NONLINEAR RESONANCES IN SYSTEMS HAVING MANY DEGREES OF FREEDOM

by

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NOMENCLATURE

A	area of cross section of the beam
A_n, A_{nm}, B_n, B_{nm}	slowly varying complex functions containing the amplitude and the phase
a_n, a_{nm}, b_n, b_{nm}	amplitudes, real
c_n, c_{nm}	damping coefficients
D	plate rigidity
D_n	differential operators; see equation (2.3d)
E	Young's modulus
e_x	axial strain in the beam
e_r, e_θ, $e_{r\theta}$	in-plane strains in the plate
F	force function in the von Karman equations
F_n, H_{nm}, H_{nmp}	constants characterizing the amplitude of the excitation
h	plate thickness
I	moment of inertia of the beam cross section.

K_n	amplitudes, real; see Section 2.2.b
M	eigenvalue in stability calculations
m	bending moment on an element of the beam; see Figure 1
N_n, N_{nm}	terms which appear in the solvability conditions due to external resonances
n	axial force in the beam; see Figure 1
P, P_n, $\hat{P}_n$	amplitudes of the excitation; constants
p	excitation; function of the spatial variables and time
Q_i, Q_{ij}	constant coefficients of terms due to internal resonance
q	shear force on an element of the beam; see Figure 1
R_n, R_{nm}	terms which appear in the solvability conditions due to the internal resonance
r	coordinate, polar, in the plate problem - also radius of gyration in the beam problem

S_j, $\hat{S}_j$	Coefficients which are functions of the A_n, B_n, A_{nm}, and B_{nm}; see Appendicies A and C.
T_n	time scales used in the method of multiple scales; see equation (2.3a)
t	time
u	axial displacement of the beam; see Figure 1
u_r, u_θ	in-plane displacements of the plate
u_n, u_{nj}, $\tilde{u}_n$	temporal functions to be determined
w, w_n	transverse deflection of the beam and plate
x	coordinate, Cartesian, in the beam problem
α_n, α_{nm}, β_n, β_{nm}	phases, real
β	linear combination of the phases and the detuning of the modes involved in the internal resonance
ε	small dimensionless perturbation parameter

Γ_{nmpq}, γ_{ij}	constant coefficients of the nonlinear terms
η_m, η_{nm}	roots of the characteristic equations associated with the linear free-oscillations of beams and plates.
θ	coordinate, polar, in the plate problem - also slope of the deformed beam
κ_n, κ_{nm}	normalizing coefficients of the linear free-oscillation modes
Λ_j, $\hat{\Lambda}_j$	frequency combinations; see Appendicies A and C
λ	frequency of the excitation; a constant
μ, μ_n	linear combination of the phases and the detuning of the excitation; associated with an external resonance
ρ	material density of the beam and plate
σ_1	detuning of the frequencies involved in the internal resonance

σ, σ_2	detuning of the frequencies involved in an external resonance
ϕ_m, ϕ_{nm}	linear free-oscillation modes of the beam and plate
ψ_m, ψ_{nm}	functions which define the radial variation of the force function F
ω_m, ω_{nm}	natural frequencies

1. INTRODUCTION

Here attention is focused on deterministic systems having many degrees of freedom which are governed by a set of second-order nonlinear ordinary differential equations having constant coefficients. The nonlinear terms involved are usually either quadratic or cubic or both, these being the first two nonlinear terms in a Taylor series expansion of the nonlinearity about a static equilibrium configuration. The nonlinear terms are usually considered small in some sense and the system is said to be weakly nonlinear. The interest is in obtaining the response of the system to harmonic excitations having a single frequency (the frequency and the amplitudes of the excitations are constants).

There are various resonances associated with forced responses. Denoting the frequency of excitation by λ and the natural frequencies by ω_i, $i = 1,2,3,\ldots$, one can classify these resonances as follows:

a. external resonances

(1) main or harmonic $\qquad \lambda \approx \omega_k$;

(2) superharmonic $\qquad n\lambda \approx \omega_k$;

(3) subharmonic $\qquad \lambda \approx n\omega_k$;

(4) combination $\qquad n\lambda \approx m_1\omega_1 + m_2\omega_2 + \ldots + m_i\omega_i$;

(5) rational $\qquad \lambda \approx (m/n)\omega_k$;

b. internal resonances

$$m_1\omega_1 + m_2\omega_2 + \ldots + m_i\omega_i \approx 0 ;$$

where n, m and m_i are integers which depend on the order of the nonlinearity of the system. In the following, a number of works are cited as examples of the extensive literature available on the study of nonlinear resonances. All the studies were concerned with periodic solutions and their stability. In many cases analytical, experimental, and analogue-computer results were presented.

1.1 Literature Review

a. General Dynamical Systems

Several books such as Hayashi [1] can be found in the field. An analysis leading to the classification of a class of dynamical systems with cubic nonlinearities was presented by Sethna [2]. However, the response analysis was restricted to systems having two degrees of freedom. In a later paper [3], he studied the superharmonic and subharmonic resonances in systems having two degrees of freedom and quadratic nonlinearities. In both papers, Sethna paid particular attention to internal resonances. A thorough investigation of the different nonlinear resonances, including internal resonances, in a system having two degrees of freedom can be found in the book on rotor dynamics by Tondl [4]. A system having six degrees of freedom was considered by Efstathiades and Williams [5] in their study of vibration-isolating system. Plotnikova [6] obtained the condition for the stability of periodic solutions under main resonance for rather general systems having two degrees of freedom.

Mettler [7] gives an excellent survey of the nonlinear vibration problems in mechanical systems including applications to elastic bodies subjected to gyroscopic and non-conservative follower forces. Combination and subharmonic resonances in systems having both quadratic and cubic nonlinearities were studied by Yamamoto and Hayashi [8]. Much of their analysis was concerned with systems having two degrees of freedom. However, they did present some more general results but did not include the effects of internal resonances. Szemplinska-Stupnicka, in a number of papers [9,10,11,12], presented analyses of the various nonlinear resonances in systems having many degrees of freedom. She also made a comparitive study of the different approximate methods used in the analysis of nonlinear vibrations. An earlier work on such a comparitive study is due to Newland [13].

Most of the works cited above are concerned with discrete mechanical systems. However, as noted by Mettler, an analysis of the vibrations of elastic bodies leads to an infinite set of nonlinear differential equations, the nonlinearities being essentially quadratic and/or cubic. Some studies of the nonlinear resonances occuring in the nonlinear vibrations of structural elements, such as beams and plates, are discussed in the next section.

b. Structural Elements-Beams and Plates

The transverse vibrations of linearly elastic beams and plates supported in such a way as to restrict movement at the ends and along the edges are accompanied by stretching of the midplane. This stretching is taken in account by nonlinear strain-displacement relations, which lead to nonlinear governing equations of motion. They are usually solved by assuming the spatial variation and then obtaining a set of coupled, nonlinear, ordinary differential equations governing the temporal variation, the nonlinearity being essentially cubic. The different methods that are used for the solution of this set of nonlinear equations are discussed in Section 1.2.

Main and superharmonic resonances of different modes in straight beams were studied by Bennett and Eisley [14] and Bennett [15]. The results of their analysis and experiment indicate the inadequacy of a single-mode analysis to describe the response fully. Tseng and Dugundji [16,17] reported on the superharmonic, subharmonic and rational resonances in straight beams and the superharmonic resonances in buckled beams. Busby and Weingarten [18] and Nayfeh, Mook and Lobitz [30] used finite-element techniques to obtain the nonlinear equations governing the temporal variation of the response of straight beams to harmonic excitations. They studied the main resonances of different modes.

The dynamic analogue of the von Karman equations has been used in several attempts to determine the effect of mid-plane stretching in the response of plates to harmonic excitation. Examples are the works of Chu and Herrmann [19], who considered rectangular plates, and Yamaki [20], who studied both rectangular and circular plates. Farnsworth and Evan-Iwanowski [21] considered small-amplitude oscillations about a large-amplitude static deflection. Huang and Sandman [22] and Sandman and Huang [23] considered clamped circular plates and annuli having a clamped outer edge and a free inner edge, respectively. Bennett [24] presented an analysis of the response of rectangular, laminated plates having simple supports.

1.2 Methods of Analyzing the Nonlinear Equations

Since exact solutions are, in general, not available for the study of nonlinear differential equations, recourse has been made to approximate methods in the studies cited above. The approximations that are invariably used in the analysis of weakly nonlinear systems can be broadly classified as: (1) the method of harmonic balance, (2) a perturbation method, usually, the method developed by Krylov, Bogoliubov and Mitropolsky, which in the first approximation is known as the method of averaging and (3) a minimizing method such as the Ritz or the Galerkin method. In a noteworthy article, Rosenberg [25] gives a detailed account of the so-called geometrical methods which are more concerned with mathematical aspects of a qualitative nature of solutions of nonlinear systems. A significant feature of these methods is that their applicability is not restricted to weakly nonlinear systems. Many of the results in [25] are taken from previously published papers by Rosenberg.

In recent years, another method which has become popular in the analysis of weakly nonlinear systems is the method of multiple scales, a perturbation method. A detailed description of this method and an exhaustive bibliography are given by Nayfeh [26]. This method was applied to the nonlinear analysis of ship motions by Nayfeh, Mook, and Marshall [27] and Mook, Marshall and Nayfeh [28]. Essentially, the systems analyzed had two degrees of freedom and quadratic nonlinearities. Various nonlinear resonances were considered in detail. Nonlinear vibrations of structural elements were studied, using the method of multiple scales, by Nayfeh, Mook, and Sridhar [29], Nayfeh, Mook and Lobitz [30], Sridhar, Mook and Nayfeh [31], Morino [32] and Atluri [33].

1.3 An Assessment of the Previous Studies and the Contribution by the Present Work

Although a large amount of literature is available on the subject of nonlinear resonances in weakly nonlinear systems having many degrees of freedom, this body of knowledge suffers from the following deficiencies:

a. Many of the studies are confined to systems having two degrees of freedom, and the results obtained from such analyses are not representative of those obtainable from a system having more than two degrees of freedom, especially when the nonlinearity involved is cubic. Further, even in studies of systems having more than two degrees of freedom, the analyses are invariably restricted to the study of some specific resonance. Thus, the available information is in some sense disjointed. A unified approach leading to the study of the various nonlinear resonances is lacking.

b. The phenomenological behaviour of systems with internal resonances has not been explored in any depth. This is especially true of studies concerned with the vibrations of structural elements. In such studies, internal resonance is usually referred to as 'coupling of modes' or 'modal interaction'. The studies in references [2-4, 27-30] are some notable efforts in the study of internal resonances. An interesting feature of the internal resonance is the fact that, it is the only nonlinear resonance which is inherent in the system and is independent of the external excitation. One might expect that the 'mechanism' of internal resonance would be the same irrespective of which of the other external resonances are occuring in the system. This aspect has not been clarified in any of the studies mentioned above.

The present work is an effort to correct the above deficiencies in a class of nonlinear systems by presenting a unified method for the analysis of external resonances (main, superharmonic, subharmonic and combination) which takes internal resonances into account. The effects of internal resonances are explored in depth.

The system chosen is of intrinsic interest, meriting study in its own right. However, the choice of the system was strongly motivated by the fact that the problem of the nonlinear transverse vibrations of structural elements can often be reduced to the study of a nonlinear system that is exactly or essentially the same as the system considered in the present work.

In Chapter 2, the system is defined and a unified method for the analysis of the various nonlinear resonances is developed.

In Chapter 3, external resonances in the absence of internal resonances are analyzed in detail.

In Chapter 4, the effects of an internal resonance are carefully evaluated.

In Chapter 5, the problem of the nonlinear vibrations of beams is reduced to a study of the system considered in Chapter 2. Also numerical examples involving main, superharmonic, subharmonic, combination, and internal resonances in a hinged-clamped beam are presented.

In Chapter 6, the general problem of the nonlinear vibrations (including asymmetric vibrations and travelling waves) of circular plates is reduced to the study of essentially the same system considered in Chapter 2. Main and internal resonances are considered.

In Chapter 7, the general analysis of Chapter 6 is specialized to symmetric responses. Also numerical examples involving main and internal resonances are presented. The internal resonance differs from the one considered in Chapter 5.

In Chapter 8, a summary of the present work is presented.

2. PROBLEM FORMULATION AND METHOD OF SOLUTION

In this chapter, the nonlinear system that is chosen for study is defined and a unified method for the analysis of the various nonlinear resonances is developed.

2.1 Problem Formulation

The system chosen is governed by a set of equations having the form

$$\frac{d^2u_n}{dt^2} + \omega_n^2 u_n = \varepsilon \left[- 2c_n \frac{du_n}{dt} + \sum_{m,p,q} \Gamma_{nmpq} u_m u_p u_q \right] + P_n \cos \lambda t,$$

$$n = 1,2,\ldots \tag{2.1}$$

where ω_n are the distinct natural frequencies of the corresponding linear problem; ε is a dimensionless parameter; c_n are the modal viscous damping coefficients; Γ_{nmpq} are constant coefficients of the cubic terms; P_n are the amplitudes of excitation; and λ is the frequency of excitation. Both P_n and λ are constants.

The system is assumed to have one internal resonance combination involving four modes of the form

$$\omega_a + \omega_b + \omega_c \approx \omega_d \tag{2.2}$$

If an attempt is made to obtain straightforward perturbation expansions for u_n, for small ε, then the expansions so generated are not uniformly valid for large t due to the appearance of so-called secular terms. In the present work a modification of the straightforward procedure, the method of multiple scales, is used to construct the first terms in the asymptotic expansions of the u_n which are uniformly valid for small ε and large t. The study is concerned with periodic solutions for large time (i.e., steady-state responses) and the determination of their stability. Attention is focused on main, superharmonic, subharmonic, combination and internal resonances.

2.2 Method of Solution

According to the derivative-expansion version of the method of multiple scales, the single time scale t is replaced by a number of time scales which are defined by

$$T_j = \varepsilon^j t \quad , \quad j = 0,1,2, \ldots \tag{2.3a}$$

Introducing these time scales results in the derivatives with respect to t being transformed into expansions in terms of the derivatives with respect to the new scales as follows:

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \ldots \tag{2.3b}$$

and

$$\frac{d}{dt^2} = D_0^2 + \varepsilon 2D_0 D_1 + \ldots \tag{2.3c}$$

where

$$D_j = \frac{\partial}{\partial T_j} \quad . \tag{2.3.d}$$

Expansions for the u_n are assumed to be of the form

$$u_n(t;\varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^j u_{nj}(T_0,T_1,\ldots). \quad j = 0,1,2,\ldots \tag{2.4}$$

Substituting equations (2.3) and (2.4) into (2.1) and balancing powers of ε yields a series of problems governing the u_{nj}.

Two cases arise, depending on the order of magnitude of the P_n. The case when $P_n = O(\varepsilon)$ is associated with main resonances; that is, λ is near one of the ω_n. The case when $P_n = O(1)$ is associated with superharmonic, subharmonic and combination resonances; that is, λ is away from all ω_n.

a. <u>Main Resonance</u>

In this case, we let $P_n = \varepsilon\hat{P}_n$ where $\hat{P}_n = O(1)$. The equations governing the u_{nj} are

$$D_0^2 u_{n0} + \omega_n^2 u_{n0} = 0 \quad , \tag{2.5}$$

$$D_0^2 u_{n1} + \omega_n^2 u_{n1} = -\,2D_0 D_1 u_{n0} - 2c_n D_0 u_{n0} + \hat{P}_n \cos \lambda T_0$$

$$+ \sum_{m,p,q} \Gamma_{nmpq} u_{m0} u_{p0} u_{q0} \,, \quad n = 1, 2, \ldots \tag{2.6}$$

etc.

The solution of the zeroth-order problem defined by equations (2.5) can be written as

$$u_{n0} = A_n(T_1, \ldots)\exp(i\omega_n T_0) + cc, \quad n = 1, 2, \ldots \tag{2.7}$$

where cc represents the complex conjugate of the preceding terms. At this point, the complex functions A_n are unknown. They are determined from the so-called solvability conditions at the next level of approximation. It is noted that in the first approximation the response is of lower order than the excitation.

Substituting equations (2.7) into (2.6) leads to

$$D_0^2 u_{n1} + \omega_n^2 u_{n1} = -2i\omega_n(D_1 A_n + c_n A_n)\exp(i\omega_n T_0) + \frac{1}{2}\hat{P}_n \exp(i\lambda T_0)$$

$$+ \sum_{m,p,q} \Gamma_{nmpq} \left[\sum_{j=1}^{4} S_j \exp(i\Lambda_j T_0)\right] + cc,\ n = 1,2,\ldots \qquad (2.8)$$

where Λ_j are linear combinations of the natural frequencies and S_j are functions of the A_n; both are listed in Appendix A.

b. Superharmonic, Subharmonic and Combination Resonances

In this case, we let $P_n = O(1)$. Then the equations governing the u_{nj} are

$$D_0^2 u_{n0} + \omega_n^2 u_{n0} = P_n \cos \lambda T_0 \quad , \qquad (2.9)$$

$$D_0^2 u_{n1} + \omega_n^2 u_{n1} = -2D_0 D_1 u_{n0} - 2c_n D_0 u_{n0} + \sum_{m,p,q} \Gamma_{nmpq} u_{m0} u_{p0} u_{q0},$$

$$n = 1,2,\ \ldots \qquad (2.10)$$

etc.

The solution of equations (2.9) can be written as

$$u_{n0} = A_n(T_1,\ldots)\exp(i\omega_n T_0) + K_n \exp(i\lambda T_0) + cc,\ n = 1,2,\ldots \quad (2.11)$$

where

$$K_n = \frac{1}{2} P_n(\omega_n^2 - \lambda^2)^{-1}$$

and cc represents the complex conjugate of the preceding terms. At this point, the A_n are unknown. They are determined by the solvability conditions at the next level of approximation. We note that in the first approximation the response is of the same order as the excitation.

Substituting equations (2.11) into (2.10) leads to

$$D_0^2 u_{n1} + \omega_n^2 u_{n1} = - 2i\omega_n(D_1 A_n + c_n A_n)\exp(i\omega_n T_0) - 2ic_n \lambda K_n \exp(i\lambda T_0)$$

$$+ \sum_{m,p,q} \Gamma_{nmpq}\left[\sum_{j=1}^{27} S_j \exp(i\Lambda_j T_0)\right] + cc, \quad n = 1, 2, \ldots \qquad (2.12)$$

where the Λ_j and S_j are listed in Appendix A.

Equations (2.8) or (2.12) are a set of linear nonhomogeneous uncoupled equations which can be solved for the functions u_{n1}. However, unless certain conditions are imposed on the nonhomogeneous terms in the equations, the u_{n1} will contain 'secular terms' of the form $T_0 \exp(\pm i\omega_n T_0)$. These terms can be eliminated by simply requiring that the coefficients of $\exp(\pm i\omega_n T_0)$ in equations (2.8) and (2.12) be equal to zero. This leads to a set of nonlinear coupled equations, the so-called solvability conditions, which can be solved for the A_n.

The solvability conditions involve the first term as well as all other terms for which Λ_j is equal to or nearly equal to ω_n on the right-hand side of equations (2.8) and (2.12). An investigation of Appendix A shows that Λ_1 through Λ_7 are linear combinations of the natural frequencies only and that it is always possible for Λ_2 through Λ_7 to equal ω_n. For example, $\Lambda_5 = \omega_n$ when $m = n$, while $\Lambda_2 = \omega_n$ when $m = n$ and $p = q$.

Consequently, S_2 through S_4, which are cubic in the A_n, will always enter the solvability conditions in the case of main resonance. In addition, for the case of superharmonic, subharmonic and combination resonances S_5 through S_7, which are linear in the A_n, will always enter the solvability conditions.

An internal resonance is said to exist if any other combination of natural frequencies is approximately equal to ω_n; that is, if the natural frequencies are commensurable. Because the nonlinearity is cubic, the combination of natural frequencies associated with an internal resonance have the form

$$\sum_i m_i\omega_i \approx 0 \text{ where } \sum_i |m_i| = 4 \ .$$

For example, $(\omega_2 \approx 3\omega_1)$, $(\omega_3 \approx 2\omega_1 + \omega_2)$ and $(\omega_4 \approx \omega_1 + \omega_2 + \omega_3)$ are combinations of natural frequencies associated with internal resonances.

In the case of main resonance, an external resonance is said to exist when λ is near ω_n. In the case of superharmonic, subharmonic and combination resonances, an external resonance is said to exist when Λ_j for $j \geqslant 8$ is approximately equal to ω_n. Because the nonlinearity is cubic, Λ_j for $j \geqslant 8$ satisfies the relation

$$n\lambda \approx \sum_i m_i\omega_i \quad \text{where } |n| + \sum_i |m_i| = 4 \ .$$

For external resonances, the combinations of natural frequencies always contain λ.

A mode is said to be 'directly excited' if the corresponding natural frequency is involved in an external resonant combination with λ. For example, when λ is near ω_n, the mode corresponding to ω_n is said to be directly excited; and when λ is near $(\omega_m \pm \omega_p \pm \omega_q)$, the modes corresponding to ω_m, ω_p and ω_q are said to be directly excited.

In general, when there are external and internal resonances, the solvability conditions have the form

$$- 2i\omega_n(D_1A_n + c_nA_n) + A_n \sum_j \gamma_{nj}A_j\bar{A}_j + 2H_{nn}A_n + R_n + N_n = 0$$

$$n = 1, 2, \ldots \tag{2.13}$$

where R_n is due to internal resonances, if any; N_n is due to external resonances, if any; and

$$\gamma_{nj} = \begin{Bmatrix} 3\Gamma_{nnnn} \quad , & n = j \\ 2(\Gamma_{nnjj} + \Gamma_{njnj} + \Gamma_{njjn}), & n \neq j \end{Bmatrix} \tag{2.14a}$$

$$H_{nk} = \sum_{i,j} (\Gamma_{nkij} + \Gamma_{nikj} + \Gamma_{nijk})K_iK_j \quad , \tag{2.14b}$$

and

$$H_{nk} = 0 \quad , \text{ when } P_n = O(\varepsilon) \quad ; \tag{2.14c}$$

i.e., in the case of main resonance. The specific forms of R_n and N_n depend on the types of internal and external resonances. It is noted that direct excitation of the n^{th} mode implies that $N_n \neq 0$. Various cases are considered in the following chapters.

2.3 Summary

A method is developed for obtaining the first approximation to the resonant response of the system chosen for study [i.e., equations (2.1)]. The method is 'unified' in the sense that the solvability conditions, equations (2.13), can be used as a starting point for the analysis of main, superharmonic, subharmonic, combination and internal resonances. With the present approach the various nonlinear resonant possibilities can be readily ascertained by an inspection of Appendix A. For any specific case, the terms due to the external and internal resonances, in the solvability conditions, can be obtained essentially by an investigation of Appendix A.

3. EXTERNAL RESONANCES IN THE ABSENCE OF INTERNAL RESONANCES

In this chapter, the various external resonances are analyzed in detail, in the absence of internal resonances [i.e., $R_n = 0$ for all n in equations (2.13)]. Steady-state solutions are obtained and a method for determining their stability is discussed.

3.1 The Case of No External Resonances

For those modes which are not involved in an external resonance, $N_n = 0$ in equations (2.13). We let

$$A_n = \frac{1}{2} a_n(T_1)\exp[i\alpha_n(T_1)] \qquad \text{no resonance} \tag{3.1}$$

where a_n, the amplitude, and α_n, the phase, are real functions of the slow time scale T_1. Because the objective is to obtain only the first terms in the asymptotic expansions for the u_n, all T_n for $n > 1$ are considered to be constants. Substituting equation (3.1) into (2.13) and separating the result into real and imaginary parts yields

$$\omega_n(a_n' + c_n a_n) = 0 \tag{3.2a}$$

and

$$\omega_n a_n \alpha_n' + \frac{1}{8} a_n \sum_j \gamma_{nj} a_j^2 + H_{nn} a_n = 0 \tag{3.2b}$$

where primes denote differentiation with respect to T_1. Equation (3.2a) shows that $a_n \to 0$ as $T_1 \to \infty$, and the steady-state amplitude $a_n = 0$. Thus, the steady-state solution has the form

$$u_n = O(\varepsilon), \text{ when } P_n = O(\varepsilon) \tag{3.3a}$$

i.e., in the case of main resonance, and

$$u_n = P_n(\omega_n^2 - \lambda^2)^{-1} \cos \lambda t + O(\varepsilon), \text{ when } P_n = O(1) \tag{3.3b}$$

i.e., in the case of superharmonic, subharmonic and combination resonances.

3.2 The Case of λ Near ω_k

In this case the only resonance is due to λ being near ω_k (main resonance). A detuning parameter, σ, is used to express the nearness of λ to ω_k as follows:

$$\lambda = \omega_k + \varepsilon\sigma \tag{3.4}$$

Then

$$N_k = \frac{1}{2}\hat{P}_k \exp(i\sigma T_1) \tag{3.5}$$

Substituting equations (3.5) and (3.1) into (2.13) and separating the result into real and imaginary parts, one obtains, for n = k

$$-\omega_k(a_k' + c_k a_k) + \frac{1}{2}\hat{P}_k \sin \mu = 0 \quad , \tag{3.6a}$$

$$\omega_k a_k \alpha_k' + \frac{1}{8} a_k \sum_j \gamma_{kj} a_j^2 + \frac{1}{2}\hat{P}_k \cos \mu = 0 \tag{3.6b}$$

where

$$\mu = \sigma T_1 - \alpha_k \quad , \tag{3.6c}$$

and for $n \neq k$, the amplitudes and phases are governed by equations (3.2).

The steady-state solution corresponds to $a_n' = 0$ and $\mu' = 0$; thus, equations (3.6) can be reduced to

$$- \omega_k c_k a_k + \frac{1}{2} \hat{P}_k \sin \mu = 0 \tag{3.7a}$$

and

$$\omega_k a_k \sigma + \frac{1}{8} \gamma_{kk} a_k^3 + \frac{1}{2} \hat{P}_k \cos \mu = 0 \quad . \tag{3.7b}$$

It is noted that $a_k = 0$ is not a solution of equations (3.7). Solving for a_k and μ from equations (3.7) and using equations (3.1), (3.4) and (3.6c), one obtains

$$A_k \exp(i\omega_k T_0) = \frac{1}{2} a_k \exp(\lambda t - \mu) \tag{3.8}$$

Substituting equation (3.8) into (2.7), and using equations (3.3a) and (2.4) yields

$$u_n = O(\varepsilon), \quad n \neq k \tag{3.9a}$$

and

$$u_k = a_k \cos (\lambda t - \mu) + O(\varepsilon) \tag{3.9b}$$

for the steady-state response. The solution given by equation (3.9b) is essentially the k^{th} mode of the linear free oscillation. The difference between the solution and the linear free-oscillation mode is a shift in the frequency so that

$$\omega_k + \varepsilon\alpha_k' = \lambda$$

We note that equation (3.9b) is the steady-state solution of the Duffing equation; see reference [35]. Hence, there is a range of frequency of the excitation for which two stable solutions and one unstable solution exist. In this frequency range, the initial conditions determine which branch represents the actual response.

3.3 The Case of 3λ Near ω_k

In this case, the only resonance is due to 3λ being near ω_k (superharmonic resonance). The detuning parameter, σ, is used to express the nearness as follows:

$$3\lambda = \omega_k + \varepsilon\sigma \ . \tag{3.10a}$$

Then,

$$N_k = F_k \exp(i\sigma T_1), \; F_k = \sum_{m,p,q} \Gamma_{kmpq} K_m K_p K_q \ . \tag{3.10b}$$

After separating equation (2.13) into real and imaginary parts, one obtains

$$-\omega_k(a_k' + c_k a_k) + F_k \sin\mu = 0 \ , \tag{3.11a}$$

$$\omega_k a_k \alpha_k' + \frac{1}{8} a_k \sum_j \gamma_{kj} a_j^2 + H_{kk} a_k + F_k \cos\mu = 0 \tag{3.11b}$$

where

$$\mu = \sigma T_1 - \alpha_k \ , \tag{3.11c}$$

and for $n \neq k$, the amplitudes and phases are governed by equations (3.2).

The steady-state solution corresponds to all $a_n' = 0$ and $\mu' = 0$; thus, equations (3.11) can be reduced to

$$-\omega_k c_k a_k + F_k \sin\mu = 0 \tag{3.12a}$$

and

$$\omega_k a_k \sigma + \frac{1}{8} \gamma_{kk} a_k^3 + H_{kk} a_k + F_k \cos \mu = 0 \quad . \tag{3.12b}$$

It is noted that $a_k = 0$ is not a solution of equations (3.12); thus, solving for a_k and μ and substituting the result into equations (3.1) (2.11) and (2.4) can only yield a solution of the form

$$u_n = P_n(\omega_n^2 - \lambda^2)^{-1} \cos \lambda t + O(\varepsilon), \; n \neq k \tag{3.13a}$$

and

$$u_k = P_k(\omega_k^2 - \lambda^2)^{-1} \cos \lambda t + a_k \cos(3\lambda t - \mu) + O(\varepsilon) \; . \tag{3.13b}$$

The last term in equation (3.13b) is essentially the k^{th} mode of the linear, homogeneous solution; the difference between this term and the actual mode lies in the frequency, which the nonlinearity slightly adjusts so that

$$\omega_k + \varepsilon\alpha_k' = 3\lambda \quad .$$

Because the frequencies ω_k and λ are commensurable, this mode interacts with the excitation through the nonlinear terms in equation (2.1) and hence forms part of the steady-state solution in spite of the presence of damping.

3.4 The Case of 2λ Near $(\omega_m \pm \omega_k)$

In this case, the only resonance for which the details are presented is due to 2λ being near $(\omega_m + \omega_k)$. The results for 2λ near $(\omega_m - \omega_k)$ can be obtained from those presented below by simply changing the sign of ω_k.

The detuning parameter, σ, is used to express the nearness of 2λ to $(\omega_m + \omega_k)$ as follows:

$$2\lambda = \omega_m + \omega_k + \varepsilon\sigma \quad . \tag{3.14a}$$

Then, $N_n = 0$ for $n \neq m$ and k, while

$$N_m = H_{mk}\bar{A}_k \exp(i\sigma T_1) \text{ and } N_k = H_{km}\bar{A}_m \exp(i\sigma T_1) \tag{3.14b,c}$$

where H_{mk} and H_{km} are given by equation (2.14b). After separating equation (2.13) into real and imaginary parts, one obtains

$$-\omega_k(a_k' + c_k a_k) + \frac{1}{2} H_{km} a_m \sin\mu = 0 \quad , \tag{3.15a}$$

$$-\omega_m(a_m' + c_m a_m) + \frac{1}{2} H_{mk} a_k \sin\mu = 0 \quad , \tag{3.15b}$$

$$\omega_k a_k \alpha_k' + \frac{1}{8} a_k \sum_j \gamma_{kj} a_j^2 + H_{kk} a_k + \frac{1}{2} H_{km} a_m \cos\mu = 0 \, , \tag{3.15c}$$

$$\omega_m a_m \alpha_m' + \frac{1}{8} a_m \sum_j \gamma_{mj} a_j^2 + H_{mm} a_m + \frac{1}{2} H_{mk} a_k \cos\mu = 0 \tag{3.15d}$$

where

$$\mu = \sigma T_1 - \alpha_k - \alpha_m \quad , \tag{3.15e}$$

and for $n \neq k$, m, the amplitudes and phases are governed by equations (3.2).

The steady-state solution corresponds to all $a_n' = 0$ and $\mu' = 0$. In contrast with the superharmonic resonance, it is noted that $a_m = a_k = 0$ is a steady-state solution of equations (3.15). For a nontrivial solution neither a_m nor a_k is zero, and equations (3.15) can be reduced to

$$- \omega_k c_k a_k + \frac{1}{2} H_{km} a_m \sin \mu = 0 \quad , \tag{3.16a}$$

$$- \omega_m c_m a_m + \frac{1}{2} H_{mk} a_k \sin \mu = 0 \quad , \tag{3.16b}$$

and

$$\sigma + \frac{1}{8}\left(\frac{\gamma_{kk}}{\omega_k} + \frac{\gamma_{mk}}{\omega_m}\right)a_k^2 + \frac{1}{8}\left(\frac{\gamma_{mm}}{\omega_m} + \frac{\gamma_{km}}{\omega_k}\right)a_m^2 + \frac{H_{kk}}{\omega_k} + \frac{H_{mm}}{\omega_m}$$

$$+ \frac{1}{2}\left(\frac{H_{km}a_m}{\omega_k a_k} + \frac{H_{mk}a_k}{\omega_m a_m}\right) \cos \mu = 0 \quad . \tag{3.16c}$$

Solving equations (3.16) yields a_k, a_m, and μ which when substituted into equations (3.15c) and (3.15d), give α_k' and α_m'. Hence, the steady-state solution has the following form:

$$u_n = P_n(\omega_m^2 - \lambda^2)^{-1} \cos \lambda t + O(\varepsilon), \; n \neq m \text{ and } k, \tag{3.17a}$$

$$u_m = P_m(\omega_m^2 - \lambda^2)^{-1} \cos \lambda t + a_m \cos[(\omega_m + \varepsilon\alpha_m')t + \tau_m] + O(\varepsilon), \tag{3.17b}$$

and

$$u_k = P_k(\omega_k^2 - \lambda^2)^{-1} \cos \lambda t + a_k \cos[(\omega_k + \varepsilon\alpha_k')t + \tau_k] + O(\varepsilon) \tag{3.17c}$$

where τ_m and τ_k are constants depending on the initial conditions. The solution corresponding to $a_m = a_k = 0$ is given by equation (3.17a) for all n.

The last terms in equations (3.17b) and (3.17c) appear as a result of the resonance in spite of the presence of damping. The nonlinearity adjusts the frequencies so that

$$\omega_m + \varepsilon\alpha_m' + \omega_k + \varepsilon\alpha_k' = \omega_m + \omega_k + \varepsilon\sigma = 2\lambda$$

If both solutions ($a_m = a_k = 0$ and a_m, a_k nonzero) are stable, then the initial conditions determine which solution represents the response.

3.5 The Case of λ Near $(\omega_m \pm \omega_p \pm \omega_k)$

The case of λ being near $(\omega_m + \omega_p + \omega_k)$ is considered first. The results for this case are then specialized to yield the results for λ being near $(2\omega_k + \omega_m)$ and λ being near $3\omega_k$ (subharmonic resonance).

In this case, the detuning is introduced as follows:

$$\lambda = \omega_m + \omega_p + \omega_k + \varepsilon\sigma \ . \tag{3.18a}$$

Then, $N_n = 0$, $n \neq m$, p and k, while

$$N_m = H_{mpk}\bar{A}_p\bar{A}_k \exp(i\sigma T_1) \quad , \tag{3.18b}$$

$$N_p = H_{pkm}\bar{A}_m\bar{A}_k \exp(i\sigma T_1) \quad , \tag{3.18c}$$

and

$$N_k = H_{kmp}\bar{A}_m\bar{A}_p \exp(i\sigma T_1) \tag{3.18d}$$

where

$$H_{mpk} = \sum_j(\Gamma_{mpkj} + \Gamma_{mkpj} + \Gamma_{mpjk} + \Gamma_{mkjp} + \Gamma_{mjpk} + \Gamma_{mjkp})K_j . \tag{3.18e}$$

Then, equation (2.13) yields

$$-\omega_k(a_k' + c_k a_k) + \frac{1}{4} H_{kmp} a_m a_p \sin\mu = 0 \ , \tag{3.19a}$$

$$-\omega_p(a_p' + c_p a_p) + \frac{1}{4} H_{pkm} a_k a_m \sin\mu = 0 \ , \tag{3.19b}$$

$$-\omega_m(a_m' + c_m a_m) + \frac{1}{4} H_{mpk} a_p a_k \sin\mu = 0 \ , \tag{3.19c}$$

$$\omega_k a_k \alpha_k' + \frac{1}{8} a_k \sum_j \gamma_{kj} a_j^2 + H_{kk} a_k + \frac{1}{4} H_{kmp} a_m a_p \cos\mu = 0 \tag{3.19d}$$

$$\omega_p a_p \alpha_p' + \frac{1}{8} a_p \sum_j \gamma_{pj} a_j^2 + H_{pp} a_p + \frac{1}{4} H_{pkm} a_k a_m \cos\mu = 0 \,, \qquad (3.19e)$$

$$\omega_m a_m \alpha_m' + \frac{1}{8} a_m \sum_j \gamma_{mj} a_j^2 + H_{mm} a_m + \frac{1}{4} H_{mpk} a_p a_k \cos\mu = 0 \,, \qquad (3.19f)$$

where

$$\mu = \sigma T_1 - \alpha_m - \alpha_p - \alpha_k \,, \qquad (3.19g)$$

and for $n \neq m$, p and k, the amplitudes and phases are governed by equations (3.2).

The steady-state response corresponds to all $a_n' = 0$ and $\mu' = 0$. The trivial solution, $a_m = a_p = a_k = 0$ is possible. For a nontrivial solution, equations (3.19) can be reduced to

$$-\omega_k c_k a_k + \frac{1}{4} H_{kmp} a_m a_p \sin\mu = 0 \,, \qquad (3.20a)$$

$$-\omega_p c_p a_p + \frac{1}{4} H_{pkm} a_k a_m \sin\mu = 0 \,, \qquad (3.20b)$$

$$-\omega_m c_m a_m + \frac{1}{4} H_{mpk} a_p a_k \sin\mu = 0 \,, \qquad (3.20c)$$

and

$$\sigma + \frac{1}{8}\left(\frac{\gamma_{kk}}{\omega_k} + \frac{\gamma_{pk}}{\omega_p} + \frac{\gamma_{mk}}{\omega_m}\right)a_k^2 + \frac{1}{8}\left(\frac{\gamma_{pp}}{\omega_p} + \frac{\gamma_{kp}}{\omega_k} + \frac{\gamma_{mp}}{\omega_m}\right)a_p^2$$
$$+ \frac{1}{8}\left(\frac{\gamma_{mm}}{\omega_m} + \frac{\gamma_{pm}}{\omega_p} + \frac{\gamma_{km}}{\omega_k}\right)a_m^2 + \frac{H_{kk}}{\omega_k} + \frac{H_{pp}}{\omega_p} + \frac{H_{mm}}{\omega_m}$$
$$+ \frac{1}{4}\left(\frac{H_{kmp} a_p a_m}{\omega_k a_k} + \frac{H_{pkm} a_k a_m}{\omega_p a_p} + \frac{H_{mpk} a_p a_k}{\omega_m a_m}\right)\cos\mu = 0 \qquad (3.20d)$$

Solving equations (3.20) yields a_k, a_m, a_p and μ which, when substituted into equations (3.19d) - (3.19f), give α_k', α_m' and α_p'. Hence, the steady-state solution has the following form:

$$u_n = P_n(\omega_n^2 - \lambda^2)^{-1} \cos \lambda t + O(\varepsilon),\ n \neq m,\ p, \text{ and } k \tag{3.21a}$$

$$u_m = P_m(\omega_m^2 - \lambda^2)^{-1} \cos \lambda t + a_m \cos[(\omega_m + \varepsilon\alpha_m')t + \tau_m] + O(\varepsilon), \tag{3.21b}$$

$$u_p = P_p(\omega_p^2 - \lambda^2)^{-1} \cos \lambda t + a_p \cos[(\omega_p + \varepsilon\alpha_p')t + \tau_p] + O(\varepsilon), \tag{3.21c}$$

and

$$u_k = P_k(\omega_k^2 - \lambda^2)^{-1} \cos \lambda t + a_k \cos[(\omega_k + \varepsilon\alpha_k')t + \tau_k] + O(\varepsilon) \tag{3.21d}$$

where τ_m, τ_p and τ_k are constants depending on the initial conditions. The solution corresponding to $a_m = a_p = a_k = 0$ is given by equation (3.21a) for all n.

The last terms in equations (3.21b) - (3.21d) appear as a result of the resonance in spite of the presence of damping. The nonlinearity adjusts the frequencies so that

$$\omega_m + \varepsilon\alpha_m' + \omega_p + \varepsilon\alpha_p' + \omega_k + \varepsilon\alpha_k' = \omega_m + \omega_p + \omega_k + \varepsilon\sigma = \lambda .$$

If both solutions ($a_m = a_p = a_k = 0$ and a_m, a_p, a_k nonzero) are stable, then the initial conditions determine which solution represents the response.

The cases $\lambda \approx (\omega_m + \omega_p - \omega_k)$ and $\lambda \approx (\omega_m - \omega_p + \omega_k)$ can be obtained from the above results by simply changing the sign of ω_k and ω_p, respectively.

The steady-state response for the case $\lambda \approx (2\omega_k + \omega_m)$ can be obtained by letting $\gamma_{ip} = 0$ when $i \neq p$, then setting $p = k$ in equations (3.20). The result is (for a_m and a_k nonzero)

$$- \omega_{\kappa} c_k a_k + \frac{1}{4} H_{kkm} a_k a_m \sin \mu = 0 \ , \qquad (3.22a)$$

$$- \omega_m c_m a_m + \frac{1}{4} H_{mkk} a_k^2 \sin \mu = 0 \ , \qquad (3.22b)$$

and

$$\sigma + \frac{1}{8} \left(\frac{2\gamma_{kk}}{\omega_k} + \frac{\gamma_{mk}}{\omega_m}\right) a_k^2 + \frac{1}{8}\left(\frac{2\gamma_{km}}{\omega_k} + \frac{\gamma_{mm}}{\omega_m}\right) a_m^2 + \frac{2H_{kk}}{\omega_k} + \frac{H_{mm}}{\omega_m}$$

$$+ \frac{1}{4} \left(\frac{2H_{kkm} a_m}{\omega_k} + \frac{H_{mkk} a_k^2}{\omega_m a_m}\right) \cos \mu = 0 \qquad (3.22c)$$

where

$$\mu = \sigma T_1 - 2\alpha_k - \alpha_m \ . \qquad (3.22d)$$

The steady-state solution has the following form:

$$u_n = P_n(\omega_n^2 - \lambda^2)^{-1} \cos \lambda t + O(\varepsilon), \ n \neq m \text{ and } k \qquad (3.23a)$$

$$u_m = P_m(\omega_m^2 - \lambda^2)^{-1} \cos \lambda t + a_m \cos[(\omega_m + \varepsilon\alpha_m')t + \tau_m] + O(\varepsilon) \ (3.23b)$$

and

$$u_k = P_k(\omega_k^2 - \lambda^2)^{-1} \cos \lambda t + a_k \cos[(\omega_k + \varepsilon\alpha_k')t + \tau_k] + O(\varepsilon) \ (3.23c)$$

whose τ_m and τ_k are constants depending on the initial conditions. The solution corresponding to $a_m = a_k = 0$ is given by equation (3.23a) for all n.

The nonlinearity adjusts the frequencies so that

$$2(\omega_k + \varepsilon\alpha_k') + \omega_m + \varepsilon\alpha_m' = 2\omega_k + \omega_m + \varepsilon\sigma = \lambda$$

If both solutions ($a_k = a_m = 0$ and a_k, a_m nonzero) are stable, then the initial conditions determine which solution represents the response.

The cases $\lambda \approx (2\omega_k - \omega_m)$ and $\lambda \approx (\omega_m - 2\omega_k)$ can be obtained from the above results by changing the sign of ω_m and ω_k, respectively.

The steady-state response for the subharmonic resonant case $\lambda \approx 3\omega_k$ can be obtained from equations (3.20) by first setting $\gamma_{ij} = 0$ when $i \neq j$ and then letting $p = m = k$. The result is (for a_k nonzero)

$$- \omega_k c_k a_k + \frac{1}{4} F_k a_k^2 \sin \mu = 0 \tag{3.24a}$$

and

$$\omega_k \sigma + \frac{3}{8} \gamma_{kk} a_k^2 + \frac{3}{4} F_k a_k \cos \mu + 3H_{kk} = 0 \tag{3.24b}$$

where

$$F_k = H_{kkk} \text{ and } \mu = \sigma T_1 - 3\alpha_k \tag{3.24c,d}$$

and H_{kkk} is given by equation (3.18e). The steady-state solution has the following form:

$$u_n = P_n(\omega_n^2 - \lambda^2)^{-1} \cos \lambda t + O(\varepsilon), \; n \neq k \tag{3.25a}$$

and

$$u_k = P_k(\omega_k^2 - \lambda^2)^{-1} \cos \lambda t + a_k \cos[\tfrac{1}{3}(\lambda t - \mu)] + O(\varepsilon) \tag{3.25b}$$

The solution corresponding to $a_k = 0$ is given by equation (3.25a) for all n.

The nonlinearity adjusts the frequency of the k^{th} mode of the linear homogeneous solution so that it is precisely one third of the frequency of the excitation. If both solutions ($a_k = 0$ and a_k nonzero) are stable then the initial conditions determine which solution represents the response.

3.6 Stability of the Steady-State Solutions

Generally, not all the steady-state solutions obtained are stable. The stability is studied by determining the response of the system to an infinitesimal perturbation away from the steady-state solution. To do this we put

$$a_n = \tilde{a}_n + \Delta a_n \tag{3.26a}$$

and

$$\mu = \tilde{\mu} + \Delta\mu \tag{3.26b}$$

where tilde indicates the steady-state value. Substituting equations (3.26) into (3.6) or (3.11) or (3.15) or (3.19) and retaining only the linear terms in the perturbations leads to a set of linear first-order equations with constant coefficients governing the Δa_n and $\Delta\mu$. These equations have a solution proportional to $\exp(MT_1)$ where M is an eigenvalue of the coefficient matrix. If all the eigenvalues have negative real parts the solution is considered stable. It is to be noted that instability of a solution, as defined here, does not imply unbounded motion but merely indicates that the periodic motion corresponding to that solution is not physically realizable in the system.

In the numerical examples in Chapters 5 and 7 the stability of the various steady-state solutions will be determined by using the method outlined above.

In many of the previous stability studies, the small disturbances were introduced into the governing equations (2.1). This leads to a set of coupled equations of the Mathieu type and generally requires more effort to determine the stability.

3.7 Summary

The method developed in Chapter 2 is used to analyze various external resonances, in the absence of internal resonances. A procedure for determining the stability of the steady-state solution is outlined.

In the absence of internal resonances, the steady-state response, in the first approximation, exhibits the following features:

(1) Only the directly excited modes can appear in the steady-state response.

(2) The directly excited modes always appear in the steady-state response in the cases of main and superharmonic resonances.

(3) The directly excited modes may not appear in the steady-state response in the cases of subharmonic and combination resonances.

(4) The directly excited mode which does appear in the steady-state response is essentially the same as the linear free-oscillation mode but with a slight adjustment of the frequency. The nonlinearity adjusts or 'tunes' the frequencies involved in the external resonance so that they are exactly commensurable or 'perfectly tuned'.

We note that the linear free-oscillation mode (with its frequency adjusted) appears in the steady-state response in spite of the presence of damping. This is in contrast with linear analyses where the linear free-oscillation modes are all damped out in the steady-state response in the presence of damping.

(5) In the case of a main resonance the steady-state response is a single harmonic, the directly excited mode. In the case of a superharmonic, subharmonic or combination resonance, in addition to the directly excited modes, an additional harmonic having its frequency equal to the frequency of the excitation appears in the steady-state response.

(6) When more than one directly excited mode appears in the steady-state response (i.e. in the case of a combination resonance), initial conditions are needed to determine the phases in the solution.

(7) When more than one stable steady-state solution exists, for a given damping and excitation, the initial conditions determine which solution represents the response.

4. EFFECTS OF AN INTERNAL RESONANCE

For a given frequency of excitation, the modal content of the steady-state response depends on the internal resonances present in the system. In this chapter, the far-reaching effects of an internal resonance involving four modes [i.e., the combination of natural frequencies as given by equation (2.2)] are evaluated. Systems having internal resonances involving three modes ($\omega_b + 2\omega_c \approx \omega_d$) and two modes ($3\omega_c \approx \omega_d$) are treated as special cases.

In order to express the approximation in equation (2.2) quantitatively, a detuning parameter, σ_1, is introduced as follows:

$$\omega_a + \omega_b + \omega_c + \varepsilon\sigma_1 = \omega_d \quad . \tag{4.1}$$

An investigation of Appendix A (Λ_1 through Λ_4) shows that the contribution to equations (2.13) due to the internal resonance is

$$R_a = Q_a\bar{A}_b\bar{A}_c A_d \quad \exp(i\sigma_1 T_1) \tag{4.2a}$$

$$R_b = Q_b\bar{A}_c A_d\bar{A}_a \quad \exp(i\sigma_1 T_1) \tag{4.2b}$$

$$R_c = Q_c A_d\bar{A}_a\bar{A}_b \quad \exp(i\sigma_1 T_1) \tag{4.2c}$$

$$R_d = Q_d A_a A_b A_c \quad \exp(-i\sigma_1 T_1) \tag{4.2d}$$

where the Q_n are constants involving the Γ_{nmpq}.

Substituting equations (3.1) and (4.2) into (2.13) and separating the result into real and imaginary parts yields the following solvability conditions:

$$- \omega_n(a_n' + c_n a_n) + \frac{1}{8}\hat{R}_n \sin\beta + N_n^{(1)} = 0 \quad , \tag{4.3a}$$

$$\omega_n a_n \alpha_n' + \frac{1}{8} a_n \sum_j \gamma_{nj} a_j^2 + H_{nn} a_n + \frac{1}{8}\hat{R}_n \cos\beta + N_n^{(2)} = 0 \tag{4.3b}$$

for n = a, b, and c;

$$- \omega_d(a_d' + c_d a_d) - \frac{1}{8}\hat{R}_d \sin\beta + N_d^{(1)} = 0 \quad , \tag{4.4a}$$

$$\omega_d a_d \alpha_d' + \frac{1}{8} a_d \sum_j \gamma_{dj} a_j^2 + H_{dd} a_d + \frac{1}{8}\hat{R}_d \cos\beta + N_d^{(2)} = 0 \tag{4.4b}$$

for n = d;

$$- \omega_n(a_n' + c_n a_n) + N_n^{(1)} = 0 \; , \tag{4.5a}$$

$$\omega_n a_n \alpha_n' + \frac{1}{8} a_n \sum_j \gamma_{nj} a_j^2 + H_{nn} a_n + N_n^{(2)} = 0 \tag{4.5b}$$

for $n \neq$ a, b, c and d;

where

$$\beta = \sigma_1 T_1 - \alpha_a - \alpha_b - \alpha_c + \alpha_d \tag{4.6}$$

and

$$\hat{R}_a = Q_a a_b a_c a_d \quad , \quad \hat{R}_b = Q_b a_c a_d a_a \quad , \tag{4.7a,b}$$

$$\hat{R}_c = Q_c a_d a_a a_b \quad , \quad \hat{R}_d = Q_d a_a a_b a_c \quad . \tag{4.7c,d}$$

Because $N_n^{(1),(2)}$ appear as a result of external resonances, they can be functions of the a_n and α_n (see Chapter 3); hence, the precise form of the solvability conditions depends on the external resonance involved. However, setting $a_n' = 0$ for the steady-state condition in equations (4.3a), (4.4a) and (4.5a) leads to

$$- \omega_a c_a a_a + \frac{1}{8} Q_a a_b a_c a_d \sin\beta + N_a^{(1)} = 0 \; , \tag{4.8a}$$

$$- \omega_b c_b a_b + \frac{1}{8} Q_b a_c a_d a_a \sin \beta + N_b^{(1)} = 0 \ , \tag{4.8b}$$

$$- \omega_c c_c a_c + \frac{1}{8} Q_c a_d a_a a_b \sin \beta + N_c^{(1)} = 0 \ , \tag{4.8c}$$

$$- \omega_d c_d a_d - \frac{1}{8} Q_d a_a a_b a_c \sin \beta + N_d^{(1)} = 0 \ , \tag{4.8d}$$

and

$$- \omega_n c_n a_n + N_n^{(1)} = 0 \tag{4.9}$$

for $n \neq$ a, b, c and d.

Several possibilities are considered next.

4.1 The Case of No External Resonances

In this case $N_n = 0$ for all n. Thus equation (4.9) leads directly to

$$a_n = 0 \ , \quad \text{for } n \neq a,b,c \text{ and } d.$$

Assuming nontrivial solutions for a_a, a_b, a_c and a_d, one finds from equations (4.8) that

$$\left(\frac{a_a}{a_d}\right)^2 = - \frac{\omega_d c_d Q_a}{\omega_a c_a Q_d} \ , \tag{4.10a}$$

$$\left(\frac{a_b}{a_d}\right)^2 = - \frac{\omega_d c_d Q_b}{\omega_b c_b Q_d} \ , \tag{4.10b}$$

and

$$\left(\frac{a_c}{a_d}\right)^2 = - \frac{\omega_d c_d Q_c}{\omega_c c_c Q_d} \ . \tag{4.10c}$$

However, for mechanical systems and structural elements, nontrivial solutions cannot exist in the absence of external excitations and in the presence of linear viscous damping; that is, the systems cannot be self-excited. Consequently, the signs of Q_a, Q_b, Q_c and Q_d must be the same so that the relationships given by equations (4.10) are impossible and thus $a_a = a_b = a_c = a_d = 0$.

4.2 The Case of An External Resonance

In the following subsections, various possibilities of directly exciting the modes are considered. (We recall from Chapter 2 that the n^{th} mode is said to be directly excited when $N_n \neq 0$.) We will refer to the a^{th}, b^{th} and c^{th} modes as the 'lower modes' and the d^{th} mode as the 'highest mode'.

a. No Direct Excitation of the Modes Involved in the Internal Resonance

If none of the modes involved in the internal resonance are directly excited, then

$$N_a = N_b = N_c = N_d = 0 \text{ , and}$$

it follows that

$$a_a = a_b = a_c = a_d = 0$$

in the steady-state solution, which is governed by equations (4.5). The internal resonance has no influence on the solution, which can be obtained as in Chapter 3.

b. Direct Excitation of One of the Lower Modes Involved in the Internal Resonance

Consider, for example, the a^{th} mode to be the one which is directly excited. Then

$$N_a \neq 0 \text{ and } N_n = 0 \text{ for } n \neq a \text{ .}$$

It follows from equations (4.9) that

$$a_n = 0 \quad \text{for } n \neq a,b,c \text{ and } d$$

and from equations (4.8b) - (4.8d), (4.10b) and (4.10c) that

$$a_b = a_c = a_d = 0 \ .$$

Therefore equations (4.7) yield

$$\hat{R}_n = 0 \quad \text{for } n = a,b,c \text{ and } d.$$

Thus, the internal resonance has no influence on the solution, which can be obtained as in Chapter 3.

Direct excitation of either the b^{th} mode or the c^{th} mode leads to similar results.

c. Direct Excitation of Two of the Lower Modes Involved in the Internal Resonance

Consider, for example, the a^{th} and the b^{th} modes to be directly excited. Then

$$N_a \neq 0, \ N_b \neq 0, \text{ and } N_n = 0 \text{ for } n \neq a \text{ and } b.$$

It follows from equations (4.9) that

$$a_n = 0 \text{ for } n \neq a, b, c \text{ and } d.$$

and from equations (4.8c), (4.8d) and (4.10c) that

$$a_c = a_d = 0.$$

Therefore, equations (4.7) yield

$$\hat{R}_n = 0 \text{ for } n = a, b, c \text{ and } d.$$

Again the internal resonance has no influence on the solution, which can be obtained as in Chapter 3.

Similar results are obtained if either the b^{th} and c^{th} modes or the a^{th} and the c^{th} modes are directly excited.

d. <u>Direct Excitation of All the Lower Modes Involved in the Internal Resonance</u>

When all the lower modes involved in the internal resonance are directly excited

$$N_a \neq 0,\ N_b \neq 0,\ N_c \neq 0 \quad \text{and } N_n = 0 \text{ for } n \neq a, b, \text{ and } c.$$

It follows from equations (4.9) that

$$a_n = 0 \quad \text{for } n \neq a, b, c \text{ and } d.$$

Depending on the type of external resonance, the amplitudes of the directly excited modes may be either zero or nonzero. Specifically, the directly excited modes must have nonzero amplitudes in the cases of main and superharmonic resonances but may have zero amplitudes in the cases of combination and subharmonic resonances (see Chapter 3).

If the amplitudes, a_a, a_b and a_c, of the directly excited lower modes are nonzero, it can be seen from equation (4.8d) that the amplitude, a_d, of the highest mode cannot be zero, though the d^{th} mode is not directly excited. The solution is governed by equations (4.3) and (4.4).

We note that for internal resonances involving four modes and three modes, direct excitation of all the lower modes can be effected only through combination resonances and hence, the highest mode may or may not participate in the response.

For an internal resonance involving two modes, if the lower mode is directly excited through either a combination or a subharmonic resonance then, the higher mode may or may not appear in the response. However, if the direct excitation of the lower mode is through a main or superharmonic resonance then, the higher mode is always drawn into the response.

e. Direct Excitation of the Highest Mode Involved in the Internal Resonance

In this case, the d^{th} mode is the only mode involved in the internal resonance to be directly excited so that

$$N_d \neq 0 \quad \text{and} \quad N_n = 0 \quad \text{for } n \neq d \ .$$

It follows from equations (4.9) that

$$a_n = 0 \quad \text{for } n \neq a, b, c \text{ and } d.$$

Depending on the type of external resonance, a_d may be either zero or nonzero.

If the d^{th} mode is directly excited through a main or superharmonic resonance then, there are two possibilities:

(1) $a_d \neq 0$ and $a_a = a_b = a_c = 0$;

(2) a_d, a_a, a_b and a_c are nonzero.

For the first subcase, the solution can be obtained as in Chapter 3 and for the second subcase, from equations (4.3) and (4.4).

If the d^{th} mode is directly excited through a combination or subharmonic resonance then, there are three possibilities:

(1) $a_d = 0$ and thus $a_a = a_b = a_c = 0$;

(2) $a_d \neq 0$ and $a_a = a_b = a_c = 0$;

(3) a_d, a_a, a_b and a_c are nonzero.

For the first and second subcases, the solution can be obtained as in Chapter 3. For the third subcase, the solution is governed by equations (4.3) and (4.4).

f. <u>Direct Excitation of the Highest Mode and One of the Lower Modes Involved in the Internal Resonance</u>

This case is essentially a combination of the cases discussed in Sections 4.2.b and 4.2.e. We note that direct excitation of two modes can be effected only through a combination resonance.

Consider, for example, the d^{th} and the a^{th} modes to be directly excited. There are three possibilities

(1) $a_d = a_a = 0$ and thus $a_b = a_c = 0$;

(2) $a_d \neq 0$, $a_a \neq 0$ and $a_b = a_c = 0$;

(3) a_a, a_b, a_c and a_d are nonzero.

Similar results are obtained if either the d^{th} and the b^{th} modes or the d^{th} and the c^{th} modes are directly excited.

g. Direct Excitation of the Highest Mode and Two of the Lower Modes Involved in the Internal Resonance

This case is essentially a combination of the cases discussed in Sections 4.2.c and 4.2.e. We note that direct excitation of three modes can be effected only through a combination resonance.

Consider, for example, the d^{th}, the a^{th} and the b^{th} modes to be directly excited. Here again there are three possibilities

(1) $a_d = a_a = a_b = 0$ and thus $a_c = 0$;

(2) $a_d \neq 0$, $a_a \neq 0$, $a_b \neq 0$ and $a_c = 0$;

(3) a_a, a_b, a_c and a_d are nonzero.

Similar results are obtained if either the d^{th}, b^{th} and c^{th} modes or the d^{th}, a^{th} and c^{th} modes are directly excited.

4.3 Other Possibilities

There may be more than one internal resonant combination in a given system. The modes involved in these combinations can be directly excited by external resonances in various ways. The analysis of this chapter can be extended to the study of such problems in a straightforward manner. The modal content of the response can be determined by essentially a superposition of the various cases analyzed in Sections 4.1 and 4.2.

4.4 Summary

The effects of an internal resonance involving four modes is evaluated for various cases of the direct excitation of the modes involved.

In the presence of the internal resonance, the steady-state response, in the first approximation, exhibits the following features:

(1) It is possible for modes other than those that are directly excited to appear in the steady-state response.

(2) If the highest mode involved in the internal resonance is directly excited and appears in the steady-state response, then all or none of the lower modes are drawn into the steady-state response.

(3) If all the lower modes involved in the internal resonance are directly excited and appear in the steady-state response then, the highest mode is always drawn into the steady-state response.

(4) If not more than two of the lower modes in an internal resonance involving four modes (not more than one of the lower modes in an internal resonance involving three modes)

are (is) directly excited, then none of the other modes involved in the internal resonances appears in the steady-state response.

It is noted that in the case of an internal resonance involving two modes, there is only one lower mode. If this mode is directly excited and appears in the steady-state response, then the other higher mode is always drawn into the steady-state response; but the converse is not true.

5. NONLINEAR VIBRATIONS OF BEAMS

The transverse vibrations of beams supported in such a way as to restrict axial movement at the ends are accompanied by stretching of the mid-plane. One accounts for this stretching by using nonlinear strain-displacement relations. Consequently, the governing equations are nonlinear. In this chapter, the problem of the nonlinear transverse vibrations of beams is reduced to the study of the nonlinear system considered in Chapter 2.

5.1 Problem Formulation

Referring to Figure 1, we write the pertinent equations of motion as follows (vibrations in the XZ plane only):

$$\frac{\partial}{\partial x}(n\cos\theta - q\sin\theta) = \rho A\frac{\partial^2 u}{\partial t^2}\quad, \tag{5.1a}$$

$$\frac{\partial}{\partial x}(n\sin\theta + q\cos\theta) + p - c\frac{\partial w}{\partial t} = \rho A\frac{\partial^2 w}{\partial t^2}\quad, \tag{5.1b}$$

and

$$-\frac{\partial m}{\partial x} + n[(1+\frac{\partial u}{\partial x})\sin\theta - \frac{\partial w}{\partial x}\cos\theta]$$
$$+ q[(1+\frac{\partial u}{\partial x})\cos\theta + \frac{\partial w}{\partial x}\sin\theta] = \rho I\frac{\partial^2\theta}{\partial t^2} \tag{5.1c}$$

The force-displacement and the strain-displacement relations are

$$n = EAe_x\quad,\quad m = -EI\frac{\partial\theta}{\partial x} \tag{5.2a,b}$$

and

$$e_x = [(1+\frac{\partial u}{\partial x})^2 + (\frac{\partial w}{\partial x})^2]^{\frac{1}{2}} - 1\ . \tag{5.2c}$$

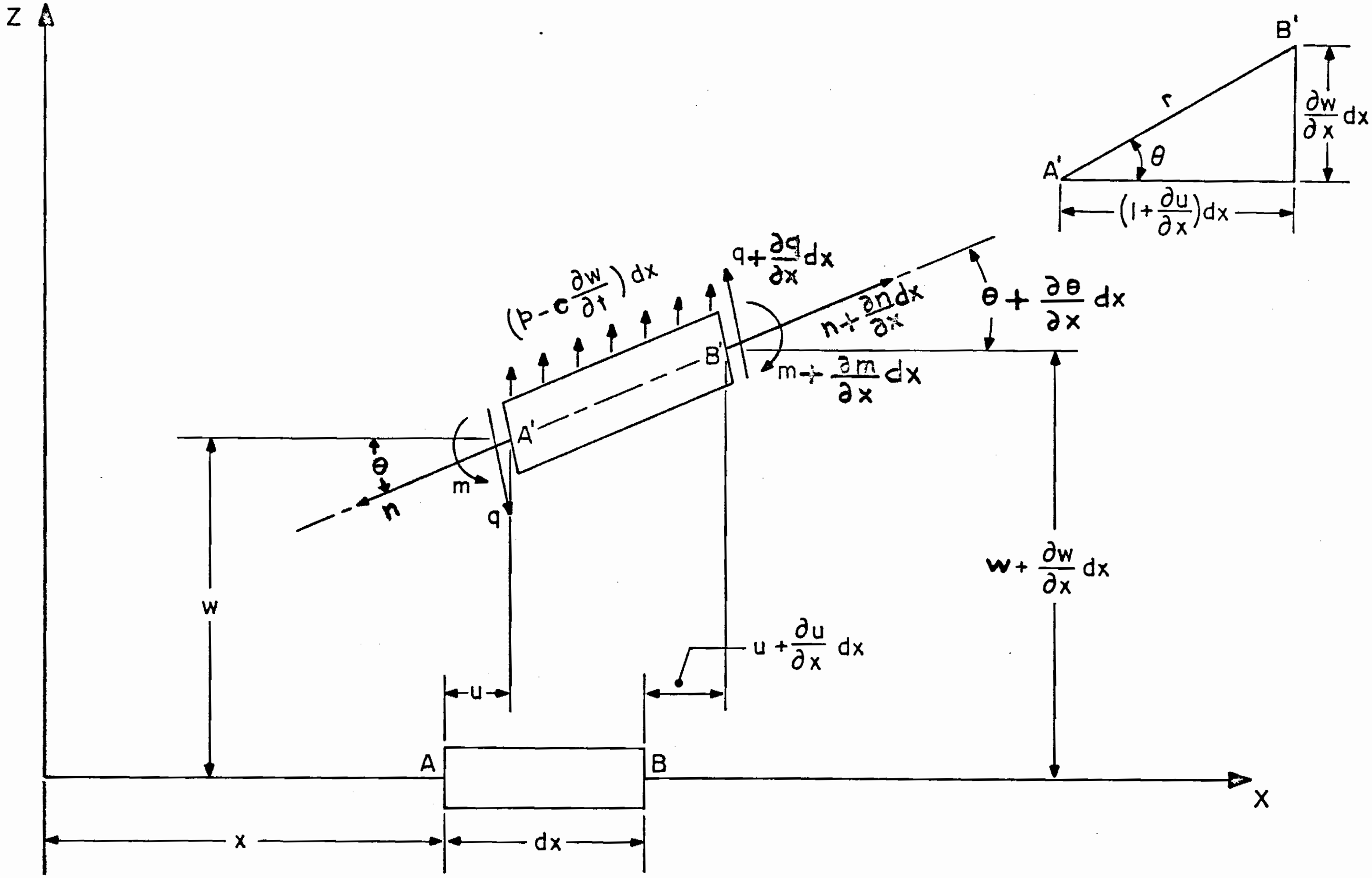

Figure 1 An Element of the Beam

Here E is Young's modulus, A and I are the area and moment of inertia of the cross section, ρ is the density of the beam, and c is the damping coefficient.

It is convenient to rewrite the equations in terms of non-dimensional variables, which are denoted by overbars and defined as follows:

$$x = \bar{x}L \ , \quad t = \bar{t}(\rho L^4/Er^2)^{\frac{1}{2}} \quad , \quad w = \bar{w}(r^2/L) \ , \ u = \bar{u}(r^4/L^3) \ ,$$

$$p = \bar{p}L^5/(r^4 EA) \text{ and } c = \bar{c}L^4/[2r^3 A(\rho E)^{\frac{1}{2}}]$$

where r is the radius of gyration of the cross-sectional area, and L, the characteristic length, may be the actual length of the beam or a wavelength of a linear transverse oscillation.

Substituting the above definitions into equations (5.1) and (5.2), combining equations (5.1b) and (5.1c), and dropping overbars in the final result, one obtains

$$\frac{\partial e_x}{\partial x} = 0(\varepsilon) \tag{5.3a}$$

and

$$\frac{\partial^2 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = \varepsilon[\frac{\partial}{\partial x}(e_x \frac{\partial w}{\partial x}) + p - 2c \frac{\partial w}{\partial t}] \tag{5.3b}$$

where $\varepsilon = r^2/L^2$.

It is noted that the terms which account for shear deformation and rotary inertia have been neglected. These linear terms, if they had been retained, would have appeared in equations (5.3) because they are formally of the same order [i.e., $0(\varepsilon)$] as the nonlinear term which accounts for mid-plane stretching. The nonlinear term can alter the character of the solution drastically whereas the linear terms can only affect the solution slightly; and hence they have been neglected.

Further, we note that, when ε is small, the deflection, w, is much smaller than the radius of gyration, r. Had w been the same order as r (say, $w = r\bar{w}$), then no small parameter would have appeared in the equation (5.3b) and the linear and the nonlinear terms would have been the same order. Hence, the present approach must be viewed as one which provides corrections for the small-deflection theory (for which w is much smaller than r) and not as one which provides a solution for the large-deflection theory (for which w is the same order as r). This means that typical nonlinear phenomena, such as jump phenomena, modal interactions, etc. can be part of the small deflection theory.

Integrating equation (5.3a) leads to

$$e_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 = f_1(t) \quad ,$$

and thus

$$u = f_2(t) + x f_1(t) - \frac{1}{2}\int_0^x \left(\frac{\partial w}{\partial x}\right)^2 dx$$

where f_1 and f_2 are arbitrary functions to be determined from the in-plane boundary conditions:

$$u(0,t) = 0 \quad \text{and} \quad k u(\ell,t) + e_x(\ell,t) = 0$$

where ℓ is the length of the beam, and k is a constant which depends on the type of axial restraint; k is zero for no restraint and infinity for a rigid restraint. It follows that

$$f_2 = 0 \ , \quad f_1 = \hat{k}\int_0^{\ell} \left(\frac{\partial w}{\partial x}\right)^2 dx$$

where

$$\hat{k} = k/[2(1 + k\ell)] \ ,$$

and

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = \varepsilon \left\{ - 2c \frac{\partial w}{\partial t} + \hat{k} \left[\int_0^{\ell} \left(\frac{\partial w}{\partial x}\right)^2 dx \right] \frac{\partial^2 w}{\partial x^2} \right\} + p(x,t) \ . \qquad (5.4)$$

The linear homogeneous version of equation (5.4), that is,

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = 0 \ ,$$

together with the boundary conditions (taken to be homogeneous) on w leads to a complete set of orthonormal eigenfunctions (linear free-oscillation modes) and the corresponding eigenvalues (natural frequencies).

For the nonlinear problem defined by equation (5.4) and homogeneous boundary conditions, we assume an expansion of the form

$$w(x,t) = \sum_m \phi_m(x) u_m(t) \qquad (5.5)$$

where $\phi_m(x)$ are the linear free-oscillation modes. Substituting equation (5.5) into (5.4), multiplying by $\phi_n(x)$ and integrating from $x = 0$ to ℓ, we obtain

$$\frac{d^2 u_n}{dt^2} + \omega_n^2 u_n = \varepsilon \left[- 2c_n \frac{du_n}{dt} + \sum_{m,p,q} \Gamma_{nmpq} u_m u_p u_q \right] + P_n \cos \lambda t,$$

$$\text{for } n = 1, 2, \ldots \ , \qquad (5.6)$$

where

$$\Gamma_{nmpq} = \hat{k}\left[\int_0^\ell \phi_n \frac{d^2\phi_q}{dx^2}\, dx\right]\left[\int_0^\ell \frac{d\phi_m}{dx}\,\frac{d\phi_p}{dx}\, dx\right] , \tag{5.7}$$

modal damping is assumed and the forcing function is assumed to be harmonic; that is

$$p(x,t) = P(x) \cos \lambda t,$$

and

$$P_n = \int_0^\ell P(x)\phi_n(x)dx \quad .$$

Generally, one cannot obtain the linear modes, $\phi_m(x)$, analytically for beams having complicated boundaries and/or composition. However, numerical methods can be used to determine the linear modes and hence the coefficients, Γ_{nmpq}, of the nonlinear terms in equations (5.6); see reference [30].

We note that equations (5.6) are identical to equations (2.1). Therefore, the analyses and results of Chapters 2, 3 and 4 can be applied to the problem of the nonlinear vibrations of beams.

5.2 Numerical Examples

We begin by considering a beam with one end hinged and the other end clamped. No axial movement can occur so that $\hat{k} = 1/(2\ell)$. The linear free-oscillation modes are

$$\phi_m = \kappa_m[\sin(\eta_m x) - \frac{\sin(\eta_m \ell)}{\sinh(\eta_m \ell)} \sinh(\eta_m x)] \tag{5.8a}$$

where κ_m are the normalizing coefficients chosen such that

$$\int_0^{\ell} \phi_m^2 dx = 1 \quad , \tag{5.8b}$$

and η_m are the roots of

$$\tan(\eta_m \ell) = \tanh(\eta_m \ell). \tag{5.8c}$$

The non-dimensional length of the beam is chosen to be equal to 2, and the five lowest natural frequencies for $\ell = 2$ are

$$\omega_1 = 3.8545,\ \omega_2 = 12.491,\ \omega_3 = 26.062,\ \omega_4 = 44.568,\ \omega_5 = 68.007,$$

where $\omega_m = \eta_m^2$.

It is noted that ω_2 and ω_1 are nearly in the ratio of three to one. This relationship is a special case of equation (2.2) with a = b = c = 1 and d = 2. Thus, this is the case of an internal resonance involving two modes. The nearness of ω_2 to $3\omega_1$ is expressed quantitatively by a detuning parameter, σ_1, as follows:

$$\omega_2 - 3\omega_1 = 0.9275 = \varepsilon\sigma_1 \tag{5.9}$$

The equations governing the amplitudes and phases of the response are obtained by letting a = b = c = 1 and d = 2 in equations (4.3) - (4.5). In general, the steady-state equations are nonlinear in nature and hence recourse has to be made to numerical methods for their solution. In this chapter, all the steady-state equations were solved by using a Newton-Raphson procedure. The stability of the various steady-state solutions was determined by using the method in Section 3.6.

Numerical results are presented for main, superharmonic, subharmonic and combination resonances. For convenience, all calculations were made with $c_1 = c_2 = c_3 = c$. Values of the constant coefficients appearing in the numerical examples are listed in Appendix B.

a. The Case of λ Near ω_k; $k = 1, 2$(Main Resonances)

Here the steady-state response is obtained for the main resonances of the 1st and the 2nd modes. (We recall that $P_n = \varepsilon \hat{P}_n$ and $H_{nn} = 0$ for main resonances.)

To express the nearness of λ to $\omega_k(k = 1 \text{ or } 2)$, we let

$$\lambda = \omega_k + \varepsilon\sigma_2 , \tag{5.10}$$

Thus,

$$N_k = \frac{1}{2}\hat{P}_k \exp(i\sigma_2 T_1)$$

$$N_k^{(1)} = \frac{1}{2}\hat{P}_k \sin\mu , \quad N_k^{(2)} = \frac{1}{2}\hat{P}_k \cos\mu \tag{5.11}$$

where

$$\mu = \sigma_2 T_1 - \alpha_k ,$$

and

$$N_n = 0 \quad \text{for } n \neq k .$$

The equations governing the steady-state solution are obtained by substituting equations (5.11) into (4.3) - (4.5) and letting $a_n' = \beta' = \mu' = 0$.

(1) λ Near ω_1 (k = 1)

Equations (4.3) - (4.5) can be reduced to

$$-\omega_1 c_1 a_1 + \frac{1}{8} Q_1 a_1^2 a_2 \sin\beta + \frac{1}{2}\hat{P}_1 \sin\mu = 0\ , \tag{5.12a}$$

$$-\omega_2 c_2 a_2 - \frac{1}{8} Q_2 a_1^3 \sin\beta = 0\ , \tag{5.12b}$$

$$\omega_2(\sigma_1 - 3\sigma_2)a_2 - \frac{1}{8}(\gamma_{22}a_2^3 + \gamma_{21}a_2a_1^2) - \frac{1}{8} Q_2 a_1^3 \cos\beta = 0 \tag{5.12c}$$

$$\omega_1\sigma_2 a_1 + \frac{1}{8}(\gamma_{11}a_1^3 + \gamma_{12}a_1a_2^2) + \frac{1}{8} Q_1 a_1^2 a_2 \cos\beta + \frac{1}{2}\hat{P}_1 \cos\mu = 0, \tag{5.12d}$$

and

$$a_n = 0 \quad \text{for } n > 2$$

where

$$\beta = \sigma_1 T_1 - 3\alpha_1 + \alpha_2 \tag{5.12e}$$

and

$$\mu = \sigma_2 T_1 - \alpha_1\ . \tag{5.12f}$$

We note that neither a_1 nor a_2 can be zero. This is in agreement with the comments of Section 4.2.d. The steady-state solution can be written as

$$u_1 = a_1 \cos(\lambda t - \mu) + O(\varepsilon), \tag{5.13a}$$

$$u_2 = a_2 \cos(3\lambda t - 3\mu + \beta) + O(\varepsilon), \tag{5.13b}$$

and

$$u_n = O(\varepsilon), \quad n > 2\ . \tag{5.13c}$$

It is noted that the nonlinearity adjusts the frequencies of the second and the first modes such that they are precisely in the ratio of three to one and the frequency of the first mode is equal to the frequency of the excitation.

For some arbitrary values of the amplitude of excitation and the damping coefficients, Figures 2a and 2b show the variation of a_1 and a_2 with $(\varepsilon\sigma_2/\omega_1)$ for several values of $\varepsilon\hat{P}_1$. In addition, the original equations (5.6) were integrated numerically by using a Runga-Kutta procedure and the steady-state amplitudes plotted on these figures. Although a_2 cannot be zero, it is small compared with a_1 and the response is dominated by the first mode.

In Figures 3a and 3b, the variation of a_1 with $\varepsilon\hat{P}_1$ is shown for two values of $(\varepsilon\sigma_2/\omega_1)$. The log-log plot in Figure 3b can be qualitatively compared with the plot of experimental data obtained by Jacobson and van der Heyde [34], though they experimented with honeycomb panels.

The result for a_1, given by Figures 2a, 3a and 3b, is similar to the solution of the Duffing equation for a 'hardening type' of nonlinearity. These figures illustrate that, when the amplitude of the excitation is small, the solution is practically indistinguishable from the solution of the linear problem. The phenomena of 'jumps' associated with the variation of the frequency of the excitation and with the variation of the amplitude of the excitation are illustrated in Figures 2a and 3a, respectively. Both are typical nonlinear effects and are discussed by Stoker [35], for example.

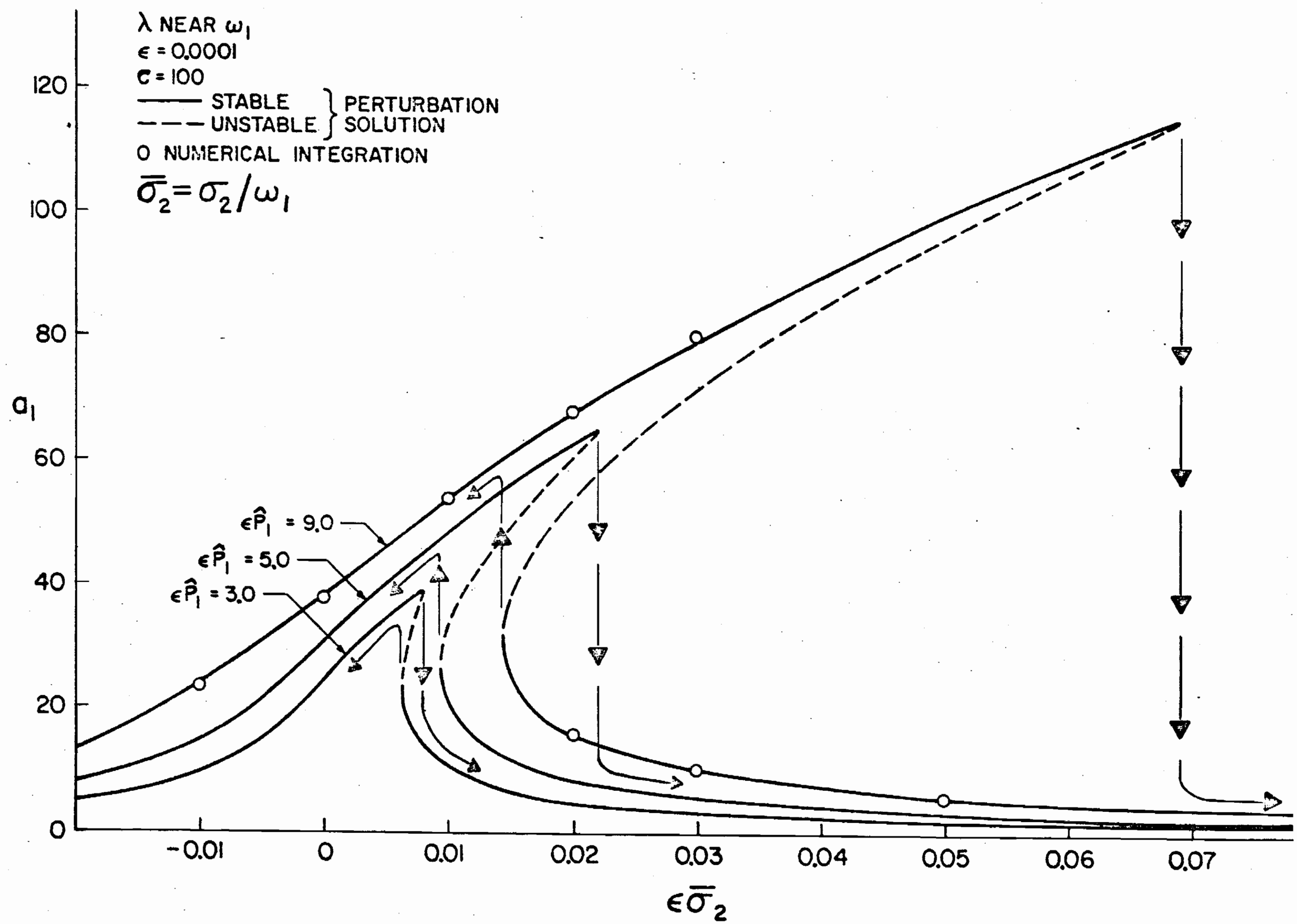

Figure 2a Variation of First Mode Amplitude with Detuning of Excitation

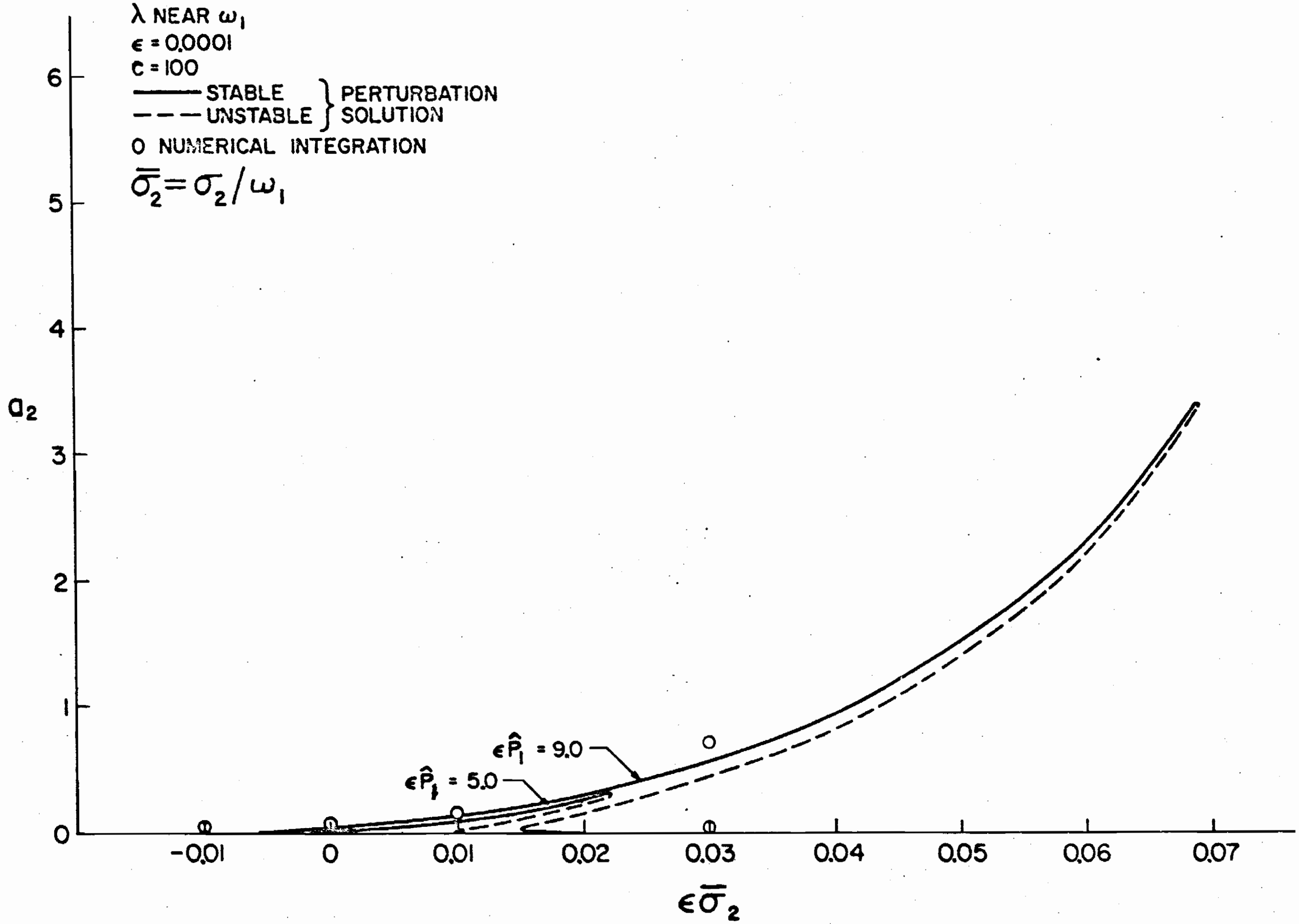

Figure 2b Variation of Second Mode Amplitude with Detuning of Excitation

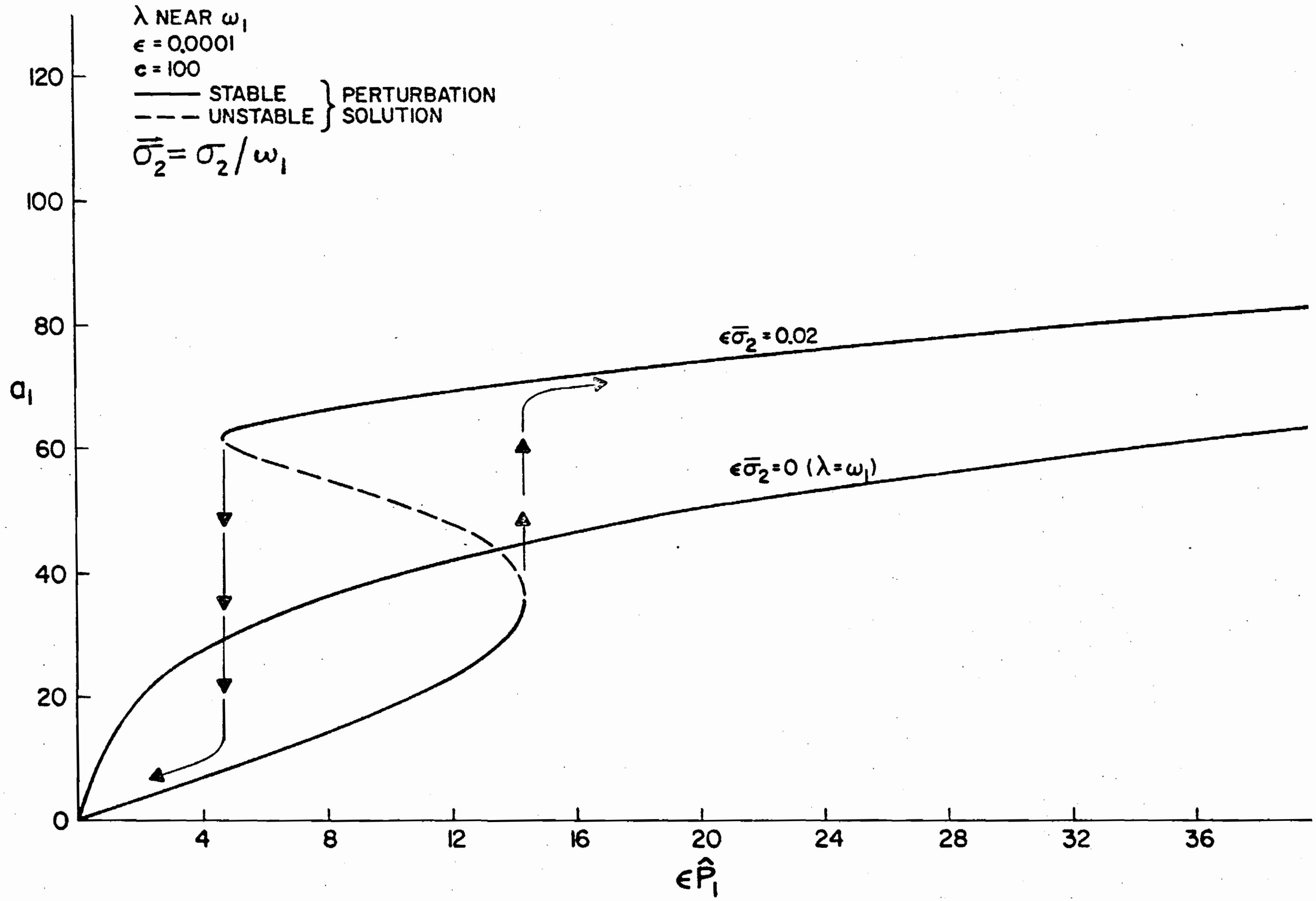

Figure 3a Variation of First Mode Amplitude with Amplitude of Excitation

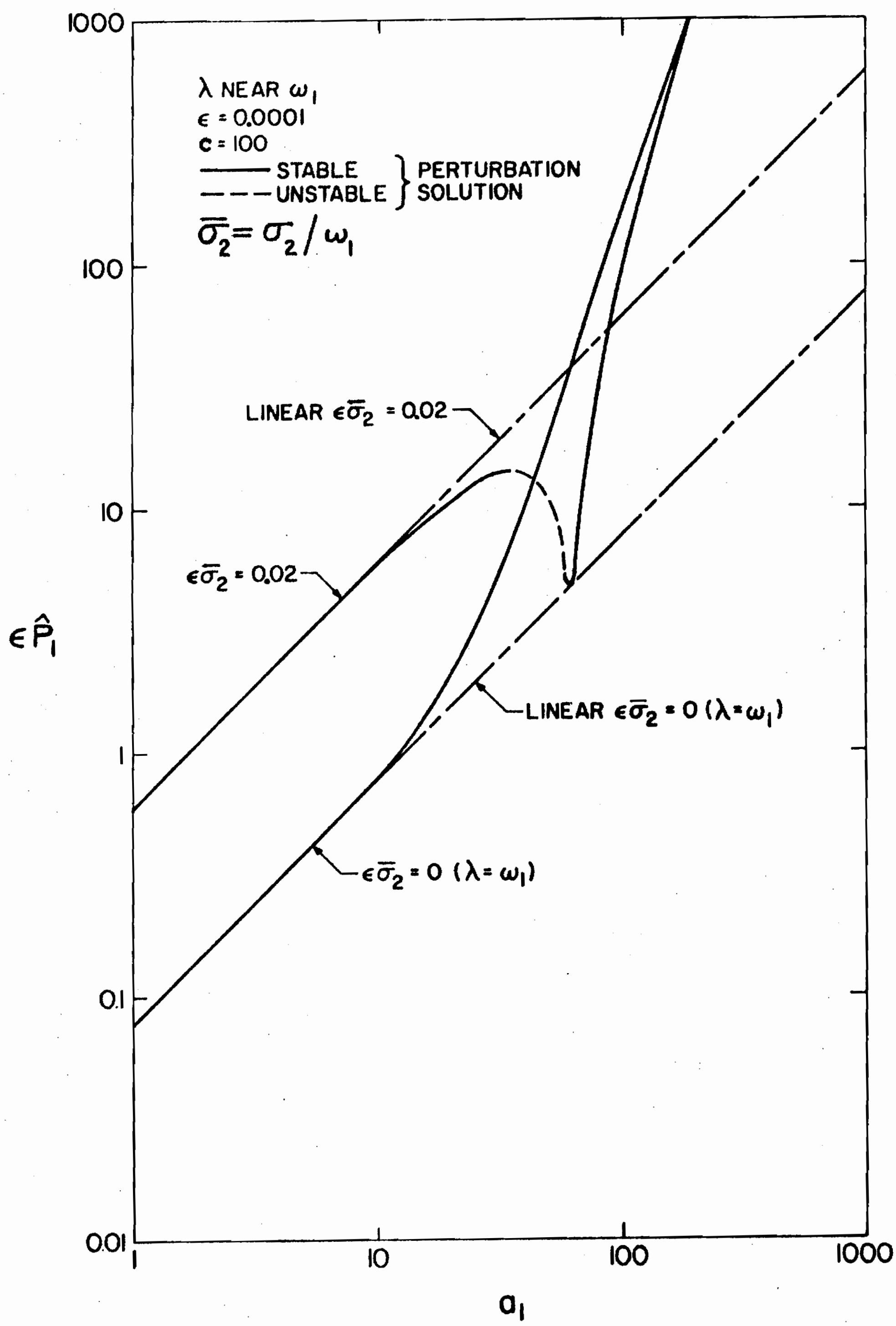

Figure 3b Log-Log Plot of the Variation of First Mode Amplitude with Amplitude of Excitation

(2) λ Near $\omega_2 (k = 2)$

Equations (4.3) - (4.5) can be reduced to

$$- \omega_1 c_1 a_1 + \frac{1}{8} Q_1 a_1^2 a_2 \sin \beta = 0 \quad , \tag{5.14a}$$

$$- \omega_2 c_2 a_2 - \frac{1}{8} Q_2 a_1^3 \sin \beta + \frac{1}{2} \hat{P}_2 \sin \mu = 0 \quad , \tag{5.14b}$$

$$\omega_1 (\sigma_1 + \sigma_2) a_1 + \frac{3}{8}(\Upsilon_{11} a_1^3 + \Upsilon_{12} a_1 a_2^2) + \frac{3}{8} Q_1 a_1^2 a_2 \cos \beta = 0 \ , \tag{5.14c}$$

$$\omega_2 \sigma_2 a_2 + \frac{1}{8}(\Upsilon_{22} a_2^3 + \Upsilon_{21} a_2 a_1^2) + \frac{1}{8} Q_2 a_1^3 \cos \beta + \frac{1}{2} \hat{P}_2 \cos \mu = 0, \tag{5.14d}$$

and

$$a_n = 0 \quad \text{for } n > 2$$

where

$$\beta = \sigma_1 T_1 - 3\alpha_1 + \alpha_2 \tag{5.14e}$$

and

$$\mu = \sigma_2 T_1 - \alpha_2 \ . \tag{5.14f}$$

We note that $a_2 = 0$ is not a possible solution. However, $a_1 = 0$ is a possible solution. Thus, there are two possible solutions; this is in agreement with the comments of Section 4.2.e.

When a_1 is zero, the steady-state solution is given by

$$u_n = O(\varepsilon) \quad , n \neq 2 \tag{5.15a}$$

and

$$u_2 = a_2 \cos(\lambda t - \mu) + O(\varepsilon) \quad . \tag{5.15b}$$

When a_1 is nonzero, the steady-state solution has the form

$$u_1 = a_1 \cos\left[\tfrac{1}{3}(\lambda t - \mu - \beta)\right] + O(\varepsilon), \tag{5.16a}$$

$$u_2 = a_2 \cos(\lambda t - \mu) + O(\varepsilon)\ , \tag{5.16b}$$

and the remaining u_n are given by equation (5.15a). We note from equations (5.16) that the nonlinearity adjusts the frequencies of the second and the first modes so that they are precisely in the ratio of three to one and the frequency of the second mode is equal to the frequency of the excitation.

The variation of a_1 and a_2 with $\varepsilon\sigma_2/\omega_2$ is shown in Figure 4a for the case when a_1 is nonzero and in Figure 4b for the case when $a_1 = 0$. For comparison, the stable portions of Figure 4a are replotted in Figure 4b. The variation of a_1 and a_2 with $\varepsilon\hat{P}_2$ is shown in Figures 5a for the case when a_1 is nonzero and in Figure 5b for the case when $a_1 = 0$. The stable portions of Figure 5a are replotted in Figure 5b. Also shown in Figures 4b and 5b are the results of the numerical integration of equations (5.6).

It is interesting to note that when a_1 is nonzero, it can be much larger than a_2; that is, the response can be dominated by the first mode though the frequency of the excitation is near the second natural frequency. Using equations (5.15), (5.16) and (5.5), one can write the deflection as

$$w = \phi_2 a_2 \cos(\lambda t - \mu) + O(\varepsilon) \tag{5.17}$$

when $a_1 = 0$ and

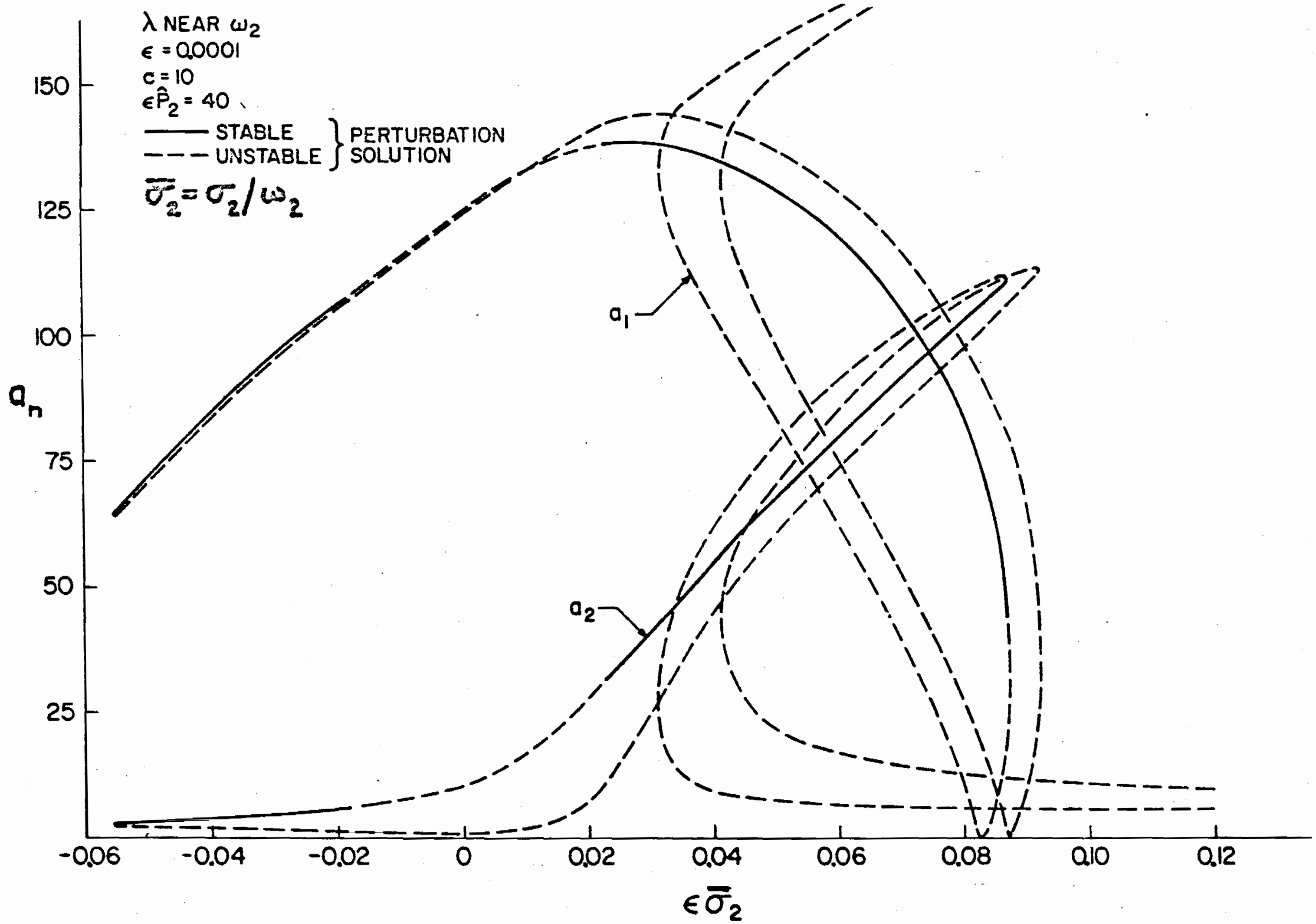

Figure 4a Variation of First and Second Mode Amplitudes with Detuning of Excitation

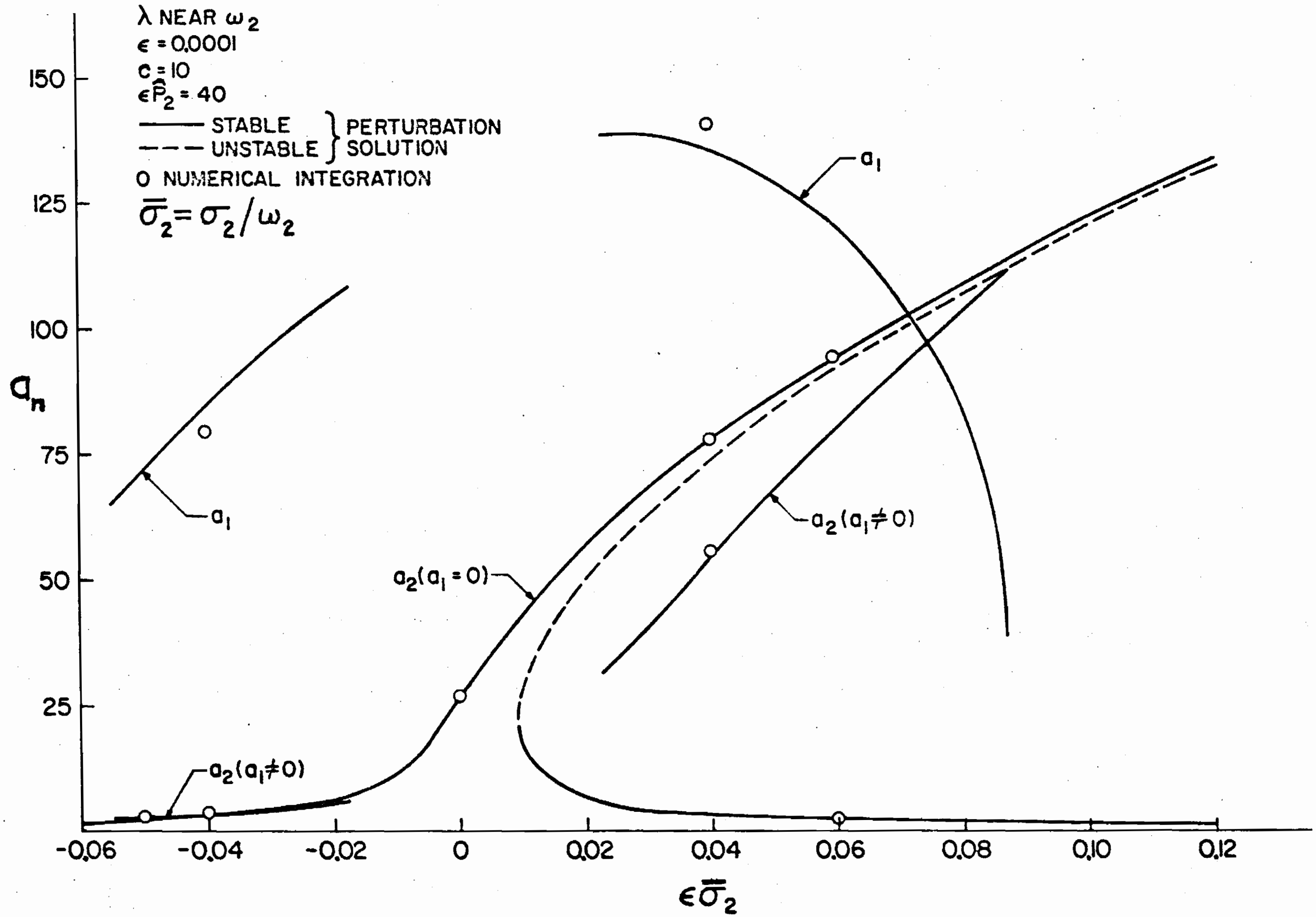

Figure 4b Variation of Second Mode Amplitude with Detuning of Excitation

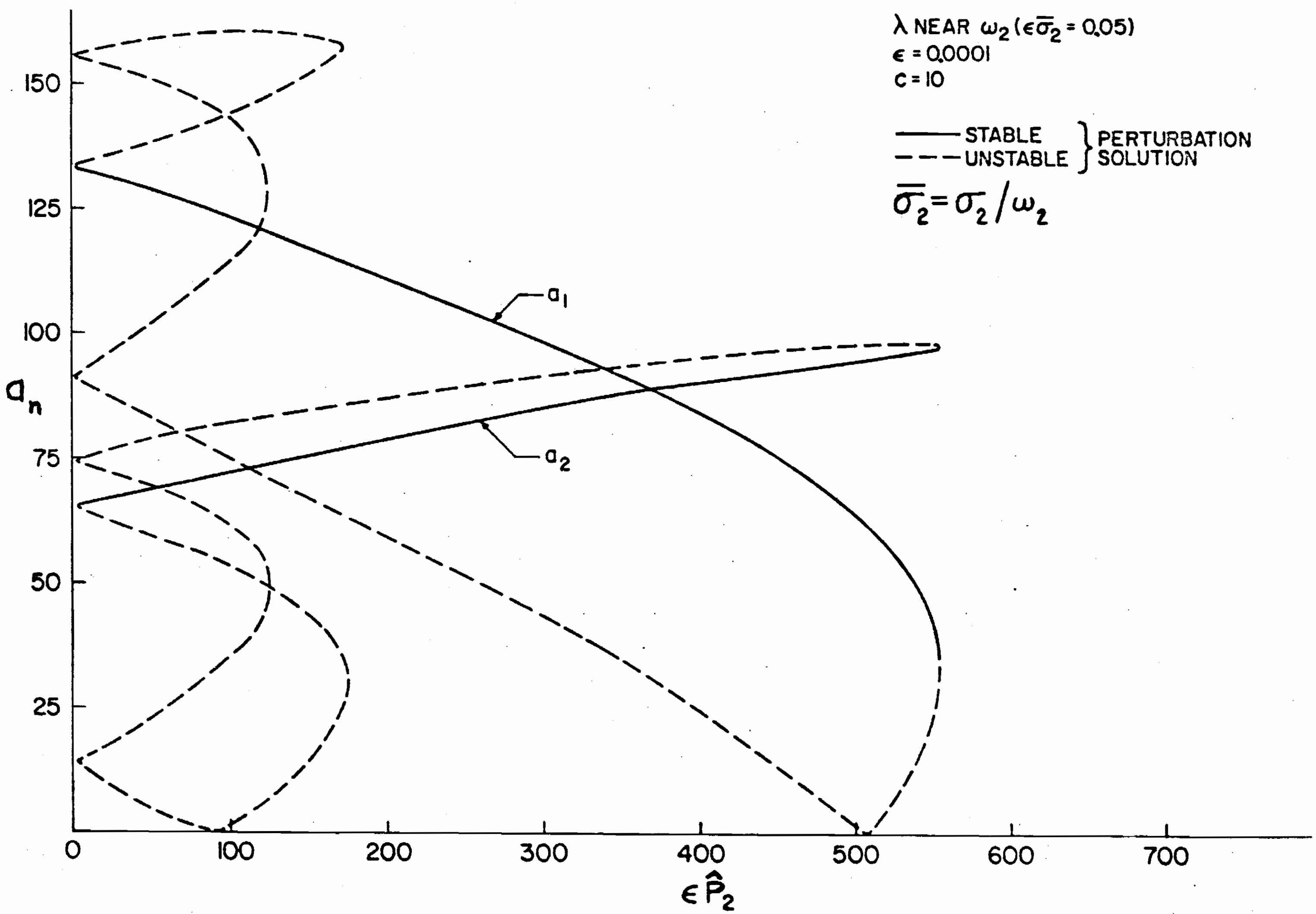

Figure 5a Variation of First and Second Mode Amplitudes with Amplitude of Excitation

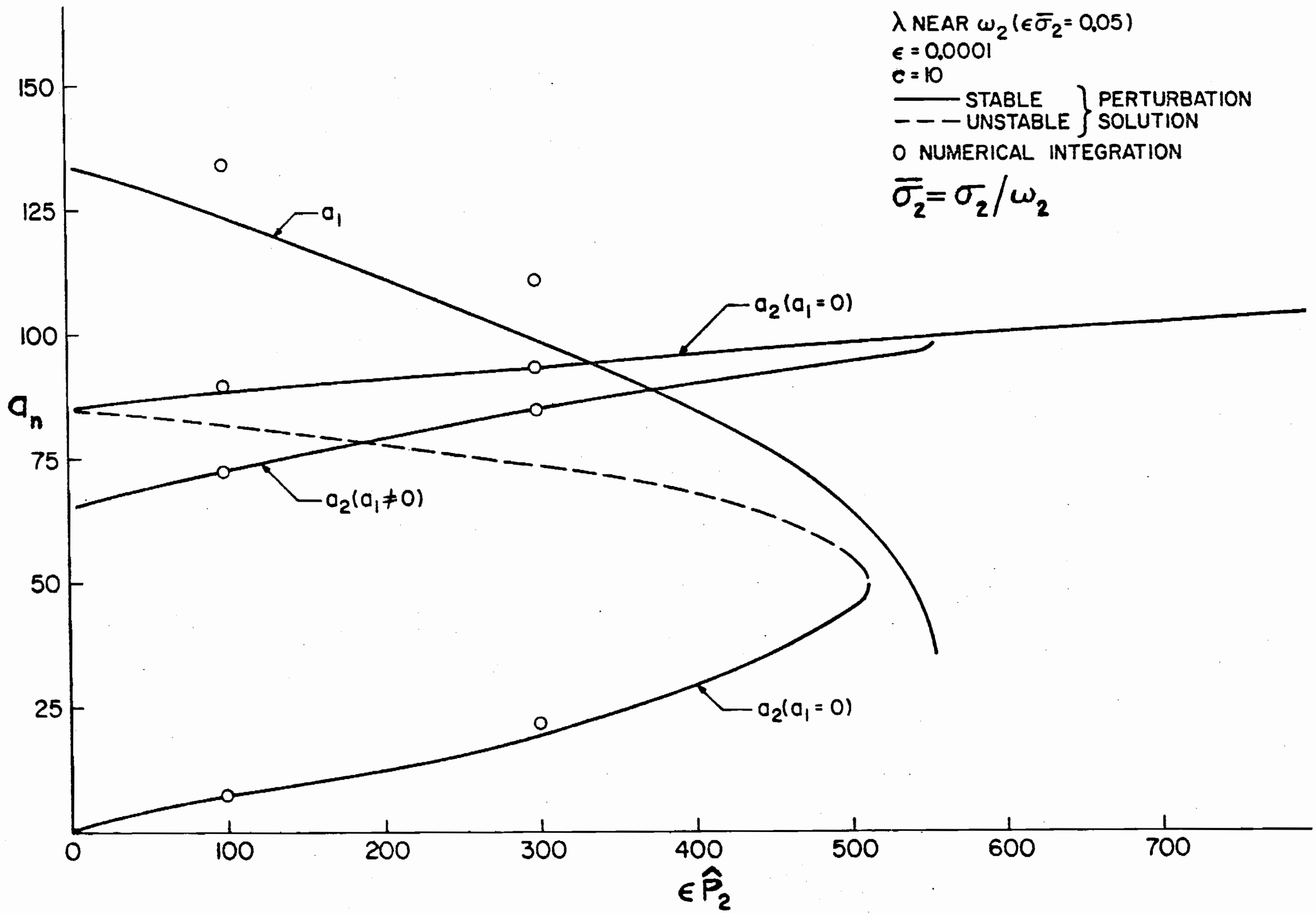

Figure 5b Variation of Second Mode Amplitude with Amplitude of Excitation

$$w = \phi_1 a_1 \cos[\tfrac{1}{3}(\lambda t - \mu - \beta)] + \phi_2 a_2 \cos(\lambda t - \mu) + O(\varepsilon) \qquad (5.18)$$

when $a_1 \neq 0$.

For some typical values of the frequency and amplitude of the excitation, the two possible mode shapes, as given by equations (5.17) and (5.18) are plotted for increasing values of time in Figures 6a and 6b, respectively. Figure 6b clearly shows the strong resemblence of the response to the first mode, thus, illustrating the far-reaching effects of the internal resonance.

b. <u>The Case of 3λ Near ω_1(Superharmonic Resonance)</u>

In this case, the detuning parameter, σ_2, is defined as follows:

$$3\lambda = \omega_1 + \varepsilon\sigma_2 \quad .$$

Thus,

$$N_1 = F_1 \exp(i\sigma_2 T_1), \; F_1 = \sum_{m,p,q} \Gamma_{1mpq} K_m K_p K_q \quad ,$$

and

$$N_n = 0 \quad \text{for } n > 1 \; .$$

Equations (4.3) - (4.5) can be reduced to

$$-\omega_1 c_1 a_1 + \frac{1}{8} Q_1 a_1^2 a_2 \sin\beta + F_1 \sin\mu = 0 \; , \qquad (5.19a)$$

$$-\omega_2 c_2 a_2 - \frac{1}{8} Q_2 a_1^3 \sin\beta = 0 \; , \qquad (5.19b)$$

$$\omega_2(\sigma_1 - 3\sigma_2)a_2 - \frac{1}{8}(\gamma_{22}a_2^3 + \gamma_{21}a_2 a_1^2) - H_{22}a_2 - \frac{1}{8}Q_2 a_1^3 \cos\beta = 0 \; , \qquad (5.19c)$$

$$\omega_1\sigma_2 a_1 + \frac{1}{8}(\gamma_{11}a_1^3 + \gamma_{12}a_1 a_2^2) + H_{11}a_1 + \frac{1}{8}Q_1 a_1^2 a_2 \cos\beta + F_1 \cos\mu = 0 \; , \qquad (5.19d)$$

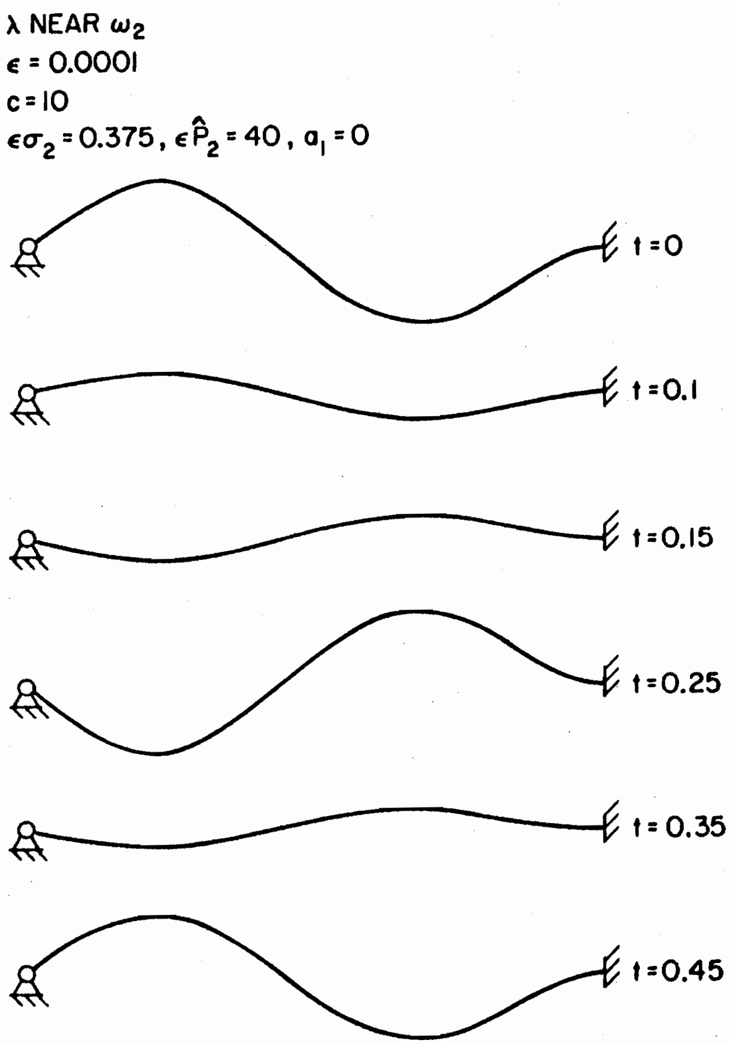

Figure 6a Mode Shape with Increasing Time - No Influence of Internal Resonance

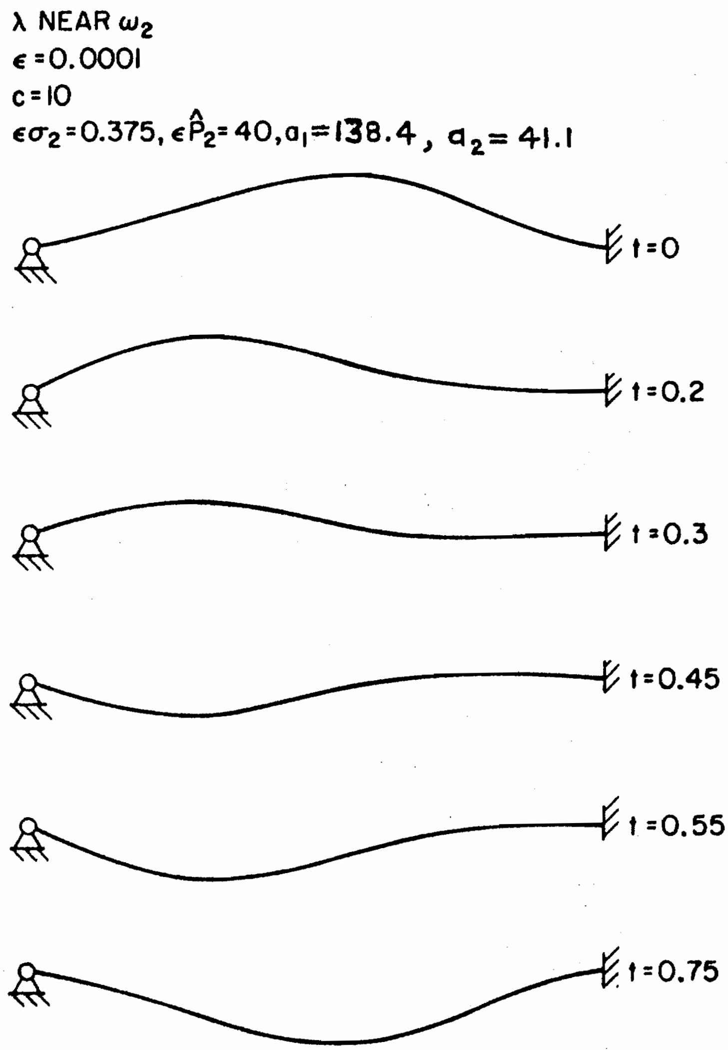

Figure 6b Mode Shape with Increasing Time - With the Effects of Internal Resonance

and

$$a_n = 0 \quad \text{for } n > 2 \ ,$$

where

$$\beta = \sigma_1 T_1 - 3\alpha_1 + \alpha_2 \ , \tag{5.19e}$$

and

$$\mu = \sigma_2 T_1 - \alpha_1 \quad . \tag{5.19f}$$

Because F_1 is independent of a_1, it follows from equations (5.19) that neither a_1 nor a_2 can be zero. This is in agreement with the comments of Section 4.2.d. The steady-state solution has the form

$$u_1 = P_1(\omega_1^2 - \lambda^2)^{-1} \cos \lambda t + a_1 \cos(3\lambda t - \mu) + O(\varepsilon) \ , \tag{5.20a}$$

$$u_2 = P_2(\omega_2^2 - \lambda^2)^{-1} \cos \lambda t + a_2 \cos(9\lambda t - 3\mu + \beta) + O(\varepsilon) \ , \tag{5.20b}$$

and

$$u_n = P_n(\omega_n^2 - \lambda^2)^{-1} \cos \lambda t + O(\varepsilon) \ , \ n > 2. \tag{5.20c}$$

It is noted that the nonlinearity adjusts the frequencies of the second and the first modes such that they are precisely in the ratio of three to one and the frequency of the first mode is precisely three times that of the excitation.

For some arbitrary values of the amplitude of the excitation and the damping coefficients, the variation of a_1 and a_2 are plotted in Figure 7 as functions of $\varepsilon\sigma_2$. For the sake of clarity, only the stable portions of the complete solution are shown in this figure as well as in all those that follow in this chapter. It is noted that a_2 is always smaller than a_1.

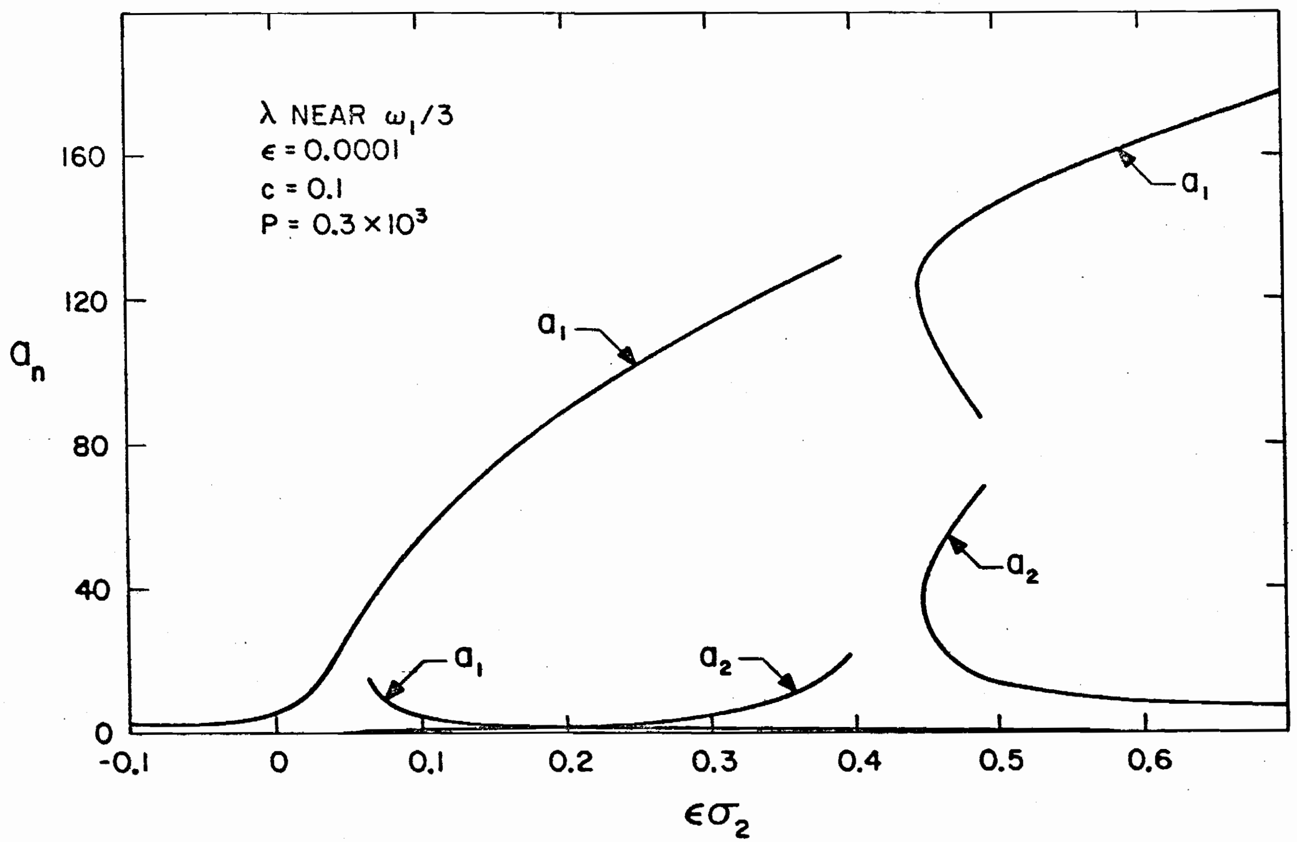

Figure 7 Variation of First and Second Mode Amplitudes with Detuning of Excitation

c. The Case of λ Near $3\omega_2$ (Subharmonic Resonance)

In this case, the detuning parameter, σ_2, is defined as follows:

$$\lambda = 3\omega_2 + \varepsilon\sigma_2 \quad .$$

Thus,

$$N_2 = F_2\bar{A}_2^2 \exp(i\sigma_2 T_1) \ , \quad F_2 = H_{222} \quad ,$$

where H_{222} is given by equation (3.18e) and

$$N_n = 0 \quad \text{for} \quad n = 1, 3, 4, \ldots \quad .$$

Equations (4.3) - (4.5) can be reduced to

$$- \omega_1 c_1 a_1 + \frac{1}{8} Q_1 a_1^2 a_2 \sin\beta = 0 \quad , \tag{5.21a}$$

$$- \omega_2 c_2 a_2 - \frac{1}{8} Q_2 a_1^3 \sin\beta + \frac{1}{4} F_2 a_2^2 \sin\mu = 0 \quad , \tag{5.21b}$$

$$\omega_1(\sigma_1 + \frac{1}{3}\sigma_2)a_1 + \frac{3}{8}(\Upsilon_{11}a_1^3 + \Upsilon_{12}a_1 a_2^2) + 3H_{11}a_1 + \frac{3}{8}Q_1 a_1^2 a_2 \cos\beta = 0 \ , \tag{5.21c}$$

$$\omega_2\sigma_2 a_2 + \frac{3}{8}(\Upsilon_{22}a_2^3 + \Upsilon_{21}a_2 a_1^2) + 3H_{22}a_2 + \frac{3}{8} Q_2 a_1^3 \cos\beta$$
$$+ \frac{3}{4} F_2 a_2^2 \cos\mu = 0 \quad , \tag{5.21d}$$

and

$$a_n = 0 \quad \text{for} \quad n > 2$$

where

$$\beta = \sigma_1 T_1 - 3\alpha_1 + \alpha_2 \tag{5.21e}$$

and

$$\mu = \sigma_2 T_1 - 3\alpha_2 \quad . \tag{5.21f}$$

Equations (5.21) reveal that there are three possible solutions. This is in agreement with the comments of Section 4.2.e.

When a_1 and a_2 are zero, the steady-state solution is given by

$$u_n = P_n(\omega_n^2 - \lambda^2)^{-1} \cos \lambda t + O(\varepsilon) \tag{5.22}$$

for all n.

When $a_1 = 0$ and $a_2 \neq 0$, the steady-state solution has the form

$$u_2 = P_2(\omega_2^2 - \lambda^2)^{-1} \cos \lambda t + a_2 \cos[\tfrac{1}{3}(\lambda t - \mu)] + O(\varepsilon) \tag{5.23}$$

and the remaining u_n are given by equation (5.22). It is noted that the nonlinearity adjusts the frequency of the second mode such that it is precisely one third of that of the excitation.

When a_1 and a_2 differ from zero, the steady-state solution has the form

$$u_1 = P_1(\omega_1^2 - \lambda^2)^{-1} \cos \lambda t + a_1 \cos [\tfrac{1}{9}(\lambda t - \mu) - \tfrac{1}{3} \beta] + O(\varepsilon). \tag{5.24a}$$

$$u_2 = P_2(\omega_2^2 - \lambda^2)^{-1} \cos \lambda t + a_2 \cos[\tfrac{1}{3}(\lambda t - \mu)] + O(\varepsilon) , \tag{5.24b}$$

and the remaining u_n are given by equation (5.22). It is noted that the nonlinearity adjusts the frequencies of the second and the first modes such that they are precisely in the three to one ratio and the frequency of the second mode is precisely one third of that of the excitation.

The first case [equation (5.22)] is of little interest and no results are presented.

For the second subcase [equation (5.23)], a_2 is plotted as a function of $\varepsilon\sigma_2$ in Figure 8a and as a function of the amplitude of the excitation, P, in Figure 8b (see Appendix B for the definition of P). As one might expect, this result resembles the solution of the Duffing equation for subharmonic resonance.

For the third case [equations (5.24)], a_1 and a_2 are plotted as functions of $\varepsilon\sigma_2$ in Figure 9a and as function of P in Figure 9b. It is noted that, when a_1 is not zero, it is greater than a_2 over a considerable range of $\varepsilon\sigma_2$ and P.

d. <u>The Case of 2λ Near $(\omega_2 + \omega_3)$ (Combination Resonance)</u>

In this case, the detuning parameter, σ_2, is defined as follows:

$$2\lambda = \omega_2 + \omega_3 + \varepsilon\sigma_2 \quad .$$

Thus,

$$N_2 = H_{23}\bar{A}_3\exp(i\sigma_2 T_1) \ , \quad N_3 = H_{32}\bar{A}_2\exp(i\sigma_2 T_1)$$

where H_{23} and H_{32} are given by equation (2.14b) and

$$N_n = 0 \quad \text{for } n = 1, 4, 5, \ldots \quad .$$

Equations (4.3) - (4.5) can be reduced to

$$-\omega_1 c_1 a_1 + \frac{1}{8} Q_1 a_1^2 a_2 \sin\beta = 0 \ , \tag{5.25a}$$

$$-\omega_2 c_2 a_2 - \frac{1}{8} Q_2 a_1^3 \sin\beta + \frac{1}{2} H_{23} a_3 \sin\mu = 0 \ , \tag{5.25b}$$

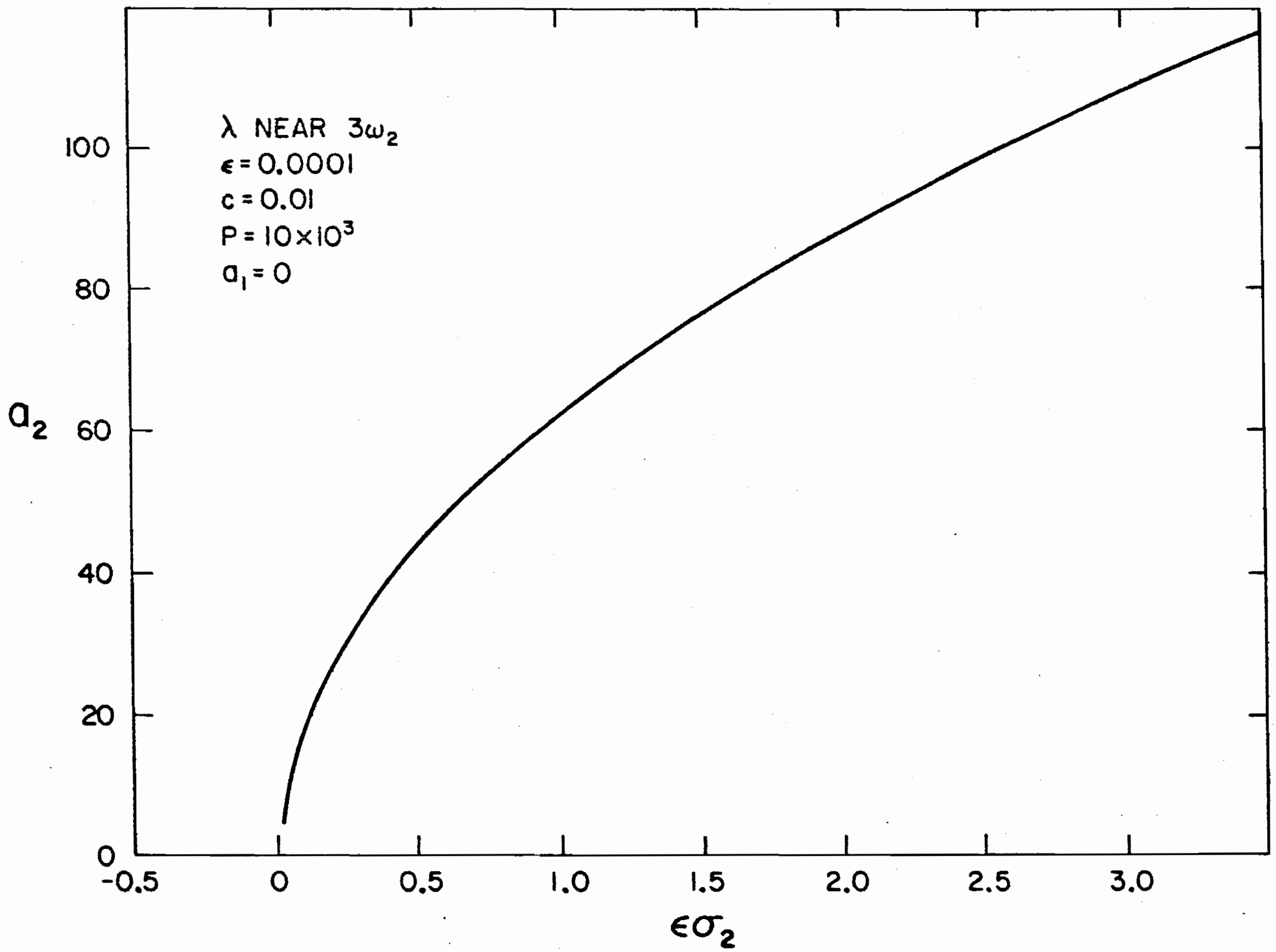

Figure 8a Variation of Second Mode Amplitude with Detuning of Excitation

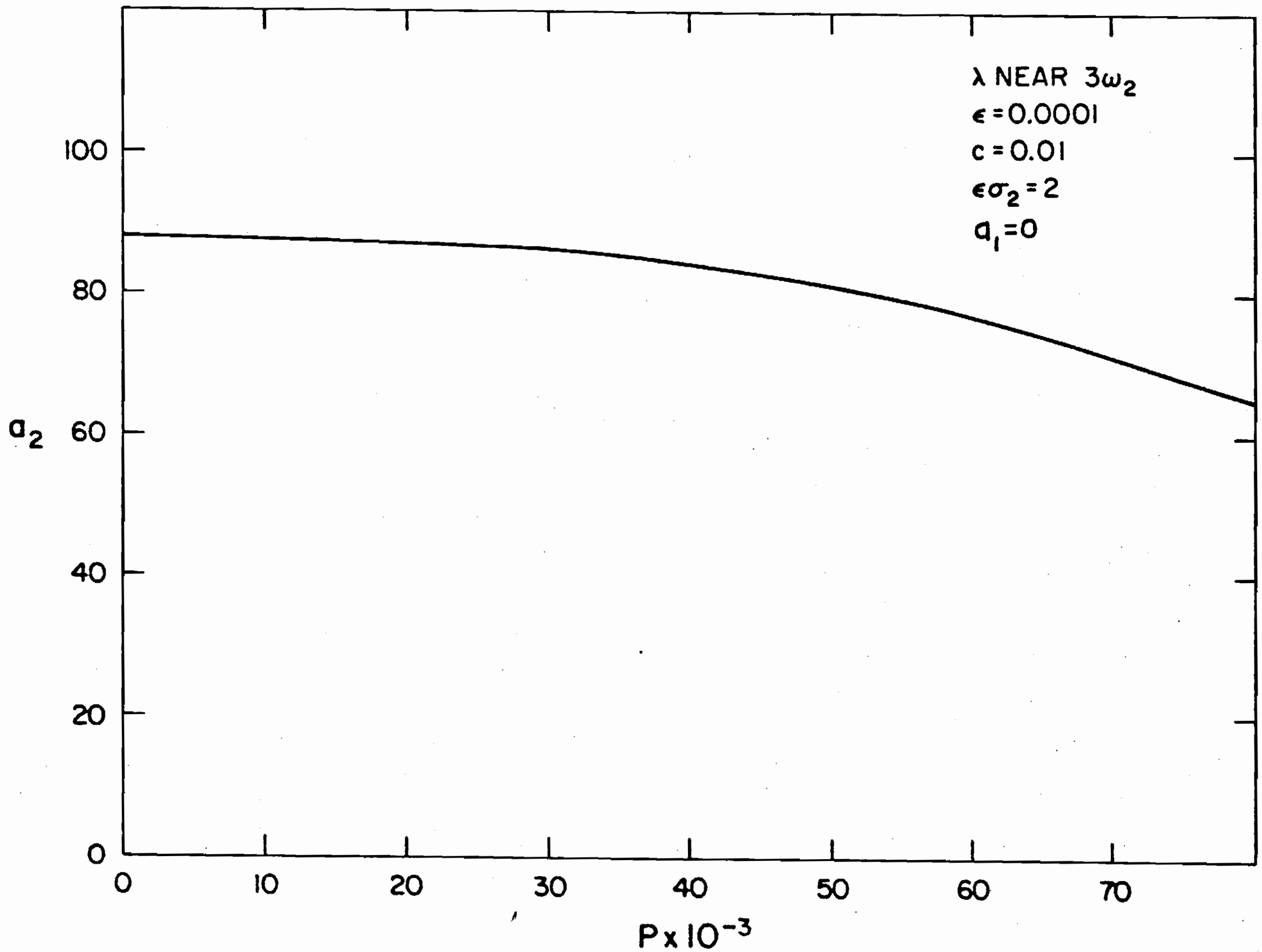

Figure 8b Variation of Second Mode Amplitude with Amplitude of Excitation

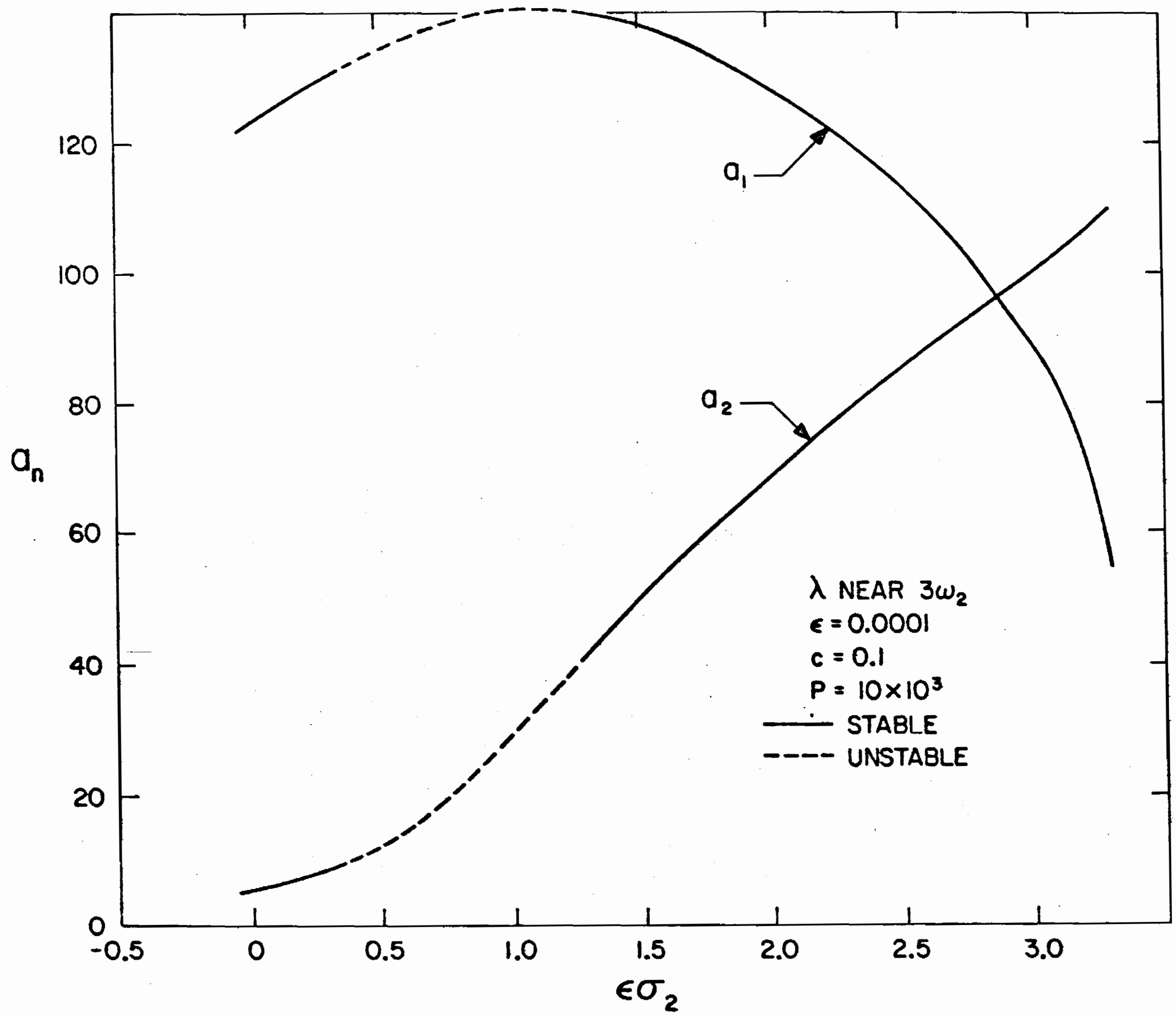

Figure 9a Variation of First and Second Mode Amplitudes with Detuning of Excitation

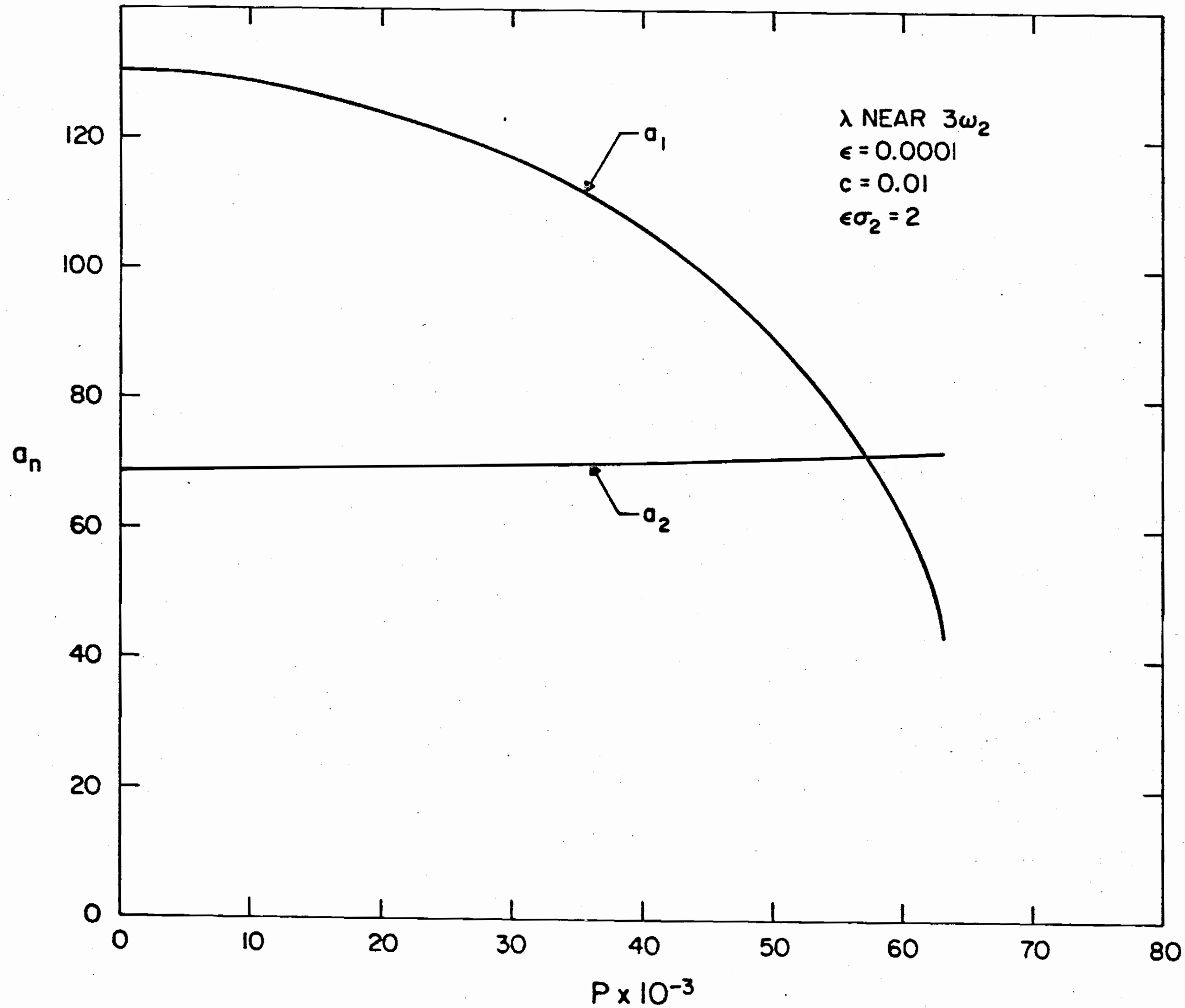

Figure 9b Variation of First and Second Mode Amplitudes with Amplitude of Excitation

$$-\omega_3 c_3 a_3 + \frac{1}{2} H_{32} a_2 \sin\mu = 0 \quad , \tag{5.25c}$$

and for a nontrivial solution of a_1, a_2 and a_3

$$\sigma_1 + \frac{1}{8}\left(\frac{3\Upsilon_{11}}{\omega_1} - \frac{\Upsilon_{21}}{\omega_2}\right)a_1^2 + \frac{1}{8}\left(\frac{3\Upsilon_{12}}{\omega_1} - \frac{\Upsilon_{22}}{\omega_2}\right)a_2^2 + \frac{1}{8}\left(\frac{3\Upsilon_{13}}{\omega_1} - \frac{\Upsilon_{23}}{\omega_2}\right)a_3^2$$
$$+ \frac{3H_{11}}{\omega_1} - \frac{H_{22}}{\omega_2} + \frac{1}{8}\left(\frac{3Q_1}{\omega_1} a_1 a_2 - \frac{Q_2 a_1^3}{\omega_2 a_2}\right)\cos\beta - \frac{1}{2}\frac{H_{23}a_3}{\omega_2 a_2}\cos\mu = 0 \tag{5.25d}$$

$$\sigma_2 + \frac{1}{8}\left(\frac{\Upsilon_{21}}{\omega_2} + \frac{\Upsilon_{31}}{\omega_3}\right)a_1^2 + \frac{1}{8}\left(\frac{\Upsilon_{22}}{\omega_2} + \frac{\Upsilon_{32}}{\omega_3}\right)a_2^2 + \frac{1}{8}\left(\frac{\Upsilon_{23}}{\omega_2} + \frac{\Upsilon_{33}}{\omega_3}\right)a_3^2 + \frac{H_{22}}{\omega_2}$$
$$+ \frac{H_{33}}{\omega_3} + \frac{1}{8}\frac{Q_2 a_1^3}{\omega_2 a_2}\cos\beta + \frac{1}{2}\left(\frac{H_{23}a_3}{\omega_2 a_2} + \frac{H_{32}a_2}{\omega_3 a_3}\right)\cos\mu = 0 \tag{5.25e}$$

where

$$\beta = \sigma_1 T_1 - 3\alpha_1 + \alpha_2 \tag{5.25f}$$

and

$$\mu = \sigma_2 T_1 - \alpha_2 - \alpha_3 \quad . \tag{5.25g}$$

In this case also, there are three possible solutions.

When a_1, a_2 and a_3 are zero, the steady-state solution is given by equation (5.22) for all n.

When $a_1 = 0$ and a_2 and a_3 differ from zero, the steady-state solution is obtained by solving equations (5.25b, c and e) for a_2, a_3 and μ, after setting $a_1 = 0$, and then obtaining α_2' and α_3' from equations (4.3b) and (4.4b), which become

$$\omega_2 a_2 \alpha_2' + \frac{1}{8} a_2(\Upsilon_{22}a_2^2 + \Upsilon_{23}a_3^2) + H_{22}a_2 + \frac{1}{8} Q_2 a_1^3 \cos\beta$$
$$+ \frac{1}{2} H_{23} a_3 \cos\mu = 0 \; , \tag{5.26a}$$

and

$$\omega_3 a_3 \alpha_3' + \frac{1}{8} a_3(\gamma_{32}a_2^2 + \gamma_{33}a_3^2) + H_{33}a_3 + \frac{1}{2} H_{32}a_2 \cos\mu = 0. \quad (5.26b)$$

Then, the solution is given by

$$u_2 = P_2(\omega_2^2 - \lambda^2)^{-1} \cos\lambda t + a_2 \cos[(\omega_2 + \varepsilon\alpha_2')t + \tau_2] + O(\varepsilon), \quad (5.27a)$$

$$u_3 = P_3(\omega_3^2 - \lambda^2)^{-1} \cos\lambda t + a_3 \cos[(\omega_3 + \varepsilon\alpha_3')t + \tau_3] + O(\varepsilon), \quad (5.27b)$$

and the remaining u_n are given by equation (5.22). It is noted that the nonlinearity adjusts the frequencies of the second and third modes such that the resonant frequency combination is satisfied exactly; that is,

$$\omega_2 + \varepsilon\alpha_2' + \omega_3 + \varepsilon\alpha_3' = \omega_2 + \omega_3 + \varepsilon\sigma_2 = 2\lambda .$$

When a_1, a_2 and a_3 are not zero, the steady-state solution is obtained by solving equations (5.25) for a_1, a_2, a_3, β and μ, and then obtaining α_1', α_2' and α_3' from

$$\omega_1 a_1 \alpha_1' + \frac{1}{8} a_1(\gamma_{11}a_1^2 + \gamma_{12}a_2^2 + \gamma_{13}a_3^2) + H_{11}a_1 + \frac{1}{8} Q_2 a_1^2 a_2 \cos\beta = 0, \quad (5.28a)$$

$$\omega_2 a_2 \alpha_2' + \frac{1}{8} a_2(\gamma_{21}a_1^2 + \gamma_{22}a_2^2 + \gamma_{23}a_3^2) + H_{22}a_2 + \frac{1}{8} Q_2 a_1^3 \cos\beta + \frac{1}{2} H_{23}a_3 \cos\mu = 0 , \quad (5.28b)$$

and

$$\omega_3 a_3 \alpha_3' + \frac{1}{8} a_3(\gamma_{31}a_1^2 + \gamma_{32}a_2^2 + \gamma_{33}a_3^2) + H_{33}a_3 + \frac{1}{2} H_{32}a_2 \cos\mu = 0 . \quad (5.28c)$$

Then, the solution is given by

$$u_1 = P_1(\omega_1^2 - \lambda^2)^{-1} \cos \lambda t + a_1 \cos[(\omega_1 + \varepsilon\alpha_1')t + \tau_1] + O(\varepsilon) , \tag{5.29a}$$

$$u_2 = P_2(\omega_2^2 - \lambda^2)^{-1} \cos \lambda t + a_2 \cos[(\omega_2 + \varepsilon\alpha_2')t + \tau_2] + O(\varepsilon), \tag{5.29b}$$

$$u_3 = P_3(\omega_3^2 - \lambda^2)^{-1} \cos \lambda t + a_3 \cos[(\omega_3 + \varepsilon\alpha_3')t + \tau_3] + O(\varepsilon) , \tag{5.29c}$$

and the remaining u_n are given by equation (5.22). It follows from equations (5.25f and g) and (5.29) that the nonlinearity adjusts the frequencies such that the frequencies of the first and second modes are precisely in the ratio of one to three and the sum of the frequencies of the second and third modes is precisely 2λ; that is,

$$3(\omega_1 + \varepsilon\alpha_1') = 3\omega_1 + \varepsilon\sigma_1 + \varepsilon\alpha_2' = \omega_2 + \varepsilon\alpha_2' ,$$

and

$$\omega_2 + \varepsilon\alpha_2' + \omega_3 + \varepsilon\alpha_3' = \omega_2 + \omega_3 + \varepsilon\sigma_2 = 2\lambda .$$

The first subcase [equation (5.22)], is of little interest, and the results are not presented.

For the second case [equations (5.27)], a_2 and a_3 are plotted as functions of $\varepsilon\sigma_2$ in Figure 10a and as functions of P (see Appendix B) in Figure 10b. We note that a_2 is always greater than a_3.

For the third case [equations (5.29)], a_1, a_2 and a_3 are plotted as functions of $\varepsilon\sigma_2$ in Figure 11a and as functions of P in Figure 11b. It is noted that a_1 is greater than a_2 and a_3 over a wide range of $\varepsilon\sigma_2$ and P.

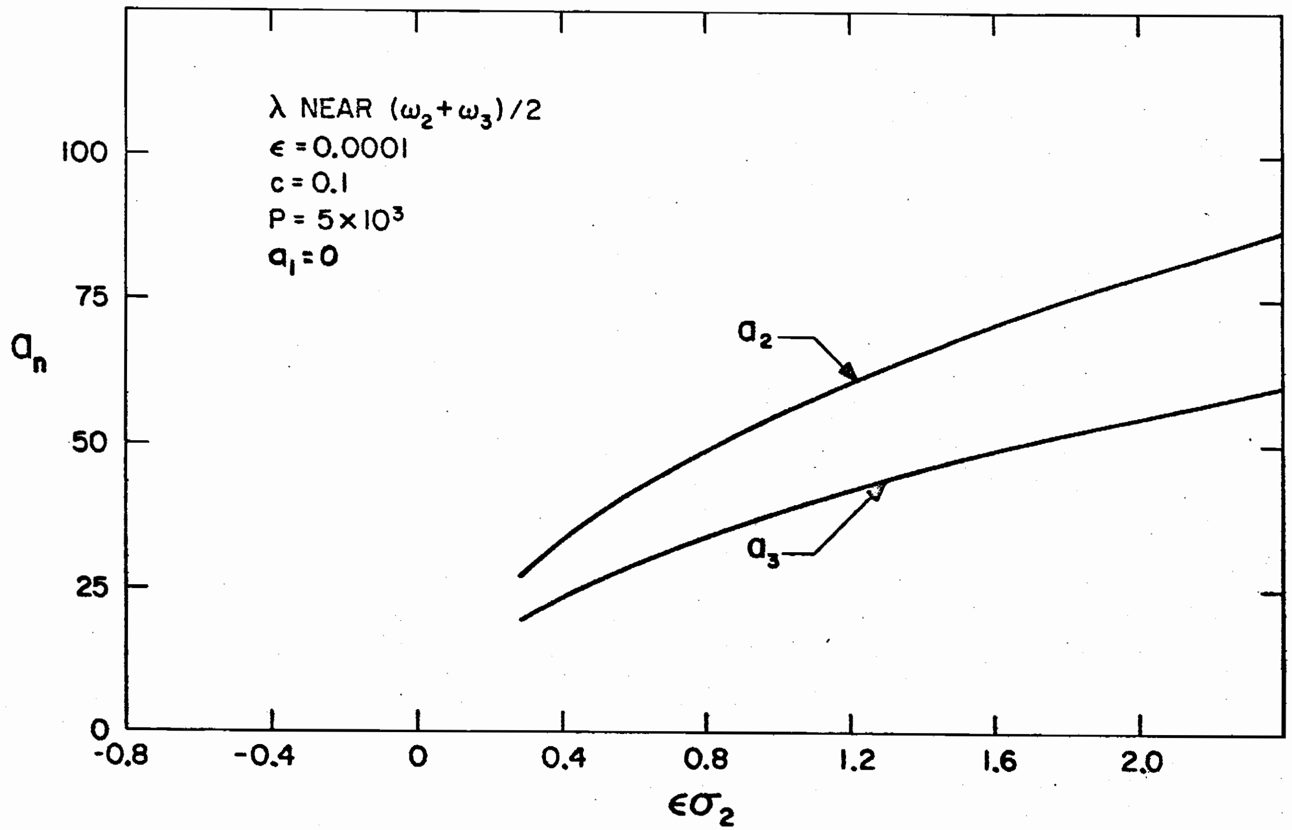

Figure 10a Variation of Second and Third Mode Amplitudes with Detuning of Excitation

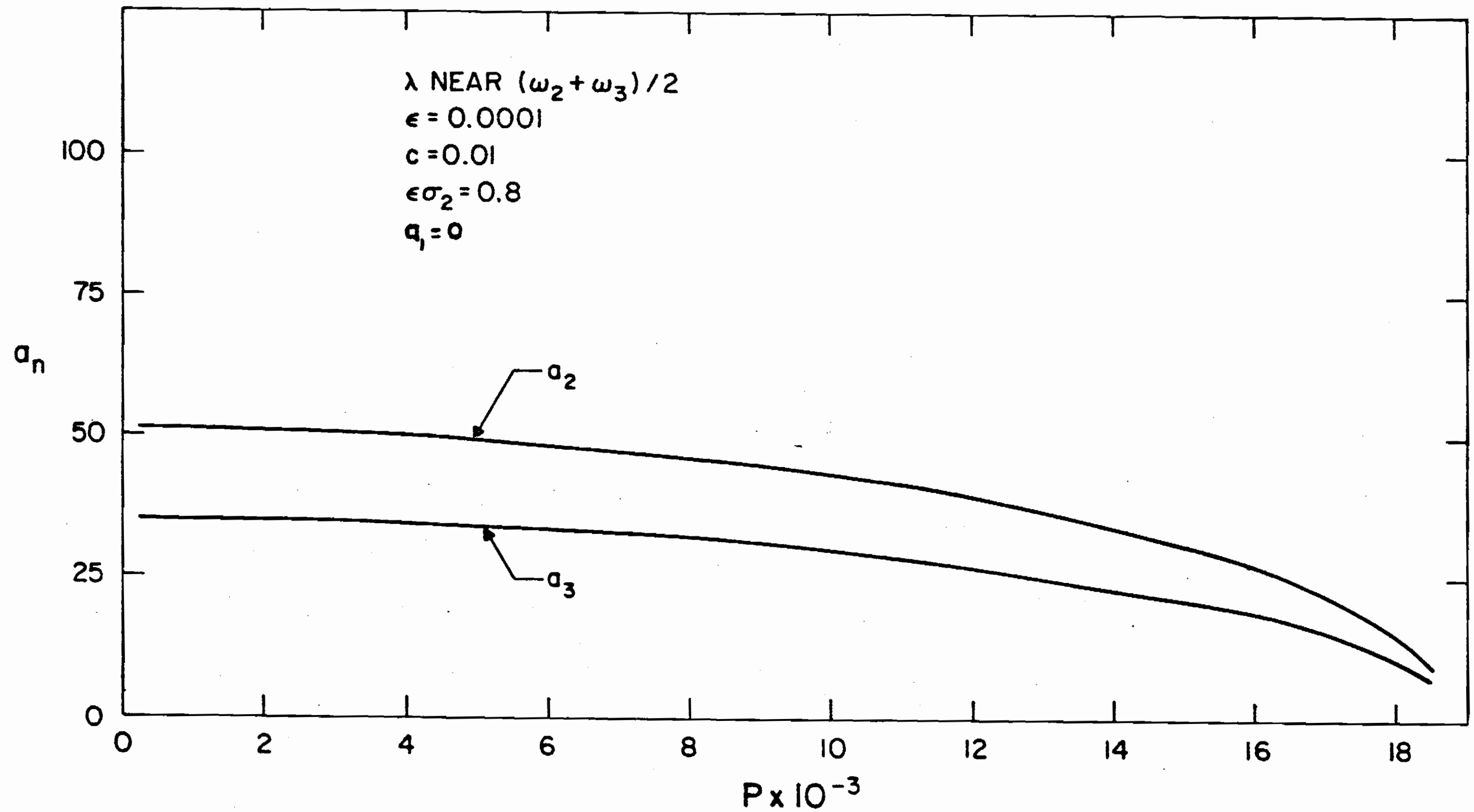

Figure 10b Variation of Second and Third Mode Amplitudes with Amplitude of Excitation

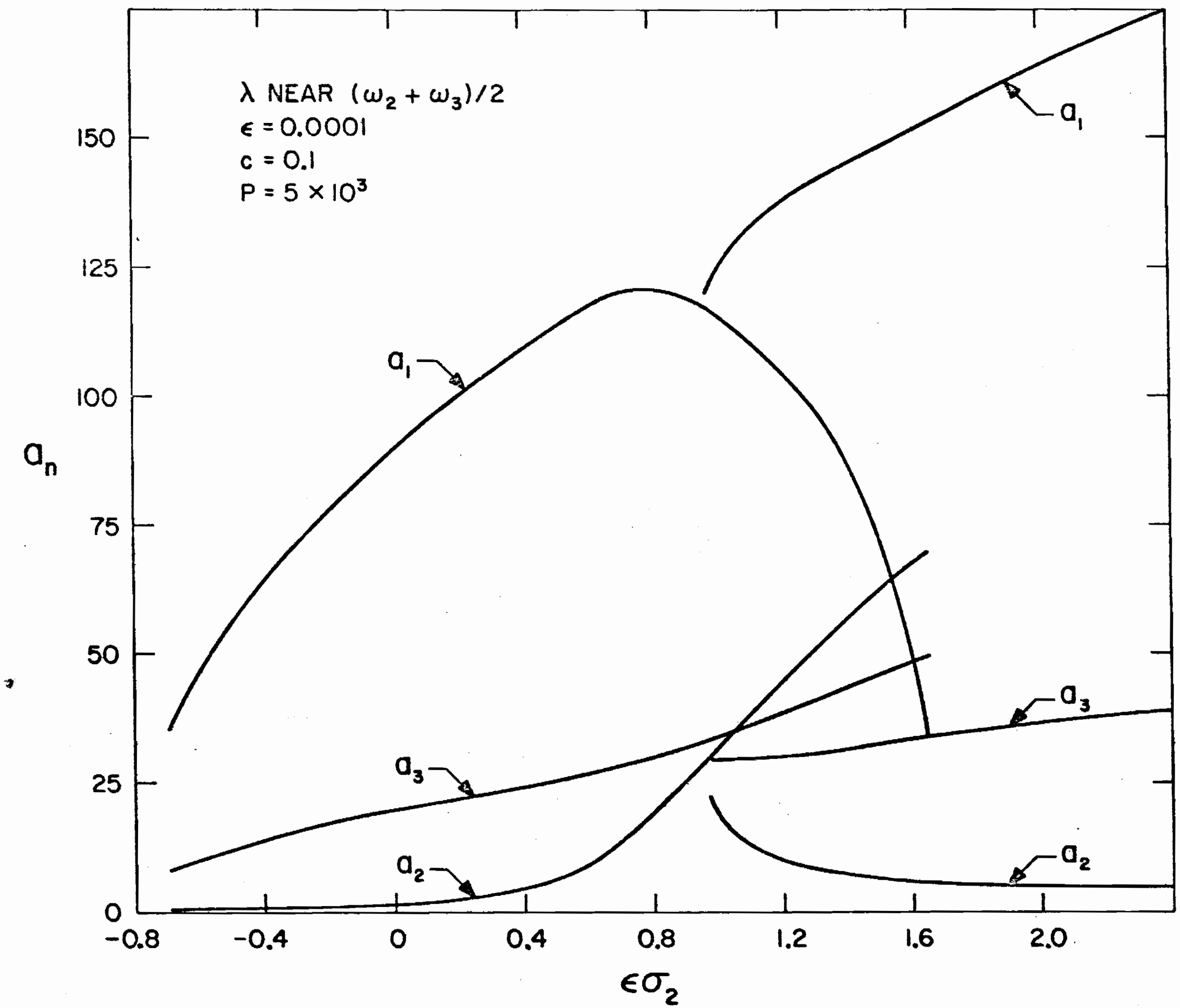

Figure 11a Variation of First, Second and Third Mode Amplitudes with Detuning of Excitation

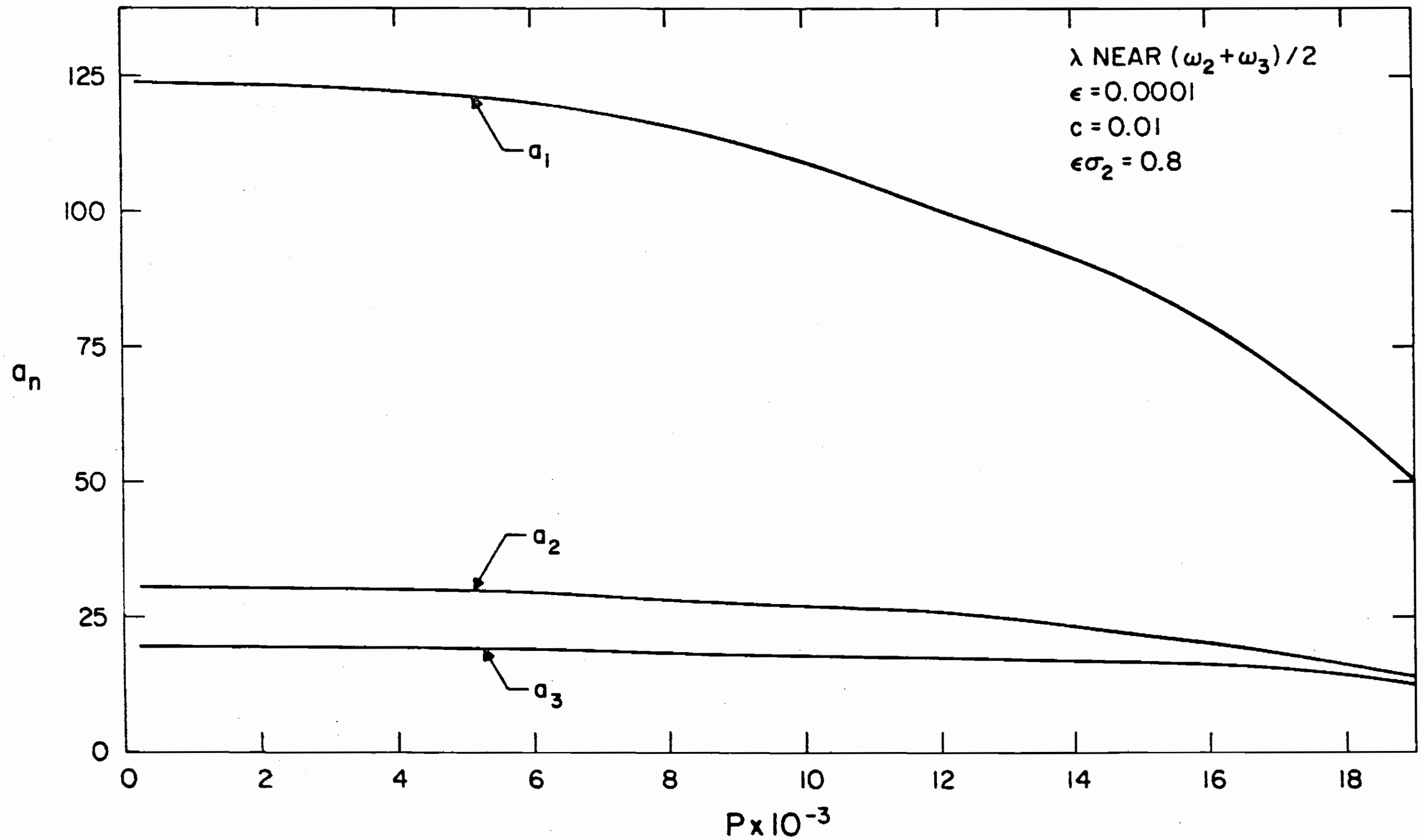

Figure 11b Variation of First, Second and Third Mode Amplitudes with Amplitude of Excitation

5.3 Summary

The problem of the nonlinear transverse vibrations of beams is reduced to the study of the system considered in Chapter 2. Numerical examples of the various nonlinear resonances in a hinged-clamped beam are presented, which illustrate that stable solutions of the type discussed in Chapters 3 and 4 do exist. The two lowest natural frequencies of such a beam are nearly in the ratio of three to one; i.e., the first and the second modes are in an internal resonance.

The effects of the internal resonance are clearly shown in a series of figures. These figures, in addition to exhibiting the typical nonlinear phenomena, illustrate that when more than one mode appears in the steady-state response, the lowest mode involved is likely to dominate the response. Consequently, the steady-state response can strongly resemble the response which occurs when the frequency of the excitation is near that of a lower mode, though the frequency of the excitation is, in fact, far away from that of the lower mode (see Figures 6a and 6b).

The accuracy of the present method is illustrated by comparing the solution with that obtained by a direct numerical integration of the original equations governing the temporal variation of the response (see Figures 2a, 2b, 4b and 5b).

6. NONLINEAR VIBRATIONS OF CLAMPED CIRCULAR PLATES-GENERAL PROBLEM

The dynamic analogue of the von Karman equations has been used in several attempts to determine the effect of mid-plane stretching on the response of clamped circular plates to harmonic excitations. The governing equations are two nonlinear coupled fourth-order partial differential equations. In this chapter, the general problem, including asymmetric vibrations and travelling waves is reduced to the study of essentially the same system considered in Chapter 2. For the sake of clarity, the analysis of this chapter is restricted to main resonances only. The steady-state response is obtained in the absence of internal resonances. The influence of an internal resonance involving four modes on the steady-state response is evaluated.

6.1 Problem Formulation

The equations governing the free, undamped oscillations of non-uniform circular plates were derived by Efstathiades [36]. We simplify these equations to fit the special case of uniform plates and add damping and forcing terms. The result is

$$\rho h \frac{\partial^2 w}{\partial t^2} + D\nabla^4 w = L_1(w,F) - c\frac{\partial w}{\partial t} + p(r,\theta,t) \tag{6.1a}$$

and

$$\nabla^4 F = Eh\, L_2(w) \tag{6.1b}$$

where

$$L_1(w,F) = \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r}\frac{\partial^2 F}{\partial \theta^2}\right) + \frac{\partial^2 F}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)$$

$$- 2\left(\frac{1}{r}\frac{\partial^2 F}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial F}{\partial \theta}\right)\left(\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right) ,$$

$$L_2(w) = \left(\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right)^2 - \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) ,$$

ρ is the density, h is the thickness, $D = Eh^3/[12(1-\nu^2)]$, c is the damping coefficient, p is the forcing function, E is Young's modulus, ν is Poisson's ratio, w is the deflection of the middle surface, F is the force function which satisfies the in-plane equilibrium conditions (in-plane inertia is neglected) and

$$\nabla^4 \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 .$$

The relationships between F, w and the in-plane displacements, u_r and u_θ, are given by

$$e_r = \frac{1}{Eh}\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} - \nu\frac{\partial^2 F}{\partial r^2}\right) , \tag{6.2a}$$

$$e_\theta = \frac{1}{Eh}\left[\frac{\partial^2 F}{\partial r^2} - \nu\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right)\right] , \tag{6.2b}$$

and

$$e_{r\theta} = \frac{2(1+\nu)}{Eh}\left[\frac{1}{r^2}\frac{\partial F}{\partial \theta} - \frac{1}{r}\frac{\partial^2 F}{\partial r \partial \theta}\right] , \tag{6.2c}$$

where

$$e_r = \frac{\partial u_r}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 , \tag{6.3a}$$

$$e_\theta = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 , \tag{6.3b}$$

and

$$e_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta} . \tag{6.3c}$$

It is convenient to rewrite these equations in terms of non-dimensional variables, denoted by overbars, which are defined as follows

$$r = a\bar{r}\ ,\ t = a^2\left(\frac{\rho h}{D}\right)^{\frac{1}{2}}\bar{t}\ ,\quad w = \frac{h^2}{a}\bar{w}\ ,\ (u_r,u_\theta) = \frac{h^4}{a^3}(\bar{u}_r,\bar{u}_\theta)\ ,$$

$$c = \frac{24(1-\nu^2)}{a^4}\ (\rho h^5 D)^{\frac{1}{2}}\ \bar{c},\ p = \frac{12(1-\nu^2)Dh^4}{a^7}\ \bar{p}\ ,\ \text{and}\ F = \frac{Eh^5}{a^2}\ \bar{F}$$

where "a" is the radius of the plate. We are concerned with generating an approximate solution which is valid as h/a approaches zero; each of the non-dimensional variables defined above is presumed to be O(1) in this limit. In addition, we define $\bar{e}_r$, $\bar{e}_\theta$, and $\bar{e}_{r\theta}$, which are also presumed to be O(1) as h/a approaches zero, as follows:

$$(e_r,\ e_\theta,\ e_{r\theta}) = \frac{h^2}{a^4}\ (\bar{e}_r,\bar{e}_\theta,\bar{e}_{r\theta})$$

Substituting these definitions into equations (6.1) and (6.2) and dropping the overbars in the result, one obtains

$$\frac{\partial^2 w}{\partial t^2} + \nabla^4 w = \varepsilon[L_1(w_1 F) - 2c\ \frac{\partial w}{\partial t} + p] \tag{6.4a}$$

and

$$\nabla^4 F = L_2(w) \tag{6.4b}$$

where

$$\varepsilon = 12(1-\nu^2)h^2/a^2\ ,$$

$$e_r = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} - \nu\ \frac{\partial^2 F}{\partial r^2}\ , \tag{6.5a}$$

$$e_\theta = \frac{\partial^2 F}{\partial r^2} - \nu\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right)\ , \tag{6.5b}$$

$$e_{r\theta} = 2(1+\nu)\left(\frac{1}{r^2}\frac{\partial F}{\partial \theta} - \frac{1}{r}\frac{\partial^2 F}{\partial r \partial \theta}\right)\ , \tag{6.5c}$$

and the form of equations (6.3) is not changed.

We develop the boundary conditions for plates which are clamped along a circular edge. For all t, and θ,

$$w = 0 \quad , \frac{\partial w}{\partial r} = 0 \quad , \tag{6.6a}$$

and

$$u_r = 0 \quad , u_\theta = 0 \tag{6.6b}$$

at r = a. It follows from equations (6.3), (6.5) and (6.6) that, for all t and θ,

$$e_\theta = 0 \quad , \frac{\partial}{\partial r}(re_\theta) - e_r - \frac{\partial}{\partial \theta}(e_{r\theta}) = 0 \quad ,$$

$$\frac{\partial^2 F}{\partial r^2} - \nu\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right) = 0 \quad , \tag{6.7a}$$

and

$$\frac{\partial^3 F}{\partial r^3} + \frac{1}{r}\frac{\partial^2 F}{\partial r^2} - \frac{1}{r^2}\frac{\partial F}{\partial r} + \frac{(2+\nu)}{r^2}\frac{\partial^3 F}{\partial r \partial \theta^2} - \frac{(3+\nu)}{r^3}\frac{\partial^2 F}{\partial \theta^2} = 0 \tag{6.7b}$$

at r = a. In addition, it is necessary to require the solution to be bounded at r = 0.

We note that, when ε is small, w is much smaller than h. Had w been the same order as h (say, $w = h\bar{w}$), then no small parameter would have appeared in equation (6.4a) and the linear and nonlinear terms would have been the same order. Hence, the present approach must be viewed as one which provides corrections for the small-deflection theory (for which w is much smaller than h) and not as one which provides a solution for the large-deflection theory (for which w is the same order as h). This means that some typical nonlinear phenomena, such as jump phenomena, modal interactions, etc., can be part of the small-deflection theory (see Chapter 5 for analogous remarks on the vibrations of beams).

Further we note that, w is a function of r, θ and t and the solution may contain travelling waves. Equations (6.4) do not lend themselves to a straightforward separation of the spatial and temporal variables as in Chapter 5. However, by using the method of multiple scales, an asymptotic expansion of the solution of equations (6.4) can still be constructed. The expansion is to be uniformly valid for small ε and large t.

6.2 Solution

Following the derivative-expansion version of the method of multiple scales, we assume expansions of the form

$$w(r,\theta,t;\varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^j w_j(r,\theta,T_0,T_1,\ldots) \tag{6.8a}$$

and

$$F(r,\theta,t;\varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^j F_j(r,\theta,T_0,T_1,\ldots) \tag{6.8b}$$

where

$$T_n = \varepsilon^n t \ .$$

Substituting equations (6.8) and (2.3) into equations (6.4) and balancing powers of ε, we obtain

$$D_0^2 w_0 + \nabla^4 w_0 = 0 \tag{6.9}$$

$$\nabla^4 F_0 = \left(\frac{1}{r}\frac{\partial^2 w_0}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial w_0}{\partial\theta}\right)^2 - \frac{\partial^2 w_0}{\partial r^2}\left(\frac{1}{r}\frac{\partial w_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_0}{\partial\theta^2}\right) \tag{6.10}$$

$$\begin{aligned} D_0^2 w_1 + \nabla^4 w_1 = &- 2D_0 D_1 w_0 - 2c\, D_0 w_0 + p \\ &+ \frac{\partial^2 w_0}{\partial r^2}\left(\frac{1}{r}\frac{\partial F_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F_0}{\partial\theta^2}\right) + \frac{\partial^2 F_0}{\partial r^2}\left(\frac{1}{r}\frac{\partial w_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_0}{\partial\theta^2}\right) \\ &- 2\left(\frac{1}{r}\frac{\partial^2 F_0}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial F_0}{\partial\theta}\right)\left(\frac{1}{r}\frac{\partial^2 w_0}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial w_0}{\partial\theta}\right) \end{aligned} \tag{6.11}$$

etc. where $D_n = \frac{\partial}{\partial T_n}$.

Substituting equations (6.8) into equation (6.6a) and (6.7) and balancing powers of ϵ, we obtain (letting a = 1)

$$w_j = 0 \quad , \quad \frac{\partial w_j}{\partial r} = 0 \ , \tag{6.12a,b}$$

$$\frac{\partial^2 F_j}{\partial r^2} - \nu\left(\frac{\partial F_j}{\partial r} + \frac{\partial^2 F_j}{\partial\theta^2}\right) = 0 \tag{6.13a}$$

and

$$\frac{\partial^3 F_j}{\partial r^3} + \frac{\partial^2 F_j}{\partial r^2} - \frac{\partial F_j}{\partial r} + (2+\nu)\frac{\partial^3 F_j}{\partial r\partial\theta^2} - (3+\nu)\frac{\partial^2 F_j}{\partial\theta^2} = 0 \tag{6.13b}$$

for all j, θ, and t at r = 1. In addition, it is necessary to require w_j and F_j, for all j, to be bounded at r = 0.

It follows from equations (6.9) and (6.12) that

$$\begin{aligned} w_0 = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \phi_{nm}(r) \Big\{ & A_{nm}\exp[i(\omega_{nm}T_0 + n\theta)] \\ & + B_{nm}\exp[i(\omega_{nm}T_0 - n\theta)] + cc \Big\} \end{aligned} \tag{6.14}$$

where the $\phi_{nm}(r)$ are the linear, free-oscillation modes given by

$$\phi_{nm}(r) = \kappa_{nm}[J_n(\eta_{nm}r) - \frac{J_n(\eta_{nm})}{I_n(\eta_{nm})} I_n(\eta_{nm}r)] \quad ,$$

the κ_{nm} are chosen so that

$$\int_0^1 r\phi_{nm}^2(r)dr = 1 \quad ,$$

the η_{nm} are the roots of

$$I_n(\eta)J_n'(\eta) - I_n'(\eta)J_n(\eta) = 0 \quad ,$$

$$\omega_{nm} = \eta_{nm}^2 \quad ,$$

the A_{nm} and the B_{nm} are complex functions of all the T_n for $n \geq 1$ which are to be determined from the solvability conditions at the next level of approximation, and cc represents the complex conjugate of the preceding terms.

We note that the solution given by equation (6.14) contains both travelling and standing waves depending on the relative values of the A_{nm} and B_{nm}. The solution can also be written in the following equivalent form:

$$w_0 = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \phi_{nm}(r)u_{nm}(T_0,T_1,\ldots)\exp(in\theta) \tag{6.15}$$

where

$$u_{nm} = A_{nm}\exp(i\omega_{nm}T_0) + \overline{B}_{nm}\exp(-i\omega_{nm}T_0) \quad , \tag{6.16}$$

$$\phi_{-nm} = \phi_{nm} \quad ,$$

and

$$\omega_{-nm} = \omega_{nm} \quad .$$

Because w_0 is real,

$$A_{-nm} = B_{nm} \text{ and } B_{-nm} = A_{nm} \tag{6.17}$$

Substituting equation (6.15) into equation (6.10) leads to

$$\nabla^4 F_0 = \sum_{n,p=-\infty}^{\infty} \sum_{m,q=1}^{\infty} E(nm,pq) u_{nm} u_{pq} \exp[i(n+p)\theta] \tag{6.18}$$

where

$$E(nm,pq) = \frac{-np}{r^2} \left(\phi'_{nm} - \frac{\phi_{nm}}{r}\right) \left(\phi'_{pq} - \frac{\phi_{pq}}{r}\right)$$

$$- \frac{1}{2r} \left(\phi'_{nm} \phi'_{pq}\right)' + \frac{1}{2r^2} \left(p^2 \phi''_{nm} \phi_{pq} + n^2 \phi''_{pq} \phi_{nm}\right)$$

and primes denote differentiation with respect to r.

We assume an expansion for F_0 as follows

$$F_0 = \sum_{n=-\infty}^{\infty} H_n(r, T_0, T_1, \ldots) \exp(in\theta) \tag{6.19}$$

Substituting equation (6.19) into equation (6.18), multiplying the result by $\exp(-ia\theta)$, and integrating from $\theta = 0$ to $\theta = 2\pi$, we obtain

$$\nabla_a^4 H_a = \sum_{n=-\infty}^{\infty} \sum_{m,q=1}^{\infty} E(nm,pq) u_{nm} u_{pq} \tag{6.20}$$

where

$$p = a - n \tag{6.21}$$

and

$$\nabla_a^4 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{a^2}{r^2}\right)^2$$

Then we further assume an expansion for H_a as follows

$$H_a = \sum_{n=1}^{\infty} v_{an}(T_0, T_1, \ldots) \psi_{an}(r) \tag{6.22}$$

where the ψ_{an} are the eigenfunctions of the following problem:

$$(\nabla_a^4 - \xi_{an}^4)\psi_{an} = 0 \quad \text{in } r = [0,1]$$

where ψ_{an} is bounded at $r = 0$ and , from equations (6.13),

$$\psi''_{an} - \nu(\psi'_{an} - a^2\psi_{an}) = 0$$

and

$$\psi'''_{an} + \psi''_{an} - \psi'_{an} - a^2[(2+\nu)\psi'_{an} - (3+\nu)\psi_{an}] = 0$$

for all θ and t at $r = 1$. It follows that

$$\psi_{an} = \tilde{\kappa}_{an}[J_a(\xi_{an}r) - \tilde{c}_{an}I_a(\xi_{an}r)] \tag{6.23}$$

where the $\tilde{\kappa}_{an}$ are chosen so that

$$\int_0^1 r\,\psi_{an}^2\,dr = 1 \quad ,$$

$$\tilde{c}_{an} = \frac{[a(a+1)(\nu+1)-\xi_{an}^2]J_a(\xi_{an})-\xi_{an}(\nu+1)J_{a-1}(\xi_{an})}{[a(a+1)(\nu+1) + \xi_{an}^2]I_a(\xi_{an}) - \xi_{an}(\nu+1)I_{a-1}(\xi_{an})} \quad ,$$

and ξ_{an} are the roots of

$$a^2(a+1)(\nu+1)[J_a(\xi_{an}) - \tilde{c}_{an}I_a(\xi_{an})]$$

$$- a\xi^2{}_{an}(\nu+1)[J_{a-1}(\xi_{an}) - \tilde{c}_{an}I_{a-1}(\xi_{an})]$$

$$+ a\xi_{an}^2[J_a(\xi_{an}) + \tilde{c}_{an}I_a(\xi_{an})]$$

$$- \xi_{an}^3[J_{a-1}(\xi_{an}) + \tilde{c}_{an}I_{a-1}(\xi_{an})] = 0 \quad .$$

Substituting equation (6.22) into equation (6.20), multiplying the result by $r\psi_{ab}$, and then integrating from r = 0 to r = 1, we obtain

$$v_{ab}(T_0,T_1,\ldots) = \sum_{n=-\infty}^{\infty} \sum_{m,q=1}^{\infty} G(nm,pq;ab)u_{nm}u_{pq} \tag{6.24}$$

where

$$G(nm,pq;ab) = \xi_{ab}^{-4} \int_0^1 r\psi_{ab}E(nm,pq)dr \tag{6.25}$$

and p, a, and n are related according to equation (6.21). It follows from equations (6.24), (6.22) and (6.19) that

$$F_0 = \sum_{a,n=-\infty}^{+\infty} \sum_{b,m,q=1}^{\infty} \psi_{ab}G(nm,pq;\ ab)u_{nm}u_{pq}\exp(ia\theta)\ . \tag{6.26}$$

where p = a-n.

Substituting equations (6.26) and (6.14) into equation (6.11) leads to

$$\begin{aligned} D_0^2w_1 + \nabla^4w_1 = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} & -2i\omega_{nm}\phi_{nm}[(D_1A_{nm} + c_{nm}A_{nm})\exp(i\omega_{nm}T_0) \\ & - (D_1\overline{B}_{nm} + c_{nm}\overline{B}_{nm})\exp(-i\omega_{nm}T_0)]\exp(in\theta) \\ & + [\sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \hat{P}_{nm}\phi_{nm}\exp(in\theta)]\cos \lambda T_0 \\ & + \sum_{a,n,c=-\infty}^{\infty} \sum_{b,m,d,q=1}^{\infty} G(nm,pq;ab)\hat{E}(cd,ab)u_{cd}u_{pq}u_{nm}\exp[i(a+c)\theta] \end{aligned} \tag{6.27}$$

where we assume modal damping and p has been expanded as

$$p(r,\theta,t) = [\sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \hat{P}_{nm}\phi_{nm}\exp(in\theta)]\cos \lambda T_0$$

and

$$\hat{E}(cd,ab) = \frac{\phi''_{cd}}{r} (\psi'_{ab} - \frac{a^2}{r}\psi_{ab}) + \frac{\psi''_{ab}}{r} (\phi'_{cd} - \frac{c^2}{r}\phi_{cd})$$

$$+ \frac{2ac}{r^2} (\psi'_{ab} - \frac{1}{r}\psi_{ab})(\phi'_{cd} - \frac{1}{r}\phi_{cd}).$$

Because w_1 and w_0 satisfy the same boundary conditions, we assume an expansion for w_1 in the form

$$w_1 = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \tilde{u}_{nm}(T_0,T_1,\ldots)\phi_{nm}(r) \exp(in\theta) \tag{6.28}$$

Substituting equation (6.28) into (6.27), multiplying by $r\phi_{k\ell}(r)\exp(-ik\theta)$, and integrating the result from $r = 0$ to 1 and $\theta = 0$ to 2π, we obtain (for main resonances)

$$\begin{aligned} D_0^2\tilde{u}_{k\ell} + \omega_{k\ell}^2\tilde{u}_{k\ell} = & - 2i\omega_{k\ell}[(D_1A_{k\ell} + c_{k\ell}A_{k\ell})\exp(i\omega_{k\ell}T_0) \\ & - (D_1\overline{B}_{k\ell} + c_{k\ell}\overline{B}_{k\ell})\exp(-i\omega_{k\ell}T_0)] + \hat{P}_{k\ell}[\exp(i\lambda T_0) \\ & + \exp(-i\lambda T_0)] + \sum_{n,c=-\infty}^{\infty} \sum_{d,m,q=1}^{\infty} \Gamma(k\ell,cd,nm,pq) \\ & [\sum_{j=1}^{8} \hat{S}_j\exp(i\hat{\Lambda}_jT_0)] , \quad k = 1,2,\ldots, \ \ell = 1,2,\ldots \end{aligned} \tag{6.29}$$

where

$$\Gamma(k\ell,cd,nm,pq) = \sum_{b=1}^{\infty} G(nm,pq;ab) \int_0^1 r\phi_{k\ell}\hat{E}(cd,ab)dr \quad , \tag{6.30a}$$

$$a = k - c \tag{6.30b}$$

$$p = k - c - n \quad , \tag{6.30c}$$

$\hat{\Lambda}_j$ are frequency combinations, and $\hat{S}_j$ are functions of A_{nm} and B_{nm}. Both $\hat{\Lambda}_j$ and S_j are listed in Appendix C.

It is noted that equations (6.29) are essentially the same (for $B_{nm} = A_{nm}$, exactly the same) as equations (2.8).

The solvability conditions can be obtained by requiring the coefficients of $\exp(\pm i\omega_{k\ell}T_0)$ to vanish from the right-hand sides of equation (6.29). In general the solvability conditions can be written as

$$- 2i\omega_{k\ell}(D_1 A_{k\ell} + c_{k\ell}A_{k\ell}) + A_{k\ell} \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \gamma_{k\ell nm}(A_{nm}\bar{A}_{nm} + B_{nm}\bar{B}_{nm})$$

$$+ N_{k\ell}^{A} + R_{k\ell}^{A} = 0 \quad , \tag{6.31a}$$

and

$$2i\omega_{k\ell}(D_1 \bar{B}_{k\ell} + c_{k\ell}\bar{B}_{k\ell}) + \bar{B}_{k\ell} \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \gamma_{k\ell nm}(A_{nm}\bar{A}_{nm} + B_{nm}\bar{B}_{nm})$$

$$+ N_{k\ell}^{B} + R_{k\ell}^{B} = 0 \quad , \tag{6.31b}$$

where $R_{k\ell}^{A,B}$ are terms due to internal resonances if any, $N_{k\ell}^{A,B}$ are terms due to the external excitation if any, and $\gamma_{k\ell nm}$ are constants. We made use of equations (6.17) and (6.30c) to arrive at the summation terms in equations (6.31).

We note that equations (6.31) are comparable to equations (2.13), for the case of main resonances.

6.3 The Case of No Internal Resonance

In the absence of internal resonances $R^{A,B}_{k\ell} = 0$. Because we are dealing with main resonances, the analysis here closely follows that of Section 3.2. Thus, when λ is near ω_{rs}

$$\lambda = \omega_{rs} + \epsilon\sigma \quad , \tag{6.32}$$

$$N^A_{rs} = \frac{1}{2}\hat{P}_{rs}\exp(i\sigma T_1) \; , \quad N^B_{rs} = \frac{1}{2}\hat{P}_{rs}\exp(-i\sigma T_1) \tag{6.33a,b}$$

and

$$N^{A,B}_k = 0, \quad \text{for } k\ell \neq rs \tag{6.33c}$$

Next we let

$$A_{nm} = \frac{1}{2}a_{nm}\exp(i\alpha_{nm}) \text{ and } B_{nm} = \frac{1}{2}b_{nm}\exp(i\beta_{nm}) \tag{6.34a,b}$$

where a_{nm}, b_{nm}, α_{nm} and β_{nm} are real functions of T_1.

From the results of Section 3.1 (only the directly excited modes can appear in the steady-state response) we obtain

$$a_{k\ell} = b_{k\ell} = 0 \quad , \quad \text{for } k\ell \neq rs \tag{6.35}$$

$$-\omega_{rs}c_{rs}a_{rs} + \frac{1}{2}\hat{P}_{rs}\sin\mu^a_{rs} = 0 \; , \tag{6.36a}$$

$$\omega_{rs}a_{rs}\sigma + \frac{1}{8}\gamma_{rsrs}(a^2_{rs} + b^2_{rs})a_{rs} + \frac{1}{2}\hat{P}_{rs}\cos\mu^a_{rs} = 0 \tag{6.36b}$$

$$\omega_{rs}c_{rs}b_{rs} - \frac{1}{2}\hat{P}_{rs}\sin\mu^b_{rs} = 0 \quad , \tag{6.36c}$$

and

$$\omega_{rs}b_{rs}\sigma + \frac{1}{8}\gamma_{rsrs}(a^2_{rs} + b^2_{rs})b_{rs} + \frac{1}{2}\hat{P}_{rs}\cos\mu^b_{rs} = 0 \tag{6.36d}$$

where

$$\mu_{rs}^{a} = \sigma T_1 - \alpha_{rs} \text{ and } \mu_{rs}^{b} = \sigma T_1 - \beta_{rs} \ . \qquad (6.37a,b)$$

It follows immediately from equations (6.36) that neither a_{rs} nor b_{rs} can be zero. These equations can be rewritten as

$$\omega_{rs} c_{rs} = (\hat{P}_{rs}/2a_{rs}) \sin \mu_{rs}^{a} \quad , \qquad (6.38a)$$

$$\omega_{rs}\sigma + \frac{1}{8}\gamma_{rsrs}(a_{rs}^2 + b_{rs}^2) = -(\hat{P}_{rs}/2a_{rs}) \cos \mu_{rs}^{a} \qquad (6.38b)$$

$$\omega_{rs} c_{rs} = (\hat{P}_{rs}/2b_{rs}) \sin \mu_{rs}^{b} \quad , \qquad (6.39a)$$

and

$$\omega_{rs}\sigma + \frac{1}{8}\gamma_{rsrs}(a_{rs}^2 + b_{rs}^2) = -(\hat{P}_{rs}/2b_{rs}) \cos \mu_{rs}^{b} \ . \qquad (6.39b)$$

Squaring and adding equations (6.38) and comparing the result with that obtained by squaring and adding equations (6.39), we obtain

$$b_{rs} = a_{rs} \quad \text{and} \quad \mu_{rs}^{b} = \mu_{rs}^{a} \quad . \qquad (6.40a,b)$$

Therefore, using equations (6.37), (6.34) and (6.14), one can write the steady-state response as

$$w = 2\phi_{rs} a_{rs} \cos(\lambda t - \mu_{rs}^{a}) \cos r\theta + O(\varepsilon) \qquad (6.41)$$

Consequently, in the absence of internal resonances, the steady-state forced response consist of standing waves only. One can describe the response with the single mode having a frequency equal to that of the excitation, as several investigators have done previously, the solution being essentially that of the Duffing equation.

6.4 Effects of An Internal Resonance

In this section consideration is given to the effects of an internal resonance invloving four modes [i.e., combination of frequencies given by equation (2.2)] of the form

$$\omega_{CD} + \omega_{NM} + \omega_{PQ} \approx \omega_{KL} \quad . \tag{6.42}$$

Further, we assume that these frequencies are such that equation (6.30c) is satisfied; that is

$$K = C + N + P \tag{6.43}$$

To characterize the approximation in equation (6.42), we introduce a detuning parameter, σ_1, as follows:

$$\omega_{CD} + \omega_{NM} + \omega_{PQ} + \varepsilon\sigma_1 = \omega_{KL} \tag{6.44}$$

The terms due to the internal resonance, $R^{A,B}_{k\ell}$, appearing in the solvability conditions (6.31), which can be obtained by considering Appendix C and equations (6.17), (6.43) and (6.44) are

$$R^A_{KL} = Q_{KL}(A_{CD}A_{NM}A_{PQ} + B_{CD}B_{NM}B_{PQ})\exp(-i\sigma_1 T_1)$$

$$R^A_{PQ} = Q_{PQ}(A_{KL}\bar{A}_{CD}\bar{A}_{NM} + B_{KL}\bar{B}_{CD}\bar{B}_{NM})\exp(i\sigma_1 T_1)$$

$$R^A_{NM} = Q_{NM}(A_{KL}\bar{A}_{PQ}\bar{A}_{CD} + B_{KL,}\bar{B}_{PQ}\bar{B}_{CD})\exp(i\sigma_1 T_1)$$

$$R^A_{CD} = Q_{CD}(A_{KL}\bar{A}_{NM}\bar{A}_{PQ} + B_{KL}\bar{B}_{NM}\bar{B}_{PQ})\exp(i\sigma_1 T_1)$$

and

$$R^A_{k\ell} = 0 \quad \text{for } k\ell \neq KL,\ PQ,\ NM,\ CD\ ,$$

where the Q's are constants. The expressions for $R^B_{k\ell}$ can be obtained from those of $R^A_{k\ell}$ by replacing $A_{k\ell}$ by $\overline{B}_{k\ell}$, $B_{k\ell}$ by $\overline{A}_{k\ell}$ and σ_1 by $-\sigma_1$.

Substituting equations (6.34) and the expressions for $R^{A,B}_{k\ell}$ and $N^{A,B}_{k\ell}$ into equations (6.31) and separating the result into real and imaginary parts leads to the following solvability conditions:

$$\omega_{k\ell}(a'_{k\ell} + c_{k\ell}a_{k\ell}) - \frac{1}{8}Q_{k\ell}S^1_{k\ell} - N_{k\ell}\sin\mu^a_{k\ell} = 0\ , \tag{6.45a}$$

$$\omega_{k\ell}(b'_{k\ell} + c_{k\ell}b_{k\ell}) - \frac{1}{8}Q_{k\ell}S^1_{k\ell} - N_{k\ell}\sin\mu^b_{k\ell} = 0\ , \tag{6.45b}$$

$$\omega_{k\ell}a_{k\ell}\alpha'_k + \frac{1}{8}a_{k\ell}s_{k\ell} + \frac{1}{8}Q_{k\ell}S^2_{k\ell} + N_{k\ell}\cos\mu^a_{k\ell} = 0\ , \tag{6.46a}$$

$$\omega_{k\ell}b_{k\ell}\beta'_{k\ell} + \frac{1}{8}b_{k\ell}s_{k\ell} + \frac{1}{8}Q_{k\ell}S^2_{k\ell} + N_{k\ell}\cos\mu^b_{k\ell} = 0\ , \tag{6.46b}$$

for $k\ell$ = CD, NM, PQ ,

$$\omega_{KL}(a'_{KL} + c_{KL}a_{KL}) + \frac{1}{8}Q_{KL}S^1_{KL} - N_{KL}\sin\mu^a_{KL} = 0\ , \tag{6.47a}$$

$$\omega_{KL}(b'_{KL} + c_{KL}b_{KL}) + \frac{1}{8}Q_{KL}S^1_{KL} - N_{KL}\sin\mu^b_{KL} = 0\ , \tag{6.47b}$$

$$\omega_{KL}a_{KL}\alpha'_{KL} + \frac{1}{8}a_{KL}s_{KL} + \frac{1}{8}Q_{KL}S^2_{KL} + N_{KL}\cos\mu^a_{KL} = 0\ , \tag{6.48a}$$

$$\omega_{KL}b_{KL}\beta'_{KL} + \frac{1}{8}b_{KL}s_{KL} + \frac{1}{8}Q_{KL}S^2_{KL} + N_{KL}\cos\mu^b_{KL} = 0\ , \tag{6.48b}$$

for $k\ell$ = KL and

$$\omega_{k\ell}(a'_{k\ell} + c_{k\ell}a_{k\ell}) - N_{k\ell}\sin\mu^a_{k\ell} = 0\ , \tag{6.49a}$$

$$\omega_{k\ell}(b'_{k\ell} + c_{k\ell}b_{k\ell}) - N_{k\ell}\sin\mu^b_{k\ell} = 0 \tag{6.49b}$$

$$\omega_{k\ell}a_{k\ell}\alpha'_{k\ell} + \frac{1}{8}\, a_{k\ell}s_{k\ell} + N_{k\ell}\cos\, \mu^{a}_{k\ell} = 0\ , \qquad (6.50a)$$

$$\omega_{k\ell}b_{k\ell}\beta'_{k\ell} + \frac{1}{8}\, b_{k\ell}s_{k\ell} + N_{k\ell}\cos\, \mu^{b}_{k\ell} = 0\ , \qquad (6.50b)$$

for $k\ell \neq$ CD, NM, PQ and KL, where

$$s_{k\ell} = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \gamma_{k\ell nm}(a^2_{nm} + b^2_{nm})\ ,$$

$$N_{k\ell} = \frac{1}{2}\, \hat{P}_{k\ell} \text{ when } \lambda \text{ is near } \omega_{k\ell}\ , \qquad (6.51a)$$

$$N_{k\ell} = 0 \text{ when } \lambda \text{ is away from } \omega_{k\ell}\ , \qquad (6.51b)$$

$$\mu^{a}_{k\ell} = \sigma_2 T_1 - \alpha_{k\ell}\ , \qquad (6.52a)$$

$$\mu^{b}_{k\ell} = \sigma_2 T_1 - \beta_{k\ell}\ , \qquad (6.52b)$$

the detuning parameter σ_2 being defined by

$$\lambda = \omega_{k\ell} + \epsilon\sigma_2,$$

when λ is near $\omega_{k\ell}$;

$$S^{1}_{CD} = a_{NM}a_{PQ}a_{KL}\sin\, \tilde{\mu}_A + b_{NM}b_{PQ}b_{KL}\, \sin\, \tilde{\mu}_B,$$

$$S^{2}_{CD} = a_{NM}a_{PQ}a_{KL}\cos\, \tilde{\mu}_A + b_{NM}b_{PQ}b_{KL}\cos\, \tilde{\mu}_B,$$

$$S^{1}_{NM} = a_{PQ}a_{KL}a_{CD}\cos\, \tilde{\mu}_A + b_{PQ}b_{KL}b_{CD}\cos\, \tilde{\mu}_B,$$

$$S^{2}_{NM} = a_{PQ}a_{KL}a_{CD}\cos\, \tilde{\mu}_A + b_{PQ}b_{KL}b_{CD}\cos\, \tilde{\mu}_B,$$

$$S^{1}_{PQ} = a_{KL}a_{CD}a_{NM}\sin\, \tilde{\mu}_A + b_{KL}b_{CD}b_{NM}\sin\, \tilde{\mu}_B,$$

$$S^{2}_{PQ} = a_{KL}a_{CD}a_{NM}\cos\, \tilde{\mu}_A + b_{KL}b_{CD}b_{NM}\cos\, \tilde{\mu}_B,$$

$$S^1_{KL} = a_{CD}a_{NM}a_{PQ}\sin\tilde{\mu}_A + b_{CD}b_{NM}b_{PQ}\sin\tilde{\mu}_B ,$$

$$S^2_{KL} = a_{CD}a_{NM}a_{PQ}\cos\tilde{\mu}_A + b_{CD}b_{NM}b_{PQ}\cos\tilde{\mu}_B ,$$

$$\tilde{\mu}_A = \sigma_1 T_1 - \alpha_{CD} - \alpha_{NM} - \alpha_{PQ} + \alpha_{KL}$$

and

$$\tilde{\mu}_B = \sigma_1 T_1 - \beta_{CD} - \beta_{NM} - \beta_{PQ} + \beta_{KL} .$$

For a steady-state solution, all $a_{k\ell}$, $b_{k\ell}$, $\tilde{\mu}_A$, $\tilde{\mu}_B$, $\mu^a_{k\ell}$ and $\mu^b_{k\ell}$ are constants. This leads to

$$\omega_{k\ell}c_{k\ell}a_{k\ell} - \frac{1}{8}Q_{k\ell}S^1_{k\ell} - N_{k\ell}\sin\mu^a_{k\ell} = 0 \tag{6.53a}$$

$$\omega_{k\ell}c_{k\ell}b_{k\ell} - \frac{1}{8}Q_{k\ell}S^1_{k\ell} - N_{k\ell}\sin\mu^b_{k\ell} = 0 \tag{6.53b}$$

for $k\ell$ = CD, NM, and PQ,

$$\omega_{KL}c_{KL}a_{KL} + \frac{1}{8}Q_{KL}S^1_{KL} - N_{KL}\sin\mu^a_{KL} = 0 , \tag{6.54a}$$

$$\omega_{KL}c_{KL}b_{KL} + \frac{1}{8}Q_{KL}S^1_{KL} - N_{KL}\sin\mu^b_{KL} = 0 , \tag{6.54b}$$

for $k\ell$ = KL, and

$$\omega_{k\ell}c_{k\ell}a_{k\ell} - N_{k\ell}\sin\mu^a_{k\ell} = 0 , \tag{6.55a}$$

$$\omega_{k\ell}c_{k\ell}b_{k\ell} - N_{k\ell}\cos\mu^b_{k\ell} = 0 , \tag{6.55b}$$

for $k\ell \neq$ CD, NM, PQ and KL ,

$$\tilde{\mu}'_A = \sigma_1 - \alpha'_{CD} - \alpha'_{NM} - \alpha'_{PQ} + \alpha'_{KL} = 0 , \tag{6.56a}$$

$$\tilde{\mu}'_B = \sigma_1 - \beta'_{CD} - \beta'_{NM} - \beta'_{PQ} + \beta'_{KL} = 0 , \tag{6.56b}$$

where $\alpha'_{k\ell}$ and $\beta'_{k\ell}$ are given by equations (6.46) for $k\ell$ = CD, NM and PQ and by equations (6.48) for $k\ell$ = KL,

$$\mu^{a'}_{rs} = \sigma_2 - \alpha'_{rs} = 0 \quad , \tag{6.57a}$$

and

$$\mu^{b'}_{rs} = \sigma_2 - \beta'_{rs} = 0 \tag{6.57b}$$

when λ is near ω_{rs}.

When λ is near ω_{rs}, it follows from equations (6.53) - (6.55) and (6.51) that

$$a_{k\ell} = b_{k\ell}, \text{ for } k\ell \neq rs \tag{6.58a}$$

and from equations (6.46), (6.48), (6.50) and (6.51) that

$$\alpha'_{k} = \beta'_{k} \quad , \text{ for } k\ell \neq rs \tag{6.58b}$$

It is noted that equations (6.45) - (6.50) are analogous to equations (4.3) - (4.5). Also equations (6.53) - (6.55) are analogous to equations (4.8) and (4.9). Recalling that in this chapter the interest is in main resonances only, we note that the results of Chapter 4 that are applicable to main resonances are relevant here. Different possibilities are considered next.

a. The Case of λ Near $\omega_{k\ell}$, $k\ell \neq$ CD, NM, PQ, KL

From equations (6.58) we obtain

$$a_{k\ell} = b_{k\ell} \quad \text{for } k\ell = \text{CD, NM, PQ and KL} \quad .$$

It follows from the results of Section 4.2.a (i.e., internal resonance has no influence on the response), the solution is governed by equations (6.49) and (6.50). These equations are identical in structure to equations (6.36). Hence the steady-state response is a standing wave of the form

$$w = 2\phi_{k\ell}a_{k\ell}\cos(\lambda t - \mu^{a}_{k\ell})\cos k\theta + O(\varepsilon) \; . \tag{6.59}$$

b. <u>The Case of λ Near ω_{CD}</u>

From equations (6.58) we obtain

$$a_{k\ell} = b_{k\ell} \quad \text{for } k\ell = NM, PQ \text{ and } KL \; .$$

It follows from the results of Sections 4.2b (i.e. the internal resonance has no influence on the response) that

$$S^1_{CD} = S^2_{CD} = 0 \; . \tag{6.60}$$

Substituting equation (6.60) into (6.45) and (6.46), one obtains the equations governing the solution. These equations are identical in structure to equations (6.36) and hence the steady-state response is a standing wave of the form

$$w = 2\phi_{CD}a_{CD}\cos(\lambda t - \mu^{a}_{CD})\cos C\theta + O(\varepsilon) \; . \tag{6.61}$$

Similar results are obtained for the cases of λ near ω_{NM} and λ near ω_{PQ}.

c. <u>The Case of λ Near ω_{KL}</u>

From equations (6.58) we obtain

$$a_{k\ell} = b_{k\ell} \quad \text{for } k = CD, NM \text{ and } PQ \ .$$

It follows from the results of Section 4.2.e that there are two possibilities as follows:

(1) <u>$a_{CD} = a_{NM} = a_{PQ} = 0$</u>

Therefore,

$$S^1_{KL} = S^2_{KL} = 0 \qquad (6.62)$$

Substituting equation (6.62) into (6.47) and (6.48), we obtain the equations governing the solution. Again these equations are identical in structure to equations (6.36); that is, the steady-state response is a standing wave of the form

$$w = 2\phi_{KL} a_{KL} \cos(\lambda t - \mu^a_{KL}) \cos K\theta + O(\varepsilon) \ . \qquad (6.63)$$

(2) <u>a_{CD}, a_{NM} and a_{PQ} are nonzero</u>

Here one cannot arrive at the result given by equation (6.62) and hence conclude that the only possible steady-state response is a standing wave. The highest mode involved in the internal resonance can appear in the response either as a standing wave (i.e., $a_{KL} = b_{KL}$) of the form

$$2\phi_{KL} a_{KL} \cos(\lambda t - \mu^a_{KL}) \cos K\theta$$

or as a travelling wave (i.e., $a_{KL} \neq b_{KL}$) of the form

$$\phi_{KL}[a_{KL} \cos(\lambda t - \mu^a_{KL} + K\theta) + b_{KL} \cos(\lambda t - \mu^b_{KL} - K\theta)] .$$

Thus, the steady-state response is described by either

(1) a superposition of the standing wave components of all the modes involved in the internal resonance

or

(2) a superposition of the standing wave components of all the lower modes and the travelling wave component of the highest mode in the internal resonance.

6.5 Summary

The usefulness of the method of analysis developed in Chapters 2, 3 and 4 is demonstrated by an application to the study of the nonlinear vibrations of a clamped circular plate. The general problem, including asymmetric vibrations and travelling waves, is a difficult exercise in analysis and the present approach is shown to provide a great deal of clarity and insight into the nature of the nonlinear forced response. The effects of an internal resonance involving four modes are evaluated.

The steady-state forced response, in the first approximation, for the case of main resonances exhibits the following features:

(1) In the absence of internal resonances or when the frequency of excitation is near one of the lower modes involved in the

internal resonance, the steady-state response can only have the form of a standing wave.

(2) When the frequency of the excitation is near the highest mode involved in the internal resonance, the steady-state response is given by one of two forms given below.

(a) a superposition of the standing wave components of all the modes involved in the internal resonance

(b) a superposition of the standing wave components of all the lower modes and the travelling wave component of the highest mode involved in the internal resonance.

7. NONLINEAR VIBRATIONS OF CLAMPED CIRCULAR PLATES - SYMMETRIC RESPONSES

In this chapter, the general analysis of Chapter 6 is specialized to the case of symmetric responses. In order to clearly exhibit the far-reaching effects of an internal resonance in the system, numerical examples involving main resonances are presented.

7.1 Problem Formulation and Solution

For the symmetric responses of circular plates, the non-dimensional governing equations (6.4) of Chapter 6 reduce to

$$\frac{\partial^2 w}{\partial t^2} + \nabla^4 w = \varepsilon\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial r}\frac{\partial F}{\partial r}\right) - 2c\,\frac{\partial w}{\partial t} + p\right] \quad , \qquad (7.1a)$$

and

$$\nabla^4 F = -\frac{1}{2r}\frac{\partial}{\partial r}\left[\left(\frac{\partial w}{\partial r}\right)^2\right] \quad , \qquad (7.1b)$$

where $\varepsilon = 12(1-\nu^2)h^2/a^2$ and $\nabla^4 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)^2$.

The boundary conditions (6.6a) and (6.7) reduce to

$$w = 0 \quad , \quad \frac{\partial w}{\partial r} = 0 \quad , \qquad (7.1c)$$

$$\frac{\partial^2 F}{\partial r^2} - \frac{\nu}{r}\frac{\partial F}{\partial r} = 0 \quad , \qquad (7.1d)$$

and

$$\frac{\partial^3 F}{\partial r^3} + \frac{1}{r}\frac{\partial^2 F}{\partial r^2} - \frac{1}{r^2}\frac{\partial F}{\partial r} = 0 \qquad (7.1e)$$

for all t at r = a. In additon the solution must be bounded at r = 0.

Because of the nature of the responses (i.e., symmetric) under consideration the possibility of travelling-wave solutions is eliminated. It is possible to reduce the problem defined by equations (7.1) exactly to the system given by equations (2.1) by a straightforward separation of the spatial and temporal variables. However, in this chapter, we choose to specialize the analysis of Chapter 6.

Following the procedure in Section (6.2), we obtain the specialized versions of equations (6.9) - (6.13) as

$$D_0^2 w_0 + \nabla^4 w_0 = 0 , \tag{7.2}$$

$$\nabla^4 F_0 = -\frac{1}{2r}\frac{\partial}{\partial r}\left[\left(\frac{\partial w_0}{\partial r}\right)^2\right] \tag{7.3}$$

$$D_0^2 w_1 + \nabla^4 w_1 = -2D_0 D_1 w_0 - 2cD_0 w_0 + p + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial F_0}{\partial r}\frac{\partial w_0}{\partial r}\right) \tag{7.4}$$

etc.,

$$w_j = 0 , \quad \frac{\partial w_j}{\partial r} = 0 \quad \text{at } r = 1 , \tag{7.5a,b}$$

$$\frac{\partial^2 F_j}{\partial r^2} - \nu \frac{\partial F_j}{\partial r} = 0 , \quad \frac{\partial^3 F_j}{\partial r^3} + \frac{\partial^2 F_j}{\partial r^2} - \frac{\partial F_j}{\partial r} = 0 \quad \text{at } r = 1 . \tag{7.6a,b}$$

In addition, one must require the solution to be bounded at $r = 0$.

The solution of equations (7.2) and (7.5) can be written as

$$w_0 = \sum_{m=1}^{\infty} \phi_m(r) u_m(T_0, T_1, \ldots) \tag{7.7}$$

where

$$u_m = A_m \exp(i\omega_m T_0) + cc , \tag{7.8a}$$

$$\phi_m = \kappa_m\left[J_0(\eta_m r) - \frac{J_0(\eta_m)}{I_0(\eta_m)} I_0(\eta_m r)\right] , \tag{7.8b}$$

κ_m is chosen so that

$$\int_0^1 r\phi_m^2 \, dr = 1 \quad , \tag{7.8c}$$

η_m are the roots of

$$I_0(\eta_m)J_0'(\eta_m) - I_0'(\eta_m)J_0(\eta_m) = 0 \quad , \tag{7.8d}$$

$$\omega_m = \eta_m^2 \quad ,$$

the A_m are unknown complex functions of all T_n for $n \geq 1$ at this point, but they are to be determined from the solvability conditions at higher levels of approximations; and the cc represents the complex conjugate of the preceding terms. The ϕ_m are the linear, free-oscillation modes, and the ω_m are the natural frequencies.

Equations (7.3) - (7.6) suggest that it may be more convenient to solve for $\frac{\partial F_0}{\partial r}$ instead of F_0. Such an analysis is indeed possible and can be found in reference [31]. However, we again choose to specialize the analysis of Chapter 6.

Equations (6.18) becomes

$$\nabla^4 F_0 = \sum_{m,q=1}^{\infty} E(m,q) u_{nm} u_{pq} \tag{7.9}$$

where

$$E(m,q) = -\frac{1}{2r}(\phi_m' \phi_q')' \quad .$$

We further assume an expansion for F of the form

$$F_0 = \sum_{n=1}^{\infty} v_n(T_0, T_1, \ldots)\psi_n(r) \tag{7.10}$$

where ψ_n are the eigenfunctions of the following problem:

$$(\nabla^4 - \xi_n^4)\psi_n = 0 \quad \text{on } r = [0,1]$$

where ψ_n is bounded at $r = 0$ and from equations (7.6)

$$\psi_n'' - \nu\psi_n' = 0$$

and

$$\psi_n''' + \psi_n'' - \psi_n' = 0$$

for all t at $r = 1$. It follows that

$$\psi_n = \tilde{\kappa}_n[J_0(\xi_n r) - \tilde{c}_n I_0(\xi_n r)] \tag{7.11a}$$

where $\tilde{\kappa}_n$ are chosen so that

$$\int_0^1 r\psi_n^2 \, dr = 1 \ ,$$

$$\tilde{c}_n = \frac{-\xi_n J_0(\xi_n) + (\nu+1)J_1(\xi_n)}{\xi_n I_0(\xi_n) - (\nu+1)I_1(\xi_n)} \ , \tag{7.11b}$$

and ξ_n are the roots of

$$J_1(\xi_n) - \tilde{c}_n I_1(\xi_n) = 0 \ . \tag{7.11c}$$

The first twelve roots of equation (7.11c) are listed in Appendix D.

Equation (6.26) reduces to

$$F_0 = \sum_{b,m,q=1}^{\infty} \psi_b G(m,q;b) u_m u_q \tag{7.12}$$

where

$$G(m,q;b) = \xi_b^{-4} \int_0^1 r\psi_b \, E(m,q) dr \ .$$

Substituting equations (7.12) and (7.7) into (7.4) leads to

$$D_0^2 w_1 + \nabla^4 w_1 = \sum_{m=1}^{\infty} - 2i\omega_m \phi_m [(D_1 A_m + c_m A_m) \exp(i\omega_m T_0)$$
$$- (D_1 \bar{A}_m + c_m \bar{A}_m) \exp(-i\omega_m T_0)] + \sum_{m=1}^{\infty} P_m \phi_m \cos\lambda T_0$$
$$+ \sum_{b,m,d,q=1}^{\infty} G(m,q;b) \hat{E}(d,b)\ u_d u_m u_q \tag{7.13}$$

where

$$\hat{E}(d,b) = \frac{1}{r}(\phi_d' \psi_b')' \quad .$$

Assuming an expansion for w_1 of the form

$$w_1 = \sum_{m=1}^{\infty} \tilde{u}_m(T_0, T_1, \ldots)\phi_m(r) \ , \tag{7.14}$$

substituting into equation (7.13), multiplying by $r\phi_\ell$ and integrating the result from $r = 0$ to 1, we obtain

$$D_0^2 \tilde{u}_\ell + \omega_\ell^2 \tilde{u}_\ell = - 2i\omega_\ell (D_1 A_\ell + c_\ell A_\ell) \exp(i\omega_\ell T_0) + \frac{1}{2} \hat{P}_\ell \exp(i\lambda T_0)$$
$$+ \sum_{d,m,q=1}^{\infty} \Gamma_{\ell dmq} \left[\sum_{j=1}^{4} S_j \exp(i\Lambda_j T_0)\right] + cc \ , \quad \ell = 1, 2, \ldots \tag{7.15}$$

where

$$\Gamma_{\ell dmq} = \sum_{b=1}^{\infty} G(m,q;b) \int_0^1 r\phi_\ell \hat{E}(d,b) dr, \tag{7.16}$$

and S_j and Λ_j are listed in Appendix A. (Note that the subscripts have been redefined.)

We note that equations (7.15) are the specialized forms of equations (6.29) and are precisely of the same form as equations (2.8).

7.2 Numerical Examples

For a clamped circular plate, the first five natural frequencies (see, for example, Leissa [37]) are

$$\omega_1 = 10.2158, \quad \omega_2 = 39.7710, \quad \omega_3 = 89.1040,$$
$$\omega_4 = 158.1830, \quad \text{and} \quad \omega_5 = 247.0050.$$

An internal resonance exists because

$$\omega_1 + 2\omega_2 = 89.7578 \approx \omega_3. \tag{7.17a}$$

To express the nearness of this approximation quantitatively, we introduce a detuning parameter, σ_1, defined as follows:

$$\omega_1 + 2\omega_2 - \omega_3 = 0.6538 = \varepsilon\sigma_1 \quad . \tag{7.17b}$$

It is noted that equation (7.17a) is a special case of equation (2.2) with a = 1, b = c = 2 and d = 3. Thus, this is the case of an internal resonance involving three modes.

The equations governing the amplitudes and phases of the response are obtained either by specializing equations (6.45) - (6.50) to the case of symmetric responses or from equations (4.3) - (4.5). They can be written as

$$\omega_1(a_1' + c_1 a_1) + \frac{1}{8} Q_1 a_2^2 a_3 \sin\beta - \frac{1}{2}\delta_{1N}\hat{P}_1 \sin\mu_1 = 0, \tag{7.18a}$$

$$\omega_1 a_1' \alpha_1' + \frac{1}{8}(\gamma_{11} a_1^3 + \gamma_{12} a_1 a_2^2 + \gamma_{13} a_1 a_3^2 + Q_1 a_2^2 a_3 \cos\beta$$
$$+ a_1 \sum_{j=4}^{\infty} \gamma_{1j}\, a_j^2) + \frac{1}{2}\delta_{1N}\hat{P}_1 \cos\mu_1 = 0 \; ; \tag{7.18b}$$

for k = 1,

$$\omega_2(a_2' + c_2a_2) + \frac{1}{8}\,Q_2a_1a_2a_3 \sin\beta - \frac{1}{2}\,\delta_{2N}\hat{P}_2 \sin\mu_2 = 0 \,, \tag{7.19a}$$

$$\omega_2a_2\alpha_2' + \frac{1}{8}(\gamma_{22}a_2^3 + \gamma_{21}a_2a_1^2 + \gamma_{23}a_2a_3^2 + Q_2a_1a_2a_3 \cos\beta$$

$$+ a_2 \sum_{j=4}^{\infty} \gamma_{2j}a_j^2) + \frac{1}{2}\,\delta_{2N}\hat{P}_2 \cos\mu_2 = 0 \,; \tag{7.19b}$$

for k = 2,

$$\omega_3(a_3' + c_3a_3) - \frac{1}{8}\,Q_3a_1a_2^2 \sin\beta - \frac{1}{2}\,\delta_{3N}\hat{P}_3 \sin\mu_3 = 0, \tag{7.20a}$$

$$\omega_3a_3\alpha_3' + \frac{1}{8}(\gamma_{33}a_3^3 + \gamma_{31}a_1^2a_3 + \gamma_{32}a_2^2a_3 + Q_3a_1a_2^2 \cos\beta$$

$$+ a_3 \sum_{j=4}^{\infty} \gamma_{3j}a_j^2) + \frac{1}{2}\,\delta_{3N}\hat{P}_3 \cos\mu_3 = 0 \,; \tag{7.20b}$$

for k = 3,

$$\omega_k(a_k' + c_ka_k) - \frac{1}{2}\,\delta_{kN}\hat{P}_k \sin\mu_k = 0 \,, \tag{7.21a}$$

$$\omega_ka_k\alpha_k' + \frac{1}{8}\,a_k \sum_{j=1}^{\infty} \gamma_{kj}\,a_j^2 + \frac{1}{2}\,\delta_{kN}\hat{P}_k \cos\mu_k = 0 \tag{7.21b}$$

for k > 3; where

$$\mu_n = \sigma_2T_1 - \alpha_n \quad , \tag{7.22a}$$

$$\beta = \sigma_1T_1 + \alpha_1 - 2\alpha_2 - \alpha_3, \tag{7.22b}$$

δ_{ij} is the Kronecker delta and σ_2 is a second detuning parameter defined as follows:

$$\lambda = \omega_N + \varepsilon\sigma_2 \quad . \tag{7.22c}$$

For the steady-state solution, all the a_n, β and the appropriate μ_n are constants. As in Chapter 5, the steady-state equations are in general nonlinear and were solved by a Newton-Raphson procedure. The stability of the various solutions was determined by the method in Section 3.6. Numerical results are presented for the main resonances of the different modes involved in the internal resonance. For convenience, all the calculations were made with $c_1 = c_2 = c_3 = c$. Values of the constant coefficients appearing in equations (7.18) - (7.20) are listed in Appendix D.

a. The Case of λ Not Near ω_1, ω_2 or ω_3 $(N > 3)$

In this case none of the modes involved in the internal resonance are directly excited. It follows from the results of Sections 4.2.a and 3.2 that the steady-state response is given by

$$u_n = O(\varepsilon) \ , \ n \neq k \tag{7.23a}$$

and

$$u_k = a_k \cos (\lambda t - \mu_k) + O(\varepsilon) \ . \tag{7.23b}$$

b. The Case of λ Near $\omega_1 (N = 1)$

Here one of the lower modes involved in the internal resonance is directly excited. The results of Section 4.2.b and 3.2 are applicable, so that the steady-state response is given by

$$u_n = O(\varepsilon) \quad , \quad n \neq 1 \tag{7.24a}$$

and

$$u_1 = a_1 \cos(\lambda t - \mu_1) + O(\varepsilon) . \qquad (7.24b)$$

For some arbitrary values of the amplitude of excitation and the damping coefficient, the amplitude, a_1, is plotted as a function of the detuning of the excitation, $\varepsilon\sigma_2$, in Figure 12.

c. The Case of λ Near $\omega_2 (N = 2)$

This case is similar to the previous case of λ near ω_1 and the steady-state response is given by

$$u_n = O(\varepsilon) , \quad n \neq 2 \qquad (7.25a)$$

and

$$u_2 = a_2 \cos(\lambda t - \mu_2) + O(\varepsilon). \qquad (7.25b)$$

The amplitude, a_2, is plotted as a function of the detuning of the excitation, $\varepsilon\sigma_2$, in Figure 13.

d. The Case of λ Near $\omega_3 (N = 3)$

Here the highest mode involved in the internal resonance is directly excited. The results of Section 4.2.e that are applicable to main resonances are relevant in this case. There are two possiblities as follows:

(1) $a_3 \neq 0$, and $a_1 = a_2 = 0$

The steady-state response is given by

$$u_n = O(\varepsilon) \quad , n \neq 3 \qquad (7.26a)$$

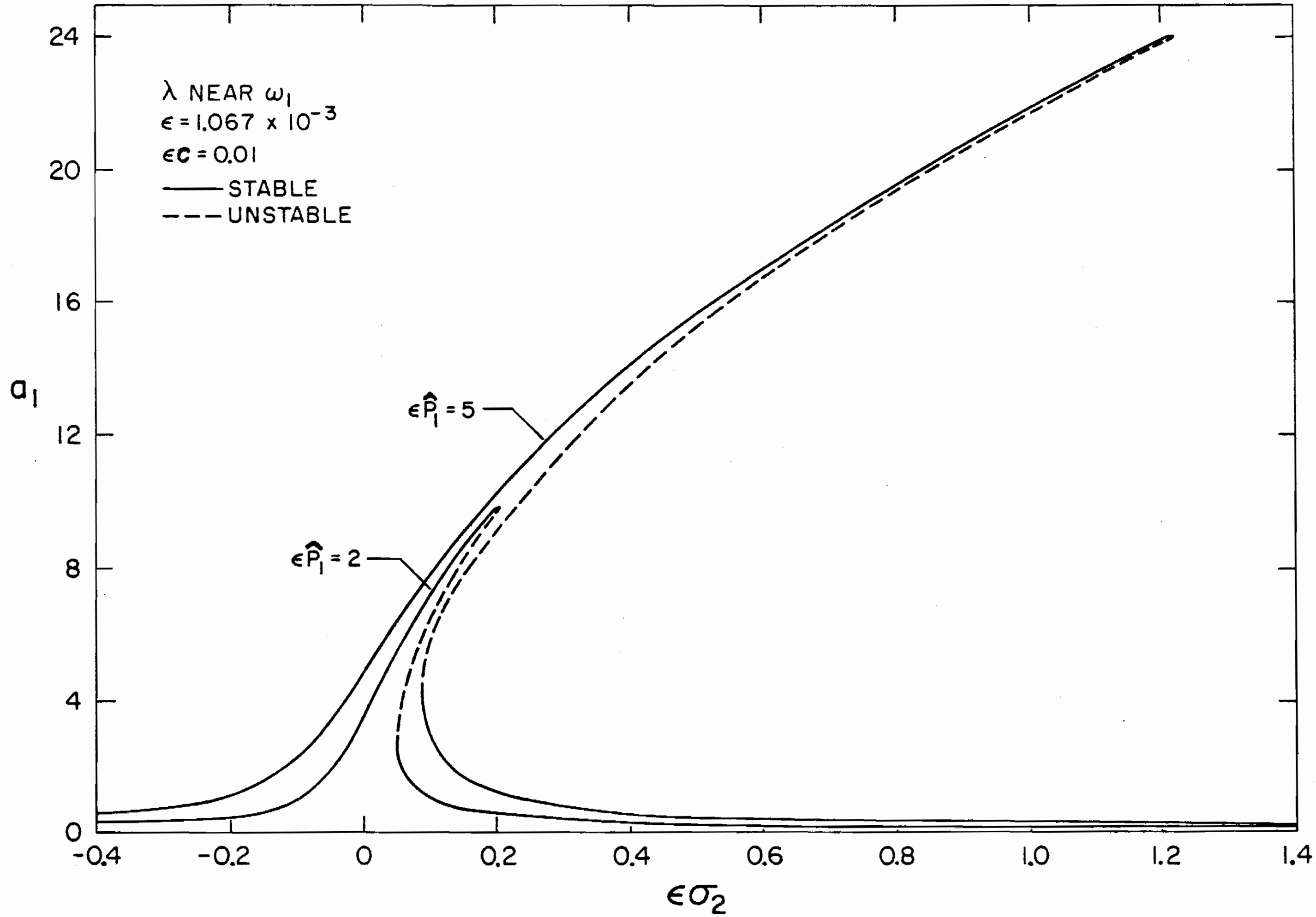

Figure 12 Variation of First Mode Amplitude with Detuning of Excitation

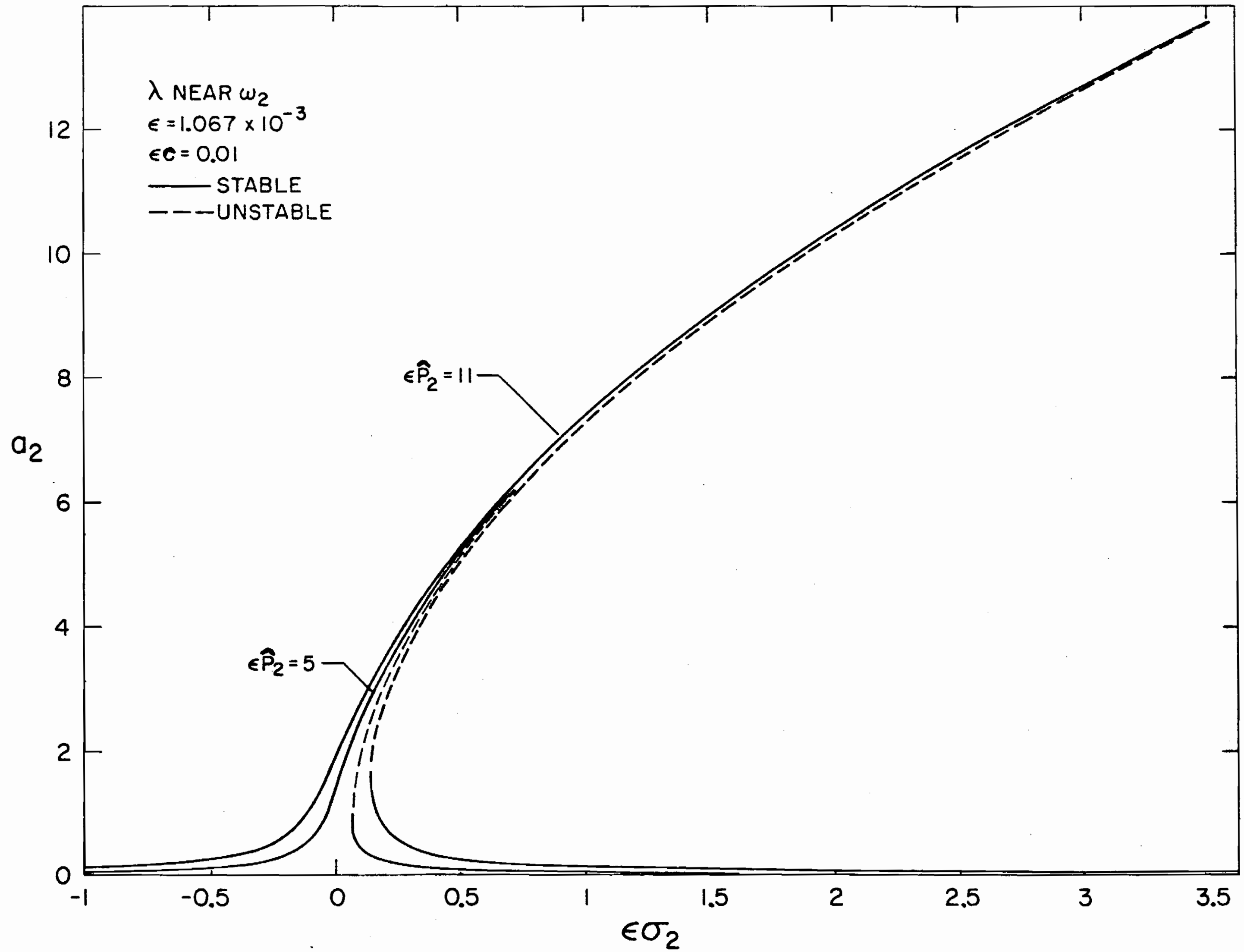

Figure 13 Variation of Second Mode Amplitude with Detuning of Excitation

and

$$u_3 = a \cos(\lambda t - \mu_3) + O(\varepsilon) . \tag{7.26b}$$

The amplitude, a_3, is plotted as a function of the detuning of the excitation, $\varepsilon\sigma_2$, in Figure 14a and as a function of the amplitude of the excitation, $\varepsilon\hat{P}_3$, in Figure 14b.

The results for a_1, a_2 and a_3, as given in Figures 12 - 14b, are similar to the solution of the Duffing equation for a 'hardening' type of nonlinearity. These figures illustrate the phenomena of 'jumps' [see Figures 2a and 3a] as well as the fact that when the amplitude of the excitation becomes small the solution approaches the solution of the linear problem.

(2) <u>a_1, a_2, a_3 are nonzero</u>

In this case, the solution has the following form:

$$u_n = O(\varepsilon) \quad , n > 3 \tag{7.27a}$$

$$u_1 = a_1 \cos[(\omega_1 + \varepsilon\alpha_1')t + \tau_1] + O(\varepsilon) , \tag{7.27b}$$

$$u_2 = a_2 \cos[(\omega_2 + \varepsilon\alpha_2')t + \tau_2] + O(\varepsilon) , \tag{7.27c}$$

and

$$u_3 = a_3 \cos(\lambda t - \mu_3) + O(\varepsilon) . \tag{7.27d}$$

In this case, a_1, a_2, a_3, α_1', α_2', and μ_3 are obtained as follows. One finds from equation (7.22a) that

$$\alpha_3' = \sigma_2 \tag{7.28a}$$

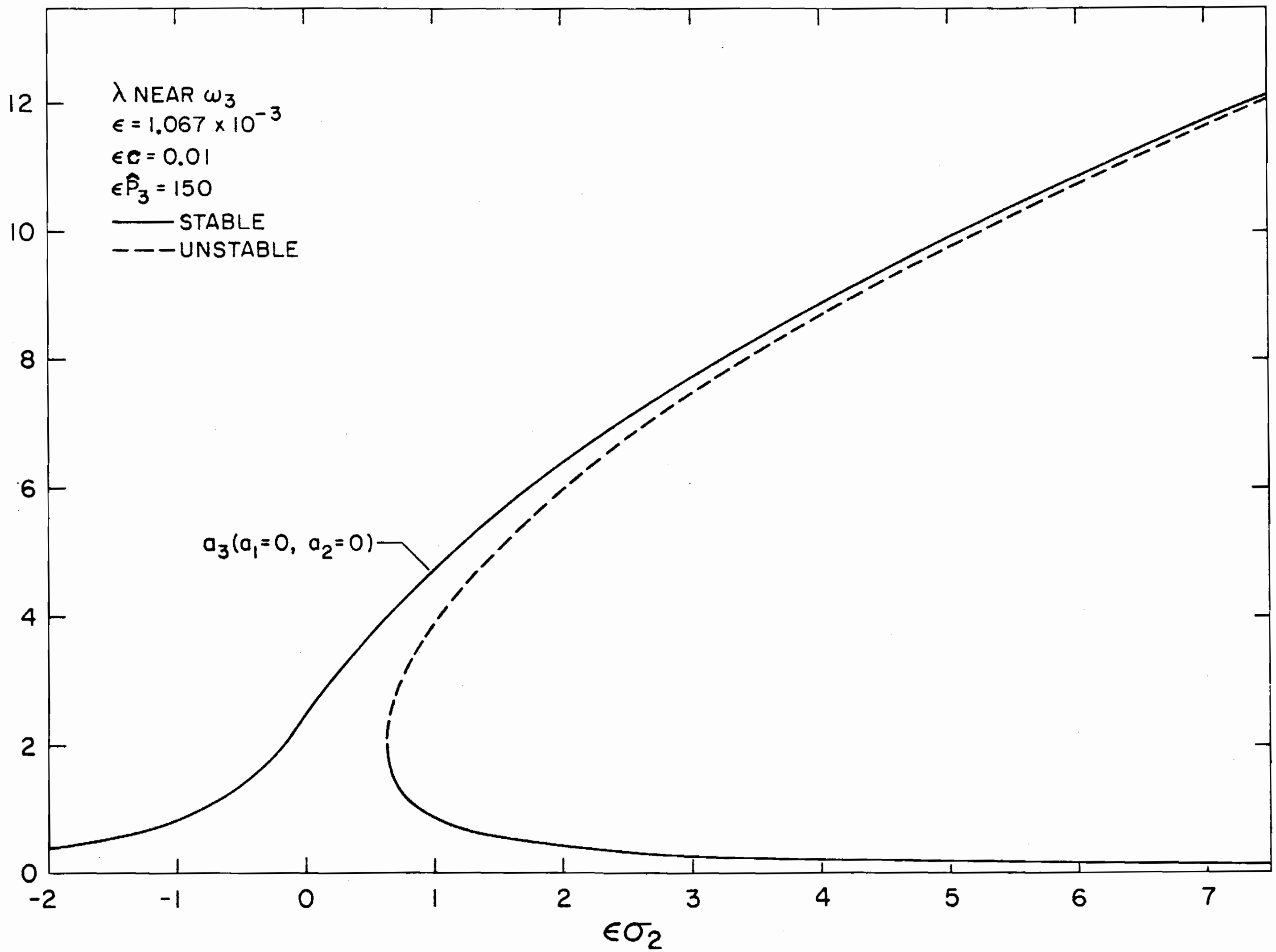

Figure 14a Variation of Third Mode Amplitude with Detuning of Excitation

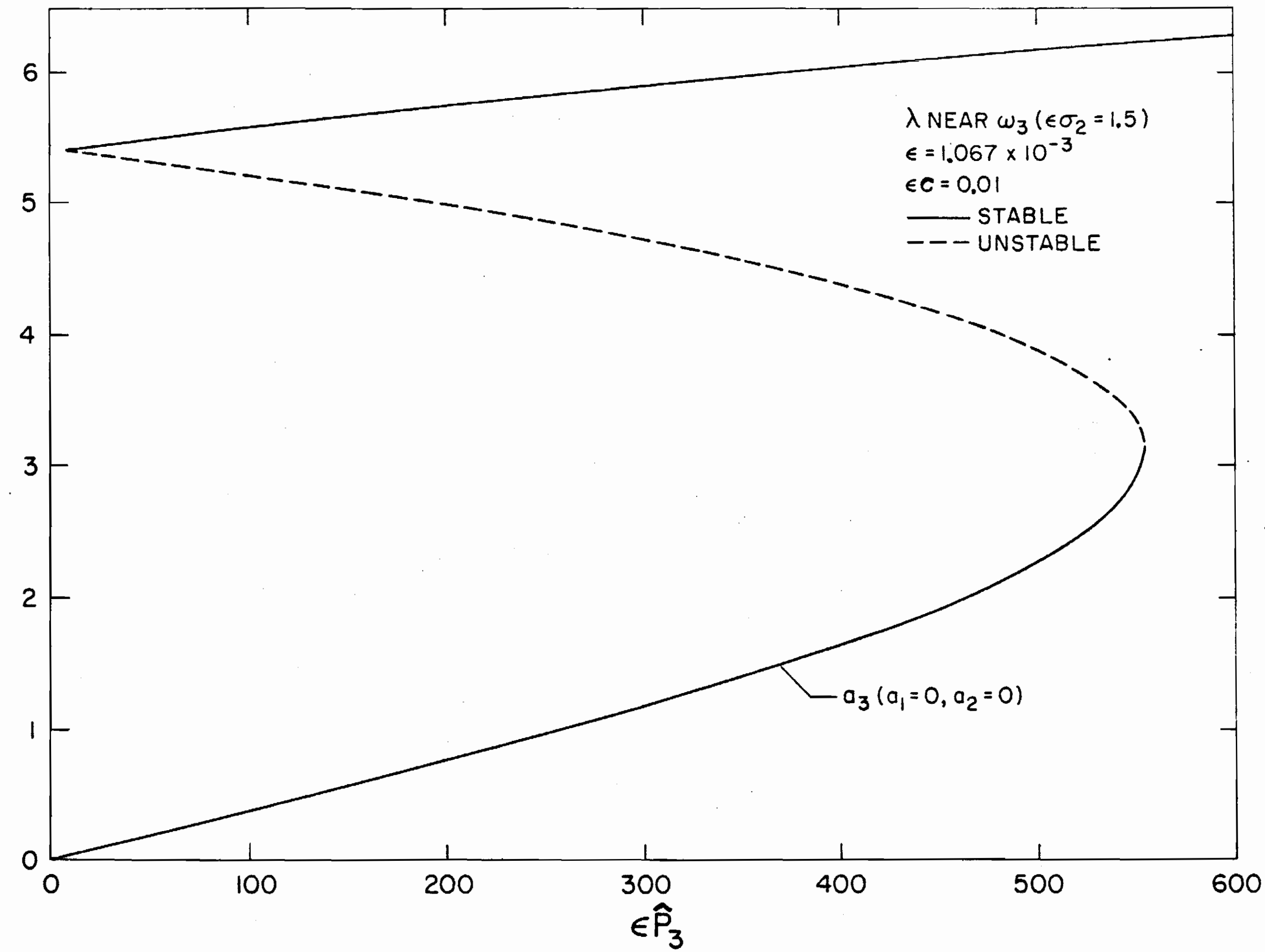

Figure 14b Variation of Third Mode Amplitude with Amplitude of Excitation

and hence, from equation (7.22b), that

$$\sigma_2 - \sigma_1 = \alpha_1' + 2\alpha_2' \quad . \tag{7.28b}$$

Then one can combine equations (7.18) - (7.20) to obtain

$$c_1 a_1 + \frac{Q_1}{8\omega_1} a_2^2 a_3 \sin\beta = 0 \, , \tag{7.29a}$$

$$c_2 a_2 + \frac{Q_2}{8\omega_2} a_1 a_2 a_3 \sin\beta = 0 \, , \tag{7.29b}$$

$$c_3 a_3 - \frac{Q_3}{8\omega_3} a_1 a_2^2 \sin\beta - \frac{\hat{P}_3}{2\omega_3} \sin\mu_3 = 0 \, , \tag{7.29c}$$

$$\sigma_2 a_3 + \frac{1}{8\omega_3} (\Upsilon_{31} a_3 a_1^2 + \Upsilon_{32} a_3 a_2^2 + \Upsilon_{33} a_3^3 + Q_3 a_1 a_2^2 \cos\beta)$$

$$+ \frac{\hat{P}_3}{2\omega_3} \cos\mu_3 = 0 \, , \tag{7.29d}$$

and

$$\sigma_2 - \sigma_1 + \frac{1}{8} [(\frac{\Upsilon_{11}}{\omega_1} + \frac{2\Upsilon_{21}}{\omega_2}] a_1^2 + (\frac{\Upsilon_{12}}{\omega_1} + \frac{2\Upsilon_{22}}{\omega_2}) a_2^2 + (\frac{\Upsilon_{13}}{\omega_1} + \frac{2\Upsilon_{23}}{\omega_2}) a_3^2$$

$$+ (\frac{Q_1}{\omega_1} \frac{a_2^2 a_3}{a_1} + \frac{2Q_2}{\omega_2} a_1 a_3) \cos\beta] = 0 \, . \tag{7.29e}$$

One can solve these equations numerically for a_1, a_2, a_3, β and μ_3. After this is done, one can obtain α_1' and α_2' from equations (7.18b) and (7.19b); α_1 and α_2, and hence the constants τ_1 and τ_2 appearing in equations (7.27), cannot be determined from this analysis. This phasing depends on the initial conditions.

The nonlinearity adjusts the frequencies so that the third mode is tuned to the forcing frequency exactly and the frequencies of the three modes involved in the internal resonance are exactly commensurable. To explain this, we recall equations (7.28) and note the following:

$$\omega_1 + \varepsilon\alpha_1' + 2(\omega_2 + \varepsilon\alpha_2') = \omega_1 + 2\omega_2 + \varepsilon(\sigma_2 - \sigma_1) = \lambda$$

The amplitudes a_1, a_2, and a_3 are plotted as functions of the detuning of the excitation, $\varepsilon\sigma_2$, in Figure 15a and as functions of $\varepsilon\hat{P}_3$ in Figure 15b. Generally, when a_1 and a_2 are not zero, a_1 is larger than a_2 and a_2 is much larger than a_3. Consequently, the solution is dominated by the first mode.

Figure 16 illustrates that decreasing the damping coefficient has the effect of extending the range of frequency for which a_1 and a_2 can be different from zero. Figure 17 illustrates that increasing the amplitude of the excitation has the same effect as decreasing the damping coefficients. In these two figures, only the stable branches of the solution are plotted.

Finally figure 18 is a log-log plot of the amplitude of the excitation as a function of the amplitude of the response. For this graph, λ is exactly ω_3; consequently, it is not possible for a_1 and a_2 to differ from zero. This graph illustrates that the present results are qualitatively in agreement with the experimental data obtained by Jacobson and van der Heyde [34], though they experimented with honeycomb panels.

7.3 Summary

By specializing the general results of Chapter 6, one obtains the nonlinear resonant symmetric response of a clamped circular plate. The lowest three natural frequencies of such a plate are involved in

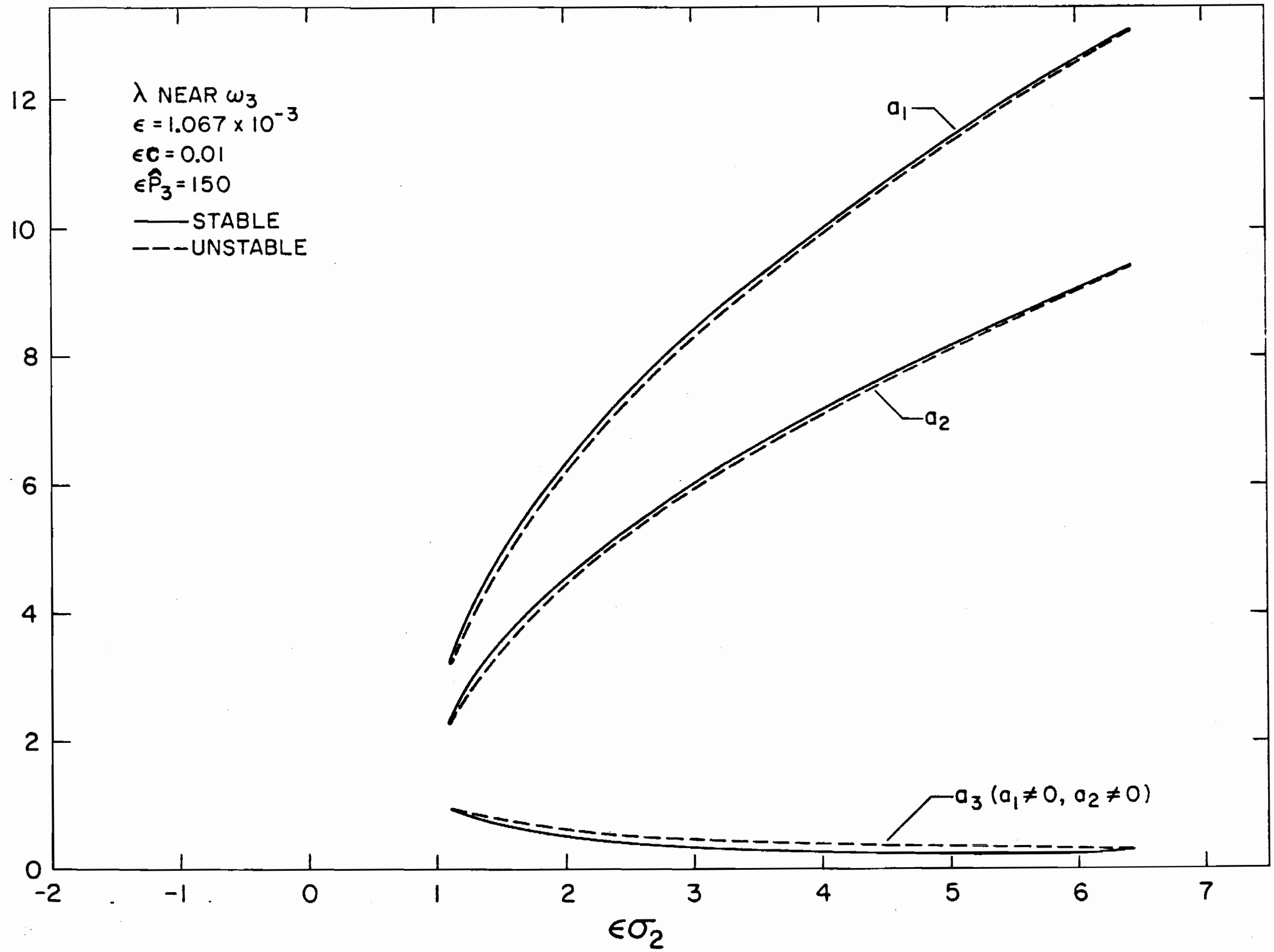

Figure 15a Variation of First, Second and Third Mode Amplitudes with Detuning of Excitation

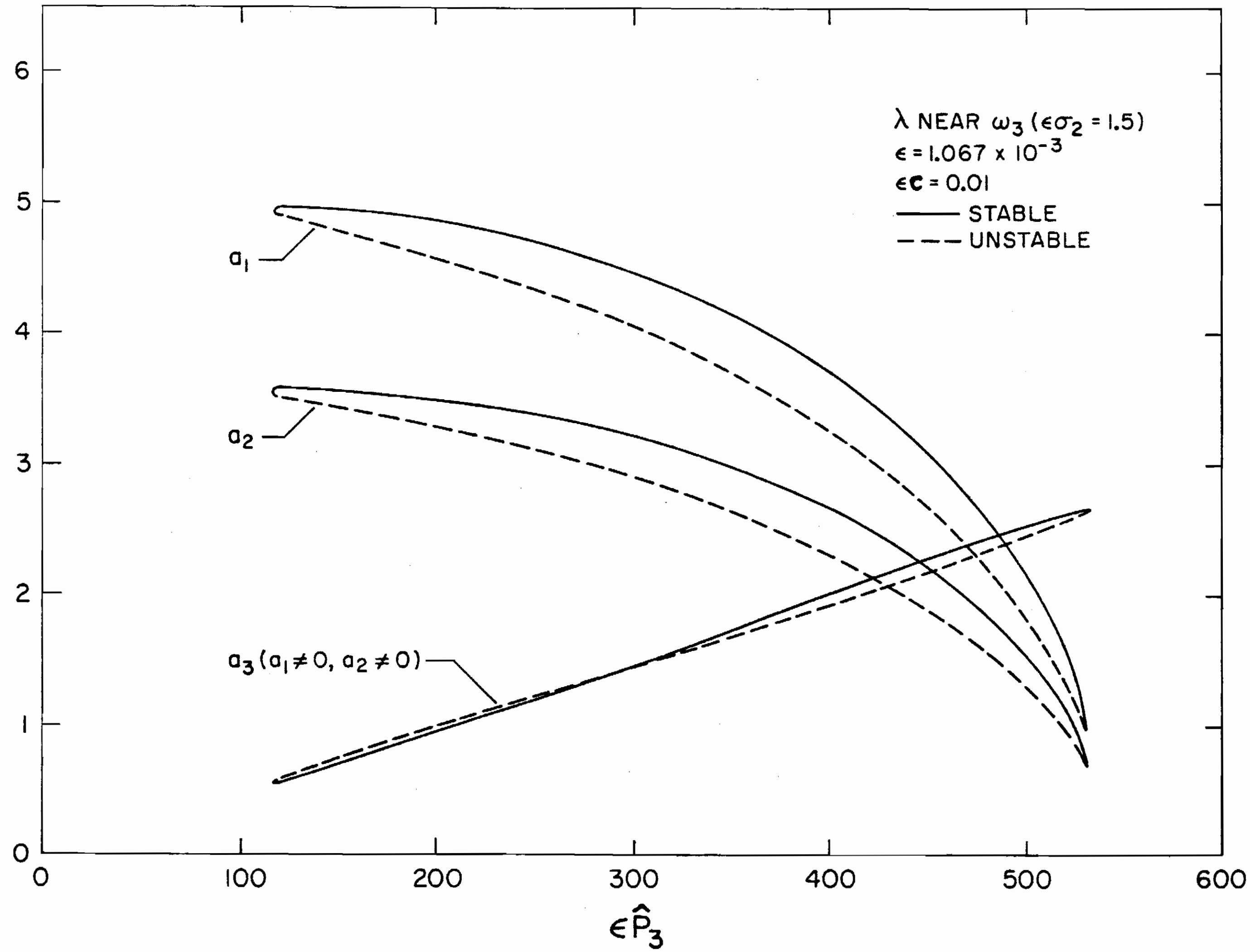

Figure 15b Variation of First, Second and Third Mode Amplitudes with Amplitude of Excitation

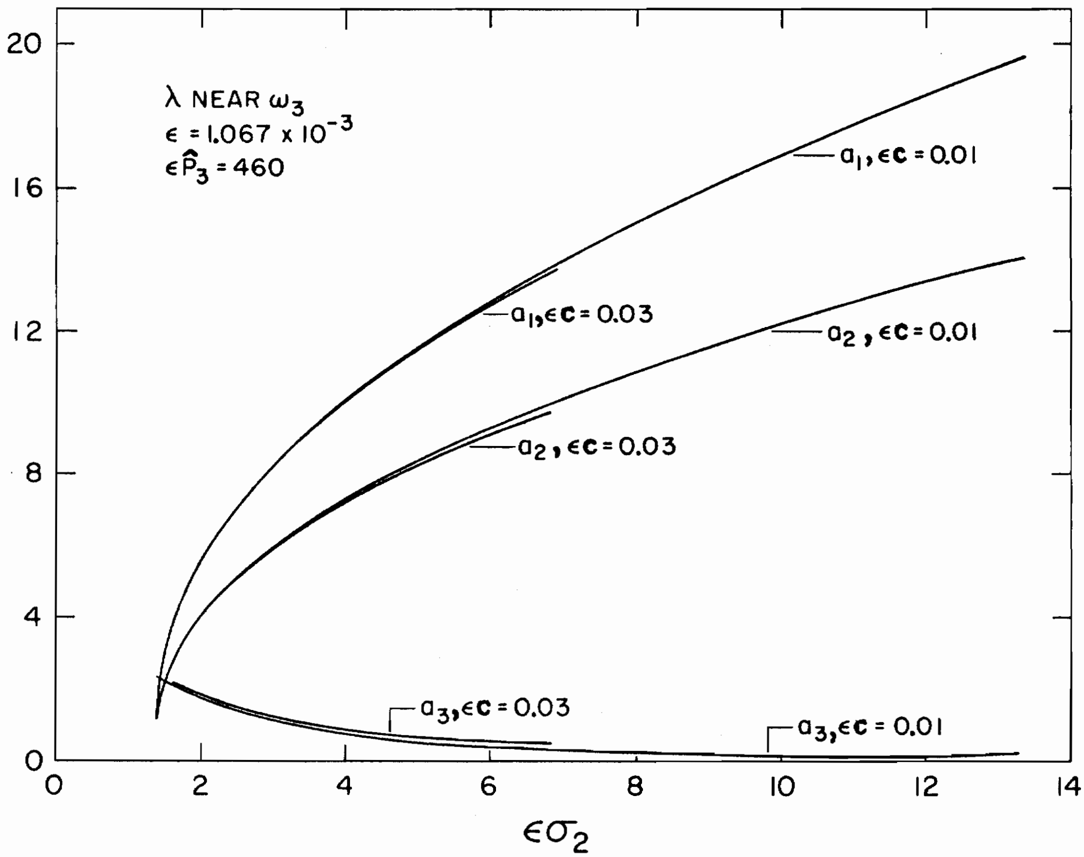

Figure 16 Variation of First, Second and Third Mode Amplitudes with Detuning of Excitation, for Two Values of Damping

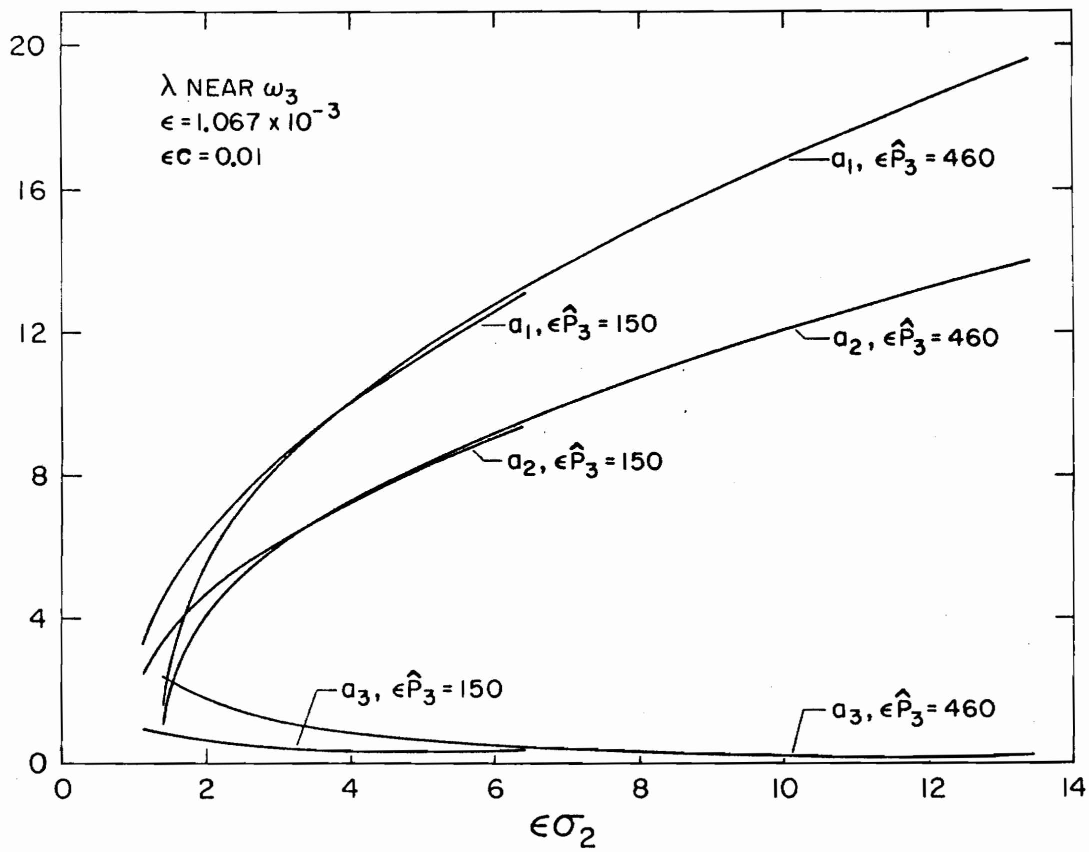

Figure 17 Variation of First, Second and Third Mode Amplitudes with Detuning of Excitation, for Two Values of the Amplitude of Excitation

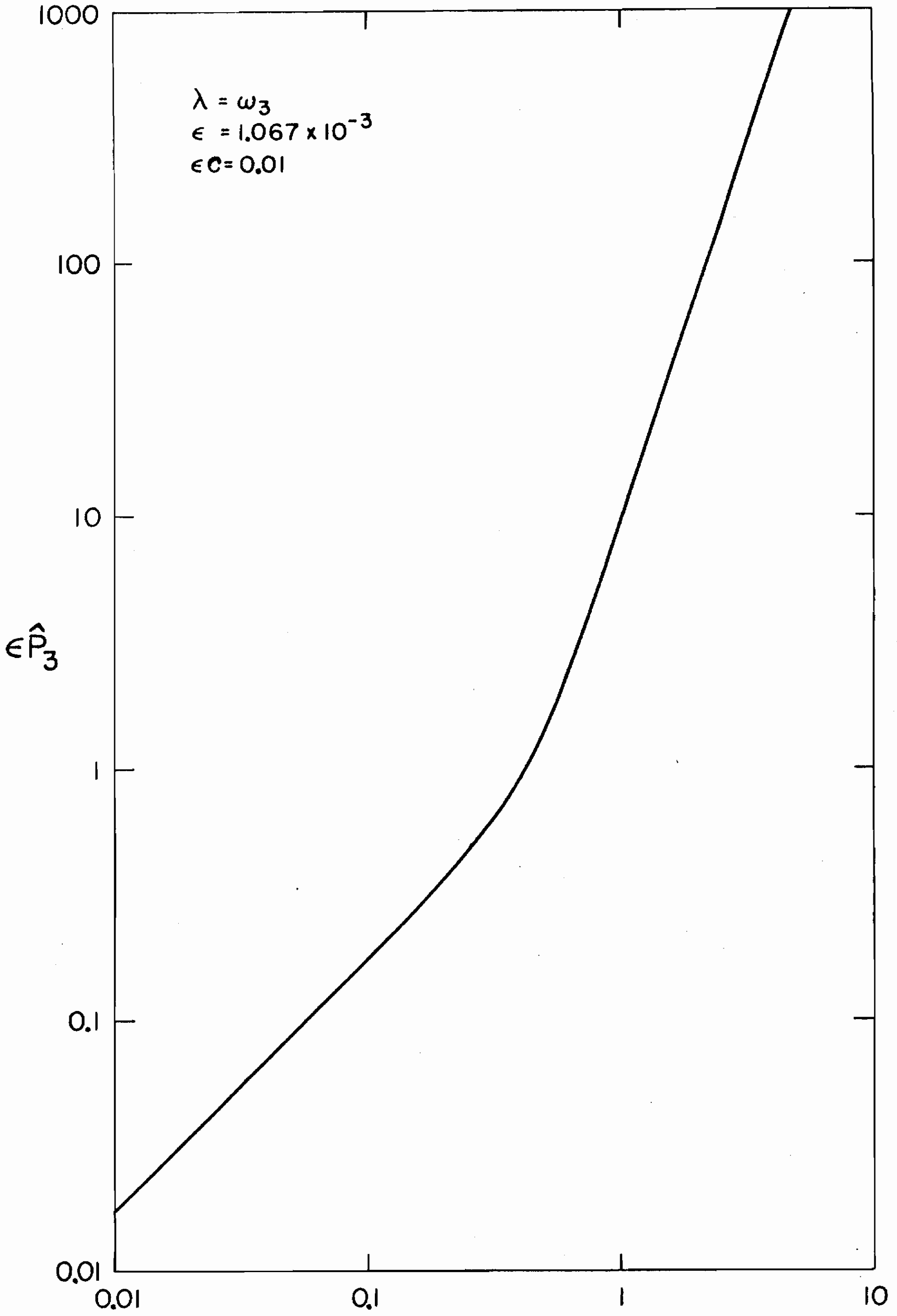

Figure 18 Log-Log Plot of the Variation of Third Mode Amplitude with Amplitude of Excitation

an internal resonance. It is noted that in a hinged-clamped beam only two modes are involved in the internal resonance.

Numerical examples are presented of the main resonances of the different modes involved in the internal resonance and the results shown in a series of figures. As in the numerical examples involving the beam, the results for the plate also exhibit typical nonlinear phenomena and illustrate that when more than one mode appears in the steady-state response, the lowest mode involved is likely to dominate the response.

8. CONCLUSIONS

The present work on the nonlinear resonances in systems having many degrees of freedom can be summarized as follows:

(1) A unified method for the analysis of the various nonlinear resonances is presented and the influence of an internal resonance is explored in depth.

(2) The success of the present work can largely be attributed to the method of multiple scales, a perturbation technique. It is an elegant and reliable method for systematically obtaining approximate solutions. The first approximation extracts the dominant features of the response and expresses them in terms of elementary functions. Moreover, the results obtained from the first approximation compare favourably with those obtained by a numerical integration of the governing nonlinear differential equations. It is noted that the numerical solution of the steady-state equations is far less demanding, in time and cost, than the numerical integration of the governing differential equations.

(3) The effectiveness of the approach developed in this work in providing clarity and insight into nonlinear phenomena is illustrated by an application to the general problem of the nonlinear vibrations (including asymmetric vibrations and travelling waves) of a clamped circular plate.

(4) In the presence of an internal resonance, it is possible for energy to be transferred among the modes involved. The numerical examples involving beams and plates confirm such possibilities and also indicate that, there can be significant transfer of energy down from the highest mode to the lower modes but not vice versa.

(5) The governing equations which are used in the problems of the nonlinear vibrations of beams and plates are the same as those used by several previous investigators, and all the single-mode solutions in the present work are essentially the same as the solutions obtained by them. However, all the earlier studies apparently failed to reveal the additional, multi-mode solutions which can exist in the presence of an internal resonance.

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APPENDIX A

Coefficients S_j and Frequency Combinations Λ_j in Equations (2.8) and (2.12)

j	S_j	Λ_j	j	S_j	Λ_j
1	$A_m A_p A_q$	$\omega_m+\omega_p+\omega_q$	15	$A_q K_m K_p$	$-2\lambda+\omega_q$
2	$A_m A_p \bar{A}_q$	$\omega_m+\omega_p-\omega_q$	16	$A_m A_p K_q$	$\lambda+\omega_m+\omega_p$
3	$A_m \bar{A}_p A_q$	$\omega_m-\omega_p+\omega_q$	17	$A_m A_p K_q$	$-\lambda+\omega_m+\omega_p$
4	$\bar{A}_m A_p A_q$	$-\omega_m+\omega_p+\omega_q$	18	$A_p A_q K_m$	$\lambda+\omega_p+\omega_q$
5	$2A_m K_p K_q$	ω_m	19	$A_p A_q K_m$	$-\lambda+\omega_p+\omega_q$
6	$2A_p K_q K_m$	ω_p	20	$A_q A_m K_p$	$\lambda+\omega_q+\omega_m$
7	$2A_q K_m K_p$	ω_q	21	$A_q A_m K_p$	$-\lambda+\omega_q+\omega_m$
8	$3K_m K_p K_q$	λ	22	$A_m \bar{A}_p K_q$	$\lambda+\omega_m-\omega_p$
9	$K_m K_p K_q$	3λ	23	$A_m \bar{A}_p K_q$	$-\lambda+\omega_m-\omega_p$
10	$A_m K_p K_q$	$2\lambda+\omega_m$	24	$A_p \bar{A}_q K_m$	$\lambda+\omega_p-\omega_q$
11	$A_m K_p K_q$	$-2\lambda+\omega_m$	25	$A_p \bar{A}_q K_m$	$-\lambda+\omega_p-\omega_q$
12	$A_p K_q K_m$	$2\lambda+\omega_p$	26	$A_q \bar{A}_m K_p$	$\lambda+\omega_q-\omega_m$
13	$A_p K_q K_m$	$-2\lambda+\omega_p$	27	$A_q \bar{A}_m K_p$	$-\lambda+\omega_q-\omega_m$
14	$A_q K_m K_p$	$2\lambda+\omega_q$			

APPENDIX B

Constant Coefficients in the Numerical Examples in Chapter 5

Equation (5.7) can be reduced to

$$\Gamma_{nmpq} = -\hat{k}\left[\int_0^\ell \phi_n'\phi_q'\,dx\right]\left[\int_0^\ell \phi_m'\phi_p'\,dx\right]$$

where primes denote differentiation with respect to x,

$\hat{k} = 0.25$, for $\ell = 2$ and ϕ_n is given by equation (5.8a). The first three roots of equation (5.8c), for $\ell = 2$ are

$\eta_1 = 1.9635$, $\eta_2 = 3.5345$, and $\eta_3 = 5.105$.

The γ_{ij} and Q_i are combinations of the Γ_{nmpq} and are obtained by numerical quadrature. They are as follows:

$$\gamma_{11} = 3\Gamma_{1111} = -6.213$$

$$\gamma_{22} = 3\Gamma_{2222} = -86.26$$

$$\gamma_{33} = 3\Gamma_{3333} = -414.5$$

$$\gamma_{12} = 2(2\Gamma_{1122} + \Gamma_{1221}) = \gamma_{21} = -16.58$$

$$\gamma_{13} = 2(2\Gamma_{1133} + \Gamma_{1331}) = \gamma_{31} = -34.73$$

$$\gamma_{23} = 2(2\Gamma_{2233} + \Gamma_{2332}) = \gamma_{32} = -129.9$$

$$Q_1 = 2\Gamma_{1121} + \Gamma_{1112} = -2.3108$$

$$Q_2 = \Gamma_{2111} = -0.77027$$

The spatial variation of the forcing function was taken to be a constant; that is

$$P_n = \int_0^{\ell} P(x)\phi_n(x)dx = P\int_0^{\ell} \phi_n(x)dx \; .$$

Typical values of the constant coefficients in Sections 5.2.b, 5.2.c and 5.2.d are as follows:

Section 5.2.b

For $P = 0.3 \times 10^3$ and $\varepsilon\sigma_2 = 0.1$,

$$F_1 = -\,9459, \quad H_{11} = -\,1710, \quad H_{22} = -\,2264 \; .$$

Section 5.2.c

For $P = 10 \times 10^3$ and $\varepsilon\sigma_2 = 2$,

$$F_1 = 41.37, \quad H_{11} = -\,899.2, \quad H_{22} = -\,2713 \; .$$

Section 5.2.d

For $P = 5 \times 10^3$ and $\varepsilon\sigma_2 = 1.5$,

$$H_{11} = -\,819.1, \quad H_{22} = -\,1914, \quad H_{33} = -\,9921, \quad H_{23} = H_{32} = -\,1054.$$

APPENDIX C

Coefficients $\hat{S}_j$ and Frequency Combinations $\hat{\Lambda}_j$ in Equations (6.29)

j	$\hat{S}_j$	$\hat{\Lambda}_j$
1	$A_{cd}A_{nm}A_{pq}$	$\omega_{cd} + \omega_{nm} + \omega_{pq}$
2	$A_{cd}A_{nm}\bar{B}_{pq}$	$\omega_{cd} + \omega_{nm} - \omega_{pq}$
3	$A_{cd}\bar{B}_{nm}A_{pq}$	$\omega_{cd} - \omega_{nm} + \omega_{pq}$
4	$\bar{B}_{cd}A_{nm}A_{pq}$	$-\omega_{cd} + \omega_{nm} + \omega_{pq}$
5	$\bar{B}_{cd}\bar{B}_{nm}\bar{B}_{pq}$	$-\omega_{cd} - \omega_{nm} - \omega_{pq}$
6	$\bar{B}_{cd}\bar{B}_{nm}A_{pq}$	$-\omega_{cd} - \omega_{nm} + \omega_{pq}$
7	$\bar{B}_{cd}A_{nm}\bar{B}_{pq}$	$-\omega_{cd} + \omega_{nm} - \omega_{pq}$
8	$A_{cd}\bar{B}_{nm}\bar{B}_{pq}$	$\omega_{cd} - \omega_{nm} - \omega_{pq}$

APPENDIX D

Constant Coefficients in the Numerical Examples in Chapter 7

Equation (7.16) can be reduced to

$$\Gamma_{\ell dmq} = -\frac{1}{2}\sum_{b=1}^{\infty} \xi_b^{-4} \left[\int_0^1 \phi_m'\phi_q'\psi_b' \, dr\right] \left[\int_0^1 \phi_\ell'\phi_d'\psi_b' \, dr\right]$$

where primes denote differentiation with respect to r, ϕ_m is given by equation (7.8b) and ψ_m is given by equation (7.11a).

The first three roots of equation (7.8d) are

$\eta_1 = 3.196$, $\eta_2 = 6.306$ and $\eta_3 = 9.439$.

The first twelve roots of equation (7.11c) are

$\xi_1 = 2.6629$, $\xi_2 = 6.0763$, $\xi_3 = 9.2912$, $\xi_4 = 12.468$

$\xi_5 = 15.629$ $\xi_6 = 18.784$, $\xi_7 = 21.935$, $\xi_8 = 25.084$

$\xi_9 = 28.231$, $\xi_{10} = 31.377$, $\xi_{11} = 34.522$, $\xi_{12} = 37.667$.

The γ_{ij} and Q_i are combinations of the $\Gamma_{\ell dmq}$ and are obtained by numerical quadrature. They are as follows:

Coefficient	Values		
[Number of terms in equation (7.16):	2	10	11]
$\gamma_{11} = 3\Gamma_{1111}$	-162.22	-166.22	-166.22
$\gamma_{12} = 2(2\Gamma_{1212} + \Gamma_{1122}) =$	-873.26	-883.80	-883.80
$\gamma_{13} = 2(2\Gamma_{1313} + \Gamma_{1133}) =$	-1460.3	-1644.8	-1644.8

Coefficient	Values		
[Number of terms in equation (7.16):	2	10	11]
$\gamma_{22} = 3\Gamma_{2222}$	-5155.2	-5552.1	-5552.1
$\gamma_{23} = 2(2\Gamma_{3232} + \Gamma_{2233}) = \gamma_{32}$	- 11181	- 14220	- 14220
$\gamma_{33} = 3\Gamma_{3333}$	- 28831	- 34401	- 34401
$Q_1 = 2\Gamma_{1223} + \Gamma_{1322} = Q_3$	-414.93	-556.77	-556.77
$Q_2 = 2Q_1$	-662.08	-1113.5	-1113.5

VITA

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S. Sridhar

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NONLINEAR RESONANCES IN SYSTEMS

HAVING MANY DEGREES OF FREEDOM

by

Seshadri Sridhar

(ABSTRACT)

An analysis is presented of the main, superharmonic, subharmonic, combination and internal resonances in a weakly nonlinear system having many degrees of freedom. The system has cubic nonlinearities, modal linear viscous damping and is subject to harmonic excitations. The method of multiple scales, a perturbation technique, is used to develop a unified method for the study of the various resonances. The effects of an internal resonance are explored in depth.

The first approximation obtained by the method of multiple scales extracts the dominant features of the response and expresses them in terms of elementary functions. It is shown that in the absence of internal resonances, the steady-state response can contain only the modes which are resonantly excited. In the presence of an internal resonance, modes other than those that are resonantly excited can appear in the response.

The usefulness of the method developed in this work in providing clarity and insight into nonlinear phenomena is illustrated by applications to the nonlinear vibrations of beams and circular plates. Numerical examples show that the results obtained by the first approximation compare favorably with the results obtained by a numerical integration

of the governing differential equations. The numerical examples indicate that a significant transfer of energy can take place from the highest mode involved in the internal resonance to the lower modes but not vice versa.