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INTERPOLATION BY RATIONAL MATRIX FUNCTIONS
WITH MINIMAL MC MILLAN DEGREE

by

Jeongook Kim Kang

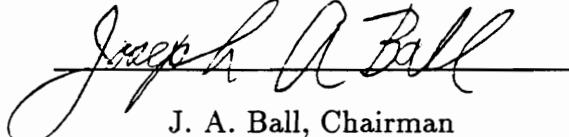
Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

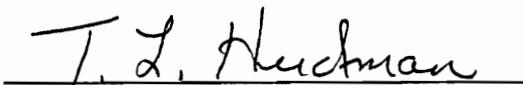
Mathematics

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December, 1990

Blacksburg, Virginia

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Abstract

Interpolation conditions on rational matrix functions expressed in terms of residues are studied. As a compact way of expressing tangential interpolation conditions of arbitrarily high multiplicity possibly from both sides simultaneously, interpolation conditions are represented in terms of residues. The minimal possible complexity, measured by the McMillan degree, of interpolants is found in terms of the controllability and the observability indices of certain pairs of matrices which are part of given data. An interpolant of such complexity is obtained in realization form. This leads to another approach to the partial realization problem. As a generalization of the well-known Lagrange interpolation problem for scalar polynomials, the problem of seeking for a matrix polynomial interpolant of low complexity is studied. The main tool is state space methods borrowed from systems theory. After adoption of state space methods, problems concerning rational matrix functions are reduced to the realm of linear algebra.

Acknowledgement

I would like to express my heartfelt thanks to Professor Joseph A. Ball, my advisor and teacher, for his guidance and patience. I am grateful to Professors Kenneth B. Hannsgen, Terry L. Herdman, Werner E. Kohler, and James E. Thomson for kindly consenting to be on my committee. I also wish to acknowledge the support and help given to me by my family. Finally, but not least, I would like to acknowledge Donna Noll and Jin Seo for typing this thesis.

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0. Introduction

Interpolation problems for rational matrix functions play an important role in systems theory, network theory and control theory. A classical paper on the occurrence of interpolation problems in network and systems theory is [YS]. The partial realization problem of linear systems theory introduced by Kalman can be viewed as a special case of rational interpolation problem at one point (conventionally taken to be the point at infinity)(see [Kal]).

Perhaps the simplest classical interpolation problem is that of Lagrange, where one is given n distinct points z_1, \dots, z_n in the complex plane \mathbb{C} and n complex numbers w_1, \dots, w_n and seeks a polynomial $l(z)$ for which

$$l(z_i) = w_i$$

for $i = 1, \dots, n$ and which has degree at most $n - 1$. Such a solution exists, is unique and is given by

$$l(z) = \sum_{k=1}^n w_k \prod_{j \neq k} \frac{z - z_j}{z_k - z_j} \quad (0.1)$$

(see[Da]). A more general problem is the so called *Cauchy interpolation problem* which can be formulated as follows: given n distinct points z_1, \dots, z_n in the complex plane \mathbb{C} and equally many complex numbers w_1, \dots, w_n and two integers l, m satisfying $l + m = n - 1$, find a rational function

$$f(z) := \frac{p(z)}{q(z)} = \frac{p_0 + p_1 z + \dots + p_l z^l}{q_0 + q_1 z + \dots + q_m z^m}, \quad q(z) \neq 0 \quad (0.2)$$

for which

$$f(z_i) = w_i, \quad i = 1, \dots, n. \quad (0.3)$$

The condition $l + m = n - 1$ is imposed so that the number of constraints and the number of parameters are the same. If we choose $l = n - 1, m = 0$, then this problem

is reduced to that of Lagrange. But unlike the Lagrange interpolation problem, this problem is not always solvable. The question of the existence of a solution has been investigated from various algebraic aspects; Euclidean algorithm, continued fraction, Páde approximation, etc. (see[M]).

If we represent a rational function $f(z)$ as

$$f(z) := \frac{\tilde{p}(z)}{\tilde{q}(z)}$$

for relatively coprime polynomials $\tilde{p}(z)$ and $\tilde{q}(z)$, then the McMillan degree of $f(z)$, denoted by $\delta(f)$, is

$$\delta(f) = \max\{\deg \tilde{p}(z), \deg \tilde{q}(z)\}.$$

Among the rational functions $f(z)$ satisfying the interpolation conditions, one of the minimal possible McMillan degree is of special interest because physically such a function corresponds to a system of minimal complexity. The problem of finding a rational function which satisfies (0.3) and has the minimal possible McMillan degree has been studied in [AA1],[B],[Do] by analyzing the *Löwner matrix*. It turns out that there are two cases. If there exists a solution with McMillan degree less than $\frac{n}{2}$, say, d , then the solution can be obtained by solving a system of linear equations involving the coefficients of $p(z)$ and $q(z)$. In this case, d is the minimal possible McMillan degree among all the solutions and the solution of McMillan degree d is unique (see[B]). In the other case, there exists no solution of McMillan degree less than $\frac{n}{2}$ and the related difficulty is referred to as the *inaccessible point* case. That is, although the coefficients of the numerator and denominator of the candidate solution are deduced from the prescribed interpolation nodes and ordinates by solving linear equations, the resulting polynomials have common zeros so that the candidate solution may not take the prescribed values at the corresponding points. This difficulty is surmounted in [AA1] by analyzing the *Löwner matrix* more systematically.

A matricial analogue of the interpolation problem is not only interesting from the purely mathematical point of view as a generalization of classical function theory to rational matrix functions but also relevant to many applications such as linear systems theory or the active area of H^∞ -control. Generalization to the matrix case can be done in many ways. One such way is the *full matrix-valued interpolation problem* which can be stated as follows: given n distinct points in the complex plane \mathbb{C} and $M \times N$ matrices W_1, \dots, W_n , find an $M \times N$ rational matrix function $W(z)$ for which

$$W(z_j) = W_j, \quad j = 1, \dots, n.$$

A more interesting one is the following tangential interpolation problem. For n_ζ distinct points z_1, \dots, z_{n_ζ} in a subset σ of the complex plane \mathbb{C} , x_1, \dots, x_{n_ζ} nonzero $1 \times M$ row vectors and y_1, \dots, y_{n_ζ} $1 \times N$ row vectors, find an $M \times N$ rational matrix function $W(z)$ with no poles in σ for which

$$x_i W(z_i) = y_i, \tag{0.4}$$

for $i = 1, \dots, n_\zeta$. The analogous right tangential interpolation condition is

$$W(w_j) u_j = v_j \tag{0.5}$$

($j = 1, \dots, n_\pi$) where w_1, \dots, w_{n_π} are given distinct points in σ , u_1, \dots, u_{n_π} are given $N \times 1$ nonzero column vectors and v_1, \dots, v_{n_π} are given $M \times 1$ column vectors. When considering problems of finding rational matrix functions which satisfy both conditions (0.4) and (0.5), in the case that some point z_i coincides with a point w_j , it is natural to impose a third type of interpolation condition

$$x_i W'(z_i) u_j = \rho_{ij}. \tag{0.6}$$

These conditions can be represented more compactly using residues as follows.

Let

$$A_\zeta := \begin{bmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_{n_\zeta} \end{bmatrix}, \quad B_+ := \begin{bmatrix} x_1 \\ \vdots \\ x_{n_\zeta} \end{bmatrix}, \quad B_- := - \begin{bmatrix} y_1 \\ \vdots \\ y_{n_\zeta} \end{bmatrix}. \quad (0.7)$$

$$A_\pi := \begin{bmatrix} w_1 & & 0 \\ & \ddots & \\ 0 & & w_{n_\pi} \end{bmatrix}, \quad C_+ := [v_1, \dots, v_{n_\pi}], \quad C_- := [u_1, \dots, u_{n_\pi}]. \quad (0.8)$$

Then, the conditions (0.4) and (0.5) can be written in residue form as

$$\sum_{z_0 \in \sigma} \text{Res}_{z=z_0} (zI - A_\zeta)^{-1} B_+ W(z) = -B_- \quad (0.9)$$

$$\sum_{z_0 \in \sigma} \text{Res}_{z=z_0} W(z) C_- (zI - A_\pi)^{-1} = C_+. \quad (0.10)$$

To describe the condition (0.6) introduce the $n_\zeta \times n_\pi$ matrix Γ defined by $\Gamma = [\Gamma_{ij}]$ for $1 \leq i \leq n_\zeta, 1 \leq j \leq n_\pi$ and

$$\Gamma_{ij} = \begin{cases} \frac{x_i v_j - y_i u_j}{w_j - z_i} & , \text{ if } z_i \neq w_j \\ \rho_{ij} & , \text{ if } z_i = w_j. \end{cases}$$

It is easy to see that (0.6) is equivalent to

$$\sum_{z_0 \in \sigma} \text{Res}_{z=z_0} (zI - A_\zeta)^{-1} W(z) C_- (zI - A_\pi)^{-1} = \Gamma. \quad (0.11)$$

In [BGR4], it is proved that solutions to the problem (0.9)-(0.11) always exist and all solutions are described via a linear fractional formula

$$W = (\Theta_{11}P + \Theta_{12}Q)(\Theta_{21}P + \Theta_{22}Q)^{-1}, \quad (0.12)$$

where the rational matrix function

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$$

is fixed and P, Q are arbitrary matrix functions without poles in σ and subject to the certain condition (see Theorem 1.4.1 for the precise statement). The method of solution involves constructing the rational matrix valued function $\Theta(z)$ which is the solution of another kind of interpolation problem (homogeneous interpolation problem). This problem is studied in [BR1,2], [GK1,2], [GKR1,2]. Various interpolation problems in the form of (0.9)–(0.11) with additional metric constraints on the solution have been considered in numerous papers. With interpolation conditions in the form of (0.9) these problems were studied by [N1,2] with metric constraints. In [BGR2], the problem of (0.9)–(0.11) with metric constraints such as H^∞ –norm on the imaginary axis less than 1 are also considered. A complete and unified treatment of these interpolation problems for rational matrix functions will appear in the book [BGR6].

An important idea basic to the approach developed in [BGR6] is the state space method introduced by Kalman in 1960. A model for a linear dynamical system is a system of vector differential equations

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) &= Cx(t) + Du(t).\end{aligned}$$

When one assumes $x_0 = 0$, then after Laplace transformation the relation between the input function $\hat{u}(z)$ and output function $\hat{y}(z)$ is given by

$$\hat{y}(z) = W(z)\hat{u}(z)$$

where

$$W(z) = D + C(zI - A)^{-1}B \tag{0.13}$$

is the *transfer function* of the system. The converse problem is to express a given proper rational matrix function $W(z)$ in terms of four matrices A, B, C, D as in (0.13).

Then, zero and pole data for $W(z)$ is encoded in a pair of matrices $((C, A)$, or (A, B)) and the smallest possible size of such a matrix A is called the *McMillan degree* of $W(z)$. Work up to 1970 on rational matrix functions generally dealt directly with polynomial coefficients matrix entries, expressed many ideas in terms of determinants and minors, developed solution algorithms using row and column operations, or took the linear dynamical system as the main object of study rather than the rational matrix function itself. Here, we elaborate further on the approach from [BGR6] of using the state space method as a tool for developing the theory of rational matrix functions.

In this thesis, we consider the problem of finding the minimal possible McMillan degree of the solutions of a “*two-sided residue interpolation problem*” (TRIP) in which the conditions (0.9)-(0.11) (without norm constraints) are generalized so that the derivatives of $W(z)$ are taken into account. The (TRIP) is a generalization of the well-known partial realization problem. For the partial realization problem, the minimal possible McMillan degree is found in [Kal] (scalar case) and [GK2] (matrix case). The following problems are solved here: (a) find the minimal possible McMillan degree of solutions of (TRIP), (b) find a solution with the minimal possible McMillan degree, (c) find all nonnegative integers d such that there exists a solution of McMillan degree d , (d) parametrize all solutions with McMillan degree d for a given d satisfying (c). Moreover, a matrix polynomial solution with relatively small McMillan degree is constructed and as a byproduct a type of Euclidean algorithm for matrix polynomials and McMillan degree is obtained. The starting points for this thesis are the parametrization of all solutions of (TRIP) due to [BGR4] and the explicit formula for the matrix function

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$$

which provides the coefficients $\Theta_{ij}(z)$ for the linear fractional formula (0.12) due to

[GKR1,2].

This thesis is divided into four chapters (besides the introduction). The first chapter is of a preliminary character; the necessary notions about rational matrix functions such as realization theory, the null pole structure of a rational matrix function, the definition and the properties of a column reduced matrix function are introduced. Also, in the last section, the parametrization due to [BGR4] is given. In Chapter II, a homogeneous interpolation problem is studied. Especially, we build an $(M + N) \times (M + N)$ rational matrix function $\Theta(z)$ in (0.12) which has zeros and poles in the complex plane \mathbb{C} so that

$$\Theta \mathcal{P}_{M+N} = \left\{ \begin{bmatrix} C_+ \\ C_- \end{bmatrix} (zI - A_\pi)^{-1} x + h(z) \mid x \in \mathbb{C}^{n\pi}, h \in \mathcal{P}_{M+N} \right. \\ \left. \text{such that } \sum_{z_0 \in \mathbb{C}} \text{Res}_{z=z_0} (zI - A_\zeta)^{-1} [B_+ \ B_-] h(z) = \Gamma x \right\} \quad (0.14)$$

where the matrices $C_+, C_-, A_\pi, A_\zeta, B_+, B_-, \Gamma$ are given by (0.7) (0.8) and (0.11). Here \mathcal{P}_{M+N} denotes the set of polynomials with coefficients in \mathbb{C}^{M+N} . Also we require that the matrix $\Theta(z)$ to be column reduced at infinity. The column indices of $\Theta(z)$ which satisfies (0.14) and is column reduced at infinity are given by the controllability indices and the observability indices of certain pairs of matrices which are constructed from the given data. By the index of a rational column vector function is meant the highest power of z occurring in the Laurent series expansion at infinity of the rational vector function.

Chapter III concerns a generalization of Lagrange interpolation problem to the matrix case, i.e., matrix polynomial solutions of tangential interpolation conditions of low McMillan degree. A matrix polynomial of McMillan degree $n - 1$ which satisfies n tangential interpolation conditions is constructed in state space realization form. A corollary is a type of Euclidean algorithm for matrix polynomials and McMillan

degree. To get the result of this chapter, any $\Theta(z)$ satisfying (0.12) can be used for the parametrization of all solutions (0.14). But here we choose $\Theta(z)$ which has the minimal possible McMillan degree among all $(M + N) \times (M + N)$ rational matrix functions satisfying (0.12).

In Chapter IV, the questions (a)-(d) for (TRIP) are answered. Here we want $\Theta(z)$ to satisfy (0.14) and also to be column reduced at infinity. This extra latter constraint on $\Theta(z)$ is very crucial in determining the minimal possible McMillan degree of all solutions of (TRIP).

I. Rational matrix functions

In this chapter, we present some notions and auxiliary results on rational matrix functions which will be used extensively in the sequel. By an $M \times N$ rational matrix function, we understand an $M \times N$ matrix function with rational entries and shall regard it as a meromorphic matrix function over the extended complex plane \mathbb{C}^∞ . A rational matrix function W is said to be *regular* if the size of W is square and $\det W$ is not identically zero.

Rational matrix functions have been studied extensively during the last decade motivated by many applications in various branches in engineering. In particular, realization theory and the null-pole structure of a rational matrix function are indispensable features in those studies. The structure of poles and zeros of a matrix function is more complicated than that of a scalar function, for example, a rational matrix function may have a zero and a pole at the same point. The starting point of the study of interpolation theory of rational matrix functions is to understand the null-pole structure of them.

The notion of column reducedness of a rational matrix function is also introduced in this chapter. One of the important features of a column reduced rational matrix function is the relation to Wiener-Hopf factorization (see Chapter II). Some properties of column reduced rational matrix functions which are well known for the matrix polynomial case are presented.

Throughout this thesis, we study a generalization of the interpolation conditions (0.9)-(0.11), which is called *two-sided residue interpolation problem* (TRIP) (in some places in the literature, (TRIP) is also called the *two-sided Lagrange-Sylvester interpolation problem*). A parametrization of all solutions of (TRIP) due to [BGR4] is introduced in the last section.

In this chapter, the proofs of well known results are omitted and they can be found [BGR4,6] or [Kai].

1.1. Realization and McMillan degree

For an $M \times N$ proper (i.e., analytic at infinity) rational matrix function $W(z)$, we define a *realization* of $W(z)$ to be a representation of the form

$$W(z) = D + C(zI - A)^{-1}B, \quad z \notin \sigma(A), \quad (1.1.1)$$

where A, B, C, D are matrices of sizes $n \times n, n \times N, M \times n, M \times N$, respectively. Since $D = \lim_{z \rightarrow \infty} W(z)$, we may identify a realization (1.1.1) with the triple of matrices (A, B, C) (for a given $W(z)$).

A realization W is said to be *minimal* if (C, A) is a null-kernel pair and (A, B) is a full-range pair, that is,

$$\mathcal{K}(C, A) := \bigcap_{j=0}^{n-1} \text{Ker } CA^j = \{0\} \quad (1.1.2)$$

$$\mathcal{I}(A, B) := \sum_{j=0}^{n-1} \text{Im } A^j B = \mathbb{C}^n. \quad (1.1.3)$$

If the conditions (1.1.2) and (1.1.3) are not both fulfilled, the realization (A, B, C) may be reduced to a minimal one in the following way. Put $X_1 = \mathcal{K}(C, A)$, let X_0 be a direct complement of $X_1 \cap \mathcal{I}(A, B)$ in $\mathcal{I}(A, B)$ and choose X_2 such that $\mathbb{C}^n = X_1 \oplus X_0 \oplus X_2$. Then, relative to the decomposition $\mathbb{C}^n = X_1 \oplus X_0 \oplus X_2$, the operators A, B and C partition as follows:

$$A = \begin{bmatrix} * & * & * \\ 0 & A_0 & * \\ 0 & 0 & * \end{bmatrix} \quad B = \begin{bmatrix} * \\ B_0 \\ 0 \end{bmatrix}, \quad C = [0 \quad C_0 \quad *]$$

and (A_0, B_0, C_0) is minimal.

By definition the *McMillan degree* of $W(z)$ is $\dim X_0$, i.e., the smallest possible size of A in a realization of $W(z)$. By $\delta(W)$, we will denote the McMillan degree of W . Here, we observe that

$$\delta(W) = \text{rank}_{1 \leq j \leq n-1} \text{col } CA^j \quad \text{if} \quad \sum_{j=0}^{n-1} \text{Im } A^j B = \mathbb{C}^n. \quad (1.1.4)$$

If D is invertible in (1.1.1), then

$$W^{-1}(z) = D^{-1} - D^{-1}C(zI - (A - BD^{-1}C))^{-1}BD^{-1} \quad (1.1.5)$$

is a realization of $W^{-1}(z)$ and (1.1.1) is minimal if and only if (1.1.5) is.

When W has singularities at infinity, the realization (1.1.1) cannot be used and is replaced by

$$W(z) = D + C_F(zI - A_F)^{-1}B_F + zC_\infty(I - zA_\infty)^{-1}B_\infty \quad (1.1.6)$$

with $\sigma(A_\infty) = \{0\}$. Here, a set of matrices $(A_F, B_F, C_F, D, A_\infty, B_\infty, C_\infty)$ of sizes $n_F \times n_F$, $n_F \times N$, $M \times n_F$, $M \times N$, $n_\infty \times n_\infty$, $n_\infty \times N$, $M \times n_\infty$ is said to be a realization of W . Realizations for a rational matrix function always exist. Moreover, any two minimal realizations for $W(z)$, $(A_F, B_F, C_F, D, A_\infty, B_\infty, C_\infty)$ and $(\tilde{A}_F, \tilde{B}_F, \tilde{C}_F, \tilde{D}, \tilde{A}_\infty, \tilde{B}_\infty, \tilde{C}_\infty)$ are similar, i.e., there exist invertible matrices S and T for which

$$\begin{aligned} \tilde{C}_F &= C_F S, & \tilde{A}_F &= S^{-1} A_F S, & \tilde{B}_F &= S^{-1} B_F \\ \tilde{C}_\infty &= C_\infty T, & \tilde{A}_\infty &= T^{-1} A_\infty T, & \tilde{B}_\infty &= T^{-1} B_\infty, & \tilde{D} &= D. \end{aligned}$$

A realization of the form (1.1.6) has the disadvantage that the value of W at infinity or even at any point $\alpha \in \mathbb{C}$ is not displayed in an obvious way. This situation can be alleviated by changing the form of (1.1.6). For $\alpha \notin \sigma(A_F)$,

$$\begin{aligned} W(z) &= D_\alpha - (z - \alpha)C_F(zI - A_F)^{-1}(\alpha I - A_F)^{-1}B_F \\ &\quad + (z - \alpha)C_\infty(I - zA_\infty)^{-1}(I - \alpha A_\infty)^{-1}B_\infty, \end{aligned} \quad (1.1.7)$$

with $D_\alpha = W(\alpha)$. The realization in the form of (1.1.7) is called a *realization for W centered at α* . If we put

$$\begin{aligned} G &:= \begin{bmatrix} I & 0 \\ 0 & A_\infty \end{bmatrix}, & A &:= \begin{bmatrix} A_F & 0 \\ 0 & I \end{bmatrix}, \\ B &:= \begin{bmatrix} -(\alpha I - A_F)^{-1}B_F \\ (I - \alpha A_\infty)^{-1}B_\infty \end{bmatrix}, & C &:= [C_F \quad C_\infty], \end{aligned}$$

then (1.1.7) can be represented as

$$W(z) = D_\alpha + (z - \alpha)C(zG - A)^{-1}B. \quad (1.1.8)$$

If D_α is invertible in (1.1.8), then

$$W(z)^{-1} = D_\alpha^{-1} - (z - \alpha)D_\alpha^{-1}C(zG^\times - A^\times)^{-1}BD_\alpha^{-1}, \quad (1.1.9)$$

where

$$A^\times = A + \alpha BD_\alpha^{-1}C$$

$$G^\times = G + BD_\alpha^{-1}C.$$

By choosing an appropriate basis, we may decompose $zG^\times - A^\times$ and BD^{-1} in the following way:

$$zG^\times - A_F^\times = \begin{bmatrix} zI - A^\times & 0 \\ 0 & zA_\infty^\times - I \end{bmatrix} \quad \text{and} \quad BD^{-1} = \begin{bmatrix} Y_F \\ Y_\infty \end{bmatrix}$$

with $\sigma(A_\infty^\times) = \{0\}$. The realizations (1.1.8) and (1.1.9) are said to be *minimal* if the size of $zG - A$ (and hence of $zG^\times - A^\times$) is as small as possible, and in this case, $\delta(W) = \text{the size of } (zG - A)$.

We finish this section with an observation on the McMillan degree of a matrix polynomial. By an $M \times N$ matrix polynomial, we understand the case $n_F = 0$ in (1.1.6). The more familiar form for an $M \times N$ matrix polynomial L is

$$L(z) = \sum_{i=1}^n A_i z^i,$$

where A_0, A_1, \dots, A_n are $M \times N$ matrices.

In the next proposition the McMillan degree of a matrix polynomial $L(z)$ is expressed in terms of the coefficients of $L(z)$.

Proposition 1.1.1. Suppose an $M \times N$ matrix polynomial $L(z)$ is given by

$$L(z) = \sum_{i=1}^n A_i z^i.$$

Then

$$\delta(L) = \text{rank} \begin{bmatrix} A_n & A_{n-1} & \dots & A_1 \\ \ddots & & & \vdots \\ 0 & \ddots & & A_{n-1} \\ & & & A_n \end{bmatrix}.$$

Proof. A matrix polynomial $L(z)$ can be represented in a realization form as

$$L(z) = D_L + zC_L(I - zA_L)^{-1}B_L$$

with

$$A_L := \begin{bmatrix} 0 & I_n & & 0 \\ \ddots & \ddots & & \\ 0 & \ddots & I_n & \\ & & & 0 \end{bmatrix} \in \mathbb{C}^{nN \times nN}, \quad B_L := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_N \end{bmatrix} \in \mathbb{C}^{nN \times N},$$

$$C_L := [A_1 \ \dots \ A_n] \in \mathbb{C}^{M \times nN}, \quad D_L := A_0.$$

Upon applying (1.1.4) to

$$\mathcal{I}(A_L, B_L) = \text{Im} [B_L \ A_L B_L \ \dots \ A_L^{n-1} B_L] = \mathbb{C}^{nN},$$

we can conclude that

$$\delta(L) = \text{rank}_{\substack{\text{col} \\ 1 \leq j \leq n-1}} C_L A_L^j = \text{rank} \begin{bmatrix} A_n & A_{n-1} & \dots & A_1 \\ \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & A_{n-1} \\ & & & A_n \end{bmatrix}.$$

□

The next Corollary is a straightforward consequence of Proposition 1.1.1.

Corollary 1.1.2. *If $L(z)$ is an $M \times N$ matrix polynomial given by*

$$L(z) = \sum_{j=0}^n A_j z^j$$

with $A_n \neq 0$, then

- (i) $\delta(L) \geq n$
- (ii) $\delta(L) = n \quad \text{if } M = N = 1$
- (iii) $\delta(L) = nM \quad \text{if } M = N \text{ and } A_n \text{ is invertible.}$

1.2. Null and pole structure

Throughout this section, $W(z)$ represents an $M \times N$ rational matrix function. Poles and zeros of $W(z)$ are defined as follows: in a neighborhood of $\alpha \in \mathbb{C}$, one can represent $W(z)$ as

$$W(z) = E_\alpha(z) D_\alpha(z) F_\alpha(z), \quad (1.2.1)$$

where $E_\alpha^{\pm 1}$ and $F_\alpha^{\pm 1}$ are analytic near α and

$$D_\alpha(z) = \text{diag}((z - \alpha)^{\kappa_1^\alpha}, \dots, (z - \alpha)^{\kappa_r^\alpha}) \oplus 0.$$

The representation (1.2.1) and integers $\kappa_1^\alpha, \dots, \kappa_r^\alpha$ are known as the local *Smith-McMillan form* and *partial multiplicities* of $W(z)$ at α . The positive (negative) indices κ_j^α are said to be *partial multiplicities of α as a zero (pole) of $W(z)$* .

For $\alpha = \infty$, by considering the local Smith- McMillan form of $W(\frac{1}{z})$ at $z = 0$, one can represent $W(z)$ as

$$W(z) = E_\infty(z) D_\infty(z) F_\infty(z),$$

where $E_\infty^{\pm 1}$ and $F_\infty^{\pm 1}$ are analytic in a neighborhood of infinity and

$$D_\infty(z) = \text{diag}(z^{\kappa_1^\infty}, \dots, z^{\kappa_r^\infty}) \oplus 0. \quad (1.2.2)$$

The positive (negative) indices κ_j^∞ are said to be *partial multiplicities at infinity as a pole (zero)* of $W(z)$.

By the number of zeros (poles) at $\alpha \in \mathbb{C}^\infty$ is meant the sum of the partial multiplicities at α as a zero (pole).

The number r of (1.2.1) is called the *normal rank* of $W(z)$ and W is said to be full column rank if $r = N$. When W is full column rank, the *defect* of W can be defined as follows:

$$\begin{aligned} \text{def } W &:= \text{the defect of } W(z) \\ &= \sum_{\alpha \in \mathbb{C}^\infty} (\# \text{ of poles at } \alpha - \# \text{ of zeros at } \alpha). \end{aligned}$$

A null-kernel pair (C_π, A_π) of sizes $M \times n_\pi$, $n_\pi \times n_\pi$ is said to be a *(right) σ -pole pair* of W if $\sigma(A_\pi) \subset \sigma$ and there exists an $n_\pi \times N$ matrix \tilde{B} such that (A_π, \tilde{B}) is a full-range pair and

$$W(z) - C_\pi(zI - A_\pi)^{-1}\tilde{B}$$

is analytic on σ .

In general, the null structure of $W(z)$ is more complicated. But, when W is regular, a zero of W is defined to be a pole of W^{-1} and a full-range pair (A_ζ, B_ζ) of sizes $n_\zeta \times n_\zeta$, $n_\zeta \times M$ said to be a *(left) σ -null pair* for $W(z)$ if $\sigma(A_\zeta) \subset \sigma$ and there exists a $M \times n_\zeta$ matrix \tilde{C} such that (\tilde{C}, A_ζ) is a null-kernel pair and

$$W^{-1}(z) - \tilde{C}(zI - A_\zeta)^{-1}B_\zeta$$

is analytic in σ . A $\{\infty\}$ -pole pair, $\{\infty\}$ -null pair for $W(z)$ can be defined by considering $W(\frac{1}{z})$ with $\sigma = \{0\}$.

To characterize the null structure of an $M \times N$ rational matrix function W which is not regular we need more notions. Two of them are right and left Kronecker indices which are defined via the *Forney minimal bases* in the following way.

Let $\mathcal{K}_R(W)$ be a subspace of \mathcal{R}_N over the (scalar) rational functions which is defined by

$$\mathcal{K}_R(W) := \{ \varepsilon(z) \in \mathcal{R}_N \mid W(z)\varepsilon(z) = 0 \},$$

where \mathcal{R}_N denotes the set of N -dimensional rational column vector functions. A *minimal basis* of $\mathcal{K}_R(W)$ is a basis $\{ \rho_1(z), \dots, \rho_\mu(z) \}$ consisting of N -dimensional vector polynomials with degree r_i as small as possible. The degrees $\{ r_1, \dots, r_\mu \}$ are called the *right indices* of $W(z)$, and are invariants of $W(z)$ even though the minimal basis is not unique. A set of *left indices* $\{ \ell_1, \dots, \ell_\nu \}$ can be similarly formed, using a minimal basis for

$$\mathcal{K}_L(W) := \{ \eta(z) \in \mathcal{R}_M \mid \eta^T(z)W(z) = 0 \}.$$

It is known that

$$\text{def } W = \sum_{j=1}^{\mu} r_j + \sum_{j=1}^{\nu} \ell_j \quad (1.2.3)$$

(see [VK]).

The next proposition will be used later.

Proposition 1.2.1. *If W is an $M \times N$ rational matrix function with full column rank and T is a regular $N \times N$ rational matrix function, then*

$$\text{def}(WT) = \text{def } W.$$

Proof. By (1.2.3), it is enough to show that $\mathcal{K}_R(WT) = \mathcal{K}_R(W)$ and $\mathcal{K}_L(WT) = \mathcal{K}_L(W)$. Since WT and W are full column rank, $\mathcal{K}_R(WT) = \mathcal{K}_R(W) = \{0\}$. Suppose there exists a $\eta(z) \in \mathcal{R}_M$ such that

$$\eta^T(z)W(z)T(z) \equiv 0. \quad (1.2.5)$$

Noting that T is regular means $\mathcal{K}_R(T) = \mathcal{K}_L(T) = \{0\}$, from (1.2.5) we get

$$\eta^T(z)W(z) \equiv 0,$$

i.e., $\mathcal{K}_L(WT) \subset \mathcal{K}_L(W)$. The opposite inclusion is straightforward. \square

In the rest of this section, we assume $M = N$ and $W(z)$ is regular and σ is a subset of \mathbb{C} . We denote the σ -null-pole subspace for W by $\mathcal{S}_\sigma(W)$ which is defined by

$$\begin{aligned}\mathcal{S}_\sigma(W) &:= W\mathcal{R}_M(\sigma) \\ &= \{ W(z)h(z) \mid h(z) \in \mathcal{R}_M(\sigma) \},\end{aligned}$$

where $\mathcal{R}_M(\sigma)$ is a set of a vector functions in \mathcal{R}_M which has no poles on σ . The σ -null-pole subspace $\mathcal{S}_\sigma(W)$ turns out to be expressible explicitly in terms of a σ -pole pair (C_π, A_π) , a σ -null pair (A_ζ, B_ζ) and an additional matrix Γ called the *null-pole coupling matrix*. The explicit description is given by

$$\begin{aligned}\mathcal{S}_\sigma(W) &= \{ C_\pi(zI - A_\pi)^{-1}x + h(z) \mid x \in \mathbb{C}^{n_\pi}, h(z) \in \mathcal{R}_M(\sigma), \\ &\quad \sum_{z_0 \in \sigma} Res_{z=z_0}(zI - A_\zeta)^{-1}B_\zeta h(z) = \Gamma x \}.\end{aligned}$$

The null-pole coupling matrix Γ is uniquely determined by the matrix function W together with a σ -null pair (A_ζ, B_ζ) , σ -pole pair (C_π, A_π) for W and is a solution of the Sylvester equation

$$\Gamma A_\pi - A_\zeta \Gamma = B_\zeta C_\pi.$$

If (C_π, A_π) is a σ -pole pair, (A_ζ, B_ζ) is a σ -null pair and Γ is the associated null-pole coupling matrix for W , we refer to the whole collection $\{ (C_\pi, A_\pi), (A_\zeta, B_\zeta), \Gamma \}$ as a σ -null-pole triple for W . If $W(\infty) = I$, then all these objects are computable from a realization for $W(z)$. That is, if $W(z)$ is given by

$$W(z) = I + C(zI - A)^{-1}B,$$

then $(A_\zeta, B_\zeta) = (A^\times|_{\text{Im } P^\times}, P^\times B)$, $(C_\pi, A_\pi) = (C|_{\text{Im } P}, A|_{\text{Im } P})$ and $\Gamma = P^\times|_{\text{Im } P}$, where $A^\times = A - BC$, $P(P^\times)$ is the Riesz projection of $A(A^\times)$ corresponding to the eigenvalues in σ .

Let $\mathcal{R}_{M \times N}$ denote the set of $M \times N$ rational matrix functions and $\mathcal{R}_{M \times N}(\sigma)$ be the set of functions in $\mathcal{R}_{M \times N}$ which have no poles in σ . The following gives an alternative characterization of σ -null-pole-triples.

Proposition 1.2.2. *If $W_1, W_2 \in \mathcal{R}_{M \times M}$ and $\sigma \subset \mathbb{C}$, the followings are equivalent.*

- (i) W_1, W_2 have the same σ -null-pole-triple
- (ii) $W_1^{-1}W_2$ has no zeros or poles on σ
- (iii) $W_1\mathcal{R}_M(\sigma) = W_2\mathcal{R}_M(\sigma)$
- (iv) $W_1 = W_2Q_2, W_2 = W_1Q_1$ for some $Q_1, Q_2 \in \mathcal{R}_{M \times M}(\sigma)$.

1.3. Column reducedness

Assume that $W(z) \in \mathcal{R}_{M \times N}$ has full column rank and makes a Laurent series expansion at $\alpha \in \mathbb{C}$:

$$\begin{aligned} W(z) &= R_{-\ell}(z - \alpha)^{-\ell} + \cdots + R_0 + R_1(z - \alpha) + \cdots \\ &= [W_0 + W_1(z - \alpha) + \cdots]D_\alpha(z) \\ &= W_\alpha(z)D_\alpha(z), \end{aligned} \tag{1.3.1}$$

where

$$D_\alpha(z) = \text{diag}((z - \alpha)^{\beta_i^\alpha})_{i=1}^N \tag{1.3.2}$$

and

$\beta_i^\alpha =$ the lowest power of $(z - \alpha)$ occurring in the i^{th} column of $W(z)$.

We shall say that $W(z)$ is *column reduced at α* if W_0 has full column rank. If W is column reduced at α , then $\{ -\beta_i^\alpha \mid \beta_i^\alpha < 0 \}$ are the pole multiplicities and $\{ \beta_i^\alpha \mid \beta_i^\alpha > 0 \}$ are the zero multiplicities in the Smith-McMillan form of $W(z)$ at α . If $\alpha = \infty$, we simply replace $(z - \alpha)$ by z^{-1} everywhere (1.3.1) and (1.3.2). Then $W(z)$ is

represented as

$$W(z) = W_\infty(z)D_\infty(z) \quad (1.3.3)$$

with

$$W(z) = W_0 + \frac{1}{z}W_1 + \frac{1}{z^2}W_2 + \dots \quad (1.3.4)$$

and

$$D_\infty(z) = \text{diag}(z^{\beta_i^\infty})_{i=1}^N, \quad (1.3.5)$$

where

$$\beta_i^\infty = \text{the highest power of } z \text{ in the } i^{\text{th}} \text{ column of } W(z). \quad (1.3.6)$$

The integer β_i^∞ in (1.3.6) is called the i^{th} *column index* of $W(z)$ and $W(z)$ is said to be *column reduced at infinity* if W_0 in (1.3.4) has full column rank. If $W(z)$ is column reduced at infinity, then $\{-\beta_i^\infty \mid \beta_i^\infty < 0\}$ are the zero multiplicities and $\{\beta_i^\infty \mid \beta_i^\infty > 0\}$ are the pole multiplicities of $W(z)$ at infinity. Sometimes we represent (1.3.3) as

$$W(z) = W_{hc}D_\infty(z) + R(z), \quad (1.3.7)$$

where $W_{hc} = W_0$ and $R(z)$ represent the terms of lower degree in z . We call W_{hc} the *leading coefficients* of W .

A matrix polynomial is said to be column reduced, provided the matrix polynomial is column reduced at infinity. The significance of column reduced matrix polynomials is that they have no zeros at infinity.

The following two theorems assert that the well-known predictable degree property and invariance of column indices of column reduced matrix polynomials can be extended to rational matrix functions which are column reduced at infinity.

Theorem 1.3.1. *Let $W(z) \in \mathcal{R}_{M \times N}$ have full column rank. Then $W(z)$ is column reduced at infinity if and only if for any polynomial vector $p(z)$,*

$$\text{column index of } W(z)p(z) = \max_{i: p_i(z) \neq 0} \{ \deg p_i(z) + \gamma_i \}$$

where $p_i(z)$ is the i^{th} entry of $p(z)$ and γ_i is the i^{th} column index of $W(z)$.

For the proof, see Theorem 7.3-13 of [Kai]. Although Theorem 7.3-13 of [Kai] is stated only for column reduced matrix polynomials, the same technique is used for the proof of Theorem 1.3.1.

Theorem 1.3.2. *Let $W(z), \tilde{W}(z) \in \mathcal{R}_{M \times N}$ be column reduced at infinity. If*

$$\tilde{W}(z) = W(z)U(z)$$

for an unimodular matrix function $U(z)$, then $W(z)$ and $\tilde{W}(z)$ have the same column indices.

For the proof, see Theorem 6.3-14 of [Kai]. The comment following Theorem 1.3.1 also applies for Theorem 1.3.2. It is worthwhile to note that Theorem 1.3.2 tells us that the column indices of a regular rational matrix function which is column reduced at infinity are uniquely determined by its \mathbb{C} -null-pole triple.

We will close this section with a theorem due to [VK].

Theorem 1.3.3. *For a given $W(z) \in \mathcal{R}_{M \times N}$ and $\alpha \in \mathbb{C}^\infty$, there exists an unimodular matrix $V_\alpha(z)$ for which*

$$\hat{W}(z) := W(z)V_\alpha(z)$$

is column reduced at α .

1.4. Two-sided residue interpolation problem

A generalization of interpolation conditions (0.9)-(0.11) can be stated as follows:

(TRIP) Given matrices A_ζ, B_+, B_- of sizes $n_\zeta \times n_\zeta, n_\zeta \times M, n_\zeta \times N$, respectively, matrices C_+, C_-, A_π of sizes $M \times n_\pi, N \times n_\pi, n_\pi \times n_\pi$ and an $n_\zeta \times n_\pi$ matrix Γ . Find an $M \times N$ rational matrix function $W \in \mathcal{R}_{M \times N}(\sigma)$ which satisfies

$$\sum_{z_0 \in \sigma} \text{Res}_{z=z_0} (zI - A_\zeta)^{-1} B_+ W(z) = -B_- \quad (1.4.1)$$

$$\sum_{z_0 \in \sigma} \text{Res}_{z=z_0} W(z) C_- (zI - A_\pi)^{-1} = C_+ \quad (1.4.2)$$

$$\sum_{z_0 \in \sigma} \text{Res}_{z=z_0} (zI - A_\zeta)^{-1} B_+ W(z) C_- (zI - A_\pi)^{-1} = \Gamma, \quad (1.4.3)$$

where σ is a subset of the complex plane \mathbb{C} .

We call the above *two-sided residue interpolation problem* (TRIP). Without loss of generality, we can assume

$$(A_\zeta, B_+) \text{ is a full-range pair, } \sigma(A_\zeta) \subset \sigma, \quad (1.4.4)$$

$$(C_-, A_\pi) \text{ is a null-kernel pair, } \sigma(A_\pi) \subset \sigma, \quad (1.4.5)$$

and

$$\Gamma \text{ satisfies the Sylvester equation} \quad (1.4.6)$$

$$\Gamma A_\pi - A_\zeta \Gamma = B_+ C_+ + B_- C_-.$$

More precisely, if (1.4.4)-(1.4.6) are not satisfied initially and solutions exist, then there is a new set of data with the same solutions for which (1.4.4)-(1.4.6) are satisfied (see [BGR4]). Hence throughout this thesis, when we mention (TRIP), we assume (1.4.4)-(1.4.6).

We now give specific examples to illustrate the various types of interpolation conditions which can be put in the form of (1.4.1)-(1.4.3).

Example 1

Suppose C_+, C_-, A_π and Γ are all vacuous (i.e., $n_\pi = 0$) and A_ζ, B_+, B_- have the form

$$A_\zeta := \begin{bmatrix} z_1 I_M & & 0 \\ & \ddots & \\ 0 & & \ddots & z_n I_M \end{bmatrix} \in \mathbb{C}^{nM \times nM},$$

$$[B_+ \ B_-] := \begin{bmatrix} I_M & -W_1 \\ \vdots & \vdots \\ I_M & -W_n \end{bmatrix} \in \mathbb{C}^{Mn \times (M+N)},$$

where $W_i \in \mathbb{C}^{M \times N}$. Then, (1.4.1)-(1.4.3) collapse to

$$Y(z_i) = W_i, \quad i = 1, \dots, n.$$

Example 2

Suppose C_+, C_-, A_π and Γ are as in Example 1 and A_ζ, B_+, B_- have the form

$$A_\zeta := \begin{bmatrix} z_1 I_{r_1} & & 0 \\ & \ddots & \\ 0 & & \ddots & z_m I_{r_m} \end{bmatrix} \in \mathbb{C}^{r \times r},$$

$$[B_+ \ B_-] := \begin{bmatrix} X_1 & -Y_1 \\ \vdots & \vdots \\ X_m & -Y_m \end{bmatrix} \in \mathbb{C}^{r \times (M+N)},$$

where I_{r_i} denotes the $r_i \times r_i$ identity matrix, $r := r_1 + \dots + r_m$, X_i and Y_i are of sizes $r_i \times M$, $r_i \times N$. Then, (1.4.1) collapses to

$$X_i W(z_i) = Y_i, \quad i = 1, 2, \dots, n.$$

Example 3

Suppose A_ζ, B_+, B_- and Γ are vacuous (i.e., $n_\zeta = 0$) and C_+, C_-, A_π are given by

$$\begin{bmatrix} C_+ \\ C_- \end{bmatrix} := \begin{bmatrix} V_1 & \dots & \dots & V_n \\ U_1 & \dots & \dots & U_n \end{bmatrix} \in \mathbb{C}^{(M+N) \times s}$$

$$A_\pi := \begin{bmatrix} w_1 I_{s_1} & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & w_n I_{s_n} \end{bmatrix} \in \mathbb{C}^{s \times s}$$

where $s := s_1 + \dots + s_n$, U_j, V_j are matrices of sizes $N \times s_j, M \times s_j$, respectively.

Then, the condition (1.4.2) collapses to

$$W(w_j)U_j = V_j, \quad j = 1, 2, \dots, n.$$

Example 4

This example combines Example 2 and Example 3. We assume that A_ζ, B_+, B_- are as in Example 2 and C_+, C_-, A_π , are as in Example 3. As noted in Example 2 and Example 3, the interpolation condition (1.4.1) amounts to

$$X_i W(z_i) = Y_i, \quad i = 1, 2, \dots, m$$

and (1.4.2) reduces to

$$W(w_j)U_j = V_j, \quad j = 1, 2, \dots, n.$$

In considering the condition (1.4.3) we can think of two cases.

Case I: $z_i \neq w_j$ for $i = 1, \dots, m, j = 1, \dots, n$.

It is known that in this case there exists a unique solution Γ of the Sylvester equation (1.4.6). On the other hand, it can be easily checked that the matrix Γ which is given by (1.4.3) satisfies the Sylvester equation (1.4.6) if W is any solution of (1.4.1) and (1.4.2). Hence the condition (1.4.3) is automatic from the necessary condition (1.4.6).

Case II: $z_i = w_j$ for some i, j .

In this case, the $(i, j)^{\text{th}}$ block of (1.4.3) is

$$\rho_{ij} = X_i W'(z_i) U_j.$$

Thus the additional set of interpolation conditions generated by (1.4.3) is to specify matrices Γ_{ij} for each index i, j for which $z_i = w_j$ and to demand

$$X_i W'(z_i) U_j = \Gamma_{ij} \quad \text{if } z_i = w_j.$$

Hence, the condition (1.4.3) collapses to

$$\Gamma_{ij} = \begin{cases} \frac{Y_i U_j - X_i V_j}{w_j - z_i}, & \text{if } z_i \neq w_j \\ \rho_{ij}, & \text{if } z_i = w_j \end{cases}$$

for $i = 1, \dots, m, j = 1, \dots, n$.

Example 5

Let

$$\begin{aligned} A_\zeta &:= \begin{bmatrix} z_0 & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & z_0 \end{bmatrix} \in \mathbb{C}^{n_\zeta \times n_\zeta}, \\ [B_+, B_-] &:= \begin{bmatrix} x_1 & -y_1 \\ \vdots & \vdots \\ x_{n_\zeta} & -y_{n_\zeta} \end{bmatrix} \in \mathbb{C}^{n_\zeta \times (M+N)} \\ \begin{bmatrix} C_+ \\ C_- \end{bmatrix} &:= \begin{bmatrix} v_1 & \dots & \dots & v_{n_\pi} \\ u_1 & \dots & \dots & u_{n_\pi} \end{bmatrix} \in \mathbb{C}^{(M+N) \times n_\pi}, \\ A_\pi &:= \begin{bmatrix} w_0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & \ddots & 1 \\ & & & w_0 \end{bmatrix} \in \mathbb{C}^{n_\pi \times n_\pi}, \end{aligned}$$

where $z_0 \neq w_0$, $x_i \in \mathbb{C}^{1 \times M}$, $y_i \in \mathbb{C}^{1 \times N}$ and $v_j \in \mathbb{C}^{M \times 1}$, $u_j \in \mathbb{C}^{N \times 1}$. Then the relevant interpolation conditions (1.4.1) and (1.4.2) become

$$x(z)W(z) = y(z) + (z - z_0)^{n_\zeta} \cdot [\text{analytic at } z_0]$$

$$W(z)u(z) = v(z) + (z - w_0)^{n_\pi} \cdot [\text{analytic at } w_0],$$

where we set

$$\begin{aligned}x(z) &= x_1 + x_2(z - z_0) + \cdots + x_{n_\zeta}(z - z_0)^{n_\zeta-1} \\y(z) &= y_1 + y_2(z - z_0) + \cdots + y_{n_\zeta}(z - z_0)^{n_\zeta-1} \\v(z) &= v_1 + v_2(z - w_0) + \cdots + v_{n_\pi}(z - w_0)^{n_\pi-1} \\u(z) &= u_1 + u_2(z - w_0) + \cdots + u_{n_\pi}(z - w_0)^{n_\pi-1}.\end{aligned}$$

Since $z_0 \neq w_0$, the matrix Γ in (1.4.3) is uniquely determined by the Sylvester equation in (1.4.6) and (1.4.3) follows from (1.4.1) and (1.4.2).

By [BGR4], all solutions of (TRIP) are parametrized as follows.

Theorem 1.4.1. *Let σ be a subset of \mathbb{C} and $\sigma(A_\pi) \cup \sigma(A_\zeta) \subset \sigma$. There exist rational matrix functions $W \in \mathcal{R}_{M \times N}(\sigma)$ which are solutions of (TRIP). Moreover, if $\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$ is any $(M+N) \times (M+N)$ matrix function having the set*

$$\tilde{\tau} := \left(\begin{bmatrix} C_+ \\ C_- \end{bmatrix}, A_\pi; A_\zeta, [B_+ B_-]; \Gamma \right)$$

as a σ -null-pole triple and φ^{-1} is a regular rational $N \times N$ matrix function having the set

$$\tilde{\tau}_- := (C_-, A_\pi; 0, 0; 0)$$

as a σ -null-pole triple. Then $W \in \mathcal{R}_{M \times N}(\sigma)$ is a solution of (TRIP) if and only if W has the following form. There exists rational matrix functions $P \in \mathcal{R}_{M \times N}(\sigma)$, $Q \in \mathcal{R}_{N \times N}(\sigma)$ for which the function

$$\varphi(\Theta_{21}P + \Theta_{22}Q)$$

has no zeros or poles in σ such that

$$W = (\Theta_{11}P + \Theta_{12}Q)(\Theta_{21}P + \Theta_{22}Q)^{-1}. \quad (1.4.7)$$

Theorem 1.4.1 suggests the study of the inverse homogeneous interpolation problem, namely: for a given σ -null-pole triple τ , find a rational matrix function for which Θ has τ as a σ -null-pole triple. This problem is studied in the next chapter. We close this chapter with a lemma and a remark which will be needed in the sequel.

Lemma 1.4.2. *Let $\sigma, \Theta, \varphi^{-1}$ be given as in Theorem 1.4.1. Then*

$$[\Theta_{21} \Theta_{22}] \mathcal{R}_{M+N}(\sigma) = \varphi^{-1} \mathcal{R}_N(\sigma).$$

For the proof, see [BGR4].

Remark 1.4.3. In theorem 1.4.1, for each solution $W \in \mathcal{R}_{M \times N}(\sigma)$, W can be represented in the form of (1.4.7) for $P \in \mathcal{R}_{M \times N}(\sigma)$ and $Q \in \mathcal{R}_{N \times N}(\sigma)$ satisfying

$$\varphi(\Theta_{21}P + \Theta_{22}Q) = I$$

(see [BGR4]).

II. Homogeneous interpolation problem

For a given subset of \mathbb{C} , denoted by σ , a collection of matrices

$$\tau = (C, A_\pi; A_\zeta, B; \Gamma) \quad (2.1)$$

is said to be a σ -admissible *Sylvester data set* if

$$\sigma(A_\pi) \cup \sigma(A_\zeta) \subset \sigma \quad (2.2)$$

$$(C, A_\pi) \text{ is a null-kernel pair of sizes } n \times n_\pi, n_\pi \times n_\pi \quad (2.3)$$

$$(A_\zeta, B) \text{ is a full-range pair of sizes } n_\zeta \times n_\zeta, n_\zeta \times n \quad (2.4)$$

$$\text{an } n_\zeta \times n_\pi \text{ matrix } \Gamma \text{ satisfies the following Sylvester equation} \quad (2.5)$$

$$\Gamma A_\pi - A_\zeta \Gamma = BC.$$

Here, we note that a σ -null-pole triple for a regular rational matrix function is a σ -admissible Sylvester data set.

A natural inverse problem is to construct a regular rational matrix function with a given σ -admissible Sylvester data set τ as its σ -null-pole triple. For the scalar case, the problem is to find a rational function which has zeros at z_1, \dots, z_r and poles at w_1, \dots, w_p with their multiplicities k_1, \dots, k_r and n_1, \dots, n_p . In this case, the condition $z_j \neq w_k$ for all j and k ($1 \leq j \leq r, 1 \leq k \leq p$) should be fulfilled for a solution to exist. If this condition holds, then a solution is given by

$$r(z) = c \frac{\prod_{j=1}^r (z - z_j)^{k_j}}{\prod_{i=1}^p (z - w_i)^{n_i}},$$

where c is a nonzero complex number. For the matrix case, recently there has appeared a number of papers on this question (see [BCR], [BR1,2], [GK1,2], [GKLR1] and [GKR1,2]; a complete treatment will appear in [BGR6]). It turns out that, for the matrix case, this problem is always solvable. Specifically, by [GKR2], there exists

an $n \times n$ regular rational matrix function Θ for which

$$\Theta \text{ has } \tau \text{ as its } \mathbb{C}\text{-null-pole triple} \quad (2.6)$$

$$\Theta \text{ has the minimal possible McMillan degree.} \quad (2.7)$$

In this chapter, this result is refined by demanding that $\Theta(z)$ fulfill (2.6), (2.7) and also the condition

$$\Theta \text{ is column reduced at infinity.} \quad (2.8)$$

Here, we note two things. Firstly, due to [BRk], the condition (2.7) is satisfied automatically if the conditions (2.6) and (2.8) are fulfilled. Secondly, if Γ in τ is square and invertible, then a rational matrix function Θ given by

$$\Theta(z) := I + C(zI - A_\pi)^{-1}\Gamma^{-1}B$$

satisfies (2.6)–(2.8).

This chapter consists of three sections. In the first section, some results of [GKR1] are introduced; that is, for a given σ -null-pole triple τ , a *minimal complement* τ_0 of τ is constructed. In the last section, a regular rational matrix function $\Theta(z)$ satisfying (2.6)–(2.8) is obtained; these results for the general rational case appeared for the first time in [ABKW]. Also, the column indices of such a $\Theta(z)$ are found in terms of a given spectral data set τ . The second section plays the role of stepping stone to Sections 1 and 3. There, a proper rational matrix function which has the prescribed σ -null-pole triple and has a special form of *Wiener-Hopf factorization* is found; this refines results of [GKR1] and extends work of [GLeR] to the rational case.

2.1. A minimal complement

Suppose a σ -admissible Sylvester data set τ as in (2.1) is given with $\sigma \subsetneq \mathbb{C}$. Let

$$\tau_0 := (C_0, A_{\pi 0}; A_{\zeta 0}, B_0; \Gamma_0)$$

be a ε -admissible Sylvester data set for ε , a subset of \mathbb{C} , satisfying $\varepsilon \cap \sigma = \emptyset$. We call τ_0 a *complement* to τ if the matrix

$$\tilde{\Gamma} := \begin{bmatrix} \Gamma & \Gamma_{12} \\ \Gamma_{21} & \Gamma_0 \end{bmatrix}$$

is square and invertible, where Γ_{12} and Γ_{21} are the unique solutions of

$$\Gamma_{12}A_{\pi 0} - A_\zeta\Gamma_{12} = BC_0 \quad (2.1.1)$$

$$\Gamma_{21}A_\pi - A_{\zeta 0}\Gamma_{21} = B_0C. \quad (2.1.2)$$

The complement will be called *minimal* if and only if among all complements of τ , the size of the matrix $\tilde{\Gamma}$ is as small as possible. If τ_0 is a minimal complement to τ , then the function

$$\Theta(z) = I + [C \ C_0] \begin{bmatrix} (zI - A_\pi)^{-1} & 0 \\ 0 & (zI - A_{\pi 0})^{-1} \end{bmatrix} \tilde{\Gamma}^{-1} \begin{bmatrix} B \\ B_0 \end{bmatrix}$$

has τ as a σ -null-pole triple and has the minimal possible McMillan degree among such functions.

To describe such a minimal complement, first we need to introduce some notions. Let N be a complement of $\text{Ker}\Gamma$ in \mathbb{C}^{n_π} and K be a complement of $\text{Im}\Gamma$ in \mathbb{C}^{n_ζ} , i.e.,

$$\mathbb{C}^{n_\pi} = \text{Ker}\Gamma \dot{+} N$$

$$\mathbb{C}^{n_\zeta} = \text{Im}\Gamma \dot{+} K.$$

Let ρ_π be the projection onto $\text{Ker}\Gamma$ along N and ρ_ζ be the projection onto K along $\text{Im}\Gamma$. Further, let η_π be the embedding of $\text{Ker}\Gamma$ into \mathbb{C}^{n_π} and η_ζ be the embedding of K into \mathbb{C}^{n_ζ} .

The controllability indices of a full-range pair can be defined in many ways. Here the controllability indices of the pair $(\rho_\zeta A|_K, \rho_\zeta B)$ are introduced through the following incoming subspaces. Let

$$H_0 := \text{Im}\Gamma \quad (2.1.3)$$

$$H_j := \text{Im}\Gamma + \text{Im}A_\zeta B + \cdots + \text{Im}A_\zeta^{j-1}B, \quad j = 1, 2, \dots \quad (2.1.4)$$

We define the incoming indices $\omega_1 \geq \dots \geq \omega_s$ by

$$s := \dim(H_1/H_0) \quad (2.1.5)$$

and

$$\omega_j := \#\{k \mid \dim(H_k/H_{k-1}) \geq j\} \quad j = 1, \dots, s. \quad (2.1.6)$$

Then, the numbers $\omega_1 \geq \dots \geq \omega_s$ are the nonzero controllability indices of $(\rho_\zeta A_\zeta|_K, \rho_\zeta B)$. Similarly, the observability indices of the null-kernel pair $(C|_{\text{Ker}\Gamma}, \rho_\pi A_\pi|_{\text{Ker}\Gamma})$ are defined through outgoing subspaces

$$K_0 := \text{Ker}\Gamma$$

$$K_j := \text{Ker}\Gamma \cap \text{Ker}CA_\pi \cap \cdots \cap \text{Ker}CA_\pi^{j-1}, \quad j = 1, 2, \dots$$

We also define outgoing indices $\alpha_1 \geq \dots \geq \alpha_t$ by $t = \dim(K_0/K_1)$ and

$$\alpha_j = \#\{\ell \mid \dim(K_{\ell-1}/K_\ell) \geq j\}, \quad j = 1, \dots, t.$$

Then $\alpha_1 \geq \dots \geq \alpha_t$ are the nonzero observability indices of the pair $(C|_{\text{Ker}\Gamma}, \rho_\pi A_\pi|_{\text{Ker}\Gamma})$.

Choose a point $\varepsilon \notin \sigma$. Let $\{e_{jk}\}_{k=1, j=1}^{\alpha_j, t}$ be an outgoing basis for $\text{Ker}\Gamma$ and $\{f_{jk}\}_{k=1, j=1}^{\omega_j, s}$ be an incoming basis for K . This means $\{f_{j1}\}_{j=1}^s$ forms a basis of a complement of $\text{Im}\Gamma$ in $\text{Im}\Gamma + \text{Im}B$

$$(A_\zeta - \varepsilon I)f_{jk} - f_{j, k+1} \in \text{Im}\Gamma + \text{Im}B \quad \text{for all } j, k \quad (f_{j, \omega_j+1} = 0)$$

$$(A_\pi - \varepsilon I)e_{jk} = e_{j, k+1}, \quad k = 1, \dots, \alpha_j - 1, \quad j = 1, \dots, t$$

$$\{e_{jk}\}_{k=1, j=1}^{\alpha_j-1, t} \quad \text{forms a basis for } \text{Ker}\Gamma \cap \text{Ker}C.$$

Such a basis can be constructed (see [BGK2]).

The next theorem gives a minimal complement of τ . For the proof, see [GKR1].

Theorem 2.1.1. *Let $\tau = (C, A_\pi; A_\zeta, B; \Gamma)$ be a σ -admissible Sylvester data set. Then a minimal complement to τ is given by*

$$\tau_0 = (-CX - F, T; S, -YB + G; Y\Gamma X - Y\eta_\zeta - \rho_\pi X).$$

Here $S : \text{Ker}\Gamma \rightarrow \text{Ker}\Gamma$ and $T : K \rightarrow K$ are given by

$$\begin{aligned} (S - \varepsilon)e_{jk} &= e_{j,k+1} & (e_{j,\alpha_j+1} = 0), \\ (T - \varepsilon)f_{jk} &= f_{j,k+1} & (f_{j,\omega_j+1} = 0). \end{aligned} \tag{2.1.7}$$

Furthermore, G and F are defined as follows. Let

$$z_j = Ce_{j,\alpha_j}, \tag{2.1.9}$$

and choose vectors y_j such that

$$f_{j1} - By_j \in \text{Im}\Gamma.$$

Choose vectors w_1, \dots, w_{n-s-t} so that an $n \times n$ constant matrix

$$E := [z_1, \dots, z_t, w_1, \dots, w_{n-s-t}, y_1, \dots, y_s] \tag{2.1.9}$$

has full rank. Then $G : \mathbb{C}^n \rightarrow \text{Ker}\Gamma$ is defined by

$$Gz_j = \rho_\pi(A_\pi - \varepsilon I)e_{j,\alpha_j}, \quad j = 1, \dots, t \tag{2.1.10}$$

$$Gy = 0, \quad y \in \text{span } \{w_j\}_{j=1}^{n-s-t} + \text{span } \{y_j\}_{j=1}^s \tag{2.1.11}$$

The operator $F : K \rightarrow \mathbb{C}^n$ is given by

$$F f_{jk} = u_{jk}$$

where $u_{jk} \in \text{span } \{y_i\}_{i=1}^s$ is such that

$$(A_\zeta - T)f_{jk} + Bu_{jk} \in \text{Im}\Gamma.$$

Finally,

$$X = \sum_{\nu=1}^{\omega_1} (A_\pi - \varepsilon)^{-\nu} \Gamma^+ (A_\zeta - BF)(T - \varepsilon)^{\nu-1} \quad (2.1.12)$$

$$Y = \sum_{\nu=1}^{\alpha_1} (S - \varepsilon)^{\nu-1} \rho_\pi (A_\pi - GC) \Gamma^+ (A_\zeta - \varepsilon)^{-\nu} \quad (2.1.13)$$

where Γ^+ is a generalized inverse of Γ such that $\Gamma \Gamma^+ = I - \rho_\zeta$, $\Gamma^+ \Gamma = I - \rho_\pi$, $\text{Ker} \Gamma^+ = K$ and $\text{Im} \Gamma^+ = N$.

Remark 2.1.2. (a) From (2.1.10) and (2.1.11) it follows

$$\rho_\pi (A_\pi - GC)|_{\text{Ker} \Gamma} = S$$

by (2.1.7)–(2.1.19).

(b) In Theorem 2.1.1, a $\sigma \cup \{\varepsilon\}$ -admissible Sylvester data set $\tau \oplus \tau_0$ is given by

$$\left([C - CX - F], \begin{bmatrix} A_\pi & 0 \\ 0 & T \end{bmatrix}; \begin{bmatrix} A_\zeta & 0 \\ 0 & S \end{bmatrix}, \begin{bmatrix} B \\ -YB + G \end{bmatrix}; \tilde{\Gamma} \right)$$

where the null-pole coupling matrix $\tilde{\Gamma}$ is an $(n_\pi + n_\zeta - \text{rank } \Gamma) \times (n_\pi + n_\zeta - \text{rank } \Gamma)$ invertible matrix given by

$$\tilde{\Gamma} = \begin{bmatrix} \Gamma & -\Gamma X + \eta_\zeta \\ -Y\Gamma + \rho_\pi & Y\Gamma X - \rho_\pi - Y\eta_\zeta \end{bmatrix} \quad (2.1.14)$$

and

$$\tilde{\Gamma}^{-1} = \begin{bmatrix} \eta_\pi Y + X\rho_\zeta + \Gamma^+ & \eta_\pi \\ \rho_\zeta & 0 \end{bmatrix} \quad (2.1.15)$$

In this case, an $n \times n$ rational matrix function $\tilde{\Theta}(z)$ with $\tilde{\Theta}(\infty) = I$ and having τ , as its σ -null-pole triple is given by

$$\tilde{\Theta}(z) = I + C(zI - A_\pi)^{-1} \{ (\Gamma^+ + X\rho_\zeta)B + \eta_\pi G \} - (CX + F)(zI - T)^{-1}B$$

and

$$\tilde{\Theta}^{-1}(z) = I - \{ C(\Gamma^+ + \eta_\pi Y) - F\rho_\zeta \} (zI - A_\zeta)^{-1}B - C(zI - S)^{-1}(-YB + G).$$

For more details, see [GKR1].

2.2. Wiener-Hopf factorization and local spectral data

By a *Wiener-Hopf factorization* of an $n \times n$ regular rational matrix function $W(z)$ with respect to a simple closed curve γ , we mean a factorization of $W(z)$ in the form

$$W(z) = W_-(z)D(z)W_+(z), \quad (2.2.1)$$

where $W_-(z)$ has no poles or zeros outside γ , $W_+(z)$ has no poles or zeros inside γ and

$$D(z) = \text{diag}\left(\left(\frac{z - \varepsilon_1}{z - \varepsilon_2}\right)^{\nu_j}\right)_{j=1}^n, \quad (2.2.2)$$

where the point ε_1 is inside γ , the point ε_2 is outside γ and ν_1, \dots, ν_n are integers which are called *Wiener-Hopf factorization indices*. In general an arbitrary rational matrix function does not admit a Wiener-Hopf factorization, but if $W(z)$ has no zeros or poles on γ , then $W(z)$ admits a Wiener-Hopf factorization with respect to γ . A related notion is that of Wiener-Hopf factorization at infinity. By this we mean a factorization of a regular $n \times n$ rational matrix function in the form (2.2.1), where $W_+(z)$ and $W_+^{-1}(z)$ are proper, $D(z)$ has the form $D(z) = \text{diag}(z^{\mu_j})_{j=1}^n$ for some integers $\mu_1 \leq \dots \leq \mu_n$ and $W_+(z)$ is unimodular (i.e., $W_+(z)$ has no zeros or poles in \mathbb{C}). By comparing (2.2.1) with (1.3.3), we note that a regular rational matrix function $W(z)$ is column reduced at infinity if and only if $W(z)$ has a Wiener-Hopf factorization with $W_+(z) \equiv I$.

Remember that in this chapter we want to find a rational matrix function $\Theta(z)$ satisfying (2.6) and (2.8) for a prescribed σ -admissible Sylvester data set τ . As an intermediate step, this section devotes to a biproper function which has the prescribed

σ -null-pole triple and has a Wiener-Hopf factorization (2.2.1) with $W_+(z) \equiv I$. Let

$$W(z) = I + C(zI - A)^{-1}B \quad (2.2.4)$$

be a minimal realization for $W \in \mathcal{R}_{n \times n}$ and γ be a simple closed curve on which $W(z)$ has no poles or zeros. Let P be the spectral projection of A corresponding to the region inside γ (denoted by γ_+) and P^\times be the spectral projection of $A^\times := A - BC$ to the same region. The image of P will be denoted by M and we write M^\times for the kernel of P^\times . Let

$$\tilde{\tau} := (\tilde{C}, \tilde{A}; \tilde{A}^\times, \tilde{B}; \tilde{\Gamma}) \quad (2.2.5)$$

where $\tilde{C} := C|_M$, $\tilde{A} := A|_M$, $\tilde{A}^\times := A^\times|_{\text{Im } P^\times}$, $\tilde{B} := P^\times B$, $\tilde{\Gamma} := P^\times|_M : M \rightarrow \text{Im } P$. By $\beta_1 \geq \dots \geq \beta_\ell$ we denote the nonzero observability indices of the pair $(\tilde{C}|_{\text{Ker } \tilde{\Gamma}}, \rho_\pi \tilde{A}|_{\text{Ker } \tilde{\Gamma}})$ and by $\sigma_1 \geq \dots \geq \sigma_m$ we denote the nonzero controllability indices of the pair $(\rho_\zeta \tilde{A}|_K, \rho_\zeta \tilde{B})$, where K is a complement of $\text{Im } \tilde{\Gamma}$ and ρ_ζ the projection onto K along $\text{Im } \tilde{\Gamma}$ and ρ_π be a projection onto $\text{Ker } \tilde{\Gamma}$. The next two theorems are due to [GKR1]; for the proofs, see Theorem 4.1 and Theorem 4.2 of [GKR1].

Theorem 2.2.1. *Let an $n \times n$ rational matrix function $W(z)$ be given by (2.2.4) and γ be a simple closed curve on which $W(z)$ has no poles or zeros. Then there exists an $n \times n$ rational matrix function $\tilde{\Theta}(z)$ for which*

- (a) $\tilde{\Theta}(z)W^{-1}(z)$ has no poles or zeros in γ_+
- (b) $\tilde{\Theta}(z)$ factors as $\tilde{\Theta}(z) = \tilde{\Theta}_-(z)\tilde{D}(z)$,

where γ_+ denotes the region inside γ and $\tilde{\Theta}_-(z)$ has no poles and zeros outside γ .

The rational matrix function $\tilde{D}(z)$ is in the form

$$\tilde{D}(z) = \text{diag}\left(\left(\frac{z - \varepsilon_1}{z - \varepsilon_2}\right)^{\kappa_j}\right)_{j=1}^n$$

where ε_1 is inside γ , ε_2 is outside γ and

$$\kappa_j = \begin{cases} -\beta_j, & 1 \leq j \leq \ell \\ 0, & \ell + 1 \leq j \leq n - m \\ \sigma_{n-j+1}, & n - m + 1 \leq j \leq n. \end{cases}$$

To have a realization formula for $\tilde{\Theta}(z)$ in Theorem 2.2.1, we consider a minimal complement of $\tilde{\tau}_0$ to $\tilde{\tau}$ given by (2.2.5). Let

$$\tilde{\tau}_0 := (-\tilde{C}X - F, T; S, -Y\tilde{B} + G; Y\tilde{\Gamma}X - Y\eta_\zeta - \rho_\pi X)$$

be a minimal complement to $\tilde{\tau}$ built as in Theorem 2.1.1.

Theorem 2.2.2. *Let an $n \times n$ constant matrix E be constructed as in (2.1.9) with $\tilde{\tau}$ in (2.2.5). Then a rational matrix function $\tilde{\Theta} \in \mathcal{R}_{n \times n}$ satisfying (a),(b) of Theorem 2.2.1 is given by*

$$\tilde{\Theta}(z) = E + \tilde{C}(zI - \tilde{A})^{-1} \{ (\tilde{\Gamma}^+ + X\rho_\zeta)\tilde{B} + \eta_\pi G \} E - (\tilde{C}X + F)(zI - T)^{-1}\rho_\zeta\tilde{B}E,$$

where $\tilde{\Gamma}^+$ is a generalized inverse of $\tilde{\Gamma}$.

2.3. A column reduced rational matrix function

Suppose a σ -admissible Sylvester data set

$$\tau = (C, A_\pi; A_\zeta, B; \Gamma) \quad (2.3.1)$$

is given and α is a complex number satisfying $\alpha \notin \sigma(A_\pi) \cup \sigma(A_\zeta) \cup \{0\}$. Let

$$\psi(z) := \alpha + \frac{1}{z}$$

and

$$\hat{\sigma} := \{ \psi(z) \mid z \in \sigma(A_\pi) \cup \sigma(A_\zeta) \}.$$

Then

$$\hat{\tau} := (-C(A_\pi - \alpha I)^{-1}, (A_\pi - \alpha I)^{-1}; (A_\zeta - \alpha I)^{-1}, (A_\zeta - \alpha I)^{-1}B; \Gamma) \quad (2.3.2)$$

is a $\hat{\sigma}$ -admissible Sylvester data set.

In the following theorem, a realization formula for a rational matrix function satisfying (2.6), (2.8) and the column indices of such a function are given.

Theorem 2.3.1. *Suppose a σ -admissible Sylvester data set $\tau = (C, A_\pi; A_\zeta, B; \Gamma)$ is given. For a complex number $\alpha \notin \sigma(A_\pi) \cup \sigma(A_\zeta) \cup \{0\}$, let*

$$\hat{\tau}_0 := (-C(A_\pi - \alpha I)^{-1}X - F, T; S, -Y(A_\zeta - \alpha I)^{-1}B + G; Y\Gamma X - Y\eta_\zeta - \rho_\pi X)$$

be a minimal complement of $\hat{\tau}$ in (2.3.2) constructed as in Theorem 2.1.1 with $\varepsilon = 0$ and E be an $n \times n$ constant matrix constructed by (2.1.9) with $\tau = \hat{\tau}, \varepsilon = 0$. Then

$$\begin{aligned} \Theta(z) &:= E - (z - \alpha)[-C(A_\pi - \alpha I)^{-1}X + F](I + \alpha T)^{-1}] \\ &\quad \times \begin{bmatrix} (zI - A_\pi)^{-1} & 0 \\ 0 & \{I - zT(I + \alpha T)^{-1}\}^{-1} \end{bmatrix} \begin{bmatrix} \Gamma^x + X\rho_\zeta + \eta_\pi Y & \eta_\pi \\ \rho_\zeta & 0 \end{bmatrix} \\ &\quad \times \begin{bmatrix} (A_\zeta - \alpha I)^{-1}B \\ -Y(A_\zeta - \alpha I)^{-1}B + G \end{bmatrix} E \end{aligned} \quad (2.3.3)$$

satisfies (2.6) and (2.8). Moreover,

$$\text{the } j^{\text{th}} \text{ column index of } \Theta = \begin{cases} -\alpha_j, & 1 \leq j \leq t \\ 0, & t+1 \leq j \leq n-s \\ \omega_{n-j+1}, & n-s+1 \leq j \leq n, \end{cases} \quad (2.3.4)$$

where $\alpha_1 \geq \dots \geq \alpha_t$ are the nonzero observability indices of the pair $(\rho_\pi A_\pi|_{Ker\Gamma}, C|_{Ker\Gamma})$ and $\omega_1 \geq \dots \geq \omega_s$ are the nonzero controllability indices of the pair $(\rho_\zeta A_\zeta|_K, \rho_\zeta B)$.

Proof. Let $\hat{\gamma}$ be a simple closed curve for which

$$\sigma((A_\pi - \alpha I)^{-1}) \cup \sigma((A_\zeta - \alpha I)^{-1}) \cup \{\frac{1}{\alpha}\} \text{ is inside } \hat{\gamma} \quad (2.3.5)$$

and

$$0 \text{ is outside } \hat{\gamma}. \quad (2.3.6)$$

Then $\hat{\tau}$ given by (2.3.2) is a $\hat{\gamma}_+$ -admissible Sylvester data set, where $\hat{\gamma}_+$ is the region inside $\hat{\gamma}$. By Remark 2.1.2.(b), there exists an $n \times n$ rational matrix function $\tilde{\Theta}(z)$ for which

$$\tilde{\Theta}(\infty) = I$$

and

$$\hat{\tau} \text{ is a } \hat{\gamma}_+ \text{-null-pole triple for } \tilde{\Theta}(z). \quad (2.3.7)$$

Then by Theorem 2.2.1 and Theorem 2.2.2, a rational matrix function

$$\begin{aligned} \hat{\Theta}(z) := & E - C(A_\pi - \alpha I)^{-1} \{ zI - (A_\pi - \alpha I)^{-1} \}^{-1} \{ (\Gamma^+ + X\rho_\zeta)(A_\zeta - \alpha I)^{-1} B \\ & + \eta_\pi G \} E - \{ C(\alpha I - A_\pi)^{-1} X + F \} (zI - T)^{-1} \rho_\zeta (A_\zeta - \alpha I)^{-1} B E \end{aligned} \quad (2.3.8)$$

satisfies the following properties:

$$\hat{\Theta}(z) \text{ has } \hat{\tau} \text{ as its } \hat{\gamma}_+ \text{-null-pole triple} \quad (2.3.9)$$

and $\hat{\Theta}(z)$ factors as $\hat{\Theta}(z) = \hat{\Theta}_-(z)\hat{D}(z)$, where

$$\hat{\Theta}_-(z) \text{ has no poles or zeros outside } \hat{\gamma} \quad (2.3.10)$$

and

$$\hat{D}(z) = \text{diag}\left(\left(\frac{z - \varepsilon_1}{z - \varepsilon_2}\right)^{\kappa_j}\right)_{j=1}^n,$$

where ε_1 is inside $\hat{\gamma}$, ε_2 is outside $\hat{\gamma}$ and

$$\kappa_j = \begin{cases} -\beta_j, & 1 \leq j \leq \ell \\ 0, & \ell + 1 \leq j \leq n - m \\ \sigma_{n-j+1}, & n - m + 1 \leq j \leq n. \end{cases} \quad (2.3.11)$$

Here $\beta_1 \geq \dots \geq \beta_\ell$ are the nonzero observability indices of the pair $(-C(A_\pi - \alpha I)^{-1}|_{\text{Ker}\Gamma}, \rho_\pi(A_\pi - \alpha I)^{-1}|_{\text{Ker}\Gamma})$ and $\sigma_1 \geq \dots \geq \sigma_m$ are the nonzero controllability indices of the pair $(\rho_\zeta(A_\zeta - \alpha I)^{-1}|_K, \rho_\zeta(A_\zeta - \alpha I)^{-1}B)$. If we choose $\varepsilon_1 = -\frac{1}{\alpha}$, $\varepsilon_2 = 0$ for $\hat{D}(z)$, then

$$\hat{D}(z) = \text{diag}\left(\left(\frac{1}{\alpha} \frac{\alpha z + 1}{z}\right)^{\kappa_j}\right)_{j=1}^n. \quad (2.3.12)$$

Let $\Theta(z) := \hat{\Theta}(\frac{1}{z-\alpha})$. Then from (2.3.8), $\Theta(z)$ is given by

$$\begin{aligned} \Theta(z) = & E + (z - \alpha)C(zI - A_\pi)^{-1} \{ (\Gamma^+ + X\rho_\zeta)(A_\zeta - \alpha I)^{-1} B + \eta_\pi G \} E \\ & - (z - \alpha) \{ C(\alpha I - A_\pi)^{-1} X + F \} (I + \alpha T)^{-1} \{ I - zT(I + \alpha T)^{-1} \}^{-1} \\ & \rho_\zeta(A_\zeta - \alpha I)^{-1} B E \end{aligned}$$

which is equivalent to (2.3.3).

To prove $\Theta(z)$ has τ as its \mathbb{C} -null-pole triple, let us consider a simple closed curve

$$\gamma := \left\{ \frac{1}{z - \alpha} \mid z \in \hat{\gamma} \right\}.$$

Then, by (2.3.5) and (2.3.6),

$$\sigma(A_\pi) \cup \sigma(A_\zeta) \cup \{0\} \text{ is inside } \gamma \quad (2.3.13)$$

and

$$\{\infty\} \text{ is outside } \gamma. \quad (2.3.14)$$

Upon applying Theorem 5.1.3 of [BGR6] to (2.3.7) we know that $\Theta(z)$ has τ as its γ_+ -null-pole triple for any simple closed curve γ satisfying (2.3.13) and (2.3.14). Hence, it is forced that $\Theta(z)$ has τ as its \mathbb{C} -null-pole triple.

Now, we will prove that $\Theta(z)$ is column reduced at infinity. Let

$$\Theta_-(z) := \hat{\Theta}_-\left(\frac{1}{z - \alpha}\right) \text{diag}(\alpha^{-\kappa_j})_{j=1}^n \quad (2.3.15)$$

and

$$D(z) := \text{diag}(\alpha^{\kappa_j})_{j=1}^n \cdot \hat{D}\left(\frac{1}{z - \alpha}\right). \quad (2.3.16)$$

Then,

$$\Theta(z) = \Theta_-(z)D(z)$$

and upon substituting (2.3.12) in (2.3.16), (2.3.16) is reduced to

$$D(z) = \text{diag}(z^{\kappa_j})_{j=1}^n. \quad (2.3.17)$$

From (2.3.9), we know that $\Theta_-(z)$ has no poles or zeros inside γ for any simple closed curve γ satisfying (2.3.13) and (2.3.14). Hence, it is forced that

$$\Theta_-(z) \text{ is unimodular.} \quad (2.3.18)$$

By (2.3.17) and (2.3.18), $\Theta(z)$ is column reduced at infinity with the column indices $\kappa_1 \leq \cdots \leq \kappa_n$, where $\kappa_1, \dots, \kappa_n$ are given by (2.3.11).

The only thing left is to prove (2.3.4). Now, we shall show that the controllability indices of the pair $(\rho_\zeta A_\zeta|_K, \rho_\zeta B)$ and those of $(\rho_\zeta(A_\zeta - \alpha I)^{-1}|_K, \rho_\zeta(A_\zeta - \alpha I)^{-1}B)$ are the same. Since the controllability indices are defined by (2.1.3)–(2.1.6), it is enough to show that

$$\dim H_j = \dim \tilde{H}_j, \quad j = 1, 2, \dots,$$

where H_0, H_j are given by (2.1.3) (2.1.4) and

$$\tilde{H}_j := \text{Im}\Gamma + \text{Im}(A_\zeta - \alpha I)^{-1}B + \cdots + \text{Im}(A_\zeta - \alpha I)^{-j}B, \quad j = 1, 2, \dots.$$

For $j = 1, 2, \dots$,

$$\begin{aligned} \tilde{H}_j &= (A_\zeta - \alpha I)^{-j} \{ \text{Im}(A_\zeta - \alpha I)^j\Gamma + \text{Im}(A_\zeta - \alpha I)^{j-1}B + \cdots + \text{Im}B \} \\ &\subset (A_\zeta - \alpha I)^{-j} \{ \text{Im}(A_\zeta - \alpha I)^j\Gamma + \text{Im}B + \cdots + \text{Im}A_\zeta^{j-1}B \}. \end{aligned} \tag{2.3.19}$$

Using a binomial expansion and the Sylvester equation $\Gamma A_\pi - A_\zeta\Gamma = BC$ repeatedly, it is found that

$$\text{Im}(A_\zeta - \alpha I)^j\Gamma \subset \text{Im}\Gamma + \text{Im}B + \cdots + \text{Im}A_\zeta^{j-1}B. \tag{2.3.20}$$

Substituting (2.3.20) into (2.3.19) we see that

$$\tilde{H}_j \subset (A_\zeta - \alpha I)^{-j}H_j.$$

Hence

$$\dim \tilde{H}_j \leq \dim H_j. \tag{2.3.21}$$

To prove the opposite inequality, we observe that

$$\begin{aligned} H_j &= \text{Im}\Gamma + \text{Im}B + \text{Im}A_\zeta B + \cdots + \text{Im}A_\zeta^{j-1}B \\ &\subset \text{Im}\Gamma + \text{Im}B + \text{Im}(A_\zeta - \alpha I)B + \cdots + \text{Im}(A_\zeta - \alpha I)^{j-1}B. \end{aligned}$$

Equivalently,

$$H_j \subset (A_\zeta - \alpha I)^j \{ \text{Im}(A_\zeta - \alpha I)^{-j} \Gamma + \text{Im}(A_\zeta - \alpha I)^{-j} B + \cdots + \text{Im}(A_\zeta - \alpha I)^{-1} B \}. \quad (2.3.22)$$

By applying the Sylvester equation

$$\Gamma(A_\zeta - \alpha I)^{-1} - (A_\zeta - \alpha I)^{-1} \Gamma = -(A_\zeta - \alpha I)^{-1} BC(A_\pi - \alpha I)^{-1}$$

to the term $(A_\zeta - \alpha I)^{-j} \Gamma$ of (2.3.22), (2.3.22) is reduced to

$$\begin{aligned} H_j &\subset (A_\zeta - \alpha I)^j \{ \text{Im}\Gamma + \text{Im}(A_\zeta - \alpha I)^{-1} B + \cdots + \text{Im}(A_\zeta - \alpha I)^{-j} B \} \\ &\subset (A_\zeta - \alpha I)^j \tilde{H}_j. \end{aligned}$$

Thus,

$$\dim H_j \leq \dim \tilde{H}_j. \quad (2.3.23)$$

From (2.3.21) and (2.3.23), we have

$$\dim H_j = \dim \tilde{H}_j.$$

To prove the other half of (2.3.4), we go through the previous argument with the transpose of the pairs $(C|_{\text{Ker}\Gamma}, \rho_\pi A_\pi|_{\text{Ker}\Gamma})$ and $(-C(A_\pi - \alpha I)^{-1}|_{\text{Ker}\Gamma}, \rho_\pi(A_\pi - \alpha I)^{-1}|_{\text{Ker}\Gamma})$. This completes the proof. \square

Remark 2.3.2. As we mentioned in the previous section, a rational matrix function W is column reduced at infinity if and only if W has a Wiener-Hopf factorization (2.2.1) at infinity with $W_+(z) \equiv I$. This connection implies that the vast literature on Wiener-Hopf factorization can be brought to bear on the construction of column reduced rational matrix functions. Indeed, the proof of Theorem 2.3.1 is one instance of this; another is [GLeR] where a form of Theorem 2.3.1 for the polynomial case appears but expressed in the context of Wiener-Hopf factorization at infinity.

In the rest of this section, a realization formula for a rational matrix function $\Theta(z)$ satisfying (2.6) and (2.8) for the σ -null-pole triple Γ given by (2.3.1) with $n_\zeta = 0$ or $n_\pi = 0$ is found. The following corollary provides an $n \times n$ regular rational matrix function $\varphi^{-1}(z)$ for which

$$\tau = (C, A_\pi; 0, 0; 0) \text{ is a } \mathbb{C}\text{-null-pole triple for } \varphi^{-1}(z) \quad (2.3.24)$$

$$\varphi^{-1}(z) \text{ is column reduced at infinity.} \quad (2.3.25)$$

Corollary 2.3.3. *Let*

$$\hat{\tau}_0 = (0, 0; S, G; 0) \quad (2.3.26)$$

be a minimal complement to $\hat{\tau} = (-C(A_\pi - \alpha I)^{-1}, (A_\pi - \alpha I)^{-1}; 0, 0; 0)$ as in Theorem 2.3.1 and a complex number α and an $n \times n$ matrix E be given as in Theorem 2.3.1 with τ given by (2.3.24). Then a rational matrix function

$$\varphi^{-1}(z) := E + (z - \alpha)C(zI - A_\pi)^{-1}(I - \alpha S_\infty)^{-1}G_\infty E \quad (2.3.27)$$

satisfies (2.3.24) (2.3.25) and $\varphi(z)$ is given by

$$\varphi(z) := E^{-1} + (z - \alpha)E^{-1}C(A_\pi - \alpha I)^{-1}(I - zS_\infty)^{-1}G_\infty, \quad (2.3.28)$$

where $G_\infty \in \mathbb{C}^{n_\pi \times n}$, $S_\infty \in \mathbb{C}^{n_\pi \times n_\pi}$ are defined by

$$S_\infty := (I + \alpha S)^{-1}S \quad (2.3.29)$$

$$G_\infty := (I + \alpha S)^{-1}G. \quad (2.3.30)$$

In this case,

the j^{th} column index of $\varphi^{-1}(z) = -\alpha_j$, $j = 1, \dots, n$,

where $\alpha_1 \geq \dots \geq \alpha_n$ are the observability indices of the pair (C, A_π) .

Proof. By substituting (2.3.29) (2.3.30) in (2.3.27), we get

$$\varphi^{-1}(z) := E + (z - \alpha)C(zI - A_\pi)^{-1}GE \quad (2.3.31)$$

to which $\Theta(z)$ given by (2.3.3) is reduced if $n_\zeta = 0$. Thus, by Theorem 2.3.1, every assertion except (2.3.28) is proved.

To prove (2.3.28), we apply (1.1.9) to (2.3.27) and get

$$\varphi(z) := E^{-1} + (z - \alpha)E^{-1}C(zH^\times - A_\pi^\times)G \quad (2.3.32)$$

where

$$H^\times = I + GC, \quad A_\pi^\times = A_\pi + \alpha GC.$$

For this case, Remark 2.1.2.(a) implies that

$$(A_\pi - \alpha)^{-1} + GC(A_\pi - \alpha I)^{-1} = S$$

or equivalently

$$I + GC = S(A_\pi - \alpha I). \quad (2.3.33)$$

Upon applying the above equality to H^\times and A_π^\times , they are reduced to

$$H^\times = S(A_\pi - \alpha I), \quad A_\pi^\times = (I + \alpha S)(A_\pi - \alpha I).$$

By plugging the above in (2.3.32) and applying the identities (2.3.29) and (2.3.30), (2.3.28) is obtained. \square

Next, we suppose $n_\pi = 0$. Then τ in (2.3.1) is given by

$$\tau = (0, 0; A_\zeta, B; 0) \quad (2.3.34)$$

and a minimal complement $\hat{\tau}_0$ in Theorem 2.3.1 is given by

$$\hat{\tau}_0 = (F, T; 0, 0; 0).$$

The next corollary is dual to Corollary 2.3.3. The proof is omitted.

Corollary 2.3.4. Let (A_ζ, B) be a given full-range pair of sizes $n_\zeta \times n_\zeta$, $n_\zeta \times n$, respectively, and a complex number α and an $n \times n$ matrix E are as in Theorem 2.3.1 with τ given by (2.3.34). Then an $n \times n$ matrix polynomial

$$\psi(z) := E - (z - \alpha)F_\infty(I - zT_\infty)^{-1}(A_\zeta - \alpha I)^{-1}BE$$

has (A_ζ, B) as its \mathbb{C} -null pair and is column reduced at infinity with the j^{th} column index ω_{n-j+1} , where $\omega_1 \geq \dots \geq \omega_n$ are the controllability indices of (A_ζ, B) and

$$F_\infty := F(I + \alpha T)^{-1}, \quad T_\infty := T(I + \alpha T)^{-1}.$$

We close this chapter with the following remark.

Remark 2.3.5. In Corollary 2.3.3, the matrices S_∞, A_π, G satisfy the following identity

$$S_\infty A_\pi - I = G_\infty C. \quad (2.3.35)$$

Indeed, upon substituting (2.3.29) and (2.3.3) in (2.3.35), it turns out that (2.3.35) is equivalent to (2.3.33).

III. A matrix polynomial solution of low McMillan degree

In the scalar case, the well-known Lagrange interpolation problem seeks a polynomial $l(z)$ for which $l(z_j) = w_j$ for given points z_1, \dots, z_n and w_1, \dots, w_n in the complex plane, where $z_i \neq z_j$ if $i \neq j$. A solution of degree at most $n - 1$ exists, is unique, and is given by

$$l(z) = \sum_{k=1}^n w_k \prod_{j \neq k} \frac{(z - z_j)}{(z_k - z_j)}.$$

In this chapter, we consider a problem of constructing a matrix polynomial function with low McMillan degree satisfying the two-sided residue interpolation problem given by (1.4.1)–(1.4.3) with $\sigma = \mathbb{C}$. For example, suppose A_ζ, B_+, B_- are vacuous (i.e., $n_\zeta = 0$) and C_+, C_-, A_π are given as follows:

$$\begin{bmatrix} C_+ \\ C_- \end{bmatrix} = \begin{bmatrix} v_1, \dots, v_{n_\pi} \\ u_1, \dots, u_{n_\pi} \end{bmatrix}; \quad A_\pi = \begin{bmatrix} w_1 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & w_{n_\pi} \end{bmatrix},$$

where the size of v_k is $M \times 1$, u_k is $N \times 1$ nonzero column vector for $k = 1, \dots, n_\pi$. Then the problem (1.4.1)–(1.4.3) is reduced to finding a matrix polynomial $L \in \mathcal{P}_{M \times N}$ for which

$$L(w_k)u_k = v_k, \quad k = 1, \dots, n_\pi. \quad (3.1)$$

By a reduction to the scalar case, we can see that there exists a matrix polynomial solution $L(z)$ with polynomial degree at most $n_\pi - 1$. Indeed, choose an $M \times N$ matrix M_k satisfying $M_k u_k = v_k$; this can be done since u_k is a nonzero column vector by assumption. Then, we can generate solutions by solving the problem

$$L(w_k) = M_k, \quad k = 1, \dots, n_\pi.$$

Suppose the unknown $L(z)$ has the matrix entries $l_{ij}(z)$ and $m_{ij}^{(k)}$ is the $(i, j)^{\text{th}}$ entry of M_k ($1 \leq i \leq M$, $1 \leq j \leq N$). If one finds MN scalar Lagrange polynomials $l_{ij}(z)$

for which

$$l_{ij}(w_k) = m_{ij}^{(k)}$$

for $k = 1, \dots, n_\pi$, then $L(z) = (l_{ij}(z))_{i=1}^M {}_{j=1}^N$ is a unique $M \times N$ matrix polynomial with polynomial degree at most $n_\pi - 1$ which solves (3.1). In this way, we solve the problem with an $M \times N$ matrix polynomial of polynomial degree at most $n_\pi - 1$. Of course, the solution depends on the choice of matrix M_k satisfying $M_k u_k = v_k$.

Another measure of complexity for matrix polynomials is the McMillan degree. As we see in Corollary 1.1.2, the McMillan degree of a matrix polynomial is greater than or equal to the polynomial degree. It seems that the existence of a solution of McMillan degree at most $n_\pi - 1$ does not follow by a reduction to a collection of scalar problems. However, the main result of this chapter concludes that there exists a matrix polynomial which solves (3.1) and is of McMillan degree at most $n_\pi - 1$.

There are four sections in this chapter. The first and the second sections are parallel; the first one concerns the one-sided problem and the second one the two-sided problem. In each of these two sections, a matrix polynomial solution of low McMillan degree is obtained in realization form. The remaining two sections concern applications of the results of the previous two sections. In the third section, the simultaneous interpolation problem is considered. If certain necessary and sufficient conditions for the existence of solutions (which are known already) are fulfilled, there exists a matrix polynomial solution with low McMillan degree. In the last section, the interpolation problem is formulated in divisor-remainder form and a Euclidean algorithm with respect to the McMillan degree is given. The main results of this chapter appear also in [BK].

3.1 One-sided problem

Consider the following tangential interpolation problem.

Given s distinct complex numbers w_1, \dots, w_s , given s column polynomials $u_1(z), \dots, u_s(z)$ of size $N \times 1$ for which $u_k(w_k) \neq 0$ for $k = 1, \dots, s$ and given s column vector polynomials $v_1(z), \dots, v_s(z)$ of size $M \times 1$, find an $M \times N$ matrix polynomial $L(z)$ for which

$$\frac{d^{\alpha-1}}{dz^{\alpha-1}}\{L(z)u_k(z)\}\Big|_{z=w_k} = \frac{d^{\alpha-1}}{dz^{\alpha-1}}v_k(z)\Big|_{z=w_k} \quad (3.1.1)$$

for $1 \leq k \leq s$, $1 \leq \alpha \leq m_k$. In (3.1.1), if we let $u_k(z)$ and $v_k(z)$ have Taylor expansions

$$u_k(z) = \sum_{j=1}^{\infty} u_{kj}(z - w_k)^{j-1}$$

$$v_k(z) = \sum_{j=1}^{\infty} v_{kj}(z - w_k)^{j-1}$$

at w_k , and the unknown matrix polynomial $L(z)$ have Taylor form

$$L(z) = \sum_{j=1}^{m_k} L_{kj}(z - w_k)^{j-1},$$

then the interpolation condition (3.1.1) can be expressed in terms of Taylor coefficients as

$$[L_{k1}, \dots, L_{km_k}] \begin{bmatrix} u_{k1} & \cdots & \cdots & u_{km_k} \\ \ddots & & \ddots & \vdots \\ 0 & & \ddots & u_{k1} \end{bmatrix} = [v_{k1}, \dots, v_{km_k}] \quad (3.1.2)$$

for $1 \leq k \leq s$. If we put

$$A_\pi = \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_s \end{bmatrix}$$

$$C_+ = [v_{11}, \dots, v_{1m_1}, \dots, v_{s1}, \dots, v_{sm_s}]$$

$$C_- = [u_{11}, \dots, u_{1m_1}, \dots, u_{s1}, \dots, u_{sm_s}]$$

where J_i is $m_i \times m_i$ Jordan matrix with eigenvalue w_i , then the interpolation condition (3.1.2) takes the form

$$\sum_{z_0 \in \mathbb{C}} \text{Res}_{z=z_0} L(z) C_- (zI - A_\pi)^{-1} = C_+ \quad (3.1.3)$$

or equivalently, the form

$$\frac{1}{2\pi i} \int_{\gamma} L(z) C_- (zI - A_\pi)^{-1} dz = C_+ \quad (3.1.4)$$

for a simple closed contour γ such that all eigenvalues of A_π are inside γ . It is more convenient for us to work with the interpolation problem in the latter formulation (3.1.3) or (3.1.4) than in the formulation (3.1.1), and in this thesis the formulation (3.1.3) will be adopted. Since in the formulation (3.1.3) the interpolation conditions are described in terms of the residue of a matrix-valued function and matrix polynomial solutions are sought, we shall call the problem (3.1.3) the *residue interpolation problem for matrix polynomials* (RIPP).

Let us restate the (RIPP).

(RIPP) Given matrices C_+, C_-, A_π of sizes $M \times n_\pi$, $N \times n_\pi$ and $n_\pi \times n_\pi$ respectively, find an $M \times N$ matrix polynomial $L(z)$ for which

$$\sum_{z_0 \in \mathbb{C}} \text{Res}_{z=z_0} L(z) C_- (zI - A_\pi)^{-1} = C_+.$$

The dual problem of (RIPP) is stated as follows:

(RIPP_d) Given matrices A_ζ, B_+, B_- of sizes $n_\zeta \times n_\zeta$, $n_\zeta \times M$ and $n_\zeta \times N$ respectively, find an $M \times N$ matrix polynomial $L(z)$ for which

$$\sum_{z_0 \in \mathbb{C}} \text{Res}_{z=z_0} (zI - A_\zeta)^{-1} B_+ L(z) = -B_-.$$

Here without loss of generality, we can assume that (C_-, A_π) is a null-kernel pair and (A_ζ, B_+) is a full-range pair (see Section 1.4).

Before we state the first theorem in this chapter, we introduce some notions. Let $\mathcal{P}_{M \times N}$ (\mathcal{P}_N) denote the set of all $M \times N$ matrix ($N \times 1$ vector) polynomials. Throughout this section

$$\tilde{\tau} = \left(\begin{bmatrix} C_+ \\ C_- \end{bmatrix}, A_\pi; 0, 0; 0 \right) \quad (3.1.5)$$

and

$$\tilde{\tau}_- = (C_-, A_\pi; 0, 0; 0), \quad (3.1.6)$$

where the matrices C_+, C_-, A_π are given as in (RIPP). Then $\tilde{\tau}$ and $\tilde{\tau}_-$ are σ -admissible Sylvester data sets.

A particular solution to (RIPP) is found in the following theorem.

Theorem 3.1.1. *Let C_+, C_-, A_π be given as in (RIPP) and S_∞ and G_∞ be given as in Corollary 2.3.3 with $\tilde{\tau}_-$ in place of τ . Then*

$$K_0(z) = -C_+G_\infty - zC_+S_\infty(I - zS_\infty)^{-1}G_\infty \quad (3.1.7)$$

is a particular solution to (RIPP) of McMillan degree at most $n_\pi - 1$.

Proof. Let α be a complex number which is not in $\sigma(A_\pi) \cup \{0\}$ and

$$\varphi^{-1}(z) = I + (z - \alpha)C_-(zI - A_\pi)^{-1}(I - \alpha S_\infty)^{-1}G_\infty. \quad (3.1.8)$$

Then, by Corollary 2.3.3, $\varphi^{-1} \in \mathcal{R}_{N \times N}$ has $\tilde{\tau}_-$ as its \mathbb{C} -null-pole triple and $\varphi(z)$ is given by

$$\varphi(z) = I + (z - \alpha)C_-(A_\pi - \alpha I)^{-1}(I - zS_\infty)^{-1}G_\infty. \quad (3.1.9)$$

Equivalently, by (1.1.6) and (1.1.7),

$$\varphi^{-1}(z) = I + C_-(I - \alpha S_\infty)^{-1}G_\infty - C_-(zI - A_\pi)^{-1}(\alpha I - A_\pi)(I - \alpha S_\infty)^{-1}G_\infty \quad (3.1.10)$$

and

$$\varphi(z) = I - \alpha C_-(A_\pi - \alpha I)^{-1} G_\infty + z C_-(A_\pi - \alpha I)^{-1} (I - z S_\infty)^{-1} (I - \alpha S_\infty) G_\infty. \quad (3.1.11)$$

If we take $\sigma = \mathbb{C}$ in Theorem 1.4.1, then Theorem 1.4.1 and Remark 1.4.3 say that $L \in \mathcal{P}_{M \times N}$ satisfies (RIPP) if and only if $L(z)$ has the following form: There exist $Q_1 \in \mathcal{P}_{M \times N}$, $Q_2 \in \mathcal{P}_{N \times N}$ for which

$$\varphi(\Theta_{21}Q_1 + \Theta_{22}Q_2) = I \quad (3.1.12)$$

such that

$$L = (\Theta_{11}Q_1 + \Theta_{12}Q_2)(\Theta_{21}Q_1 + \Theta_{22}Q_2)^{-1}, \quad (3.1.13)$$

where an $(M+N) \times (M+N)$ rational matrix function

$$\Theta(z) := \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$$

has $\tilde{\tau} = \left(\begin{bmatrix} C_+ \\ C_- \end{bmatrix}, A_\pi; 0, 0; 0 \right)$ as its \mathbb{C} -null-pole triple. The conditions (3.1.12) and (3.1.13) can be put together as

$$\begin{bmatrix} L \\ I \end{bmatrix} \varphi^{-1} = \Theta \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}. \quad (3.1.14)$$

On the other hand, by remembering that Θ has $\tilde{\tau}$ as its \mathbb{C} -null-pole triple, we can see that there exist $B \in \mathbb{C}^{n_\pi \times N}$, $H \in \mathcal{P}_{M \times N}$, $K \in \mathcal{P}_{N \times N}$ satisfying

$$\Theta \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} C_+ \\ C_- \end{bmatrix} (zI - A_\pi)^{-1} B + \begin{bmatrix} H(z) \\ K(z) \end{bmatrix} \quad (3.1.15)$$

for some $Q_1 \in \mathcal{P}_{M \times N}$, $Q_2 \in \mathcal{P}_{N \times N}$. From (3.1.14) and (3.1.15), we can restate the result of Theorem 1.4.1 as follows: $L \in \mathcal{P}_{M \times N}$ is a solution to the (RIPP) if and only if

$$\begin{bmatrix} L \\ I \end{bmatrix} \varphi^{-1} = \begin{bmatrix} C_+ \\ C_- \end{bmatrix} (zI - A_\pi)^{-1} B + \begin{bmatrix} H(z) \\ K(z) \end{bmatrix} \quad (3.1.16)$$

for some $B \in \mathbb{C}^{N \times n_\pi}$, $H \in \mathcal{P}_{M \times N}$, $K \in \mathcal{P}_{N \times N}$. From the second row of (3.1.16) and (3.1.10), we get

$$\begin{aligned} & I + C_-(I - \alpha S_\infty)^{-1} G_\infty - C_-(zI - A_\pi)^{-1} (\alpha I - A_\pi)(I - \alpha S_\infty)^{-1} G_\infty \\ &= C_-(zI - A_\pi)^{-1} B + K(z). \end{aligned}$$

Equivalently,

$$\begin{aligned} & I + C_-(I - \alpha S_\infty)^{-1} G_\infty - K(z) \\ &= C_-(zI - A_\pi)^{-1} \{B + (\alpha I - A_\pi)(I - \alpha S_\infty)^{-1} G_\infty\}. \end{aligned}$$

Since $I + C_-(I - \alpha S_\infty)^{-1} G_\infty - K(z)$ is analytic in \mathbb{C} and $C_-(zI - A_\pi)^{-1} \{B + (\alpha I - A_\pi)(I - \alpha S_\infty)^{-1} G_\infty\}$ is analytic with value 0 at infinity,

$$I + C_-(I - \alpha S_\infty)^{-1} G_\infty - K(z) = 0$$

and

$$C_-(zI - A_\pi)^{-1} \{B + (\alpha I - A_\pi)(I - \alpha S_\infty)^{-1} G_\infty\} = 0. \quad (3.1.17)$$

Because (C_-, A_π) is a null-kernel pair, (3.1.17) implies that

$$B = -(\alpha I - A_\pi)(I - \alpha S_\infty)^{-1} G_\infty. \quad (3.1.18)$$

Upon combining (3.1.16) and (3.1.18), we get $L \in \mathcal{P}_{M \times N}$ is a solution to (RIPP) if and only if

$$L(z) = C_+(zI - A_\pi)^{-1} (A_\pi - \alpha I)(I - \alpha S_\infty)^{-1} G_\infty \varphi(z) + H(z) \varphi(z) \quad (3.1.19)$$

for some $H \in \mathcal{P}_{M \times N}$. Let K_0 denote a solution corresponding to the choice of $H \equiv 0$ in (3.1.19). Then,

$$K_0(z) = C_+(zI - A_\pi)^{-1} (A_\pi - \alpha I)(I - \alpha S_\infty)^{-1} G_\infty \varphi(z). \quad (3.1.20)$$

By substituting (3.1.11) in (3.1.20), we get

$$\begin{aligned}
 K_0(z) = & C_+(zI - A_\pi)^{-1}(A_\pi - \alpha I)(I - \alpha S_\infty)^{-1}\{I - \alpha G_\infty C_-(A_\pi - \alpha I)^{-1}\}G_\infty \\
 & + zC_+(zI - A_\pi)^{-1}(A_\pi - \alpha I)(I - \alpha S_\infty)^{-1}G_\infty C_-(A_\pi - \alpha I)^{-1}(I - zS_\infty)^{-1} \\
 & \times (I - \alpha S_\infty)G_\infty.
 \end{aligned} \tag{3.1.21}$$

To represent $K_0(z)$ in realization form, we apply the identity (2.2.35)

$$S_\infty A_\pi - I = G_\infty C_- \tag{3.1.22}$$

to the first term of (3.1.21) to obtain

$$\begin{aligned}
 & C_+(zI - A_\pi)^{-1}(A_\pi - \alpha I)(I - \alpha S_\infty)^{-1}\{A_\pi - \alpha I - \alpha(S_\infty A_\pi - I)\}(A_\pi - \alpha I)^{-1}\}G_\infty \\
 = & C_+(zI - A_\pi)^{-1}A_\pi G_\infty.
 \end{aligned} \tag{3.1.23}$$

Again, we insert (3.1.22) into the second term of (3.1.21), but we change the expression as

$$S_\infty(A_\pi - \alpha I) - (I - \alpha S_\infty) = G_\infty C_-$$

to get

$$zC_+(zI - A_\pi)^{-1}(A_\pi - \alpha I)\{(I - \alpha S_\infty)^{-1}S_\infty - (A_\pi - \alpha I)^{-1}\}(I - zS_\infty)^{-1}(I - \alpha S_\infty)G_\infty.$$

Since $(I - zS_\infty)^{-1}$ and $(I - \alpha S_\infty)$ commute, the above expression can be written as

$$\begin{aligned}
 & zC_+(zI - A_\pi)^{-1}\{(A_\pi - \alpha I)S_\infty - (I - \alpha S_\infty)\}(I - zS_\infty)^{-1}G_\infty \\
 = & zC_+(zI - A_\pi)^{-1}(A_\pi S_\infty - I)(I - zS_\infty)^{-1}G_\infty.
 \end{aligned} \tag{3.1.24}$$

By adding (3.1.23) and (3.1.24) we have

$$\begin{aligned}
K_0(z) &= C_+(zI - A_\pi)^{-1} \{A_\pi(I - zS_\infty) + z(A_\pi S_\infty - I)\}(I - zS_\infty)^{-1}G_\infty \\
&= -C_+(I - zS_\infty)^{-1}G_\infty \\
&= -C_+ \sum_{k=0}^{n_\pi-1} (zS_\infty)^k G_\infty \\
&= -C_+ G_\infty - zC_+ S_\infty \sum_{k=0}^{n_\pi-1} (zS_\infty)^k G_\infty \\
&= -C_+ G_\infty - zC_+(I - zS_\infty)^{-1}G_\infty. \tag{3.1.25}
\end{aligned}$$

To compute the McMillan degree of K_0 , we use (3.1.25). Since, (S_∞, G_∞) is a full-range pair, by (1.1.4),

$$\begin{aligned}
\delta(K_0) &= \text{rank} \begin{bmatrix} C_+ S_\infty \\ \vdots \\ C_+ S_\infty^{n_\pi-1} \end{bmatrix} \\
&= \text{rank} \left(\begin{bmatrix} C_+ \\ C_+ S_\infty \\ \vdots \\ C_+ S_\infty^{n_\pi-2} \end{bmatrix} S_\infty \right) \\
&\leq \text{rank } S_\infty \\
&\leq n_\pi - 1. \square
\end{aligned}$$

In some instances, it can be shown that $n_\pi - 1$ is the minimal possible McMillan degree for an interpolant as the following simple example shows.

Example 3.1.2. Find an 2×2 matrix polynomial $L(z)$ satisfying

$$L(0) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3.1.26}$$

$$L(1) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3.1.27}$$

and has the minimal possible McMillan degree.

It is clear that there exists no 2×2 constant matrix satisfying (3.1.26) and (3.1.27). By Theorem 3.1.1 the minimal possible McMillan degree for this example is 1. Since

the McMillan degree of a matrix polynomial is greater than or equal to the polynomial degree, we expect a solution $L(z)$ with McMillan degree 1 to be

$$L(z) = A_1 z + A_0$$

where A_0, A_1 is 2×2 matrices. The condition (3.1.26) forces

$$A_0 = \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} \quad (3.1.28)$$

for some $a, b \in \mathbb{C}$. Upon combining (3.1.27) and (3.1.28) we get

$$A_1 = \begin{bmatrix} -1 & \alpha \\ 1 & \beta \end{bmatrix}$$

for some $\alpha, \beta \in \mathbb{C}$. Since by Proposition 1.1.1,

$$\delta(L) = \text{rank } A_1$$

and we want an interpolant of McMillan degree 1, we have

$$\beta = -\alpha.$$

Hence,

$$L(z) = \begin{bmatrix} -1 & \alpha \\ 1 & -\alpha \end{bmatrix} z + \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}$$

is a solution with $\delta(L) = 1$ for any $a, b, \alpha \in \mathbb{C}$. \square

Unlike the scalar case, we do not know that in general $K_0(z)$ is the unique solution of McMillan degree at most $n_\pi - 1$. In fact, as Example 1.3.2 shows, even in the case that $n_\pi - 1$ is the minimal possible McMillan degree for solutions, there may be more than one solution of McMillan degree $n_\pi - 1$.

For the scalar case, by the uniqueness of the interpolant with McMillan degree (=polynomial degree, in this case) at most $n_\pi - 1$, $K_0(z)$ in Theorem 1.3.1 must be the Lagrange interpolant. We can also see this directly.

Example 3.1.3. For the scalar Lagrange interpolation problem K_0 in Theorem 3.1.1 is the well-known Lagrange interpolant. The scalar Lagrange interpolation problem is as follows:

For given z_1, \dots, z_n distinct n points, and n complex numbers w_1, \dots, w_n , find a polynomial $l(z)$ for which

$$l(z_i) = w_i$$

for $i = 1, \dots, n$.

Put

$$A_\pi = \begin{bmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_n \end{bmatrix}; \quad C_+ = [w_1, \dots, w_n]; \quad C_- = [1, \dots, 1].$$

Then a rational function φ^{-1} has $(C_-, A_\pi; 0, 0; 0)$ as a \mathbb{C} -null-pole triple if and only if

$$\varphi(z) = c \prod_{j=1}^n (z - z_j)$$

for some nonzero complex number c . Without loss of generality, we can assume $c = 1$. Then,

$$\begin{aligned} \varphi^{-1}(z) &= \frac{1}{\prod_{j=1}^n (z - z_j)} \\ &= \sum_{j=1}^n \frac{c_j}{z - z_j} \end{aligned} \tag{3.1.29}$$

where

$$c_j = \frac{1}{\varphi'(z_j)} = \frac{1}{\prod_{i=1, i \neq j}^n (z_j - z_i)}. \tag{3.1.30}$$

On the other hand, suppose

$$(I - \alpha S_\infty)^{-1} G_\infty = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$$

in (3.1.10), then

$$\begin{aligned}
& \varphi^{-1}(z) \\
&= 1 + \beta_1 + \cdots + \beta_n - [1, \dots, 1] \left(zI - \begin{bmatrix} z_1 & & 0 \\ 0 & \ddots & \\ & & z_n \end{bmatrix} \right)^{-1} \\
&\quad \times \begin{bmatrix} \alpha - z_1 & & 0 \\ 0 & \ddots & \\ & & \alpha - z_n \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} \\
&= 1 + \beta_1 + \cdots + \beta_n - \sum_{j=1}^n \frac{\alpha - z_j}{z - z_j} \beta_j. \tag{3.1.31}
\end{aligned}$$

Upon comparing (3.1.29) and (3.1.31), we have

$$\beta_i = -\frac{c_i}{\alpha - z_i}$$

for $i = 1, \dots, n$. Hence,

$$(I - \alpha s_\infty)^{-1} G_\infty = \left[\frac{c_1}{z_1 - \alpha}, \dots, \frac{c_n}{z_n - \alpha} \right]^T,$$

where c_j is given as in (3.1.30). Plug φ , C_+ , C_- , A_π and $(I - \alpha S_\infty)^{-1} G_\infty$ into (3.1.20) to get

$$\begin{aligned}
K_0(z) &= \left[\sum_{j=1}^n \frac{w_j(z_j - \alpha)}{z - z_j} \cdot \frac{c_j}{z_j - \alpha} \right] \cdot \prod_{i=1}^n (z - z_i) \\
&= \sum_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n w_j \frac{(z - z_i)}{(z_j - z_i)},
\end{aligned}$$

which is the Lagrange interpolant. \square

Using the explicit formula of a particular solution K_0 in Theorem 3.1.1 and φ in Corollary 2.3.3, we can describe all solutions of (RIPP) in realization form.

Theorem 3.1.4. *Let C_+ , C_- , A_π , S_∞ , G_∞ be given as in Theorem 3.1.1 and α be a complex number not in $\sigma(A_\pi) \cup \{0\}$. Then $L \in \mathcal{P}_{M \times N}$ is a solution of (RIPP) if and only if*

$$L(z) = \hat{D} + z \hat{C} (I - z \hat{A})^{-1} \hat{B},$$

where

$$\begin{aligned}\hat{A} &= \begin{bmatrix} A & BC_-(A_\pi - \alpha I)^{-1} \\ 0 & S_\infty \end{bmatrix} \\ \hat{B} &= \begin{bmatrix} B - \alpha BC_-(A_\pi - \alpha I)^{-1}G_\infty \\ (I - \alpha S_\infty)G_\infty \end{bmatrix} \\ \hat{C} &= [C \quad DC_-(A_\pi - \alpha I)^{-1} - C_+S_\infty(I - \alpha S_\infty)^{-1}] \\ \hat{D} &= -C_+G_\infty + D - \alpha DC_-(A_\pi - \alpha I)^{-1}G_\infty.\end{aligned}$$

Here $A : X \rightarrow X$, subject to $\sigma(A) = \{0\}$, $B : \mathbb{C}^N \rightarrow X$, $C : X \rightarrow \mathbb{C}^M$, $D : \mathbb{C}^N \rightarrow \mathbb{C}^M$ are arbitrary, and X is a finite dimensional linear space.

Proof. By (3.1.19) and (3.1.20), $L \in \mathcal{P}_{M \times N}$ is a solution to (RIPP) if and only if

$$L = K_0 + Q\varphi \quad (3.1.32)$$

for $Q \in \mathcal{P}_{M \times N}$.

To parameterize all the solutions in realization form, we work with formula (3.1.32). Suppose a realization for Q is given by

$$Q(z) = D + zC(I - zA)^{-1}B,$$

for $A : X \rightarrow X$, $B : \mathbb{C}^N \rightarrow X$, $C : X \rightarrow \mathbb{C}^M$, $D : \mathbb{C}^N \rightarrow \mathbb{C}^M$ where X is finite dimensional linear space, and $\sigma(A) = \{0\}$. Formula (3.1.11) for φ can be used to compute $Q\varphi$. By [BGK1],

$$\begin{aligned}Q(z)\varphi(z) &= D - \alpha DC_-(A_\pi - \alpha I)^{-1}G_\infty + z \begin{bmatrix} C & DC_-(A_\pi - \alpha I)^{-1} \end{bmatrix} \\ &\quad \times \left\{ I - z \begin{bmatrix} A & BC_-(A_\pi - \alpha I)^{-1} \\ 0 & S_\infty \end{bmatrix} \right\}^{-1} \begin{bmatrix} B\{I - \alpha C_-(A_\pi - \alpha I)^{-1}G_\infty\} \\ (I - \alpha S_\infty)G_\infty \end{bmatrix}.\end{aligned} \quad (3.1.33)$$

Since $(I - zS_\infty)^{-1}$ and $(I - \alpha S_\infty)$ commute, $K_0(z)$ in Theorem 3.1.1 can be written as

$$K_0(z) = -C_+G_\infty - zC_+S_\infty(I - \alpha S_\infty)^{-1}(I - zS_\infty)^{-1}(I - \alpha S_\infty)G_\infty. \quad (3.1.34)$$

By combining (3.1.33) and (3.1.34), we get

$$K_0(z) + Q(z)\varphi(z) = \hat{D} + z\hat{C}(I - z\hat{A})^{-1}\hat{B}$$

with

$$\begin{aligned}\hat{A} &= \begin{bmatrix} A & BC_-(A_\pi - \alpha I)^{-1} \\ 0 & S_\infty \end{bmatrix} \\ \hat{B} &= \begin{bmatrix} B - \alpha BC_-(A_\pi - \alpha I)^{-1}G_\infty \\ (I - \alpha S_\infty)G_\infty \end{bmatrix} \\ \hat{C} &= [C \quad DC_-(A_\pi - \alpha I)^{-1} - C_+S_\infty(I - \alpha S_\infty)^{-1}] \\ \hat{D} &= -C_+G_\infty + D - \alpha DC_-(A_\pi - \alpha I)^{-1}G_\infty.\end{aligned}$$

This completes the proof. \square

In the rest of this section, we state the dual version of previous theorems for $(RIPP_d)$ but we omit the proofs. Analogously to $(RIPP)$ we assume that (A_ζ, B_+) is a full-range pair.

Theorem 3.1.5. *Let A_ζ, B_+, B_- be given as in $(RIPP_d)$ and F_∞, T_∞ be constructed as in Corollary 2.3.4 with $\tilde{\tau}_- = (0, 0; A_\zeta, B; 0)$ in place of τ . Then*

$$H_0(z) = F_\infty B_- + zF_\infty(I - zT_\infty)^{-1}T_\infty B_- \tag{3.1.35}$$

is a solution of $(RIPP_d)$ with McMillan degree at most $n_\zeta - 1$.

Theorem 3.1.6. *Let matrices $A_\zeta, B_+, B_-, F_\infty, T_\infty$ be given as in Theorem 3.1.5 and α be a complex number such that $\alpha \notin \sigma(A_\zeta) \cup \{0\}$. Then, L is a solution to $(RIPP_d)$ if and only if*

$$L(z) = \tilde{D} + z\tilde{C}(I - z\tilde{A})^{-1}\tilde{B},$$

where

$$\begin{aligned}\tilde{A} &= \begin{bmatrix} A \\ (A_\zeta - \alpha I)^{-1}B_+C & 0 \end{bmatrix} \\ \tilde{B} &= \begin{bmatrix} B \\ (A_\zeta - \alpha I)^{-1}B_+D + (I - \alpha T_\infty)^{-1}T_\infty B_- \end{bmatrix} \\ \tilde{C} &= [C - F_\infty(A_\zeta - \alpha I)B_+C \quad F_\infty(I - \beta T_\infty)] \\ \tilde{D} &= F_\infty B_- + D - F_\infty(A_\zeta - \alpha I)^{-1}B_+D\end{aligned}$$

with $A : X \rightarrow X$, subject to $\sigma(A) = \{0\}$, $B : \mathbb{C}^N \rightarrow X$, $C : X \rightarrow \mathbb{C}^M$, $D : \mathbb{C}^N \rightarrow \mathbb{C}^M$ for any finite linear space X .

Remark 3.1.7. As we have seen in Chapter II, for a given full-range pair (A_ζ, B) , there exists a matrix polynomial Θ with (A_ζ, B) as its (left) null pair and of minimal possible McMillan degree. Here we consider the special case where $B = [B_+ B_-]$ with (A_ζ, B_+) a full range pair and find such a matrix polynomial in a block triangular form. Indeed, if we put

$$\Theta = \begin{bmatrix} \psi & H_0 \\ 0 & I \end{bmatrix} \quad (3.1.36)$$

where ψ and H_0 are the matrix polynomials constructed as in Corollary 2.2.4 and Theorem 3.1.5, then Θ is a regular matrix polynomial with $(A_\zeta, [B_+ B_-])$ as its (left) null pair and has minimal possible McMillan degree n_ζ . But a matrix polynomial Θ given by (3.1.36) is obviously not column reduced in general. Conversely, if a regular matrix polynomial Θ has a block triangular form as

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ 0 & \Theta_{22} \end{bmatrix}$$

and Θ_{22} has no zeros in \mathbb{C} , then one can show that there exists a left null pair $(A_\zeta, [B_+ B_-])$ for Θ with (A_ζ, B_+) a full-range pair.

3.2 Two-sided problem

In this section, we consider the following two-sided residue interpolation problem for matrix polynomials (TRIPP).

(TRIPP) *Given are matrices A_ζ, B_+, B_- of sizes $n_\zeta \times n_\zeta$, $n_\zeta \times M$, $n_\zeta \times N$ respectively, matrices C_+, C_-, A_π of sizes $M \times n_\pi$, $N \times n_\pi$, $n_\pi \times n_\pi$, and an $n_\zeta \times n_\pi$ matrix Γ . Find an $M \times N$ matrix polynomial $L(z)$ which satisfies*

$$\sum_{z_0 \in \mathbb{C}} \text{Res}_{z=z_0} (zI - A_\zeta)^{-1} B_+ L(z) = -B_- \quad (3.2.1)$$

$$\sum_{z_0 \in \mathbb{C}} \text{Res}_{z=z_0} L(z) C_- (zI - A_\pi)^{-1} = C_+ \quad (3.2.2)$$

$$\sum_{z_0 \in \mathbb{C}} \text{Res}_{z=z_0} (zI - A_\zeta)^{-1} B_+ L(z) C_- (zI - A_\pi)^{-1} = \Gamma. \quad (3.2.3)$$

As we mentioned in Section 1.4, in the discussion of (TRIPP), we assume that

$$(C_-, A_\pi) \text{ is a null-kernel pair} \quad (3.2.4)$$

$$(A_\zeta, B_+) \text{ is a full-range pair} \quad (3.2.5)$$

$$\Gamma \text{ satisfies the Sylvester equation} \quad (3.2.6)$$

$$\Gamma A_\pi - A_\zeta \Gamma = B_+ C_+ + B_- C_-.$$

A collection of matrices

$$\omega = (C_+, C_-, A_\pi; A_\zeta, B_+, B_-; \Gamma) \quad (3.2.7)$$

is said to be a σ -admissible TRIP data set if the given matrices C_+ , C_- , A_π , A_ζ , B_+ , B_- , Γ have sizes as in (TRIPP) and satisfy (3.2.4)–(3.2.6) with $\sigma(A_\pi) \cup \sigma(A_\zeta) \subset \sigma$ where σ is a non-empty subset of \mathbb{C} . If a collection of matrices ω given by (3.2.7) is a σ -admissible TRIP data set, then

$$\tau := \left(\begin{bmatrix} C_+ \\ C_- \end{bmatrix}, A_\pi; A_\zeta, [B_+ \ B_-]; \Gamma \right)$$

is a σ -admissible Sylvester data set.

By Theorem 1.4.1, there exists a solution to (TRIPP). In the next theorem, a particular solution with low McMillan degree of (TRIPP) is obtained.

Theorem 3.2.1. Suppose matrices C_+ , C_- , A_π , A_ζ , B_+ , B_- , Γ are given as in (TRIPP) and G_∞ , S_∞ are constructed as in Theorem 3.1.1 and F_∞, T_∞ are constructed as in Theorem 3.1.5 for a given complex number $\alpha \notin \sigma(A_\pi) \cup \sigma(A_\zeta) \cup \{0\}$. Then

$$L_0(z) = D_L + zC_L(I - zA_L)^{-1}B_L \quad (3.2.8)$$

is a particular solution to (TRIPP) of McMillan degree at most $n_\pi + n_\zeta - 1$, where

$$\begin{aligned} A_L &= \begin{bmatrix} T_\infty & T_\infty \tilde{B}_- C_- (A_\pi - \alpha I)^{-1} \\ 0 & S_\infty \end{bmatrix} \\ B_L &= \begin{bmatrix} T_\infty \tilde{B}_- \{I - C_- (A_\pi - \alpha I)^{-1} G_\infty\} \\ (I - \alpha S_\infty) G_\infty \end{bmatrix} \\ C_L &= [F_\infty \quad F_\infty \tilde{B}_- (A_\pi - \alpha I) - C_+ S_\infty (I - \alpha S_\infty)^{-1}] \\ D_L &= -C_+ G_\infty + F \tilde{B}_- \{I - C_- (A_\pi - \alpha I)^{-1} G_\infty\} \end{aligned}$$

with $\tilde{B}_- = B_- + \{B_- C_- - \Gamma(A_\pi - \alpha I)\}(I - \alpha S_\infty)^{-1}G_\infty$.

Proof. Let φ be an $N \times N$ regular matrix polynomial constructed as in Corollary 2.3.3 with $\tilde{\tau}_- := (C_-, A_\pi; 0, 0; 0)$ in place of τ and let ψ be an $M \times M$ regular matrix polynomial constructed as in Corollary 2.3.4 with $\tilde{\tau}_+ := (0, 0; A_\zeta, B_+; 0)$ in place of τ . By Theorem 1.4.1, there exists a solution to (TRIPP) and $L \in \mathcal{P}_{M \times N}$ is a solution to (TRIPP) if and only if

$$\begin{bmatrix} L \\ I \end{bmatrix} \varphi^{-1} = \Theta \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \quad (3.2.9)$$

for some $Q_1 \in \mathcal{P}_{M \times N}$, $Q_2 \in \mathcal{P}_{N \times N}$, where an $(M+N) \times (M+N)$ rational matrix function Θ has

$$\tau = \left(\begin{bmatrix} C_+ \\ C_- \end{bmatrix}, A_\pi; A_\zeta, [B_+ \quad B_-]; \Gamma \right)$$

as its \mathbb{C} -null-pole triple. Thus, by the definition of null-pole subspace, the right hand side of (3.2.9) is expressed as

$$\Theta \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} C_+ \\ C_- \end{bmatrix} (zI - A_\pi)^{-1} B + \begin{bmatrix} H(z) \\ K(z) \end{bmatrix}, \quad (3.2.10)$$

where $B \in \mathbb{C}^{n_\pi \times N}$, $H \in \mathcal{P}_{M \times N}$, $K \in \mathcal{P}_{N \times N}$ satisfying

$$\sum_{z_0 \in \mathbf{t}} \text{Res}_{z=z_0} (zI - A_\zeta)^{-1} [B_+ \quad B_-] \begin{bmatrix} H(z) \\ K(z) \end{bmatrix} = \Gamma B. \quad (3.2.11)$$

Combining (3.2.10) and (3.2.11), we get another characterization of solutions as follows: $L \in \mathcal{P}_{M \times N}$ is a solution to (TRIPP) if and only if

$$\begin{bmatrix} L \\ I \end{bmatrix} \varphi^{-1} = \begin{bmatrix} C_+ \\ C_- \end{bmatrix} (zI - A_\pi)^{-1} B + \begin{bmatrix} H(z) \\ K(z) \end{bmatrix} \quad (3.2.12)$$

for some $B \in \mathbb{C}^{n_\pi \times N}$, $H \in \mathcal{P}_{M \times N}$, $K \in \mathcal{P}_{N \times N}$ satisfying (3.2.11). Since from (3.1.10) φ^{-1} is given by

$$\varphi^{-1}(z) = I + C_-(I - \alpha S_\infty)^{-1} G_\infty - C_-(zI - A_\pi)^{-1} (\alpha I - A_\pi)(I - \alpha S_\infty)^{-1}$$

the second row of (3.2.12) implies that

$$\begin{aligned} I + C_-(I - \alpha S_\infty)^{-1} G_\infty - K(z) \\ = C_-(zI - A_\pi)^{-1} \{B + (\alpha I - A_\pi)^{-1} (I - \alpha S_\infty)^{-1} G_\infty\}. \end{aligned}$$

By the same argument as in Theorem 3.1.1, we obtain

$$K(z) = I + C_-(I - \alpha S_\infty)^{-1} G_\infty \quad (3.2.13)$$

$$B = (A_\pi - \alpha I)(I - \alpha S_\infty)^{-1} G_\infty. \quad (3.2.14)$$

Substitute (3.2.13) and (3.2.14) in (3.2.11) to get

$$\sum_{z_0 \in \mathbf{t}} \text{Res}_{z=z_0} (zI - A_\zeta)^{-1} B_+ H(z) = -\tilde{B}_- \quad (3.2.15)$$

with

$$\begin{aligned} \tilde{B}_- &= -\Gamma(A_\pi - \alpha I)(I - \alpha S_\infty)^{-1} G_\infty + B_- \{I + C_-(I - \alpha S_\infty)^{-1} G_\infty\} \\ &= B_- + \{B_- C_- - \Gamma(A_\pi - \alpha I)\}(I - \alpha S_\infty)^{-1} G_\infty. \end{aligned}$$

By Theorem 3.1.5,

$$H_0(z) = F_\infty \tilde{B}_- + z F_\infty (I - z T_\infty)^{-1} T_\infty \tilde{B}_-$$

is a solution of (3.2.15).

Let L_0 be a solution of (TRIPP) corresponding to the choice of $H = H_0$ in (3.2.12). Then from the first row of (3.2.12), we get

$$L_0(z) = C_+ (zI - A_\pi)^{-1} (A_\pi - \alpha I) (I - \alpha S_\infty)^{-1} G_\infty \varphi(z) + H_0(z) \varphi(z).$$

In Theorem 3.1.1, it was shown that

$$\begin{aligned} K_0(z) &= C_+ (zI - A_\pi)^{-1} (A_\pi - \alpha I) (I - \alpha S_\infty)^{-1} G_\infty \varphi(z) \\ &= -C_+ G_\infty - z C_+ S_\infty (I - z S_\infty)^{-1} G_\infty. \end{aligned}$$

Since $(I - z S_\infty)^{-1}$ and $(I - \alpha S_\infty)$ commute,

$$K_0(z) = -C_+ G_\infty - z C_+ S_\infty (I - \alpha S_\infty)^{-1} (I - z S_\infty)^{-1} (I - \alpha S_\infty) G_\infty. \quad (3.2.18)$$

Now, we calculate $H_0 \varphi$. Remember that by (3.1.11)

$$\begin{aligned} \varphi(z) &= I - \alpha C_- (A_\pi - \alpha I)^{-1} G_\infty \\ &\quad + C_- (A_\pi - \alpha I)^{-1} (I - z S_\infty)^{-1} (I - \alpha S_\infty) G_\infty. \end{aligned} \quad (3.2.19)$$

Using again the product rule of two matrix polynomials H_0 and φ given as (3.2.16) and (3.2.19) respectively, we get

$$\begin{aligned} H_0(z) \varphi(z) &= F_\infty \tilde{B}_- \{I - \alpha C_- (A_\pi - \alpha I)^{-1} G_\infty\} + z \left[\begin{matrix} F_\infty & F_\infty \tilde{B}_- C_- (A_\pi - \alpha I)^{-1} \end{matrix} \right] \\ &\quad \times \left\{ zI - \left[\begin{matrix} T_\infty & T_\infty \tilde{B}_- C_- (A_\pi - \alpha I)^{-1} \\ 0 & S_\infty \end{matrix} \right] \right\}^{-1} \\ &\quad \times \left[\begin{matrix} T_\infty \tilde{B}_- \{I - \alpha C_- (A_\pi - \alpha I)^{-1} G_\infty\} \\ (I - \alpha S_\infty) G_\infty \end{matrix} \right]. \end{aligned} \quad (3.2.20)$$

We substitute (3.2.19) and (3.2.20) into (3.2.17) to obtain

$$L_0(z) = D_L + zC_L(I - zA_L)^{-1}B_L$$

where

$$\begin{aligned} A_L &= \begin{bmatrix} T_\infty & T_\infty \tilde{B}_- C_- (A_\pi - \alpha I)^{-1} \\ 0 & S_\infty \end{bmatrix} \\ B_L &= \begin{bmatrix} T_\infty \tilde{B}_- \{I - C_- (A_\pi - \alpha I)^{-1} G_\infty\} \\ (T - \alpha S_\infty) G_\infty \end{bmatrix} \\ C_L &= [F_\infty \quad F_\infty \tilde{B}_- (A_\pi - \alpha I) - C_+ S_\infty (I - \alpha S_\infty)^{-1}] \\ D_L &= -C_+ G_\infty + F \tilde{B}_- \{I - C_- (A_\pi - \alpha I)^{-1} G_\infty\} \end{aligned} \quad (3.2.21)$$

with

$$\tilde{B}_- = B_- + \{B_- C_- - \Gamma(A_\pi - \alpha I)\}(I - \alpha S_\infty)^{-1}G_\infty.$$

To compute the McMillan degree of L_0 , we assume $n_\zeta \neq 0$ and $n_\pi \neq 0$. Indeed, if $n_\zeta = 0$ or $n_\pi = 0$, then (TRIPP) is reduced to (RIPP) or (RIPP_d) and it is proved in Theorem 3.1.1 and Theorem 3.1.5 that the McMillan degree of L_0 is at most $n_\pi + n_\zeta - 1$. In the rest of this proof, we use * to represent the terms or matrices which are irrelevant.

Remember that

$$\delta(L_0) \leq \text{rank} \left[B_L \quad A_L B_L \dots A_L^{n_\pi+n_\zeta-1} B_L \right]. \quad (3.2.22)$$

Since B_L is given by (3.2.21) and

$$A_L^i = \begin{bmatrix} T_\infty^i & T_\infty^i * + T_\infty^{i-1} * S_\infty + \dots + * S_\infty^i \\ 0 & S_\infty^i \end{bmatrix},$$

(3.2.22) is equivalent to

$$\begin{aligned} \delta(L_0) &\leq \text{rank} \left[\begin{bmatrix} T_\infty * & T_\infty * \dots T_\infty * \\ * & * \dots * \end{bmatrix} \right] \\ &= \text{rank} \left(\left[\begin{bmatrix} T_\infty & 0 \\ 0 & I_{n_\pi} \end{bmatrix} \right] [*] \right) \\ &\leq \text{rank} \left(\left[\begin{bmatrix} T_\infty & 0 \\ 0 & I_{n_\pi} \end{bmatrix} \right] \right) \\ &\leq n_\pi + n_\zeta - 1. \end{aligned}$$

The last inequality holds since $\text{rank } T_\infty$ is less than the largest controllability index of the pair (A_ζ, B_+) . This completes the proof. \square

The next example shows that in some instances the minimal possible McMillan degrees is $n_\pi + n_\zeta - 1$.

Example 3.2.2. Find a 2×2 matrix polynomial $P(z)$ for which

$$[1 \ 0] P(0) = [1 \ 0] \quad (3.2.23)$$

$$P(1) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (3.2.24)$$

It is clear that there exists no constant 2×2 matrix satisfying (3.2.23) and (3.2.24).

\square

Remarks analogous to those made in the first section can also be made here. For a prescribed σ -admissible Sylvester data set

$$\tau = (C, A_\pi; A_\zeta, B; \Gamma)$$

a rational matrix function Θ which has τ as a \mathbb{C} -null-pole triple and is column reduced at infinity is given in Chapter II. Here we consider the special case where $B = [B_+ \ B_-]$ with (A_ζ, B_+) a full-range pair and $C = \begin{bmatrix} C_+ \\ C_- \end{bmatrix}$ with (C_-, A_π) a null-kernel pair and show that there exists an interpolant Θ which has a block triangular form

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ 0 & \Theta_{22} \end{bmatrix}$$

so that (A_ζ, B_+) is a (left) null pair for Θ_{11} and (C_-, A_π) is a (right) pole pair for Θ_{22} . Indeed, for φ^{-1} in Corollary 2.3.3, ψ in Corollary 2.3.4 and L_0 in Theorem 3.2.1

$$\Theta = \begin{bmatrix} \psi & L_0 \varphi^{-1} \\ 0 & \varphi^{-1} \end{bmatrix}$$

is such a rational matrix function. Since the McMillan degree of Θ is $n_\pi + n_\zeta$ by its construction the above Θ does not have the minimal possible McMillan degree which is $n_\pi + n_\zeta - \text{rank } \Gamma$ except for the case $\Gamma = 0$.

3.3 Simultaneous interpolation problem

Consider the following interpolation problem for matrix polynomials for which the data are given in terms of several equations involving residues.

(SRIPP) *Given a collection of matrices $C_{+k}, C_{-k}, A_{\pi k}$ ($k = 1, \dots, r$) of sizes $M \times n_{\pi k}$, $N \times n_{\pi k}$, and $n_{\pi k} \times n_{\pi k}$ respectively, such that $(C_{-k}, A_{\pi k})$ is a null-kernel pair for each k , find a matrix polynomial $L \in \mathcal{P}_{M \times N}$ for which*

$$\sum_{z_0 \in \mathbb{C}} \text{Res}_{z=z_0} L(z) C_{-k}(zI - A_{\pi k})^{-1} = C_{+k} \quad k = 1, \dots, r. \quad (3.3.1)$$

This interpolation problem will be termed the *simultaneous residue interpolation problem for matrix polynomials* (SRIPP). Unlike the single residue interpolation problem, a solution to (SRIPP) may fail to exist because the conditions for different k may not be consistent. The conditions which guarantee the existence of a solution are given in Theorem 3.4 of [BGR5]. By using the theorem of [BGR5], we can extend Theorem 3.1.1 to (SRIPP). Before stating the result we need some definitions from [GKLR2] (see also [GLR1]).

An ordered pair of matrices (C, A) where C is $N \times p$ and A is $p \times p$ we call a *right pair*. We refer to N as the *base dimension* and p as the *order* of (C, A) . In the ensuing discussion the base dimension N is fixed but the order p is a variable dimension.

Two right pairs (C_1, A_1) and (C_2, A_2) are *similar* if there exists a nonsingular matrix S such that $C_1 = C_2S$ and $A_1 = S^{-1}A_2S$. Given two right pairs (C_1, A_1) and (C_2, A_2) of orders p_1 and p_2 respectively, we say that (C_1, A_1) is an *extension* of (C_2, A_2) (or, equivalently, (C_2, A_2) is a *restriction* of (C_1, A_1)) if there exists an injective linear transformation $S : \mathbb{C}^{p_1} \rightarrow \mathbb{C}^{p_2}$ such that

$$C_1S = C_2 \text{ and } A_1S = SA_2.$$

The right pair (C, A) is said to be a *common extension* of $(C_1, A_1), \dots, (C_r, A_r)$ if

(C, A) is an extension of each (C_i, A_i) , $i = 1, \dots, r$. We say that (C_0, A_0) is a *least common extension* of $(C_1, A_1), \dots, (C_r, A_r)$ if (C_0, A_0) is a common extension of $(C_1, A_1), \dots, (C_r, A_r)$, and any common extension of $(C_1, A_1), \dots, (C_r, A_r)$ is an extension of (C_0, A_0) .

Theorem 3.3.1. Suppose a collection of matrices $C_{+k}, C_{-k}, A_{\pi k}$ ($k = 1, \dots, r$) is given as in (SRIPP). Suppose (C_-, A_π) is a least common extension of the collection $\{(C_{-k}, A_{\pi k})|k = 1, \dots, r\}$ of sizes $N \times n_\pi$, $n_\pi \times n_\pi$ with associated injective linear maps $S_k : \mathbb{C}^{n_{\pi k}} \rightarrow \mathbb{C}^{n_\pi}$, and suppose that there exists $C_+ : \mathbb{C}^{n_\pi} \rightarrow \mathbb{C}^M$ satisfying

$$C_+ S_k = C_{+k}, \quad k = 1, \dots, r.$$

Then there exists a $L \in \mathcal{P}_{M \times N}$ satisfying (3.3.1) of McMillan degree at most $n_\pi - 1$.

Proof. By Theorem 3.4 of [BGR5] there exists a solution L to (SRIPP) and L satisfies (3.3.1) if and only if L satisfies the single contour integral interpolation condition

$$\sum_{z_0 \in \mathbf{t}} \text{Res}_{z=z_0} L(z) C_- (zI - A_\pi)^{-1} = C_+. \quad (3.3.2)$$

By Theorem 3.1.1, there exists a $L \in \mathcal{P}_{M \times N}$ satisfying (3.3.2) with $\delta(L) \leq n_\pi - 1$. This completes the proof. \square

A result dual to Theorem 3.3.1 holds for (SRIPP). This result follows by simply applying the map $L(z) \rightarrow L(z)^T$ in Theorem 3.3.1. In general we say that an ordered pair of matrices (A, B) where A is $p \times p$ and B is $p \times M$ is a left pair of base dimension M and order p . Two left pairs (A_1, B_1) and (A_2, B_2) are said to be *similar* if $A_1 S = S A_2$ and $B_1 = S B_2$ for some nonsingular matrix S . Given two left pairs (A_1, B_1) and (A_2, B_2) of the same base dimension we say that (A_1, B_1) is a *corestriction* of (A_2, B_2) (or equivalently (A_2, B_2) is a *coextension* of (A_1, B_1)) if there exists a surjective linear transformation S such that

$$A_1 S = S A_2 \text{ and } B_1 = S B_2.$$

Clearly the left pair (A_2, B_2) is a coextension of the left pair (A_1, B_1) with associated surjection S if and only if the right pair (B_2^T, A_2^T) is an extension of the right pair (B_1^T, A_1^T) with associated injective mapping S^T . Via this connection, we can define the notions of common coextension and least common coextension for left pairs as simply the transposed version of common extension and least common extension of a collection of right pairs.

We now state the result dual to Theorem 3.3.1 for (SRIPP) and omit the proof.

Theorem 3.3.2. Suppose a collection of matrices $A_{\zeta k}, B_{+k}, B_{-k}$ ($k = 1, \dots, n$) is given, where for each k , $A_{\zeta k}, B_{+k}, B_{-k}$ have respective sizes $n_{\zeta k} \times n_{\zeta k}, n_{\zeta k} \times M, n_{\zeta k} \times N$ and $(A_{\zeta k}, B_{+k})$ is a full range pair. Let (A_ζ, B_+) be a least common coextension of the collection $\{(A_{\zeta k}, B_{+k})|k = 1, \dots, n\}$ of full range left pairs, with associated surjective linear maps $T_k : \mathbb{C}^{n_{\zeta k}} \rightarrow \mathbb{C}^{n_{\zeta k}}$ (thus, the size of A_ζ is $n_\zeta \times n_\zeta$ and B_+ is $M \times n_\zeta$). Suppose there exists $B_- : \mathbb{C}^N \rightarrow \mathbb{C}^{n_\zeta}$ satisfying

$$T_k B_- = B_{-k}, \quad k = 1, \dots, n.$$

Then there exists $L \in \mathcal{P}_{M \times N}$ for which

$$\sum_{z_0 \in \mathbb{C}} \text{Res}_{z=z_0} (zI - A_{\zeta k})^{-1} B_{+k} L(z) = -B_{-k}, \quad k = 1, \dots, n \quad (3.3.3)$$

with McMillan degree at most $n_\zeta - 1$.

In the rest of this section we consider the problem of *simultaneous two-sided residue interpolation problem* for matrix polynomials (STRIPP) which is stated as follows:

(STRIPP) For a given collection of σ -admissible interpolation data sets $\tilde{\tau} = (C_{+j}, C_{-j}, A_{\pi j}; A_{\zeta j}, B_{+j}, B_{-j}; \Gamma_j)$ $1 \leq j \leq m$, where the matrices have sizes $M \times n_{\pi j}, N \times n_{\pi j}, n_{\pi j} \times n_{\pi j}, n_{\zeta j} \times M, n_{\zeta j} \times N, n_{\zeta j} \times n_{\pi j}$ respectively, find a matrix polynomial

$L \in \mathcal{P}_{M \times N}$ which satisfies the interpolation conditions

$$\sum_{z_0 \in \mathbb{C}} \text{Res}_{z=z_0} L(z) C_{-j}(zI - A_{\pi j})^{-1} = C_{+j} \quad (3.3.4)$$

$$\sum_{z_0 \in \mathbb{C}} \text{Res}_{z=z_0} (zI - A_{\zeta j})^{-1} B_{+j} L(z) = -B_{-j} \quad (3.3.5)$$

$$\sum_{z_0 \in \mathbb{C}} \text{Res}_{z=z_0} (zI - A_{\zeta j})^{-1} B_{+j} L(z) C_{-j}(zI - A_{\pi j})^{-1} = \Gamma_j \quad (3.3.6)$$

for $j = 1, \dots, m$.

As we mentioned before a solution to (STRIPP) does not always exist since the conditions for different j can be contradictory. In [BGR5], the necessary and sufficient conditions for the existence of solutions of (STRIPP) problem are given and it is shown that (STRIPP) can be reduced to (TRIPP) when the existence of the solution to (STRIPP) is guaranteed.

The next theorem asserts that if some consistency conditions are fulfilled, then there exists a solution to (STRIPP) with low McMillan degree.

Theorem 3.3.3. *If there exists a solution to (STRIPP), then there exists a solution of McMillan degree at most $n_\pi + n_\zeta - 1$, where (C_-, A_π) is a least common extension of $\{(C_{-j}, A_{\pi j})|j = 1, \dots, m\}$ of sizes $N \times n_\pi$, $n_\pi \times n_\pi$ respectively, and (A_ζ, B_+) is a least common coextension of $\{(A_{\zeta j}, B_{+j})|j = 1, \dots, m\}$ of sizes $n_\zeta \times n_\zeta$, $n_\zeta \times M$ respectively.*

Proof. Suppose a collection

$$\tilde{\tau}_j = (C_{+j}, C_{-j}, A_{\pi j}; A_{\zeta j}, B_{+j}, B_{-j}; \Gamma_j) \quad 1 \leq j \leq m$$

of σ -admissible (TRIP) data sets are given. Let (C_-, A_π) be a minimal common extension of the collection $\{(C_{-j}, A_{\pi j})|1 \leq j \leq m\}$ of null-kernel pairs with associated injective linear maps $T_{\pi j} : \mathbb{C}^{n_\pi} \rightarrow \mathbb{C}^{n_\pi}$ (here, n_π is the size of A_π) and let (A_ζ, B_+) be a minimal common coextension of the collection $\{(A_{\zeta j}, B_{+j})|1 \leq j \leq m\}$ of full-range

pairs with associated surjective maps $T_{\zeta k} : \mathbb{C}^{n_\zeta} \rightarrow \mathbb{C}^{n_\pi}$ (n_ζ is the size of A_ζ). It is proved in [BGR5] that there exists a solution $L \in \mathcal{P}_{M \times N}$ if and only if

- (i) There exists a solution C_+ to the system of equations

$$C_+ T_{\pi j} = C_{+j}, \quad 1 \leq j \leq m$$

- (ii) There exists a solution B_- to the system of equations

$$B_{-j} = T_{\zeta j} B_-, \quad 1 \leq j \leq m$$

and

- (iii) There exists a solution Γ to the system of equations

$$T_{\zeta i} \Gamma_{\pi j} = \Gamma_{ij}, \quad 1 \leq i, j \leq m$$

where

$$\Gamma_{ii} = \Gamma_i$$

and for $i \neq j$, Γ_{ij} is any solution of the Sylvester equation

$$\Gamma_{ij} A_{\pi j} - A_{\zeta i} \Gamma_{ij} = B_{+i} C_{+j} + B_{-i} C_{-j}.$$

Also it is proved in [BGR5] that in this case $L \in \mathcal{P}_{M \times N}$ satisfies (3.3.4)–(3.3.6) if and only if $L \in \mathcal{P}_{M \times N}$ satisfies the conditions

$$\begin{aligned} \sum_{z_0 \in \mathfrak{C}} \text{Res}_{z=z_0} (zI - A_\zeta)^{-1} B_+ L(z) &= -B_- \\ \sum_{z_0 \in \mathfrak{C}} \text{Res}_{z=z_0} L(z) C_- (zI - A_\pi)^{-1} &= C_+ \\ \sum_{z_0 \in \mathfrak{C}} \text{Res}_{z=z_0} (zI - A_\zeta)^{-1} B_+ L(z) C_- (zI - A_\pi)^{-1} &= \Gamma, \end{aligned}$$

for some choice of Γ as in (iii).

By applying Theorem 3.2.1, we conclude that there exists a $L \in \mathcal{P}_{M \times N}$ satisfying (3.3.4)–(3.3.6) of McMillan degree at most $n_\pi + n_\zeta - 1$. \square

3.4 Divisor-remainder form

In this section, we shall formulate an interpolation problem in divisor remainder form and state the implications of Theorem 3.1.1 and Theorem 3.2.1 for the interpolation problem in divisor-remainder form.

We start with scalar polynomials. Let $p(z)$, $d(z)$ be scalar polynomials. By the division of scalar polynomials, we understand a representation in the form

$$p(z) = q(z)d(z) + r(z), \quad (3.4.1)$$

where $q(z)$ (the quotient) and $r(z)$ (the remainder) are polynomials and

$$\text{the degree of } r(z) < \text{the degree of } d(z), \quad (3.4.2)$$

where the degree of zero is taken to be $-\infty$. The division is always possible and the quotient $q(z)$ and the remainder $r(z)$ are uniquely determined.

The (right) division of matrix polynomials can be defined similarly. Let $D(z)$ and $L(z)$ be given matrix polynomials with sizes $N \times N$ and $M \times N$ respectively. By the division of matrix polynomials, we understand a representation in the form

$$L(z) = Q(z)D(z) + R(z), \quad (3.4.3)$$

where $Q(z)$ (the quotient) and $R(z)$ (the remainder) are $M \times N$ matrix polynomials and

$$\text{the polynomial degree of } R(z) < \text{the polynomial degree of } D(z), \quad (3.4.4)$$

where as in the scalar case, the polynomial degree of the zero matrix is taken to be $-\infty$. When $D(z)$ is regular, there always exist matrix polynomials $Q(z)$ and $R(z)$ satisfying (3.4.3) and (3.4.4). Indeed, express the rational matrix function $L(z)D^{-1}(z)$ as

$$L(z)D^{-1}(z) = Q(z) + W(z),$$

where $Q(z)$ is an $M \times N$ matrix polynomial and $W(z)$ is an $M \times N$ strictly proper rational matrix function (i.e., $W(\infty) = 0$). If we put

$$R(z) := W(z)D(z)$$

then, $R(z)$ is an $M \times N$ matrix polynomial for which

$$L(z) = Q(z)D(z) + R(z).$$

Moreover, by the choice of $W(z)$, $R(z)D^{-1}(z)$ is strictly proper. Hence

the j^{th} column index of $R(z) < \text{the } j^{\text{th}} \text{ column index of } D(z)$

for $j = 1, \dots, n$. The above inequality implies (3.4.4).

On the other hand, we can generalize the constraint (3.4.2) to the matrix polynomial case as

$$\delta(R) < \delta(D) \tag{3.4.5}$$

since in scalar case, (3.4.2) is equivalent to $\delta(r) < \delta(d)$. The division of matrix polynomials (3.4.2) is always possible. To prove this, first we state the following theorem which characterizes the solutions of (RIPP) in *divisor-remainder form*; the proof is found in [BGR4].

Theorem 3.4.1. *Let C_+, C_-, A_π be given as in (RIPP) in the Section 3.1, and assume that (C_-, A_π) is a null-kernel pair. Suppose φ is an $N \times N$ regular matrix polynomial for which $\tilde{\tau}_- = (C_-, A_\pi; 0, 0; 0)$ is a \mathbb{C} -null-pole triple for φ^{-1} . Let $K \in \mathcal{P}_{M \times N}$ be any particular solution of (RIPP). Then $L \in \mathcal{P}_{M \times N}$ is a solution to (RIPP) if and only if*

$$L = K + Q\varphi \tag{3.4.6}$$

for some $Q \in \mathcal{P}_{M \times N}$.

The next theorem says that the division of matrix polynomials (3.4.2) with the condition (3.4.5) is always possible for a regular matrix function $D(z)$.

Theorem 3.4.2. Suppose an $N \times N$ regular matrix polynomial $D(z)$ and an $M \times N$ matrix polynomial $L(z)$ are given. Then there exist $Q \in \mathcal{P}_{M \times N}$, $R \in \mathcal{P}_{M \times N}$ for which

$$L = QD + R$$

and

$$\delta(R) < \delta(D).$$

Proof. Let (C_-, A_π) be a (right) \mathbb{C} -null-pair for $D(z)$. Then we can construct a full-range pair (S_∞, G_∞) as in Corollary 2.2.3 with $\tau = (C_-, A_\pi; 0, 0; 0)$. Let

$$C_+ := \sum_{z_0 \in \mathbb{C}} \text{Res}_{z=z_0} L(z) C_-(zI - A_\pi)^{-1}$$

and

$$R(z) := -C_+ G_\infty - z C_+ C_\infty (I - z S_\infty)^{-1} G_\infty.$$

Then, by Theorem 3.1.1, $R(z)$ satisfies the interpolation condition (3.4.7) with $R(z)$ in the place of $L(z)$ and

$$\delta(R) \leq [\text{the size of } A_\pi] - 1. \quad (3.4.8)$$

Since the McMillan degree of a regular matrix function is at least the number of zeros in \mathbb{C} , $\delta(D) \geq [\text{the size of } A_\pi]$ and in turn (3.4.8) is reduced to

$$\delta(R) < \delta(D).$$

Finally, by applying Theorem 3.4.1 and recalling the fact that $R(z)$ and $L(z)$ both satisfy the interpolation condition (3.4.7), we conclude that there exists $Q \in \mathcal{P}_{M \times N}$ for which

$$L = QD + R.$$

□

Now we consider the following *several divisor-remainder interpolation problem* (SDRI).

(SDRI) Let $L_1(z), \dots, L_r(z)$ be given $N \times N$ regular matrix polynomials respectively and $R_1(z), \dots, R_r(z)$ be given $M \times N$ matrix polynomials. Find a matrix polynomial $A \in \mathcal{P}_{M \times N}$ such that

$$A(z) = S_i(z)L_i(z) + R_i(z) \quad (3.4.9)$$

for some $S_i \in \mathcal{P}_{M \times N}$, $i = 1, \dots, r$.

By Theorem 3.4.1, the interpolation conditions (3.4.9) are equivalent to

$$\sum_{z_0 \in \mathbf{t}} \text{Res}_{z=z_0} A(z) C_{-i}(zI - A_{\pi i})^{-1} = C_{+i}, \quad i = 1, \dots, r \quad (3.4.10)$$

where $(C_{-i}, A_{\pi i})$ is a right null pair for L_i and

$$C_{+i} = \sum_{z_0 \in \mathbf{t}} \text{Res}_{z=z_0} R_i(z) C_{-i}(zI - A_{\pi i})^{-1}$$

for each $i = 1, \dots, r$. Let (C_-, A_π) be a least common extension of the collection $\{(C_{-i}, A_{\pi i}) | i = 1, \dots, r\}$ of sizes $N \times n_\pi$, $n_\pi \times n_\pi$ respectively with associated injective linear map $S_i : \mathbb{C}^{n_{\pi i}} \rightarrow \mathbb{C}^{n_\pi}$. Then by Theorem 16.5.3 of [BGR6], there exists $C_+ : \mathbb{C}^{n_\pi} \rightarrow \mathbb{C}^M$ satisfying

$$C_+ S_i = C_{+i} \quad i = 1, \dots, r \quad (3.4.11)$$

if and only if there exists $A \in \mathcal{P}_{M \times N}$ satisfying (3.4.10) and in this case $A \in \mathcal{P}_{M \times N}$ satisfies (3.4.10) if and only if $A \in \mathcal{P}_{M \times N}$ satisfies the single contour integral interpolation condition

$$\sum_{z_0 \in \mathbf{t}} \text{Res}_{z=z_0} A(z) C_-(zI - A_\pi)^{-1} = C_+. \quad (3.4.12)$$

Since (3.4.9) and (3.4.10) are equivalent the existence of C_+ satisfying (3.4.11) is the necessary and sufficient condition for the existence of a solution $A(z)$ of (3.4.9). By Theorem 3.1.1, there exists a solution to (3.4.12) of McMillan degree at most $n_\pi - 1$.

On the other hand, it is proved (see Theorem 9.8 of [GLR1]) that the number of zeros (counting multiplicities) of a least common multiple of L_1, \dots, L_r is n_π .

We have proved the following theorem.

Theorem 3.4.3. *If there exists a solution to (SDRI) problem, then there exists a solution of McMillan degree at most $n_\pi - 1$, where n_π is the number of zeros (counting multiplicities) of the least common multiple of L_1, \dots, L_r .*

Suppose $M = N$ and L_1, \dots, L_r are monic with polynomial degrees p_1, \dots, p_r respectively in the (SDRI) problem. Without loss of generality we can assume that the polynomial degree of R_i is less than p_i for $i = 1, \dots, r$ (see [GLR1], Chapter 3). Under these assumptions, it is proved in [GKLR3] that the (SDRI) problem has a solution of polynomial degree $\leq m$ if and only if

$$\text{Ker } \tilde{V}_m(L_1, \dots, L_r) \subset \text{Ker row}([R_{i0} R_{i1} \dots R_{ip_i-1}])_{i=1}^r$$

where $R_i(z) = \sum_{j=0}^{p_i-1} R_{ij} z^j$ and

$$\tilde{V}_m(L_1, \dots, L_r) = \begin{bmatrix} C_{-1}U_1 & \cdots & C_{-r}U_r \\ C_{-1}A_{\pi 1}U_1 & \cdots & C_{-r}A_{\pi r}U_r \\ \vdots & & \vdots \\ C_{-1}A_{\pi 1}^{m-1}U_1 & \cdots & C_{-r}A_{\pi r}^{m-1}U_r \end{bmatrix}$$

with

$$U_i = (\text{col}(C_{-i} A_{\pi i}^j)_{j=0}^{p_i-1})^{-1}.$$

For this special case, we have the following theorem which states the consistency conditions in terms of the given data.

Theorem 3.4.4. *Suppose, in the (SDRI) problem, $M = N$ and L_1, \dots, L_r are monic matrix polynomials of polynomial degrees p_1, \dots, p_r respectively and the polynomial degrees of R_1, \dots, R_r are less than p_1, \dots, p_r respectively. Let $m = N(p_1 + \dots + p_r) - 1$ and $R_i(z) = \sum_{j=0}^{p_i-1} R_{ij} z^j$. Then the following are equivalent.*

- (i) There exists a matrix polynomial $A \in \mathcal{P}_{M \times N}$ which solves the (SDRI) problem.
- (ii) There exists a matrix polynomial $A \in \mathcal{P}_{M \times N}$ of McMillan degree $\leq m$ which solves the (SDRI) problem.
- (iii) There exists a matrix polynomial $A \in \mathcal{P}_{M \times N}$ of polynomial degree $\leq m$ which solves the (SDRI) problem.
- (iv) $\text{Ker } \tilde{V}_m(L_1, \dots, L_r) \subset \text{Ker row } ([R_{i0} \dots R_{ip_i-1}])_{i=1}^r$,

Proof. (i) \Rightarrow (ii) Let $(C_{-i}, A_{\pi i})$ be a right null pair for L_i for each $i = 1, \dots, r$ and (C_-, A_π) be a least common extension of $\{(C_{-i}, A_{\pi i}) | i = 1, \dots, r\}$. Since L_i is monic,

$$\text{the size of } A_{\pi i} = \delta(L_i) = p_i N \quad (3.4.13)$$

for each $i = 1, \dots, r$. On the other hand, by the property of the least common extension,

$$[\text{the size of } A_\pi] \leq \sum_{k=1}^r [\text{the size of } A_{\pi k}]. \quad (3.4.14)$$

Upon combining (3.4.13) and (3.4.14), we get

$$[\text{the size of } A_\pi] \leq N(p_1 + \dots + p_r).$$

By applying Theorem 3.4.3, we conclude that there exists a solution to the (SDRI) problem of McMillan degree at most $n_\pi - 1$.

Since the polynomial degree is less than or equal to the McMillan degree (ii) \Rightarrow (iii) is clear and (iii) \Rightarrow (iv), (iv) \Rightarrow (i) are the results of [GKLR3]. \square

The statements dual to Theorem 3.4.2 and Theorem 3.4.3 are as follows:

Theorem 3.4.5. Suppose an $M \times M$ regular matrix polynomial $\tilde{D}(z)$ and an $M \times N$ matrix polynomial $L(z)$ are given. Then there exist $\tilde{Q} \in \mathcal{P}_{M \times N}$, $\tilde{R} \in \mathcal{P}_{M \times N}$ for which

$$L = \tilde{D}\tilde{Q} + \tilde{R}$$

and

$$\delta(\tilde{R}) < \delta(\tilde{D}).$$

Theorem 3.4.6. *Let $M_1(z), \dots, M_r(z)$ be given $M \times M$ regular matrix polynomials and $R_1(z), \dots, R_r(z)$ be given $M \times N$ matrix polynomials. If there exists a matrix polynomial $A \in \mathcal{P}_{M \times N}$ such that*

$$A(z) = M_i(z)S_i(z) + R_i(z) \quad (3.4.15)$$

for some $S_i \in \mathcal{P}_{M \times N}$, $i = 1, \dots, r$, then there exists an $M \times N$ matrix polynomial of McMillan degree at most $n_\zeta - 1$ which solves (3.4.15) where n_ζ is the number of zeros (counting multiplicities) of a least common multiple of M_1, \dots, M_r .

Next we consider two-sided interpolation problems in divisor-remainder form. As in the one-sided residue interpolation problem, all solutions of (TRIPP) can be characterized in divisor-remainder form as follows. The proof is found in [BGR4].

Theorem 3.4.7. *Suppose $\tilde{\tau} = (C_+, C_-, A_\pi; A_\zeta, B_+, B_-; \Gamma)$ is a \mathbb{C} -admissible (TRIP) data set. Suppose that ψ is an $M \times M$ regular matrix polynomial which has (A_ζ, B_+) as its (left) null pair and φ is an $N \times N$ regular matrix polynomial which has (C_-, A_π) as its (right) null pair and $K \in \mathcal{P}_{M \times N}$ is any particular solution of (TRIPP). Then $L \in \mathcal{P}_{M \times N}$ is also a solution of (TRIPP) if and only if*

$$L = K + \psi Q \varphi \quad (3.4.16)$$

for some $Q \in \mathcal{P}_{M \times N}$.

By combining Theorem 3.4.7 and Theorem 3.2.1, we obtain the following division formula for matrix polynomials in which divisors on both sides are involved.

Theorem 3.4.8. *For a given $M \times M$ regular matrix polynomial \tilde{D} , an $N \times N$ regular matrix polynomial D , and $L \in \mathcal{P}_{M \times N}$, there exist $Q, R \in \mathcal{P}_{M \times N}$ for which*

$$L = \tilde{D} Q D + R$$

with

$$\delta(R) < \delta(\tilde{D}) + \delta(D).$$

Proof. Let (A_ζ, B_+) be a left null pair for \tilde{D} and (C_-, A_π) be a right null pair for D . Then

$$\delta(\tilde{D}) \geq n_\zeta \quad (3.4.17)$$

and

$$\delta(D) \geq n_\pi, \quad (3.4.18)$$

where n_ζ and n_π are the sizes of A_ζ and A_π respectively. If we define the matrices B_-, C_+, Γ by

$$-B_- = \sum_{z_0 \in \mathbf{t}} \text{Res}_{z=z_0} (zI - A_\zeta)^{-1} B_+ L(z) \quad (3.4.19)$$

$$C_+ = \sum_{z_0 \in \mathbf{t}} \text{Res}_{z=z_0} L(z) C_- (zI - A_\pi)^{-1} \quad (3.4.20)$$

and

$$\Gamma = \sum_{z_0 \in \mathbf{t}} \text{Res}_{z=z_0} (zI - A_\zeta)^{-1} B_+ L(z) C_- (zI - A_\pi)^{-1}, \quad (3.4.21)$$

then, it can be easily shown that $\tilde{\tau} = (C_+, C_-, A_\pi; A_\zeta, B_+, B_-; \Gamma)$ is a \mathbb{C} -admissible (TRIP) data set.

By Theorem 3.2.1, there exists a matrix polynomial $R \in \mathcal{P}_{M \times N}$ satisfying (3.4.19)–(3.4.21) with $R(z)$ in place of $L(z)$ and satisfying

$$\delta(R) \leq n_\pi + n_\zeta - 1. \quad (3.4.22)$$

Since both L and R are solutions of the (TRIPP), by Theorem 3.4.7, there exists $Q \in \mathcal{P}_{M \times N}$ for which

$$L = \tilde{D} Q D + R.$$

By combining (3.4.17), (3.4.18) and (3.4.22), we get

$$\delta(R) < \delta(\tilde{D}) + \delta(D).$$

□

The rest of this section is devoted to the interpolation problem in the form of several two-sided divisor-remainder form constraints which is stated as follows.

Let $L_i(z)$, $K_i(z)$ be given $N \times N$ and $M \times M$ regular matrix polynomials respectively and let $R_i(z)$ be a given $M \times N$ matrix polynomial for $1 \leq i \leq r$. Find a matrix polynomial $A \in \mathcal{P}_{M \times N}$ such that

$$A(z) = K_i(z)S_i(z)L_i(z) + R_i(z) \quad (3.4.23)$$

for some $S_i \in \mathcal{P}_{M \times N}$ for $1 \leq i \leq r$.

Theorem 3.4.9. *If there exists an $M \times N$ matrix polynomial satisfying (3.4.23), then there exists a $M \times N$ matrix polynomial satisfying (3.4.23) of McMillan degree at most $n_\pi + n_\zeta - 1$, where n_π is the number of finite zeros of a least common multiple of L_1, \dots, L_r and n_ζ is the number of finite zeros of a least common multiple of K_1, \dots, K_r .*

Proof. Let $(C_{-j}, A_{\pi j})$ be a right null pair for L_j and $(A_{\zeta j}, B_{+j})$ be a left null pair for $K_j(z)$ for $j = 1, \dots, r$. Since Theorem 3.4.7 implies that there exists a matrix polynomial $A \in \mathcal{P}_{M \times N}$ satisfying (3.4.23) if and only if there exists a matrix polynomial $A \in \mathcal{P}_{M \times N}$ satisfying (3.3.4)–(3.3.6) (with A in place of L) with

$$\begin{aligned} C_{+j} &= \sum_{z_0 \in \mathbb{C}} \text{Res}_{z=z_0} R_j(z) C_{-j}(zI - A_{\pi j})^{-1} \\ -B_{-j} &= \sum_{z_0 \in \mathbb{C}} \text{Res}_{z=z_0} (zI - A_{\zeta j})^{-1} B_{+j} R_j(z) \\ \Gamma_j &= \sum_{z_0 \in \mathbb{C}} \text{Res}_{z=z_0} (zI - A_{\zeta j})^{-1} B_{+j} R_j(z) C_{-j}(zI - A_{\pi j})^{-1} \end{aligned}$$

for $j = 1, \dots, r$. By assumption and Theorem 3.3.3, if there exists any matrix polynomial solution $A \in \mathcal{P}_{M \times N}$ of (3.4.23), then there exists such a solution of McMillan degree at most $n_\pi + n_\zeta - 1$, where (C_-, A_π) is a least common extension of $\{(C_{-j}, A_{\pi j}) | j = 1, \dots, r\}$ of sizes $N \times n_\pi$, $n_\pi \times n_\pi$ respectively and (A_ζ, B_+) is a least common coextension of $\{(A_{\zeta j}, B_{+j}) | j = 1, \dots, r\}$ of sizes $n_\zeta \times n_\zeta$, $n_\zeta \times M$ respectively.

It is known (see [GLR1]) that the number of finite zeros (counting the multiplicities) of a least common multiple of $L_1, \dots, L_r(K_1, \dots, K_r)$ is $n_\pi(n_\zeta)$. \square

IV. A solution of minimal possible McMillan degree

Our concern in this chapter is to find a rational solution to (TRIP) (as defined in Section 1.4) of minimal possible McMillan degree. The scalar version of (TRIP) defined by (1.4.1)–(1.4.3) is the following. Given distinct points z_1, \dots, z_n in the complex plane \mathbb{C} , and given complex numbers $\{w_{ij}\}_{i=1}^{\mu_j} {}_{j=1}^n$, find all rational functions

$$f(z) := \frac{n(z)}{d(z)}, \quad \gcd(n, d) = 1 \quad (4.1)$$

such that

$$f^{(i-1)}(z_j) = w_{ij}, \quad i = 1, \dots, \mu_j, \quad j = 1, \dots, n, \quad (4.2)$$

where \gcd represents the greatest common divisor of polynomials. In particular, we are interested in a solution of minimal possible McMillan degree. In the scalar case, the McMillan degree of $f(z)$ in (4.1) is defined as

$$\delta(f) := \max\{\deg n, \deg d\}. \quad (4.3)$$

Some special cases of the problem were studied by Belevitch (1970) and Donoghue (1974) and recently the general case was understood by Antoulas and Anderson (1986). In the latter approach, the *Löwner matrix* is a key notion. For simplicity assume $\mu_i = 1$ for $i = 1, \dots, n$ in (4.2) and $n = 2m + 1$. Then the associated Löwner matrix L is given by

$$L := \left[\frac{w_{m+1+i} - w_j}{z_{m+1+i} - z_j} \right]_{1 \leq i \leq m, 1 \leq j \leq m+1}. \quad (4.4)$$

In the first section, the already existing results for the scalar case are introduced with assumptions $\mu_i = 1$ for all i and $n = 2m + 1$. The minimal possible McMillan degree for solutions is q or $n - q$, where q is the rank of the matrix L given by (4.4). This section follows [AA1]. In the second section, some efforts to extend the approach

of the first section to the matrix case are made. Here the null-pole coupling matrix Γ introduced in Section 1.4 plays the role of the Löwner matrix for the matrix case. Even though the problem considered in this section is a nice special one, it turns out that the approach which depends on the Löwner matrix is not very enlightening. One of the difficulties is the collapse of Theorem 4.1.1 (compare Theorem 4.1.1 and Theorem 4.2.1). So, we go back to the scalar case in Section 3 and come up with a new point of view for which the extension to the matrix case is possible. The discussion in Section 2 appears here for the first time; Section 3 follows [ABKW] but more explanations are given.

The remaining two sections present the main results of this chapter. The fourth section is about the one-sided residue interpolation problem (RIP) and the last section is about the two-sided residue interpolation problem (TRIP). The following aspects are discussed for (RIP) and (TRIP) respectively: (a) the minimal possible McMillan degree, (b) a realization formula for a solution of minimal possible McMillan degree, (c) all admissible degrees of complexity, and (d) a parameterization of all solutions of McMillan degree n for a given admissible degree n . These last two sections follow mostly [ABKW] but some details are done differently or are added.

4.1 The scalar case

The problem considered in this section is the following. Given distinct points z_1, \dots, z_n in the complex plane \mathbb{C} and given complex numbers $\{w_j\}_{j=1}^n$ find a rational function $f(z)$ in the form of (4.1) which has the minimal possible McMillan degree among the rational functions satisfying the interpolating conditions

$$f(z_j) = w_j, \quad j = 1, \dots, n. \quad (4.1.1)$$

To fix the notation, suppose

$$n = 2m + 1$$

for some nonnegative integer m .

The classical approach to this problem is based on the naive (but in general false) hope that the interpolation problem (4.1.1) can be solved by a rational function of McMillan degree at most m . Indeed, if we put

$$f(z) := \frac{n(z)}{d(z)}$$

with

$$n(z) := a_0 + a_1 z + \cdots + a_m z^m$$

$$d(z) := b_0 + b_1 z + \cdots + b_m z^m$$

then, from (4.1.1) the following system of linear equations are derived;

$$n(z_j) - w_j d(z_j) = 0, \quad j = 1, \dots, n \quad (4.1.2)$$

or equivalently,

$$\begin{bmatrix} 1 & z_1, \dots, z_1^m, w_1, w_1 z_1, \dots, w_1 z_1^m \\ 1 & z_2, \dots, z_2^m, w_2, w_2 z_2, \dots, w_2 z_2^m \\ \vdots & \vdots \\ 1 & z_n, \dots, z_n^m, w_n, w_n z_n, \dots, w_n z_n^m \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \\ -b_0 \\ -b_1 \\ \vdots \\ -b_m \end{bmatrix} = 0. \quad (4.1.3)$$

The system of linear equations (4.1.2) is called the modified interpolation problem in [M] and has often been studied as an intermediate tool in solving the interpolation problem (4.1.1). The system of linear equations (4.1.2) or (4.1.3) always has a non-trivial solution since the system of linear equations to be solved has $(2m+1)$ equations and $(2m+2)$ variables. But, in this approach, the difficulty arises if the solution of

the modified interpolation problem (4.1.2) $d(z)$ has a zero at one of the prescribed points z_1, \dots, z_n . Then, the polynomials $n(z)$ and $d(z)$ have a common zero at the point and the function $f(z) = \frac{n(z)}{d(z)}$ may not interpolate at the point. Those points are called *unattainable* or *inaccessible* points. By the *generic case*, we refer to the case where there is no inaccessible point. Hence, in this case, a solution of McMillan degree at most m exists. In [B], the problem of finding a solution of McMillan degree at most m is analyzed more systematically by using the Löwner matrix L defined by (4.4). To introduce his approach, we start with the following theorem which is crucial in his analysis. For the proof see [AA1] or [B].

Theorem 4.1.1. *Let L be an $p \times l$ Löwner matrix with $p \geq q$, $l \geq q$, built on finite values taken by a rational function*

$$f(z) := \frac{n(z)}{d(z)}, \quad \gcd(n, d) = 1$$

of McMillan degree q at $p + l$ points. Then,

$$\text{rank } L = q.$$

Let L be the $m \times (m + 1)$ Löwner matrix given by (4.4) and

$$q := \text{rank } L.$$

Then a solution of the modified interpolation problem (4.1.2) or (4.1.3) $n(z)$, $d(z)$ with $\deg n(z) \leq m$, $\deg d(z) \leq m$ can be constructed in the following way. Let \bar{L} be an $(n - q - 1) \times (q + 1)$ Löwner matrix defined by

$$\bar{L} = \left[\frac{w_{q+1+i} - w_j}{z_{q+1+i} - z_i} \right]_{1 \leq i \leq n-q-1, 1 \leq j \leq q+1}. \quad (4.1.5)$$

By Corollary 2.24 of [AA1] which states that any $r \times l$ Löwner matrix constructed from the same data for L with $r \geq q$, $l \geq q$ has $\text{rank } q$,

$$\text{rank } \bar{L} = q. \quad (4.1.6)$$

Since the $(N - q - 1) \times (q + 1)$ matrix \bar{L} has rank q , there exists a $(q + 1)$ -dimensional vector

$$c := [c_1, \dots, c_{q+1}]^T$$

satisfying

$$\bar{L}c = 0 \quad (4.1.7)$$

and

$$c_j \neq 0 \text{ for } j = 1, \dots, q + 1. \quad (4.1.8)$$

The conditions (4.1.7) are rewritten as

$$\sum_{j=1}^{q+1} \frac{c_j(w_k - w_j)}{z_k - z_j} = 0, \quad k = q + 2, q + 3, \dots, n. \quad (4.1.9)$$

Let

$$\bar{d}(z) := \sum_{j=1}^{q+1} c_j \prod_{\substack{i=1 \\ i \neq j}}^{q+1} (z - z_i) \quad (4.1.10)$$

$$\bar{n}(z) := \sum_{j=1}^{q+1} c_j w_j \prod_{\substack{i=1 \\ i \neq j}}^{q+1} (z - z_i) \quad (4.1.11)$$

and $\bar{f}(z)$ be a rational function satisfying

$$\bar{f}(z)\bar{d}(z) = \bar{n}(z). \quad (4.1.12)$$

Since $\bar{d}(z_j) \neq 0$ for $j = 1, \dots, q + 1$ by our choice of vector c , $\bar{f}(z)$ in (4.1.12) is well-defined, and

$$w_k \bar{d}(z_k) - \bar{n}(z_k) = 0 \text{ for } k = 1, \dots, n. \quad (4.1.13)$$

Indeed, for $k = 1, \dots, q + 1$, the equality (4.1.13) is obtained as soon as we plug $z = z_k$ in (4.1.10) and (4.1.11). For $k \geq q + 2$, (4.1.13) is derived from (4.1.9) by multiplying both sides by $\prod_{j=1}^{q+1} (z_k - z_j)$.

The following theorem tells us in which case the solution $\bar{n}(z), \bar{d}(z)$ of the modified interpolation problem gives a solution to the interpolation problem (4.1.1).

Theorem 4.1.2. Let $\bar{f}(z)$ be a rational function given by (4.1.12). Then the following statements are equivalent.

- (a) $\bar{f}(z)$ is a solution to (4.1.1).
- (b) $\delta(\bar{f}) = q$.
- (c) $\bar{d}(z_i) \neq 0$ for $i = 1, \dots, n$.

Proof. (a) \Rightarrow (b): Upon observing that $\delta(f) \leq q$ by its construction, we get $\delta(f) = q$ by Theorem 4.1.1. The implications (b) \Rightarrow (c), (c) \Rightarrow (a) are obvious from (4.1.10)–(4.1.13). \square

As we will see in the proof of the next theorem, if there exists a solution of McMillan degree less than $\frac{n}{2}$, $\bar{f}(z)$ is the solution. The following theorem due to [AA1] gives a solution of minimal possible McMillan degree when $\bar{f}(z)$ in (4.1.12) is not a solution of (4.1.1). We will sketch the proof, but the proof is revised by the author.

Theorem 4.1.3. Let L be an $m \times (m + 1)$ Löwner matrix given by (4.4) and $\bar{d}(z)$ and $\bar{f}(z)$ be given as in (4.1.11), (4.1.12).

- (a) If $\bar{d}(z)$ has no zeros at z_j , $j = 1, \dots, n$, then the minimal possible McMillan degree for the solutions to the interpolation problem (4.1.1) is $\text{rank } L$, and $\bar{f}(z)$ is the unique solution of $\text{rank } L$.
- (b) Otherwise, $n - \text{rank } L$ is the minimal possible McMillan degree and there is more than one solution of McMillan degree $n - \text{rank } L$.

Sketch of the Proof. By Theorem 4.1.1, there is no solution of McMillan degree less than $\text{rank } L = q$. First, we show that if $f(z)$ is a solution of McMillan degree at most m , then $f(z) = \bar{f}(z)$. Suppose $\bar{f}(z)$ interpolates $n - \alpha$ ($= 2m + 1 - \alpha$) points ($\alpha = 0$ if and only if $\bar{f}(z)$ is a solution). Then, by Theorem 4.1.2, there exist points

$z_{i_1}, \dots, z_{i_\alpha}$ for which

$$\bar{d}(z_{i_j}) = 0, \quad j = 1, \dots, \alpha. \quad (4.1.14)$$

Because of (4.1.12), (4.1.14) forces that

$$\bar{n}(z_{i_j}) = 0, \quad j = 1, \dots, \alpha. \quad (4.1.15)$$

Upon combining (4.1.14) and (4.1.15), we can see that $\delta(\bar{f}) \leq q - \alpha$. Hence a rational function $f(z) - \bar{f}(z)$ has degree no more than $m + q - \alpha$ and has at least $n - \alpha$ zeros. But, the fact that $m + q - \alpha \leq 2m - \alpha < n - \alpha$ forces that $f(z) = \bar{f}(z)$. Hence, the uniqueness of (a) is proved.

(b) Suppose $\bar{f}(z)$ is not a solution. By the previous argument, there is no solution of McMillan degree at most m . Suppose there exists a solution $\tilde{f}(z)$ of McMillan degree $\tilde{q} > m$. Let \tilde{L} be a $l \times n$ Löwner matrix constructed from $\tilde{f}(z)$ so that $l \geq \tilde{q}$, $n \geq \tilde{q}$ and L is a submatrix of \tilde{L} . Then by Theorem 4.1.1, any $\tilde{q} \times \tilde{q}$ submatrix of \tilde{L} is invertible and in turn by Section 2.1, the minimal possible such \tilde{q} is $n - \text{rank } L = n - q$.

□

Now, we describe a solution of McMillan degree $n - q$. Let \hat{L} be a $(q-1) \times (n-q+1)$ Löwner matrix given by

$$\hat{L} = \left[\begin{array}{c} \frac{w_{n-q+1+i} - w_j}{z_{n-q+1+i} - z_j} \\ \vdots \end{array} \right]_{1 \leq i \leq q-1, \quad 1 \leq j \leq n-q+1} \quad (4.1.16)$$

and \hat{c} be a $(n - q + 1)$ -dimensional vector satisfying the following:

$$\hat{L}\hat{c} = 0 \quad (4.1.17)$$

and

$$\hat{d}(z) := \sum_{j=1}^{n-q+1} \hat{c}_j \prod_{\substack{i=1 \\ i \neq j}}^{n-q+1} (z - z_i) \quad (4.1.18)$$

has no zeros at z_j , $j = 1, \dots, n$, where

$$\hat{c} := [\hat{c}_1, \dots, \hat{c}_{n-q+1}]^T.$$

Indeed, there exists infinitely many such vectors \hat{c} . Let

$$\hat{n}(z) := \sum_{j=1}^{n-q+1} \hat{c}_j w_j \prod_{\substack{i=1 \\ i \neq j}}^{n-q+1} (z - z_i) \quad (4.1.19)$$

and

$$\hat{f}(z) := \frac{\hat{n}(z, c)}{\hat{d}(z, c)}. \quad (4.1.20)$$

Then, by its construction (4.1.18) (4.1.19)

$$w_j \hat{d}(z_j, \hat{c}) = \hat{n}(z_j, \hat{c}). \quad (4.1.21)$$

Since \hat{c} is chosen so that $\hat{d}(z_j, \hat{c}) \neq 0$ for any $j = 1, \dots, n$, from (4.1.20), (4.1.21), we conclude that $\hat{f}(z)$ is a solution of McMillan degree at most $n - q$. But, we already have proved that the minimal possible McMillan degree for solutions other than $\bar{f}(z)$ (if $\bar{f}(z)$ is a solution) is $n - q$. This completes the proof. \square

Remark 4.1.4. (a) For the multiple point interpolation problem (4.1), see [AA1].
(b) In the proof of Theorem 4.1.3 (b), we applied Remark 2.1.2 (b) to get $\tilde{q} \geq n - \text{rank } L$ since in this case the Löwner matrix L can be recovered from the null-pole-coupling matrix Γ associated with the bitangential setting of this scalar problem:

$$A_\zeta = \begin{bmatrix} z_{m+2} & & 0 \\ & \ddots & \\ 0 & & z_n \end{bmatrix}, \quad B = [B_+ \ B_-] := \begin{bmatrix} 1 & -w_{m+2} \\ \vdots & \vdots \\ \vdots & \vdots \\ 1 & -w_n \end{bmatrix}$$

$$C = \begin{bmatrix} C_+ \\ C_- \end{bmatrix} = \begin{bmatrix} w_1, \dots, w_{m+1} \\ 1, \dots, 1 \end{bmatrix}, \quad A_\pi = \begin{bmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_{m+1} \end{bmatrix}$$

are the proper choice for the null pair (A_ζ, B) , and the pole pair (C, A_π) of Section 2.1. The same reasoning can be used for the multiple point case since the generalized Löwner matrix (see [AA1]) is a special case of the matrix Γ with appropriately chosen matrices A_ζ, B, C, A_π .

(c) If n is even and unless the rank of the $\frac{n}{2} \times \frac{n}{2}$ Löwner matrix L is $\frac{n}{2}$, all the arguments are the same. But, if $\text{rank } L = \frac{n}{2}$, then $\frac{n}{2}$ is the minimal possible McMillan degree for the solution and a solution of McMillan degree $\frac{n}{2}$ is obtained by the construction (4.1.16)–(4.1.20) with $q = \frac{n}{2}$. \square

4.2 A matrix analogue of the scalar generic case

In this section, we consider a generalization of the interpolation conditions (4.1.1) to the matrix case and try to find a solution of minimal possible McMillan degree. Here, basically we adapt the approach used in the previous section, but the parametrization of all solutions of (TRIP) given by Theorem 1.4.1 plays an important role.

The problem we consider in this section is the following. Given distinct points $z_1, \dots, z_n, w_1, \dots, w_{n+N}$ in the complex plane \mathbb{C} , and nonzero vectors x_1, \dots, x_n in $\mathbb{C}^{1 \times M}$, vectors y_1, \dots, y_n in $\mathbb{C}^{1 \times N}$, nonzero vectors u_1, \dots, u_{n+N} in $\mathbb{C}^{N \times 1}$, and vectors v_1, \dots, v_{n+N} in $\mathbb{C}^{M \times 1}$. Find an $M \times N$ rational matrix function $W(z)$ for which

$$x_i W(z_i) = y_i, \quad i = 1, \dots, n \quad (4.2.1)$$

$$W(w_j) u_j = v_j, \quad j = 1, \dots, n + N. \quad (4.2.2)$$

Then, we can generate $MN + (M + N)n$ equations from the conditions (4.2.1) and (4.2.2). Indeed, for each i in (4.2.1) we have a $1 \times N$ vector equation and for each j in (4.2.2), we get an $M \times 1$ vector equation. On the other hand, it is known that an $M \times N$ rational matrix function of McMillan degree d is determined by $MN + (M + N)d$ parameters (see [BF]). Hence, by comparing the number of constraints and the number of parameters, we can expect there may be a solution of (4.2.1) and (4.2.2) which has McMillan degree at most n . But, as in the scalar case, this is not always the case.

Here we study the case where there exists a solution of McMillan degree at most n .

Let Γ be an $n \times (n + N)$ matrix whose $(i, j)^{\text{th}}$ component, Γ_{ij} , is given by

$$\Gamma_{ij} = \frac{x_i v_j - y_i u_j}{w_j - z_i} = x_i \frac{W(w_j) - W(z_i)}{w_j - z_i} u_j, \quad (4.2.3)$$

where vectors x_i, y_i, u_j, v_j are the same as in (4.2.1) (4.2.2). Then, Γ is a matrix analogue of the Löwner matrix L in the previous section. The next theorem is a counterpart of Theorem 4.1.1.

Theorem 4.2.1. *Let $W(z)$ be a given $M \times N$ rational matrix function with $\delta(W) = q$ and let*

$$\Gamma := \left[x_i \frac{W(w_j) - W(z_i)}{w_j - z_i} u_j \right]_{1 \leq i \leq n, 1 \leq j \leq l}, \quad (4.2.4)$$

where $n \geq q$, $l \geq q$ and $z_1, \dots, z_n, w_1, \dots, w_l$ are distinct points in the complex plane at which $W(z)$ is analytic and x_1, \dots, x_n are nonzero $1 \times M$ vectors, u_1, \dots, u_l are $N \times 1$ nonzero vectors. Then,

$$\text{rank } \Gamma \leq \delta(W).$$

Proof. Let

$$W(z) = D + C(zG - A)^{-1}B \quad (4.2.5)$$

be a minimal realization for $W(z)$. Upon substituting (4.2.5) in (4.2.4), we get

$$\Gamma = \left[x_i \frac{C\{(w_j G - A)^{-1} - (z_i G - A)^{-1}\}B}{w_j - z_i} u_j \right]_{1 \leq i \leq n, 1 \leq j \leq l}. \quad (4.2.6)$$

Since

$$(w_j G - A)^{-1} - (z_i G - A)^{-1} = (z_i G - A)^{-1}(z_i - w_j)G(w_j G - A)^{-1},$$

(4.2.6) reduces to

$$\begin{aligned}\Gamma &= -[x_i C(z_i G - A)^{-1} G(w_j G - A)^{-1} B u_j]_{1 \leq i \leq n, 1 \leq j \leq l} \\ &= -\begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{bmatrix} \begin{bmatrix} C(z_1 G - A)^{-1} \\ \vdots \\ C(z_n G - A)^{-1} \end{bmatrix} G [(w_1 G - A)^{-1}, \dots, (w_n G - A)^{-1}] \\ &\times \begin{bmatrix} u_1 & & 0 \\ & \ddots & \\ 0 & & u_{n+1} \end{bmatrix}.\end{aligned}$$

Thus, we have

$$\text{rank } \Gamma \leq \text{rank } G.$$

By the minimality of the realization of $W(z)$ given in (4.2.5), the size of the square matrix G is q . Hence, we conclude that

$$\text{rank } \Gamma \leq q. \quad \square$$

The following example shows that the inequality

$$\text{rank } \Gamma \leq \delta(W)$$

which we have in Theorem 4.2.1 is the best estimation, unlike the scalar case. Let

$$W(z) = \begin{bmatrix} \frac{1}{z} & 0 \\ 1 & 1 \end{bmatrix}$$

and $z_1 = 1$, $w_1 = -1$ and $x_1 = [1 \ 0]$, $u_1 = [0 \ 1]^T$. Then, for this case, Γ given by (4.2.4) is

$$\begin{aligned}\Gamma &= x_1 \frac{W(w_1) - W(z_1)}{w_1 - z_1} u_1 \\ &= -\frac{1}{2} [1 \ 0] \left(\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= -\frac{1}{2} [1 \ 0] \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= 0.\end{aligned}$$

Thus, $\text{rank } \Gamma = 0$. But the McMillan degree of $W(z)$ is 1.

To use the parametrization of solutions given in Theorem 1.4.1, we introduce the following matrices. Let

$$A_\zeta := \begin{bmatrix} z_1 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & z_n \end{bmatrix} \quad (4.2.7)$$

$$\begin{aligned} [B_+ \ B_-] &:= \begin{bmatrix} x_1 & -y_1 \\ \vdots & \vdots \\ \vdots & \vdots \\ x_n & -y_n \end{bmatrix} \\ \begin{bmatrix} C_+ \\ C_- \end{bmatrix} &:= \begin{bmatrix} v_1, \dots, v_{n+N} \\ u_1, \dots, u_{n+N} \end{bmatrix}, \end{aligned} \quad (4.2.8)$$

$$A_\pi := \begin{bmatrix} w_1 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & w_{n+N} \end{bmatrix}.$$

Then, the data set $\tilde{\tau} = (C_+, C_-, A_\pi; A_\zeta, B_+, B_-; \Gamma)$ satisfy the normalization conditions (1.4.4)–(1.4.6), where Γ is given by (4.2.3). Moreover, (4.2.1) and (4.2.2) are equivalent to

$$\sum_{z_0 \in \mathbb{C}} \text{Res}_{z=z_0} (zI - A_\zeta)^{-1} B_+ W(z) = -B_- \quad (4.2.9)$$

$$\sum_{z_0 \in \mathbb{C}} \text{Res}_{z=z_0} W(z) C_- (zI - A_\pi)^{-1} = C_+, \quad (4.2.10)$$

and in this case the condition

$$\sum_{z_0 \in \mathbb{C}} \text{Res}_{z=z_0} (zI - A_\zeta)^{-1} B_+ W(z) C_- (zI - A_\pi)^{-1} = \Gamma \quad (4.2.11)$$

is equivalent to the Sylvester equation (1.4.6).

Let B be an $(n + N) \times N$ matrix satisfying

$$\Gamma B = 0 \text{ and } \text{rank } B = N \quad (4.2.12)$$

and

$$\sigma = \{z_i\}_{i=1}^n \cup \{w_j\}_{j=1}^{n+N}.$$

Define an $N \times N$ rational matrix function

$$\varphi^{-1}(z) = C_-(zI - A_\pi)^{-1}B. \quad (4.2.13)$$

The following theorem gives a sufficient condition for the existence of a solution of McMillan degree n . In this context, we see that the genericity condition analogous to $\bar{d}(z)$ having no zeros in σ in Theorem 4.1.2 is that φ^{-1} be regular with σ -null-pole triple equal to $\tilde{\tau}_- := (C_-, A_\pi; 0, 0; 0)$.

Theorem 4.2.2. *Let matrices Γ , C_+ , C_- , A_π be given by (4.2.3) and (4.2.8) and an $N \times N$ rational matrix function $\varphi^{-1}(z)$ be given by (4.2.13). If $\text{rank } \Gamma = n$ and $\varphi^{-1}(z)$ is regular and has*

$$\tilde{\tau}_- := (C_-, A_\pi; 0, 0; 0)$$

as its σ -null-pole triple, then there exists a solution of McMillan degree n . If we choose $\alpha \in \mathbb{C}$ so that $\varphi^{-1}(\alpha)$ is invertible, then a solution of McMillan degree n is given by

$$W_0(z) = C_+(z\tilde{G} - \tilde{A})^{-1}BD^{-1}, \quad (4.2.14)$$

where the matrix B is given by (4.2.12) and

$$\begin{aligned} D &= \varphi^{-1}(\alpha) = C_-(\alpha I - A_\pi)^{-1}B \\ \tilde{G} &= I - BD^{-1}C_-(\alpha I - A_\pi)^{-1} \\ \tilde{A} &= A_\pi - \alpha BD^{-1}C_-(\alpha I - A_\pi)^{-1}. \end{aligned}$$

Proof. Let matrices A_ζ , B_+ , B_- , C_+ , C_- , A_π be given by (4.2.7) (4.2.8) and

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$$

be an $(M + N) \times (M + N)$ rational matrix function which has

$$\tilde{\tau} := \left(\begin{bmatrix} C_+ \\ C_- \end{bmatrix}, A_\pi; A_\zeta, [B_+ B_-]; \Gamma \right)$$

as its σ -null-pole triple. Suppose $\varphi^{-1}(z)$ has $\tilde{\tau}_-$ as its σ -null-pole triple. Then by Theorem 1.4.1 and Remark 1.4.3, $W \in \mathcal{R}_{M \times N}(\sigma)$ satisfies the interpolation conditions (4.2.9)–(4.2.11) if and only if

$$W(z) = (\Theta_{11}P + \Theta_{12}Q)(\Theta_{21}P + \Theta_{22}Q)^{-1} \quad (4.2.15)$$

where $P \in \mathcal{R}_{M \times N}(\sigma)$, $Q \in \mathcal{R}_{N \times N}(\sigma)$ are rational matrix functions for which

$$\varphi(\Theta_{21}P + \Theta_{22}Q) = I. \quad (4.2.16)$$

Upon combining (4.2.15) and (4.2.16), we parametrize the solutions as follows: $W \in \mathcal{R}_{M \times N}(\sigma)$ is a solution if and only if

$$\begin{bmatrix} W \\ I \end{bmatrix} \varphi^{-1} = \Theta \begin{bmatrix} P \\ Q \end{bmatrix}. \quad (4.2.17)$$

On the other hand, by noting that Θ has $\tilde{\tau}$ as its σ -null-pole triple, we see that there exist an $(n + N) \times N$ matrix \tilde{B} and rational functions $H \in \mathcal{R}_{M \times N}(\sigma)$, $K \in \mathcal{R}_{N \times N}(\sigma)$ for which

$$\Theta \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} C_+ \\ C_- \end{bmatrix} (zI - A_\pi)^{-1} \tilde{B} + \begin{bmatrix} H(z) \\ K(z) \end{bmatrix}, \quad (4.2.18)$$

where $H(z)$ and $K(z)$ satisfy the equation

$$\sum_{z_0 \in \mathbf{C}} \text{Res}_{z=z_0} (zI - A_\zeta)^{-1} [B_+ B_-] \begin{bmatrix} H(z) \\ K(z) \end{bmatrix} = \Gamma \tilde{B}. \quad (4.2.19)$$

From (4.2.17) and (4.2.18), we get

$$\varphi^{-1}(z) = C_-(zI - A_\pi)^{-1} \tilde{B} + K(z). \quad (4.2.20)$$

Since $\varphi^{-1}(z)$ and $C_-(zI - A_\pi)^{-1} \tilde{B}$ have poles only inside σ and $K(z)$ has poles only outside σ (4.2.20) forces $K(z) = 0$ and

$$\varphi^{-1}(z) - C_-(zI - A_\pi)^{-1} \tilde{B} = C_-(zI - A_\pi)^{-1} (B - \tilde{B}) = 0.$$

By the fact that the pair (C_-, A_π) is a null-kernel pair, we get $B = \tilde{B}$ from the above identity. Thus, (4.2.18) and (4.2.19) are reduced to

$$\Theta \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} C_+ \\ C_- \end{bmatrix} (zI - A_\pi)^{-1} B + \begin{bmatrix} H(z) \\ 0 \end{bmatrix}, \quad (4.2.21)$$

where

$$\sum_{z_0 \in \mathbb{C}} \text{Res}_{z=z_0} (zI - A_\zeta)^{-1} B_+ H(z) = 0. \quad (4.2.22)$$

By combining (4.2.17) (4.2.21) and (4.2.22), all solutions are reparametrized as follows: $W \in \mathcal{R}_{M \times N}(\sigma)$ is a solution if and only if

$$\begin{bmatrix} W \\ I \end{bmatrix} \varphi^{-1} = \begin{bmatrix} C_+ \\ C_- \end{bmatrix} (zI - A_\pi)^{-1} B + \begin{bmatrix} H(z) \\ 0 \end{bmatrix},$$

where $H \in \mathcal{R}_{M \times N}(\sigma)$ is subject to the constraint (4.2.22). Let $W_0(z)$ be the solution corresponding to the choice of $H(z) = 0$. Then,

$$W_0(z) = C_+ (zI - A_\pi)^{-1} B \varphi(z). \quad (4.2.23)$$

Now, we find the McMillan degree of $W_0(z)$. We note that all poles of $W_0(z)$ come from those of $\varphi(z)$ since, by our construction, $W_0(z)$ has no poles inside σ which contains $\sigma(A_\pi)$ and all poles of $\varphi(z)$ are outside σ . But, there may be some cancellation of poles of $\varphi(z)$ by premultiplying $R(z) := C_+ (zI - A_\pi)^{-1} B$. Indeed, at infinity, at least N poles of $\varphi(z)$ are cancelled by premultiplying $R(z)$. To see this, we apply a Möbius transformation to $R(z)$ and $\varphi^{-1}(z)$ and get

$$R \left(\frac{1}{z} \right) = z C_+ (I - z A_\pi)^{-1} B$$

and

$$\varphi^{-1} \left(\frac{1}{z} \right) = z C_- (I - z A_\pi)^{-1} B.$$

Thus, if we premultiply $\varphi \left(\frac{1}{z} \right)$ by $R \left(\frac{1}{z} \right)$, the factor $\frac{1}{z} I_N$ in $\varphi \left(\frac{1}{z} \right)$ is cancelled out, that is, N poles of $\varphi(z)$ at infinity are cancelled out. Hence,

$$\delta(W_0) \leq \delta(\varphi) - N = n.$$

On applying Theorem 4.2.1, we get $\delta(W_0) = n$.

Lastly, we express $W_0(z)$ in a realization form. By the assumption that $\varphi^{-1}(z)$ is regular, there exists $\alpha \in \mathbb{C}$ for which

$$D := \varphi^{-1}(\alpha)$$

is invertible. Then, $\varphi^{-1}(z)$ can be expressed in a realization form centered at α as follows:

$$\varphi^{-1}(z) = D - (z - \alpha)C_{-}(zI - A_{\pi})^{-1}(\alpha I - A_{\pi})^{-1}B,$$

and $\varphi(z)$ is given by

$$\varphi(z) = D^{-1} + (z - \alpha)D^{-1}C_{-}(zG^{\times} - A^{\times})^{-1}(\alpha I - A_{\pi})^{-1}BD^{-1}, \quad (4.2.24)$$

where

$$G^{\times} = I - (\alpha I - A_{\pi})^{-1}BD^{-1}C_{-} \quad (4.2.25)$$

$$A^{\times} = A_{\pi} - \alpha(\alpha I - A_{\pi})^{-1}BD^{-1}C_{-}. \quad (4.2.26)$$

Substitute (4.2.24) in (4.2.23) to get

$$\begin{aligned} W_0(z) &= C_{+}(zI - A_{\pi})^{-1}BD^{-1} + (z - \alpha)C_{+}(zI - A_{\pi})^{-1}B \\ &\quad \times D^{-1}C_{-}(zG^{\times} - A^{\times})^{-1}(\alpha I - A_{\pi})^{-1}BD^{-1} \\ &= C_{+}(zI - A_{\pi})^{-1}\{(\alpha I - A_{\pi})^{-1}(zG^{\times} - A^{\times}) - (z - \alpha)BD^{-1}C_{-}\} \\ &\quad \times (zG^{\times} - A^{\times})^{-1}(\alpha I - A_{\pi})^{-1}BD^{-1}. \end{aligned} \quad (4.2.27)$$

Upon plugging (4.2.25) and (4.2.26) into $(\alpha I - A_{\pi})(zG^{\times} - A^{\times}) - (z - \alpha)BD^{-1}C_{-}$, it is reduced to $(zI - A_{\pi})^{-1}(\alpha I - A_{\pi})$. Hence, (4.2.27) is

$$\begin{aligned} W_0(z) &= C_{+}(\alpha I - A_{\pi})^{-1}(zG^{\times} - A^{\times})^{-1}(\alpha I - A_{\pi})^{-1}BD^{-1} \\ &= C_{+}\{(\alpha I - A_{\pi})^{-1}(zG^{\times} - A^{\times})(\alpha I - A_{\pi})^{-1}\}^{-1}BD^{-1}. \end{aligned}$$

Again we substitute (4.2.25) and (4.2.26) in the above identity to get

$$W_0(z) = C_+(z\tilde{G} - \tilde{A})^{-1}BD^{-1},$$

where

$$\begin{aligned}\tilde{G} &= I - BD^{-1}C_-(\alpha I - A_\pi)^{-1} \\ \tilde{A} &= A_\pi - \alpha BD^{-1}C_-(\alpha I - A_\pi)^{-1}.\end{aligned}$$

This completes the proof. \square

We close this section with some remarks.

Remark 4.2.3. (a) In Theorem 4.2.2, the null-pole structure of $\varphi^{-1}(z)$ is independent of the choice of the matrix B . If \tilde{B} is a $(n + N) \times N$ matrix satisfying

$$L\tilde{B} = 0 \quad \text{and} \quad \text{rank } \tilde{B} = N,$$

then there exist an $N \times N$ invertible matrix V so that $\tilde{B} = BV$. Thus, $\tilde{\tau}_- = (C_-, A_\pi; 0, 0; 0)$ is a σ -null-pole triple for $\varphi^{-1}(z) = C_-(zI - A_\pi)^{-1}B$ if and only if $\tilde{\tau}_-$ is a σ -null-pole triple for $C_-(zI - A_\pi)^{-1}\tilde{B}$.

Unlike the scalar case, the condition that the matrix Γ has full rank is needed so that the null pole structure of φ^{-1} is independent of the choice of the matrix B . Remember that in the scalar case, if the $m \times (m + 1)$ Löwner matrix L given by (4.4) is not of full rank then we reproduce a Löwner matrix \bar{L} (see 4.1.5) so that the dimension of $\text{Ker } \bar{L}$ is 1 by rearranging the given data. By doing so, the null-pole structure of $\bar{d}(z)$ given by (4.1.10) is independent of the choice of a vector c satisfying $\bar{L}c = 0$. But, in the problem (4.2.1) and (4.2.2), we cannot rearrange the data so that the size of the new Γ is changed. This point and Theorem 4.2.1 where we have inequality instead of equality of scalar case are the main difficulties which we have in extending the results of the scalar case to the matrix interpolation problem (4.2.1) and (4.2.2).

(b) Even for the very special case of the matrix interpolation problem which we consider in this section, it seems hard to go beyond Theorem 4.2.2. In [AA2], the minimal possible McMillan degree for the solutions of the full-matrix valued interpolation problem (e.g. Example 1 in section 1.4) is found by extending the approach of section 4.1 to the matrix case under some additional restrictive hypothesis on the structure of the Löwner matrix. For the pure multiple-point case at infinity (i.e. the partial realization problem) where the Löwner matrix collapses to a Hankel matrix, a complete solution is obtained in [GKL], but extending these ideas to the general case appears to be difficult. It seems the approach using the Löwner matrix as a key notion is not fruitful for the matrix case. So, in the next section, we go back to the scalar case and understand the results from another point of view so that a generalization to the matrix case is possible.

4.3 The scalar case revisited

The aim of this section is to understand Theorem 4.1.3 from another point of view. First, we introduce a 2×2 matrix polynomial

$$\tilde{\Theta}(z) := \begin{bmatrix} \bar{n}(z) & \hat{n}(z) \\ \bar{d}(z) & \hat{d}(z) \end{bmatrix}, \quad (4.3.1)$$

where $\bar{n}(z)$, $\bar{d}(z)$, $\hat{n}(z)$, $\hat{d}(z)$ are polynomials given by (4.1.10) (4.1.11) (4.1.18) (4.1.19) respectively. Remember that those polynomials are constructed so that

$$(a) w_i \bar{d}(z_i) = \bar{n}(z_i), \quad (b) w_i \hat{d}(z_i) = \hat{n}(z_i), \quad i = 1, \dots, n \quad (4.3.2)$$

and

$$(a) \text{ the index of } \begin{bmatrix} \bar{n}(z) \\ \bar{d}(z) \end{bmatrix} \leq q, \quad (b) \text{ the index of } \begin{bmatrix} \hat{n}(z) \\ \hat{d}(z) \end{bmatrix} \leq n - q, \quad (4.3.3)$$

where $\{z_i\}_{i=1}^n$, $\{w_j\}_{j=1}^n$ are the same as in (4.1.1) and q represents the rank of the Löwner matrix (4.4). Here we note that whether $\frac{\bar{n}(z)}{\bar{d}(z)}$ is a solution to the interpolation

problem (4.1.1) or not, we can find $\hat{n}(z)$ and $\hat{d}(z)$ satisfying the conditions (4.3.2b) (4.3.3b) and

$$\hat{n}(z)\bar{d}(z) - \bar{n}(z)\hat{d}(z) \neq 0. \quad (4.3.4)$$

Let

$$A_\zeta := \begin{bmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_n \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -w_1 \\ \vdots & \vdots \\ 1 & -w_n \end{bmatrix}. \quad (4.3.5)$$

Then, $\tilde{\Theta}(z)$ satisfies the following;

$$\tilde{\Theta}(z) \text{ has } (A_\zeta, B) \text{ as its } \sigma\text{-null-pair} \quad (4.3.6)$$

$$\tilde{\Theta}(z) \text{ is column reduced,} \quad (4.3.7)$$

where $\sigma = \{z_i\}_{i=1}^n$. Indeed, it is straightforward from (4.3.2) that (A_ζ, B) is at least a corestriction of a left null pair for $\tilde{\Theta}(z)$. To prove (4.3.7), it is enough to show that

$$\deg \det \tilde{\Theta}(z) = \text{sum of the column indices.} \quad (4.3.8)$$

This will then also imply that (A_ζ, B) is in fact precisely a left null pair for $\tilde{\Theta}(z)$. From (4.3.3), $\deg \det \tilde{\Theta}(z) \leq n$, but, the conditions (4.3.4) (4.3.6) imply that the nonzero polynomial $\det \tilde{\Theta}(z)$ has at least n zeros. Hence,

$$\deg \det \tilde{\Theta}(z) = n \quad (4.3.9)$$

and, in turn,

$$\text{the column index of } \begin{bmatrix} \bar{n}(z) \\ \bar{d}(z) \end{bmatrix} = q \quad (4.3.10)$$

and

$$\text{the column index of } \begin{bmatrix} \hat{n}(z) \\ \hat{d}(z) \end{bmatrix} = n - q. \quad (4.3.11)$$

From Remark (2.3.5), the column indices of a column reduced matrix polynomial are the controllability indices of its left null pair. Hence, we have

$$q = \alpha_1, \quad (4.3.12)$$

where α_1 is the smaller controllability index of the pair (A_ζ, B) given by (4.3.5). The next theorem restates Theorem 4.1.4 without using the Löwner matrix.

Theorem 4.3.1. *Let $n \times n$ matrix A_ζ , $n \times 2$ matrix B be given by (4.3.5) and*

$$\Theta := \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$$

be any 2×2 matrix polynomial satisfying (4.3.6) (4.3.7) with the column indices in nondecreasing order. Suppose $\alpha_1 \leq \alpha_2$ are the controllability indices of the pair (A_ζ, B) .

- (a) *If $\Theta_{21}(z_i) \neq 0$ for $i = 1, \dots, n$, then the minimal possible McMillan degree is α_1 and $f(z) := \frac{\Theta_{11}(z)}{\Theta_{21}(z)}$ is a unique solution of McMillan degree α_1 .*
- (b) *Otherwise, α_2 is the minimal possible McMillan degree of solutions to (4.1.1) and there is more than one solution of McMillan degree α_2 .*

Proof. Let $\Theta(z)$ be a 2×2 matrix polynomial satisfying (4.3.6) and (4.3.7) with Θ in place of $\tilde{\Theta}$ and let the column indices of $\Theta(z)$ be in nondecreasing order. Then, by Remark 2.3.5, the column indices of Θ are $\alpha_1 \leq \alpha_2$. Since $\tilde{\Theta}(z)$ given by (4.3.1) satisfies (4.3.6) and (4.3.7), there exists a 2×2 unimodular matrix $V(z) := \begin{bmatrix} v_{11}(z) & v_{12}(z) \\ v_{21}(z) & v_{22}(z) \end{bmatrix}$

$$\Theta(z) = \tilde{\Theta}(z)V(z). \quad (4.3.13)$$

By the predictable degree property of a column reduced matrix polynomial (see Theorem 1.3.1), and by (4.3.10) and (4.3.11),

$$\text{the column index of } \begin{bmatrix} \Theta_{11} \\ \Theta_{21} \end{bmatrix} \geq q + \deg v_{11}, \text{ unless } v_{11} \neq 0 \quad (4.3.14)$$

$$\text{the column index of } \begin{bmatrix} \Theta_{11} \\ \Theta_{21} \end{bmatrix} \geq (n - q) + \deg v_{21}, \text{ unless } v_{21} \neq 0. \quad (4.3.15)$$

Because the column index of $\begin{bmatrix} \Theta_{11} \\ \Theta_{21} \end{bmatrix} = \alpha_1 = q$, and $n - q > q$ in this case, (4.3.15) forces $v_{21} = 0$ and (4.3.14) forces v_{11} is a nonzero constant c . By plugging $v_{11} =$

$c, v_{21} = 0$ in (4.3.13), we get

$$\bar{n}(z) = \frac{1}{c}\Theta_{11}(z), \quad c \neq 0 \quad (4.3.16)$$

$$\bar{d}(z) = \frac{1}{c}\Theta_{21}(z), \quad c \neq 0. \quad (4.3.17)$$

Upon substituting (4.3.16) and (4.3.17) in Theorem 4.1.1, the proof is completed. \square

Remark 4.3.2. For the multiple point case or for the distinct point case with even n , the same result as (4.3.12) can be obtained. Of course, for the multiple point case, the matrix L is a generalized Löwner matrix (see [AA1]) and the pair (A_ζ, B) is in a different form. \square

In the next two sections, the approach developed in Theorem 4.3.1 is generalized to the one-sided and two-sided residue interpolation problem (1.4.1)–(1.4.3).

4.4 One-sided residue interpolation problem

Consider the following *residue interpolation problem* (RIP).

(RIP) Given matrices A_ζ, B_+, B_- of sizes $n_\zeta \times n_\zeta, n_\zeta \times M, n_\zeta \times N$ respectively.

Find an $M \times N$ rational matrix function $W \in \mathcal{R}_{M \times N}(\sigma)$ for which

$$\sum_{z_0 \in \sigma} \text{Res}_{z=z_0} (zI - A_\zeta)^{-1} B_+ W(z) = -B_-,$$

where σ is a subset of \mathbb{C} .

Without loss of generality, we can assume that (A_ζ, B_+) is a full range pair and $\sigma(A_\zeta) \subset \sigma$ (see section 1.4). In this section, we start with Theorem 1.4.1 (the parametrization of all solutions) and determine the following: (a) the minimal possible McMillan degree for solutions of (RIP) (b) a solution of minimal possible McMillan degree (c) admissible degrees for solutions of (RIP), i.e. nonnegative integers d for which there exists a solution of McMillan degree d . The dual problem of (RIP_d) is stated as follows.

(RIP_d) Given matrices C_+, C_-, A_π of sizes $M \times n_\pi, N \times n_\pi, n_\pi \times n_\pi$ respectively, find an $M \times N$ rational matrix function $W \in \mathcal{R}_{M \times N}(\sigma)$ for which

$$\sum_{z_0 \in \mathbb{C}} \text{Res}_{z=z_0} W(z) C_- (zI - A_\pi)^{-1} = C_+,$$

where σ is a subset of \mathbb{C} .

As in (RIP), we can assume that $\sigma(A_\pi) \subset \sigma$ and (C_-, A_π) is a null-kernel pair (see section 1.4). We study (RIP) in this section. The parallel results for (RIP_d) can be found in the next section.

To get an idea about how to determine the McMillan degree of an interpolant, we present the following theorem. The proof will be given in Theorem 4.5.1 which is about the parallel results for more general case.

Theorem 4.4.1. Let $N(z)$ and $D(z)$ be matrix polynomials of sizes $M \times N, N \times N$ and $D(z)$ be regular. If

$$N(z), D(z) \text{ are coprime} \quad (4.4.1)$$

and

$$\begin{bmatrix} N(z) \\ D(z) \end{bmatrix} \text{ is column reduced,} \quad (4.4.2)$$

then

$$\delta(ND^{-1}) = \sum_{i=1}^n \gamma_i,$$

where γ_i is the i^{th} column index of the matrix polynomial $\begin{bmatrix} N(z) \\ D(z) \end{bmatrix}$.

Here we note that matrix polynomials $N(z), D(z)$ are (right) coprime if and only if $\begin{bmatrix} N(z) \\ D(z) \end{bmatrix}$ has full column rank for any $z \in \mathbb{C}$. The following theorem shows that all solutions of (RIP) can be parametrized in the form $N(z)D(z)^{-1}$ for matrix polynomials $N(z)$ and $D(z)$ satisfying the conditions (4.4.1) (4.4.2) and one more constraint (4.4.6). The proof is given in theorem 4.5.1.

Theorem 4.4.2. Let

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$$

be an $(M + N) \times (M + N)$ matrix polynomial for which

$$(A_\zeta, [B_+ \ B_-]) \text{ is a } \mathbb{C}\text{-null-pole for } \Theta(z) \quad (4.4.3)$$

$$\Theta(z) \text{ is column reduced at infinity.} \quad (4.4.4)$$

Then, $W \in \mathcal{R}_{M \times N}(\sigma(A_\zeta))$ is a solution of (RIP) if and only if there exist matrix polynomials $P(z)$ and $Q(z)$ of sizes $M \times N, N \times N$ for which

$$W = (\Theta_{11}P + \Theta_{12}Q)(\Theta_{21}P + \Theta_{22}Q)^{-1} \quad (4.4.5)$$

$$\Theta_{21}P + \Theta_{22}Q \text{ has no zeros on } \sigma(A_\zeta) \quad (4.4.6)$$

$$P, Q \text{ are right coprime} \quad (4.4.7)$$

$$\Theta \begin{bmatrix} P \\ Q \end{bmatrix} \text{ is column reduced at infinity.} \quad (4.4.8)$$

It is worthwhile to note that in the above theorem the conditions (4.4.6) and (4.4.7) guarantee that $\Theta_{11}P + \Theta_{12}Q, \Theta_{21}P + \Theta_{22}Q$ are right coprime. Now, by Theorem 4.4.1 and Theorem 4.4.2, the problem of finding a solution of the minimal possible McMillan degree is reduced to the problem of finding matrix polynomials $P(z), Q(z)$ for which (4.4.6)–(4.4.8) are satisfied and the sum of the column indices of $\Theta \begin{bmatrix} P \\ Q \end{bmatrix}$ is as small as possible. The next lemma shows that there exist such constant parameters P and Q .

Lemma 4.4.3. Let $\Theta(z)$ be as in Theorem 4.4.2 with the column indices in nondecreasing order. Then there exists a full column rank $(M + N) \times N$ upper echelon constant matrix Δ with leading 1's occurring in rows $i_1 < i_2 < \dots < i_N$ such that

$$\tilde{\Theta}_I(z) := \Theta(z)\Delta$$

satisfies the properties

- (a) $[0 \ I_N]\tilde{\Theta}_I(z)$ has no zeros in $\sigma(A_\zeta)$.
- (b) among all choices of such Δ for which (a) is satisfied

$$\sum_{j=1}^N \nu_{ij} \text{ is minimal,}$$

where ν_j denotes the j^{th} column index of $\Theta(z)$.

Proof. We will construct N columns $\tilde{\theta}_{i_1}(z), \dots, \tilde{\theta}_{i_N}(z)$ of $\tilde{\Theta}_I(z)$ recursively so that (a) and (b) are satisfied. Suppose $\sigma(A_\zeta)$ consists of m points $\{z_1, \dots, z_m\}$. In this proof, by \underline{k} for integer k , we mean the set $\{1, \dots, k\}$. Let

$$[\Theta(z_k)]_l$$

denote the first l columns of $\Theta(z)$ evaluated at $z = z_k$, let

$$i_1 := \min\{l \mid \text{rank}([0 \ I][\Theta(z_k)]_l) \geq 1, k \in \underline{m}\} \quad (4.4.9)$$

and let f_{oj} denote the j^{th} column of $\Theta(z)$.

(step I) First, we prove that there exists a set of complex numbers $\{\beta_j\}_{j=1}^{i_1-1}$ such that

$$\tilde{\theta}_{i_1}(z) := \left\{ f_{oi_1}(z) + \sum_{j=1}^{i_1-1} \beta_j f_{oj}(z_k) \right\} \quad (4.4.10)$$

satisfies

$$[0 \ I]\tilde{\theta}_{i_1}(z_k) \neq 0 \quad \text{for } k \in \underline{m}. \quad (4.4.11)$$

Consider the following linear transformation

$$h_{ok} : \mathbb{C}^{i_1} \rightarrow \mathbb{C}^N$$

which is defined by

$$h_{ok}([c_1, \dots, c_{i_1}]^T) = [0 \ I] \left(\sum_{j=1}^{i_1} c_j f_{oj}(z_k) \right), \quad k \in \underline{m}.$$

By (4.4.10) and (4.4.11), $h_{ok} \neq 0$, i.e.,

$$\text{Ker } h_{ok} \text{ is a proper subspace of } \mathbb{C}^{i_1}, \quad k \in \underline{m}. \quad (4.4.12)$$

Let

$$S_0 = \bigcup_{k=1}^m \text{Ker } h_{ok}.$$

Then, by (4.4.12)

$$S_0^c \text{ is dense in } \mathbb{C}^{i_1},$$

where S_0^c denotes the complement of S_0 in \mathbb{C}^{i_1} . Hence, there exists

$$[c_1^0, \dots, c_{i_1}^0]^T \in S_0^c$$

satisfying $c_{i_1}^0 \neq 0$. Let

$$\tilde{\theta}_{i_1}(z) := \frac{1}{c_{i_1}^0} \left(\sum_{j=1}^{i_1} c_j^0 f_{oj}(z) \right).$$

Putting $\beta_j = \frac{c_j^0}{c_{i_1}^0}$, $j = 1, \dots, i_1 - 1$, we have

$$\tilde{\theta}_{i_1}(z) = f_{oi_1}(z) + \sum_{j=1}^{i_1-1} \beta_j f_{oj}(z)$$

and

$$[0 \ I] \tilde{\theta}_{i_1}(z_k) \neq 0, \quad k \in \underline{m}$$

by the choice of β_j 's.

(step II) Assume a set of integers $\{i_1, \dots, i_n\}$ and a set of $(M + N)$ -dimensional rational vector functions $\{\tilde{\theta}_{i_1}(z), \dots, \tilde{\theta}_{i_n}(z)\}$ are chosen for $n < N$ so that

$$i_1 < i_2 < \dots < i_n \quad (4.4.13)$$

$$\tilde{\theta}_{i_n}(z) := f_{n-1, i_n}(z) + \sum_{j=1}^{i_n-1} \gamma_j f_{n-1, j}(z) \text{ for some constants } \gamma_1, \dots, \gamma_{i_n-1} \quad (4.4.14)$$

$$\text{rank} \left([0 \ I] [\tilde{\theta}_{i_1}(z_k) \dots \tilde{\theta}_{i_n}(z_k)] \right) = n, \quad k \in \underline{m} \quad (4.4.15)$$

where $f_{n-1,j}(z)$ is the j^{th} column of

$$\Theta_{n-1}(z) := \Theta(z)V_0 \dots V_{n-1}, \quad (V_0 = I),$$

where V_j is a $(M + N) \times (M + N)$ constant upper triangular matrix with main diagonal entries equal to 1.

(step III) Define

$$\Theta_n(z) := \Theta_{n-1}(z)V_n,$$

where V_n is a $(M + N) \times (M + N)$ constant upper triangular matrix satisfying

$$\text{the } j^{\text{th}} \text{ column of } (I_{M+N} - V_n) = \begin{cases} [-\gamma_1 \dots -\gamma_{i_{n-1}} 0 \dots 0]^T & , j = i_n \\ 0 & , j \neq i_n \end{cases} \quad (4.4.16)$$

with γ_j 's as in (4.4.14). Let $f_{nj}(z)$ denote the j^{th} column of $\Theta_n(z)$ and define

$$i_{n+1} := \min\{l | \text{rank}([0 \ I][\Theta_n(z_k)]_l) \geq n+1, k \in \underline{m}\}. \quad (4.4.17)$$

Here, it is worthwhile to note that there exists i_{n+1} satisfying (4.4.17) for $n < m$ by Lemma 1.4.3 (take $\varphi^{-1} = I_N$ for this case). Now, we prove there exists a set of complex numbers $\{\epsilon_j\}_{j=1}^{i_{n+1}-1}$ for which

$$\text{rank}[0 \ I][\tilde{\theta}_{i_1}(z_k) \dots \tilde{\theta}_{i_{n+1}}(z_k)] = n+1, k \in \underline{m},$$

where

$$\tilde{\theta}_{i_{n+1}}(z) := f_{n,i_{n+1}}(z) + \sum_{j=1}^{i_{n+1}-1} \epsilon_j f_{nj}(z).$$

Let

$$\mathfrak{N}_k := \bigvee_{j=1}^{i_{n+1}} \{[0 \ I]f_{nj}(z_k)\}, \quad k \in \underline{m}$$

and

$$\mathfrak{M}_k := \bigvee_{j=1}^n \{[0 \ I]\theta_{i_j}(z_k)\}, \quad k \in \underline{m},$$

where $\bigvee_{j=1}^n \{v_j\}$ denotes the linear span of vectors v_1, v_2, \dots, v_n . Then, by (4.4.15) and (4.4.17),

$$\dim \left(\frac{\mathfrak{N}_k}{\mathfrak{M}_k} \right) \geq 1, \quad k \in \underline{m}. \quad (4.4.18)$$

For each $k \in \underline{m}$, consider the following maps:

$$L_k : \mathbb{C}^{i_{n+1}} \rightarrow \mathfrak{N}_k, \quad L([c_1 \dots c_{i_{n+1}}]^T) := [0 \ I] \left(\sum_{j=1}^{i_{n+1}} c_j f_{n_j}(z_k) \right)$$

$$Q_k := \mathfrak{N}_k \rightarrow \frac{\mathfrak{N}_k}{\mathfrak{M}_k}, \quad Q_k(u) := u + \mathfrak{M}_k \text{ for } u \in \mathfrak{N}_k$$

and

$$H_k : \frac{\mathfrak{N}_k}{\mathfrak{M}_k} \rightarrow \mathfrak{M}_k^\perp, \quad H_k(u + \mathfrak{M}_k) := u|_{\mathfrak{M}_k^\perp},$$

where \mathfrak{M}_k^\perp denotes the subspace of \mathfrak{N}_k which is orthogonal to \mathfrak{M}_k and $u|_{\mathfrak{M}_k^\perp}$ represent the projection of u on \mathfrak{M}_k^\perp along \mathfrak{M}_k . Then,

$$F_k := H_k \circ Q_k \circ L_k, \quad k \in \underline{m}$$

is a nonzero linear transformation from $\mathbb{C}^{i_{n+1}}$ to \mathfrak{N}_k since, by (4.4.18)

$$\dim \left(\frac{\mathfrak{N}_k}{\text{Ker } F_k} \right) \geq 1, \quad k = \underline{m}.$$

Thus,

$$S_n^c := \left(\bigcup_{k \in \underline{m}} \text{Ker } F_k \right)^c$$

is dense in $\mathbb{C}^{i_{n+1}}$ and therefore there exists

$$[c_1^n \dots c_{i_{n+1}}^n]^T \in S_n^c$$

satisfying $c_{i_{n+1}}^n \neq 0$. By the construction of F_k ,

$$F_k([c_1^n \dots c_{i_{n+1}}^n]^T) \neq 0, \quad k \in \underline{m},$$

equivalently,

$$[0 \ I] \left(\sum_{j=1}^{i_{n+1}} c_j^n f_{n_j}(z_k) \right) \notin \mathfrak{M}_k, \quad k \in \underline{m}.$$

Putting

$$\tilde{\theta}_{i_{n+1}}(z) := f_{n,i_{n+1}}(z) + \sum_{j=1}^{i_{n+1}-1} \varepsilon_j f_j(z)$$

with $\varepsilon_j = \frac{c_j^n}{c_{i_{n+1}}^n}$, we get

$$\text{rank} \left([0 \ I] [\tilde{\theta}_{i_1}(z_k) \dots \tilde{\theta}_{i_{n+1}}(z_k)] \right) = n+1, \quad k \in \underline{m}$$

by (4.4.15) and (4.4.19).

Let

$$V := V_0 V_1 \dots V_{N-1}, \quad (4.4.20)$$

$$\Delta := V \cdot [e_{i_1} e_{i_2} \dots e_{i_N}], \quad (4.4.21)$$

and $\tilde{\Theta}_I(z) := \Theta(z)\Delta$, where $\{e_j\}_{j=1}^{M+N}$ denotes the usual standard basis for \mathbb{C}^{M+N} . Then, Δ is a full-rank $(M+N) \times N$ upper echelon constant matrix with leading 1's occurring in rows $i_1 < i_2 < \dots < i_N$ since V is an upper triangular matrix with 1's on its main diagonal by the construction. Thus, $\tilde{\Theta}_I(z)$ is given by

$$\tilde{\Theta}_I(z) = [\tilde{\theta}_{i_1}(z), \dots, \tilde{\theta}_{i_N}(z)] \quad (4.4.22)$$

from (4.4.14) and (4.4.16). Then, by our choice of i_j 's, (a), (b) are fulfilled. This completes the proof. \square

Remarks 4.4.4. (a) Let

$$\begin{bmatrix} P_0 \\ Q_0 \end{bmatrix} := \Delta,$$

where Δ is given by (4.4.21). Then it turns out that the parameters P_0, Q_0 satisfy the properties (4.4.7) (4.4.8) besides (a) and (b) in Theorem 4.4.3. Indeed, (4.4.7) is straightforward since Δ has full-rank. To see (4.4.8), we consider an $(M+N) \times (M+N)$ matrix polynomial

$$\tilde{\Theta}(z) := \Theta(z)V,$$

where V is an $(M+N) \times (M+N)$ constant matrix given by (4.4.20). If $\tilde{\Theta}(z)$ is column reduced at infinity, then $\tilde{\Theta}_I(z)$, which is a collection of certain N columns of $\tilde{\Theta}(z)$, is also column reduced at infinity. Because, a column reduced matrix polynomial $\Theta(z)$ postmultiplied by an invertible upper triangular matrix V is again column reduced with the same column indices in the same order. Hence, we conclude that

$$\text{the sum of the column indices of } \tilde{\Theta}_I(z) = \sum_{j=1}^N \nu_{i_j},$$

where ν_j is the j^{th} column index of $\Theta(z)$.

(b) By Corollary 2.3.4, $\nu_j = \omega_{M+N-j+1}$, where $\omega_1 \geq \dots \geq \omega_{M+N}$ are the controllability indices of the pair $(A_\zeta, [B_+ \ B_-])$. Thus, the column indices of $\Theta(z)$, $\nu_1 \leq \dots \leq \nu_{M+N}$, are the controllability indices of the pair $(A_\zeta [B_+ \ B_-])$.

The following is the main theorem of this section.

Theorem 4.4.5. *Let $\Theta(z)$ be as in Lemma 4.4.3. Then there exist constant matrices P_0, Q_0 of sizes $M \times N, N \times N$ for which*

$$W^{\min}(z) := (\Theta_{11}(z)P_0 + \Theta_{12}(z)Q_0)(\Theta_{21}(z)P_0 + \Theta_{22}(z)Q_0)^{-1}$$

is a solution of (RIP) which has the minimal possible McMillan degree. The McMillan degree of $W^{\min}(z)$ is given by

$$\delta(W^{\min}) = \sum_{j=1}^N \nu_{i_j},$$

where $\nu_1 \leq \dots \leq \nu_{M+N}$ are the controllability indices of the pair $(A_\zeta, [B_+ \ B_-])$ and $i_1 < \dots < i_N$ are as in Lemma 4.4.3.

Proof. Let

$$\begin{bmatrix} P_0 \\ Q_0 \end{bmatrix} := \Delta,$$

where the $(M+N) \times N$ matrix Δ is as in Lemma 4.4.3. Then, every claim of Theorem 4.4.5 is clear from Lemma 4.4.3 and Remark 4.4.4 except the minimality of $\delta(W^{\min})$.

Let $W \in \mathcal{R}_{M \times N}$ be a solution of (RIP). Then, there exist matrix polynomials $P(z)$, $Q(z)$ satisfying (4.4.5)–(4.4.8). Let μ_j denote the j^{th} column index of $\Theta \begin{bmatrix} P \\ Q \end{bmatrix}$. Then,

$$\delta(W) = \sum_{j=1}^N \mu_j.$$

Without loss of generality we can assume that $\mu_1 \leq \dots \leq \mu_N$. Hence, to prove the minimality of $\delta(W^{\min})$, it is enough to show that

$$\mu_j \geq \nu_{i_j} \quad \text{for } j = 1, \dots, N. \quad (4.4.25)$$

Suppose there exists a μ_j for which $\mu_j < \nu_{i_j}$. Let

$$l := \min\{j \mid \mu_j < \nu_{i_j}, j = 1, \dots, N\}.$$

By the predictable degree property (Theorem 1.3.1) the above implies that the bottom $M + N - i_l + 1$ elements of the first l columns of $\begin{bmatrix} P \\ Q \end{bmatrix}$ are zero. In turn, this implies that the rank of the first l columns of $[0 \ I] \Theta(z_k) \begin{bmatrix} P(z_k) \\ Q(z_k) \end{bmatrix}$ is less than l for some $z_k \in \sigma(A_\zeta)$ by the choice of i_l in Lemma 4.4.3. This contradicts to (4.4.6). Hence, we conclude that $\mu_j \geq \nu_{i_j}$ for $j = 1, \dots, N$. The proof is completed. \square

Next, a realization formula for the solution $W^{\min}(z)$ will be found by using the realization formula for $\Theta(z)$ which is obtained in Chapter II. Let α be a complex number which is not in $\sigma(A_\zeta) \cup \{0\}$ and E, T_∞, F_∞ be the matrices constructed as in Corollary 2.3.4 with $B = [B_+ \ B_-]$. Then by Corollary 2.3.4,

$$\Theta(z) = E + (z - \alpha)F_\infty(I - zT_\infty)^{-1}(A_\zeta - \alpha I)^{-1}BE \quad (4.4.26)$$

satisfies the properties (4.4.3) (4.4.4). Let

$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} := E \begin{bmatrix} P_0 \\ Q_0 \end{bmatrix}, \quad \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} := F_\infty, \quad \tilde{B} = (A_\zeta - \alpha I)^{-1}B \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}, \quad (4.4.27)$$

where $\begin{bmatrix} P_0 \\ Q_0 \end{bmatrix}$ is the same as in Theorem 4.4.5 and $D_1, F_1 \in \mathbb{C}^{M \times N}$, $D_2, F_2 \in \mathbb{C}^{N \times N}$.

Without loss of generality we can assume that D_2 is invertible.

Corollary 4.4.6. A realization formula for $W^{\min}(z)$ is given by

$$W^{\min}(z) = D_1 D_2^{-1} + (z - \alpha)(D_1 D_2^{-1} F_2 + F_1)(A^\times - z T^\times)^{-1} \tilde{B} D_2^{-1},$$

where the matrices $D_1, D_2, F_1, F_2, \tilde{B}$ are given by (4.4.27) and $A^\times = I + \alpha \tilde{B} D_2^{-1} F_2$, $T^\times = T_\infty + \tilde{B} D_2^{-1} F_2$.

Proof. Remember that $W^{\min}(z)$ is given by

$$W^{\min}(z) = (\Theta_{11} P_0 + \Theta_{12} Q_0)(\Theta_{21} P_0 + \Theta_{22} Q_0)^{-1}. \quad (4.4.28)$$

From (4.2.26) and (4.2.27), we get

$$\Theta_{11} P_0 + \Theta_{12} Q_0 = D_1 + (z - \alpha) F_1 (I - z T_\infty)^{-1} \tilde{B} \quad (4.4.49)$$

and

$$\Theta_{21} P_0 + \Theta_{22} Q_0 = D_2 + (z - \alpha) F_2 (I - z T_\infty)^{-1} \tilde{B}.$$

Then,

$$(\Theta_{21} P_0 + \Theta_{22} Q_0)^{-1} = D_2^{-1} + (z - \alpha) D_2^{-1} F_2 (A^\times - z T^\times)^{-1} \tilde{B} D_2^{-1}, \quad (4.4.30)$$

where

$$A^\times = I + \alpha \tilde{B} D_2^{-1} F_2, \quad T^\times = T_\infty + \tilde{B} D_2^{-1} F_2. \quad (4.4.31)$$

Plug (4.4.29) (4.4.30) in (4.4.28) to have

$$\begin{aligned} W^{\min}(z) &= D_1 D_2^{-1} + (z - \alpha) F_1 (I - z T_\infty)^{-1} \tilde{B} D_2^{-1} \\ &\quad + (z - \alpha) D_1 D_2^{-1} F_2 (A^\times - z T^\times)^{-1} \tilde{B} D_2^{-1} \\ &\quad + (z - \alpha)^2 F_1 (I - z T_\infty)^{-1} \tilde{B} D_2^{-1} F_2 (A^\times - z T^\times)^{-1} \tilde{B} D_2^{-1}. \end{aligned} \quad (4.4.32)$$

Adding the second and the last terms, we obtain

$$(z - \alpha) F_1 (I - z T_\infty)^{-1} \{A^\times - z T^\times + (z - \alpha) \tilde{B} D_2^{-1} F_2\} (A^\times - z T^\times)^{-1} \tilde{B} D_2^{-1}.$$

Plug (4.4.31) in $\{A^\times - zT^\times + (z - \alpha)\tilde{B}D_2^{-1}F_2\}$ to reduce the above to

$$(z - \alpha)F_1(A^\times - zT^\times)^{-1}\tilde{B}D_2^{-1}.$$

Upon substituting the above in (4.4.32), we get

$$W^{\min}(z) = D_1D_2^{-1} + (z - \alpha)(D_1D_2^{-1}F_2 + F_1)(A^\times - zT^\times)^{-1}\tilde{B}D_2^{-1}.$$

and this completes the proof. \square

Next, we specify all the admissible degrees of complexity. Let

$$\nu_* := \nu_{i_1} + \nu_{i_2} + \cdots + \nu_{i_N}.$$

Theorem 4.4.7. *The admissible degrees of complexity are as follows: If $i_j = j$ for $j \in \underline{N}$ and $\nu_N < \nu_{N+1}$, then the admissible degrees are*

$$\nu_*, \nu_{**}, \nu_{**} + 1, \nu_{**} + 2, \dots,$$

where $\nu_{**} = \nu_1 + \cdots + \nu_{N-1} + \nu_{N+1}$. In this case, there exists a unique solution of minimal possible McMillan degree $\nu_* = \nu_1 + \cdots + \nu_N$. If $i_j > j$ for some $j \in \underline{N}$ or $\nu_N = \nu_{N+1}$, then the admissible degrees are

$$\nu_*, \nu_* + 1, \nu_* + 2, \dots.$$

The proof is the same as that of Theorem 4.5.6 upon choosing $\varphi(z) = I$, $n_\pi = 0$.

Hence, it is omitted here.

4.5 Two-sided residue interpolation problem

The problem which we consider in this section is the following two-sided residue interpolation problem (TRIP).

(TRIP) Given matrices A_ζ, B_+, B_- of sizes $n_\zeta \times n_\zeta$, $n_\zeta \times M$, $n_\zeta \times N$ respectively, matrices C_+, C_-, A_π of sizes $M \times n_\pi$, $n_\pi \times N$, $n_\pi \times n_\pi$ and an $n_\zeta \times n_\pi$ matrix Γ . Find an $M \times N$ rational matrix function $W \in \mathcal{R}_{M \times N}(\sigma)$ which satisfies

$$\begin{aligned} \sum_{z_0 \in \sigma} \text{Res}_{z=z_0} (zI - A_\zeta)^{-1} B_+ W(z) &= -B_- \\ \sum_{z_0 \in \sigma} \text{Res}_{z=z_0} W(z) C_-(zI - A_\pi)^{-1} &= C_+ \\ \sum_{z_0 \in \sigma} \text{Res}_{z=z_0} (zI - A_\zeta)^{-1} B_+ W(z) C_-(zI - A_\pi)^{-1} &= \Gamma, \end{aligned}$$

where σ is a subset of \mathbb{C} .

Here we also suppose the normalization conditions (1.4.4)–(1.4.6). The parallel results of the previous section for (TRIP) are developed here. Throughout this section,

$$\tilde{\tau} = \left(\begin{bmatrix} C_+ \\ C_- \end{bmatrix}, A_\pi; A_\zeta, [B_+ \ B_-]; \Gamma \right) \quad (4.5.1)$$

$$\tilde{\tau}_- = (C_-, A_\pi; 0, 0; 0) \quad (4.5.2)$$

and $\Theta(z)$ represents for the rational matrix function as in Theorem 2.3.1 with $\tau = \tilde{\tau}$ and $n = M + N$. Remember that this $\Theta(z)$ satisfies the following properties:

$$\Theta(z) \text{ has } \tilde{\tau} \text{ as its } \mathbb{C}\text{-null-pole triple} \quad (4.5.3)$$

$$\Theta(z) \text{ is column reduced at infinity} \quad (4.5.4)$$

$$\kappa_j = \begin{cases} -\alpha_j, & 1 \leq j \leq t \\ 0, & t+1 \leq j \leq M+N-s \\ \omega_{M+N-j+1}, & M+N-s+1 \leq j \leq M+N, \end{cases} \quad (4.5.5)$$

where κ_j is the j^{th} column index of $\Theta(z)$ and $\alpha_1 \geq \dots \geq \alpha_t$, $\omega_1 \geq \dots \geq \omega_s$ are the same as in (2.3.4) with $n = M + N$, $B = [B_+ \ B_-]$, $C = \begin{bmatrix} C_+ \\ C_- \end{bmatrix}$.

The next theorem refines Theorem 1.4.1 so that it indicates how to determine the McMillan degree of an interpolant in terms of the corresponding parameters.

Theorem 4.5.1. Let $\Theta(z)$ be as in Theorem 2.3.1 with $\tau = \tilde{\tau}$ given by (4.5.1) and $n = M + N$. Then $W \in \mathcal{R}_{M \times N}(\sigma(A_\pi) \cup \sigma(A_\zeta))$ is a solution of (TRIP) if and only if

there exist polynomial matrices $P(z), Q(z)$ of sizes $M \times N, N \times N$ for which

$$W = (\Theta_{11}P + \Theta_{12}Q)(\Theta_{21}P + \Theta_{22}Q)^{-1} \quad (4.5.6)$$

$$\tilde{\tau}_- \text{ is a } \sigma(A_\pi) \cup \sigma(A_\zeta)\text{-null-pole triple for } (\Theta_{21}P + \Theta_{22}Q) \quad (4.5.7)$$

$$P, Q \text{ are right coprime} \quad (4.5.8)$$

$$\Theta \begin{bmatrix} P \\ Q \end{bmatrix} \text{ is column reduced at infinity,} \quad (4.5.9)$$

where $\tilde{\tau}_-$ is given by (4.5.2). If (4.5.6)–(4.5.9) are satisfied, then

$$\delta(W) = n_\pi + \sum_{j=1}^N \gamma_j, \quad (4.5.10)$$

where γ_j is the j^{th} column index of $\Theta \begin{bmatrix} P \\ Q \end{bmatrix}$.

Proof. Let $\Theta(z)$ be the rational matrix function described in Theorem 2.3.1 with $n = M + N$ and τ of special form $\tilde{\tau}$ of (4.5.1). Also, we suppose $\tilde{\tau}_-$ is given by (4.5.2) and $\sigma := \sigma(A_\pi) \cup \sigma(A_\zeta)$. By Thoerem 1.4.1, $W \in \mathcal{R}_{M \times N}(\sigma)$ is a solution if and only if there exist $\tilde{P} \in \mathcal{R}_{M \times N}(\sigma)$, $\tilde{Q} \in \mathcal{R}_{N \times N}(\sigma)$ for which

$$W = (\Theta_{11}\tilde{P} + \Theta_{12}\tilde{Q})(\Theta_{21}\tilde{P} + \Theta_{22}\tilde{Q})^{-1} \quad (4.5.11)$$

where

$$\Theta_{21}\tilde{P} + \Theta_{22}\tilde{Q} \text{ has } \tilde{\tau}_- \text{ as its } \sigma\text{-null-pole triple.} \quad (4.5.12)$$

Without loss of generality, we can assume \tilde{P}, \tilde{Q} are polynomial matrices.

In [VK], it is proved that for any rational matrix function G and for any $q \in \mathbb{C}^\infty$ there exists a unimodular square matrix polynomial V such that $G(z)V(z)$ is column reduced at $q \in \mathbb{C}^\infty$. To construct matrix polynomials P, Q satisfying additional properties (4.5.8) (4.5.9) from \tilde{P}, \tilde{Q} , we shall go through the following steps.

(step I) Find an unimodular matrix $V_0(z)$ at every $z_0 \in \mathbb{C} \setminus \sigma$ where $\text{rank} \begin{bmatrix} \tilde{P}(z_0) \\ \tilde{Q}(z_0) \end{bmatrix} < N$ so that $\begin{bmatrix} \tilde{P}(z) \\ \tilde{Q}(z) \end{bmatrix} V_0(z)$ is column reduced at z_0 .

(step II) Shift the zeros of $\begin{bmatrix} \tilde{P}(z) \\ \tilde{Q}(z) \end{bmatrix} V_0(z)$ at $z = z_0$ to infinity: since $\begin{bmatrix} \tilde{P}(z) \\ \tilde{Q}(z) \end{bmatrix} V_0(z)$ is column reduced at $z = z_0$,

$$\begin{bmatrix} \tilde{P}(z) \\ \tilde{Q}(z) \end{bmatrix} V_0(z) = L(z)D(z),$$

where $D(z) = \text{diag}((z - z_0)^{\gamma_j})_{j=1}^N$ and L is a matrix polynomial which has no zeros or poles at $z = z_0$. Let

$$\begin{bmatrix} \tilde{P}_0(z) \\ \tilde{Q}_0(z) \end{bmatrix} = \begin{bmatrix} \tilde{P}(z) \\ \tilde{Q}(z) \end{bmatrix} V_0(z) D^{-1}(z).$$

Then, \tilde{P}_0, \tilde{Q}_0 are matrix polynomials having no zeros at $z = z_0$.

(step III) repeat (step I) and (step II) until the resulting matrix has no zeros in $\mathbb{C} \setminus \sigma$.

(step IV) Find an unimodular matrix $\tilde{V}(z)$ so that

$$\Theta(z) \begin{bmatrix} \tilde{P}_l(z) \\ \tilde{Q}_l(z) \end{bmatrix} \tilde{V}(z) \text{ is column reduced at infinity,}$$

where $\begin{bmatrix} \tilde{P}_l(z) \\ \tilde{Q}_l(z) \end{bmatrix}$ is the resulting matrix of (step III). Let

$$\begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} := \begin{bmatrix} \tilde{P}_l(z) \\ \tilde{Q}_l(z) \end{bmatrix} \tilde{V}(z).$$

Then, (4.5.6) and (4.5.9) are straightforward.

Now, we prove (4.5.7). Since $V_0^{\pm 1}, V^{\pm 1}, D^{\pm 1}$ have no zeros or poles in σ , from (4.5.12), we get

$$\tilde{\tau}_- \text{ is } \sigma\text{-null-pole triple for } (\Theta_{21}P + \Theta_{22}Q). \quad (4.5.13)$$

To establish (4.5.8), we shall show that $\begin{bmatrix} P \\ Q \end{bmatrix}$ has no zeros in \mathbb{C} . It is enough to show that $\begin{bmatrix} P \\ Q \end{bmatrix}$ has no zeros in σ by (step III) and (step IV). Suppose $\begin{bmatrix} P \\ Q \end{bmatrix}$ has a zero at $z_0 \in \sigma$. Then, there exists a rational matrix function $U \in \mathcal{R}_{N \times N}(\sigma)$ for which

$$\begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} = \begin{bmatrix} \bar{P}(z) \\ \bar{Q}(z) \end{bmatrix} U(z)$$

with $\overline{P}, \overline{Q}$ matrix polynomials and

$$\det U(z_0) = 0. \quad (4.5.14)$$

Since (4.5.14) implies $U\mathcal{R}_N(\sigma) \subsetneq \mathcal{R}_N(\sigma)$,

$$\begin{bmatrix} P \\ Q \end{bmatrix} \mathcal{R}_N(\sigma) \subsetneq \begin{bmatrix} \bar{P} \\ \bar{Q} \end{bmatrix} \mathcal{R}_N(\sigma) \subset R_{M+N}(\sigma).$$

Let $\varphi^{-1}(z)$ be an $N \times N$ rational matrix function which has $\tilde{\tau}_-$ as its σ -null-pole triple. By Lemma 1.4.3, we see that $\varphi[\Theta_{21} \Theta_{22}]$ is onto and therefore one-to-one. Upon applying $\varphi[\Theta_{21} \Theta_{22}]$ to the above, we get

$$\varphi[\Theta_{21} \Theta_{22}] \begin{bmatrix} P \\ Q \end{bmatrix} \mathcal{R}_N(\sigma) \subsetneq \varphi[\Theta_{21} \Theta_{22}] \mathcal{R}_{M+N}(\sigma). \quad (4.5.15)$$

By applying Lemma 1.4.3 to the right hand side of (4.5.15), we get

$$\varphi[\Theta_{21} \Theta_{22}] \begin{bmatrix} P \\ Q \end{bmatrix} \mathcal{R}_N(\sigma) \subsetneq \mathcal{R}_N(\sigma)$$

which contradicts (4.5.13); namely, $\Theta_{21}P + \Theta_{22}Q$ and φ^{-1} have the same σ -null-pole triple.

To prove (4.5.10), represent $\begin{bmatrix} W \\ I \end{bmatrix}$ as

$$\begin{bmatrix} W \\ I \end{bmatrix} = \Theta \begin{bmatrix} P \\ Q \end{bmatrix} (\Theta_{21}P + \Theta_{22}Q)^{-1}$$

to get

$$\text{def} \left(\begin{bmatrix} W \\ I \end{bmatrix} \right) = \text{def} \left(\Theta \begin{bmatrix} P \\ Q \end{bmatrix} \right), \quad (4.5.16)$$

by Proposition 1.2.1. On the other hand, by the definition of the defect,

$$\text{def} \left(\begin{bmatrix} W \\ I \end{bmatrix} \right) = \delta \left(\begin{bmatrix} W \\ I \end{bmatrix} \right) \quad (4.5.17)$$

since the McMillan degree of a rational matrix function is defined to be the number of poles in \mathbb{C}^∞ and $\begin{bmatrix} W \\ I \end{bmatrix}$ has no zeros in \mathbb{C}^∞ . From (4.5.16) and (4.5.17),

$$\delta(W) = \delta \left(\begin{bmatrix} W \\ I \end{bmatrix} \right) = \text{def} \left(\Theta \begin{bmatrix} P \\ Q \end{bmatrix} \right). \quad (4.5.18)$$

To find the defect of $\Theta \begin{bmatrix} P \\ Q \end{bmatrix}$, remember that (4.5.7) implies

$$\Theta \begin{bmatrix} P \\ Q \end{bmatrix} \text{ has no zeros in } \sigma$$

and

$$\Theta \begin{bmatrix} P \\ Q \end{bmatrix} \text{ has } \left(\begin{bmatrix} C_+ \\ C_- \end{bmatrix}, A_\pi \right) \text{ as its } \sigma\text{-pole pair.}$$

But, by our choice of Θ and $\begin{bmatrix} P \\ Q \end{bmatrix}$, they have no zeros or poles on $\mathbb{C} \setminus \sigma$. Hence, we conclude

$$\Theta \begin{bmatrix} P \\ Q \end{bmatrix} \text{ has no zeros in } \mathbb{C} \quad (4.5.19)$$

$$\Theta \begin{bmatrix} P \\ Q \end{bmatrix} \text{ has } \left(\begin{bmatrix} C_+ \\ C_- \end{bmatrix}, A_\pi \right) \text{ as its } \mathbb{C} - \text{pole pair.} \quad (4.5.20)$$

Moreover, by (4.5.9)

$$\begin{aligned} & \# \text{ of poles of } \Theta \begin{bmatrix} P \\ Q \end{bmatrix} \text{ at infinity} - \# \text{ of zeros of } \Theta \begin{bmatrix} P \\ Q \end{bmatrix} \text{ at infinity} \\ &= \text{sum of the column indices of } \Theta \begin{bmatrix} P \\ Q \end{bmatrix}. \end{aligned} \quad (4.5.21)$$

From (4.5.19)–(4.5.21), we conclude that

$$\text{def} \left(\Theta \begin{bmatrix} P \\ Q \end{bmatrix} \right) = n_\pi + \sum_{i=1}^N \gamma_i,$$

where $\{\gamma_i\}_{i=1}^N$ are the column indices of $\Theta \begin{bmatrix} P \\ Q \end{bmatrix}$. This completes the proof. \square

For the proof of Theorem 4.4.1 and Theorem 4.4.2, we take $\varphi(z) = I$ and $n_\pi = 0$ in the proof of the previous theorem. The following Lemma is the counterpart of Lemma 4.4.3.

Lemma 4.5.2. *Let Θ and $\tilde{\tau}_-$ be as in Theorem 4.5.1. Then there exists a full column rank $(M+N) \times N$ upper echelon constant matrix Δ with leading 1's occurring $i_1 < i_2 < \dots < i_N$ such that*

$$\tilde{\Theta}_I(z) := \Theta(z)\Delta$$

satisfies the properties:

- (a) $\tilde{\tau}_-$ is a $\sigma(A_\pi) \cup \sigma(A_\zeta)$ -null-pole triple for $[0 \ I_N]\tilde{\Theta}_I(z)$.
- (b) among all choices of Δ for which (a) is satisfied,

$$\sum_{j=1}^N \kappa_{i_j} \text{ is minimal,}$$

where κ_j is the j^{th} column index of $\Theta(z)$.

Proof. Let $\varphi^{-1}(z)$ be any regular $N \times N$ rational matrix function for which $\tilde{\tau}_-$ is a \mathbb{C} -null-pole triple for $\varphi^{-1}(z)$. Suppose $\sigma(A_\pi) \cup \sigma(A_\zeta)$ consists of m points z_1, \dots, z_m . Then by Lemma 1.4.3

$$\varphi(z)[0 \ I_N]\Theta(z) \text{ has no zeros or poles at } z_k, \ k = 1, \dots, m. \quad (4.5.22)$$

By going through the same argument of Lemma 4.4.3 with $\varphi(z)[0 \ I_N]\Theta(z)$ in place of $[0 \ I_N]\Theta(z)$, we can construct an $(M + N) \times N$ upper echelon constant matrix Δ with leading 1's occurring $i_1 < \dots < i_N$ such that $\tilde{\Theta}_I(z) := \Theta(z)\Delta$ satisfies

$$\text{rank } \left(\varphi(z_k)[0 \ I_N]\tilde{\Theta}_I(z_k) \right) = N \text{ for } k = 1, \dots, m \quad (4.5.23)$$

and (b) holds. Upon considering (4.5.22), (4.5.23) implies that

$$\varphi(z)[0 \ I_N]\tilde{\Theta}(z) \text{ has no zeros and poles on } \sigma(A_\pi) \cup \sigma(A_\zeta)$$

which is equivalent to (a). So, the theorem is proved. \square

Remark 4.4.4 (a) applies to this section; but the conclusion is changed to

$$\text{the sum of the column indices of } \tilde{\Theta}_I(z) = n_\pi + \sum_{j=1}^N \kappa_{i_j},$$

where κ_j is the j^{th} column index of $\Theta(z)$ given by (4.5.5). The next theorem specifies the minimal possible McMillan degree for the solutions of (TRIP).

Theorem 4.5.3. Let $\Theta(z)$ be as in Lemma 4.5.2. Then there exist constant matrices P_0, Q_0 of sizes $M \times N, N \times N$ for which

$$W^{\min}(z) := (\Theta_{11}(z)P_0 + \Theta_{12}(z)Q_0)(\Theta_{21}(z)P_0 + \Theta_{22}(z)Q_0)^{-1}$$

is a solution of (TRIP) which has the minimal possible McMillan degree. The McMillan degree of $W^{\min}(z)$ is given by

$$\delta(W^{\min}) = n_\pi + \sum_{j=1}^N \kappa_{i_j},$$

where n_π is the size of A_π in $\tilde{\tau}$, κ_j is the j^{th} column index of $\Theta(z)$ and the integers $i_1 < \dots < i_N$ are as in Lemma 4.5.2.

Proof. Let

$$\begin{bmatrix} P_0 \\ Q_0 \end{bmatrix} := \Delta,$$

where the $(M + N) \times N$ matrix Δ is as in Lemma 4.4.3. Then, by the choice of Δ , every assertion is obvious except the minimality of $\delta(W^{\min})$. The minimality of $\delta(W^{\min})$ can be proved as in Theorem 4.4.5 with the following minor changes:

$$\begin{aligned} (\text{RIP}) \rightarrow & (\text{TRIP}), (4.4.5)-(4.4.8) \rightarrow (4.5.6)-(4.5.9), \sum_{j=1}^N \mu_j \rightarrow n_\pi + \sum_{j=1}^N \mu_j, \nu_j \rightarrow \kappa_j, \\ [0 \quad I] \Theta(z_k) \begin{bmatrix} P(z_k) \\ Q(z_k) \end{bmatrix} \rightarrow & \varphi(z_k) [0 \quad I] \Theta(z_k) \begin{bmatrix} P(z_k) \\ Q(z_k) \end{bmatrix}, \sigma(A_\zeta) \rightarrow \sigma(A_\zeta) \cup \sigma(A_\pi), \\ (4.4.6) \rightarrow & (4.5.7). \square \end{aligned}$$

If the matrix Γ is invertible, then the column indices of $\Theta(z)$ are all zero and, in turn, $\sum_{j=1}^N \kappa_{i_j} = 0$. But, in general, by (4.5.5)

$$-\dim \text{Ker } \Gamma \leq \sum_{j=1}^N \kappa_{i_j} \leq \dim K,$$

where K is a complement of $\text{Im } \Gamma$ in \mathbb{C}^{n_ζ} . Upon observing that $\dim \text{Ker } \Gamma = n_\pi - \text{rank } \Gamma$ and $\dim K = n_\zeta - \text{rank } \Gamma$, the following corollary is derived from the previous theorem.

Corollary 4.5.4. Let Γ be the given $n_\zeta \times n_\pi$ matrix in (4.5.1). Then,

$$\text{rank } \Gamma \leq \delta(W^{\min}) \leq n_\pi + n_\zeta - \text{rank } \Gamma.$$

In particular, if Γ is invertible, then

$$\delta(W^{\min}) = \text{rank } \Gamma = n_\zeta = n_\pi.$$

Now, we describe $W^{\min}(z)$ in a realization formula. Let $\Theta(z)$ be as in Theorem 2.3.3 with $\tau = \tilde{\tau}$ given by (4.5.1). Then $\Theta(z)$ is given by

$$\Theta(z) = E + (z - \alpha) \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (zH - A)^{-1} \tilde{B} E, \quad (4.5.24)$$

$$\text{where } \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} := \left[\begin{bmatrix} C_+ \\ C_- \end{bmatrix} \quad \left\{ \begin{bmatrix} C_+ \\ C_- \end{bmatrix} (A_\pi - \alpha I)^{-1} X + \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \right\} (I + \alpha T)^{-1} \right] \quad (4.5.25)$$

with

$$\begin{aligned} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} &:= F, & H &:= \begin{bmatrix} I & 0 \\ 0 & -T(I + \alpha T)^{-1} \end{bmatrix}, \\ A &:= \begin{bmatrix} A_\pi & 0 \\ 0 & -I \end{bmatrix}, & \tilde{B} &:= \begin{bmatrix} (\Gamma^+ + X\rho_\zeta)(A_\zeta - \alpha I)^{-1} B + \eta_\pi G \\ \rho_\zeta(A_\zeta - \alpha I)^{-1} B \end{bmatrix}. \end{aligned}$$

Let

$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} := E \begin{bmatrix} P_0 \\ Q_0 \end{bmatrix}. \quad (4.5.26)$$

From (4.5.24),

$$\Theta(z) \begin{bmatrix} P_0 \\ Q_0 \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} + (z - \alpha) \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (zH - A)^{-1} \tilde{B} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}.$$

Without loss of generality we can assume that D_2 is invertible, and obtain

$$(\Theta_{21}P_0 + \Theta_{22}Q_0)^{-1} = D_2^{-1} - (z - \alpha)D_2^{-1}C_2(zH^\times - A^\times)^{-1}\tilde{B} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} D_2^{-1},$$

where

$$A^\times = \begin{bmatrix} A_\pi & 0 \\ 0 & -I \end{bmatrix} + \alpha \tilde{B} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} D_2^{-1} C_2 \quad (4.5.27)$$

and

$$H^\times = \begin{bmatrix} I & 0 \\ 0 & -T(I + \alpha T)^{-1} \end{bmatrix} + \tilde{B} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} D_2^{-1} C_2. \quad (4.5.28)$$

It now follows that $W^{\min}(z)$ has the representation

$$\begin{aligned} W^{\min}(z) &= (\Theta_{11}P_0 + \Theta_{12}Q_0)(\Theta_{21}P_0 + \Theta_{22}Q_0)^{-1} \\ &= D_1 D_2^{-1} + (z - \alpha)C_1(zH - A)^{-1}\tilde{B} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} D_2^{-1} - (z - \alpha)D_1 D_2^{-1} C_2 \\ &\quad \times (zH^\times - A^\times)^{-1}\tilde{B} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} D_2^{-1} - (z - \alpha)^2 C_1(zH - A)^{-1}\tilde{B} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} D_2^{-1} C_2 \\ &\quad \times (zH^\times - A^\times)^{-1}\tilde{B} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} D_2^{-1}. \end{aligned}$$

The sum of the second and the last terms of the above is given by

$$\begin{aligned} &(z - \alpha)C_1(zH - A)^{-1} \left\{ zH^\times - A^\times - (z - \alpha)\tilde{B} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} D_2^{-1} C_2 \right\} \\ &\quad \times (zH^\times - A^\times)^{-1}\tilde{B} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} D_2^{-1} \\ &= (z - \alpha)C_1(zH^\times - A^\times)^{-1}\tilde{B} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} D_2^{-1}. \end{aligned}$$

Consequently,

$$W^{\min}(z) = D_1 D_2^{-1} + (z - \alpha)(C_1 - D_1 D_2^{-1} C_2)(zH^\times - A^\times)^{-1}\tilde{B} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} D_2^{-1}$$

and we have proved the following corollary.

Corollary 4.5.5. *Let $\Theta(z)$ be given as in (4.5.24). Then, a realization formula for $W^{\min}(z)$ in Theorem 4.5.3 is given by*

$$W^{\min}(z) = D_1 D_2^{-1} + (z - \alpha)(C_1 - D_1 D_2^{-1} C_2)(zH^\times - A^\times)^{-1}\tilde{B} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} D_2^{-1},$$

where $D_1, D_2, C_1, C_2, H^\times, A^\times, \tilde{B}$ are given by (4.5.25)–(4.5.28).

Let

$$\kappa_* = \kappa_{i_1} + \cdots + \kappa_{i_N}, \quad \kappa_{**} = \kappa_1 + \cdots + \kappa_{N-1} + \kappa_{N+1}$$

where the integers $i_1 < \cdots < i_N$ are as in Lemma 4.5.2, and $\kappa_1 \leq \cdots \leq \kappa_{M+N}$ are given by (4.5.5).

Theorem 4.5.6. *The admissible degrees of complexity are as follows. If $i_j = j$ for $j = 1, \dots, N$ and $\kappa_N < \kappa_{N+1}$, then the admissible degrees are*

$$n_\pi + \kappa_*, n_\pi + \kappa_{**}, n_\pi + \kappa_{**} + 1, \dots$$

In this case $\kappa_ = \sum_{j=1}^N \kappa_j$ and there exists a unique solution of minimal possible McMillan degree $n_\pi + \sum_{j=1}^N \kappa_j$. If $i_j > j$ for some j or $\kappa_N = \kappa_{N+1}$, then the admissible degrees are*

$$n_\pi + \kappa_*, n_\pi + \kappa_* + 1, n_\pi + \kappa_* + 2, \dots$$

and in this case there is more than one solution of the minimal possible McMillan degree.

Proof. The proof is divided into three cases:

(Case I) $i_j = j$ for any $j \in \underline{N}$ and $\kappa_N < \kappa_{N+1}$

(Case II) $i_j = j$ for any $j \in \underline{N}$ and $\kappa_N = \kappa_{N+1}$

(Case III) $i_j > j$ for some $j \in \underline{N}$.

Let $\varphi^{-1}(z)$ be an $N \times N$ rational matrix function which has $\tilde{\tau}_- = (C_-, A_\pi; 0, 0, ; 0)$ as its \mathbb{C} -null-pole triple and let $\sigma(A_\pi) \cup \sigma(A_\zeta) = \{z_1, \dots, z_n\}$ and let

$$v_{jk} := \text{the } j^{\text{th}} \text{ column of } \varphi(z_k)[0 \ I]\Theta(z_k). \quad (4.5.29)$$

We start with Case III. Let

$$J := \min\{j \in \underline{N} | i_j > j\}$$

and β be a complex number which is chosen so that

$$\langle v_{i_j k} + (z_k - \beta)^l v_{i_j-1, k}, v_{i_j k} \rangle \neq 0, \quad k \in \underline{n} \quad (4.5.30)$$

where

$$l := \kappa_{i_j} - \kappa_{i_j-1} + K$$

for some nonnegative integer K and $\langle u, v \rangle$ represents the usual inner product of vectors in \mathbb{C}^N . Let $\begin{bmatrix} P(z) \\ Q(z) \end{bmatrix}$ be an $(M+N) \times N$ matrix polynomial for which

$$\text{the } j^{\text{th}} \text{ column of } \begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} = \begin{cases} e_j & , j \neq J \\ e_{i_J} + (z - \beta)^l e_{i_J-1} & , j = J, \end{cases} \quad (4.5.31)$$

where $\{e_j\}_{j=1}^{M+N}$ is the usual standard basis of \mathbb{C}^{M+N} . Then

$$\text{the } j^{\text{th}} \text{ column index of } \Theta \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{cases} \kappa_{i_J} & , j \neq J \\ \kappa_{i_J} + K & , j = J \end{cases}$$

so that the sum of the column indices is $\kappa_* + K$. If the matrix polynomial $\begin{bmatrix} P(z) \\ Q(z) \end{bmatrix}$ satisfies the conditions (4.5.7)–(4.5.9), then by (4.5.10), we conclude that there exists a solution

$$W(z) := (\Theta_{11}P + \Theta_{12}Q)(\Theta_{21}P + \Theta_{22}Q)^{-1}$$

with $\delta(W) = n_\pi + \kappa_* + K$ for any nonnegative integer K .

Now, we show that the conditions (4.5.7)–(4.5.9) are satisfied. From (4.5.31), it is obvious that $\begin{bmatrix} P \\ Q \end{bmatrix}$ is coprime. To see that $\Theta \begin{bmatrix} P \\ Q \end{bmatrix}$ is column reduced at infinity, we compare the leading coefficient matrix of $\Theta \begin{bmatrix} P \\ Q \end{bmatrix}$. If $j \neq J$, the j^{th} column of $\Theta \begin{bmatrix} P \\ Q \end{bmatrix}$ is θ_{i_J} , while if $j = J$, the j^{th} column of $\Theta \begin{bmatrix} P \\ Q \end{bmatrix}$ is $\theta_{i_J} + \beta z^l \theta_{i_J-1}$. The leading coefficient of the j^{th} column of $\Theta \begin{bmatrix} P \\ Q \end{bmatrix}$ is therefore given by

$$\begin{cases} [\theta_{i_J}]_j & , \text{ if } j \neq J \\ [\theta_{i_J}]_h + [\theta_{i_J-1}]_h & , \text{ if } j = J \text{ and } K = 0 \\ [\theta_{i_J}]_h & , \text{ if } j = J \text{ and } K > 0, \end{cases} \quad (4.5.32)$$

where $[r]_h$ denotes the leading coefficient of the rational vector function $r(z)$ in its Laurent expansion at infinity and θ_j is the j^{th} column of $\Theta(z)$. Since we are in Case III, $i_{J-1} < i_J - 1$ by the choice of J . Hence from the fact that Θ is column reduced at infinity, $\Theta \begin{bmatrix} P \\ Q \end{bmatrix}$ is column reduced at infinity.

To prove (4.5.7), we show that $\varphi[0 \ I]\Theta \begin{bmatrix} P \\ Q \end{bmatrix}$ has no zeros or poles on $\sigma(A_\pi) \cup \sigma(A_\zeta)$.

By Lemma 1.4.3, $\varphi[0 \ I]\Theta$ has no poles on $\sigma(A_\pi) \cup \sigma(A_\zeta)$ hence so does $\varphi[0 \ I]\Theta \begin{bmatrix} P \\ Q \end{bmatrix}$.

Moreover,

$$\text{rank } \left(\varphi(z_k)[0 \ I]\Theta(z_k) \begin{bmatrix} P(z_k) \\ Q(z_k) \end{bmatrix} \right) = N, \quad k \in \underline{n}$$

by the choice of the complex number β . Remember that the j^{th} column of $\varphi(z_k)[0 \ I]\Theta(z_k) \begin{bmatrix} P(z_k) \\ Q(z_k) \end{bmatrix}$ is given by

$$\begin{cases} v_{i,j,k} & , j \neq J \\ v_{i,j,k} + (z_k - \beta)^l v_{i,J-1,k} & , j = J \end{cases}$$

for the complex number β satisfying (4.5.30). Thus,

$$\begin{aligned} & \det \left(\varphi(z_k)[0 \ I]\Theta(z_k) \begin{bmatrix} P(z_k) \\ Q(z_k) \end{bmatrix} \right) \\ &= \det ([v_{i_1,k}, \dots, v_{i_N,k}]) \neq 0, \quad \text{for } k \in \underline{n}. \end{aligned}$$

This completes the proof of Case III.

Case I: In this case, we get a solution $W^{\min}(z)$ of McMillan degree $n_\pi + \sum_{j=1}^N \kappa_j$ by the parameter $\begin{bmatrix} P \\ Q \end{bmatrix}$ to be $\begin{bmatrix} I_N \\ 0 \end{bmatrix}$. To get a solution other than this particular W^{\min} , we must find a parameter $\begin{bmatrix} P \\ Q \end{bmatrix}$ satisfying (4.5.7)–(4.5.9) such that

$$\begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} \neq \begin{bmatrix} I_N \\ 0 \end{bmatrix} V(z) \tag{4.5.33}$$

for any $N \times N$ unimodular matrix polynomial $V(z)$. Upon considering the condition (4.5.8) we see that (4.5.33) forces

$$[0 \ I_M] \begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} \neq 0,$$

that is the j^{th} row of $\begin{bmatrix} P \\ Q \end{bmatrix}$ is nonzero for some $j > N$. Then the sum of the column indices of $\Theta \begin{bmatrix} P \\ Q \end{bmatrix}$ is at least

$$\kappa_{**} = \kappa_1 + \dots + \kappa_{N-1} + \kappa_{N+1}.$$

This proves that in Case I the interpolant W^{\min} of minimal McMillan degree is unique and that there are no interpolants of McMillan degree $n_\pi + \kappa_* + 1, n_\pi + \kappa_* + 2, \dots, n_\pi + \kappa_{**} - 1$.

Now, we prove the existence of interpolants of McMillan degree $n_\pi + \kappa_{**} + K$ for any nonnegative integer K . Let $\begin{bmatrix} P(z) \\ Q(z) \end{bmatrix}$ be an $(M+N) \times N$ matrix polynomial for which

$$\text{the } j^{\text{th}} \text{ column of } \begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} = \begin{cases} e_j & j = 1, \dots, N-1 \\ (z - \beta)^l e_N + e_{N+1} & , j = N, \end{cases}$$

where the complex number β is chosen so that

$$<(z_k - \beta)^l v_{Nk} + v_{N+1,k}, v_{Nk}> \neq 0, \quad k \in \underline{n}$$

where

$$l := k_{N+1} - k_N + K.$$

Then by going through the same argument of Case III, it can be shown that $\begin{bmatrix} P \\ Q \end{bmatrix}$ satisfies (4.5.7)–(4.5.9) and hence there exists a solution of McMillan degree $n_\pi + \kappa_{**} + K$.

Case II. In this case we repeat the argument of Case I which is used to show the existence of solutions of McMillan degree $n_\pi + \kappa_{**} + K$. Since $\kappa_{**} = \kappa_*$ in this case, this will do it. The analysis shows that the minimal possible McMillan degree of an interpolant is

$$\delta(W^{\min}) = n_\pi + \sum_{j=1}^N \kappa_j$$

but in this case $W^{\min}(z)$ is not unique. \square

We close this section with the following corollary which parametrizes the set of all interpolants of a given admissible degree of complexity. The proof is omitted.

Corollary 4.5.7. Let n be an admissible degree of complexity. Then all the solutions with McMillan degree n are parametrized as follows;

$$W = (\Theta_{11}P + \Theta_{12}Q)(\Theta_{21}P + \Theta_{22}Q)^{-1},$$

where $\begin{bmatrix} P \\ Q \end{bmatrix}$ is an $(M+N) \times N$ matrix polynomial satisfying (4.5.7)–(4.5.9) and where

the sum of the column indices of $\Theta \begin{bmatrix} P \\ Q \end{bmatrix} = n - n_\pi$.

more explicitly, represent the j^{th} column of $\begin{bmatrix} P \\ Q \end{bmatrix}$ as

$$v_j(z) := [v_{1j}(z), \dots, v_{M+N,j}(z)]^T.$$

Then in addition to (4.5.7)–(4.5.9), $\begin{bmatrix} P \\ Q \end{bmatrix}$ should satisfy

$$\sum_{j=1}^N \left[\max_l \{ \kappa_l + \deg v_{lj}(z) \} \right] = n - n_\pi, \quad (4.5.34)$$

where the degree of the zero polynomial is taken to be $-\infty$.

For the case where $n = \delta(W^{\min})$, (4.5.34) collapses to

$$\deg v_{lj}(z) \leq \kappa_{i_j} - \kappa_l, \quad (4.5.35)$$

where $\{i_j\}_{j=1}^N$ is the same as in Lemma 4.5.2 and where (4.5.7)–(4.5.9) forces equality to hold in (4.5.35) for at least one l .

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List of Symbols

\mathbb{C}	the field of complex numbers
\mathbb{C}^∞	the extended complex plane, $\mathbb{C} \cup \{\infty\}$
\mathbb{C}^n	the set of n -dimensional complex column vectors
$\mathbb{C}^{m \times n}$	the set of $m \times n$ matrices with entries in \mathbb{C}
$col_{1 \leq j \leq n} A_j$	$\begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix}$ a block column matrix
$\delta(W)$	the McMillan degree of a rational matrix function W
$diag(\alpha_j)_{j=1}^n$	the diagonal matrix with the elements $\alpha_1, \dots, \alpha_n$ along the main diagonal
$\mathfrak{M} + \mathfrak{N}$	direct sum of subspaces \mathfrak{M} and \mathfrak{N}
\mathcal{P}_N	the set of N -dimensional column vectors with polynomial entries
$\mathcal{P}_{M \times N}$	the set of $M \times N$ matrices with polynomial entries
\mathcal{R}_N	the set of N -dimensional column vectors with rational function entries
$\mathcal{R}_{M \times N}$	the set of $M \times N$ matrices with rational function entries
σ	a subset of the complex plane \mathbb{C}
$\mathcal{R}(\sigma)$	the set of rational functions analytic in the subset σ of \mathbb{C}
$\mathcal{R}_N(\sigma)$	the set of N -dimensional column vectors with entries in $\mathcal{R}(\sigma)$
$\mathcal{R}_{M \times N}(\sigma)$	the set of $M \times N$ matrices with entries in $\mathcal{R}(\sigma)$
\underline{n}	the set $\{1, 2, \dots, n\}$ for an integer n
$A _{\mathfrak{M}}$	the restriction of a linear transformation on the subspace \mathfrak{M}

$Ker A$	the kernel of a linear transformation A
$Im A$	the image of a linear transformation A
$\sum_{z_0 \in \mathbb{C}} Res_{z=z_0} W(z)$	sum of the residues of a function $W(z)$ in \mathbb{C}
$\sigma(A)$	the spectrum of a matrix A
$\#S$	the number of the elements of a set S
$v^T(A^T)$	the transpose of a vector v (a matrix A)
e_i	the i^{th} usual standard basis in \mathbb{C}^n $[0, \dots, 0, 1, 0, \dots 0]$ (with 1 in the i^{th} place); its size n will be clear from the context
$[a_{ij}]_{1 \leq i \leq m, 1 \leq j \leq n}$	$m \times n$ matrix whose entry in the (i, j) place is a_{ij}
I	identity matrix (the size of I is understood from the context)
$\bigvee_{i=1}^n \{v_i\}$	the subspace spanned by vectors v_1, \dots, v_n
□	end of a proof or a remark

Vita

Jeongook Kang was born in Kwang-Ju, Korea on June 1, 1959. She attended high school in Seoul, Korea. She studied at Seoul National University, where she received a Bachelor of Science degree in Mathematics Education in 1983 and a Master of Art in Mathematics Education in 1985. Since September 1985, she has been a graduate student in the Department of Mathematics at Virginia Polytechnic Institute and State University. She married Sungkwon Kang in 1985 and has one son. Her maiden name is Kim.

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