TRANSCENDENCE DEGREE IN POWER SERIES RINGS

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I. INTRODUCTION

Let D be a commutative integral domain (with identity) with quotient field K and denote by D[[X]] the ring of formal power series with coefficients in D. If J is an overring of D (that is, if $D \subsetneq J \subseteq K$), then we are interested in studying the relation between J[[X]] and D[[X]]. The questions considered in this dissertation are motivated by results obtained by Gilmer in [3] and by Sheldon in [8]. In [3], Gilmer determines necessary and sufficient conditions in order that D[[X]] and K[[X]] have the same quotient field. Specifically, Gilmer proves that the following conditions are equivalent if we take J = K:

(1) For each sequence $\{\xi_i\}_{i=1}^{\infty}$ of elements of J, there is a nonzero element d of D such that $d\xi_i \varepsilon D$ for each positive integer i.

(2) $J[[X]] \subseteq D[[X]]_{D^*}$, where D* is the multiplicative system of nonzero elements of D.

(3) D[[X]] and J[[X]] have the same quotient field.

Sheldon has observed [8, Theorem 3.8] that these same three conditions are equivalent if $J = D_S$ is any quotient overring of D. Further, if a is a nonzero element of D such that $\bigcap_{n=1}^{\infty} (a^n) = (0)$, then Sheldon shows [8, Theorem 2.1] that tr.d.(D[[X/a]]/D[[X]]) = ∞. (We shall write tr.d.(J[[X]]/D[[X]]) to denote the transcendence degree of the quotient field of J[[X]] over the quotient field of D[[X]].) As an immediate consequence of Sheldon's result, we see that if J = D[1/a], where a is some nonzero element of D, then either conditions (1)-(3) above hold, or tr.d.$(J[[X]]/D[[X]]) = \infty$. Thus, we are led to consider the following conditions:

(A)

(1) For each sequence $\{\xi_i\}_{i=1}^{\infty}$ of elements of J, there is a nonzero element d of D such that $d\xi_i \in D$ for each positive integer i.

(2) $J[[X]] \subseteq D[[X]]_{D*}$.

(3) D[[X]] and J[[X]] have the same quotient field.

(4) J[[X]] is algebraic over D[[X]].

(5) tr.d.$(J[[X]]/D[[X]]) < \infty$.

Our main goal is to determine when these five conditions are equivalent. In general, we have (1) $\leftrightarrow$ (2) $\rightarrow$ (3) $\rightarrow$ (4) $\rightarrow$ (5) (Theorem 1). As we have observed above, Sheldon's results show that these conditions are equivalent when J = D[1/a] for some nonzero element a of D.

The main result of this dissertation is Theorem 3, which shows that if D is integrally closed, then conditions (1)-(5) are equivalent for any overring J of D. In section IV, we give an example which shows that these conditions are not equivalent in general; specifically we show that (4) $\not\rightarrow$ (3). We prove a number of other results relating these five conditions. For example, if $J = D[\xi_1,\xi_2,\ldots,\xi_n]$ for some finite set $\{\xi_1,\xi_2,\ldots,\xi_n\}$ of elements of K, and if D and its integral closure $\bar{D}$ have the same complete integral closure, then conditions (1)-(5) are equivalent (Theorem 7). As a consequence, if D is

Noetherian, then conditions (1)-(5) are equivalent for finitely generated overrings J of D.

II. MAIN THEOREM

In this section we shall consider the case in which J is an arbitrary overring of D. The key result of this paper, Lemma 2, is a rather technical result, but yields our main theorem, Theorem 3, as an immediate corollary. Further results are obtained in Section III as corollaries of Lemma 2. Before proceeding with the proof of Lemma 2, we prove the following theorem.

THEOREM 1. _The following implications hold between the five conditions given in_ (A):

(1) $\leftrightarrow$ (2) $\rightarrow$ (3) $\rightarrow$ (4) $\rightarrow$ (5).

Further, condition (2) _implies that_ J _is contained in the complete integral closure of_ D.

Proof: (1) $\rightarrow$ (2): Let $\sum_{n=0}^{\infty} b_n X^n \in J[[X]]$. Then, by (1), there is a nonzero element d of D such that $db_i \in D$ for each i. It follows that $d\sum_{n=0}^{\infty} b_n X^n = \sum_{n=0}^{\infty} db_n X^n$ is an element of D[[X]]. Therefore $\sum_{n=0}^{\infty} b_n X^n \in D[[X]]_{D*}$, so that $J[[X]] \subseteq D[[X]]_{D*}$.

(2) $\rightarrow$ (1): Let $\{\xi_i\}_{i=1}^{\infty}$ be a sequence of elements of J. Then $\sum_{n=1}^{\infty} \xi_n X^n \in J[[X]] \subseteq D[[X]]_{D*}$ so there is an element d of D* such that $d\sum_{n=1}^{\infty} \xi_n X^n = \sum_{n=1}^{\infty} d\xi_n X^n \in D[[X]]$. Therefore $d\xi_i \in D$ for each positive integer i.

(2) $\to$ (3): If $J[[X]] \subseteq D[[X]]_{D^*}$, then clearly $J[[X]]$ is contained in the quotient field of $D[[X]]$. But since $D \subseteq J$, $D[[X]] \subseteq J[[X]]$, so that $D[[X]]$ is contained in the quotient field of $J[[X]]$. It follows that $D[[X]]$ and $J[[X]]$ have the same quotient field.

(3) $\to$ (4): Clear

(4) $\to$ (5): If $J[[X]]$ is algebraic over $D[[X]]$, then the transcendence degree of $J[[X]]$ over $D[[X]]$ is zero, hence finite.

Further, if (2) holds and if $\xi \in J$, then by (1), there is a nonzero element d of D such that $d\xi^n \in D$ for each positive integer n. Therefore ξ is almost integral over D, and hence J is contained in the complete integral closure of D.

In studying the relation between $J[[X]]$ and $D[[X]]$, our general approach is to consider a particular set $\{a_i\}_{i=1}^{\infty}$ of power series in $J[[X]]$, to assume that this set is algebraically dependent over $D[[X]]$, and then to consider the implications of this dependence relation on the coefficients of the a_i. Following Sheldon's lead [8], we note that the key step is defining the a_i to have "sufficiently large gaps." We now proceed to define such a sequence.

Let ω denote the set of positive integers, and let $h\colon \omega \times \omega \to \omega \cup \{0\}$ be a function satisfying the following two conditions:

1. $h(n,m) = 0$ if and only if $n > m$.
2. $h(n,m) > m \left[\sum_{j=1}^{m-1} \sum_{i=1}^{j} h(i,j) + \sum_{j=1}^{n-1} h(j,m) \right]$, if $n \leq m$.

The set $\{h(n,m) \mid n \leq m\}$ may be viewed as a sequence whose ordering is that induced by the reverse lexicographic order on $\omega \times \omega$. Thus, the function h is merely an infinite upper triangular matrix such that each nonzero term h(n,m) is larger than m times the sum of those nonzero entries preceding it in the sequence.

Let K denote the quotient field of D and denote the integral closure of D by $\bar{D}$. For a sequence $\{\xi_i\}_{i=1}^{\infty}$ of elements of K, we define a sequence $\{a_n\}_{n=1}^{\infty}$ of power series in $D[\{\xi_i\}_{i=1}^{\infty}][[X]]$ by $a_n = 1 + \sum_{j=1}^{\infty} (\xi_j X)^{h(n,j)}$.

With this notation, we state the following lemma.

LEMMA 2: <u>If the set</u> $\{a_n\}_{i=1}^{\infty}$ <u>is algebraically dependent over</u> D[[X]], <u>then there is a nonzero element</u> d, <u>in</u> D, <u>such that</u> $d\xi_i \in \bar{D}$ <u>for each</u> $i \in \omega$.

Before proving Lemma 2, we state and prove our main result, which is an immediate consequence of Lemma 2.

THEOREM 3: <u>If</u> D <u>is integrally closed</u>, <u>conditions</u> (1)-(5) <u>of</u> (A) <u>are equivalent</u>.

<u>Proof</u>: In view of Theorem 1, it suffices to show that (5) $\rightarrow$ (1). Thus suppose that tr.d.(J[[X]]/D[[X]]) is finite and let $\{\xi_i\}_{i=1}^{\infty}$ be a sequence of elements of J. The sequence $\{a_n\}_{n=1}^{\infty}$ as defined in Lemma 2 is in J[[X]] and must be algebraically dependent over D. Since D is integrally closed, Lemma 2 implies the existence of a nonzero element d of D such that $d\xi_i \in D$ for each $i \in \omega$. Therefore

(5) implies (1).

Proof of Lemma 2: If $\{a_i\}_{i=1}^{\infty}$ is an algebraically dependent set over $D[[X]]$, then there is a positive integer n and a polynomial $f \in D[[X]]\,[\{Y_i\}_{i=1}^{n}]$ different from zero such that $f(a_1, a_2, \ldots, a_n) = 0$. For each i, $1 \leq i \leq n$, let M_i denote the degree of f in Y_i and set $M = \max_{1 \leq i \leq n} \{M_i\}$. Denote by $(N_1, N_2, \ldots, N_n)$ the largest element of the set $T = \{(m_1, m_2, \ldots, m_n) \mid Y_1^{m_1} Y_2^{m_2} \ldots Y_n^{m_n}$ appears in f with a nonzero coefficient$\}$, where T is given the reverse lexicographic ordering. $N_n = M_n$, and $N_i \leq M_i$ for $1 \leq i \leq n - 1$. For each $(m_1, m_2, \ldots, m_n) \in T$, let $f_{(m_1, m_2, \ldots, m_n)} \in D[[X]]$ denote the nonzero coefficient of $Y_1^{m_1} Y_2^{m_2} \ldots Y_n^{m_n}$ in f, and suppose that t is the order of $f_{(N_1, N_2, \ldots, N_n)}$. (An element $g(X) \in D[[X]]$ has order t provided we may write $g(X) = X^t g_1(X)$, where $g_1(X)$ has nonzero constant term).

Choose a positive integer k such that $k > M + t + n$ and let d_1 be a nonzero element of D such that $d_1 \xi_i \in D$ for $1 \leq i \leq k - 1$. If d_0 is the nonzero coefficient of X^t in $f_{(N_1, N_2, \ldots, N_n)}$, then certainly $d_0 d_1 \xi_i \in D$ for $1 \leq i \leq k - 1$. By induction, we show that $d_0 d_1 \xi_i \in \bar{D}$ for each $i \in \omega$. Since $M \geq 1$ and $n \geq 1$, we have $k > 2$, so $d_0 d_1 \xi_1 \in D \subseteq \bar{D}$. Let $s \geq k$, and assume that $d_0 d_1 \xi_i \in \bar{D}$ for $1 \leq i \leq s - 1$.

We now wish to determine the form of the coefficient of $X^{\sum_{i=1}^{n} N_i h(i,s) + t}$ in $f(a_1,a_2,\ldots,a_n)$. We shall divide our argument into three parts which we label (i), (ii), and (iii) for future reference.

(i) The coefficient of $X^{\sum_{i=1}^{n} N_i h(i,s) + t}$ does not involve ξ_u if $u \geq s + 1$.

To prove (i) note that the smallest power to which ξ_u appears in any a_i, $1 \leq i \leq n$, is $h(1,u)$. From the defining properties of h and from the fact that $s \geq k > M + n + t$, we have $h(1,u) \geq h(1,s+1)$

$$> (s+1) \left[\sum_{j=1}^{s} \sum_{i=1}^{j} h(i,j) \right]$$

$$= (s+1) \sum_{i=1}^{s} h(i,s) + (s+1) \left[\sum_{j=1}^{s-1} \sum_{i=1}^{j} h(i,j) \right]$$

$$> M \sum_{i=1}^{n} h(i,s) + t\left[\sum_{j=1}^{s-1} \sum_{i=1}^{j} h(i,j) \right] .$$

$$\geq \sum_{i=1}^{n} M_i h(i,s) + t\, h(1,1)$$

$$\geq \sum_{i=1}^{n} M_i h(i,s) + t$$

$$\geq \sum_{i=1}^{n} N_i h(i,s) + t.$$

Since the smallest power to which $\xi_u X$ appears in any a_i is larger than $\sum_{i=1}^{n} N_i h(i,s) + t$, it follows that ξ_u cannot appear in the coefficient of $X^{\sum_{i=1}^{n} N_i h(i,s) + t}$.

(ii) In the expression $f_{(N_1,N_2,\ldots,N_n)}\, a_1^{N_1} a_2^{N_2} \ldots a_n^{N_n}$ the coefficient of $X^{\sum_{i=1}^{n} N_i h(i,s) + t}$ has the form $d_0\, \xi_s^{\sum_{i=1}^{n} N_i h(i,s)}$ $+ g(\xi_1,\xi_2,\ldots,\xi_s)$, where g is a polynomial in s variables over D with total degree less than $\sum_{i=1}^{n} N_i h(i,s)$.

To prove this, note that in $a_i^{N_i}$ the term $(\xi_s X)^{N_i h(i,s)}$ appears. Thus, since t is the order of $f_{(N_1,N_2,\ldots,N_n)}$ and d_0 is the coefficient of X^t, it is clear that $d_0 \xi_s^{\sum_{i=1}^{n} N_i h(i,s)}$ appears in the coefficient of $X^{\sum_{i=1}^{n} N_i h(i,s) + t}$. From (i), we see that only $\xi_1,\xi_2,\ldots,\xi_s$ are involved in the coefficient of $X^{\sum_{i=1}^{n} N_i h(i,s) + t}$ in

$f_{(N_1,N_2,\ldots,N_n)}\, a_1^{N_1} a_2^{N_2} \ldots a_n^{N_n}$. Since $f_{(N_1,N_2,\ldots,N_n)}$ has order t, any monomial in the coefficient of $X^{\sum_{i=1}^{n} N_i h(i,s) + t}$ in $f_{(N_1,N_2,\ldots,N_n)}\, a_1^{N_1} a_2^{N_2} \ldots a_n^{N_n}$ has total degree in $\xi_1,\xi_2,\ldots,\xi_s$ at most $\sum_{i=1}^{n} N_i h(i,s)$. To see that $d_0 \xi_s^{\sum_{i=1}^{n} N_i h(i,s)}$ is the only such monomial with total degree exactly $\sum_{i=1}^{n} N_i h(i,s)$ in $\xi_1,\xi_2,\ldots,\xi_s$, let $d'\xi_1^{P_1} \xi_s^{P_2} \ldots \xi_s^{P_s}$ be another monomial in this coefficient, with $P_s < \sum_{i=1}^{n} N_i h(i,s)$. Then $P_s \leq \sum_{i=1}^{n} N_i h(i,s) - h(1,s)$, since $h(1,s)$ is the smallest exponent to which ξ_s appears in any a_i. Since $s \geq k > M$, we have $h(1,s) > s[\sum_{j=1}^{s-1} \sum_{i=1}^{j} h(i,j)] > M[\sum_{j=1}^{s-1} \sum_{i=1}^{j} h(i,j)]$.

In $a_1^{N_1} a_2^{N_2} \ldots a_n^{N_n}$, ξ_j appears with an exponent no larger than $\sum_{i=1}^{n} N_i h(i,j)$ since the largest exponent to which ξ_j appears in a_i is $h(i,j)$. If $n > j$, $h(i,j) = 0$ for $j < i \leq n$, so that $\sum_{i=1}^{n} N_i h(i,j) = \sum_{i=1}^{j} N_i h(i,j)$. If $n \leq j$, $\sum_{i=1}^{n} N_i h(i,j) \leq \sum_{i=1}^{j} N_i h(i,j)$.

It follows, then, that $P_j \leq \sum_{i=1}^{j} N_i h(i,j) \leq \sum_{i=1}^{j} M_i h(i,j) \leq M \sum_{i-1}^{j} h(i,j)$. But then $\sum_{n=1}^{s-1} P_j \leq \sum_{j=1}^{s-1} M \sum_{i=1}^{j} h(i,j) = M \sum_{j=1}^{s-1} \sum_{i=1}^{j} h(i,j) < h(1,s)$. Therefore $\sum_{j=1}^{s} P_j < h(1,s) + P_s \leq h(1,s) + (\sum_{i=1}^{n} N_i h(i,s) - h(1,s)) = \sum_{i=1}^{n} N_i h(i,s)$. It follows that the total degree of $d'\xi_1^{P_1}\xi_2^{P_2} \ldots \xi_s^{P_s}$ is less than $\sum_{i=1}^{n} N_i h(i,s)$.

(iii) Let $(m_1,m_2,\ldots,m_n)$ be in T, with $(m_1,m_2,\ldots,m_n) \neq (N_1,N_2,\ldots,N_n)$. In the expression $f_{(m_1,m_2,\ldots,m_n)} a_1^{m_1} a_2^{m_2} \ldots a_n^{m_n}$ the coefficient of $X^{\sum_{i=1}^{n} N_i h(i,s) + t}$ has the form $g(\xi_1,\xi_2,\ldots,\xi_s)$, where g is a polynomial in s variables over D with total degree less than $\sum_{i=1}^{n} N_i h(i,s)$.

To prove (iii), let $(m_1,m_2,\ldots,m_n) \in T$, with $(m_1,m_2,\ldots,m_n) \neq (N_1,N_2,\ldots,N_n)$ and let e be the largest integer such that

$m_e \neq N_e$. Since $(N_1, N_2, \ldots, N_n)$ is the largest element of T, where T is given the reverse lexicographic order, we have $m_e < N_e$. Thus $m_i \leq M_i$ for $1 \leq i \leq e-1$, $m_e \leq N_e - 1$, and $m_i = N_i$ for $e+1 \leq i \leq n$. Since $s > M$, the definition of h implies that

$$h(e,s) > s\left[\sum_{j=1}^{s-1} \sum_{i=1}^{j} h(i,j) + \sum_{j=1}^{e-1} h(j,s)\right]$$

$$\geq s\left[\sum_{j=1}^{e-1} h(j,s)\right]$$

$$> M \sum_{j=1}^{e-1} h(j,s)$$

$$\geq \sum_{j=1}^{e-1} M_j h(j,s).$$

It follows that

$$N_e\, h(e,s) > (N_e - 1)\, h(e,s) + \sum_{j=1}^{e-1} M_j h(j,s)$$

$$\geq m_e\, h(e,s) + \sum_{j=1}^{e-1} m_j h(j,s) = \sum_{j=1}^{e} m_j h(j,s).$$

Therefore, $\displaystyle\sum_{i=1}^{n} N_i h(i,s) \geq N_e\, h(e,s) + \sum_{i=e+1}^{n} N_i h(i,s)$

$$> \sum_{j=1}^{e} m_j h(j,s) + \sum_{i=e+1}^{n} N_i h(i,s)$$

$$= \sum_{j=1}^{n} m_j h(j,s),$$

and (iii) follows.

Since $f(a_1,a_2,\ldots,a_n) = 0$, we see from (i), (ii), and (iii) that by considering the coefficient of $X^{\sum_{i=1}^{n} N_i h(i,s) + t}$ we get an expression of the form

$$(*) \quad d_0 \xi_s^{\sum_{i=1}^{n} N_i h(i,s)} + \sum_{j=1}^{z} g_j(\xi_1,\xi_2,\ldots,\xi_{s-1})\xi_s^{q_j} = 0,$$

where $q_j < \sum_{i=1}^{n} N_i h(i,s)$, and the total degree of g_j in $\xi_1,\xi_2,\ldots,\xi_{s-1}$ is less than $\sum_{i=1}^{n} N_i h(i,s) - q_j$. When we multiply (*) through by $d_1(d_1d_0)^{\sum_{i=1}^{n} N_i h(i,s) - 1}$, we get $(**)\ (d_1d_0\xi_s)^{\sum_{i=1}^{n} N_i h(i,s)}$

$$+ \sum_{j=1}^{z} d_1(d_1d_0)^{\sum_{i=1}^{n} N_i h(i,s) - 1 - q_j} g_j(\xi_1,\xi_2,\ldots,\xi_{s-1})\ (d_1d_0\xi_s)^{q_j} = 0.$$

By assumption, $d_1 d_0 \xi_i \in \bar{D}$ for $1 \leq i \leq s-1$, and the total degree of g_j in $\xi_1, \xi_2, \ldots, \xi_{s-1}$ is no greater than $\sum_{i=1}^{n} N_i h(i,s) - q_j - 1$, so we have $(d_1 d_0)^{\sum_{i=1}^{n} N_i h(i,s) - q_j - 1} g_j(\xi_1, \xi_2, \ldots, \xi_{s-1}) \in \bar{D}$ for $1 \leq j \leq z$. It follows that (**) is an equation of integral dependence of $d_1 d_0 \xi_s$ over $\bar{D}$. But clearly $\bar{D}$ is integrally closed, so $d_1 d_0 \xi_s \in \bar{D}$.

By induction on i, we have shown that $d_0 d_1 \xi_i \in \bar{D}$ for each $i \in \omega$.

As an immediate corollary to Theorems 1 and 3, we have the following result.

COROLLARY 4: _Let_ D _be an integrally closed domain and let_ J _be an overring of_ D. _If_ D[[X]] _and_ J[[X]] _have the same quotient field, then_ J _is contained in the complete integral closure of_ D.

Proof: By Theorem 3, conditions (1)-(5) of (A) are equivalent. Since we are assuming that condition (3) holds, it follows that condition (2) also holds, so by Theorem 1, J is contained in the complete integral closure of D.

In the case when D is integrally closed, Theorem 3 shows that either D[[X]] and J[[X]] have the same quotient field or tr.d. $(J[[X]]/D[[X]]) = \infty$. Corollary 4 leads to the question whether D[[X]] and J[[X]] have the same quotient field if and only if J is contained in the complete integral closure of D. That this is not the case can be seen by taking D to be a valuation ring that has countably many prime ideals and no minimal (nonzero) prime ideal. In this case, the complete integral closure of D is K. But Gilmer's result [3, Theorem 1] shows that tr.d. $(K[[X]]/D[[X]]) = \infty$.

We now give a characterization of those domains D with the property that tr.d. $(J[[X]]/D[[X]]) = \infty$ for each proper overring J of D.

THEOREM 5: *An integral domain* D *is completely integrally closed if and only if* tr.d. $(J[[X]]/D[[X]])$ *is infinite for each proper overring* J *of* D.

Proof: Sheldon has shown [8, Theorem 3, 4] that D is completely integrally closed if and only if D[[X]] and J[[X]] have different quotient fields for each proper overring J of D.

If D is completely integrally closed, it is integrally closed, so that by Corollary 4, tr.d. $(J[[X]]/D[[X]]) = \infty$ for each proper overring J of D.

Conversely, if tr.d. $(J[[X]]/D[[X]]) = \infty$ for each proper overring J of D, then D[[X]] and J[[X]] have different quotient fields, hence D is completely integrally closed.

As we have already mentioned, we show in Example 1 of Section IV that conditions (1)-(5) of (A) are not, in general, equivalent. The example given is an integral domain D which is not root closed (D is said to be root closed provided every element of K which is a root of an element of D is already in D). Thus, we ask whether conditions (1)-(5) of (A) are equivalent for any overring J of D provided D is root closed. We are unable to answer this question; however, the following Theorem provides a partial answer.

THEOREM 6: *If D is a root closed domain, conditions (1)-(3) of (A) are equivalent.*

Proof: As a result of Theorem 1, we need only show that (3) implies (1). Therefore, let J be an overring of D such that J[[X]] and D[[X]] have the same quotient field. Let $\{\xi_i\}_{i=1}^{\infty}$ be a sequence of elements of J. Consider the power series $f = \sum_{i=0}^{\infty} \xi_i^{i+1} X^i$. Since f is in the quotient field of D[[X]], there is a nonzero element d in D such that $d^{i+1} (\xi_i^{i+1}) \in D$ for each i [7, Proposition 2.5]. Since D is root closed and $(d\xi_i)^{i+1} \in D$ for each i, $d\xi_i \in D$ for each positive integer i.

III. FINITE RING EXTENSIONS

In this section we shall consider the case in which $J = D[\xi_1, \xi_2, \ldots, \xi_n]$, where $\xi_1, \xi_2, \ldots, \xi_n$ are elements of the quotient field K of D. Our next result shows that in this special case we are able to weaken the hypothesis that D be integrally closed and still get the equivalence of conditions (1)-(5) of (A).

THEOREM 7: _Suppose that D and its integral closure $\bar{D}$ have the same complete integral closure. If $J = D[\xi_1, \xi_2, \ldots, \xi_n]$, where $\xi_1, \xi_2, \ldots, \xi_n$ are in K, then conditions (1)-(5) of (A) are equivalent to the condition the each ξ_i is almost integral over D._

Proof: Assume that each ξ_i, $1 \leq i \leq n$, is almost integral over D. Then there is a nonzero element d in D such that $d\xi_i^m \in D$ for $1 \leq i \leq n$, and for each positive integer m. Therefore $dJ \subseteq D$, and hence condition (1) holds.

If some ξ_i is not almost integral over D, then ξ_i is not in the complete integral closure of $\bar{D}$; that is, there is no nonzero element d in $\bar{D}$ such that $d\xi_i^n \in \bar{D}$ for each positive integer n. As in Lemma 2, we construct a sequence $\{a_i\}_{i=1}^{\infty}$ of power series in $J[[X]] = D[\xi_1, \xi_2, \ldots, \xi_n]\,[[X]]$ so that $\text{tr.d.}\,(J[[X]]/D[[X]]) = \infty$. Thus we take $a_k = 1 + \sum_{j=1}^{\infty} (\xi_i^j X)^{h(k,j)}$, where h is the function in section II. From Lemma 2 we see that either the set $\{a_i\}_{i=1}^{\infty}$ is algebraically independent over $D[[X]]$ or there exists a nonzero element d in D such that $d\xi_i^j \in \bar{D}$ for each j. We have just observed that the latter does not occur, so it follows that tr.d. $(J[[X]]/D[[X]]) = \infty$. It follows by contrapositive that condition (5) implies that each ξ_i almost integral over D.

COROLLARY 8: <u>If</u> D <u>is Noetherian and</u> $J = D[\xi_1,\xi_2,\ldots,\xi_n]$, <u>then conditions</u> (1)-(5) <u>of</u> (A) <u>are equivalent to the condition that each</u> ξ_i <u>is almost integral over</u> D.

<u>Proof</u>: Since D is Noetherian, its integral closure $\bar{D}$ is completely integrally closed [6, Theorem 33.10, and 4, page 524]. Since integrity implies almost integrity, this result follows from Theorem 7.

THEOREM 9: <u>Let</u> D <u>be a countable integral domain with quotient field</u> K. <u>If</u> $\xi_1,\xi_2,\ldots,\xi_n \in K$ <u>and</u> $J = D[\xi_1,\xi_2,\ldots,\xi_n]$, <u>then conditions</u> (1) - (3) <u>of</u> (A) <u>are equivalent</u>.

<u>Proof</u>: It is sufficient to show that condition (3) implies that $dJ[[X]] \subseteq D[[X]]$ for some nonzero element d of D, or that $dJ \subseteq D$ for some nonzero element d of D. It is sufficient, then to show that each ξ_i is almost integral over D; for if $d_i\xi_i^{\,m} \in D$ for each $m \in \omega$, then $d = d_1d_2\ldots d_n \in D$ is such that $d\xi_i^m \in D$ for $1 \leq i \leq n$ and for each $m \in \omega$, and thus $dJ \subseteq D$.

Suppose, by contrapositive, that ξ_i is not almost integral over D for some i. Let $D = \{0\} \cup \{d_i\}_{i=1}^{\infty}$, $d_i \neq 0$. Since ξ_i is not almost integral over D, define a sequence of positive integers $\{k_i\}_{i=1}^{\infty}$ as follows. Let $k_1 \in \omega$ be such that $d_1\xi_i^{\,k_1} \notin D$. If $k_1,k_2,\ldots,k_{j-1}$ are chosen, chose $k_j > k_{j-1}$ such that $d_j^j\xi_i^{\,k_j} \notin D$. Let $f = \sum_{j=1}^{\infty} \xi_i^{k_j} X^{j-1}$

$\varepsilon\ J[[X]]$. f is not in the quotient field of $D[[X]]$, for there is no nonzero element d in D such that $d^j \xi_i^{k_j} \in D$ for each j [7, Proposition 2.5].

IV. EXAMPLES

Our first example shows that, in general, conditions (3) and (4) of (A) are not equivalent.

Example 1: Let Z_p denote the field of integers modulo p, a prime. If $\{t_i\}_{i=1}^{\infty}$ is a countably infinite set of indeterminates over Z_p, let $J = Z_p[\{t_i\}_{i=1}^{\infty}]$ and $D = Z_p[\{t_i^p, t_i^{p+1}\}_{i=1}^{\infty}]$. Since J has characteristic p, if $f(t_1,t_2,\ldots,t_n) \in J$, then $[f(t_1,t_2,\ldots,t_n)]^p = g(t_1^p,t_2^p,\ldots,t_n^p) \in D$, where the coefficients of g are the $p^{\underline{th}}$ powers of the corresponding coefficients of f. Therefore, if $\sum_{i=0}^{\infty} a_i X^i \in J[[X]]$, then $\left(\sum_{i=0}^{\infty} a_i X^i\right)^p = \sum_{i=0}^{\infty} a_i^p X^{ip} \in D[[X]]$, so that J[[X]] is integral, and hence algebraic, over D[[X]]. Also, J is an overring of D, since $t_i = t_i^{p+1}/t_i^p$.

To see that D[[X]] and J[[X]] have different quotient fields, we show that $\sum_{i=1}^{\infty} t_i X^i \in J[[X]]$ is not in the quotient field of D[[X]]. Let $f = f(t_1^p,t_2^p,\ldots,t_n^p,t_1^{p+1},t_2^{p+1},\ldots,t_n^{p+1})$ be an arbitrary element of D. For $k > n$, $f^{k+1}\, t_k \notin D$, so by Proposition 2.5 of [7], $\sum_{i=1}^{\infty} t_i X^i$ is not in the quotient field of D[[X]]. Therefore condition

(4) does not imply condition (3).

Our next example shows that if D is not integrally closed, and $\bar{D}$ is its integral closure, then $\bar{D}[[X]]$ may have infinite transcendence degree over D[[X]].

Example 2: Let F be a field, and let s and t be indeterminates over F. Let $D = F[t, \{s^k t^m \mid m \geq -k^2, k \text{ any integer}\}]$. (This appears as an example in [6]). We show that $\bar{D}[[X]]$ has infinite transcendence degree over D[[X]] by showing that $D[[X]] \subsetneq (D + sD[1/t])[[X]] \subsetneq \bar{D}[[X]]$, and that $(D + sD[1/t])[[X]]$ has infinite transcendence degree over D[[X]]. We first observe that $\bigcap_{n=1}^{\infty} t^n D = (0)$. If this were not the case, then there is a nonzero polynomial $f(t,s) \in F[t,s] \subsetneq D$ such that $f(t,s) \in \bigcap_{n=1}^{\infty} t^n D$. (If $f_1 \in D$, $f_1 \in \bigcap_{n=1}^{\infty} t^n D$, then there is a smallest negative integer k such that t^k appears in f_1. Then $f = t^{-k} f_1 \in F[t,s]$, and $f \in \bigcap_{n=1}^{\infty} t^n D$). Therefore, for each positive integer n, there is a $d_n \in D$ such that $f(t,s) = d_n t^n$. It then follows that $t^{-n} f(t,s) \in D$ for each positive integer n. If $\alpha t^m s^k$ is any nonzero monomial of f(t,s), then $m \geq -k^2$. Let $n \geq k^2 + m + 1$. Then $t^{-n} t^m s^k = t^{m-n} s^k$ must be an element of D. But this is impossible, since $m - n \leq -k^2 - 1 < -k^2$. It follows that $\bigcap_{n=1}^{\infty} t^n D = (0)$. But then D[1/t][[X]] has infinite transcendence degree over D[[X]] [8, Theorem 2.1]. If $\{a_i\}_{i=1}^{\infty}$ is an infinite transcendence set in

$D[1/t][[X]]$ over $D[[X]]$, then so is $\{sa_i\}_{i=1}^{\infty}$. But $sa_i \in sD[1/t][[X]]$ for each i. Therefore, $(D + sD[1/t])[[X]]$ has infinite transcendence degree over $D[[X]]$. Clearly $D \subseteq \bar{D}$, and $sD[1/t] \subseteq \bar{D}$, since $(st^{-k})^k = s^k t^{-k^2} \in D$, so that st^{-k} is integral over D, hence in $\bar{D}$ for each k. Therefore, $\bar{D}[[X]]$ has infinite transcendence degree over $D[[X]]$.

Our final example shows that if D is an integral domain and $\bar{D}$ its integral closure, then $\bar{D}[[X]]$ may be contained in $D[[X]]_{D*}$ and have infinite Krull dimension, while $\dim D[[X]] = 3$.

<u>Example 3</u>: Let Z denote the ring of integers. Gilmer gives an example of a field K in which the integral closure $\bar{Z}$ of Z in K is almost Dedekind, but not Dedekind [5, p. 586]. Let $V = K + M$ be a rank one discrete valuation ring and let $D = Z + M$. Thus $\bar{D} = \bar{Z} + M$ [5, Appendix 2, Theorem A]. Arnold shows [1, Example 2] that dim $\bar{D}[[X]] = \infty$. Since $m\bar{D} \subseteq D$ for each $m \in M$, it follows that $m\bar{D}[[X]] \subseteq D[[X]]$, and therefore $\bar{D}[[X]] \subseteq D[[X]]_{D*}$.

We now show that $\dim D[[X]] = 3$. Since $mV[[X]] \subseteq D[[X]]$, it follows that $V[[X]]_{V*} = D[[X]]_{D*}$. Thus $\dim D[[X]]_{D*} = \dim V[[X]]_{V*} = 1$. Therefore if $(0) \subset Q_1 \subset Q_2$ is a chain of prime ideals in $D[[X]]$, then $Q_2 \cap D \neq (0)$, that is, $M \subseteq Q_2 \cap D$. Thus $MD[[X]] \subseteq Q_2$, so that $\sqrt{MD[[X]]} \subseteq Q_2$. But $M^2 \subseteq mD \subseteq M$, where $M = mV$, so by Theorem 1 of [1], $M[[X]] = \sqrt{mD[[X]]} \subseteq Q_2$. However, $M[[X]] \neq$

Q_2, since $M[[X]]$ is minimal in $D[[X]]$. To see this, we show that there is no minimal prime ideal Q of $V[[X]]$ such that $Q \cap D[[X]] \subseteq M[[X]]$ if $Q \nsubseteq M[[X]]$. Let $\xi(X) = \sum_{i=0}^{\infty} \xi_i X^i \in Q - M[[X]]$, and let r be the smallest integer such that $\xi_r \notin M$. Then ξ_r is a unit of V, so we may assume that $\xi_r = 1$. Therefore, $u(X) = \sum_{i=0}^{\infty} \xi_{r+i} X^i$ is a unit of $V[[X]]$, and $\xi(X)\, u(X)^{-1} = \sum_{i=0}^{r-1} \xi_i X^i u(X)^{-1} + X^r \in Q \cap D[[X]]$. Hence $M[[X]]$ is minimal in $D[[X]]$, so that $M[[X]] \subset Q_2$. If $(0) \subset Q_1 \subset Q_2 \subset Q_3$ is a chain of prime ideals of $D[[X]]$, then $Q_2/M[[X]] \subset Q_3/M[[X]]$ is a chain of prime ideals of $D[[X]]/M[[X]] = (D/M)[[X]] = Z[[X]]$ which has dimension 2. Thus, any such chain $(0) \subset Q_1 \subset Q_2 \subset Q_3$ of prime ideals of $D[[X]]$ is a maximal one, so that $\dim D[[X]] = 3$.

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TRANSCENDENCE DEGREE IN POWER SERIES RINGS

by

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(ABSTRACT)

Let D[[X]] be the ring of formal power series over the commutative integral domain D. Gilmer has shown that if K is the quotient field of D, then D[[X]] and K[[X]] have the same quotient field if and only if $K[[X]] \subseteq D[[X]]_{D-(0)}$. Further, if a is any nonzero element of D, Sheldon has shown that either D[1/a][[X]] and D[[X]] have the same quotient field, or the quotient field of D[1/a][[X]] has infinite transcendence degree over the quotient field of D[[X]]. In this paper, the relationship between D[[X]] and J[[X]] is investigated for an arbitrary overring J of D. If D is integrally closed, it is shown that either J[[X]] and D[[X]] have the same quotient field, or the quotient field of J[[X]] has infinite transcendence degree over the quotient field of D[[X]]. It is shown further, that D is completely integrally closed if and only if the quotient field of J[[X]] has infinite transcendence degree over the quotient field of D[[X]] for each proper overring J of D. Several related results are given; for example, if D is Noetherian, and if J is a finite ring extension of D, then either J[[X]] and D[[X]] have the same quotient field or the quotient field of J[[X]] has infinite transcendence degree over the

quotient field of D[[X]]. An example is given to show that if D is not integrally closed, J[[X]] may be algebraic over D[[X]] while J[[X]] and D[[X]] have different quotient fields.