partial}{\partial t}z(x,t) \quad = \quad \epsilon\frac{\partial^2}{\partial x^2}z(x,t), \qquad x \in (0,\ell) \;\; t \geq 0 \tag{2.2.6}$$

with boundary control and boundary observation as given in (2.2.2)–(2.2.3) and (2.2.5), respectively.

In order to put the problem (2.2.3)–(2.2.4) and (2.26) into the abstract setting of the proceeding section, we introduce the following operators and spaces. We let $z(t)(\cdot) = z(\cdot,t)$ and $H = L^2(0,\ell)$. We define the operator A_ϵ on $H = L^2(0,\ell)$ by

$$A_\epsilon\phi = \epsilon\phi'' \tag{2.2.7}$$

for all $\phi \in \mathcal{D}(A) = \{\phi \in H^2(0,\ell) : \phi'(0) = \phi'(\ell) = 0\}$. Here $U = \mathbb{R}^2 = Y$. It is well known that A_ϵ is a self-adjoint operator that generates an analytic semigroup $S(t)$ on H. Moreover, the spectrum $\sigma(A_\epsilon)$ of A_ϵ consists of all eigenvalues λ_n, $n = 0,1,2,\cdots$, given by $\lambda_n = -\frac{\epsilon n^2\pi^2}{\ell^2}$ and the corresponding eigenfunctions ϕ_n given by

$$\phi_0(x) = 1, \qquad \phi_n(x) = \sqrt{\frac{2}{\ell}}\,\cos(\frac{n\pi x}{\ell}) \qquad \text{for } n = 1,2,\cdots. \tag{2.2.8}$$

In addition, since the operator A_ϵ is self-adjoint, the semigroup $S(t)$ can be presented by the following formula

$$S(t)z = < z,\phi_0 > \phi_0 + \sum_{n=1}^{\infty} e^{-(\frac{\epsilon n^2\pi^2}{\ell^2})t} < z,\phi_n > \phi_n \tag{2.2.9}$$

for all $z \in H$, where ϕ_n' s are defined by equation (2.2.8). Also, it is easy to see that the uncontrolled system (2.2.2)–(2.24) i.e., $u_1 = u_2 = 0$, is not asymptotically stable. Indeed, the function $z_o(x) = 1$ is in the domain of A_ϵ since it is $H^2(0, \ell)$ and satisfies the boundary conditions and the differential equation for all time, but the norm does not go to zero for increasing time since in this case by using equation (2.2.9) we can show that

$$\|z(t)\|_H = \|S(t)z_o\|_H = \|S(t)\|_H = 1, \qquad t \geq 0. \tag{2.2.10}$$

Next, we define the Hilbert space V to be the completion of $\mathcal{D}(A_\epsilon)$ with respect to a new inner product $< \cdot, \cdot >_V$ that will be defined below. Thus, if we introduce a sesquilinear form $a(\cdot, \cdot) : V \times V \to \mathbb{C}$ defined by

$$a(z, w) = \int_0^\ell \epsilon z'(x) w'(x) dx, \qquad \forall z, w \in V \tag{2.2.11}$$

where $V = H^1(0, \ell)$ then, it is easy to see that the sesquilinear form $a(\cdot, \cdot)$ is V-coercive,i.e.,

$$|a(z, w)| \leq \epsilon \|z\|_V \|w\|_V \qquad \text{(Continuity)} \tag{2.2.12}$$

$$\text{Re } a(z, z) + \lambda \|z\|_H^2 \geq \epsilon \|z\|_V^2 \tag{2.2.13}$$

for all $z, w \in V$ and $\lambda > 0$. It follows from the above inequalities that there exists a unique operator $\tilde{A}_\epsilon \in \mathcal{L}(V, V^*)$ such that

$$a(z, w) = \left\langle -\tilde{A}_\epsilon z, w \right\rangle_{V^*, V} \tag{2.2.14}$$

$$\overline{a(z, w)} = \left\langle -\tilde{A}_\epsilon^* w, z \right\rangle_{V^*, V} \quad \text{for all } \; z, w \in V \tag{2.2.15}$$

and in this case, the inner product on V is defined by

$$\langle v_1, v_2 \rangle_V = \left\langle -\tilde{A}_\epsilon v_1, v_2 \right\rangle_H, \quad \text{for } \; v_1, v_2 \in \mathcal{D}(A_\epsilon).$$

Now, to lift system (2.2.6) to a variational differential equation in V', we make use of Green's formula. In particular, for all $v \in V = H^1(0, \ell)$

$$\begin{aligned}\int_\Omega A_\epsilon z(t) v dx &= -a(z,v) + \epsilon \int_\Gamma \frac{\partial}{\partial x} z(t) v d\Gamma \\ &= -a(z,v) + \epsilon \int_\Gamma u(t) v d\Gamma\end{aligned}$$

where in this case $\Omega = (0, \ell)$ and $\Gamma = (\{0\}, \{\ell\})$. Thus, for all $v \in V$

$$\int_\Omega \frac{\partial}{\partial t} z(t) v dx = -a(z,v) + \epsilon \int_\Gamma u(t) v d\Gamma + \int_\Omega f(z(t)) v dx \tag{2.2.16}$$

$$z(0) = z_o \tag{2.2.17}$$

where $f(z(t)) = zz'$ is defined on the space $V = H^1(\Omega)$. Here, we define the control operator $B : U \to V'$ as follows

$$< Bu, \phi >_V = \epsilon \int_\Gamma u \phi|_\Gamma d\Gamma \qquad u \in U, \ \phi \in V. \tag{2.2.18}$$

Notice that not only is B in $\mathcal{L}(U, V')$ but also B belongs to $\mathcal{L}(U, \left[H^{\frac{1}{2}+2\rho}(\Omega)\right]')$, for all $\rho > 0$, since the trace operator

$$\phi \to \phi|_\Gamma \ : \ H^{\frac{1}{2}+2\rho}(\Omega) \ \to L^2(\Gamma) = U \tag{2.2.19}$$

is continuous. Moreover, if we let $\bar{A}_\epsilon = -A_\epsilon + wI$ as in Section 2.1, then

$$H^{\frac{1}{2}+2\rho}(\Omega) = \mathcal{D}(\bar{A}_\epsilon^{(\frac{1}{4}+\rho)}) \qquad \text{and} \quad B \in \mathcal{L}\left(U, \left[\mathcal{D}(\bar{A}_\epsilon^{*(\frac{1}{4}+\rho)})\right]'\right). \tag{2.2.20}$$

Therefore, (H-1-a) is satisfied with the choice of $\gamma = \frac{1}{4} + \rho$. Since

$$< Bu, \phi >_V = < u, B^* \phi >_U = \epsilon \int_\Gamma u \phi|_\Gamma d\Gamma, \tag{2.2.21}$$

it follows that $B^*\phi = \epsilon\phi|_\Gamma$ and $B^* \in \mathcal{L}\left(\mathcal{D}(\bar{A}_\epsilon^{(\frac{1}{4}+\rho)}), U'\right)$. Also, $C\phi = \phi|_\Gamma = \frac{1}{\epsilon}B^*\phi$ for all $\phi \in V$ and $Y = U$, implying that $C \in \mathcal{L}\left(\mathcal{D}(\bar{A}_\epsilon^{(\frac{1}{4}+\rho)}), Y\right)$ and that (H-1-b) is satisfied with $r = \frac{1}{4} + \rho$ and moreover, $\gamma + r < 1$ for small $\rho > 0$.

From above, we can formulate equations (2.2.1) and (2.2.5) as the abstract Cauchy problem

$$\begin{aligned} \frac{d}{dt}z(t) &= A_\epsilon z(t) + Bu(t) + f(z(t)), \qquad z(0) = z_o && (2.2.22) \\ y(t) &= Cz(t) && (2.2.23) \end{aligned}$$

in the space $H = L^2(\Omega) = L^2(0,\ell)$, where $A_\epsilon = \epsilon\phi''$ for all $\phi \in \mathcal{D}(A_\epsilon)$, $B \in \mathcal{L}(U, Z)$ and $C \in \mathcal{L}(W, Y)$ where $W = Z' = \mathcal{D}(\bar{A}_\epsilon^{(\frac{1}{4}+\rho)})$ and $V = H^1(\Omega)$. With these definitions, one has the imbeddings

$$\mathcal{D}(A) \subseteq V \hookrightarrow W \hookrightarrow H = H' \hookrightarrow Z = W' \hookrightarrow V'. \qquad (2.2.24)$$

Now we consider the following weighted linear quadratic-Gaussian problem for the linearized Burgers' equation

$$\begin{aligned} \frac{d}{dt}z(t) &= A_\epsilon z(t) + Bu(t), \qquad z(0) = z_o && (2.2.25) \\ y(t) &= Cz(t) && (2.2.26) \end{aligned}$$

where A_ϵ, B and C are defined as above.

$(\mathbf{LQG})_\alpha$: Design an optimal infinite-dimensional compensator for the linearized system (2.2.25) by minimizing the following performance index

$$J_\alpha(u) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{E}\left[R_1 \|z(s)\|_H^2 + R_2 \|u(s)\|_U^2\right] e^{2\alpha s} ds \qquad (2.2.27)$$

such that the closed-loop system is exponentially stable.

Remark 2.2.1 The weight function $e^{2\alpha t}$ in the definition of the cost functional $J_\alpha(.)$ will play an important role in the exponential decay rate of the closed-loop system.

For the control problem $(\mathrm{LQG})_\alpha$, we introduced an "α -shifted" control system. Let $\hat{z}(t) = z(t)e^{\alpha t}$, $\hat{u}(t) = u(t)e^{\alpha t}$, $\hat{y}(t) = y(t)e^{\alpha t}$, we then have the following modified LQG problem

$(\widehat{\mathrm{LQG}})$: Find an admissible controller $\hat{u}(.) \in L^2(0,\infty;U)$ minimizing the cost functional

$$\hat{J}(\hat{u}) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{E}\left[R_1 \|\hat{z}(s)\|_H^2 + R_2 \|\hat{u}(s)\|_U^2\right] ds \tag{2.2.28}$$

subject to

$$\begin{aligned} \frac{d}{dt}\hat{z}(t) &= (A_\epsilon + \alpha I)\hat{z}(t) + B\hat{u}(t), \quad \hat{z}(0) = z_o \\ \hat{y}(t) &= C\hat{z}(t) \qquad t \geq 0 \end{aligned} \tag{2.2.29}$$

where $\hat{u}(.)$ is the output estimate of the observer system given by

$$\begin{aligned} \frac{d}{dt}\hat{z}_c(t) &= (A_c + \alpha I)\hat{z}_c(t) + B_c\hat{y}(t), \\ \hat{u}(t) &= C_c\hat{z}_c(t) \end{aligned} \tag{2.2.30}$$

and $\hat{z}_c(t) = z_c(t)e^{\alpha t}$.

If we solve $(\widehat{\mathrm{LQG}})$ and apply

$$u_\alpha(t) = \hat{u}(t)e^{-\alpha t}, \qquad t \geq 0 \tag{2.2.31}$$

to the original control system (2.2.25), then the resulting optimal trajectories pair $(z^\alpha(t), z_c^\alpha(t))$ will satisfy the inequality

$$\left\| \begin{bmatrix} z^\alpha(t) \\ z_c^\alpha(t) \end{bmatrix} \right\|_{\mathcal{H}} \leq Me^{-\alpha t} \left\| \begin{bmatrix} z_o \\ z_{c_o} \end{bmatrix} \right\|_{\mathcal{H}} \tag{2.2.32}$$

where $M \geq 1$ is a constant and $\alpha > 0$ is the desired degree of stability.

Remark 2.2.2 We can see that the spectrum $\sigma(A_\epsilon + \alpha I)$ of the infinitesimal generator $A_\epsilon + \alpha I$ consists of all eigenvalues $\lambda_{\alpha,n}$, $n = 0, 1, 2, \cdots$, given by

$$\lambda_{\alpha,n} = \alpha - \frac{\epsilon n^2 \pi^2}{\ell^2} \tag{2.2.33}$$

and for each n the eigenfunction $\phi_{\alpha,n}$ corresponding to $\lambda_{\alpha,n}$ are given by

$$\phi_{\alpha,0}(x) = 1, \qquad \phi_{\alpha,n}(x) = \sqrt{\frac{2}{\ell}}\ \cos(\frac{n\pi x}{\ell}) \qquad \text{for } n = 1, 2, \cdots. \tag{2.2.34}$$

Let $\alpha > 0$ be given and let

$$n_\alpha = \max\left\{ n \in \mathbb{N} : \lambda_{\alpha,n} = \alpha - \frac{\epsilon n^2 \pi^2}{\ell^2} \geq 0 \right\} \tag{2.2.35}$$

Since A_ϵ is self-adjoint and the set $\{\phi_{\alpha,n} : n = 0, 1, \cdots\}$ is a basis for $H = L^2(0, \ell)$, we can express the spaces W and Z as

$$W = \left\{ z \in H : \sum_{n=0}^{\infty} \gamma_n \left| < z, \phi_{\alpha,n} > \right|^2 < \infty \right\} \tag{2.2.36}$$

$$Z' = \left\{ z \in H : \sum_{n=0}^{\infty} \beta_n^{-1} \left| < z, \phi_{\alpha,n} > \right|^2 < \infty \right\} \tag{2.2.37}$$

where $\beta_0 = \gamma_0 = 1$ and $\gamma_n = \beta_n^{-1} = n$, for $n = 1, 2, \cdots$, since $W = Z'$, and the injection $H \hookrightarrow Z$ is given by identifying $z \in H$ with the sequence $\left\{ < z, \phi_{\alpha,n} > \right\}_{n \in \mathbb{N}} \in Z$. Assume that the sequences $b_n \in U,\ c_n \in Y$ satisfy

$$\sum_{n=0}^{\infty} \beta_n \left\| b_n \right\|_U^2 < \infty, \qquad \sum_{n=0}^{\infty} \gamma_n^{-1} \left\| c_n \right\|_Y^2 < \infty \tag{2.2.38}$$

and that the operators $B \in \mathcal{L}(U,Z)$ and $C \in \mathcal{L}(W,Y)$ are given by

$$Cz = \sum_{n=0}^{\infty} c_n < z, \phi_{\alpha,n} >, \quad \text{and} \quad Bu = \{< b_n, u >\}_{n \in \mathbb{N}}. \tag{2.2.39}$$

In this case it is easy to see that for each $n = 0, 1, \cdots,$

$$c_n = \sum_{j=0}^{\infty} < \phi_{\alpha,n}|_\Gamma, \phi_{\alpha,j} >= C(\phi_{\alpha,n}) = \phi_{\alpha,n}|_\Gamma. \tag{2.2.40}$$

Since $c_n \in Y = \mathbb{R}^2$, we have that $c_n = [c_n^1, c_n^2]$ where each pair of components has the form

$$c_0 = [1,1], \quad c_n = \sqrt{\frac{2}{\ell}}\, [1, (-1)^n], \quad \text{for all} \quad n = 1, 2, \cdots. \tag{2.2.41}$$

Similar arguments leads to $b_n = \epsilon c_n^T$.

Lemma 2.2.3 For each $T > 0$, both hypotheses (H-2) and (H-3) hold true for the α–shifted system $(\widehat{LQG})$, where $W = Z' = \mathcal{D}(\bar{A}_\epsilon^{\frac{1}{4}+\rho})$, $H = L^2(0,\ell)$ and $\hat{S}(t)$ is the analytic semigroup generated by $A_\epsilon + \alpha I$.

Proof:

Since the infinitesimal generator $A_\epsilon + \alpha I$ of $\hat{S}(t)$ is self-adjoint, the semigroup $\hat{S}(t)$ can be represented by

$$\hat{S}(t) = \sum_{n=0}^{\infty} e^{\lambda_{\alpha,n}} < z, \phi_{\alpha,n} >_H \phi_{\alpha,n}. \tag{2.2.42}$$

By the Schwartz inequality

$$\begin{aligned}
\left\| \int_0^T \hat{S}(T-s)B\hat{u}(s)ds \right\|_W^2 &= \sum_{n=0}^{\infty} \gamma_n \left(\int_0^T e^{\lambda_{\alpha,n}(T-s)} b_n \hat{u}(s)ds \right)^2 \\
\leq & \left(\sum_{n=0}^{n_\alpha} \gamma_n \left(\int_0^T e^{2\lambda_{\alpha,n}s} ds \right) \|b_n\|^2 + \sum_{n=n_\alpha+1}^{\infty} \frac{\gamma_n}{2|\lambda_{\alpha,n}|} \|b_n\|^2 \right) \|\hat{u}\|^2_{L^2(0,T;U)} \\
\leq & \left(\sum_{n=0}^{n_\alpha} \frac{\gamma_n^2}{2\lambda_{\alpha,n}} \left(e^{2\lambda_{\alpha,n_\alpha}T} - 1 \right) \frac{\|b_n\|^2}{\gamma_n} + \frac{(n_\alpha+1)^2}{2|\lambda_{\alpha,n_\alpha+1}|} \sum_{n=n_\alpha+1}^{\infty} \frac{\|b_n\|^2}{\gamma_n} \right) \|\hat{u}\|^2_{L^2(0,T;U)} \\
\leq & \left[\frac{n_\alpha^2}{2\lambda_{\alpha,n_\alpha}} \left(e^{2\lambda_{\alpha,n_\alpha}T} - 1 \right) + \frac{(n_\alpha+1)^2}{2|\lambda_{\alpha,n_\alpha+1}|} \sum_{n=n_\alpha+1}^{\infty} \beta_n \|b(\cdot)\|^2 \right] \|\hat{u}\|^2_{L^2(0,T;U)} \\
\leq & \hat{b}(T) \|\hat{u}\|^2_{L^2(0,T;U)}
\end{aligned}$$

where we used the facts that $\beta_n = \frac{1}{\gamma_n}$ and that

$$\sum_{n=n_\alpha+1}^{\infty} \frac{n^2}{|\lambda_{\alpha,n}|} \frac{\|b_n\|^2}{\gamma_n} \leq \frac{(n_\alpha+1)^2}{|\lambda_{\alpha,n}|} \sum_{n=n_\alpha+1}^{\infty} \beta_n \|b_n\|^2 < \infty. \tag{2.2.43}$$

Finally, the duality between (H-2) and (H-3) implies (H-3) holds.∎

Remark 2.2.4 Lemma 2.2.3 together with the fact that

$$\sum_{n=n_\alpha+1}^{\infty} \frac{\|b_n\| \, \|c_n\|}{|\lambda_{\alpha,n}|^{\frac{1}{2}}} < \infty \tag{2.2.44}$$

show that the system (2.2.25)–(2.2.26) characterized by the operators A_ϵ, B and C defined above, is of Pritchard–Salamon–class with respect to the Hilbert spaces Z and W where the operators B and C are defined, respectively. Hence, in order to solve the shifted linear quadratic-Gaussian problem $(\widehat{LQG})$, it suffices to show that the system (2.2.29) is stabilizable and detectable.

The following lemma is an application of stability and detectability results of Pritchard and Salamon [27]. Note that in [5] Burns and Kang used the results of Triggiani in [22] to establish stabilizability for the Dirichlet boundary control problem which, unlike system (2.2.29), is not defined by a system of Pritchard–Salamon class.

Lemma 2.2.5 (i) The system (2.2.29) is stabilizable in the space Z if and only if $b_n \neq 0$ for each $n = 0, 1, 2, \cdots, n_\alpha$.

(ii) The system (2.2.29) is detectable through the unbounded output operator $C : W \to Y$ if and only if $c_n \neq 0$ for each $n = 0, 1, 2, \cdots, n_\alpha$.

Proof:

(i) We know that the spectrum $\sigma(A_\epsilon + \alpha I)$ of $A_\epsilon + \alpha I$ consists of all eigenvalues $\lambda_{\alpha,n} = \alpha - \varepsilon n^2\pi^2/\ell > 0$ for all $n = 0, 1, 2, \cdots, n_\alpha$. Let H_u be the linear space spanned by the eigenfunctions $\phi_{\alpha,0}, \cdots, \phi_{\alpha,n_\alpha}$. Then the dimension of H_u is $n_\alpha + 1$ and hence the system (2.2.29) is stabilizable if and only if the projected system (2.2.29) on H_u is controllable, i.e., if and only if $b_n =< b(\cdot), \phi_{\alpha,n} >\neq 0$ for $n = 0, 1, \cdots, n_\alpha$.

(ii) By the duality statements of (i) system (2.2.29) is detectable through $C \in \mathcal{L}(W, Y)$ if and only if $c_n \neq 0$ for $n = 0, 1, \cdots, n_\alpha$.

Hence, the controllability and stabilizability of system (2.2.29) follow from the representations (2.2.41) of $b_n \in U$ and $c_n \in Y$, respectively.∎

Now we return to the original weighted problem $(\mathrm{LQG})_\alpha$ and state the main results.

Theorem 2.2.6 Let $\alpha > 0$ be fixed and assume that (H-1)–(H-5) and (C-1) are satisfied. Then there is a unique optimal dynamic compensator for the problem $(\mathrm{LQG})_\alpha$ such that

$$
\begin{aligned}
\frac{d}{dt} z_c^\alpha(t) &= A_c z_c^\alpha(t) + B_c y^\alpha(t) \\
u^\alpha(t) &= C_c z_c^\alpha(t)
\end{aligned}
\tag{2.2.45}
$$

where

$$
A_c = A_\epsilon + BC_c - B_c C \tag{2.2.46}
$$

$$
B_c = Q_\alpha C^* V_2^{-1} \tag{2.2.47}
$$

$$
C_c = -R_2^{-1} B^* P_\alpha \tag{2.2.48}
$$

and P_α, Q_α are the unique nonnegative self-adjoint operators satisfying the following algebraic Riccati equations

$$
\begin{aligned}
0 &= < P_\alpha x, (A_\epsilon + \alpha I) y >_H + < (A_\epsilon + \alpha I) x, P_\alpha y >_H + < R_1 x, y >_H - \\
&\quad < P_\alpha B R_2^{-1} B^* P_\alpha x, y >_H \qquad \forall (x, y) \in \mathcal{D}(A_\epsilon)
\end{aligned}
\tag{2.2.49}
$$

$$
\begin{aligned}
0 &= < Q_\alpha x, (A_\epsilon + \alpha I)^* y >_H + < (A_\epsilon + \alpha I)^* x, Q_\alpha y >_H + < V_1 x, y >_H - \\
&\quad < Q_\alpha C^* V_2^{-1} C Q_\alpha x, y >_H \qquad \forall (x, y) \in \mathcal{D}(A_\epsilon^*).
\end{aligned}
\tag{2.2.50}
$$

Moreover, the closed-loop semigroup $\mathcal{S}_\alpha(t) \in \mathcal{L}(\mathcal{H})$ satisfies the following stability property

$$
\|\mathcal{S}_\alpha(t)\|_{\mathcal{L}(\mathcal{H})} \leq M e^{-(\alpha + w)t}, \qquad t \geq 0 \tag{2.2.51}
$$

for some constants $M = M(\alpha, \epsilon) \geq 1$ and $w = w(\alpha, \epsilon) > 0$.

Proof:

By Lemmas 2.2.3 and 2.2.5 we know that the α–shifted control system (2.2.29) satisfies all the hypotheses (H-1)–(H-5) and (C-1) with $z(t)$, $y(t)$, $u(t)$, A_ϵ, $S(t)$, $z_c(t)$ and $J(\cdot)$ replaced by $\hat{z}(t)$, $\hat{y}(t)$, $\hat{u}(t)$, $A_\epsilon + \alpha I$, $\hat{S}(t)$, $\hat{z}_c(t)$ and $\hat{J}(\cdot)$, respectively. Hence, by Theorem 2.1.2 there is a unique optimal compensator for the $(\widehat{\text{LQG}})$ problem and the corresponding closed-loop semigroup $\hat{\mathcal{S}}(t)$ decays exponentially, i.e., there exist $\hat{M} = \hat{M}(\alpha, \epsilon) \geq 1$ and $w = w(\alpha, \epsilon) > 0$ such that

$$\left\| \hat{\mathcal{S}}(t) \right\|_{\mathcal{L}(\mathcal{H})} \leq \hat{M} e^{-wt}, \qquad t \geq 0. \tag{2.2.52}$$

Now, since the semigroup $\hat{\mathcal{S}}(t)$ is generated by

$$\hat{\mathcal{A}} = \mathcal{A} + \alpha \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{2.2.53}$$

and the infinitesimal generator of the closed-loop system (2.2.25) is $\mathcal{A}$, then $\mathcal{S}_\alpha(t) = \hat{\mathcal{S}}(t) e^{-\alpha t}$ and by the relation (2.2.52), $\mathcal{S}_\alpha(t)$ satisfies the inequality (2.2.51) with $M = \hat{M}$. Moreover, the optimal controller $u^\alpha(t)$ for the $(\text{LQG})_\alpha$ is given by the formula (2.2.45), since

$$u^\alpha(t) = e^{\alpha t} \hat{u}(t) = C_c \hat{z}_c(t) e^{\alpha t} = C_c z_c^\alpha(t) \tag{2.2.54}$$

where $z_c^\alpha(t) = \hat{z}_c(t) e^{\alpha t}$ is the corresponding optimal trajectory estimate for the original system (2.2.25). This completes the proof. ∎

The optimal dynamic compensator is characterized by the feedback and observer operators C_c, $B_c \in \mathcal{L}(H)$ given in equations (2.2.47) and (2.2.48), respectively. We recall that

$H = L^2(0,\ell)$ and $U = Y = \mathbb{R}^2$. Thus, it follows from the Ritz Representation Theorem that there is a unique feedback gain function $K_\alpha(\cdot) = [k_0^\alpha(\cdot), k_1^\alpha(\cdot)] \in L^2(0,\ell) \times L^2(0,\ell)$ given by

$$K_\alpha z = \begin{bmatrix} \int_0^\ell k_0^\alpha(s) z(s) ds \\ \\ \int_0^\ell k_1^\alpha(s) z(s) ds \end{bmatrix} \tag{2.2.55}$$

for all $z \in H = L^2(0,\ell)$. Similarly, by duality there is a unique observer function gain $L_\alpha(\cdot) = [\ell_0^\alpha(\cdot), \ell_1^\alpha(\cdot)] \in L^2(0,\ell) \times L^2(0,\ell)$ such that

$$L_\alpha y = \ell_0^\alpha(x) y_1 + \ell_1^\alpha(x) y_2 \tag{2.2.56}$$

for all $y = [y_1, y_2]^T \in \mathbb{R}^2$.

Corollary 2.2.7 Let $\alpha > 0$ be given and let K_α, $L_\alpha \in L^2(0,\ell) \times L^2(0,\ell)$ be given by the formulas (2.2.55) and (2.2.56). For any initial pair of data z_o, $z_{c_o} \in L^2(0,\ell)$ there is a unique solution of the closed–loop system

$$\begin{aligned} \frac{\partial}{\partial t} z(x,t) &= \epsilon \frac{\partial^2}{\partial x^2} z(x,t) \\ -\frac{\partial}{\partial x} z(0,t) &= -\int_0^\ell k_0^\alpha(s) z_c(s,t) ds \qquad (2.2.57) \\ \frac{\partial}{\partial x} z(\ell,t) &= -\int_0^\ell k_1^\alpha(s) z_c(s,t) ds \\ z(x,0) &= z_o(x) \in L^2(0,\ell) \qquad (2.2.58) \\ y(t) &= \begin{bmatrix} z(0,t) \\ \\ z(\ell,t) \end{bmatrix} \qquad (2.2.59) \end{aligned}$$

where $z_c(x,t)$ satisfies the observer system given by

$$\begin{aligned}
\frac{\partial}{\partial t} z_c(x,t) &= \epsilon \frac{\partial^2}{\partial x^2} z_c(x,t) - \ell_0^\alpha(x) z_c(0,t) - \ell_1^\alpha z_c(\ell,t) + \ell_0^\alpha z(0,t) + \ell_1^\alpha z(\ell,t) \\
-\frac{\partial}{\partial x} z_c(0,t) &= -\int_0^\ell k_0^\alpha(s) z_c(s,t) ds \\
\frac{\partial}{\partial x} z_c(\ell,t) &= -\int_0^\ell k_1^\alpha(s) z_c(s,t) ds \\
z_c(x,0) &= z_{c_o}(x) \in L^2(0,\ell).
\end{aligned} \tag{2.2.60}$$

Moreover, the solution pair $\begin{bmatrix} z(t) \\ z_c(t) \end{bmatrix}(\cdot) = \begin{bmatrix} z(\cdot,t) \\ z_c(\cdot,t) \end{bmatrix}$ satisfies the inequality

$$\left\| \begin{bmatrix} z(t)(\cdot) \\ z_c(t)(\cdot) \end{bmatrix} \right\|_{\mathcal{H}} \leq M_\alpha e^{-(\alpha+w)t} \left\| \begin{bmatrix} z_o \\ z_{c_o} \end{bmatrix} \right\|_{\mathcal{H}} \tag{2.2.61}$$

for some constant $M_\alpha = M(\alpha, \epsilon) \geq 1$ and $w = w(\alpha, \epsilon) > 0$.

Proof:

If $e(x,t) = z(x,t) - z_c(x,t)$ denotes the estimate error, then from equations (2.2.57) and (2.2.60) we obtain

$$\begin{aligned}
\frac{\partial}{\partial t} e(x,t) &= \epsilon \frac{\partial^2}{\partial x^2} e(x,t) - \ell_0^\alpha e(0,t) - \ell_1^\alpha e(\ell,t) \\
-\frac{\partial}{\partial x} e(0,t) &= 0 \\
\frac{\partial}{\partial x} e(\ell,t) &= 0 \\
e(x,0) &= z(x) - z_{c_o}(x) = e_o(x) \in L^2(0,\ell).
\end{aligned} \tag{2.2.62}$$

With this estimator error $e(x,t)$, it is easy to show that the closed-loop system is equivalent to (2.2.62) coupled with

$$\begin{aligned}
\frac{\partial}{\partial t}z(x,t) &= \epsilon\frac{\partial^2}{\partial x^2}z(x,t) \\
-\frac{\partial}{\partial x}z(0,t) &= -\int_0^\ell k_0^\alpha(s)z(s,t)ds + \int_0^\ell k_0^\alpha(s)e(s,t)ds \\
\frac{\partial}{\partial x}z(\ell,t) &= -\int_0^\ell k_1^\alpha(s)z(s,t)ds + \int_0^\ell k_1^\alpha(s)e(s,t)ds.
\end{aligned} \tag{2.2.63}$$

Therefore, the existence and uniqueness of a strong solution $z(t)(\cdot) = z(\cdot,t)$ follow from the fact that the closed-loop system of the coupled equations (2.2.62) and (2.2.63) is characterized by the semigroup $S_{L_\alpha}(t)$ generated by $A_\epsilon - L_\alpha C$, which is strongly (uniformly exponentially) stable and that the system (2.2.63) generates an exponential stable analytic semigroup $S_{K_\alpha}(t)$ on H. The stability results in (2.2.61) follows from the inequality given by equation (2.2.51).∎

2.3 Linear Quadratic Differential Game Problem

In the previous section, we treated the linear quadratic-Gaussian problem, which arises when the disturbance signals either are fixed or have a fixed power spectrum. In this section we discuss the design of a robust controller for the worst case disturbance signals. The optimal control problem to be considered here is the infinite dimensional version of the so-called "soft-constraint quadratic differential game" or "MinMax/H^∞ problem" as described in [1]. The game theoretic analogy is intuitively appealing for, in the MinMax problem, the disturbance input and the control input can be viewed as strategies employed by opposing players in a game. The disturbance input is chosen to maximize the norm of the output while the control input is chosen to minimize it. The basic assumptions needed for this problem are the same as those required for the LQG problem. However, there are significant differences that reflect the fact that MinMax criterion corresponds to designing for the worst case disturbance signals. One important difference here is that the algebraic Riccati equations that arise in the theory of linear quadratic differential game approach contains terms that are not sign definite. Therefore, we can not guarantee the solvability of those Riccati equations unless sufficient additional conditions are imposed.

The main objective of this section is to design an infinite-dimensional dynamic compensator, in the presence of a worst–case disturbance signal $\eta(\cdot)$, of the form

$$\begin{aligned} \frac{d}{dt}z_c(t) &= A_c z_c(t) + B_c y(t) \\ u(t) &= C_c z_c(t) \end{aligned} \tag{2.3.1}$$

which minimizes the steady-state "disturbance–augmented" cost functional

$$J_\infty(u,\eta) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{E}\left[R_1 \|z(s)\|_H^2 + R_2 \|u(s)\|_U^2 \; - \gamma^2 \|\eta(s)\|_{\aleph}^2\right] ds \tag{2.3.2}$$

subject to

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t) + H_1\eta(t) \qquad z(0) = z_o \in H & (2.3.3)\\ y(t) &= Cz(t) + H_2\eta(t) & (2.3.4)\\ w(t) &= E_1 z(t) + E_2 u(t). & (2.3.5) \end{aligned}$$

Here $\gamma \in \mathbb{R}$ is a fixed positive constant. This parameter is selected so that the design produces an exponentially stable closed-loop system and such that the closed-loop transfer function from the output $w(t)$ to the input $\eta(t)$ denoted by $T_{w\eta}$ satisfies the H^∞–norm bound, $\|T_{w\eta}\|_\infty < \gamma$. Defining the spaces $\bar{U} = L^2(0,\infty;U)$ and $\bar{\aleph} = L^2(0,\infty;\aleph)$, the linear quadratic differential game (or MinMax problem) is to find

$$\inf_{u\,\in\,\bar{U}}\ \sup_{\eta\,\in\,\bar{\aleph}}\quad J_\infty(\mathrm{u},\eta) \tag{2.3.6}$$

subject to dynamics governed by (2.3.3) and (2.3.4).

Definition 2.3.1 A solution $(u^{\mathrm{opt}},\eta^{\mathrm{opt}})$ is called a saddle point of $J_\infty(u,\eta)$ if and only if

$$J_\infty(u^{\mathrm{opt}},\eta) \le J_\infty(u^{\mathrm{opt}},\eta^{\mathrm{opt}}) \le J_\infty(u,\eta^{\mathrm{opt}}), \qquad \forall (u,\eta) \in \bar{U}\times\bar{\aleph} \tag{2.3.7}$$

Remark 2.3.2 (i) All assumptions made in Section 2.1 will hold unless otherwise noted. (ii) It will be understood from the context that the term admissible controller will be taken to mean that a dynamic compensator exists such that the closed-loop is exponentially stable.

The MinMax problem for the linearized Burgers' equation can be stated as follows:

(MM): Find an admissible controller for system (2.3.3)–(2.3.4) such that $\|T_{w\eta}\|_{\infty} < \gamma$.

Next, in the following theorem we state sufficient conditions which guarantee the existence of an admissible controller to the MinMax problem outlined above.

Theorem 2.3.3 Assume (H-1)-(H-5) and (C-1) hold. Then, there exists a critical value $\gamma_c > 0$ such that

a) If $0 < \gamma < \gamma_c$, then taking the supremum in $\eta(\cdot)$ as in (2.3.6) leads to $+\infty$, i.e., there is no finite solution of the game theory problem.

b) If $\gamma > \gamma_c$ then, there exists an admissible controller such that $\|T_{w\eta}\|_{\infty} < \gamma$ only if

(i) There exist unique nonnegative self-adjoint operators, $P = P^* \in \mathcal{L}(H)$ and $Q = Q^* \in \mathcal{L}(H)$, as solutions to the following algebraic Riccati equations $(\text{ARE})_\gamma$

$$\begin{aligned} 0 &= \langle Px, Ay \rangle_H + \langle Ax, Py \rangle_H + \langle R_1 x, y \rangle_H - \\ &\quad \langle PBR_2^{-1}B^*Px, y \rangle_H + \gamma^{-2} \langle PV_1Px, y \rangle_H \qquad \forall (x,y) \in \mathcal{D}(A) \qquad (2.3.8) \end{aligned}$$

$$\begin{aligned} 0 &= \langle Qx, A^*y \rangle_H + \langle A^*x, Qy \rangle_H + \langle V_1 x, y \rangle_H - \\ &\quad \langle QC^*V_2^{-1}CQx, y \rangle_H + \gamma^{-2} \langle QR_1Qx, y \rangle_H \qquad \forall (x,y) \in \mathcal{D}(A^*) \qquad (2.3.9) \end{aligned}$$

(ii) and such that $P - \gamma^2 Q^{-1} < 0$ or $\rho(QP) < \gamma^2$.

Moreover, when these conditions hold, the unique saddle point for the differential game problem is given by

$$u^{\text{opt}}(t) = -R_2^{-1}B^*Pz_c(t) = C_cz_c(t) \tag{2.3.10}$$

$$\eta^{\text{opt}}(t) = \gamma^{-2}H_1^*Pz_c(t) \tag{2.3.11}$$

where $z_c(t)$ is the optimal dynamic compensator given by

$$\dot{z}_c(t) = A_cz_c(t) + \bar{M}B_cy(t) \tag{2.3.12}$$

and

$$A_c = A_\epsilon + \gamma^{-2}H_1H_1^*P + BC_c - \bar{M}B_cC \tag{2.3.13}$$

$$B_c = QC^*V_2^{-1} \tag{2.3.14}$$

$$C_c = -R_2^{-1}B^*P \quad \text{and} \qquad \bar{M} = (I - \gamma^{-2}QP)^{-1}. \tag{2.3.15}$$

Remark 2.3.4 To emphasize the relationship between the MinMax and the LQG controller formulas, the MinMax controller can be written as

$$\dot{z}_c(t) = A_\epsilon z_c(t) + H_1\bar{\eta}_{\text{worst}}(t) + B\bar{u}(t) + \bar{M}B_c\left(y(t) - Cz_c(t)\right) \tag{2.3.16}$$

$$\bar{u}(t) = C_cz_c(t), \qquad \bar{\eta}_{\text{worst}}(t) = \gamma^{-2}H_1^*Pz_c(t). \tag{2.3.17}$$

These equations have the structure of an observer-based compensator. The obvious questions that arise, when those formulas are compared to LQG's ones, are as follows:

1) Where does the term $H_1\bar{\eta}_{\text{worst}}(t)$ come from?

2) Why $\bar{M}B_c$ instead of B_c as observer gain?

The formula $\eta_{\text{worst}}(t) = \gamma^{-2}H_1^*Pz(t)$ is, in some sense, a worst-case disturbance input that maximizes the quantity $\|w\|^2 - \gamma^2\|\eta\|^2$ in $J(u, \eta)$ for the minimizing value of $u(t) = C_cz(t)$. Furthermore, $\bar{M}B_c$ is actually the optimal observer gain for estimating $C_cz(t)$, which is the optimal feedback control input, in the presence of this worst-case disturbance input. It is therefore not surprising that $\bar{M}B_c$ should enter in the controller equation instead of B_c as in the LQG problem where the resulting equations are much simpler. This is because there is no such worst-case disturbance in $J(u)$.

Proof of Theorem 2.3.3:

Here we first need to show that under the above conditions the closed-loop system generates an exponential stable semigroup on $\mathcal{H}$. If we define new disturbance and control variables $r = \eta - \gamma^{-2}H_1^*Pz(t)$, $v = u + B^*Pz(t)$, and the estimate error $e(t) = z(t) - z_c(t)$, then the systems (2.3.3) and (2.3.12) can be written as

$$\frac{d}{dt}\begin{bmatrix} z(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A_{tem} + BC_c & -BC_c \\ 0 & A_{tem} - \bar{M}B_cC \end{bmatrix}\begin{bmatrix} z(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} H_1 \\ H_1 - \bar{M}B_cH_2 \end{bmatrix} r(t) \qquad (2.3.18)$$

where $A_{tem} = A + \gamma^{-2}H_1H_1^*P$. Therefore, it is sufficient to prove that both $(A_{tem} + BC_c)$ and $(A_{tem} - \bar{M}B_cC)$ generates an exponential semigroup on H, then by perturbation theory it can be concluded that the closed-loop system given by equations (2.3.3) and (2.3.12) is the infinitesimal of an exponential semigroup on $\mathcal{H}$. First, the controller $(\text{ARE})_\gamma$ given by (2.3.8) can be written as

$$0 = \langle Px, A_Fy\rangle_H + \langle A_Fx, Py\rangle_H + \gamma^{-2}\langle PH_1H_1^*Px, y\rangle_H + \langle C_F^*C_Fx, y\rangle_H\,, \qquad (2.3.19)$$

where $A_F = A + BC_c$ and $C_F = E_1 + E_2C_c$. Then, since (A, H_1) is stabilizable and by virtue of the fact that P exists and is a nonnegative self-adjoint operator, we can apply Lemma 1.4 in [10] and obtain that $A_F + \gamma^{-2}H_1H_1^*P = A_{tem} + BC_c$ is exponentially stable.

Next, since $\rho(QP) < \gamma^2$ holds, then the inverse $(I-\gamma^{-2}QP)^{-1} = \bar{M}$ is well-defined. If we let $Y_{tem} = \bar{M}Q$ then we know that Y_{tem} is a nonnegative self-adjoint operator. Moreover, it can be verified that Y_{tem} satisfies the following modified observer algebraic Riccati equation

$$\begin{aligned} 0 \; = \; & < Y_{tem}x, (A_{tem} - Y_{tem}C^*V_2^{-1}C)^*y >_H + < (A_{tem} - Y_{tem}C^*V_2^{-1}C)^*x, Y_{tem}y >_H + \\ & \gamma^{-2} < Y_{tem}C_c^*R_2^{-1}C_cY_{tem}x, y >_H + < B_LB_L^*x, y >_H, \end{aligned} \tag{2.3.20}$$

where $B_L = H_1 - \bar{M}B_c^*H_2$. Using duality and Lemma 1.4 in [10], we conclude that $A_{tem} - Y_{tem}C^*V_2^{-1}C = A_{tem} - \bar{M}B_cC$ is exponentially stable. To complete the proof of this theorem we need to show that the transfer function of the closed-loop system satisfies the bound $\|T_{w\eta}\|_\infty < \gamma$. Note that since $Q^{-1} = Y_{tem}^{-1} + \gamma^{-2}P$ the operator

$$\mathcal{K} \stackrel{\text{def}}{=} \begin{bmatrix} Q^{-1} & -Y_{tem}^{-1} \\ -Y_{tem}^{-1} & Y_{tem}^{-1} \end{bmatrix}$$

can be written as

$$\mathcal{K} = \begin{bmatrix} Y_{tem}^{-1} \\ -Y_{tem}^{-1} \end{bmatrix} \begin{bmatrix} Y_{tem}^{-1} & -Y_{tem}^{-1} \end{bmatrix} + \gamma^{-2} \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}.$$

Moreover, it can be verified that $\mathcal{K}$ satisfies the following closed–loop algebraic Riccati equation for all $Z \in \mathcal{H}$

$$0 =< \mathcal{K}Z, \mathcal{A}Z >_{\mathcal{H}} + < \mathcal{A}Z, \mathcal{K}Z >_{\mathcal{H}} + < \mathcal{K}\mathcal{B}\mathcal{B}^*\mathcal{K}Z, Z >_{\mathcal{H}} + \gamma^{-2} < \mathcal{C}^*\mathcal{C}Z, Z >_{\mathcal{H}} \tag{2.3.21}$$

where $\mathcal{B} = [H_1 \;\; \bar{M}B_cH_2]^T$ and $\mathcal{C} = [E_1 \;\; E_2C_c]$. Then the transfer function from the disturbance $\eta(\cdot)$ to the controlled output $w(\cdot)$ is given by

$$T_{w\eta} = \mathcal{C}\,(s\mathcal{I} - \mathcal{A})^{-1}\,\mathcal{B}. \tag{2.3.22}$$

However, we proved that $\mathcal{A}$ is stable and the operator $\mathcal{K}$ satisfying the algebraic Riccati equation (2.3.21), is a nonnegative self-adjoint operator, therefore from Theorem 1.1 in [10] we have

$$\|T_{w\eta}\|_{\infty} < \gamma. \tag{2.3.23}$$

This completes the proof.▮

Now, as in the case of the LQG problem we consider an "α–shifted" MinMax problem for the linearized Burgers' equation, stated as follows

$(\widehat{MM})$: Find an admissible controller $\hat{u}(\cdot) \in L^2(0,\ell)$ minimizing the disturbance- augmented cost functional

$$\hat{J}_{\infty}(\hat{u},\hat{\eta}) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{E}\left[R_1\,\|\hat{z}(s)\|_H^2 + R_2\,\|\hat{u}(s)\|_U^2 \;-\; \gamma^2\,\|\hat{\eta}(s)\|_{\aleph}^2\right] ds \tag{2.3.24}$$

subject to

$$\begin{aligned} \frac{d}{dt}\hat{z}(t) &= (A_\epsilon + \alpha I)\hat{z}(t) + B\hat{u}(t) + H_1\hat{\eta}(t) \\ \hat{y}(t) &= C\hat{z}(t) + H_2\hat{\eta}(t) \end{aligned} \tag{2.3.25}$$

where $\hat{u}(.)$ is the output estimate of the observer system given by

$$\begin{aligned} \frac{d}{dt}\hat{z}_c(t) &= (A_c + \alpha I)\hat{z}_c(t) + \bar{M}B_c\hat{y}(t) \\ \hat{u}(t) &= C_c\hat{z}_c(t) \end{aligned} \tag{2.3.26}$$

where $\hat{z}_c(t) = z_c(t)e^{\alpha t}$ such that $\|T_{w,\eta}\|_\infty < \gamma$.

If we solve $\widehat{MM}$ and apply

$$u_\alpha(t) = \hat{u}(t)e^{-\alpha t}, \qquad t \geq 0 \tag{2.3.27}$$

to the original control system (2.3.3), then the resulting optimal trajectories pair $(z^\alpha(t), z_c^\alpha(t))$ will satisfy the inequality

$$\left\| \begin{bmatrix} z^\alpha(t) \\ z_c^\alpha(t) \end{bmatrix} \right\|_{\mathcal{H}} \leq Me^{-\alpha t} \left\| \begin{bmatrix} z_o \\ z_{c_o} \end{bmatrix} \right\|_{\mathcal{H}} \tag{2.3.28}$$

where $M \geq 1$ is a constant and $\alpha > 0$ is the desired degree of stability.

2.4 Optimal Fixed-Finite-Dimensional Compensator

In this section we consider the problem of designing a fixed-finite-dimensional compensator for a class of distributed parameter system governed by Burgers' equation. We shall present a short summary of one approach (the optimal projection method due to Bernstein and Hyland) and show how this approach can be used in conjunction with standard numerical schemes to produce a realizable low order controller.

2.4.1 A Theoretical Existence Result

We consider the following abstract Cauchy problem

$$\dot{z}(t) = Az(t) + Bu(t), \qquad z(0) = z_o \in H \tag{2.4.1}$$

$$y(t) = Cz(t) \qquad t \geq 0 \tag{2.4.2}$$

where H is a Hilbert space, $u(\cdot) \in L^2(0,\ T; \mathrm{IR}^m)$, $y(\cdot) \in L^2(0,\ T; \mathrm{IR}^\ell)$, and A is the infinitesimal generator of analytic semigroup $S(t)$ on H. Also, assume that all hypotheses and assumptions made in the previous sections hold here.

We now give sufficient conditions which imply that the system (2.4.1)–(2.4.2) can be stabilized by a finite-dimensional compensator of the form

$$\dot{z}_c(t) = A_c z_c(t) + B_c y(t) \qquad z_c(0) = z_{c_o} \tag{2.4.3}$$

$$u(t) = C_c z_c(t) \tag{2.4.4}$$

where $A_c \in \mathrm{IR}^{N_c \times N_c}$, $B_c \in \mathrm{IR}^{N_c \times m}$, and $C_c \in \mathrm{IR}^{\ell \times N_c}$ are suitably chosen matrices.

In addition to hypotheses (H-1)-(H-5) and (C-1), we assume:

(H-6) Assume that there exists a finite-dimensional subspace $\mathcal{X} \subset W$, with $\dim \mathcal{X} \leq N_c$ such that

(i) $S_K(t)\mathcal{X} \subset \mathcal{X}$, for all $t \geq 0$,

(ii) Range $G \subset \mathcal{X}$,

(iii) $\mathcal{X} \subseteq \mathcal{D}(A_K)$.

Moreover, there exist linear maps i: $\mathbb{R}^{N_c} \to \mathcal{X}$, $\pi : \mathcal{X} \to \mathbb{R}^{N_c}$ such that

$$\pi i = I_{N_c}, \qquad i\pi x = x \qquad \text{for } \ x \in \mathcal{X}. \tag{2.4.5}$$

Note that (H-6) implies that $\pi A_K i$ is a well defined linear map on $\mathbb{R}^{N_c}$. We will show that the system

$$\dot{z}_c(t) = \pi\,(A_K - GC)\,i\,z_c(t) + \pi G y(t), \qquad z_c(0) = z_{c_o} \tag{2.4.6}$$

$$u(t) = F\,i\,z_c(t) \tag{2.4.7}$$

defines a stabilizing compensator for the Cauchy problem (2.4.1)–(2.4.2). The following result is an extension of Theorem 2.5 in [11] for unbounded inputs and outputs.

Theorem 2.4.1 If (H-1)-(H-6) are satisfied, then the closed-loop system defined by (2.4.1)–(2.4.2) and (2.4.6)–(2.4.7) is exponentially stable.

Proof: Note that without loss of generality we can assume that dim $\mathcal{X} = N_c$. By Proposition 2.1.3 it follows that the closed-loop system is a well-posed Cauchy problem. Let

$z_o \in W$, $z_{c_o} \in \mathbb{R}^{N_c}$ and $z(t)$, $z_c(t)$ be defined by (2.4.1)–(2.4.2) and (2.4.6)–(2.4.7), respectively. Since $z(t) \in W$, if $x(t)$ is defined by

$$x(t) \quad = \quad iz_c(t) - z(t) \qquad t \geq 0,$$

then $x(t)$ belongs to W and it is straightforward to show that

$$\dot{z}_c(t) = \pi\, A_K\, i\, z_c(t) - \pi\, GC\, x(t). \tag{2.4.8}$$

Therefore,

$$\begin{aligned} x(t) &= i\,\pi\, S_K(t)\, i\, z_{c_o} - \int_0^t i\,\pi\, S_K(t-s)\, i\,\pi\, GC\, x(s)\, ds \;-\; z(t) \\ &= S_K(t)\, i\, z_{c_o} - \int_0^t S_K(t-s)\, GC\, x(s)\, ds \;-\; z(t) \\ &= S(t)\, i\, w_o - \int_0^t S(t-s)\, [B\, K\, i\, z_c(s) + GC\, x(s)]\, ds \\ &\qquad - S(t)\, z_0 + \int_0^t S(t-s)\, B\, u(s)\, ds \\ &= S(t)\, x(0) - \int_0^t S(t-s)\, GC\, x(s)\, ds, \end{aligned}$$

which implies that $x(t) = S_G(t)\, x(0)$. The stability of $x(t)$, $z_c(t)$ and $z(t)$ follows.∎

2.4.2 Optimal Projection Theory

Consider the steady-state fixed-order dynamic compensator problem, defined by the infinite-dimensional control system

$$\dot{z}(t) = Az(t) + Bu(t) + H_1\eta(t) \tag{2.4.9}$$

with measurements

$$y(t) = Cz(t) + H_2\eta(t). \tag{2.4.10}$$

The objective is to design a finite-dimensional fixed-order dynamic compensator

$$\dot{z}_c(t) = A_c z_c(t) + B_c y(t) \tag{2.4.11}$$

$$u(t) = C_c z_c(t) \tag{2.4.12}$$

which minimizes the steady-state performance criterion

$$\begin{aligned} J(A_c, B_c, C_c) &\stackrel{\text{def}}{=} \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{E}\Big[\langle R_1 z(s)\,,\, z(s)\rangle \,+\, \langle R_2 u(s), u(s)\rangle\Big]\, ds \qquad (2.4.13) \\ &= \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{E}\left[R_1 \left\|z(s)\right\|_H^2 + R_2 \left\|u(s)\right\|_U^2\right] ds \end{aligned}$$

where all operators satisfy all the assumptions given in section 2.1. The compensator will be assumed to be of fixed, finite order N_c (i.e., $z_c(t) \in \mathbb{R}^{N_c}$) and the optimization is performed over $A_c \in \mathbb{R}^{N_c \times N_c}$, $B_c \in \mathbb{R}^{N_c \times \ell}$ and $C_c \in \mathbb{R}^{m \times N_c}$.

If one introduces the augmented state space $\mathcal{H} = H \times \Re^{N_c}$, then the closed–loop system becomes a linear system on $\mathcal{H}$. Consequently, define the closed-loop operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subseteq \mathcal{H} \to \mathcal{H}$ on the dense domain $\mathcal{D}(\mathcal{A}) \equiv \{(z, z_c) \in \mathcal{H} : Az + BC_c z_c \in H;\ B_c Cz + A_c z_c \in H\}$ by

$$\mathcal{A} = \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & BC_c \\ B_c C & A_c \end{bmatrix}.$$

Since the operator

$$\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} : \mathcal{D}(\mathcal{A}) \to \mathcal{H},$$

generates an analytic semigroup

$$\begin{bmatrix} e^{At} & 0 \\ 0 & I_{N_c} \end{bmatrix} \qquad t \geq 0,$$

and that A_c, BC_c, B_cC are of finite rank, then $\mathcal{A}$ is also closed and generates an analytic semigroup $e^{\mathcal{A}t}$ on $\mathcal{H}$ (see [15]). To guarantee that J is finite and independent of initial conditions we restrict our attention to the set of admissible compensators defined by

$$\mathcal{S} = \{(A_c, B_c, C_c) \ : \ e^{\mathcal{A}t} \text{ is exponentially stable}\}. \tag{2.4.14}$$

If $(A_c, B_c, C_c) \in \mathcal{S}$, then there exist $M \geq 1$ and $w > 0$ such that

$$\|e^{\mathcal{A}t}\| \leq Me^{-wt} \qquad t \geq 0. \tag{2.4.15}$$

Moreover, we know from Theorem 2.4.1 above that $\mathcal{S}$ is non-empty. We now state some results found in [3] and [4].

Lemma 2.4.2 If $\hat{Q}$ and $\hat{P} \ \in \mathcal{L}(H)$ have finite rank and are nonnegative definite, then $\hat{Q}\hat{P}$ is nonnegative semi-simple. Furthermore, if rank $(\hat{Q}\hat{P}) = N_c$, then there exist G and $\Gamma \in \mathcal{L}(H, \mathbb{R}^{N_c})$ and a positive semi-simple matrix $M \in \mathbb{R}^{N_c \times N_c}$ such that

$$\hat{Q}\hat{P} \ = \ G^* M \Gamma \tag{2.4.16}$$

$$\Gamma G^* \ = \ I_{N_c}. \tag{2.4.17}$$

Proof: Bernstein and Hyland give a complete proof of this result in [3]. Here we outline their proof in order to illustrate the form of the factorization of $\hat{Q}\hat{P}$ and to provide a description of the operators G and Γ. Since $\hat{Q}$ and $\hat{P}$ have finite rank, there exists a finite dimensional subspace $\mathcal{Z} \subset H$ such that $\hat{Q}\mathcal{Z} \subset \mathcal{Z}$, $\hat{Q}\mathcal{Z}^{\perp} = 0$, $\hat{P}\mathcal{Z} \subset \mathcal{Z}$ and $\hat{P}\mathcal{Z}^{\perp} = 0$. Hence there exists an orthonormal basis for H and in this basis $\hat{Q}$ and $\hat{P}$ have the infinite matrix representations

$$\hat{Q} = \begin{bmatrix} \hat{Q}_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \hat{P} = \begin{bmatrix} \hat{P}_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where $\hat{Q}_1$, $\hat{P}_1 \in \mathbb{R}^{r\times r}$ and $r = \dim \mathcal{Z}$. Consequently, there exists an invertible $\Phi \in \mathbb{R}^{r\times r}$ such that $\hat{\Lambda} = \Phi^{-1}\hat{Q}_1\hat{P}_1\Phi$ is nonnegative and diagonal and $\hat{Q}\hat{P}$ is nonnegative and semisimple. If rank $(\hat{Q}\hat{P}) = N_c$, then it is clear that Φ can be chosen so that

$$\hat{\Lambda} = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix}$$

where $\Lambda \in \mathbb{R}^{N_c\times N_c}$ is positive and diagonal. Hence,

$$\hat{Q}\hat{P} = \begin{bmatrix} \Phi & 0 \\ 0 & I_\infty \end{bmatrix} \begin{bmatrix} \begin{bmatrix} I_{N_c} \\ 0 \end{bmatrix} \\ 0 \end{bmatrix} \Lambda \begin{bmatrix} \begin{bmatrix} I_{N_c} & 0 \end{bmatrix} & 0 \end{bmatrix} \begin{bmatrix} \Phi^{-1} & 0 \\ 0 & I_\infty \end{bmatrix},$$

and if we define G, M and Γ by

$$\begin{aligned} G &= \begin{bmatrix} \begin{bmatrix} S^{\mathrm{T}} & 0 \end{bmatrix} & 0 \end{bmatrix} \begin{bmatrix} \Phi^{\mathrm{T}} & 0 \\ 0 & I_\infty \end{bmatrix} \\ \Gamma &= \begin{bmatrix} \begin{bmatrix} S^{-1} & 0 \end{bmatrix} & 0 \end{bmatrix} \begin{bmatrix} \Phi^{-1} & 0 \\ 0 & I_\infty \end{bmatrix} \\ M &= S^{-1}\Lambda S, \end{aligned}$$

for any invertible $S \in \mathbb{R}^{N_c\times N_c}$, then G, Γ and M provide the desired factorization and this completes the proof.$\blacksquare$

Throughout the paper we will refer to G, Γ and M satisfying the above lemma as a $(G-M-\Gamma)$–factorization of $\hat{Q}\hat{P}$. For convenience we define $\Sigma = BR_2^{-1}B^*$ and $\overline{\Sigma} = C^*V_2^{-1}C$ and let I_{N_c} and I_H denote respectively the $N_c \times N_c$ identity matrix and the identity operator on H, respectively. We state Bernstein's and Hyland's main theorem which provides a set of necessary conditions that characterize the optimal steady-state fixed order dynamic compensator for bounded input and output operators (see [3]).

Theorem 2.4.3 Let B and C be bounded operators and let N_c be given and suppose that there exists a controllable and observable dynamic compensator $(A_c, B_c, C_c) \in \mathcal{S}$ of order N_c which minimizes J given by (2.4.13), then there exist nonnegative definite operators Q, P, $\hat{Q}$, and $\hat{P}$ on H such that A_c, B_c, and C_c are given by

$$A_c = \Gamma(A - Q\overline{\Sigma} - \Sigma P)G^* \tag{2.4.18}$$

$$B_c = \Gamma Q C^* V_2^{-1} \tag{2.4.19}$$

$$C_c = -R_2^{-1}B^*PG^* \tag{2.4.20}$$

for some $(G-M-\Gamma)$–factorization of $\hat{Q}\hat{P}$ and such that, with $\tau = G^*\Gamma \in \mathcal{L}(H)$, the following conditions are satisfied:

$$\begin{array}{llll} Q : \mathcal{D}(A^*) \to \mathcal{D}(A) & & P : \mathcal{D}(A) \to \mathcal{D}(A^*) & \\ \hat{Q} : H \to \mathcal{D}(A) & & \hat{P} : H \to \mathcal{D}(A^*) & \\ \text{rank } (\hat{Q}) = \text{rank } (\hat{P}) & & = \text{rank } (\hat{Q}\hat{P}) = N_c & \end{array}$$

and

$$0 = (A - \tau Q\overline{\Sigma})Q + Q(A - \tau Q\overline{\Sigma})^* + V_1 + \tau Q\overline{\Sigma}Q\tau^* \tag{2.4.21}$$

$$0 = (A - \Sigma P\tau)^*P + P(A - \Sigma P\tau) + R_1 + \tau^* P\Sigma P\tau \tag{2.4.22}$$

$$0 = \left[(A - \Sigma P)\hat{Q} + \hat{Q}(A - \Sigma P)^* + Q\overline{\Sigma}Q\right]\tau^* \tag{2.4.23}$$

$$0 = \left[(A - Q\overline{\Sigma})^*\hat{P} + \hat{P}(A - Q\overline{\Sigma}) + P\Sigma P\right]\tau. \tag{2.4.24}$$

Note that these necessary conditions consist of a system of four operator equations, including a pair of modified Riccati equations and a pair of modified Lyapunov equations which are coupled by the operator $\tau \in \mathcal{L}(H)$. The operator τ is idempotent, since $\tau^2 = \tau\tau = G^*\Gamma G^*\Gamma = G^* I_{N_c}\Gamma = G^*\Gamma = \tau$. In general τ is an oblique projection and may not be orthogonal since there is no requirement that τ be self-adjoint. Moreover, we note that in view of Lemma 2.4.2, Theorem 2.4.3 applies to $(SA_cS^{-1}, SB_c, C_cS^{-1})$ for any invertible $S \in \mathbb{R}^{N_c \times N_c}$, since the $(G - M - \Gamma)$–factorization of $\hat{Q}\hat{P}$, used to determine A_c, B_c and C_c, is not unique. However, the operator τ remains invariant over the class of factorizations. An easy computation yields the following identities:

$$\hat{Q} = \tau\hat{Q} \qquad \text{and} \quad \hat{P} = \hat{P}\tau. \tag{2.4.25}$$

It is helpful to have an alternative form of the optimal projection equations to actually compute the optimal fixed-order compensator of the approximating finite-dimensional plant. The following result for bounded input bounded output operators may be found in [3].

Proposition 2.4.4 If B and C are bounded, then the optimal projection equations (2.4.21)–(2.4.24) are equivalent, respectively, to

$$
\begin{aligned}
0 &= AQ + QA^* + V_1 - Q\overline{\Sigma}Q + \tau_\perp Q\overline{\Sigma}Q\tau_\perp^* && (2.4.26)\\
0 &= A^*P + PA + R_1 - P\Sigma P + \tau_\perp^* P\Sigma P\tau_\perp && (2.4.27)\\
0 &= A_p\hat{Q} + \hat{Q}A_p + Q\overline{\Sigma}Q - \tau_\perp Q\overline{\Sigma}Q\tau_\perp^* && (2.4.28)\\
0 &= A_q^*\hat{P} + \hat{P}A_q + P\Sigma P - \tau_\perp^* P\Sigma P\tau_\perp && (2.4.29)
\end{aligned}
$$

where

$$\tau_\perp = I_H - \tau, \qquad A_p = A - \Sigma P \ \text{ and } \ A_q = A - Q\overline{\Sigma}. \tag{2.4.30}$$

This form of the optimal projection equations shows that there is a connection between Theorem 2.4.3 and the standard LQG result when dim $H = N < \infty$. In this case, we note that the $(G - M - \Gamma)$-factorization of $\hat{Q}\hat{P}$ when $N_c = N$ is given by $G = \Gamma = I_N$ and $M = \hat{Q}\hat{P}$. Since $\tau = I_N$ and $\tau_\perp = 0$, it follows that (2.4.26)–(2.4.27) reduce to the standard observer and regulator Riccati equations.

To obtain a geometric interpretation of the optimal projection we introduce the "quasi-full-state" estimate

$$\hat{z}(t) = G^* z_c(t) \in H, \tag{2.4.31}$$

so that $\tau\hat{z}(t) = \hat{z}(t)$ and $z_c(t) = \Gamma\hat{z}(t)$. Hence, the closed-loop system can be written as

$$
\begin{aligned}
\dot{z}(t) &= Az(t) + B\hat{C}_c\tau\hat{z}(t) && (2.4.32)\\
\dot{\hat{z}}(t) &= \tau(A + B\hat{C}_c - \hat{B}_cC)\tau\hat{z}(t) + \tau\hat{B}_cCz(t) && (2.4.33)
\end{aligned}
$$

where

$$\hat{B}_c = QC^*V_2^{-1} \qquad \text{and} \quad \hat{C}_c = -R_2^{-1}B^*P. \tag{2.4.34}$$

This shows that the geometric structure of the quasi-full-order compensator is dictated by the projection τ. Sensor inputs $\tau\hat{B}_cCz$ are annihilated unless they are contained in $\mathcal{R}(\tau^*) = \mathcal{N}(\tau)^{\perp}$, while $\tau\hat{z}$ employed in the control input is contained in $\mathcal{R}(\tau)$. Consequently, $\mathcal{R}(\tau)$ and $\mathcal{R}(\tau^*)$ are the control and observation subspaces of the compensator, respectively.

Remark 2.4.5 Also, in the case of fixed-finite-dimensional compensator, we solved the α–shifted problem, then we applied the controller to the original problem with the weighted performance criterion.

Chapter 3

CONVERGENCE AND STABILITY OF THE APPROXIMATING SYSTEM

3.1 Introduction

The main goal of this chapter is to construct a finite-dimensional dynamic compensator z_c^h, which will be based on a finite element approximation of the original problem, such that the finite-dimensional control feedback

$$u_h(t) = C_c^h z_c^h(t) \tag{3.1.1}$$

once inserted into the original system, would produce the solutions which are uniformly exponentially stable in h.

In order to establish the appropriate convergence and stability results for the compensator, the techniques recently developed in the context of finite element approximations of Riccati Equations with unbounded control input and output operators (see [21]) will be used in an essential way and will be discussed in Section 3.2. In Section 3.3, we will apply these approximations to the Burgers' equation and present the finite-dimensional matrix representations that will be used for the numerical experiments of the three approaches discussed in Chapter II.

3.2 Approximations of Spaces and Operators

We suppose that H, W, Z, U, Y and $\tilde{H}$ are separable Hilbert spaces, that $A : \mathcal{D}(A) \subset H \to H$ is the infinitesimal generator of a semigroup $S(t)$ on H and that

$$B \in \mathcal{L}(U, Z) \qquad \text{and} \qquad C \in \mathcal{L}(W, Y) \tag{3.2.1}$$

satisfying hypotheses (H-1)-(H-3).

Consider a control problem given by

$$\begin{aligned} \frac{d}{dt} z(t) &= Az(t) + Bu(t) + H_1 \eta(t) \\ y(t) &= Cz(t) + H_2 \eta(t) \\ w(t) &= E_1 z(t) + E_2 u(t) \end{aligned} \tag{3.2.2}$$

and an associated steady-state performance index

$$J(u) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \mathbf{E} \left[R_1 \|z(s)\|_H^2 + R_2 \|u(s)\|_U^2 \right] ds. \tag{3.2.3}$$

The abstract linear quadratic-Gaussian problem is stated as

($\mathcal{P}$): Find an admissible controller $u(\cdot) \in L^2(0, \infty; U)$ minimizing $J(u)$ subject to the system (3.2.2) such that the closed-loop is exponentially stable.

The following theorem can be viewed as a revised version of Theorem 2.1.2.

Theorem 3.2.1 Assume that (A, B) and (A, H_1) are stabilizable and that (A, C) and (A, E_1) are detectable. Then the minimum of J in (3.2.3) is attained by the optimal feedback control

$$u(t) = -R_2^{-1}B^*Pz_c(t) \tag{3.2.4}$$

where P is the unique self-adjoint solution of the algebraic Riccati equation

$$\begin{aligned} 0 &= \; < Px, Ay >_H + < Ax, Py >_H + < R_1x, y >_H - \\ & \quad < PBR_2^{-1}B^*Px, y >_H \qquad \forall (x, y) \in \mathcal{D}(A). \end{aligned} \tag{3.2.5}$$

$z_c(t)$ is defined by

$$\dot{z}_c(t) = (A - QC^*V_2^{-1}C - BR_2^{-1}B^*P)z_c(t) + QC^*V_2^{-1}y(t) \tag{3.2.6}$$

where Q is the unique self-adjoint solution of

$$\begin{aligned} 0 &= \; < Qx, A^*y >_H + < A^*x, Qy >_H + < V_1x, y >_H - \\ & \quad < QC^*V_2^{-1}CQx, y >_H \qquad \forall (x, y) \in \mathcal{D}(A^*). \end{aligned} \tag{3.2.7}$$

Moreover, the semigroup $\mathcal{S}(t)$ generated by the closed-loop system operator $\mathcal{A}$ is exponentially stable.

We next formulate a sequence of approximate linear quadratic-Gaussian problems and present a convergence result for the corresponding Riccati operators. In order to study convergence and stability of the approximating compensator we introduce a family of approximating subspaces $V_h \subset H \cap \mathcal{D}(B^*) \cap \mathcal{D}(\bar{A}^r)$, where h is a parameter of discretization which tends to zero, $h \leq h_0$. Let π_h be the orthogonal projection of V into V_h with the usual approximating property:

$$|\pi_h z - z|_H < Nh^s \, |z|_{\mathcal{D}(A)}, \qquad \text{for some} \quad s > 0 \;\text{ and constant }\; N > 0. \tag{3.2.8}$$

We then consider the family of linear quadratic-Gaussian problems

$(\mathcal{P}^h)$: Find an admissible controller $u(\cdot) \in L^2(0,\infty;U)$ for the control system

$$\begin{aligned} \frac{d}{dt}z^h(t) &= A^h z^h(t) + B^h u(t) + H_1^h \eta(t) \\ y^h(t) &= C^h z^h(t) + H_2^h \eta(t) \\ w^h(t) &= E_1^h z^h(t) + E_2^h u(t) \end{aligned} \tag{3.2.9}$$

by minimizing

$$J^h(u) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{E}\left[R_1^h \left\|z^h(s)\right\|_H^2 + R_2 \|u(s)\|_U^2\right] ds \tag{3.2.10}$$

so that the closed–loop system is exponentially stable.

Remark 3.2.2 If, for any $h \le h_0$, (A^h, B^h) and (A^h, H_1^h) are stabilizable and (A^h, C^h) and (A^h, E_1^h) are detectable, then by Theorem 3.2.1, there is a unique optimal controller given by

$$u(t) = -R_2^{-1}(B^h)^* P^h z_c^h(t) \tag{3.2.11}$$

where P^h is the unique self-adjoint solution of the algebraic Riccati equation

$$0 = (A^h)^* P^h + P^h A^h - P^h B^h R_2^{-1} (B^h)^* P^h + R_1^h. \tag{3.2.12}$$

$z_c^h(t)$ is defined by

$$\dot{z}_c^h(t) = [A^h - Q^h(C^h)^* V_2^{-1} C^h - B^h R_2^{-1}(B^h)^* P^h] z_c^h(t) + Q^h (C^h)^* V_2^{-1} y^h(t) \tag{3.2.13}$$

where Q^h is the unique self-adjoint solution of

$$0 = A^h Q^h + Q^h (A^h)^* - Q^h (C^h)^* V_2^{-1} C^h Q^h + V_1^h. \tag{3.2.14}$$

For the finite-dimensional approximation systems, it is not clear that (A^h, B^h) is stabilizable even if the original system (A, B) is stabilizable. Similar observations can be made for the detectability conditions, by duality. Therefore, the question we have to consider here is the convergence of the approximates P^h and Q^h and subsequently, the convergence of the approximate control $u^h(t)$ to the infinite-dimensional solutions P, Q and $u(t)$, respectively. These issues are at the heart of all approximation theories that have been developed for control. We shall use the theory developed by Lasiecka [20] for parabolic problems. We summarize her main results here and refer to [20] for proofs.

Let $A_h : V_h \to V_h$ be approximations of A satisfying:

(A-1) There exist $\mathrm{w} \in \mathbb{R}$ and $\theta_o, \pi/2 < \theta_o < \pi$ such that the spectrum $\sigma(A)$ of A contains a sector S_{w,θ_o} defined by

$$S_{\mathrm{w},\theta_o} = \{\lambda \in \mathbb{C} : \lambda \neq \mathrm{w} \quad \text{and} \quad |\arg\,(\lambda - \mathrm{w})| < \theta_o\} \tag{3.2.15}$$

and then there exists h_{W} such that for all $0 \leq h \leq h_{\mathrm{W}}$ we have $\sigma(A_h) =$ spectrum of $A_h \subset S_{\mathrm{w},\theta_o}$

$$\left|R(\lambda, A_h)\bar{A}_h^{\mu}\pi_h\right|_{\mathcal{L}(H)} \leq \frac{N}{|\lambda - \mathrm{w}|^{1-\mu}} \qquad \forall\lambda \notin S_{\mathrm{w},\theta_o}$$

(A-2) $\left|\pi_h\bar{A}^{-1} - \bar{A}_h^{-1}\pi_h\right|_{\mathcal{L}(H)} < Nh^s.$

In addition, we assume that B and C satisfy:

(A-3) (i) $|B^* z_h|_U \leq Nh^{-\gamma s}\,|z_h|_H\,, \qquad z_h \in V_h$

(ii) $|\bar{A}^r z_h|_H \leq Nh^{-rs}\,|z_h|_H$

(A-4) (i) $|B^*(\pi_h - I)z|_U \leq Nh^{(1-\gamma)s}\,|z|_{\mathcal{D}(A^*)}$

(ii) $\left|\bar{A}^r(\pi_h - I)z\right|_H \leq Nh^{(1-r)s}\,|z|_{\mathcal{D}(A)}$

(A-5) (i) $|B^*\pi_h z|_U \leq N\left|\bar{A}^{*\gamma}z\right|_H$

(ii) $\left|\bar{A}^*\pi_h z\right|_H \leq N\left|\bar{A}^r z\right|_H$.

Lasiecka also requires the following conditions to hold for the approximations of operators B_c and C_c

(A-6) The operator $C_c^h : V_h \to U$ satisfies one of the following conditions:

(i) either $C_c^h\pi_h \to C_c$ strongly and $B^*R(\lambda_0, A^*)$ is compact

(ii) or else $\left|C_c^h\pi_h - C_c\right|_{\mathcal{L}(H,U)} \longrightarrow 0 \qquad \text{as } h \to 0$

(A-7) Similarly, for $B_c : Y \to V_h$

(i) either $B_c^h \longrightarrow B_c$ strongly and $CR(\lambda_0, A)$ is compact

(ii) or else $\left|B_c^h - B_c\right|_{\mathcal{L}(Y,H)} \longrightarrow 0 \qquad \text{as } h \to 0$

Consider the following finite-dimensional dynamic compensator

$$\dot{z}_c(t) = (A_h\pi_h + \pi_h BC_c^h\pi_h - B_c^h C)z_c(t) + B_c^h Cz(t) \tag{3.2.16}$$

when coupled with the associated control system

$$\begin{aligned} \dot{z}(t) &= Az(t) + BC_c^h\pi_h z(t) \\ z(0) &= z_o \in H. \end{aligned} \tag{3.2.17}$$

The following result may be found in [20].

Theorem 3.2.3 Assume hypotheses (H-1)-(H-5) and (C-1) hold along with the approximation conditions (A-1)-(A-7). Moreover, assume $\gamma + r < 1$. Then, the systems (3.2.16) and (3.2.17) represented by

$$\mathcal{A}_h \equiv \begin{bmatrix} A & BC_c^h\pi_h \\ B_c^hC & A_h\pi_h + \pi_h BC_c^h\pi_h - B_c^hC \end{bmatrix} \tag{3.2.18}$$

generate an uniformly exponentially stable analytic semigroup on $\mathcal{H}$ with

$$\left\| e^{\mathcal{A}_h t} \right\|_{\mathcal{L}(\mathcal{H})} \leq Me^{-w_0 t}, \tag{3.2.19}$$

where M and w_0 are independent of h.

We now return to the problem $(\mathcal{P}^h)$ where we recall recent results from [21] and [22] on solvability and approximations of the algebraic Riccati equations arising in control dynamics with unbounded input/output operators.

By the virtue of Theorem 3.2.1 under hypotheses (H-1)-(H-5) and (C-1), we know that there exist unique solutions P and $Q \in \mathcal{L}(H)$ such that $B^*P \in \mathcal{L}(H,U)$ and $QC^* \in \mathcal{L}(Y,H)$ and that both $(A - BR_2^{-1}B^*P)$ and $(A - QC^*V_2^{-1}C)$ generate analytic and exponentially stable semigroups. On the other hand, the approximating properties (A-1)–(A-7) guarantee (see [21]) that the approximating ARE corresponding to the the infinite-dimensional ARE given by (3.2.5) and (3.2.7) are uniquely solvable with P_h and $Q_h \in \mathcal{L}(V_h)$. Moreover, if we take $B^h = \pi_h B$ and $C^h = C\pi_h$ then we will have $B^*P_h\pi_h \in \mathcal{L}(H,U)$ and $CQ_h\pi_h \in \mathcal{L}(H,Y)$ uniformly in $h > 0$ and

$$|P_h\pi_h - P|_{\mathcal{L}(H)} + |B^*(P_h\pi_h - P)|_{\mathcal{L}(H,U)} \longrightarrow 0 \quad \text{as} \quad h \to 0 \tag{3.2.20}$$

$$|C(Q_h\pi_h - Q)|_{\mathcal{L}(H,Y)} \longrightarrow 0 \qquad \text{as} \quad h \to 0. \tag{3.2.21}$$

Thus, we are in a position to apply the results of Theorem 3.2.3 with the specific feedback law

$$C_c^h = -R_2^{-1}B^*P_h \qquad \text{and} \qquad B_c^h = Q_h\pi_h C^* V_2^{-1}. \tag{3.2.22}$$

Indeed, part (ii) of assumptions (A-6)-(A-7) are satisfied by the convergence results given by equations (3.2.20) and (3.2.21) with $C_c = -R_2^{-1}B^*P$ and $B_c = QC^*V_2^{-1}$ and C_c^h, B_c^h as in (3.2.22). Thus we have the following result.

Corollary 3.2.4 If hypotheses (H-1)–(H-5), (C-1), assumptions (A-1)–(A-7) hold and $\gamma + r < 1$, then the conclusions of Theorem 3.2.3 hold with C_c^h and B_c^h given by (3.2.22).

Remark 3.2.5 For approximations of the MinMax problem, one can follow the same procedure since B_c and C_c have the structure as that of the (LQG) problems and the new operator $\bar{M}$ is given as a function of the operators P and Q. Although the MinMax problem produces more complex Riccati equations, condition b-(ii) in Theorem 2.3.3 reduces the problem to a form where a straight forward generalization of Lasiecka's proofs yields convergence.

3.3 Approximations of Burgers' Equation

In this section, we apply the approximation results discussed in the previous section to the linearized Burgers' equation. The governing equation is given by

$$
\begin{aligned}
\frac{\partial}{\partial t} z(x,t) &= \epsilon \frac{\partial^2}{\partial x^2} z(x,t) + \hbar(x)\eta(t) \\
-\frac{\partial}{\partial x} z(0,t) &= u_1(t) \\
\frac{\partial}{\partial x} z(\ell,t) &= u_2(t) \\
z(x,0) &= z_o(x) \in H,
\end{aligned}
\tag{3.3.1}
$$

and the output is given by

$$
y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} z(0,t) \\ z(\ell,t) \end{bmatrix} + H_2\eta(t) \tag{3.3.2}
$$

where $H = L^2(0,\ell)$. In Section 2.2, we formulated the above system as an abstract system

$$
\begin{aligned}
\frac{d}{dt} z(t) &= A_\epsilon z(t) + Bu(t) + H_1\eta(t) \\
y(t) &= Cz(t) + H_2\eta(t)
\end{aligned}
\tag{3.3.3}
$$

in the space $H = L^2(0,\ell)$. Here, $A_\epsilon\phi = \epsilon\phi''$ for all $\phi \in H^2(0,\ell) \cap H^1(0,\ell)$, $B \in \mathcal{L}(U,Z)$ and $C \in \mathcal{L}(W,Y)$, where spaces $W = Z' = \mathcal{D}(\bar{A}_\epsilon^{(\frac{1}{4}+\rho)})$ and $B^*\phi = \epsilon C\phi = \epsilon\phi|_\Gamma$ for all $\phi \in V = H^1(0,\ell) \subset W \subset H$.

To apply the approximation framework above, we introduce $V_h \subset V = H^1(0,\ell)$ to be the space of piecewise linear splines which comply with the usual approximation properties:

$$
|\pi_h z - z|_{H^k(0,\ell)} \le N h^{s-k} |z|_{H^s}, \qquad s \le 2, \qquad s-k \ge 0, \;\; 0 \le k \le 1 \tag{3.3.4}
$$

$$
|z_h|_{H^\alpha} \le N\,|z_h|_{L^2(0,\ell)}, \qquad 0 \le \alpha < 1 \tag{3.3.5}
$$

$$
\left|D^k(z - \pi_h z)\right|_{L^2(\Gamma)} \le N h^{s-k-\frac{1}{2}} |z|_{H^s}, \qquad \frac{3}{2} < s \le 2, \;\; k = 0,1 \tag{3.3.6}
$$

$$\left|D^k z_h\right|_{L^2(\Gamma)} \leq N h^{-k-\frac{1}{2}} \left|z_h\right|_{L^2(0,\ell)}, \qquad k = 0, 1 \tag{3.3.7}$$

The approximating finite-dimensional representation A_ϵ^h of A_ϵ is obtained by restricting the sesquilinear form $a(\cdot,\cdot)$ (presented in section 2.2) to $V_h \times V_h$. We obtain a representation A_ϵ^h of A_ϵ satisfying

$$a(z, w) = < -A_\epsilon^h z, w > \tag{3.3.8}$$

and

$$a(z, w) = < -(A_\epsilon^h)^* w, z > \tag{3.3.9}$$

for all z, $w \in V_h$. It is well known from the condition of uniform analyticity and classical results on approximation of elliptic problems that assumptions (A-1)–(A-2) hold with $s = 2$. Next, since $B^* z = \epsilon z|_\Gamma$, property (i) of assumptions (A-3)–(A-4) follow from (3.3.7) applied to the case $k = 0$. Also, (A-5)(i) holds, since

$$\left|B^* \pi_h z\right|_{L^2(\Gamma)} \leq \epsilon \left|\pi_h z\right|_\Gamma \leq N \left|\pi_h z\right|_{H^{\frac{1}{2}+2\rho}(\Omega)} \leq N \left|z\right|_{\mathcal{D}(\bar{A}_\epsilon^{\frac{1}{4}+\rho})}, \tag{3.3.10}$$

where we have used (3.3.4) and the trace theorem (see [2]). Since

$$\left|\bar{A}_\epsilon^r z\right|_{L^2(\Omega)} \simeq \left|z\right|_{H^{2r}(\Omega)}, \tag{3.3.11}$$

the part (ii) of (A-3)–(A-5) follows from (3.3.4) and (3.3.5). Assumptions (A-6)–(A-7) hold with C_c^h and B_c^h taken as the projections of C_c and B_c respectively, i.e., $C_c^h = C_c \pi_h$, $B_c^h = \pi_h B_c$. Therefore, all of the hypotheses in Theorem 3.2.3 are satisfied.

Now, consider the matrix representations of operators on the space V_h. Throughout the rest of the section we assume that $\ell = 1$ and the discretization parameter h is taken to be

$h = \frac{1}{N+1}$ for convenience. We divide the unit interval [0, 1] into $N+1$ equal subintervals $[x_i, x_{i+1}]$, $x_i = \frac{i}{N+1}$, $i = 0, 1, 2, \cdots, N+1$. For each i, $0 \le i \le N+1$, let $h_i^N(x)$ denote the linear spline basis function defined by

$$h_0^N(x) = \begin{cases} -(N+1)(x - x_1), & x_0 \le x \le x_1 \\ 0, & \text{otherwise} \end{cases} \tag{3.3.12}$$

$$h_i^N(x) = \begin{cases} (N+1)(x - x_{i-1}), & x_{i-1} \le x \le x_i \\ -(N+1)(x - x_{i+1}), & x_i \le x \le x_{i+1} \\ 0, & \text{otherwise.} \end{cases} \tag{3.3.13}$$

for $1 \le i \le N$ and

$$h_{N+1}^N(x) = \begin{cases} (N+1)(x - x_N), & x_N \le x \le x_{N+1} \\ 0, & \text{otherwise.} \end{cases} \tag{3.3.14}$$

Let V^N be the $(N+2)$-dimensional finite element space given by

$$V^N = \left\{ \sum_{i=0}^{N+1} z_i h_i^N(x) : z_i \in \mathbb{R},\ i = 0, 1, 2, \cdots, N+1 \right\}. \tag{3.3.15}$$

The approximate solution $z^N(x,t)$ of $z(x,t)$ on V^N is given by

$$z^N(x,t) = \sum_{i=0}^{N+1} z_i^N(t) h_i^N(x), \tag{3.3.16}$$

with $z_i^N(t) \in \mathbb{R}$, $i = 0, 1, \cdots, N+1$. Standard finite element/Galerkin procedures applied to equations (3.3.3) and (3.3.16) yield the following finite dimensional ODE system

$$\left[\Psi^N\right] \frac{d}{dt}\{z^N(t)\} = \left[\tilde{A}^N\right]\{z^N(t)\} + \left[\tilde{B}^N\right] u(t) + \left[\tilde{H}_1^N\right] \eta(t) \tag{3.3.17}$$

where $\{z^N(t)\} = [z_0^N(t), \cdots, z_{N+1}^N(t)]^T$,

$$\left[\Psi^N\right] = \left[< h_j^N, h_i^N >\right]_{(N+2)\times(N+2)}$$

$$= \frac{1}{6(N+1)} \begin{bmatrix} 2 & 1 & 0 & 0 & & \cdots & 0 \\ 1 & 4 & 1 & 0 & & \cdots & 0 \\ 0 & 1 & 4 & 1 & & \cdots & 0 \\ & & \ddots & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & & & & 1 & 4 & 1 \\ 0 & & \cdots & & 0 & 1 & 2 \end{bmatrix}_{(N+2)\times(N+2)}, \tag{3.3.18}$$

$$\left[\tilde{A}^N\right] = \epsilon(N+1) \begin{bmatrix} -1 & 1 & 0 & 0 & & \cdots & 0 \\ 1 & -2 & 1 & 0 & & \cdots & 0 \\ 0 & 1 & -2 & 1 & & \cdots & 0 \\ & & \ddots & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & & & & 1 & -2 & 1 \\ 0 & & \cdots & & 0 & 1 & -1 \end{bmatrix}_{(N+2)\times(N+2)}, \tag{3.3.19}$$

$$\left[\tilde{B}^N\right] = \epsilon\left[C^N\right]^T = \epsilon\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T_{2\times(N+2)}, \tag{3.3.20}$$

$$\left[\tilde{H}_1^N\right] = \left[< \hbar, h_0^n >, < \hbar, h_1^N >, \cdots, < \hbar, h_{N+1}^N >\right]^T, \tag{3.3.21}$$

where $< \hbar, h_i^N >= \int_0^1 \hbar(x)h_i^N(x)dx$, $0 \leq i \leq N+1$. Since $\left[\Psi^N\right]$ is invertible, by multiplying $\left[\Psi^N\right]^{-1}$ from the left side of equation (3.3.17), we get

$$\frac{d}{dt}\{z^N(t)\} = [A^N]z^N(t) + [B^N]u(t) + [H_1^N]\eta(t), \tag{3.3.22}$$

$$y^N(t) = [C^N]\{z^N(t)\} + [H_2^N]\eta(t), \tag{3.3.23}$$

$$\{z^N(0)\} = \{z_o^N\}, \tag{3.3.24}$$

where

$$[A^N] = [\Psi^N]^{-1}[\tilde{A}^N], \quad [B^N] = [\Psi^N]^{-1}[\tilde{B}^N], \quad [H_1^N] = [\Psi^N]^{-1}[\tilde{H}_1^N] \tag{3.3.25}$$

$$\{z_o^N\} = [\Psi^N]^{-1}[< z_o, h_0^N >, \cdots, < z_o, h_{N+1}^N >]^T. \tag{3.3.26}$$

Since the computational algorithm for the LQG and MinMax problems is different from the computational method needed for the optimal projection problem, we will discuss their matrix representations separately.

3.3.1 Approximation of the LQG and MinMax Compensator

We consider here the MinMax problem, since the LQG problem can be obtained by letting the disturbance attenuation parameter γ approach $+\infty$. From above, the approximating α–shifted MinMax problem can be stated as

(MM)$_\alpha^N$: Find a finite dimensional controller that minimizes the performance criterion

$$J_\alpha^N(u_\alpha, \eta_\alpha) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{E}\left[R_1^N \left\|z_\alpha^N(s)\right\|_{V^N}^2 + R_2 \left\|u_\alpha(s)\right\|_U^2 \; -\gamma^2 \left\|\eta_\alpha(s)\right\|_{\aleph}^2\right] ds \tag{3.3.27}$$

subject to

$$\begin{aligned} \frac{d}{dt} z_\alpha^N(t) &= (A^N + \alpha I^N) z_\alpha^N(t) + B^N u_\alpha(t) + H_1^N \eta_\alpha(t), \\ y_\alpha^N(t) &= C^N z_\alpha^N(t) + H_2^N \eta_\alpha(t), \\ z_\alpha^N(0) &= z_o^N, \end{aligned} \tag{3.3.28}$$

such that $\|T_{w\eta}\|_\infty < \gamma$ for a suitably chosen γ.

Here, $\alpha > 0$ and every parameter with subscript α is defined by multiplying it by $e^{\alpha t}$, e.g., $z_\alpha^N(t) = z^N(t)e^{\alpha t}$, where $z^N(t)$ is the solution of the original problem. Therefore, we can write the finite dimensional Riccati operator equations as

$$0 = (A^N + \alpha I^N)^* P_\alpha^N + P_\alpha^N (A^N + \alpha I^N) - P_\alpha^N [B^N R_2^{-1} (B^N)^* - \gamma^{-2} V_1^N] P_\alpha^N + R_1^N \quad (3.3.29)$$

$$0 = (A^N + \alpha I^N) Q_\alpha^N + Q_\alpha^N (A^N + \alpha I^N)^* - Q_\alpha^N [(C^N)^* V_2^{-1} C^N - \gamma^{-2} R_1^N] Q_\alpha^N + V_1^N \quad (3.3.30)$$

and the approximating optimal controller which is characterized by the feedback gain, observer gain and operator A_c^N, given by

$$\begin{aligned} C_c^N &= -R_2^{-1} (B^N)^* P_\alpha^N, \qquad \bar{M}^N B_C^N = (I^N - \gamma^{-2} Q_\alpha P_\alpha)^{-1} Q_\alpha (C^N)^* V_2^{-1} & (3.3.31) \\ A_c^N &= A + \gamma^{-2} H_1^N (H_1^N)^* P_\alpha^N + B^N C_c^N - \bar{M}^N B_c^N C^N. & (3.3.32) \end{aligned}$$

Recall that adjoints have the matrix representations

$$\begin{aligned} (A^N)^* &= (\Psi^N)^{-1} (A^N)^T \Psi^N \quad & (B^N)^* &= (B^N)^T \Psi^N \\ (C^N)^* &= (\Psi^N)^{-1} (C^N)^T & (H_1^N)^* &= (H_1^N)^T \Psi^N \qquad (3.3.33) \\ (E_1^N)^* &= (\Psi^N)^{-1} (E_1^N)^T. \end{aligned}$$

Therefore, if we define

$$\tilde{P}_\alpha^N = \Psi^N P_\alpha^N, \; \tilde{Q}_\alpha^N = Q_\alpha^N (\Psi^N)^{-1}, \; \tilde{R}_1^N = (E_1^N)^T E_1^N, \; \tilde{V}_1^N = H_1^N (H_1^N)^T, \quad (3.3.34)$$

then the Riccati operator equations (3.3.29)–(3.3.30) are equivalent to the Riccati matrix equations that are solved numerically:

$$0 = (A^N + \alpha I^N)^T \tilde{P}_\alpha^N + \tilde{P}_\alpha^N (A^N + \alpha I^N) - \tilde{P}_\alpha^N [B^N R_2^{-1} (B^N)^T - \gamma^{-2} \tilde{V}_1^N] \tilde{P}_\alpha^N + \tilde{R}_1^N \quad (3.3.35)$$

$$0 = (A^N + \alpha I^N) \tilde{Q}_\alpha^N + \tilde{Q}_\alpha^N (A^N + \alpha I^N)^T - \tilde{Q}_\alpha^N [(C^N)^T V_2^{-1} C^N - \gamma^{-2} \tilde{R}_1^N] \tilde{Q}_\alpha^N + \tilde{V}_1^N. \quad (3.3.36)$$

The corresponding gains and operator A_c^N are given by

$$\tilde{C}_c^N = -R_2^{-1}(B^N)^T\tilde{P}_\alpha^N, \quad \tilde{M}^N\tilde{B}_c^N = (I^N - \gamma^{-2}\tilde{Q}_\alpha\tilde{P}_\alpha)^{-1}\tilde{Q}_\alpha(C^N)^T V_2^{-1} \tag{3.3.37}$$

$$\tilde{A}_c^N = A^N + \gamma^{-2}H_1^N(H_1^N)^T\tilde{P}_\alpha + B\tilde{C}_c^N - \tilde{M}^N\tilde{B}_c^N C^N. \tag{3.3.38}$$

Moreover, since the functional gains K_α and L_α are elements of $V^N \times V^N$, they can be written as

$$K_\alpha^N(x) = \sum_{i=0}^{N+1}\begin{bmatrix} k_i^0 \\ \\ k_i^1 \end{bmatrix} h_i^N(x), \quad L_\alpha^N(x) = \sum_{i=0}^{N+1}\begin{bmatrix} \ell_i^0 \\ \\ \ell_i^1 \end{bmatrix} h_i^N(x). \tag{3.3.39}$$

Then the coefficients k_i^0, k_i^1, ℓ_i^0 and ℓ_i^1 in equation (3.3.39) can be determined, using the numerical solutions of $\tilde{C}_c^N$ and $\tilde{M}\tilde{B}_c^N$ given by equation (3.3.37), by the formulas

$$\begin{bmatrix} k_0^0, & \cdots, & k_{N+1}^0 \\ \\ k_0^1, & \cdots, & k_{N+1}^1 \end{bmatrix}^T = (\Psi^N)^{-1}\tilde{P}_\alpha^N B^N R_2^{-1}, \tag{3.3.40}$$

$$\begin{bmatrix} \ell_0^0, & \cdots, & \ell_{N+1}^0 \\ \\ \ell_0^1, & \cdots, & \ell_{N+1}^1 \end{bmatrix}^T = (I^N - \gamma^{-2}\tilde{Q}_\alpha^N\tilde{P}_\alpha^N)^{-1}\tilde{Q}_\alpha^N(C^N)^T V_2^{-1}. \tag{3.3.41}$$

Finally, we apply the control laws defined by the functional gains in(3.3.39) obtained from the shifted MinMax problem $(\mathrm{MM})_\alpha$ to Burgers' equation (3.3.3). The resulting finite dimensional (nonlinear) approximating closed–loop system becomes

$$\frac{d}{dt}\begin{bmatrix} z^N(t) \\ \\ z_c^N(t) \end{bmatrix} = \begin{bmatrix} A^N & B^N\tilde{C}_c^N \\ \\ \tilde{M}\tilde{B}_c^N C^N & \tilde{A}_c^N \end{bmatrix}\begin{bmatrix} z^N(t) \\ \\ z_c^N(t) \end{bmatrix} + \begin{bmatrix} H_1^N \\ \\ \tilde{M}\tilde{B}_c H_2^N \end{bmatrix}\eta(t) + \begin{bmatrix} \mathcal{F}^N(z^N(t)) \\ \\ \mathcal{F}^N(z_c^N(t)) \end{bmatrix} \tag{3.3.42}$$

with initial data

$$z^N(0) = z_o^N, \qquad z_c^N(0) = z_{c_o}^N$$

where the nonlinear term in the observer is given by

$$\mathcal{F}^N(z^N(t)) = (\Psi^N)^{-1}\tilde{\mathcal{F}}^N(z^N(t)), \tag{3.3.43}$$

with

$$\tilde{\mathcal{F}}^N(z^N(t)) = \frac{-1}{6}\begin{bmatrix} -2(z_0^N(t))^2 + z_0^N(t)z_1^N(t) + (z_1^N(t))^2 \\ -(z_0^N(t))^2 - z_0^N(t)z_1^N(t) + z_1^N(t)z_2^N(t) + (z_2^N(t))^2 \\ \vdots \\ -(z_{N-1}^N(t))^2 - z_{N-1}^N(t)z_N^N(t) + z_N^N(t)z_{N+1}^N(t) + (z_{N+1}^N(t))^2 \\ -(z_N^N(t))^2 - z_N^N(t)z_{N+1}^N(t) + 2(z_{N+1}^N(t))^2 \end{bmatrix}. \tag{3.3.44}$$

The closed–loop system is defined by substituting $z_c^N(t)$ into (3.3.43) yielding (3.3.42). Theorem 3.2.3 implies that for sufficiently large N the finite dimensional LQG controller will stabilize the (linearized) problem defined by the heat equation. Although this has not be shown to stabilize the nonlinear system, we shall present numerical results that suggest that the linear controller with nonlinear observer will perform well on the nonlinear problem.

3.3.2 Approximations of the Fixed-Order Compensator

In general, the optimal projection equations (2.4.26)–(2.4.29) are infinite dimensional operator equations. In order to use these equations to compute the optimal fixed-finite-order compensator, a finite dimensional approximation is needed (see [4] for details).

Let $A^N \in \mathcal{L}(V^N)$, $B^N \in \mathcal{L}(\mathbb{R}^m,\ V^N)$, $C^N \in \mathcal{L}(V^N,\ \mathbb{R}^\ell)$, $R_1^N \in \mathcal{L}(V^N)$ and $V_1^N \in \mathcal{L}(V^N)$ be given as before. Consider the approximating system

$$\dot{z}^N(t) = A^N z^N(t) + B^N u^N(t) + H_1^N \eta(t) \tag{3.3.45}$$

$$y^N(t) \;=\; C^N z^N(t) + H_2^N \eta(t). \tag{3.3.46}$$

The goal is to design a sequence of finite-dimensional dynamic compensators of fixed order N_c of the form

$$\dot{z}_c^N(t) \;=\; A_c^N z_c^N(t) + B_c^N y^N(t) \tag{3.3.47}$$

$$u^N(t) \;=\; C_c^N z^N(t), \tag{3.3.48}$$

which minimizes the performance criterion

$$J^N(A_c^N, B_c^N, C_c^N) \stackrel{\text{def}}{=} \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{E}\Big[\langle R_1^N z^N(s)\,,\, z^N(s)\rangle \,+\, u(s)^T R_2 u(s)\Big] e^{2\alpha t}\, ds. \tag{3.3.49}$$

Here, the observations made above for the LQG and MinMax problems hold. In particular,

$$\begin{aligned}
(A^N)^* &= (\Psi^N)^{-1}(A^N)^\top\Psi^N & (B^N)^* &= (B^N)^\top\Psi^N \\
(C^N)^* &= (\Psi^N)^{-1}(C^N)^\top & (\Sigma^N) &= B^N R_2^{-1}(B^N)^\top\Psi^N \\
(\tau_\perp^N)^* &= (\Psi^N)^{-1}(\tau_\perp^N)^\top\Psi^N & (\overline{\Sigma}^N) &= (\Psi^N)^{-1}(C^N)^\top V_2^{-1} C^N
\end{aligned} \tag{3.3.50}$$

and if we define the $(N+2)\times(N+2)$ nonnegative definite matrices

$$\begin{aligned}
Q_0^N &\stackrel{\text{def}}{=} Q^N(\Psi^N)^{-1} & P_0^N &\stackrel{\text{def}}{=} \Psi^N P^N \\
\hat{Q}_0^N &\stackrel{\text{def}}{=} \hat{Q}^N(\Psi^N)^{-1} & \hat{P}_0^N &\stackrel{\text{def}}{=} \Psi^N \hat{P}^N \\
V_0^N &\stackrel{\text{def}}{=} V_1^N(\Psi^N)^{-1} & R_0^N &\stackrel{\text{def}}{=} \Psi^N R_1^N \\
\Sigma_0^N &\stackrel{\text{def}}{=} B^N R_2^{-1}(B^N)^\top & \overline{\Sigma}_0^N &\stackrel{\text{def}}{=} (C^N)^\top V_2^{-1} C^N,
\end{aligned} \tag{3.3.51}$$

then the matrix equivalence of the operator equations (2.4.26)–(2.4.29) become

$$0 = (A^N+\alpha I^N)Q_0^N + Q_0^N(A^N+\alpha I^N)^\top + V_0^N - Q_0^N\overline{\Sigma}_0^N Q_0^N + \tau_\perp^N Q_0^N\overline{\Sigma}_0^N Q_0^N(\tau_\perp^N)^\top \tag{3.3.52}$$

$$0 = (A^N+\alpha I^N)^\top P_0^N + P_0^N(A^N+\alpha I^N) + R_0^N - P_0^N\Sigma_0^N P_0^N + (\tau_\perp^N)^\top P_0^N\Sigma_0^N P_0^N\tau_\perp^N \tag{3.3.53}$$

$$0 = (A_{p_o}^N + \alpha I^N \hat{Q}_0^N + \hat{Q}_0^N (A_{p_o}^N + \alpha I^N)^\top + Q_0^N \overline{\Sigma}_0^N Q_0^N - \tau_\perp^N Q_0^N \overline{\Sigma}_0^N Q_0^N (\tau_\perp^N)^\top \tag{3.3.54}$$

$$0 = (A_{q_o}^N + \alpha I^N)^\top \hat{P}_0^N + \hat{P}_0^N (A_{q_o}^N + \alpha I^N) + P_0^N \Sigma_0^N P_0^N - (\tau_\perp^N)^\top P_0^N \Sigma_0^N P_0^N \tau_\perp^N. \tag{3.3.55}$$

The approximating optimal dynamic compensator (A_c^N, B_c^N, C_c^N) of order N_c is then given by

$$\begin{aligned} A_c^N &= \Gamma_0^N (A^N - Q_0^N \overline{\Sigma}_0^N - \Sigma_0^N P_0^N)(G_0^N)^\top && (3.3.56) \\ B_c^N &= \Gamma_0^N Q_0^N (C^N)^\top V_2^{-1} && (3.3.57) \\ C_c^N &= -R_2^{-1} (B^N)^\top P_0^N (G_0^N)^\top && (3.3.58) \end{aligned}$$

where Γ_0^N, $G_0^N \in \mathrm{IR}^{N_c \times (N+2)}$ and $M_0^N \in \mathrm{IR}^{N_c \times N_c}$ provide a $(G_0^N - M_0^N - \Gamma_0^N)$–factorization of $\hat{Q}_0^N \hat{P}_0^N$. The resulting approximating nonlinear closed-loop system is given by

$$\frac{d}{dt} \begin{bmatrix} z^N(t) \\ z_c^N(t) \end{bmatrix} = \begin{bmatrix} A^N & B^N C_c^N \\ B_c^N C^N & A_c^N \end{bmatrix} \begin{bmatrix} z^N(t) \\ z_c^N(t) \end{bmatrix} + \begin{bmatrix} \mathcal{F}^N(z^N(t)) \\ \Gamma_0^N [\mathcal{F}^N (G_0^T z_c^N(t))] \end{bmatrix} \tag{3.3.59}$$

$$z^N(0) = z_o^N, \qquad z_c^N(0) = \Gamma_0^N z_{c_o}^N \tag{3.3.60}$$

where $\mathcal{F}^N(\cdot) = (\Psi^N)^{-1} \tilde{\mathcal{F}}^N(\cdot)$ is defined by (3.3.44).

Remark 3.3.1 We shall apply this reduced order controller to Burgers' equation. In particular, numerical results will be given that indicate that a result similar to Theorem 3.2.3 holds, if one uses the optimal projection method to construct a finite–dimensional control law. However, one must either develop an alternative approach to the proof of convergence, or else one must show that B_c^N and C_c^N in (3.3.56)–(3.3.57) satisfy (A-6) and (A-7), respectively. We shall not address this issue here.

Chapter 4

NUMERICAL RESULTS

4.1 Introduction

In this chapter numerical results will be presented for the LQG, MinMax and Fixed-Order compensations discussed in the preceding chapters. The numerical schemes were implemented on an IBM PC computer. Most of the numerical experiments were done using MATLAB and various MATLAB routines from the Control Toolbox for computing the functional gains by solving the Riccati equations and for solving the nonlinear ODE systems using the 4th-5th order Runge–Kutta method. Also, in the case of the Fixed-order compensator experiment, we used FORTRAN Codes interfaced with the HOMPACK subroutines in solving the four optimal projection equations (3.3.52)–(3.3.55) since they are coupled by the oblique projection matrix and the well-known HOMPACK subroutines use the homotopic continuation method to handle these kind of problems. In Section 4.2, four feedback schemes are introduced. Numerical results will be shown in Section 4.3 where a comparison of the three methods will be made.

4.2 Approximation Schemes

From Sections 2.1, 2.3 and 3.3, we know that the optimal control $u(\cdot) \in L^2(0,1)$, in the cases of the linear quadratic-Gaussian and quadratic differential game problems, is given by a feedback form

$$u(t) = -Kz_c(t) \tag{4.2.1}$$

and the feedback operator $K \in \mathcal{L}(H, U)$ can be presented by

$$Kz_c(t) = \begin{bmatrix} \int_0^1 k^0(s) z_c(s) ds \\ \\ \int_0^1 k^1(s) z_c(s) ds \end{bmatrix} \tag{4.2.2}$$

for some $k^0(\cdot)$, $k^1(\cdot) \in H$, where $H = L^2(0,1)$, $U = Y = \mathbb{R}^2$ and $z_c(t)(\cdot) = z_c(\cdot, t)$ is the corresponding optimal observer trajectory. Next, we introduce four numerical schemes for approximating the feedback law defined by the optimal L^2–gain functions $k^0(\cdot)$, $k^1(\cdot)$ in equation (4.2.2). We use step functions averaging $k^0(\cdot)$, $k^1(\cdot)$ on each interval $I_j = (\tilde{x}_{j-1}, \tilde{x}_j)$ and acting on the optimal state trajectory $z(x,t)$ where $\tilde{x}_0 = 0$, $\tilde{x}_{M+1} = 1$, the average value of $k^0(\cdot)$, $k^1(\cdot)$ on each center point $\hat{x}_j$ of I_j and acting on the optimal state trajectory $z(\hat{x}_j, t)$ evaluated at that center point of each interval, and finally, we use the third approximations except we use a compensator to estimate the states at $\hat{x}_j$.

Scheme 1. First, we consider the feedback form (4.2.2), say K_{opt}, corresponding to the optimal control $u_{\text{opt}}(t)$. Then, on V^N, K^N_{opt} is determined by

$$\begin{aligned} K^N_{\text{opt}} &= \begin{bmatrix} \gamma_0^N, \gamma_1^N, \cdots, \gamma_{N+1}^N \\ \\ \beta_0^N, \beta_1^N, \cdots, \beta_{N+1}^N \end{bmatrix} \\ &= R_2^{-1} [B^N]^T P^N_\alpha \end{aligned} \tag{4.2.3}$$

and hence

$$u_{\mathrm{opt}}^N(t) = -K_{\mathrm{opt}}^N z_c^N(t) = -\sum_{i=0}^{N+1} \begin{bmatrix} \gamma_i^N \\ \\ \beta_i^N \end{bmatrix} z_{ci}^N(t). \tag{4.2.4}$$

Remark 4.2.1 Numerical results presented for Scheme 1 include all three methods discussed in Chapter II. Schemes 2–4 are designed only for the MinMax problem. Of course, those schemes can be applied to the other methods too.

Scheme 2. Consider the following feedback form K_a:

$$K_a = \int_0^1 \left(\sum_{j=1}^{M+1} \begin{bmatrix} a_j^0 \\ \\ a_j^1 \end{bmatrix} \chi_{I_j}(s) \right) z(t,s)ds \tag{4.2.5}$$

where $\tilde{x}_0 = 0,\ \tilde{x}_{M+1} = 1$, for $j = 1, \cdots, M+1$ and $l = 0, 1$

$$\begin{aligned} a_j^l &= \frac{1}{(\tilde{x}_j - \tilde{x}_{j-1})} \int_{I_j} k^l(s)ds \\ &= \text{the average of } k^l(x) \text{ over the interval } I_j, \\ \chi_{I_j} &= \begin{cases} 1, & x \in I_j \\ \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{4.2.6}$$

where for each j, $0 \le j \le M+1$, $\tilde{x}_j$ is the location of a sensor measuring the flow $z(\tilde{x}_j, t)$. Then, on V^N, the control function $u_a^N(t)$ becomes

$$\begin{aligned} u_a^N(t) &= -K_a^N z^N(t) \\ &= -\int_0^1 \left(\sum_{j=1}^{M+1} \begin{bmatrix} a_j^0 \\ \\ a_j^1 \end{bmatrix} \chi_{I_j}^N(s) \right) z^N(t,s)ds \\ &= -\sum_{i=0}^{N+1} \begin{bmatrix} u_{0,i}^N \\ \\ u_{1,i}^N \end{bmatrix} z_i^N(t) \end{aligned} \tag{4.2.7}$$

where

$$u_{l,i}^N = \sum_{j=1}^{M+1} a_j^l \int_{I_j} h_i^N(x)dx, \quad a_j^l = \frac{1}{\tilde{x}_j - \tilde{x}_{j-1}} \sum_{n=0}^{N+1} k_n^l (\int_{I_j} h_n^N(x)dx), \tag{4.2.8}$$

for $l = 0, 1$. Thus, the feedback operator K_a^N can be represented by

$$K_a^N = \begin{bmatrix} u_{0,0}^N, u_{0,1}^N, \cdots, u_{0,N+1}^N \\ \\ u_{1,0}^N, u_{1,1}^N, \cdots, u_{1,N+1}^N \end{bmatrix} \tag{4.2.9}$$

and the finite dimensional closed–loop system is given by

$$\begin{aligned} \frac{d}{dt} z^N(t) &= (A^N - B^N K_a^N) z^N(t) + H_1^N \eta^N(t) \\ z^N(x, 0) &= z_o^N. \end{aligned} \tag{4.2.10}$$

Scheme 3. Consider the feedback form

$$K_{ap} z = \sum_{j=1}^{M+1} \begin{bmatrix} a_j^0 \\ \\ a_j^1 \end{bmatrix} z(\hat{x}_j, t) \tag{4.2.11}$$

where a_j^l's, for $l = 0, 1$, are the same as Scheme 2 and for each j, $1 \leq j \leq M+1$, $\hat{x}_j$ is the center point of the interval $I_j = (\tilde{x}_{j-1}, \tilde{x}_j)$ and $z(\hat{x}_j, t) = \int_{\tilde{x}_{j-1}}^{\tilde{x}_j} z(s, t)ds$. Then, it follows that on V^N, we have

$$\begin{aligned} u_{ap}^N(t) &= -K_{ap}^N z^N(t) \\ &= -\sum_{j=1}^{M+1} \begin{bmatrix} a_j^0 \\ \\ a_j^1 \end{bmatrix} z(\hat{x}_j, t) \\ &= -\sum_{i=0}^{N+1} z_i^N(t) \sum_{j=1}^{M+1} \begin{bmatrix} a_j^0 \\ \\ a_j^1 \end{bmatrix} \int_{\tilde{x}_{j-1}}^{\tilde{x}_j} h_i^N(s)ds \end{aligned}$$

$$= -\sum_{i=0}^{N+1} z_i^N(t) \sum_{j=1}^{M+1} \begin{bmatrix} a_j^0 \\ \\ a_j^1 \end{bmatrix} h_i^N(\hat{x}_j)\Delta\tilde{x}_j$$

$$= -\sum_{i=0}^{N+1} \begin{bmatrix} v_{0,i}^N \\ \\ v_{1,i}^N \end{bmatrix} z_i^N(t) \tag{4.2.12}$$

where

$$v_{l,i}^N = \sum_{j=1}^{M+1} a_j^l h_i^N(\hat{x}_j)\Delta\tilde{x}_j \qquad \text{and} \quad \Delta\tilde{x}_j = (\tilde{x}_j - \tilde{x}_{j-1}). \tag{4.2.13}$$

Hence,

$$K_{ap}^N = \begin{bmatrix} v_{0,0}^N, v_{0,1}^N, \cdots, v_{0,N+1}^N \\ \\ v_{1,0}^N, v_{1,1}^N, \cdots, v_{1,N+1}^N \end{bmatrix} \tag{4.2.14}$$

and the finite dimensional closed–loop system becomes

$$\begin{aligned} \frac{d}{dt} z^N(t) &= (A^N - B^N K_{ap}^N) z^N(t) + H_1^N \eta^N(t) \\ z^N(x,0) &= z_o^N. \end{aligned} \tag{4.2.15}$$

Scheme 4. Here we use the approximations in Scheme 3, except we use the compensator to estimate the states at $\hat{x}_j$. In particular, we have

$$\hat{u}_{ap}^N(t) = -K_{ap} z_c^N(t) = -\sum_{i=0}^{N+1} v_i^N z_{ci}^N(t). \tag{4.2.16}$$

4.3 Numerical Results

In this section, we discuss numerical results for the disturbed Burgers' equation, with Neumann boundary control and Dirichlet observation. First, we fixed certain parameters and used the selected values throughout all computations. Namely, we chose Re $= 60$, $r_1 = v_1 = 1$ and $r_2 = v_2 = 10^{-3}$, that is, $R_1 = E_1^* E_1 = r_1 I_H, R_2 = E_2^* E_2 = r_2 I_2, V_1 = v_1 H_1 H_1^*$ and $V_2 = H_2 H_2^* = v_2 I_2$ and the disturbance input function was chosen to be $\hbar(x) = x + 1$. We also worked with two different initial conditions, namely $\cos(\pi x)$ and $-\cos(\pi x)$. As for the desired degree of stability α, we set $\alpha = 0.15$.

Plots of the trajectories for open–loop and closed–loop solutions, use the N^{th} order approximation for all cases and methods. The corresponding trajectories are plotted from $t = 0.0$ to $t = 10.0$. Figures 4.3.1–4.3.4 illustrate the response of the nonlinear system with no controls and in the absence of disturbance. In these cases, we can see that all solutions converge to zero except for the case when the initial condition is $z_o = \cos(\pi x)$. In Schemes 2–4, we have chosen $M = 3$ where the end points of each subinterval I_j were set to $\tilde{x}_j = 0.15,\ 0.5,\ 0.85$ for $j = 1, 2, 3$ along with the two end points of the domain. That is, the center points of each interval I_j are set to $\hat{x}_j = 0.075,\ 0.325,\ 0.675,\ 0.925$.

Remark 4.3.1 With the above choices, it is easy to check that the hypotheses and conditions that were assumed throughout this paper hold. In fact, we know that the hypotheses (H-1)–(H-5) are satisfied with our parameters choice. Also, we know that part of condition (C-1) is satisfied,i.e., the detectability of the pair (A, E_1), since we chose the identity operator for E_1, but on the other hand we have to make sure that the pair (A, H_1) is stabilizable. Indeed, $\hbar(x) = x + 1$ satisfies the stabilizability condition for any desired degree of stability $\alpha > 0$, since as we did for the coefficients b_n, $n = 0, 1, \cdots$, the coefficients $\hbar_n$ representing the disturbance input function $\hbar(\cdot)$ are not zero,i.e.,

$$\hbar_n =< \hbar(\cdot), \phi_{\alpha,n}(\cdot) >_{L^2(0,1)}= \int_0^1 (x+1)\phi_{\alpha,n}(x)dx \neq 0 \qquad (4.3.1)$$

for all $n = 0, 1, \cdots$, where $\phi_{\alpha,n}$ are the eigenfunctions presented in Section 2.2.

Example 4.3.1 (LQG/MinMax Problems)

All control laws "converged" for $N = 32$ elements. The converged feedback functional gain $K_\alpha(\cdot)$ and observer functional gain $L_\alpha(\cdot)$ are given in Figures 4.3.5 and 4.3.6, respectively. Since we are controlling the flux at each end point $x = 0$ and $x = 1$, we have two feedback functional gains. From these plots, one can easily see that the control actions are concentrated on the location of the boundary control points where the solid lines represent flux control gains obtained from the MinMax problem and the dashed lines are the flux control obtained from the LQG problem. Similarly, since we are sensing the flow at the origin and the end point, we have two observer gains (solid lines for the MinMax observer gains and dashed lines for the LQG observer gains). We also compared the maximum singular value of the transfer functions of the LQG and MinMax linear closed–loop system in Figure 4.3.7. This figure shows that the MinMax controller yields a closed–loop system satisfying the H^∞ disturbance attenuation bound $\gamma = 4.6$. This value is obtained by decreasing γ until solutions to the Riccati equations (3.3.36)–(3.3.37) with the required properties fail to exist. Next, these controllers were then applied to the disturbed Burgers' equation resulting in the nonlinear closed-loop trajectories depicted in Figures 4.3.8 – 4.3.11. For each initial condition, we observe that the resulting solutions (Figures 4.3.8–4.3.9) obtained by the linear quadratic Gaussian controllers are not stable in the presence of disturbances. On the other hand, Figures 4.3.10–4.3.11 show that the MinMax controllers provide stability margins sufficient to accommodate the degree of disturbances introduced. Figure 4.3.12 shows the control function $u(t)$ for the LQG and MinMax problems. These preliminary results indicate that the control laws designed using the linear quadratic differential game outperform the standard LQG controllers in the presence of disturbance signals. This is not surprising because disturbances were not introduced in the performance criterion as in the case of MinMax design.

Example 4.3.2. (Finite-Fixed-Order Compensator)

In the fixed-order case, we considered the accuracy of the impulse and step responses of the various reduced order compensator designs compared to the corresponding responses of the full order LQG design. Figure 4.3.13 illustrates the linear closed-loop impulse response for the full-order LQG and reduced order compensator (of order $N_c = 16$) designs. The impulse response of the linear closed-loop system for the 16th-order compensator is in perfect agreement with the LQG response. Note that in Figure 4.3.13 we see only one plot for both designs because both plots are essentially the same. Similar trends are seen (Figure 4.3.14) in the comparisons of the step responses (for the same design case) with the corresponding LQG responses.

For the nonlinear closed-loop response, the 16th-order compensator was applied to Burgers' equation and we see (in Figures 4.3.15–4.3.16) excellent agreement with the full order closed-loop trajectory response. Hence, replacing the 32nd-order optimal LQG controller by a 16th-order compensator produces a closed-loop system with minor performance degradation. We also compared the performances of the closed-loop system of the 4th-order compensator with the full order LQG responses. Figures 4.3.17 and 4.3.18 are the impulse and step responses of the linear closed-loop system, respectively. If one compares these responses with the corresponding responses for the full order LQG controller shown in Figures 4.3.13 and 4.3.14, then it is clear that the 4th-order compensator performs almost as well as the full order LQG controller. Similar comments hold for the nonlinear closed-loop responses. For example, the 4th-order compensator response (Figures 4.3.19 and 4.3.20) compares well to the LQG response, especially after time $T = 1.0$.

Example 4.3.3. (Suboptimal Feedback Designs)

We considered the feedback functional gains obtained from the MinMax problem and applied Schemes 2–4. The closed–loop solutions are given in Figures 4.3.22–4.3.27. Figures 4.3.22–4.3.24 illustrate the average of the gain functions in Figure 4.3.5 and the corresponding closed–loop solutions (Scheme 2). The average values were estimated on each subinterval $(\tilde{x}_{j-1}, \tilde{x}_j)$, where $\tilde{x}_j = 0,\ 0.15,\ 0.5,\ 0.85,\ 1.0$ for $j = 0, 1, 2, 3, 4$, respectively. Also, Schemes 3 and 4 were applied to obtain Figures 4.3.25 and 4.3.27. The basic observation to be made here is that each one of the Schemes 2, 3 and 4 works well compared with Scheme 1, all drive the state to the zero equilibrium position and reduce the gradient of the solution in the presence of disturbances. On the other hand, from a practical point of view, Scheme 3 is the most efficient, since in this case we require only data that can be obtained from the sensors placed at the boundary. Indeed, if we compare the trajectory obtained from

Scheme 3 (Figure 4.3.25), to the one from Scheme 1 (Figure 4.3.10), we can see that they are almost identical.

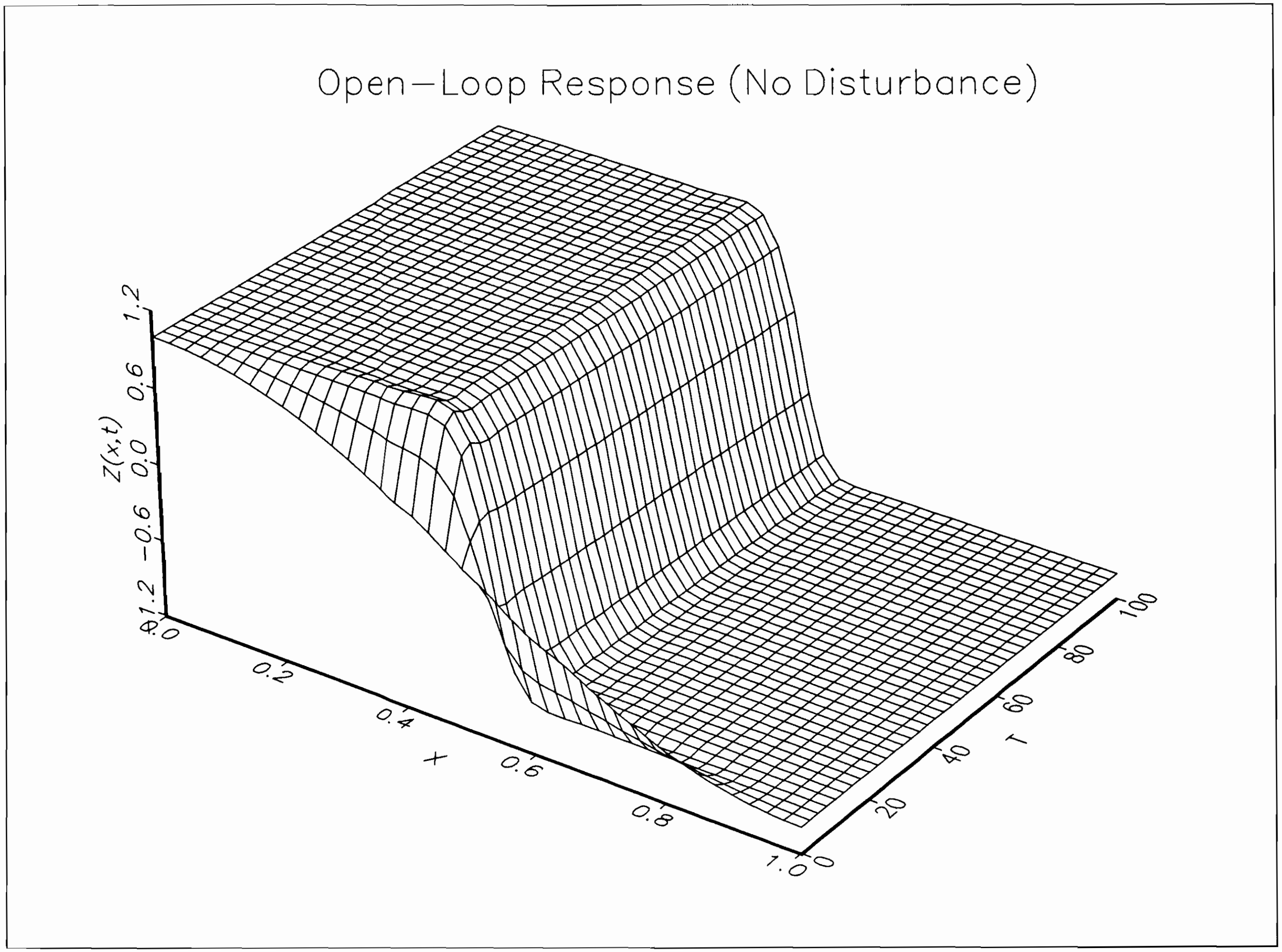

Figure 4.3.1: Open Loop Response (No Disturbance) ($z_o(x) = \cos(\pi x)$)

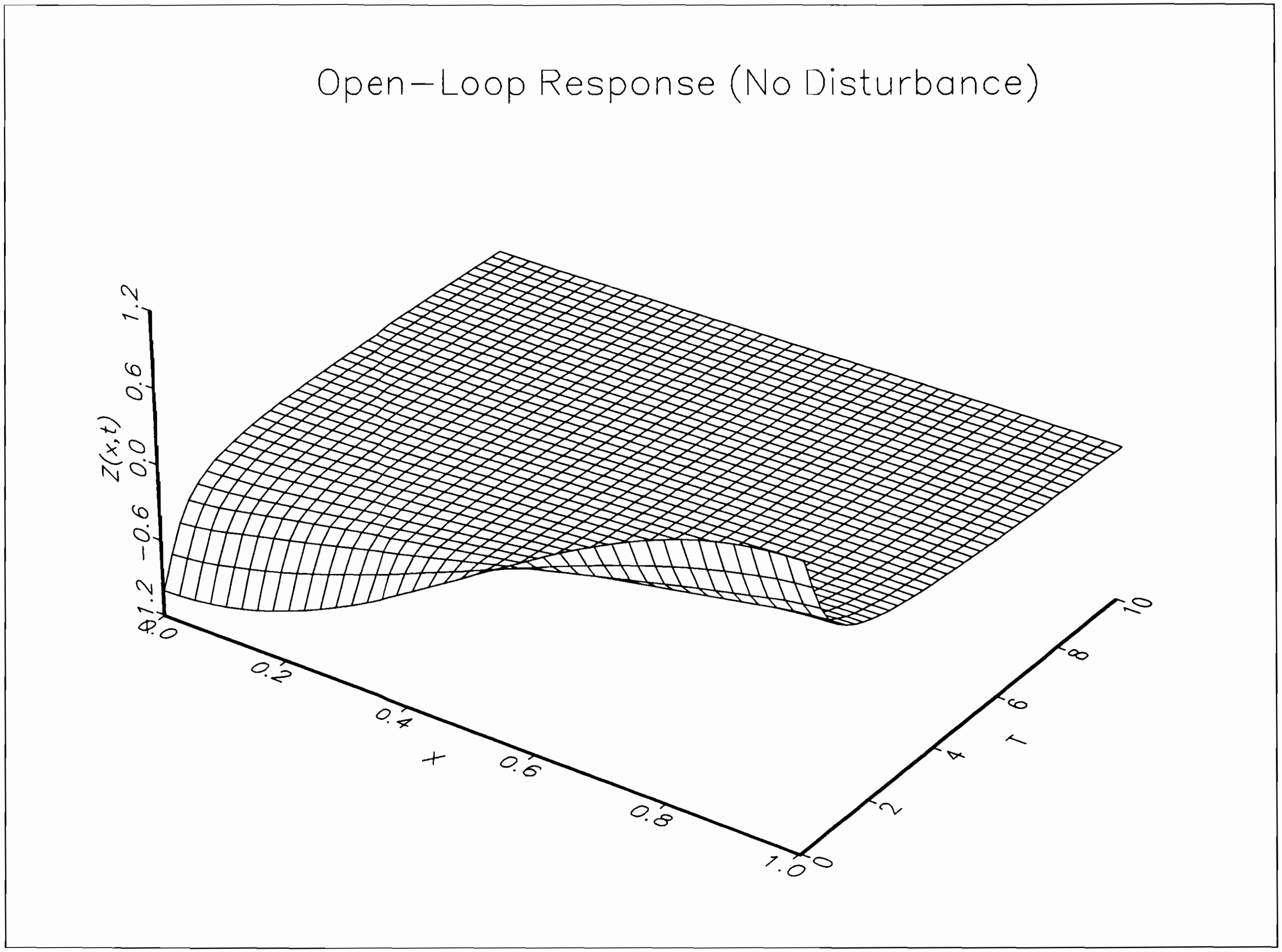

Figure 4.3.2: Open Loop Response (No Disturbance) ($z_o(x) = -\cos(\pi x)$)

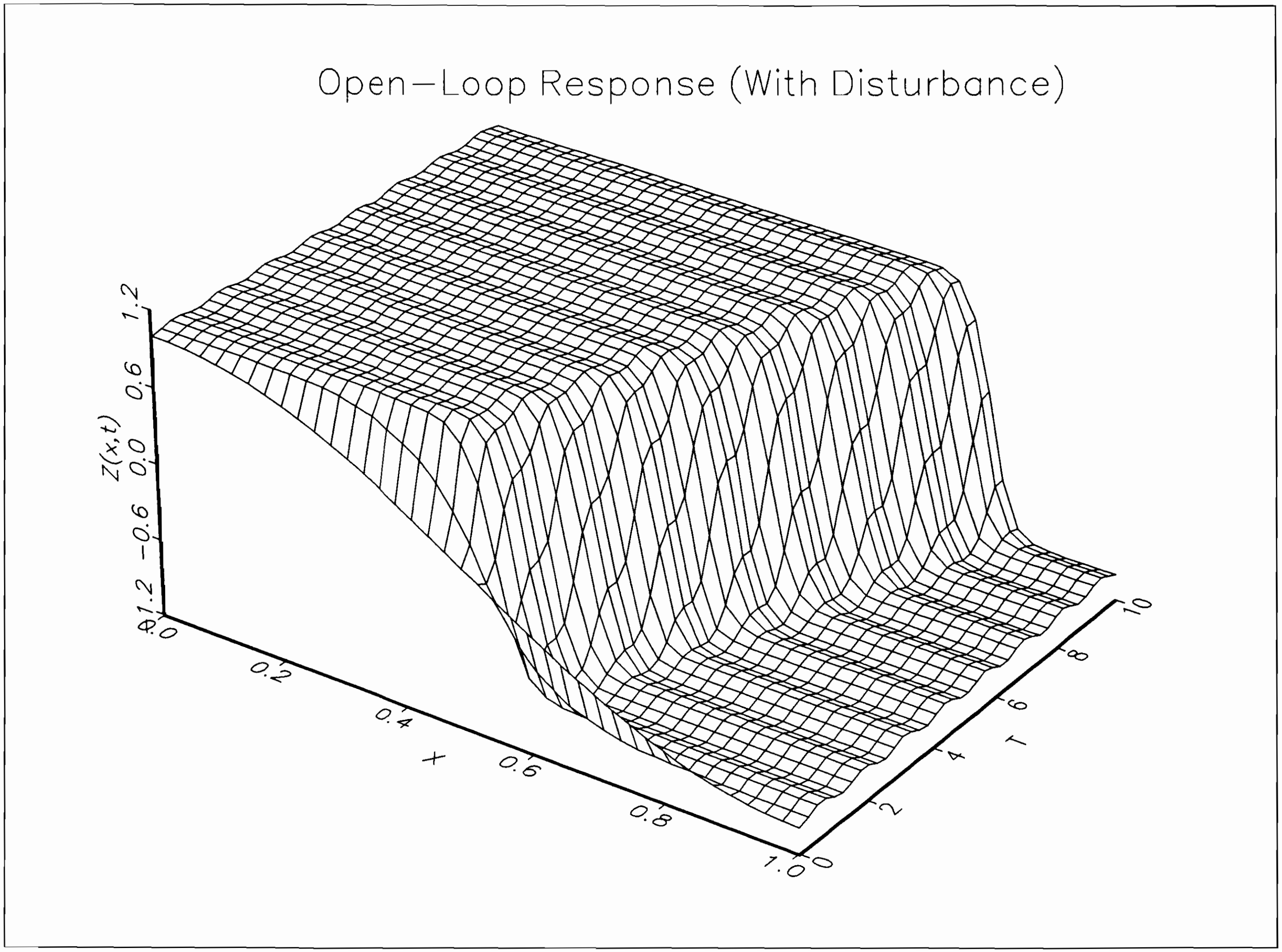

Figure 4.3.3: Open Loop Response (With Disturbance) ($z_o(x) = \cos(\pi x)$)

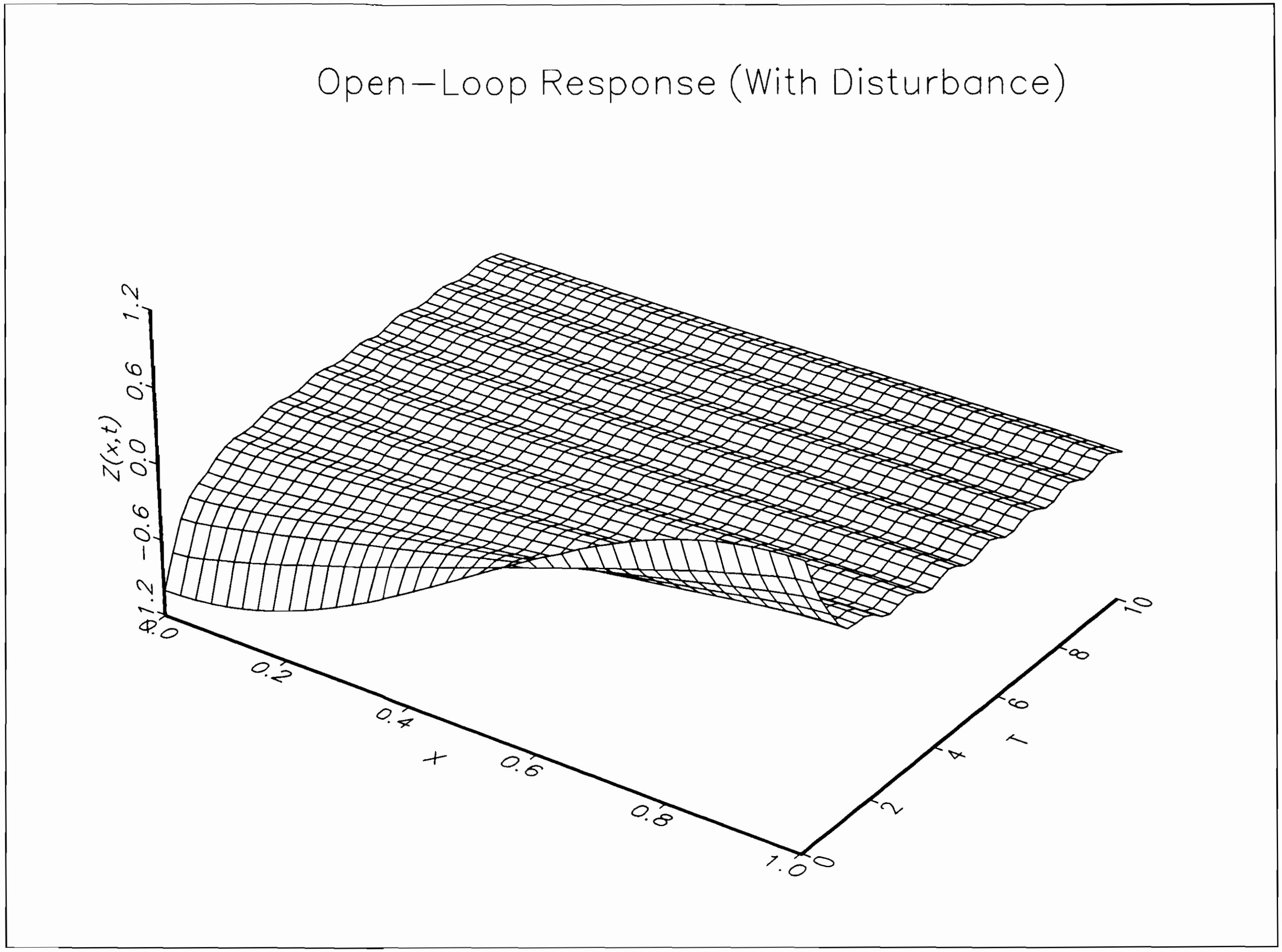

Figure 4.3.4: Open Loop Response (With Disturbance) ($z_o(x) = -\cos(\pi x)$)

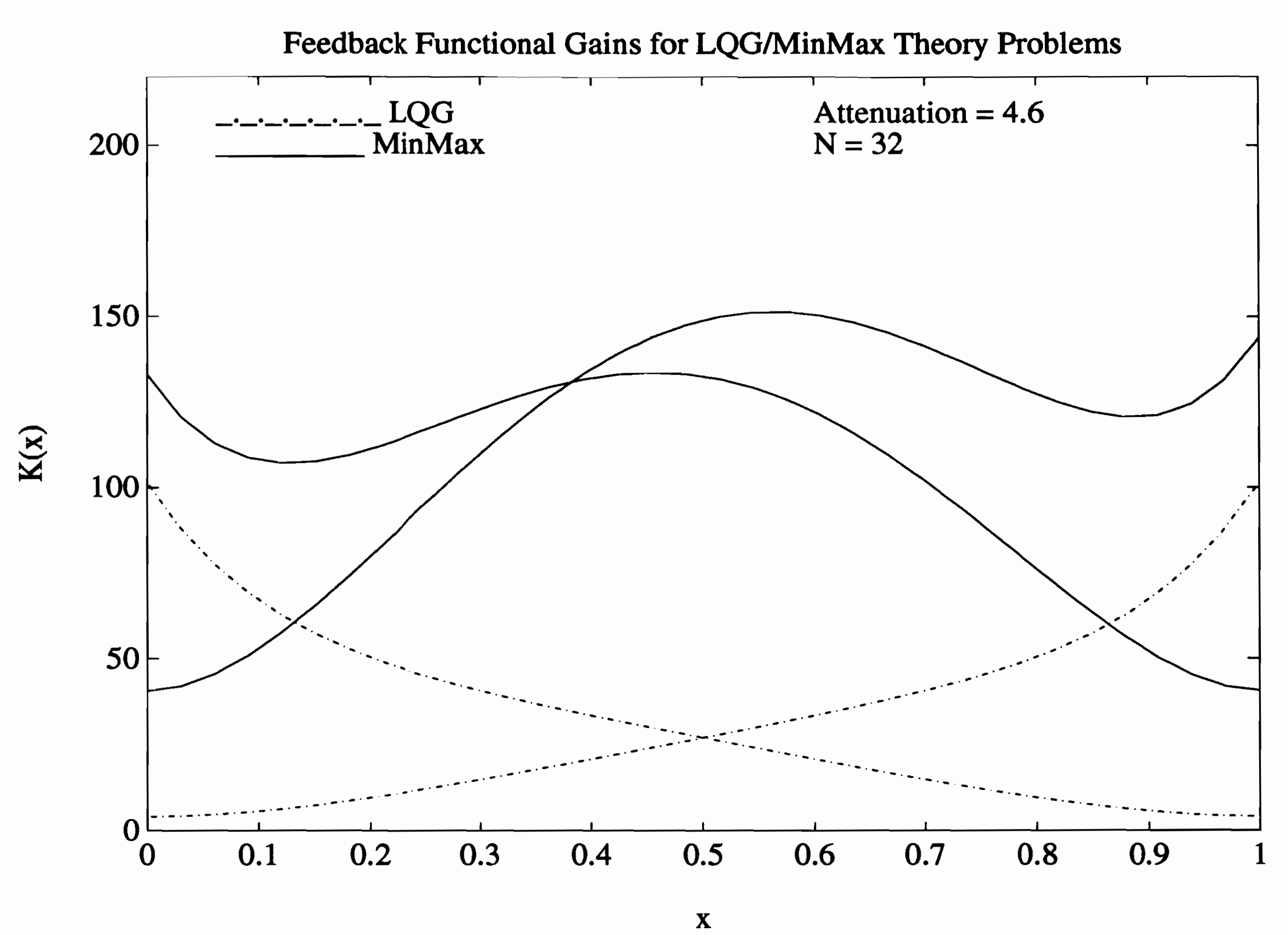

Figure 4.3.5: LQG & MinMax Functional Control Gains

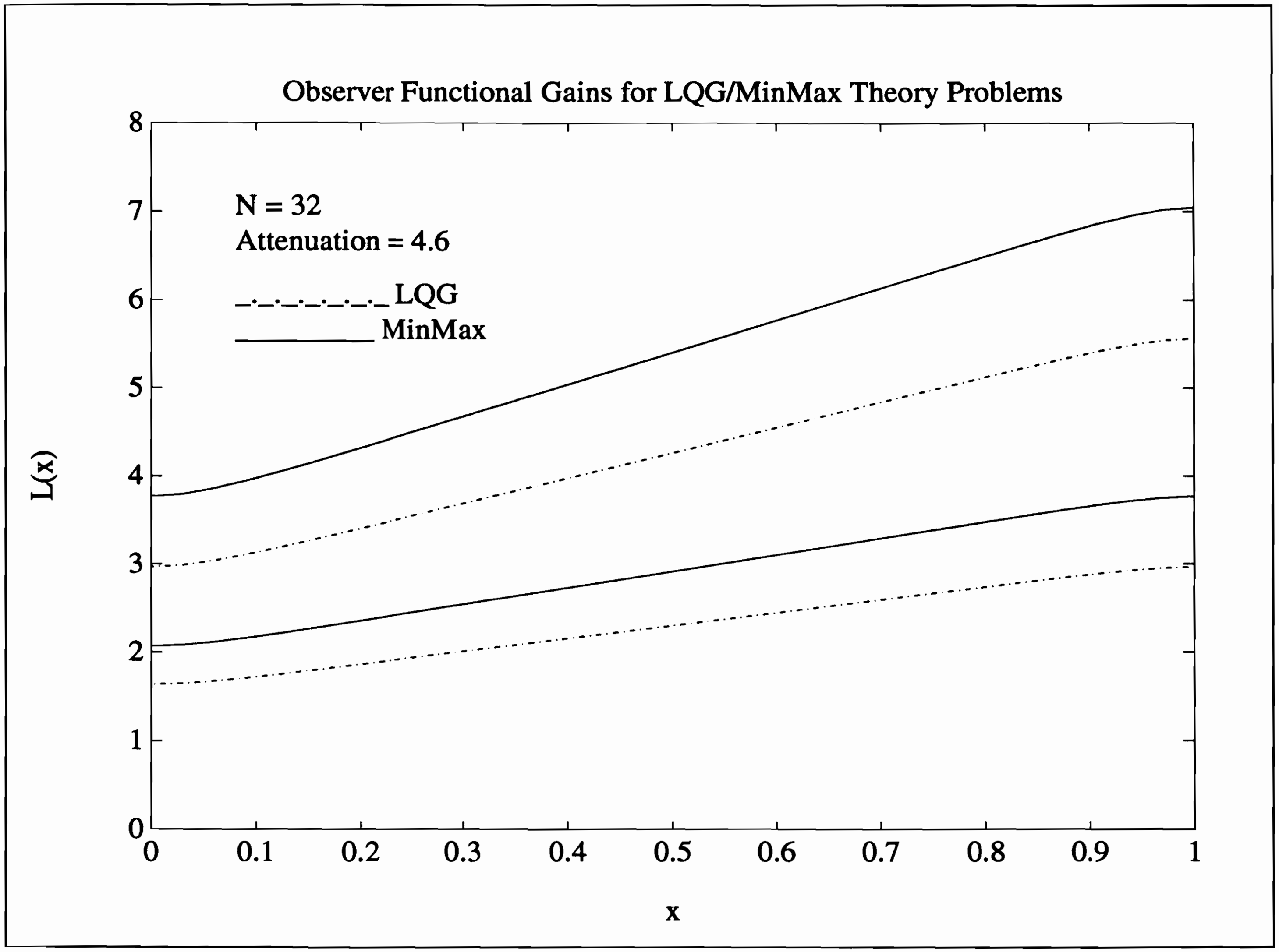

Figure 4.3.6: LQG & MinMax Functional Observer Gains

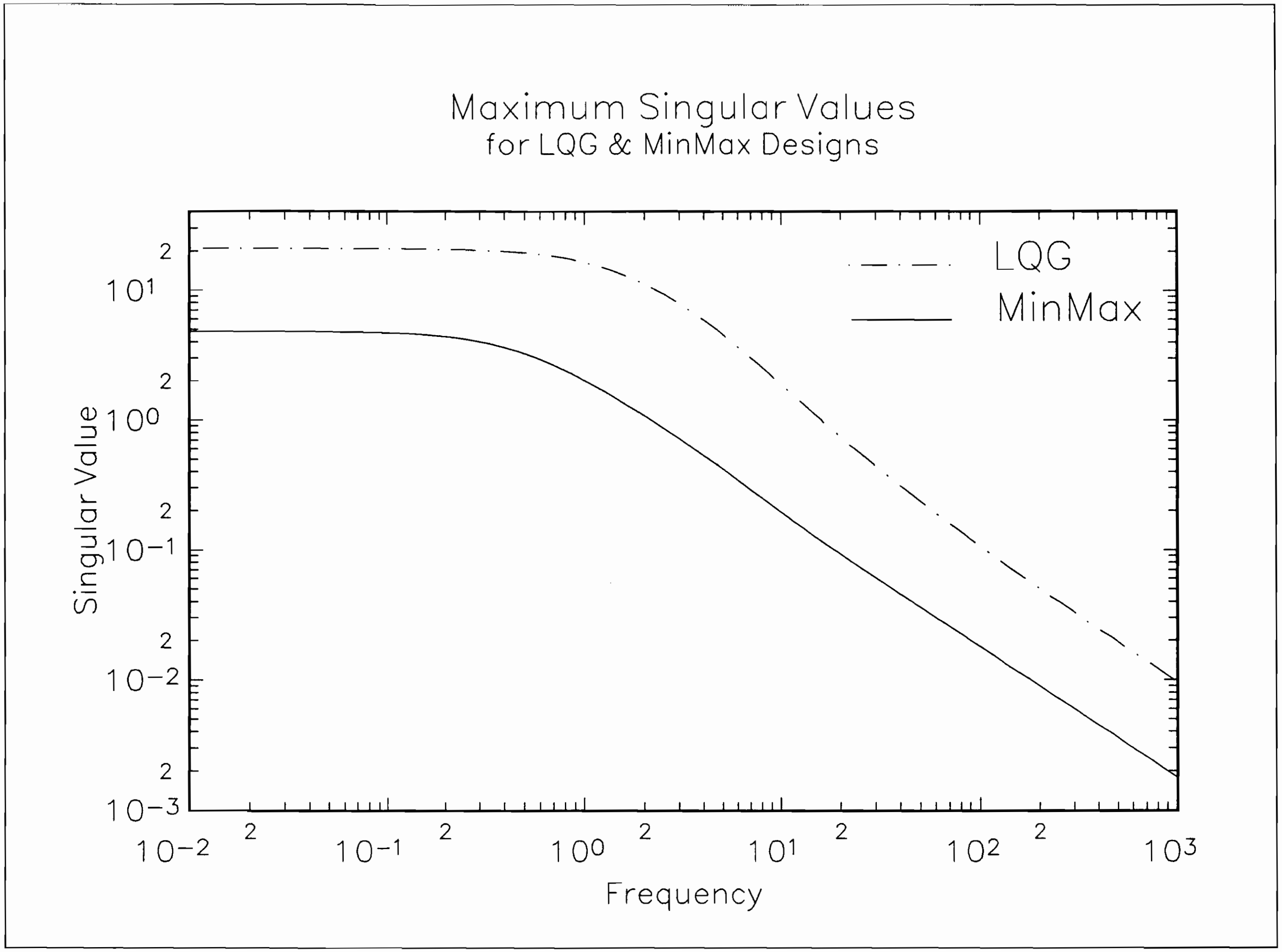

Figure 4.3.7: LQG & MinMax Largest Singular Value of the Transfer Function

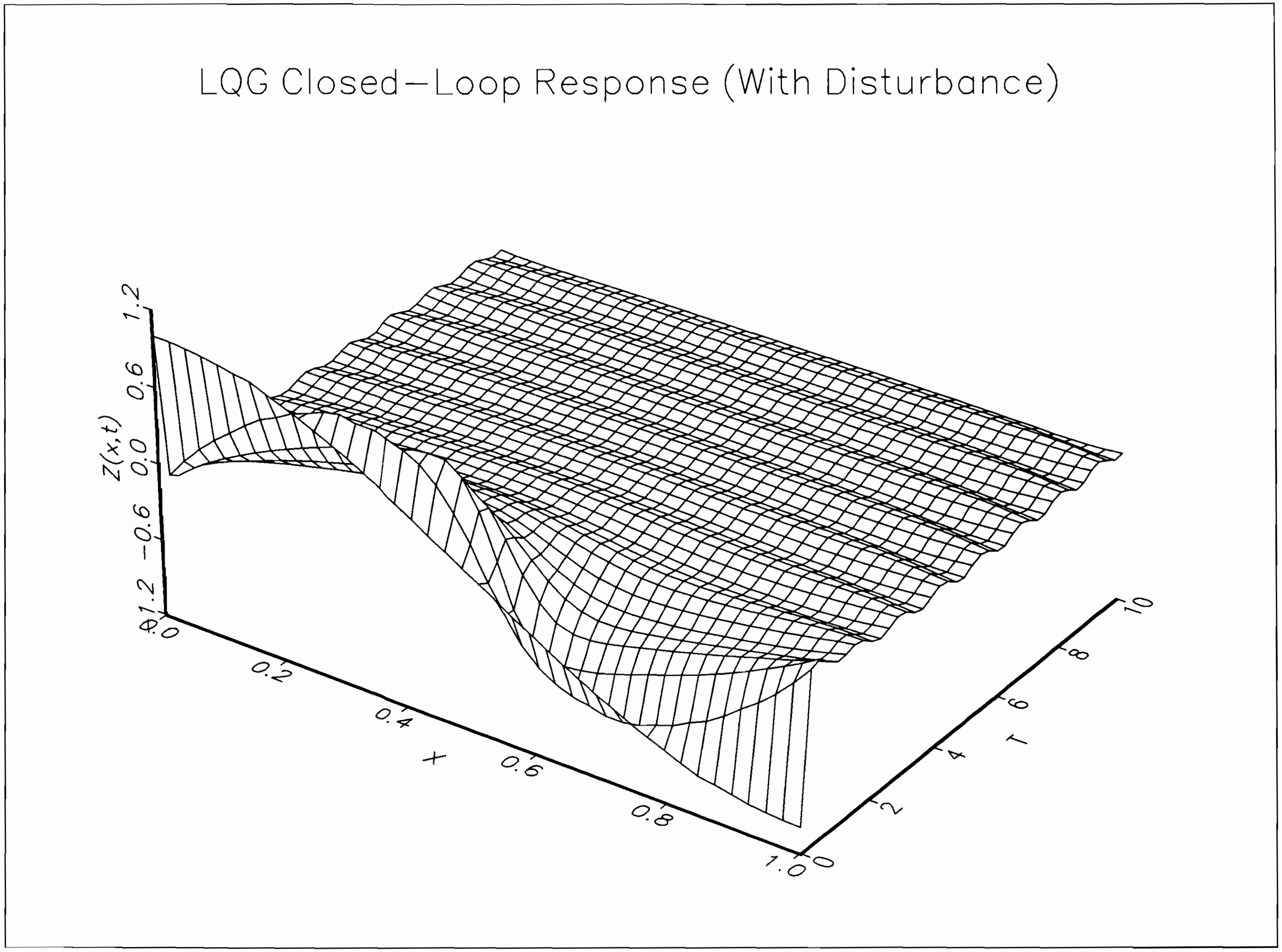

Figure 4.3.8: LQG Closed Loop Response ($z_o(x) = \cos(\pi x)$)

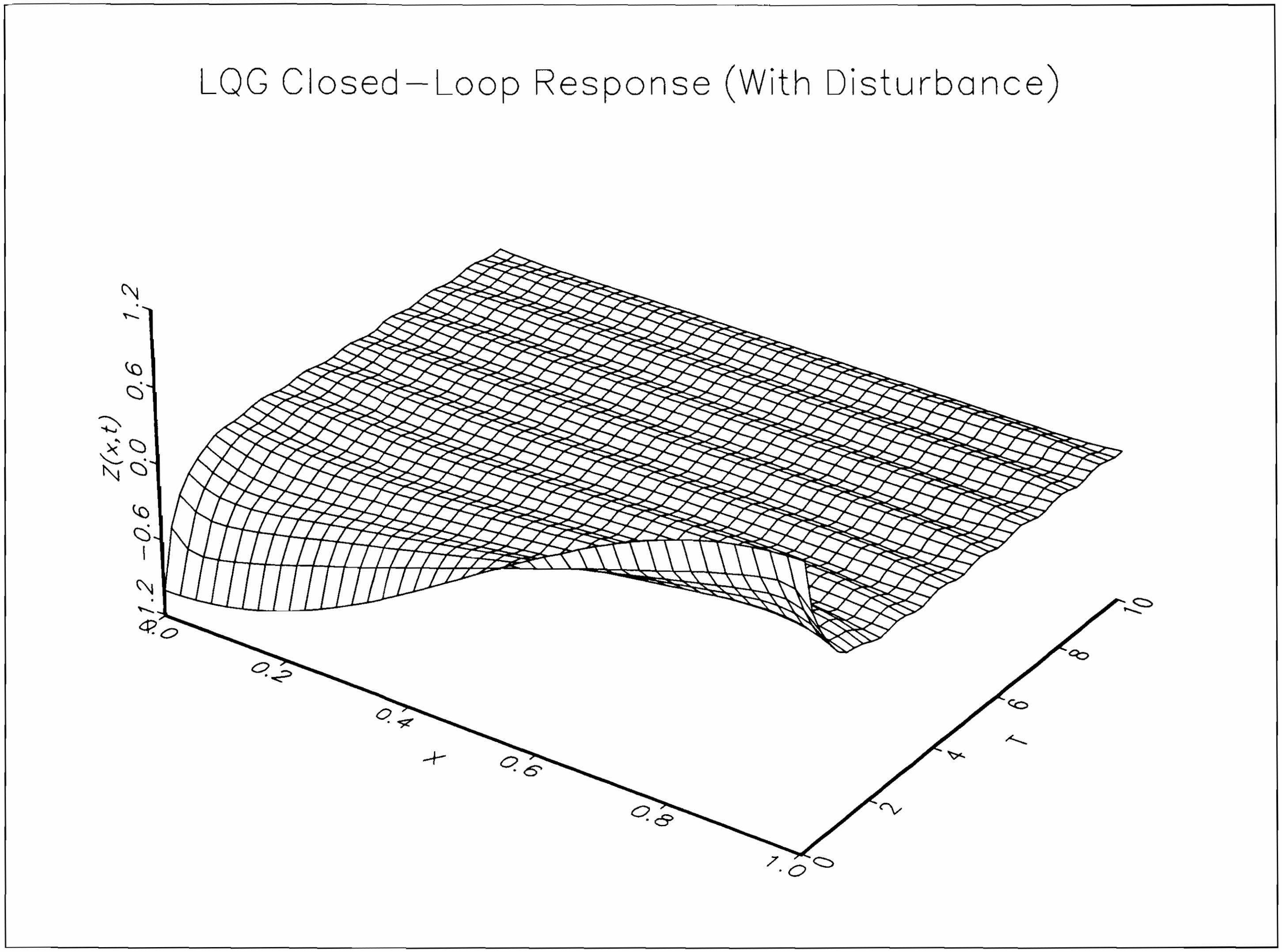

Figure 4.3.9: LQG Closed Loop Response ($z_o(x) = -\cos(\pi x)$)

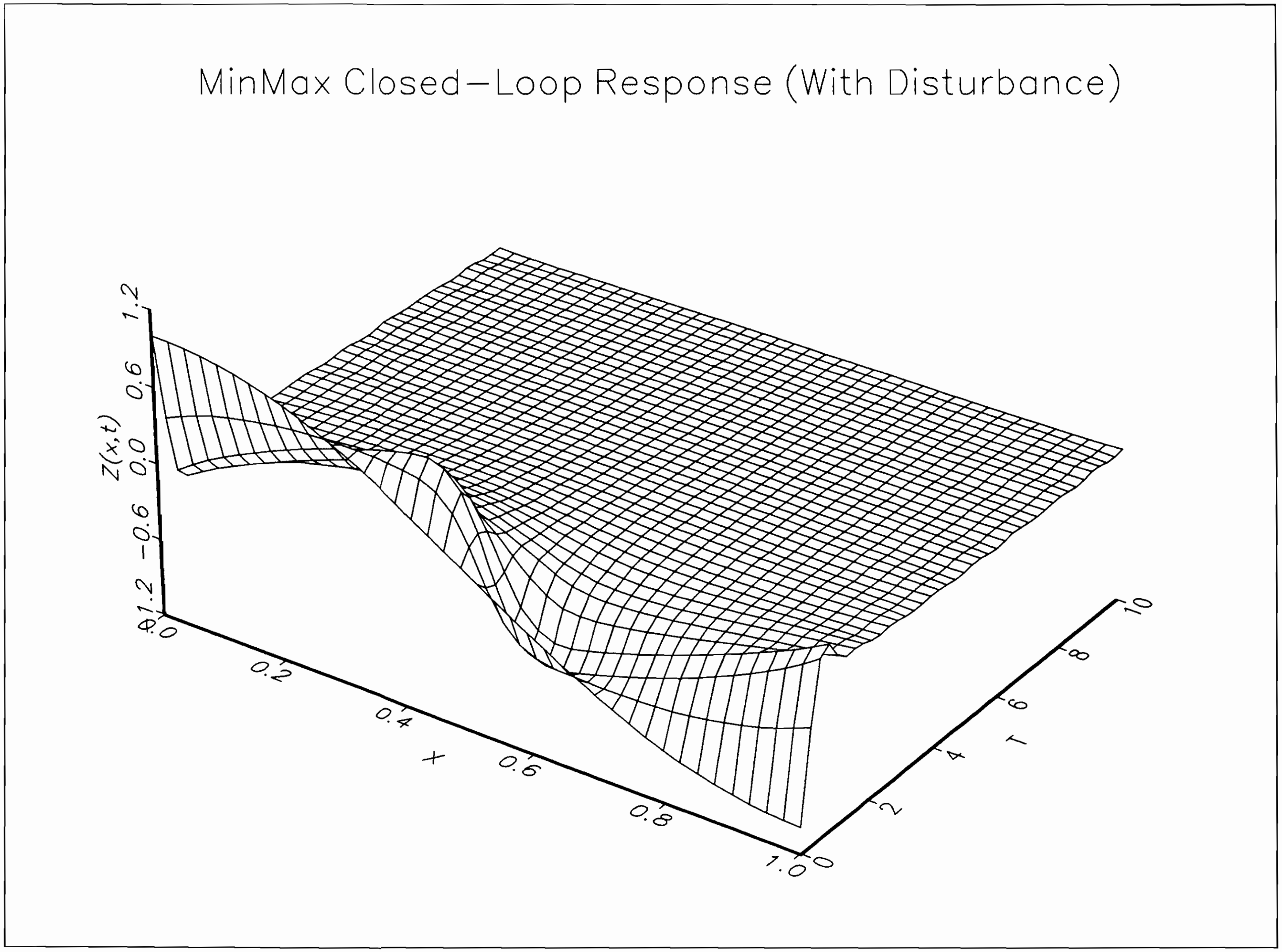

Figure 4.3.10: MinMax Closed Loop Response ($z_o(x) = \cos(\pi x)$)

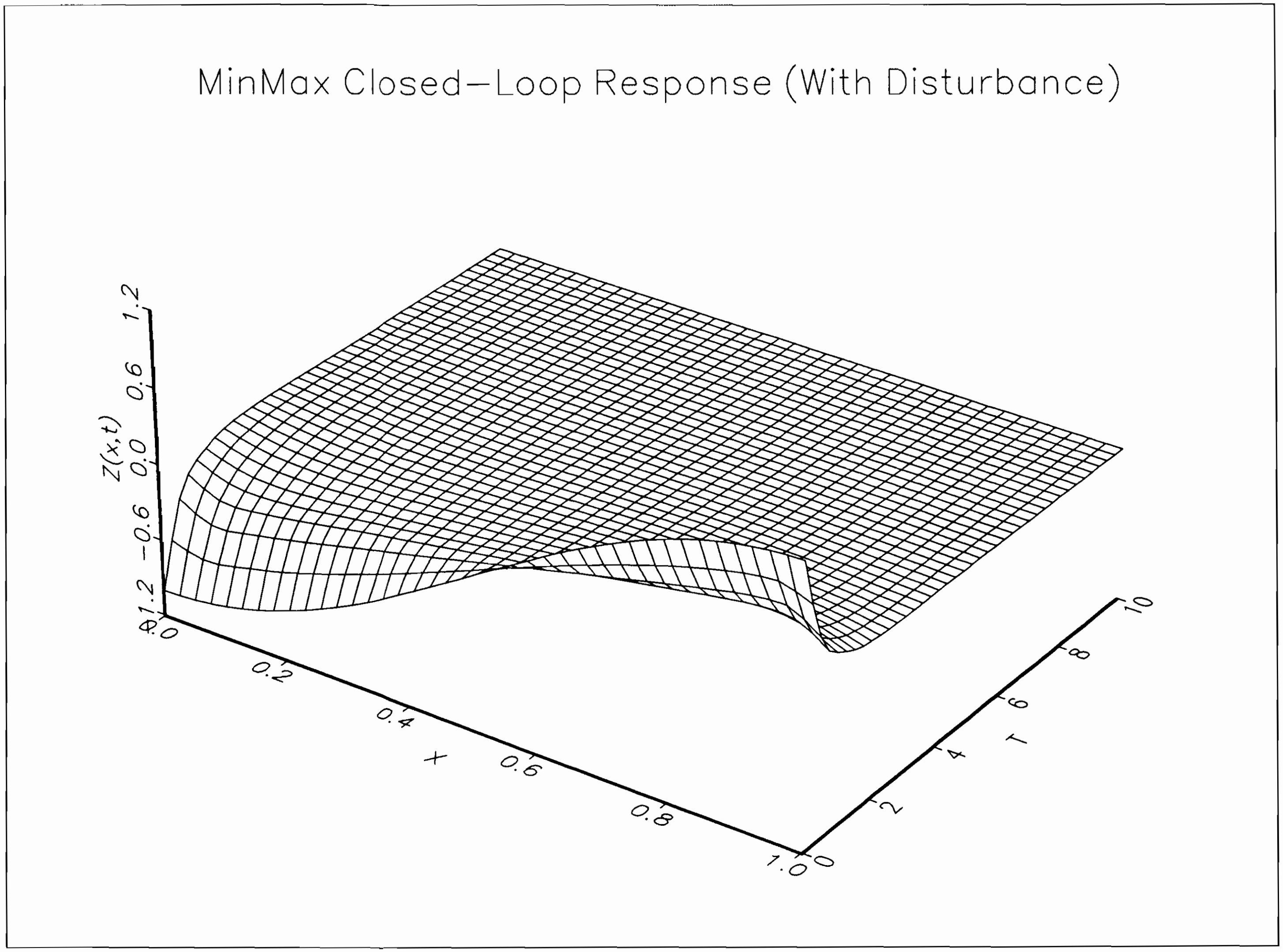

Figure 4.3.11: MinMax Closed Loop Response ($z_o(x) = -\cos(\pi x)$)

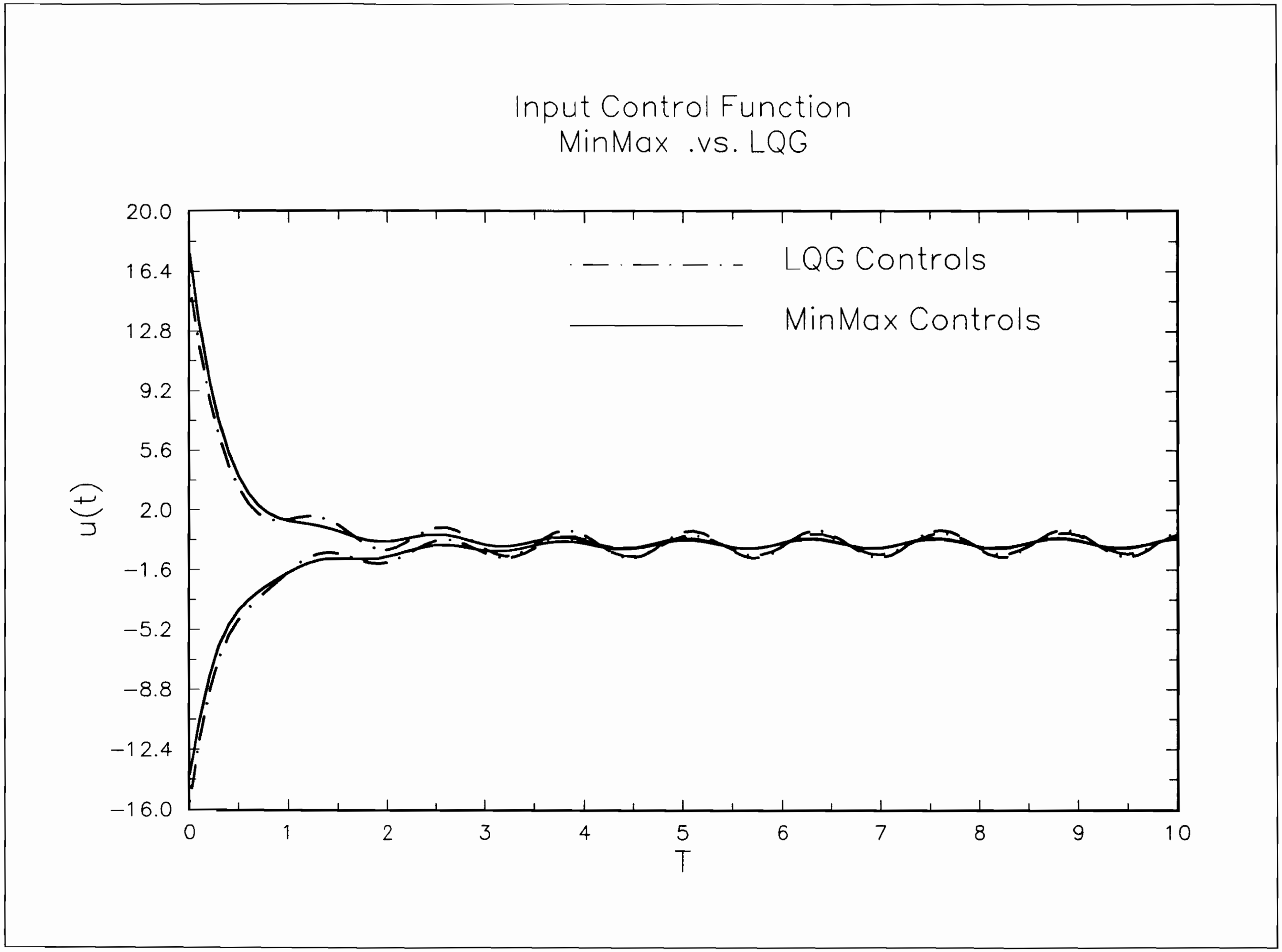

Figure 4.3.12: LQG & MinMax Control Function u(t)

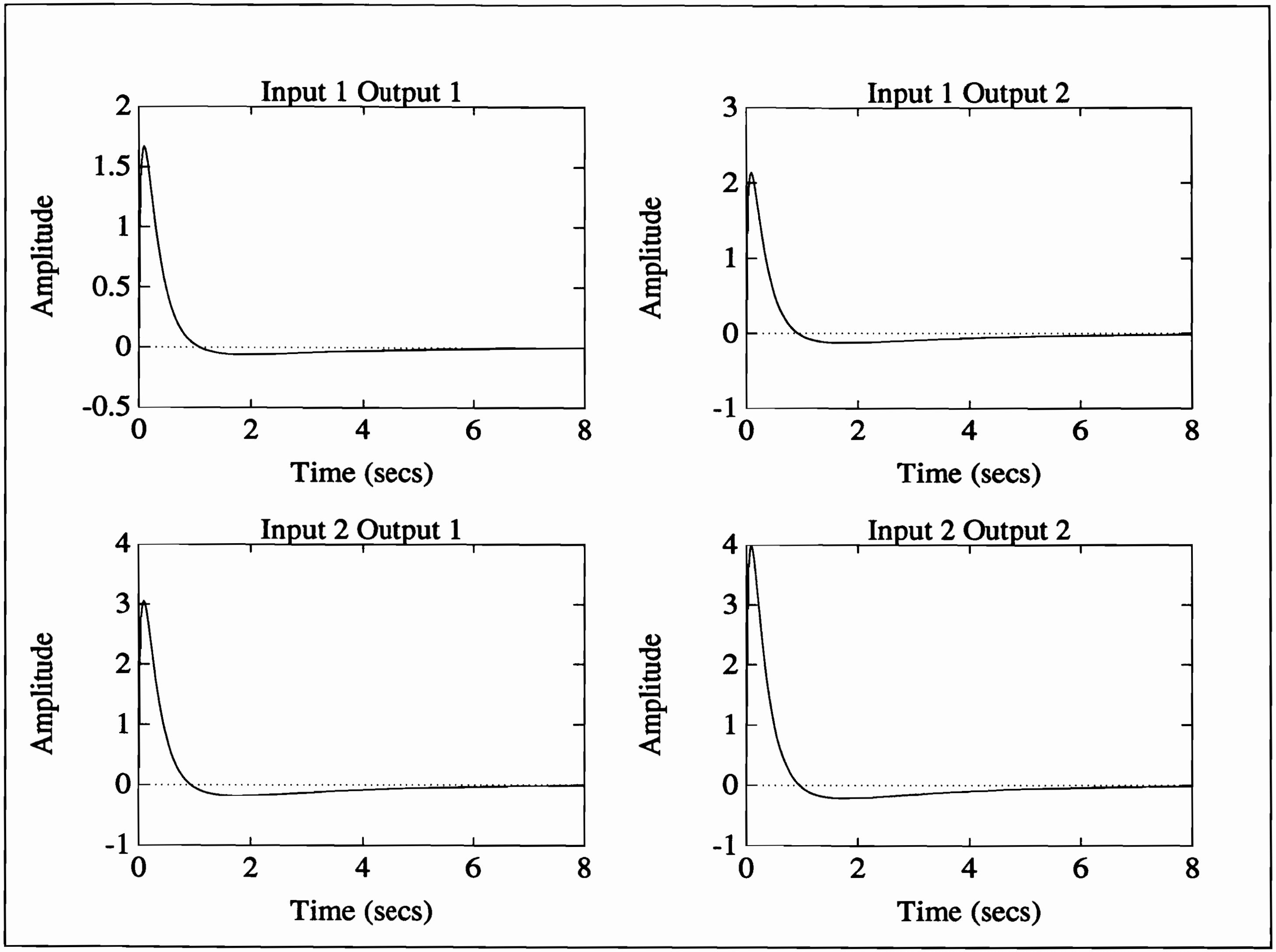

Figure 4.3.13: Impulse Response for the LQG & Fixed-Order Cases, N=32, N_c=16

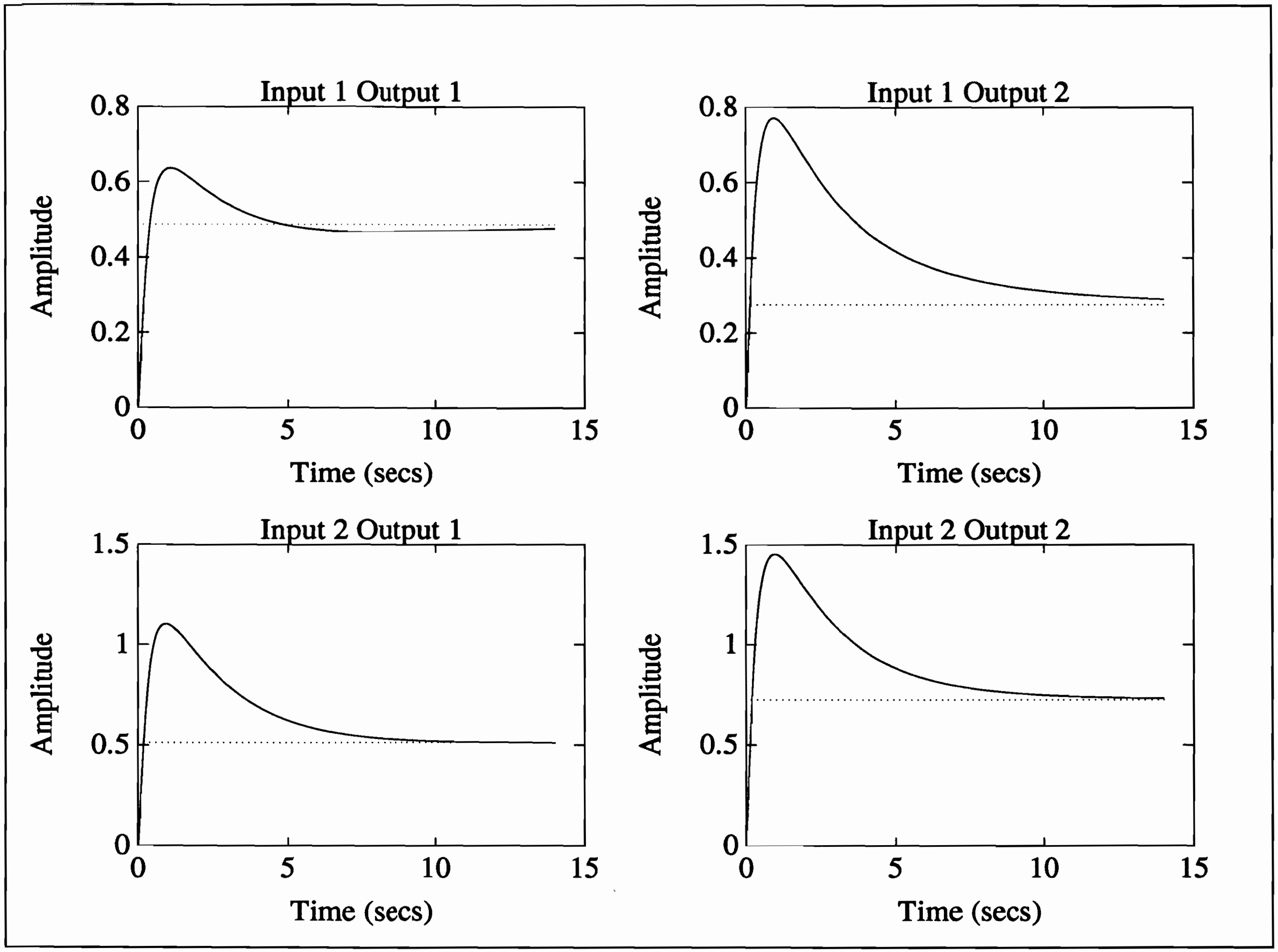

Figure 4.3.14: Step Response for the LQG & Fixed-Order Cases, N=32, N_c=16

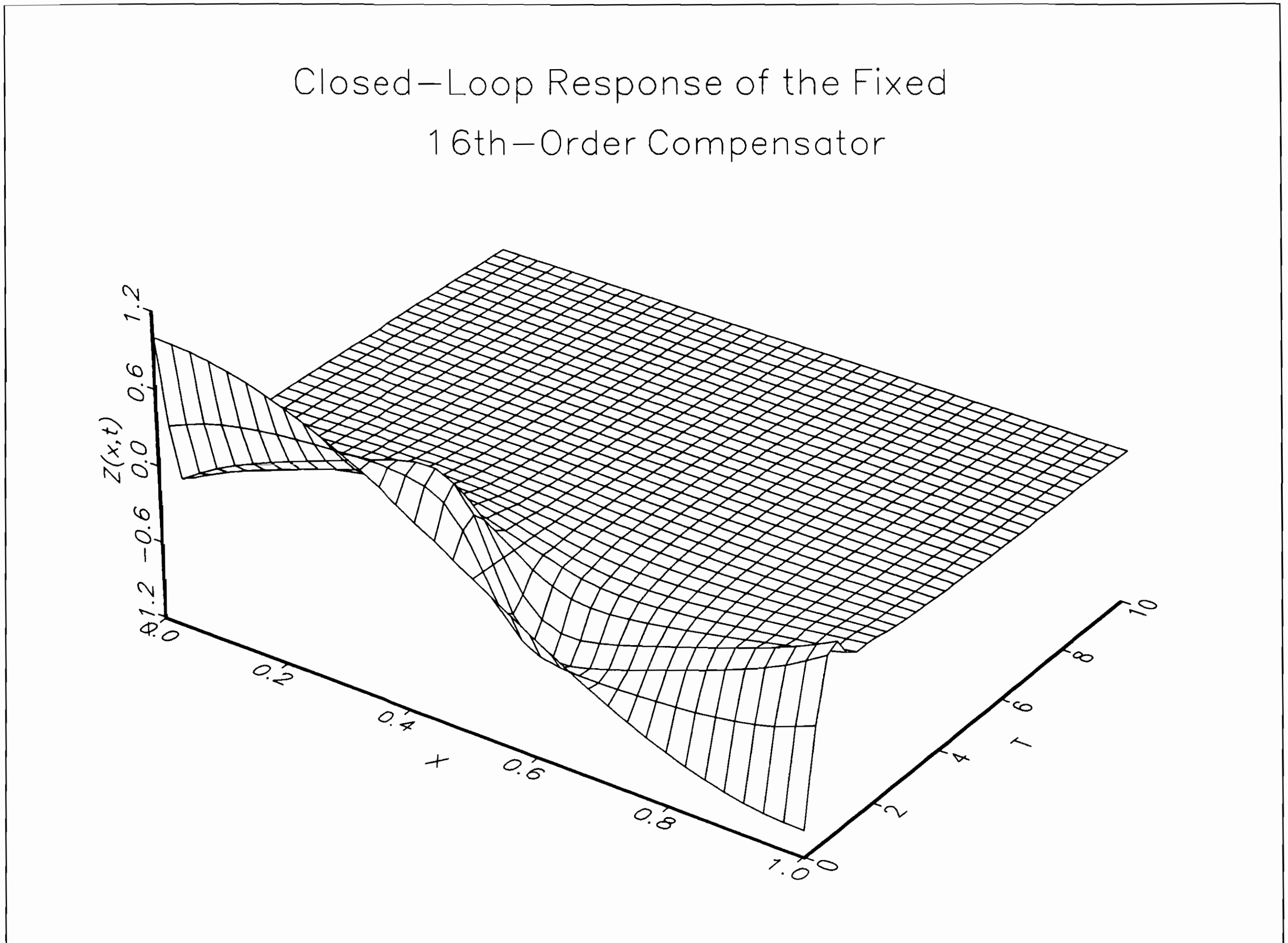

Figure 4.3.15: Closed-Loop for the Fixed-Order Cases, (N_c=16, $z_o(x) = \cos(\pi x)$)

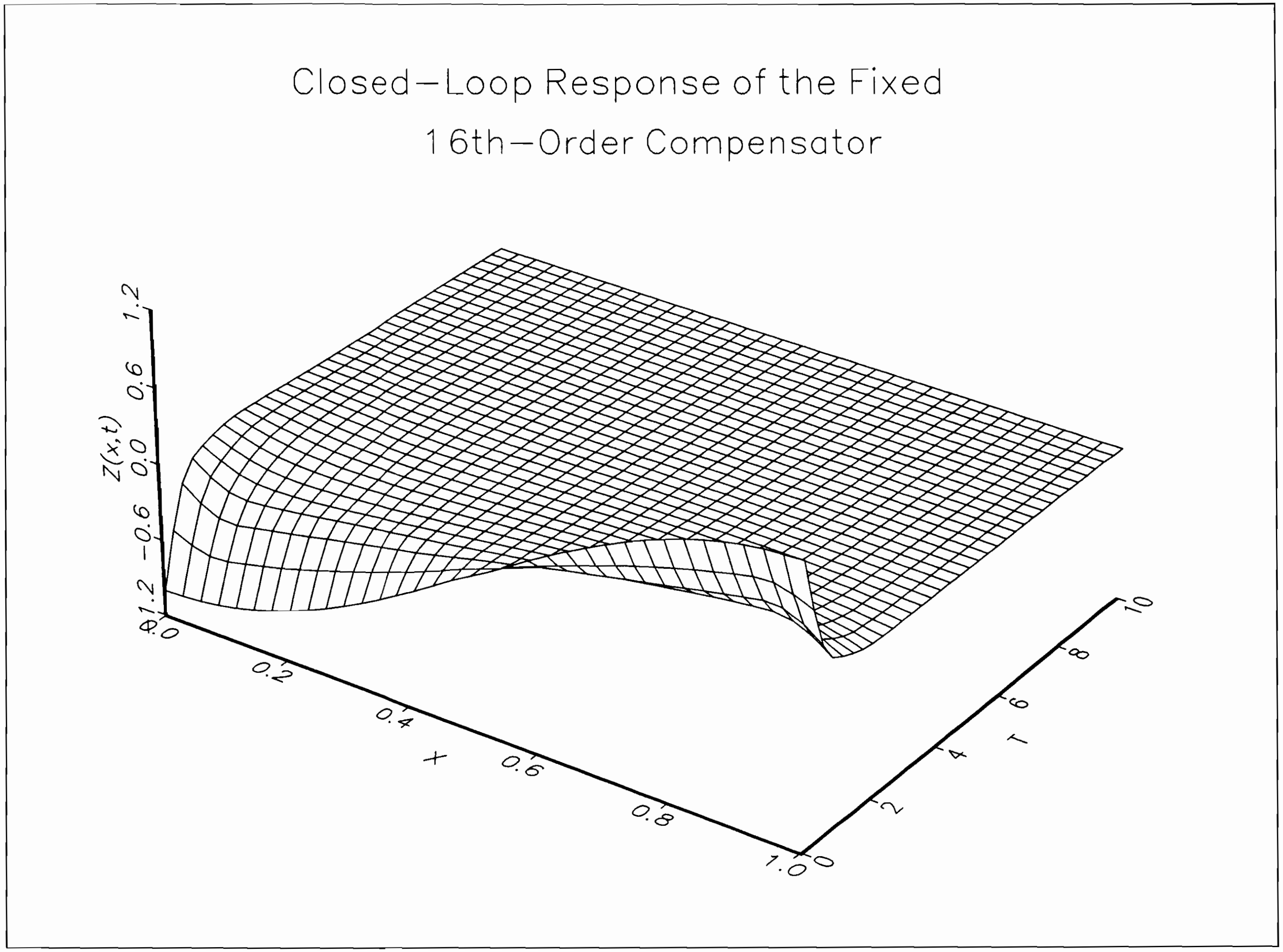

Figure 4.3.16: Closed-Loop for the Fixed-Order Cases, (N_c=16, $z_o(x) = -\cos(\pi x)$)

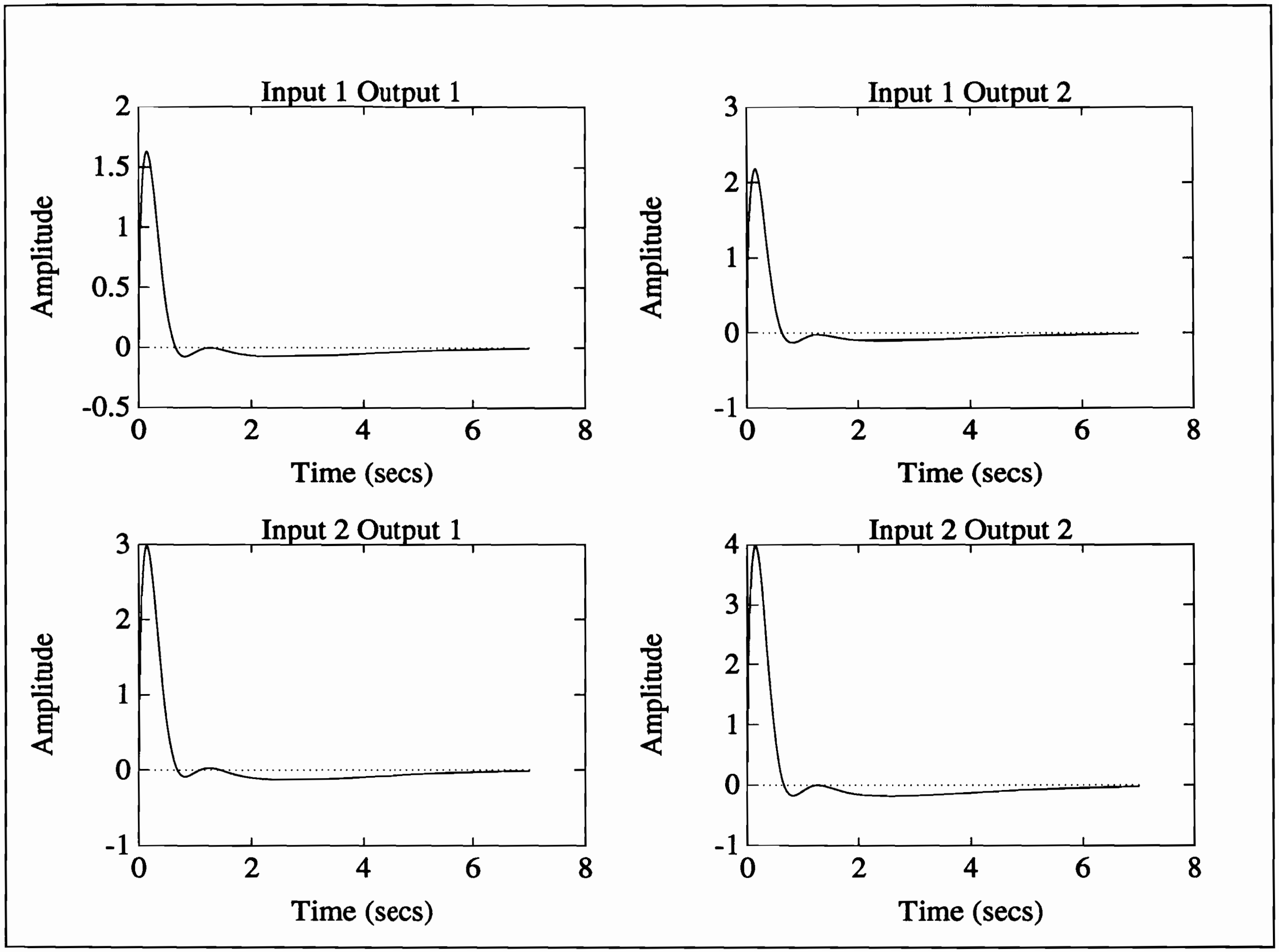

Figure 4.3.17: Impulse Response for the LQG & Fixed-Order Case, N=32, N_c=4

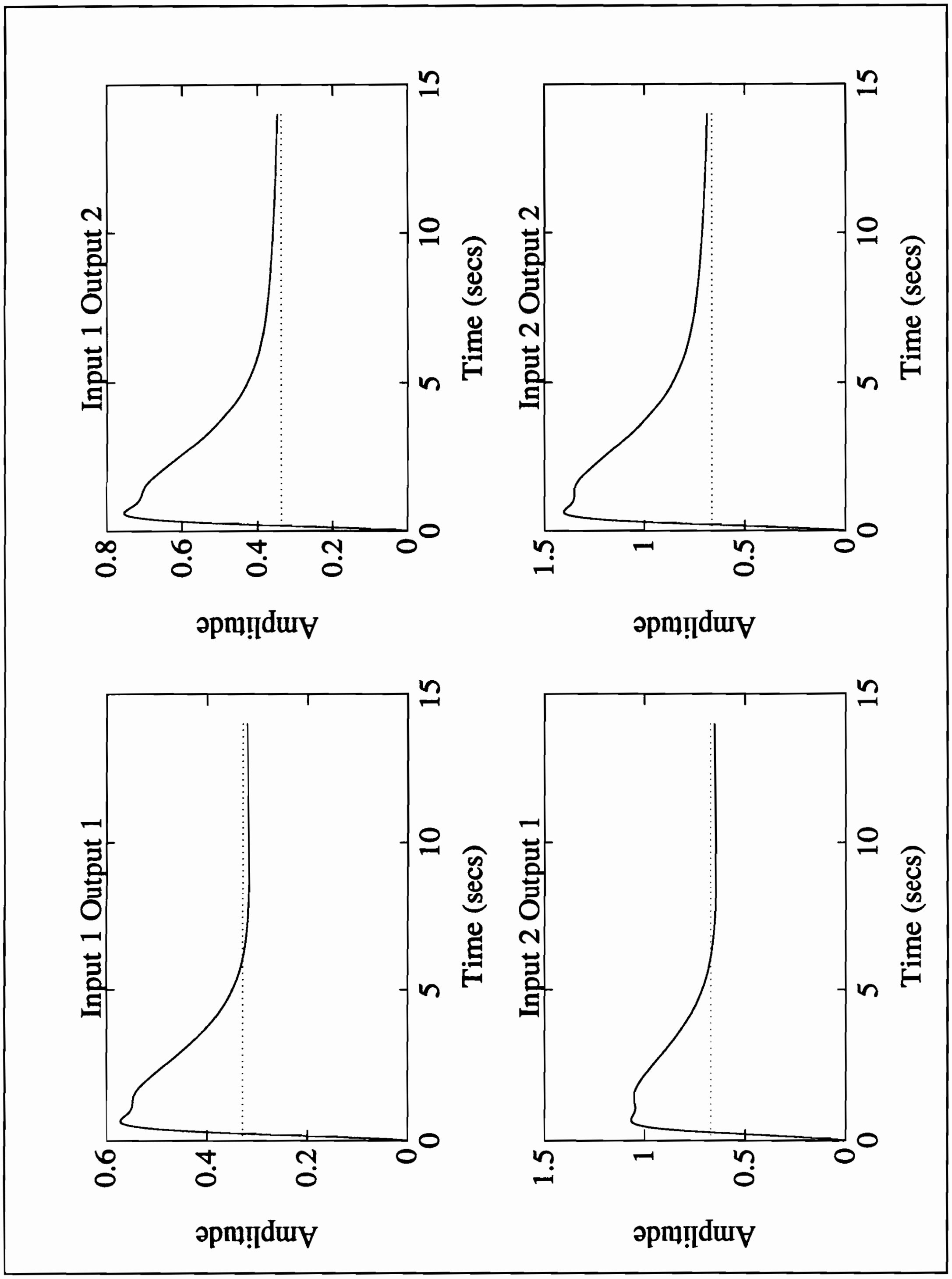

Figure 4.3.18: Step Response for the LQG & Fixed-Order Case, N=32, N_c=4

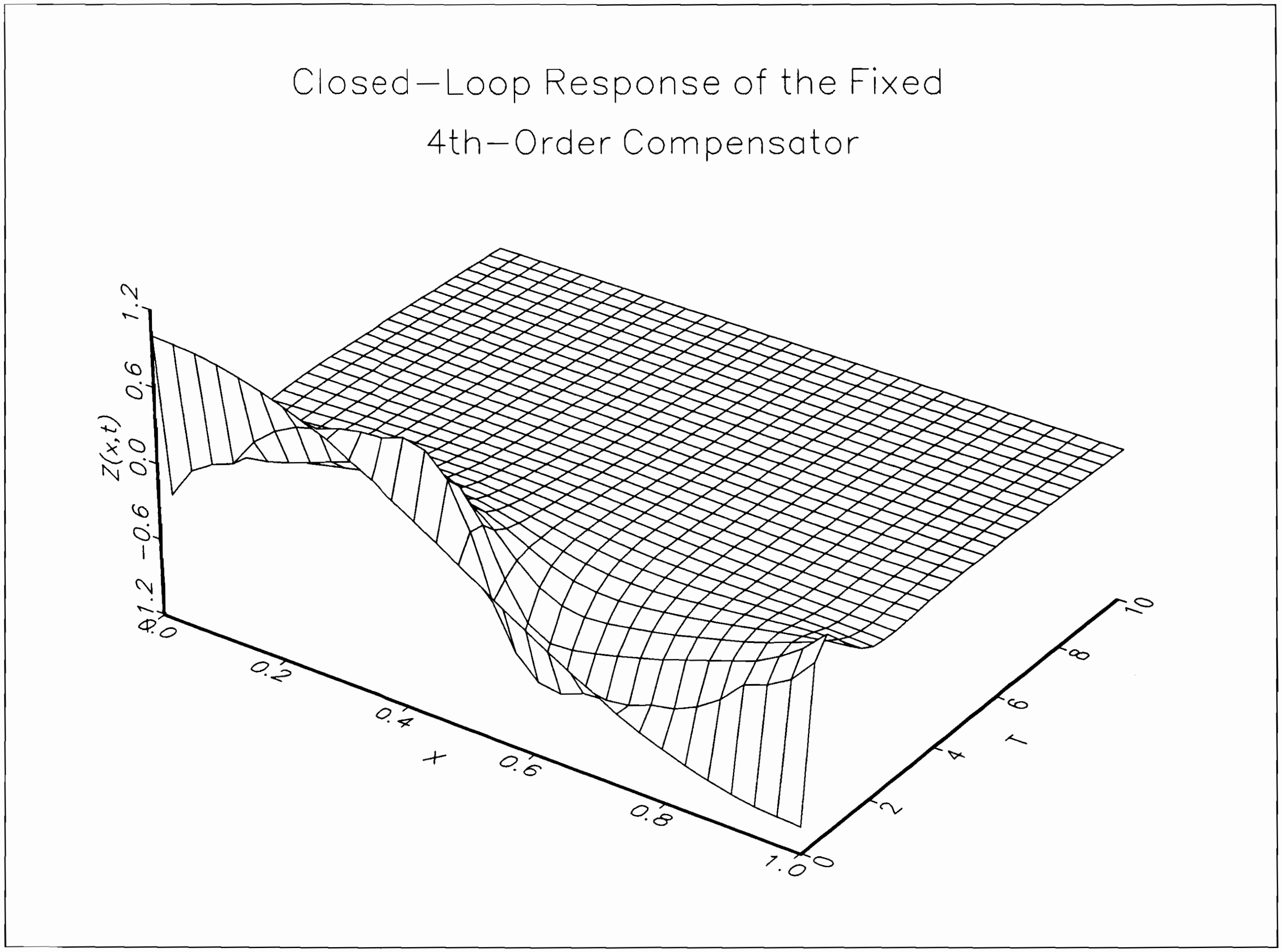

Figure 4.3.19: Closed-Loop for the Fixed-Order Cases, (N_c=4, $z_o(x) = \cos(\pi x)$)

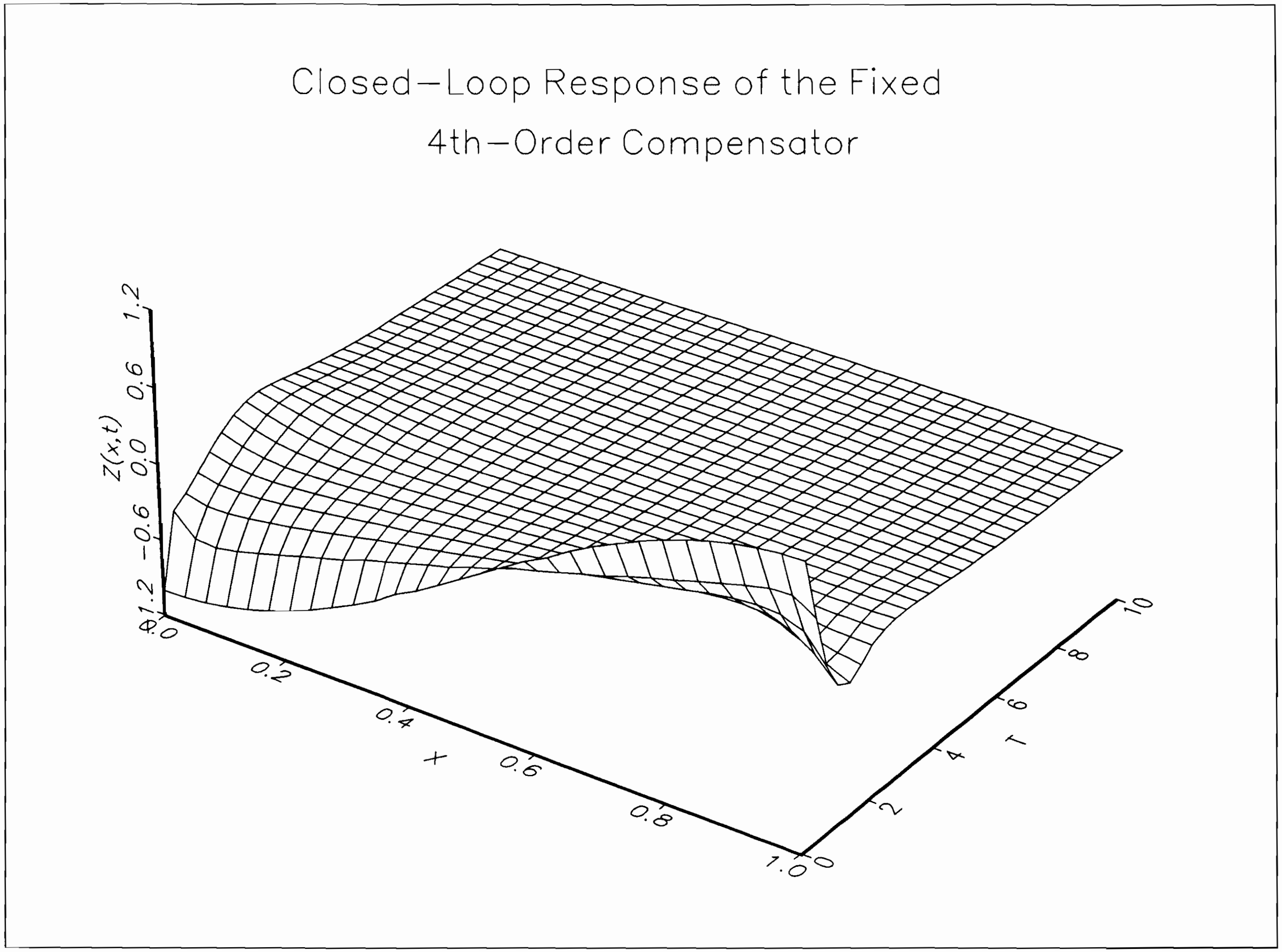

Figure 4.3.20: Closed-Loop for the Fixed-Order Cases, (N_c=4, $z_o(x) = -\cos(\pi x)$)

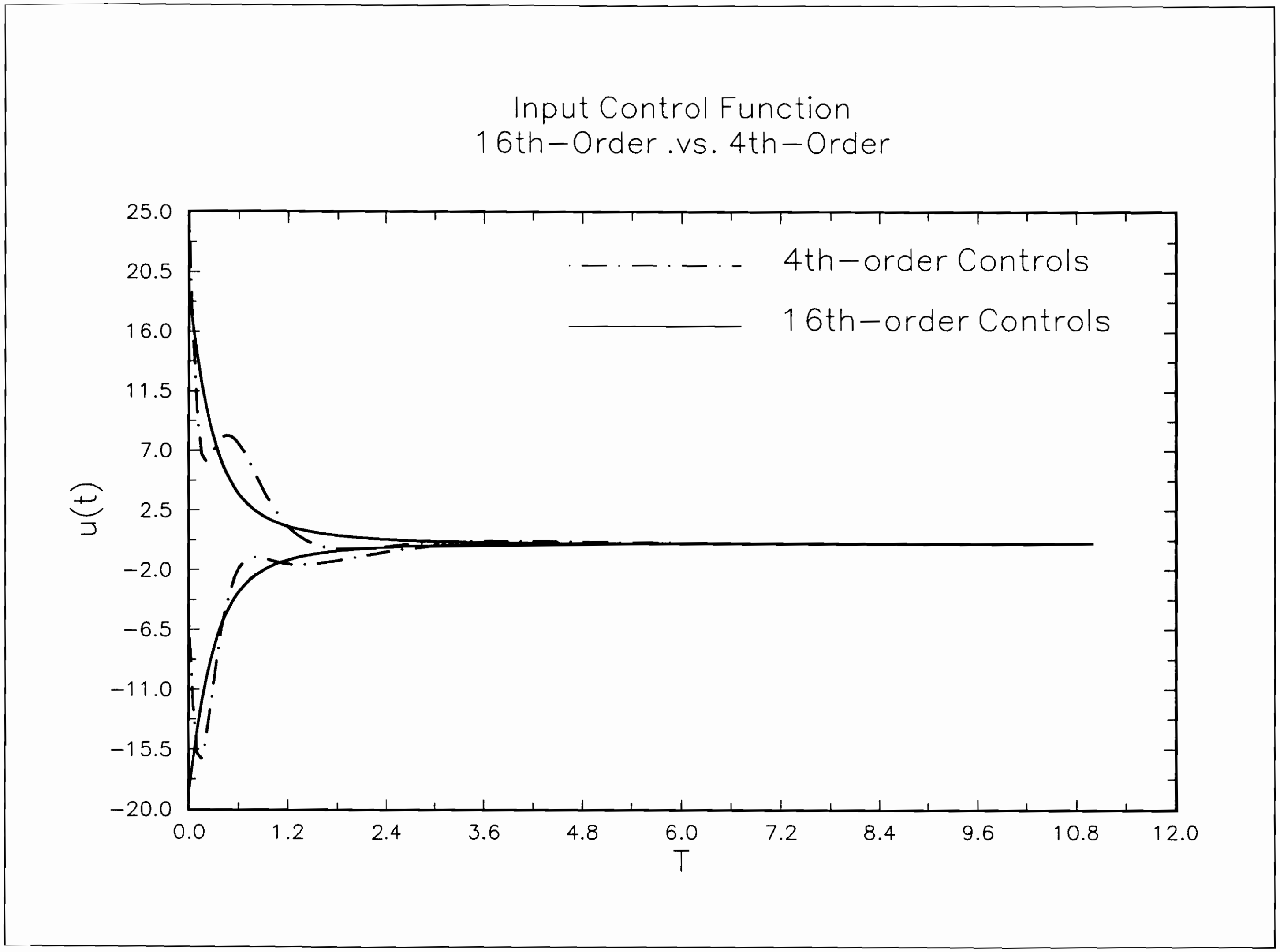

Figure 4.3.21: Fixed-order Control Function u(t), $N_c = 4$ and $N_c = 16$

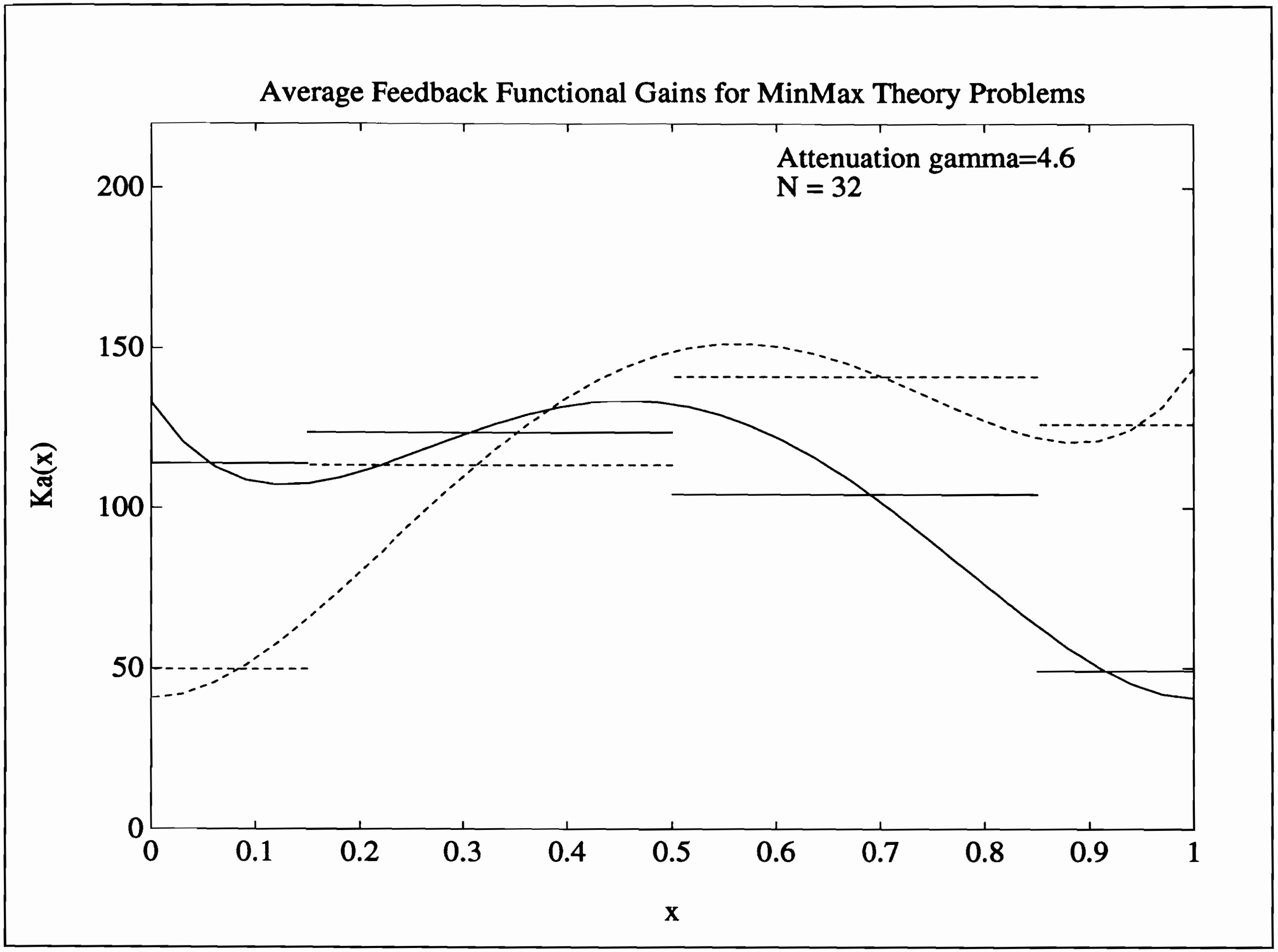

Figure 4.3.22: Average Step Functional Gains K_a of K_{opt} (Scheme 2)

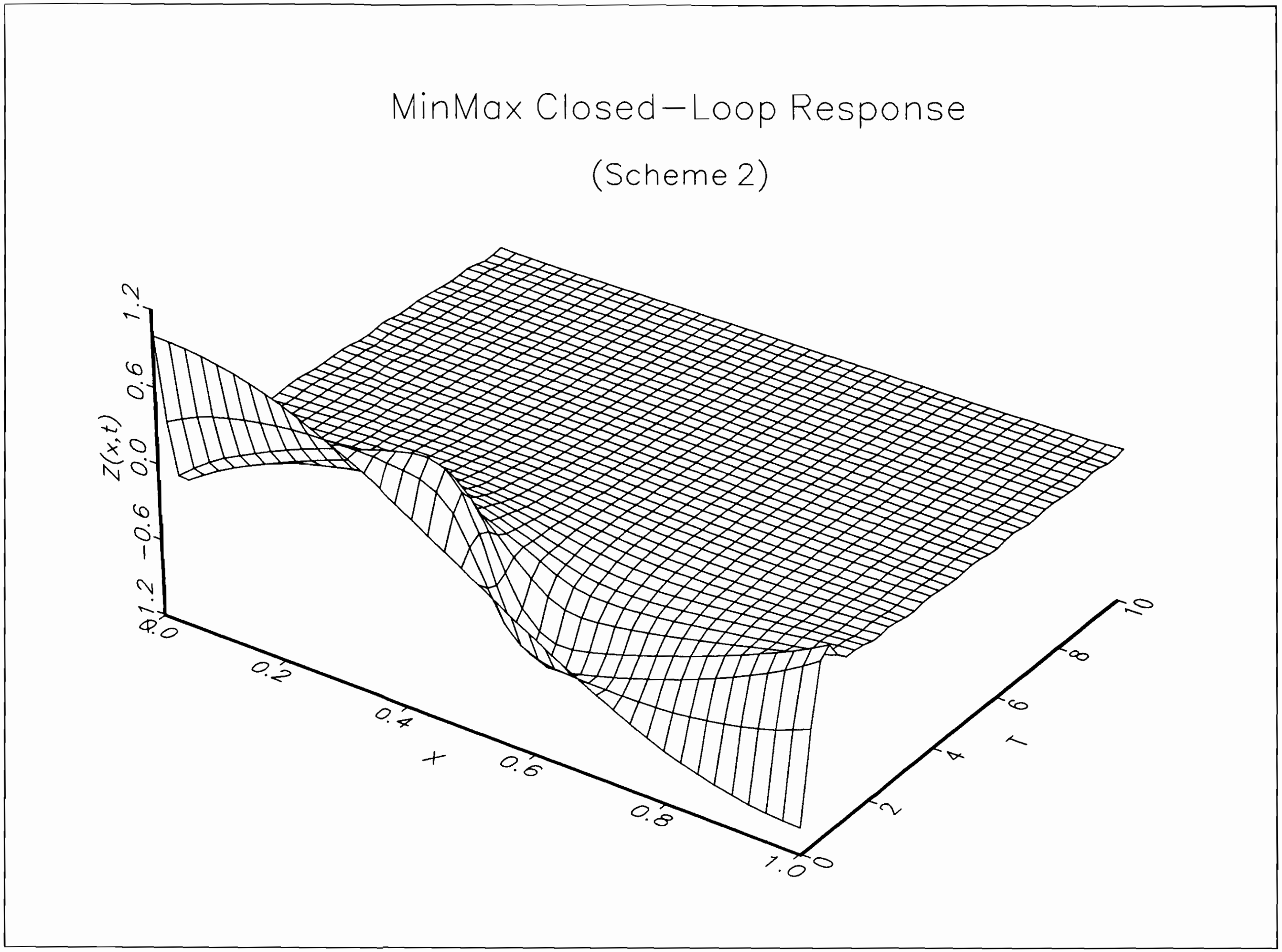

Figure 4.3.23: MinMax Closed Loop Response (Scheme 2), ($z_o(x) = \cos(\pi x)$)

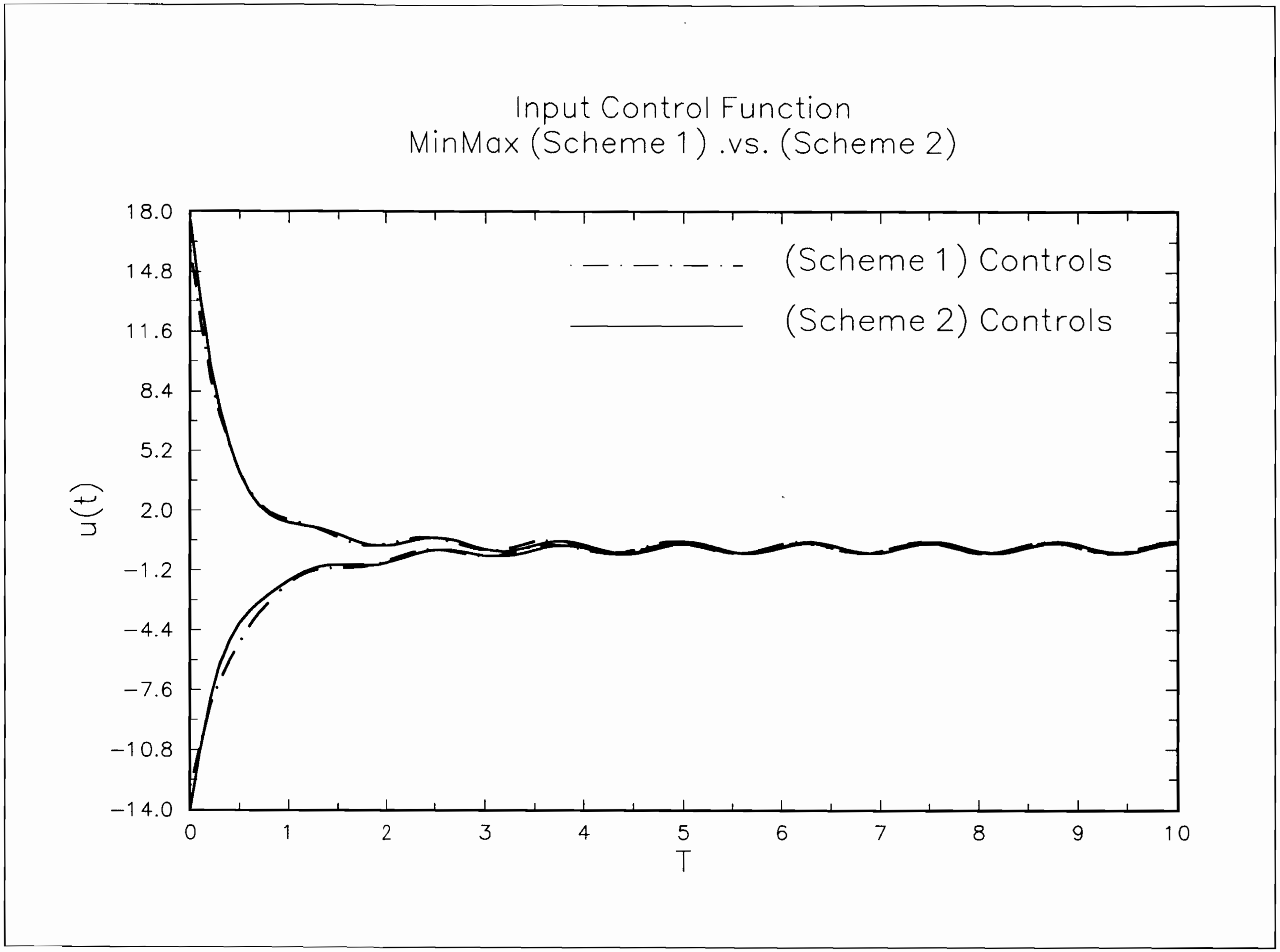

Figure 4.3.24: MinMax Control Function u(t), (Scheme 2)

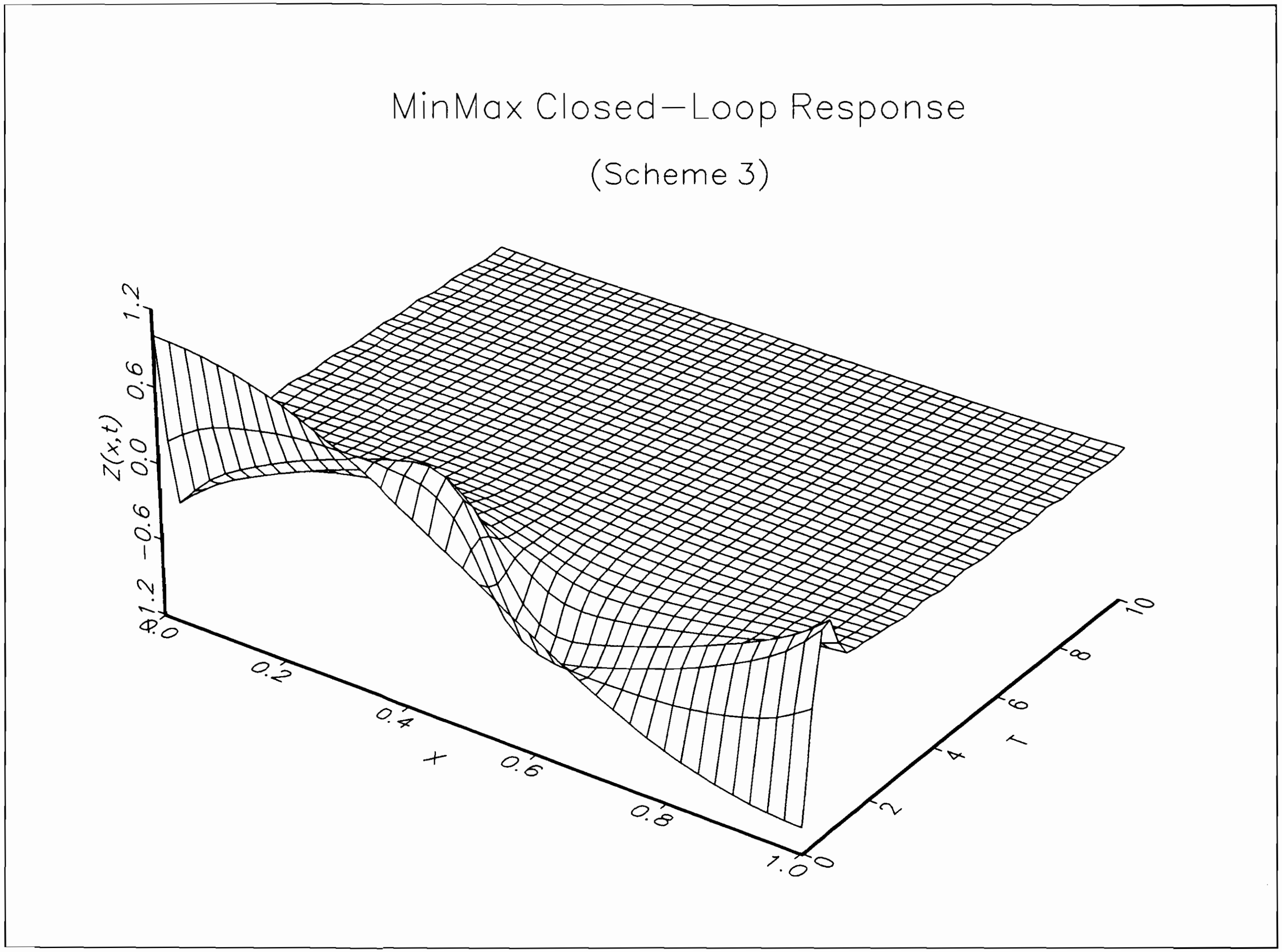

Figure 4.3.25: MinMax Closed Loop Response (Scheme 3), $(z_o(x) = \cos(\pi x))$

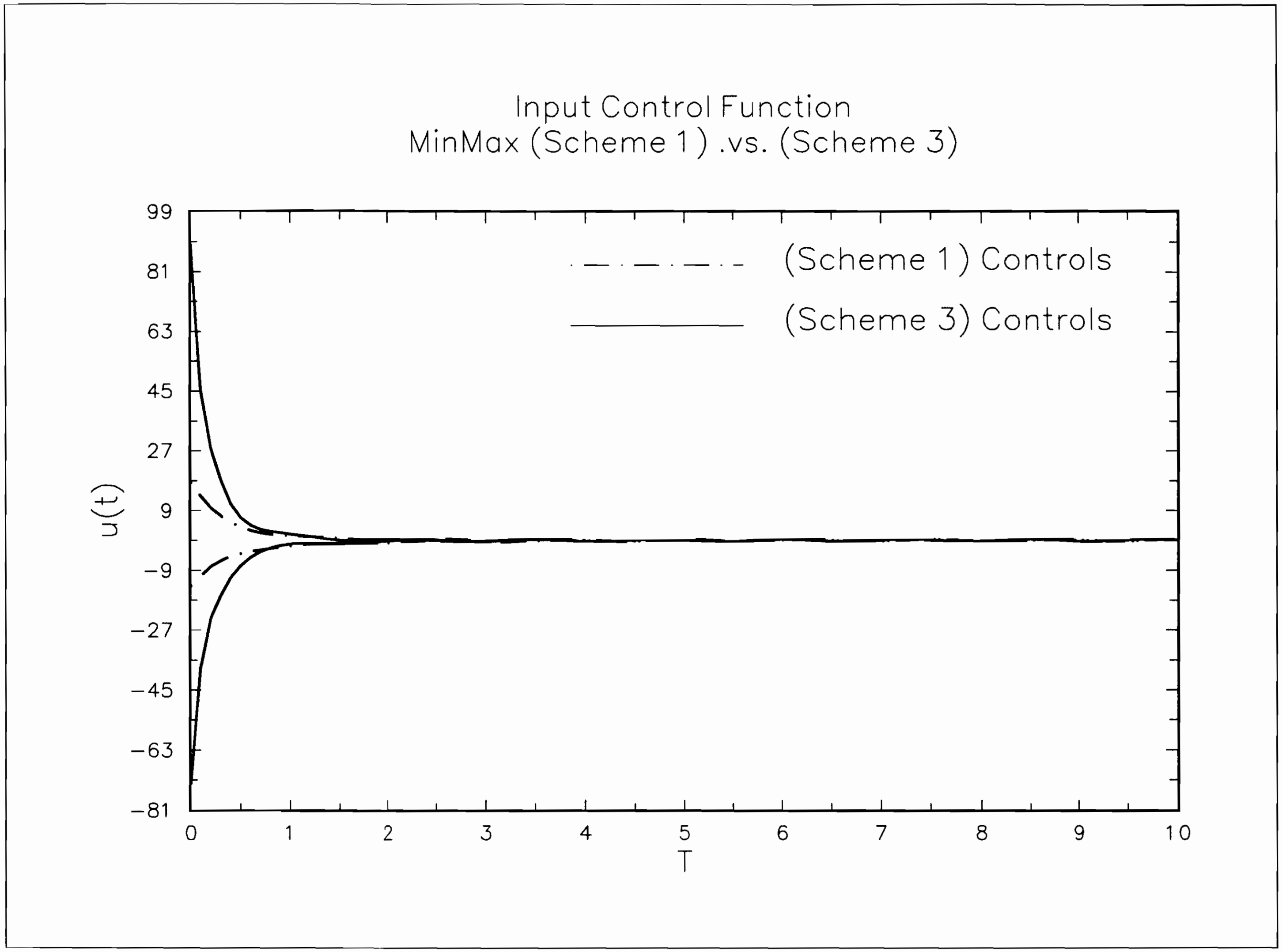

Figure 4.3.26: MinMax Control Function u(t), (Scheme 3)

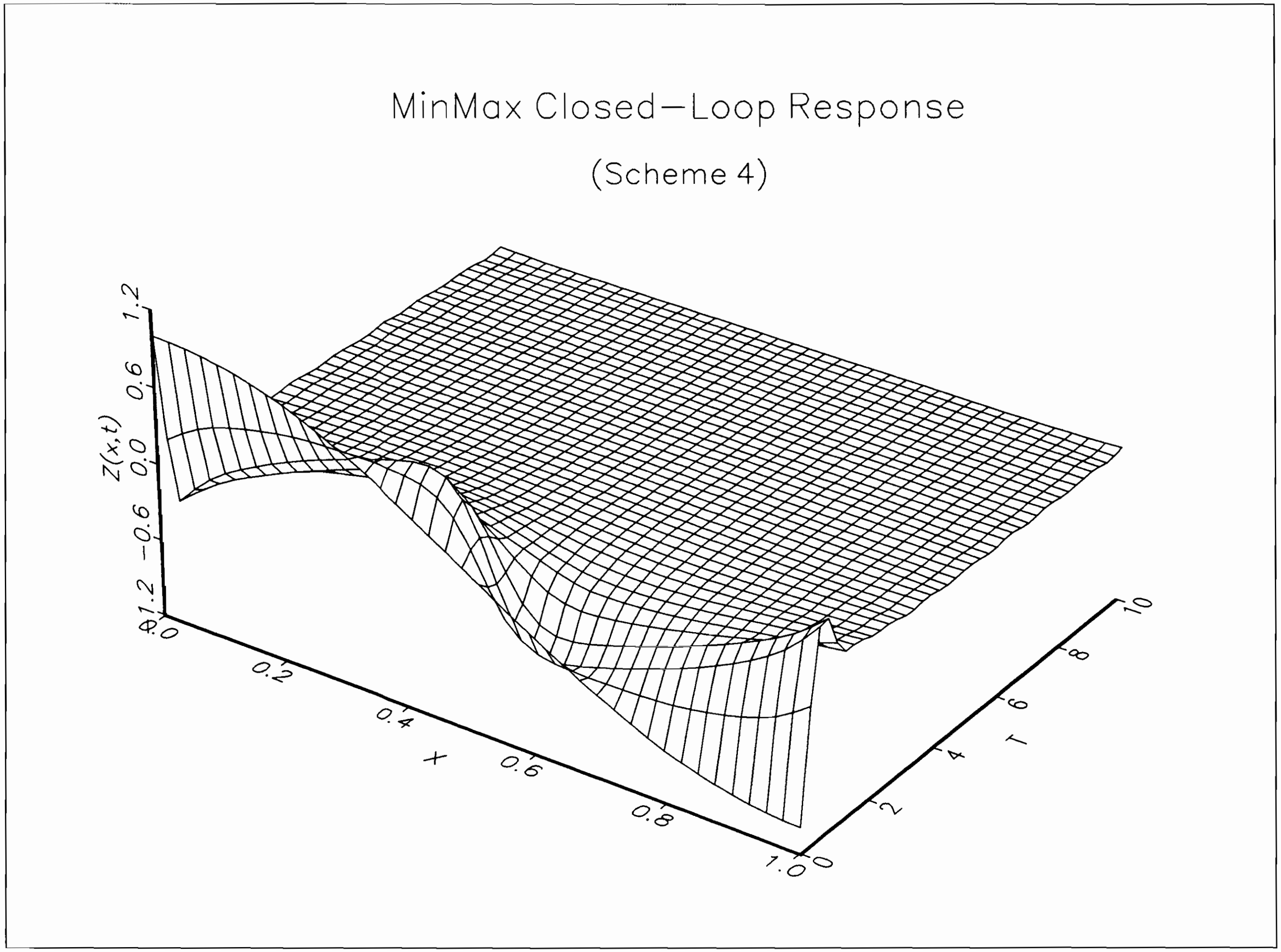

Figure 4.3.27: MinMax Closed Loop Response (Scheme 4), $(z_o(x) = \cos(\pi x))$

4.4 Summary

In this paper, we showed that dynamic compensators can be used to control a nonlinear partial differential equation. In particular, we selected a boundary control problem governed by Burgers' equation with boundary observations. The controllers were based on a linearization of the nonlinear equation. We discussed three different approaches. The infinite dimensional Linear Quadratic-Gaussian (LQG) and linear quadratic differential game theory (H^∞/MinMax) approaches presented in Sections 2.1 and 2.3 respectively, yield infinite dimensional compensators which are then approximated by finite dimensional compensators. In contrast, the optimal projection theory presented in Section 2.4 fixes the (finite) order of the compensator prior to the design process. The numerical results indicate that the LQG, MinMax and optimal projection approaches can work well in practice. However, the theory for the boundary control problems considered here is not complete.

A "shifted quadratic cost" problem was used to construct feedback and observer gains which produce a fixed decay rate. In particular, this approach produces a closed-loop system satisfying the estimate

$$\left\| \begin{bmatrix} z(x,t;z_o) \\ z_c(x,t;z_{c_o}) \end{bmatrix} \right\|_{\mathcal{H}} \leq \mathcal{M}_\alpha e^{-\alpha t} \left\| \begin{bmatrix} z_o \\ z_{c_o} \end{bmatrix} \right\|_{\mathcal{H}}$$

where $\alpha > 0$ does not depend on the Reynolds number. However, $\mathcal{M}_\alpha = \mathcal{M}(\alpha, \epsilon)$ will, in general, depends on the viscosity parameter $\epsilon > 0$.

We developed a numerical scheme for computing the feedback and observer functional gains and several numerical experiments were performed. The following observations were made:

1) The nonlinear closed–loop system was stabilized by all of the nonlinear controllers (linear feedback and nonlinear observer).

2) A comparison between LQG and MinMax designs shows that the latter outperforms the former (in disturbance attenuation) and provides a stability margin sufficient to accommodate the degree of disturbances introduced.

3) One does not have to sense the entire "flow" to control the "flow". The numerical experiments based on Schemes 2-4 illustrate that by a proper location of sensors and the use of low order observers one can control a distributed parameter system with a finite number of controllers and measurements.

These preliminary results provided some evidence that robust "linear" controllers can be used to control complex nonlinear distributed parameter systems. On the other hand, there are several interesting questions that need further study. In particular, we have established the following:

- The existence of MinMax controllers for systems of Pritchard–Salamon Class (Theorem 2.3.3).
- The stability of the approximating closed–loop system (Theorem 3.2.3).
- The existence of optimal projection controllers for systems of Pritchard–Salamon Class (Theorem 2.4.1).

On the other hand, there are several areas that require further work. In particular, one needs;

- A theory for problems that are not of Pritchard–Salamon Class. In particular, a theory is needed for the MinMax and optimal projection approaches.
- To establish the convergence of finite element approximations of the MinMax and optimal projection problem.
- To find alternative conditions (other than those given by Hypothesis (H–6) in Section 2.4) that guarantee the optimal projection equations have solutions.
- To prove the stability of the nonlinear approximating closed–loop system.

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VITA

Hamadi Marrekchi was born in Sfax, Tunisia on June 15, 1962. He graduated from "15 Novembre 1955" High School in Sfax, Tunisia in 1981 and went to the United States of America to continue his education. He received a B.S. degree in Electrical Engineering from Michigan State University in 1985, a M.S. degree in Electrical Engineering from Pennsylvania State University in 1987, a M.S. and a Ph.D. degree in Mathematics from Virginia Polytechnic Institute & State University in 1990 and 1993, respectively.

He will be working with the Interdisciplinary Center of Applied Mathematics (ICAM) until January, then he will start work with the University of Tunisia in Tunis, Tunisia in January 1994.