PROBABILISTIC FORMULATIONS OF SOME FACILITY LOCATION PROBLEMS

by

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Dissertation submitted to the Graduate Faculty of the

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Industrial Engineering and Operations Research

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January 1975

Blacksburg, Virginia

ACKNOWLEDGMENTS

The author wishes to express his appreciation to Dr. J. A. White, his major advisor, for the time, guidance, encouragement and patience he provided during the course of this research.

The author is grateful to Dr. D. M. Miller and Dr. I. M. Besieris for suggesting and providing several references which contributed much towards accomplishing this research.

Gratitude is also expressed to Dr. P. M. Ghare for contributing his knowledge and time to the author throughout the research effort.

An expression of gratitude is extended to Dr. J. M. Moore for his encouragement and understanding during the preparation of this research.

Appreciation is also extended to Ms. Margie Strickler for her patience and excellent typing of the manuscript under time limitations.

A special word of appreciation is given to Dr. M. A. Hamdan and Dr. D. R. Jensen in the Department of Statistics for their unselfish contributions in various aspects of the research.

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Chapter 1

INTRODUCTION

1.1 Subject of the Research

Since 1960 over five hundred papers have been published on the subject of facilities location. Typical examples of location problems which have been studied include the locations of fire stations, health outreach clinics, ambulances, warehouses, plants, machine tools, schools, solid waste receptacles, controls on a control panel, and military bases. Thus, both public and private sector location problems have been studied in recent years.

Due to the wide variety of problems involving the location of one or more facilities, many different disciplines have developed an interest in location problems. Among the academic disciplines which have been involved in the study of location problems are architecture, industrial engineering, operations research, regional science, mathematics, computer science, management science, systems engineering, urban planning, transportation, and economics.

To date, the study of location problems has been restricted primarily to deterministic formulations of the problem. In this research effort, the effect of random variation on the location decision will be studied. The research objectives are threefold. Namely, probabilistic formulations of a variety of location problems are developed, solution procedures are obtained, and computation experience is provided. Applications of the various formulations are cited to motivate the relevancy of the research.

The intent of the research is to explore the effects of random variation on the location problem. Typical questions to be considered concern the complexity of the resulting formulation and associated solution procedure and the optimum solutions obtained. Conceivably, a consideration of random variation could simplify the analysis; likewise, by considering random variation the resulting formulation might become sufficiently complex that solutions cannot be obtained efficiently. Additionally, the optimum location for a facility might not be affected significantly by considering random variation.

1.2 Sources of Random Variation in Location Problems

A taxonomy of facilities location problems, based on one proposed by Francis and White [32], is depicted in Figure 1.1. As indicated in Figure 1.1, the elements of a facilities location problem which might be reasonably expected to be random variables are the locations of the existing facilities and the amount of interaction between new and existing facilities.

As an illustration of a location problem in which random variation can exist, consider the location of a maintenance department for material handling equipment in an industrial plant. Maintenance performed is of two types, scheduled and unscheduled. Unscheduled maintenance arises when the material handling equipment fails and a repairman is dispatched to the job site to perform the necessary repairs. The location of the equipment when it fails is a random variable. Additionally, the number of times a particular piece of equipment fails during a year is a random variable. The determination of the location of the maintenance

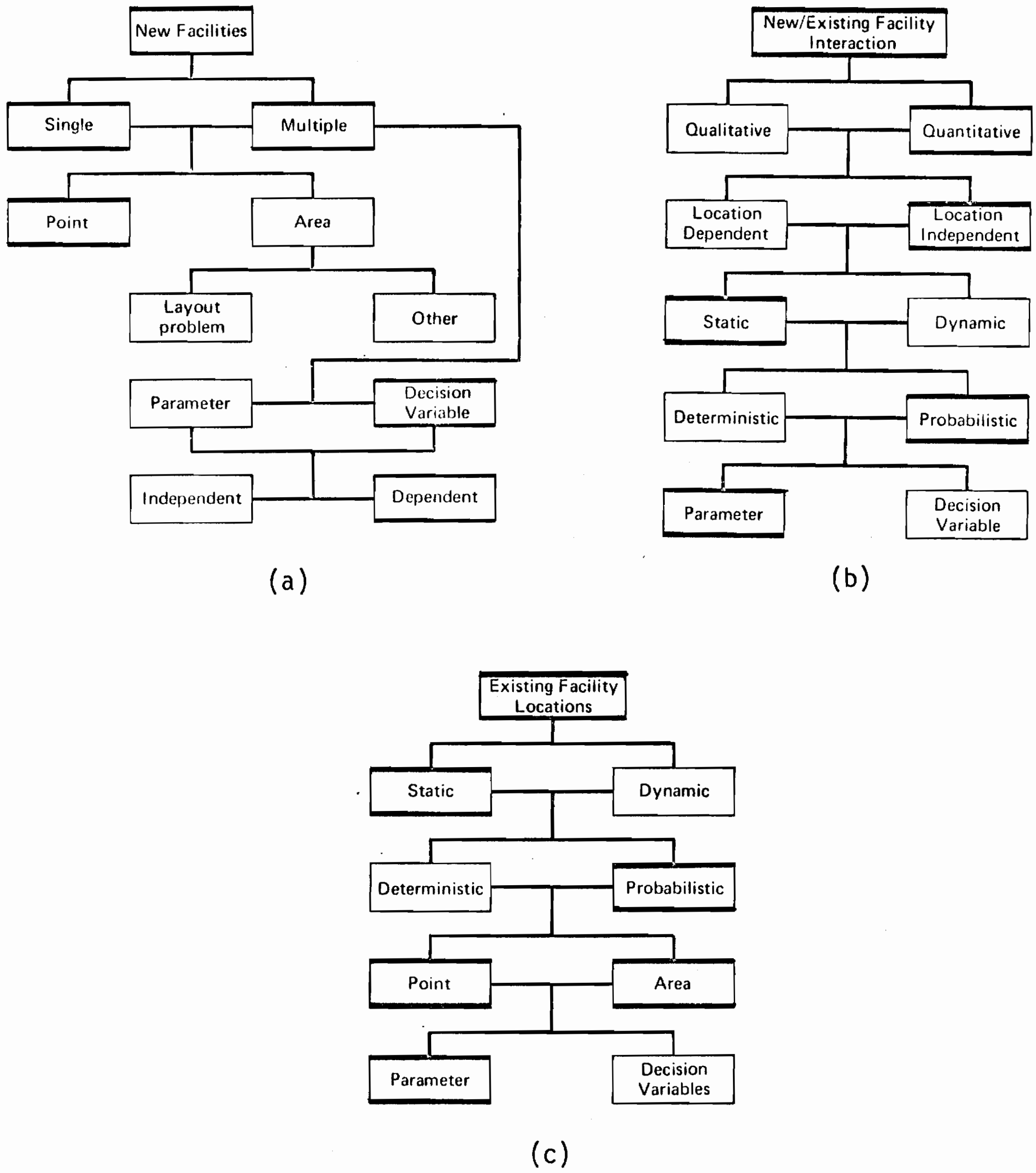
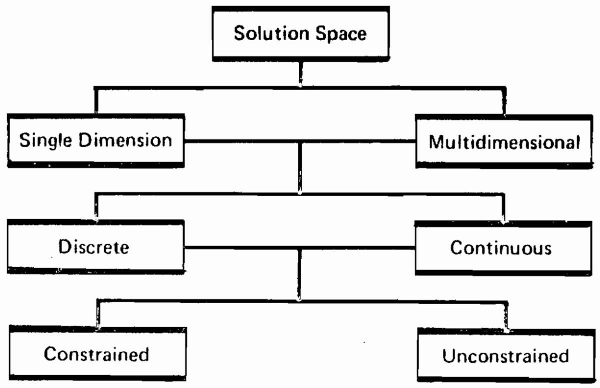


Figure 1.1. Classification of facility layout and location problems.



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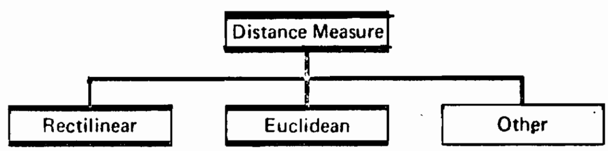

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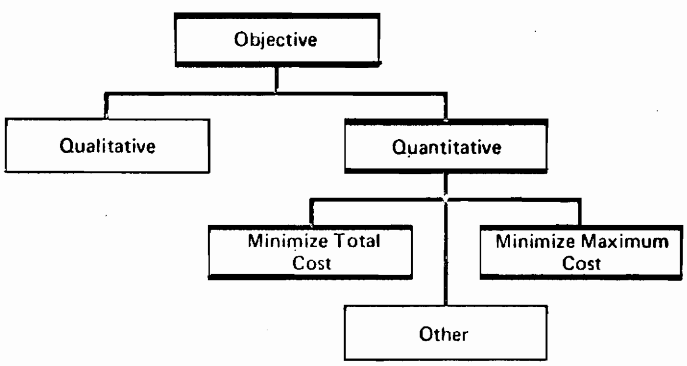


(f)

Figure 1.1. (continued)

department based on the random variation involved is typical of the location problems considered in the research. Other illustrations of location problems possessing random variation are presented throughout the sequel to motivate the particular formulation under study.

1.3 Principles of Choice

In modeling a real world decision problem, Morris [73] suggests three alternatives are available. The problem can be modeled as either a decision under assumed certainty, a decision under risk, or a decision under uncertainty. The research to date on location problems has concentrated on modeling the problem as a decision under assumed certainty. Thus, deterministic approaches were taken. The present research effort concentrates on modeling location problems as decisions under risk. The third alternative of modeling the location problem as a decision under uncertainty will not be considered.

In a decision under risk it is assumed that the probability distributions are known for all random variables. Also, a number of alternate principles of choice are possible. Sengupta and Portillò-Cambell [84] suggest four possible optimization criteria under conditions of risk:

a) expected value criterion
b) portfolio criterion
c) aspiration level criterion
d) fractile criterion.

The expected value criterion involves the determination of the location of the new facilities such that an appropriately defined expected cost function is minimized. The portfolio criterion seeks the location which minimizes the variance of cost, subject to a constraint on the expected cost produced. An aspiration level criterion is used when the facilities are to be located such that the probability of cost being less than some specified value is maximized. The fractile criterion is difficult to express verbally; it involves the location of the new facilities such that one minimizes the value on the cumulative distribution function of total cost which represents the acceptable probability of cost exceeding that value. Mathematically, the fractile criterion can be stated as,

$$\text{Minimize } z$$
$$\text{subject to: } \Pr(f(\underset{\sim}{x}) \leq z) \geq \alpha$$

where $f(\underset{\sim}{x})$ is the total cost resulting from $\underset{\sim}{x}$, the vector of coordinate locations on the new facilities, z is the cost below which total cost occurs with at least a probability of α.

Depending upon the situation considered and the preferences of the decision maker, additional constraints can be added to the four basic principles of choice. As an illustration, one might wish to minimize the total expected distance traveled between all facilities, with the restriction that the probability of the total distance exceeding 100 miles must be less than 0.10.

The aspiration level and fractile criteria require that the probability distribution for total cost be known; whereas, the expected value and portfolio criteria require a knowledge of at most the first two moments of the distribution of total cost. Consequently, in order to provide sufficient information to model a specific location problem using the appropriate principle of choice, the probability distribution for total cost will be developed for the situations considered.

Geoffrion [36] discussed the above criteria and the relationships between them. Sengupta and Portillo-Campbell [84] and Hazell [43] supported Geoffrion's observations with computational results.

1.4 Location Problems Considered

Two classes of location problems are considered in this research. Deterministic formulations of the problems treated are referred to as the generalized Weber problem, and the emergency service facilities location problem.

The generalized Weber problem involves the location of one or more new facilities in the plane relative to several existing facilities such that the total cost of item movement per unit time is minimized. The location problem can be formulated deterministically as follows:

$$\text{Minimize} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ji} d(X_j, P_i) + \sum_{1 \leq j < k \leq n} v_{jk} d(X_j, X_k)$$

where

m = number of existing facilities

n = number of new facilities

X_j = (x_j, y_j), the coordinate location of the jth new facility

P_i = (a_i, b_i), the coordinate location of the ith existing facility

$d(U,V)$ = distance between the points U and V

w_{ji} = cost per unit time per unit distance traveled between existing facility i and new facility j

v_{jk} = cost per unit time per unit distance traveled between new facilities j and k.

One version of the emergency service facilities location problem which is treated can be formulated deterministically as a set covering problem. Namely, it is desired that the minimum number of emergency service facilities be located so that all customers are covered. Mathematically the problem can be stated as follows:

$$\text{Minimize} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to:} \quad \sum_{j=1}^{n} a_{ij} x_j \geq 1 \qquad \text{for all } i$$

$$x_j = (0,1) \qquad \text{for all } j$$

where

a_{ij} = 1, if customer i is covered by a facility located at site j

= 0, otherwise

c_j = cost of locating a facility at site j

x_j = 1, if a facility is located at site j

= 0, otherwise.

A second version of the problem of locating emergency service facilities is the location-allocation problem. The location-allocation problem treated in this research is formulated as follows:

$$\text{Minimize} \quad \sum_{j=1}^{n} \sum_{i=1}^{m} z_{ij} w_i \, d(X_j, P_i)$$

$$\text{subject to:} \quad \sum_{j=1}^{n} z_{ij} = 1 \qquad \text{for all } i$$

$$z_{ij} = (0,1)$$

where the notation of the generalized Weber problem is employed and z_{ij} is the allocation decision variable defined as

$$z_{ij} = \begin{cases} 1, & \text{if existing facility } i \text{ is allocated to new facility } j \\ 0, & \text{otherwise.} \end{cases}$$

1.5 Scope and Limitations

The analysis of the effects of random variation on the facilities location decision is limited to a consideration of the two classes of location problems cited previously. Additionally, constrained and unconstrained versions of the expected value criterion will be employed. Both discretely and continuously distributed random variables are treated; however, for the case of continuously distributed random variables, the normal distribution is emphasized. Only two sources of random variation are considered: the locations of existing facilities and the "weights" between facilities.

1.6 Order of Presentation

To facilitate the presentation of the research the dissertation is organized as follows. Due to the magnitude of the analysis of the generalized Weber problem, Chapter 2 contains the research findings for the case of a single new facility when the locations of existing facilities are random variables; both constrained and unconstrained formulations are considered. The research is generalized in Chapter 3 to the case of multiple new facilities and the treatment of both the locations of existing facilities and the "weights" between facilities being random variables. Thus, the results obtained in Chapter 3 can be applied to the single facility location problem treated in Chapter 2. The emergency service facilities location problem is studied in Chapter 4. The research is summarized, conclusions are drawn, and recommendations are made in Chapter 5.

In each chapter, the related research literature is surveyed; alternative probabilistic formulations of the specific class of location problems are developed; applications of the formulations are cited; appropriate solution procedures are obtained; and computational experience with the solution procedures is provided.

Chapter 2

SINGLE FACILITY LOCATION PROBLEMS

2.1 Introduction

The deterministic formulation of the single facility location problem has been studied extensively in the literature. The problem is to determine the location of a new facility with respect to a number of existing facilities so that a total cost function is minimized. A general formulation of the unconstrained problem is given by D2.1.

D2.1 $$\underset{X}{\text{minimize}}\ f(X) = \sum_{i=1}^{m} w_i\ |X-P_i|_\ell$$

where

X = location of the new facility, $X \in E^2$

m = number of existing facilities

P_i = location of existing facility i, $i = 1, \ldots, m$, $P_i \in E^2$

w_i = annual cost per unit distance between the new facility and existing facility i

$|X-P_i|_\ell$ = distance between the locations of the new facility and existing facility i, i.e., the ℓ-norm of the distance between the two vectors X and P_i

$f(X)$ = total cost as a function of X.

In D2.1, if $\ell = 1$ for the distance representation, then the distance is measured by the rectilinear norm. If $\ell = 2$, then the distance is measured by the Euclidean norm.

When the Euclidean norm is used, the problem is known in the literature as the Weber problem. A special case of D2.1 is the Fermat problem in which all the weights are equal to one. Weber [98] introduced a method of solution when there are only three existing facilities. Several solution procedures depending on an iterative scheme were developed by Cooper [9], Kuhn and Kuenne [62], Miehle [71] and Palermo [74]. Kuhn [61], Katz [52], and Weizfeld [99] supplied a proof for the convergence to the optimal solution when the iterative algorithm is used. Subsequently, Eyster, et al. [22] and Wesolosky and Love [105] employed a hyberbolic approximation to the distance measure. A non-linear programming alogirthm for the multi-facility problem was proposed by Love [67].

When the rectilinear norm is used, Wesolosky and Love [103] observed that the problem could be solved as a linear programming problem. Independently, Cabot, et al. [3] solved the multi-facility problem by converting it to a linear programming problem. The optimal solution is obtained by Cabot, et al. by formulating the dual problem and converting it to a network flow problem.

A generalization of the unconstrained problem D2.1 to accommodate norm constraints has received limited attention. The problem arises when an upper bound is required on the distance traveled between the new facility and existing facilities. This requirement, when the new facility is a central facility, is essential to reduce the distance traveled by the customer. For the emergency location problem, the maximum time to reach an incident (call) is very vital; in practical problems local governments might impose limits on the time required to answer an alarm.

The constrained problem can be formulated mathematically as,

$$\text{D2.2} \quad \underset{X}{\text{minimize}}\ f(X) = \sum_{i=1}^{m} w_i \, |X-P_i|_\ell$$

$$\text{subject to:} \quad |X-P_i|_\ell \leq \alpha_i \qquad \text{for all } i,\ i = 1, \ldots, m$$

where α_i is a preassigned value representing the upper bound on the distance traveled.

When rectilinear distances are used, the constraints in D2.2 can be augmented to the linear programming problem formulation; however, the dimensionality of the problem will be increased drastically. For the case of Euclidean distance, the same treatment is used to lump the constraints with the objective function, forming an unconstrained penalty function which could be optimized using a convex programming algorithm. Love [67] solved the constrained Euclidean problem using SUMT. Schaefer and Hurter [83] used the concepts of Lagrange multipliers to convert the constrained problem to an unconstrained problem. Planchart [76], using a geometric programming dual formulation, developed an algorithm to solve the constrained problem. For more details about the deterministic single facility location problems, consult El-shafei and Haley [16] and Francis and White [32].

In this chapter, the probabilistic versions of the single facility location problems are studied. In all formulations it is assumed that w_i is known deterministically and the only random variable is P_i. An equivalent probabilistic formulation to D2.1 is to minimize the total expected cost, where each P_i is treated as a continuously distributed random variable. Also, a probabilistic formulation equivalent to D2.2

is to minimize the total expected cost where there is a predetermined upper bound imposed on the expected distance traveled between the new facility and each existing facility. A third formulation is presented for the constrained problem where, instead of the upper bound imposed on the expected distance traveled, a chance constraint is introduced to provide a confidence interval on the random distance as discussed in Chapter 1.

Throughout the chapter, the normal distribution is used as the probability density function for the random variables; the discrete distribution formulations are presented in a separate section. For the rectilinear case the exponential distribution is treated in Appendix A to study the impact of different distributions on the location problem. The problem of "location on a line," ($X \varepsilon E^1$), is studied for the constrained problem to aid in understanding the general problem ($X \varepsilon E^2$). A solution procedure is proposed for each model; numerical examples are solved to show the impact of the probabilistic formulations and their relationship to the deterministic formulations.

2.2 Probabilistic Formulations

As discussed in Chapter 1, the probabilistic formulation for the unconstrained problem is to obtain the expected value of the random cost of location. Other formulations will not be treated in this research effort, since the portfolio criterion is considered as a constrained problem. Some authors treated the unconstrained problem using a fractile criterion which employed chance constraints to satisfy a predetermined aspiration level. However, the pros and cons for each formulation are not clear unless extensive computational experience accompanies each.

For a critique of the effectiveness of the fractile and aspiration formulations refer to Hazell[43].

Using the expected value criterion, a probabilistic formulation for the unconstrained facility location problem, D2.1, is given by,

$$\text{P2.1} \quad \underset{X}{\text{minimize}}\ Z = \sum_{i=1}^{m} w_i\ E[|X-P_i|_\ell]$$

$$= \sum_{i=1}^{m} w_i \int_{b_i} \int_{a_i} |X-P_i|_\ell\ f(P_i)\ da_i\ db_i$$

where

$$f(a_i) \sim N(\mu_{a_i}, \sigma^2_{a_i}), \quad \text{for all } i$$

$$f(b_i) \sim N(\mu_{b_i}, \sigma^2_{b_i}), \quad \text{for all } i$$

and

$$f(P_i) \sim \text{bivariate normal distribution}$$

where $f(P_i) = f(a_i) \cdot f(b_i)$, for all i.

The distance in P2.1 is measured either by the rectilinear or the Euclidean norm.

The constrained problem may be formulated similarly to the deterministic model D2.2 when an upper bound on the individual distance is required. The problem is given:

$$\text{P2.2} \quad \underset{X}{\text{minimize}}\ Z = \sum_{i=1}^{m} w_i \int_{b_i} \int_{a_i} |X-P_i|_\ell\ f(P_i)\ da_i\ db_i$$

$$\text{subject to:} \quad E[|X-P_i|_\ell] \leq \alpha_i \qquad \text{for all } i,\ i = 1, \ldots, m$$

where α_i is the predetermined upper bound. To give more flexibility and to account for the probabilistic element without losing its variation, chance constraints are employed. Mathematically the problem is formulated as:

$$\text{P2.3} \quad \underset{X}{\text{minimize}}\ Z = \sum_{i=1}^{m} w_i \int_{b_i} \int_{a_i} |X-P_i|_\ell \, f(P_i) \, da_i \, db_i$$

$$\text{subject to:} \quad \Pr(|X-P_i|_\ell \leq \alpha_i) \geq \beta_i \ ,$$

where α_i is the preassigned constant representing the upper bound on the distance between the new facility and the existing facility. The service level (aspiration level) is denoted by β_i, i.e., $0 \leq \beta \leq 1$; in this research β_i is chosen between 0.5-0.99, depending on the importance of the activity between X and P_i.

In both P2.2 and P2.3, it is assumed that α_i is chosen so that the constraint is a binding constraint since, if α_i is arbitrarily large, the constraint may be dropped and the problem solved as an unconstrained problem.

2.3 Related Work

Cooper [12], to determine the location of a new facility, has employed a convergent, iterative search technique which minimizes expected distance traveled. In Cooper's model, the locations of the existing facilities, P_i, are considered random variables with a bivariate normal probability density function; also, the distances are in the Euclidean norm. From the necessary and sufficient conditions for an optimal solution, Cooper derives the following iteration formulas,

$$x_1^{k+1} = \frac{\displaystyle\sum_{i=1}^{m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{w_i a_i \; f(P_i)}{\sqrt{(x_1{}^k - a_i)^2 + (x_2{}^k - b_i)^2}} \; da_i \; db_i}{\displaystyle\sum_{i=1}^{m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{w_i \; f(P_i)}{\sqrt{(x_1{}^k - a_i)^2 + (x_2{}^k - b_i)^2}} \; da_i \; db_i},$$

$$x_2^{k+1} = \frac{\displaystyle\sum_{i=1}^{m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{w_i b_i \; f(P_i)}{\sqrt{(x_1{}^k - a_i)^2 + (x_2{}^k - b_i)^2}} \; da_i \; db_i}{\displaystyle\sum_{i=1}^{m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{w_i \; f(P_i)}{\sqrt{(x_1{}^k - a_i)^2 + (x_2{}^k - b_i)^2}} \; da_i \; db_i}$$

where

$$f(P_i) = \frac{1}{2\pi \; \sigma_{a_i} \sigma_{b_i}} \; e^{-\frac{1}{2}\left[\left(\frac{a_i - \mu_{a_i}}{\sigma_{a_i}}\right)^2 + \left(\frac{b_i - \mu_{b_i}}{\sigma_{b_i}}\right)^2\right]}$$

$$X = (x_1, x_2)$$

$$P_i = (a_i, b_i)$$

Cooper indicated that this iterative scheme will converge to the global optimal solution, when $X^{k+1} = X^k = X^*$. A comparison is provided between the probabilistic solutions and the deterministic solutions (when $P_i = (\mu_{a_i}, \mu_{b_i})$, i.e., its location is known deterministically).

Katz and Cooper [51] supplied a proof for the convergence of the algorithm and stated that the order of convergence is linear. In a recent article Katz and Cooper [52] employed the same iterative algorithm to treat both exponential and symmetrical exponential distributions as a probability density function for the random variable P_i. Some test problems were solved. They also introduced a lemma to establish conditions such that the optimal solution X^* would fall in the convex hull of the means, i.e., $x_1^* \varepsilon \{\mu_{a_i}\}$, $x_2^* \varepsilon \{\mu_{b_i}\}$.

When P_i is considered as a random variable in the unconstrained single facility location problem, the literature is restricted to a treatment of Euclidean distances. In the subsequent sections of this chapter, the unconstrained single facility location problem under rectilinear distance will be formulated and a solution approach will be proposed. This will be developed for the case when P_i is distributed according to the normal distributions. The same model of Cooper [12] will be approached differently, resulting in a much simpler iterative scheme, which will be helpful in the general model of Chapter 3. The constrained single facility location problem will be treated for either norm type constraints and chance constraints.

2.4 Rectilinear-Distance Location Problems: Unconstrained

The rectilinear distance single facility location problem may be stated mathematically as

$$\text{P2.4} \quad \underset{X}{\text{minimize}}\ f(X) = \sum_{i=1}^{m} \int_{P_i} w_i(|x_1-a_i| + |x_2-b_i|)\ f(P_i)dP_i$$

where

$X = (x_1,x_2)$

$P_i = (a_i,b_i)$, $i = 1, \ldots, m$

$f(P_i)$ = joint probability density function of location P_i, $i = 1, \ldots, m$

$f(a_i)$ = marginal probability density function of a_i,
$i = 1, \ldots, m$

$f(b_i)$ = marginal probability density function of b_i,
$i = 1, \ldots, m$.

It is assumed, henceforth, that for a given i the random variables, a_i, b_i are independent. Also, all m locations are considered independent of each other. In the location analysis the probability density functions, $f(P_i)$, are given by the bivariate normal distribution. The case of the bivariate exponential distribution is treated in Appendix A.

2.4.1 Rectilinear-Distance Location Problem with a Bivariate Normal Density Function

When the bivariate normal distribution is applied to P2.4, the resulting problem is given by,

$$\text{P2.4.1} \quad \underset{X}{\text{minimize}}\ f(X) = \sum_{i=1}^{m} w_i \int_{b_i} \int_{a_i} (|x_1-a_i|+|x_2-b_i|)\ f(P_i)da_i db_i$$

where

$$f(a_i) = \frac{1}{\sqrt{2\pi}\ \sigma_{a_i}}\ e^{-\frac{1}{2}\left(\frac{a_i-\mu_{a_i}}{\sigma_{a_i}}\right)^2}, \quad -\infty < a_i < \infty,\ i = 1, \ldots, m$$

$$f(b_i) = \frac{1}{\sqrt{2\pi}\ \sigma_{b_i}}\ e^{-\frac{1}{2}\left(\frac{b_i-\mu_{b_i}}{\sigma_{b_i}}\right)^2}, \quad -\infty < b_i < \infty,\ i = 1, \ldots, m$$

$$f(P_i) = f(a_i)\cdot f(b_i),$$

since a_i and b_i are independent. Hence,

$$f(P_i) = \frac{1}{2\pi\ \sigma_{a_i}\sigma_{b_i}}\ e^{-\frac{1}{2}\left[\left(\frac{a_i-\mu_{a_i}}{\sigma_{a_i}}\right)^2+\left(\frac{b_i-\mu_{b_i}}{\sigma_{b_i}}\right)^2\right]}\ ,\ i = 1,\ \ldots,\ m.$$

Using the independence property, problem P2.4.1 can be further simplified using the linear-loss integral formulations developed by Raiffa and Schlaifer [79]. Following their approach, let

$$Z_i(X) = \int_{b_i}\int_{a_i} (|x_1-a_i|+|x_2-b_i|)\ f(P_i)\ da_i db_i\ .$$

Hence

$$Z_i(X) = \int_{b_i} f(b_i)db_i \int_{a_i} |x_1-a_i|\ f(a_i)da_i$$

$$+ \int_{a_i} f(a_i)da_i \int_{b_i} |x_2-b_i|\ f(b_i)db_i$$

since $\int_{b_i} f(b_i)db_i = \int_{a_i} f(a_i)da_i = 1$, then

$$Z_i(X) = \int_{a_i} |x_1-a_i|\ f(a_i)da_i + \int_{b_i} |x_2-b_i|\ f(b_i)db_i$$

$$= E[|x_1-a_i|] + E[|x_2-b_i|]$$

$$= Z_i(x_1) + Z_i(x_2) \qquad (2.4.1)$$

For simplicity, only one term in (2.4.1) will be considered, since the other term may be obtained by replacing x_2 in place of x_1 or vice versa. Also, since the procedure is the same for all i, let $a_i = a$, $\mu_{a_i} = \mu$, $\sigma_{a_i} = \sigma$, $x_1 = x$, $Z(x) = Z_i(x_1)$. Thus,

$$Z(x) = \int_a |x-a| \; f(a)da$$

$$= \int_{-\infty}^{x} (x-a)f(a)da + \int_{x}^{\infty} (a-x)f(a)da$$

$$= x[\int_{-\infty}^{x} f(a)da - \int_{x}^{\infty} f(a)da] + [\int_{x}^{\infty} a\; f(a)da - \int_{-\infty}^{x} a\; f(a)da]$$

Since

$$\int_{x}^{\infty} f(a) = 1 - \int_{-\infty}^{x} f(a)da \;, \qquad \int_{x}^{\infty} a\; f(a)da = E[a] - \int_{-\infty}^{x} a\; f(a)da$$

then

$$Z(x) = x(2\; F(x)-1) + 2\int_{x}^{\infty} a\; f(a)da - E[a] \tag{2.4.2}$$

where F(x) is the cumulative distribution function. For the normal distribution,

$$F(x) = \frac{1}{\sqrt{2\pi}\;\sigma} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{a-\mu}{\sigma}\right)^2} da = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\left(\frac{x-\mu}{\sigma}\right)} e^{-\frac{1}{2}w^2} dw \;.$$

For the normal distribution

$$\Phi(x) = \int_{-\infty}^{x} \phi(w)dw \;, \qquad \text{where } \phi(w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}w^2} \;.$$

Therefore, $F(x) = \Phi(\frac{x-\mu}{\sigma})$ and (2.4.2) may be written as

$$Z(x) = 2x\,\Phi(\frac{x-\mu}{\sigma}) - (x+\mu) + 2\int_x^\infty a\,f(a)da \tag{2.4.3}$$

where $\mu = E[a]$.

Evaluating the integral in (2.4.3), first

$$\int_x^\infty a\,f(a)da = \frac{1}{\sqrt{2\pi}\,\sigma}\int_x^\infty a\,e^{-\frac{1}{2}(\frac{a-\mu}{\sigma})^2} da$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\frac{x-\mu}{\sigma}}^\infty (w_1\sigma+\mu)\,e^{-\frac{1}{2}w_1^2} dw_1$$

$$= \frac{\sigma}{\sqrt{2\pi}}\int_{\frac{x-\mu}{\sigma}}^\infty w_1\,e^{-\frac{1}{2}w_1^2} dw_1$$

$$+ \frac{\mu}{\sqrt{2\pi}}\int_{\frac{x-\mu}{\sigma}}^\infty e^{-\frac{1}{2}w_1^2} dw_1 \tag{2.4.4}$$

but the second integral may be written as

$$\frac{\mu}{\sqrt{2\pi}}\int_{\frac{x-\mu}{\sigma}}^\infty e^{-\frac{1}{2}w_1^2} dw_1 = \mu[1 - \Phi(\frac{x-\mu}{\sigma})] . \tag{2.4.5}$$

Let $w_2 = w_1^2$, thus $dw_2 = 2w_1dw_1$, and the first integral will become

$$\frac{\sigma}{\sqrt{2\pi}} \int_{\frac{x-\mu}{\sigma}}^{\infty} w_1 \, e^{-\frac{1}{2} w_1^2} dw_1 = \frac{\sigma}{2\sqrt{2\pi}} \int_{(\frac{x-\mu}{\sigma})^2}^{\infty} e^{-\frac{1}{2} w_2} dw_2$$

$$= \frac{\sigma}{\sqrt{2\pi}} \left[e^{-\frac{1}{2} w_2}\right]_{\infty}^{(\frac{x-\mu}{\sigma})^2}$$

$$= \frac{\sigma}{\sqrt{2\pi}} \, e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} = \sigma \, \phi(\frac{x-\mu}{\sigma}) \qquad (2.4.6)$$

Substituting (2.4.5) and (2.4.6) into (2.4.4), then (2.4.3) is given by

$$Z(x) = (x-\mu)[2\Phi\,(\frac{x-\mu}{\sigma}) - 1] + 2\sigma \, \phi(\frac{x-\mu}{\sigma}) \qquad (2.4.7)$$

Equation 2.4.7 represents the expected distance between a_i and x_1 or b_i and x_2. Hence, (2.4.1) is given as

$$Z_i(X) = (x_1-\mu_{a_i})[2\Phi(\frac{x_1-\mu_{a_i}}{\sigma_{a_i}})-1] + 2\sigma_{a_i} \, \phi(\frac{x_1-\mu_{a_i}}{\sigma_{a_i}})$$

$$+ (x_2-\mu_{b_i})[2\Phi(\frac{x_2-\mu_{b_i}}{\sigma_{b_i}})-1] + 2\sigma_{b_i} \, \phi(\frac{x_2-\mu_{b_i}}{\sigma_{b_i}}) \qquad (2.4.8)$$

and total expected cost is given by

$$f(X) = \sum_{i=1}^{m} w_i Z_i(X) \qquad (2.4.9)$$

Before solving the minimization problem P2.4.1, some mathematical properties of the function f(X) are presented.

Theorem 2.4.1: The function $f(X)$ is a convex function of $X \in E^2$.†

Proof: Examine first the function $Z_i(X)$. From its definition, note that $f(P_i) = f(a_i)f(b_i)$ is a probability density function, Therefore $f(P_i) \geq 0$. From the definition of the rectilinear norm, it is clear that

$$|X-P_i| = |x_1-a_i| + |x_2-b_i|$$

is a convex function. Let X^1, $X^2 \in E^2$ and $X^1 \neq X^2$, hence X is expressed as

$$X = \alpha X^1 + (1-\alpha)X^2 \quad , \; 0 \leq \alpha \leq 1 \; .$$

By the convexity property,

$$|X-P_i| \leq \alpha|X^1-P_i| + (1-\alpha)|X^2-P_i| \tag{2.4.10}$$

Multiplying (2.4.10) by $f(P_i) \geq 0$, yields

$$|X-P_i|f(P_i) \leq \alpha|X^1-P_i|f(P_i) + (1-\alpha)|X^2-P_i|f(P_i)$$

Integrating both sides with respect to P_i,

$$\int_{P_i} |X-P_i|f(P_i)dP_i \leq \alpha \int_{P_i} |X^1-P_i|f(P_i)dP_i + (1-\alpha) \int_{P_i} |X^2-P_i|f(P_i)dP_i \tag{2.4.11}$$

From the definition of $Z_i(X)$, inequality (2.4.11) is rewritten as

$$Z_i(X) \leq \alpha \; Z_i(X^1) + (1-\alpha) \; Z_i(X^2)$$

---

†The proof parallels that given by Cooper [12].

Hence, $Z_i(X)$ is a convex function. Using the property that the sum of convex functions is a convex function, then $f(X)$ is a convex function. The proof of the theorem is complete.

Theorem 2.4.2: The function $f(X)$ given in P2.4.1 is a strictly convex function of $X \varepsilon E^2$.

Proof: From Theorem 2.4.1 $f(X)$ is a convex function. To prove that it is strictly convex, it is sufficient to establish that at least one function under the summation is strictly convex. Observe the function $Z(x)$ defined by (2.4.7). Since $Z(x)$ is a function of one variable, then a sufficient condition for it to be strictly convex is to have a positive second derivative for all values of x. From (2.4.2),

$$\frac{d\,Z(x)}{dx} = \frac{d}{dx}\left(x(2F(x)-1) + 2\int_x^\infty a\,f(a)da - E[a]\right)$$

$$= 2F(x) - 1 + 2x\,f(x) - 2x\,f(x)$$

$$= 2F(x) - 1 \qquad (2.4.12)$$

The second derivative is given as

$$\frac{d^2Z(x)}{dx^2} = 2f(x) \geq 0$$

For the normal distribution, $f(x) > 0$ for all x in the open set, $-\infty < x < \infty$. Thus, $Z(x)$ is a strictly convex function, i.e., $Z_i(x_1)$ and $Z_i(x_2)$ are strictly convex functions over all $x_1, x_2 \varepsilon E^1$. Therefore, $Z_i(X)$ is also a strictly convex function. Consequently, $f(X)$ is a strictly convex function over $X \varepsilon E^2$.

From Theorem 2.4.2, f(X) is a strictly convex function; therefore, there is a unique minimum, and the necessary and sufficient condition for the solution is given by

$$\nabla f(X^*) = 0 \; . \tag{2.4.13}$$

The optimality condition (2.4.13) can be used to construct a solution procedure based either on a first order derivative method (steepest descent) or a second order derivative method (Newton, Quasi-Newton). Naturally any direct search technique like the pattern search employed by Hooke and Jeeves [47] may be used as a solution procedure since $\Phi(X)$ is a monotonic increasing function in X. But by checking the second derivatives of f(X), one may utilize an efficient second order derivative algorithm. The property of the Hessian matrix is given by the following.

<u>Property 2.4.1</u>: The inverse of the Hessian of f(X) in P2.4.1 is positive definite.

<u>Proof</u>: From 2.4.12 it is seen that

$$\frac{\partial f(X)}{\partial x_1} = \sum_{i=1}^{m} w_i (2\Phi \left(\frac{x_1-\mu_{a_i}}{\sigma_{a_i}}\right)-1) \; ,$$

$$\frac{\partial f(X)}{\partial x_2} = \sum_{i=1}^{m} w_i \; (2\Phi \left(\frac{x_2-\mu_{b_i}}{\sigma_{b_i}}\right)-1)$$

Hence

$$\frac{\partial^2 f(X)}{\partial x_1^2} = 2 \sum_{i=1}^{m} \frac{w_i}{\sigma_{a_i}} \phi\left(\frac{x_1-\mu_{a_i}}{\sigma_{a_i}}\right) = A_1 ,$$

$$\frac{\partial^2 f(X)}{\partial x_2^2} = 2 \sum_{i=1}^{m} \frac{w_i}{\sigma_{b_i}} \phi\left(\frac{x_2-\mu_{b_i}}{\sigma_{b_i}}\right) = A_2 ,$$

and

$$\frac{\partial^2 f(X)}{\partial x_1 \partial x_2} = \frac{\partial^2 f(X)}{\partial x_2 \partial x_1} = 0 .$$

Therefore, the Hessian and its inverse are constructed as,

$$H(X) = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, [H(X)]^{-1} = \begin{bmatrix} \frac{1}{A_1} & 0 \\ 0 & \frac{1}{A_2} \end{bmatrix}$$

Notice that A_1, A_2 are positive for any values of x_1, x_2 since $\phi(\cdot) \geq 0$. Therefore, $[H(X)]^{-1}$ is positive definite.

Using an iterative algorithm depending on this Hessian matrix, it is guaranteed that the matrix will behave well when X is far from X^*. An iterative scheme based on Newton's method is used to find the minimum of f(X), X^*. As stated in Himmelblau [44], the iteration formula is given by

$$X^{k+1} = X^k - \rho^k S^k \tag{2.4.14}$$

where $-S^{(k)}$ is the direction of steepest descent, and $\rho^{(k)}$ is the step size taken in the direction of steepest descent, $\rho^{(k)} \geq 0$. For Newton's method, S^k is chosen as

$$S^{(k)} = [H(X^{(k)})]^{-1} \nabla f(X^{(k)}) / \left| [H(X^{k}]^{-1} \nabla f(X^{(k)}) \right| \qquad (2.4.15)$$

where $|\cdot|$ is the norm of the vector $(\cdot)$. Thus, $S^{(k)}$ is a directional vector.

From the above property and since $[H(X^k)]^{-1}$ is positive definite, $-S^{(k)}$ must be along the direction of steepest descent. Substituting the values of the gradient vector $\nabla f(X^k)$, and the Hessian into (2.4.15), $S^{(k)}$ is written unnormalized as

$$S^{(k)} = \begin{bmatrix} \dfrac{\sum_{i=1}^{m} w_i [2\Phi(\frac{x_1^{(k)} - \mu_{a_i}}{\sigma_{a_i}}) - 1]}{2 \sum_{i=1}^{m} \frac{w_i}{\sigma_{a_i}} \phi(\frac{x_1^{(k)} - \mu_{a_i}}{\sigma_{a_i}})} \\[2em] \dfrac{\sum_{i=1}^{m} w_i [2\Phi(\frac{x_2^{(k)} - \mu_{b_i}}{\sigma_{b_i}}) - 1]}{2 \sum_{i=1}^{m} \frac{w_i}{\sigma_{b_i}} \phi(\frac{x_2^{(k)} - \mu_{b_i}}{\sigma_{b_i}})} \end{bmatrix} \qquad (2.4.16)$$

The optimum step size, $\rho^{(k)}$, at each iteration may be found in the direction of $-S^{(k)}$ for the value of $\rho^{*(k)}$ which minimizes the function $f(X^{(k+1)})$, where $X^{(k+1)}$ is as given in (2.4.14). As shown in Luenberger [69] the modified Newton's method when $\rho^{*(k)} \neq 1$ converges with order two, which is better than the steepest descent method where the convergence is linear.

Applying the above iterative scheme to f(X), the method will converge to the global optimal solution X^*. Notice that to use the algorithm a starting point X^o is required. In the next section an approximate solution for P2.4.1, hopefully close enough to the optimal to reduce computation time, will be found. Also, if the starting point is close to the optimal, the step size $\rho^{(k)} \approx 1$ facilitates the computations drastically, and in this case there is no need to normalize the vector $S^{(k)}$. As a good initial feasible solution use,

$$X^{(o)} = \begin{bmatrix} \dfrac{\sum_{i=1}^{m} w_i \mu_{a_i}/\sigma_{a_i}}{\sum_{i=1}^{m} w_i/\sigma_{a_i}} \\ \\ \dfrac{\sum_{i=1}^{m} w_i \mu_{b_i}/\sigma_{b_i}}{\sum_{i=1}^{m} w_i/\sigma_{b_i}} \end{bmatrix}$$

2.4.2 Rectilinear-Distance Location Problem with a Bivariate Normal Probability Distribution: Approximate Solution

The previous section concentrated on determining the optimal solution for the rectilinear-distance problem involving the normal distribution. In some practical situations, accuracy of the solution can be sacrificed to obtain reductions in solution time. An approximate solution to the rectilinear distance location problem is developed in this section for the case of a bivariate normal probability distribution.

Due to the fact that the function f(X) defined in P2.4.1 is strictly convex, a necessary and sufficient condition for the optimal solution is given as

$$\nabla f\,(X^*) = \left(\frac{\partial f(X^*)}{\partial x_1}\ ,\ \frac{\partial f(X^*)}{\partial x_2}\right) = 0\ . \tag{2.4.17}$$

Using this condition, it is seen that

$$\frac{\partial f(X^*)}{\partial x_1^*} = \sum_{i=1}^{m} w_i\left[2\Phi\left(\frac{x_1^* - \mu_{a_i}}{\sigma_{a_i}}\right) - 1\right] = 0$$

and

$$\frac{\partial f(X^*)}{\partial x_2^*} = \sum_{i=1}^{m} w_i\left[2\Phi\left(\frac{x_2^* - \mu_{b_i}}{\sigma_{b_i}}\right) - 1\right] = 0$$

which yields,

$$\sum_{i=1}^{m} w_i\ \Phi\left(\frac{x_1^* - \mu_{a_i}}{\sigma_{a_i}}\right) = \frac{\sum_{i=1}^{m} w_i}{2} \tag{2.4.18}$$

$$\sum_{i=1}^{m} w_i\ \Phi\left(\frac{x_2^* - \mu_{b_i}}{\sigma_{b_i}}\right) = \frac{\sum_{i=1}^{m} w_i}{2}\ . \tag{2.4.19}$$

Expanding the function $\Phi(w)$ in a Taylor's series, Hodges and Lehman [45] demonstrated that the following approximation holds to a very close degree,

$$\Phi(w) = \frac{1}{2} + \frac{w}{\sqrt{2\pi}} - \frac{w^3}{6\sqrt{2\pi}} + \ldots \tag{2.4.20}$$

Using the linear term of this approximation and discarding terms with higher order yields

$$\sum_{i=1}^{m} w_i\left[\frac{1}{2} + \frac{1}{\sqrt{2\pi}}\left(\frac{x_1^* - \mu_{a_i}}{\sigma_{a_i}}\right)\right] = \frac{\sum_{i=1}^{m} w_i}{2} .$$

Thus, the optimal value $\tilde{x}_1^*$ is given by

$$\tilde{x}_1^* = \frac{\sum_{i=1}^{m} \frac{w_i \mu_{a_i}}{\sigma_{a_i}}}{\sum_{i=1}^{m} \frac{w_i}{\sigma_{a_i}}} \tag{2.4.21}$$

By symmetry,

$$\tilde{x}_2^* = \frac{\sum_{i=1}^{m} \frac{w_i \mu_{b_i}}{\sigma_{b_i}}}{\sum_{i=1}^{m} \frac{w_i}{\sigma_{b_i}}} \tag{2.4.22}$$

The simplicity of this solution encourages its use as a starting feasible solution $X^{(o)}$ for the optimal iterative procedure discussed in Section 2.4.1.

2.4.3 Results and Conclusions on the Rectilinear-Distance Location Problems

In this section, a study of the properties of the optimal solution obtained in the previous sections will provide more insights to the problem.

By checking the approximate solution given by (2.4.21) and (2.4.22), the following lemma yields a well-known result in the theory of location.

Lemma 2.4.1: Given that the variances of the normal probability density functions associated with the existing facilities P_i are equal along the same coordinate for all i. The optimal solution may be interpreted as the weighted average of means for the bivariate normal distribution.

Proof: The proof is straightforward; if $\sigma_{a_i} = \sigma_a$, $\sigma_{b_i} = \sigma_b$ for all i, then (2.4.21) and (2.4.22) may be written as

$$\tilde{x}_1^* = \frac{\sum_{i=1}^{m} w_i \mu_{a_i}}{\sum_{i=1}^{m} w_i} ,$$

$$\tilde{x}_2^* = \frac{\sum_{i=1}^{m} w_i \mu_{b_i}}{\sum_{i=1}^{m} w_i}$$

and represent the center of gravity of the collection of means of the bivariate normal distributions. The lemma provides a comparison between the center of gravity solution for the squared-Euclidean case and this special case of the probabilistic rectilinear solution. Notice that the only difference between the deterministic gravity solution and its probabilistic counterpart is that (μ_{a_i}, μ_{b_i}) replaces (a_i, b_i). Hence the existing facilities are considered to be located at their mean and the variance has no effect at all in this case in obtaining the probabilistic solution.

From the necessary and sufficient conditions for the case of normal distribution, (2.4.18) may be rewritten as

$$\sum_{i=1}^{m} \theta_i \; \Phi\left(\frac{x_1^* - \mu_{a_i}}{\sigma_{a_i}}\right) = \frac{1}{2}$$

where $\theta_i = w_i / \sum_{i=1}^{m} w_i$. Hence,

$$\sum_{i=1}^{m} \theta_i = 1 \quad , \; 0 < \theta_i < 1 \text{ for all } i.$$

Since $\Phi(0) = \frac{1}{2} = \Phi(x_1 - x_1)$

$$\sum_{i=1}^{m} \theta_i \; \Phi\left(\frac{x_1^* - \mu_{a_i}}{\sigma_{a_i}}\right) = \Phi(x_1^* - x_1^*) \; . \tag{2.4.23}$$

From (2.4.23), notice that $\Phi(0)$ is written as a convex combination of all $\Phi(Z_i)$ where $Z_i \sigma_{a_i} = x_1^* - \mu_{a_i}$. Therefore, x^* is chosen such that the convex combination of the probabilities $\Phi(Z_i)$ lies always at the median of the normal probability distribution. This result is different from the one obtained for the deterministic case in the sense that a different mapping is used for the new and existing facilities. As illustrated in Figure 2.1, in the probability domain x^* corresponds to the median of

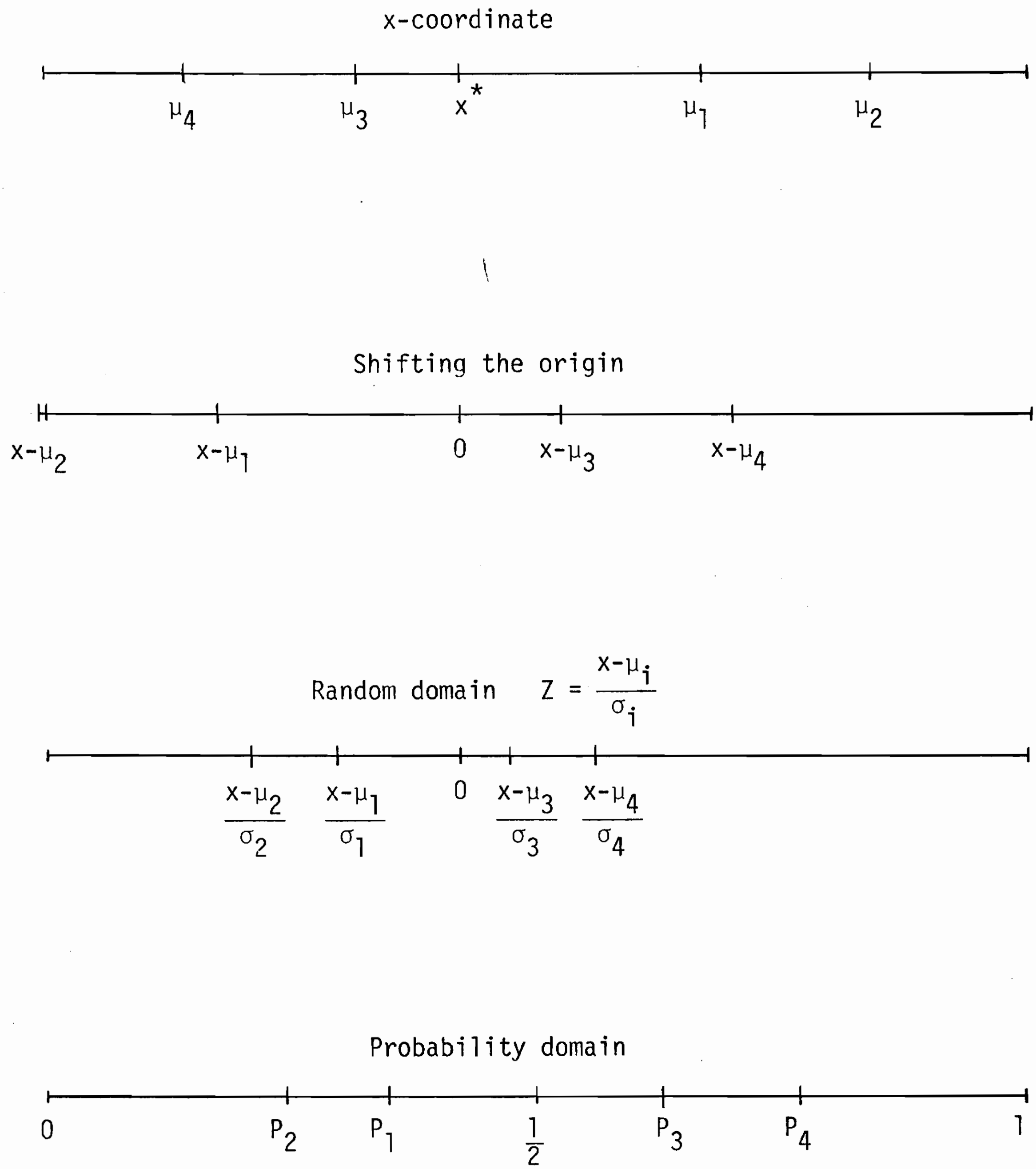


Figure 2.1. The optimal location x^* and its mapping in the probability domain.

the normal distribution; whereas, in the deterministic case, x^* is the median of locations in the original domain. In the random domain, x^* corresponds to the origin which is the median of the value taken by the random variable $Z \sim N(0,1)$.

The above argument is very helpful in determining if x^* lies in the convex hull of the means. Wendell and Hurter [102] proved that x^* lies in the convex hull of the locations of the existing facilities when the distance is either rectilinear or Euclidean and the space is of dimension two. Katz and Cooper [52] arrived at the same result through their lemma for the case of Euclidean distance and the special case of normal distributions when $\sigma_{a_i} = \sigma_{b_i} = \sigma_i$. For the rectilinear and probabilistic case the following theorem can be stated.

Theorem 2.4.3: The optimal solution for P2.4.1 belongs to the convex hull of the means of the distributions corresponding to the random existing facilities.

Proof: Let $\{\mu_i\}$ denote the convex hull of the means. Then the theorem is proved by contradiction. Assume that $x^* \notin \{\mu_i\}$, thus all μ_i lie either to the right or to the left of x^* (see Figure 2.1). In the random domain since x^* corresponds to the origin, then all μ_i correspond to values of Z_i which is either non-negative or non-positive but not both, providing that $\sigma_i \geq 1$ which is true for practical problems, or may be achieved by rescaling the inputs. Assuming they are all non-negative, then the corresponding probabilities $\Phi_i > \frac{1}{2}$ (since $\Phi_i \neq \frac{1}{2}$). Let $\Phi_i = \frac{1}{2} + \varepsilon_i$, where $\varepsilon_i > 0$ for all i. From (2.4.23), the convex combination of Φ_i is given as

$$\Phi_o = \sum_{i=1}^{m} \theta_i \Phi_i = \sum_{i=1}^{m} \theta_i (\frac{1}{2} + \varepsilon_i) = \frac{1}{2} + \sum_{i=1}^{m} \theta_i \varepsilon_i \qquad (2.4.24)$$

Since θ_i, $\varepsilon_i > 0$, then $\Phi_o > \frac{1}{2}$, which is a contradiction.

Katz and Cooper [52] showed that, for the exponential distribution, the optimal solution does not necessarily belong to the convex hull of the means, and actually the optimal solution lies outside the convex hull most of the time.

Another interesting relation may be derived to relate the total cost of both probabilistic and deterministic problems. This may be shown in the following lemma.

Lemma 2.4.2: The optimal total cost obtained from the deterministic version of P2.4.1, in which the existing facilities are located at the means of P_i, provides a lower bound to the probabilistic solution.

Proof: From (2.4.7), let $(x-\mu) = t$

$$z(t) = t\ [(\Phi(\frac{t}{\sigma}) - (1-\Phi(\frac{t}{\sigma}))] + 2\sigma \quad \phi(\frac{t}{\sigma})$$

i) If $x > \mu$, then $t > 0$ and

$$z(t) = t\ \phi(\frac{t}{\sigma})\ [\frac{\Phi(\frac{t}{\sigma})}{\phi(\frac{t}{\sigma})} - \frac{1-\Phi(\frac{t}{\sigma})}{\phi(\frac{t}{\sigma})}] + 2\sigma$$

Since $\frac{1-\Phi(t)}{\phi(t)} = R(t) =$ Mills' ratio, as developed in Mills [72], then

$$Z(t) = t\ \phi(\frac{t}{\sigma})\ [\frac{1}{\phi(\frac{t}{\sigma})} - 2R(t)] + 2\sigma$$

Using the bounds discussed by Gordon [37], where

$$\frac{t}{t^2+1} < R(t) < \frac{1}{t}\,, \qquad t > 0,$$

hence, z(t) can be bound from below, using the right inequality, as

$$Z(t) > t \;\; \phi(\tfrac{t}{\sigma}) \; [\frac{1}{\phi(\frac{t}{\sigma})} - \frac{2}{t}] + 2\sigma \;\; = t + 2(\sigma - 1)\phi(\tfrac{t}{\sigma}) \;.$$

Since $\sigma \geq 1$ and $\phi(\cdot) > 0$ for all arguments, then

$$Z(t) > t$$

or

$$Z(x) > (x-\mu) \tag{2.4.25}$$

ii) If $x < \mu$, let $t = (\mu-x)$ and from (2.4.7)

$$Z(t) = -t[1 - \Phi(\tfrac{t}{\sigma}) - \Phi(\tfrac{t}{\sigma})] + 2\sigma \;\; \phi(\tfrac{t}{\sigma})$$

where $\Phi(-\frac{t}{\sigma}) = 1 - \Phi(\frac{t}{\sigma})$ and $\phi(-\frac{t}{\sigma}) = \phi(\frac{t}{\sigma})$.

Therefore,

$$Z(t) = -t \;\phi(\tfrac{t}{\sigma}) \; [2R - \frac{1}{\phi(\frac{t}{\sigma})}] + 2\sigma$$

Hence Z(t) can be bounded from below using the left inequality, as

$$Z(t) > -t \;\phi(\tfrac{t}{\sigma}) \; [\frac{2t}{t^2+1} - \frac{1}{\phi(\frac{t}{\sigma})}] + 2\sigma$$

$$= t + 2(\sigma \;\; - \frac{t^2}{t^2+1}) \;\phi(\tfrac{t}{\sigma})$$

Since $\frac{t^2}{t^2+1} < 1$, and $\phi(\frac{t}{\sigma}) > 0$, then

$$Z(t) > t$$

or it can be written as

$$Z(x) > (\mu - x) \; . \qquad (2.4.26)$$

From inequalities (2.4.25) and (2.4.26) it can be concluded that

$$Z(x) > |x-\mu| \; .$$

Therefore,

$$Z_i(X) = Z_i(x_1) + Z_i(x_2) > |x_1-\mu_{a_i}| + |x_2-\mu_{b_i}|$$

and the total cost value is obtained from (2.4.9),

$$f(X) = \sum_{i=1}^{m} w_i Z_i(X) > \sum_{i=1}^{m} w_i(|x_1-\mu_{a_i}|+|x_2-\mu_{b_i}|) = \overline{f}(X) \quad (2.4.27)$$

where $\overline{f}(X)$ is the equivalent deterministic objective function when the existing facilities are placed at the means of P_i. From (2.4.27) can be derived the fact that the probabilistic model, when compared to the deterministic model, overestimates the total cost incurred.

2.5 Squared Euclidean Distance Location Problem: Unconstrained

In this section, the squared Euclidean distance location problem is studied. Also referred to as the gravity problem, the cost of item movement between the new facility and an existing facility is proportional to the square of the Euclidean distance between the facilities. Although there do exist situations in which the squared Euclidean distance formulation is a close approximation of the true cost structure for a location problem, the primary motivation for treating the gravity problem in this research is to facilitate the subsequent analysis of the Euclidean distance location problem.

The objective will be assumed to be the determination of the location of a single new facility such that the expected cost of item movement is minimized. The probability density function describing the location of the existing facilities will be assumed to be the bivariate normal density function; it will also be assumed that the x_1 coordinate and x_2 coordinate locations of an existing facility are statistically independent.

An expected cost formulation of the probabilistic gravity problem is given as follows:

$$\text{P2.5} \quad \underset{X}{\text{minimize}}\ f(X) = \sum_{i=1}^{m} w_i \int_{P_i} [(x_1-a_i)^2+(x_2-b_i)^2] f(P_i)\, dP_i$$

where $f(P_i)$ is the bivariate normal probability density function, i.e.,

$$f(P_i) = \frac{1}{2\pi\,\sigma_{a_i}\sigma_{b_i}}\, e^{-\frac{1}{2}\left[\left(\frac{x_1-\mu_{a_i}}{\sigma_{a_i}}\right)^2+\left(\frac{x_2-\mu_{b_i}}{\sigma_{b_i}}\right)^2\right]}$$

$$= f(a_i)\ f(b_i)$$

where $$f(a_i) = \frac{1}{\sqrt{2\pi}\,\sigma_{a_i}}\, e^{-\frac{1}{2}\left(\frac{x_1-\mu_{a_i}}{\sigma_{a_i}}\right)^2}, \quad -\infty < a_i < \infty,\ i = 1, \ldots, m$$

and $$f(b_i) = \frac{1}{\sqrt{2\pi}\,\sigma_{b_i}} e^{-\frac{1}{2}\left(\frac{x_2-\mu_{b_i}}{\sigma_{b_i}}\right)^2}, \quad -\infty < b_i < \infty,\ i = 1, \ldots, m$$

with the assumption that the coordinates a_i, b_i are independent random variables. To minimize P2.5, two different approaches are used to obtain the same result.

2.5.1 Solution Procedure I

From previous analysis it can be seen that the function f(X) is separable over the variables x_1, x_2. Therefore,

$$\min_X f(X) = \min_{x_1} f(x_1) + \min_{x_2} f(x_2) \ .$$

Hence, the minimum of $f(x_1)$ will be developed and the same procedure will hold for $f(x_2)$

$$f(x_1) = \sum_{i=1}^{m} w_i \int_{a_i} (x_1-a_i)^2 f(a_i)\, da_i$$

$$= \sum_{i=1}^{m} w_i (x_1^2 - 2x_1 \int_{a_i} a_i\, f(a_i) da_i + \int_{a_i} a_i^2\, f(a_i) da_i)$$

$$= \sum_{i=1}^{m} w_i (x_1^2 - 2x_1\mu_{a_i} + E[a_i^2])$$

but $E[a_i^2] = \sigma_{a_i}^2 + \mu_{a_i}^2$. Thus,

$$f(x_1) = \sum_{i=1}^{m} w_i [(x_1-\mu_{a_i})^2 + \sigma_{a_i}^2] \ . \tag{2.5.1}$$

To minimize $f(x_1)$ the derivative of (2.5.1) with respect to x_1 is computed and set equal to zero to obtain

$$\frac{df(x_1^*)}{dx_1^*} = 2 \sum_{i=1}^{m} w_i(x_1^* - \mu_i) = 0 \tag{2.5.2}$$

Hence, the following unique solution is obtained,

$$x_1^* = \frac{\sum_{i=1}^{m} w_i \mu_{a_i}}{\sum_{i=1}^{m} w_i} \tag{2.5.3}$$

Similarly, x_2^* is given by,

$$x_2^* = \frac{\sum_{i=1}^{m} w_i \mu_{b_i}}{\sum_{i=1}^{m} w_i} . \tag{2.5.4}$$

From the above results the following is concluded.

Property 2.5.1: The function $f(x_1)$ defined in (2.5.1) is a strictly convex function over $x_1 \varepsilon E^1$.

Proof: The first derivative of $f(x_1)$ with respect to x_1 is given by (2.5.2); the second derivative is given as follows

$$\frac{d^2f(x_1)}{dx_1^{\,2}} = 2 \sum_{i=1}^{m} w_i > 0, \text{ since all } w_i > 0 \text{ for all } i.$$

Hence, the sufficient condition for the function to be strictly convex is satisfied. Therefore, condition (2.5.2) is both necessary and sufficient for a global minimum. Notice that the optimal solutions given by (2.5.3) and (2.5.4) are interpreted as the weighted average of the two coordinates of the existing facilities.

2.5.2 Solution Procedure II

In this section, the total expected cost function is derived in a different manner than before. Certainly the result obtained will be similar to the one derived in the previous section, but this result is required subsequently.

Theorem 2.5.1: Let the coordinates of the existing facility be two independent random variables, and let each correspond to a normal distribution, i.e., $a \sim N(\mu_a, \sigma_a^2)$, $b \sim N(\mu_b, \sigma_b^2)$. Let R^2 denote the statistic given by $R^2 = (x_1-a)^2 + (x_2-b)^2$. Assuming that $\sigma_a^2 = \sigma_b^2 = \sigma^2$, then the probability density function of R^2 is as follows,

$$g_{R^2}(y) = \frac{1}{2\sigma^2} e^{-\frac{1}{2\sigma^2}(y+\lambda^2)} I_o\left(\frac{\lambda\sqrt{y}}{\sigma^2}\right), \quad 0 < y < \infty$$

where

$$\lambda^2 = (x_1-\mu_a)^2 + (x_2-\mu_b)^2$$

and

I_n = the modified Bessel function of the first kind and of order n.

Proof: Since $a \sim N(\mu_a, \sigma^2)$ and $b \sim N(\mu_b, \sigma^2)$, then $(x_1-a) \sim N(x_1-\mu_a, \sigma^2)$ and $(x_2-b) \sim N(x_2-\mu_b, \sigma^2)$. Let Z_1, Z_2 denote (x_1-a), (x_2-b) and μ_1, μ_2

denote $(x_1-\mu_a)$, $(x_2-\mu_b)$, respectively. The cumulative distribution function of R^2 is given by

$$G(y) = Pr(Z_1^{\ 2} + Z_2^{\ 2} \leq y) \ .$$

If $y < 0$, then $G(y) = 0$. However, if $y \geq 0$, then

$$G(y) = \int_A\int \frac{1}{2\pi\ \sigma^2}\ e^{-\frac{1}{2\sigma^2}[(z_1-\mu_1)^2+(z_2-\mu_2)^2]}\ dz_1dz_2 \qquad (2.5.6)$$

where

$$A = \{z_1,z_2|z_1^{\ 2}+z_2^{\ 2} \leq y\},$$

i.e., the set of all points contained within a circle of radius R, i.e., $R = \sqrt{y}$. To solve the double integral in (2.5.6), polar coordinates will be used.

Let

$$z_1 = r\cos\theta\ ,\ z_2 = r\sin\theta\ ,\ \text{where } r \geq 0,\ 0 \leq \theta < 2\pi$$

$$\mu_1 = \overline{r}\cos\overline{\theta}\ ,\ \mu_2 = \overline{r}\sin\overline{\theta}\ ,\ \text{where } \overline{r} \geq 0,\ 0 \leq \overline{\theta} < 2\pi.$$

From (2.5.6) it is seen that the exponential power, $(z_1-\mu_1)^2 + (z_2-\mu_2)^2$ is just the distance between the two vectors $Z = (z_1,z_2)$ and $\mu = (\mu_1,\mu_2)$. In polar coordinates,

$$(z_1-\mu_1)^2 + (z_2-\mu_2)^2 = r^2 + \overline{r}^2 - 2r\ \overline{r}\cos(\theta-\overline{\theta})$$

where $r^2 = z_1^{\ 2} + z_2^{\ 2}$, $\overline{r} = \mu_1^{\ 2} + \mu_2^{\ 2}$. If it is assumed that the vector μ is in the direction of the polar axis, then $\overline{\theta} = 0$. Hence,

$$dz_1dz_2 = r\ dr\ d\theta \quad ,\ \text{where } 0 \leq r \leq \sqrt{y} \ \text{ and }\ 0 \leq \theta < 2\pi\ .$$

Now, integrating over half the circle, (2.5.6) is written as

$$G_{R^2}(y) = 2\int_0^{\sqrt{y}} \frac{r}{2\pi\,\sigma^2}\, e^{-\frac{1}{2\sigma^2}[r^2+\bar{r}^2]} \int_0^{\pi} e^{(\frac{r\,\bar{r}}{\sigma^2})\cos\theta}\, d\theta\, dr\,, \qquad (2.5.7)$$

but from [1, p. 376], it is seen that

$$\frac{1}{\pi}\int_0^{\pi} e^{z\,\cos\theta}\, d\theta = I_o(z) \qquad (2.5.8)$$

where $I_o(z)$ is the modified Bessel function of the first kind and order zero. Using (2.5.8), G(y) is given by

$$G_{R^2}(y) = 2\int_0^{\sqrt{y}} \frac{r}{2\pi\,\sigma^2}\, e^{-\frac{1}{2\sigma^2}[r^2+\bar{r}^2]} \cdot\pi\, I_o(\frac{r\,\bar{r}}{\sigma^2})\, dr\,.$$

Let $w = r^2$ so that $dw = 2r\,dr$, then

$$G_{R^2}(y) = \int_0^{y} \frac{1}{2\sigma^2}\, e^{-\frac{1}{2\sigma^2}[w+\bar{r}^2]}\, I_o(\frac{\sqrt{w}\,\bar{r}}{\sigma^2})\, dw \qquad (2.5.9)$$

Since R^2 is a random variable of a continuous type, the probability density function of R^2, i.e., $g_{R^2}(y) = dG\,(y)/dy$. Differentiating (2.5.9) with respect to y and letting $\lambda^2 = \bar{r}^2$, gives

$$g_{R^2}(y) = \frac{1}{2\sigma^2}\, e^{-\frac{1}{2\sigma^2}[y+\lambda^2]}\, I_o(\frac{\lambda\sqrt{y}}{\sigma^2})$$

which is equivalent to (2.5.5) and completes the proof of the theorem.

Theorem 2.5.1 will be used throughout the rest of this chapter. If the variance σ^2 equals unity in Theorem 2.5.1, Johnson and Kotz [49] indicated that the distribution is known as the non-central chi square distribution. The expected value of R^2 is given by Theorem 2.5.2.

<u>Theorem 2.5.2</u>: If the probability density function of R^2 is given by (2.5.5), then the expected value of R^2 is given by,

$$E[R^2] = 2\sigma^2 + \lambda^2 . \tag{2.5.10}$$

<u>Proof</u>: The expected value of R^2 is derived as follows,

$$E[R^2] = \int_0^\infty y \; g_{R^2}(y)dy$$

$$= \int_0^\infty \frac{y}{2\sigma^2} \; e^{-\frac{1}{2\sigma^2}(y+\lambda^2)} \; I_o\left(\frac{\lambda\sqrt{y}}{\sigma^2}\right) dy$$

Let $\bar{\lambda}^2 = \frac{\lambda^2}{2\sigma^2}$, $\frac{y}{2\sigma^2} = w$, then $dw = \frac{1}{2\sigma^2} dy$

$$E[R^2] = 2\sigma^2 \; e^{-\bar{\lambda}^2} \int_0^\infty we^{-w} \; I_o(2\bar{\lambda} \sqrt{w}) \; dw \tag{2.5.11}$$

But from [1, p. 375],

$$I_o(z) = \sum_{k=0}^\infty \frac{\left(\frac{z^2}{4}\right)^k}{(k!)^2} = \sum_{k=0}^\infty \frac{\left(\frac{z}{2}\right)^{2k}}{(k!)^2} \tag{2.5.12}$$

Substituting (2.5.12) in (2.5.11) yields

$$E[R^2] = 2\sigma^2 \; e^{-\bar{\lambda}^2} \sum_{k=0}^\infty \frac{(\bar{\lambda})^{2k}}{(k!)^2} \int_0^\infty we^{-w} \; (\sqrt{w})^{2k} \; dw .$$

It can be seen that the integral is a gamma function where

$$\Gamma(z) = \int_0^\infty t^{z-1}\, e^{-t}\, dt$$

from [1, p. 255]. Therefore, the integral equals

$$\int_0^\infty w^{k+1}\, e^{-w}\, dw = \Gamma(k+2) = (k+1)(k!) \tag{2.5.13}$$

Substituting (2.5.13) for the integral,

$$E[R^2] = 2\sigma^2\, e^{-\bar{\lambda}^2} \sum_{k=0}^{\infty} \frac{(\bar{\lambda}^2)^k}{k!} \cdot (k+1)$$

$$= 2\sigma^2\, e^{-\bar{\lambda}^2} \left[\sum_{k=0}^{\infty} \frac{(\bar{\lambda}^2)^k}{k!} + \sum_{k=1}^{\infty} \frac{k(\bar{\lambda}^2)^k}{k!}\right] .$$

Since $e^z = \sum_{k=0}^{\infty} \frac{(z)^k}{k!}$, then

$$E[R^2] = 2\sigma^2\, e^{-\bar{\lambda}^2} \left[e^{\bar{\lambda}^2} + (\bar{\lambda}^2)\, e^{\bar{\lambda}^2}\right]$$

$$= 2\sigma^2 + 2\sigma^2\, \bar{\lambda}^2$$

Since $\bar{\lambda}^2 = \frac{\lambda^2}{2\sigma^2}$, then

$$E[R^2] = 2\sigma^2 + \lambda^2 ,$$

which is equivalent to the form given by (2.5.10).

The form obtained for the expected distance in Theorem 2.5.2 may be used to solve problem P2.5. Problem P2.5 is now written as

$$\min_{x_1,x_2} f(X) = \sum_{i=1}^{m} w_i \; E[R_i{}^2]$$

$$= \sum_{i=1}^{m} w_i (2\sigma_i^2 + \lambda_i^2)$$

$$= 2 \sum_{i=1}^{m} w_i \sigma_i^2 + \sum_{i=1}^{m} w_i [(x_1 - \mu_{a_i})^2 + (x_2 - \mu_{b_i})^2] \qquad (2.5.14)$$

Any solution (x_1^*, x_2^*) that minimizes (2.5.14) must satisfy the conditions

$$\nabla f(X^*) = \left(\frac{\partial f(X^*)}{\partial x_1^*}, \frac{\partial f(X^*)}{\partial x_2^*}\right) = (0,0) \qquad (2.5.15)$$

Computing the partial derivatives of (2.5.14) with respect to x_1 and x_2, and setting them equal to zero, the following unique solution results,

$$x_1^* = \frac{\sum_{i=1}^{m} w_i \mu_{a_i}}{\sum_{i=1}^{m} w_i} \; , \qquad x_2^* = \frac{\sum_{i=1}^{m} w_i \mu_{b_i}}{\sum_{i=1}^{m} w_i} \; , \qquad (2.5.16)$$

where the solution given by (2.5.16) is equivalent to the one given by (2.5.3) and (2.5.4) using the first approach.

The following interesting property, which ties together the probabilistic and deterministic total cost values, can be derived from equation (2.5.14).

Lemma 2.5.1: The optimal solution for the probabilistic squared Euclidean location problem is equivalent to the optimal solution of the deterministic variation where the existing facilities are located at the means of P_i. Also, the total cost of the deterministic equivalent provides a lower bound on the total cost of the probabilistic variation.

Proof: The proof follows immediately from (2.5.14), since for the deterministic equivalent $\overline{f}(X)$ is given by

$$\overline{f}(X) = \sum_{i=1}^{m} w_i[(x_1-\mu_{a_i})^2+(x_2-\mu_{b_i})^2]$$

Therefore, from (2.5.14),

$$f(X) = 2\sum_{i=1}^{m} w_i\sigma_i^2 + \overline{f}(X) \tag{2.5.17}$$

To prove the first part, minimize the function given by (2.5.17),

$$\min_X f(X) = \min_X \overline{f}(X) .$$

The proof of the second part is obtained directly from (2.5.17). Since

$$f(X^*) = 2\sum_{i=1}^{m} w_i\sigma_i^2 + \overline{f}(X^*)$$

then

$$f(X^*) > \overline{f}(X^*) .$$

Therefore, the value of the total cost obtained from the probabilistic model is greater than the total cost obtained from the comparable deterministic model.

2.6 Euclidean Distance Location Problem: Unconstrained

As the above sections covered the rectilinear and gravity problems, this section deals with the Euclidean distance as a distance measure for the single facility location problem. Although Cooper [12] and Katz and Cooper [51] studied the problem, the approach outlined in this section simplifies the treatment of the problem. The following helps to reduce the effort required to treat the generalized multi-facility location problem treated in Chapter 3.

2.6.1 Modeling and Properties

The Euclidean problem may be defined as

$$\text{P2.6} \quad \underset{X}{\text{minimize}}\ f(X) = \sum_{i=1}^{m} w_i \int_{P_i} [(x-a_i)^2+(y-b_i)^2]^{1/2} f(P_i)\,dP_i$$

where

$$f(P_i) = f(a_i)\cdot f(b_i)$$

$$f(a_i) = \frac{1}{\sqrt{2\pi}\,\sigma_i}\, e^{-\frac{1}{2}\left(\frac{a_i-\mu_{a_i}}{\sigma_i}\right)^2}, \quad -\infty < a_i < \infty, \text{ for all } i$$

$$f(b_i) = \frac{1}{\sqrt{2\pi}\,\sigma_i}\, e^{-\frac{1}{2}\left(\frac{b_i-\mu_{b_i}}{\sigma_{b_i}}\right)^2}, \quad -\infty < b_i < \infty, \text{ for all } i$$

Thus, $f(P_i)$ is the bivariate normal distribution, and its probability density function is given by

$$f(P_i) = \frac{1}{2\pi\sigma_i^2} e^{-\frac{1}{2}\left[\left(\frac{a_i-\mu_{a_i}}{\sigma_i}\right)^2+\left(\frac{b_i-\mu_{b_i}}{\sigma_i}\right)^2\right]}$$

Throughout this analysis it is assumed that $\sigma_{a_i} = \sigma_{b_i} = \sigma_i$ for all i, where the generality of the model is not lost when the variance is equal along both coordinates.

The solution of problem P2.6 will parallel the approach used in the previous section. First, the probability density function of the distance statistic is developed.

Theorem 2.6.1: Given that the random variable P behaves according to the normal distribution, i.e., $a \sim N(\mu_a,\sigma^2)$, $b \sim N(\mu_b,\sigma^2)$. Let R denote the statistic given by $R = [(x_1-a)^2+(x_2-b)^2]^{1/2}$. Then the probability density function of R is

$$\overline{g}_R(r) = \frac{r}{\sigma^2} e^{-\frac{1}{2\sigma^2}(r^2+\lambda^2)} I_o\left(\frac{\lambda r}{\sigma^2}\right) \quad , 0 < y < \infty \tag{2.6.1}$$

where

$\lambda^2 = (x_1-\mu_a)^2 + (x_2-\mu_b)^2$,

I_n = the modified Bessel function of the first kind and order n.

Proof: In Theorem 2.5.1, the probability density function of R^2 is developed. Since the statistic R equals the square root of the statistic R^2, then a change of variables in $g_{R^2}(y)$ will yield the required distribution. Since $g_{R^2}(y)$ is known, then

$$\overline{g}_R(r) = g_R(\sqrt{y}) \cdot |\frac{dy}{dr}|$$

where $r = \sqrt{y}$, thus $2rdr = dy$ and the Jacobian $|\frac{dy}{dr}| = 2r$. Therefore,

$$\overline{g}_R(r) = 2r \frac{1}{2\sigma^2} e^{-\frac{1}{2\sigma^2}(r^2+\lambda^2)} I_o(\frac{\lambda r}{\sigma^2}) \quad , 0 < y < \infty$$

which gives the probability density function shown in (2.6.1).

In P2.6, since f(X) represents the total expected cost, the expected Euclidean distances are required before finding the optimal solution.

<u>Theorem 2.6.2</u>: If the probability density function of R is given by (2.6.1), then the expected value of R is given as

$$E[R] = \sqrt{\frac{\pi}{2}} \; e^{-\frac{\lambda^2}{2\sigma^2}} M(\frac{3}{2}, 1, \frac{\lambda^2}{2\sigma^2}) \tag{2.6.2}$$

where

M(a,b,z) = the confluent hypergeometric function.

<u>Proof</u>: Given the probability density function of R, the expected value is derived as,

$$E[R] = \int_0^\infty r \, \overline{g}_R(r) \, dr$$

$$= \int_0^\infty \frac{r^2}{\sigma^2} e^{-\frac{1}{2\sigma^2}(r^2+\lambda^2)} I_o(\frac{\lambda r}{\sigma^2}) \, dr$$

Using the expansion of $I_o(z)$ given in [1, p. 375], where

$$I_o(z) = \sum_{k=0}^{\infty} \frac{(\frac{z}{2})^{2k}}{(k!)^2}$$

E[R] is written as

$$E[R] = \frac{e^{-\frac{\lambda^2}{2\sigma^2}}}{\sigma^2} \int_0^{\infty} r^2 \sum_{k=0}^{\infty} \frac{(\frac{\lambda r}{2\sigma^2})^{2k}}{(k!)^2} e^{-\frac{r^2}{2\sigma^2}} dr .$$

Let $\frac{r^2}{2\sigma^2} = w$, then $r = \sqrt{2}\,\sigma\,\sqrt{w}$, $dr = \frac{\sigma}{\sqrt{2}} w^{-1/2} dw$, giving

$$E[R] = \frac{e^{-\frac{\lambda^2}{2\sigma^2}}}{\sigma^2} \sum_{k=0}^{\infty} \frac{(\frac{\lambda}{2\sigma^2})^{2k}}{(k!)^2} \int_0^{\infty} (\sqrt{2}\,\sigma)^{2k+2} w^{k+1} e^{-w} \frac{\sigma}{\sqrt{2}} w^{-\frac{1}{2}} dw$$

$$= \sqrt{2}\,\sigma\, e^{-\frac{\lambda^2}{2\sigma^2}} \sum_{k=0}^{\infty} \frac{(\frac{\lambda}{\sqrt{2}\,\sigma})^{2k}}{(k!)^2} \int_0^{\infty} w^{k+\frac{1}{2}} e^{-w} dw \qquad (2.6.3)$$

But the integral is a gamma function [1, p. 255], thus

$$\int_0^{\infty} w^{k+\frac{1}{2}} e^{-w} dw = \Gamma(k+\tfrac{3}{2})$$

Substituting the value of the integral in (2.6.3),

$$E[R] = \sqrt{2}\,\sigma\, e^{-\frac{\lambda^2}{2\sigma^2}} \sum_{k=0}^{\infty} \frac{(\frac{\lambda^2}{2\sigma^2})^{k}}{k!} \frac{\Gamma(k+\frac{3}{2})}{\Gamma(k+1)} \cdot \frac{\Gamma(1)}{\Gamma(\frac{3}{2})} \cdot \frac{\Gamma(\frac{3}{2})}{\Gamma(1)} \qquad (2.6.4)$$

But from [1, p. 256]

$$(Z)_n = Z(Z+1)(Z+2)\ \ldots\ (Z+n-1) = \frac{\Gamma(n+z)}{\Gamma(z)}\ ,\ (Z)_o = 1 \qquad (2.6.5)$$

and also from [1, p. 504]

$$M(a,b,z) = \sum_{k=0}^{\infty} \frac{(a)_n\ z^n}{(b)_n\ n!} \qquad (2.6.6)$$

where $(a)_n$, $(b)_n$ are as defined in (2.6.5). Using both (2.6.5) and (2.6.6) to simplify (2.6.4),

$$E[R] = \sqrt{2}\ \sigma\ \Gamma(\tfrac{3}{2})\ e^{-\frac{\lambda^2}{2\sigma^2}}\ M(\tfrac{3}{2}, 1, \frac{\lambda^2}{2\sigma^2})\ .$$

But from [1, p. 255], $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$, therefore,

$$E[R] = \sqrt{\frac{\pi}{2}}\ \sigma\ e^{-\frac{\lambda^2}{2\sigma^2}}\ M(\tfrac{3}{2}, 1, \frac{\lambda^2}{2\sigma^2})$$

which is the desired result.

From Theorem 2.6.2, a closed form for the expected distance is obtained, and the total expected cost function in problem P2.6 may be written as

$$f(X) = \sum_{i=1}^{m} w_i\ E[R_i]$$

$$= \sqrt{\frac{\pi}{2}}\ \sum_{i=1}^{m} w_i \sigma_i\ e^{-\frac{\lambda_i^2}{2\sigma_i^2}}\ M(\tfrac{3}{2}, 1, \frac{\lambda_i^2}{2\sigma_i^2}) \qquad (2.6.7)$$

The function f(X) expressed in (2.6.7) may be further simplified by using the Kummer Transformation [1, p. 505] which is stated as

$$M(a,b,z) = e^{z}\, M(b-a,b,-z)$$

Application of the Kummer Transformation to (2.6.7) yields the following modification to P2.6,

$$\overline{\text{P}}2.6 \qquad \underset{X}{\text{minimize}}\ f(X) = \sqrt{\frac{\pi}{2}} \sum_{i=1}^{m} w_i \sigma_i\, M\left(-\frac{1}{2},\ 1,\ -\frac{\lambda_i^2}{2\sigma_i^2}\right)$$

Comparing problems P2.6 and $\overline{\text{P}}$2.6 reveals the simplicity of working with the modified version. Instead of evaluating the integrals, a single value for the confluent hypergeometric function is needed. The latter is well tabulated and can be programmed easily. Before solving $\overline{\text{P}}$2.6, a study of the behavior of the function f(X) and its property near the optimal is needed.

Theorem 2.6.3: The function f(X) is a strictly convex function of $X \varepsilon E^2$.

Proof: Observing the function f(X) in P2.6, $f(P_i) \geq 0$ since it is defined as a probability density function. Also, the Euclidean norm contains the convexity property; hence, the conditions for convexity are the same as the ones given in Theorem 2.4.1. Therefore, f(X) is a convex function. To prove that f(X) is a strictly convex function, it is sufficient to prove that each term under the summation is a strictly convex function.

Since w_i and σ_i are assumed to be greater than zero, it remains to prove that $M(-\frac{1}{2}, 1, -\frac{\lambda^2}{2\sigma^2})$ is a strictly convex function by checking the second order partial derivatives.

From [1, p. 507], the derivative of the confluent hypergeometric function is given as

$$M'(a,b,z) = \frac{a}{b} M(a+1,b+1,z)\cdot z' \tag{2.6.9}$$

Therefore, the partial derivatives of $g = M(-\frac{1}{2}, 1, -\frac{\lambda^2}{2\sigma^2})$ with respect to x_1 and x_2 are

$$\frac{\partial g}{\partial x_1} = -\frac{1}{2} \cdot -\frac{(x_1-\mu_a)}{\sigma^2} M(\tfrac{1}{2},2,-z) = \frac{1}{2\sigma^2} [(x_1-\mu_a)\ M(\tfrac{1}{2},2,-z)] \tag{2.6.10}$$

and

$$\frac{\partial g}{\partial x_2} = \frac{1}{2\sigma^2} [(x_2-\mu_b)\ M(\tfrac{1}{2},2,-z)] \tag{2.6.11}$$

The second partial derivatives are derived from (2.6.10) and (2.6.11),

$$\frac{\partial^2 g}{\partial x_1{}^2} = \frac{1}{2\sigma^2} [M(\tfrac{1}{2},2,-z) - \frac{1}{4} \frac{(x_1-\mu_a)^2}{\sigma^2} M(\tfrac{3}{2},3,-z)] ,$$

$$\frac{\partial^2 g}{\partial x_2{}^2} = \frac{1}{2\sigma^2} [M(\tfrac{1}{2},2,-z) - \frac{1}{4} \frac{(x_2-\mu_b)^2}{\sigma^2} M(\tfrac{3}{2},3,-z)] ,$$

and

$$\frac{\partial^2 g}{\partial x_1 \partial x_2} = \frac{1}{2\sigma^2} [-\frac{1}{4} (x_1-\mu_a)(x_2-\mu_b)\ M(\tfrac{3}{2},3,-z)]$$

To check the Hessian of f(X), the first principle minor

$|A_1| = \frac{\partial^2 g}{\partial x_1^2} > 0$ from the proof of the positivity of the second principle minor $|A_2|$ which follows. To check the second minor $|A_2|$ which is given by

$$|A_2| = \frac{\partial^2 g}{\partial x_1^2} \cdot \frac{\partial^2 g}{\partial x_2^2} - \left(\frac{\partial^2 g}{\partial x_1 \partial x_2}\right)^2$$

but

$$\frac{\partial^2 g}{\partial x_1^2} \cdot \frac{\partial^2 g}{\partial x_2^2} = \frac{1}{4\sigma^4} [M^2(\tfrac{1}{2},2,-z) + \frac{M^2(\tfrac{3}{2},3,-z)}{16\sigma^4} (x_1-\mu_a{}^2)(x_2-\mu_b{}^2)$$

$$- \frac{1}{4\sigma^2} M(\tfrac{1}{2},2,-z) \cdot M(\tfrac{3}{2},3,-z)((x_1-\mu_a)^2+(x_2-\mu_b)^2)] \quad (2.6.12)$$

and

$$\left(\frac{\partial^2 g}{\partial x_1 \partial x_2}\right)^2 = \frac{1}{4\sigma^4} [\frac{M^2(\tfrac{3}{2},3,-z)}{16\sigma^4} (x_1-\mu_a)^2(x_2-\mu_b)^2] \quad (2.6.13)$$

From (2.6.12) and (2.6.13), $|A_2|$ is positive if,

$$K = M^2(\tfrac{1}{2},2,-z) - \frac{\lambda^2}{4\sigma^2} M(\tfrac{1}{2},2,-z)\, M(\tfrac{3}{2},3,-z) > 0 \quad (2.6.14)$$

Using the Kummer Transformation on (2.6.14) and simplifying

$$K = e^{-z} M(\tfrac{1}{2},2,-z)[M(\tfrac{3}{2},2,z) - \frac{\lambda^2}{4\sigma^2} M(\tfrac{3}{2},3,z)] \quad (2.6.15)$$

Since $z = \frac{\lambda^2}{2\sigma^2}$, and using the expansion of the confluent hypergeometric function defined in (2.6.6), (2.6.15) gives

$$K = e^{-z} M(\tfrac{1}{2},2,-z)[\sum_{n=0}^{\infty} \frac{(a)_n z^n}{(2)_n n!} - \frac{z}{2} \sum_{n=0}^{\infty} \frac{(a_n) z^n}{(3)_n n!}] . \qquad (2.6.16)$$

The term in the square bracket is further simplified as

$$\sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!} (\frac{1}{(2)_n} - \frac{1}{2(3)_n}) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!(n+2)!} (n+2-z) \qquad (2.6.17)$$

From (2.6.17) the infinite series converges to a positive value. Also, $e^{-z} M(\frac{1}{2},2,-z) = e^{-2z} M(\frac{3}{2},2,z)$ from (2.6.8) and is a positive value for all arguments z. Combining this result with (2.6.16) gives $K > 0$, i.e., $|A_2|$ is positive. The proof that $|A_1|$ is positive is straightforward from (2.6.14) since $\lambda^2 > (x_1-\mu_a)^2$. Since the Hessian is positive definite, it is a sufficient condition to prove that f(X) is a strictly convex function.

2.6.2 Solution Procedure

From Theorem 2.6.4, f(X) is a strictly convex function of $X \varepsilon E^2$. Therefore, f(X) has a unique minimum on E^2. Also, a necessary and sufficient condition for the existence of the optimal solution is given by

$$\nabla f(X^*) = (\frac{\partial f(X^*)}{\partial x_1^*}, \frac{\partial f(X^*)}{\partial x_2^*}) = 0 \qquad (2.6.18)$$

Obtaining the partial derivatives of the objective function in $\overline{P}2.6$,

$$\frac{\partial f(X^*)}{\partial x_1^*} = \frac{\pi}{2} \sum_{i=1}^{m} w_i \sigma_i \frac{x_1^* - \mu_{a_i}}{2\sigma_i^2} M(\tfrac{1}{2}, 2, -\frac{\lambda_i^2}{2\sigma_i^2}) = 0 \qquad (2.6.19)$$

$$\frac{\partial f(X^*)}{\partial x_2^*} = \sqrt{\frac{\pi}{2}} \sum_{i=1}^{m} w_i \sigma_i \frac{x_2-\mu_{b_i}}{2\sigma_i^2} M(\frac{1}{2}, 2, -\frac{\lambda_i^2}{2\sigma_i^2}) = 0 \qquad (2.6.20)$$

Solving for x_1, x_2 from (2.6.19) and (2.6.20),

$$x_1^* = \frac{\sum_{i=1}^{m} \frac{w_i}{\sigma_i} M(\frac{1}{2}, 2, -\frac{\lambda_i^2}{2\sigma_i^2}) \mu_{a_i}}{\sum_{i=1}^{m} \frac{w_i}{\sigma_i} M(\frac{1}{2}, 2, -\frac{\lambda_i^2}{2\sigma_i^2})} \qquad (2.6.21)$$

$$x_2^* \quad \frac{\sum_{i=1}^{m} \frac{w_i}{\sigma_i} M(\frac{1}{2}, 2, -\frac{\lambda_i^2}{2\sigma_i^2}) \mu_{b_i}}{\sum_{i=1}^{m} \frac{w_i}{\sigma_i} M(\frac{1}{2}, 2, -\frac{\lambda_i^2}{2\sigma_i^2})} \qquad (2.6.22)$$

The definition of λ_i shows it is a function of the decision variables x_1, x_2. Therefore, (2.6.21) and (2.6.22) are used in an iterative scheme similar to the one used by Kuhn and Kuenne [62] and Cooper [10]. Let $x_1^{(k)}$, $x_2^{(k)}$ and $\lambda_i^{(k)}$ denote the values obtained at the kth iteration,

$$x_1^{(k+1)} = \frac{\sum_{i=1}^{m} \frac{w_i}{\sigma_i} M(\frac{1}{2}, 2, -\frac{\lambda_i^{2(k)}}{2\sigma_i^2}) \mu_{a_i}}{\sum_{i=1}^{m} \frac{w_i}{\sigma_i} M(\frac{1}{2}, 2, -\frac{\lambda_i^{2(k)}}{2\sigma_i^2})} \qquad (2.6.23)$$

$$x_2^{(k+1)} = \frac{\sum_{i=1}^{m} \frac{w_i}{\sigma_i} M(\frac{1}{2}, 2, -\frac{\lambda_i^{2(k)}}{2\sigma_i^2}) \mu_{b_i}}{\sum_{i=1}^{m} \frac{w_i}{\sigma_i} M(\frac{1}{2}, 2, -\frac{\lambda_i^{2(k)}}{2\sigma_i^2})} \qquad (2.6.24)$$

To employ the iterative algorithm, an initial, feasible solution is required. As mentioned above, any solution (x_1, x_2) which lies in the convex hull of the means of $\{\mu_{a_i}, \mu_{b_i}\}$ could be used as a starting solution, but it may take more iterations to converge to the optimal solution. In the next section, an approximate solution will be suggested as a good initial solution X^o.

2.6.3 The Euclidean Problem: Approximate Solution

Problem $\overline{P}2.6$ may be solved easily if an approximation to the confluent hypergeometric function M is introduced. From the expansion of M given in (2.6.6),

$$M(a,b,z) = 1 + \frac{b}{a} z + \frac{b(b+1)}{a(a+1)} \frac{z^2}{2!} + \ldots \tag{2.6.25}$$

Retaining the linear term in the expansion (2.6.25) and discarding terms of higher order, then

$$M(a,b,z) \simeq 1 + \frac{b}{a} z \tag{2.6.26}$$

Substituting (2.6.26) in problem $\overline{P}2.6$, yields

$$\begin{aligned} f(X) &= \sqrt{\frac{\pi}{2}} \sum_{i=1}^{m} w_i \sigma_i \left(1 + \frac{\lambda_i^2}{2\sigma_i^2}\right) \\ &= \sqrt{\frac{\pi}{2}} \sum_{i=1}^{m} w_i \sigma_i \left[1 + \frac{(x_1-\mu_{a_i})^2+(x_2-\mu_{b_i})^2}{2\sigma_i^2}\right] \end{aligned} \tag{2.6.27}$$

To minimize f(X) as expressed in (2.6.27), compute the partial derivatives of (2.6.27) with respect to x_1, x_2 and set them to zero. The optimal unique solution will be given as

$$\tilde{x}_1^* = \frac{\sum_{i=1}^{m} \frac{w_i}{\sigma_i} \mu_{a_i}}{\sum_{i=1}^{m} \frac{w_i}{\sigma_i}} \tag{2.6.28}$$

and

$$\tilde{x}_2^* = \frac{\sum_{i=1}^{m} \frac{w_i}{\sigma_i} \mu_{b_i}}{\sum_{i=1}^{m} \frac{w_i}{\sigma_i}} \tag{2.6.29}$$

which are easily computed for given values of w_i, σ_i, and μ_i. The solution given by (2.6.28) and (2.6.29) is used as a starting solution in the iterative scheme discussed in Section 2.6.2. Notice that this solution is the same one given by (2.4.21) and (2.4.22) for the rectilinear case when $\sigma_{a_i} = \sigma_{b_i} = \sigma_i$.

From the solution obtained in (2.6.21) and (2.6.22) an interesting result concerning the location of X relative to the existing facilities P_i may be derived.

<u>Lemma 2.6.1</u>[†]: The optimal location of the new facility X^* lies within the convex hull of the means of the existing facilities P_i.

<u>Proof</u>: The proof is straightforward by defining the parameter θ_i as

[†]Katz and Cooper [52] arrived at the same result, but used a different approach.

$$\theta_i = \frac{\frac{w_i}{\sigma_i} M(\frac{1}{2}, 2, -\frac{\lambda_i^2}{2\sigma_i^2})}{\sum_{i=1}^{w_i} \frac{w_i}{\sigma_i} M(\frac{1}{2}, 2, -\frac{\lambda_i^2}{2\sigma_i^2})}$$

Hence, from (2.6.21) and (2.6.22)

$$x_1^* = \sum_{i=1}^{m} \theta_i \mu_{a_i} \tag{2.6.30}$$

and

$$x_2^* = \sum_{i=1}^{m} \theta_i \mu_{b_i} \tag{2.6.31}$$

From the definition of θ_i, it is clear that,

$$1 \geq \theta_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{m} \theta_i = 1 .$$

Therefore, there exist values for θ_i such that x_1^* and x_2^* are expressed as convex combinations of μ_{a_i} and μ_{b_i}, respectively. The existence of the multipliers θ_i proves that $x_1^* \in \{\mu_{a_i}\}$ and $x_2^* \in \{\mu_{b_i}\}$ where $\{\mu_i\}$ is the convex hull containing all μ_i. Thus X^* lies in the convex hull of $\{\mu_{a_i}, \mu_{b_i}\}$.

The above lemma implies that whatever the value of the variance associated with P_i, the optimal solution always lies in the convex hull of the means of P_i.

2.6.4 Lower and Upper Bounds for f(X)

In the iterative procedure developed in Section 2.6.2, the approximate solution obtained in Section 2.6.3 or the centriod solution of Section 2.5 may be used as a starting point. In the deterministic case Pritsker and Ghare [78] showed how to bound the objective function value of the optimum Euclidean solution. In this section, similar bounds are developed using the rectilinear solution.

I. A lower bound for f(X): Recalling that the objective function of the total expected cost for the rectilinear case is

$$z = \sum_{i=1}^{m} w_i [\int_{a_i} |x_1 - a_i| \; f(a_i) da_i + \int_{b_i} |x_2 - b_i| \; f(b_i) db_i]$$

and applying Schwartz's Inequality for each separate term gives

$$\int_{a_i} |x_1 - a_i| \; f(a_i) da_i \leq [\int_{a_i} (x_1 - a_i)^2 f(a_i) da_i]^{\frac{1}{2}} \cdot [\int_{a_i} f(a_i) da_i]^{\frac{1}{2}}$$

However, $\int_{a_i} f(a_i) da_i = 1$ from the definition of the probability density function.

Substituting the definition of $z_i(x_1)$, $z_i(x_2)$ from (2.4.7),

$$z_i(x_1) = \int_{a_i} |x_1 - a_i| \; f(a_i) da_i \leq [\int_{a_i} (x_1 - a_i)^2 f(a_i) da_i]^{\frac{1}{2}}$$

$$= E[(x_1 - a_i)^2]^{\frac{1}{2}} \qquad (2.6.32)$$

Multiplying both sides of (2.6.32) by $w_i > 0$ and summing over all i,

$$\sum_{i=1}^{m} w_i z_i(x_1) \leq \sum_{i=1}^{m} w_i \; E[(x_1-a_i)^2]^{\frac{1}{2}} \tag{2.6.33}$$

Using Cauchy's Inequality [1, p. 11] and assuming that all w_i are normalized, i.e., $\overline{w}_i = \frac{w_i}{|w_i|}$ where $|w_i|$ is the norm of the vector $W = (w_1, \ldots, w_n)$, (2.6.33) is written as,

$$\sum_{i=1}^{m} \overline{w}_i z_i(x_1) \leq (\sum_{i=1}^{m} \overline{w}_i \; E[(x_1-a_i)^2])^{\frac{1}{2}} \cdot (\sum_{i=1}^{m} \overline{w}_i)^{\frac{1}{2}} \tag{2.6.34}$$

Since $\sum_{i=1}^{m} \overline{w}_i = 1$, after squaring both sides of (2.6.34),

$$(\sum_{i=1}^{m} \overline{w}_i z_i(x_1))^2 \leq \sum_{i=1}^{m} \overline{w}_i \; E[(x_1-a_i)^2] \tag{2.6.35}$$

Similarly,

$$(\sum_{i=1}^{m} \overline{w}_i z_i(x_2))^2 \leq \sum_{i=1}^{m} \overline{w}_i \; E[(x_2-b_i)^2] \tag{2.6.36}$$

Let $R(x) = \sum_{i=1}^{m} \overline{w}_i z_i(x)$, and by combining both (2.6.35) and (2.6.36),

$$R^2(x_1) + R^2(x_2) \leq \sum_{i=1}^{m} \overline{w}_i \; E[(x_1-a_i)^2+(x_2-b_i)^2] \tag{2.6.37}$$

Taking the square root of both sides in the inequality (2.6.37), and applying Jensen's Inequality,

$$[R^2(x_1)+R^2(x_2)]^{\frac{1}{2}} \leq [\sum_{i=1}^{m} \overline{w}_i \; E[(x_1-a_i)^2+(x_2-b_i)^2]^{\frac{1}{2}}$$

$$\leq \sum_{i=1}^{m} \overline{w}_i \; E[((x_1-a_i)^2+(x_2-b_i)^2)^{\frac{1}{2}}] \qquad (2.6.38)$$

Let X^* be the optimal solution to the rectilinear problem solved in Section 2.4, and let X^o be the optimal solution of the Euclidean problem; hence, from (2.6.38),

$$[R^2(x_1^*) + R^2(x_2^*)]^{\frac{1}{2}} \leq f(X^o) \qquad (2.6.39)$$

II. <u>An upper bound on f(X)</u>: If X^o is defined as above,

$$f(X) \geq \min f(X) = f(X^o)$$

and

$$f(X^*) \geq f(X^o) \qquad (2.6.40)$$

Combining both (2.6.39) and (2.6.40), the required lower and upper bounds are given as follows,

$$f(X^*) \geq f(X^o) \geq [z^2(x_1^*) + z^2(x_2^*)]^{\frac{1}{2}} \qquad (2.6.41)$$

The inequality given by (2.6.41) is helpful if the rectilinear solution is available and the bounds are tight. Depending on the particular application, it may be sufficient to consider that the rectilinear solution (X^*) is close enough to the optimal Euclidean solution (X^o).

2.7 A Single Facility Location Problem: Norm Constraints

As described in Section 2.2, some restriction may be imposed on the location of the new facility with respect to the location of each existing facility. For example, an upper bound might be placed on the expected distance traveled to each existing facility. In this section both rectilinear and Euclidean problems under norm constraints are presented. The analysis is restricted to the situation in which P_i is normally distributed.

The first problem treated employs the rectilinear norm as the measure of distance. The resulting optimization problem is written as

$$\text{P2.7.1} \quad \underset{X}{\text{minimize}}\ f(X) = \sum_{i=1}^{m} w_i \int_{b_i} \int_{a_i} [|x_1-a_i|+|x_2-b_i|] \; f(a_i)f(b_i)da_i db_i$$

$$\text{subject to:} \quad E[|x_1-a_i|+|x_2-b_i|] \leq \xi_i \quad \text{for all } i,\ i = 1, \ldots, m.$$

Employing the results obtained in (2.4.8) and (2.4.9), P2.7.1 is given as

$$\overline{\text{P}}2.7.1 \quad \underset{X}{\text{minimize}}\ f(X) = \sum_{i=1}^{m} \sum_{j=1}^{2} w_i[(x_j-\mu_{ij})(2\Phi(\frac{x_j-\mu_{ij}}{\sigma_{ij}})-1) + 2\sigma_{ij}\ \phi(\frac{x_j-\mu_{ij}}{\sigma_{ij}})]$$

$$\text{subject to:} \quad \sum_{j=1}^{2} [(x_j-\mu_{ij})(2\Phi(\frac{x_j-\mu_{ij}}{\sigma_{ij}})-1) + 2\sigma_{ij}\ \phi(\frac{x_j-\mu_{ij}}{\sigma_{ij}}) \leq \xi_i$$

$$\text{for all } i$$

where ξ_i is the upper bound on the individual distance between the new facility and existing facility i.

The second constrained problem involving the Euclidean norm is used as a measure of the distance, and P2.2 is written as

$$\text{P2.7.2} \quad \underset{X}{\text{minimize}}\ f(X) = \sum_{i=1}^{m} w_i \int_{b_i}\int_{a_i} [(x_1-a_i)^2+(x_2-b_i)^2]^{\frac{1}{2}} f(a_i)f(b_i)da_i db_i$$

$$\text{subject to:} \quad E[((x_1-a_i)^2+(x_2-b_i)^2)^{\frac{1}{2}}] \le \xi_i \quad \text{for all } i,\ i = 1, \ldots, m$$

From $\overline{\text{P}}2.6$, the valuesof the expected values are substituted in P2.7.2, resulting in

$$\overline{\text{P}}2.7.2 \quad \underset{X}{\text{minimize}}\ f(X) = \sum_{i=1}^{m} \sqrt{\frac{\pi}{2}}\ w_i\sigma_i\ M(-\frac{1}{2}, 1, -\frac{\lambda_i^2}{2\sigma_i^2})$$

$$\text{subject to:} \quad \sqrt{\frac{\pi}{2}}\ w_i\sigma_i\ M(-\frac{1}{2}, 1, -\frac{\lambda_i^2}{2\sigma_i^2}) \le \xi_i \quad \text{for all } i,\ i = 1, \ldots, m$$

where α_i is the upper bound as defined above.

As can be seen from previous sections, $\overline{\text{P}}2.7.1$ and $\overline{\text{P}}2.7.2$ are convex programming problems. It has been shown that the objective functions in $\overline{\text{P}}2.7.1$ and $\overline{\text{P}}2.7.2$ are strictly convex; the same is true for each constraint. Thus, the following properties are satisfied:

1. The functions $f(X)$, $f_i(X)$ are twice continuously differentiable.
2. The function $f(X)$ is strictly convex.
3. The set of constraints is a convex set.

Hence, any convex programming algorithm will converge globally to the optimal solution, where the Kuhn-Tucker necessary and sufficient conditions are satisfied. Therefore, SUMT [25], the sequential unconstrained minimization technique, may be used to solve the above problems.

2.8 Rectilinear-Distance Location Problems: Chance Constraints

In this section, the chance constrained facility problem is studied. As discussed in Chapter 1, the chance constrained approach is more appropriate than, say, expected value constraints. The probability distribution used is the normal distribution; as an alternative, the exponential distribution, is treated in Appendix A.

2.8.1 Normally Distributed Chance Constrained Location Problem

The problem presented in Section 2.4.1 is formulated in this section as a chance constrained programming problem. The model is given by,

$$\text{P2.8.1} \quad \underset{X}{\text{minimize}}\ f(X) = \sum_{i=1}^{m} w_i \int_{b_i} \int_{a_i} (|x_1-a_i|+|x_2-b_i|)\, f(a_i)f(b_i)\,da_i\,db_i$$

$$\text{subject to:} \quad Pr(|x_1-a_i|+|x_2-b_i| \le \xi_i) \ge \gamma_i \qquad \text{for all } i$$

where $f(a_i)$, $f(b_i)$ are defined as in P2.4.1. The independency between the random variables a_i, b_i still holds for all values of i, and γ_i is the assigned service level.

To solve problem P2.8.1, the chance constraints have to be changed to equivalent deterministic constraints. Consequently, the probability density function of the rectilinear distance $v_i = |x_1-a_i| + |x_2-b_i|$ is required. This is obtained through the following theorem.

Theorem 2.8.1: Given that a, b are both mutually independent, random variables with probability density function $N(\mu_a,\sigma^2)$ and $N(\mu_b,\sigma^2)$, respectively, then the probability density of $v = |x_1-a| + |x_2-b|$ is given by,

$$g(v) = \frac{1}{2\sqrt{\pi}\,\sigma} \sum_{j=1}^{2} \{ e^{-\frac{1}{2}\left(\frac{v+(-1)^j y_j}{\sqrt{2}\,\sigma}\right)^2} [\Phi(\frac{v+(-1)^{j+1} y_{3-j}}{\sqrt{2}\,\sigma})$$

$$- \Phi(\frac{-v+(-1)^{j+1} y_{3-j}}{\sqrt{2}\,\sigma})] + e^{-\frac{1}{2}\left(\frac{v+(-1)^j y_{3-j}}{\sqrt{2}\,\sigma}\right)^2}$$

$$[\Phi(\frac{v+(-1)^{j+1} y_j}{\sqrt{2}\,\sigma}) - \Phi(\frac{-v+(-1)^{j+1} y_j}{\sqrt{2}\,\sigma})]\} \qquad 0 < v < \infty \qquad (2.8.1)$$

where

$$y_1 = x + y$$

$$y_2 = x - y$$

and

$$\Phi(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{w} e^{-\frac{z^2}{2}} dz .$$

Proof: Since $a \sim N(\mu_a,\sigma^2)$, $b \sim N(\mu_b,\sigma^2)$, by letting A denote (x_1-a), and B denote (x_2-b), then, both A and B have normal distributions given as $A \sim N(x_1-\mu_a,\sigma^2)$ and $B \sim N(x_2-\mu_b,\sigma^2)$. For simplicity, let $x = x_1 - \mu_a$, $y = x_2 - \mu_b$. To develop the probability density function of the absolute value, notice that

$$\Pr(|A| < z) = \Pr(-z < A < z)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-z}^{z} e^{-\frac{1}{2}\left(\frac{A-x}{\sigma}\right)^2} dA \quad . \tag{2.8.2}$$

Let $\frac{A-x}{\sigma} = w$, then $dA = \sigma\, dw$ and (2.8.2) may be written as

$$\Pr(|A| < z) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{z-x}{\sigma}}^{\frac{z-x}{\sigma}} e^{-\frac{1}{2}w^2} dw$$

$$= \Phi\left(\frac{z-x}{\sigma}\right) - \Phi\left(\frac{-z-x}{\sigma}\right) \tag{2.8.3}$$

Equation 2.8.3 gives the distribution function of $|A|$. To obtain the probability density function,(2.8.3) is differentiated to obtain

$$g(z) = \frac{1}{\sqrt{2\pi}\,\sigma}\left[e^{-\frac{1}{2}\left(\frac{z-x}{\sigma}\right)^2} + e^{-\frac{1}{2}\left(\frac{z+x}{\sigma}\right)^2}\right] \quad , 0 \leq z < \infty \tag{2.8.4}$$

Therefore, both $g_1(|A|)$ and $g_2(|B|)$ have the form of (2.8.4). To obtain the probability density function of $v = |A| + |B|$, the joint distribution of $|A|$ and $|B|$ must be obtained as,

$$g(|A|, |B|) = g_1(|A|)\cdot g_2(|B|) \ .$$

Define L_1 and L_2 as

$$L_1 = |A| + |B|$$

$$L_2 = |B|$$

Hence, $|A| = L_1 - L_2$ and $|B| = L_2$; the Jacobian of the transformation is given by $J = 1$. The joint density function of L_1 and L_2 follows directly,

$$g(L_1,L_2) = \frac{1}{2\pi\sigma^2}\,[e^{-\frac{1}{2\sigma^2}\{(L_1-L_2)-x\}^2} + e^{-\frac{1}{2\sigma^2}\{(L_1-L_2)+x\}^2}]\cdot$$

$$[e^{-\frac{1}{2\sigma^2}\{L_2-y\}^2} + e^{-\frac{1}{2\sigma^2}\{L_2+y\}^2}]\,. \tag{2.8.5}$$

Using (2.8.5) the marginal density of $L_1 = v$ is obtained,

$$f(L_1) = \int_0^{L_1} g(L_1,L_2)\,dL_2 \tag{2.8.6}$$

$$= \frac{1}{2\pi\sigma^2}\,[\int_0^{L_1} e^{-\frac{1}{2\sigma^2}\{(L_1-L_2)+x\}^2+\{L_2+y\}^2}\,dL_2$$

$$+ \int_0^{L_1} e^{-\frac{1}{2\sigma^2}\{(L_1-L_2)-x\}^2+\{L_2+y\}^2}\,dL_2$$

$$+ \int_0^{L_1} e^{-\frac{1}{2\sigma^2}\{(L_1-L_2)+x\}^2+\{L_2-y\}^2}\,dL_2$$

$$+ \int_0^{L_1} e^{-\frac{1}{2\sigma^2}\{(L_1-L_2)-x\}^2+\{L_2-y\}^2}\,dL_2] \tag{2.8.7}$$

The first integral in (2.8.7) is evaluated as follows,

$$\frac{1}{2\pi\sigma^2}\int_0^{L_1} e^{-\frac{1}{2\sigma^2}\{(L_1-L_2)+x\}^2+\{L_2+y\}^2}\,dL_2$$

$$= \frac{1}{2\pi\sigma^2}\,e^{-\frac{1}{2\sigma^2}\{L_1^2+x^2+2xL_1+y_2\}}\int_0^{L_1} e^{-\frac{1}{2\sigma^2}\{2L_2^2-2(L_1+x-y)L_2\}}\,dL_2$$

By completing the square of the power of the integrand,

$$\int_0^{L_1} g_1(L_1,L_2)dL_2 = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}\{\frac{L_1+x+y}{\sqrt{2}}\}^2} \cdot \frac{1}{\sqrt{2\pi}\,\sigma} \int_0^{L_1} e^{-\frac{1}{2\sigma^2}\{\frac{2L_2-L_1-x+y}{\sqrt{2}}\}^2} dL_2 \tag{2.8.8}$$

Let $w = \dfrac{2L_2-L_1-x+y}{\sqrt{2}\,\sigma}$ so that $dw = \frac{\sqrt{2}}{\sigma} dL_2$ and (2.8.8) is written as

$$\int_0^{L_1} g_1(L_1,L_2)dL_2 = \frac{1}{2\sqrt{\pi}\,\sigma} e^{-\frac{1}{2}(\frac{L_1+x+y}{\sqrt{2}\,\sigma})^2} \cdot \int_{-(\frac{L_1+x-y}{\sqrt{2}\,\sigma})}^{(\frac{L_1-x+y}{\sqrt{2}\,\sigma})} e^{-\frac{w^2}{2}} dw$$

$$= \frac{1}{2\sqrt{\pi}\,\sigma} e^{-\frac{1}{2}(\frac{L_1+x+y}{\sqrt{2}\,\sigma})^2} [\Phi(\frac{L_1-x+y}{\sqrt{2}\,\sigma}) - \Phi(\frac{-L_1-x+y}{\sqrt{2}\,\sigma})] \tag{2.8.9}$$

After evaluating the remaining integrals in (2.8.7), the marginal density function of v given in (2.8.1) is obtained.

To develop the distribution function of v, F(v), (2.8.1) is integrated with respect to v,

$$Pr(v \leq \xi) = F(\xi) = \int_0^{\xi} g(v)\, dv \tag{2.8.10}$$

Using the above results, the optimization problem, P2.8.1, may be written as

$$\overline{P}2.8.1 \quad \underset{x_1,x_2}{\text{minimize}}\ f(X) = \sum_{i=1}^{m} \sum_{j=1}^{2} w_i[(x_j-\mu_j)(2\ (\frac{x_j-\mu_j}{\sigma_i})-1)$$
$$+ 2\sigma_i \quad (\frac{x_j-\mu_j}{\sigma_i})]$$

subject to: $F(\xi_i) \geq \gamma_i$ for all i, i = 1, ..., m

where (ξ_i) is as defined in (2.8.10). Problem $\overline{P}2.8.1$ has a strictly convex objective function, but the set of constraints is not identified as a convex set, since the concavity of $F(\xi_i)$ over $X \varepsilon E^2$ is not obvious.

To solve $\overline{P}2.8.1$, any nonlinear programming algorithm used in solving a convex programming problem, e.g., SUMT, may be employed. Since Kuhn-Tucker necessary conditions are satisfied, a local optimum will be achieved. The sufficient condition may be checked at the local solution X^* to determine if the local optimum is a global optimum.

2.9 Location on Line: A Chance Constrained Problem

In the above sections, the location problem was formulated as a two-dimensional (planar) location. In some situations, the new facility is constrained to be located on a line. Additionally, a study of one-dimensional location problems provides insight concerning the more general planar location problem. In this section, the problem of locating a single new facility on a line is formulated subject to chance constraints.

The chance constrained problem may be written as,

$$P2.9 \quad \underset{x}{\text{minimize}}\ f(x) = \sum_{i=1}^{m} w_i \int_{a_i} |x-a_i| f(a_i) da_i \quad , x \varepsilon E^1$$

subject to: $\Pr(|x-a_i| \leq \xi_i) \geq \gamma_i$ for all i, i = 1, ..., m.

When the normal distribution is used as the continuous probability density function of the existing locations a_i, then, from (2.4.8) and (2.4.9), the problem is expressed as

$$\text{P2.9.1} \quad \underset{x_1}{\text{minimize}}\ f(x_1) = \sum_{i=1}^{m} (x_1-\mu_{a_i})(2\Phi\ (\frac{x_1-\mu_{a_i}}{\sigma_i})-1) + 2\sigma_i\ \phi(\frac{x_1-\mu_{a_i}}{\sigma_i})$$

subject to: $\Pr(|x_1-a_i| \leq \xi_i) \geq \gamma_i$ for all i.

From (2.8.3), the probability distribution of $|x-a_i|$ is developed. Hence, the chance constraints can be given as

$$F(\xi_i) = \Phi(\frac{\xi_i-x}{\sigma_i}) - \Phi(\frac{-\xi_i-x}{\sigma_i}) \geq \gamma_i \qquad \text{for all } i$$

where $x = x_1 - \mu_a$ is the mean of the distribution of (x_1-a).

<u>Lemma 2.9.1</u>: The function $F(\xi)$ is a concave function over all values of $x_1 \in E^1$.

<u>Proof</u>: From the differentiability conditions,

$$\frac{dF}{dx_1} = -\frac{1}{\sigma}\ \phi(\frac{\xi-x}{\sigma}) + \frac{1}{\sigma}\ \phi(\frac{\xi+x}{\sigma})$$

and

$$\frac{d^2F}{dx_1^{\ 2}} = -\frac{1}{\sigma^2}\ (\frac{\xi-x}{\sigma})\ \phi(\frac{\xi-x}{\sigma}) - \frac{1}{\sigma^2}\ (\frac{\xi+x}{\sigma})\ \phi(\frac{\xi+x}{\sigma}) \tag{2.9.1}$$

From Chebyshev's Inequality,

$$\Pr\left(|z| > \xi\right) \le \frac{E(z^2)}{\xi^2} \; . \tag{2.9.2}$$

Hence, $F(\xi) \ge \frac{\xi^2 - E(z^2)}{\xi^2} = \frac{\xi^2 - (\sigma^2+\mu^2)}{\xi^2}$ and since $F(\xi) \ge 0$, then $\xi^2 \ge \sigma^2 + \mu^2 > \mu^2$, since $\sigma^2 > 0$ and the following is obtained

$$\xi \ge \pm \mu = |\mu| = |x| \; . \tag{2.9.3}$$

Equation 2.9.3 implies that $(\frac{\xi - x}{\sigma}) \ge 0$, $(\frac{\xi + x}{\sigma}) \ge 0$. Observing this with (2.9.1), then $\frac{d^2F}{dx_1^2} \le 0$ and the sufficient condition for concavity is established.

Applying Lemma 2.9.1 in problem P2.9.1, it is seen that the constraint set is a convex set. Therefore, using any convex programming algorithm will yield a global optimum solution, and both sufficient and necessary conditions of Kuhn-Tucker are satisfied.

2.10 Euclidean Distance Location Problem: Chance Constraints

In this section the effect of adding chance constraints to the Euclidean problem of Section 2.6 is studied. As discussed above, the chance constraint represents a bound on the probability that the distance traveled is within a preassigned value. This added constraint gives the decision makers more flexibility in locating the new facility. The normal probability density function is used throughout the section to represent the distribution of the random locations. The mathematical formulation is given as,

$$\text{P2.10}\quad \underset{X}{\text{minimize}}\ f(X) = \sum_{i=1}^{m} w_i \int_{b_i}\int_{a_i} \sqrt{(x^1-a_i)^2+(x_2-b_i)^2}\, f(a_i)f(b_i)da_i db_i$$

$$\text{subject to: } \Pr([(x_1-a_i)^2+(x_2-b_i)^2]^{\frac{1}{2}} \le \xi_i) \ge \gamma_i \quad \text{for all } i,\ i = 1, \ldots, m$$

where

$$f(a_i) = \frac{1}{\sqrt{2\pi}\,\sigma_i} e^{-\frac{1}{2}\left(\frac{a_i-\mu_{a_i}}{\sigma_i}\right)^2}, \quad -\infty < a_i < \infty$$

$$f(b_i) = \frac{1}{\sqrt{2\pi}\,\sigma_i} e^{-\frac{1}{2}\left(\frac{b_i-\mu_{b_i}}{\sigma_i}\right)^2}, \quad -\infty < b_i < \infty$$

and γ_i is the lower bound on the probability.

In Theorem 2.6.1, the probability density function of $R = [(x_1-a)^2+(x_2-b)^2]^{1/2}$ was obtained. In the following theorem, the cumulative distribution function of R is provided.

Theorem 2.10.1: Given that the probability density function of $R = [(x_1-a)^2+(x_2-b)^2]^{1/2}$ is given by (2.6.1), then the cumulative distribution of R is expressed as,

$$\Pr(R \le r) = F(r) = e^{-\frac{1}{2}[r^2+\lambda^2]} \sum_{n=1}^{\infty} \left(\frac{r}{\lambda}\right)^n I_n(\lambda r) \qquad (2.10.1)$$

where

I_n = the modified Bessel function of the first kind and order n

$$\lambda^2 = (x_1-\mu_a)^2 + (x_2-\mu_b)^2.$$

Proof: From (2.6.1), the probability density function is known; hence, the distribution function is given by,

$$F(r) = \int_0^r \overline{g}(z)dz = \int_0^r \frac{z}{\sigma^2} e^{-\frac{1}{2\sigma^2}(z^2+\lambda^2)} I_o(\frac{z}{2})dz \tag{2.10.2}$$

Let $w = \frac{r}{\sigma}$, $\overline{\lambda} = \frac{\lambda}{\sigma}$, then $dw = \frac{1}{\sigma}dr$, and (2.10.2) is written as,

$$F(r) = \int_0^r g(w)ds = \int_0^r w\, e^{-\frac{1}{2}(w^2+\lambda^2)} I_o(\lambda w)dw \tag{2.10.3}$$

The integral in (2.10.3) is evaluated through integration by parts as follows,

let $e^{-\frac{1}{2}(w^2+\lambda^2)} = u$ and $w\, I_o(\lambda w)dw = dv$,

thus $du = -we^{-\frac{1}{2}(w^2+\lambda^2)}dw$.

From [1, p. 484]

$$\int w^n I_{n-1}(\lambda w)dw = \frac{w^n}{\lambda} I_n(\lambda w) \tag{2.10.4}$$

From (2.10.4), $v = \frac{w}{\lambda} I_1(\lambda w)$, and the integral is written as

$$\int we^{-\frac{1}{2}(w^2+\lambda^2)} I_o(\lambda w)dw = \int u\, dv = uv - \int v\, du$$

$$= e^{-\frac{1}{2}(w^2+\lambda^2)} \cdot \frac{w}{\lambda} I_1(\lambda w) + \int e^{-\frac{1}{2}(w^2+\lambda^2)} \frac{w^2}{\lambda} I_1(\lambda w)dw$$

$$= e^{-\frac{1}{2}(w^2+\lambda^2)} \cdot \frac{w}{\lambda} I_1(\lambda w) + e^{-\frac{1}{2}(w^2+\lambda^2)} \cdot \frac{w^2}{\lambda^2} I_2(\lambda w)$$

$$+ \int e^{-\frac{1}{2}(w^2+\lambda^2)} \frac{w^3}{\lambda^2} I_2(\lambda w)\, dw \tag{2.10.5}$$

From (2.10.5), repeated integration by parts will yield,

$$F(r) = e^{-\frac{1}{2}[r^2+\lambda^2]} \sum_{n=1}^{\infty} \left(\frac{r}{\lambda}\right)^n I_n(\lambda r) \tag{2.10.6}$$

Substituting (2.10.6) in problem P2.10, a deterministic equivalence for the chance constraint is obtained. Additionally, an asympotic expansion for the infinite series may be used. As an example see [1, p. 375]. However, from the property of the chance constraint a relatively simple expression can be derived for F(r). Knowing that for a random variable x, $x \geq 0$,

$$\Pr(x^2 \leq \alpha^2) = \Pr(-\alpha \leq x \leq \alpha)$$

hence,

$$\Pr(x^2 \leq \alpha^2) = \Pr(-\alpha \leq x < 0) + \Pr(0 \leq x < \alpha)$$

$$= \Pr(x < \alpha) \tag{2.10.7}$$

Using (2.10.7), all chance constraints in P2.10 may be converted to the following form,

$$\Pr((x_1-a)^2+(x_2-b)^2 \leq \xi_i^2) \geq \gamma_i \quad \text{for all } i,\ i = 1, \ldots, m \tag{2.10.8}$$

The probability density function of R^2, given in Theorem 2.5.1, is identified in the literature [49] as the noncentral chi square distribution with two degrees of freedom. From (2.5.5), the non-centrality parameter, $\overline{\lambda}^2$ is defined as $\overline{\lambda}^2 = \frac{\lambda^2}{\sigma^2}$. Letting $\overline{\xi}$ denote $\frac{\xi}{\sigma}$, (2.10.8) can be written as,

$$\Pr(\chi_2'^2(\overline{\lambda}_i^2) \leq \overline{\xi}_i^2) = F(\overline{\xi}_i^2, 2, \overline{\lambda}_i^2) \geq \gamma_i \quad \text{for all } i \qquad (2.10.9)$$

Patnaik [75] suggested a good approximation to the noncentral χ^2, which consists of replacing $\chi_2^2(\lambda^2)$ by a multiple of a central χ^2, $c\chi_v^2$, where c and v (degrees of freedom) are defined as follows,

$$c = \frac{2(1+\overline{\lambda}^2)}{(2+\overline{\lambda}^2)}, \qquad v = \frac{(2+\overline{\lambda}^2)^2}{2(1+\overline{\lambda}^2)} \qquad (2.10.10)$$

Therefore, if the $\chi_2'^2(\overline{\lambda}_i^2)$ distribution in (2.10.9) is replaced by its approximation

$$\Pr(\chi_2'^2(\overline{\lambda}_i^2) \leq \overline{\xi}_i^2) \simeq \Pr(\chi_v^2 \leq \frac{\overline{\xi}_i^2}{c})$$

and the constraints of problem P2.10 are written as

$$F_{\chi_v^2}(\overline{\xi}_i^2/c) \geq \gamma_i \qquad \text{for all } i,\ i = 1, \ldots, m \qquad (2.10.11)$$

Another approximation which is relatively accurate is to approximate the noncentral χ^2 as a standard normal distribution. Johnson and Kotz [49] referred to the following approximation,

$$F(\bar{\xi}_i^{\,2},\ 2,\ \bar{\lambda}_i^{\,2}) \simeq \Phi\left(\frac{\bar{\xi}_i^{\,2}-(2+\bar{\lambda}_i^{\,2})}{2(1+\bar{\lambda}_i^{\,2})^{\frac{1}{2}}}\right) \qquad (2.10.12)$$

Therefore, if (2.10.12) is used, then the constraints are written as

$$\Phi\left(\frac{\bar{\xi}_i^{\,2}-(2+\bar{\lambda}_i^{\,2})}{2(1+\bar{\lambda}_i^{\,2})^{\frac{1}{2}}}\right) \geq \gamma_i \qquad \text{for all } i,\ i = 1,\ \ldots,\ m \qquad (2.10.13)$$

Before solving P2.10 the properties of the constraints given by either (2.10.11) or (2.10.13) are studied in the next two theorems.

<u>Theorem 2.10.2</u>: The constraints given by (2.10.11) form a convex set.

<u>Proof</u>: In the χ_v^2 distribution defined by (2.10.11), the degrees of freedom v are expressed as a function of $\bar{\lambda}_i^{\,2}$, i.e., it is a function of x_1, x_2. Hence, for a given location (x_1,x_2), v is defined. Let $F^{-1}(\gamma) = \pi$; hence, any value of $\chi_v^2 = \pi_1$, where $\pi_1 \geq \pi$ must correspond to a probability value greater than or equal to γ. Therefore, an equivalence for the constraint, $F(\xi) \geq \gamma$, will be,

$$\xi \geq F^{-1}(\gamma) = \pi \qquad (2.10.14)$$

Using the relation (2.10.14), (2.10.11) is written as

$$F^{-1}_{\chi_v^2}(\gamma_i) = \pi_i \leq \frac{\bar{\xi}_i^{\,2}}{c} \qquad \text{for all } i \qquad (2.10.15)$$

Substituting the value of c given by (2.10.10) in (2.10.15)

$$\pi_i \leq \frac{(2+\bar{\lambda}_i^{\,2})}{2(1+\bar{\lambda}_i^{\,2})}\ \bar{\xi}_i^{\,2}$$

After simplification,

$$\overline{\lambda}_i^{\,2} \leq \frac{2(\overline{\xi}_i^{\,2}-\pi_i)}{(2\pi_i-\overline{\xi}_i^{\,2})} = \overline{\theta}_i \tag{2.10.16}$$

To check the sign of θ_i, recall that,

$$c = \frac{2 + 2\overline{\lambda}^2}{2 + \overline{\lambda}^2} = 1 + \frac{\overline{\lambda}^2}{2 + \overline{\lambda}^2} = 1 + \varepsilon$$

where $0 \leq \varepsilon < 1$, i.e., a fraction; hence, from (2.10.15),

$$\overline{\xi}_i^{\,2} \geq c\,\pi_i = \pi_i + \varepsilon\,\pi_i \tag{2.10.17}$$

Observing (2.10.17), $\overline{\xi}_i^{\,2}$ is bounded as follows,

$$2\pi_i > \overline{\xi}_i^{\,2} \geq \pi_i \tag{2.10.18}$$

Using (2.10.18), the conclusion can be drawn that $\theta_i \geq 0$. Therefore,

$$\overline{\lambda}_i^{\,2} = \frac{(x_1-\mu_{a_i})^2+(x_2-\mu_{b_i})^2}{\sigma_i^{\,2}} \leq \overline{\theta}_i$$

or

$$(x_1-\mu_{a_i})^2 + (x_2-\mu_{b_i})^2 \leq \theta_i \quad \text{for all } i,\ i = 1, \ldots, m \tag{2.10.19}$$

where $\theta_i = \sigma_i^{\,2}\overline{\theta}_i$.

Equation 2.10.19 represents all interior and boundary points contained within a circle of radius $(\sqrt{\overline{\theta}_i}\,\sigma_i)$. Therefore, (2.10.19) is a convex function over all $X \varepsilon E^2$. Hence, the set of all constraints is a convex set. The following theorem will check the property of the constraints of (2.10.13).

Theorem 2.10.3: The constraints given by (2.10.13) form a convex set.

Proof: From the property of the standard normal distribution, the function $\Phi(w)$ is both concave and convex over its domain. From the first and second derivatives,

$$\frac{d\Phi(w)}{dw} = \phi(w) ,$$

$$\frac{d^2\Phi(w)}{dw^2} = -w\ \phi(w)$$

Since $\phi(w) > 0$ for all w, then the second derivative is nonpositive only if $w \geq 0$. Therefore, the function $\Phi(w)$ is concave over the domain, $0 \leq w < \infty$. From Markov's Inequality,

$$F(w) \geq 1 - \frac{E(x)}{w} = \frac{w - E(x)}{w} \qquad (2.10.20)$$

From Theorem 2.5.2, $E[R^2] = \sigma^2(2 + \frac{\lambda^2}{\sigma^2}) = \sigma^2(2+\bar{\lambda}^2)$. Using this result with (2.10.20),

$$\bar{\xi}_i^{\,2} \geq \sigma_i^{\,2}(2+\bar{\lambda}_i^{\,2}) > \bar{\lambda}_i^{\,2} \qquad (2.10.21)$$

Using (2.10.21), the argument of Φ in (2.10.13) is nonnegative, implying that the function is concave over its argument. Since $\bar{\xi}_i^{\,2} \geq 0$, (2.10.21) is a convex function over $X \in E^2$. Therefore, the set of constraints is a convex set.

2.10.1 Solution Procedure

The result given by (2.10.13) may be formulated in the same fashion as (2.10.19). Let $\Phi^{-1}(\gamma_i) = \pi_i$, then assuming $\gamma_i \geq 0.50$, it is true that $\pi_i \geq 0$. Hence, (2.10.13) can be written as

$$\overline{\xi}_i^{\,2} - (2+\overline{\lambda}_i^{\,2}) \geq 2\pi_i \sqrt{1+\overline{\lambda}_i^{\,2}} \tag{2.10.22}$$

or

$$\lambda_i^2 + \sigma_i\sqrt{\sigma_i^2+\lambda_i^2} \leq \delta_i \qquad \text{for all } i,\ i = 1, \ldots, m \tag{2.10.23}$$

where $\delta_i = \dfrac{\xi_i^2}{2\pi_i} - 2\sigma_2^2$.

As in problem $\overline{\text{P}}$2.6, problem P2.10 may be formulated in one of the following forms,

$\overline{\text{P}}$2.10.1 $\quad \underset{X}{\text{minimize}}\ f(X) = \sqrt{\dfrac{\pi}{2}} \displaystyle\sum_{i=1}^{m} w_i\sigma_i\ M\left(-\frac{1}{2},\ 1,\ -\frac{\lambda_i^2}{2\sigma_i^2}\right)$

subject to: $\lambda_i^2 \leq \theta_i \qquad$ for all i, $i = 1, \ldots, m$

where

$$\lambda_i^2 = (x_1-\mu_{a_i})^2 + (x_2-\mu_{b_i})^2$$

$$\theta_i = \frac{2\xi_i^2 - 2\sigma_i^2\pi_i}{2\pi_i - \dfrac{\xi_i^2}{\sigma_i^2}}$$

and

$$\pi_i = F^{-1}_{\chi_v^2}(\gamma_i)\ .$$

Alternately, P2.10 may be written

$$\overline{P}2.10.2 \quad \underset{X}{\text{minimize}}\ f(X) = \sqrt{\frac{\pi}{2}} \sum_{i=1}^{m} w_i \sigma_i\ M(-\frac{1}{2}, 1, -\frac{\lambda_i^2}{2\sigma_i^2})$$

$$\text{subject to:} \quad \lambda_i^2 + \sigma_i\sqrt{\sigma_i^2+\lambda_i^2} \leq \delta_i \quad \text{for all } i,\ i = 1, \ldots, m$$

where

$$\lambda_i^2 = (x_1-\mu_{a_i})^2 + (x_2-\mu_{b_i})^2$$

$$\delta_i = \frac{\xi_i^2}{2\pi_i} - 2\sigma_i^2$$

and

$$\pi_i = \Phi^{-1}(\gamma_i).$$

From Theorems 2.10.2 and 2.10.3, problems $\overline{P}2.10.1$ and $\overline{P}2.10.2$ are convex programming problems. In problem $\overline{P}2.10.1$ a cumulative χ_v^2 distribution is required and interpolations are performed in estimating the parameter π_i, since v is calculated at each step from the feasible solution (x_1,x_2). Note that problem $\overline{P}2.10.2$ depends on the normal distribution (error function) which is well-tabulated for computer computations. Also, the nonlinear constraints are of a simple form. Therefore, the use of the second formulation ($\overline{P}2.10.2$) with the aid of an efficient convex programming algorithm is recommended. Note that a global optimal solution is guaranteed by satisfying both the necessary and sufficient conditions of optimality.

2.11 Discrete Distribution Formulations

Previously, P_i has been considered to be a continuously distributed random variable. Although such an assumption is valid for a number of location problems, in some cases the locations of existing and new facilities are restricted to having discrete values. For example, in emergency facility services (police, fire and ambulance), the distance between two locations might be measured by the number of blocks. Physicists studying Brownian motion face the problem of identifying the location of a physical particle which moves randomly along a grid depending on the number of molecular collisions happening. When the locations of the existing facilities can be modeled as a random walk, a discrete probability distribution is appropriate. As an example, if an existing facility can undergo a unit change in either the x or y coordinate direction each time interval, then the location of the particle after n steps is always random. In particular, the location of the facility can be modeled as a Markov chain. In this section, discrete distributions are employed in modeling the single facility location problem where distances are measured in rectilinear norm.

2.11.1 Rectilinear-Distance Location Problem: Discrete Distributions

The single facility location problem can be formulated as follows for the case of discrete probability mass functions and rectilinear distances.

$$\text{P11.1} \quad \underset{X}{\text{minimize}}\ f(X) = \sum_{i=1}^{m} w_i E[|X-P_i|] = \sum_{i=1}^{m} w_i E[|x_1-a_i|+|x_2-b_i|]$$

$$= \sum_{i=1}^{m} w_i [\int_{a_i} |x_1 - a_i| p(a_i) + \int_{b_i} |x_2 - b_i| p(b_i)]$$

$$a_i = 0, 1, \ldots, n_{a_i}$$
$$b_i = 0, 1, \ldots, n_{b_i}$$

where w_i, a_i, and b_i are defined as in Section 2.2, $p(a_i)$, $p(b_i)$ are the probability mass functions associated with a_i, b_i, and n_{a_i}, n_{b_i} are the maximum values attained by the random variables a_i and b_i, respectively. For a problem similar to the random walk problem, n_{a_i} will be the number of steps taken by a_i. In the above formulation, a_i and b_i may have lower bounds other than zero; this will be the case when the rectangular distribution is employed over the ranges $\alpha_{a_i} \leq a_i \leq \beta_{a_i}$, and $\alpha_{b_i} \leq b_i \leq \beta_{b_i}$. The probability mass function should be selected according to the nature of the location problem under study.

Since the objective function of P11.1 is a separable function in both x_1 and x_2,

$$\min_X f(X) = \min_{x_1} f(x_1) + \min_{x_2} f(x_2)$$

where

$$f(x_1) = \sum_{i=1}^{m} \sum_{a_i} w_i |x_1 - a_i| p(a_i) \tag{2.11.1}$$

and

$$f(x_2) = \sum_{i=1}^{m} \sum_{b_i} w_i |x_2 - b_i| p(b_i) \tag{2.11.2}$$

Since (2.11.1) and (2.11.2) have the same form, any procedure that applies to minimizing $f(x_1)$ will also apply to minimizing $f(x_2)$.

To minimize $f(x_1)$, (2.11.1) is written as

$$f(x_1) = \sum_{i=1}^{m} \sum_{k=0}^{n_i} w_{ik}|x_1-a_{ik}| \tag{2.11.3}$$

where a_{ik} = kth value of the random variable a_i, $n_i = n_{a_i}$, and $w_{ik} = w_i p_{a_i}(k)$.

Using the formulation in (2.11.3), $f(x_1)$ can be optimized as a linear programming problem by using the following transformation of variables,

$$|x_1-a_{ik}| = r_{ik} + s_{ik} \tag{2.11.4}$$

where

$$x_1-a_{ik}-r_{ik}+s_{ik} = 0 , \quad r_{ik}\cdot s_{ik} = 0 \quad \text{and } r_{ik},s_{ik} \geq 0.$$

The corresponding linear programming formulation is given by,

$\overline{\text{P}}11.1$ (Primal) $\underset{x_1}{\text{minimize}}\ Z = \sum_{i=1}^{m} \sum_{k=0}^{n_i} w_{ik}(r_{ik}+s_{ik})$

subject to: $x_1 - r_{ik} + s_{ik} = a_{ik}$, $i = 1, \ldots, m$; $k = 0, 1, \ldots, n_i$

$r_{ik}\cdot s_{ik} = 0$, for all i, k

$r_{ik},s_{ik} \geq 0$ for all i, k

Shanno and Weil [87] proved that the nonlinear constraints in $\overline{\text{P}}11.1$ may be dropped if the simplex algorithm is used to solve the problem. Using this fact, $\overline{\text{P}}11.1$ is a regular linear programming problem with $(m\ n_i)$ variables and $(m\ n_i)$ constraints, where the simplex technique is very efficient in solving it. However, $\overline{P}11.1$ may be further simplified by formulating the dual of $\overline{P}11.1$.

$$\overline{\overline{P}}11.1 \text{ (Dual)} \quad \text{maximize } Z = \sum_{i=1}^{m} \sum_{k=0}^{n_i} a_{ik} u_{ik}$$

$$\text{subject to: } \sum_{i=1}^{m} \sum_{k=0}^{n_i} u_{ik} = 0$$

$$|u_{ik}| \leq w_{ik} \qquad , i = 1, \ldots, m; \; k = 0, 1, \ldots, n_i$$

Notice that the bounded variable constraints may be handled efficiently using a bounded variable linear programming algorithm. Further solution efficiency can be obtained if $\overline{\overline{P}}11.1$ is converted to a network flow problem. See Cabot, et al. [3].

2.11.2 A Median Solution Approach

Francis [26] showed that the deterministic single facility location problem with rectilinear distances may be solved by applying median conditions. The median conditions state that $f(x_1)$ is minimized if the facility is located at the point where there is no more than one half of the weight to the left of the new facility location and no more than one half the weight is to the right of the new facility. The same condition was derived by Francis and White [32] from the dual formulation of the deterministic problem. Following the same pattern as in [32], it is easily seen that the median conditions apply for the dual formulation, $\overline{\overline{P}}11.1$. These conditions are given by,

$$\sum_{k=0}^{t} \sum_{i=1}^{m} w_{ik} \geq \frac{1}{2} \sum_{k=0}^{N} \sum_{i=1}^{m} w_{ik} \tag{2.11.5}$$

$$\sum_{k=0}^{t-1} \sum_{i=1}^{m} w_{ik} \leq \frac{1}{2} \sum_{k=0}^{N} \sum_{i=1}^{m} w_{ik} \tag{2.11.6}$$

where N is defined as the maximum cardinality of the set $\{n_i\}$, i.e., $N = \max_i \{n_i\}$.

Using (2.11.5) and (2.11.6), the probabilistic problem defined by P11.1 is solved directly if the probability mass functions are known. Notice that the size of the problem increases with a high value of n_i, but in most discrete problems, n_i will be within a reasonable range. For Poisson distributions with large means, a normal approximation to the Poisson can be used and the results obtained in the previous sections can be applied.

2.12 Numerical Examples

In this section, numerical examples are solved to illustrate the effect of random variation on the location decision. The examples emphasize the unconstrained formulations since the optimal solution to the unconstrained problem may be a feasible solution to the constrained problem. As suggested above, if the optimal solution for the unconstrained problem does not satisfy the constrained problem, then the problem must be solved using any available convex programming algorithm, e.g., SUMT.

To demonstrate the effect of random variation on the solution obtained, a probabilistic problem is solved as a deterministic problem by considering that the random existing facilities are located at their corresponding means. In this way, the optimal solution may be compared with that obtained from the probabilistic models.

2.12.1 <u>Rectilinear Distance Single Facility Location Example Problem</u>

It is desired to locate a tool crib in a maintenance shop. The supervisor wishes to locate the tool crib so that the total expected walking distance per day for the employees is minimized. It is assumed that the location of each employee is a random variable with a bivariate normal distribution. The location means, standard deviations and the number of trips made by each employee to the tool crib are given in Table 2.1. The tool crib is to be located such that the expected rectilinear distance walked per day is minimized.

To solve the problem using the exact iterative procedure developed in Section 2.4, the algorithm has been programmed in Fortran IV (see Appendix B). The iterations are given as follows,

$$(x_1^{(1)}, x_2^{(1)}) = (7.688324,\ 5.097736)$$

$$(x_1^{(2)}, x_2^{(2)}) = (7.694024,\ 5.104512)$$

$$(x_1^{(3)}, x_2^{(3)}) = (7.694025,\ 5.104517)$$

The optimal location is $(x_1^*, x_2^*) = (7.694025, 5.104517)$. Francis and White [32] in treating the expected values as deterministic values of the coordinates, obtained the "optimum" location (10,5).

Comparing both solutions, an obvious difference is noticed even though the standard deviations used are not very large. Therefore, locating the tool crib by interpreting the expected values as deterministic locations can lead to a location significantly different from that which minimizes the expected total distance traveled. Notice that

TABLE 2.1. Input data for the tool crib example using rectilinear distance.

Employee i	x_1-coordinates (μ_{a_i},σ_{a_i})	x_2-coordinates (μ_{b_i},σ_{b_i})	Number of trips/day (w_i)
1	(4,2)	(4,3)	4
2	(4,2)	(10,4)	4
3	(6,3)	(5,2)	2
4	(10,5)	(5,2)	3
5	(10,4)	(9,3)	5
6	(12,3)	(3,1)	6

the minimum total expected distance is 138.9, but the expected total distance from the point (10,5) is 143.1. Thus, a 3% increase in the cost occurred by failing to explicitly account for the random variation.

2.12.2 Euclidean Distance Single Facility Location Example Problem

In the same context as the above problem, assume that the distance traveled is measured in the Euclidean norm. The location means, standard deviations (let $\sigma_i = \sigma_{a_i}$ for all i), and weights are given as in Table 2.2.

To solve the Euclidean problem, the exact iterative procedure developed in Section 2.6 is programmed in Fortran IV (see Appendix B). The iterations are given as follows,

$$(x_1^{(1)}, x_2^{(1)}) = (4.373089,\ 3.219747)$$

$$(x_1^{(2)}, x_2^{(2)}) = (4.266038,\ 2.954257)$$

$$\vdots$$

$$(x_1^{(5)}, x_2^{(5)}) = (4.256475,\ 2.850864)$$

$$\vdots$$

$$(x_1^{(9)}, x_2^{(9)}) = (4.256942,\ 2.848327)$$

The optimal location is $(x_1^*, x_2^*) = (4.256942,\ 2.848327)$.

When the expected values are considered as coordinates in a deterministic problem, Francis and White [32] obtained an optimal location of $(x_1^*, x_2^*) = (3.995,\ 2.011)$. Consequently, the deterministic problem obtained by assuming that the existing facilities are located at their

TABLE 2.2. Input data for the tool crib example using Euclidean distance

Employee i	x_1-coordinates (μ_{a_i},σ_{a_i})	x_2-coordinates (μ_{b_i},σ_{b_i})	Number of trips/day (w_i)
1	(0,4.4)	(0,2)	1
2	(0,4)	(10,3)	1
3	(5,2)	(0,3)	1
4	(12,4.3)	(6,4)	1

expected values yields significantly different locations. The minimum total expected distance is 29.88; the expected total distance at the point (3.995, 2.011) is 31.01. Thus, a 4% increase in the cost occurred by failing to explicitly account for the random variation.

In addition, suppose that it is desired that the distance traveled per trip by each employee not exceed five distance units with probability of .85. The feasibility of the optimal unconstrained solution when the chance constraints are imposed is checked first. Using the formulation given by (2.10.13), the chance constraints are written as,

$$\Phi\left(\frac{\left(\frac{5}{4.4}\right)^2-(2+\bar{\lambda}_1^{\,2})}{2(1+\bar{\lambda}_1^{\,2})^{1/2}}\right) \geq .85$$

$$\Phi\left(\frac{\left(\frac{5}{4}\right)^2 - (2+\bar{\lambda}_2^{\,2})}{2(1+\bar{\lambda}_2^{\,2})^{1/2}}\right) \geq .85$$

$$\Phi\left(\frac{\left(\frac{5}{2}\right)^2 - (2+\bar{\lambda}_3^{\,2})}{2(1+\bar{\lambda}_3^{\,2})^{1/2}}\right) \geq .85$$

$$\Phi\left(\frac{\left(\frac{5}{4.3}\right)^2-(2+\bar{\lambda}_4^{\,2})}{2(1+\bar{\lambda}_4^{\,2})^{1/2}}\right) \geq .85$$

Since $\bar{\lambda}_i^{\,2}$ is defined by

$$\bar{\lambda}_i^{\,2} = \frac{(x_1-\mu_{a_i})^2+(x_2-\mu_{b_i})^2}{\sigma_i^{\,2}}$$

the values of $\overline{\lambda}_i^{\,2}$ for the given solution (x,y) = (5.35,.539), are found to be $\overline{\lambda}_1^{\,2}$ = 1.56, $\overline{\lambda}_2^{\,2}$ = 29.53, $\overline{\lambda}_3^{\,2}$ = .03, and $\overline{\lambda}_4^{\,2}$ = 3.82. Substituting the values of $\overline{\lambda}_i^{\,2}$ into the above constraints and obtaining the corresponding probabilities from the cumulative normal distribution tables gives

$$\Phi(-0.71) = .24 \not\geq .85$$

$$\Phi(-2.7) = .003 \not\geq .85$$

$$\Phi(2.09) = .98 \geq .85$$

$$\Phi(-1.01) = .1562 \not\geq .85$$

where the third constraint is the only constraint which is not violated. Therefore, the unconstrained optimal solution is an infeasible solution to the constrained problem and the constrained problem must be solved using a nonlinear programming algorithm.

Assuming that each employee is located at the corresponding expected value, if the same upper bound is imposed on the distance traveled per trip, the deterministic constraints will be,

$$|x_1-0| + |x_2-0| \leq 5$$

$$|x_1-0| + |x_2-10| \leq 5$$

$$|x_1-5| + |x_2-0| \leq 5$$

$$|x_1-12| + |x_2-6| \leq 5$$

Upon substituting the value of (x_1^*, x_2^*) for the unconstrained problem, it is easily seen that the second constraint is the only constraint satisfied.

2.13 Summary

In this chapter, probabilistic formulations of the single facility location problem have been presented. In all formulations the random variation was considered to be due to the location of the existing facilities. It was assumed that the coordinates of each location are independent random variables in two dimensional space; also, the locations of all existing facilities are mutually independent. Both constrained and unconstrained formulations were analyzed.

Using the rectilinear norm as a measure of distance traveled, and the bivariate normal density function as a representation of the random variation associated with the location of an existing facility, the expected cost of item movements was minimized using Newton's iterative technique. Since the convergence of Newton's method is not efficient when the starting point is not contained in a close neighborhood of the optimum, an approximate solution procedure was developed to provide a good starting solution for the iterative scheme. Also, a single dimensional search was performed to determine optimal step size. It was shown that the optimal location is contained in the convex hull of the mean value of all locations.

The gravity problem was investigated. The optimal solution was shown to have a unique value which depends only on the weights and the mean values of the random locations.

Using results obtained in Section 2.5, the probabilistic Euclidean distance location problem was formulated and an iterative scheme was proposed for solving the problem. An approximate solution procedure was obtained for developing a starting point for the iterative algorithm. In case the rectilinear solution is known, lower and upper bounds on the objective function were developed using the optimal rectilinear solution.

In Section 2.7, the constrained problem was treated when norm type constraints are added to the original objective function. A Lagrange multipliers approach was recommended when there are few constraints; otherwise, a convex programming algorithm is needed.

Chance constrained formulations have been presented for both rectilinear and Euclidean distances. In each case the distribution function of distance was obtained and was used to transform the chance constraints to deterministic equivalent constraints. When the rectilinear distance is used, it was recommended that the equivalent deterministic formulation be solved by applying any convex programming algorithm, e.g., SUMT. However, global convergence has not been guaranteed.

For the case of Euclidean distances, the chance constraints were transformed to deterministic equivalent constraints in two ways. First, a chi-square distribution was used to approximate the actual probability distribution obtained for the distance traveled. Second, the distribution was approximated by the standard normal distribution. The latter approach appeared easier to handle.

The solution procedure for each formulation of the unconstrained problem was programmed and a sample problem was solved. A comparison between probabilistic and deterministic formulations was performed.

When the set of possible values for the random variables are countable, discrete distributions are employed. It was shown that median conditions, similar to those obtained for the deterministic case, may be used to obtain the optimal solution; otherwise, a linear programming problem must be solved.

Chapter 3

MULTIFACILITY PROBABILISTIC LOCATION PROBLEMS

3.1 Introduction

In the previous chapter the single facility probabilistic location problem was treated. The analysis was concerned with locating a single new facility relative to a number of existing facilities. The locations of the existing facilities were treated as random variables. In this chapter, the analysis is extended to cover the problem of locating more than one facility with respect to multiple existing facilities. Before introducing the probabilistic multifacility problem, a presentation of the deterministic case will be helpful.

The deterministic formulation of the multifacility (generalized Weber) problem is given by D3,

$$\text{D3.}\quad \underset{X_j}{\text{minimize}}\ f(X_1,\ldots,X_n) = \sum_{1\leq j<k\leq n} v_{jk}|X_j-X_k|_\ell + \sum_{j=1}^{n}\sum_{i=1}^{m} w_{ji}|X_j-P_i|_\ell$$

where

X_j = location of new facility j, j = 1, ..., n

P_i = location of existing facility i, i = 1, ..., m

v_{jk} = annual cost per unit distance between new facilities j and k, for all j, k.

w_{ji} = annual cost per unit distance between new facility j and existing facility i, for all j, i.

$|X_j - P_i|_\ell$ = distance between the points X_j and P_i measured in the ℓ norm

$f(X_1,\ldots,X_n)$ = total annual cost as a function of $X_1, \ldots, X_n$

ℓ = type of norm used, $\ell = 1$ represents the rectilinear norm, and $\ell = 2$ represents the Euclidean norm.

In D3, the objective is to determine the locations of the new facilities in order to minimize total annual cost. In the subsequent discussion, it is always assumed for each value of j that v_{jk} is non-zero for at least one value of k. Additionally, it is assumed that all new facilities are chained to existing facilities [31].

As indicated in Chapter 2, several research efforts have been directed toward a study of D3. For the rectilinear case, Cabot, et al. [3], formulated D3 equivalently as a network flow problem; Wesolowsky and Love [103] proposed a linear programming solution. For the case of Euclidean distances, Love [67] employed a non-linear programming algorithm to seek the optimal solution, and Eyster, et al. [22] solved the problem using a hyperbolic approximation iterative technique (HAP). Francis and Cabot [31] obtained a dual formulation for the Euclidean problem. Wendell [100] developed a geometric programming duality formulation with mixed type of norms. For a more complete review of previous research on P3, see Francis and White [32].

In this chapter, the treatment of the probabilistic variation of D3 includes the possibility of P_i, v_{jk}, and w_{ji} being random variables. The reasons for considering P_i as a random variable parallel those given in Chapter 2. The weights can easily be random variables. For example, if the weights represent the cost per unit distance, then for a long planning horizon its randomness is natural. In another application, such as locating emergency units (centers), the weight, interpreted as time per unit distance, is random due to the traveling speed, road and weather conditions, and time of the day. In locating central facilities such as banks, shopping centers and post offices, the weight may be considered as the frequency of travel between two facilities or level of business which depends on the demand of customers or their availability; thus, it is most likely to be random. For transporting commodities between locations, the weights may denote the volume of goods, which can be considered to be random variables.

The normal distribution is employed throughout the subsequent models, since P_i, w_{ji}, and v_{jk} correspond in most situations to the normal distribution; otherwise, the distribution could be approximated by the normal using some Central Limit Theorem. Two types of probabilistic problems are studied. In the first case it is assumed that, for a given realization of P_i, a realization of w_i occurs. Once the location of the existing facility (customer) i is known, all subsequent trips between new facility j and existing facility i will share the same distance $|X_j-P_i|_\ell$, and the weight attached to the trip will be w_{ji}. From a probability point of view, the cost of transportation incurred between new facility j and existing facility i is expressed as a multiplication of the random variables w_{ji} and $|X_j-P_i|_\ell$. As discussed above the distance is treated as a random variable due to the randomness of P_i. In the second case considered, for each trip included in the weight w_{ji}, the distance between new facility j and existing facility i can be different.

There are w_{ji} trips during the planning horizon under investigation and existing facility i changes its location during this planning horizon independent of the weight w_{ji}. For convenience, let P_{ih} denote the location of existing facility i on trip h. Thus, on trip h the distance traveled is determined from the value of the random variable P_{ih}. The "weight" or number of trips per unit time is considered to be independent of the location of each existing facility. Thus, the cost of transportation can be represented as a random sum of random variables.

To motivate the two cases considered, suppose new warehouses are to be located across the country. The sources of goods shipped to the warehouses and the destinations of goods shipped from the warehouses are not known a priori. However, after the warehouses become operational the locations of suppliers and customers will become known. The number of shipments per month from the suppliers to the warehouses and from the warehouses to customers is not known exactly, but can be expressed in the form of a probability distribution. Since all shipments from supplier i to warehouse j will be from the point P_i, once the value of P_i becomes known, the location problem can be formulated as the weighted sum of the products of the random variables P_i and w_{ji}.

As an illustration of the second case considered, suppose a military hospital is to be located to provide medical treatment for personnel wounded in combat. Patients are brought from the combat area to the hospital in helicopters. There are m combat areas, the number of helicopter trips to and from combat area i is a random variable w_i. The location of a wounded soldier in combat area i is a random variable denoted by P_i. Thus, each of the w_i trips can be to a different location in combat area i. In this case, the location problem is formulated as a random sum of random variables.

In the subsequent sections, both problems are formulated and solution procedures are introduced. In analyzing the models, two types of norms are used: rectilinear and Euclidean. Both will be used to solve the unconstrained case. For the constrained case, only the Euclidean norm is employed since the rectilinear norm involves more tedious algebra and makes the model more cumbersome to solve, as discussed in Chapter 2.

3.2 Probabilistic Formulations

In this section, the two problems discussed above will be formulated mathematically. The first problem is identified as the case when the expected cost is the product of the random variables P_i and w_{ji}; the second one is associated with the case when the expected cost is a random sum of random variables. In the sequel both unconstrained and constrained formulations are given.

First, for the case of the product of the random variables, the expected total cost function is given by

$$\text{P3.1} \quad \underset{X_j}{\text{minimize}}\ E[f(X_1,\ldots,X_n)] = E\Big[\sum_{1\le j<k\le n} v_{jk}|X_j - X_k|_\ell + \sum_{j=1}^{n}\sum_{i=1}^{m} w_{ji}|X_j - P_i|_\ell\Big]$$

$$= \sum_{1\le j<k\le n} E[v_{jk}]|X_j - X_k|_\ell + \sum_{j=1}^{n}\sum_{i=1}^{m} E[w_{ji}]E|X_j - P_i|_\ell]$$

For the case of a random sum of random variables the problem of minimizing expected total cost is written as

$$\text{P3.2}\quad \underset{X_j}{\text{minimize}}\ E[f(X_1,\ldots,X_n) = E[\sum_{1\le j<k\le n} v_{jk}|X_j-X_k|_\ell + \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{h=1}^{w_{ji}}|X_j-P_{ih}|_\ell]$$

$$= \sum_{1\le j<k\le n} E[v_{jk}]|X_j-X_k|_\ell + \sum_{j=1}^{n}\sum_{i=1}^{m} E[\sum_{h=1}^{w_{ji}} |X_j-P_{ih}|_\ell]$$

A comparison of both P3.1 and P3.2 discloses that the first summation is the same, but the second one is different. The expected value of the random sum of identically distributed random variables is given by

$$\sum_{i=1}^{N} X_i = E[N]\ E[X] \tag{3.2.1}$$

Thus, if the second summation in P3.2 may be written as

$$\sum_{j=1}^{n}\sum_{i=1}^{m} E[\sum_{h=1}^{w_{ji}}|X_j-P_{ih}|_\ell] = \sum_{j=1}^{n}\sum_{i=1}^{m} E[w_{ji}]\ E[|X_j-P_i|_\ell]$$

then it is identical to the second summation in P3.1. Therefore, optimizing either one implies the optimization of the other. However, different solutions to P3.1 and P3.2 are anticipated when the constrained problem is solved.

In a number of applications it is not enough to minimize the total expected cost. Rather, some upper bound constraints may be imposed on the individual expected cost elements. Specifically, it is quite common to encounter situations in which an aspiration level is recommended in the form of a confidence interval on the random cost. Thus, a chance constraint might be included in the formulation of the location problem. A constrained version of P3.1 is

$$\text{P3.1.1} \quad \underset{X_j}{\text{minimize}}\ E[f(X_1,\ldots,X_n)] = \sum_{1\leq j<k\leq n} E[v_{jk}]|X_j-X_k|_\ell + \sum_{j=1}^{n}\sum_{i=1}^{m} E[w_{ji}]\, E[|X_j-P_i|_\ell]$$

$$\text{subject to:} \quad E[v_{jk}]|X_j-X_k|_\ell \leq \xi_{jk} \quad \text{for all } j,\ k = 1,\ \ldots,\ n \tag{3.2.2}$$

$$Pr(w_{ji}|X_j-P_i|_\ell \leq \xi_{ji}) \geq \gamma_{ji} \quad \begin{array}{l} j = 1,\ \ldots,\ n \\ i = 1,\ \ldots,\ m \end{array} \tag{3.2.3}$$

where ξ_{jk} and ξ_{ji} are specified upper bounds on the cost of transportation between locations (j,k) and (j,i), respectively, and γ_{ji} represents the required service level, $0 < \gamma_{ji} < 1$. The constraints (3.2.2) may also be replaced by chance constraints of the form

$$Pr[v_{jk}|X_j-X_k|_\ell \leq \xi_{jk}) \geq \bar{\gamma}_{jk} \quad \text{for all } j,\ k = 1,\ \ldots,\ m \tag{3.2.4}$$

In the subsequent discussion, both types of constraints are treated.

A constrained version of P3.2 is given by

$$\text{P3.2.1} \quad \underset{X_j}{\text{minimize}}\ E[f(X_1,\ldots,X_n)] = \sum_{1\leq j<k\leq n} E[v_{jk}]|X_j-X_k|_\ell + \sum_{j=1}^{n}\sum_{i=1}^{m} E[\sum_{h=1}^{w_{ji}} |X_j-P_{ih}|_\ell]$$

subject to: $E[v_{jk}]|X_j - X_k|_\ell \leq \xi_{jk}$ for all j, k = 1, ..., n

$$Pr\left(\sum_{h=1}^{w_{ji}} |X_j - P_{ih}|_\ell \leq \xi_{ji}\right) \geq \gamma_{ji} \quad , \begin{array}{l} j = 1, \ldots, n \\ i = 1, \ldots, m \end{array} \quad (3.2.5)$$

As in P3.1.1, chance constraints of the type shown in (3.2.4) may replace the expected value constraints (3.2.2) on the new facilities.

3.3 Related Work

The only probabilistic formulation of the generalized Weber problem (P3) appears to be the chance constrained formulation of Seppälä [86]. In his model, v_{jk} and w_{ji} are treated as random variables, but P_i is known deterministically and the Euclidean norm is used to measure distances. He employs the fractile criterion described in Chapter 1 and applied by Sengupta and Portillo-Campbell [84]. Using the approach developed by Charnes, et al. [7] to convert the chance constraint to its deterministic equivalent, Seppälä obtained a non-linear objective function. To solve his model, the CHAPS algorithm developed by Seppälä [85] is used to convert the non-linear objective function to a linear objective function augmented by some non-linear convex constraints. A linear approximation algorithm similar to MAP, introduced by Griffith and Stewart [38], is employed to solve the resulting formulation.

In considering the cost per unit distance (w_i) as a random variable, Hurter and Prawda [48] solved the Euclidean single facility location problem when the quantity of service demanded is a random variable. They formulated the problem as a chance-constrained programming problem, but the constraints were used to bound w_i instead of bounding the cost of transportation, which is a function of the distance. In the analysis by Hurter and Prawda [48] the locations of the existing facilities are assumed to be deterministic when the probabilistic problem is changed to a deterministic one, using the approach of Charnes, et al. [7]. Hurter and Prawda [48] showed that any existing algorithm to solve the deterministic single facility problem can be used for their chance-constrained problem. In [33], Frank studied the problem of optimum locations on graphs when the demand at the existing facilities (nodes) are considered random variables with a normal probability density function. He generalized the results obtained by Hakimi [41, 42] about the center and the median of the graph. In a later paper [34], Frank modified the problem to accommodate the case when the random variables are correlated. Even though both problems are limited to network location problems, the formulations are not simple and numerical solutions are required.

From the above survey it appears that no previous research has been devoted to the study of the generalized Weber problem when P_i, v_{jk}, and w_{ji} are treated simultaneously as random variables. Through the current research effort, the general problem will be explored such that any special case may be obtained easily from the proposed solution methods.

3.4 Rectilinear Distance Generalized Weber Problem: Unconstrained

As discussed in Section 3.2, problems P3.1 and P3.2 are equivalent. Therefore, where P3.1 is used with a rectilinear norm as a measure for distances, the problem may be written as,

$$\text{P3.4} \quad \underset{X_j}{\text{minimize}}\ E[f(X_1,\ldots,X_n)] = \sum_{1\le j<k\le n} E[v_{jk}]|X_j - X_k| + \sum_{j=1}^{n}\sum_{i=1}^{m} E[w_{ji}]\ E[|X_j - P_i|]$$

with the following assumptions:

i. $v_{jk} \sim N(\mu_{jk}, \sigma_{jk}^{\ 2})$, where $\mu_{jk} > 0$, for all j, k = 1, ..., n;

ii. $w_{ji} \sim N(\overline{\mu}_{ji}, \overline{\sigma}_{ji}^{\ 2})$, where $\overline{\mu}_{ji} > 0$, for all $j = 1, \ldots, n$; $i = 1, \ldots, m$;

iii. $a_i \sim N(\mu_{a_i}, \sigma_{a_i}^{\ 2})$, for all i, i = 1, ..., m;

iv. $b_i \sim N(\mu_{b_i}, \sigma_{b_i}^{\ 2})$, for all i, i = 1, ..., m;

v. a_i, b_i are considered independent for the same i;

vi. $P_i = (a_i, b_i)$ is an independent random variable for all i, where P_i corresponds to a bivariate normal distribution;

vii. for a single value of j, all v_{jk} are uncorrelated random variables for all k, k = 1, ..., n;

viii. for a single value of j, all w_{ji} are uncorrelated random variables for all i, i = 1, ..., m.

From the above assumptions, the expected value of weights (v_{jk}, w_{ji}) are all known. In Chapter 2, the expected distance from the location of the new facility, X, to any location P_i was developed. Using (2.4.8) and the above assumptions in P3.4, the following results,

$$E[f(X_1,\ldots,X_n)] = \sum_{1\le j<k\le n} \mu_{jk}|X_j-X_k| + \sum_{j=1}^{n}\sum_{i=1}^{m} \bar{\mu}_{ji}[(x_{j1}-\mu_{a_i})$$

$$(2\Phi(\frac{x_{j1}-\mu_{a_i}}{\sigma_{a_i}})-1) + 2\sigma_{a_i}\ \phi(\frac{x_{j1}-\mu_{a_i}}{\sigma_{a_i}})$$

$$+ (x_{j2}-\mu_{b_i})(2\Phi(\frac{x_{j2}-\mu_{b_i}}{\sigma_{b_i}})-1) + 2\sigma_{b_i}\ \phi(\frac{x_{j2}-\mu_{b_i}}{\sigma_{b_i}})] \tag{3.4.1}$$

where $X_j = (x_{j1},x_{j2})$ for all $j = 1, \ldots, n$.

But since the rectilinear distance between the new facilities can be decomposed to its coordinates,

$$\sum_{1\le j<k\le n} \mu_{jk}|X_j-X_k| = \sum_{1\le j<k\le n} \mu_{jk}|x_{j1}-x_{k1}| + \sum_{1\le j<k\le n} \mu_{jk}|x_{j2}-x_{k2}| \tag{3.4.2}$$

Using (3.4.1) and (3.4.2), the expected total cost function is expressed as

$$E[f(X_1,\ldots,X_n)] = E[f_1(x_{11},\ldots,x_{n1})] + E[f_2(x_{12},\ldots,x_{n2})] \tag{3.4.3}$$

where

$$E[f_1(x_{11},\ldots,x_{n1})] = \sum_{1\le j<k\le n} \mu_{jk}|x_{j1}-x_{k1}| + \sum_{j=1}^{n}\sum_{i=1}^{m}$$

$$+ \sum_{j=1}^{n}\sum_{i=1}^{m} \bar{\mu}_{ji}[(x_{j1}-\mu_{a_i})(2\ (\frac{x_{j1}-\mu_{a_i}}{\sigma_{a_i}})-1)$$

$$+ 2\sigma_{a_i}\ \phi(\frac{x_{j1}-\mu_{a_i}}{\sigma_{a_i}})] \tag{3.4.4}$$

and

$$E[f_2(x_{12},\ldots,x_{n2})] = \sum_{1\le j<k\le n} \mu_{jk}|x_{j2}-x_{k2}| + \sum_{j=1}^{n}\sum_{i=1}^{m} \bar{\mu}_{ji}[(x_{j2}-\mu_{b_i})$$

$$(2\Phi(\frac{x_{j2}-\mu_{b_i}}{\sigma_{b_i}})-1) + 2\sigma_{b_i}\ \phi(\frac{x_{j2}-\mu_{b_i}}{\sigma_{b_i}})] \qquad (3.4.5)$$

Since $E[f(X_1,\ldots,X_n)$ is separable in the variables x_{j1}, x_{j2} (from (3.4.3)), then it follows that

$$\underset{X_j}{\text{minimize}}\ E[f(X_1,\ldots,X_n)] = \underset{x_{j1}}{\text{minimize}}\ E[f_1(x_{11},\ldots,x_{n1})]$$

$$+ \underset{x_{j2}}{\text{minimize}}\ E[f_2(x_{12},\ldots,x_{n2})] \qquad (3.4.6)$$

Hence, optimum x_{j1} coordinates of the new facilities can be obtained independently of the optimum x_{j2} coordinates. Also, any procedure developed for minimizing $E[f_1]$ will also apply to $E[f_2]$ on replacing x_{j1} by x_{j2}, μ_{a_i} by μ_{b_i}, and σ_{a_i} by σ_{b_i}. Before optimizing $E[f_1]$, its properties are studied.

<u>Theorem 3.4.1</u>: The function $E[f_1]$ defined by (3.4.4) is a strictly convex function over $x_{j1} \in E^1$.

<u>Proof</u>: From Theorem 2.4.2, it has been proven that $\sum_{i=1}^{m} \bar{\mu}_{ji}\ E[|x_{j1}-a_i|]$ is a strictly convex function. Thus, summing this function over all values of j yields a strictly convex function. The first term in $E[f_1]$ is a summation of a multiple of the rectilinear norm. Since the rectilinear norm is convex (by the properties of the norm), the summation

over j yields another convex function. Therefore, $E[f_1]$ is a strictly convex function, due to the fact that it is the summation of two convex functions and one of them is strictly convex.

3.4.1 Solution Procedure

Shanno and Weil [87] suggested a solution procedure by solving an optimization problem which is a function of absolute values. Employing the change of variables

$$p_{jk} = x_{j1} - x_{k1} \quad \text{if } x_{j1} \geq x_{k1}$$

$$q_{jk} = x_{k1} - x_{j1} \quad \text{if } x_{k1} > x_{j1}$$

then

$$x_{j1} - x_{k1} - p_{jk} + q_{jk} = 0 \quad \text{and } p_{jk} \cdot q_{jk} = 0 \quad \text{for all } j,\ k \tag{3.4.7}$$

Thus, the first term in $E[f_1]$ is written as

$$\sum_{1 \leq j < k \leq n} \mu_{jk} |x_{j1} - x_{k1}| = \sum_{1 \leq j < k \leq n} \mu_{jk}(p_{jk} + q_{jk}) \tag{3.4.8}$$

with the addition of the set of constraints defined by (3.4.7). Substituting (3.4.8) in (3.4.5) and letting $x_{j1} = x_j$, $x_{k1} = x_k$, the following modified problem is obtained.

$$\text{P3.4.1} \quad \text{minimize} \sum_{1 \leq j < k \leq n} \mu_{jk}(p_{jk} + q_{jk}) + \sum_{j=1}^{n} \sum_{i=1}^{m} \bar{\mu}_{ji} [(x_j - \mu_i)$$

$$(2\Phi(\frac{x_j - \mu_i}{\sigma_i}) - 1) + 2\sigma_i \ \phi(\frac{x_j - \mu_i}{\sigma_i})$$

subject to: $x_j - x_k - p_{jk} + q_{jk} = 0 \quad , 1 \leq j < k \leq n$

$$p_{jk} \cdot q_{jk} = 0 \quad , 1 \leq j < k \leq n$$

$$p_{jk}, \; q_{jk} \geq 0 \quad , 1 \leq j < k \leq n$$

In P3.4.1, let $f_j(x_j)$ denote the second term in the objective function, where $f_j(x_j)$ is a nonlinear function over x_j and $\sum_{j=1}^{n} f_j(x_j)$ is a separable function in x_j. Therefore, a separable programming approach is well-suited for solving this problem. From Theorem 2.4.4, it is known that the optimal value of x_j is contained in the convex hull of μ_i. Therefore, an upper and lower bound on x_j are obtained. If $x_j < 0$ for some j, we may change the coordinates by shifting the origin so that all values of x_j become nonnegative. Having an upper bound available on the value of x_j allows the use of the separable programming technique discussed in Hadley [39]. First, divide the interval of x_j into r_j subintervals, then define the following variables,

$$x_j = \sum_{\ell=0}^{r_j} \lambda_{\ell j} x_{\ell j}$$

where

$$\sum_{\ell=0}^{r_j} \lambda_{\ell j} = 1, \quad \lambda_{\ell j} \geq 0 \quad \text{for all } \ell, j$$

and for a given j, no more than two adjacent $\lambda_{\ell j}$ are allowed to be positive. Using the change of variables, problem P3.4.1 is transformed to,

$$\overline{\text{P}}3.4.1 \quad \text{minimize} \sum_{1 \leq j < k \leq n} \mu_{jk}(p_{jk}+q_{jk}) + \sum_{j=1}^{n} \sum_{\ell=0}^{r_j} f_{\ell j} \lambda_{\ell j}$$

$$\text{subject to:} \quad x_j - x_k - p_{jk} + q_{jk} = 0 \quad , 1 \leq j < k \leq n$$

$$\sum_{\ell=0}^{r_j} \lambda_{\ell j} = 1 \quad , j = 1, \ldots, n$$

$$p_{jk} \cdot q_{jk} = 0 \quad , 1 \leq j < k \leq n$$

$$p_{jk}, \ q_{jk} \geq 0 \quad , 1 \leq j < k \leq n$$

$$\lambda_{\ell j} \geq 0 \quad , \begin{array}{l} j = 1, \ldots, n \\ \ell = 0, \ldots, r_j \end{array}$$

where it is also required that no more than two adjacent $\lambda_{\ell j}$ be positive. Except for the last constraint and the nonlinear constraints $p_{jk} \cdot q_{jk} = 0$, the problem is a linear programming problem and the simplex method may be applied easily. Including the nonlinear constraints poses no substantial difficulty if "restricted basis entry" is employed in order to satisfy the constraints. However, the property of the objective function $E[f_1]$ and the matrix of coefficients for the set of constraints allow us to ignore these constraints.

Hadley [39] proved that if the original nonlinear objective function is a convex function and the set of constraints is a convex set, an optimal solution to the approximate problem is a global optimum to the original problem. Also, he proved that, if these conditions hold, the global optimum is obtained through the simplex method without restricting the entry of the variable $\lambda_{\ell j}$. Shanno and Weil [87] demonstrated that if p_{jk} is the basic feasible solution, q_{jk} will not be, and vice versa. This is due to the fact that the column vectors corresponding to p_{jk} and p_{jk} (with the assumption that all $\mu_{jk} > 0$)

are linearly dependent; therefore, p_{jk} and q_{jk} cannot both be included in a basic feasible solution.

Using the above results, all restrictions on variables entering the basis are dropped and the problem will be

$$\bar{\bar{P}}3.4.1 \quad \text{minimize} \sum_{1\leq j<k\leq n} \mu_{jk}(p_{jk}+q_{jk}) + \sum_{j=1}^{n} \sum_{=0}^{r_j} f_{\ell j}\lambda_{\ell j}$$

$$\text{subject to:} \quad x_j - x_k - p_{jk} + q_{jk} = 0 \quad , 1 \leq j < k \leq n$$

$$\sum_{=0}^{r_j} \lambda_{\ell j} = 1 \quad , j = 1, \ldots, n$$

$$p_{jk}, \; q_{jk} \geq 0 \quad , 1 \leq j < k \leq n$$

$$\lambda_{\ell j} \geq 0 \quad , j = 1, \ldots, n; \; \ell = 0, \ldots, r_j$$

Thus, the rectilinear, unconstrained generalized Weber problem can be solved as a linear programming problem with $(5n+\sum_{j=1}^{n} r_j)$ variables and 3n constraints. Depending on the number of variables and constraints, it appears $\bar{\bar{P}}3.4.1$ can be solved easily using the simplex method.

3.5 Squared Euclidean Distance Generalized Weber Problem: Unconstrained

In this section, the squared Euclidean distance problem discussed in Chapter 2 is extended to the multifacility case. Since the study of the squared Euclidean distance, single facility problem yielded valuable information about the Euclidean problem, the multifacility variation of the gravity problem is considered. When the objective function involves a weighted sum of squared Euclidean distances, problem P3.1 may be written as,

$$\text{P3.5} \quad \underset{X_j}{\text{minimize}}\ E[f(X_1,\ldots,X_n)] = \sum_{1\le j<k\le n} E[v_{jk}][(x_{j1}-x_{k1})^2 + (x_{j2}-x_{k2})^2] + \sum_{j=1}^{n}\sum_{i=1}^{m} E[w_{ji}]\, E[(x_{j1}-a_i)^2+(x_{j2}-b_i)^2]$$

where assumptions (i-viii) in P3.4 are assumed to hold for P3.5; it is further assumed that $\sigma_{a_i}^{\ 2} = \sigma_{b_i}^{\ 2} = \sigma_i^{\ 2}$ for all i.

In Theorem 2.5.2, the expected distance with respect to a single facility is given by (2.5.10). Using the notations of Francis and White [32] for the weights v_{jk}, let

$$\hat{v}_{jk} = \begin{cases} v_{jk} & , k > j \\ v_{kj} & , k \le j \end{cases} \tag{3.5.1}$$

Substituting (2.5.10) and (3.5.1) in P3.5,

$$\overline{\text{P}}3.5 \quad \underset{X_j}{\text{minimize}}\ E[f(X_1,\ldots,X_n)] = \sum_{k=1}^{n} E[\hat{v}_{jk}](x_{j1}-x_{k1})^2 + \sum_{k=1}^{n} E[\hat{v}_{jk}](x_{j2}-x_{k2})^2 + \sum_{j=1}^{n}\sum_{i=1}^{m} E[w_{ji}](x_{j1}-\mu_{a_i})^2 + \sum_{j=1}^{n}\sum_{i=1}^{m} E[w_{ji}](x_{j2}-\mu_{b_i})^2 + 2\sum_{j=1}^{n}\sum_{i=1}^{m} E[w_{ji}]\,\sigma_i^{\ 2}$$

Observation of $\overline{P}3.5$ discloses that the function is separable in x_{j1}, x_{j2}. Hence, the optimization procedure is employed twice; first over x_{j1} and then over x_{j2}. For simplicity let $x_j = x_{j1}$ and $\mu_{a_i} = \mu_i$. To obtain the optimal x_j^*, partial derivatives of the objective function with respect to each variable are computed and then set to zero. Computing the partial derivative of expected total cost with respect to x_j gives

$$\frac{\partial E[f]}{\partial x_j} = 2 \sum_{k=1}^{n} E[\hat{v}_{jk}](x_j - x_k) + 2 \sum_{i=1}^{m} E[w_{ji}](x_j - \mu_i) \qquad (3.5.2)$$

for all j, j = 1, ..., n

Setting (3.5.2) equal to zero and rearranging yields,

$$x_j^*\left(\sum_{k=1}^{n} E[\hat{v}_{jk}] + \sum_{i=1}^{m} E[w_{ji}]\right) - \sum_{k=1}^{n} E[\hat{v}_{jk}]x_k^* = \sum_{i=1}^{m} E[w_{ji}]\mu_i \qquad (3.5.3)$$

for all j, j = 1, ..., n

Equation 3.5.3 represents a system of n linear equations in n variables to be solved to determine the optimum locations for the new facilities. To write (3.5.3) in a closed form, let $\alpha_{j1}' = \left(\sum_{k=1}^{n} E[\hat{v}_{jk}] + \sum_{i=1}^{m} E[w_{ji}]\right)$, $\alpha_{jk} = E[\hat{v}_{jk}]$, and $b_j' = \sum_{i=1}^{m} E[w_{ji}]\mu_i$, then (3.5.4) will represent the optimal value of x_{j1}^* for all j,

$$x_{j1}^*\alpha_j' - \alpha_{jk}x_{k1}^* = b_j' \qquad \text{for all } j,\ j = 1, \ldots, n \qquad (3.5.4)$$

Similarly, the optimal value of x_{j2}^* for all j,

$$x_{j2}^*\alpha_j' - \alpha_{jk}x_{k2}^* = b_j^2 \qquad \text{for all } j,\ j = 1, \ldots, n \qquad (3.5.5)$$

As Francis and White [32] indicated for the deterministic case, the matrix of coefficients is the same for (3.5.4) and (3.5.5). Therefore, its inverse is required only once.

Notice that the function in $\overline{\mathrm{P}}3.5$ is a strictly convex function. Therefore, the necessary and sufficient conditions are satisfied by the solution to (3.5.4). From (3.5.3) the following is derived

$$x_{j1} = \frac{\sum_{k=1}^{n} E[\hat{v}_{jk}]x_{k1} + \sum_{i=1}^{m} E[w_{ji}]\mu_{a_i}}{\sum_{k=1}^{m} E[\hat{v}_{jk}] + \sum_{i=1}^{m} E[w_{ji}]} \tag{3.5.6}$$

which indicates that the location of new facility j is the gravity location with respect to all other facilities, new and existing. The same result was obtained by Francis and White [32] on replacing the weights with their expected values. Consequently, the variance σ_i^2 has no effect on the gravity solution when using an unconstrained expected value criterion.

3.6 Euclidean Distance Generalized Weber Problem: Unconstrained

In this section, the Euclidean distance problem of Chapter 2 is extended to the multifacility case, and a comparison is made with the deterministic solution developed by Eyster, et al. [22]. Problem P3.1 may be formulated as,

$$\text{P3.6}\quad \underset{X_j}{\text{minimize}}\ E[f(X_1,\ldots,X_n)] = \sum_{1\le j<k\le n} E[v_{jk}][(x_{j1}-x_{k1})^2 + (x_{j2}-x_{k2})^2]^{\frac{1}{2}}$$

$$+ \sum_{j=1}^{n} \sum_{i=1}^{m} E[w_{ji}]E[(x_{j1}-a_i)^2+(x_{j2}-b_i)^2]^{\frac{1}{2}}$$

All of the assumptions of P3.5 are applied to the study of P3.6.

In Theorem 2.6.2, the expected Euclidean distance is obtained and employed in $\overline{P}2.6$. Substituting the expected Euclidean distance in P3.6 yields

$$\overline{P}3.6 \quad \underset{X_j}{\text{minimize}} \sum_{1\leq j<k\leq n} \mu_{jk}[(x_{j1}-x_{k1})^2+(x_{j2}-x_{k2})^2]^{\frac{1}{2}}$$
$$+\sqrt{\frac{\pi}{2}} \sum_{j=1}^{n} \sum_{i=1}^{m} \overline{\mu}_{ji}\sigma_i \; M(-\frac{1}{2}, 1, -\frac{\lambda_{ji}^2}{2\sigma_i^2})$$

where M is the confluent hypergeometric function defined by (2.6.6) and λ_{ji}^2 is defined as,

$$\lambda_{ji}^2 = (x_{j1}-\mu_{a_i})^2 + (x_{j2}-\mu_{b_i})^2$$

Before solving $\overline{P}3.6$, the following property of the objective function is established.

<u>Theorem 3.6.1</u>: The objective function defined in $\overline{P}3.6$ is a strictly convex function over $X_j \in E^2$.

<u>Proof</u>: Theorem 2.6.4 demonstrates that the expected Euclidean distance for X_j is a strictly convex function. Thus, its summation over j is a strictly convex function. The first summation in $\overline{P}3.6$ represents a positive combination of Euclidean norms. Since the Euclidean norm is

convex, then the combination is a convex function. Hence, the function in $\overline{P}3.6$ is strictly convex, since it is a summation of a strictly convex function and a convex function.

3.6.1 Solution Procedure

Theorem 3.6.1 provides sufficient conditions for $\overline{P}3.6$ to have a unique solution. Also, from the differentiability conditions at the optimum, the necessary conditions are obtained. The partial derivatives of $E[f(X_1,\ldots,X_n)]$ in $\overline{P}3.6$ are computed with respect to all X_j and set equal to zero; thus,

$$\frac{\partial E[f]}{\partial x_{j1}} = 0 = \sum_{\substack{k=1\\ \neq j}}^{n} \frac{\hat{\mu}_{jk}(x_{j1}-x_{k1})}{D_{jk}} + \frac{1}{2}\sqrt{\frac{\pi}{2}} \sum_{i=1}^{m} \overline{\mu}_{ji}\left(\frac{x_{j1}-\mu_{a_i}}{\sigma_i}\right) M\left(\frac{1}{2}, 2, -\frac{\lambda_{ji}^2}{2\sigma_i^2}\right) \quad , \quad j = 1, \ldots, n \tag{3.6.1}$$

and

$$\frac{\partial E[f]}{\partial x_{ja}} = 0 = \sum_{\substack{k=1\\ \neq j}}^{n} \frac{\hat{\mu}_{jk}(x_{j2}-x_{k2})}{D_{jk}} + \frac{1}{2}\sqrt{\frac{\pi}{2}} \sum_{i=1}^{m} \overline{\mu}_{ji}\left(\frac{x_{j2}-\mu_{b_i}}{\sigma_i}\right) M\left(\frac{1}{2}, 2, -\frac{\lambda_{ji}^2}{2\sigma_i^2}\right) \quad , \quad j = 1, \ldots, n \tag{3.6.2}$$

where

$$\hat{\mu}_{jk} = \begin{cases} \mu_{jk} \ , & k > j \\ \mu_{kj} \ , & k < j \end{cases}$$

and

$$D_{jk} = [(x_{j1}-x_{k1})^2+(x_{j2}-x_{k2})^2]^{\frac{1}{2}} \quad \text{for all } j, k \tag{3.6.3}$$

Unfortunately, if any two new facilities j and k have the same location at any time, then $D_{jk} = 0$ and the partial derivatives in (3.6.1) and (3.6.2) are undefined. Kuhn [60] introduced a modified gradient method to overcome this difficulty. Also, Eyster, et al. [22] developed a hyperbolic approximation procedure (HAP) to eliminate this situation. To adopt their approach, introduce a positive constant ε under the square root in D_{jk}; consequently, the partial derivatives always exist. Let $\hat{\lambda}_{jk}$ denote the modified D_{jk}, i.e.,

$$\hat{\lambda}_{jk} = [(x_{j1}-x_{k2})^2+(x_{j2}-x_{k2})^2+\varepsilon]^{\frac{1}{2}} \tag{3.6.4}$$

Substituting (3.6.4) in both (3.6.1) and (3.6.2) and setting the derivatives to zero, the following iterative expressions result,

$$x_{j1}^{(h+1)} = \frac{\sum\limits_{\substack{k=1\\ \neq j}}^{n} \frac{\hat{\mu}_{jk}x_{k1}^{(h)}}{\hat{\lambda}_{jk}^{(h)}} + \frac{1}{2}\sqrt{\frac{\pi}{2}} \sum\limits_{i=1}^{m} \frac{\bar{\mu}_{ji}}{\sigma_i} \mu_{a_i} \, M(\frac{1}{2}, 2, -\frac{\lambda_{ji}^{2(h)}}{\sigma_i^{2}})}{\sum\limits_{\substack{k=1\\ \neq j}}^{n} \frac{\hat{\mu}_{jk}}{\hat{\lambda}_{jk}^{(h)}} + \frac{1}{2}\sqrt{\frac{\pi}{2}} \sum\limits_{i=1}^{m} \frac{\bar{\mu}_{ji}}{\sigma_i} M(\frac{1}{2}, 2, -\frac{\lambda_{ji}^{2(h)}}{2\sigma_i^{2}})} \tag{3.6.5}$$

and

$$x_{j2}^{(h+1)} = \frac{\sum_{\substack{k=1\\ \neq j}}^{n} \frac{\hat{\mu}_{jk} x_{k2}^{(h)}}{\hat{\lambda}_{jk}^{(h)}} + \frac{1}{2}\sqrt{\frac{\pi}{2}} \sum_{i=1}^{m} \frac{\bar{\mu}_{ji}}{\sigma_i} \mu_{b_i} M(\frac{1}{2}, 2, -\frac{\lambda_{ji}^{2(h)}}{2\sigma_i^2})}{\sum_{\substack{k=1\\ \neq j}}^{n} \frac{\hat{\mu}_{jk}}{\hat{\lambda}_{jk}^{(h)}} + \frac{1}{2}\sqrt{\frac{\pi}{2}} \sum_{i=1}^{m} \frac{\bar{\mu}_{ji}}{\sigma_i} M(\frac{1}{2}, 2, -\frac{\lambda_{ji}^{2(h)}}{2\sigma_i^2})} \tag{3.6.6}$$

Both (3.6.5) and (3.6.6) form an iterative scheme to obtain the optimal solution vector. Eyster, et al. [22] observed that the larger the value of ε the faster the convergence to the optimum value to the approximating function. However, large values of ε can produce an approximating function which has a significantly different optimum solution than the original problem. Hence, a successive reduction in the value of ε is employed after each iteration.

3.7 Euclidean Distance Generalized Weber Problem: Constrained

In the previous sections, the generalized Weber problem was treated for the case of rectilinear and Euclidean distances, but without any additional constraints. In this section the formulations involving the products of random variables and the random sum of random variables are treated separately.

3.7.1 Constrained Generalized Weber Problem: Case I

The first case discussed here is the one involving products of random variables. From P3.1.1 the problem is written for the Euclidean distance as

$$\text{P3.7.1} \quad \underset{X_j}{\text{minimize}}\ Z = \sum_{1\le j<k\le n} \mu_{jk} D_{jk} + \sqrt{\frac{\pi}{2}} \sum_{j=1}^{n} \sum_{i=1}^{m} \bar{\mu}_{ji}\sigma_i \, M\left(-\frac{1}{2}, 1, -\frac{\lambda_{ji}^{2}}{2\sigma_i^{2}}\right)$$

subject to: $D_{jk} \le \xi_{jk}$ for all j, k (3.7.1)

$$Pr(w_{ji}R_{ji} \le \xi_{ji}) \ge \gamma_{ji} \quad , \begin{array}{l} j = 1, \ldots, n \\ i = 1, \ldots, m \end{array} \qquad (3.7.2)$$

where

$$D_{jk} = [(x_{j1}-x_{k1})^2+(x_{j2}-x_{k2})^2]^{\frac{1}{2}}, \quad R_{ji} = [(x_{j1}-a_i)^2+(x_{j2}-b_i)^2]^{\frac{1}{2}}$$

ξ_{jk}, ξ_{ji}, and γ_{ji} are as defined in Section 3.1. Observing the constraints (3.7.1) discloses that they are deterministic and form a convex set. So, the only probabilistic element in P3.7.1 comes from the chance-constraints (3.7.2). To find their deterministic equivalent, the following result is employed.

Result 3.7.1: Let the random variable w correspond to a normal distribution with mean μ_w and variance σ_w^2, and let the random variable R^2 correspond to a probability density function given by (2.5.5). Assuming that w and R^2 are independent, then the probability density function of the new random variable $y = w^2R^2$ is given approximately by

$$g(z) = \frac{z^{\frac{v_1+v_2}{2}} K_{(v_1-v_2)/2}(z)}{2^{\frac{v_1+v_2}{2}-2}\,\Gamma(\frac{v_1}{2})\Gamma(\frac{v_2}{2})} \qquad 0 < y < \infty$$

where

$R^2 = (x_1-a)^2 + (x_2-b)^2$, as in Theorem 2.5.1

$K_n(y)$ = the modified Bessel function of the second kind and order n

and

v_1, v_2 are two known constants.

Proof: Springer and Thompson [90] employed the Mellin transform to obtain the probability density function of products of independent random variables. The Mellin transform is defined as

$$M(f(x)|s) = E[x^{s-1}] = \int_0^\infty x^{s-1} f(x)dx \qquad 0 \le x < \infty \tag{3.7.3}$$

where it is defined on the complex variable s; the inversion of this transform M^{-1} is obtained by

$$M^{-1} = f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} M(f(x)|s)ds \tag{3.7.4}$$

Epstein [20] proved that the Mellin transform of the density function of the product of two independent random variables is the product of the Mellin transforms of the density functions of the individual variables, i.e., the Mellin transform of z = xy is

$$M(g(z)|s) = M(f_1(x)|s) \cdot M(f_2(y)|s) \tag{3.7.5}$$

where $g(\cdot)$, $f_1(\cdot)$, and $f_2(\cdot)$ are the corresponding probability density functions, respectively.

From (2.5.5), the probability density function of R^2 is

$$f_1(R^2=w) = \frac{1}{2\sigma^2} e^{-\frac{1}{2\sigma^2}(w+\lambda^2)} I_o\left(\frac{\lambda\sqrt{w}}{\sigma^2}\right) \quad , 0 \leq w < \infty$$

Therefore, the Mellin transform of w, $M(f_1(w)|s)$, is derived using (3.7.3) as

$$M(f_1(w)|s) = \frac{1}{2\sigma^2} e^{-\frac{\lambda^2}{2\sigma^2}} \int_0^\infty w^{s-1} e^{-\frac{w}{2\sigma^2}} I_o\left(\frac{\lambda\sqrt{w}}{\sigma^2}\right) dw \qquad (3.7.6)$$

Expansion of $I_o(\cdot)$ in its series yields

$$M(f_1(w)|s) = \frac{1}{2\sigma^2} e^{-\frac{\lambda^2}{2\sigma^2}} \int_0^\infty w^{s-1} e^{-\frac{w}{2\sigma^2}} \sum_{k=0}^\infty \frac{\left(\frac{\lambda\sqrt{w}}{2\sigma^2}\right)^{2k}}{(\Gamma(k+1))^2} dw$$

$$= \frac{1}{2\sigma^2} e^{-\frac{\lambda^2}{2\sigma^2}} \sum_{k=0}^\infty \frac{\left(\frac{\lambda}{\sqrt{2}\,\sigma}\right)^{2k}}{(\Gamma(k+1))^2} \int_0^\infty w^{s-1} e^{-\frac{w}{2\sigma^2}} \left(\frac{\sqrt{w}}{\sqrt{2}\,\sigma}\right)^{2k} dw \qquad (3.7.7)$$

Let $\bar{\lambda}^2 = \frac{\lambda^2}{2\sigma^2}$, $\frac{w}{2\sigma^2} = z$, then $dz = \frac{1}{2\sigma^2} dw$,

$$M(f_1(z)|s) = (2\sigma^2)^{s-1} e^{-\bar{\lambda}^2} \sum_{k=0}^\infty \frac{(\bar{\lambda})^{2k}}{(\Gamma(k+1))^2} \int_0^\infty z^{s-1} e^{-z} z^k dz \quad (3.7.8)$$

but, from [1, p. 255],

$$\int_0^\infty z^{k+s-1} e^{-z} dz = \Gamma(s+k)$$

Thus

$$M(f_1(z)|s) = (2\sigma^2)^{s-1} e^{-\bar{\lambda}^2} \sum_{k=0}^\infty \frac{(\bar{\lambda})^{2k}}{(\Gamma(k+1)^2} \Gamma(s+k) \qquad (3.7.9)$$

Unfortunately, (3.7.9) is not in a closed form and more difficulties are anticipated in dealing with it in further algebraic manipulation. However, in Chapter 2 an approximation developed by Patnaik [75] was used to approximate the non-central chi square distribution. From Johnson and Kotz [49] it may be seen that R^2 is distributed as a non-central $\chi_2'^2(\lambda^2)$ with two degrees of freedom and non-centrality parameter λ^2; also, w^2 is a non-central $\chi_1'^2(\mu^2)$ with one degree of freedom and non-centrality parameter μ^2. Hence, assuming Patnaik's approximation is used, two different $\chi_v{}^2$ distributions with degrees of freedom v_1 and v_2, respectively, are obtained.

Webb [97] obtained the Mellin transform of $\chi_v{}^2$ which is given as,

$$M(f(\chi^2_{v_1}|s) = s^{2-1} \frac{\Gamma(\frac{v_1}{2} + s-1)}{\Gamma(\frac{v_1}{2})} \tag{3.7.10}$$

Substituting (3.7.10) in (3.7.5) for both distributions,

$$M(g(y)|s) = M(f(\chi^2_{v_1})|s)\cdot M(f(\chi^2_{v_2})|s)$$

$$= 2^{2(s-1)} \frac{\Gamma(\frac{v_1}{2} +s-1)\Gamma(\frac{v_2}{2} +s-1)}{\Gamma(\frac{v_1}{2})\cdot\Gamma(\frac{v_2}{2})} \tag{3.7.11}$$

In order to find the density function of y, g(y), the inverse Mellin transform of (3.7.11) must be obtained by using (3.7.4), and

$$M^{-1} = g(y) = \frac{1}{\Gamma(\frac{v_1}{2})\Gamma(\frac{v_2}{2})} \frac{1}{2\pi_i} \int_{c-i\infty}^{c+i\infty} y^{-s} 2^{2(s-1)} \Gamma(\frac{v_1}{2} +s-1)\Gamma(\frac{v_2}{2} +s-1)ds \tag{3.7.12}$$

Equation 3.7.12 can be expressed alternately in order that the inverse will be recognized easily from the tables of inverse Mellin transform. In (3.7.12), let $s' = 2s + \frac{v_1}{2} + \frac{v_2}{2} - 2$, then $ds = \frac{1}{2}\, ds'$ and (3.7.12) is written as

$$g(y) = \frac{2^{1-(\frac{v_1}{2}+\frac{v_2}{2})}\, y^{\frac{v_1}{4}+\frac{v_2}{4}-1}}{\Gamma(\frac{v_1}{2})\Gamma(\frac{v_2}{2})} \; \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (\sqrt{y})^{s'} \quad s^{s'-2}$$

$$\Gamma(\frac{s'}{2} + \frac{v_1}{4} - \frac{v_2}{4})\Gamma(\frac{s'}{2} + \frac{v_2}{4} - \frac{v_1}{4})\; ds' \qquad (3.7.13)$$

In Erdelyi, et al. [21] extensive tables for the Mellin transform and its inverse are pvovided. From [21, p. 331],

$$K_v(ax) = M^{-1}[a^{-s}2^{s-2}\Gamma(\frac{s}{2} - \frac{v}{2})\Gamma(\frac{s}{2} + \frac{v}{2})] \qquad (3.7.14)$$

where $K_v(ax)$ is the modified Bessel function of the second kind and order v. Observing the similarity between (3.7.13) and (3.7.14), it is concluded that

$$g(y) = \frac{y^{\frac{v_1}{4}+\frac{v_2}{4}-1}}{2^{\frac{v_1}{2}+\frac{v_2}{2}-1}\,\Gamma(\frac{v_1}{2})\Gamma(\frac{v_2}{2})} \; K_{(\frac{v_1-v_2}{2})}(y^{\frac{1}{2}}) \qquad (3.7.15)$$

To simplify (3.7.15), let $y^{\frac{1}{2}} = z$, then $2zdz = dy$,

$$g(z) = \frac{z^{\frac{v_1+v_2}{2}-1} K_{(v_1-v_2)/2}(z)}{2^{\frac{v_1+v_2}{2}-2}\,\Gamma(\frac{v_1}{2})\Gamma(\frac{v_2}{2})} \tag{3.7.16}$$

which has the same form as in the theorem, hence the proof is complete.

If the cumulative distribution of g(z) is required, some assumptions are to be made first to obtain a closed form for the integral of (3.7.16). It is first assumed that all weights (w_i) are such that $0 \leq w_i \leq 1$; it is done easily by dividing each weight by $\sum_{i=1}^{m} w_i$. Thus, the weights can then be defined as the fraction of the total weight. Patnaik [75] provided the following approximation to a non-central χ'^2 with degrees of freedom v and non-centrality parameter λ^2,

$$\chi'_v{}^2(\lambda^2) = c\,\chi_f{}^2 \tag{3.7.17}$$

where

$$c = \frac{v + 2\lambda}{v + \lambda}\ ; \qquad f = v + \frac{\lambda^2}{v + 2\lambda}$$

Therefore, when the distribution of $w_i{}^2$, which is $\chi_1'^2(\mu_i{}^2)$ distributed, is approximated by a central $\chi_f{}^2$, the degrees of freedom, f, is

$$f = 1 + \frac{\mu^4}{1 + 2\mu^2} \tag{3.7.18}$$

From the assumption that the random variable w_i takes values below one, then the second term in (2.7.18) is always a fraction and some interpolation is needed to obtain the probability. Given two chi square distributions, each with degrees of freedoms n_1, n_2, respectively. If $n_1 > n_2$, then

$$\Pr(\chi^2_{n_1} \leq \xi^2) \leq \Pr(\chi^2_{n_2} \leq \xi^2) \tag{3.7.19}$$

It may be seen that using n_1 instead of n_2 will underestimate the probability. Hence, there is no overestimation in the probability if we assume that f defined in (3.7.18) equals two, since the difference in this range is small for the χ^2 distribution.

Deterministic formulations for the constraints in (3.7.2) must be found. As proved in Chapter 2,

$$\Pr(w_{ji}R_{ji} \leq \xi_{ji}) = \Pr(w_{ji}{}^2R_{ji}{}^2 \leq \xi_{ji}{}^2) \tag{3.7.20}$$

Using Patnaik's approximation $(\chi_v'^2(\lambda^2) = c\ \chi_f{}^2)$, then

$$\Pr(w_{ji}{}^2R_{ji}{}^2 \leq \xi_{ji}{}^2) \simeq \Pr(w_{ji}{}^2R_{ji}{}^2 \leq \xi_{ji}{}^2 | c_1c_2)$$

where c_1 and c_2 are defined as in (3.7.17).

In Theorem 3.7.1, the probability density function of the random variable $w_{ji}{}^2R_{ji}{}^2$ is developed. Using the assumption that $f_{w_{ji}^2} = 2$, (3.7.16) gives

$$g(z) = \frac{z^{\frac{v_1}{2}}}{2^{\frac{v_1}{2}-1}\,\Gamma(\frac{v_1}{2})}\ K_{\frac{v_1}{2}-1}(z) \tag{3.7.21}$$

The cumulative distribution is obtained by integrating (3.7.21) over z, thus,

$$F(z) = \Pr(w_{ji}^2 R_{ji}^2 \leq \alpha_{ji}) = \frac{1}{2^{\frac{v_1}{2}-1}} \tag{3.7.22}$$

where $\alpha_{ji} = \xi_{ji}^2 | c_1 c_2$.

From [1, p. 255], the integral in (3.7.22) is evaluated as,

$$\int_0^w z^{n+1} K_n(z)dz = -w^{(n+1)} + K_{n+1}(w) + 2^n \Gamma(n+1) \tag{3.7.23}$$

Substituting (3.7.23) in (3.7.22) yields,

$$F(\alpha_{ji}) = 1 - \frac{(\alpha_{ji})^{\frac{v_1}{2}} K_{v_{1/2}}(\alpha_{ji})}{2^{\frac{v_1}{2}-1}\Gamma(\frac{v_1}{2})} \tag{3.7.24}$$

Therefore, the constraints (3.7.2) are written as,

$$1 - \gamma_{ji} \geq \frac{\alpha_{ji}^{\frac{v_1}{2}} K_{v_{1/2}}(\alpha_{ji})}{2^{\frac{v_1}{2}-1}\Gamma(\frac{v_1}{2})} \quad \text{for all } \begin{array}{l} j = 1, \ldots, n \\ i = 1, \ldots, m \end{array} \tag{3.7.25}$$

Note that α_{ji} is a function of c_1, and from (3.7.17) it is clear that it is a function of X_j; in the same manner v_1 is a function of X_j. Since $K_v(\cdot)$ is well tabulated and available for computer calculations, an iterative method to solve the non-linear programming problem is recommended. Problem P3.7.1 may be stated in the deterministic form as,

$$\overline{\mathrm{P}}3.7.1 \quad \underset{X_j}{\text{minimize}}\ Z = \sum_{1\le j<k\le n} \mu_{jk}D_{jk} + \sqrt{\frac{\pi}{2}} \sum_{j=1}^{n} \sum_{i=1}^{m} \bar{\mu}_{ji}\sigma_i \, M\left(-\frac{1}{2}, 1, -\frac{\lambda_{ji}^2}{2\sigma_i^2}\right)$$

subject to: $\mu_{jk}D_{jk} \le \xi_{jk}$ for all j, k

$$\frac{\alpha_{ji}^{\frac{v_1}{2}} K_{v_{1/2}}(\alpha_{ji})}{2^{\frac{v_1}{2}-1}\,\Gamma(\frac{v_1}{2})} \le 1 - \gamma_{ji} \qquad \text{for all } j, i$$

For a constant v_i, it is easily shown that the last set of constraints forms a convex set (from the definition of $K(\cdot)$) and certainly the first set of constraints forms a convex set; thus, the joint constraint set is convex. In Theorem 3.6.1, it has been proven that the objective function is strictly convex; thus, if a local optimum is achieved, it is also a global optimum. Any nonlinear programming algorithm (e.g., SUMT) may be employed to obtain the solution to $\overline{\mathrm{P}}3.7.1$. In the case that v_1 is not considered as a constraint, i.e., v_1 is a function of X_j, the convexity condition of the constraints may not hold. However, a local optimum solution is still available and it may turn out to be a global optimum solution.

It is easier to work with the constraints when v_1 is constant since $K_{v_{1/2}}(\cdot)$ will have the same order during all iterations. However, this may be accomplished without loss of generality, if the known (μ_{a_i}, μ_{b_i}) are rescaled so that any coordinate takes a value between zero and one. This will imply that λ_{ji} is bounded as $0 \le \lambda_{ji} \le 1$, from (3.7.17), $v_1 \simeq 2$ and the constraints (3.7.25) are written as follows,

$$\alpha_{ji}K_1(\alpha_{ji}) \leq 1 - \gamma_{ji} \qquad \text{for all } i,\ j$$

which will simplify the computations dramatically. Note the optimum solution has to be adjusted to its former scale so that the total cost obtained is in the correct units.

In problem P3.7.1, the constraints formed by imposing an upper bound on the expected cost are replaced by the chance constraints described in (3.2.4). Given that $v_{jk} \sim N(\mu_{jk},\sigma_{jk}^2)$; hence,

$$v_{jk}|X_j-X_k|_2 \sim N(\mu_{jk}|X_j-X_k|_2,\ \sigma_{jk}^2[|X_j-X_k|_2]^2) \qquad (3.7.26)$$

For simplicity, let $\mu_{jk}|X_j-X_k|_2 = M_{jk}$, $\sigma_{jk}^2[|X_j-X_k|_2]^2 = S_{jk}^2$. From (3.7.26), the constraints (3.2.4) may be written as,

$$\Pr\left(\frac{v_{jk}|X_j-X_k|_2 - M_{jk}}{S_{jk}} \leq \frac{\xi_{jk} - M_{jk}}{S_{jk}}\right) \geq \gamma_{jk} \quad \text{for all } j,\ k \qquad (3.7.27)$$

As discussed in Chapter 2, (3.7.27) is the same as,

$$\begin{aligned}\xi_{jk} \geq M_{jk} + S_{jk}\Phi^{-1}(\gamma_{jk}) &= \mu_{jk}|X_j-X_k|_2 + \sigma_{jk}|X_j-X_k|_2\ \Phi^{-1}(\gamma_{jk}) \\ &= [\mu_{jk} + \sigma_{jk}\Phi^{-1}(\gamma_{jk})]\ |X_j-X_k|_2 \qquad (3.7.28)\end{aligned}$$

where $\Phi^{-1}(\gamma_{ji})$ is the inverse of the standard cumulative normal distribution.

On comparing both (3.7.1) and (3.7.28) the effect on the constraints is noted when they are treated as chance constraints instead of expected value constraints. Replacing (3.7.1) by (3.7.28) does not change the solution procedure employed to solve $\overline{\text{P}}$3.7.1.

3.7.2 Constrained Generalized Weber Problem: Case II

The second problem proposed in Section 3.1 involved a random sum of random variables. For the Euclidean norm, P3.2.1 is given as,

$$\text{P3.7.2} \quad \underset{X_j}{\text{minimize}}\ Z = \sum_{1 \le j < k \le n} \mu_{jk} D_{jk} + \sqrt{\frac{\pi}{2}} \sum_{j=1}^{n} \sum_{i=1}^{m} \bar{\mu}_{ji} \sigma_i \, M\left(-\frac{1}{2}, 1, -\frac{\lambda_{ji}^2}{2\sigma_i^2}\right)$$

$$\text{subject to:} \quad \mu_{jk} D_{jk} \le \xi_{jk} \qquad \text{for all } j, k$$

$$\Pr\left(\sum_{h=1}^{w_{ji}} (R_{ji})^{(h)} \le \xi_{ji}\right) \ge \gamma_{ji} \qquad \begin{array}{l} j = 1, \ldots, n \\ i = 1, \ldots, m \end{array}$$

where D_{jk}, R_{ji}, ξ_{jk}, ξ_{ji}, and γ_{ji} are as defined in Section 3.7.1; and M, λ_{ji} are as defined in Chapter 2.

Before solving P3.7.2 the chance constraints are converted to equivalent deterministic constraints. In Chapter 2, the probability density function of R_{jk} is developed and is given by (2.5.5); also, it is known that $w_{ji} \sim N(\bar{\mu}_{ji}, \sigma_{ji}^2)$. Feller [23] discussed how to use transform methods (characteristic function and Laplace transform) to study the random variable, $Y_{ji} = \sum_{h=1}^{w_{ji}} (R_{ji})^{(h)}$. Once its transform is obtained, the inverse transform provides the desired probability density function. Unfortunately, this procedure is very tedious for the given distributions and it involves very cumbersome mathematics.

Based on the Central Limit Theorem for the sum of a random number of independent random variables, it may be concluded that the probability density function of Y_{ji} will be approximately normally distributed under very general conditions. For a detailed discussion of the conditions required, see Blum, et al. [108] and Rènyi [109,110].

As shown by Hadley and Whitin [40], the mean and variance of the random sum of random variables is given by,

$$M_{ji} = \bar{\mu}_{ji}\ E[R_{ji}] \qquad , \text{ for all } j,\ i \tag{3.7.29}$$

$$\begin{aligned} S_{ji}^2 &= \bar{\mu}_{ji}(E[R_{ji}^2]-E^2[R_{ji}]) + \sigma_{ji}^2 E^2[R_{ji}] \\ &= \bar{\mu}_{ji}\ E[R_{ji}^2] + (\sigma_{ji}^2-\bar{\mu}_{ji})\ E^2[R_{ji}] \quad , \text{ for all } j,\ i \end{aligned} \tag{3.7.30}$$

In Theorem 2.5.2, the expected value of the squared distance is given by (2.5.10). Also, Theorem 2.6.2 provides the expected value of the distance, expressed in (2.6.2). Substituting both equations in (3.7.29) and (3.7.30), the mean and variance are written as,

$$M_{ji} = \bar{\mu}_{ji}\left(\sqrt{\tfrac{\pi}{2}}\,\sigma_i\ M\left(-\tfrac{1}{2},\ 1, -\frac{\lambda_{ji}^2}{2\sigma_i^2}\right)\right) \qquad , \text{ for all } j,\ i \tag{3.7.31}$$

$$S_{ji}^2 = \bar{\mu}_{ji}(2\sigma_i^2+\lambda_{ji}^2) + (\sigma_{ji}^2-\bar{\mu}_{ji})\left(\sqrt{\tfrac{\pi}{2}}\,\sigma_i\ M\left(-\tfrac{1}{2},\ 1,\ -\frac{\lambda_{ji}^2}{2\sigma_i^2}\right)\right)^2 \qquad , \text{ for all } j,\ i \tag{3.7.32}$$

The chance constraints are now written in normalized form as,

$$Pr\left(\frac{Y_{ji} - M_{ji}}{S_{ji}} \leq \frac{\xi_{ji} - M_{ji}}{S_{ji}}\right) \geq \gamma_{ji} \qquad \text{for all } j,\ i \tag{3.7.33}$$

As discussed above, (3.7.33) is similar to,

$$\xi_{ji} \geq M_{ji} + S_{ji}\Phi^{-1}(\gamma_{ji}) \qquad , \begin{array}{l} j = 1, \ldots, n \\ i = 1, \ldots, m \end{array} \qquad (3.7.34)$$

The deterministic variation of P3.7.2 is given by

$$\overline{\text{P}}3.7.2 \quad \underset{X_j}{\text{minimize}}\ Z = \sum_{1\leq j<k\leq n} \mu_{jk}D_{jk} + \sqrt{\frac{\pi}{2}} \sum_{j=1}^{n} \sum_{i=1}^{m} \overline{\mu}_{ji}\sigma_i \, M\left(-\frac{1}{2}, 1, -\frac{\lambda_{ji}^{2}}{2\sigma_i^{2}}\right)$$

subject to: $\mu_{jk}D_{jk} \leq \xi_{jk}$ $\qquad 1 \leq j < k \leq n$

$$M_{ji} + S_{ji}\Phi^{-1}(\gamma_{ji}) \leq \xi_{ji} \qquad \begin{array}{l} j = 1, \ldots, n \\ i = 1, \ldots, m \end{array}$$

From the results of Chapter 2, it can be concluded that both M_{ji} and S_{ji} are convex functions. Therefore, the constraints in (3.7.34) are also convex functions; the set of constraints in $\overline{\text{P}}$3.7.2 is a convex set; and the objective function is strictly convex. Hence, a unique optimum solution is guaranteed if any convergent convex programming algorithm is used, e.g., SUMT.

3.7.3 Constrained Generalized Weber Problem: Approximate Solution

In Section 3.7.1, a solution procedure is developed for Case I, when the cost of transportation is given by the product of the two random variables, w_{ji} and R_{ji}. The method obtained is an exact one; however, we may apply a Chebyshev type inequality to obtain an approximate solution for Case I. Feller [24] introduced a similar inequality for the case of non-negative random variables (which is the case under consideration). This inequality is given by

$$Pr(x > t) \leq \frac{\sigma^2}{\sigma^2 + t^2} \tag{3.7.35}$$

where

$$E(x) = 0 \quad \text{and} \quad V(x) = \sigma^2, \; t \geq 0$$

To bound the probability, the following may be applied,

$$Pr(w_{ji}R_{ji} \leq \xi_{ji}) = Pr(w_{ji}^2 R_{ji}^2 \leq \xi_{ji}^2) \tag{3.7.36}$$

Since $w_{ji} \sim N(\bar{\mu}_{ji}, \sigma_{ji}^2)$, then from Johnson and Kotz [49] w_{ji}^2 has a non-central $\chi_1'^2$ with one degree of freedom and non centrality parameter $\bar{\mu}_{ji}^2$. Thus, the random variable Y_{ji} has a finite mean and variance as shown below,

$$E[Y_{ji}] = E[w_{ji}^2 \cdot R_{ji}^2] = E[w_{ji}^2] \cdot E[R_{ji}^2]$$

Johnson and Kotz [49] showed that $E[w_{ji}^2]$, $E[w_{ji}^4]$ are given as

$$E[w_{ji}^2] = \sigma_{ji}^2 + \bar{\mu}_{ji}^2 \tag{3.7.37}$$

$$E[w_{ji}^4] = 3\sigma_{ji}^4 + 6\sigma_{ji}^2 \, \bar{\mu}_{ji}^2 + \mu_{ji}^4 \tag{3.7.38}$$

and in Chapter 2 the expected squared distance is given by (2.5.10). Combining (3.7.37) and (2.5.10) yields

$$E[Y_{ji}] = (\sigma_{ji}^2 + \bar{\mu}_{ji}^2)(2\sigma_i^2 + \lambda_{ji}^2) \tag{3.7.39}$$

$$V(Y_{ji}) = E[(w_{ji}^2 \, R_{ji}^2)^2] - E^2[w_{ji}^2] \, E^2[R_{ji}^2]$$

$$= E[w_{ji}^4] \, E[R_{ji}^4] - (\sigma_{ji}^2 + \bar{\mu}_{ji}^2)^2 (2\sigma_i^2 + \lambda_{ji}^2)^2 \tag{3.7.40}$$

To evaluate $E[R_{ji}{}^4]$, the second moment for the random variable $R_{ji}{}^2$ is computed; this is done as follows,

$$E[R^4] = \int_0^\infty y^2 g(R^2{=}y)\ dy$$

$$= \int_0^\infty \frac{y^2}{2\sigma^2}\ e^{-\frac{1}{2\sigma^2}(y+\lambda^2)}\ I_o\left(\frac{\lambda\sqrt{y}}{\sigma^2}\right) dy \qquad (3.7.41)$$

Let $\overline{\lambda}^2 = \frac{\lambda^2}{2\sigma^2}$, $\frac{y}{2\sigma^2} = w$, then $dw = \frac{1}{2\sigma^2}\,dy$ and (3.7.41) may be written as,

$$E[R^4] = 4\sigma^4\ e^{-\overline{\lambda}^2} \int_0^\infty w^2\ e^{-w}\ I_o(2\overline{\lambda}\sqrt{w})\ dw$$

$$= 4\sigma^4\ e^{-\overline{\lambda}^2} \sum_{k=0}^\infty \frac{(\overline{\lambda}^2)^k}{(k!)^2} \int_0^\infty w^2\ e^{-w}\ w^k\ dw$$

$$= 4\sigma^4\ e^{-\overline{\lambda}^2} \sum_{k=0}^\infty \frac{(\overline{\lambda}^2)^k}{(k!)^2}\ \Gamma(k+3)$$

$$= 4\sigma^4\ e^{-\overline{\lambda}^2}\ M(3,1,\overline{\lambda}^2) \qquad (3.7.42)$$

Using the Kummer transformation, given in [1, p. 507], and expanding the confluent hypergeometric function, the following is derived

$$E[R^4] = 8\sigma^4 + 8\sigma^2\lambda^2 + 2\lambda^4 \qquad (3.7.43)$$

Substituting (3.7.38) and (3.7.43) in (3.7.40), the following is obtained,

$$V(Y_{ji}) = (3\sigma_{ji}{}^4 + 6\sigma_{ji}{}^2\overline{\mu}_{ji}{}^2 + \mu_{ji}{}^4)(8\sigma_i{}^4 + 8\sigma_i{}^2\lambda_{ji}{}^2 + 2\lambda_{ji}{}^4)$$

$$- (\sigma_{ji}{}^2 + \overline{\mu}_{ji}{}^2)^2(2\sigma_i{}^2 + \lambda_{ji}{}^2)^2 \qquad (3.7.44)$$

which, after rearranging and combining the common terms yields,

$$V(Y_{ji}) = K_{ji}(4\sigma_i^4+4\sigma_i^2\lambda_{ji}^2+\lambda_{ji}^4) \tag{3.7.45}$$

where

$$K_{ji} = (5\sigma_{ji}^4+10\sigma_{ji}^2\bar{\mu}_{ji}^2+\mu_{ji}^4)$$

Returning to the relation obtained in (3.7.36) and applying Markov's Inequality,

$$\Pr(Y_{ji} \leq \xi_{ji}^2) \geq \frac{\xi_{ji}^2 - E[Y_{ji}]}{\xi_{ji}^2} \tag{3.7.46}$$

Since $F(\cdot) \geq 0$, then

$$\xi_{ji}^2 \geq E[Y_{ji}]$$

or, after adding a slack variable S_{ji}, it becomes

$$\xi_{ji}^2 - S_{ji} = E[Y_{ji}] \tag{3.7.47}$$

From (3.7.36) it may be seen that

$$\Pr(Y_{ji} \leq \xi_{ji}^2) = \Pr(Y_{ji} - E[Y_{ji}] \leq \xi_{ji}^2 - E[Y_{ji}])$$

Substituting the value of Y_{ji} for the right hand side of the inequality and using the inequality of (3.7.35) yields, after simplification,

$$\Pr(Y_{ji} - E[Y_{ji}] \leq S_{ji}) \geq \frac{S_{ji}^2}{V(Y_{ji}) + S_{ji}^2} \tag{3.7.48}$$

Thus, the chance constraints of (3.7.2) may be written as

$$\frac{S_{ji}^{2}}{V(Y_{ji}) + S_{ji}^{2}} \geq \gamma_{ji} \qquad \text{for all } j,\ i$$

Equivalently,

$$S_{ji}^{2} \geq V(Y_{ji}) \frac{\gamma_{ji}}{1 - \gamma_{ji}} = \overline{V}(Y_{ji}) \tag{3.7.49}$$

The above demonstrates, for a fixed value of the left hand side, that (3.7.49) is not a concave function in S_{ji}. Therefore, the logarithm of both sides of (3.7.49) is taken,

$$2 \log(S_{ji}) \geq \log \overline{V}(Y_{ji}) \tag{3.7.50}$$

where (3.7.50) is now a concave function over S_{ji}. The same is true for X_j since $V(Y_{ji})$ is a function of X_j.

Problem P3.7.1 may be written as

$$\overline{\overline{P}}3.7.1 \quad \underset{X_j}{\text{minimize}}\ Z = \sum_{1 \leq j < k \leq n} \mu_{jk} D_{jk} + \sqrt{\frac{\pi}{2}} \sum_{j=1}^{n} \sum_{i=1}^{m} \overline{\mu}_{ji} \sigma_i \; M\left(-\frac{1}{2}, 1, -\frac{\lambda_{ji}^{2}}{2\sigma_i^{2}}\right)$$

subject to:

$$\mu_{jk}\, D_{jk} \leq \xi_{jk} \qquad \text{for all } j,\ k = 1, \ldots, m$$

$$\frac{1}{2} \log \overline{V}(Y_{ji}) \leq \log(S_{ji}) \qquad j = 1, \ldots, n;\ i = 1, \ldots, m$$

$$K_{ji}(4\sigma_i^{4} + 4\sigma_i^{2}\lambda_{ji}^{2} + \lambda_{ji}^{4}) \frac{\gamma_{ji}}{1 - \gamma_{ji}} = \overline{V}(Y_{ji}) \qquad j = 1, \ldots, n;\ i = 1, \ldots, m$$

$$S_{ji} \geq 0 \qquad j = 1, \ldots, n;\ i = 1, \ldots, m$$

From the above discussions, $\overline{\overline{P}}3.7.1$ has a strictly convex objective function and is defined over a convex set. Consequently, a unique optimum solution will be obtained if a convex programming algorithm is used to solve the problem. Certainly this model is much simpler than the exact one developed in Section 3.7.1.

3.8 Numerical Examples

In this section, numerical examples are solved to aid in understanding the solution procedures; they are also intended to demonstrate the effect of the probabilistic component on the location models. Examples are solved for both rectilinear and Euclidean location problems.

3.8.1 Rectilinear Distance Multifacility Location Example Problem

Suppose there are three existing facilities and their locations are considered as random variables corresponding to a bivariate normal distribution. The mean and standard deviation for the x_1-coordinates are $a_1 = (3,2)$, $a_2 = (8,5)$, and $a_3 = (15,4)$; for the x_2-coordinates, $b_1 = (4,2)$, $b_2 = (7,5)$, and $b_3 = (2,4)$. Two new facilities are to be located with respect to the existing facilities. The expected value of the interaction between the two new facilities is $\mu_{12} = 3$ and the expected values of the interaction between the new and existing facilities are given as

$$\overline{\mu} = \begin{bmatrix} 2 & 6 & 0 \\ 4 & 5 & 1 \end{bmatrix}$$

The new facilities are to be located such that the total expected rectilinear distance is minimized.

To solve the multifacility location problem when distances are measured in the rectilinear norm, the separable programming procedure described in Section 3.4 is used. The optimization procedure is performed separately for the x_1 and x_2 coordinates. To optimize $f_1(X_1,X_2)$, from Theorem 2.4.3 observe that both x_{11} and x_{21} must be less than 16. Dividing the interval of the variable X_j into eight subintervals yields nine $\lambda_{\ell j}$ variables. The values of the functions, f_j, at the appropriate mesh points, i.e., $f_{\ell j}$, are evaluated (see Appendix B for the corresponding computer program). The resulting linear programming problem is solved using the MPS/360. The same procedure is repeated to optimize $f_2(X_1,X_2)$.

The optimal locations for the new facilities are,

$$X_1^* = (6,6)$$

and

$$X_2^* = (6,5).$$

Francis and White [32], in treating the expected values as deterministic values of the coordinates, obtained the optimum locations, $X_1^* = (8,7)$ and $X_2^* = (8,7)$. Consequently, the deterministic problem obtained by assuming that the existing facilities are located at their expected values yields significantly different locations. The minimum total expected distance is 136.9; the expected total distance when the locations $X_1^* = X_2^* = (8,7)$ are used is 161.043. Thus, a 17% increase in the cost occurred by failing to account explicitly for the random variation.

3.8.2 Euclidean Distance Multifacility Location Example Problem

In the above example let $\sigma_i = \sigma_{a_i}$ for all i and assume that the distance traveled is measured in the Euclidean norm. To solve this example, the iterative procedure given by (3.6.5) and (3.6.6) is programmed in Fortran IV (see Appendix B). The iterations are given as follows,

$$X_1^{(1)} = (4.999795,\ 4.925308),\quad X_2^{(1)} = (4.306354,\ 4.744833)$$
$$X_1^{(2)} = (4.852744,\ 5.083404),\quad X_2^{(2)} = (4.723700,\ 4.958285)$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$X_1^{(13)} = (4.964707,\ 5.118733),\quad X_2^{(13)} = (4.964679,\ 5.118752)$$

The optimal locations for the new facilities are,

$$X_1^* = (4.964707,\ 5.118733)$$
$$X_2^* = (4.964679,\ 5.118572)$$

When the expected values are considered as coordinates in a deterministic problem, Francis and White [32] obtained the optimal locations $X_1^* = (8,7)$ and $X_2^* = (8,7)$. The comparison between both solutions implies that there is a great difference in locations, and using the expected values as a basis of locations will yield a solution different from the one which minimizes the expected total distance traveled. The minimum total expected distance is 84.4; the expected total distance at the points $X_1 = X_2 = (8,7)$, is 88.76.

Suppose it is desired to limit the total distance traveled between existing facility i and a new facility j within a maximum distance of

12 units, i.e., $\xi_{ji} = 12$ for all i, j, with a probability of 0.85. Assuming that the case under study is the one involving products of random variables discussed in Section 3.7.1: The feasibility of the optimal unconstrained solution obtained above when the chance constraints are imposed is checked first. From $\overline{P}3.7.1$, the chance constraints are written as,

$$\alpha_{j1}k_1(\alpha_{j1}) \leq 1 - .85$$

$$\alpha_{j2}k_1(\alpha_{j2}) \leq 1 - .85$$

$$\alpha_{j3}k_1(\alpha_{j3}) \leq 1 - .85$$

Since $X_1^* = X_2^*$, then the only difference between the set of constraints for each j is in the weights, w_{ji}. The arguments of the Bessel function are evaluated at j = 1 as follows,

$$\alpha_{11} = \frac{\xi_{11}^2(1+w_{11}^2)(2+\lambda_{11}^2)}{(1+2w_{11}^2)(2+2\lambda_{11}^2)\sigma_1^{\;2}} = \frac{144(1+4)(2+1.27)}{4(1+8)(2+2.54)} = 14.4$$

$$\alpha_{12} = \frac{\xi_{12}^2(1+w_{12}^2)(2+\lambda_{12}^2)}{(1+2w_{12}^2)(2+2\lambda_{12}^2)\sigma_2^{\;2}} = \frac{144(1+36)(2+.51)}{25(1+72)(2+1.02)} = 2.43$$

$$\alpha_{13} = \frac{\xi_{12}^2(1+w_{13}^2)(2+\lambda_{13}^2)}{(1+2w_{13}^2)(2+2\lambda_{13}^2)\sigma_3^{\;2}} = \frac{144(1+0)(2+6.9)}{16(1+0)(2+13.8)} = 5.1$$

From [1, p. 417], the values of $k_1(\alpha_{ji})$ is obtained, and the constraints are written as,

$$27 \times 10^{-6} \leq .15$$

$$0.2 \not\leq .15$$

$$.0184 \leq .15$$

where the second constraint is violated. Therefore, the unconstrained solution is not a feasible solution to the constrained problem.

When the unconstrained problem involves a random sum of random variables, then formulation given in Section 3.7.2 yields the following constraints,

$$M_{ji} + S_{ji}\ \Phi^{-1}(.85) \leq 12 \qquad \text{for all } i, j$$

where M_{ji} and S_{ji} are as defined in (3.7.31) and (3.7.32). The values of M_{ji} are computed for the second new facility, $j = 2$. Hence, M_{21} = 13.175, M_{22} = 29.2 and M_{23} = 4.755. From (3.7.32), S_{ji}^2 is calculated as follows,

$$S_{21}^2 = 4(8+5.096) + (4-4) \times 10.85 = 52.384\ , \ S_{21} = 7.23$$

$$S_{22}^2 = 5(50+12.776) + (4-5) \times 34.10 = 279.78\ , \ S_{22} = 16.73$$

$$S_{23}^2 = 1(32+110.53) + (4-1) \times 22.01 = 210.36\ , \ S_{23} = 14.5$$

Substituting the values of M_{ji} and S_{ji} into (3.7.34), then

$$13.175 + 7.23\ \Phi^{-1}(.85) \leq 12$$

$$29.2 + 16.73\ \Phi^{-1}(.85) \leq 12$$

$$4.755 + 14.5\ \Phi^{-1}(.85) \leq 12$$

where $\Phi^{-1}(.85) = 1.04$. Therefore, all three constraints are violated; the optimal unconstrained solution is not feasible for the constrained problem.

3.9 Summary

In this chapter, extensions to the single facility location problems described in Chapter 2 were presented. Both the level of interaction between facilities and the random variation in the location of the existing facilities were considered to be random variables. These assumptions generalized the problem to handle both kinds of random inputs to the location problem. Minimization of the total expected distance was the optimization criterion used throughout the chapter.

The unconstrained multifacility location problem was treated when the cost of item movement was linearly proportional to either the rectilinear or Euclidean distance, as well as the case when cost is a linear function of the squared distance.

For the rectilinear problem, the probabilistic model was transformed to its deterministic equivalent and shown to be a convex programming problem. Separable programming was used to solve the convex problem. When the Euclidean distance is used a solution procedure similar to the one developed in the preceding chapter was employed.

The constrained multifacility location problems were different from those obtained for the single facility, since the weights were considered to be random variables. Two cases were studied. In the first case, the cost of travel is considered to be given by the sum of the products of the weight and distance; in the second case, a sum of random sums of the distance traveled was considered. The normal distribution was employed as the probability density function for both random variables considered.

Chance constraints are transformed into the deterministic equivalents by developing the distribution function of the cost of travel. In both cases described above, the transformed problem was formulated as a convex programming problem. Thus, an iterative convex programming algorithm was recommended as a solution procedure. When the cost of travel is given by the product of the weight and distance, the deterministic equivalent problem was obtained using a limit theorem similar to Chebyshev's Inequality; a convex programming algorithm, e.g., SUMT, was recommended for solving the converted problem.

For the unconstrained problem, each solution procedure was programmed and a sample problem was solved. The solutions of both probabilistic and deterministic formulations were compared to measure the impact of the probabilistic formulations.

Chapter 4

EMERGENCY SERVICE FACILITIES LOCATION PROBLEMS

4.1 Introduction

The problem of locating emergency service facilities in an urban environment is the subject of this chapter. Typical illustrations of emergency service facilities are police stations, fire stations, ambulance stations, health outreach clinics, hospitals, police patrol cars, and civil defense stations. The study of location problems in the public sector is relatively new, but the subject has attracted considerable attention in the past five years.

Public and private sector formulations of location problems usually employ different types of objective functions. As indicated by ReVelle, et al. [81] the objective in the private sector is typically the minimization of cost; whereas in the public sector the objective is stated as the maximization of benefits. As surrogate measures of benefits distance traveled and response times are often used in the public sector. In the case of emergency services, the objective is often stated as the minimization of losses to the public. Consequently, in the case of police facilities protection against theft, assault, and accidents are to be provided. Fire departments are to be located in order to minimize losses resulting from fire, such as property loss, loss of lives, and psychological damages. In the case of ambulance services, accident victims are to be transported to the appropriate health facilities for treatment.

In studying the effect of random variation on the location of emergency service facilities it is assumed that the location of the incident (existing facility) is a random variable. Both continuous space and discrete space formulations are developed and solution procedures are described.

In order to motivate the study of emergency service facilities location problems the following deterministic formulations of location problems are considered.

D4.1 minimize $\sum_{j=1}^{n} c_j x_j$

subject to: $\sum_{j=1}^{n} \bar{a}_{ij} x_j \geq 1$ for all i

$x_j = (0,1)$ for all j

D4.2 minimize $\max_{i} (\min_{j \varepsilon \theta(x)} t_{ij})$

subject to: $\sum_{j=1}^{n} x_j \leq k$

$x_j = (0,1)$ for all j

D4.3 minimize $\sum_{i=1}^{m} \min_{j \varepsilon \theta(x)} t_{ij}$

subject to: $\sum_{j=1}^{n} x_j \leq k$

$x_j = (0,1)$ for all j

where

m = number of customers

n = number of sites

k = number of facilities available

$$\bar{a}_{ij} = \begin{cases} 1, & \text{if a facility located at site } j \text{ covers customer } i \\ 0, & \text{otherwise} \end{cases}$$

$$x_j = \begin{cases} 1, & \text{if a facility is to be located at site } j \\ 0, & \text{otherwise} \end{cases}$$

c_j = cost of locating a facility at site j

t_{ij} = time required to provide service to customer i from site j

$\theta(x) = \{j:\ x_j = 1,\ j = 1, \ldots, n\}$.

The formulations given above are discrete space formulations; D4.1 is referred to as the set covering problem, D4.2 is variously referred to as the p-center problem and the minimax network location problem, D4.3 is referred to as either the p-median problem, a central facilities location problem, the generalized partial cover problem, or a network location problem. For a review and comparison of the above formulations, see White and Case [106].

A continuous space formulation of an emergency facilities location problem is given by D4.4.

D4.4 Minimize $\sum_{j=1}^{n} \sum_{i=1}^{m} z_{ij} w_i |X_j - P_i|_\ell$

$$\text{subject to:} \quad \sum_{j=1}^{n} z_{ij} = 1 \qquad \text{for all } i$$

$$z_{ij} = (0,1) \qquad \text{for all } i, j$$

where

m = number of customers

n = number of new facilities to be located

w_i = number of demands for service generated per unit time by customer i

X_j = (x_{1j}, x_{2j}), coordinate location of new facility j

P_i = (a_i, b_i), coordinate location of customer i

$$z_{ij} = \begin{cases} 1, & \text{if customer } i \text{ is to be served by facility } j \\ 0, & \text{otherwise.} \end{cases}$$

D4.4 is referred to in the location literature as the location-allocation problem.

4.2 Literature Survey

The survey of the literature treating the emergency service facilities location problem begins with a brief review of the general literature, followed by a detailed consideration of the research which relates directly to the present effort. Due to the widespread interest in the problem, the literature cited in this section does not provide an exhaustive listing of literature; however, it is felt that the cited research is representative of that which has been performed.

4.2.1 Overview of Previous Research

The problem of locating emergency service facilities is covered in the literature under three classifications:

a) Design of response areas (fixed locations)

b) Location (fixed response areas)

c) Relocation.

In designing response areas, also called districting, it is assumed that the location of the service units are known and it is required to partition a region into districts such that some service level (minimum response time, workload imbalance) is achieved. The location problem differs from the districting problem, since it is assumed that the region has been previously partitioned into districts (beats) and the locations of the new facilities are to be determined. The relocation problem occurs when a unit responds to a call and leaves its station empty; hence, a decision must be made to relocate available units to provide protection for the area until the original unit returns from its assignment.

The work done on the districting problem has been concentrated at N.Y.C.-Rand Institute in the study of the N.Y.C. fire department. Carter, et al. [4] have shown that districting to achieve equal travel time dividing lines does not yield minimum response time and choosing a unit other than the closest one may reduce the average travel time. However, their research was limited to only two locations. Larson and Stevenson [65] continued the work of Carter, et al. [4] and generalized it to handle the multifacility location problem. In a related work, Kenney [53] developed an algorithm for zoning, given the location of the N facilities.

For the location problem Hogg [46] considered the problem of obtaining the best combination of N station sites from a set of M alternative sites, so as to minimize the total number of fire engine journey times for a given number of fire stations. This problem is equivalent to the p-median problem covered in the literature. Toregas, et al. [94] formulated the problem of locating emergency facilities as a set covering problem, where the objective is to minimize the number of emergency units used; they also studied emergency service applications of the p-median problem, where the number of units is fixed. Larson and Stevenson [65] studied the problem of locating one new facility relative to a single existing facility; they assumed the existing facility was fixed in location and the location of the new facility was a random variable. The new facility was to be located such that the average travel time was minimized. They concluded that the location of the new facility is insensitive to the precise location of the existing facility. Plane and Hendrick [77] formulated the problem of locating fire companies as a set covering problem and used the solution obtained as an input to a configuration information model in order to obtain a solution which satisfied other criteria.

Recently, the relocation (repositioning) problem has received attention in the literature. Swersey [91] developed an integer programming model to determine which fire houses should be empty and which should be assigned relocated vehicles. The objective function which he used was to minimize the average travel time to incidents taking into account the average time that busy units would remain busy. Kolesar and Walker [57] suggested the use of a coverage criterion instead of average travel time. They developed a heuristic algorithm to determine which vacant houses to fill, then which available units to relocate to the vacant houses. For a review paper, see Chaiken and Larson [5].

In most of the above research, the demand points are known and the only random element involved is the demand for service, which is assumed to follow a Poisson distribution. Much of the underlying analysis of different formulations depends on queuing theory. In the subsequent sections the assumption that the location of existing facilities (demand points) is not known deterministically is imposed. Initially, it is assumed that a discrete solution space exists and the problem is formulated as a chance constrained covering problem, which is treated like the regular set covering problem after converting the chance constraints to equivalent deterministic constraints. A numerical example is introduced to aid in understanding the solution procedure. Subsequently, a continuous space formulation is considered and the problem is formulated as a location-allocation problem. A numerical example is provided to illustrate the solution procedures recommended.

4.2.2 Related Research

In most of the references cited in Section 4.2.1, the travel time is known deterministically. In contexts other than location problems, some authors assumed that the response time is probabilistic. Larson [63] developed several models for the police patrol allocation problem; he assumed that the location of existing facility (incident) P_i is known but the location of the new facility (police patrol unit) is randomly distributed. In this case all t_{ij} are independent random variables for a given i; this is opposite to the definition of t_{ij} in problems formumulated below. Kolesar and Blum [56] made the same assumption introduced by Larson [63] about t_{ij} and developed a square root formula for the response time.

Swoveland, et al. [92] studied the problem of ambulance location where the probabilistic aspect is introduced by finding the probability of calls arising at district i and serviced by the qth closest ambulance; all probabilities were developed from a simulation model, then a branch-and-bound algorithm was developed to obtain the optimal assignment of units to districts. The optimization criterion used by Swoveland, et al. is the minimization of the total average travel time.

Volz [96] presented a method for the optimum location of ambulance stations such that the average response time is minimized. In his model the stochastic variation of the response time is due primarily to the route selection and the varying speed of the responding ambulance unit, but all locations of incidents are known in advance. He solved the nonlinear model using a steepest descent algorithm.

Recently, Larson [64] generalized an existing method for allocating units to accommodate the location problem. He developed a model which is basically a multi-server queuing model (M|M|N); it was generalized to include both fixed (fire and ambulance) and mobile locations. The availability of units and workload are considered. Except for the work of Larson [64] and Carter, et al. [4], a restriction has been imposed on the interdistrict interactions. Larson [64] allowed a unit to answer a call from other districts.

Chapman and White [6] introduced a chance constrained formulation, where the t_{ij} are considered as continuously distributed random variables with a given distribution. They assumed that the randomness is due to changing of traffic patterns, road and weather conditions. The problem was transformed to a (0,1) programming problem for solution.

In this research effort, a chance constrained formulation similar to the one in [6] is employed. However, the random variables are defined differently. Here it is assumed that t_{ij} is a random variable due to the randomness of the location of the demand points. This assumption forces the dependency property on the random variables and results in the development of a different formulation. Compared to the work of Larson [64], a covering criterion is chosen, in addition to the criterion of minimizing response time. Larson's model is more general than those presented herein; his work appeared during the latter stages of this research. In some location problems, the centroid of the regions is used as the location of the existing facility; such an assumption is not reasonable when the population intensity is high. Therefore, it has been assumed that demand occurs uniformly over the response region.

4.3 Discrete Space Formulations

In this section, probabilistic variations of the deterministic formulations given in Section 4.1 are presented. As discussed above, the probabilistic component arises due to the assumption that the location of an incident is a random variable occurring uniformly over a given region. The randomness of the location implies that the distance traveled from the location of the emergency unit to the location of the incident is random. Assuming that the driving speed is constant over the area considered, then the response time to the incident is treated as a random variable. A discrete set of possible locations for the emergency units is assumed to be given by the decision makers.

4.3.1 Probabilistic Formulations

As described in Chapter 1, different criteria can be used in conjunction with probabilistic formulations. For an extensive listing see Elmaghraby [15]. In this section, two criteria are considered; the first is a chance constrained formulation, and the second is an expected value formulation.

A probabilistic variation of D4.1 is defined as the probabilistic set covering problem and is given by,

$$\text{P4.3.1} \quad \underset{x_j}{\text{minimize}}\ f(x_1,\ldots,x_n) = \sum_{j=1}^{n} x_j$$

$$\text{subject to:} \quad \Pr(t_{ij} \le t_i) \ge \gamma_i \qquad \text{for some } j\varepsilon\theta(x),\ i = 1, \ldots, n$$

$$x_j = (0,1) \qquad \text{for all } j$$

where

t_{ij} = response time from an emergency facility at site j to an incident in region i

t_i = the upper bound on the response time from location j to an incident in region i

γ_i = the required service (aspiration) level, $0 \leq \gamma_i \leq 1$

$\theta(x)$ = the set of sites where a facility has been located, i.e., $\theta(x) = \{j | x_j = 1\}$

where x_j is defined as in D4.1.

In P4.3.1, the chance constraints replace the deterministic constraints in the comparable deterministic model. The aspiration level γ_i indicates the tolerance measure for admitting constraint violations; to ensure that the quality of service is almost equal over the m regions, the same value of γ_i may be used for each region. Additionally, by using chance constraints the sensitivity of the location decision to the value of γ_i can be tested. In fact, some researchers [94] have found that administrators are quite interested in the results of such parametric analyses. The cost of installing a facility at site j is considered equal for all sites ($c_j = 1$). Savas [82] showed that about 85% of the cost of ambulance services is labor. Therefore, in minimizing the number of service locations, it is assumed that labor cost is homogeneous over all regions. The rectilinear norm is used as a distance measure in P4.1 and the subsequent problems.

A probabilistic formulation of D4.2 is presented as,

$$\text{P4.3.2} \quad \min Z = \max_i \left(\min_{j \varepsilon \theta(x)} E[t_{ij}] \right)$$

$$\text{subject to:} \quad \sum_{j=1}^{n} x_j \leq k$$

$$x_j = (0,1) \qquad \text{for all } j$$

where k, x_j and $\theta(x)$ are defined as in D4.2; $E[t_{ij}]$ is the expected response time from location j to incident i. In P4.3.2, the minimax criterion is used, hence the location is based on the maximum expected response time to an incident in region i from location j. Notice that the minimax formulation is a complement of P4.3.1, where instead of imposing probability bounds on the response time, the maximum response time is minimized. Also, instead of minimizing the number of facilities to be located, an upper bound on the number of facilities to be located is given. P4.3.2 will be referred to as the probabilistic p-center problem.

A probabilistic variation of D4.3 is given by,

$$\text{P4.3.3} \quad \text{minimize } Z = \frac{1}{w} \sum_{i=1}^{m} \min_{j \varepsilon \theta(x)} E[t_{ij}]\, w_i$$

$$\text{subject to:} \quad \sum_{j=1}^{n} x_j \leq k$$

$$x_j = (0,1) \qquad \text{for all } j$$

where w_i is the expected incident rate in area i for a given time period, and w is the total expected incident rate, i.e., $w = \sum_{i=1}^{m} w_i$.

P4.3.3 is referred to as the probabilistic central facility location problem, where the objective is to locate at most k facilities such that the overall expected response time is minimized. P4.3.3 differs from P4.3.2 in the sense that the average response time over the region is minimized instead of minimizing the extreme value of response time.

4.3.2 Solution Procedure

The region under study (city, county, etc.) is partitioned into m rectangular subregions. In partitioning the region, the following assumptions are made:

a) The location of an incident is uniformly distributed over the subregion, i.e., if an incident occurs it has an equal chance of occuring anywhere within the region.

b) If there are some natural barriers (railroads, rivers, etc.) within the region, it may be subdivided further to eliminate the necessity of crossing the barriers.

c) The m subregions are disjoint areas, hence no overlapping is allowed.

Through the partitioning of a region into rectangular subregions the uniformity assumptions can be satisfied. As illustrated it is possible to apply the zoning procedure to subregions such as chemical plants, hospitals, public guildings, small retail shops, and residential areas. Notice that the partitioning of the region into m rectangles with any size provides more flexibility in satisfying the above assumptions. Previous work by Volz [96] required the partitioning of a county into square areas of equal size.

After partitioning the region, the next step is to select alternative sites for possible locations of the new facilities. This decision is made by the decision makers according to geographical, sociological, and political factors. Naturally if there are already some facilities in operation, their locations may be considered among the possible sites to avoid extra fixed cost. Figure 4.1 provides an illustration of the configurations of subregions and sites allowed.

Since the random variable t_{ij} is defined as the travel time between a fixed location $X = (x_1, x_2)$ and a random location $P_i = (a_i, b_i)$, where i denotes the subregion along which the random variable is defined, the probability density function of t_{ij}, $f(t_{ij})$, is needed before attempting to solve problem P4.1.

Theorem 4.1: Given that a random variable P is distributed uniformly over a rectangle of size M x N, where its coordinates are distributed as

$$f(a) = \begin{cases} \frac{1}{N} & , 0 < a < N \\ 0 & , \text{otherwise} \end{cases} \quad ; \quad f(b) = \begin{cases} \frac{1}{M} & , 0 < b < M \\ 0 & , \text{otherwise} \end{cases} \tag{4.3.1}$$

Assuming that the driving speed along the direction of the coordinates x_1, x_2 is the same and equals v, then the probability density function of t, the travel time between P and any fixed location X within the rectangle, is given by

$$f(t) = \frac{1}{MN} [4t - 2 \sum_{\ell=1}^{4} (t-a'_\ell)\, u(t-a'_\ell) + \sum_{\ell=5}^{8} (t-a'_\ell)\, u(t-a'_\ell)] \tag{4.3.2}$$

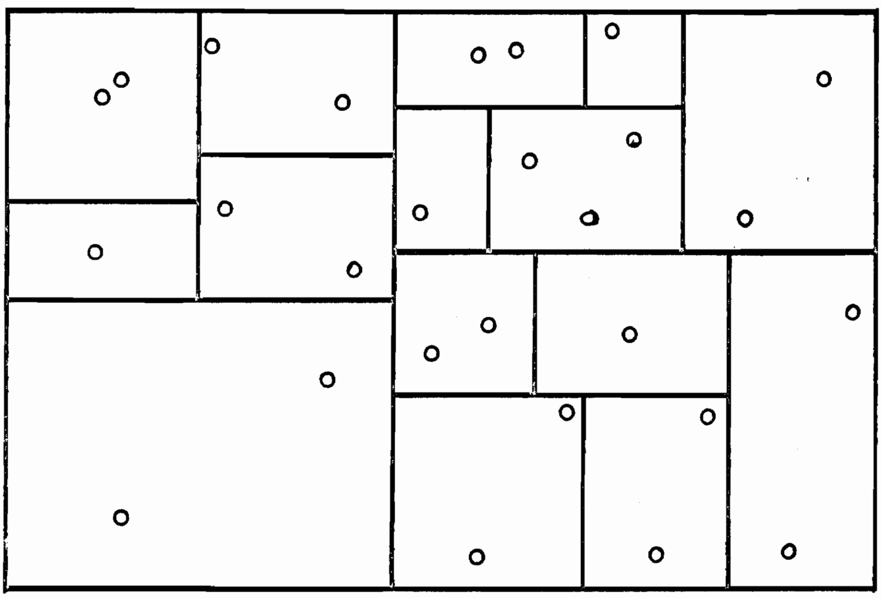
$$0 \leq t < \infty$$

where u(t-k) is the unit step function, defined as,

$$u(t-k) = \begin{cases} 0 & , t < k \\ 1 & , t > k; \end{cases}$$

$a'_1 = \frac{a}{v}$, $a'_2 = \frac{b}{v}$, $a'_3 = \frac{N-a}{v}$, $a'_4 = \frac{M-b}{v}$, $a'_5 = \frac{a+b}{v}$, $a'_6 = \frac{M+a-b}{v}$,

$a'_7 = \frac{N-a+b}{v}$, and $a'_8 = \frac{M+N-a-b}{v}$, respectively.



o denotes possible sites for location

Figure 4.1. Division of a region into m rectangular subregions.

Proof: Let d_1 and d_2 denote (x_1-a) and (x_2-b), respectively. Let $\overline{d}_1$ and $\overline{d}_2$ denote $|x_1-a|$ and $|x_2-b|$, respectively. If the probability density function of $\overline{d}_1$, $f(\overline{d}_1)$, is developed, then $f(\overline{d}_2)$ may be obtained by substituting the x_2 coordinates instead of x_1 coordinates. Since $f(a)$ is uniformly distributed, then $f(d_1)$ is uniformly distributed and its probability density function is given by

$$f(d_1) = \begin{cases} \frac{1}{N} & , -a < d < N-a \\ 0 & , \text{otherwise} \end{cases} \tag{4.3.3}$$

From (4.3.3), the probability density function of the absolute value, $f(\overline{d}_1)$, is obtained easily under two different cases:

If $a \leq \frac{N}{2}$, then

$$f(\overline{d}_1) = f(|x_1-a|) = \begin{cases} \frac{2}{N} & , 0 < \overline{d}_1 < a \\ \frac{1}{N} & , a < \overline{d}_1 < N-a \end{cases} \tag{4.3.4}$$

and

If $a > \frac{N}{2}$, then

$$f(\overline{d}_1) = f(|x_1-b|) = \begin{cases} \frac{2}{N} & , 0 < \overline{d}_1 < N-a \\ \frac{1}{N} & , N-a < \overline{d}_1 < a. \end{cases} \tag{4.3.5}$$

Let t_1, t_2 denote the travel time along the x_1 and x_2 coordinates, where

$$t_1 = \frac{\overline{d}_1}{v} \quad , \qquad t_2 = \frac{\overline{d}_2}{v}$$

Therefore, the distribution of t_1 under the two cases is derived directly from (4.3.4) and (4.3.5):

If $a \leq \frac{N}{2}$, then

$$f(t_1) = \begin{cases} \frac{2}{N} & , 0 < t_1 < a_1' \\ \frac{1}{N} & , a_1' < t_1 < a_3' \end{cases} \tag{4.3.6}$$

and

If $a > \frac{N}{2}$, then

$$f(t_1) = \begin{cases} \frac{2}{N} & , 0 < t_1 < a_3' \\ \frac{1}{N} & , a_3' < t_1 < a_1' \end{cases} \tag{4.3.7}$$

Similarly, the probability density function of t_2 is obtained:

If $b \leq \frac{M}{2}$, then

$$f(t_2) = \begin{cases} \frac{2}{M} & , 0 < t_2 < a_2' \\ \frac{1}{M} & , a_2' < t_2 < a'_4 \end{cases} \tag{4.3.8}$$

and

If $b > \frac{M}{2}$, then

$$f(t_2) = \begin{cases} \frac{2}{M} & , 0 < t_2 < a_4' \\ \frac{1}{M} & , a_4' < t_2 < a_2' \end{cases} \tag{4.3.9}$$

Since the total time $t = t_1 + t_2$, the probability density function is obtained from the convolution of $f(t_1)$ and $f(t_2)$,

$$f(t) = \int_0^t f_1(t-t_2)\cdot f_2(t_2)\ dt_2 \tag{4.3.10}$$

In performing the integration indicated by (4.3.10), a total of forty combinations are obtained; however, this may be reduced to twenty combinations if a' is replaced by b' in every distribution.

The Laplace transform of (4.3.10) is given by

$$L\{f(t)\} = L\{f(t_1)\}\cdot L\{f(t_2)\} \tag{4.3.11}$$

To facilitate the use of the Laplace transform, when $f(t_1)$ and $f(t_2)$ are as given in Figures 4.2(a) and (b), the unit step function is used. From (4.3.6) and (4.3.8), $f(t_1)$ and $f(t_2)$ may be expressed as,

$$f(t_1) = \frac{2}{N}\, u(t_1) - \frac{1}{N}\, u(t_1-a_1') - \frac{1}{N}\, u(t_1-a_3') \tag{4.3.12}$$

and

$$f(t_2) = \frac{2}{M}\, u(t_2) - \frac{1}{M}\, u(t_2-a_2') - \frac{1}{M}\, u(t_2-a_4') \tag{4.3.13}$$

where $a \leq \frac{N}{2}$, $b \leq \frac{M}{2}$.

Applying the Laplace transform to (4.3.12),

$$L\{f(t_1)\} = \frac{1}{Ns}\,[2 - e^{-a_1's} - e^{-a_3's}] \tag{4.3.13}$$

Also, for $f(t_2)$,

$$L\{f(t_2)\} = \frac{1}{Ms}\,[2 - e^{-a_2's} - e^{-a_4's}] \tag{4.3.14}$$

Substituting both (4.3.13) and (4.3.14) in (4.3.11),

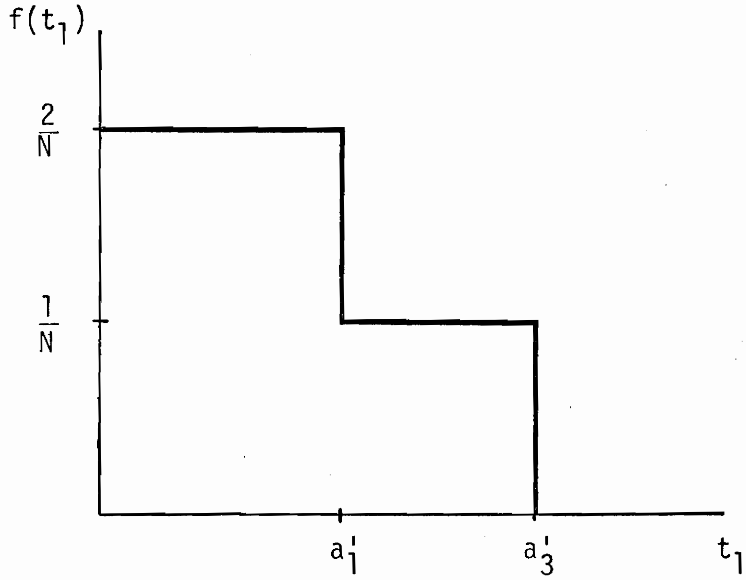


Figure 4.2(a). Probability density function of t_1.

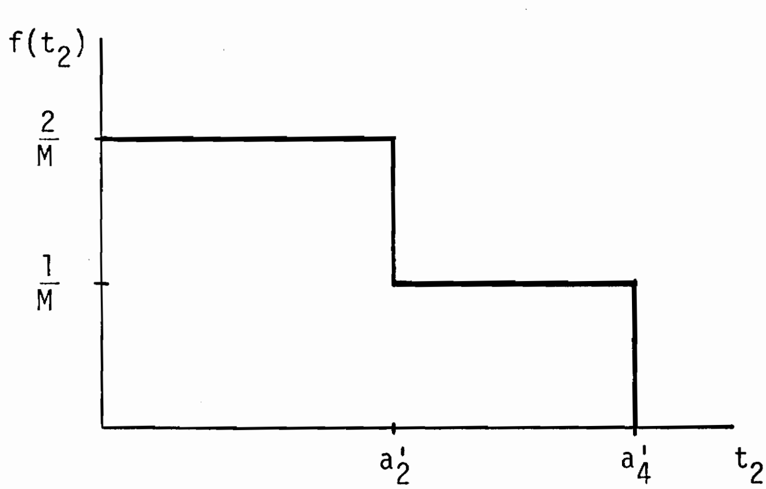


Figure 4.2(b). Probability density function of t_2.

$$L\{f(t)\} = \frac{1}{MNs^2}[4 - 2e^{-a_1's} - 2e^{-a_2's} - 2e^{-a_3's} - 2e^{-a_4's} + e^{-a_5's} + e^{-a_6's} + e^{-a_7's} + e^{-a_8's}] \quad (4.3.15)$$

Taking the inverse transform of (4.3.15),

$$L^{-1}\{f(t)\} = f(t) = \frac{1}{MN}[4t - 2(t-a_1')u(t-a_1') - 2(t-a_2')u(t-a_2') - 2(t-a_3')u(t-a_3') - 2(t-a_4')u(t-a_4') + (t-a_5')u(t-a_5') + (t-a_6')u(t-a_6') + (t-a_7')u(t-a_7') + (t-a_8')u(t-a_8')] \quad (4.3.16)$$

If (4.3.16) is written in a short form, then

$$f(t) = \frac{1}{MN}[4t - 2\sum_{\ell=1}^{4}(t-a_\ell')u(t-a_\ell') + \sum_{\ell=5}^{8}(t-a_\ell')u(t-a_\ell')]$$

which is the desired result.

Letting S_i denote the set of points belonging to region i, the probability density function developed in Theorem 4.1 is valid if $X_j \varepsilon S_i$; thus, it yields $f(t_{ij})$ for all $X_j \varepsilon S_i$. If it is assumed that there are no interdistrict interactions, then all calls generated from a subregion are served by facilities stationed in the subregion; if no facilities are available then calls are queued until a facility becomes available. From the literature, most authors impose this restriction; in certain cases the restriction is imposed to decrease interaction between political districts. However, the general case is studied here. Consequently, it is possible to define a subregion which has no potential sites for a new facility. Henceforth it is assumed that inter-district interactions are possible.

To evaluate the probability density function for the distance from a site which lies outside rectangular region i, say X_j', notice that the shortest distance traveled from X_j' until the service unit reaches the border of region i is known deterministically. Let X_j denote the point on the border of region i closest to the point X_j'; thus, the total distance between X_j' and P_i is given as,

$$|X_j'-P_i| = |X_j'-X_j| + |X_j-P_i| \tag{4.3.17}$$

Accordingly, the travel time is,

$$\begin{aligned} t_{ij'} &= \frac{|X_j'-X_j|}{v} + \frac{|X_j-P_i|}{v} \\ &= t_{j'j} + t_{ij} \end{aligned} \tag{4.3.18}$$

Since $t_{j'j}$ is known deterministically, then the probability density function of the random variable $t_{ij'}$ is obtained directly from $f(t_{ij})$ by observing the relation in (4.3.18). From (4.3.2), $f(t_{ij})$ is written as,

$$f(t_{ij}) = \frac{1}{MN} [4t_{ij} - 2\sum_{\ell=1}^{4} (t_{ij}-a_\ell')u(t_{ij}-a_\ell') + \sum_{\ell=5}^{8} (t_{ij}-a_\ell')u(t_{ij}-a_\ell')] \tag{4.3.19}$$

Since $t_{ij} = (t_{ij'}-t_{j'j})$, from (4.3.18), the probability density function of $t_{ij'}$, $f(t_{ij'})$, is obtained by substituting the value of t_{ij} in (4.3.19).

Notice that the distance traveled outside the subregion i, i.e., $|X_j'-X_j|$ is easily evaluated from the geometry of the region once their

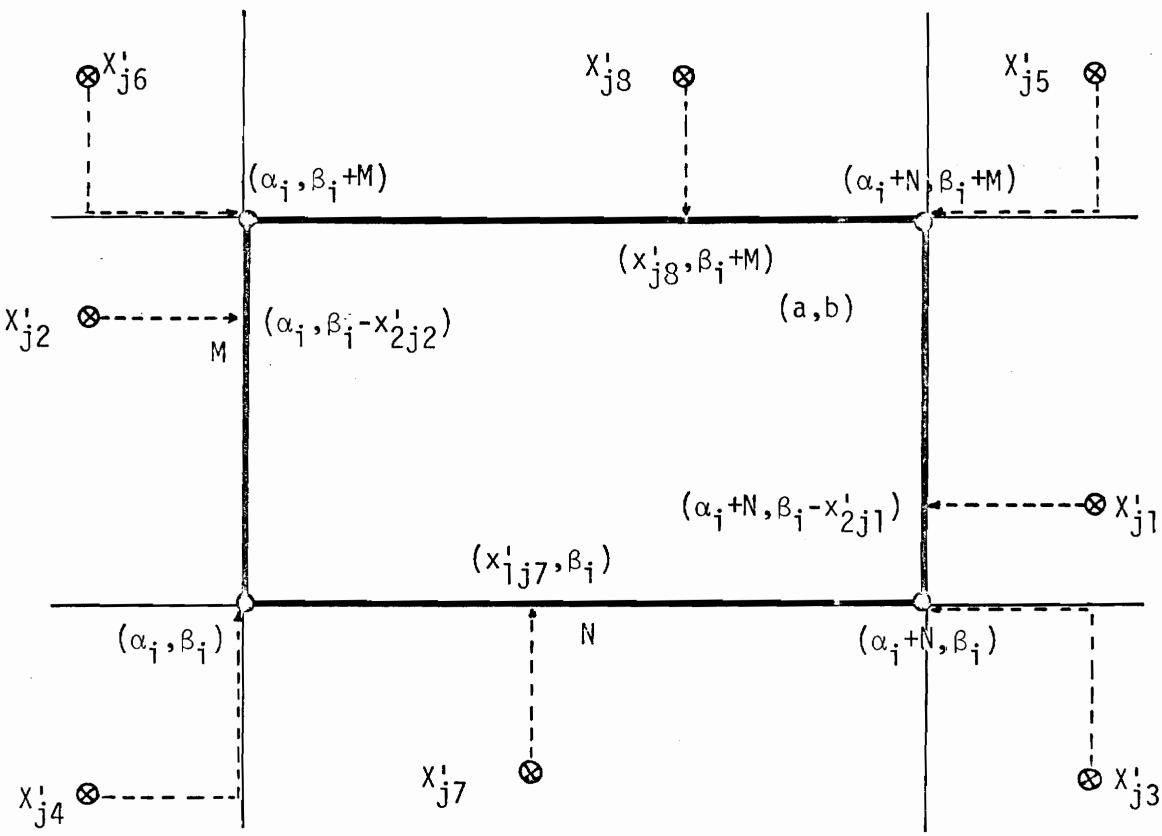


Figure 4.3. Service units located outside the subregion i, $X'_{jk} \notin S_i$.

locations are known. Let the coordinates of the origin of subregion i be (α_i, β_i); hence, the coordinates of the extreme points for S_i are (α_i, β_i), (α_{i+N}, β_i), $(\alpha_{i+N}, \beta_{i+M})$, and (α_i, β_{i+M}). From Figure 4.3, there are eight different cases for the potential location for X_j'. Let X_{jk}' denote the location in each case, k = 1, ..., 8; then using (4.3.17) the location X_j which gives the minimum $|X_{jk}' - X_j|$ is indicated in the figure for each case.

The probability density function for every fixed location may be derived using the above results. Before solving P4.3.1, the cumulative distribution function for t_{ij} is developed.

<u>Theorem 4.2</u>: If random variable t has a probability density function given by (4.3.2), then the corresponding cumulative distribution function is given by

$$F(t) = \frac{1}{MN}\left[2t^2 - \sum_{\ell=1}^{4}(t-a_\ell')^2 u(t-a_\ell') + \frac{1}{2}\sum_{\ell=5}^{8}(t-a_\ell')^2 u(t-a_\ell')\right]$$

where $u(\cdot)$ and a_ℓ' are as defined in Theorem 4.1.

<u>Proof</u>: The distribution function of t is given as

$$F(t) = \int_0^t f(y)\, dy \qquad (4.3.20)$$

Before evaluating the integral in (4.3.20), it is necessary to evaluate an integral of the form

$$\int_0^t (z-a)u(y-a)dy$$

Let $z = y - a$, then $dz = dy$ and the integral is written as

$$\int_0^t (y-a)u(y-a)dy = \int_{-a}^{t-a} u(z)z\,dz \qquad (4.3.21)$$

The integral on the right hand side of (4.3.21) is evaluated through integration by parts. Let $u = u(z)$ and $dv = zdz$, then $v = \frac{z^2}{2}$.
The derivative of the unit step function is the Dirac Delta function [107, p. 342], defined as

$$\delta(z-z_0) = \begin{cases} 0 & , z \neq z_0 \\ \infty & , z = z_0 \end{cases} \qquad (4.3.22)$$

Therefore, $du = \delta(z)dz$, and (4.3.21) is written as,

$$\int_{-a}^{t-a} u(z)zdz = \frac{1}{2} z^2u(z)\Big|_{-a}^{t-a} - \frac{1}{2}\int_{-a}^{t-a} z^2\delta(z)dz$$

$$= \frac{1}{2} [(t-a)^2u(t-a) - a^2u(-a)] - \frac{1}{2}\int_0^t (y-a)^2\delta(y-a)dy$$

From (4.3.22), $\delta(y-a) = 0$; also, $u(-a) = 0$ for negative values. Therefore,

$$F(t) = \frac{1}{MN} [\int_0^t 4y\,dy - 2\int_0^t \sum_{\ell=1}^{4} (y-a_\ell')u(y-a_\ell')dy + \int_0^t \sum_{\ell=5}^{8} (y-a_\ell')u(y-a_\ell')dy]$$

Substituting the result of (4.3.23), then the desired result is obtained,

$$F(t) = \frac{1}{MN} [2t^2 - \sum_{\ell=1}^{4} (t-a_\ell')^2u(t-a_\ell') + \frac{1}{2}\sum_{\ell=5}^{8} (t-a_\ell')u(t-a_\ell')] \qquad (4.3.24)$$

From Theorem 4.2, the distribution function of t_{ij}, where $X_j \varepsilon S_i$ is obtained from (4.3.24). For any location outside the rectangular region i, $X_j' \notin S_i$, the distribution is derived as follows. Since $F(t_i) = Pr(t_{ij} \leq t_i)$, then from (4.3.18), $t_{ij} = t_{ij'} - t_{j'j}$, and the distribution function of $t_{ij'}$ is given by

$$Pr(t_{ij} + t_{j'j} \leq t_i) = Pr(t_{ij} \leq t_i - t_{j'j}) = F(t_i - t_{j'j}) \qquad (4.3.25)$$

From (4.3.24) and (4.3.25) the distribution function of $t_{ij'}$ is obtained. The last result to be established is the expected response time to travel from any location, either X_j or X_j', to a random location within the subregion i.

Theorem 4.3: Given that a random variable P is distributed uniformly on a rectangular region of dimension M x N, where its coordinates are distributed as

$$f(a) = \begin{cases} \frac{1}{N} & , 0 < a < N \\ 0 & , \text{otherwise} \end{cases} \quad ; \quad f(b) = \begin{cases} \frac{1}{M} & , 0 < b < M \\ 0 & , \text{otherwise} \end{cases}$$

Assuming that the driving speed along the direction of the coordinates x_1, x_2 equals v, then the expected value of the travel time between P and fixed location X within the rectangular region, E[t], is given by

$$E[t] = \frac{1}{v} \left[\frac{x_1^2}{N} + \frac{x_2^2}{M} - (x_1+x_2) + \frac{1}{2}(M+N)\right] \qquad (4.3.26)$$

Proof: The expected value of the travel time is given as

$$E[t] = \frac{1}{v} \int_b \int_a |x_1-a| + |x_2-b| \; f(a)f(b) \; da \; db$$

Substituting the corresponding values of f(a) and f(b), then

$$E[t] = \frac{1}{v} [\frac{1}{N} \int_a |x_1-a| \; da + \frac{1}{M} \int_b |x_2-b| \; db]$$

$$= \frac{1}{v} [\frac{1}{N} \int_0^{x_1} (x_1-a)da + \int_{x_1}^{N} (a-x_1)da)$$

$$+ \frac{1}{M}(\int_0^{x_2} (x_2-b)db + \int_{x_2}^{M} (b-x_2)db)]$$

$$= \frac{1}{v} [\frac{1}{N} (x_1{}^2 + \frac{N^2}{2} - Nx_1) + \frac{1}{M} (x_2{}^2 + \frac{M^2}{2} - Mx_2)]$$

$$= \frac{1}{v} [\frac{x_1^2}{N} + \frac{x_2^2}{M} + \frac{1}{2} (N+M) - (x_1+x_2)]$$

which is the desired result.

Corollary 4.1: The minimum expected response time to subregion i occurs when the service facility is stationed at the center of the subregion, i.e., $X_j = (\frac{N}{2}, \frac{M}{2})$.

Proof: From (4.5.26), compute the partial derivatives with respect to x_1 and x_2 and set them to zero. Thus,

$$\frac{\partial E[t]}{\partial x_1} = \frac{1}{v} [-1 + \frac{2x_1}{N}] = 0$$

and

$$\frac{\partial E[t]}{\partial x_2} = \frac{1}{v} [-1 + \frac{2x_2}{M}] = 0$$

From the second partial derivatives the expected response time is strictly convex. Thus the optimum solution is

$$x_1^* = \frac{N}{2}, \qquad x_2^* = \frac{M}{2}.$$

The expected response time from a location $X_j' \notin S_i$, where the coordinates of the lower left hand corner are (0,0), is easily obtained from (4.3.26). From Figure 4.3 the eight cases are developed and summarized as follows:

$$E[t_{ij'1}] = \frac{1}{v}\left[\frac{x_{2j1}'^2}{M} + (x_{1j1}'-x_{2j1}') - \frac{1}{2}(N-M)\right]$$

$$E[t_{ij'2}] = \frac{1}{v}\left[\frac{x_{2j2}'^2}{M} - (x_{1j2}'+x_{2j2}') + \frac{1}{2}(N+M)\right]$$

$$E[t_{ij'3}] = \frac{1}{v}\left[(x_{1j3}'-x_{2j3}') - \frac{1}{2}(N-M)\right]$$

$$E[t_{ij'4}] = \frac{1}{v}\left[-(x_{1j4}'+x_{2j4}') + \frac{1}{2}(N+M)\right] \tag{4.3.27}$$

$$E[t_{ij'5}] = \frac{1}{v}\left[(x_{1j5}'+x_{2j5}') - \frac{1}{2}(N+M)\right]$$

$$E[t_{ij'6}] = \frac{1}{v}\left[-(x_{1j6}'-x_{2j6}') + \frac{1}{2}(N-M)\right]$$

$$E[t_{ij'7}] = \frac{1}{v}\left[\frac{x_{1j7}'^2}{N} - (x_{1j7}'-x_{2j7}') + \frac{1}{2}(N-M)\right]$$

$$E[t_{ij'8}] = \frac{1}{v}\left[\frac{x_{1j8}'^2}{N} - (x_{1j8}'+x_{2j8}') + \frac{1}{2}(N+M)\right]$$

Given (4.3.26) and (4.3.27), the expected response time from any location X_j to any subregion i is calculated.

All the tools required to solve problems P4.3.1, P4.3.2, and P4.3.3 are available. To solve P4.3.1, define the variable a_{ij} as follows

$$a_{ij} = \begin{cases} 1 & , \text{ if } Pr(t_{ij} \leq t_i) \geq \gamma_i \\ 0 & , \text{ if } Pr(t_{ij} \leq t_i) < \gamma_i \end{cases}$$

where γ_i is the minimum allowable probability that any incident occuring in subregion i is covered by some service unit stationed at location j. Assuming that at least one unit at site j is available, a positive value of a_{ij} indicates that the unit in site j covers an incident occuring in subregion i. The chance constraints in P4.3.1 imply that any incident in i must be covered for some $j\varepsilon\theta(x)$. Therefore the problem may be formulated as follows,

$$\overline{P}4.3.1 \quad \underset{x_j}{\text{minimize}} \quad \sum_{j=1}^{n} x_j$$

$$\text{subject to:} \quad \sum_{j=1}^{n} a_{ij}x_j \geq 1 \qquad \text{for all } i$$

$$x_j = (0,1) \qquad \text{for all } j$$

where $\overline{P}4.3.1$ is identified now as the set covering problem studied extensively in the literature, e.g., Garfinkel and Nemhauser [35], Bellmore and Ratliff [2], Lawler [66], and Khumawala [54]. The solution to $\overline{P}4.3.1$ may often be obtained by reduction techniques, where the rows and columns are deleted until the optimal solution is obtained. The use of reduction techniques is demonstrated subsequently with a numerical example.

To solve P4.3.2, the expected value of response time from each site j to each subregion j is calculated from (4.3.26) and (4.3.27). Therefore, the problem is transformed to problem D4.2, where it has been studied by Hakimi [41], Singer [89], and Christofides and Viola [8]. According to the computational experience obtained by Christofides and Viola, their iterative algorithm is more efficient than other existing ones. Therefore, it is recommended for solving P4.3.2.

Since the expected response times are available, P4.3.3 is solved using the same procedure designed to solve D4.3. Several algorithms are available for solving the central facilities location problem; among them are the ones studied by ReVelle and Swain [80], Curry and Skeith [13], and Shannon and Ignizio [88]. If the problem is formulated as a network location problem, also referred to as the p-median problem, the algorithms developed by Hakimi [41, 42], Maranzana [70], Wendell and Hurter [101], Teitz and Bart [93], and Singer [89] may be used to solve the problem.

4.4 Continuous Space Formulations

Although the majority of the research dealing with emergency service facilities location problems has concentrated on discrete space formulations, a number of continuous space formulations have been suggested. Specifically, the minimax location problem studied by Dearing and Francis [14] and Elzinga and Hearn [18, 19], among others, has been suggested as an appropriate formulation for the emergency service facilities location problem.

The class of location problems which can be considered as continuous space counterparts of the discrete space formulations considered previously is referred to as location-allocation problems. As originally formulated by Cooper [9] the location-allocation problem involves the determination of the number and locations of new facilities, as well as the allocation of customers to the new facilities. The allocation aspect of the location-allocation problem is especially appealing in modeling emergency service location problems; districts or regions are normally assigned to emergency facilities to denote primary responsibility for providing service to the districts.

4.4.1 Probabilistic Formulations

Probabilistic formulations of emergency service location problems can be provided in continuous space by modifying the location-allocation problem. For purposes of this research it will be assumed that the number of new facilities is a parameter, rather than a decision variable. The probabilistic formulations to be treated are stated mathematically as follows:

$$\text{P4.4.1} \quad \text{minimize} \quad \sum_{j=1}^{n} \sum_{i=1}^{m} z_{ij} f_i(X_j)$$

$$\text{subject to:} \quad \sum_{j=1}^{n} z_{ij} = 1 \qquad \text{for all } i$$

$$\sum_{i=1}^{m} w_i z_{ij} \leq W_j \qquad \text{for all } j$$

$$z_{ij} = (0,1) \quad \text{for all } i, j$$

P4.4.2 minimize $[\max_{i,j} z_{ij} g_i(X_j)]$

subject to: $\sum_{j=1}^{n} z_{ij} = 1$ for all i

$\sum_{i=1}^{m} A_i z_{ij} \leq C_j$ for all j

$z_{ij} = (0,1)$ for all i, j

where

n = number of new facilities to be located

m = number of regions

w_i = expected number of demands for service per unit time for region i

A_i = area of region i

W_j = upper bound on the expected number of demands for service per unit time to be assigned to facility j

C_j = upper bound on the allowable area to be assigned to facility j

$$z_{ij} = \begin{cases} 1, & \text{if region } i \text{ is assinged to facility } j \\ 0, & \text{otherwise} \end{cases}$$

X_j = (x_{1j}, x_{2j}), coordinate location of facility j

S_i = set of coordinate points belonging to region i

P_i = (a_i, b_i), an element of the set S_i

$f_i(X_j)$ = expected distance traveled per unit time between region i and X_j

$g_i(X_j)$ = maximum distance from X_j to any $P_i \varepsilon S_i$.

In P4.4.1 the n emergency service facilities are to be located in such a way that the total expected distance traveled is minimized. Exactly one facility is assigned to a given region. In P4.4.2 the n facilities are to be assigned so that the maximum distance between the location of a facility and any point in its region of responsibility is minimized. In both P4.4.1 and P4.4.2, more than one region can be assigned to a facility as long as the total demand for service and the total area served by the facility do not exceed the quantities W_j and C_j, respectively.

The minimax formulation, P4.4.2, is designed to model the preferences of a very conservative decision maker. The effects of the worst possible situation are to be minimized. Consequently, the term in the objective function $g_i(X_j)$ represents the extreme value of the random variation in distance traveled to region i from the point X_j. Since each point in region i is visited with equal probability, the extreme value for the random variable is represented by the distance from X_j to the most distant point in region i. Thus, the probabilistic aspects of the formulation P4.4.2 are quite subtle.

4.4.2 Solution Procedure

In order to solve P4.4.1 and P4.4.2 both the allocation problem and the location problem must be solved. Unfortunately, there does not exist an efficient exact method for solving the location-allocation problem, since the two sub-problems are not separable. Given the allocations, the location problem is easily solved. Likewise, given the locations, the associated allocation problem can be solved. Previous research on the location-allocation problem has concentrated on the development of heuristics [9], [10], [11] and branch-and-bound methods [17], [55], [58], [59]. In most cases, the continuous space assumption was replaced by a discrete space assumption in which sites for new facilities coincided with the locations of existing facilities.

In this research effort, the existing facilities are represented by rectangular areas, rather than points. Thus, the location problem involves the location of points relative to a number of rectangular areas. Previous research on the point/area location problem includes that of Francis [27], [28], [29], Love [68], and Wesolowsky and Love [104], among others. Love [68] employed SUMT in determining the location of a single new facility relative to several rectangular regions in order to minimize the expected Euclidean distance traveled; Wesolowsky and Love [104] studied a multifacility version of the problem using rectilinear distances.

In solving P4.4.1 the following procedure is suggested:

1) Determine the set of <u>feasible</u> allocations using total enumeration.
2) For a given allocation, let $Y_j = \{i: \; z_{ij} = 1, \; i = 1, \; \ldots, \; m\}$ and $R_j = \bigcup_{i \varepsilon Y_j} S_i$.
3) Determine the location X_j such that $\sum_{i \varepsilon Y_j} f_i(X_j)$ is minimized.
4) Compute the value of the objective function in P4.4.1 for each allocation and determine the optimum allocation.

Depending on the size of the problem it might not be feasible to enumerate all possible allocations. Consequently, heuristics might be employed to reduce the number of combinations considered. As an illustration, only allocations producing R_j's which are connected sets would be considered. The capacity constraint to achieve balanced assignments between facilities will also filter out a number of possible allocations.

In solving P4.4.2 a similar procedure is recommended.

1) Determine the set of feasible allocations using total enumeration.
2) For a given allocation, Let $Y_j = \{i: \; z_{ij} = 1, \; i = 1, \ldots, m\}$
3) Determine the location X_j such that $\max_{i \varepsilon Y_j} g_i(X_i)$ is minimized.
4) Determine the value of the objective function in P4 for each allocation and determine the optimum allocation.

Since the present research effort is concerned with location problems, no effort has been made to develop an efficient procedure for treating the allocation aspects of the problem. However, it is felt that the branch-and-bound procedure employed by Kuenne and Soland [59] in solving the deterministic location-allocation problem can be modified to accommodate the probabilistic location-allocation problems P4.4.1 and P4.4.2.

4.5 Numerical Examples

In this section, numerical examples are solved to show the impact of random variation on the location decision. First, an example is solved based on a discrete location space; problem formulation P4.3.1 is used and the same example problem is solved when the randomness is eliminated by considering that the location of customers is always at the centroid of the region. The same upper bound on the travel time is used with the chance constraints.

When the location problem is formulated in continuous space, an example is given for the location-allocation problems presented in P4.4.1 and P4.4.2. When the minimum total expected time is the optimization criterion, the deterministic counterpart is solved and the difference between the solutions is discussed.

4.5.1 Discrete Space Example Problem

It is desired to locate ambulance stations over an area consisting of five districts. The geometry of the districts and the potential sites available for locating the stations are shown in Figure 4.4. It is assumed that incidents are distributed uniformly over the district; distances between the centroids of the districts and the eight potential sites are given in Table 4.1. It is desired that an ambulance station be located within a given distance of an incident occuring in district i. The maximum distance traveled and the service level required for each district are shown in Table 4.1. The minimum number of stations are to be determined such that the travel distance requirements are met.

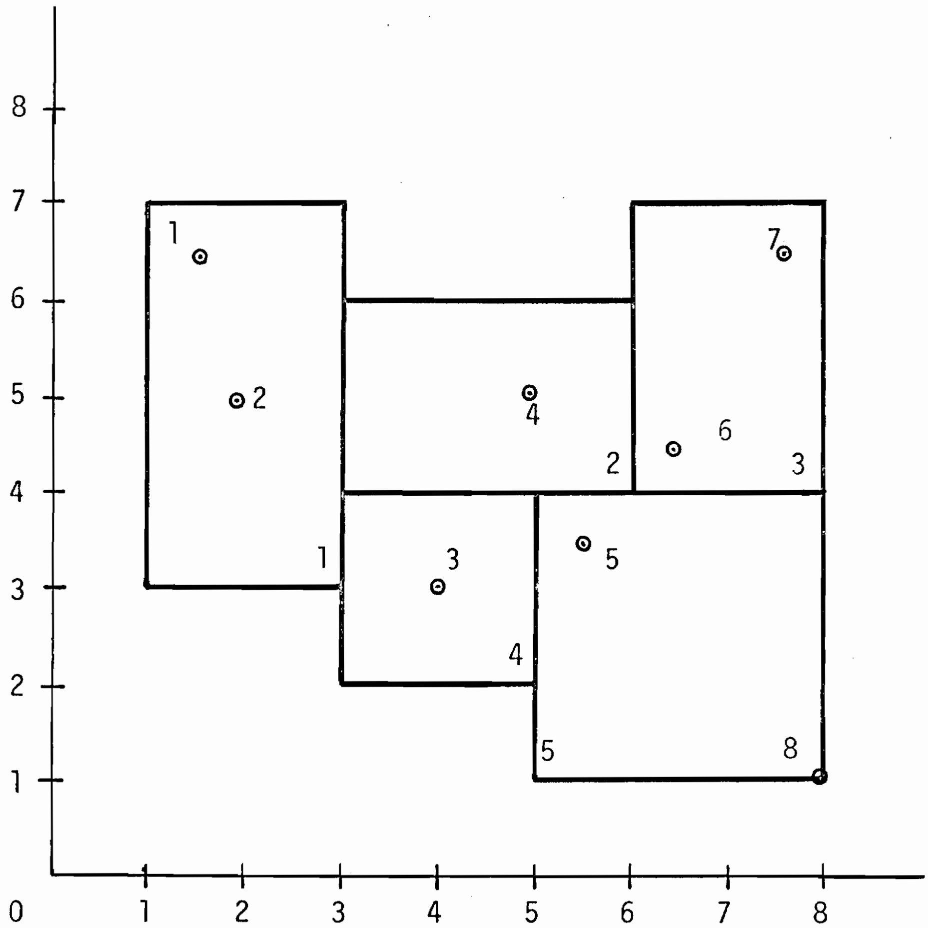


Figure 4.4. Partitioned districts and potential sites for locating ambulance stations.

TABLE 4.1. Maximum travel distance requirements, distances between sites and incident locations, and service levels.

District i	Maximum distance (d_i)	Service level (γ_i)	Sites 1	2	3	4	5	6	7	8
1	2.5	.85	2	0	4	3	5.5	5	7	10
2	2.5	.85	4.5	2.5	2.5	.5	3	2.5	4.5	7.5
3	3	.85	7.5	5.5	5.5	2.5	4	1.5	1.5	5.5
4	1.5	.85	6	4	0	3	1.5	4	7	6
5	4	.85	9	7	3	4	1.5	2	5	3

Assuming the average driving speed is constant, the distance traveled will be proportional to the response time. Therefore, distances are used throughout the example to simplify computations. The example is solved using the solution procedure recommended for problem P4.3.1. First, the covering probabilities are computed for all regions and sites using (4.3.24) and (4.3.25) with a_ℓ replacing a'_ℓ; let $F_j(d_i)$ denote the probability that an ambulance located at site j "covers" region i. The result is shown in Table 4.2. Problem P4.3.1 is formulated as $\overline{\text{P}}$4.3.1 by transforming the probability matrix to a cover matrix by testing the feasibility of the chance constraints. The resulting cover matrix is given as follows:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \tag{4.5.1}$$

Using (4.5.1), the set cover problem is solved using the reduction technique (the solution is trivial in this case) and the optimal solution is given as: $X_2 = 1$, $X_3 = 1$, $X_4 = 1$, and $X_6 = 1$. Therefore, only four out of eight sites are chosen. For the decision makers, the optimal solution guarantees a cover with a probability of 0.85.

Suppose now that the probabilistic formulation given by P4.3.1 is ignored and the problem is solved deterministically. This is done by considering that the locations of incidents coincide with the centroid

TABLE 4.2 Probability matrix of covering region i from site j, $F_j(d_i)$.

	Site j							
Region i	1	2	3	4	5	6	7	8
1	.53	.94	.14	.03	0	0	0	0
2	.02	.30	.35	.95	.29	.13	.02	0
3	0	0	0	.58	.19	.92	.92	0
4	0	0	.88	.03	.25	0	0	0
5	0	0	.72	.50	.98	.88	.19	.78

of each region. After eliminating the random component, the problem is solved using the maximum allowable travel distance and the distance matrix shown in Table 4.1. The cover matrix is given by:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \qquad (4.5.2)$$

The set cover problem is solved using (4.5.2); several optimal solutions may be obtained, e.g., $X_2 = 1$, $X_3 = 1$, and $X_4 = 1$ or $X_1 = 1$, $X_3 = 1$, and $X_4 = 1$. Thus, any optimal solution will involve only three sites; however, noticing the covering probability matrix, any optimal solution has a small chance of covering certain regions. For example, if the solution is given by $X_2 = 1$, $X_3 = 1$, and $X_4 = 1$, the five regions will be covered with probabilities .93, .95, .58, .87, and .72, respectively. Thus region 3 has relatively little chance of being covered. The probabilistic formulation may require a higher number of assigned sites in order to assure that all sites will be covered at least 85% of the time.

4.5.2 Continuous Space Example Problem

In order to illustrate the recommended procedures for solving P4.4.1 and P4.4.2, an example problem is presented. Consider the five regions depicted in Figure 4.5(a). Two facilities are to provide service to

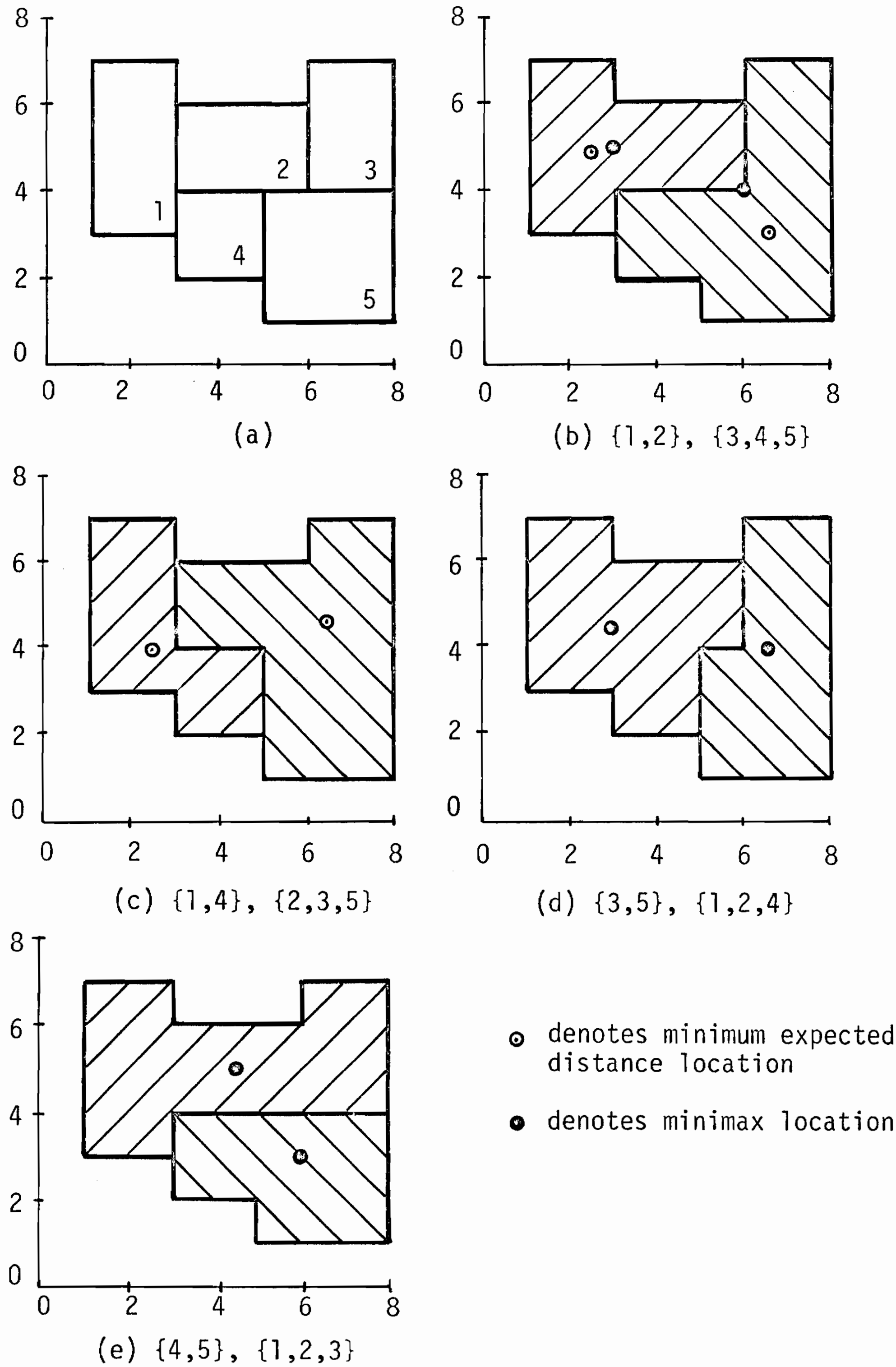


Figure 4.5. Feasible allocations and the corresponding optimal locations.

the regions. Management wishes to allocate regions in such a way that a contiguous area is served by a facility. With five regions and two facilities, the Stirling number of the second kind [9] indicates 15 allocation combinations are possible.

The expected number of demands per unit time and the area for each region are given in Table 4.3. In solving P4.4.1 it is desired that a reasonable balance in the workload for the facilities be maintained. Consequently, it is required that the expected demand for service per unit time not exceed 15. For P4.4.2 the workload is expressed in terms of the area served; thus, it is specified that the service area for a facility not exceed 20. As indicated in Table 4.4 there are 2 feasible allocations to be considered for P4.4.1 and 3 feasible allocations to be considered for P4.4.2.

Using the solution procedure developed by Wesolowsky and Love [104] the locations of the two facilities are determined for each allocation combination. The expected distance traveled per unit time is given by

$$\sum_{j=1}^{n} \sum_{i \varepsilon Y_j} \frac{w_i}{A_i} \iint_{S_i} |x_{1j}-a_i| + |x_{2j}-b_i| \, da_i db_i \tag{4.5.3}$$

Since (4.5.3) is separable in x_{1j} and x_{2j} the expected distance traveled per unit time in the x_1 direction is expressed as

$$\sum_{j=1}^{n} \sum_{i \varepsilon Y_j} \frac{w_i}{A_i} \iint_{S_i} |x_{1j}-a_i| \, da_i db_i \tag{4.5.4}$$

Using a solution procedure similar to that employed for the single facility, rectilinear location problem, (4.5.4) is minimized by taking

TABLE 4.3. The expected number of demands per unit time and the area for each region.

Region i	Expected number of demands w_i	Area of region i
1	8	8
2	4	6
3	6	6
4	4	4
5	5	9

TABLE 4.4. Allocation combinations for the continuous space example problem.

Allocation combination	Y_1	Y_2	Feasible P4.4.1	Feasible P4.4.2	Comment
1	{1}	{2,3,4,5}	No	No	Workload imbalance for P4.4.1 and P4.4.2
2	{2}	{1,3,4,5}	No	No	Workload imbalance for P4.4.1 and P4.4.2
3	{3}	{1,2,4,5}	No	No	Workload imbalance for P4.4.1 and P4.4.2
4	{4}	{1,2,3,5}	No	No	Workload imbalance for P4.4.1 and P4.4.2
5	{5}	{1,2,3,4}	No	No	Workload imbalance for P4.4.1 and P4.4.2
6	{1,2}	{3,4,5}	Yes	Yes	
7	{1,3}	{2,4,5}	No	No	Disjoint service area
8	{1,4}	{2,3,5}	Yes	No	Workload imbalance for P4.4.2
9	{1,5}	{2,3,4}	No	No	Disjoint service area
10	{2,3}	{1,4,5}	No	No	Workload imbalance for P4.4.1 and P4.4.2
11	{2,4}	{1,3,5}	No	No	Disjoint service area
12	{2,5}	{1,3,4}	No	No	Disjoint service area
13	{3,4}	{1,2,5}	No	No	Disjoint service area
14	{3,5}	{1,2,4}	No	Yes	Workload imbalance for P4.4.1
15	{4,5}	{1,2,3}	No	Yes	Workload imbalance for P4.4.1

the partial derivatives, ordering the coordinates of the regions, and accumulating weights (derivatives) until a median condition is satisfied. Applying the solution procedure the two new facilities should be located as shown in Figures 4.5(b) and (c) for the 2 feasible allocations. The total expected distances for the allocations are 61.08 and 59.39, respectively. Consequently, the optimal allocation is {1,4}, {2,3,5} and the optimal locations are $(x^*_{11}, x^*_{21}) = (2.5, 4)$ and $(x^*_{12}, x^*_{22}) = (6.4, 4.625)$, respectively.

To illustrate the impact of random variation on the location decision, P4.4.1 and P4.4.2 are solved as deterministic problems. The deterministic formulation is obtained when the location of customers is given by the centroid of the region and the weight w_i is attached to the corresponding centroid location. P4.4.1 is solved as a regular minisum rectilinear problem, discussed extensively in Francis and White [32]. For the two feasible allocations shown in Figures 4.5(b) and (c) the "optimum" locations yield total distances of 40.5 and 42.5, respectively. Consequently, the optimal allocation is {1,2}, {3,4,5} and the optimum deterministic locations are $(x^*_{11}, x^*_{21}) = (2,5)$ and $(x^*_{12}, x^*_{22}) = (6.5,3)$.

Comparing both the probabilistic and the deterministic results it is obvious that the optimal allocation and locations are different. The expected distance traveled from the points (2,5) and (6.5,3) totals 62.68; thus, failing to account explicitly for random variations yields a 6% increase in expected distance traveled.

A minimax location for a single facility relative to a number of regions is obtained by determining the center of the smallest diamond which contains all points in the allocated regions. Using the minimax solution procedure for a single facility described in [30], the two new facilities should be located as shown in Figures 4.5(b), (d), and (e) for the 3 feasible allocations. The maximum distances for the allocations are 5, 4.5, and 5.5, respectively. Consequently, the optimum allocation is {3,5}, {1,2,4} and the optimum locations are (x_{11}^*, x_{21}^*) = (6.5,4) and (x_{12}^*, x_{22}^*) = (3,4,5), respectively.

If the centroids of each rectangular region are used to represent deterministic locations of five existing facilities, the optimum allocation obtained by solving the deterministic, minimax location-allocation problem is {3,5}, {1,2,4}. Thus, the same allocation is obtained using probabilistic and deterministic approaches. The "deterministic" location obtained for {3,5} is the line segment connecting the points (3.25,4.25) and (4.0,5.0); whereas, the "probabilistic" location is the point (3.0,4.5). For the set of regions {1,2,4}, the "deterministic" location is the line segment connecting the points (6.5,2.5) and (7.0,5.5); whereas, the "probabilistic" location is the point (6.5,4.0). Locating at the "deterministic" solution yields a maximum of 2 distance units to the centroid of a region and a maximum of 5 distance units to any point in the assigned regions.

4.6 Summary

In this chapter, the problem of locating emergency service facilities in an urban environment was studied. The location of each incident was considered to be a random variable occuring uniformly over a given region. The location problem was considered in both discrete space and continuous space.

In discrete space, the location of possible sites for the emergency facilities was assumed to be given by the decision makers. Chance constraints were introduced to bound the probability that the response time is within a specified limit. After developing the distribution function for response time a deterministic formulation was obtained and recognized to be a set cover problem. Probabilistic variations of the p-median and p-center problems were also presented.

In continuous space, the problem investigated is similar to the location-allocation (L-A) problem. Random variation was introduced by considering that customers (incidents) are uniformly distributed over a given region; whereas, in the deterministic (L-A) formulations the locations are taken as the centroids of the regions.

Complete enumeration was used to evaluate all feasible allocations; then the locations were determined optimally using a median type approach. The optimization criterion employed was the minimization of the total distance traveled.

A minimax criterion was considered for the probabilistic problem in order to ensure that the extreme values of the random variables are satisfied. Noticing that the extreme values lie on the boundary of the region, a procedure similar to that used for the deterministic counterpart was applied. Solved examples are provided to emphasize the impact of the probabilistic formulations on the location decision.

Chapter 5

SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

5.1 Introduction

In this chapter, a summary of the research performed on the probabilistic formulations of location problems is presented. Remarks are provided and conclusions are drawn based on the theoretical and computational results obtained in the preceding chapters. Recommendations for future research efforts are given to assist in identifying research topics and in extending the current research effort.

5.2 Summary

In this research effort three location problems were studied, the single facility location problem, the multifacility location problem, and the emergency service location problem. The primary objective of this research effort was to investigate the effect of random variation on the location decision.

The first two problems treated are defined as the generalized Weber problem, where the concern is to locate one or more new facilities in the plane relative to several existing facilities such that the total expected cost of item movement is minimized. The cost was considered to be a linear function of either the expected rectilinear or Euclidean distance, and is a quadratic function of the expected Euclidean distance (gravity problem).

Random variation was introduced in the problems by considering that the locations of the existing facilities was not known deterministically; also, the item movement between two locations was considered to be a random variable.

In Chapter 2, the single facility location problem was studied. Random variation was assumed to exist in the location of the existing facilities; the weights attached to the movement were assumed to be known deterministically. Different formulations were derived for the unconstrained problem; for each formulation, possible applications were discussed, related literature was surveyed, theoretical properties were developed, and a solution procedure was provided. Each algorithm was programmed and optimal solutions were obtained for several problems. A comparison between the probabilistic and deterministic solutions was provided.

In the constrained case, norm type constraints and chance constraints were employed. The deterministic equivalent of each probabilistic formulation was obtained and the properties of the problem were studied before suggesting a solution procedure.

In Chapter 3, the multifacility location problem was studied. It was assumed that both weights and existing locations were random variables. Two formulations of total expected cost of movement were given, the first type involved the product of the random variables, weight and distance; the second type involved the random sum of each individual distance traveled. Solution procedures were provided for the unconstrained problem. Each was programmed and tested for a sample example, and previous research was surveyed. For the constrained problem, the deterministic equivalent was obtained for each formulation. Its properties were studied, and a solution procedure was suggested.

In Chapter 4, the emergency service facilities location problem was introduced. A literature survey was provided only for the work dealing directly with the location problem. Random variation was assumed to be present due to the assumption that the location of an incident is a random variable occuring uniformly over a given region. The problem of locating new facilities in both discrete space and continuous space were considered. For the discrete case, the properties of the probabilistic formulation were discussed and the deterministic equivalent were solved as integer programming problems. For the continuous case, the probabilistic formulations were solved as a location-allocation problem. An example was solved for each case to show the impact of considering the probabilistic aspects of the location problem.

5.3 Conclusions

A number of conclusions can be drawn from the research effort. Throughout the detailed consideration of specific location problems in Chapters 2, 3, and 4 conclusions were drawn concerning applicable probabilistic formulations of the location problems, theoretical properties of the models obtained, and appropriate solution procedures. It was found that random variation can produce results significantly different from those obtained using deterministic formulations. Consequently, it appears appropriate for the analyst to consider random variation when studying location problems. Additionally, it was found that the consideration of random variation did not produce formulations too complex to solve. In many cases, solution procedures similar to those employed for the deterministic location problem were recommended. The usual tradeoffs of ease of solution, clarity of cause-effect relationships, and degree of realism found in modeling real world problems indicates a consideration of random variation increases the degree of realism for the model without significant sacrifices in ease of solution and clarity of cause-effect relationships.

The joint consideration in Chapter 3 of random variation in the locations of the existing facilities and the level of interaction between new and existing facilities is unique to the present research effort. Yet, there exist real world location problems in which both forms of random variation occur. Further, a number of real world situations can be modeled as the random sum of random variables, as well as the product of random variables.

In Chapter 4, the primary contributions of the research were the development of the probability distribution for response time and the development of probabilistic formulations for the location-allocation problem based on both minimum and minimax objectives.

5.4 Recommendations for Future Research

A number of areas for further study were encountered during the research and are given below:

1) Perform sensitivity analyses for each model obtained,
2) Study the effect of different types of probability distributions on the location decision,

3) Determine if the set of solutions to P2.8.1 is convex,
4) Model the location problem as a decision under uncertainty,
5) Develop a branch-bound procedure for solving the location-allocation problems treated in Chapter 4,
6) Combine the location decision with the queuing aspects of the emergency service facilities location problem,
7) Apply multi-criteria objective function optimization approaches to the location problem,
8) Extend the consideration of random variation to other location problems, e.g., quadratic assignment problems,
9) Model the layout problem as a decision under risk,
10) Formulate the location problem as a dynamic, probabilistic decision problem,
11) Apply Bayesian approaches in studying the relocation problem,
12) Extend the consideration of discretely distributed random variables to other location problems.

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APPENDIX A

The effect of non-symmetric probability density functions on the location decision is studied in this appendix. Since the exponential distribution is a relatively simple non-symmetric distribution to work with, it is used throughout the analysis.

A.1 Rectilinear-Distance Location Problem with Bivariate Exponential Distributions

When the bivariate exponential is used in Problem P2.4, the model is given by

$$\text{PA.1} \quad \underset{X}{\text{Minimize}}\ f(X) = \sum_{i=1}^{m} w_i \int_{b_i} \int_{a_i} (|x_1-a_i|+|x_2-b_i|)\ f(P_i)da_i dP_i$$

where

$$f(a_i) = \alpha_i e^{-\alpha_i a_i} \quad , 0 \le a_i < \infty,\ \alpha_i > 0,\ i = 1,\dots,m$$

$$f(b_i) = \beta_i e^{-\beta_i b_i} \quad , 0 \le b_i < \infty,\ \beta_i > 0,\ i = 1,\dots,m$$

$$f(P_i) = \alpha_i \beta_i e^{-(\alpha_i a_i+\beta_i b_i)} \quad , 0 \le a_i < \infty,\ 0 \le b_i < \infty,\ i = 1,\ \dots,m$$

To solve PA.1 we can use the result of (2.4.2) to evaluate z(x), where $a_i = a$, $\alpha_i = \alpha$, and $x_1 = x$. Hence,

$$z(x) = 2x\ F(x) - x - \frac{1}{\alpha} + \int_x^\infty a\ f(a)da$$

where $E[a] = \frac{1}{\alpha}$, since the distribution is exponential; also,

$$F(x) = \int_0^\infty \alpha\ e^{-\alpha w}\ dw = 1 - e^{-\alpha x}$$

To evaluate the integral,

$$\int_x^\infty a\ f(a)da = \int_x^\infty a\alpha\ e^{-\alpha a}da = -\frac{e^{-\alpha a}}{\alpha}(\alpha a+1)\Big|_x^\infty = \frac{e^{-\alpha x}}{\alpha}(\alpha x+1) \qquad \text{(A.1.1)}$$

Substituting (A.1.1) and the value of F(x) into z(x), gives

$$z(x) = x + \frac{2}{\alpha} e^{-\alpha x} - \frac{1}{\alpha} \qquad \text{(A.1.2)}$$

From (A.1.2), the total cost function is written as,

$$f(X) = \sum_{i=1}^{m} w_i[(x_1 + \frac{2}{\alpha_i} e^{-\alpha_i x_1} - \frac{1}{\alpha_i}) + (x_2 + \frac{2}{\beta_i} e^{-\beta_i x_2} - \frac{1}{\beta_i})] \qquad \text{(A.1.3)}$$

Before solving (A.1.3) for the optimal X^* it must be established that f(X) is a convex function. This is proved in Theorem 2.4.1, based on the property that the probability density function is greater than or equal to zero. That f(X) is strictly convex is established by the following theorem.

Theorem A.1.1: The function f(X) given in PA.1 is a strictly convex function of $X \varepsilon E^2$.

Proof: In order to prove that f(X) is strictly convex, it is sufficient to establish that at least one function under the summation is a strictly convex function. To study the function z(x) defined by (A.1.2), recall the result of Theorem 2.4.2, where $\frac{d^2z}{dx^2} = 2f(x) \geq 0$.

For the exponential distribution, $f(x) > 0$ for all x such that $0 \leq x < \infty$; thus, the second derivative is positive, implying that $z(x)$ is strictly a convex function over all $x_1, x_2 \in E^1$. Consequently, $f(X)$ is a strictly convex function and the Hessian of $f(X)$ is positive definite.

Property A.1.1: The inverse of the Hessian of $f(X)$ in (A.1.3) is positive definite.

Proof: Obtaining the first and second derivatives of $z(x)$ in (A.1.2),

$$\frac{\partial f(x)}{\partial x_1} = \sum_{i=1}^{m} w_i(1-2e^{-\alpha_i x_1})$$

and

$$\frac{\partial f(X)}{\partial x_2} = \sum_{i=1}^{m} w_i(1-2e^{-\beta_i x_2})$$

Hence, the second derivatives are

$$\frac{\partial^2 f(X)}{\partial x_1^2} = 2 \sum_{i=1}^{m} w_i \alpha_i e^{-\alpha_i x_1} = \bar{A}_1$$

$$\frac{\partial^2 f(X)}{\partial x_2^2} = 2 \sum_{i=1}^{m} w_i \beta_i e^{-\beta_i x_2} = \bar{A}_2$$

and

$$\frac{\partial f(X)}{\partial x_1 \partial x_2} = \frac{\partial f(X)}{\partial x_2 \partial x_1} = 0 \ .$$

Therefore, the Hessian and its inverse are constructed as follows,

$$H(X) = \begin{bmatrix} \bar{A}_1 & 0 \\ 0 & \bar{A}_2 \end{bmatrix} \quad \text{and} \quad [H(X)]^{-1} = \begin{bmatrix} \frac{1}{\bar{A}_1} & 0 \\ 0 & \frac{1}{\bar{A}_2} \end{bmatrix}$$

Since $\bar{A}_1$, $\bar{A}_2$ are positive over all x_1, x_2, then $[H(X)]^{-1}$ is positive definite.

To optimize the function f(X) given in (A.1.3), the same iterative method (Newton) employed in Chapter 2 is used. The iteration formula is the same as in (2.4.14), except that $S^{(k)}$ is obtained from

$$S^{(k)} = [H(X^k)]^{-1} \nabla f(X^k) = \begin{bmatrix} \dfrac{\sum_{i=1}^{m} w_i(1-2e^{-\alpha_i x_1})}{2\sum_{i=1}^{m} w_i \alpha_i e^{-\alpha_i x_1}} \\ \\ \dfrac{\sum_{i=1}^{m} w_i(1-2e^{-\beta_i x_2})}{2\sum_{i=1}^{m} w_i \beta_i e^{-\beta_i x_2}} \end{bmatrix} \tag{A.1.4}$$

As proven in the next section, an approximate solution is given as,

$$\tilde{X}^* = \begin{bmatrix} \dfrac{\sum_{i=1}^{m} w_i}{2\sum_{i=1}^{m} w_i \alpha_i} \\ \\ \dfrac{\sum_{i=1}^{m} w_i}{2\sum_{i=1}^{m} w_i \beta_i} \end{bmatrix} \tag{A.1.5}$$

Therefore, it will be used as a starting feasible solution for the iterative procedure, hoping that the solution converges faster to its optimal. Using (2.4.14), (A.1.4), and (A.1.5) the optimal solution is obtained through the above iterative scheme.

A.2 Rectilinear Distance Location Problem with a Bivariate Exponential Distribution: Approximate Solution

Following the same approach as in Section 2.4.2, the necessary and sufficient conditions for the optimal of f(X) defined in (A.1.3) are

$$\frac{\partial f(X)}{\partial x_1} = \sum_{i=1}^{m} w_i(1-2e^{-\alpha_i x_1^*}) = 0$$

and

$$\frac{\partial f(X)}{\partial x_2} = \sum_{i=1}^{m} w_i(1-2e^{-\beta_i x_2^*}) = 0$$

which are translated as

$$\sum_{i=1}^{m} w_i e^{-\alpha_i x_1^*} = \frac{\sum_{i=1}^{m} w_i}{2} \tag{A.2.1}$$

and

$$\sum_{i=1}^{m} w_i e^{-\beta_i x_2^*} = \frac{\sum_{i=1}^{m} w_i}{2} \tag{A.2.2}$$

Expanding the exponential function in its series, retaining the first linear term only, and substituting in (A.2.1)

$$\sum_{i=1}^{m} w_i(1-\alpha_i x_1^*) = \frac{\sum_{i=1}^{m} w_i}{2}$$

x_1^* and x_2^* are obtained as

$$\tilde{x}_1^* = \frac{\sum_{i=1}^{m} w_i}{2 \sum_{i=1}^{m} w_i \alpha_i} \tag{A.2.3}$$

and

$$\tilde{x}_2^* = \frac{\sum_{i=1}^{m} w_i}{2 \sum_{i=1}^{m} w_i \beta_i} \tag{A.2.4}$$

The simplicity of this solution makes it a good starting solution for the iterative procedure recommended in Section A.1.

A.3 Exponentially Distributed Planer Location Problem: Chance Constraints

The single facility location problem will be treated when there are chance constraints on the location of the new facility relative to the location of the existing facilities. The chance constrained single facility location problem with exponential distribution is expressed as

$$\text{PA.3} \quad \underset{X}{\text{minimize}}\ f(X) = \sum_{i=1}^{m} w_i \int_{b_i} \int_{a_i} (|x_1-a_i|+|x_2-b_i|)\ f(a_i)f(b_i)da_i db_i$$

$$\text{subject to:} \quad \Pr(|x_1-a_i|+|x_2-b_i| \leq \xi_i) \geq \gamma_i \qquad \text{for all } i$$

where

$$f(a_i) = \alpha_i e^{-\alpha_i a_i} \qquad , 0 < a_i < \infty$$

$$f(b_i) = \beta_i e^{-\beta_i b_i} \qquad , 0 < b_i < \infty$$

To solve problem PA.2 the chance constraints are replaced by their deterministic equivalents. As in the approach of Section 2.8, the probability density function of the rectilinear distance $v_i = |x_1-a_i| + |x_2-b_i|$ is developed as follows.

Theorem A.3.1: Given that the independent random variables a, b are distribution exponentially, where $f(a) = \alpha e^{-\alpha a}$, $f(b) = \beta e^{-\beta b}$. The probability density function of the rectilinear distance $v = |x_1-a| + |x_2-b|$ is given by,

$$
g(v) = \begin{cases}
\frac{\alpha\beta}{\alpha^2-\beta^2} e^{-\alpha x_1-\beta x_2}[4\alpha\sinh(\alpha v) - 4\beta\sinh(\beta v)] & 0 \le v < x_2 \\[2ex]
\frac{\alpha\beta}{\alpha^2-\beta^2} e^{-\alpha x_1}[4\alpha\sinh(\alpha v-\beta x_2) - 2\beta\cosh(\alpha v-\alpha x_2) & \\
\qquad - 2\alpha\sinh(\alpha v-\alpha x_2) + 2\beta e^{-\beta(v+x_2)}] & x_2 \le v < x_1 \\[2ex]
\frac{\alpha\beta}{\alpha^2-\beta^2} [(\alpha+\beta)(e^{-\beta(x_1+x_2-v)} - e^{-\alpha(x_1+x_2-v)}) + (\alpha-\beta)(e^{-\alpha(x_1-x_2+v)} & \\
\qquad + e^{-\beta(v-x_1+x_2)}) - 2e^{-\alpha x_1-\beta x_2}(\alpha e^{-\alpha v}-\beta e^{-\beta v})] & \\
 & x_1 \le v < x_1+x_2 \\[2ex]
\frac{\alpha\beta}{\alpha^2-\beta^2} [(\alpha-\beta)(e^{-\alpha(x_1-x_2+v)} + e^{-\beta(v-x_1+x_2)}) - 2e^{-\alpha x_1-\beta x_2} & \\
\qquad (\alpha e^{-\alpha v}-\beta e^{-\beta v})] & x_1+x_2 \le v < \infty
\end{cases}
\tag{A.3.1}
$$

where $x_1 \ge x_2$ and $\alpha \ne \beta$.

Proof: Given that $f(a) = \alpha e^{-\alpha a}$, $f(b) = \beta e^{-\beta b}$, define A, B as follows

$$A = (x_1 - a) \quad , \quad B = (x_2 - b) \; .$$

Hence, the probability density functions are given as

$$f(A) = \alpha e^{\alpha(A+x_1)} \quad , \qquad f(B) = \beta e^{\beta(B+x_2)}$$

To obtain the probability density function of the absolute values $|A|$ and $|B|$, the procedure is developed for $|A|$, and $|B|$ will be obtained by replacing its parameter with the parameter of $|A|$. Two cases may occur:

Case I: $0 < z \leq x_1$

$$\Pr(|A| \leq z) = F(z) = \int_{-z}^{z} \alpha e^{-\alpha(A+x_1)} dA$$

$$= e^{-\alpha x_1}[e^{\alpha z} - e^{-\alpha z}] \tag{A.3.2}$$

Case II: $z > x_1$

$$\Pr(|A| \leq z) = \int_{-x_1}^{z} \alpha e^{-\alpha(A+x_1)} dA$$

$$= 1 - e^{-\alpha(x_1+z)} \tag{A.3.3}$$

From (A.3.2) and (A.3.3), the probability density function is obtained by differentiating both equations with respect to z to get,

$$f(|A|) = \begin{cases} \alpha e^{-\alpha x_1}\,[e^{\alpha|A|}+e^{-\alpha|A|}] & , 0 < |A| \leq x_1 \\ \alpha e^{-\alpha x_1}\,[e^{-\alpha|A|}] & , x_1 < |A| < \infty \end{cases} \qquad (A.3.4)$$

Similarly,

$$f(|B|) = \begin{cases} \beta e^{-\beta x_2}\,[e^{\beta|B|}+e^{-\beta|B|}] & , 0 < |B| \leq x_2 \\ \beta e^{-\beta x_2}\,[e^{-\beta|B|}] & , x_2 < |B| < \infty \end{cases} \qquad (A.3.5)$$

The joint probability density function is obtained from (A.3.4) and (A.3.5),

$$f(|A|, |B|) = f(|A|)\cdot f(|B|) \qquad (A.3.6)$$

To obtain the marginal density function g(v), the following transformation is performed. Let

$$L_1 = |A| + |B|$$

and

$$L_2 = |B|$$

Hence $|A| = L_1 - L_2$, $|B| = L_2$ and the Jacobian of the transformation, $|J| = 1$. The joint density function of L_1, L_2, $g(L_1,L_2)$, is given by

$$g(L_1,L_2) = \begin{cases} \alpha\beta e^{-\alpha x_1-\beta x_2}[e^{\alpha L_1-\alpha L_2+\beta L_2}+e^{-\alpha L_1+\alpha L_2+\beta L_2}+e^{\alpha L_1-\alpha L_2-\beta L_2} \\ \qquad + e^{-\alpha L_1+\alpha L_2-\beta L_2}] & , 0 < L_2 \le x_2 \\ & \quad L_2 < L_1 < x_1+L_2 \\ \alpha\beta e^{-\alpha x_1-\beta x_2}[e^{-\alpha L_1+\alpha L_2+\beta L_2}+e^{-\alpha L_1+\alpha L_2-\beta L_2} & \\ & , 0 < L_2 < x_2 \\ & \quad L_2+x_1 < L_1 < \infty \\ \alpha\beta e^{-\alpha x_1-\beta x_2}[e^{\alpha L_1-\alpha L_2-\beta L_2}+e^{-\alpha L_1+\alpha L_2-\beta L_2}] & \\ & , x_2 < L_2 \\ & \quad L_2 \le L_1 < x_1+L_2 \\ \alpha\beta e^{-\alpha x_1-\beta x_2}[e^{-\alpha L_1+\alpha L_2-\beta L_2}] & , x_2 < L_2 \\ & \quad L_2+x_1 < L_1 \end{cases}$$

(A.3.7)

The marginal density function of the rectilinear distance $v = L_1$ is derived by integrating (A.3.7) with respect to L_2, i.e.,

$$g_{L_1}(v) = \int_0^{L_1} g(L_1,L_2)\, dL_2 \qquad \text{(A.3.8)}$$

The evaluation of the integral in (A.3.8) is not a simple one, but the idea is to integrate over the four areas depicted in Figure A.1 under the assumption that $x_1 \ge x_2$.

Integrating over area I,

1. $\int_0^{L_1} g(L_1,L_2)\, dL_2 \quad , 0 < L_1 < x_2$

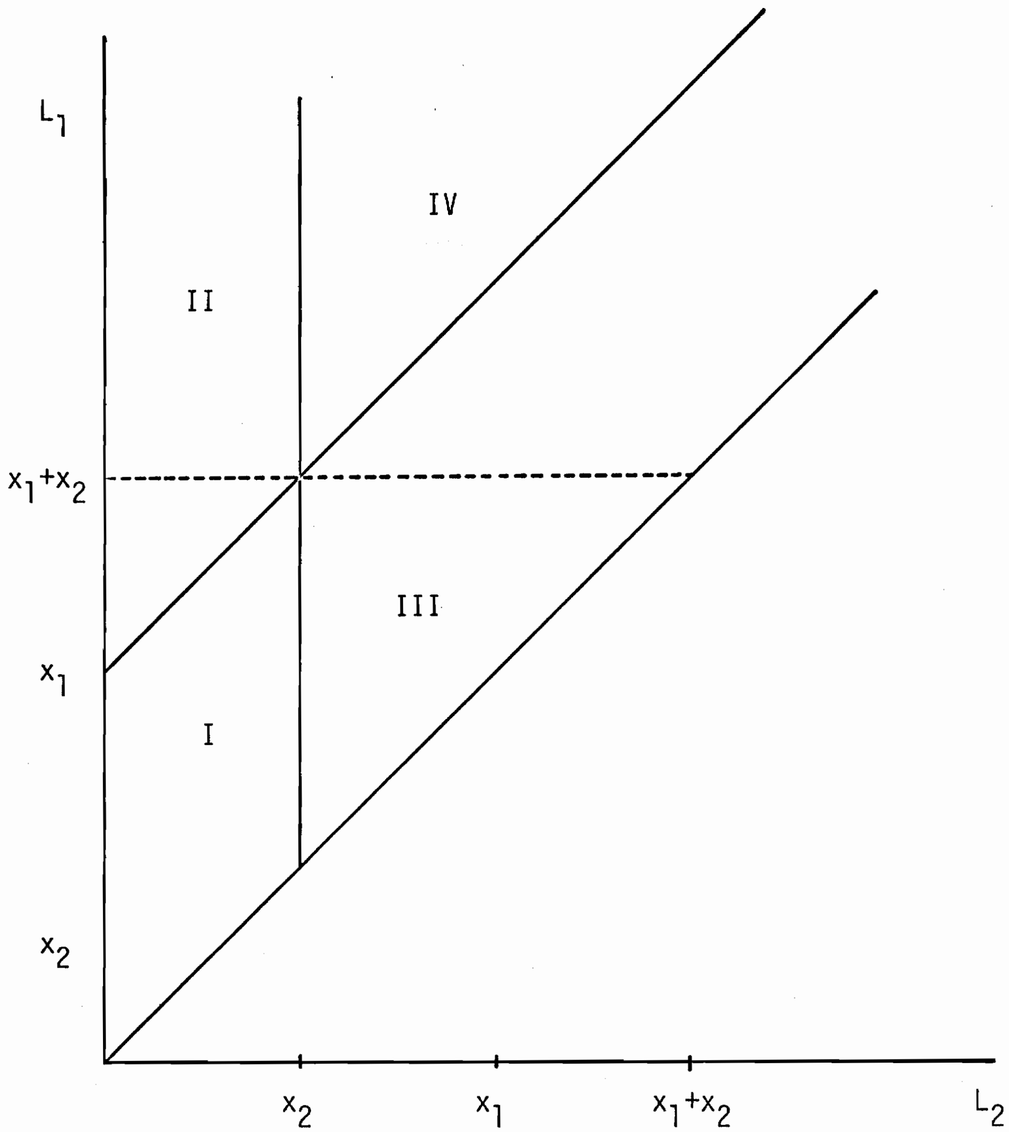

Figure A.1. Integrating the joint density function, $g(L_1,L_2)$.

2. $\int_{0}^{x_1} g(L_1,L_2)dL_2 \quad , x_2 \leq L_1 < x_1$

3. $\int_{L_1-x_1}^{x_2} g(L_1,L_2)dL_2 \quad , x_1 \leq L_1 < x_1+x_2$

Integrating over area II,

1. $\int_{0}^{L_1-x_1} g(L_1,L_2)dL_2 \; , x_1 < L_1 < x_1+x_2$

2. $\int_{0}^{x_2} g(L_1,L_2)dL_2 \quad , x_1+x_2 \leq L_1 < \infty$

Integrating over area III,

1. $\int_{x_2}^{L_1} g(L_1,L_2)dL_2 \quad , x_2 < L_1 < x_1+x_2$

2. $\int_{L_1-x_1}^{L_1} g(L_1,L_2)dL_2 \quad , x_1+x_2 \leq L_1 < \infty$

Integrating over area IV,

$$\int_{x_2}^{L_1-x_1} g(L_1,L_2)dL_2 \quad , x_1+x_2 < L_1 < \infty$$

Combining all probabilities within the same interval results in the probability density function of g(v) given by (A.3.1).

The distribution function $F(\xi) = Pr(v \leq \xi)$ is obtained by integrating (A.3.1) with respect to v. Hence,

$$F(\xi) = \begin{cases} F_1(\xi) = \frac{4\alpha\beta}{\alpha^2-\beta^2} e^{-\alpha x_1-\beta x_2}[\cosh(\alpha\xi)-\cosh(\beta\xi)] & , 0 \leq \xi < x_2 \\[2ex] F_2(\xi) = F_1(x_2) + \frac{2\alpha\beta}{\alpha^2-\beta^2} e^{-\alpha x_1}[e^{-\beta x_2}(2\cosh(\alpha\xi)-e^{-\beta\xi}) & \\ \qquad - \cosh(\alpha(\xi-x_2)) - \frac{\beta}{\alpha}\sinh(\alpha(\xi-y))] & , x_2 \leq \xi < x_1 \\[2ex] F_3(\xi) = F_2(x_1) + \frac{2\alpha\beta}{\alpha^2-\beta^2} [e^{-\beta x_2}\cosh(\beta(\xi-x_1)) - e^{-\alpha x_1}\cosh(\alpha(\xi-x_2)) & \\ \qquad + \frac{\alpha}{\beta} e^{-\beta x_2}\sinh(\beta(\xi-x_1)) - \frac{\beta}{\alpha} e^{-\alpha x_1}\sinh(\alpha(\xi-x_2)) & \\ \qquad + e^{-\alpha x_1-\beta x_2}(e^{-\alpha\xi}-e^{-\beta\xi})] & , x_1 \leq \xi < x_1+x_2 \\[2ex] F_4(\xi) = F_3(x_1+x_2) + \frac{\alpha\beta}{\alpha^2-\beta^2} [-\frac{(\alpha-\beta)}{\alpha} e^{-\alpha(x_1-x_2+\xi)} & \\ \qquad -\frac{(\alpha-\beta)}{\beta} e^{-\beta(x_2-x_1+\xi)} + 2e^{-\alpha x_1-\beta x_2}(e^{-\alpha\xi}-e^{-\beta\xi})] & \\ & , x_1+x_2 \leq \xi < \infty \end{cases}$$

$F(\xi)$ is written as a function of x_1 and x_2. Hence, PA.3 is written as,

$$\overline{\text{PA}}.3 \quad \underset{x_1,x_2}{\text{minimize}}\ f(X) = \sum_{i=1}^{m} w_i[x_1+x_2+2(\frac{e^{-\alpha_i x_1}}{\alpha_i} + \frac{e^{-\beta_i x_2}}{\beta_i}) - (\frac{\alpha_i+\beta_i}{\alpha_i\beta_i})]$$

subject to: $F(\xi_i) \geq \gamma_i$ for all i, i = 1, ..., m

$x_1, x_2 \geq 0$

where γ_i is the service level, $0 \leq \gamma_i \leq 1$.

The probability distribution $F(\xi_i)$ can be determined if the values of ξ_i, x_1, and x_2 are known. To solve $\overline{P}$A.3 any one of the existing iterative methods for nonlinear programming may be employed. The objective function is a strictly convex function, but the constraints set is not a convex set which leads to the fact that a global optimum is not guaranteed. Unfortunately, the functions $F(\xi_i)$ are not concave for all i. However, under certain conditions the sufficient condition for a global optimum may be obtained.

Lemma 2.8.1: If the local optimal solution obtained is such that $x_1^*+x_2^* \leq \xi_i$ for all i, then the local optimum is a global optimum.

Proof: If $x_1^*+x_2^* \leq \xi_i$ for all i, then

$$F(\xi_i) = F_4(\xi_i) = K + \frac{\alpha\beta}{\alpha^2-\beta^2} [-\frac{(\alpha-\beta)}{\alpha} e^{-\alpha(x_1-x_2+\xi)} - \frac{(\alpha-\beta)}{\beta} e^{-\beta(x_2-x_1+\xi)}$$

$$+ 2e^{-\alpha_1 x_1-\beta x_2} (e^{-\alpha\xi}-e^{-\beta\xi})]$$

where k is a constant with a non-negative value. Let $\alpha > \beta$, hence $e^{-\alpha\xi} < e^{-\beta\xi}$ and $e^{-\alpha\xi}$, $e^{-\beta\xi}$ are of positive values. Thus,

$$F(\xi_i) = K - K_1 e^{-\alpha(x_1-x_2)} - K_2 e^{-\beta(x_2-x_1)} - K_3 e^{-\alpha_1 x_1-\beta x_2} \tag{A.3.10}$$

Each exponential function is convex over x_1, x_2. Hence, the negative combination forms a concave function. Therefore, $F(\xi_i)$ is a concave function over x_1, x_2 if the above conditions are satisfied. From the Kuhn-Tucker sufficient condition, this implies that the local optimum is global.

A.4 Exponentially Distributed Line Location Problem

If the exponential distribution is used for the probability density function of the existing locations a_i, the problem may be formulated as,

$$\text{PA.4} \quad \underset{x_1}{\text{minimize}}\ f(x_1) = \sum_{i=1}^{m} w_i\left(x_1 + \frac{2e^{-\alpha_i x_1} - 1}{\alpha_i}\right)$$

$$\text{subject to:} \quad \Pr(|x_1 - a_i| \leq \xi_i) \geq \gamma_i \qquad \text{for all } i = 1, \ldots, m$$

$$x_1 \geq 0$$

In (A.3.2) and (A.3.3), the probability distribution of $|x_1 - a|$ is developed and is given by,

$$F(\xi) = \begin{cases} 2e^{-\alpha x_1} \sinh(\alpha\xi) & , \ 0 \leq \xi > x_1 \\ 1 - e^{-\alpha(x_1 + \xi)} & , \ x_1 \leq \xi < \infty \end{cases} \tag{A.4.1}$$

By observing the behavior of $F(\xi)$ in (A.4.1) with respect to $x_1 \in E^1$, the function is convex in the interval $0 \leq \xi < x_1$ and is a concave function in the interval $x_1 \leq \xi$. Hence, there is an inflection point at $\xi = x_1$.

From (A.4.1), the deterministic equivalence of PA.4 is

$$\overline{\text{PA}}.4 \quad \underset{x_1}{\text{minimize}}\ f(x_1) = \sum_{i=1}^{m} w_i\left(x_1 + \frac{2e^{-\alpha_i x_1} - 1}{\alpha_i}\right)$$

$$\text{subject to:} \quad F(\xi_i) \geq \gamma_i \qquad \text{for all } i, \ i = 1, \ldots, m$$

The problem $\overline{P}A.4$ is solved optimally using an iterative convex programming algorithm, e.g., SUMT, but the Kuhn-Tucker necessary condition guarantees a local optimum only. The local optimum is a global one if the following condition is satisfied.

Lemma A.4.1: Given a local optimum solution x^*, if $x_i^* \leq \xi_i$ for all i, then the local optimum is a global.

Proof: The proof is direct since, when $x_i^* \leq \xi_i$ for all i, problem $\overline{P}A.4$ becomes a well behaved convex programming problem where the global optimum is certain.

APPENDIX B

```
C
C     RECTILINEAR-DISTANCE SINGLE FACILITY LOCATION PROBLEM
C     WITH A BIVARIATE NORMAL DENSITY FUNCTION
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION UA(5),UB(5),SIGA(5),SIGB(5),W(5),X1(20),X2(20)
      READ(5,100)M,ITMAX
  100 FORMAT(5X,2I5)
      DO 1 I=1,M
    1 READ(5,101)UA(I),UB(I),SIGA(I),SIGB(I),W(I)
  101 FORMAT (5X,5F10.5)
      DO 2 I=1,M
    2 WRITE(6,200)I,UA(I),UB(I),SIGA(I),SIGB(I),W(I)
  200 FORMAT(2X,'I=',I2,3X,'UA=',F10.5,3X,'UB=',F10.5,3X,'SIGA=',F10.5,
     13X,'SIGB=',F10.5,3X,'W=',F10.5/)
      WRITE(6,301)
  301 FORMAT('1')
      WRITE(6,202)
  202 FORMAT(10X,'ITER',10X,'X1',15X,'DX1',15X,'X2',15X,'DX2'/)
C
C     CALCULATION OF THE INITIAL SOLUTION
C
      SUM1N=0.D0
      SUM1D=0.D0
      SUM2N=0.D0
      SUM2D=0.D0
      DO 7 I=1,M
      A1=W(I)/SIGA(I)
      A2=W(I)/SIGB(I)
      SUM1N=SUM1N+A1*UA(I)
      SUM1D=SUM1D+A1
      SUM2N=SUM2N+A2*UB(I)
```

```
    7 SUM2D=SUM2D+A2
      X1(1)=SUM1N/SUM1D
      X2(1)=SUM2N/SUM2D
      SQP2=1.2533141D0
      DO 8 IT=1,ITMAX
      IT1=IT+1
      SUM1N=0.D0
      SUM1D=0.D0
      SUM2N=0.D0
      SUM2D=0.D0
      DO 9 I=1,M
      A1=(X1(IT)-UA(I))/SIGA(I)
      A1A1=-0.5D0*A1*A1
      A1S2=A1/1.414214D0
      IF(A1S2)54,55,55
   54 A1S2=-A1S2
      SUM1N=SUM1N-W(I)*DERF(A1S2)
      GO TO 56
   55 SUM1N=SUM1N+W(I)*DERF(A1S2)
   56 SUM1D=SUM1D+W(I)*DEXP(A1A1)/SIGA(I)
      B1=(X2(IT)-UB(I))/SIGB(I)
      B1B1=-0.5D0*B1*B1
      B1S2=B1/1.414214D0
      IF(B1S2)64,65,65
   64 B1S2=-B1S2
      SUM2N=SUM2N-W(I)*DERF(B1S2)
      GO TO 9
   65 SUM2N=SUM2N+W(I)*DERF(B1S2)
    9 SUM2D=SUM2D+W(I)*DEXP(B1B1)/SIGB(I)
      X1(IT1)=X1(IT)-SQP2*SUM1N/SUM1D
      X2(IT1)=X2(IT)-SQP2*SUM2N/SUM2D
      DX1=X1(IT1)-X1(IT)
```

```
      DX2=X2(IT1)-X2(IT)
      WRITE(6,203)IT   ,X1(IT1),DX1,X2(IT1),DX2
  203 FORMAT(11X,I2,4(4X,D14.7))
      IF(DABS(DX1).LT.0.0001D0.AND.DABS(DX2).LT.0.0001D0)GO TO 10
    8 CONTINUE
      WRITE(6,204)
  204 FORMAT(10X,'UNSUCCESSFUL CONVERGENCE'//)
      GO TO 12
   10 WRITE(6,205)
  205 FORMAT(10X,'SUCCESSFUL CONVERGENCE'//)
```

```
C
C     EVALUATING THE OBJECTIVE FUNCTION FOR THE OPTIMAL SOLUTION
C     AND THE SOLUTION OBTAINED FROM THE DETERMINISTIC PROBLEM
C
      L=1.0
  400 CONTINUE
      SIGN=1.0
      SUMM=0.0
      DO 20 I =1,M
      A2=X1(IT1)-UA(I)
      Z1=A2/SIGA(I)
      Z1Z1=-0.5D0*Z1*Z1
      Z1S2=Z1/1.414214D0
      IF(Z1S2)21,22,22
   21 Z1S2=-Z1S2
      SIGN=-1.0
   22 S1 = (A2*DERF(Z1S2)*SIGN)+(0.79788456D0*SIGA(I)*DEXP(Z1Z1))
      B2=X2(IT1)-UB(I)
      Z2=B2/SIGB(I)
      Z2Z2=-0.5D0*Z2*Z2
      Z2S2=Z2/1.414214D0
      IF(Z2S2)23,24,24
   23 Z2S2=-Z2S2
      SIGN=-1.0
   24 S2 = (B2*DERF(Z2S2)*SIGN)+(0.79788456D0*SIGB(I)*DEXP(Z2Z2))
      S = S1+S2
      FUNI =W(I)*S
   20 SUMM =SUMM+FUNI
      IF(L.EQ.2) GO TO 601
      WRITE(6,600)
  600 FORMAT(10X,'PROBABILISTIC SOLUTION :')
      WRITE(6,207)X1(IT1),X2(IT1),SUMM
```

```
207 FORMAT  (/10X,'OPTIMAL X1 =',F10.5/10X,'OPTIMAL X2 ='
   1,F10.5/16X,'F(X) =',F10.5//)
    IF(L.EQ.1) GO TO 603
601 WRITE(6,602)
602 FORMAT(10X,'DETERMINISTIC SOLUTION :')
    WRITE(6,604) X1(IT1),X2(IT1),SUMM
604 FORMAT  (/10X,'OPTIMAL X1 =',F10.5/10X,'OPTIMAL X2 ='
   1,F10.5/16X,'F(X) =',F10.5/1H1)
    IF(L.EQ.2.) GO TO 12
603 X1(IT1)=0.
    X2(IT1)=0.
    L=L+1.
    GO TO 400
 12 STOP
    END
```

```
C
C     EUCLIDEAN-DISTANCE SINGLE FACILITY LOCATION PROBLEM
C     WITH A BIVARIATE NORMAL DENSITY FUNCTION
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION UA(5),UB(5),SIGA(5),SIGB(5),W(5),X1(40),X2(40)
      READ(5,100)M,ITMAX
  100 FORMAT(5X,2I5)
      DO 1 I=1,M
    1 READ(5,101)UA(I),UB(I),SIGA(I),SIGB(I),W(I)
  101 FORMAT (5X,5F10.5)
      DO 2 I=1,M
    2 WRITE(6,200)I,UA(I),UB(I),SIGA(I),SIGB(I),W(I)
  200 FORMAT(2X,'I=',I2,3X,'UA=',F10.5,3X,'UB=',F10.5,3X,'SIGA=',F10.5,
     13X,'SIGB=',F10.5,3X,'W=',F10.5/)
      WRITE(6,202)
  202 FORMAT(10X,'ITER',10X,'X1',15X,'DX1',15X,'X2',15X,'DX2'/)
      DO 13 IT=1,ITMAX
      SUM1N=0.D0
      SUM1D=0.D0
      SUM2N=0.D0
      IT1=IT+1
      DO 14 I=1,M
      AM=1.D0
      X1(1)=UA(I)
      X2(1)=UB(I)
      SLMDA=(X1(IT)-UA(I))*(X1(IT)-UA(I))+(X2(IT)-UB(I))*(X2(IT)-UB(I))
      Z= 0.5D0*SLMDA/(SIGA(I)*SIGA(I))
      CALL HYPR(1.5D0,2.D0,Z,AM,AN)
      Z1=-Z
      A1=W(I)*AN*DEXP(Z1)/SIGA(I)
      SUM1N=SUM1N+A1*UA(I)
```

```
      SUM1D=SUM1D+A1
   14 SUM2N=SUM2N+A1*UB(I)
      X1(IT1)=SUM1N/SUM1D
      X2(IT1)=SUM2N/SUM1D
      DX1=X1(IT1)-X1(IT)
      DX2=X2(IT1)-X2(IT)
      WRITE(6,203)IT   ,X1(IT1),DX1,X2(IT1),DX2
  203 FORMAT(11X,I2,4(4X,D14.7))
      IF(DABS(DX1).LT.0.0001D0.AND.DABS(DX2).LT.0.0001D0)GO TO 15
   13 CONTINUE
      WRITE(6,204)
  204 FORMAT(10X,'UNSUCCESSFUL CONVERGENCE'//)
      GO TO 20
   15 WRITE(6,205)
  205 FORMAT(10X,'SUCCESSFUL CONVERGENCE'//)
```

```
C
C     EVALUATING THE OBJECTIVE FUNCTION FOR THE OPTIMAL SOLUTION
C     AND THE SOLUTION OBTAINED FROM THE DETERMINISTIC PROBLEM
C
      L=1.0
  400 CONTINUE
      SUM=0.0
      DO 16 I=1,M
      AM=1.0
      SLMDA=(X1(IT1)-UA(I))*(X1(IT1)-UA(I))+(X2(IT1)-UB(I))
     1*(X2(IT1)-UB(I))
      Z=0.5D0*SLMDA/(SIGA(I)*SIGA(I))
      CALL HYPR(1.5D0,1.D0,Z,AM,AN)
      Z1=-Z
      A2=1.2533141D0*W(I)*SIGA(I)*AN*DEXP(Z1)
   16 SUM=SUM+A2
      IF(L.EQ.2) GO TO 601
      WRITE(6,600)
  600 FORMAT(10X,'PROBABILISTIC SOLUTION :')
      WRITE(6,206)X1(IT1),X2(IT1),SUM
  206 FORMAT  (/10X,'OPTIMAL X1 =',F10.5/10X,'OPTIMAL X2 =',
     1F10.5/16X,'F(X) =',F10.5/)
      IF(L.EQ.1) GO TO 603
  601 WRITE(6,602)
  602 FORMAT(10X,'DETERMINISTIC SOLUTION :')
      WRITE(6,604) X1(IT1),X2(IT1),SUM
  604 FORMAT  (/10X,'OPTIMAL X1 =',F10.5/10X,'OPTIMAL X2 ='
     1F10.5/16X,'F(X) =',F10.5/1H1)
      IF(L.EQ.2.) GO TO 20
  603 X1(IT1)=4.
      X2(IT1)=2.
      L=L+1.
```

```
      GO TO 400
   20 STOP
      END
```

```
      SUBROUTINE HYPR(A,B,Z,AM,AN)
      IMPLICIT REAL*8(A-H,O-Z)
      AM1=1.D0
      TERM=1.D0
      DO 1 I=1,20
      V=DFLOAT(I-1)
      TERM=(A+V)*Z*TERM/((B+V)*DFLOAT(I))
      AM=AM+TERM
      DAM=AM-AM1
      AN=AM
      IF(DABS(DAM).LT.0.0001D0)GO TO 2
      AM1=AM
    1 CONTINUE
    2 RETURN
      END
```

```
C
C     EUCLIDEAN DISTANCE MULTIFACILITY LOCATION PROBLEM
C     WITH A BIVARIATE NORMAL DENSITY FUNCTION
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION UA(6),UB(6),SIG(6),W(6,6),V(6,6),X1(6,40),X2(6,40),
     1DX1(6),DX2(6)
      READ(5,100)M,N,ITMAX
  100 FORMAT(5X,3I5)
      DO 1 I=1,M
    1 READ(5,101)UA(I),UB(I),SIG(I)
  101 FORMAT(5X,3F10.5)
      DO 2 J=1,N
    2 READ(5,102)(V(J,L),L=1,J)
  102 FORMAT(5X,6F10.5)
      DO 3 L=1,N
      IF(L.EQ.N)GO TO 17
      L1=L+1
      DO 3 J=L1,N
      V(L,J)=V(J,L)
    3 CONTINUE
   17 DO 4 J=1,N
    4 READ(5,102)(W(J,I),I=1,M)
      DO 5 I=1,M
    5 WRITE(6,200)I,UA(I),UB(I),SIG(I)
  200 FORMAT(2X,'I=',I2,3X,'UA=',F10.5,3X,'UB=',F10.5,3X,'SIG=',F10.3/)
      WRITE(6,201)
  201 FORMAT(/2X,'NEW FACILITY INTERACTIONS')
      DO 6 J=1,N
    6 WRITE(6,102)(V(J,L),L=1,N)
      WRITE(6,202)
  202 FORMAT(/2X,'NEW AND EXISTING FACILITY INTERACTIONS')
```

```
      DO 7 J=1,N
    7 WRITE(6,102)(W(J,I),I=1,M)
      DO 19 J=1,N
      X1(J,1)=DFLOAT(J)*2.
      X2(J,1)=DFLOAT(J)*2.
   19 CONTINUE
      DO 8 IT=1,ITMAX
      IT1=IT+1
      SUM1D=0.0
      SUM11N=0.0
      SUM12N=0.0
      SUM2D=0.0
      SUM21N=0.0
      SUM22N=0.0
      TOTMAX=0.0
      DO 9 J=1,N
      DO 10 K=1,N
      IF(K.NE.J)GO TO 13
      IF(K.EQ.J)D=1.0
      GO TO 12
   13 D=DSQRT((X1(J,IT)-X1(K,IT))*(X1(J,IT)-X1(K,IT))+(X2(J,IT)-X2(K,
     1IT))*(X2(J,IT)-X2(K,IT)))
   12 A1=V(J,K)/D
      SUM1D=SUM1D+A1
      SUM11N=SUM11N+A1*X1(K,IT)
      SUM12N=SUM12N+A1*X2(K,IT)
   10 CONTINUE
      DO 11 I=1,M
      AM=1.0
      SLMDA=(X1(J,IT)-UA(I))*(X1(J,IT)-UA(I))+(X2(J,IT)-UB(I))*
     1(X2(J,IT)-UB(I))
      Z= 0.50*SLMDA/(SIG(I)*SIG(I))
```

```
      CALL HYPR(1.5D0,2.D0,Z,AM,AN)
      Z1=-Z
      A2=W(J,I)*AN*0.62665707D0*DEXP(Z1)/SIG(I)
      SUM2D=SUM2D+A2
      SUM21N=SUM21N+A2*UA(I)
      SUM22N=SUM22N+A2*UB(I)
   11 CONTINUE
      SUMD=SUM1D+SUM2D
      SUM1N=SUM11N+SUM21N
      SUM2N=SUM12N+SUM22N
      X1(J,IT1)=SUM1N/SUMD
      X2(J,IT1)=SUM2N/SUMD
      DX1(J)=DABS(X1(J,IT1)-X1(J,IT))
      DX2(J)=DABS(X2(J,IT1)-X2(J,IT))
      DELMAX=DMAX1(DX1(J),DX2(J))
      TOTMAX=DMAX1(DELMAX,TOTMAX)
      WRITE(6,250)X1(J,IT1),DX1(J),X2(J,IT1),DX2(J)
  250 FORMAT(/5X,4(4X,D14.7))
    9 CONTINUE
      IF(TOTMAX.LT.0.0001D0)GO TO 20
      WRITE(6,251)
  251 FORMAT(///)
    8 CONTINUE
      WRITE(6,252)
  252 FORMAT(5X,'UNSUCCESSFUL CONVERGANCE'/)
      GO TO 21
   20 WRITE(6,253)
  253 FORMAT(5X,'SUCCESSFUL CONVERGANCE'/)
```

```
C
C     EVALUATING THE OBJECTIVE FUNCTION FOR THE OPTIMAL SOLUTION
C     AND THE SOLUTION OBTAINED FROM THE DETERMINISTIC PROBLEM
C
      L=1.0
  400 CONTINUE
      SUM=0.0
      SUM1=0.0
      SUM2=0.0
      DO 30 J=1,N
      DO 31 K=J,N
      D =DSQRT((X1(J,IT1)-X1(K,IT1))*(X1(J,IT1)-X1(K,IT1))
     1+(X2(J,IT1)-X2(K,IT1))*(X2(J,IT1)-X2(K,IT1)))
      A1=V(J,K)*D
      SUM1=SUM1+A1
   31 CONTINUE
      DO 32 I=1,M
      AM=1.
      SLMDA= (X1(J,IT1)-UA(I))*(X1(J,IT1)-UA(I))+
     1(X2(J,IT1)-UB(I))*(X2(J,IT1)-UB(I))
      Z=0.5D0*SLMDA/(SIG(I)*SIG(I))
      CALL HYPR(1.5D0,1.D0,Z,AM,AN)
      Z1=-Z
      A2=1.2533141D0*W(J,I)*AN*DEXP(Z1)*SIG(I)
   32 SUM2=SUM2+A2
   30 CONTINUE
      SUM=SUM1+SUM2
      IF(L.EQ.2) GO TO 601
      WRITE(6,600)
  600 FORMAT(/5X,'PROBABILISTIC SOLUTION :')
  604 WRITE(6,500)
  500 FORMAT (/10X,'J',10X,'X1',10X,'X2'/)
```

```
      DO 300 J=1,N
  300 WRITE(6,501)J,X1(J,IT1),X2(J,IT1)
  501 FORMAT(9X,I2,6X,F10.5,2X,F10.5)
      WRITE(6,502)SUM
  502 FORMAT(//6X,'F(X) =',F10.5//)
      IF(L.EQ.2) GO TO 605
      IF(L.EQ.1) GO TO 603
  601 WRITE(6,602)
  602 FORMAT(5X,'DETERMINISTIC SOLUTION :')
      GO TO 604
  603 X1(1,IT1)=8.0
      X2(1,IT1)=7.0
      X1(2,IT1)=8.0
      X2(2,IT1)=7.0
      L=L+1.0
      GO TO 400
  605 WRITE(6,606)
  606 FORMAT('1')
   21 STOP
      END
```

```
      SUBROUTINE HYPR(A,B,Z,AM,AN)
      IMPLICIT REAL*8(A-H,O-Z)
      AM1=1.D0
      TERM=1.D0
      DO 1 I=1,20
      V=DFLOAT(I-1)
      TERM=(A+V)*Z*TERM/((B+V)*DFLOAT(I))
      AM=AM+TERM
      DAM=AM-AM1
      AN=AM
      IF(DABS(DAM).LT.0.0001D0)GO TO 2
      AM1=AM
    1 CONTINUE
    2 RETURN
      END
```

```
C
C     RECTILINEAR-DISTANCE MULTIFACILITY LOCATION PROBLEM
C     WITH A BIVARIATE NORMAL DENSITY FUNCTION
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION UA(6),SIGA(6),W(6,6),X(6),F(6),UB(6),SIGB(6)
      READ(5,100)M,N
  100 FORMAT(5X,2I5)
      DO 1 I=1,M
    1 READ(5,101)UA(I),UB(I),SIGA(I),SIGB(I)
  101 FORMAT(5X,4F10.5)
      DO 11 J=1,N
   11 READ(5,102)(W(J,I),I=1,M)
  102 FORMAT(5X,6F10.5)
      DO 2 I=1,M
    2 WRITE(6,200)I,UA(I),UB(I),SIGA(I),SIGB(I)
  200 FORMAT(2X,'I =',I2,3X,'UA =',F10.5,3X,'UB =',F10.5,3X,
     1'SIGA =',F10.5,3X,'SIGB =',F10.5/)
      WRITE(6,205)
  205 FORMAT(/2X,'NEW AND EXISTING FACILITY INTERACTIONS')
      DO 13 J=1,N
   13 WRITE(6,102)(W(J,I),I=1,M)
      WRITE(6,202)
  202 FORMAT(/21X,'X11',15X,'F11',15X,'X21',15X,'F21'/)
      K =1.0
   14 DO 22 J=1,N
   22 X(J)=0.0
    9 CONTINUE
      DO 23 J=1,N
   23 F(J)=0.0
      DO 7 J=1,N
      DO 7 I=1,M
```

```
      SIGN=1.0
      AR=X(J)-UA(I)
      AR1=AR/SIGA(I)
      AR1S=-0.5D0*AR1*AR1
      B1=0.79788456D0*SIGA(I)*DEXP(AR1S)
      AR2=AR1/1.414214D0
      IF(AR2)50,51,51
   50 AR2=-AR2
      SIGN=-1.D0
   51 B2=AR*W(J,I)*DERF(AR2)*SIGN
      B=B1+B2
      F(J)=F(J)+B
    7 CONTINUE
      WRITE(6,203)X(1),F(1),X(2),F(2)
  203 FORMAT(11X,4(4X,D14.7))
      IF(K.GT.1.D0) GO TO 25
      X(1)=X(1)+0.2D0
      X(2)=X(2)+0.2D0
      IF(X(1).LE.1.6D0.AND.X(2).LE.1.6D0) GO TO 9
      WRITE(6,206)
  206 FORMAT(//21X,'X12',15X,'F12',15X,'X22',15X,'F22'/)
      DO 20 I=1,M
      UA(I)=UB(I)
   20 SIGA(I)=SIGB(I)
      K=K+1
      GO TO 14
   25 X(1)=X(1)+0.1D0
      X(2)=X(2)+0.1D0
      IF(X(1).LE.0.8D0.AND.X(2).LE.0.8D0) GO TO 9
      WRITE(6,600)
  600 FORMAT('1')
      STOP
```

END

VITA

Adel Ahmed Aly was born October 30, 1944 in Fayoum, Egypt. He received his elementary and secondary education in Cairo, Egypt, graduating from Orman Secondary School in 1961.

In 1966, he received the Bachelor of Science degree in Production Engineering from the University of Cairo.

From 1966 to 1967, the author was employed by the Management Consulting Center, National Institute of Management Development, Cairo, as a junior consultant. From 1967 to 1969, he was teaching at the University of Cairo. In 1972, he received the Master of Science degree with a major in Industrial Engineering from North Carolina State University at Raleigh. He held a teaching assistantship in the Industrial Engineering Department, North Carolina State University at Raleigh from 1969 to 1971. From 1972 to 1974, the author held teaching and research assistantships in the Department of Industrial Engineering and Operations Research, VPI & SU; he is presently employed as an Assistant Professor in the School of Industrial Engineering at the University of Oklahoma.

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PROBABILISTIC FORMULATIONS

OF SOME LOCATION PROBLEMS

by

Adel Ahmed Aly

(ABSTRACT)

The area of facilities location covers a wide variety of problems involving both public and private sector applications. To date, the study of location problems has been restricted primarily to deterministic formulations of the problem. The present research effort investigates the effect of random variation on the location decision.

Three location problems are considered: the single facility location problem, the multifacility location problem, and the emergency service location problem. The first two problems treated are defined as the generalized Weber problem, where the concern is to locate one or more new facilities in the plane relative to several existing facilities such that the expected total cost of item movements is minimized. The total cost function is considered to be a linear function of either the expected rectilinear or the Euclidean distance, as well as a quadratic function of the expected Euclidean distance.

In the generalized Weber problem the locations of the existing facilities and the item movement between facilities are considered to be random variables. Two expected total cost formulations are presented; the first involves the product of the random variables, weight and distance; the second involves the random sum of each individual distance traveled. For each formulation, possible applications are discussed,

theoretical properties are developed, and a solution procedure is provided. Each algorithm is programmed and optimal solutions are obtained for several example problems. A comparison between the probabilistic and deterministic solutions is provided. Both discretely and continuously distributed random variables are treated; however, for the case of continuously distributed random variables, the normal distribution is emphasized. Both constrained and unconstrained formulations are considered.

In formulating the emergency service facilities location problems which are studied, random variation is assumed to be present due to the assumption that the location of an incident is a random variable occuring uniformly over a given region. Both discrete space and continuous space formulations are considered. For the discrete case, a covering criterion is employed and the deterministic equivalent problem is solved as a set cover problem. For the continuous case, the problem is solved as a location-allocation problem. In all formulations, the rectilinear norm is used to measure the distance traveled. An example is solved for each case to illustrate the impact of probabilistic aspects on the location decision.