Analysis and Numerical Approximations of Exact Controllability Problems for Systems Governed by Parabolic Differential Equations

by

Yanzhao Cao

Dissertation submitted to the faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

APPROVED:

Max D. Gunzburger, Chairman

Eugene M. Cliff

Tao Lin

Janet S. Peterson

Robert C. Rogers

June, 1996

Blacksburg, Virginia

Analysis and Numerical Approximations of Exact Controllability Problems for Systems Governed by Parabolic Differential Equations

by

Yanzhao Cao

Committee Chairman: Max D. Gunzburger

Department of Mathematics

(ABSTRACT)

The exact controllability problems for systems modeled by linear parabolic differential equations and the Burger's equations are considered. A condition on the exact controllability of linear parabolic equations is obtained using the optimal control approach. We also prove that the exact control is the limit of appropriate optimal controls. A numerical scheme of computing exact controls for linear parabolic equations is constructed based on this result. To obtain numerical approximation of the exact control for the Burger's equation, we first construct another numerical scheme of computing exact controls for linear parabolic equations by reducing the problem to a hypoelliptic equation problem. A numerical scheme for the exact zero control of the Burger's equation is then constructed, based on the simple iteration of the corresponding linearized problem. The efficiency of the computational methods are illustrated by a variety of numerical experiments.

ACKNOWLEDGEMENTS

I can not use my words to express my deep gratitude to my advisor, Max Gunzburger. Whatever success I have gotten is due to his support and guidance.

Expressions of sincere appreciations and gratitude go to Tao Lin for his great deal of help and encouragement. I could not have completed the numerical experiment part of my work without his help.

I would like to thank Eugene Cliff, Janet Peterson and Bob Rogers for kindly serving on my committee.

My gratitude also extends to David Russell and Matthias Heinkenschloss for their willingness to share their time and expertise. Their suggestions and contributions have been invaluable.

Deepest appreciations are extended to John Burkardt for his help on coding, latex, unix, xfig,$\cdots$.

Special thanks to all people in ICAM for the wonderful atomsphere they created and in particular to J. Burns, T. Herdman, Jeff Borggaard , Justin Appel, Jennifer Deang, and Hyesuk Kwon.

Finally I would like to express my gratitude to the Air Force Office of Scientific Research for the suport of this work under grant number of AFOSR-93-1-0280.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

Chapter 1

Introduction

The goal of this thesis is to investigate the theoretical and computational issues associated with controllability problems for systems modeled by parabolic partial differential equations. First, we study the exact controllability problem of linear parabolic differential equations with Neumann boundary conditions using optimal control methods. The method is different from the traditional harmonic analysis method. Second, we consider a finite element method for the exact boundary controllability problem of linear parabolic differential equations with Dirichlet boundary conditions. Finally we study numerical approximations of the exact controllability problems for the Burger's equation.

1.1 Controllability via Optimal Control

For a control problem, we are given the following data.

1. A control u belonging to a set $\mathcal{U}_{ad}$ which is at our disposition.

2. For a given control u, the state $y(u)$ of the system given by the solution of an equation

$$\Lambda y(u) = \text{given function of u}$$

where Λ is an operator which specifies the system to be controlled.

3. A given state $\hat{y}$ and the observation $z(u)$ which is a function of $y(u)$.

Definition 1.1 The system is said to be exactly controllable if there exists a control u

such that

$$z(u) = \hat{y}.$$

The main difficulty in studying the exact controllability problem is that it is an ill-posed problem. First, for a given state $\hat{y}$ in a certain Banach space, exact control may not exist. Second, even if an exact control exists, it may not be unique.

This problem has been studied extensively in the last 30 years. A very significant early contribution is due to Egorov ([6], [7]). Though primarily interested in optimal control, he found it necessary, as part of this study, to characterize a class of reachable states. Another early paper is due to Gal'chuk ([12]). Many developments in the controllability theory of the linear parabolic equations and hyperbolic equations are due to Fattorini and Russell ([8], [9], [30], [31]). Using a harmonic analysis method, they obtained results such as approximate controllability and conditions on $\hat{y}$ for the exact controllability of systems governed by hyperbolic and parabolic equations. More recently Lions ([24]) developed the Hilbert Uniqueness Method for the exact controllability problem of hyperbolic equations. The numerical implementation of this method is discussed in [14]. Another recent development on the exact controllability problems of distributed parameter systems is due to Fursikov and Imanuvilov ([11]). We will have more to say concerning their work in the next section.

A related but well-posed problem is the optimal control problem. To define an optimal control problem, we first define a cost function $J(u)$ by

$$J(u, \epsilon) = d(z(u) - \hat{y}) + \epsilon g(u) \tag{1.1}$$

where ϵ is a positive parameter and d and g are positive functions, usually norm functions on certain Banach spaces. Then the optimal control problem is to find a control $u = u(\epsilon) \in \mathcal{U}_{ad}$, such that

$$J(u(\epsilon),\epsilon) = \inf_{u\in\mathcal{U}_{ad}} J(u,\epsilon). \tag{1.2}$$

Optimal control is a rapidly developing subject. It dates back to the revolutionary treatment of an old mathematical subject, the Calculus of Variations, by Pontryagin ([28], [29]). Optimal control theory of distributed parameter systems has been studied by Lions ([20], [21], [22]); he addresses the question of existence of an optimal control and the derivation of necessary conditions. A Galerkin approximation of the optimal control problem for parabolic equations is studied by Winther in [37]; error estimates are obtained there.

In this part of the thesis we attempt to study the exact controllability problem by examining the limit behavior of the optimal control problem (1.2) as $\epsilon \to 0$. Notice that if $\epsilon = 0$, then the solution(s) of problem (1.2) is an exact control. We are concerned with the following three issues:

1. The limit behavior of $d(z(u,\epsilon)-\hat{y})$ as $\epsilon \to 0$;

2. The limit behavior of $y(\epsilon)$ as $\epsilon \to 0$ where $y(\epsilon) = y(u(\epsilon))$;

3. The limit behavior of $u(\epsilon)$ as $\epsilon \to 0$.

In Chapter 2 we consider a system modeled by linear parabolic equations. By studying the above three issues, we obtained a new condition on the exact controllability of the system and a method of numerically computing the exact control that has minimum L^2 norm. The problem can be described as follows.

Let Ω be a bounded domain in R^d. For a fixed T, let $Q = [0,T]\times\Omega$ and $\Sigma = (0,T)\times\partial\Omega$. On the domain Ω, let A be the second order elliptic differential operator

$$Ay = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{i,j}(x)\frac{\partial y}{\partial x_j}) + c(x)y.$$

We will assume that $c > 0$ on $\bar{\Omega}$ and that the matrix $(a_{i,j}(x))$ is symmetric and positive definite.

Consider now the parabolic initial boundary value problem

$$\begin{aligned} \frac{\partial}{\partial t}y + Ay &= 0 \quad \text{on } Q, \\ \frac{\partial y}{\partial \nu} &= u, \quad \text{on } \Sigma, \\ y(0) &= v, \quad \text{on } \Omega. \end{aligned} \tag{1.3}$$

Here $\frac{\partial}{\partial \nu} = \sum_{i,j=1}^{d} a_{i,j} n_i (\frac{\partial}{\partial x_j})$ where n_i is the $i-th$ component of the outward unit normal vector on $\partial\Omega$. If the components of the data v and u are given such that $v \in L^2(\Omega)$ and $u \in L^2(\Sigma)$, then problem (1.3) has a unique weak solution y ([23]). We now define a cost function J as

$$J(\epsilon, y, u) = \frac{1}{2}\int_{\Omega}(y(T) - \hat{y})^2 dx + \frac{\epsilon}{2}\int_{\Sigma} u^2(t,x)dxdt$$

where $(y,u) = (y(t), u(t))$ satisfies (1.3). $u(\epsilon) = u(\epsilon, t)$ is said to be an optimal control if it satisfies

$$J(\epsilon, y(\epsilon), u(\epsilon)) = \inf_{(y,u) \text{ satisfies } (1.3)} J(\epsilon, y, u). \tag{1.4}$$

Problem (1.4) is analyzed by Lions in [20]. It is proved that (1.4) has a unique solution and this solution can be characterized by a system of two parabolic equations.

In Chapter 2, using a symmetric, nonnegative and invertible compact operator R, we represent $y(\epsilon, T) - \hat{y}$ as

$$(y(\epsilon, T) - \hat{y}) = \epsilon(\epsilon + R)^{-1}(\hat{y} - E(T)v) \tag{1.5}$$

where $E = E(t)$ is the solution operator of problem (1.3) that will be specified in Chapter 2. Using the above equality we prove that

$$\lim_{\epsilon \to 0} y(\epsilon, T) = \hat{y}, \quad \text{in } L^2(\Omega).$$

As we shall point out in Chapter 2 this implies that system (1.3) is approximately controllable. Also by using (1.5) we obtain the following condition on $\hat{y}$ for the exact controllability of system (1.3):

$$\hat{y} - E(T)v = \sum_{j=1}^{\infty} y_j e_j \text{ satisfies } \sum_{j=1}^{\infty} \frac{y_j^2}{\lambda_j^2} < \infty$$

where $(\{e_j\}_{j=1}^{\infty}, \{\lambda_j\}_{j=1}^{\infty})$ is the eigensystem of R.

The above results are obtained without the assumption that the system is exactly controllable. If the system is indeed exactly controllable, we prove in Chapter 2 that

$$\begin{aligned} \lim_{\epsilon \to 0} y(\epsilon) &= y, \quad \text{in } L^2(0,T;H^1(\Omega)), \\ \lim_{\epsilon \to 0} u(\epsilon) &= u, \quad \text{in } L^2(\Sigma), \end{aligned}$$

where u is the exact control that has minimum $L^2(\Sigma)$ norm and y is the corresponding state. The significance of this result is that it enables us to find numerical solutions of exact controls by solving the corresponding optimal control problem numerically for small ϵ.

1.2 A Finite Element Method for Boundary Controllability Problems of Parabolic Equations with Dirichlet Boundary Conditions

We consider a control system where evolution in time is described by a function $y = y(t,x)$ defined in $(t,x) \in Q = [0,T] \times \Omega, \Omega \subset R^d$, that satisfies the linear parabolic equation

$$Ly = \partial_t y - \Delta y + \sum_{i=1}^{d} a_i \partial_{x_i} y = f \quad \text{in } Q, \tag{1.6}$$

with the initial condition

$$y(0,\cdot) = y_0, \quad \text{in } \Omega \tag{1.7}$$

and the boundary condition

$$y = u, \quad \text{on } \Sigma = [0,T] \times \partial\Omega. \tag{1.8}$$

The real valued functions u is interpreted as a boundary control function. According to Definition 1.1, the system is exactly controllable for a given state $\hat{y}$ if there exists $u \in L^2(\Sigma)$ such that the solution y of problem (1.6)-(1.8) also satisfies

$$y(T, \cdot) = \hat{y}.$$

As we have pointed out in the last section, for a general function $\hat{y} \in L^2(\Omega)$, the system may not be exactly controllable. An important special case is $\hat{y} = 0$. It is proved both by Egorov ([6]) and Fattorini and Russell ([9]) that if $\hat{y} = 0$, then the linear parabolic system is exactly controllable. The exact boundary control u in this case is also called the exact zero boundary control. Recently, Fursikov and Imanuvilov ([11]) developed a new method to treat this problem. The method differs from the others in that the problem is reduced to solving the following fourth order hypoelliptic equation

$$L^* cLp = f_0, \quad \text{in } Q \tag{1.9}$$

with boundary conditions

$$p = \frac{\partial p}{\partial \nu} = 0, \quad \text{on } \Sigma, \tag{1.10}$$

$$cLp(t, \cdot)|_{t=0} = cLp(t, \cdot)|_{t=T} = 0, \quad \text{in } \Omega \tag{1.11}$$

where $c = c(t, x)$ is a weight function that will be specified in Chapter 3. Once p is solved from (1.9)-(1.11), the state y and the control u are obtained through

$$y = y_1 + cL^* p \tag{1.12}$$

where y_1 is a function obtained by solving a heat equation. The advantages of this method are that it can be used to study the controllability problem for systems governed by Burgers equations or Navier-Stokes equations and that numerical approximations of the exact control u may be obtained by solving problem (1.9)-(1.11) numerically.

In Chapter 3 we present a finite element method for obtaining numerical solutions for problem (1.9)-(1.11) which yields numerical approximations to the state y and the control u through (1.12). Some modifications on Furiskov and Immanulov's theory are made to meet our needs of numerical approximations. We derive the finite element scheme for the problem (1.9)-(1.11). We demonstrate the convergence of the finite element approximation p_h to the exact solution p and the convergence of the approximate solution y_h to the exact solution y of problem (1.6)-(1.8) through (1.12) with minimum regularity requirements. The convergence of the approximate control u_h to the control u are proved under higher regularity assumptions. We performed a variety of numerical experiments using two different choices of weight functions c appearing in (1.9). These experiments agree with our error estimates.

1.3 A Numerical Scheme for the Controllability Problem of the Burger's Equation

As one can expect, the controllability problem for nonlinear systems is much more complicated than that of linear systems. Fursikov and Imanuvilov ([11]) constructed an example showing that in general a system governed by Burger's equations may not even be approximately controllable.

As in the linear case the study of controllability of nonlinear systems started with finite dimensional systems. Some sufficient conditions were obtained by Cirina ([4]) using inverse function methods. Using the hypoelliptic equation method described in §1.2, Fursikov and Imanuvilov ([10], [11]) prove the exact zero controllability of Burger's equations and the Navier-Stokes equations.

In Chapter 4 we study a numerical scheme for finding the exact zero controls for systems governed by Burger's equations. The scheme is based on a simple iteration of the corresponding linearized problem which is studied in Chapter 3.

Chapter 2

Controllability via Optimal Control

2.1 Statement of the Problem, Notation, and Preliminaries

Let Ω be a bounded domain in R^{d}. For a fixed T, let $Q = [0,T]\times\Omega$ and $\Sigma = (0,T)\times\partial\Omega$. On the domain Ω, let A be the second order elliptic differential operator

$$Ay = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{i,j}(x)\frac{\partial y}{\partial x_j}) + c(x)y.$$

We will assume that $c > 0$ on $\bar{\Omega}$ and that the matrix $(a_{i,j}(x))$ is symmetric and positive definite.

Consider now the parabolic initial boundary value problem

$$\begin{aligned} \frac{\partial y}{\partial t} + Ay &= 0, \quad \text{on } Q, \\ \frac{\partial y}{\partial \nu} &= u, \quad \text{on } \Sigma, \\ y(0) &= v, \quad \text{in } \Omega. \end{aligned} \tag{2.1}$$

Here $\frac{\partial}{\partial\nu} = \sum_{i,j=1}^{d} a_{i,j}n_i(\frac{\partial}{\partial x_j})$ where n_i is the $i-th$ component of the outward unit normal on $\partial\Omega$. Define a functional

$$J(y,u) = \frac{1}{2}\int_\Omega (y(T)-\hat{y})^2 dx + \frac{\epsilon}{2}\int_\Sigma u^2(t,x)dxdt.$$

We consider the following optimal control problem: find $(y(\epsilon),u(\epsilon)) = (y(\epsilon,t),u(\epsilon,t))$ such that

$$J(y(\epsilon,t),u(\epsilon,t)) = \inf_{y,u \text{ satisfies } (2.1)} J(\epsilon,y,u). \tag{2.2}$$

We call $(y(\epsilon,t),u(\epsilon,t))$ the optimal solution for problem (2.2). In this chapter we study the limit behavior of $(y(\epsilon,t),u(\epsilon,t))$ as $\epsilon\to 0$. In §2.2 we prove that

$$\lim_{\epsilon\to 0} y(\epsilon,T)=\hat{y} \quad \text{in } L^2(\Omega).$$

In §2.3 we obtain a sufficient condition on the exact controllability of the system using the same technique. In §2.4, we study the limit of $(y(\epsilon),u(\epsilon))$ as $\epsilon\to 0$ under the assumption that system (2.1) is exactly controllable. In §2.5, using the result obtained in §2.4, we construct a numerical algorithm to compute the exact control with minimum L^2 norm. In the rest of this section we introduce some notation and some fundmentals for linear parabolic differential equations that will be used in the sequel.

For a Banach space X, define

$$L^2(0,T;X)=\{f:(0,T)\to X;\ \int_0^T \|f(t)\|_X^2 dt<\infty\}.$$

It is a Hilbert space with the norm

$$\|f\|_{L^2(0,T;X)}=(\int_0^T \|f(t)\|_X^2 dt)^{\frac{1}{2}}.$$

Let $L^2(\Omega)$ and $L^2(\partial\Omega)$ be the usual L^2 function spaces. We shall use the notation

$$(\phi,\psi)=\int_\Omega \phi\psi dx$$

and

$$<\phi,\psi>=\int_{\partial\Omega}\phi\psi d\omega$$

for the inner products and the associated norms will be denoted by $\|\cdot\|$ and $|\cdot|$, respectively. We also define

$$W(0,T)=\{f;f\in L^2(0,T;H^1(\Omega)),\frac{df}{dt}\in L^2(0,T;H^{-1}(\Omega))\},$$

where $\frac{d}{dt}$ is taken in the sense of distributions on $(0,T)$ with values in $H^1(\Omega)$. $W(0,T)$ is a Hilbert space with the norm

$$\|f\|_{W(0,T)} = (\|f\|^2_{L^2(0,T;H^1(\Omega))} + \|\frac{df}{dt}\|^2_{L^2(0,T;H^{-1}(\Omega))})^{\frac{1}{2}}.$$

For p, $q > 0$, let also

$$H^{p,q}(Q) = L^2(0,T;H^q(\Omega)) \cap H^p(0,T;L^2(\Omega)).$$

These spaces are described in [23], and their norms are defined by

$$\|f\|_{H^{p,q}(Q)} = (\|f\|^2_{L^2(0,T;H^q(\Omega))} + \|f\|^2_{H^p(0,T;L^2(\Omega))})^{\frac{1}{2}}.$$

The spaces $H^{p,q}(\Sigma)$ with associated norms $\|\cdot\|_{H^{p,q}(\Sigma)}$ are defined similarly by replacing Ω by Σ above.

We recall ([23]) that if $q > \frac{1}{2}$ and $p = q - \frac{1}{2}$, then there is a constant c such that for any $f \in H^{p,\frac{p}{2}}(Q)$,

$$\|f\|_{H^{p,\frac{p}{2}}(\Sigma)} \leq \|f\|_{H^{q,\frac{q}{2}}(Q)},$$

and if $q > 1$, $g \in H^{q,\frac{q}{2}}(\Sigma)$ and $t \in [0,T]$ then

$$\|f(t)\|_{H^{p-1}(\Omega)} \leq c\|f\|_{H^{p,\frac{p}{2}}(Q)}$$

and

$$|g|_{H^{p-1}(\partial\Omega)} \leq c\|g\|_{H^{q,\frac{q}{2}}(\Sigma)}.$$

Define the bilinear form B: $H^1(\Omega) \times H^1(\Omega) \to R$ by

$$B(\phi,\psi) = \int_\Omega \{\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial\phi}{\partial x_i}\frac{\partial\psi}{\partial x_j} + c(x)\phi\psi\}dx.$$

Note that from the properties of the operator A we have that there exist constants c_1, $c_2 > 0$ such that

$$|B(\phi, \psi)| \leq c_1 ||\phi||_1 ||\psi||_1$$

and

$$B(\phi, \phi) \geq c_2 ||\phi||_1^2$$

for all ϕ, $\psi \in H^1(\Omega)$.

We now define the operators for weak solutions of elliptic boundary value problems. For $f \in H^{-1}(\Omega)$, let $Tf \in H^1(\Omega)$ denote the unique solution of the problem

$$B(Tf, \phi) = (f, \phi) \quad \text{for } \phi \in H^1(\Omega).$$

T is a linear operator on $H^{-1}(\Omega)$. Also since

$$(Tf, \phi) = B(Tf, T\phi) = (f, T\phi)$$

for any f, $\phi \in L^2(\Omega)$, T is self adjoint on $L^2(\Omega)$. Next we introduce the eigenvalue problem

$$A\phi = \Lambda\phi, \quad \text{in } \Omega,$$

$$\frac{\partial \phi}{\partial \nu} = 0, \quad \text{on } \partial\Omega.$$

It is well known ([23]) that this problem has a system of eigenfunctions $\{\phi_j\}_{j=1}^{\infty}$, forming a complete orthonormal set in $L^2(\Omega)$, with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to \infty$ as $j \to \infty$. Note that if we let $\nu_j = \lambda_j^{-1}$, then

$$T\phi_j = \nu_j \phi_j \quad j = 1, 2, ...$$

W now consider the weak formulation of problem (2.1)

$$\begin{aligned} (\tfrac{dy}{dt}, \phi) + B(y, \phi) &= < u, \phi >, \quad \text{for } \phi \in H^1(\Omega), \\ y(0, \cdot) &= v. \end{aligned} \tag{2.3}$$

It is well known that if $v \in L^2(\Omega)$ and $u \in L^2(\Sigma)$ then (2.3) has a unique solution y in $W(0,T)$, and there is a constant C, independent of v and u, such that

$$\|y\|_{W(0,T)} \leq C(\|v\| + \|u\|_{L^2(\Sigma)}).$$

If $u = 0$, then the exact solution of (2.3) can be represented by

$$y(t,x) = \sum_{j=1}^{\infty}(v, \phi_j)e^{-\lambda_j t}\phi_j(x)$$

The following lemma can be found in [33].

Lemma 2.1 *Let $\phi \in H^p(\Omega)$ for $-1 \leq p \leq 1$. Then*

$$|||\phi||| = (\sum_{j=1}^{\infty}(\phi, \phi_j)^2 \lambda^p)^{\frac{1}{2}}$$

is an equivalent norm in $H^p(\Omega)$.

2.2 Convergence of the Terminal State

2.1.1 An operator representation of the terminal state. In the following lemma we state the optimality system for the problem (2.2). It will play a central role in obtaining the main results of this section and the next section. We refer to [20] for a proof of this lemma.

Lemma 2.2 $(y(\epsilon), u(\epsilon)) = (y(\epsilon, t), u(\epsilon, t)) \in H^{1,2}(Q) \times L^2(\Sigma)$ *is the solution of (2.2) if and only if there exists $p(\epsilon) = p(\epsilon, t) \in W(0,T)$ such that p satisfies the following parabolic differential equation with initial and boundary conditions:*

$$\begin{aligned} -\tfrac{dp}{dt} + Ap &= 0, \quad \text{in } Q, \\ \tfrac{\partial p(\epsilon)}{\partial \nu} &= 0, \quad \text{on } \Sigma, \\ p(\epsilon, T) &= y(\epsilon, T) - \hat{y}, \quad \text{in } \Omega, \\ p &= \epsilon u, \quad \text{on } \Sigma. \end{aligned} \tag{2.4}$$

The weak formulation of the optimality system can be written as follows.

$$\begin{aligned}
(\tfrac{dy}{dt},\phi)+B(y,\phi) &= \tfrac{1}{\epsilon}<p,\phi>, \quad \text{for } \phi\in H^1(\Omega),\\
-(\tfrac{dp}{dt},\phi)+B(p,\phi) &= 0, \quad \text{for } \phi\in H^1(\Omega),\\
y(0) &= v,\\
p(T) &= \hat{y}-y(\epsilon,T).
\end{aligned} \tag{2.5}$$

To represent the terminal state $y(\epsilon, T)$ by an operator, we consider, for a given $z \in L^2(\Omega)$, the system

$$\begin{aligned}
(\tfrac{dy_1}{dt},\phi)+B(y_1,\phi) &= <p_1,\phi>, \quad \text{for } \phi\in H^1(\Omega),\\
-(\tfrac{dp_1}{dt},\phi)+B(p_1,\phi) &= 0, \quad \text{for } \phi\in H^1(\Omega),\\
y_1(0) &= 0,\\
p_1(T) &= z.
\end{aligned} \tag{2.6}$$

This system has a unique solution $(y_1(t), p_1(t)) \in W(0,T)$. We now define an operator R: $L^2(\Omega) \to L^2(\Omega)$ by

$$Rz = y_1(T). \tag{2.7}$$

Comparing (2.5) with (2.6), we have that

$$\frac{1}{\epsilon}R(y(T)-\hat{y}) = -E(T)v + y(T) \tag{2.8}$$

or

$$(\epsilon + R)(y(T)-\hat{y}) = \epsilon(\hat{y} - E(T)v) \tag{2.9}$$

where $E(t)$, $0 < t \le T$, is the solution operator of problem (2.1) for with u=0.

2.2.2. Properties of the operator R.

Lemma 2.3 $R \in L(L^2(\Omega), H^1(\Omega))$. *Therefore R is a compact operator in $L^2(\Omega)$.*

Proof: Consider the mapping

$$z \to p_1(t,\cdot) = E(T-t)z.$$

We prove that this mapping is continuous from $L^2(\Omega)$ into $H^{\frac{1}{2},1}(Q)$. By standard theory for parabolic differential equations ([24]) this mapping is continuous from $H^1(\Omega)$ into $H^{1,2}(Q)$. If $z \in H^{-1}(\Omega)$ then

$$\begin{aligned}\int_0^T \|p_1(t,\cdot)\|^2 dt &= \int_0^T \sum_{j=1}^{\infty}(z,\phi_j)^2 e^{-2\lambda_j(T-t)}dt \\ &= \frac{1}{2}\sum_{j=1}^{\infty}(z,\phi_j)^2\lambda_j^{-1}(1-e^{-2\lambda_j T}) \\ &\leq C\|z\|_{-1}.\end{aligned}$$

Hence the mapping is continuous from $H^{-1}(\Omega)$ into $L^2(Q)$. By interpolation ([1]) we have that R is a continuous mapping from $L^2(\Omega)$ to $H^{\frac{1}{2},1}(Q)$.

By (2.6), we have that $\frac{\partial y_1}{\partial \nu}|_\Sigma = p_1|_\Sigma$. Thus

$$\|Rz\|_1 = \|y_1(T)\|_1 \leq C\|y_1\|_{H^{1,2}(Q)} \leq C\|z\|.$$

The proof is complete. □

Lemma 2.4 *R is symmetric and semi-definite.*

Proof: For given z, $z_0 \in L^2(\Omega)$, let (y,p), (y_0,p_0) be the corresponding solutions of (2.6), respectively. Then

$$\begin{aligned}(\frac{dy}{ds},p_0) + B(y,p_0) &= < p,p_0 >, \\ (\frac{dy_0}{ds},p) + B(y_0,p) &= < p_0,p >, \\ B(y,p_0) &= (y,\frac{d}{ds}p_0), \\ B(y_0,p) &= (y_0,\frac{d}{ds}p).\end{aligned}$$

We therefore have that

$$(\frac{d}{ds}y(s),p_0(s)) + (y(s),\frac{d}{ds}p_0(s)) = (\frac{d}{ds}y_0(s),p(s)) + (y_0(s),\frac{d}{ds}p(s))$$

or

$$\frac{d}{ds}(y_0(s), p(s)) = \frac{d}{ds}(y(s), p_0(s)).$$

Integrating the above equality from 0 to T, noting that $y(0) = y_0(0) = 0$, we have that

$$\begin{aligned}
(y(T), p_0(T)) &= \int_0^T \frac{d}{ds}(y(s), p_0(s))ds \\
&= \int_0^T \frac{d}{ds}(y_0(s)p(s))ds \\
&= (y_0(T), p(T))
\end{aligned}$$

or

$$(Rz, z_0) = (Rz_0, z).$$

Thus R is self-adjoint. We also have that

$$\begin{aligned}
|p|^2 &= (\frac{d}{ds}y, p) + B(y, p) \\
&= (\frac{d}{ds}y, p) + (y, \frac{d}{ds}p) \\
&= \frac{d}{ds}(y, p).
\end{aligned}$$

Integrating the above equality, we have that

$$(Rz, z) = \int_0^T |p|^2 ds \geq 0.$$

So R is semi-definite. The proof is complete. □

Lemma 2.5 *Assume that* Σ *is analytic. Then* $\mathrm{Ker} R = 0$.

Proof: Under the given assumption we know that p is analytic in Q and on Σ. Assume that $Rz = 0$. Then from the proof of Lemma 2.4 we have that

$$0 = (Rz, z) = \int_0^T |p|^2 ds.$$

Thus $p = 0$ on Σ. Hence the data corresponding to the Cauchy problem is zero on Σ which implies according to the Cauchy-Kowaleska theorem (since p is analytic) that $p = 0$ on Q. In particular $z = p(T) = 0$. The proof is complete. $\square$

2.2.3 Convergence of the terminal state.

Theorem 2.6 *Let $\hat{y} \in L^2(\Omega)$ and $(y(\epsilon), u(\epsilon)) = (y(\epsilon, t), u(\epsilon, t))$ be the solution of problem (2.4). Then*

$$\lim_{\epsilon \to 0} \|y(\epsilon, T) - \hat{y}\|_{L^2(\Omega)} = 0.$$

Proof: By Lemma 2.2, Lemma 2.3 and Lemma 2.4, and the Gilbert-Schmidt ([23]) theorem, the operator R has a system of eigenfunctions $\{e_j\}_{j=1}^{\infty}$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_j \to 0$ as $j \to \infty$. Moreover, $\{e_j\}_{j=1}^{\infty}$ forms an orthonormal basis in $L^2(\Omega)$. For $\hat{y} \in L^2(\Omega)$ let

$$\hat{y} - E(T)v = \sum_{j=1}^{\infty} y_j e_j.$$

Then

$$R(\hat{y} - E(T)v) = \sum_{j=1}^{\infty} \lambda_j y_j e_j.$$

Using (2.9) we have that

$$y(\epsilon, T) - \hat{y} = \sum_{j=1}^{\infty} \frac{\epsilon y_j e_j}{\epsilon + \lambda_j}.$$

Thus

$$\begin{aligned} \|y(\epsilon, T) - \hat{y}\|_{L^2(\Omega)}^2 &= \sum_{j=1}^{\infty} \frac{\epsilon^2 y_j^2}{(\epsilon + \lambda_j)^2} \\ &\leq \sum_{j=1}^{N} \frac{\epsilon^2 y_j^2}{(\epsilon + \lambda_j)^2} + \sum_{j=N+1}^{\infty} y_j^2. \end{aligned}$$

For any given $\delta > 0$, there exists N such that

$$\sum_{j=N+1}^{\infty} y_j^2 < \frac{\delta}{2}.$$

For this fixed N there exists $\beta > 0$ such that

$$\sum_{j=1}^{N} \frac{\epsilon^2 y_j^2}{(\epsilon + \lambda_j)^2} < \frac{\delta}{2}$$

for $0 < \epsilon < \beta$. Thus

$$\|y(\epsilon, T) - \hat{y}\|_{L^2(\Omega)}^2 < \delta$$

for $0 < \epsilon < \beta$. The proof is complete. □

Definition 2.1 System (2.1) is said to be approximately controllable if, for any given $\hat{y}$ and $\delta > 0$, there exists a control $u(t)$ and a function $y(t)$ such that $(y(t), u(t))$ is a solution of (2.1) and

$$\|y(T) - \hat{y}\| < \delta.$$

The following theorem is an immediate consequence of Theorem 2.6.

Theorem 2.7 *System (2.1) is approximately controllable.* □

Approximate controllability has been proved in [8]. Our proof is different in that it is a constructive proof.

2.3 A Condition for Exact Controllability

In this section we give a condition on $\hat{y}$ for the exact controllability of system (2.1). The main tool is the operator R introduced in §2.2.

Theorem 2.8 *Let R be the operator defined in §2.2 and $(\{e_j\}_{j=1}^{\infty}, \{\lambda_j\}_{j=1}^{\infty})$ be the eigen-system of R. Assume that $\hat{y} - E(T)v = \sum_{j=1}^{\infty} y_j e_j$ satisfies*

$$\sum_{j=1}^{\infty} \frac{y_j^2}{\lambda_j^2} < \infty. \tag{2.10}$$

Then system (2.1) is exactly controllable.

Proof: By the proof of Theorem 2.6 we have that

$$\|y(\epsilon, T) - \hat{y}\|_{L^2(\Omega)}^2 = \sum_{j=1}^{\infty} \frac{\epsilon^2 y_j^2}{(\epsilon + \lambda_j)^2}.$$

Hence

$$\begin{aligned} \|y(\epsilon, T) - \hat{y}\|_{L^2(\Omega)} &= (\sum_{j=1}^{\infty} \frac{\epsilon^2 y_j^2}{(\epsilon + \lambda_j)^2})^{\frac{1}{2}} \\ &\leq \epsilon(\sum_{j=1}^{\infty} \frac{y_j^2}{\lambda_j^2})^{\frac{1}{2}} \leq C\epsilon. \end{aligned}$$

Let $\{\epsilon_k\}_{k=1}^{\infty}$ be a sequence such that $\epsilon_k \to 0$ as $k \to \infty$. From the optimality system (2.4) and the proof of Lemma 2.3 we have that

$$\begin{aligned} \|y(\epsilon_k)\|_{H^{1,2}(Q)} &\leq C(\|u(\epsilon_k)\|_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} + \|v\|_{H^1(\Omega)}) \\ &= C(\|\frac{p(\epsilon_k)}{\epsilon_k}\|_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} + \|v\|_{H^1(\Omega)}) \\ &\leq C(\|\frac{p(\epsilon_k)}{\epsilon_k}\|_{H^{\frac{1}{2},1}(Q)} + \|v\|_{H^1(\Omega)}) \\ &\leq C(\|\frac{y(\epsilon_k, T) - \hat{y}}{\epsilon_k}\|_{L^2(\Omega)} + \|v\|_{H^1(\Omega)}) \\ &\leq C_1\|v\|_{H^1(\Omega)} + C_2. \end{aligned}$$

Thus there exists a subsequence of $\{\epsilon_k\}$, still denoted as $\{\epsilon_k\}$, and $y \in H^{1,2}(Q)$ such that

$$\lim_{k\to\infty} y(\epsilon_k) = y \quad \text{weakly in } H^{1,2}(Q).$$

Using the trace theorem ([1]) and the compact embedding theorem ([1]) in Sobolev spaces we also have that there exists $u \in L^2(\Sigma)$ such that

$$\lim_{k\to\infty} u(\epsilon_k) = u \quad \text{in } L^2(\Sigma).$$

Also by the trace theorem ([1]) we have that $y(T) = \hat{y}$ and $y(0) = v$. We now prove that (y, u) is a solution of (2.1). Since $(y(\epsilon, t), u(\epsilon, t))$ is a solution of (2.2)

$$(\frac{dy(\epsilon_k)}{dt}, \phi) + B(y(\epsilon_k), \phi) =< u(\epsilon_k), \phi >, \quad \text{for } \phi \in H^1(\Omega).$$

Passing to the limit in the above equality we obtain

$$(\frac{dy}{dt}, \phi) + B(y, \phi) =< u, \phi >, \quad \text{for } \phi \in H^1(\Omega).$$

Thus $(y, u) \in H^{1,2}(Q) \times L^2(\Sigma)$ is a solution of (2.1). This proves that u is an exact control. The proof is complete. □

Remark 2.1. Let R^{-1} be the inverse of operator R. Then the condition (2.10) is equivalent to

$$\hat{y} - E(T)v \in D(R^{-1}).$$

Remark 2.2. From the proof of Theorem 2.8 we see that if (2.10) holds, then

$$\|y(\epsilon, T) - \hat{y}\|_{L^2(\Omega)} \leq C\epsilon.$$

It is easy to show that (2.10) is actually a necessary condition for the above inequality to hold. In fact, assume that the above inequality holds. Then by the proof of Theorem 2.8 we have that

$$\|y(\epsilon, T) - \hat{y}\|^2_{L^2(\Omega)} = \sum_{j=1}^{\infty} \frac{\epsilon^2 y_j^2}{(\epsilon + \lambda_j)^2} \leq C\epsilon^2.$$

Thus

$$\sum_{j=1}^{\infty} \frac{y_j^2}{(\epsilon + \lambda_j)^2} \le C.$$

Letting $\epsilon \to 0$, we obtain (2.10).

Remark 2.3. In [30], Russell gave a condition on exact controllability of system (2.1) using the harmonic analysis method. The condition is

$$\hat{y} - E(T)v \in e^{C\sqrt{A}}$$

where $C > 0$ is a constant and A is the elliptic operator that appears in (2.1). For $d = 1$, this condition implies that $\hat{y} - E(T)v$ is at least a C^∞ function. Though it is difficult to check if our condition is weaker or stronger, it does seem that a function does not have to be infinitely differentiable to satisfy our condition.

2.4 The Convergence of the Optimal Control to the Exact Control

Assume that the system is exactly controllable for a given function $\hat{y} \in L^2(\Omega)$. Then by Definition 1.1 there exists $(y, u) = (y(t), u(t)) \in H^{1,2}(Q) \times L^2(\Sigma)$ such that (y, u) satisfies (2.1) and

$$y(T) = \hat{y}.$$

Let

$$\mathcal{U}_{ex} = \{u \in L^2(\Sigma); (y, u) \text{ is a solution of (2.1) and } y(T) = \hat{y}\}.$$

In other words, $\mathcal{U}_{ex}$ is a set consisting of all exact controls. We now consider the problem: find $\bar{u} \in \mathcal{U}_{ex}$ such that

$$\bar{J}(\bar{u}) = \min_{u \in \mathcal{U}_{ex}} \bar{J}(u) = \int_\Sigma u^2 dx dt. \tag{2.11}$$

The next lemma, stated as a theorem in [20], will be used in the proof of the existence and uniqueness of the solution for (2.11)

Lemma 2.9 *([20]) Assume that $\mathcal{U}$ is a Hilbert space and $\mathcal{U}_{ad} \subset \mathcal{U}$ is closed and convex. Let $\pi(u, v)$ be a continuous bilinear form on $\mathcal{U}$ which satisfies*

$$\pi(v, v) \geq C\|v\|^2_{\mathcal{U}}, \quad \text{for } v \in \mathcal{U},\ C > 0.$$

Let also

$$J(u) = \pi(u, u).$$

Then there exists a unique element u in $\mathcal{U}_{ad}$ such that

$$J(u) = \inf_{v \in \mathcal{U}_{ad}} J(v).$$

□

The following lemma is about the existence and uniqueness of the solution for problem (2.11).

Lemma 2.10 *There exists a unique solution $u \in \mathcal{U}_{ex}$ of problem (2.11).*

Proof: Let $\mathcal{U} = L^2(\Sigma), \mathcal{U}_{ad} = \mathcal{U}_{ex}$. Define

$$\pi(u, v) = \int_\Sigma uv d\sigma.$$

It is clear that $\mathcal{U}_{ad}$ is a convex set and $\pi(u, v)$ is a continuous coercive bilinear form. To apply Lemma 2.9, we only need to verify that $\mathcal{U}_{ex}$ is a closed set. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence such that $u_n \in \mathcal{U}_{ex}$ and $u_n \to u$ in $L^2(\Sigma)$ as $n \to \infty$. By the definition of $\mathcal{U}_{ex}$ there exist $y_n \in W(0, T)$, $n = 1, 2, \ldots$, such that

$$\begin{aligned} (\frac{dy_n}{dt}, \phi) + B(y_n, \phi) &= < u_n, \phi >, \quad \text{for } \phi \in H^1(\Omega), \\ y_n(0) &= v, \\ y_n(T) &= \hat{y}. \end{aligned}$$

Let $y \in W(0,T)$ be the solution of the problem

$$\begin{aligned} \frac{\partial y}{\partial t} + Ay &= 0, \quad \text{on } Q, \\ \frac{\partial y}{\partial \nu} &= u, \quad \text{on } \Sigma, \\ y(0) &= v, \quad \text{on } \Omega \end{aligned}$$

and let $\delta_n = y_n - y$. Then we have that

$$\begin{aligned} \frac{\partial \delta_n}{\partial t} + A\delta_n &= 0, \quad \text{on } Q, \\ \frac{\partial \delta_n}{\partial \nu} &= u_n - u, \quad \text{on } \Sigma, \\ \delta_n(0) &= 0, \quad \text{on } \Omega. \end{aligned}$$

Thus

$$(\frac{d\delta_n}{dt}, \delta_n) + B(\delta_n, \delta_n) = < u - u_n, \delta_n > .$$

Integrating the above equality from 0 to T, we obtain

$$\frac{1}{2}||y_n(T) - y(T)||^2 + \int_0^T B(\delta_n, \delta_n) = \int_0^T < u_n - u, \delta_n > dt.$$

Recall that there exists a constant $c_2 > 0$ such that

$$B(w,w) \geq c_2^2 ||w||^2_{H^1(\Omega)} \quad \text{for } w \in H^1(\Omega).$$

Also by the trace theorem there exists a constant $c > 0$ such that

$$||w||_{L^2(\partial\Omega)} \leq c||w||_{H^1(\Omega)} \quad \text{for } w \in H^1(\Omega).$$

Using the Schwartz inequality we have that

$$\frac{1}{2}||y_n(T) - y(T)||^2 + \int_0^T B(\delta_n, \delta_n) \quad \leq \quad \int_0^T ||u_n - u||_{L^2(\partial\Omega)} ||\delta_n||_{L^2(\partial\Omega)} dt.$$

$$\begin{aligned} &\leq c\int_0^T \|u_n - u\|_{L^2(\partial\Omega)}\|\delta_n\|_{H^1(\Omega)}dt \\ &\leq \frac{c}{c_2}\int_0^T \|u_n - u\|_{L^2(\partial\Omega)}\sqrt{B(\delta_n,\delta_n)}dt \\ &\leq \frac{2c^2}{c_2^2}\|u_n - u\|^2_{L^2(\Sigma)} + \frac{1}{2}\int_0^T B(\delta_n,\delta_n)dt. \end{aligned}$$

Thus we have that

$$\frac{1}{2}\|y_n(T) - y(T)\|^2 + \frac{1}{2}\int_0^T B(\delta_n,\delta_n) = \frac{2c^2}{c_2^2}\|u_n - u\|_{L^2(\Sigma)} \to 0.$$

Therefore

$$\lim_{n\to\infty} y_n(T) = y(T).$$

But $y_n(T) = \hat{y}$, thus $y(T) = \hat{y}$. This implies that $u \in \mathcal{U}_{ex}$. The proof is complete. □

As in §2.2, let $(y(\epsilon), u(\epsilon)) = (y(\epsilon,t), u(\epsilon,t))$ be the solution of the optimal control problem (2.2). Assume that system (2.1) is exactly controllable. We then have that

$$\begin{aligned} \|y(\epsilon,T) - \hat{y}\|^2 + \epsilon\|u(\epsilon)\|^2_{L^2(\Sigma)} &\leq \|y(T) - \hat{y}\|^2 + \epsilon\|u\|^2_{L^2(\Sigma)} \\ &= \epsilon\|u\|^2_{L^2(\Sigma)}. \end{aligned} \tag{2.12}$$

Therefore

$$\lim_{\epsilon\to 0} \|y(\epsilon,T) - \hat{y}\| = 0,$$

and, for each $\epsilon > 0$,

$$\|u(\epsilon)\|_{L^2(\Sigma)} \leq \|u\|_{L^2(\Sigma)}.$$

As a consequence, $\|u(\epsilon)\|_{L^2(\Sigma)}$ is uniformly bounded. We are now in a position to prove the main theorem of this section.

Theorem 2.11 *Assume that (2.1) is exactly controllable for $\hat{y} \in L^2(\Omega)$. Let $(y(\epsilon), u(\epsilon)) = (y(\epsilon,t), u(\epsilon,t))$ be the solution of the optimal control problem (2.2). Then*

$$\lim_{\epsilon\to 0} y(\epsilon) = y, \quad \text{in } L^2(0,T;H^1(\Omega))$$

and

$$\lim_{\epsilon \to 0} u(\epsilon) = u, \quad \text{in } L^2(\Sigma)$$

where (y, u) is the solution of problem (2.1) and u is the solution of problem (2.11).

Proof: Let $\{\epsilon_n\}_{n=1}^{\infty}$ be a sequence such that $\epsilon_n \to 0$ as $n \to \infty$. Since $\|u(\epsilon_n)\|_{L^2(\Sigma)})$ is uniformly bounded, there exists a subsequence of $\{\epsilon_n\}$, still denoted as $\{\epsilon_n\}$, such that

$$\lim_{n \to \infty} u(\epsilon_n) = u \quad \text{weakly in } L^2(\Sigma).$$

First we prove that u is an exact control. Since $(y(\epsilon), u(\epsilon))$ is a solution of problem (2.2), we have that

$$\begin{aligned} (\tfrac{dy(\epsilon_n)}{dt}, \phi) + B(y(\epsilon_n), \phi) &= < u(\epsilon_n), \phi >, \quad \text{for } \phi \in H^1(\Omega), \\ y(\epsilon_n, 0) &= v. \end{aligned} \tag{2.13}$$

Choosing $\phi = y(\epsilon_n)$, we have that

$$(\frac{dy(\epsilon_n)}{dt}, y(\epsilon_n)) + B(y(\epsilon_n), y(\epsilon_n)) =< u(\epsilon_n), y(\epsilon_n) >,$$

that is

$$\frac{1}{2}\frac{d}{dt}\|y(\epsilon_n)\|^2 + B(y(\epsilon_n), y(\epsilon_n)) =< u(\epsilon_n), y(\epsilon_n) > .$$

Integrating the above equality from 0 to T and following the same argument as we have used in the proof of Lemma 2.10, we obtain

$$\frac{1}{2}\|\hat{y}\|^2 + \int_0^T B(y(\epsilon_n), y(\epsilon_n))dt \le C_0\|u(\epsilon_n)\|^2_{L^2(\Sigma)} + \|v\|^2 \le C + \|v\|^2$$

where C_0 and C are constants. Using the property of B we obtain

$$\|\hat{y}\|^2 + c\int_0^T \|y(\epsilon_n)\|^2_{H^1(\Omega)} dt \le C + \|v\|^2.$$

Therefore $y(\epsilon_n)$ ranges in a bounded set in $L^2(0,T;H^1(\Omega))$ and we may extract a subsequence of $\{y(\epsilon_n)\}$, still denoted as $\{y(\epsilon_n)\}$, such that

$$y(\epsilon_n) \to y \quad \text{weakly in } L^2(0,T;H^1(\Omega)).$$

Multiply both sides of (2.13) by $\psi(t)$ where

$$\psi(t) \in C^1[0,T], \quad \psi(T) = 0,$$

and integrate over $[0,T]$. Setting $\xi(t) = \psi(t)\phi$, we have that

$$\int_0^T [-(y(\epsilon_n), \frac{d\xi(t)}{dt}) + B(y(\epsilon_n), \xi(t))]dt = \int_0^T < \xi(t), u(\epsilon_n) > dt + (v, \phi)\psi(0). \tag{2.14}$$

By virtue of the fact that $y(\epsilon_n) \to y$ weakly in $L^2(0,T;H^1(\Omega))$ and $u(\epsilon_n) \to u$ weakly in $L^2(\Sigma)$, we can pass to the limit in (2.14) and obtain

$$\int_0^T [-(y(t), \frac{d\xi(t)}{dt}) + B(y(t), \xi(t))]dt = \int_0^T < \xi(t), u(t) > dt + (v, \phi)\psi(0).$$

But the above equality is true for any $\psi(t) \in \mathcal{D}(0,T)$. Therefore we have that

$$(\frac{dy}{dt}, \phi) + B(y, \phi) =< u, \phi >, \quad \text{for } \phi \in H^1(\Omega). \tag{2.15}$$

In particular we have that

$$(\frac{dy}{dt}, \phi) + B(y, \phi) = 0, \quad \text{for } \phi \in \mathcal{D}(\Omega).$$

Thus

$$\frac{dy}{dt} + Ay = 0.$$

By virtue of the fact that A is an operator from $L^2(0,T;H^1(\Omega))$ to $L^2(0,T;H^{-1}(\Omega))$, $\frac{dy}{dt} = Ay \in L^2(0,T;H^{-1}(\Omega))$ ([20]). Thus $y \in W(0,T)$. We can easily derive that $\frac{dy}{d\nu}|_\Sigma = u$. Taking into account (2.14) and (2.15) we obtain

$$(y(0), \phi)\psi(0) = (v, \phi)\psi(0), \quad \forall \phi \in H^1(\Omega), w \in C^1[0,T], \; w(T) = 0.$$

Thus $y(0) = v$. Let $\delta_n = y - y(\epsilon_n)$. Then

$$(\frac{d\delta_n}{dt}, \phi) + B(\delta_n, \phi) = 0, \quad \text{for } \phi \in \mathcal{D}(\Omega).$$

Integrating this equality from 0 to T, we obtain

$$(\delta_n(T), \phi) + \int_0^T B(\delta_n, \phi)dt = 0.$$

Since $\delta_n \to 0$ weakly in $L^2(0, T; H^1(\Omega))$, we have that

$$\lim_{n\to\infty} y(\epsilon_n) \to y(T) \quad \text{weakly in } L^2(\Omega).$$

But from (2.13) we have that $y(\epsilon_n) \to \hat{y}$ strongly in $L^2(\Omega)$. Thus $y(T) = \hat{y}$. This proves that u is an exact control.

We now return to the proof of the convergence of $u(\epsilon)$ to u in $L^2(\Sigma)$. By the fact that the norm function in a Hilbert space is weakly lower semi-continuous ([22]) and the inequality (2.12), we have that

$$||u|| \leq \liminf_{n\to\infty} ||u(\epsilon_n)|| \leq \limsup_{n\to\infty} ||u(\epsilon_n)|| \leq ||u||.$$

Thus

$$\lim_{n\to\infty} ||u(\epsilon_n)|| \to ||u|| \quad \text{in } L^2(\Sigma).$$

This, together with the fact that $u(\epsilon_n)$ weakly converges to u implies that

$$\lim_{n\to\infty} u(\epsilon_n) = u \quad \text{in } L^2(\Sigma).$$

From the inequality (2.12) we deduce that u is the solution of problem (2.11). Since u is unique and $\{\epsilon_n\}$ is arbitrary, we conclude that

$$\lim_{\epsilon\to 0} u(\epsilon) \to u \text{ in } L^2(\Sigma).$$

Finally we turn to the proof of the convergence of $y(\epsilon)$ in $L^2(0,T;H^1(\Omega))$. Let $\delta_\epsilon = y(\epsilon) - y$. We have that

$$\frac{1}{2}\frac{d}{dt}\|\delta_\epsilon\|^2 + B(\delta_\epsilon, \delta_\epsilon) =< u - u(\epsilon), \delta_\epsilon > .$$

Integrating this equality and using the same argument as we have used in the proof of Lemma 2.10, we obtain

$$\int_0^T B(\delta_\epsilon, \delta_\epsilon) \leq C\|u - u(\epsilon)\|^2_{L^2(\Sigma)},$$

where C is a constant. Thus

$$\|\delta_\epsilon\|_{L^2(0,T;H^1(\Omega))} \leq C \int_0^T B(\delta_\epsilon, \delta_\epsilon) = C\|u - u(\epsilon)\|_{L^2(\Sigma)} \to 0.$$

The proof is complete. □

Remark 2.4. As we have pointed out in §1.2, if $\hat{y} = 0$, then (2.1) is always exactly controllable. Thus for $\hat{y} = 0$, the optimal control $u(\epsilon)$ always converges to the exact control with minimum L^2 norm.

Remark 2.5. If there exists a sequence $\{\epsilon_k\}_{k=1}^{\infty}$ such that $\epsilon_k \to 0$ as $k \to \infty$ and $\{u(\epsilon_k)\}$ is bounded, then we may use the technique in the proof Theorem 2.11 to prove that system (2.11) is exactly controllable. This implies if the system is not exactly controllable, then

$$\lim_{\epsilon \to 0} \|u(\epsilon)\|_{L^2(\Sigma)} = \infty.$$

2.5 Numerical Computation of Exact Controls

If the system (2.1) is exactly controllable, then thanks to Theorem 2.11 we may find approximate solutions for exact controls by solving the optimal control problem (2.2) numerically for small ϵ.

2.5.1. Galerkin approximation of the state equations. To start with, we consider, in the notation of §2.1, the state y given by

$$\begin{aligned} (\tfrac{dy}{dt}, \phi) + B(y, \phi) &= < u, \phi >, \quad \text{for } \phi \in H^1(\Omega), \\ y(0, \cdot) &= v. \end{aligned} \tag{2.16}$$

Assume that we are given a family of subspaces H^h of $H^1(\Omega)$. A Galerkin semi-discretization of (2.16) is as follows: seek $y_h(t) \in H^h$ for each $0 < t \leq T$ such that

$$\begin{aligned} (\tfrac{dy_h}{dt}, \phi_h) + B(y_h, \phi_h) &= < u, \phi_h >, \quad \text{for } \phi_h \in H^1(\Omega), \\ y_h(0, \cdot) &= v_h \in H^h. \end{aligned} \tag{2.17}$$

(2.17) is a system of ordinary differential equations.

The next step is to discretize in t. Let us denote

$$\begin{aligned} y_h^n &= \text{approximation of } y_h \text{ at } n\Delta t, \\ u^n &= u(n\Delta t) \end{aligned}$$

where Δt is the step size in t direction. There are various discretization schemes available. From a practical point of view the backward Euler and Crank-Nicolson schemes are two of the most popular ones. For the backward Euler scheme, one approximates (2.17) by

$$\begin{aligned} (y_h^n, \phi_h) + \Delta t B(y_h^n, \phi_h) &= (y_h^{n-1}, \phi_h) + \Delta t < u^n, \phi_h >, \quad \text{for } \phi_h \in H^h, \\ y_h^0 &= v_h \end{aligned} \tag{2.18}$$

for $n = 1, 2, ..., N$ where $N\Delta t = T$; For the Crank-Nicolson scheme, one approximates (2.17) by

$$\begin{aligned} (y_h^n, \phi_h) + \tfrac{1}{2}\Delta t B(y_h^n, \phi_h) &= (y_h^{n-1}, \phi_h) - \tfrac{1}{2}\Delta t B(y_h^{n-1}, \phi_h) \\ &\quad + \Delta t < u^{n+1/2}, \phi_h >, \text{ for } \phi_h \in H^h, \\ y_h^0 &= v_h \end{aligned} \tag{2.19}$$

for $n = 1, 2, ..., N$ where $N\Delta t = T$ and $u^{n+1/2} = u((n + 1/2)\Delta t)$. We refer to [34] for a discussion of the error estimates of the above scheme. In our numerical experiments we shall use the Crank-Nicolson scheme.

2.5.2. Approximation of the optimal control problem. We consider the optimal control problem introduced in §2.1: Find $u \in L^2(\Sigma)$ such that

$$J(u) = \frac{1}{2}\int_\Omega (y(T) - \hat{y})^2 dx + \frac{\epsilon}{2}\int_\Sigma u^2(t,x)dxdt \tag{2.20}$$

is minimized.

We now introduce a family U_r of the subspaces of $L^2(\Sigma)$

$$U_r \subset L^2(\Sigma)$$

where r denotes a discretization parameter. Of course we assume that U_r approximates $L^2(\Sigma)$. The approximate state $\{y^n_{h,r}\}$ is then obtained by replacing $u^{n+1/2}$ in (2.19) by an approximation u^n_r in U_r.

The approximate cost function is then defined by

$$J^{\Delta t}_{h,r}(u_r) = \frac{1}{2}\int_\Omega (y^N_{h,r}(x) - \hat{y})^2 dx + \frac{\epsilon}{2}\Delta t \sum_{n=1}^{N} \|u^n_r\|^2_{L^2(\partial\Omega)}. \tag{2.21}$$

The approximate control problem consists in minimizing $J^{\Delta t}_{h,r}(u_r)$ over U_r. The above method is also called output least squares method. For convergence issues of the method, we refer to [2].

2.5.3. Numerical experiments: a finite element algorithm.

Example 2.1.

We consider the heat equation in Ω which is the interval $(0,\pi) \subset R^1$. We take

$$v = \sin(x), \quad \hat{y} = e^{-1}\sin(x), \quad T = 1.$$

The weak formulation of (2.1) is

$$(\frac{dy}{dt}, \phi) + (y_x, \phi_x) = y_x(t,\pi)\phi(1) - y_x(t,0)\phi(0) \quad \text{for } \phi \in H^1(\Omega),$$

$$y(0,x) = \sin(x)$$

Let $u_0(t) = -y_x(t,0)$, $u_1(t) = y_x(t,\pi)$. Then $u = (u_0, u_1)$ is our control function. So $L^2(\Sigma)$ here is equivalent to $L^2(0,1) \times L^2(0,1)$. The system is exactly controllable since $y(t,x) = e^{-t}\sin(x)$ is an exact solution of the parabolic equation problem. The corresponding exact control is $u = (u_0, u_1) = (-y_x(t,0), y_x(t,\pi)) = (-e^{-t}, -e^{-t})$. Define H^h as

$$H^h = \{v \in C[0,\pi]; v|_{[x_i,x_{i+1}]} \in \mathcal{P}_1, h = x_{i+1} - x_i, x_0 = 0, x_N = \pi\},$$

that is, H^h is the continuous piecewise linear function space on a uniform partition of $(0,\pi)$. Also define $U_r = U_h \subset R^N \times R^N$ as

$$U_h = \{u_h = (u_h^0, u_h^1); u_h^0, u_h^1|_{[t_i,t_{i+1}]} \in \mathcal{P}_0, h = t_{i+1} - t_i, t_0 = 0, t_N = 1\}.$$

Here we have chosen Δt, h, r such that $r = h = \Delta t = \frac{1}{N}$. For the sake of notational convenience, we denote u_h^0, u_h^1 by

$$\begin{aligned} u_h^0 &= (u_1^0, u_2^0, ..., u_N^0), \\ u_h^1 &= (u_1^1, u_2^1, ..., u_N^1). \end{aligned}$$

Let $\{\phi_j\}_{j=1}^N$ be the hat function basis in H^h. Let also

$$C = (\phi_i, \phi_j), \quad D = (\nabla\phi_i, \nabla\phi_j).$$

C and D are the so called mass matrix and stiffness matrix, respectively. Now we can write the Crank-Nicolson scheme (2.19) in the matrix form

$$Cy_h^n + \frac{1}{2}\Delta t D y_h^n = \frac{1}{2}Cy_h^{n-1} - \frac{1}{2}\Delta t D y_h^{n-1} + g^n, \quad n = 1, \cdots N. \tag{2.22}$$

Notice that

$$\phi_i(0) = \begin{cases} 1, & i = 1, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\phi_i(1) = \begin{cases} 1, & \text{if } i = N, \\ 0, & \text{otherwise.} \end{cases}$$

Thus g^n is given by

$$g_i^n = \begin{cases} u_n^0, & \text{if } i = 1, \\ u_n^1 & \text{if } i = N, \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, \cdots, n$. Let y_I be the interpolant of $\hat{y} = e^{-1}\sin(x)$ in H^h,

$$y_I = \sum_{i=1}^{N} y_i \phi_i,$$

where $y_i = e^{-1}\sin(x_i)$. Assume that the solution y_h^n of (2.22) is given by

$$y_h^n = \sum_{i=1}^{N} f_i \phi_i.$$

The approximate cost function is then given by

$$\begin{aligned} J_h^{\Delta t}(u_h) = J_h(u_h^0, u_h^1) &= \frac{1}{2}\int_\Omega (y_h^N(x) - y_I(x))^2 dx + \frac{\epsilon}{2}\Delta t \sum_{i=1}^{N}((u_i^0)^2 + (u_i^1)^2) \\ &= \frac{1}{2}\int_\Omega (\sum_{i=1}^{N}(f_i - y_i)\phi_i)^2 dx + \frac{\epsilon}{2}\Delta t \sum_{i=1}^{N}((u_i^0)^2 + (u_i^1)^2) \\ &= (D(\mathbf{f} - \mathbf{y}, \mathbf{f} - \mathbf{y}) + \frac{\epsilon}{2}\Delta t \sum_{i=1}^{N}((u_i^0)^2 + (u_i^1)^2) \end{aligned}$$

where

$$\begin{aligned} \mathbf{f} &= (f_1, f_2, \ldots f_N)^T, \\ \mathbf{y} &= (y_1, y_2, \ldots y_N)^T. \end{aligned}$$

So essentially what we need to do is to find the minimizer of a quadratic function defined in $R^N \times R^N$. We accomplish this part of the computation using an optimization code called "optimization 611" created by Gay ([13]).

We report the numerical results for cases N=10, 20, 30, 40, 50 and ϵ=10^{-1}, 10^{-2}, 10^{-3}, 10^{-4} and 10^{-5}. The L^2 errors of $y_h(T)$ from $\hat{y}$ are shown in Table 2.1. In Figure 2.1 and 2.2 we plot the the approximate controls u_h^0.

Table 2.1: Convergence of y_h^N to y_I

h	ϵ	$\|\|y_h^N - y_I\|\|$	h	ϵ	$\|\|y_h^N - y_I\|\|$
0.10E+00	0.10E+00	0.10E+00	0.33E-01	0.10E+00	0.10E+00
0.10E+00	0.10E-01	0.15E-01	0.33E-01	0.10E-01	0.15E-01
0.10E+00	0.10E-02	0.19E-02	0.33E-01	0.10E-02	0.21E-02
0.10E+00	0.10E-03	0.27E-03	0.33E-01	0.10E-03	0.33E-03
0.50E-01	0.10E+00	0.10E+00	0.25E-01	0.10E+00	0.10E+00
0.50E-01	0.10E-01	0.15E-01	0.25E-01	0.10E-01	0.15E-01
0.50E-01	0.10E-02	0.21E-02	0.25E-01	0.10E-02	0.21E-02
0.50E-01	0.10E-03	0.32E-03	0.25E-01	0.10E-03	0.34E-03

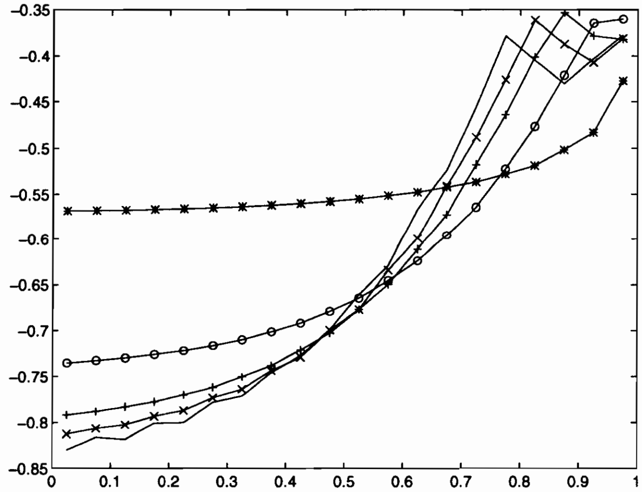

Figure 2.1: The graph of u_h^0 for N = 20, **: $\epsilon = 10^{-1}$, oo: $\epsilon = 10^{-2}$, ++: $\epsilon = 10^{-3}$, xx: $\epsilon = 10^{-4}$, —: $\epsilon = 10^{-5}$.

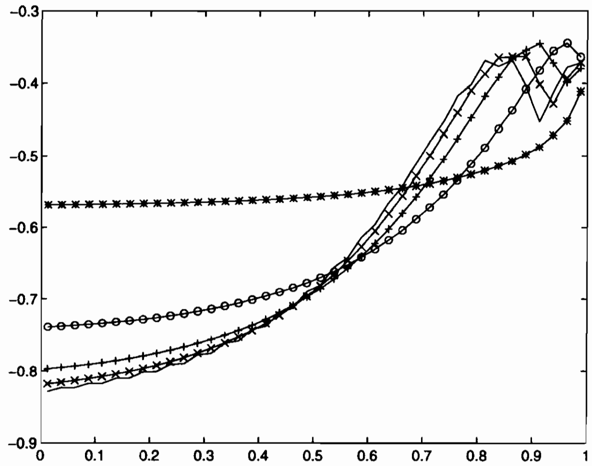

Figure 2.2: The graph of u_h^0 for $N = 40$, **: $\epsilon = 10^{-1}$, oo: $\epsilon = 10^{-2}$, ++: $\epsilon = 10^{-3}$, xx: $\epsilon = 10^{-4}$, —: $\epsilon = 10^{-5}$.

2.5.4. Numerical experiments: regularization method. Table 2.1 shows the convergence of $y_h(\epsilon, T)$ to y_I as $\epsilon \to 0$. But we find out immediately from Figure 2.1 and Figure 2.2 that the approximate controls do not converge, which is inconsistent with our theoretical analysis in §2.4. We believe that numerical instability of the optimization problem for small ϵ is the cause for this convergence failure.

One remedy is to add a regularization term in the optimization functional. This treatment is also called the Tikhonov regularization method ([36]). More specifically, we consider the following modified optimal control problem

$$J(u) = \frac{1}{2}\int_\Omega (y(T) - \hat{y})^2 dx + \frac{\epsilon}{2}\int_\Sigma u^2(t,x)dxdt + \frac{\delta}{2}\|\frac{\partial u}{\partial t}\|^2_{L^2(\Sigma)}.$$

The corresponding discretization is given by

$$\begin{aligned}
J_h^{\Delta t}(u_r) &= B(\mathbf{f}-\mathbf{v},\mathbf{f}-\mathbf{v}) + \tfrac{\epsilon}{2}\Delta t \textstyle\sum_{i=1}^{N}((u_i^0)^2 + (u_i^1)^2) \\
&+ \delta\Delta t \textstyle\sum_{i=2}^{N}((\frac{u_i^0-u_{i-1}^0}{\Delta t})^2 + (\frac{u_i^1-u_{i-1}^1}{\Delta t})^2) \\
&+ 2\delta\Delta t((\tfrac{u_1^0-1}{\Delta t})^2 + (\tfrac{u_N^0-e^{-1}}{\Delta t})^2 + (\tfrac{u_1^1+1}{\Delta t})^2 + (\tfrac{u_N^1+e^{-1}}{\Delta t})^2 \\
&= B(\mathbf{f}-\mathbf{v},\mathbf{f}-\mathbf{v}) + + \tfrac{\epsilon}{2}\Delta t \textstyle\sum_{i=1}^{N}((u_i^0)^2 + (u_i^1)^2) \\
&+ \delta \textstyle\sum_{i=2}^{N}((u_i^0-u_{i-1}^0)^2 + (u_i^1-u_{i-1}^1)^2))/\Delta t \\
&+ 2\delta((u_1^0-1)^2 + (u_N^0-e^{-1})^2 + (u_1^1+1)^2 + (u_N^1+e^{-1})^2)/\Delta t.
\end{aligned} \tag{2.24}$$

Example 2.2.

In this example, all the data are the same as in Example 2.1; but we use (2.24) as the approximate optimization functional. We report numerical results for cases $N = 10,\ 20,\ 30,\ 40$ and $(\epsilon,\delta) = (10^{-1},10^{-2}),\ (10^{-2},10^{-3}),\ (10^{-3},10^{-4})$ and $(10^{-4},10^{-5})$. The L^2 errors of $y_h(T)$ from $\hat{y}$ are listed in Table 2.2. In Figure 2.3 we plot the approximate controls u_h^0 for $N = 20$ and the function $-e^{-t}$. As we can see from Figure 2.3, the Tikhonov regularization method does give us convergent approximate controls.

Remark 2.6. From Table 2.2 we can see that

$$\|y(\epsilon,T) - \hat{y}\| \le \epsilon.$$

By Remark 2.2, the system considered in this example satisfies the condition of Theorem 2.8.

Remark 2.7. It is interesting to notice from Figure 2.3 that the approximate controls actually converge to $u(t) = (e^{-t}, -e^{-t})$, which is the the Neumann boundary value of solution $y(t,x) = e^{-t}\sin(x)$ of (2.1). By Theorem 2.11, $u = (-e^{-t}, e^{-t})$ has minimum $L^2(\Sigma)$ norm among all the exact controls.

Table 2.2: Convergence of y_h^N to y_I

h	$\epsilon(\delta = \frac{\epsilon}{10})$	$\|\|y_h^N - y_I\|\|$
	0.10000E+00	0.69424E-01
	0.10000E-01	0.97533E-02
0.10000E+00	0.10000E-02	0.11662E-02
	0.10000E-03	0.12403E-03
	0.10000E-04	0.16334E-04
	0.10000E+00	0.70642E-01
	0.10000E-01	0.10348E-01
0.33333E-01	0.10000E-02	0.13808E-02
	0.10000E-03	0.18258E-03
	0.10000E-04	0.27712E-04
	0.10000E+00	0.70721E-01
	0.10000E-01	0.10395E-01
0.20000E-01	0.10000E-02	0.13993E-02
	0.10000E-03	0.19024E-03
	0.10000E-04	0.30347E-04

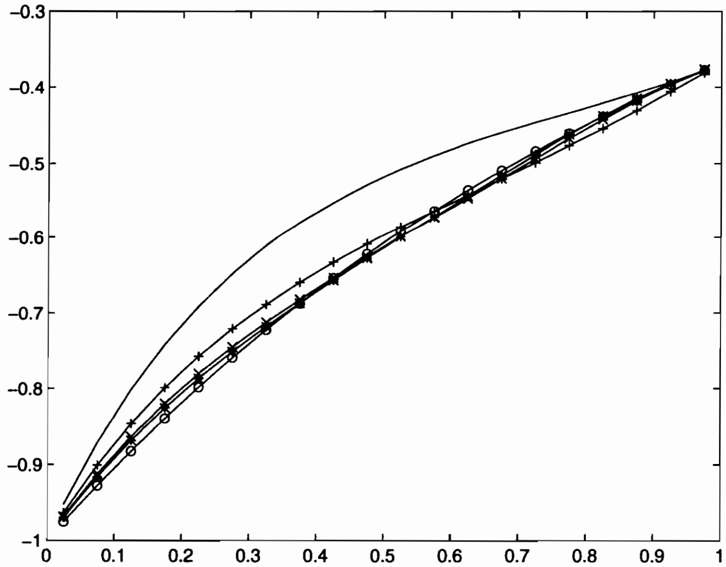

Figure 2.3: The graph of u_h^0 for $N = 20$, -: $(\epsilon, \delta) = (10^{-1}, 10^{-2})$, ++: $(\epsilon, \delta) = (10^{-2}, 10^{-3})$, xx: $(\epsilon, \delta) = (10^{-3}, 10^{-4})$, **: $(\epsilon, \delta) = (10^{-4}, 10^{-5})$. oo: $-e^{-t}$

Example 2.3.

In this example we consider the zero controllability problem for system (2.1). All the data are the same as in Example 2.1 except that $\hat{y} = 0$ and $\Omega = (0,1)$. As we have pointed out in Remark 2.4, the system is always controllable in this case. Numerical results are obtained for $N = 20$, $(\epsilon, \delta) = (10^{-3}, 10^{-4}), (10^{-4}, 10^{-5})$. The L^2 norms of $y_h(T)$ are listed in Table 2.3. In Figure 2.4 we plot the the approximate controls u_0^h for $N = 20$, $(\epsilon, \delta) = (10^{-3}, 10^{-4}), (10^{-4}, 10^{-5})$. In Figure 2.4 we plot the graph of $y_h(t, x)$ for $(\epsilon, \delta) = (10^{-4}, 10^{-5})$.

Table 2.3: The L^2 norms of y_h^N

h	ϵ	$\|y_h^N - y_I\|$
0.50E-01	0.10E-03	0.16E-02
0.50E-01	0.10E-04	0.84E-03

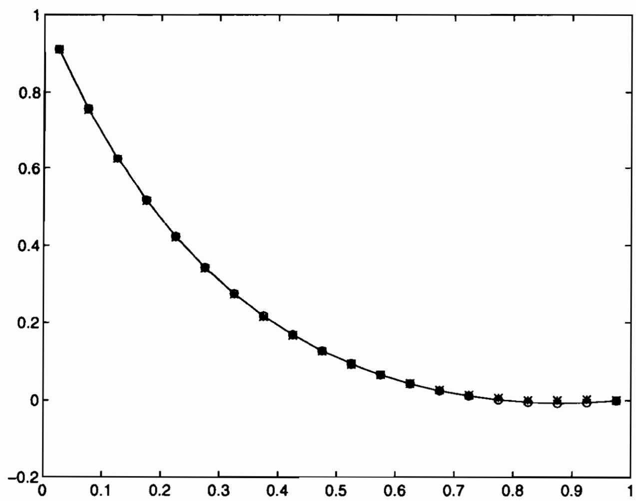

Figure 2.4: The graph of u_h^0 for $N = 20$, **: $(\epsilon, \delta) = (10^{-3}, 10^{-4})$, oo: $(\epsilon, \delta) = (10^{-4}, 10^{-5})$.

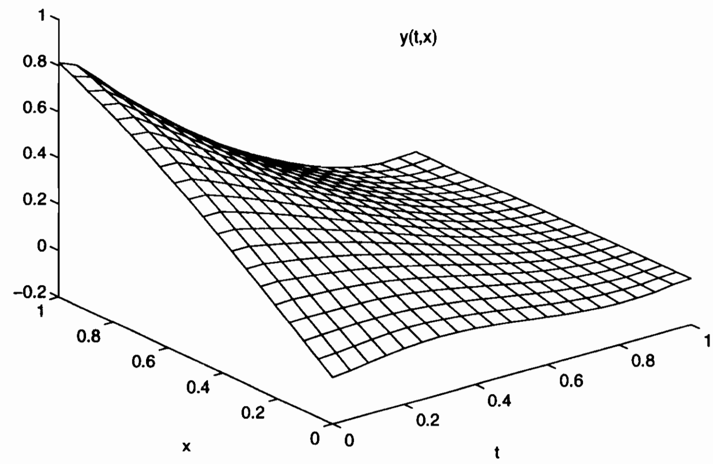

Figure 2.5: The graph of the approximate state $y_h(t,x)$ for $N = 20$ and $(\epsilon, \delta) = (10^{-5}, 10^{-6})$.

Example 2.4.

In this example we consider a system that is not exactly controllable. Here we choose $\hat{y}$ as

$$\hat{y}(x) = \begin{cases} x, & \text{if } 0 < x < 0.5, \\ 1 - x, & \text{if } 0.5 \leq x < 1. \end{cases}$$

According to Remark 2.5, if a system is not exactly controllable, then

$$\lim_{\epsilon \to 0} ||u(\epsilon)||_{L^2(\Sigma)} = \infty.$$

In the numerical computation we use the optimization functional (2.24). In Figure 2.5, we plot the graphs of the approximate controls for $N = 20$ and $(\epsilon, \delta) = (10^{-1}, 10^{-2})$, $(10^{-2}, 10^{-3}$, $(10^{-3}, 10^{-4})$, $(10^{-4}, 10^{-5})$. As we can see from the graphs, the behavior of the approximate controls does agree with our theoretical analysis and we conclude that the system is indeed not exactly controllable.

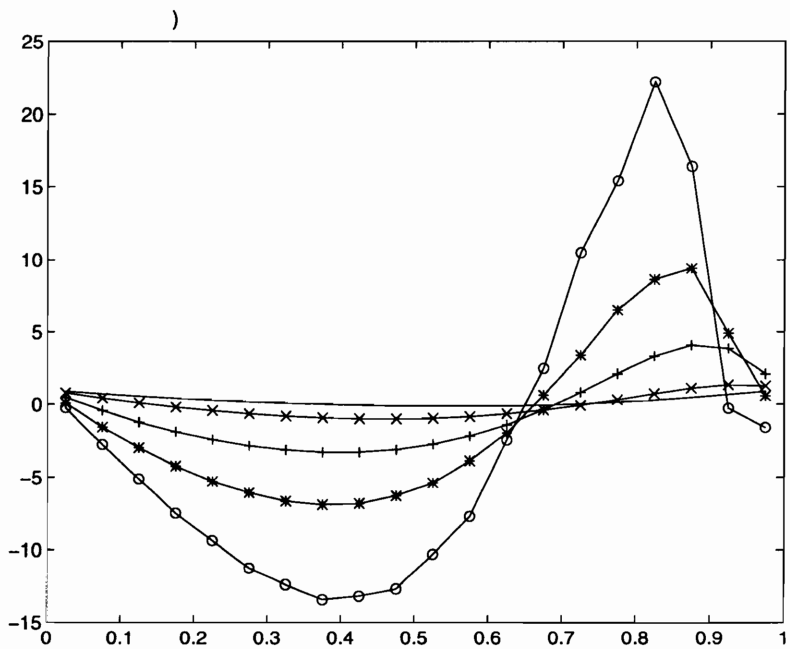

Figure 2.6: The graph of u_h^0 for$N$=20, –: $(\epsilon, \delta) = (10^{-1}, 10^{-2})$, xx: $(\epsilon, \delta) = (10^{-2}, 10^{-3})$, ++: $(\epsilon, \delta) = (10^{-3}, 10^{-4})$, **: $(\epsilon, \delta) = (10^{-4}, 10^{-5})$, oo: $(\epsilon, \delta) = (10^{-5}, 10^{-6})$.

Chapter 3

A Finite Element Method for Boundary Controllability Problems for Parabolic Equations

3.1 Statement of the Problem and Notation

We consider a control system where evolution in time is described by a function $y = y(t,x)$ defined in $0 \le t \le T$, $0 \le x \le 1$ that satisfies the parabolic equation

$$Ly(t,x) = (\partial_t - \partial_{xx} + a\partial_x)y(t,x) = f(t,x), \quad 0 < t < T,\ 0 < x < 1 \tag{3.1}$$

and the boundary conditions

$$y(t,0) = v_0(t), \quad y(t,1) = v_1(t), \quad 0 < t < T. \tag{3.2}$$

The real valued functions v_0 and v_1 are interpreted as boundary control functions. If we are given an initial state

$$y(0,x) = y_0(x) \in L^2[0,1] \tag{3.3}$$

then for f, v_0, v_1 and y_0 in appropriate function spaces the mixed initial-boundary value problem (3.1), (3.2), (3.3) has a unique solution $y(t,x)$ in $0 \le t \le T$, $0 \le x \le 1$ such that ([23])

$$\frac{\partial y}{\partial x}(t,\cdot) \quad , \frac{\partial^2 y}{\partial^2 x}(t,\cdot) \in L^2[0,1]$$

for $t > 0$. According to Definition 1.1, the exact zero boundary controllability problem for the system (3.1), (3.2) can be formulated as follows. Let an initial condition (3.3) and a terminal condition

$$y(T, x) = 0 \tag{3.4}$$

be given. We ask: do there exist controls v_0, v_1 in certain function spaces such that the solution of (3.1), (3.2), (3.3) also satisfies (3.4)?

As we have pointed out in both Chapter 1 and Chapter 2, the answer for the above question is positive. In this chapter we study a numerical method of solving this problem. The method is based on a constructive proof of the existence of exact controls by Fursikov and Immanulov [11]. In their approach the problem is reduced to solving the following fourth order hypoelliptic equation

$$L^* cLp(t, x) = f_0(t, x), \quad 0 < t < T, \ \ 0 < x < 1, \tag{3.5}$$

with boundary conditions

$$p(t, x)|_{x=0,1} = p_x(t, x)|_{x=0,1} = 0, \quad 0 < t < T, \tag{3.6}$$

$$cLp(t, x)|_{t=0} = cLp(t, x)|_{t=T} = 0, \quad 0 < x < 1 \tag{3.7}$$

where $c = c(t, x)$ is a weight function that will be specified in the following sections. Once p is solved from (3.5), (3.6), (3.7), the state y and the controls v_0 and v_1 are obtained through

$$\begin{aligned} y = & \quad y_1 + cL^* p \\ v_0 = y(\cdot, 0), & \quad v_1 = y(\cdot, 1) \end{aligned} \tag{3.8}$$

where y_1 is a function obtained by solving a heat equation problem.

We shall present a finite element method for obtaining numerical solutions for the problem (3.5)-(3.7) thus obtaining numerical approximations to the state y and controls v_0 and v_1 through (3.8). In §3.2 we study a simplified model in which we assume that the initial data y_0 vanishes in (3.3). The general case is discussed in §3.3.

We close this section by introducing some notation that will be used in the sequel. Let $\Omega = [0,1]$, $Q = [0,T] \times \Omega$. $H^s(\Omega)$, $H^s(Q)$ denote the Sobolev spaces of order s defined on Ω and Q, respectively. Define Sobolev spaces

$$L^2(0,T;H^s(\Omega)) = \{y \in L^2(Q); \int_0^T \|y\|_{H^s(\Omega)} dt < \infty\}$$

with the norm

$$\|y\|_{L^2(0,T;H^s(\Omega))} = \int_0^T \|y\|_{H^s(\Omega)} dt$$

and

$$H^m(0,T;L^2(\Omega)) = \{y \in L^2(Q); \partial_t y, ..., \partial_t^{(m)} y \in L^2(Q)\}$$

with the norm

$$\|y\|_{H^m(0,T;L^2(\Omega))} = \sum_{i=0}^{m} \|\partial_t^{(i)} y\|_{L^2(Q)}.$$

Also define

$$H^{m,s}(Q) = L^2(0,T;H^s(\Omega)) \cap H^m(0,T;L^2(\Omega))$$

which is a Hilbert space with the norm

$$\|y\|_{H^{m,s}(Q)} = \|y\|_{L^2(0,T;H^s(\Omega))} + \|y\|_{H^m(0,T;L^2(\Omega))}.$$

We refer to [23] for a discussion of the space $H^{m,s}(Q)$.

3.2 Homogeneous Initial Conditions

3.2.1. Reduction to A Hypoelliptic Equation Problem. In this section we assume that the initial data $y_0 = 0$. Thus the exact controllability problem is equivalent to the following problem: Solve for y from

$$Ly(t,x) = f(t,x), \quad (t,x) \in Q, \tag{3.9}$$

$$y(t,x)|_{t=0} = y(t,x)|_{t=T} = 0, \quad x \in \Omega. \tag{3.10}$$

The solution of problem (3.9)-(3.10) may not be unique. Thus we consider the following extremal problem

$$J(y) = \frac{1}{2}\int_Q y^2(t,x)dxdt \longrightarrow \inf \tag{3.11}$$

on the set of functions y satisfying (3.9), (3.10). Using the standard Lagrange multiplier argument, we can derive

Proposition 3.1 *([11]) the optimality system for problem (3.11) is given by*

$$L^*p(t,x) = y(t,x), \quad (t,x) \in Q, \tag{3.12}$$

$$p(t,x)|_{x=0,1} = p_x(t,x)|_{x=0,1} = 0, \quad 0 < t < T, \tag{3.13}$$

$$y \text{ satisfies } (3.9), (3.10). \tag{3.14}$$

□

Applying operator L on (3.12) and taking into account (3.13)-(3.14), we obtain

$$LL^*p(t,x) = f(t,x), \quad (t,x) \in Q, \tag{3.15}$$

$$p(t,x)|_{x=0,1} = p_x(t,x)|_{x=0,1} = 0, \quad 0 < t < T, \tag{3.16}$$

$$L^*p(t,x)|_{t=0} = L^*p(t,x)|_{t=T} = 0, \quad x \in \Omega. \tag{3.17}$$

Note that (3.15) is a special case of (3.5) with $c = 1$.

Proposition 3.2 *p is a solution of the optimality system (3.12)-(3.14) if and only if p is a solution of problem (3.15)-(2.17).*

Proof: The proof is straightforward and we omit it. □

The operator LL^* is a fourth order differential operator. It belongs to a class of the so called hypoelliptic differential operators. We refer to [16] for a discussion of such operators.

Suppose that the solution of problem (3.15)-(3.17) exists. Then we can find the exact boundary control through the following procedure.

1. Solve for p from (3.15)-(3.17);

2. Find the state: $y = L^*p$;

3. Find the exact zero boundary control: $v_0(t) = y(t,0)$, $v_1(t) = y(t,1)$.

3.2.2 Existence of exact boundary controls. Define

$$\Phi = \{\phi \in L^2(Q); \|L^*\phi\| < \infty, \phi|_{x=0,1} = \phi_x|_{x=0,1} = 0\}. \tag{3.18}$$

Here and in the following $\|\cdot\|$ denotes the usual norm in $L^2(Q)$.

Proposition 3.3 *Φ is a Hilbert space with the inner product defined by*

$$< p, q >= (L^*p, L^*q). \tag{3.19}$$

Proof: We only need to prove that (3.19) defines an inner product in Φ. Assume that $< p,p >= 0$. Then by an a priori estimate in [11], we have that

$$\begin{aligned}&\int_Q e^{-\frac{k}{(T-t)^2}}((\partial_t p)^2 + (\partial_{xx}p)^2 + (\partial_x p)^2 + p^2)dxdt \\ &+ \int_\Omega (\partial_x p(\tau,x)^2 + p^2(\tau,x))e^{-\frac{k}{T-\tau)^2}}dx \le C\int_Q (L^*p)^2 dxdt = 0, \quad 0 \le \tau < T.\end{aligned} \tag{3.20}$$

Hence $p = 0$. Obviously $< \cdot,\cdot >$ satisfies the other properties of an inner product. □

The weak formulation of problem (3.15)-(3.17) is defined as follows: seek $p \in \Phi$ such that

$$(L^*p, L^*\phi) = (f,\phi), \quad \forall \phi \in \Phi. \tag{3.21}$$

p is called a generalized solution of problem (3.15), (3.16), (3.17) if it satisfies (3.21). The following theorem follows from Lax-Milgram theorem.

Theorem 3.4 *([11]) Let $f \in \Phi^*$. Then there exists a unique generalized solution $p \in \Phi$ of problem (3.15), (3.16), (3.17). Also $p \in H^{2,4}(\bar{Q})$ for any smooth subdomain $\bar{Q}$ of Q.* $\square$

Using (3.8) and the trace theorem, we have

Theorem 3.5 *There exist $v_0, v_1 \in L^2(0,T)$, $y \in H^{1,2}(Q)$ that satisfy (3.1)-(3.4). Aslo if $p \in H^{3,6}(Q)$, then $y \in H^{2,4}(Q)$ and $v_0, v_1 \in H^{\frac{3}{2}}[0,T]$.* $\square$

From (3.20) we can see that if $f \in L^2(Q)$ vanishes in a neighborhood of $t = T$, $T_0 < t < T$, then $f \in \Phi^*$. In fact, if f satisfies such condition then we have, for $p \in \Phi$, that

$$\begin{aligned}
|\int_Q fpdxdt| &= |\int_{[0,T_0]} dt \int_\Omega fpdx| \\
&\leq C\|f\|_{L^2(Q)}(\int_{[0,T_0]}\int_\Omega e^{-\frac{k}{(T-t)}}p^2dxdt)^{\frac{1}{2}} \\
&\leq C\|f\|_{L^2(Q)}(\int_Q e^{-\frac{k}{(T-t)}}p^2dxdt)^{\frac{1}{2}} \\
&\leq C\|f\|_{L^2(Q)}\|L^*p\|_{L^2(Q)} = C\|f\|_{L^2(Q)}\|p\|_\Phi
\end{aligned}$$

Thus $f \in \Phi^*$.

3.3.3. Finite element approximations. Let Φ^h $(h > 0)$ be finite dimensional subspaces of Φ. Finite element approximations of (3.21) are defined as follows: seek $p_h \in \Phi^h$ such that

$$(L^*p_h, L^*\phi_h) = (f, \phi_h), \quad \forall \phi_h \in \Phi^h. \tag{3.22}$$

The approximations y_h, v_0^h, and v_1^h to y, v_0, and v_1, respectively, are defined by

$$y_h = L^*p_h, \; v_0^h = y_h|_{x=0}, \; v_1^h = y_h|_{x=1}.$$

We assume the approximation property on Φ^h: For $q \in H^{k,2k}(Q)$, $k > r \geq 0$, there exists $q_h \in \Phi^h$ such that

$$\sum_{i=0}^{r} h^{i-r} \|q - q_h\|_{H^{i,2i}(Q)} \leq C h^{k-r} \|q\|_{H^{k,2k}(Q)}. \tag{3.23}$$

Also assume that the following inverse inequality holds in Φ^h:

$$\|\phi_h\|_{H^r(Q)} \leq C h^{-r} \|\phi_h\| \tag{3.24}$$

for $r > 0$, $\phi_h \in \Phi^h$.

Theorem 3.6 *Let p be the solution of (3.21) and p_h be the solution of (3.22). Assume that $p \in H^{k+1,2(k+1)}(Q)$ for $k > 0$. Then*

$$\|y - y_h\| \leq C h^k \|p\|_{H^{k+1,2(k+1)}(Q)}, \tag{3.25}$$

$$\|v_0 - v_0^h\|_{L^2(\Omega)} \leq C h^{k-\frac{1}{2}-\epsilon} \|p\|_{H^{k+1,2(k+1)}(Q)}, \tag{3.26}$$

$$\|v_1 - v_1^h\|_{L^2(\Omega)} \leq C h^{k-\frac{1}{2}-\epsilon} \|p\|_{H^{k+1,2(k+1)}(Q)} \tag{3.27}$$

where $\epsilon > 0$.

Proof: From (3.21) we have that

$$(L^*p, L^*v_h) = (f, v_h), \quad \forall v_h \in \Phi^h.$$

Subtracting (3.22) from the above equality, we have that

$$(L^*(p - p_h), L^*v_h) = 0, \quad \forall v_h \in \Phi^h.$$

Thus

$$\begin{aligned} \|L^*p - L^*p_h\|^2 &= (L^*(p - p_h), L^*(p - p_h)) \\ &= (L^*(p - p_h), L^*(p - v_h)) \\ &\leq \|L^*p - L^*p_h\| \|L^*p - L^*v_h\|, \quad \forall v_h \in \Phi^h. \end{aligned}$$

Hence

$$\|L^*p - L^*p_h\| \leq \inf_{v_h \in \Phi^h} \|L^*p - L^*v_h\|$$

or

$$\|y - y_h\| \leq \inf_{v_h \in \Phi^h} \|L^*p - L^*v_h\|.$$

By the approximation property (3.23) we obtain

$$\|y - y_h\| \leq Ch^k \|p\|_{H^{k+1,2(k+1)}}.$$

This proves (3.25).

We now turn to the proof of (3.26) and (3.27). From the inverse inequality (3.24) we have that for $\phi_h \in \Phi^h$

$$\begin{aligned}
\|y - y_h\|_{H^r(Q)} &\leq \|y - \phi_h\|_{H^r(Q)} + \|\phi_h - y_h\|_{H^r(Q)} \\
&\leq \|y - \phi_h\|_{H^r(Q)} + Ch^{-r}\|\phi_h - y_h\| \\
&\leq \|y - \phi_h\|_{H^r(Q)} + Ch^{-r}(\|\phi_h - y\| + \|y - y_h\|).
\end{aligned}$$

Using (3.23) and (3.25) we obtain

$$\begin{aligned}
\|y - y_h\|_{H^r(Q)} &\leq C \inf_{\phi_h \in \Phi^h} (\|y - \phi_h\|_{H^r(Q)} + h^{-r}\|\phi_h - y\|) + Ch^{-r}\|y - y_h\| \\
&\leq Ch^{k-r}\|p\|_{H^{k+1,2(k+1)}(Q)}
\end{aligned}$$

Letting $r = \frac{1}{2} + \epsilon$ and using the trace theorem, we have that

$$\|v_0 - v_0^h\|_{L^2[0,T]} \leq C\|y - y_h\|_{H^{\frac{1}{2}+\epsilon}(Q)} \leq Ch^{k-\frac{1}{2}-\epsilon}\|p\|_{H^{k+1,2(k+1)}(Q)},$$

$$\|v_1 - v_1^h\|_{L^2[0,T]} \leq C\|y - y_h\|_{H^{\frac{1}{2}+\epsilon}(Q)} \leq Ch^{k-\frac{1}{2}-\epsilon}\|p\|_{H^{k+1,2(k+1)}(Q)}.$$

The proof is complete. □

3.2.4. Numerical experiments.

Example 3.1. In this example we compute the numerical solutions for the problem (3.1)-(3.4) with $T = 1$, $y_0 = 0$ and $f = LL^*p$ where

$$p(t,x) = \begin{cases} 10^4 t^4 (0.8-t)^4 (1-x)^2 \sin^2 x, & \text{if } t < 0.8, \\ 0 & \text{otherwise.} \end{cases}$$

using finite element methods. The boundary conditions (3.16) are treated as essential boundary conditions and (3.17) are treated as natural boundary conditions. The finite element subspaces Φ^h are constructed as follows

$$\Phi^h = L_t^h \otimes C_x^h$$

where

$$L_t^h := \{v \in C[0,T]; v|_{[t_i,t_{i+1}]} \in \mathcal{P}_1, h = t_{i+1} - t_i, t_0 = 0, t_n = T\},$$

$$C_x^h = \{v \in C^1[0,1]; v|_{[x_i,x_{i+1}]} \in \mathcal{P}_3, h = x_{i+1} - x_i, x_0 = 0, x_n = 1\}.$$

We have, by the standard approximation theory of finite element spaces ([27]), that for $q \in H^{2,4}(Q)$, there exists $q_h \in \Phi^h$, such that

$$h^{-1}\|q - q_h\| + \|q - q_h\|_{H^{1,2}(Q)} \le h\|q\|_{H^{2,4}(Q)},$$

that is (3.23) is valid for $k = 1$. Hence we expect the first order convergence of y_h to y according to (3.25). Figure 3.1 displays the graph of $v_0 = y(\cdot,0) = L^*p(\cdot,0)$ and the graph $v_0^h = y_h(\cdot,0) = L^*p_h(\cdot,0)$ for the indicated h. Figure 3.2 displays the graph of $v_1 = y(\cdot,1) = L^*p(\cdot,1)$ and the graph of $v_1^h = y_h(\cdot,1) = L^*p_h(\cdot,1)$. Table 3.1 lists the L_2 errors of the approximations of p_h to p and y_h to y. The rate of convergence of y_h to y esitmated by linear regression is 0.9875, which matches our theoretical error estimate.

Table 3.1: L^2 Errors

h	$\|\|p_h - p\|\|$	$\|\|y_h - y\|\|$
1/9	0.19D-02	0.25D+00
1/19	0.34D-03	0.12D+00
1/29	0.14D-03	0.79D-01
1/39	0.74D-04	0.59D-01
1/49	0.47D-04	0.47D-01
1/59	0.32D-04	0.39D-01

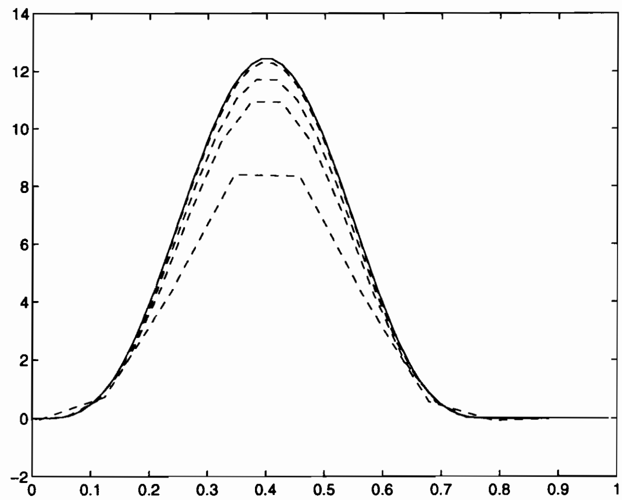

Figure 3.1: The graphs of v_0 and v_0^h for $h = 1/9, 1/29, 1/39, 1/59$.

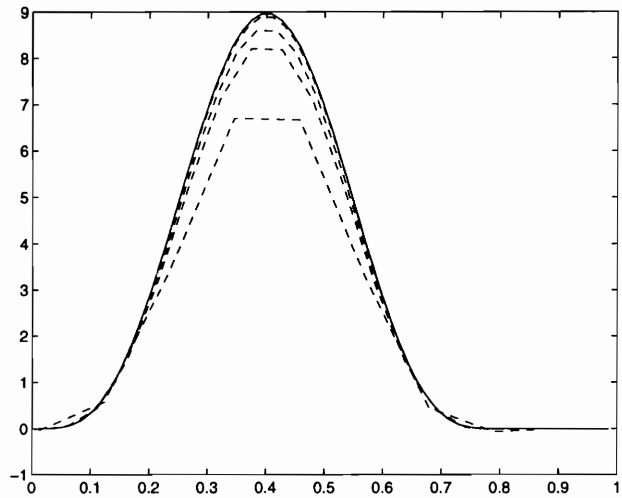

Figure 3.2: The graphs of v_1 and v_1^h for $h = 1/9, 1/29, 1/39, 1/59$.

Example 3.2. In this example the problem is the same as in Example 3.1. But the finite element spa:es Φ^h are chosen as $\Phi^h = C_t^h \otimes C_x^h$ (Bogner-Fox-Schmit element). Again using the standard approximation theory on finite element spaces we have that (3.23) is valid for $k = 2$. Hence we expect the second order convergence of y_h to y and nearly $\frac{3}{2}$ order convergence of v_h to v. Figure 3.3, Figure 3.4 and Table 3.2 provide the information similar to that of Figure 3.1, Figure 3.2 and Table 3.1, respectively. The rate of convergence of y_h to y esitmated by linear regression is 1.9918, which matches our theoretical error estimate.

Table 3.2: L^2 Errors

h	$\|\|p_h - p\|\|$	$\|\|y_h - y\|\|$
1/9	0.80D-04	0.33D-01
1/19	0.57D-05	0.76D-02
1/29	0.11D-05	0.32D-02
1/39	0.39D-06	0.18D-02
1/49	0.16D-06	0.11D-02
1/59	0.82D-07	0.80D-03

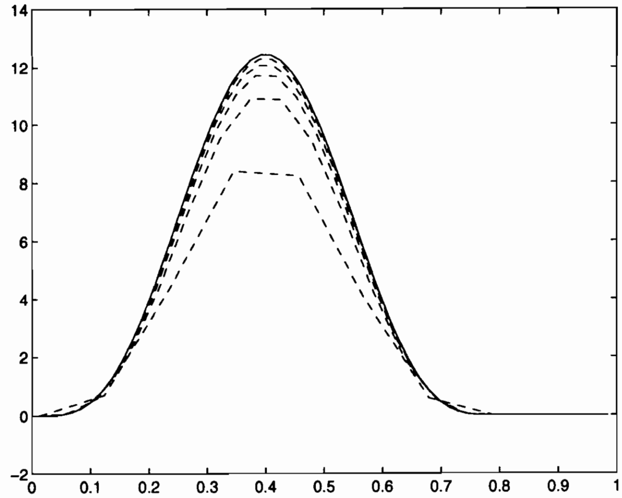

Figure 3.3: The graphs of v_0 and v_0^h for $h = 1/9, 1/19, 1/29, 1/39, 1/49$.

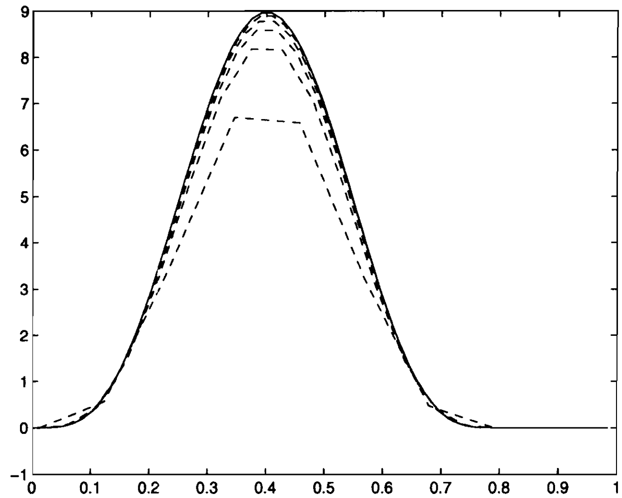

Figure 3.4: The graphs of v_1 and v_1^h for $h = 1/9, 1/19, 1/29, 1/39, 1/49$.

Example 3.3. In this example we solve problem (3.1)-(3.4) with $y_0 = 0$, $T = 1$ and $f(t,x) = 100\sin(tx)(1-t)^2(1+x^2)$ for $0 < t < T$, $0 < x < 1$ using a finite element method. The finite element space is the same as in Example 3.1. The exact solution is not available in this example. Figure 3.5 displays the graphs of $\|p_h(t,\cdot)\|_{L^2[0,1]}$ and Figure 3.6 displays the graphs of $\|y_h(t,\cdot)\|_{L^2[0,1]}$.

Remark 3.1. For the solution of (3.15)-(3.17) to exist, f has to be an element of Φ^* where Φ is defined by (3.18). It is easy to verify, using (3.20), that $f \in \Phi^*$ if $f \in L^2(Q)$ vanishes in a neighborhood of $t = T$. But in general it is difficult to check if a function belongs to Φ^*. This example shows that for a fairly general function f the solution of problem (3.15)-(3.17) may still exist.

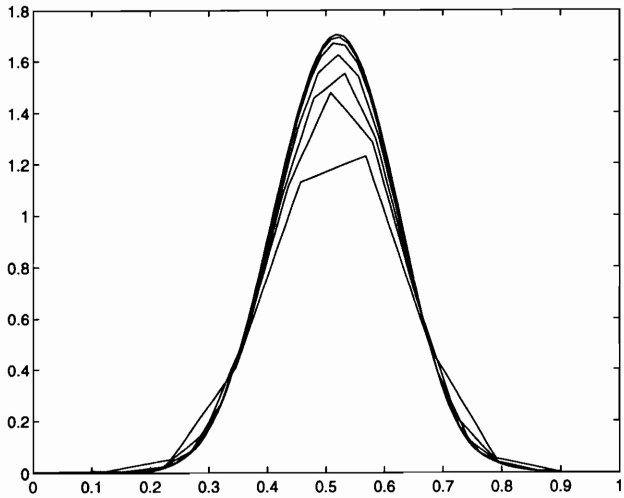

Figure 3.5: The graphs of $\|p_h(t,\cdot)\|$ for $h = 1/10, 1/15, 1/20, 1/30, 1/50, 1/70$.

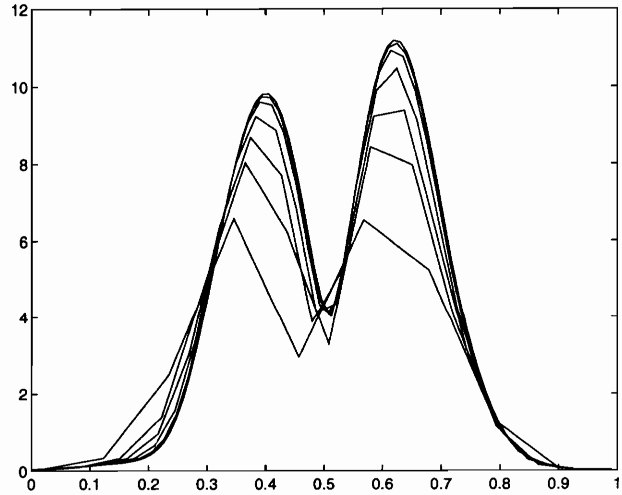

Figure 3.6: The graphs of $\|y_h(t,\cdot)\|$ for $h = 1/10, 1/15, 1/20, 1/30, 1/50, 1/70$.

3.3 Inhomogeneous Initial Conditions

3.3.1. Reduction to homogeneous initial conditions

Let $\chi(t,x)$ be the solution of the problem (3.1)-(3.3) with $v_0 = v_1 = 0$. Assume that $y_0 \in H_0^1(\Omega)$, where $H_0^1(\Omega)$ is defined by

$$H_0^1(\Omega) = \{w \in H^1(\Omega);\ w|_{\partial\Omega} = 0\}.$$

Then χ belongs to $H^{1,2}(Q)$ ([23]). Let $\varphi(t) \in C^4[0,T]$ such that $\varphi(t) = 1$ for $t \in (0,t_1)$ and $\varphi(t) = 0$ for $t \in (t_2,T)$ for $0 < t_1 < t_2 < T$. Let

$$y_1(t,x) = \chi(t,x)\varphi(t) \tag{3.28}$$

and

$$f_0(t,x) = -Ly_1(t,x) + f(t,x). \tag{3.29}$$

Let

$$w(t,x) = y(t,x) - y_1(t,x). \tag{3.30}$$

Proposition 3.7 *A function $y \in L^2(Q)$ is a solution of the problem (3.1)-(3.4) if and only if the function w defined in (3.30) satisfies*

$$Lw(t,x) = f_0(t,x), \quad (t,x) \in Q, \tag{3.31}$$

$$w(t,x)|_{t=0} = w(t,x)|_{t=T} = 0, \quad x \in \Omega. \tag{3.32}$$

□

Thus with transform (3.28) we have reduced the problem to a problem with homogeneous initial condition.

3.3.2. Further reduction to an optimization problem. Similar to §3.2, we consider the following extremal problem

$$\min J(w) = \frac{1}{2} \int_Q w^2(t,x)dxdt \tag{3.33}$$

on the set of functions w satisfying (3.31), (3.32). Using the standard Lagrange multiplier argument, we obtain the following result

Proposition 3.8 *The optimality system for the problem (3.33) is given by*

$$L^*p(t,x) = w(t,x), \quad (t,x) \in Q, \tag{3.34}$$

$$p(t,x)|_{x=0,1} = p_x(t,x)|_{x=0,1} = 0, \quad 0 < t < T, \tag{3.35}$$

$$w \text{ satisfies } (3.31), (3.32). \tag{3.36}$$

□

Applying operator L on (3.34) and taking into account (3.31), we obtain

$$LL^*p(t,x) = f_0(t,x), \quad (t,x) \in Q, \tag{3.37}$$

$$p(t,x)|_{x=0,1} = p_x(t,x)|_{x=0,1} = 0, \quad 0 < t < T, \tag{3.38}$$

$$L^*p(t,x)|_{t=0} = L^*p(t,x)|_{t=T} = 0, \quad x \in \Omega. \tag{3.39}$$

Notice that (3.37) is a special case of (3.5) with $c = 1$. Similar to Proposition 3.2, we have the following result.

Proposition 3.9 *p is a solution of the optimality system (3.34)-(3.36) if and only if p is a solution of problem (3.37)-(3.39).*

3.3.3. Existence of boundary controls. Recall that the Hilbert space Φ is defined by

$$\Phi = \{\phi \in L^2(Q); ||L^*\phi|| < \infty, \phi|_{x=0,1} = \phi_x|_{x=0,1} = 0\}. \tag{3.40}$$

The weak formulation of problem (3.37)-(3.39) is then defined as follows: seek $p \in \Phi$ such that

$$(L^*p, L^*\phi) = (f_0, \phi), \quad \forall \phi \in \Phi. \tag{3.41}$$

p is called a generalized solution of problem (3.37), (3.38), (3.39) if it satisfies (3.41). The following theorem follows from the Lax-Milgram theorem.

Theorem 3.10 *Let $f \in \Phi^*$ and $y_0 \in L^2(0,1)$. Then there exists a unique generalized solution $p \in \Phi$ of the problem (3.37), (3.38), (3.39). Also $p \in H^{2,4}(\bar{Q})$ for any smooth subdomain $\bar{Q}$ of Q.* □

From (3.30), we have

Theorem 3.11 *There exist functions $v_0, v_1 \in L^2(0,T)$, $y \in H^{1,2}(Q)$ that satisfy (3.1), (3.2), (3.3), (3.4). Also if $p \in H^{3,6}(Q)$, $y_0 \in H^2(\Omega)$, then $y \in H^{2,4}(Q)$ and $v_0, v_1 \in H^{\frac{3}{2}}[0,T]$.* □

3.3.4. Finite elment approximation. Recall that $f_0 = -L(\varphi\chi) + f$ in (3.41) where χ is the solution of the problem (3.1)-(3.3) with $v_0(t) = v_1(t) = 0$. So to solve for p numerically from (3.41) χ has to be solved for numerically first. Let Φ^h be the same as in §3.2.3 with the properties (3.23) and (3.24) and let $S^h \subset H^1(0,1)$ be a finite dimensional subspace with the following property: for $u \in H^k(0,1)$, there exists $u_h \in S^h$ such that

$$||u - u_h||_{H^r(0,1)} \leq Ch^{k-r}||u||_{H^k(0,1)}.$$

Also assume that the following inverse property holds in S^h,

$$\|u_h\|_{H^r(0,1)} \leq Ch^{-r}\|u_h\| \quad \forall u_h \in S^h.$$

The finite element approximation of problem (3.41) is defined as follows: seek $\chi_h \in S^h$, $p_h \in \Phi^h$ such that

$$(\partial_t \chi_h, \psi_h) + (\partial_x \chi_h, \partial_x \psi_h) + (a\partial_x \chi_h, \psi_h) = 0, \quad \forall \psi_h \in S^h, \tag{3.42}$$

$$\chi_h(0, \cdot) = P_h y_0, \tag{3.43}$$

$$(L^* p_h, L^* \phi_h) = -(L(\varphi \chi_h), \phi_h) + (f, \phi_h) \quad \forall \phi_h \in \Phi^h, \tag{3.44}$$

where P_h is the L^2 projection from $H^1(0,1)$ to S^h. The approximations y_h, v_0^h and v_1^h to y, v_0 and v_1, respectively, are then defined as

$$y_h = L^* p_h + \varphi \chi_h, \ v_0^h = y_h|_{x=0}, \ v_1^h = y_h|_{x=1}.$$

Theorem 3.12 *Assume that $y_0 \in H^k(0,1)$, $p \in H^{k+1,2(k+1)}(Q)$. Then there exists a constant C, independent of h, such that*

$$\|y - y_h\| \leq Ch^k(\|p\|_{H^{k+1,2(k+1)}(Q)} + \|\chi\|_{H^{k,2k}(Q)} + \|y_0\|_{H^k(\Omega)}), \tag{3.45}$$

$$\|v_0 - v_0^h\|_{L^2(\Omega)} \leq Ch^{k-\frac{1}{2}-\epsilon}(\|p\|_{H^{k+1,2(k+1)}(Q)} + \|\chi\|_{H^{k,2k}(Q)} + \|y_0\|_{H^k(\Omega)}), \tag{3.46}$$

$$\|v_1 - v_1^h\|_{L^2(\Omega)} \leq Ch^{k-\frac{1}{2}-\epsilon}(\|p\|_{H^{k+1,2(k+1)}(Q)} + \|\chi\|_{H^{k,2k}(Q)} + \|y_0\|_{H^k(\Omega)}) \tag{3.47}$$

where $\frac{1}{2} > \epsilon > 0$.

Before proving the theorem, we state the following lemma which is an immediate consequence of Theorem 3.6.

Lemma 3.13 *Let $w = L^*p$ and $w_h^* = L^*p_h^*$ where p_h^* satisfies*

$$(L^*p_h^*, L^*\phi_h) = (f_0, \phi_h), \quad \forall \phi_h \in \Phi^h. \tag{3.48}$$

Then

$$\|w - w_h^*\| \le Ch^k \|p\|_{H^{k+1,2(k+1)}}. \tag{3.49}$$

The proof of Theorem 3.12. Let $w_h = L^*p_h$. Subtracting (3.48) from (3.44) and integrating by parts, we have that

$$(L^*(p_h - p_h^*), L^*\phi_h) = (\varphi(\chi_h - \chi), L^*\phi_h) + (y_0 - \chi_h(0,\cdot), \phi_h(0,\cdot))_{L^2(\Omega)}$$

for all $\phi_h \in \Phi^h$. Choosing $\phi_h = p_h - p_h^*$ we obtain

$$\begin{aligned}\|w_h - w_h^*\|^2 &= (\phi(\chi - \chi_h), w_h - w_h^*) + (y_0 - \chi_h(0,\cdot), p_h(0,\cdot) - p_h^*(0,\cdot))_{L^2(\Omega)} \\ &\le C\|\chi - \chi_h\|\|w_h - w_h^*\| + \|y_0 - \chi_h(0,\cdot)\|_{L^2(\Omega)}\|p_h(0,\cdot) - p_h^*(0,\cdot)\|_{L^2(\Omega)}.\end{aligned}$$

From the a priori estimate (3.20) we have that

$$\|p_h(0,\cdot) - p_h^*(0,\cdot)\|_{L_2(\Omega)} \le C\|w_h - w_h^*\|.$$

Hence

$$\|w_h - w_h^*\|^2 \le C(\|\chi - \chi_h\| + \|y_0 - \chi_h(0,\cdot)\|_{L_2(\Omega)})\|w_h - w_h^*\|.$$

Thus

$$\|w_h - w_h^*\| \le C(\|\chi - \chi_h\| + \|y_0 - \chi_h(0,\cdot)\|_{L_2(\Omega)}).$$

From the error estimates of the standard Galerkin approximations of parabolic equations ([34]), we have that

$$\|w_h - w_h^*\| \le Ch^k(\|\chi\|_{H^{k,2k}(Q)} + \|y_0\|_{H^k(\Omega)}).$$

Thus using Lemma 3.13 we obtain

$$\begin{aligned}
\|y - y_h\| &\le \|\phi(\chi - \chi_h)\| + \|w - w_h\| \\
&\le C\|\chi - \chi_h\| + \|w - w_h^*\| + \|w_h - w_h^*\| \\
&\le Ch^k(\|p\|_{H^{k+1,2(k+1)}(Q)} + \|\chi\|_{H^{k,2k}(Q)} + \|y_0\|_{H^k(\Omega)}).
\end{aligned}$$

This proves (3.45). We now turn to the proof of (3.46) and (3.47). For $r > 0$ we have that,

$$\|y - y_h\|_{H^r(Q)} \le \|w - w_h\|_{H^r(Q)} + C\|\chi - \chi_h\|_{H^r(Q)}.$$

From the inverse inequality (3.24) we have that for $\phi_h \in \Phi^h$

$$\begin{aligned}
\|w - w_h\|_{H^r(Q)} &\le \|w - \phi_h\|_{H^r(Q)} + \|\phi_h - w_h\|_{H^r(Q)} \\
&\le \|w - \phi_h\|_{H^r(Q)} + Ch^{-r}\|\phi_h - w_h\| \\
&\le \|w - \phi_h\|_{H^r(Q)} + Ch^{-r}(\|\phi_h - w\| + \|w - w_h\|).
\end{aligned}$$

Thus by (3.41) and the proof of (3.45) we have that

$$\begin{aligned}
\|w - w_h\|_{H^r(Q)} &\le C \inf_{\phi_h \in \Phi^h}(\|w - \phi_h\|_{H^r(Q)} + h^{-r}\|\phi_h - w\|) + Ch^{-r}\|w - w_h\| \\
&\le Ch^{k-r}(\|p\|_{H^{k+1,2(k+1)}(Q)} + \|\chi\|_{H^{k,2k}} + \|y_0\|_{H^k(\Omega)}).
\end{aligned}$$

Similarly we have the estimate

$$\|\chi - \chi_h\|_{H^r(Q)} \le Ch^{k-r}\|\chi\|_{H^{k,2k}(Q)}.$$

Hence

$$\|y - y_h\|_{H^r(Q)} \le Ch^{k-r}(\|p\|_{H^{k+1,2(k+1)}(Q)} + \|\chi\|_{H^{k,2k}} + \|y_0\|_{H^k(\Omega)}).$$

Letting $r = \frac{1}{2} + \epsilon$ and using the trace theorem, we have that

$$\|v_0 - v_0^h\|_{L^2[0,T]} \le C\|y - y_h\|_{H^{\frac{1}{2}+\epsilon}(Q)} \le Ch^{k-\frac{1}{2}-\epsilon}(\|p\|_{H^{k+1,2(k+1)}(Q)} + \|\chi\|_{H^{k,2k}(Q)} + \|\chi\|_{H^{k,2k}}(Q)),$$

$$\|v_0 - v_0^h\|_{L^2[0,T]} \le C\|y - y_h\|_{H^{\frac{1}{2}+\epsilon}(Q)} \le Ch^{k-\frac{1}{2}-\epsilon}(\|p\|_{H^{k+1,2(k+1)}(Q)} + \|\chi\|_{H^{k,2k}(Q)} + \|\chi\|_{H^{k,2k}}(Q)).$$

The proof is complete □.

3.3.5. Numerical experiments.

Example 4.4. In this example we solve the problem (3.1)-(3.4) with $T = 1$, $f = 0$ and $y_0(x) = \sin(\pi \mathrm{x})$. We first construct a function φ as

$$\varphi(t) = \begin{cases} 1, & \text{if } t > 0.1, \\ \frac{(t-0.1)^4+1)(0.9-t)^4 \sin(t)}{0.8^4 \sin(0.1)}, & \text{if } 0.1 < t < 0.9, \\ 0; & \text{otherwise.} \end{cases}$$

Following the theory of §§3.3.1-3.3.3, the solution y of (3.1)-(3.4) can be written as

$$y = -L^* p + \varphi\chi,$$

where p satisfies (3.37)-(3.39) and χ is the solution of (3.1)-(3.3) with $v_0(t) = v_1(t) = 0$ which is given exactly by $\chi(t,x) = e^{-\pi^2 t}\sin(\pi x)$. The finite element approximation p_h to p is obtained by solving (3.42)-(3.44). For finite element spaces we choose $\Phi^h = L_t^h \times C_x^h$ and $S^h = L_x^h$. We solve for χ_h in (3.42)-(3.43) using the backward Euler method in t direction. Once we find χ_h and p_h, the approximation y_h of y is then given by

$$y_h = -L^* p_h + \varphi\chi_h.$$

Figure 3.7 displays the graph of $\|y_h(t,.)\|_{L^2[0,1]}$. Figure 3.8 and Figure 3.9 display the graph of $v_0^h(t) = y_h(t,0)$ and the graph of $v_1^h(t) = y_h(t,1)$, respectively, for $h = 1/80$.

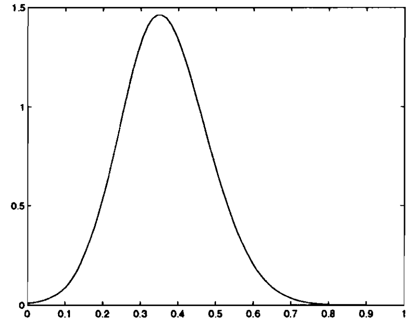

Figure 3.7: The graph of $y_h(t, \cdot)$ for $h = 1/80$.

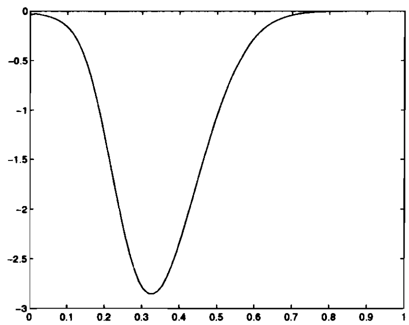

Figure 3.8: The graph of v_0^h for $h = 1/80$.

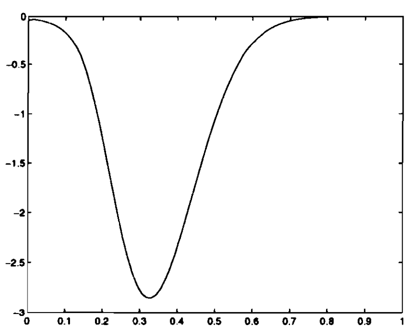

Figure 3.9: The graph of v_1^h for $h = 1/80$.

3.3.6. The weight function $c(t,x) = e^{\frac{s}{(T-t)^2}}$**.** Here instead of (3.33), we consider the following extremal problem

$$J(w) = \int_Q c(t,x)w^2(t,x)dxdt \longrightarrow \inf \tag{3.50}$$

on the set of functions w satisfying (3.9), (3.10). Here $c(t,x) = e^{\frac{s}{(T-t)^2}}$ and s is a sufficiently large number. Analogous to §3.2 and §3.3, the optimality system for problem (3.50) is given by

$$cL^*p(t,x) = w(t,x), \quad (t,x) \in Q, \tag{3.51}$$

$$p(t,x)|_{x=0,1} = p_x(t,x)|_{x=0,1} = 0, \quad 0 < t < T, \tag{3.52}$$

$$w \text{ satisfies } (3.31), (3.32). \tag{3.53}$$

p is a solution of the optimality system (3.2)-(3.4) if and only if p is a solution of the following problem

$$L^*cLp(t,x) = f_0(t,x), \quad (t,x) \in Q, \tag{3.54}$$

$$p(t,x)|_{x=0,1} = p_x(t,x)|_{x=0,1} = 0, \quad 0 < t < T, \tag{3.55}$$

$$cL^*p(t,x)|_{t=0} = cL^*p(t,x)|_{t=T} = 0, \quad x \in \Omega. \tag{3.56}$$

The solution y of problem (3.1)-(3.4) is then obtained through

$$y = cL^*p + y_1 = w + y_1$$

where y_1 is given by (3.28). We set

$$\Phi_c = \{f \in L^2(Q); \int_Q c(t,x)(L^*f)^2 dxdt < \infty\}.$$

p is said to be a generalized solution of problem (3.54)-(3.56) if it satisfies

$$(e^{c(t,x)}L^*p, L^*z) = (f_0, z) \quad \forall z \in \Phi_c.$$

The next two theorems can be found in [10].

Theorem 3.14 *There exists a unique generalized solution $p \in \Phi_c$ of problem (3.54)-(3.56). Also there exists a constant C such that*

$$\|L^* p\|_{L^2(Q)} \le C\|cf_0\|_{L^2(Q)}, \quad 0 < t < T.$$

□

Theorem 3.15 *There exist a function $y \in H^{1,2}(Q)$ and controls $v_0, v_1 \in L^2[0,T]$ that satisfy (3.1)-(3.4). Also there exists a constant C such that*

$$\|y(t,.)\|_{L^2(\Omega)} \le Ce^{-\frac{s}{(T-t)^2}}, \quad 0 \le t < T.$$

□

From the above theorem we see that the exponential weight function makes it possible to obtain an exponential rate of approach of y to the terminal state $y(T,x) = 0$. On the other hand our numerical experiments and the a priori estimate in [11] indicate that the solution p of the problem (3.54)-(3.56) may blow up as t approaches the terminal time T. To circumvent this flaw of the model, we introduce the transformation

$$q(t,x) = e^{-\frac{s}{2(T-t)^2}} p(t,x). \tag{3.57}$$

Substituting (3.57) into (3.54), (3.55), (3.56) we obtain

$$e^{-\frac{s}{2(T-t)^2}}(\partial_t - \partial_{xx} - \frac{s}{(T-t)^3})(\partial_t + \partial_{xx} + \frac{s}{(T-t)^3})q(t,x) = f_0(t,x), \quad (t,x) \in Q, \tag{3.58}$$

$$q(t,x)|_{x=0,1} = q_x(t,x)|_{x=0,1} = 0, \quad 0 < t < T, \tag{3.59}$$

$$e^{-\frac{s}{2(T-t)^2}}(\partial_t + \partial_{xx} + \frac{s}{(T-t)^3})q(t,x)|_{t=0,T} = 0, \quad x \in \Omega. \tag{3.60}$$

Then in terms of q, $w = -cL^*p$ is written as

$$w(t,x) = e^{-\frac{s}{2(T-t)^2}}(\partial_t + \partial_{xx} + \frac{s}{(T-t)^3})q(t,x).$$

We report here a numerical example of solving problem (3.58)-(3.60) using a finite element method. The problem is the same as in Example 1. Finite element spaces Φ^h are chosen as $\Phi^h = L_t^h \times C_x^h$. For a finite element approximation we seek $q_h \in \Phi^h$ such that

$$((\partial_t - \partial_{xx} - \frac{1}{(1-t)^3})q_h, (\partial_t - \partial_{xx} - \frac{1}{(1-t)^3})(e^{\frac{-1}{(1-t)^2}}\phi_h)) = (f, \phi_h) \quad \forall \phi_h \in \Phi^h.$$

Figure 10, Figure 11, Figure 12 and Figure 13 display the graphs of $||q_h(t,\cdot)||$, $||y_h(t,\cdot)||$, $v_0^h(t)$, and $v_1^h(t)$, respectively, for $h = 1/9, 1/19, 1/29, 1/39$. We can see that $q_h(t,x)$ does not blow up as t approaches to $T = 1$.

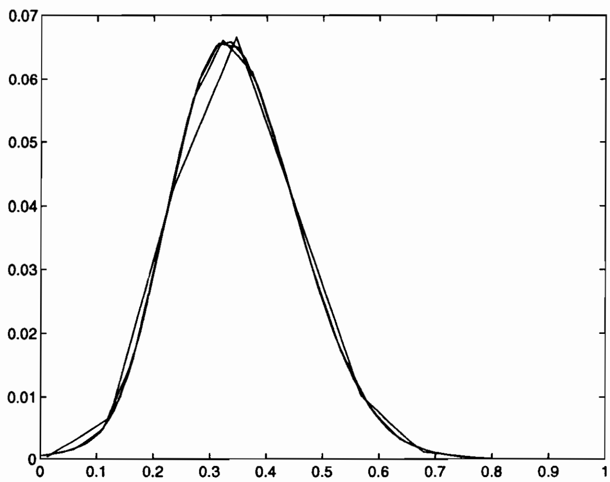

Figure 3.10: The graphs of $||q_h(t,\cdot)||$ for $h = 1/9, 1/19, 1/29, 1/39$.

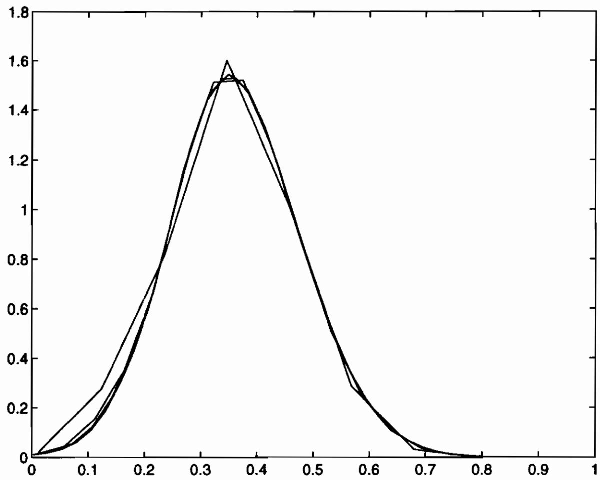

Figure 3.11: The graphs of $\|y_h(t,\cdot)\|$ for $h = 1/9, 1/19, 1/29, 1/39$.

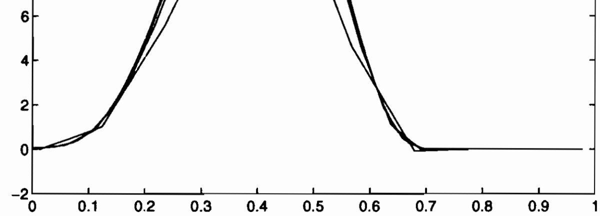

Figure 3.12: The graphs of $v_0^h(t)$ for $h = 1/9, 1/19, 1/29, 1/39$.

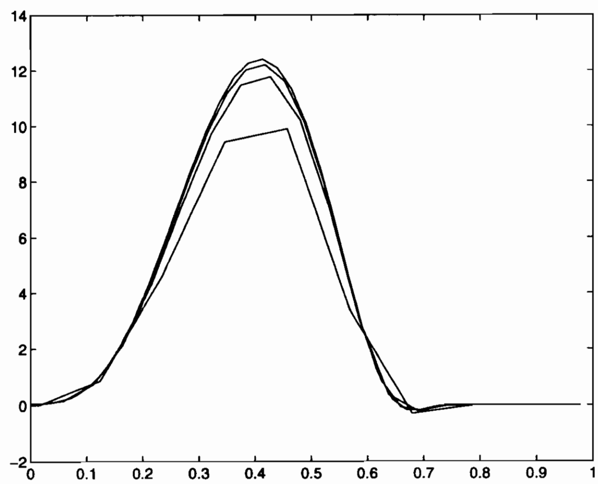

Figure 3.13: The graphs of $v_1^h(t)$ for $h = 1/9, 1/19, 1/29, 1/39$

Chapter 4

A Numerical Scheme for the Exact Zero Boundary Controllability Problem for Burger's Equations

4.1 The Statement of the Problem

We consider Burger's equation

$$\partial_t y - \partial_{xx} y + \partial_x y^2 = f, \quad \text{in } (0,T) \times (0,1) \tag{4.1}$$

with the initial and boundary conditions

$$y(0,x) = y_0, \quad 0 < x < 1, \tag{4.2}$$

$$y(t,0) = v_0(t), \; y(t,1) = v_1(t), \quad 0 < t < T. \tag{4.3}$$

The exact zero boundary controllability problem is to seek v_0, v_1 in (4.3) such that the solution y of (4.1)-(4.3) also satisfies

$$y(T,x) = 0, \quad 0 < x < 1. \tag{4.4}$$

4.2 The Continuous Problem

4.2.1. The linearized problem. To study the exact controllability problem (4.1)-(4.4) we first consider the following linearized problem. Let z, f be given. We seek v_0, v_1

such that the unique solution y of the problem

$$L(z)y = \partial_t y - \partial_{xx} y + \partial_x(zy) = f, \quad \text{in } (0,T) \times (0,1), \tag{4.5}$$

$$y(0,x) = 0, \quad 0 < x < 1, \tag{4.6}$$

$$y(t,0) = v_0(t), \; y(t,1) = v_1(t), \quad 0 < t < T \tag{4.7}$$

also satisfies

$$y(T,x) = 0, \quad 0 < x < 1. \tag{4.8}$$

Denote $Q = (0,T) \times (0,1)$. Define

$$\Phi_z = \{g \in L^2(Q); ||L^*(z)g||_{L(Q)} < \infty\}.$$

It is a Hilbert space with the inner product defined as

$$< p,q >= \int_Q L^*pL^*qdxdt.$$

In the following we state two propositions proved in [11].

Proposition 4.1 *([11]) Let $w \in L^2(Q)$ satisfy the relations*

$$Lw = f, \tag{4.9}$$

$$w|_{t=0} = 0. \tag{4.10}$$

Then the following estimate for w holds.

$$\int_Q e^{-\frac{1}{(T-t)^2}}(w^2 + \partial_t w^2 + \partial_x w^2 + \partial_{xx} w^2)dxdt \leq r(1 + ||z||_{H^{1,2}(Q)})||f||^2_{L^2(Q)} \tag{4.11}$$

where r is a monotonically continuous function.

Proposition 4.2 *([11]) Let $z \in L^2(Q)$, $f \in \Phi^*$. Then there exists a solution $y \in H^{1,2}(Q)$ of problem (4.5)-(4.8). Besides the map $\theta : z \to y$ is compact from $W_2^{1,2}(Q)$ to itself.*

4.2.2. The existence of exact boundary controls for the Burger's equations.

Theorem 4.3 *If $f \in \Phi^*$ and $||f||_{L^2(Q)}$ is sufficiently small, then there exist controls v_0, $v_1 \in C[0,T]$ such that the solution y of (4.1)-(4.3) belongs to $W_2^{1,2}$ and satisfies (4.4).*

The sketch of the proof (see [11] for the details). For a function $\delta \in H^{1,2}(Q)$, we consider the mapping θ: $\delta \to \psi$ where ψ satisfies

$$\begin{aligned} \psi &= L^*p, \\ L(\delta)L^*(\delta)p &= f, \\ L^*(\delta)p|_{t=0} = L^*(\delta)p|_{t=T} &= 0, \\ p|_{x=0,1} = p_x|_{x=0,1} &= 0. \end{aligned}$$

By Proposition 4.2 θ is a compact operator from $H^{1,2}(Q)$ to itself. Also by Proposition 2.1 we have that

$$||\theta(\delta)||_{W_2^{1,2}} \leq r(1 + ||\delta||_{w_2^{1,2}(Q)})||f||^2_{L^2(Q)}.$$

Hence for $||f||$ sufficiently small, the operator θ maps the unit ball

$$B_1 = \{\delta; ||\delta||_{W_2^{1,2}} < 1\}$$

into itself. Hence by Shauder fixed point theorem there exists $\psi \in B_1$ such that

$$\psi = \theta(\psi).$$

This function is the solution of problem (4.1)-(4.4).□

The solution of problem (4.1)-(4.4) may not be unique. In the following we show that the solution obtained in the above proof has the minimum L^2 norm among all the solutions. We consider the minimization problem

$$\inf_{w \ satisfies (4.1),(4.2),(4.4)} J(w) = \int_Q w^2 dxdt. \tag{4.12}$$

Theorem 4.4 *The optimality system of problem (4.12) is given by*

$$\partial_t p + \partial_{xx} - w\partial_x p = w, \tag{4.13}$$

$$p_{x=0,1} = p_x|_{x=0,1} = 0, \tag{4.14}$$

$$w \ satisfies \ (4.1),\ (4.2),\ (4.4). \tag{4.15}$$

We refer to [17] fcr a rigorous proof of the theorem. In the following we show that the first variation of J is zero at w if w satisfies (4.13)-(4.15). From (4.12) we have that

$$J'(w)\delta w = 2\int_Q w\delta w.$$

Since w satisfies (4.13), we have that

$$J'(w)\delta w = 2\int_Q (\partial_t p + \partial_{xx} p - w\partial_x p)\delta w dx dt. \tag{4.16}$$

But δw satisfies

$$\partial_t \delta w - \partial_{xx}\delta w + \partial_x(w\delta w) = 0,$$

$$\delta w|t = 0 = \delta w|_{t=T} = 0.$$

Integrating by parts on (4.16) we have that

$$J'(w)\delta w = 2[\int_0^T [p_x(t,1)\delta w(t,1) - p_x(t,0)\delta w(t,0)]dt - \int_0^T (pw\delta w(t,1) - (pw\delta w(t,0)dt] = 0.$$

□

Theorem 4.5 *The solution obtained in the proof of Theorem 4.3 is a solution of the optimality system (4.13)-(4.15) for problem (4.12).*

Proof: Let $y^n = \theta(y^{n-1})$ where θ is defined in §4.2, that is

$$L(y^{n-1})y_n = \partial_t y^n - \partial_{xx} y^n + \partial_x(y^{n-1}y^n) = f \text{ in } \quad Q = (0,T) \times (0,1).$$

Let p^n be the solution of the problem

$$\partial_t p^n + \partial_{xx} p^n - y^{n-1} \partial_x p^n = y^n,$$

$$p^n|_{x=0,1} = p^n_x|_{x=0,1} = 0.$$

By the proof of Theorem 4.3, we have that

$$\lim_{n \to \infty} y^n = y \quad \text{in } W_2^{1,2}(Q).$$

Let $\delta^{mn} = p^m - p^n$, $d^{mn} = y^m - y^n$. Then

$$\lim_{m,n \to \infty} d^{mn} = 0$$

and

$$\partial_t \delta^{mn} + \partial_{xx} \delta^{mn} - y^n \partial_x \delta^{mn} = (p^n_x + 1)\delta^{mn}.$$

By an estimate in [11] there exists a constant C, independent of n, such that

$$\|p^n_x\| \le C.$$

Hence from an a priori estimate in [11] we have that

$$\begin{aligned}
&\int_Q e^{-\frac{1}{(T-t)^2}} ((\partial_t \delta^{mn})^2 + (\partial_{xx} \delta^{mn})^2 + (\partial_x \delta^{mn})^2 + (\delta^{mn})^2 dx dt \\
&\le r(\|y^n\|_{W_2^{1,2}(Q)}) \int_Q \|p^n_x d^{mn}\|^2 dx dt \\
&\le C r(\|y^n\|_{W_2^{1,2}(Q)}) \|d^{mn}\|_{W_2^{mn}(Q)} \to 0.
\end{aligned}$$

Thus

$$\|p^m - p^n\|_{W_2^{1,2}(Q)} \to 0, \quad m,\ n \to 0$$

for $Q_t = (0,t) \times (0,T)$, $0 < t < T$. Let

$$p = \lim_{n \to \infty} p^n \quad \text{in } W_2^{1,2}(Q).$$

Thus $\{p,\ y\}$ is a solution of the optimality system (3.2)-(3.4). $\square$

4.3 A Numerical Algorithm Using a Finite Element Method

4.3.1. The simple iteration algorithm. We use the finite element method developed in §3.3.3 to implement the iteration procedure described in the proof of Theorem 4.3. Before applying a finite element method, we first summarize the iteration in algorithm form.

Step 0: Initialization

$$
\begin{aligned}
&y_0 \text{ is given;}\\
&\text{Solve for } p_0 :\\
&L(y_0)L^*(y_0)p_0 = f,\\
&L^*(y_0)p_0|_{t=0} = L^*(y_0)p_0|_{t=T} = 0,\\
&p_0|_{x=0,1} = \partial_x p_0|_{x=0,1} = 0.
\end{aligned}
$$

Then for $n \geq 0$, assuming that y_n is known, compute y_{n+1} as follows:

Step 1: Simple iteration. First solve for p_{n+1} from

$$
\begin{aligned}
&L(y_n)L^*(y_n)p_{n+1} = f,\\
&L^*(y_n)p_{n+1}|_{t=0} = L^*(y_n)p_{n+1}|_{t=T} = 0,\\
&p_{n+1}|_{x=0,1} = \partial_x p_{n+1}|_{x=0,1} = 0.
\end{aligned}
$$

Then y_{n+1} is computed as

$$y_{n+1} = L^*(y_n)p_{n+1}.$$

Step 2: Test of convergence. Compute $e_n = \|y_{n+1} - y_n\|_{L^2(Q)}$. if $e_n = 0$ or is small, then stop; if not, set $n = n + 1$ andgo to **Step 1**.

4.3.2. Finite element discretization. We use the finite element method to implement the above procedure. To this end let Φ^h be finite dimensional subspaces of Φ_{y_n}. The finite element discretization of the simple iteration algorithm is as follows.

Step 0: Initialization

y_0 is given;

Solve for $p_0^h \in \Phi^h$ from

$(L^*(y_0)p_0^h, L^*(y_0)\phi_h) = (f, \phi_h) \quad \forall \phi_h \in \Phi^h.$

Then for $n \geq 0$, assuming that y_n^h is known, compute y_{n+1}^h, as follows:

Step 1: Simple iteration. First solve for p_{n+1} from

$$(L^*(y_n^h)p_{n+1}^h, L^*(y_n^h)\phi_h) = (f, \phi_h) \quad \forall \phi_h \in \Phi^h.$$

Then y_{n+1}^h is computed as

$$y_{n+1}^h = L^*(y_n^h)p_{n+1}^h.$$

Step 2: Test of convergence. Compute $e_n^h = \|y_{n+1}^h - y_n^h\|_{L^2(Q)}$. If $e_n = 0$ or is small, then stop; if not, set $n = n + 1$ go to **Step 1**.

4.3.3. A numerical example. We solve problem (4.1)-(4.4) with $T = 1$, $y_0 = 0$ and $f = g_t - g_{xx}$ where

$$g(t,x) = \begin{cases} 10^4 t^4 (0.8-t)^4 (1-x)^2 \sin^2 x, & \text{if } t < 0.8, \\ 0, & \text{otherwise} \end{cases}$$

using the algorithm described above. The finite element subspaces Φ^h are constructed as follows:

$$\Phi^h = C_t^h \otimes C_x^h$$

where

$$C_t^h := \{v \in C^1[0,T]; v|_{[t_i,t_{i+1}]} \in \mathcal{P}_3, h = t_{i+1} - t_i, t_0 = 0, t_n = 1\},$$

$$C_x^h =: \{v \in C^1[0,1]; v|_{[x_i,x_{i+1}]} \in \mathcal{P}_3, h = x_{i+1} - x_i, x_0 = 0, x_n = 1\}$$

where $\mathcal{P}_3$ is the set of polynomials of degree 3.

We perform the computations for h=1/9, 1/19, 1/29, 1/39. For each h the iteration takes 5 steps for tolerance $e = 10^{-7}$. So it appears that the number of iterations is independent of h. Figure 4.1 through Figure 4.4 display the graphs of $\|p^h(t,\cdot)\|$, $\|y^h(t,\cdot)\|$, $v_0^h = y^h(t,0)$ and $v_1^h(t,1)$. From the graphs we can see the convergence of approximate solutions.

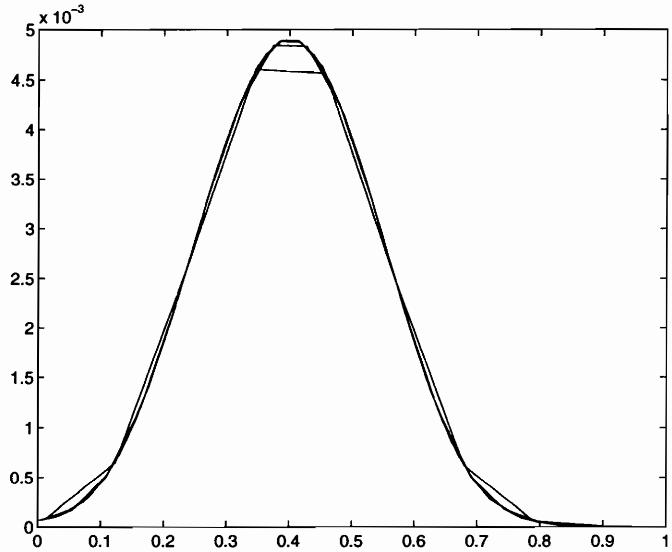

Figure 4.1: The graph of $\|p^h(t,\cdot)\|$ for h=1/9, 1/19, 1/29, 1/39.

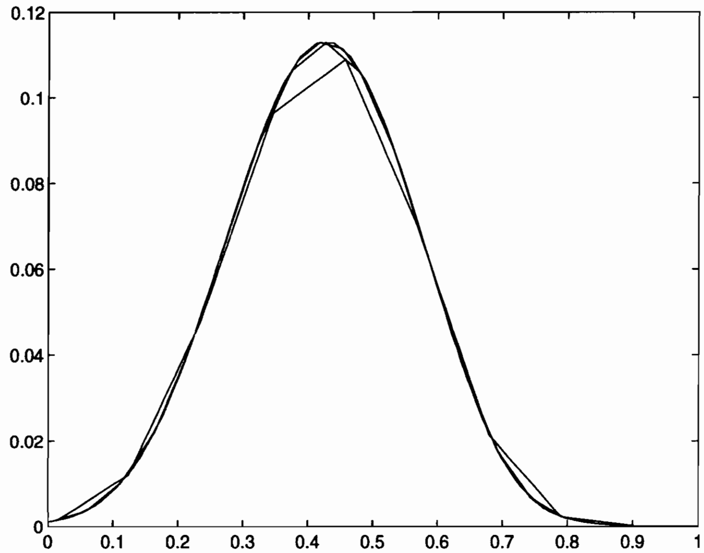

Figure 4.2: The graph of $||y^h(t,\cdot)||$ for h=1/9, 1/19, 1/29, 1/39.

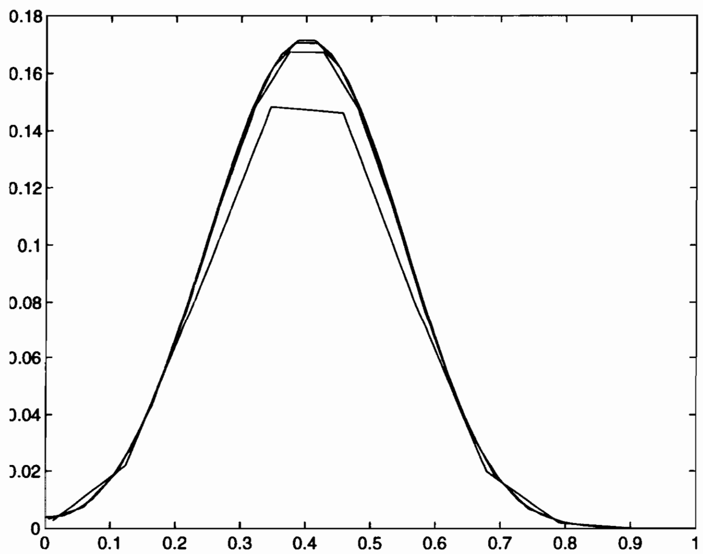

Figure 4.3: The graph of $v_0^h(t)$ for h=1/9, 1/19, 1/29, 1/39.

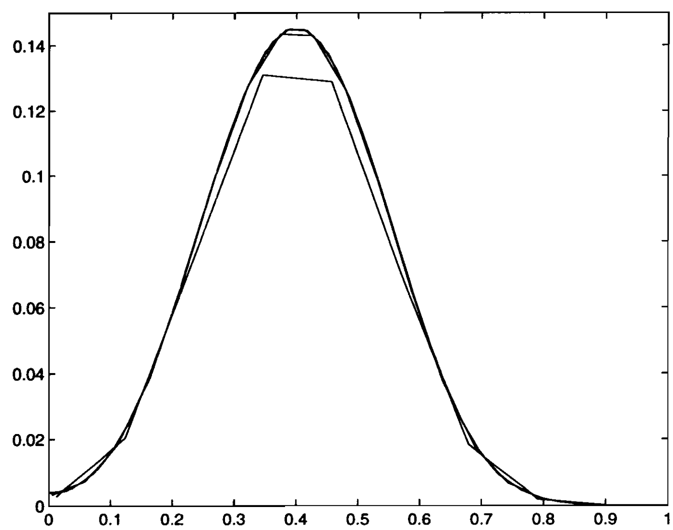

Figure 4.4: The graph of $v_1^h(t)$ for h=1/9, 1/19, 1/29, 1/39.

REFERENCES

1. Adams, **Sobolev Spaces**, Academic, New York, 1975.

2. A. Bensoussan, A. Bossavit and J. C. Nedelec, Approximation des probémes de contrôle optimal, **Cahiers de l' IRIA**, No 2, pages 107-176, 1970.

3. John Burkardt, **Sensitivity Analysis and Computational Shape Optimization for Incompressible Flows**, Ph.D Thesis, Virginia Polytechnic and State University, Department of Mathematics, 1995.

4. M. Cirina, Boundary controllability of nonlinear hyperbolic systems, **IBid,** pages 198-212, 1969

5. F. Colonius, and K. Kunisch, Output Least squares stability in elliptic systems, **Appl. Math. Optim.,** 19, pages 33-63, 1989.

6. Yu, V. Egorov, Some problems in the theory of optimal control, **Soviet Math.**, 3, pages 1080-1084, 1962.

7. Yu, V. Egorov, Z. Vycisl, **Mat. i Mat. Fiz.,** 5, 1963.

8. H. O. Fattcrini, Control in finite time of differential equations in Banach space, **Comm. Pure Appl. Math.**, 19, pages 17-34, 1966.

9. H. O. Fatorini and D. L. Russell, Exact controllability theorems for linear parabolic equations in one space dimension, **Arch. Rational Mech. Anal.**, 49, pages 272-292, 1971.

10. A. V. Fursikov and O. Yu. Imanuvilov, On exact boundary zero controllability of two-dimensional Navier-Stokes equations, **Acta Appl. Math.**, 36, 1994, 1-10.

11. A. V. Fursikov and O. Yu. Imanuvilov, On controllability of certain systems stimulating a fluid flow, in:**Flow Control**, IMA Volumes in Mathematics and Its Applications, Vol. 68, 1995.

12. L. I. Gal'chuk, Optimal control of systems described parabolic equations, **SIAM. J. Control**, 7, pages 546-558, 1969.

13. D. Gay, Algorithm 611, Subroutines for unconstrained minimization using a model/trust region Approach, **ACM transactions on mathematical software**, Volume 9, Number 4, pages 503-524, December 1983.

14. R. Glowinski, C. H. Lee and J. -L. Lions, A numerical approach to the exact boundary controllability of the wave equation (I) Dirichlet controls: Description of the numerical methods. **Japan J. Appl. Math.**, 7, 1990, 1-76.

15. M. D. Gunzburger, L. Hou, and T. P. Svobodny, Analysis and finite element approximation of optimal control problems for the stationary Navier-Stokes equations with distributed and Neumann controls, **Math. Comput.**, 57, No. 195, pages 123-151.

16. L. Hömander, **Linear Partial Differential Operators**, Springer-Verlag, New York Inc. 1969.

17. O. Yu. Imanuvilov, Optimal control problem for the backward heat equation, **Siberian Math**, 1, 181-188, 1993.

18. E. B. Lee and L. W. Markus, **Foundations of Optimal Control Theory**, John Wiley, New York, 1967.

19. H. C. Lee, **Analysis, Finite Element Approximation, and Computation of**

Optimal and Feedback Flow Control Problems, Ph.D Thesis, Virginia Polytechnic and State University, Department of Mathematics, 1994.

20. J. -L. Lions. **Optimal Control of Systems Governed by Partial Differential Equations**, Springer-Verlag, 1971.

21. J. -L. Lions. **Some Methods in the Mathematical Analysis of Systems and Their Controls**, Gordon and Breach, New York, 1981.

22. J. -L. Lions, **Some Aspect of the Optimal Control of Distributed Parameter Systems**, SIAM, Phiadephia, 1972.

23. J. -L. Lions and E. Magnenes, **Nonhomogeneous Boundary Value Problems and Applications**, Springer, Berlin, 1972.

24. J. -L. Lions, Exact controllability, stabilization and perturbations for distributed systems, **SIAM Review**, 30, pages 1-68, 1988.

25. V. A. Morozov, **Methods for Solving Incorrectly Posed Problems**, Springer-Verlag, 1984.

26. V. A. Morozov, **Regularization Methods for Ill-posed Problems**, CRC press, Boca Raton 1993.

27. J. T. Oden and J. N. Reddy, **An Introduction to the Mathematical Theory of Finite Elements**, John Wiley & Sons, New York, 1976.

28. L. S. Pontryagin, Optimal control processes, **Amer. Math. Soc. Transl.**, 18, pages, 321-339, 1961.

29. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mishchenko, **The Mathematical Theory of Optimal Processes**, Interscience, New York, 1962.

30. D. L. Russell, A unified boundary controllability theory for hyperbolic and parabolic partial differential equations, **Studies in Appl. Math.**, 52, pages 189-211, 1973.

31. D. L. Russell, Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions, **SIAM Review**, 20, No. 4, pages 639-739.

32. D. L. Russell, Some remarks on numerical aspects of coefficient identification in elliptic systems, in: **Optimal Control of Partial Differential Equations,** pages 210-228, 1984.

33. V. Thomée, Some convergence results for Galerkin methods for parabolic boundary value problems, in **Mathematical Aspects of Finite Elements in Partial Differential Equations**, Academic Press, New York, pages 55-88, 1974.

34. V. Thomée, **Galerkin Finite Element Methods for Parabolic Problems**, Springer-Verlag, 1984.

35. A. N. Tikhonov, Regularization of ill-posed problems and the regularization method, **Dokl. Akad. Nauk SSSR, Dolk.**, 4, pages 1035-1038, 1963.

36. A. N. Tikhonov and V. Y. Arsenin, **Solutions of Ill-Posed Problems,** Winston-Wiley, New York, 1977.

37. R. Winther, Error estimates for a Galerkin approximation of a parabolic control problem, **Ann. Mat. Pura. Appl.**, 4, 117, pages 173-206, 1978.

VITA

Yanzhao Cao

PERSONAL INFORMATION:

Born: 12/30/62, Heilongjiang, China.

EDUCATION:

1996	Virginia Tech,	Ph.D. Mathematics (expected)
		Thesis advisor: Max D. Gunzburger
1986	Jilin University,	M.S. Mathematics
		Research area: Numerical analysis for PDE
1983	Jilin University, (China)	B.S. Mathematics
		Specialized in computational mathematics

TEACHING EXPERIENCE:

8/94 - present	Graduate Teaching Assistant, Virginia Tech
	Taught three courses (with full responsibility) per academic year
9/92 - 12/92	Graduate Teaching Assistant, North Dakota State University
	Taught two recitation classes
9/86 - 6/91	Lecturer, Heilongjiang University, China
	Taught two courses per academic year

RESEARCH AREAS:

Numerical Analysis for Partial Differential Equations;

Numerical Analysis for Integral Equations;

Computational Methods in Control Theory.

COMPUTATIONAL SKILLS:

Matlab; Mathematica; Fortran.

PUBLICATIONS:

On finite element approximations for the bifurcation solutions of free convention problems (with M. Huang), *Proceedings on numerical methods for partial differential equations*, World Scientific, Singapore, 3-14 1992.

Singularity preserving Galerkin methods for weakly singular Fredholm integral equations, (with Y. Xu) *J. Integral Equations Appl.*, 6 303-334 (1994)

Submitted:

A nonconforming least squares finite element method for interface problems, (with M. Gunzburger) *submitted.*

Finite element approximations for exact boundary controllability problems, (with M. Gunzburger) *submitted.*

Preprints:

A collocation method for a class of neutral equations, (with C. Cerezo, T. Herdman, M. Gunzburger) *preprint.*

Singularity preserving collocation methods for weakly singular Volterra integral equations, (with Y. Xu) *preprint.*

Contributed lectures:

November 1995, Seminar, Virginia Tech, "Numerical methods for exact boundary controllability problems of parabolic equations".

March 1995, Seminar, Virginia Tech, "Regularity methods for identification problems of elliptic equations".

November 1991, Colloquium, North Dakota State University, "High order Galerkin methods for weakly singular Fredholm integral equations".

May 1991, Conference, Tianjin, China, "Finite element approximations for the bifurcation solutions of free convention problems".