CORRELATION BETWEEN ARRIVAL AND SERVICE PATTERNS AS A MEANS OF QUEUE REGULATION

by

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PART I

THEORETICAL ANALYSIS

CHAPTER 1
INTRODUCTION

1.1 Terminology and Review of Literature

In Queueing Theory we are concerned with the demand for, and supply of, some kind of service. In addition to the more obvious supermarket type situations which the theory first set out to describe in mathematical and probabilistic terms, there has been found to exist a vast complex of related and less obvious areas of application to which the concepts and techniques of the theory are relevant.

For the purposes of this dissertation it is sufficient to proceed in terms of the conventional model. The units demanding service are called "customers", and those which provide it are called "servers". These may be single or multiple, in series or in parallel. The simplest case is that of a single server.

In all cases it is convenient to visualize the basic physical situation as one in which waiting customers form a line, or queue, in front of the service facility, taking their turns to enter the service facility as predecessors complete service. Much of modern Queueing Theory is concerned with the effects of eccentric customer behavior (e.g.

impatience), and with variants of queue discipline (the obvious one is first-come, first served, but in some circumstances last-come, first served could be appropriate). These factors are not considered in this dissertation, and, where relevant, first-come, first served can be assumed to be the rule.

The major features of interest in the simplest queueing models are:

i) Waiting Time (i.e. how long a new arrival may have to wait);

ii) Queue Length (i.e. the number of customers in the queue); and sometimes the "System State" (queue length plus the number of customers in service - one in the case of a single server system). Here it may be remarked that we shall always suppose that there are unlimited waiting facilities, so that an indefinitely long queue could build up.

iii) Idle Time - the wait of the server from completion of one service until the beginning of the next.

iv) Busy Period - the uninterrupted time interval from the end of an idle period until the beginning of the next.

v) Output - in the literature this is defined in a number of different ways; the definition here is the interval between successive epochs of service completion.

These are, in general, stochastic processes, and depend on two fundamentals:

vi) The distribution of time intervals between arrivals (interarrival intervals), sometimes loosely referred to as "input".

vii) The distribution of service times.

The ratio of the mean service interval (i.e. mean duration of a service) to mean interarrival interval is called the congestion index or traffic intensity, and plays a key role in the theory since it represents the ratio of mean arrival rate to mean service rate, and thus whether we expect a long queue to build up or not depends on its magnitude. The most commonly considered situation is that in which interarrival intervals are independently and identically distributed (arrivals form a renewal process), and service times conform to a similar assumption. In the general case, with a single server, this is denoted by the hieroglyphic GI/G/1 introduced by D. G. Kendall [22]. This notation apparently implies no assumption about a possible correlation between arrivals and service. In the vast majority of the research carried out they are taken to be completely independent. As will be seen later, the main purpose of this dissertation is a specific treatment of cases where there is some sort of relationship between the arrival and service patterns.

The fact that interarrival intervals or service times have a negative exponential distribution is denoted by the letter M, which is substituted for GI or G. Similarly, fixed interarrival intervals or service times (appointments) are indicated by D. Thus, M/M/1 means a single server system with independent negative exponential arrivals and services, in general, with different means. Such a system is sometimes referred to as a "Poisson queue", because arrivals form a Poisson stream, as do the epochs of service completion as long as there are customers waiting to be served.

Another commonly considered fundamental distribution denoted by E_k (in honor of A. K. Erlang, a pioneer in the field) has density:

$$f(t) = \frac{1}{\Gamma(k)\beta^k} t^{k-1} e^{-t/\beta} \quad , \tag{1.1}$$

i.e. what is otherwise known as χ^2 with 2k degrees of freedom. This contains M and D as special cases.

It would be a thankless, and possibly useless, task to review completely the enormous literature of Queueing Theory, and it is indeed at least a point of view, that a substantial proportion of the research papers which have been published do little more than solve special cases without really contributing to a deep understanding of the probabilistic mechanisms at play (D. G. Kendall [24]), nor to a solution of the problems of the organization of a practical and efficient queueing system. In the remainder of this chapter we shall, therefore, review fundamental references which lead into the raison d'être of the research embodied in this dissertation.

To A. K. Erlang [15], a Danish telephone engineer, may be attributed credit for the formalization in about 1909, of what we know as Queueing Theory today. Erlang's attack on the waiting time problem was subsequently taken up by D. V. Lindley [28] and W. L. Smith [34], who showed how to formulate the GI/G/1 waiting time problem in terms of integral equations.

A. B. Clark [5] derived time dependent formulae for system state probabilities for M/M/1, expressing them explicitly in terms of modified Bessel functions. N. T. J. Bailey [2] and B. W. Conolly [6] obtained essentially the same result by different means; viz. a common denominator of the Laplace transformation of the fundamental differential difference equations followed by use of generating functions, and a direct solution of the difference equations, respectively. Perhaps more interestingly, D. G. Champernowne [4] invoked a "random walk" type argument which sheds light on the character of the solution.

The busy period attracted a good deal of interest. Formulae for its duration in time were obtained by D. G. Kendall [21] and L. Takács [35] for the system M/G/1. I. J. Good [18] in a contribution to the discussion of Kendall [21], showed for this system how the busy period can be regarded as a branching process, and his argument gives a direct explanation of the form of the result, otherwise obtained by laborious means. Later, N. U. Prabhu [31] made a direct attack on the problem of the joint probability and probability density function of the duration in customers served and time, showing how to derive the results simply by direct probabilistic argument, without the use of Laplace transforms. His method is applicable to GI/G/1 [32]. B. W. Conolly [7] was the first to solve the problem for GI/M/1, and later applied his method to GI/E_k/1 [8] and to E_k/G/1 [10].

The output process and other features of Queueing Theory received thorough investigation between the years 1930 and 1960. Also, further generalizations were made, to the extent that was analytically possible, to more complicated models, such as GI/G/1, M/M/N, and GI/G/N , by probabilists such as F. Pollaczek [30], L. Takács [36], A. Ya. Khinchin [25], B. W. Conolly [7], A. N. Kolmogorov [27], D. G. Kendall [22], and others. D. G. Kendall's 1953 paper [22] in which the concept of imbedded Markov chain was introduced, stimulated considerable interest for further research.

"Heavy Traffic" approximations, that is, expressions which may be used to approximate the behavior of the common features of interest when the congestion index is less than but very close to unity, have been investigated thoroughly by E. G. Samandarov [33], O. V. Viskov [37] and J. F. C. Kingman [26].

From the point of view of practical application N. T. Bailey [1] published several interesting papers on the role of Queueing Theory in medicine and hospital operation. D. G. Kendall [23] reviewed application to the theory of reservoirs or "dams". Other applications include the problems of inventory, machine repairs, aircraft operations, psychological flow of nervous impulses, Post Office operation, maintenance, assembly lines, physical processes, epidemic processes in biology and population growth. This list is by no means exhaustive.

It was in the process of studying applications that scientists as well as management realized how important it was, on the one hand, to determine a suitable measure of effectiveness of a queueing system, and on the other hand, to control it. Waiting time is a factor of considerable concern to customers, while idle time is wasteful from the management's point of view. The obvious means to employ in order to reduce customer's waiting time is to speed up service, possibly by increasing the number of service channels, or by training servers to function consistently more rapidly. In either case, at a given level of input or mean arrival rate, this has the effect of increasing the periods of idleness of the server, which is undesirable from the management's point of view. This reduces to the fact that as long as the server acts essentially independently of the arrival pattern someone has to pay a penalty, and it is on the particular situation and its location that the decision has to be made as to who can best afford to do so.

Another method used to remedy this situation is to control the input in one manner or another. A particular example of this is the use of a deterministic arrival distribution, or appointment system. D. V. Lindley [28] was the first to examine the waiting time for this situation and, as expected, concluded that the customer's mean waiting time does decrease, not considering any irregularities which might arise from delayed arrivals. Others suggested that an effective method of reducing waiting time would be for customers to serve themselves wherever possible. In spite of this the question remains, when it is not possible to set up an appointment system or institute self-service, how is it possible to reduce customer's waiting time without terribly hurting management economically or sacrificing the quality of service? It is an objective of this dissertation to try to answer this question.

1.2 Formulation of the Problem

All the work reviewed above has been concerned with models where there are two independent streams (usually Poisson) of events; namely that of arrival of customers and that of their services. That is, there is no correlation between customer's arrivals and their services. An exception to this is to be found in Cox and Smith [14], where arrival transitions ($n \to n+1$) are associated with a probability differential $\alpha_n \delta t$ and departure transitions ($n \to n-1$) with $\sigma_n \delta t$ ($n>0$) where n is the number of customers waiting. Another model of this nature is described by F. J. John [20]. This allows the service of the nth arrival to depend on the interval between the nth and (n+1)th arrivals. He obtains the generating function of a transform of the joint distribution of waiting time and arrival time of the nth customer. He then develops a relation between this generating function and a transform of waiting time density. He also obtains the generating function of the duration in time of the busy period. It should be pointed out here that D. V. Lindley [28] in setting up his formulation of waiting time claims that such a dependence will not much alter the method of obtaining a solution, although he does not make any attempt to solve the problem when such a dependence does exist. Instead, he remarks that such a model "would not make sense". By this he presumably means that a server would find it difficult to arrange service time on the basis of an inter-arrival interval which he might not have been able to observe in its totality. W. Feller [16] also treats the same model. More comments will be made on this later on.

Some further work was done by C. H. Harris [19]. He generalizes M/G/1 so that the service parameter becomes a stochastic process indexed at the length of the queue at the moment service is begun. In this sense his model is quite similar to that of Cox and Smith. He has developed some "general theory" and has obtained results which in his words "characterize queue behavior" using the method of imbedded Markov chain. He then cites an industrial example.

In the present work we shall define a "correlated queue" to be a queueing model in which the arrival pattern influences the service pattern. The nature of this influence in the present work is such that services could be tailored to an arrival interval which could always be observed before service begins. "Correlated queue" will also refer to models in which the arrival pattern is made to depend on service times.

In the model analyzed here the service time of the nth customer directly depends on the interarrival interval associated with him; that is the interarrival time between himself and his predecessor. To be precise we shall adopt the following notation. Customers C_1, C_2, C_3, ... respectively arrive at arrival epochs Σ_1, Σ_2, Σ_3, ... The interarrival intervals T_1, T_2, T_3, ... are formed as follows:

$$T_1 = \Sigma_1 - \Sigma_0$$

where Σ_0 is the starting time of the process.

$$T_2 = \Sigma_2 - \Sigma_1$$

and in general

$$T_n = \Sigma_n - \Sigma_{n-1} .$$

These will be assumed to be negative exponentially distributed with mean τ for most of the analysis.

The service time of C_n is denoted by S_n, and S_n directly depends on T_n in such a way that $S_n = \lambda T_n$ where λ is a non-time dependent constant which numerically has the value of the congestion index, and which we shall later show must be less than one for equilibrium.

It is evident that this model is different from that investigated by Lindley, Feller or John, where S_n may depend on T_{n+1}. For following Lindley's formulation of the waiting time problem, if we denote by w_n the waiting time of the nth customer, excluding service, then

$$\begin{aligned} w_{n+1} &= w_n + S_n - T_{n+1} && \text{if } w_n+S_n-T_{n+1} > 0 , \\ &= 0 && \text{if } w_n+S_n-T_{n+1} \leq 0 . \end{aligned}$$

Lindley writes $S_n-T_{n+1} = U_n$ and claims that whether S_n and T_{n+1} are dependent or not is irrelevant, and so long as U_n and U_{n+1} are independent his method can be applied. In our model, to start with, there is only one stream of events - that of arrivals. The services are then determined completely. Lindley's governing equation becomes:

$$w_{n+1} = w_n + \lambda T_n - T_{n+1} \quad , \quad \text{if } w_n+\lambda T_n - T_{n+1} > 0$$
$$\phantom{w_{n+1}} = 0 \quad , \quad \text{if } w_n+\lambda T_n - T_{n+1} \leq 0 \, ,$$

so that

$$U_n = \lambda T_n - T_{n+1}$$

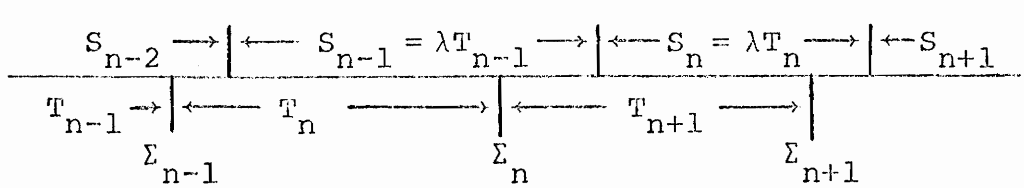
and

$$U_{n+1} = \lambda T_{n+1} - T_{n+2} \quad .$$

It is now evident that U_n and U_{n+1} are not independent and Lindley's method of obtaining waiting time distribution cannot be used.

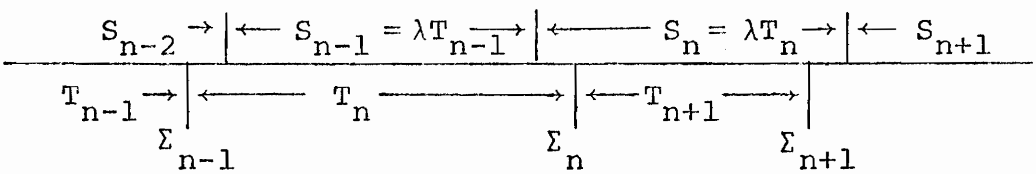
Perhaps it should be made clear here that when the service of C_n starts the interarrival interval T_n has been completed and could have been observed by the server so that $S_n = \lambda T_n$ is known. From the practical point of view, therefore, this model "makes sense". In addition we shall show later that it can be an extremely efficient method of queue regulation. The following diagrams are meant to clarify the situation:

i) The nth customer arrives before the (n-1)th completes service.



(T_i's are interarrival intervals and S_i's are service intervals). The horizontal line is the time axis.

ii) The nth customer arrives after the (n-1)th completes service.



It may be of interest at this stage to mention that the problem which suggested this model had nothing to do with queueing. For the complete statement and explanation of the original problem reference may be made to B. W. Conolly [11], but briefly, it is as follows:

Consider two runners: H (the hunter); and Q (the quarry). Starting at the origin H pursues Q along the positive X-axis. Q's motion is continuous with speed V, but H chases Q discontinuously with speed U. This curious behavior of H arises because he cannot see Q. His only knowledge of Q's whereabouts is derived from a series of position reports delivered to him at random time intervals T, each being independently distributed with a specified distribution. When H receives the nth position report, which takes the form "Q is now at Q_n" he pursues Q with a constant speed U and continues to run until he reaches Q_n, where he waits for the next report. If this arrives before H reaches Q_n, he continues to run in an effort to reach Q_{n+1}.

If we now call the nth position report I_n, and regard I_n as a customer who demands that H be moved from Q_{n-1} to Q_n, we are dealing with a queueing system which has the strange feature that the service time is a multiple of the corresponding arrival interval. For if the interarrival interval between I_{n-1} and I_n is T_n, then Q has moved a distance VT_n in that time. I_n then requires that H be moved over this distance at speed U and this occupies time VT_n/U. Letting $\lambda = V/U$ we then have a situation in which the service time of the nth customer is a constant λ multiplied by the interarrival interval associated with him. It is now evident that if $\lambda > 1$ i.e. $V > U$ the queue tends to build up, and on the other hand if $\lambda < 1$ steady state will be achieved.

No further reference will be made to this kinematic problem, and we shall consistently concentrate on the queueing aspect. All the usual features of queueing systems invite investigation. We shall here deal with waiting time, state probabilities, busy period and output processes. The exact formulation of the respective problems will be made in the beginning of each chapter.

Chapter 6 is devoted to a related model where services and interarrival intervals are independent, as in the conventional queues, however, the mean of the service time distribution does depend on the number of customers present in the queue (not in the system). Due to the nature of this correlation the exact formulation of this model will be postponed. For this model state probabilities and waiting time processes are studied analytically.

In Chapter 7 some further results are found for conventional non-correlated M/M/1 queues, which, besides their importance in their own right, are needed for comparison purposes in Part II. To be exact, the state probabilities at arrival epochs are obtained for M/M/1 which,to the author's best knowledge, do not seem to have been evaluated by means of a simple probabilistic argument before.

Chapter 8 contains some notes and comments on the queues where interarrival intervals depend directly on the previous service interval. This is a kind of dual to the previous model. It is not without application,for in assembly line or machine operation such a model would be appropriate. For, without such a system, it could happen that the objects to be processed by the machine build up into a long queue. Also, in a set of competing identical enterprises, regarded as a set of parallel channels, that enterprise which offers quicker service obtains a greater number of customers and consequently shorter interarrival intervals. It will be seen, however, that this model is more cumbersome to work with. Here, besides the dual queue, other problems are suggested which are believed to constitute further useful investigations.

In the second part of this work all the features of the correlated model are compared numerically with conventional Poisson queues. The basis of this comparison is the fact that if for the correlated model the arrivals form a Poisson stream, so will the services with a different parameter due to the nature of the correlation. It is, consequently, quite appropriate to compare the measures of effectiveness of the two systems. It is shown that in the correlated model the mean waiting time is drastically reduced; so also is the variance of waiting time. It is also demonstrated numerically that the correlated queue is shorter more often, although its probability of being empty is less, so that, in fact, the server is idle less often and is, consequently, utilized more efficiently. The busy period is also shown to have a smaller mean and variance so that there tend to be more busy periods of shorter length in the correlated queue. It is also shown that the steady state is achieved much more quickly in the correlated queue. There is also shown to be improvement on the requirements for waiting room facilities. In spite of these, the output process does not differ very much in the two models, and in fact, the mean interval between consecutive departures from the system in the steady state in both cases is the same, as it must be. The mean and variance of waiting time is also compared with the corresponding values for the system D/M/1 and M/D/1 and shown to reduce considerably, at least at moderate and heavy traffic. It is then concluded that this correlated model improves all the desired properties of a queueing system, and thus, is considered to be an efficient method of queue regulation. With this introduction, we proceed to develop the theory.

CHAPTER 2
THE WAITING TIME PROCESS

Let us here define the waiting time of the nth customer to be the time interval between the epoch of his arrival and the epoch of his departure from service (that is, the time he has to wait before entering the service facility plus his service time), and denote it by w_n. Let us further assume that the interarrival intervals T_i have a general independent probability density function (p.d.f.); say $a(T_i)$.

2.1 Integral Equation for p.d.f. of Waiting Time

We shall here proceed to derive an integro-difference equation for the p.d.f. of waiting time w_n. It is evident that:

$$w_{n+1} = \lambda T_{n+1}\,, \qquad \text{if } w_n \leq T_{n+1}\,; \tag{2.1}$$

$$= w_n - T_{n+1} + \lambda T_{n+1}\,, \qquad \text{if } w_n > T_{n+1}\,. \tag{2.2}$$

Now let

$$pr[x_n < w_n < x_n + dx_n] = f(x_n)\,dx_n\,.$$

Since w_n and T_{n+1} are statistically independent, we obtain

$$pr(dw_n, dT_{n+1}) = f(w_n)\,a(T_{n+1})\,dw_n\,dT_{n+1}\,. \tag{2.3}$$

If (2.1) is true, then the contribution to $f(w_{n+1})$ is

$$\frac{1}{\lambda}\, a\left(\frac{w_{n+1}}{\lambda}\right) \int_0^{\frac{w_{n+1}}{\lambda}} f(w_n)\,dw_n \;. \tag{2.4}$$

On the other hand, if (2.2) is true, then letting $x_{n+1} = w_n - T_{n+1}$ we have $T_{n+1} = \frac{1}{\lambda}\left(w_{n+1} - x_{n+1}\right)$ and (2.3) implies

$$pr(dx_{n+1}, dT_{n+1}) = f(x_{n+1}+T_{n+1})\,a(T_{n+1})\,dx_{n+1}\,dT_{n+1}\;,$$

or

$$pr(dx_{n+1}, dT_{n+1}) = f\left(x_{n+1} + \frac{w_{n+1}-x_{n+1}}{\lambda}\right) a\left(\frac{w_{n+1}-x_{n+1}}{\lambda}\right) dx_{n+1}\,dT_{n+1}\,,$$

which in turn implies, in this case, that the contribution to $f(w_{n+1})$ is

$$\frac{1}{\lambda}\int_0^{w_{n+1}} a\left(\frac{w_{n+1}-x_{n+1}}{\lambda}\right) f\left(x_{n+1} + \frac{w_{n+1}-x_{n+1}}{\lambda}\right) dx_{n+1}\;. \tag{2.5}$$

Thus, combining (2.4) and (2.5), and adopting the notation $f(w_n)dw_n = f_n(w)dw$, we have:

$$f_{n+1}(w) = \frac{1}{\lambda}\, a\left(\frac{w}{\lambda}\right) \int_0^{w/\lambda} f_n(x)\,dx + \frac{1}{\lambda}\int_0^{w} a\left(\frac{w-x}{\lambda}\right) f_n\left(x + \frac{w-x}{\lambda}\right) dx. \tag{2.6}$$

This is the main equation which has to be satisfied by the sequence of p.d.fs $f_n(w)$, $(n=1,2,\ldots)$. Therefore, in principle, given $a(T_i)dT_i$, we can solve for $f_n(w)$. However, as we shall see soon, there are some difficulties in obtaining a quite general solution. We, therefore, carry out the analysis with a particular interarrival interval density which is manageable, and at the same time, realistic from the practical point of view (see A. K. Erlang [15]). It should be made clear, even at the expense of repetition, that due to the nature of correlation discussed in Chapter 1,(2.6) is not by any means Lindley's [28] governing integro-difference equation.

It is then at this point that we shall make the assumption that the interarrival intervals T_i have an Erlangian distribution with parameters k and $\frac{1}{\tau}$, i.e.

$$a(T_i)\,dT_i = \frac{1}{\tau}\,\frac{1}{(k-1)!}\left(\frac{T_i}{\tau}\right)^{k-1} e^{-\frac{T_i}{\tau}}\,dT_i \;.$$

Substituting this in (2.6) we have:

$$\begin{aligned} f_{n+1}(w) &= \frac{1}{(\lambda\tau)(k-1)!}\left(\frac{w}{\lambda\tau}\right)^{k-1} e^{-\frac{w}{\lambda\tau}} \int_0^{w/\lambda} f_n(x)\,dx \;+ \\ &+ \frac{e^{-w/\lambda\tau}}{(\lambda\tau)(k-1)!}\int_0^{w}\left(\frac{w-x}{\lambda\tau}\right)^{k-1} e^{x/\lambda\tau}\, f_n\!\left(x+\frac{w-x}{\lambda}\right)dx \;. \end{aligned} \tag{2.7}$$

Now since

$$f_1(w) = \frac{1}{(\lambda\tau)(k-1)!}\left(\frac{w}{\lambda\tau}\right)^{k-1} e^{-\frac{w}{\lambda\tau}}$$

we can build up all the $f_n(x)$'s consecutively. However, this is rather tedious, and one would imagine that simpler methods exist. One of the, at least, more hopeful methods of solving (2.7) is to apply the Laplace transformation.

This is the subject of the next section.

2.2 Differential - Difference Equation for the Laplace Transform of p.d.f. of Waiting Time

We write

$$f_{n+1}(w) = \frac{w^{k-1}\, e^{-w/\lambda\tau}}{(k-1)!(\lambda\tau)^k}\, m_n(w) + \frac{1}{(k-1)!(\lambda\tau)^k}\, \ell_{nk}(w)$$

where

$$m_n(w) = \int_0^{w/\lambda} f_n(x)\,dx \quad ,$$

$$\ell_{nk}(w) = \int_0^{w} y^{k-1}\, e^{-y/\lambda\tau}\, f_n\left(w-y+\frac{y}{\lambda}\right) dy \quad ,$$

and

$$y = w - x \quad ,$$

and further let

$$f_n^*(z) = \int_0^{\infty} e^{-zw}\, f_n(w)\,dw \quad . \tag{2.8}$$

Then $f_n^*(z)$, $m_n^*(z)$ and $\ell_{nk}^*(z)$ [defined as in (2.8)] are the Laplace transforms of $f_n(w)$, $m_n(w)$ and $\ell_{nk}(w)$ respectively and we have

$$m_n^*(z) = \int_0^{\infty} e^{-zw} dw \int_0^{w/\lambda} f_n(x)\,dx = \frac{f_n^*(\lambda z)}{z} \quad ,$$

and

$$\ell^*_{nk}(z) = \int_0^\infty e^{-zw}\, \ell_{nk}(w)\,dw = \frac{1}{z}\int_0^\infty e^{-zw} \frac{\partial}{\partial w}\, \ell_{nk}(w)\,dw$$

$$= \frac{1}{z}\, L\{\frac{\partial}{\partial w}\, \ell_{nk}(w)\} \quad ,$$

where L{ } denotes the Laplace transform of the quantity inside the brackets.

Now

$$\frac{\partial}{\partial w}\, \ell_{nk}(w) = w^{k-1}\, e^{-w/\lambda\tau}\, f_n\left(\frac{w}{\lambda}\right) + \int_0^w y^{k-1}\, e^{-y/\lambda\tau}\, \frac{\partial}{\partial w}\, f_n(w-y+\frac{y}{\lambda})\,dy$$

$$= w^{k-1}\, e^{-w/\lambda\tau} f_n\left(\frac{w}{\lambda}\right) + \frac{\lambda}{1-\lambda} \int_0^w y^{k-1}\, e^{-y/\lambda\tau}\, \frac{\partial}{\partial y}\, f_n(w-y+\frac{y}{\lambda})\,dy$$

$$= \frac{1}{1-\lambda}\, w^{k-1}\, e^{-w/\lambda\tau}\, f_n\left(\frac{w}{\lambda}\right) + \frac{1}{(1-\lambda)\tau}\, \ell_{nk}(w) -$$

$$- \frac{\lambda}{(1-\lambda)}\, (k-1) \int_0^w y^{k-2}\, e^{-y/\lambda\tau}\, f_n(w-y+\frac{y}{\lambda})\,dy$$

which implies that

$$z\,\ell^*_{nk}(z) = \frac{\lambda}{1-\lambda}\, (-)^{k-1}\, \frac{\partial^{k-1}}{\partial z^{k-1}}\, f^*_n\left(\lambda z + \frac{1}{\tau}\right) +$$

$$+ \frac{1}{(1-\lambda)\tau}\, \ell^*_{nk}(z) - \frac{\lambda(k-1)}{(1-\lambda)}\, \ell^*_{n,k-1}(z) \quad ,$$

or

$$\left[z - \frac{1}{(1-\lambda)\tau}\right] \ell^*_{nk}(z) =$$

$$= \frac{\lambda(-)^{k-1}}{(1-\lambda)} \frac{\partial^{k-1}}{\partial z^{k-1}} f^*_n\left(\lambda z + \frac{1}{\tau}\right) - \frac{\lambda(k-1)}{(1-\lambda)} \ell^*_{n,k-1}(z) \quad .$$

If we now let $\left[z - \frac{1}{(1-\lambda)\tau}\right] = Y$ and observe the fact that

$$\ell^*_{n1}(z) = \frac{\lambda}{(1-\lambda)Y} \left[f^*_n\left(\lambda z + \frac{1}{\tau}\right) - f^*_n(z)\right] \; ,$$

we immediately see that

$$Y\ell^*_{n2}(z) = -\frac{\lambda}{1-\lambda} \frac{\partial}{\partial z} f^*_n\left(\lambda z + \frac{1}{\tau}\right) - \frac{\lambda^2}{(1-\lambda)^2 Y} \left[f^*_n\left(\lambda z + \frac{1}{\tau}\right) - f^*_n(z)\right] \quad ,$$

$$Y\ell^*_{n3}(z) = \frac{\lambda}{1-\lambda} \frac{\partial^2}{\partial z^2} f^*_n\left(\lambda z + \frac{1}{\tau}\right) + 2\left(\frac{\lambda}{1-\lambda}\right)^2 \frac{1}{Y} \frac{\partial}{\partial z} f^*_n\left(\lambda z + \frac{1}{\tau}\right) +$$

$$+ 2\left(\frac{\lambda}{1-\lambda}\right)^3 \frac{1}{Y^2} \left[f^*_n\left(\lambda z + \frac{1}{\tau}\right) + f^*_n(z)\right] \; ,$$

and in general

$$\begin{aligned}
Y\ell^*_{nk}(z) = {} & (-)^{k-1} \left(\frac{\lambda}{1-\lambda}\right) \frac{\partial^{k-1}}{\partial z^{k-1}} f^*_n\left(\lambda z + \frac{1}{\tau}\right) \quad + \\
& + (-)^{k-1} P(k-1,1) \left(\frac{\lambda}{1-\lambda}\right)^2 \frac{1}{Y} \frac{\partial^{k-2}}{\partial z^{k-2}} f^*_n\left(\lambda z + \frac{1}{\tau}\right) \quad + \\
& + (-)^{k-1} P(k-1,2) \left(\frac{\lambda}{1-\lambda}\right)^3 \frac{1}{Y^2} \frac{\partial^{k-3}}{\partial z^{k-3}} f^*_n\left(\lambda z + \frac{1}{\tau}\right) \quad + \\
& \vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \\
& + (-)^{k-1} P(k-1,k-1) \left(\frac{\lambda}{1-\lambda}\right)^k \frac{1}{Y^{k-1}} \left[f^*_n\left(\lambda z + \frac{1}{\tau}\right) - f^*_n(z)\right] ,
\end{aligned}$$

where $P(n,r)$ is the number of permutations of n things taken r at a time, i.e.

$$P(n,r) = \frac{n!}{(n-r)!} .$$

Thus with $\qquad \dfrac{\lambda}{(1-\lambda)Y} = \dfrac{\lambda\tau}{(1-\lambda)z\tau-1} = Z$

we have:

$$\ell^*_{nk}(z) = (-)^{k-1}\, z\, \frac{\partial^{k-1}}{\partial z^{k-1}}\, f^*_n\left(\lambda z + \frac{1}{\tau}\right) \quad +$$

$$+(-)^{k-1} \sum_{j=2}^{k} P(k-1),j-1)\; z^j\, \frac{\partial^{k-j}}{\partial z^{k-j}}\, f^*_n\left(\lambda z + \frac{1}{\tau}\right) + (k-1)!\,(-Z)^k\; f^*_n(z).$$

Now

$$f^*_{n+1}(z) = \frac{(-)^{k-1}}{(k-1)!(\lambda\tau)^k}\, \frac{\partial^{k-1}}{\partial z^{k-1}}\, \frac{f^*_n\left(\lambda z + \frac{1}{\tau}\right)}{z + \frac{1}{\lambda\tau}} + \frac{1}{(k-1)!\,(\lambda\tau)^k}\, \ell^*_{nk}(z) \quad .$$

But by Leibniz's theorem

$$\frac{\partial^{k-1}}{\partial z^{k-1}}\, \frac{f^*_n\left(\lambda z + \frac{1}{\tau}\right)}{z + \frac{1}{\lambda\tau}} = \sum_{j=0}^{k-1} (-)^j \binom{k-1}{j} \frac{j!}{\left(z + \frac{1}{\lambda\tau}\right)^{j+1}}\, \frac{\partial^{k-1-j}}{\partial z^{k-1-j}}\, f^*_n\left(\lambda z + \frac{1}{\tau}\right),$$

which implies that

$$(k-1)!\,(\lambda\tau)^k\; f^*_{n+1}(z) =$$

$$= (-)^{k-1} \sum_{j=0}^{k-1} (-)^j \binom{k-1}{j} \frac{j!}{\left(z + \frac{1}{\lambda\tau}\right)^{j+1}}\, \frac{\partial^{k-1-j}}{\partial z^{k-1-j}}\, f^*_n\left(\lambda z + \frac{1}{\tau}\right) \quad +$$

$$+(-)^{k-1}\, z\, \frac{\partial^{k-1}}{\partial z^{k-1}}\, f_n^*\left(\lambda z+\frac{1}{\tau}\right) \quad +$$

$$+(-)^{k-1}\sum_{j=2}^{k} P(k-1,j-1)\, z^j\, \frac{\partial^{k-j}}{\partial z^{k-j}}\, f_n^*\left(\lambda z+\frac{1}{\tau}\right) + (k-1)!\,(-z)^k\, f_n^*(z)\,,$$

or

$$(-)^{k-1}\,(k-1)!\,(\lambda\tau)^k\, f_{n+1}^*(z) = \left(\frac{1}{z+\frac{1}{\lambda\tau}} + z\right)\frac{\partial^{k-1}}{\partial z^{k-1}}\, f_n^*\left(\lambda z+\frac{1}{\tau}\right) +$$

$$+\sum_{j=1}^{k-1} P(k-1,j)\left(z^{j+1}+\frac{(-)^j}{\left(z+\frac{1}{\lambda\tau}\right)^{j+1}}\right)\frac{\partial^{k-1-j}}{\partial z^{k-1-j}}\, f_n^*\left(\lambda z+\frac{1}{\tau}\right) - (k-1)!\,z^k\, f_n^*(z)\,,$$

or

$$f_{n+1}^*(z) = \frac{(-)^{k-1}}{(k-1)!\,(\lambda\tau)^k}\left\{\left(z+\frac{\lambda\tau}{1+\lambda\tau z}\right)\frac{\partial^{k-1}}{\partial z^{k-1}}\, f_n^*\left(\lambda z+\frac{1}{\tau}\right) + \right.$$

$$+\sum_{j=1}^{k-1} P(k-1,j)\left(z^{j+1}+\frac{(-)^j(\lambda\tau)^{j+1}}{(1+\lambda\tau z)^{j+1}}\right)\frac{\partial^{k-1-j}}{\partial z^{k-1-j}}\, f_n^*\left(\lambda z+\frac{1}{\tau}\right) -$$

$$\left. -(k-1)!\; z^k\, f_n^*(z)\right\}. \qquad (2.9)$$

It is comforting to note that

$$f_{n+1}^*(0) = \frac{(-)^{k-1}}{(k-1)!\,(\lambda\tau)^k}\left[-(k-1)!\,(-\lambda\tau)^k\right] = 1$$

as it, of course, must.

At this point it is evident that it is hopeless to try to solve the differential difference equation (2.9) in this degree of generality. However it should be mentioned that it is of potential value. This will become evident in Chapter 3 where we shall show the time dependent state probabilities depend directly on the moments of $f_n(w)$. More comments on this will be made in Chapter 8. We are then forced to make a further simplification, namely to let $k = 1$, so that the interarrival intervals T_i have E_1, or a negative exponential distribution.

2.3 Waiting Time Density When Interarrival Intervals Have Negative Exponential Distribution

With $k = 1$ (2.9) reduces to

$$f^*_{n+1}(z) = \frac{1}{\lambda\tau}\left\{\left[\frac{\lambda\tau}{(1-\lambda)z\tau-1} + \frac{\lambda\tau}{1+\lambda\tau z}\right] f^*_n\left(\lambda z + \frac{1}{\tau}\right) - \left[\frac{\lambda\tau}{(1-\lambda)z\tau-1}\right] f^*_n(z)\right\},$$

or

$$f^*_{n+1}(z) = \frac{\lambda z\tau f^*_n\left(\lambda z + \frac{1}{\tau}\right) - \lambda(1+\lambda z\tau) f^*_n(z)}{(1+\lambda z\tau)\left[(1-\lambda)\lambda z\tau - \lambda\right]} . \qquad (2.10)$$

This was obtained directly by B. W. Conolly in [11]. His solution, together with several of its conclusions, will be quoted here with merely an indication of their proof since Chapter 3 is based on them. He defines quantities

$$s_n = \sum_{i=0}^{n} \lambda^{-i} , \qquad (2.11)$$

and shows inductively that

$$f_n^*(z) = \frac{s_0\ s_1\ \ldots\ s_{n-1}}{(s_0+\lambda z\tau)(s_1+\lambda z\tau)\ldots(s_{n-1}+\lambda z\tau)} \tag{2.12}$$

is the solution of (2.10). This is easily verified since if (2.12) is true then $f_n^*\left(\lambda z + \frac{1}{\tau}\right) = \frac{(1+\lambda z\tau)}{\lambda^n(s_n+\lambda z\tau)} f_n^*(z)$ and $\frac{1}{\lambda^n} - \lambda = s_n(1-\lambda)$. He then decomposes this into partial fractions giving:

$$f_n^*(z) = \sum_{r=0}^{n-1} \frac{g_{nr}}{(s_r+\lambda z\tau)}\ , \tag{2.13}$$

where

$$g_{n0} = \lambda^{n-1}\ s_{n-1}\ ,$$

$$g_{n1} = -\lambda^{2n-3}\ s_{n-2}\ s_{n-1}\ /\ s_0\ ,$$

and

$$g_{nr} = (-)^r\ \frac{\lambda^{(r+1)(n-\frac{1}{2}r-1)} s_{n-1}\ s_{n-2}\ \ldots\ s_{n-r-1}}{s_0\ s_1\ \ldots\ s_{r-1}}\ , \tag{2.14}$$

thus obtaining

$$f_n(y) = \frac{1}{\lambda\tau} \sum_{r=0}^{n-1} g_{nr}\ e^{-\frac{s_r}{\lambda\tau} y}\ . \tag{2.15}$$

Then, by substituting $z = \frac{w}{\lambda} - \frac{1}{\lambda\tau}$ in (2.12) with $n = n+1$, he shows that

$$f_{n+1}^*\left(\frac{w}{\lambda} - \frac{1}{\lambda\tau}\right) = \frac{\lambda^{n+1}\ s_n}{\lambda\tau}\ \frac{f_n^*(w)}{w} = \frac{\lambda^{n+1}\ s_n}{\lambda\tau}\ F_n^*(w)\ , \tag{2.16}$$

(where $F_n(y) = \int_0^y f_n(x)dx$ is the distribution function of the waiting time of the nth arrival C_n).

Consequently, upon inverting (2.16), he shows that the distribution function $F_n(y)$ is related to $f_{n+1}(y)$ by the equation

$$F_n(y) = \lambda\tau e^{y/\tau} f_{n+1}(\lambda y)/\lambda^n s_n \ . \tag{2.17}$$

We shall now prove that a non-zero steady state solution exists only if $\underline{\lambda < 1}$.

Consider the sequence of distribution functions $F_{n+1}(y)$, $n = 0,1,2,\ldots$. This sequence converges completely to a distribution function $F(y)$ up to an additive constant only if $F^*_{n+1}(z)$, $n = 0,1,2,\ldots$, the sequence of Laplace transforms of $F_{n+1}(y)$, converges to a limit (see M. Loève [29]). Now since $F^*_{n+1}(z) = \frac{f^*_n(z)}{z}$, (2.12) implies that

$$\frac{1}{F^*_{n+1}(z)} = z \prod_{j=0}^{n} \left(1 + \frac{\lambda z\tau}{s_j}\right) \ .$$

According to Theorem 59 of W. L. Ferrar's book on convergence [17] the above product converges to a finite non-zero limit only if $\sum_{j\geq 0} \frac{1}{s_j}$ converges. If this sum diverges, the above product $\to +\infty$.

To examine the convergence of $\sum_{j\geq 0} s_j^{-1}$ we consider the ratio s_j^{-1}/s_{j-1}^{-1} . This is $\left(1-1/\lambda^j s_j\right)$ and if $\lambda < 1$ it approaches

λ as $j\to\infty$. Thus, $\sum_{j\geq 0} s_j^{-1}$ certainly converges if $\lambda < 1$. If $\lambda = 1$, $\sum_{j\geq 0} s_j^{-1}$ is the harmonic series, and consequently divergent, while if $\lambda > 1$ the series is worse than the harmonic series.

Thus, the sequence $1/F^*_{n+1}(z)$ converges as $n\to\infty$ if and only if $\lambda < 1$, and under these circumstances $F_n(y)$ converges to a distribution function $F(y)$. If $\lambda \geq 1$ $F^*_{n+1}(z)\to 0$, which means that $F_n(y)\to 0$ for all y in $(0,\infty)$ and all the probability is concentrated at ∞ .

Thus, we have proved that a steady state exists only when $\lambda < 1$. This applies to the other related processes of this model as well. We can now easily show if as $n\to\infty$ $f_n(y)\to f(y)$ then,

$$f(y) = \frac{1}{\lambda\tau} \sum_{r\geq 0} g_r \, e^{-\frac{s_r}{\lambda\tau}y} , \tag{2.18}$$

where

$$g_r = \frac{(-)^r \lambda^{\frac{1}{2}r(r-1)}}{(1-\lambda)^2(1-\lambda^2)(1-\lambda^3)\ldots(1-\lambda^r)} . \tag{2.19}$$

2.4 Moments of Waiting Time

The moments can be obtained either from (2.12) by differentiation with respect to z, then putting $z = 0$, or directly from (2.15) by integration. From (2.12) we obtain

$$E_n(w) = \lambda\tau \sum_{r=0}^{n-1} s_r^{-1} \tag{2.20}$$

and

$$\mathrm{Var}_n(w) = (\lambda\tau) \sum_{r=0}^{n-1} s_r^{-2} \ , \tag{2.21}$$

with obvious notation. Rather than trying to obtain the solution of (2.9) for other particular values of k ,we shall continue to concentrate on $k = 1$,and with this choice, investigate other features of this queueing model.

This means that we are working, in a sense, in the M/M/1 framework, since arrivals follow negative exponential distribution with parameter $\frac{1}{\tau}$ as do the services with parameter $\frac{1}{\lambda\tau}$;though,of course, these are really deterministic in view of the correlation pattern chosen.

CHAPTER 3
STATE PROBABILITIES

Here we consider the process $\xi(t)$ representing the number of customers in the system at time t. First, however, we examine the sequence $\{\xi_n\}$, where $\xi_n = \xi(t_n - 0)$, and t_n is an arrival epoch. In words we consider first the number in the system immediately before an arrival takes place.

3.1 State Probabilities at Arrival Epochs

Let

$$p_k^{(n)} = \mathrm{pr}[\xi_n = k] . \tag{3.1}$$

It is evident that k can have any value between zero and $n-1$. We shall first prove that for $k=0$ we have

$$p_0^{(n)} = \frac{1}{\lambda^{n-1} S_{n-1}} , \tag{3.2}$$

and for $1 \leq k \leq n-2$

$$p_k^{(n)} = \frac{S_{n-k-1}^{(k)}}{\lambda^{n-k-1} S_{n-k-1}} - \frac{S_{n-k}^{(k-1)}}{\lambda^{n-k} S_{n-k}} , \tag{3.3}$$

where S_n is given by (2.11),

$$S_n = \sum_{i=0}^{n} s_i^{-1} , \tag{3.4}$$

and $S_n^{(k)}$ denotes the sum of all the terms in $(S_n)^k$, but with unit coefficients, i.e., $S_n^{(k)}$ is the homogeneous form of degree k of the sequence $\{s_i^{-1}\}$ $(i=1,2,\ldots n)$, and

$$p_{n-1}^{(n)} = 1 - \frac{S_1^{(n-2)}}{\lambda s_1} \quad . \tag{3.5}$$

To start with the (n+1)th arrival, C_{n+1}, does not have to wait if

$$w_n < T_{n+1} \quad . \tag{3.6}$$

Now

$$\mathrm{pr}[dw_n, dT_{n+1}] = e^{-\frac{T_{n+1}}{\tau}} f(w_n)\,dT_{n+1}\,dw_n/\tau \quad .$$

Hence

$$\begin{aligned} p_0^{(n+1)} &= \frac{1}{\tau}\int_0^\infty e^{-x/\tau}\,dx \int_0^x f_n(y)\,dy \\ &= \frac{1}{\tau}\int_0^\infty e^{-x/\tau}\,F_n(x)\,dx \, , \\ &= \frac{\lambda}{\lambda^n s_n}\int_0^\infty f_{n+1}(\lambda x)\,dx \, , \\ &= 1/\lambda^n s_n \quad , \end{aligned}$$

using (2.17). This proves (3.2). The remainder of the proof consists in finding $P_k^{(n)}$, the probability that at t_n-0 the system contains k <u>or more</u>. Then $p_k^{(n)}$ is found from $P_k^{(n)} - P_{k+1}^{(n)}$. Now

$$P_k^{(n+1)} = \mathrm{pr}[0 < T_{n-k+2} + T_{n-k+3} + \ldots + T_{n+1} < w_{n-k+1}] \quad . \tag{3.7}$$

(Recall that T_i's are interarrival intervals, and w_n is the waiting time, including service, of the nth customer.) The p.d.f. of the sum of k identically distributed T_i's is

$$\frac{1}{\tau^k (k-1)!} x^{k-1} e^{-x/\tau} .$$

Thus we have

$$P_k^{(n+1)} = \frac{1}{(k-1)!\tau^k} \int_0^\infty e^{-x/\tau} x^{k-1} dx \int_x^\infty f_{n-k+1}(y)dy$$

$$= 1 - \frac{1}{(k-1)!\tau^{k-1}\lambda^{n-k+1}s_{n-k+1}} \int_0^\infty x^{k-1} f_{n-k+2}(\lambda x) d\lambda x ,$$

again using (2.17). Now defining $E_n(\ )$ to be the expected value in the distribution of the waiting time of the nth arrival, this reduces to

$$P_k^{(n+1)} = 1 - \frac{1}{(k-1)!(\lambda\tau)^{k-1}\lambda^{n-k+1}s_{n-k+1}} E_{n-k+2}(x^{k-1}) . \quad (3.8)$$

$E_n(x^k)$, then, can easily be found by extracting the coefficient of z^k in the expansion of (2.12) in ascending powers of z and subsequently multiplying by $(-)^k k!$. It is a matter of routine to show that coefficient of z^k in (2.12) is

$$(-\lambda\tau)^k S_{n-1}^{(k)}$$

so that

$$P_k^{(n+1)} = 1 - \frac{S_{n-k+1}^{(k-1)}}{\lambda^{n-k+1} s_{n-k+1}} \quad . \tag{3.9}$$

Now since

$$p_k^{(n+1)} = P_k^{(n+1)} - P_{(k+1)}^{(n+1)} \quad \text{for} \quad 1 \leq k \leq n-1$$

we have

$$p_k^{(n+1)} = \frac{S_{n-k}^{(k)}}{\lambda^{n-k} s_{n-k}} - \frac{S_{n-k+1}^{(k-1)}}{\lambda^{n-k+1} s_{n-k+1}}$$

which proves (3.3). (3.5) follows immediately since

$$p_{n-1}^{(n)} = P_{n-1}^{(n)} \ .$$

It should be pointed out here that one can also write

$$E_n(y^k) = (\lambda\tau)^k \, k! \sum_{r=0}^{n-1} \frac{g_{nr}}{s_r^{k+1}}$$

which implies that

$$p_k^{(n)} = \frac{1}{\lambda^{n-k-1} s_{n-k-1}} \sum_{r=0}^{n-k-1} \frac{g_{n-k,r}}{s_r^{k+1}} - \frac{1}{\lambda^{n-k} s_{n-k}} \sum_{r=0}^{n-k} \frac{g_{n-k+1,r}}{s_r^{k}} \quad .$$

This formula does not, however, have much to offer except, perhaps, mathematical elegance; while from the practical point of view (3.3) is convenient for numerical calculation since $s_i^{-1} \to 0$ quite rapidly and particularly for $\lambda < \frac{1}{2}$.

3.2 Mean and Variance of the Number in the System

Let $E^{(n)}(k)$ and $\mathrm{Var}^{(n)}(k)$ represent the mean and variance of the number in the system at the moment of arrival of the nth customer. Then from (3.3) we have

$$E^{(n)}(k) = \sum_{k=1}^{n-1} kp_k^{(n)}$$

$$= \left(\frac{S_{n-2}^{(1)}}{\lambda^{n-2} s_{n-2}} - \frac{S_{n-1}^{(0)}}{\lambda^{n-1} s_{n-1}}\right) + 2\left(\frac{S_{n-3}^{(2)}}{\lambda^{n-3} s_{n-3}} - \frac{S_{n-2}^{(1)}}{\lambda^{n-2} s_{n-2}}\right)$$

$$+ 3\left(\frac{S_{n-4}^{(3)}}{\lambda^{n-4} s_{n-4}} - \frac{S_{n-3}^{(2)}}{\lambda^{n-3} s_{n-3}}\right) + \dots +$$

$$(n-2)\left(\frac{S_1^{(n-2)}}{\lambda s_1} - \frac{S_2^{(n-3)}}{\lambda^2 s_2}\right) + (n-1)\left(1 - \frac{S_1^{(n-2)}}{\lambda s_1}\right),$$

which implies that

$$E^{(n+1)}(k) = n - \sum_{j=1}^{n} \frac{S_j^{(n-j)}}{\lambda^j s_j}\ ; \tag{3.10}$$

similarly

$$E^{(n)}(k^2) = \left(\frac{S_{n-2}^{(1)}}{\lambda^{n-2}s_{n-2}} - \frac{S_{n-1}^{(0)}}{\lambda^{n-1}s_{n-1}}\right) + 4\left(\frac{S_{n-3}^{(2)}}{\lambda^{n-3}s_{n-3}} - \frac{S_{n-2}^{(1)}}{\lambda^{n-2}s_{n-2}}\right) +$$

$$+ 9\left(\frac{S_{n-4}^{(3)}}{\lambda^{n-4}s_{n-4}} - \frac{S_{n-3}^{(2)}}{\lambda^{n-3}s_{n-3}}\right) + \ldots +$$

$$+ (n^2-4n+4)\left(\frac{S_1^{(n-2)}}{\lambda s_1} - \frac{S_2^{(n-3)}}{\lambda^2 s_2}\right) + (n^2-2n+1)\left(1 - \frac{S_1^{(n-2)}}{\lambda s_1}\right),$$

or

$$E^{(n+1)}(k^2) = n^2 - \sum_{j=1}^{n} (2n+1-2j)\frac{S_j^{(n-j)}}{\lambda^j s_j} . \tag{3.11}$$

$Var^{(n)}(k)$ can then easily be obtained from (3.10) and (3.11).

3.3 State Probabilities at an Arbitrary Epoch

The problem of evaluating state probabilities at an arbitrary epoch t is rather complicated. The obvious method of approach, which works for Markov processes, would be to consider all the possible states at the time of the last arrival, and then for each such state to arrange for the appropriate number of services to be completed during the portion of an interarrival interval under consideration. This, however, requires knowledge of the epochs of all the previous arrivals and a very complicated conditioning.

Instead, we can assert that because of the fundamental properties of negative exponential distribution

$$pr[\xi(t)=k] = p_k^{(n+1)} \tag{3.12}$$

for any t in the interval

$$\Sigma_n < t < \Sigma_{n+1} \ .$$

This statement would not be true for an arbitrary inter-arrival interval distribution.

3.4 Passage to the Steady State

Non-zero steady state values come about if λ , the traffic intensity, is less than unity. In that case, steady state state probabilities could easily be obtained from (3.2) and (3.3) by letting $n \to \infty$. We shall here observe that if as $n \to \infty$ $S_n \to S$, which it does only if $\lambda < 1$, for each fixed $k > 0$

$$p_k^{(n)} \to \bar{p}_k = (1-\lambda)(S-1)^{(k)} \quad . \tag{3.13}$$

For $k = 0$

$$p_0^{(n)} \to \bar{p}_0 = 1 - \lambda \quad . \tag{3.14}$$

These follow immediately by noting that

$$\lim_{n\to\infty} \lambda^{n-m} s_{n-m} = \frac{1}{1-\lambda}$$

and

$$S^{(k)} - S^{(k-1)} = (S-1)^{(k)} \quad .$$

Note that (3.9) implies

$$\overline{P}_k = 1 - (1-\lambda)\, S^{(k-1)}$$

so that, for example,

$$\overline{P}_1 = \lambda \quad ,$$

as it should.

It should, perhaps, be mentioned here that (3.13) is quite similar to the well-known steady state result for M/M/1. We shall not make any comment about the comparison here for this will be carried out systematically in Part II. (3.14) is, in fact, exactly the same as the result for M/M/1.

A further observation which should be made here is that the methods used above to find the state probabilities are quite general. Whenever the waiting time density can be found so can $p_k^{(n)}$, for (3.7) is true regardless of the nature of the interarrival density. Hence, given the waiting time and interarrival densities, (3.6) and (3.7) lead immediately to the time dependent state probabilities.

CHAPTER 4

BUSY PERIOD

4.1 Preliminary Remarks

In Chapter 1 the Busy Period (b.p.) was defined to be the uninterrupted time interval from the end of one idle period until the beginning of the next. We shall here first define auxiliary functions which will eventually lead to the probability density function of the b.p. as such.

Let $b_n(t)$ be the joint probability and p.d.f. of the following events:

i) b.p. lasts for a time between t and $t+dt$,

ii) it is composed of the service of exactly n customers.

Consider first $b_1(t)$. This single service b.p. requires that the immediate predecessor of the initiating customer must have arrived at a time t/λ before. We set up our b.p. process on the assumption that the predecessor and all his antecedents completed service before the new arrival. The p.d.f. of an arrival interval of length t/λ is $\frac{1}{\lambda\tau} e^{-t/\lambda\tau}$, and with this we couple the requirement of no arrival in $(0,t)$ which has probability $e^{-t/\tau}$. Hence

$$b_1(t) = \frac{1}{\lambda\tau} e^{-s_1 t/\tau} , \tag{4.1}$$

where s_1 and all the s_i (which will appear) are defined by (2.11). If we then let $b_n^*(z)$ be the Laplace transform (defined by (2.8)) of $b_n(t)$, clearly

$$b_1^*(z) = \frac{s_0}{(s_1+\tau z)} \quad . \tag{4.2}$$

In the case of $b_2(t)$ suppose the first customer has service time x. Then the second has $t-x$. An arrival must occur at time $(t-x)/\lambda$ and there must be no further arrivals in the interval $[(t-x)/\lambda, t]$. But a restriction is that $(t-x)/\lambda < x$, or $x > \frac{t}{\lambda s_1}$. On the other hand x cannot exceed t. Hence we have

$$b_2(t) = \frac{1}{(\lambda\tau)^2} \int_{\frac{t}{\lambda s_1}}^{t} e^{-x/\lambda\tau}\, e^{-\frac{(t-x)}{\lambda\tau}}\, e^{-\frac{1}{\tau}\{t-(\frac{t-x}{\lambda})\}}\, dx \ ,$$

or

$$b_2(t) = \frac{1}{\lambda\tau}\left[e^{-\frac{s_2}{s_1\tau}t} - e^{\frac{s_1}{s_0\tau}t}\right], \tag{4.3}$$

the Laplace transform of which is

$$b_2^*(z) = \frac{s_0}{\lambda^2(s_2+s_1\tau z)(s_1+s_0\tau z)} \quad . \tag{4.4}$$

For $b_3(t)$ we require $t = \lambda(T_1+T_2+T_3)$ and the following inequalities:

$$0 < T_2 < \lambda T_1 \quad ,$$

$$0 < T_3 < \lambda(T_1+T_2) - T_2 \quad ,$$

which in turn imply that

$$\frac{t}{\lambda^2 s_2} < \lambda T_1 < t \quad ,$$

and

$$\max\left(0 \,,\, \frac{t}{\lambda} - \lambda s_1 T_1\right) < \lambda T_2 < \min\left(\lambda^2 T_1 \,,\, t - \lambda T_1\right) .$$

Now writing down pr(dT_1, dT_2 , dt , no arrival in $t-T_2-T_3$) and integrating out T_1 and T_2 over their proper ranges we have

$$b_3(t) =$$

$$= \frac{e^{-t/\tau}}{(\lambda\tau)^3} \left[\int_{\frac{t}{\lambda^2 s_2}}^{\frac{t}{\lambda s_1}} e^{-x/\lambda\tau}\, dx \int_{\frac{t}{\lambda} - s_1 x}^{x} dy + \int_{\frac{t}{\lambda s_1}}^{t} e^{-x/\lambda\tau}\, dx \int_0^{t-x} dy \right],$$

(where $x = \lambda T_1$ and $y = \lambda T_2$) ,

which leads to

$$b_3(t) = \frac{s_2}{\tau} e^{-\frac{s_3}{s_2\tau}t} - \frac{s_1^2}{\tau} e^{-\frac{s_2}{s_1\tau}t} + \frac{1}{\lambda\tau} e^{-\frac{s_1}{s_0\tau}t} , \tag{4.5}$$

with the Laplace transform

$$b_3^*(z) = \frac{s_0 s_1}{\lambda^3 (s_3+s_2\tau z)(s_2+s_1\tau z)(s_1+s_0\tau z)} . \tag{4.6}$$

Integrating out T_1, T_2, T_3 in pr(dt_1, dT_2, dT_3, dt, no arrival in $t-T_2-T_3-T_4$) over the following ranges:

$$\frac{t}{\lambda^4 s_3} < T_1 < \frac{t}{\lambda}$$

$$\max\left(0 , \frac{t}{\lambda^3} - T_1 s_2\right) < T_2 < \min\left(\lambda T_1 , \frac{t}{\lambda} - T_1\right)$$

$$\max\left(0 , \frac{t}{\lambda^2} - T_1 s_1 - T_2\right) < T_3 < \min[\lambda(T_1+T_2) - T_2 , \frac{t}{\lambda} - T_1 - T_2]$$

and taking the Laplace transform we again get

$$b_4^*(z) = \frac{s_0 s_1 s_2}{\lambda^4 (s_4+s_3\tau z)(s_3+s_2\tau z)(s_2+s_1\tau z)(s_1+s_0\tau z)} .$$

The previous results strongly suggest the form of the generalization for $b_n^*(z)$, and it is our intention now to prove that this generalization is valid and then, by inversion to obtain $b_n(t)$, the function desired. To carry out these steps we have found it expedient to use Prabhu's argument first employed in connection with the b.p. for the system M/G/1 [31]. This will form the subject matter of the next section.

4.2 Introduction and Evaluation of Prabhu's $g_n(x,t)$ Function

N. U. Prabhu [31] introduced a joint probability and p.d.f. $g_n(x,t)$ of the following composite event:

(i) An arrival C_1 takes place at $t = 0$;

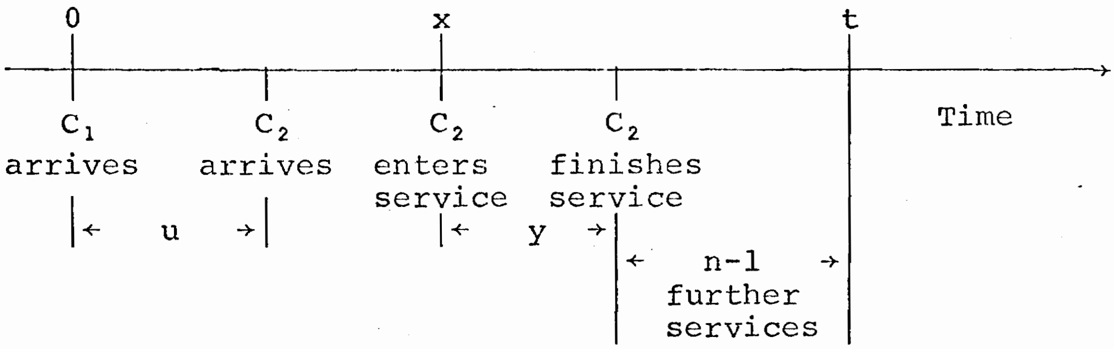
(ii) It is given that the time to serve C_1 and all his predecessors already present at $t = 0$ requires time x ;

(iii) During the interval (x,t) n further customers are served without interruption, and the busy period terminates in $(t,t+dt)$.

He then shows that $g_n(x,t)$ satisfies the following integro-difference equation

$$g_n(x,t) = \int_0^x a(u)\,du \int_0^{t-x} b(y)\, g_{n-1}(x+y-u,t-u)\,dy \, , \qquad (4.7)$$

where u is time interval separating C_1's and his successor C_2's arrival and y is C_2's service time, $a(u)$ is inter-arrival density and $b(y)$ is service density. The following diagram is meant to explain (4.7).



In our model since $u = \frac{y}{\lambda}$ and $a(u) = \frac{1}{\tau} e^{-u/\tau}$ (4.7) reduces to

$$g_n(x,t) = \frac{1}{\lambda\tau}\int_0^{\min(\lambda x,\, t-x)} e^{-y/\lambda\tau}\, g_{n-1}\left(x+y-\frac{y}{\lambda},\, t-\frac{y}{\lambda}\right)dy \; . \qquad (4.8)$$

The upper limit of integration arises because y/λ has to be less than x too, otherwise there would be a gap in service at x. Now if when C_1 arrives the system is empty we may write

$$b_n(t) = \frac{1}{\lambda\tau}\int_0^t e^{-x/\lambda\tau}\, g_{n-1}(x,t)\,dx \; . \qquad (4.9)$$

Thus knowledge of $g_n(x,t)$ leads to $b_n(t)$. We shall then find $g_n(x,t)$ here and derive $b_n(t)$ from it in the next section.

From first principles

$$g_1(x,t) = \frac{1}{\lambda\tau} e^{-t/\tau} \quad \text{for} \quad \frac{t}{\lambda s_1} < x < t \;, \qquad (4.10)$$

with $x = \lambda T_1$. No integration is required because in our system the service of the single subsequent customer after C_1 occupies $t-x$, and hence an arrival interval $t-x$ is implied. The restriction on the range of x arises from the fact that $\frac{(t-x)}{\lambda}$ must be less than x or $\frac{t}{\lambda s_1} < x$. And, in fact, for $g_n(x,t)$ since we must have n further arrivals subject to the restrictions

$$0 < T_i < x + \lambda(T_2+\ldots+T_{i-1})-(T_2+T_3+\ldots+T_{i-1}) \ , \qquad \text{for} \quad i = 2,3, \ldots n+1 \ , \tag{4.11}$$

and also

$$t = \lambda(T_1+T_2+\ldots+T_n+T_{n+1}) \ , \tag{4.12}$$

then it must follow that

$$t - x < \lambda x + \lambda^2(T_2+T_3+\ldots+T_n) \ ,$$

or by using (4.11) again with $i = n$

$$t - x < \lambda x + \lambda^2 x + \lambda^3(T_2+T_3+\ldots+T_{n-1}) \ ,$$

and consecutive usage of (4.11) with $i = n-1$, $n-2$, ... 2 , implies that

$$t - x < \lambda x + \lambda^2 x + \ldots + \lambda^n x \ .$$

So that the only possible range for x in $g_n(x,t)$ is

$$\frac{t}{\lambda^n s_n} < x < t \ . \tag{4.13}$$

The upper bound follows from (4.12). It is now convenient to introduce a δ function such that

$$\begin{aligned} \delta_{ij}(x,t) &= 1 \qquad \text{if} \qquad \frac{t}{\lambda^i s_i} < x < \frac{t}{\lambda^j s_j} \ , \\ &= 0 \qquad\qquad \text{otherwise} \quad (i > j) \ . \end{aligned} \tag{4.14}$$

Then

$$g_1(x,t) = \frac{1}{\lambda\tau} e^{-t/\tau} \delta_{10}(x,t) \ . \tag{4.15}$$

We then write

$$\sigma_i(x,t) = \frac{t}{\lambda^i} - s_i x \ , \qquad (i \geq 0) \tag{4.16}$$

and note that

$$\sigma_i(x,t) - \lambda x = \lambda \sigma_{i+1}(x,t) \ ,$$

$$\sigma_i\left(x+y-\frac{y}{\lambda}, t-\frac{y}{\lambda}\right) = \sigma_i(x,t) - y \ , \tag{4.17}$$

and also that

$$\begin{aligned} \delta_{ij}\left(x+y-\frac{y}{\lambda}, t-\frac{y}{\lambda}\right) &= 1 \quad \text{if} \quad \frac{1}{\lambda^i s_i}\left(t-\frac{y}{\lambda}\right) < x+y-\frac{y}{\lambda} < \frac{1}{\lambda^j s_j}\left(t-\frac{y}{\lambda}\right) \\ &\qquad\qquad \text{i.e. if} \quad \sigma_i(x,t) < y < \sigma_j(x,t) \\ &= 0 \qquad\qquad \text{otherwise} \ . \end{aligned} \tag{4.18}$$

For convenience we shall omit the arguments of $\sigma_i(x,t)$ and $\delta_{ij}(x,t)$ when there is no ambiguity.

We are now in a position to build up all the $g_n(x,t)$'s from (4.15) using (4.8).

$$g_2(x,t) = \frac{1}{(\lambda\tau)^2} \int_0^{\min(\lambda x, t-x)} e^{-t/\tau} \, \delta_{10}\left(x+y-\frac{y}{\lambda}, t-\frac{y}{\lambda}\right) dy$$

$$= \frac{e^{-t/\tau}}{(\lambda\tau)^2} \int_{\max(0,\sigma_1)}^{\min(\lambda x, \sigma_0)} dy \; .$$

In the range $\frac{t}{\lambda s_1} < x < t$ the region of integration is $(0,\sigma_0)$, but in $\frac{t}{\lambda^2 s_2} < x < \frac{t}{\lambda s_1}$ it is $(\sigma_1, \lambda x)$ and recalling (4.13) we observe that in $g_2(x,t)$ x cannot be less than $\frac{t}{\lambda^2 s_2}$.

Thus

$$g_2(x,t) = \frac{e^{-t/\tau}}{(\lambda\tau)^2} \sigma_0(x,t) \quad \text{for} \quad \frac{t}{\lambda s_1} < x < t \; ,$$

$$= -\frac{e^{-t/\tau}}{(\lambda\tau)^2} \lambda\sigma_2(x,t) \quad \text{for} \quad \frac{t}{\lambda^2 s_2} < x < \frac{t}{\lambda s_1} \; .$$

But

$$-\lambda\sigma_2 = \sigma_0 - s_1\sigma_1 \; .$$

Hence

$$g_2(x,t) = \frac{e^{-t/\tau}}{(\lambda\tau)^2}(\sigma_0 \ \delta_{20} - \lambda s_1 \ \sigma_1 \ \delta_{21}) \quad . \tag{4.19}$$

$g_3(x,t)$ can now be found from (4.8) and (4.19).

$$g_3(x,t) = \frac{e^{-t/\tau}}{(\lambda\tau)^3}\int_0^{\min(\lambda x,\sigma_0)} \{[\sigma_0(x,y)-y] \ \delta_{20}(x+y-\frac{y}{\lambda}, t-\frac{y}{\lambda}) -$$

$$-\lambda s_1[\sigma_1(x,y)-y] \ \delta_{21}\left(x+y-\frac{y}{\lambda}, t-\frac{y}{\lambda}\right)\}dy$$

$$= \frac{e^{-t/\tau}}{(\lambda\tau)^3}\left[\int_{\max(0,\sigma_2)}^{\min(\lambda x,\sigma_0)} (\sigma_0-y)dy - \lambda s_1 \int_{\max(0,\sigma_2)}^{\min(\lambda x,\sigma_0,\sigma_1)} (\sigma_1-y)dy\right] \quad .$$

We shall now divide the range of x into three different parts, namely $(\frac{t}{\lambda s_1}, t)$, $(\frac{t}{\lambda^2 s_2}, \frac{t}{\lambda s_1})$ and $(\frac{t}{\lambda^3 s_3}, \frac{t}{\lambda^2 s_2})$, and

further represent that part of $g_3(x,t)$ appropriate to the first, second and third range respectively by $g_3^{(1)}(x,t)$, $g_3^{(2)}(x,t)$ and $g_3^{(3)}(x,t)$. For $g_3^{(1)}(x,t)$, $\min(\lambda x,\sigma_0,\sigma_1) = \sigma_1 < \max(0,\sigma_2) = 0$; but we know y can only take positive values, hence there is no contribution from the second integral and thus

$$g_3^{(1)}(x,t) = \frac{e^{-t/\tau}}{(\lambda\tau)^3}\int_0^{\sigma_0} (\sigma_0-y)dy$$

$$= \frac{e^{-t/\tau}}{2!(\lambda\tau)^3} \sigma_0^2 \quad .$$

For $g_3^{(2)}(x,t)$ there is contribution from both integrals,

$$g_3^{(2)}(x,t) = \frac{e^{-t/\tau}}{(\lambda\tau)^3}\left[\int_0^{\lambda x}(\sigma_0-y)\,dy - \lambda s_1\int_0^{\sigma_1}(\sigma_1-y)\,dy\right]$$

$$= \frac{e^{-t/\tau}}{2!(\lambda\tau)^3}\left(\sigma_0^2-\lambda^2 s_2\ \sigma_1^2\right) .$$

Finally in the third range,

$$g_3^{(3)}(x,t) = \frac{e^{-t/\tau}}{(\lambda\tau)^3}\left[\int_{\sigma_2}^{\lambda x}(\sigma_0-y)\,dy - \lambda s_1\int_{\sigma_2}^{\lambda x}(\sigma_1-y)\,dy\right]$$

$$= \frac{e^{-t/\tau}}{(\lambda\tau)^3}\left[\int_0^{\lambda x}(\sigma_0-y)\,dy - \lambda s_1\int_0^{\sigma_1}(\sigma_1-y)\,dy + \right.$$

$$\left. + \lambda\int_0^{\sigma_2}(\sigma_2-y)\,dy - \lambda s_1\int_{\sigma_1}^{\lambda x}(\sigma_1-y)\,dy\right]$$

$$= \frac{e^{-t/\tau}}{2!(\lambda\tau)^3}\left\{\sigma_0^2-\lambda^2 s_2\sigma_1^2+\lambda^3 s_2\sigma_2^2\right\} .$$

And putting these together we have

$$g_3(x,t) = \frac{e^{-t/\tau}}{2!(\lambda\tau)^3}\left(\sigma_0^2\delta_{30}-\lambda^2 s_2\sigma_1^2\delta_{31}+\lambda^3 s_2\sigma_2^2\delta_{32}\right) . \qquad (4.10)$$

Proceeding in this manner we see that

$$g_4(x,t) = \frac{e^{-t/\tau}}{3!(\lambda\tau)^4}\left(\sigma_0^3\delta_{40} - \lambda^3 s_3\sigma_1^3\delta_{41} + \frac{\lambda^5 s_3 s_2}{s_1}\,\sigma_2^3\delta_{42} - \lambda^6 s_3\sigma_3^3\delta_{43}\right),$$

and

$$g_5(x,t) = \frac{e^{-t/\tau}}{4!(\lambda\tau)^4}\left(\sigma_0^4\delta_{50} - \lambda^4 s_4\sigma_1^4\delta_{51} + \frac{\lambda^7 s_4 s_3}{s_1}\,\sigma_2^4\delta_{52} - \frac{\lambda^9 s_4 s_3}{s_1}\,\sigma_3^4\delta_{53} + \lambda^{10} s_4\sigma_4^4\delta_{54}\right).$$

We shall now show inductively that

$$g_n(x,t) = \frac{e^{-t/\tau}}{(n-1)!(\lambda\tau)^n}\sum_{j=0}^{n-1} a_{nj}\sigma_j^{n-1}(x,t)\;\delta_{nj}(x,t)\,, \qquad (4.21)$$

where

$$a_{nj} = (-)^j\;\lambda^{j(n-1)-\frac{1}{2}j(j-1)}\;\frac{s_{n-j}\;s_{n-j+1}\cdots s_{n-1}}{s_0\;s_1\;\cdots\;s_{j-1}}\,, \qquad (4.22)$$

$$\text{for}\quad 1 \leq j \leq n-1\,,$$

and $a_{n0} = 1$.

$g_1(x,t)$ is easily verified.

From (4.8), assuming that (4.21) is true, in the light of (4.17) and (4.18) we have

$$g_{n+1}(x,t) =$$

$$= \frac{e^{-t/\tau}}{(n-1)!(\lambda\tau)^{n+1}} \sum_{j=0}^{n-1} \delta_{nj}\left(x+y-\frac{y}{\lambda}, t-\frac{y}{\lambda}\right) a_{nj} \int_0^{\min(\lambda x,\sigma_0)} [\sigma_j(x,y)-y]^{n-1}\, dy =$$

$$= \frac{e^{-t/\tau}}{(n-1)!(\lambda\tau)^{n+1}} \sum_{j=0}^{n-1} a_{nj} \int_{A_n}^{B_j} [\sigma_j(x,t)-y]^{n-1}\, dy \ , \qquad (4.23)$$

where $A_n = \max(0,\sigma_n)$,

$B_j = \min(\lambda x,\sigma_0,\sigma_j)$.

It is now convenient to denote that part of $g_n(x,t)$ appropriate to the range $t/\lambda^r s_r < x < t/\lambda^{r-1} s_{r-1}$ by $g_n^{(r)}(x,t)$ so that

$$g_n(x,t) = \sum_{r=1}^{n} g_n^{(r)}(x,t)\ \delta_{r,r-1}(x,t) \ . \qquad (4.24)$$

We further observe that

(i) $\sigma_j < 0$ unless $x < t/\lambda^j s_j$ i.e. in

$g_{n+1}^{(j+1)}(x,t)$, $g_{n+1}^{(j+2)}(x,t), \ldots$ $g_{n+1}^{(n+1)}(x,t)$

for all $j = 0, 1, \ldots n$.

(ii) $\sigma_0 < \lambda x$ only if $t/\lambda s_1 < x$ i.e. in $g_{n+1}^{(1)}(x,t)$.

(iii) $\sigma_j < \sigma_{j-1}$ for all j .

(iv) In $g_{n+1}^{(j)}(x,t)$ $\sigma_{j-1} < \lambda x < \sigma_{j-2}$.

Keeping these in mind from (4.23)

$$g_{n+1}^{(1)}(x,t) = \frac{e^{-t/\tau}}{(n-1)!(\lambda\tau)^{n+1}} \int_0^{\sigma_0} a_{n0}(\sigma_0-y)^{n-1}\, dy = \frac{a_{n0}\, e^{-t/\tau}}{n!(\lambda\tau)^{n+1}}\, \sigma_0^n ,$$

so that if $a_{n0} = 1$, so also is $a_{n+1,0}$.
Notice that only the term with $j = 0$ contributes in this range since for the other j's $\min(\lambda x, \sigma_0, \sigma_j) = \sigma_j$ is less than $\max(0, \sigma_n) = 0$, but we know y can only take positive values. Similarly

$$g_{n+1}^{(2)}(x,t) = \frac{e^{-t/\tau}}{(n-1)!(\lambda\tau)^{n+1}} \left\{ \int_0^{\lambda x} (\sigma_0-y)^{n-1}\, dy + \right.$$

$$\left. + a_{n1} \int_0^{\sigma_1} (\sigma_1-y)^{n-1}\, dy \right\}$$

$$= \frac{e^{-t/\tau}}{(n-1)!(\lambda\tau)^{n+1}} \left\{ \sigma_0^n/n + \right.$$

$$\left. + a_{n1} \int_0^{\sigma_1} (\sigma_1-y)^{n-1}\, dy - \int_{\lambda x}^{\sigma_0} (\sigma_0-y)^{n-1}\, dy \right\}$$

$$= \frac{e^{-t/\tau}}{n!(\lambda\tau)^{n+1}} \{\sigma_0^n + (a_{n1}-\lambda^n)\sigma_1^n\} .$$

(4.22) gives $a_{n1} = -\lambda^{n-1} s_{n-1}$ so that $a_{n1} - \lambda^n$ is indeed $a_{n+1,1}$. Now in general

$$g_{n+1}^{(r)}(x,t) = \frac{e^{-t/\tau}}{(n-1)!(\lambda\tau)^{n+1}} \sum_{j=0}^{\min(r-1,n-1)} a_{nj} \int_{A_n}^{B_j} [\sigma_j(x,t)-y]^{n-1}\, dy \,. \tag{4.25}$$

The limitation on the summation comes about because of (i) above and the fact that y is a positive quantity. For $1 < r \leq n$ this can be written as

$$g_{n+1}^{(r)}(x,t) = \frac{e^{-t/\tau}}{(n-1)!(\lambda\tau)^{n+1}} \sum_{j=0}^{r-2} a_{nj} \int_{A_n}^{B_j} [\sigma_j(x,t)-y]^{n-1}\, dy -$$

$$- \frac{e^{-t/\tau}\, a_{n,r-2}}{(n-1)!(\lambda\tau)^{n+1}} \int_{\lambda x}^{\sigma_{r-2}} [\sigma_{r-2}(x,t)-y]^{n-1}\, dy +$$

$$+ \frac{e^{-t/\tau}\, a_{n,r-1}}{(n-1)!(\lambda\tau)^{n+1}} \int_{0}^{\sigma_{r-1}} [\sigma_{r-1}(x,t)-y]^{n-1}\, dy$$

$$= g_{n+1}^{(r-1)}(x,t) + \frac{e^{-t/\tau}}{n!(\lambda\tau)^{n+1}} (a_{n,r-1} - \lambda^n a_{n,r-2})[\sigma_{r-1}(x,t)]^n .$$

We only have to show, then, that

$$a_{n,r-1} - \lambda^n a_{n,r-2} = a_{n+1,r-1} \,, \tag{4.26}$$

where a_{nr}'s are given by (4.22). This is easily verified since

$$(-)^{r-1}\,\lambda^{(r-1)(n-1)-\frac{1}{2}(r-1)(r-2)}\,\frac{s_{n-r+1}s_{n-r+2}\cdots s_{n-1}}{s_0 s_1\cdots s_{r-2}}\,+$$

$$+\,(-)^{r-1}\,\lambda^{n+(r-2)(n-1)-\frac{1}{2}(r-2)(r-3)}\,\frac{s_{n-r+2}s_{n-r+3}\cdots s_{n-1}}{s_0 s_1\cdots s_{r-3}}$$

$$=\,(-)^{r-1}\lambda^{(r-1)n-\frac{1}{2}(r-1)(r-2)}\,\frac{s_{n-r+2}s_{n-r+3}\cdots s_{n-1}}{s_0 s_1\cdots s_{r-2}}\left[\frac{s_{n-r+1}}{\lambda^{r-1}}+s_{r-2}\right]$$

$$=\,a_{n+1,r-1}\;.$$

This shows that (4.21) holds for $0 \le j \le n-2$ and it remains to deal with $j = n-1$. Examination of (i) above shows that every term in $g_n(x,t)$ contributes to $g_{n+1}^{(n+1)}(x,t)$. So that in the light of (ii), (iii), (iv)

$$g_{n+1}^{(n+1)}(x,t) \;=\; \frac{1}{\lambda\tau}\int_{\sigma_n}^{\lambda x} e^{-\frac{y}{\lambda\tau}}\, g_n^{(n)}\left(x+y-\frac{y}{\lambda}\,,\,t-\frac{y}{\lambda}\right)dy\;. \qquad (4.27)$$

It was shown that

$$g_1^{(1)}(x,t) \;=\; \frac{e^{-t/\tau}}{\lambda\tau}\;,$$

$$g_2^{(2)}(x,t) \;=\; -\frac{\lambda e^{-t/\tau}}{(\lambda\tau)^2}\,\sigma_2\;,$$

and using (4.27) we have

$$g_3^{(3)}(x,t) = \frac{\lambda^3 e^{-t/\tau}}{2!(\lambda\tau)^3}\sigma_3^2 ,$$

$$g_4^{(4)}(x,t) = -\frac{\lambda^6 e^{-t/\tau}}{3!(\lambda\tau)^x}\sigma_4^3 .$$

Then it is easily verified by induction that

$$g_n^{(n)}(x,t) = (-)^{n-1} \frac{\lambda^{\frac{1}{2}n(n-1)} e^{-t/\tau}}{(n-1)!(\lambda\tau)^n} \sigma_n^{n-1} . \tag{4.28}$$

We shall now use this to prove (4.21) for $j = n-1$. Putting $r = n$ in (4.25) we obtain

$$g_{n+1}^{(n)}(x,t) = \frac{1}{\lambda\tau}\int_0^{\lambda x} e^{-y/\lambda\tau}\, g_n^{(n)}\left(x+y-\frac{y}{\lambda}, t-\frac{y}{\lambda}\right)dy$$

$$-\frac{e^{-t/\tau}a_{n,n-1}}{(n-1)!(\lambda\tau)^{n+1}}\int_{\sigma_{n-1}}^{\lambda x} [\sigma_n(x,t)-y]^{n-1}dy ,$$

while (4.27) can be written as

$$g_{n+1}^{(n+1)}(x,t) = \frac{1}{\lambda\tau}\left[\int_0^{\lambda x} - \int_0^{\sigma_n}\right] e^{-y/\lambda\tau}\, g_n^{(n)}\left(x+y-\frac{y}{\lambda}, t-\frac{y}{\lambda}\right)dy$$

$$= g_{n+1}^{(n)}(x,t) + \frac{e^{-t/\tau}}{n!(\lambda\tau)^{n+1}}\sigma_n^n\left[(-)^n\lambda^{\frac{1}{2}n(n-1)} - a_{n,n-1}\lambda^n\right],$$

using (4.28). We then only have to show that

$$(-)^n \lambda^{\frac{1}{2}n(n-1)} - \lambda^n a_{n,n-1} = a_{n+1,n}$$

where the a_{nr}'s are given by (4.22). This is easily performed in exactly the same manner as for (4.26). The proof of (4.21) is, thus, completed.

4.3 Derivation of b.p. Probability Density Function from $g_n(x,t)$

We shall now show that

$$b_n^*(z) = \frac{s_0 s_1 \ldots s_{n-2}}{\lambda^n (s_1+s_0\tau z)(s_2+s_1\tau z)\ldots(s_n+s_{n-1}\tau z)} \quad . \qquad (4.29)$$

First we recall (4.9). This, in conjunction with (4.21), immediately leads to

$$b_{n+1}^*(z) =$$

$$= \frac{1}{(n-1)!(\lambda\tau)^{n+1}} L\left\{ e^{-t/\tau} \sum_{j=0}^{n-1} a_{nj} \int_{\frac{t}{\lambda^n s_n}}^{\frac{t}{\lambda^j s_j}} e^{-x/\lambda\tau} [\sigma_j(x,t)]^{n-1} dx \right\} \quad .$$

If we now let $x - \dfrac{t}{\lambda^n s_n} = y$ we get

$$b^*_{n+1}(z) =$$

$$= \frac{1}{(n-1)!(\lambda\tau)^{n+1}} L \left\{ e^{-\frac{s_{n+1}}{s_n\tau}t} \sum_{j=0}^{n-1} a_{nj} \int_0^{\frac{s_{n-j-1}t}{\lambda^j s_n s_j}} e^{-y/\lambda\tau} \left(\frac{s_{n-j-1}}{\lambda^j s_n} t - s_j y \right)^{n-1} dy \right\},$$

or

$$b^*_{n+1}(z) = \frac{1}{(\lambda\tau)^{n+1}} \sum_{j=0}^{n-1} s_j^{n-1} \left(\frac{s_{n-j-1}}{\lambda^j s_n s_j} \right)^n \frac{a_{nj}}{\left(z + \frac{s_{n+1}}{s_n\tau} \right)^n \left(z + \frac{s_{n-j-1}}{\lambda^{j+1} s_n s_j} + \frac{s_{n+1}}{s_n\tau} \right)}$$

$$= \frac{1}{\lambda^{n+1}} \sum_{j=0}^{n-1} \frac{a_{nj}\; s^n_{n-j-1}\; s_n}{\lambda^{nj} \left(s_n\tau z + s_{n+1} \right)^n \left(s_n s_j \tau z + \frac{s_{n-j-1}}{\lambda^{j+1}} + s_j s_{n+1} \right)} .$$

But

$$s_i s_{k-1} - s_{i-1} s_k = \frac{s_{k-i-1}}{\lambda^i} , \qquad \text{if} \quad i \leq k-1$$

and

$$= - \frac{s_{i-k-1}}{\lambda^k} \qquad \text{otherwise} ,$$

so that

$$b^*_{n+1}(z) = \frac{1}{\lambda^{n+1}(s_{n+1}+s_n\tau z)^n} \sum_{j=0}^{n-1} \frac{s^n_{n-j-1} a_{nj}}{\lambda^{nj}(s_{j+1}+s_j\tau z)} . \tag{4.30}$$

We shall now denote $\left(s_j+s_{j-1}\tau z\right)$ by f_j and find the coefficient of $\frac{1}{f_j}$ in $\left(f^{n-1}_{n+1} s_{n-1} s_{n-2} \cdots s_1 s_0\right)/f_1 f_2 \cdots f_n$.

The coefficient of $\frac{1}{f_j}$ in $\dfrac{f_{n+1}^{n-1} s_{n-1} s_{n-2} \cdots s_1 s_0}{f_1 f_2 \cdots f_n}$ =

$$= \frac{(s_{n+1}-s_n c_j)^{n-1}(s_{n-1}s_{n-2}\cdots s_1 s_0)}{(s_1-s_0c_j)(s_2-s_1c_j)\cdots(s_{j-1}-s_{j-2}c_j)(s_{j+1}-s_jc_j)\cdots(s_n-s_{n-1}c_j)}$$

(where $c_j = \frac{s_j}{s_{j-1}}$)

$$= \frac{\left(-\frac{s_{n-j}}{\lambda^j}\right)^{n-1} (s_{n-1}s_{n-2}\cdots s_1 s_0)}{\left(\frac{s_{j-2}}{\lambda}\right)\left(\frac{s_{j-3}}{\lambda^2}\right)\cdots\left(\frac{s_0}{\lambda^{j-1}}\right)\left(\frac{-s_0}{\lambda^j}\right)\left(\frac{-s_1}{\lambda^j}\right)\cdots\left(\frac{-s_{n-j-1}}{\lambda^j}\right)}$$

$$= (-)^{j+1}\,\lambda^{-\frac{1}{2}j(j-1)}\,\frac{s_{n-j}^n s_{n-1}s_{n-2}\cdots s_{n-j+1}}{s_0 s_1 s_2\cdots s_{j-2}},$$

which implies that the coefficient of $\frac{1}{f_{j+1}}$ =

$$= (-)^j\,\lambda^{-\frac{1}{2}j(j+1)}\,\frac{s_{n-j-1}^n s_{n-1}s_{n-2}\cdots s_{n-j}}{s_0 s_1 s_2\cdots s_{j-1}}$$

$$= \frac{s_{n-j-1}^n\, a_{nj}}{\lambda^{nj}},$$

so that

$$\sum_{j=0}^{n-1} \frac{s_{n-j-1}^{n} a_{nj}}{\lambda^{nj}} \frac{1}{f_{j+1}} = \frac{f_{n+1}^{n-1} s_{n-1} s_{n-2} \cdots s_1 s_0}{f_1 f_2 \cdots f_n} .$$

Substituting this in (4.30) completes the proof of (4.29).

It is now a matter of routine to invert (4.29) and obtain $b_n(t)$. Putting (4.29) in partial fractions we obtain

$$b_n^*(z) = \sum_{j=0}^{n-1} (-)^j \frac{\lambda^{\frac{1}{2}n(n-3)-\frac{1}{2}j(j-1)} s_{n-j-1}^{n-1} (s_{n-j-1} s_{n-j} \cdots s_{n-2})}{s_0 s_1 \cdots s_{j-1}} \times$$

$$\times \frac{1}{(s_{n-j} + s_{n-j-1} \tau z)} ,$$

which immediately leads to the main result,

$$b_n(t) = \tag{4.31}$$

$$= \frac{1}{\tau} \sum_{j=0}^{n-1} (-)^j \lambda^{\frac{1}{2}n(n-3)-\frac{1}{2}j(j-1)} s_{n-j-1}^{n-2} \frac{s_{n-j-1} s_{n-j} \cdots s_{n-2}}{s_0 s_1 s_2 \cdots s_{j-1}} e^{-\frac{s_{n-j}}{s_{n-j-1}\tau} t} .$$

This is not as complicated as it looks for most of the s_i's cancel out and in fact

$$b_n(t) = \frac{1}{\tau} \left\{ \lambda^{\frac{1}{2}n(n-3)} s_{n-1}^{n-2} e^{-\frac{s_n}{s_{n-1}\tau} t} - \lambda^{\frac{1}{2}n(n-3)} s_{n-2}^{n-1} e^{-\frac{s_{n-1}}{s_{n-2}\tau} t} + \right.$$

$$+ \lambda^{\frac{1}{2}n(n-3)-1} \frac{s_{n-2}}{s_1} s_{n-3}^{n-1} e^{-\frac{s_{n-2}}{s_{n-3}\tau}t} -$$

$$- \lambda^{\frac{1}{2}n(n-3)-3} \frac{s_{n-2}s_{n-3}}{s_1 s_2} s_{n-4}^{n-1} e^{-\frac{s_{n-3}}{s_{n-4}\tau}t} +$$

$$+ \lambda^{\frac{1}{2}n(n-3)-6} \frac{s_{n-2}s_{n-3}s_{n-4}}{s_1 s_2 s_3} s_{n-5}^{n-1} e^{-\frac{s_{n-4}}{s_{n-5}\tau}t} +$$

$$\left. + \ldots (-)^{n-1} \frac{1}{\lambda} e^{-\frac{s_1}{s_0\tau} t} \right\} .$$

The p.d.f. of the duration of a b.p., $b(t)$, however many services take place, can be obtained by summing $b_n(t)$ over all n. This leads to

$$b(t) = \frac{1}{\lambda\tau} e^{-\frac{s_1}{s_0\tau} t} +$$

$$+ \frac{1}{\lambda\tau} e^{-\frac{s_2}{s_0\tau}t} - \frac{1}{\lambda\tau} e^{-\frac{s_1}{s_1\tau}t} +$$

$$+ \frac{s_2}{\tau} e^{-\frac{s_3}{s_2\tau}t} - \frac{s_1^2}{\tau} e^{-\frac{s_2}{s_1\tau}t} + \frac{1}{\lambda\tau} e^{-\frac{s_1}{s_0\tau}t} +$$

$$+ \frac{\lambda^2 s_3^2}{\tau} e^{-\frac{s_4}{s_3\tau}t} - \frac{\lambda^2 s_2^3}{\tau} e^{-\frac{s_3}{s_2\tau}t} + \frac{\lambda s_2 s_1^2}{\tau} e^{-\frac{s_2}{s_1\tau}t} - \frac{1}{\lambda\tau} e^{-\frac{s_1}{s_0\tau}t} +$$

$$+\frac{\lambda^5 s_4^3}{\tau} e^{-\frac{s_5}{s_4\tau}t} - \frac{\lambda^5 s_3^4}{\tau} e^{-\frac{s_4}{s_3\tau}t} + \frac{\lambda^4 s_3 s_2^4}{s_1\tau} e^{-\frac{s_3}{s_2\tau}t} -$$

$$-\frac{\lambda^2 s_3 s_1^3}{\tau} e^{-\frac{s_2}{s_1\tau}t} + \frac{1}{\lambda\tau} e^{-\frac{s_1}{s_0\tau}t}$$

$$+ \vdots ,$$

which could conveniently be put in the form

$$b(t) = \sum_{m\geq 1} \sum_{n\geq 0} c_{mn} e^{-\frac{s_{n+1}}{s_n \tau}t} , \qquad (4.32)$$

where

$$c_{1n} = \lambda^{\frac{1}{2}n(n-1)-1} s_n^{n-1} , \quad (n\geq 0) ,$$

and (4.33)

$$c_{mn} = -\lambda^n \frac{s_{m+n-2} s_n}{s_{m-2}} c_{m-1,n} \quad (m\geq 2) .$$

However it should be mentioned that (4.32) is summable in one direction and not in another. So that it is not convenient for numerical calculations. For these purposes (4.31) should be summed over all $n \geq 1$.

It is, incidentally, evident that

$$b^*(z) = \sum_{n\geq 1} \frac{s_0 s_1 s_2 \ldots s_{n-2}}{\lambda^n (s_1+s_0\tau z)(s_2+s_1\tau z)\ldots(s_n+s_{n-1}\tau z)} , \qquad (4.34)$$

so that $b^*(0) = \sum_{n\geq 1} \frac{1}{\lambda^n s_{n-1} s_n}$. One expects this to be 1 and it, indeed, is since

$$1 - \frac{1}{\lambda s_1} = \frac{1}{s_1}$$

$$\frac{1}{s_1} - \frac{1}{\lambda^2 s_1 s_2} = \frac{1}{s_2}$$

$$\frac{1}{s_2} - \frac{1}{\lambda^2 s_2 s_3} = \frac{1}{s_3} ,$$

and in general

$$1 - \sum_{i=1}^{n} \frac{1}{\lambda^i s_{i-1} s_i} = \frac{1}{s_n} , \tag{4.35}$$

i.e.

$$1 - \sum_{n\geq 1} \frac{1}{\lambda^n s_{n-1} s_n} = \lim_{n\to\infty} \frac{1}{s_n} = 0 .$$

4.4 Mean and Variance of b.p.

The moments of the b.p. could conveniently be computed from (4.34). We shall here find the mean and the second non-central moment denoted respectively by b_1 and b_2 .

$-b_1$ is the coefficient of z in the expansion of (4.34) in ascending powers of z. Thus if $d_n = \frac{s_{n-1}}{s_n}\tau$ then

$$b_1 = \frac{1}{\lambda s_0 s_1}(d_1) + \frac{1}{\lambda^2 s_1 s_2}(d_1+d_2) + \frac{1}{\lambda^3 s_2 s_3}(d_1+d_2+d_3) + \cdots$$

$$= d_1\left(\frac{1}{\lambda s_0 s_1} + \frac{1}{\lambda^2 s_1 s_2} + \frac{1}{\lambda^3 s_2 s_3} + \cdots\right) +$$

$$+ d_2\left(\frac{1}{\lambda^2 s_1 s_2} + \frac{1}{\lambda^3 s_2 s_3} + \frac{1}{\lambda^4 s_3 s_4} + \cdots\right) +$$

$$+ d_3\left(\frac{1}{\lambda^3 s_2 s_3} + \frac{1}{\lambda^4 s_3 s_4} + \frac{1}{\lambda^5 s_4 s_5} + \cdots\right) +$$

$$+ \vdots \; ,$$

or

$$b_1 = d_1[b^*(0)] + d_2\left[b^*(0) - \frac{1}{\lambda s_0 s_1}\right] + d_3\left[b^*(0) - \frac{1}{\lambda s_0 s_1} - \frac{1}{\lambda^2 s_1 s_2}\right] + \cdots$$

$$= d_1 + \frac{d_2}{s_1} + \frac{d_2}{s_2} + \frac{d_3}{s_3} + \cdots ,$$

which implies that

$$b_1 = \tau \sum_{r \geq 1} s_r^{-1} . \tag{4.36}$$

b_2 , then, is found in a similar manner.

$$b_2 = d_1^2[b^*(0)] + d_2(d_1+d_2)\left[b^*(0) - \frac{1}{\lambda s_0 s_1}\right] +$$

$$+ d_3(d_3+d_2+d_1)\left[b^*(0) - \frac{1}{\lambda s_0 s_1} - \frac{1}{\lambda^2 s_1 s_2}\right] + \ldots$$

$$= \frac{\tau}{s_1}(d_1) + \frac{\tau}{s_2}(d_1+d_2) + \frac{\tau}{s_3}(d_1+d_2+d_3) + \ldots$$

$$= d_1\tau\left(\frac{1}{s_1} + \frac{1}{s_2} + \ldots\right) + d_2\tau\left(\frac{1}{s_2} + \frac{1}{s_3} + \ldots\right)$$

$$+ d_3\tau\left(\frac{1}{s_3} + \frac{1}{s_4} + \ldots\right) + \ldots ,$$

or

$$b_2 = \tau^2 \sum_{i\geq 1} \frac{s_{i-1}}{s_i} \sum_{j\geq i} s_j^{-1} . \tag{4.37}$$

It is obvious that if $\lambda < 1$ b_1 is finite. It is claimed that b_2 is also finite when $\lambda < 1$, since $\sum_{j\geq i} s_j^{-1}$ converges, say to k_i. Then applying the ratio test to the k_i's we see that

$$\lim_{i\to\infty} \frac{k_i}{k_{i-1}} = \lim_{i\to\infty} \frac{\lambda^{2i-2} s_{i-1}^2}{\lambda^{2i-2} s_i s_{i-2}} \frac{\left(\frac{1}{s_i} + \frac{1}{s_{i+1}} + \ldots\right)}{\left(\frac{1}{s_{i-1}} + \frac{1}{s_i} + \ldots\right)}$$

$$= 1 \cdot \lim_{i\to\infty} \frac{\left(\frac{1}{s_i} + \frac{1}{s_{i+1}} + \ldots\right)}{\left(\frac{1}{s_{i-1}} + \frac{1}{s_i} + \ldots\right)} ,$$

since the limits exist. Or

$$\lim \frac{k_i}{k_{i-1}} < 1$$

since $\frac{1}{s_{i-1}} > 0$ for all i .

Thus $b(t)$ does, in fact, have finite mean and variance when $\lambda < 1$.

4.5 Probability Mass Function of the Number of Customers Served in a b.p. of Unspecified Length

Let η be the number of customers served during a b.p. Then letting $z = 0$ in (4.29) we have

$$pr[\eta=n] = \left(\lambda^n s_{n-1} s_n\right)^{-1} \tag{4.38}$$

and by (4.35)

$$pr[\eta>n] = \frac{1}{s_n}$$

When $\lambda \leq 1$, as $n\to\infty$, $pr[\eta>n]\to 0$, so that with probability one the number of customers served in a b.p. is finite. However, if $\lambda > 1$, $s_n \to \frac{\lambda}{\lambda-1}$, so that $pr[\eta>n]\to 1 - \frac{1}{\lambda}$; that is, there exists a residual probability that the b.p. will consist of service of infinitely many customers. This is in accordance with expectation.

CHAPTER 5

THE IDLE TIME AND OUTPUT PROCESSES

5.1 Idle Time

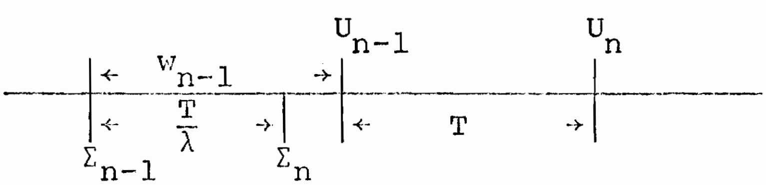
The idle time, i.e. the period between the end of one b.p. and the beginning of the next, during which the server is inevitably idle, can easily be seen to be negative exponentially distributed when arrivals occur in a Poisson stream since it is the whole, or part, of an interarrival interval.

5.2 The Derivation of the p.d.f. of the Interval T Separating Two Successive Outputs

Let $T_n = U_n - U_{n-1}$ where U_n is the epoch of completion of service of the nth customer. To find $k_{n-1}(T)dT$, the p.d.f. of T_n, we shall here distinguish two different cases illustrated by the diagrams below:

(i) C_n (the nth customer) has to wait.

That is $\qquad T/\lambda < w_{n-1}$.



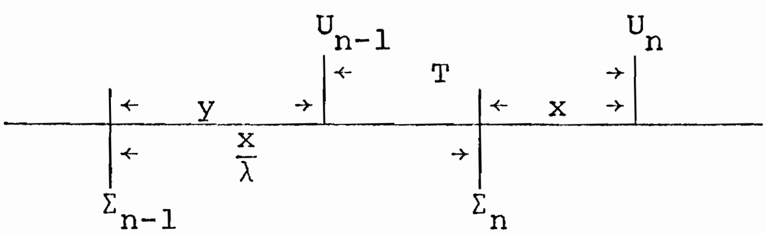
(Recall that Σ_n is the arrival epoch of the nth customer.) w_{n-1} is the waiting time of C_{n-1} . The contribution to $k_{n-1}(T)$

in this case can easily be seen to be

$$\frac{1}{\lambda\tau}\, e^{-\frac{T}{\lambda\tau}} \int_{\frac{T}{\lambda}}^{\infty} f_{n-1}(y)\,dy \;.$$

(ii) <u>C_n does not have to wait.</u>

Then $x/\lambda > w_{n-1} = y$, and C_n is served for time x .



In this case $s_1 x = T + y$, $x \leq T$ and $x/\lambda \geq y$. Thus, $0 \leq y \leq T/\lambda$. The joint p.d.f. of the arrival interval x/λ and y then is $\frac{1}{\lambda\tau}\, e^{-x/\lambda\tau}\, f_{n-1}(y)$. Substituting for x its value in terms of y then gives the contribution to $k_{n-1}(T)$ in this case, viz.

$$\frac{1}{\lambda s_1 \tau}\, e^{-\frac{T}{\lambda s_1 \tau}} \int_0^{\frac{T}{\lambda}} e^{-\frac{y}{\lambda s_1 \tau}}\, f_{n-1}(y)\,dy \;.$$

Adding the two contributions together we get

$$k_{n-1}(T) = \frac{1}{\lambda\tau}\, e^{-\frac{T}{\lambda\tau}} \int_{\frac{T}{\lambda}}^{\infty} f_{n-1}(y)\,dy \quad +$$

$$+\ \frac{1}{\lambda s_1\tau}\, e^{-\frac{T}{\lambda s_1 \tau}} \int_0^{\frac{T}{\lambda}} e^{-\frac{y}{\lambda s_1 \tau}}\, f_{n-1}(y)\,dy\ , \tag{5.1}$$

and substituting for $f_{n-1}(y)dy$ we have

$$k_{n-1}(T) = \frac{1}{\lambda\tau}\, e^{-\frac{T}{\lambda\tau}} \sum_{r=0}^{n-2} \frac{g_{n-1,r}}{s_r}\, e^{-\frac{s_r}{\lambda^2\tau}T} +$$

$$+\ \frac{1}{\lambda\tau}\, e^{-\frac{T}{\lambda s_1\tau}} \sum_{r=0}^{n-2} \frac{g_{n-1,r}}{(s_1 s_r+1)} \left[1 - e^{-\frac{s_1 s_r+1}{\lambda^2 s_1 \tau}T}\right] ,$$

or

$$k_{n-1}(T) = \frac{1}{\lambda\tau}\, e^{-\frac{T}{\lambda s_1\tau}} \sum_{r=0}^{n-2} \frac{g_{n-1,r}}{(s_1 s_r+1)} +$$

$$+\ \frac{1}{\lambda\tau} \sum_{r=0}^{n-2} \frac{g_{n-1,r}\, s_{r+1}}{s_r(s_1 s_r+1)}\, e^{-\frac{s_{r+1}}{\lambda\tau}T}\ . \tag{5.2}$$

We again expect $\int_0^\infty k_{n-1}(T)\,dT$ to be 1 and it indeed is since

$$\begin{aligned}\sum_{r=0}^{n-2} \left[\frac{s_1 g_{n-1,r}}{(s_1 s_r+1)} + \frac{g_{n-1,r}}{s_r(s_1 s_r+1)}\right] &= \sum_{r=0}^{n-2} \frac{g_{n-1,r}}{s_r} \\ &= f^*_{n-1}(0) \\ &= 1\ .\end{aligned}$$

If $\lambda < 1$ a steady state exists and we shall say that as $n\to\infty$ $k_n(T)\to k(T)$. Then,

$$k(T) = \frac{1}{\lambda\tau} e^{-\frac{T}{\lambda s_1 \tau}} \sum_{r\geq 0} \frac{g_r}{(s_1 s_r+1)} + \frac{1}{\lambda\tau} \sum_{r\geq 0} \frac{g_r s_{r+1}}{s_r(s_1 s_r+1)} e^{-\frac{s_{r+1}}{\lambda\tau}T} . \quad (5.3)$$

As might be expected this is not identical with the p.d.f. of the arrival pattern, but it does have a mean which is equal to mean time interval between arrivals when $\lambda < 1$, as it must for equilibrium. This is shown in the next section.

5.3 Mean and Variance of Output Interval

The moments of the output interval density can all be found more conveniently from the Laplace transform $k^*_{n-1}(z)$ of (5.1) with respect to z. We have

$$k^*_{n-1}(z) = \frac{1}{\lambda\tau} \frac{1-f^*_{n-1}\left(\lambda z + \frac{1}{\tau}\right)}{z + \frac{1}{\lambda\tau}} +$$

$$+ \frac{1}{\lambda s_1 \tau} \int_0^\infty e^{-\left(z + \frac{1}{\lambda s_1 \tau}\right)T} \int_0^{\frac{T}{\lambda}} e^{-\frac{y}{\lambda s_1 \tau}} f_{n-1}(y)\,dy$$

$$= \frac{1-f^*_{n-1}\left(\lambda z + \frac{1}{\tau}\right)}{1 + \lambda\tau z} + \frac{1}{\lambda+\lambda^2 s_1 \tau z} \int_0^\infty e^{-\left(\lambda z + \frac{1}{\tau}\right)\frac{T}{\lambda}} f_{n-1}\left(\frac{T}{\lambda}\right) dT ,$$

or

$$k_{n-1}^{*}(z) = \frac{1}{1+\lambda\tau z} - \frac{\tau z f_{n-1}^{*}\left(\lambda z + \frac{1}{\tau}\right)}{(1+\lambda\tau z)(1+\lambda s_{1}\tau z)} \quad . \tag{5.4}$$

We shall now find the mean and variance of the output interval from (5.4). Let $E_n(T)$, $E_n(T^2)$ and $Var_n(T)$ respectively denote mean, second noncentral moment and variance of the length of time separating departure (after service) epochs of nth and (n+1)st customers, then

$$E_{n-1}(T) = \lambda\tau + \tau f_{n-1}^{*}\left(\frac{1}{\tau}\right) \quad .$$

Recalling the form of $f_n^*(z)$, (2.12), we find that this reduces to

$$E_{n-1}(T) = \frac{\tau}{\lambda^{n-1} s_{n-1}} + \lambda\tau \ , \tag{5.5}$$

which implies in the steady state

$$E(T) = \tau \ . \tag{5.6}$$

Further

$$E_{n-1}(T^2) = \frac{\partial^2}{\partial z^2} k_{n-1}^{*}(z) \ ,$$

evaluated at $z = 0$,

or

$$E_{n-1}(T^2) = 2(\lambda\tau)^2 - 2\tau\left[-\tau\sum_{r=0}^{n-2}\frac{g_{n-1,r}}{s_{r+1}^2} - \lambda\tau\,\frac{s_1+1}{\lambda^{n-1}s_{n-1}}\right]$$

which implies

$$\mathrm{Var}_{n-1}(T) = \tau^2\left[\lambda^2 + \frac{2\lambda s_1}{\lambda^{n-1}s_{n-1}} - \left(\frac{1}{\lambda^{n-1}s_{n-1}}\right)^2 + 2\sum_{r=0}^{n-2}\frac{g_{n-1,r}}{s_{r+1}^2}\right], \qquad (5.7)$$

and in the steady state

$$\begin{aligned}\mathrm{Var}(T) &= \tau^2\left[1+2\lambda(1-\lambda)+2\sum_{r\geq 0}\frac{g_r}{s_{r+1}^2}\right] \qquad (5.8)\\ &= \tau^2\left[1+2\lambda(1-\lambda)\sum_{r\geq 0}s_r^{-1}\right].\end{aligned}$$

5.4 The Joint Probability Density Function of the Time Intervals T and U separating Three Successive Outputs

Let $h_{n-1}(T,U)$ be the joint p.d.f. of T and U, where $T = U_n - U_{n-1}$, $U = U_{n+1} - U_n$. This is found in a manner similar to that adopted in the previous section. We have to distinguish four different cases:

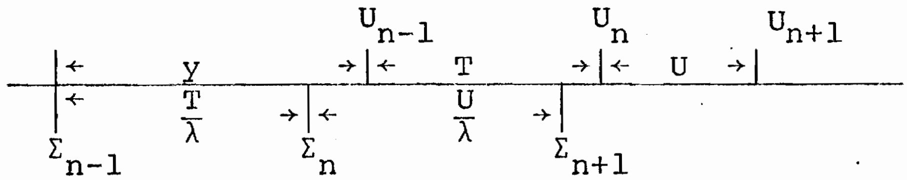
(i) the nth and (n+1)st customers (C_n and C_{n+1}) both have to wait;

(ii) C_n waits but C_{n+1} does not;

(iii) C_n does not wait, but C_{n+1} does;

(iv) neither waits.

In the first case, depicted below, the waiting time y of C_{n-1} has, on the one hand, to exceed $\frac{T}{\lambda}$ and on the other hand,



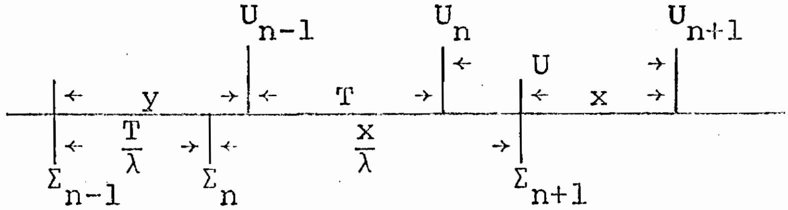
is restricted by the following inequality indirectly.

$$0 < \frac{U}{\lambda} < w_n \quad ,$$

where w_n is waiting time (including service) of C_n. Thus, since $w_n = y - \frac{T}{\lambda} + T$, y has to exceed the larger of $\frac{T}{\lambda}$ and $\frac{(T+U)}{\lambda} - T$. The contribution to $h_{n-1}(T,U)$ in this case is then

$$\frac{1}{(\lambda\tau)^2} e^{-\frac{1}{\lambda\tau}(U+T)} \int_{\max\left(\frac{T}{\lambda}\ ,\ \frac{U+T}{\lambda} - T\right)}^{\infty} f_{n-1}(y)\,dy \ . \tag{5.9}$$

For the second case the set up is shown by the following diagram:



There is now the restriction

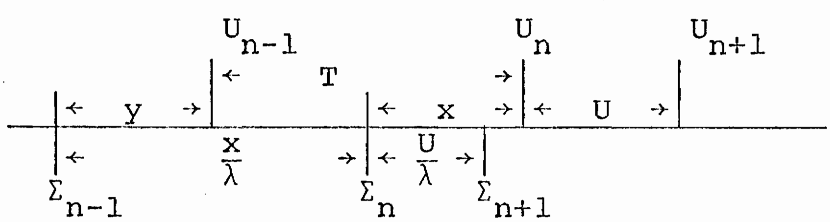
$$s_1 x + \frac{T}{\lambda} = y + T + U . \tag{5.10}$$

Moreover, obviously $y > \frac{T}{\lambda}$ and $\frac{x}{\lambda}$ has to exceed $T + y - T/x$, otherwise C_{n+1} has to wait. This with (5.10) requires y to be less than $\frac{U+T}{\lambda} - T$. Thus writing pr(dy,dT,dx) and substituting for x its value given by (5.10) we find that the contribution to $h_{n-1}(T,U)$ in this case is

$$\frac{1}{(\lambda\tau)^2 s_1} e^{-\frac{(U+2T)}{\lambda s_1 \tau}} \int_{\frac{T}{\lambda}}^{\frac{U+T}{\lambda} - T} e^{-\frac{y}{\lambda s_1 \tau}} f_{n-1}(y)\,dy , \tag{5.11}$$

as long as the upper limit exceeds the lower, i.e. as long as $U > \lambda T$. Otherwise this situation offers no contribution.

The third case is illustrated by the following diagram.

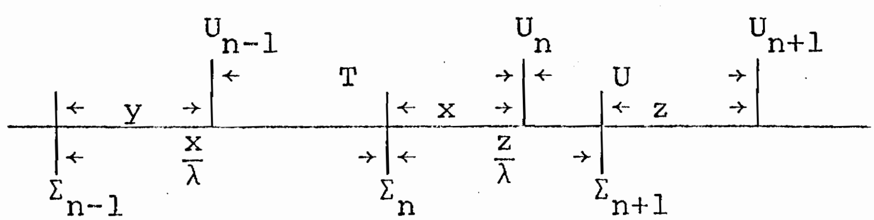


There is a restriction $s_1 x = y + T$; y can range from zero upwards but is limited (by the indirect fact that x cannot exceed T and y cannot exceed $\frac{x}{\lambda}$) by $\frac{T}{\lambda}$. But we must also have $\frac{U}{\lambda} < x$ otherwise C_{n+1} would not have to wait, which implies $\frac{U}{\lambda} < \frac{1}{s_1}(y+T)$, i.e. $\frac{s_1}{\lambda}U - T < y$. y is also a positive quantity. This then leads to $\max\left(0\ ,\ \frac{s_1}{\lambda}U-T\right) < y$, and again writing $pr(dy,dx,dU)$ and substituting for x its value in terms of y and T, the contribution to $h_{n-1}(T,U)$ in this case becomes

$$\frac{1}{(\lambda\tau)^2 s_1}\, e^{-\frac{T}{\lambda s_1 \tau} - \frac{U}{\lambda\tau}} \int_{\max\left(0\ ,\ \frac{s_1 U}{\lambda} - T\right)}^{\frac{T}{\lambda}} e^{-\frac{y}{\lambda s_1 \tau}} f_{n-1}(y)\,dy\ . \qquad (5.12)$$

Care must be taken for the case when $\frac{s_1}{\lambda}U-T > 0$, since then there is a contribution only as long as $\frac{s_1}{\lambda}U-T < \frac{T}{\lambda}$, or $U < \lambda T$. The ranges of integration are then specifically $(0\ ,\ T/x)$ for $0 < U \leq \lambda T/s_1$, and $\left(\frac{s_1}{\lambda}U-T\ ,\ T/\lambda\right)$ for $\lambda T/s_1 < U < \lambda T$.

Finally case (iv) is illustrated by:



The restrictions are:

$$s_1 x = T + y \quad \text{and} \quad s_1 z = U + x\ .$$

Now $y < \frac{x}{\lambda} < \frac{T}{\lambda}$. Further $z < U$ or $\frac{U+x}{s_1} < U$ which implies $x < \frac{U}{\lambda}$ or $\frac{1}{s_1}(T+y) < \frac{U}{\lambda}$, i.e. $y < \frac{s_1}{\lambda}U - T$. Hence y has to be less than $\min\left(\frac{s_1}{\lambda}U-T\ ,\ \frac{T}{\lambda}\right)$ but it can go down to zero. Note again that since y is a positive quantity U has to be greater than $\lambda T/s_1$ for a positive contribution from this case. Writing pr(dy,dx,dz) again and substituting for x and z their values when $U > \lambda T/s_1$ the contribution to $h_{n-1}(T,U)$ in the last case becomes

$$\frac{1}{(\lambda s_1\tau)^2}\, e^{-\frac{U}{\lambda s_1\tau}-\frac{T}{\lambda s_1\tau}\left(1+\frac{1}{s_1}\right)} \int_0^A e^{-\frac{y}{\lambda s_1\tau}\left(1+\frac{1}{s_1}\right)} f_{n-1}(y)\,dy\ , \qquad (5.13)$$

where

$$\begin{aligned} A &= T/\lambda && \text{if } U > \lambda T\ , \text{ and} \\ &= s_1U/\lambda - T && \text{if } \frac{\lambda T}{s_1} < U < T\ . \end{aligned}$$

It is now evident that if $0 \le U \le \frac{\lambda T}{s_1}$, then $\frac{T}{\lambda} > \frac{U+T}{\lambda} - T$ and $0 > \frac{s_1U}{\lambda} - T$; further if $\frac{\lambda T}{s_1} \le U \le \lambda T$, then again $\frac{T}{\lambda} > \frac{U+T}{\lambda} - T$ but $\frac{Us_1}{\lambda} - T > 0$, so that putting (5.9),(5.11), (5.12), (5.13) together we have

$$h_{n-1}(T,U) = \frac{1}{(\lambda\tau)^2}\left[e^{-(U+T)/\lambda\tau} \int_{\frac{T}{\lambda}}^{\infty} f_{n-1}(y)\,dy \; + \frac{1}{s_1}\, e^{-\frac{T}{\lambda s_1\tau}-\frac{U}{\lambda\tau}} \int_0^{\frac{T}{\lambda}} e^{-\frac{y}{\lambda s_1\tau}} f_{n-1}(y)\,dy \right]$$

$$\text{if} \quad 0 < U \le \frac{\lambda T}{s_1} ,$$

$$h_{n-1}(T,U) = \frac{1}{(\lambda\tau)^2}\left[e^{-(U+T)/\lambda\tau} \int_{\frac{T}{\lambda}}^{\infty} f_{n-1}(y)\,dy + \right.$$

$$+ \frac{1}{s_1} e^{-\frac{T}{\lambda s_1 \tau} - \frac{U}{\lambda\tau}} \int_{\frac{s_1 U}{\lambda} - T}^{\frac{T}{\lambda}} e^{-\frac{y}{\lambda s_1 \tau}} f_{n-1}(y)\,dy + \qquad (5.14)$$

$$\left. + \frac{1}{s_1^2} e^{-\frac{U}{\lambda s_1 \tau} - \frac{T}{\lambda s_1 \tau}\left(1 + \frac{1}{s_1}\right)} \int_0^{\frac{s_1 U}{\lambda} - T} e^{-\frac{y}{\lambda s_1 \tau}\left(1 + \frac{1}{s_1}\right)} f_{n-1}(y)\,dy \right]$$

$$\text{if} \quad \frac{\lambda T}{s_1} \le U \le \lambda T ,$$

and

$$h_{n-1}(T,U) = \frac{1}{(\lambda\tau)^2}\left[e^{-(U+T)/\lambda\tau} \int_{\frac{U+T}{\lambda} - T}^{\infty} f_{n-1}(y)\,dy + \right.$$

$$+ \frac{1}{s_1} e^{-(U+2T)/\lambda s_1 \tau} \int_{\frac{T}{\lambda}}^{\frac{U+T}{\lambda} - T} e^{-\frac{y}{\lambda s_1 \tau}} f_{n-1}(y)\,dy +$$

$$+\frac{1}{s_1^2}\, e^{-\frac{U}{\lambda s_1\tau}-\frac{T}{\lambda s_1\tau}\left(1+\frac{1}{s_1}\right)} \int_0^{\frac{T}{\lambda}} e^{-\frac{y}{\lambda s_1\tau}\left(1+\frac{1}{s_1}\right)} f_{n-1}(y)\,dy \Bigg]$$

$$\text{if} \quad \lambda T \leq U < \infty .$$

If we now carry out an integration with respect to U over all its possible values we obtain (5.1), the p.d.f. of T , which was found directly in the last section. This then implies that $\int_0^\infty \int_0^\infty h_{n-1}(T,U)\,dT\,dU = 1$ as we expect it to be. We can now substitute for $f_{n-1}(y)dy$ its value, if we desire, but this will not in any manner simplify the form of $h_{n-1}(T,U)$. $E_{n-1}(TU)$ (expected value of TU) can be found by lengthy and cumbersome algebraic processes and in spite of the fact that it is too complicated to try to simplify it is recorded here for completeness.

$$E_{n-1}(TU) = (\lambda\tau)^2 \sum_{r=0}^{n-2} g_{n-1,r} \Bigg\{ \frac{1}{s_r s_{r+1}^2} - \frac{2}{\lambda^2 s_r s_{r+2}^3} + \frac{2}{\lambda^2 s_r s_{r+1} s_{r+2}^3} +$$

$$+ \frac{2s_1}{\lambda^2 (s_1 s_r + 1)(s_{r+2}^3)} + \frac{s_1^2}{\lambda^2 (s_1 s_r + 1)(s_{r+2}^2)} -$$

$$- \frac{2}{\lambda^2 (s_1 s_r + 1) s_{r+1} s_{r+2}^3} - \frac{1}{\lambda^2 (s_1 s_r + 1) s_{r+1}^2 s_{r+2}^2} +$$

$$
\begin{aligned}
&+ \frac{1}{\lambda^2(s_1 s_r+1)} \left[\frac{\lambda(\lambda+3)}{s_1} - \frac{\lambda^2}{s_{r+1}^2} + \frac{2}{s_1^2 s_{r+1}} + \frac{1}{s_1^2 s_{r+1}^2} - \right. \\
&\left. - \frac{2}{s_1 s_{r+1} s_{r+2}^3} - \frac{1}{s_1^2 s_{r+1}^2 s_{r+2}^2} + \frac{2}{s_{r+2}^3} + \frac{1}{s_{r+2}^2} \right] + \\
&+ \frac{1}{\lambda^2(s_1^2 s_r + s_1 + 1)} \left[2 + s_1^2 - \frac{2}{s_1^2 s_{r+1}} - \frac{1}{s_1^2 s_{r+1}^2} + \frac{2}{s_1^2 s_{r+1} s_{r+2}^3} + \right. \\
&\left. + \frac{1}{s_1^2 s_{r+1}^2 s_{r+2}^2} - \frac{s_1^2}{s_{r+2}^2} - \frac{2 s_1^3}{s_{r+2}^3} \right] \Bigg\} . \qquad (5.15)
\end{aligned}
$$

The covariance and coefficient of correlation between two successive output intervals can now be found from (5.15) but no attempt will be made to find them here for obvious reasons.

CHAPTER 6

AN ASSOCIATED MODEL WITH CORRELATION

We recall that in the correlated model which has been discussed so far, $f_n(w)dw$, the p.d.f. of the nth arrival's waiting time (including service) possessed the Laplace transform

$$f_n^*(z) = \frac{s_0 s_1 \ldots s_{n-1}}{(s_0+\lambda\tau z)(s_1+\lambda\tau z)\ldots(s_{n-1}+\lambda\tau z)} \; .$$

This implies that

$$(s_n+\lambda\tau z)f_{n+1}^*(z) = s_n f_n^*(z) \; ,$$

and, by inversion,

$$\frac{d}{dw} f_{n+1}(w) = \frac{s_n}{\lambda\tau} f_n(w) - \frac{s_n}{\lambda\tau} f_{n+1}(w) \; . \qquad (6.1)$$

If we now define $F'_{nc}(w)$ to be the probability that the wait of the nth customer is longer than w, i.e.

$$F'_{nc}(w) = \int_w^\infty f_n(w)\,dw$$

then clearly

$$f_n^*(z) = 1 - zF'^*_{nc}(z)$$

which implies that

$$\left(\frac{s_n}{\lambda\tau} + z\right)[1-zF'^*_{n+1}(z)] = \frac{s_n}{\lambda\tau}[1-zF'^*_n(z)] ,$$

and this in turn leads to the result

$$\left(\frac{s_n}{\lambda\tau} + z\right) F'^*_{n+1}(z) = \frac{s_n}{\lambda\tau} F'^*_n(z) + 1 .$$

By inversion

$$\frac{d}{dw} F'_{n+1}(w) + \frac{s_n}{\lambda\tau} F'_{n+1}(w) = \frac{s_n}{\lambda\tau} F'_n(w) \quad \text{with } F'_n(0) = 1 \text{ for all } n ,$$

which implies

$$F'_{n+1}(t+dt) = \frac{s_n}{\lambda\tau} F'_n(t)dt + \left(1 - \frac{s_n}{\lambda\tau} dt\right) F'_{n+1}(t) . \qquad (6.2)$$

The form of (6.2) suggests a connection between this model and a slightly different one in which the probability differential associated with the completion of a service is $s_n/\lambda\tau$ when the queue contains n . This point was first observed by B. W. Conolly [11]. The connection is strong enough a reason for a separate investigation of this model which we shall introduce formally in the next section.

6.1 Introduction of the Model

Consider a queueing model in which arrivals form a Poisson stream with parameter $1/\tau$, and services also form a Poisson stream but with a variable mean which depends on the number of customers in the queue in a manner to be described. Let the interarrival intervals T_i have a negative exponential distribution with mean interval τ and the services also have a negative exponential distribution with mean $\frac{\lambda\tau}{s_n}$ when n is the number in the queue and s_n is given by (2.11). That is to say, when a customer is being served his service follows a negative exponential distribution with mean $\frac{\lambda\tau}{s_n}$ so long as the number in the queue is n ; as soon as an arrival occurs (if this is before the completion of service) this service assumes a different mean, namely $\frac{\lambda\tau}{s_{n+1}}$, but is still negative exponentially distributed. This model is, in a manner, similar to that of Cox and Smith [14] in which their $\sigma_n \delta t$ is to be identified with $\frac{s_n}{\lambda\tau}\delta t$, and their $\alpha_n \delta t$ with $\frac{1}{\tau}\delta t$. They, however, seem to discuss only the steady state state probabilities and have apparently made no attempt to obtain any further results.

With this introduction, we proceed to evaluate the state probabilities and waiting time p.d.f. for this model and to see what similarities, if any, they bear to the first correlated model, which we have already discussed.

6.2 The State Probabilities at an Arbitrary Time Instant t

The standard technique of deriving a differential difference equation for state probabilities at a given time t can be used here but as we shall see it will not produce useful results. Let $\eta(t)$ be the random variable representing the number of customers in the system at time t and let $p_k(t) = pr[\eta(t)=k]$. Then, although it is obvious, in the usual way, that

$$p_0(t+dt) = p_0(t)\left(1 - \frac{dt}{\tau}\right) + p_1(t)\frac{s_0}{\lambda\tau}dt , \tag{6.3}$$

and
$$p_n(t+dt) = p_{n-1}(t)\frac{1}{\tau}dt + p_n(t)\left(1 - \frac{dt}{\tau}\right)\left(1 - \frac{s_{n-1}}{\lambda\tau}dt\right) + \tag{6.4}$$
$$+ p_{n+1}(t)\frac{s_n}{\lambda\tau}dt \qquad \text{for} \quad n = 1,2,\ldots ,$$

we cannot so easily find a solution for $p_n(t)$. In fact, even the steady state solution does not assume a closed form. We shall substantiate these statements in the next few lines.

(6.3) and (6.4) imply that

$$\dot{p}_0(t) + \frac{1}{\tau}p_0(t) = \frac{s_0}{\lambda\tau}p_1(t) , \tag{6.5}$$

and

$$\dot{p}_n(t) + \frac{s_n}{\tau}p_n(t) = \frac{1}{\tau}p_{n-1} + \frac{s_n}{\lambda\tau}p_{n+1} , \; n = 1,2,\ldots, \tag{6.6}$$

where $\dot{p}_n(t) = \dfrac{dp_n(t)}{dt}$.

A technique which is often successful with sets like (6.5) and (6.6) is to take Laplace transforms and then to solve the resulting difference equations for $p_n^*(z)$. But in this case the derived difference equation is not linear, and this approach is not a particularly successful one. We therefore make a somewhat different attack in the next section.

First, however, we examine steady state values, implied by (6.5) and (6.6), by putting $\dot{p}_n(t) = 0$ and solving the resulting equations for $\bar{p}_n$, the value of $p_n(t)$ as $t\to\infty$. Then we obtain

$$\bar{p}_1 = \frac{\lambda}{s_0}\,\bar{p}_0 \ ,$$

$$\bar{p}_2 = \frac{\lambda^2}{s_0 s_1}\,\bar{p}_0 \ ,$$

$$\bar{p}_3 = \frac{\lambda^3}{s_0 s_1 s_2}\,\bar{p}_0 \ , \qquad \text{etc.} \ ,$$

and it is easily verified by induction (using (6.6)) that

$$\bar{p}_n = \frac{\lambda^n}{s_0 s_1 \ldots s_{n-1}}\,\bar{p}_0 \ , \qquad n \geq 1 \ . \tag{6.7}$$

However, it is when we try to find p_0 by imposing the restriction that $\sum\limits_{i\geq 0} \bar{p}_i = 1$ that we seem not to obtain elegant closed forms since

$$1 + \frac{\lambda}{s_0} + \frac{\lambda^2}{s_0 s_1} + \frac{\lambda^3}{s_0 s_1 s_2} + \ldots$$

does not, apparently, sum to a simple expression. We can, however, change it to a product which is perhaps more suitable for numerical calculations. Theorem 348 in Hardy and

Wright [38] states

$$(1+ax)(1+ax^2)\ldots(1+ax^j) = 1+ax\,\frac{1-x^j}{1-x} + a^2x^3\,\frac{(1-x^j)(1-x^{j-1})}{(1-x)(1-x^2)} + \ldots .$$

If in this we let $j\to\infty$ with $a = 1-\lambda$ and $x = \lambda$, since $|x| < 1$ (steady state) we obtain

$$\{1+\lambda(1-\lambda)\}\{1+\lambda^2(1-\lambda)\}\ldots = 1 + \frac{\lambda(1-\lambda)}{1-\lambda} + \frac{\lambda^3(1-\lambda)^2}{(1-\lambda)(1-\lambda^2)} + $$

$$+ \frac{\lambda^6(1-\lambda)^3}{(1-\lambda)(1-\lambda^2)(1-\lambda^3)} + \ldots$$

$$= 1 + \frac{\lambda}{s_0} + \frac{\lambda^2}{s_0 s_1} + \frac{\lambda^3}{s_0 s_1 s_2} + \ldots$$

or

$$\overline{p}_0 = \prod_{j=1}^{\infty} [1+\lambda^j(1-\lambda)]^{-1}$$

from which we can obtain, for example, $\overline{p}_0 = 0.62$ if $\lambda = 0.5$ or $\overline{p}_0 = 0.40$ if $\lambda = 0.9$. Nevertheless, we can easily show $\sum_{i\geq 0} \overline{p}_i/\overline{p}_0$ converges for $\lambda < 1$ as it of course must for

$$\lim_{n\to\infty} \frac{\lambda^{n+1}/s_0 s_1 \ldots s_n}{\lambda^n/s_0 s_1 \ldots s_{n-1}} = \lim_{n\to\infty} \frac{\lambda}{s_n} < 1 \quad .$$

We shall now proceed to derive "finite time" results using a different approach.

6.3 State Probabilities at Arrival Epochs

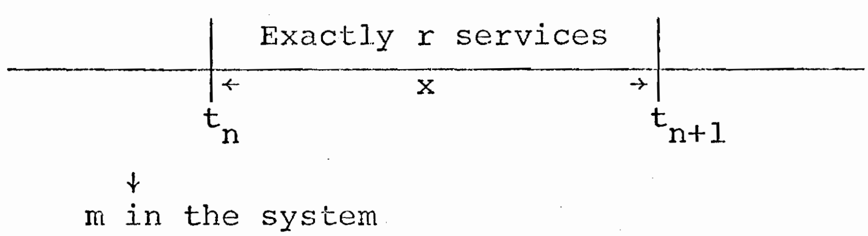
We are here again interested in the random variable $\eta_n = \eta(t_n - 0)$ which represents the number in the system immediately before t_n, where t_n is an arrival epoch. We again let $p_r^{(n)} = \mathrm{pr}[\eta_n = r]$. To find $p_r^{(n)}$ we find it convenient to introduce the auxiliary function $h_{mr}^{(n)}(x)$ defined below:

$h_{mr}^{(n)}(x)$ = Probability of exactly r customers completing service in the time interval x given that:

(i) x is the time interval separating the arrival epochs of the nth and (n+1)th customers, and

(ii) the nth customer finds m in the system upon his arrival.

That is, $h_{mr}^{(n)}(x)$ is the probability of the event depicted below.



Then clearly

$$h_{m0}^{(n)}(x) = e^{-\frac{s_m}{\lambda\tau}x}, \qquad m = 0,1,2,\ldots,n-1,$$

$$h_{mr}^{(n)}(x) = \int_0^x \frac{s_m}{\lambda\tau} e^{-\frac{s_m}{\lambda\tau}y_1} dy_1 \int_0^{x-y_1} \frac{s_{m-1}}{\lambda\tau} e^{-\frac{s_{m-1}}{\lambda\tau}y_2} dy_2 \ldots \times$$

$$\times \int_0^{x-y_1-y_2\ldots y_{r-1}} \frac{s_{m-r+1}}{\lambda\tau} e^{-\frac{s_{m-r+1}}{\lambda\tau}y_r} dy_r \times$$

$$\times\, e^{-\frac{s_{m-r}}{\lambda\tau}(x-y_1-y_2\ldots-y_r)} ,$$

for $r = 1,2,\ldots m$ and $m \leq n-1$,

and

$$h_{m,m+1}^{(n)}(x) = \int_0^x \frac{s_m}{\lambda\tau} e^{-\frac{s_m}{\lambda\tau}y_1} dy_1 \int_0^{x-y_1} \frac{s_{m-1}}{\lambda\tau} e^{-\frac{s_{m-1}}{\lambda\tau}y_2} dy_2 \ldots \times$$

$$\times \int_0^{x-y_1-y_2\ldots-y_m} \frac{s_0}{\lambda\tau} e^{-\frac{s_0}{\lambda\tau}y_{m+1}} dy_{m+1} ,$$

$$0 \leq m \leq n-1$$

so that we immediately get, with obvious notation,

$$h_{m0}^{(n)*}(z) = \frac{\lambda\tau}{(s_m+\lambda\tau z)} , \quad m = 0,1,\ldots n-1 , \tag{6.8}$$

$$h_{mr}^{(n)*}(z) = \frac{\lambda\tau(s_m s_{m-1}\ldots s_{m-r+1})}{(s_m+\lambda\tau z)(s_{m-1}+\lambda\tau z)\ldots(s_{m-r}+\lambda\tau z)} , \tag{6.9}$$

$$r = 1,2,\ldots m , \quad m \leq n-1$$

and

$$h^{(n)*}_{m,m+1}(z) = \frac{s_m s_{m-1} \cdots s_0}{(s_m+\lambda\tau z)(s_{m-1}+\lambda\tau z)\cdots(s_0+\lambda\tau z)z} \; , \qquad (6.10)$$

$$0 \le m \le n-1 \; .$$

We can now use these results to derive a difference equation for $p_r^{(n)}$. To start with, it is evident that

$$p_{n-1}^{(n)} = p_{n-2}^{(n-1)} \int_0^\infty \frac{1}{\tau} e^{-\frac{x}{\tau}} \; h_{n-2}^{(n-1)}(x)\,dx$$

$$= p_{n-2}^{(n-1)} \, \frac{1}{\tau} \, h_{n-2,0}^{(n-1)*} \left(\frac{1}{\tau}\right) \; ,$$

or, using (6.8),

$$p_{n-1}^{(n)} = p_{n-2}^{(n-1)} \, \frac{1}{s_{n-1}} \; ,$$

from which we can immediately show by induction that

$$p_{n-1}^{(n)} = \frac{1}{s_0 s_1 \cdots s_{n-1}} \; . \qquad (6.11)$$

Further

$$p_r^{(n)} = p_{r-1}^{(n-1)} \int_0^\infty \frac{1}{\tau} \, e^{-\frac{x}{\tau}} \, h_{r-1,0}^{(n-1)}(x)\,dx + p_r^{(n-1)} \int_0^\infty \frac{1}{\tau} \, e^{-\frac{x}{\tau}} \, h_{r,1}^{(n-1)}(x)\,dx +$$

$$+ \ldots + p_{n-3}^{(n-1)} \int_0^\infty \frac{1}{\tau} e^{-\frac{x}{\tau}} h_{n-3,n-2-r}^{(n-1)}(x)\,dx +$$

$$+ p_{n-2}^{(n-1)} \int_0^\infty \frac{1}{\tau} e^{-\frac{x}{\tau}} h_{n-2,n-1-r}^{(n-1)}(x)\,dx \quad \text{for}$$

$$r = 1,2,\ldots n-2 \;,$$

and

$$p_0^{(n)} = p_0^{(n-1)} \int_0^\infty \frac{1}{\tau} e^{-\frac{x}{\tau}} h_{01}^{(n-1)}(x)\,dx + p_1^{(n-1)} \int_0^\infty \frac{1}{\tau} e^{-\frac{x}{\tau}} h_{12}^{(n-1)}(x)\,dx +$$

$$+\ldots+ p_{n-3}^{(n-1)} \int_0^\infty \frac{1}{\tau} e^{-\frac{x}{\tau}} h_{n-3,n-2}^{(n-1)}(x)\,dx +$$

$$+ p_{n-2}^{(n-1)} \int_0^\infty \frac{1}{\tau} e^{-\frac{x}{\tau}} h_{n-2,n-1}^{(n-1)}(x)\,dx \;,$$

or

$$p_r^{(n)} = \sum_{j=0}^{n-r-1} p_{r+j-1}^{(n-1)} \int_0^\infty \frac{1}{\tau} e^{-\frac{x}{\tau}} h_{r+j-1,j}^{(n-1)}(x)\,dx \quad \text{for } r = 1,2,\ldots n-2,$$

i.e.

$$p_r^{(n)} = \sum_{j=0}^{n-r-1} p_{r+j-1}^{(n-1)} \frac{1}{\tau} h_{r+j-1,j}^{(n-1)*} \frac{1}{\tau} \;, \quad \text{for } r = 1,2,\ldots n-2, \quad (6.12)$$

and

$$p_0^{(n)} = \sum_{j=0}^{n-2} p_j^{(n-1)} \frac{1}{\tau} h_{j,j+1}^{(n-1)*} \left(\frac{1}{\tau}\right) .$$

Using (6.8), (6.9) and (6.10) we reduce these to

$$p_r^{(n)} = \sum_{j=0}^{n-r-1} p_{r+j-1}^{(n-1)} \frac{1}{\lambda^j s_{r+j}} , \quad \text{for } r = 1,2,\ldots n , \tag{6.13}$$

and

$$p_0^{(n)} = \sum_{j=0}^{n-2} p_j^{(n-1)} \frac{1}{\lambda^{j+1} s_{j+1}} . \tag{6.14}$$

Now, setting $p_0^{(1)} = 1$, and letting $n= 2$ in (6.14), we obtain

$$p_0^{(2)} = \frac{1}{\lambda s_1}$$

and by (6.11)

$$p_1^{(2)} = \frac{1}{s_1} .$$

These, with (6.13) and (6.14), will then give

$$p_0^{(3)} = \frac{1}{\lambda^2 s_1} \left(\frac{1}{s_1} + \frac{1}{s_2}\right) ,$$

$$p_1^{(3)} = \frac{1}{s_1} \left(\frac{1}{s_1} + \frac{1}{s_2}\right) ,$$

and using (6.11) again

$$p_2^{(3)} = \frac{1}{s_1 s_2} \quad .$$

In exactly a similar manner we obtain

$$p_0^{(4)} = \frac{1}{\lambda^3 s_1}\left[\frac{1}{s_1}\left(\frac{1}{s_1} + \frac{1}{s_2}\right) + \frac{1}{s_2}\left(\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3}\right)\right] \quad ,$$

$$p_1^{(4)} = \frac{1}{\lambda^2 s_1}\left[\frac{1}{s_1}\left(\frac{1}{s_1} + \frac{1}{s_2}\right) + \left(\frac{1}{s_2}\,\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3}\right)\right] \quad ,$$

$$p_2^{(4)} = \frac{1}{\lambda s_1 s_2}\left(\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3}\right) \quad , \text{ and}$$

$$p_3^{(4)} = \frac{1}{s_1 s_2 s_3} \quad .$$

We shall now proceed to prove by induction that

$$p_r^{(n)} = \frac{1}{\lambda^{n-r-1} s_0 s_1 \ldots s_r} \sum_{i1=1}^{r+1} s_{i1}^{-1} \sum_{i2=1}^{i1+1} s_{i2}^{-1} \ldots \sum_{ik=1}^{im+1} s_{ik}^{-1} \tag{6.15}$$

$$r = 1,2,\ldots\ n-2$$

where $\qquad k = n-r-1 \quad , \quad m = k-1 \quad ,$

and the second subscript in s_{ij} is, in fact, a subscript to the first. Further

$$p_0^{(n)} = \frac{1}{\lambda} p_1^{(n)} . \qquad (6.16)$$

(Recall $p_{n-1}^{(n)}$ is given by (6.11)).

To prove (6.15), observe that from (6.13)

$$p_r^{(n)} = \frac{1}{\lambda^{n-r-1} s_0 s_1 \ldots s_r} \sum_{i1=1}^{r} s_{i1}^{-1} \sum_{i2=1}^{i1+1} s_{i2}^{-1} \ldots \sum_{i,n-r-1=1}^{(i,n-r-2)+1} s_{i,n-r-1}^{-1} +$$

$$+ \frac{1}{\lambda^{n-r-1} s_0 s_1 \ldots s_{r+1}} \sum_{i1=1}^{r+1} s_{i1}^{-1} \sum_{i2=1}^{i1+1} s_{i2}^{-1} \ldots \sum_{i,n-r-2=1}^{(i,n-r-3)+1} s_{i,n-r-2}^{-1} +$$

$$+ \vdots$$

$$+ \frac{1}{\lambda^{n-r-1} s_0 s_1 \ldots s_{n-3}} \sum_{i1=1}^{n-3} s_{i1}^{-1} \sum_{i2=1}^{i1+1} s_{i2}^{-1} +$$

$$+ \frac{1}{\lambda^{n-r-1} s_0 s_1 \ldots s_{n-2}} \sum_{i1=1}^{n-2} s_{i1}^{-1} + \frac{1}{\lambda^{n-r-1} s_0 s_1 \ldots s_{n-1}} .$$

But

$$\frac{1}{\lambda^{n-r-1} s_0 s_1 \ldots s_{n-1}} + \frac{1}{\lambda^{n-r-1} s_0 s_1 \ldots s_{n-2}} \sum_{i1=1}^{n-2} s_{i1}^{-1} =$$

$$= \frac{1}{\lambda^{n-r-1} s_0 s_1 \ldots s_{n-2}} \sum_{i1=1}^{n-1} s_{i1}^{-1} ,$$

and

$$\frac{1}{\lambda^{n-r-1} s_0 s_1 \ldots s_{n-2}} \sum_{i1=1}^{n-1} s_{i1}^{-1} + \frac{1}{\lambda^{n-r-1} s_0 s_1 \ldots s_{n-3}} \sum_{i1=1}^{n-3} s_{i1}^{-1} \sum_{i2=1}^{i1+1} s_{i2}^{-1} =$$

$$= \frac{1}{\lambda^{n-r-1} s_0 s_1 \ldots s_{n-3}} \sum_{i1=1}^{n-2} s_{i1}^{-1} \sum_{i2=1}^{i1+1} s_{i2}^{-1} ,$$

and so on. Thus, combining the above sum (from last to first) completes the proof of (6.15). A comparison of (6.13) with $r = 1$ and (6.14) proves (6.16).

6.4 Waiting Time Probability Density Function

Let, again, $f_n(w)$ be the p.d.f. of the waiting time including service of the nth customer regardless of the time of his arrival. Due to the fact that the service parameter changes upon the arrival of a new customer, we have to first introduce a new function $\gamma_{rk}(w)$ from which we shall derive $f_n(w)dw$. Let

$\gamma_{rk}(w)$ = probability and p.d.f. of exactly r services having combined length $w(>0)$ given that at the start of the first service interval k customers were present in the system.

$\gamma_{rk}(w)$ is meaningful, regarding the waiting time, only for $r \geq 1$ and $k \geq r$.

Recalling how the service parameter changes upon the arrival of a new customer, we immediately see that

$$\gamma_{11}(w)\,dw =$$

$$e^{-\frac{w}{\tau}} \frac{s_0}{\lambda\tau} e^{-\frac{s_0}{\lambda\tau}w} dw + \int_0^w \frac{1}{\tau} e^{-\frac{y_1}{\tau}} e^{-\frac{s_0}{\lambda\tau}y} \quad e^{-\frac{w-y_1}{\tau}} \frac{s_1}{\lambda\tau} e^{-\frac{s_1}{\lambda\tau}(w-y_1)} dy_1\,dw \quad +$$

$$+\int_0^w \frac{1}{\tau} e^{-\frac{y_1}{\tau}} e^{-\frac{s_0}{\lambda\tau}y_1} \int_0^{w-y_1} \frac{1}{\tau} e^{-\frac{y_2}{\tau}} e^{-\frac{s_1}{\lambda\tau}y} \quad e^{-\frac{w-y_1-y_2}{\tau}} \frac{s_2}{\lambda\tau} e^{-\frac{s_2}{\lambda\tau}(w-y_1-y_2)} dy_2\,dw +$$

$$+ \\ \vdots \tag{6.18}$$

The first term corresponds to the case of no arrivals during the entire service interval w , the second term to exactly one arrival, the third term to two arrivals, and so on. This then gives, again with obvious notation

$$\gamma_{11}^*(z) = \frac{1}{\lambda} \sum_{i \geq 1} \frac{s_{i-1}}{(s_1+\tau z)(s_2+\tau z)\dots(s_i+\tau z)} \quad . \tag{6.19}$$

$\gamma_{12}(w)$ follows exactly the same pattern as $\gamma_{11}(w)$, except that s_0 in the first term, s_0 and s_1 in the second term, s_0, s_1 and s_2 in the third term, etc. are replaced respectively by s_1, s_1 and s_2, s_1, s_2 and s_3, etc. Thus

$$\gamma^*_{12}(z) = \frac{1}{\lambda} \sum_{i\geq 2} \frac{s_{i-1}}{(s_2+\tau z)(s_3+\tau z)\ldots(s_i+\tau z)} \quad .$$

It is then not difficult to see that

$$\gamma^*_{1k}(z) = \frac{1}{\lambda} \sum_{i\geq k} \frac{s_{i-1}}{(s_k+\tau z)(s_{k+1}+\tau z)\ldots(s_i+\tau z)} \quad . \tag{6.20}$$

Now

$$\gamma_{22}(w) = \int_0^w \left[e^{-\frac{t}{\tau}} \frac{s_1}{\lambda\tau} e^{-\frac{s_1}{\lambda\tau}t} \gamma_{11}(w-t) + \right.$$

$$+ \int_0^t \frac{1}{\tau} e^{-\frac{y_1}{\tau}} e^{-\frac{s_1}{\lambda\tau}y_1} e^{-\frac{1}{\tau}(t-y_1)} \frac{s_2}{\lambda\tau} e^{-\frac{s_2}{\lambda\tau}(t-y_1)} dy_1 \gamma_{12}(w-t) +$$

$$\left. + \ldots \right] dt \quad . \tag{6.21}$$

Again the first term corresponds to no arrivals, the second term to one arrival, etc. This implies that

$$\gamma^*_{22}(z) = \frac{1}{\lambda} \sum_{i\geq 2} \frac{s_{i-1}}{(s_2+\tau z)(s_3+\tau z)\ldots(s_i+\tau z)} \gamma^*_{1,i-1}(z) \quad ,$$

and with the proper changes in (6.21)

$$\gamma_{23}^{*}(z) = \frac{1}{\lambda} \sum_{i \geq 3} \frac{s_{i-1}}{(s_3+\tau z)(s_4+\tau z)\ldots(s_i+\tau z)} \gamma_{1,i-1}^{*}(z) \ ,$$

and in general

$$\gamma_{2k}^{*}(z) = \frac{1}{\lambda} \sum_{i \geq k} \frac{s_{i-1}}{(s_k+\tau z)(s_{k+1}+\tau z)\ldots(s_i+\tau z)} \gamma^{*}_{1,i-1}(z) \ .$$

Now that the notation and the idea is clear we can write $\gamma_{rk}(w)$ in general.

$$\gamma_{rk}(w) = \int_0^w \Bigg[e^{-\frac{t}{\tau}} \frac{s_{k-1}}{\lambda\tau} e^{-\frac{s_{k-1}}{\lambda\tau}t} \gamma_{r-1,k-1}(w-t) +$$

$$+\int_0^t \frac{1}{\tau} e^{-\frac{y_1}{\tau}} e^{-\frac{s_{k-1}}{\lambda\tau}y_1} e^{-\frac{1}{\tau}(t-y_1)} \frac{s_k}{\lambda\tau} e^{-\frac{s_k}{\lambda\tau}(t-y_1)} dy_1 \gamma_{r-1,k}(w-t)+$$

$$+\int_0^t \frac{1}{\tau} e^{-\frac{y_1}{\tau}} e^{-\frac{s_{k-1}}{\lambda\tau}y_1} \int_0^{t-y_1} \frac{1}{\tau} e^{-\frac{y_2}{\tau}} e^{-\frac{s_k}{\lambda\tau}y} \times$$

$$\times e^{-\frac{1}{\tau}(t-y_1-y_2)} \frac{s_{k+1}}{\lambda\tau} e^{-\frac{s_{k+1}}{\lambda\tau}(t-y_1-y_2)} \gamma_{r-1,k+1}(w-t)$$

$$+ \ \vdots$$

$$\Bigg] dt \ .$$

So that

$$\gamma^*_{rk}(z) = \frac{s_{k-1}}{\lambda}\frac{1}{(s_k+\tau z)}\gamma^*_{r-1,k-1}(z) + \frac{s_k}{\lambda}\frac{1}{(s_k+\tau z)(s_{k+1}+\tau z)}\gamma^*_{r-1,k}(z)$$

$$+ \frac{s_{k+1}}{\lambda}\frac{1}{(s_k+\tau z)(s_{k+1}+\tau z)(s_{k+2}+\tau z)}\gamma^*_{r-1,k+1}(z)$$

$$+ \ \ldots$$

or

$$\gamma^*_{rk}(z) = \frac{1}{\lambda}\sum_{i\geq k}\frac{s_{i-1}}{(s_k+\tau z)(s_{k+1}+\tau z)\ldots(s_i+\tau z)}\gamma^*_{r-1,i-1}(z)\ , \qquad r \geq 1\ ,\ k \geq r\ , \tag{6.23}$$

with $\gamma^*_{0k}(z) = 1$ for all k.

We can now easily write

$$f_n(w) = \sum_{r\geq 0}^{n-1} p_r^{(n)}\ \gamma_{r+1,r}(w) \qquad n \geq 2 \tag{6.24}$$

where $p_r^{(n)}$ and $\gamma_{ij}(w)$ are given by (6.14) and (6.23) respectively. For completeness it is mentioned that $f_1(w) = h_{10}(w)$. If $\gamma_{r+1,r}(w)$ is replaced by $\gamma_{rr}(w)$ in (6.24) that formula gives the waiting time excluding service.

The problems of the state probabilities and waiting time for this model are now, in principle, completely solved. It should be made clear here that the basic reason for investigating this model was to search for similarities with the correlated model which was discussed and analyzed in detail in the last chapters. It is now clear that despite the formal connection demonstrated in the introductory section of this chapter there is little which this model has in common with that discussed in the body of this dissertation. Its practical realization might also present some problems. We, therefore, abandon it at this point. It should, however, be mentioned that the techniques used here to derive the state probabilities and waiting time may well be used in other situations where the parameters change in some manner upon a new arrival.

CHAPTER 7
SOME FURTHER RESULTS FOR CONVENTIONAL M/M/1 QUEUES

7.1 State Probabilities at Arrival Epochs for M/M/1

In this chapter we derive formulae for state probabilities at arrival epochs for a conventional M/M/1 queue. Besides its importance in its own right, this is useful for comparison purposes in Part II. Let the arrival and service parameters be λ and μ respectively. We are again interested in the stochastic process η_n representing the number in the system at time (t_n-0) where t_n is the arrival epoch of the nth customer. Let

$$p_r^{(n)} = \mathrm{pr}[\eta_n=r] \;, \qquad r = 0,1,\ldots n-1 \;. \tag{7.1}$$

To find $p_r^{(n)}$ we again introduce the auxiliary function $h_{mr}^{(n)}(x)$ which, we recall, was defined as below:

$h_{mr}^{(n)}(x)$ = Probability of exactly r customers completing service in the time interval x given that:

(i) x is the time interval separating the arrival epochs of the nth and (n+1)th customers, and

(ii) the nth customer finds m in the system upon his arrival.

Then, using the "lack of memory" property of the negative exponential distribution, it follows that

$$h_{mr}^{(n)}(x) = \int_0^x \mu e^{-\mu y_1}\,dy_1 \int_0^{x-y_1} \mu e^{-\mu y_2}\,dy_2 \ldots \times$$

$$\times \int_0^{x-y_1-y_2\ldots-y_{r-1}} \mu e^{-\mu y_r} e^{-\mu(x-y_1-y_2\ldots-y_r)}\,dy_r \quad ,$$

for $r \le m \quad , \quad m \le n-1$.

Further,

$$h_{m,m+1}^{(n)}(x) = \int_0^x \mu e^{-\mu y_1}\,dy_1 \int_0^{x-y_1} \mu e^{-\mu y_2}\,dy_2 \ldots \times$$

$$\times \int_0^{x-y_1-y_2\ldots-y_m} \mu e^{-\mu y_{m+1}}\,dy_{m+1} \quad 0 \le m \le n-1$$

which, with the usual asterisk denoting a Laplace transformation, implies that

$$h_{mr}^{(n)*}(z) = \left(\frac{\mu}{\mu+z}\right)^r \Big/ (\mu+z) \quad , \qquad (7.2)$$

for $r \le m \quad , \quad m \le n-1$,

and

$$h^{(n)*}_{m,m+1}(z) = \left(\frac{\mu}{\mu+z}\right)^{m+1} \Big/ z \quad , \tag{7.3}$$

for $0 \leq m \leq n-1$.

Now, with these in mind, we observe that $p_r^{(n)}$ satisfies the following difference equation:

$$p_r^{(n)} = \sum_{j=0}^{n-r-1} p_{r+j-1}^{(n-1)} \int_0^\infty \lambda e^{-\lambda x} h_{r+j-1,j}^{(n-1)}(x)dx \ , \tag{7.4}$$

$$r = 1,2,\ldots n-1$$

and

$$p_0^{(n)} = \sum_{j=0}^{n-2} p_j^{(n-1)} \int_0^\infty \lambda e^{-\lambda x} h_{j,j+1}^{(n-1)}(x)dx \ . \tag{7.5}$$

These can then be written alternatively as

$$p_r^{(n)} = \lambda \sum_{j=0}^{n-r-1} p_{r+j-1}^{(n-1)} h_{r+j-1,j}^{(n-1)*}(\lambda) \ ,$$

and

$$p_0^{(n)} = \lambda \sum_{j=0}^{n-2} p_j^{(n-1)} h_{j,j+1}^{(n-1)*}(\lambda) \ ,$$

or

$$p_r^{(n)} = \frac{\lambda}{\mu+\lambda} \sum_{j=0}^{n-r-1} p_{r+j-1}^{(n-1)} \left(\frac{\mu}{\mu+\lambda}\right)^j \ , \tag{7.6}$$

for $r = 1,2,\ldots n-1$,

and

$$p_0^{(n)} = \sum_{j=0}^{n-2} p_j^{(n-1)} \left(\frac{\mu}{\mu+\lambda}\right)^{j+1} . \tag{7.7}$$

Now a comparison of (7.6) with $r = 1$, and (7.7) immediately gives

$$p_0^{(n)} = \frac{\mu}{\lambda} p_1^{(n)} . \tag{7.8}$$

We, therefore, only have to solve (7.6) which, since $p_0^{(1)} = 1$, with $n = 2$ leads to

$$p_1^{(2)} = \frac{\lambda}{\mu+\lambda} , \tag{7.9}$$

so that, using (7.8),

$$p_0^{(2)} = \frac{\mu}{\mu+\lambda} . \tag{7.10}$$

Now from (7.6) with $n = 3$, and using (7.9) and (7.10), we have

$$p_2^{(3)} = \left(\frac{\lambda}{\mu+\lambda}\right)^2 ,$$

and

$$p_1^{(3)} = \left(\frac{\mu}{\mu+\lambda}\right)\left(\frac{\lambda}{\mu+\lambda}\right)\left\{1 + \frac{\lambda}{\mu+\lambda}\right\} ,$$

so that using (7.8)

$$p_0^{(3)} = \left(\frac{\mu}{\mu+\lambda}\right)^2 \{1 + \frac{\lambda}{\mu+\lambda}\} \quad .$$

In exactly the same manner

$$p_3^{(4)} = \left(\frac{\lambda}{\mu+\lambda}\right)^3$$

$$p_2^{(4)} = \left(\frac{\mu}{\mu+\lambda}\right)\left(\frac{\lambda}{\mu+\lambda}\right)^2\left(1+2\left(\frac{\lambda}{\mu+\lambda}\right)\right) \quad ,$$

$$p_1^{(4)} = \left(\frac{\mu}{\mu+\lambda}\right)^2\left(\frac{\lambda}{\mu+\lambda}\right)\left(1+2\left(\frac{\lambda}{\mu+\lambda}\right)+2\left(\frac{\lambda}{\mu+\lambda}\right)^2\right) \quad \text{and}$$

$$p_0^{(4)} = \left(\frac{\mu}{\mu+\lambda}\right)^3\left(1+2\left(\frac{\lambda}{\mu+\lambda}\right)+2\left(\frac{\lambda}{\mu+\lambda}\right)^2\right) \quad ,$$

and

$$p_4^{(5)} = \left(\frac{\lambda}{\mu+\lambda}\right)^4 \quad ,$$

$$p_3^{(5)} = \left(\frac{\mu}{\mu+\lambda}\right)\left(\frac{\lambda}{\mu+\lambda}\right)^3\left(1+3\left(\frac{\lambda}{\mu+\lambda}\right)\right) \quad ,$$

$$p_2^{(5)} = \left(\frac{\mu}{\mu+\lambda}\right)^2\left(\frac{\lambda}{\mu+\lambda}\right)^2\left(1+3\left(\frac{\lambda}{\mu+\lambda}\right)+5\left(\frac{\lambda}{\mu+\lambda}\right)^2\right) \quad ,$$

$$p_1^{(5)} = \left(\frac{\mu}{\mu+\lambda}\right)^3\left(\frac{\lambda}{\mu+\lambda}\right)\left(1+3\left(\frac{\lambda}{\mu+\lambda}\right)+5\left(\frac{\lambda}{\mu+\lambda}\right)^2+5\left(\frac{\lambda}{\mu+\lambda}\right)^3\right) \quad \text{and}$$

$$p_0^{(5)} = \left(\frac{\mu}{\mu+\lambda}\right)^4\left(1+3\left(\frac{\lambda}{\mu+\lambda}\right)+5\left(\frac{\lambda}{\mu+\lambda}\right)^2+5\left(\frac{\lambda}{\mu+\lambda}\right)^3\right) \quad .$$

We shall now proceed to prove by induction that

$$p_r^{(n+1)} = (1-\gamma)^n \rho^r \sum_{i=0}^{n-r} \left(1 - \frac{i}{n}\right)\binom{n+i-1}{i} \gamma^i \qquad (7.11)$$

$$r = 1,2,\ldots n$$

where

$$\gamma = \frac{\rho}{1+\rho} \quad \text{and} \quad \rho = \frac{\lambda}{\mu} \quad .$$

$n=1$ is easily verified. Then, assuming (7.11) is true, and using (7.6), we have

$$p_r^{(n+1)} = \gamma \sum_{j=0}^{n-r} (1-\gamma)^j \, (1-\gamma)^{n-r-j} \, \gamma^{r+j-1} \times$$

$$\times \sum_{i=0}^{n-r-j} \left(1 - \frac{i}{n-1}\right)\binom{n+i-2}{i} \gamma^i$$

$$= \gamma^r (1-\gamma)^{n-r} \left[\sum_{i=0}^{n-r}\left(1 - \frac{i}{n-1}\right)\binom{n+i-2}{i} \gamma^i + \right.$$

$$+ \sum_{i=0}^{n-r-1}\left(1 - \frac{i}{n-1}\right)\binom{n+i-2}{i} \gamma^{i+1} +$$

$$+ \ \vdots \ +$$

$$\left. + \sum_{i=0}^{0}\left(1 - \frac{i}{n-1}\right)\binom{n+i-2}{i} \gamma^{i+n-2}\right] \quad .$$

A rearrangement of the above sum now leads to

$$p_r^{(n+1)} = \gamma^r (1-\gamma)^{n-r} \sum_{i=0}^{n-r} z_i \gamma^i \quad ,$$

where

$$z_0 = 1 \quad ,$$

$$z_1 = z_0 + \frac{(n-2)}{1!} \quad ,$$

$$z_2 = z_1 + \frac{(n-3)n}{2!} \quad ,$$

$$z_3 = z_2 + \frac{(n-4)n(n+1)}{3!} \quad ,$$

and in general

$$z_i = z_{i-1} + \frac{(n-i-1)n(n+1)(n+2)\ldots(n+i-2)}{i!} \quad .$$

Thus if

$$z_{i-1} = \frac{(n-i+1)(n+1)(n+2)\ldots(n+i-2)}{(i-1)!} \quad , \tag{7.12}$$

then

$$z_i = \frac{(n+1)(n+2)\ldots(n+i-2)}{i!}(n^2-n-i^2+i)$$

or

$$z_i = \frac{(n-i)(n+1)(n+2)\ldots(n+i-1)}{i!} \quad ,$$

which is (7.12) with i replaced by $i+1$. But

$$\frac{(n-i)(n+1)(n+2)\ldots(n+i-1)}{i!} = \frac{n(n+1)(n+2)\ldots(n+i-1)}{i!} -$$

$$- \frac{i}{n} \cdot \frac{n(n+1)(n+2)\ldots(n+i-1)}{i!}$$

$$= \left(1 - \frac{i}{n}\right)\binom{n+i-1}{i} \quad .$$

This completes the proof of (7.11).

It should be pointed out here that using (7.11) we can find the p.d.f. of the nth customer's waiting time regardless of the time of his arrival. For if this p.d.f. is denoted by $f_n(w)$, then it is evident that

$$f_{n+1}(w) = \sum_{r=0}^{n} p_r^{(n+1)} \frac{(\mu w)^r}{r!} \rho^{-\mu w} \quad .$$

7.2 Passage to the Steady State

The well-known fact that the steady state state probabilities for M/M/1 are given by

$$\bar{p}_r = (1-\rho)\,\rho^r \quad , \quad r = 0,1,2,\ldots \tag{7.13}$$

can be easily demonstrated from the main equation (7.6), by letting $p_r^{(n)} \to \bar{p}_r$ as $n\to\infty$. That equation, then, reduces to

$$\bar{p}_r = \frac{\rho}{1+\rho} \sum_{j\geq 0} \bar{p}_{r+j-1} \left(\frac{1}{1+\rho}\right)^j \quad . \tag{7.14}$$

Substituting (7.13) in the right hand side of (7.14), we obtain

$$\frac{\rho}{1+\rho}\left[(1-\rho)\rho^{r-1} + \frac{(1-\rho)\rho^r}{1+\rho} + \frac{(1-\rho)\rho^{r+1}}{(1+\rho)^2} + \ldots\right] =$$

$$= \frac{(1-\rho)\rho^r}{(1+\rho)}\left[1 + \frac{\rho}{1+\rho} + \left(\frac{\rho}{1+\rho}\right)^2 + \ldots\right] =$$

$$= (1-\rho)\rho^r$$

$$= \bar{p}_r \quad .$$

This then provides a further check on (7.13) which has been obtained by many authors previously.

A final remark should be made here that for M/M/1 η_n is a Markov chain whose transition probability matrix can easily be seen to be

$$P = \begin{bmatrix} q & p & 0 & 0 & \cdot & \cdot & \cdot & & & & & & \\ q^2 & pq & p & 0 & \cdot & \cdot & \cdot & & & & & & \\ q^3 & pq^2 & pq & p & 0 & 0 & \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & & & & & & & \\ q^{k+1} & pq^k & pq^{k-1} & pq^{k-2} & \cdot & \cdot & \cdot & pq & p & 0 & 0 & \cdot\cdot\cdot \\ & \cdot & & & & & & & & & & & \\ & \cdot & & & & & & & & & & & \\ & \cdot & \cdot & \cdot & & & & & & & & & \end{bmatrix}$$

where

$$p = \frac{\lambda}{\mu+\lambda} \quad ,$$

and

$$q = \frac{\mu}{\mu+\lambda} \quad .$$

From this the main equation (7.6) could be derived directly. However, this approach bypasses the basic probabilistic arguments underlying the problem which were meant to be emphasized.

CHAPTER 8

SOME FURTHER PROBLEMS WITH NOTES AND COMMENTS

8.1 The Dual Queue

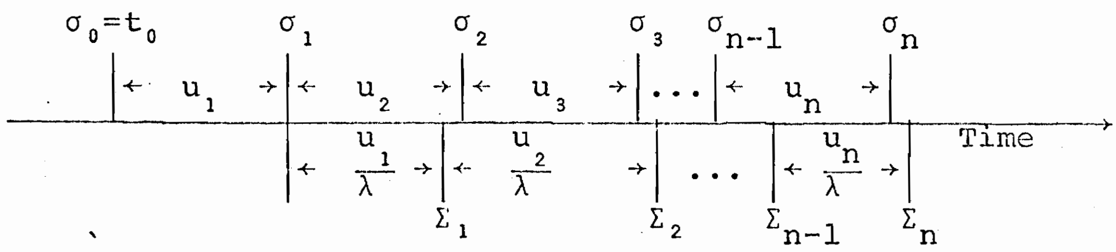
In the model analyzed in Chapters 1 through 5 the service time of C_n, the nth customer to arrive, was made to depend on the interarrival interval separating him from his predecessor C_{n-1}. It is therefore natural to investigate the counterpart of this model, namely one in which the interarrival intervals are made in some way to depend on the service intervals. Such a model might conveniently be described as the "dual" of that to which we have given principal attention in this dissertation.

In order to envisage the practical realization of the "dual" model one may imagine an uninterrupted stream of epochs σ_1, σ_2, σ_3, ... which may be regarded either as epochs of completion of service, or, perhaps more naturally, as "service opportunities". Viewed in this light, σ_n might be regarded as the epoch of arrival of a bus, or an elevator operating on some predetermined schedule, or even randomly. The service offered in this case is merely the picking up of one or more passengers, and it may happen in the framework of such a model that no customer is present at a service epoch, in which case the epoch could be denominated "imperfect". Otherwise it would be "perfect".

Another common and simple example is provided by a production line serviced by a machine through which a number of objects have to pass. If the objects are not to be delayed excessively, and if the machine is not to suffer unnecessarily long periods of idleness, the arrival pattern has, in some way, to be matched to that of the services; just as, in the original model service was matched to the arrival pattern.

It was mentioned in Chapter 1 that some work has been done on a model similar to that considered there. By contrast, absolutely no attempt has been made to give a formal mathematical model for the dual queue as defined here. Perhaps this is due to the fact that a treatment of such a model, although not terribly difficult, is algebraically very cumbersome. We do not claim here to have given a systematic treatment of this dual queue. We have, on the other hand, at least tried formally to define a suitable process, and to give some notes and comments on the different features one is usually interested in in any queueing model.

The following diagram is meant to clarify the idea of the model and to introduce some notations.



The horizontal axis is the time axis. $\sigma_0 = t_0$ is the start of the process with no customers present. σ_1 , σ_2 , ... etc. is the sequence of service epochs (completions of service, or service opportunities) such that the intervals $\sigma_1 - \sigma_0$, $\sigma_2 - \sigma_1$, ... etc. are independently and identically distributed with common probability density function b(t). We suppose that the first service interval $\sigma_1 - \sigma_0$ is an idle one and has length u_1 . Then we shall suppose that the interval from σ_1 to the arrival epoch of the first customer (Σ_1) is u_1/λ. The second interarrival is u_2/λ , etc. In this way interarrival intervals are made to depend on the observed intervals between service opportunities.

The fact that the coefficient of correlation is chosen to be $\frac{1}{\lambda}$ (rather than λ) is only to preserve consistency, so that, as long as $\lambda < 1$, at service opportunity σ_i there will be a maximum of (i-1) customers present and not more. If $\lambda > 1$ it is conceivable that quite a large number of customers will be awaiting service at service opportunity σ_i . We shall now assume that λ is in fact less than one.

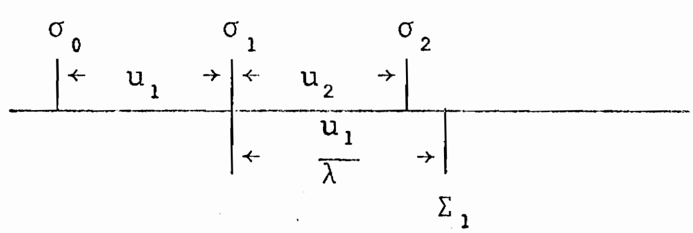
With this set up then, if a service interval is very long it will take a long time until the next customer will arrive, and, on the contrary, if a service interval is short the next customer will arrive soon. In short, arrivals are regulated on the basis of services as opposed to services being regulated on the basis of arrivals.

To carry the analogy with the original model as far as possible, let $b(t) = \frac{1}{\lambda\tau} e^{-\frac{t}{\lambda\tau}}$ so that interarrival intervals have p.d.f. $a(t) = \frac{1}{\tau} e^{-t/\tau}$.

Suppose we are now interested in the stochastic process ξ_n representing the number of customers in the system just before the service opportunity σ_n . Let again

$$p_r^{(n)} = pr[\xi_n=r] \qquad r = 0,1,2,\dots n-1 \quad .$$

Then it is clear from the following diagram



that

$$p_0^{(2)} = pr\left(u_2 < \frac{u_1}{\lambda}\right) = \frac{1}{\lambda s_1} \quad ,$$

where s_1 (and all the s_i's which will appear) are defined by (2.11). Thus

$$p_1^{(2)} = 1 - p_0^{(2)} = \frac{1}{s_1} .$$

At the third service opportunity there can be a maximum of two customers waiting. One of the following possible cases is bound to occur

(i) $\sigma_2 < \sigma_3 < \Sigma_1 < \Sigma_2$,

(ii) $\Sigma_1 < \sigma_2 < \sigma_3 < \Sigma_2$,

(iii) $\sigma_2 < \Sigma_1 < \sigma_3 < \Sigma_2$,

(iv) $\Sigma_1 < \sigma_2 < \Sigma_2 < \sigma_3$ and

(v) $\sigma_2 < \Sigma_1 < \Sigma_2 < \sigma_3$.

It is now evident that the cases (i) and (ii) contribute to $p_0^{(3)}$, (iii) and (iv) contribute to $p_1^{(3)}$ and (v) contributes to $p_2^{(3)}$. Evaluating the corresponding probabilities for each case we get

$$p_0^{(3)} = 1 - \frac{1}{\lambda s_1^2} - \frac{\lambda}{s_2} ,$$

$$p_1^{(3)} = \frac{1}{\lambda s_1^2} - \frac{\lambda}{s_1} + \frac{2\lambda}{s_2} , \qquad \text{and}$$

$$p_2^{(3)} = \frac{\lambda}{s_1} - \frac{\lambda}{s_2} = \frac{1}{\lambda s_1 s_2} .$$

We can now proceed in the same manner for $p_i^{(4)}$, $i = 0,1,2,3$. The cases that contribute to $p_0^{(4)}$ are: $\sigma_2<\sigma_3<\sigma_4<\Sigma_1<\Sigma_2<\Sigma_3$, $\sigma_2<\Sigma_1<\sigma_3<\sigma_4<\Sigma_2<\Sigma_3$, $\Sigma_1<\sigma_2<\sigma_3<\sigma_4<\Sigma_2<\Sigma_3$ and $\Sigma_1<\sigma_2<\Sigma_2<\sigma_3<\sigma_4<\Sigma_3$.

Those contributing to $p_1^{(4)}$ are: $\sigma_2<\sigma_3<\Sigma_1<\sigma_4<\Sigma_2<\Sigma_3$, $\sigma_2<\Sigma_1<\sigma_3<\Sigma_2<\sigma_4<\Sigma_3$, $\Sigma_1<\sigma_2<\sigma_3<\Sigma_2<\sigma_4<\Sigma_3$, $\sigma_2<\Sigma_1<\Sigma_2<\sigma_3<\sigma_4<\Sigma_3$ and $\Sigma_1<\sigma_2<\Sigma_2<\sigma_3<\Sigma_3<\sigma_4$. Those contributing to $p_2^{(4)}$ are: $\sigma_2<\sigma_3<\Sigma_1<\Sigma_2<\sigma_4<\sigma_3$, $\sigma_2<\Sigma_1<\sigma_3<\Sigma_2<\Sigma_3<\sigma_4$, $\Sigma_1<\sigma_2<\sigma_3<\Sigma_2<\Sigma_3<\sigma_4$ and $\sigma_2<\Sigma_1<\Sigma_2<\sigma_3<\Sigma_3<\sigma_4$. And finally the only case contributing to $p_3^{(4)}$ is $\sigma_2<\sigma_3<\Sigma_1<\Sigma_2<\Sigma_3<\sigma_4$. Evaluating the corresponding probabilities we obtain

$$p_0^{(4)} = 1 - \frac{1}{\lambda^2 s_1^3} - \frac{2\lambda}{s_1} + \frac{\lambda^2}{s_2} + \frac{\lambda}{s_2} - \frac{\lambda^3}{s_3}$$

$$p_1^{(4)} = \frac{1}{\lambda^2 s_1^3} + \frac{1}{s_1(\lambda s_2)^2} - \frac{2\lambda^2}{s_2} + \frac{2\lambda^3}{s_3} + \frac{2\lambda}{s_1} - \frac{1}{s_1^2} - \frac{\lambda}{s_2}$$

$$p_2^{(4)} = \frac{1}{s_1^2} - \frac{2}{s_1(\lambda s_2)^2} + \frac{\lambda^2}{s_2} - \frac{\lambda^3}{s_3}$$

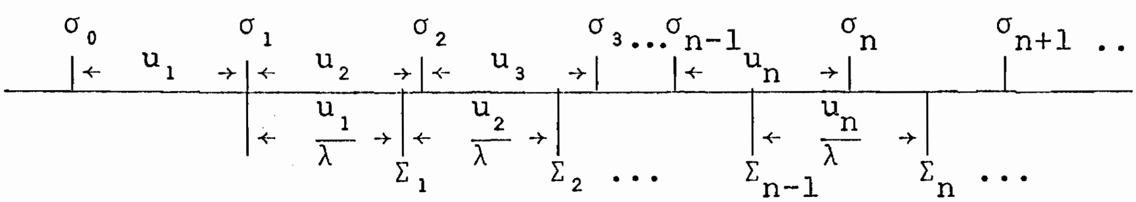
$$p_3^{(4)} = \frac{1}{s_1(\lambda s_2)^2} \quad .$$

There is no doubt that, in principle, we can find all the $p_i^{(n)}$'s in this manner. This, however, is very cumbersome since the number of cases increases very rapidly with n .

Although formulae can easily be conjectured for some of the above probabilities (for example, $p_{n-1}^{(n)} = \dfrac{1}{s_1(\lambda s_2)^{n-2}}$), we have not, as yet, been able to produce a general proof. It is believed that the analysis of this model is a major investigation in itself. We can, however, prove the following important result. Let

q_n = pr{at the 2nd, 3rd, 4th, ... up to and including the nth service opportunities there has always been exactly one customer waiting in the queue.} $(n \geq 2)$

The following diagram helps to clarify the event in question.



Notice the event whose probability is q_n is a very desirable event since it does not contain any imperfect (idle) service interval and all the customers have waited only a minimum amount of time.

We now easily observe that from first principles

$$q_2 = \frac{1}{s_1} \quad ,$$

$$q_3 = \frac{\lambda}{s_2} \quad ,$$

$$q_4 = \frac{\lambda^3}{s_3} \quad \text{etc.} \; ,$$

so that we can conjecture

$$q_{n+1} = \frac{\lambda^{\frac{1}{2}n(n-1)}}{s_n} \quad .$$

To prove this from the diagram we observe that

$$q_{n+1} = \int_0^\infty \frac{1}{\lambda\tau} e^{-\frac{u_1}{\lambda\tau}} du_1 \int_{\frac{1}{\lambda}u_1}^\infty \frac{1}{\lambda\tau} e^{-\frac{u_2}{\lambda\tau}} du_2 \quad \times$$

$$\times \int_{\frac{1}{\lambda}(u_1+u_2)-u_2}^\infty \frac{1}{\lambda\tau} e^{-\frac{u_3}{\lambda\tau}} du_3 \int_{\frac{1}{\lambda}(u_1+U_3)-U_3}^\infty \frac{1}{\lambda\tau} e^{-\frac{u_4}{\lambda\tau}} du_4 \; \ldots \; \times$$

$$\times \int_{\frac{1}{\lambda}(u_1+U_{n-1})-U_{n-1}}^\infty \frac{1}{\lambda\tau} e^{-\frac{u_n}{\lambda\tau}} du_n \int_{\frac{1}{\lambda}(u_1+U_n)-U_n}^\infty \frac{1}{\lambda\tau} e^{-\frac{u_{n+1}}{\lambda\tau}} dU_{n+1} \quad ,$$

where $U_n = \sum_{i=2}^{n} u_i$.

or

$$q_{n+1} = \int_0^\infty \frac{1}{\lambda\tau} e^{-\frac{s_1 u_1}{\lambda\tau}} du_1 \int_{\frac{1}{\lambda}u_1}^\infty \frac{1}{\lambda\tau} e^{-\frac{u_2}{\lambda^2\tau}} du_2 \quad \times$$

$$\times \int_{\frac{1}{\lambda}(u_1+u_2)-u_2}^\infty \frac{1}{\lambda\tau} e^{-\frac{u_3}{\lambda^2\tau}} \; \ldots \int_{\frac{1}{\lambda}(u_1+U_{n-1})-U_{n-1}}^\infty \frac{1}{\lambda\tau} e^{-\frac{u_n}{\lambda^2\tau}} du_n \quad ,$$

which, with the following set of transformations

$$\frac{u_1}{\lambda} = x_1 \; , \; \frac{u_2}{\lambda} = x_2 \; , \; \ldots \; , \; \frac{u_n}{\lambda} = x_n \quad ,$$

changes to

$$q_{n+1} = \lambda^{n} \int_{0}^{\infty} \frac{1}{\lambda\tau} e^{-\frac{s_1 x_1}{\tau}} dx_1 \int_{\frac{1}{\lambda}x_1}^{\infty} \frac{1}{\lambda\tau} e^{-\frac{x_2}{\lambda\tau}} dx_2 \times$$

$$\times \int_{\frac{1}{\lambda}(x_1+x_2)-x_2}^{\infty} \frac{1}{\lambda\tau} e^{-\frac{x_3}{\lambda\tau}} \cdots \int_{\frac{1}{\lambda}(x_1+X_{n-1})-X_{n-1}}^{\infty} \frac{1}{\lambda\tau} e^{-\frac{x_n}{\lambda\tau}} dx_n \quad ,$$

where $$X_n = \sum_{i=2}^{n} x_i \quad .$$

Now, we observe that if it is indeed true that

$q_n = \lambda^{\frac{1}{2}(n-1)(n-2)}/s_{n-1}$, it must be that

$$\int_{0}^{\infty} \frac{1}{\lambda\tau} e^{-\frac{u_1}{\lambda\tau}} du_1 \int_{\frac{1}{\lambda}u_1}^{\infty} \frac{1}{\lambda\tau} e^{-\frac{u_2}{\lambda\tau}} du_2 \int_{\frac{1}{\lambda}(u_1+u_2)-u_2}^{\infty} \frac{1}{\lambda\tau} e^{-\frac{u_3}{\lambda\tau}} du_3 \cdots \times$$

$$\times \int_{\frac{1}{\lambda}(u_1+U_{n-1})-U_{n-1}}^{\infty} \frac{1}{\lambda\tau} e^{-\frac{u_n}{\lambda\tau}} du_n = \frac{\lambda^{\frac{1}{2}(n-1)(n-2)}}{s_{n-1}} \quad ,$$

or, because of the fact that a Laplace transform and its inverse, if they exist, are unique, it must be that

$$\int_{\frac{1}{\lambda}u_1}^{\infty} \frac{1}{\lambda\tau} e^{-\frac{u_2}{\lambda\tau}} du \int_{\frac{1}{\lambda}(u_1+u_2)-u_2}^{\infty} \frac{1}{\lambda\tau} e^{-\frac{u_3}{\lambda\tau}} du_3 \ldots \times$$

$$\times \int_{\frac{1}{\lambda}(u_1+U_{n-1})-U_{n-1}}^{\infty} \frac{1}{\lambda\tau} e^{-\frac{u_n}{\lambda\tau}} du_n = \lambda^{\frac{1}{2}(n-1)(n-2)} e^{-\frac{s_{n-2}}{\lambda^2\tau}u_1} .$$

Thus, substituting this in the above equation for q_{n+1} , we have

$$q_{n+1} = \lambda^n \int_0^{\infty} \frac{1}{\lambda\tau} e^{-\frac{s_1 x_1}{\tau}} \lambda^{\frac{1}{2}(n-1)(n-2)} e^{-\frac{s_{n-2}}{\lambda^2\tau}x_1} dx_1 =$$

$$= \lambda^{\frac{1}{2}n(n-1)} \int_0^{\infty} \frac{1}{\tau} e^{-\frac{s_n x_1}{\tau}} dx_1 .$$

This proves the conjecture.

A remark should be made here that the function $f_n(w)$, the p.d.f. of the wait w of the nth customer, follows immediately from $p_k^{(n)}$. Thus, to obtain the results for $p_k^{(n)}$ automatically also gives $f_n(w)$.

8.2 The Original Model with General Interarrival Interval Distribution

Besides the dual queue, the solution of the original problem with a general interarrival interval distribution also constitutes a major investigation which could be undertaken. An integro-difference equation was derived in Chapter 2 for the waiting time, but a solution remains elusive. The state probabilities, we recall, depended directly on the moments of the waiting time density. Equation (3.7) is completely general and does not in any manner depend on the interarrival distribution. Thus, it is not only conceivable, but very likely, that the problem of state probabilities could be solved at least when arrivals have an E_k distribution from an equation like (2.9) without solving the equation.

8.3 Other Problems

The multiple server problem (that is, when there is more than one service channel) in itself forms an interesting, and perhaps, complicated investigation. This should be undertaken in order to evaluate the effectiveness of increasing the service facilities in a correlated system. The problems of maximum queue size and sojourn time should also be investigated. A solution of the latter problem would provide a measure of idle time during a given interval, and this helps to substantiate the claim that correlation can improve operational effectiveness, both from the point of view of customer and management.

Heavy traffic approximations (i.e. when traffic intensity is very close, but less than unity) could be sought for waiting time, state probabilities, and even busy periods. A completely new technique, of only "partially" correlating the service time, could be introduced and investigated. More restricted models in which, in addition to correlation, there is a lower and upper bound to any service interval, could, with profit, be investigated. Such a model would be much more realistic from the point of view of practical queueing problems. A further interesting model might be to give the nth customer a service time determined by the correlation mechanism we have introduced, subsequently putting him at the end of the queue again if his service is not finished in the time allotted.

In short, the study of "correlated queues" can, by no means, be considered terminated by the analysis given in this dissertation.

PART II

PRACTICAL (NUMERICAL) ANALYSIS

CHAPTER 9

COMPARISON OF THE OPERATIONAL FEATURES OF THE CORRELATED SINGLE SERVER QUEUE WITH THOSE OF M/M/1

In Part I a "correlated queue" was defined, described, and its general features were analyzed. It was asserted in Chapter 1 that such a correlation, permitting the server to regulate service time on the basis of an observable interarrival interval, could result in improved operating characteristics, deriving from a better utilization of the server's time. It is the objective of this chapter to demonstrate numerically these improved operating characteristics.

The method employed is to compare, at given levels of traffic intensity, the operational features of the correlated system with those of a conventional Poisson queue having identical arrival pattern, but completely independent service mechanism. That is, we compare the ordinary features of the correlated queue with those of M/M/1. We shall systematically compare state probabilities, waiting time, busy period and output. We also draw conclusions about the probability of the server being idle in both models. We shall also subsequently make a comparison of waiting time in the correlated model with other means of reducing waiting time, namely systems with fixed interarrival intervals and service times. Thus, we shall also make comparison with the systems D/M/1 and M/D/1.

The numerical calculations were performed on the IBM 7040 at the Virginia Polytechnic Institute, but since the programming was of a routine character, and contributes nothing to a better comprehension of the conclusions, no programming details will be included.

For the purposes of the comparison we shall fix the time scale by setting the mean interarrival interval to unity. The congestion index will be denoted by λ both for the correlated system, and for M/M/1. In order to distinguish the correlated values from noncorrelated ones, we shall add a subscript c to indicate reference to the correlated system. Thus, for example, $p_k^{(n)}$ is the probability that the system contains k customers at arrival epochs of the nth customer in M/M/1, while $p_{kc}^{(n)}$ is the corresponding probability for the correlated queue.

9.1 Comparison of State Probabilities

It was shown in Chapter 3 that

$$p_{kc}^{(n+1)} = \frac{S_{n-k}^{(k)}}{\lambda^{n-k} s_{n-k}} - \frac{S_{n-k+1}^{(k-1)}}{\lambda^{n-k+1} s_{n-k+1}} , \quad 1 \le k \le n-1$$

$$p_{nc}^{(n+1)} = 1 - \frac{S_1^{(n-1)}}{\lambda s_1} \qquad \text{and}$$

$$p_{0c}^{(n+1)} = \frac{1}{\lambda^n s_n} ,$$

with steady state values

$$\bar{p}_{kc} = (1-\lambda) S^{(k)} , \qquad k = 0,1,2...$$

where $s_i = \sum_{j=0}^{i} \lambda^{-j}$, $S_n = \sum_{i=0}^{n} s_i^{-1}$, $S = \sum_{i=1}^{\infty} s_i^{-1}$ and $S_n^{(k)}$ possesses all the terms in $(S_n)^k$, but with unit coefficients, i.e. it is the homogeneous form of degree k of the sequence $\{s_i^{-1}\},(i=1,2,...n)$.

The corresponding result in M/M/1 was shown in Chapter 7 to be

$$p_k^{(n+1)} = \frac{\lambda^k}{(\lambda s_1)^n} \sum_{i=0}^{n-k} \left(1 - \frac{i}{n}\right) \binom{n+i-1}{i} \frac{1}{s_1^i} , \qquad k = 1,2,...n ,$$

and

$$p_0^{(n+1)} = \frac{1}{\lambda} \; p_1^{(n+1)} ,$$

with steady state values

$$\bar{p}_k = (1-\lambda) \lambda^k , \qquad k = 0,1,2,... .$$

We shall now give tables of $p_k^{(n)}$ and $p_{kc}^{(n)}$ for different values of n and λ . The λ values of 0.2, 0.5, 0.9, 0.95 are chosen to provide a means of comparison at low, moderate, high and very high traffic intensity. The n values of 5, 11, 21, 31, 41, 51 are chosen to allow for a comparison at advancing stages of queue development.

From Table I to IV which give the values of $p_k^{(n)}$ and $p_{kc}^{(n)}$, we draw the following conclusions:

(i) $p_0^{(n)}$ is less in the correlated system; i.e. the correlated system tends to have a smaller probability of being empty, or the server is less likely to be idle. The difference is all the more significant as traffic intensity increases (before a steady state is reached).

(ii) The correlated system has a higher probability of containing a small number of customers, and a smaller probability of containing a large number. It contains fewer customers more often. Alternatively expressed, the queue size is more likely to remain moderate in the correlated system than in M/M/1.

One notices that for low values of λ the steady state is reached very soon in both correlated and non-correlated models. So that for example, for $\lambda = 0.2$ the probabilities are all the same after $n = 11$ to the degree of accuracy of the tabulation.

Table I(a)

$p_k^{(n)}$ and $p_{kc}^{(n)}$ $\lambda=0.2$

	n=5		n=11		n=21	
	M/M/1	Corr.	M/M/1	Corr.	M/M/1	Corr.
p_0	0.80152	0.80026	0.80001	0.80000	0.80000	0.80000
p_1	0.16030	0.16556	0.16000	0.16555	0.16000	0.16555
p_2	0.03161	0.02864	0.03200	0.02867	0.03200	0.02867
p_3	0.00579	0.00478	0.00640	0.00481	0.00640	0.00481
p_4	0.00077	0.00077	0.00128	0.00080	0.00128	0.00080
p_5			0.00025	0.00013	0.00026	0.00013
p_6			0.00005	0.00002	0.00005	0.00002
p_7			0.00001		0.00001	

Table I(b)

$p_k^{(n)}$ and $p_{kc}^{(n)}$ $\lambda=0.2$

	n=31		n=41		n=51	
	M/M/1	Corr.	M/M/1	Corr.	M/M/1	Corr.
p_0	0.80000	0.80000	0.80000	0.80000	0.80000	0.80000
p_1	0.16000	0.16555	0.16000	0.16555	0.16000	0.16555
p_2	0.03200	0.02867	0.03200	0.02867	0.03200	0.02867
p_3	0.00640	0.00481	0.00640	0.00481	0.00640	0.00481
p_4	0.00128	0.00080	0.00128	0.00080	0.00128	0.00080
p_5	0.00026	0.00013	0.00026	0.00013	0.00026	0.00013
p_6	0.00005	0.00002	0.00005	0.00002	0.00005	0.00002
p_7	0.00001		0.00001		0.00001	

Table II(a)

$p_k^{(n)}$ and $p_{kc}^{(n)}$ $\lambda=0.5$

	n=5		n=11		n=21	
	M/M/1	Corr.	M/M/1	Corr.	M/M/1	Corr.
p_0	0.54138	0.51613	0.50831	0.50024	0.50117	0.50000
p_1	0.27069	0.30673	0.25416	0.30340	0.25058	0.30335
p_2	0.12620	0.12305	0.12601	0.12630	0.12518	0.12635
p_3	0.04938	0.04175	0.06140	0.04609	0.06242	0.04615
p_4	0.01235	0.01235	0.02900	0.01594	0.03102	0.01599
p_5			0.01301	0.00538	0.01533	0.00542
p_6			0.00539	0.00179	0.00751	0.00182
p_7			0.00198	0.00059	0.00364	0.00061
p_8			0.00060	0.00019	0.00173	0.00020
p_9			0.00014	0.00006	0.00080	0.00007
p_{10}			0.00002	0.00002	0.00036	0.00002
p_{11}					0.00016	0.00001
p_{12}					0.00006	
p_{13}					0.00002	
p_{14}					0.00001	

Table II(b)

$p_k^{(n)}$ and $p_{kc}^{(n)}$ $\lambda=0.5$

	n=31		n=41		n=51	
	M/M/1	Corr.	M/M/1	Corr.	M/M/1	Corr.
p_0	0.50022	0.50000	0.50005	0.50000	0.50001	0.50000
p_1	0.25011	0.30335	0.25002	0.30335	0.25001	0.30335
p_2	0.12504	0.12635	0.12501	0.12635	0.12500	0.12635
p_3	0.06249	0.04615	0.06250	0.04615	0.06250	0.04615
p_4	0.03121	0.01599	0.03124	0.01599	0.03125	0.01599
p_5	0.01558	0.00542	0.01562	0.00542	0.01562	0.00542
p_6	0.00776	0.00182	0.00780	0.00182	0.00781	0.00182
p_7	0.00386	0.00061	0.00390	0.00061	0.00390	0.00061
p_8	0.00191	0.00020	0.00194	0.00020	0.00195	0.00020
p_9	0.00094	0.00007	0.00097	0.00007	0.00097	0.00007
p_{10}	0.00046	0.00002	0.00048	0.00002	0.00049	0.00002
p_{11}	0.00022	0.00001	0.00024	0.00001	0.00024	0.00001
p_{12}	0.00011		0.00012		0.00012	
p_{13}	0.00005		0.00006		0.00006	
p_{14}	0.00002		0.00003		0.00003	
p_{15}	0.00001		0.00001		0.00001	
p_{16}			0.00001		0.00001	

Table III(a)

$p_k^{(n)}$ and $p_{kc}^{(n)}$ $\lambda=0.9$

	n=5		n=11		n=21	
	M/M/1	Corr.	M/M/1	Corr.	M/M/1	Corr.
p_0	0.31264	0.24419	0.22200	0.14573	0.17564	0.11229
p_1	0.28138	0.33288	0.19980	0.25137	0.15808	0.21562
p_2	0.22021	0.24502	0.17211	0.24019	0.13967	0.22962
p_3	0.13543	0.12756	0.14025	0.16964	0.12076	0.18097
p_4	0.05034	0.05034	0.10657	0.09953	0.10184	0.11876
p_5			9.07413	0.05158	0.08346	0.06917
p_6			0.04603	0.02445	0.06622	0.03718
p_7			0.02458	0.01079	0.05065	0.01892
p_8			0.01063	0.00445	0.03717	0.00927
p_9			0.00333	0.00170	0.02603	0.00442
p_{10}			0.00057	0.00057	0.01728	0.00206
p_{11}					0.01080	0.00095
p_{12}					0.00629	0.00043
p_{13}					0.00338	0.00019
p_{14}					0.00165	0.00009
p_{15}					0.00072	0.00004
p_{16}					0.00027	0.00002
p_{17}					0.00008	0.00001
p_{18}					0.00002	

Table III(b)

$p_k^{(n)}$ and $p_{kc}^{(n)}$ $\lambda=0.9$

	n=21		n=41		n=51	
	M/M/1	Corr.	M/M/1	Corr.	M/M/1	Corr.
p_0	0.17564	0.11229	0.14368	0.10135	0.13585	0.10047
p_1	0.15808	0.21562	0.12931	0.20262	0.12227	0.20153
p_2	0.13967	0.22962	0.11552	0.22448	0.10944	0.22402
p_3	0.12076	0.18097	0.10233	0.18388	0.09736	0.18410
p_4	0.10184	0.11876	0.08979	0.12531	0.08602	0.12584
p_5	0.08346	0.06917	0.07798	0.07572	0.07543	0.07627
p_6	0.06622	0.03718	0.06694	0.04221	0.06560	0.04263
p_7	0.05065	0.01892	0.05676	0.02227	0.05654	0.02256
p_8	0.03717	0.00927	0.04748	0.01132	0.04827	0.01150
p_9	0.02603	0.00442	0.03914	0.00561	0.04079	0.00571
p_{10}	0.01728	0.00206	0.03178	0.00273	0.03409	0.00279
p_{11}	0.01080	0.00095	0.02537	0.00132	0.02816	0.00135
p_{12}	0.00629	0.00043	0.01990	0.00063	0.02298	0.00065
p_{13}	0.00338	0.00019	0.01531	0.00030	0.01850	0.00031
p_{14}	0.00165	0.00009	0.01155	0.00014	0.01470	0.00015
p_{15}	0.00072	0.00004	0.00853	0.00007	0.01151	0.00007
p_{16}	0.00027	0.00002	0.00615	0.00003	0.00887	0.00003
p_{17}	0.00008	0.00001	0.00433	0.00001	0.00673	0.00002
p_{18}	0.00002		0.00297	0.00001	0.00502	0.00001
p_{19}			0.00198		0.00368	
p_{20}			0.00129		0.00265	
p_{21}			0.00081		0.00187	
p_{22}			0.00049		0.00129	
p_{23}			0.00029		0.00087	
p_{24}			0.00016		0.00058	
p_{25}			0.00009		0.00037	
p_{26}			0.00004		0.00023	
p_{27}			0.00002		0.00014	
p_{28}			0.00001		0.00009	
p_{29}					0.00005	
p_{30}					0.00003	
p_{31}					0.00001	
p_{32}					0.00001	

Table IV(a)

$p_k^{(n)}$ and $p_{kc}^{(n)}$ λ=0.95

	n=5		n=11		n=21	
	M/M/1	Corr.	M/M/1	Corr.	M/M/1	Corr.
p_0	0.29230	0.22102	0.19792	0.11596	0.14884	0.07582
p_1	0.27769	0.32742	0.18802	0.22728	0.14140	0.17453
p_2	0.22772	0.25616	0.17009	0.24090	0.13139	0.21683
p_3	0.14596	0.13906	0.14495	0.18464	0.11909	0.19438
p_4	0.05633	0.05633	0.11479	0.11532	0.10498	0.14179
p_5			0.08300	0.06260	0.08972	0.08993
p_6			0.05345	0.03067	0.07408	0.05171
p_7			0.02955	0.01385	0.05886	0.02773
p_8			0.01321	0.00580	0.04480	0.01414
p_9			0.00427	0.00223	0.03250	0.00695
p_{10}			0.00075	0.00075	0.02233	0.00333
p_{11}					0.01442	0.00156
p_{12}					0.00868	0.00072
p_{13}					0.00481	0.00033
p_{14}					0.00242	0.00015
p_{15}					0.00109	0.00006
p_{16}					0.00042	0.00003
p_{17}					0.00014	0.00001
p_{18}					0.00003	
p_{19}					0.00001	

Table IV(b)

$p_k^{(n)}$ and $p_{kc}^{(n)}$ $\lambda=0.95$

	n=31		n=41		n=51	
	M/M/1	Corr.	M/M/1	Corr.	M/M/1	Corr.
p_0	0.12698	0.06281	0.11398	0.05695	0.10514	0.05394
p_1	0.12063	0.15444	0.10828	0.14476	0.09988	0.13961
p_2	0.11302	0.20445	0.10185	0.19779	0.09417	0.19406
p_3	0.10429	0.19474	0.09477	0.10413	0.08805	0.19356
p_4	0.09465	0.15047	0.08718	0.15426	0.08162	0.15616
p_5	0.08437	0.10076	0.07920	0.10603	0.07495	0.10885
p_6	0.07377	0.06099	0.07101	0.06575	0.06814	0.06837
p_7	0.06315	0.03433	0.06277	0.03786	0.06131	0.03983
p_8	0.05286	0.01835	0.05467	0.02066	0.05456	0.02197
p_9	0.04319	0.00945	0.04686	0.01084	0.04800	0.01165
p_{10}	0.03438	0.00473	0.03950	0.00554	0.04171	0.00600
p_{11}	0.02661	0.00233	0.03272	0.00277	0.03579	0.00303
p_{12}	0.01998	0.00113	0.02660	0.00137	0.03031	0.00151
p_{13}	0.01451	0.00054	0.02120	0.00067	0.02530	0.00074
p_{14}	0.01018	0.00026	0.01654	0.00032	0.02082	0.00036
p_{15}	0.00687	0.00012	0.01263	0.00016	0.01688	0.00018
p_{16}	0.00444	0.00006	0.00942	0.00008	0.01346	0.00009
p_{17}	0.00274	0.00003	0.00685	0.00004	0.01056	0.00004
p_{18}	0.00161	0.00001	0.00485	0.00002	0.00814	0.00002
p_{19}	0.00089	0.00001	0.00334	0.00001	0.00616	0.00001
p_{20}	0.00047		0.00223		0.00458	
p_{21}	0.00023		0.00145		0.00333	
p_{22}	0.00010		0.00091		0.00237	
p_{23}	0.00004		0.00055		0.00166	
p_{24}	0.00002		0.00032		0.00113	
p_{25}			0.00018		0.00075	
p_{26}			0.00009		0.00049	
p_{27}			0.00005		0.00031	
p_{28}			0.00002		0.00019	
p_{29}			0.00001		0.00011	
p_{30}					0.00006	
p_{31}					0.00004	
p_{32}					0.00002	
p_{33}					0.00001	

We shall now tabulate the steady state state probabilities and further conclude that:

(iii) In the steady state the probability of the server being idle is the same in both systems.

(iv) Here again (in fact, much more so) the correlated queue has a higher probability of containing a small number of customers, and a smaller probability of containing a large number.

In the correlated queue the last (up to five decimal places) non-zero probability is associated with a much smaller n. For example, for $\lambda = 0.9$, $\bar{p}_{18c} = 0.00001$ and the subsequent probabilities are all zero up to five decimal places. The corresponding number for M/M/1 is approximately 86. This reflects directly on the waiting room facility about which we shall make further comments later on.

Table V

$\bar{p}_k$ and $\bar{p}_{kc}$

$k \backslash \lambda$	0.2		0.5		0.90		0.95	
	M/M/1	Corr.	M/M/1	Corr.	M/M/1	Corr.	M/M/1	Corr.
$\bar{p}_0$	0.80000	0.80000	0.50000	0.50000	0.10000	0.10000	0.05000	0.05000
$\bar{p}_1$	0.16000	0.16555	0.25000	0.30334	0.09000	0.20095	0.04750	0.13265
$\bar{p}_2$	0.03200	0.02867	0.12500	0.12635	0.08100	0.22375	0.04513	0.18876
$\bar{p}_3$	0.00640	0.00481	0.06250	0.04615	0.07290	0.18418	0.04287	0.19249
$\bar{p}_4$	0.00128	0.00080	0.03125	0.01599	0.06561	0.12609	0.04073	0.15851
$\bar{p}_5$	0.00026	0.00013	0.01562	0.00542	0.05905	0.07654	0.03869	0.11260
$\bar{p}_6$	0.00005	0.00002	0.00781	0.00182	0.05314	0.04284	0.03675	0.07196
$\bar{p}_7$	0.00001		0.00391	0.00061	0.04783	0.02270	0.03492	0.04258
$\bar{p}_8$			0.00195	0.00020	0.04305	0.01158	0.03317	0.02382
$\bar{p}_9$			0.00098	0.00007	0.03874	0.00576	0.03151	0.01279
$\bar{p}_{10}$			0.00049	0.00002	0.03487	0.00282	0.02994	0.00667
$\bar{p}_{11}$			0.00024	0.00001	0.03138	0.00136	0.02844	0.00341
$\bar{p}_{12}$			0.00012		0.02824	0.00065	0.02702	0.00171
$\bar{p}_{13}$			0.00006		0.02542	0.00031	0.02567	0.00085
$\bar{p}_{14}$			0.00003		0.02288	0.00015	0.02438	0.00042
$\bar{p}_{15}$			0.00002		0.02054	0.00007	0.02316	0.00021
$\bar{p}_{16}$			0.00001		0.01853	0.00003	0.02201	0.00010
$\bar{p}_{17}$					0.01668	0.00002	0.02091	0.00005
$\bar{p}_{18}$					0.01501	0.00001	0.01986	0.00002
$\bar{p}_{19}$					0.01351		0.01887	0.00001
$\bar{p}_{20}$					0.01216		0.01792	0.00001
$\bar{p}_{21}$					0.01094		0.01703	
$\bar{p}_{22}$					0.00985		0.01618	
$\bar{p}_{23}$					0.00886		0.01587	

An important question about steady state probabilities is how fast they are achieved. We here include Table VI from which we conclude:

(v) The steady state probabilities are achieved very much more rapidly in the correlated system than they are in their M/M/1 counterpart.

In fact, for as small a λ as 0.2 the steady state probabilities are already achieved at the arrival epoch of the 7th customer for the correlated system, while for M/M/1 they are achieved at that of the 11th customer. This difference, of course, increases as λ approaches unity.

Table VI(a)
Comparison of Time Dependent and Steady State Probabilities in the Correlated Queue and M/M/1 $\lambda=0.5$

k	n=5 $\|p_k^{(n)}-\bar{p}_k\|$	n=5 $\|p_{kc}^{(n)}-\bar{p}_{kc}\|$	n=11 $\|p_k^{(n)}-\bar{p}_{kc}\|$	n=11 $\|p_k^{(n)}-\bar{p}_{kc}\|$	n=21 $\|p_k^{(n)}-\bar{p}_k\|$	n=21 $\|p_{kc}^{(n)}-\bar{p}_{nc}\|$
0	0.04138	0.01613	0.00831	0.00024	0.00117	0.00000
1	0.02069	0.00338	0.00416	0.00006	0.00058	..
2	0.00120	0.00331	0.00101	0.00005	0.00018	..
3	0.01312	0.00440	0.00110	0.00006	0.00008	..
4	0.01890	0.00365	0.00225	0.00005	0.00023	..
5			0.00261	0.00004	0.00029	
6			0.00242	0.00003	0.00030	
7			0.00193	0.00002	0.00027	
8			0.00135	0.00001	0.00022	
9			0.00084	0.00001	0.00017	
10			0.00047	0.00001	0.00013	
11					0.00009	
12					0.00006	
13					0.00004	
14					0.00002	
15					0.00001	
16					0.00001	

Table VI(b)
Comparison of Time Dependent and Steady State Probabilities in the Correlated Queue and M/M/1 $\lambda=0.9$

k	n=5 $\|p_k^n-\bar{p}_k\|$	n=5 $\|p_{kc}^{(n)}-\bar{p}_{kc}\|$	n=21 $\|p_k^{(n)}-\bar{p}_k\|$	n=21 $\|p_{kc}^{(n)}-\bar{p}_{kc}\|$	n=51 $\|p_k^{(n)}-\bar{p}_k\|$	n=51 $\|p_{kc}^{(n)}-\bar{p}_{kc}\|$
0	0.21264	0.14419	0.07564	0.01229	0.03585	0.00047
1	0.19138	0.13193	0.06808	0.01467	0.03227	0.00058
2	0.13921	0.02127	0.05868	0.00587	0.02844	0.00027
3	0.06253	0.05662	0.04786	0.00321	0.02446	0.00008
4	0.01527	0.07575	0.03623	0.00733	0.02041	0.00025
5			0.02441	0.00737	0.01638	0.00027
6			0.01307	0.00566	0.01246	0.00021
7			0.00282	0.00378	0.00871	0.00014
8			0.00588	0.00232	0.00523	0.00009
9			0.01272	0.00134	0.00205	0.00005
10			0.01759	0.00075	0.00078	0.00003
11			0.02058	0.00041	0.00322	0.00002
12			0.02195	0.00022	0.00527	0.00001
13			0.02204	0.00012	0.00692	0.00000
14			0.02123	0.00006	0.00818	..
15			0.01987	0.00003	0.00908	..
16			0.01827	0.00002	0.00966	..
17			0.01659	0.00001	0.00995	

Obviously more extensive tables could be produced, but from Tables VI(a) and VI(b) the point is rather clear.

We shall next compare expected values and variances of the number in the system which it was easy to compute at the same time as $p_k^{(n)}$ and p_{kc}^n . Table VII gives $E(k)$, $E(k)_c$, $Var(k)$ and $Var(k)_c$. From this we conclude that:

(vi) The means and variances of the number in the correlated system are, on the whole, very much less than their M/M/1 counterpart. The reduction is much more significant as time (n) increases and traffic becomes heavier.

Again, one observes that for low values of λ steady state means and variances are achieved very rapidly; much more so for the correlated system than for M/M/1 .

Table VII
Mean and Variance of Number in System

n \ λ	0.2				0.5			
	$E(k)$	$E(k)_c$	$Var(k)$	$Var(k)_c$	$E(k)$	$E(k)_c$	$Var(k)$	$Var(k)_c$
5	0.2440	0.2403	0.2917	0.2777	0.7206	0.7275	0.8982	0.8430
11	0.2499	0.2414	0.3122	0.2825	0.9238	0.8020	1.5887	1.0817
21	0.2500	0.2414	0.3125	0.2825	0.9871	0.8033	1.9133	1.0877
31	0.2500	0.2414	0.3125	0.2825	0.9974	0.8033	1.9802	1.0877
41	0.2500	0.2414	0.3125	0.2825	0.9994	0.8033	1.9952	1.0877
51	0.2500	0.2414	0.3125	0.2825	0.9999	0.8033	1.9988	1.0877
n \ λ	**0.9**				**0.95**			
5	1.3295	1.4070	1.4191	1.2869	1.3963	1.4822	1.4538	1.3078
11	2.3306	2.1755	4.1451	2.7762	2.5164	2.3923	4.3909	2.9499
21	3.3588	2.5490	8.6880	3.5597	3.7459	2.9552	9.6003	3.9987
31	4.0642	2.6555	12.9700	3.7700	4.6483	3.1921	14.8339	4.3973
41	4.6034	2.6904	16.9649	3.8376	5.3784	3.3125	19.9956	4.5907
51	5.0383	2.7023	20.6902	3.8605	5.9981	3.3787	25.0591	4.6946

Table VIII below is included to show that:

(vii) Conclusion (vi) remains valid (actually with more strength) in the steady state.

Ratios of means and variances are also given to emphasize the respective magnitudes.

Table VIII
Steady State Expected Value and Variance of Number in System and Their Ratios

λ	$E(k)$	$E(k)_c$	$\frac{E(k)_c}{E(k)}$	$Var(k)$	$Var(k)_c$	$\frac{Var(k)_c}{Var(k)}$
0.2	0.2500	0.2414	0.9656	0.3125	0.2825	0.9040
0.5	1.0000	0.8033	0.8033	2.0000	1.0877	0.5438
0.7	2.3333	1.4267	0.6115	7.7777	2.0388	0.2621
0.9	9.0000	2.7077	0.3008	90.0000	3.8701	0.0430
0.95	19.0000	3.4682	0.1825	380.0000	4.8341	0.0127
0.97	32.3333	4.0105	0.1240	1077.7778	5.4761	0.0051
0.99	99.0000	5.1391	0.0519	9900.0000	6.7520	0.0007

The "waiting room" requirement is a factor of considerable concern in many queueing systems. Before closing the discussion on state probabilities, we find it desirable here to include Table IX which shows clearly that in the correlated queue the waiting room size can be considerably smaller than in M/M/1. We choose an arbitrary probability level of 5% and tabulate a number n for both systems which is such that in the steady state the probability that the state of the system is n or more is less than 5%.

Table IX
Steady State Values of Number in System Which Has Less Than 5% Chance of Being Equalled or Exceeded

λ	M/M/1	Corr.
0.2	2	2
0.5	5	4
0.7	9	5
0.9	29	7
0.95	59	8
0.97	100	9
0.99	300	11

The advantage of the correlated system scarcely needs comment.

9.2 Comparison of Waiting Time

The steady state waiting time density (including service) in the system M/M/1 can easily be shown to be

$$f(w) = \frac{(1-\lambda)}{\lambda} e^{-\frac{(1-\lambda)}{\lambda} w}$$

when the mean arrival interval is unity and the mean service interval is λ. (See for example D. V. Lindley [28].) The corresponding result for the correlated queue was shown originally by B. W. Conolly [11] to be

$$f_c(w) = \frac{1}{\lambda} \sum_{r \geq 0} g_r \, e^{-\frac{s_r}{\lambda} w} ,$$

where

$$g_r = \frac{(-)^r \, \lambda^{\frac{1}{2}r(r-1)}}{(1-\lambda)^2 (1-\lambda^2)(1-\lambda^3)\ldots(1-\lambda^r)} .$$

We shall now plot these densities for different values of λ (0.2,0.5,0.9,0.95) and observe the following important conclusion:

(viii) The waiting time density of the correlated queue is very much more dense in the lower range of wait than the corresponding M/M/1 density. The difference is more noticeable for higher values of λ (heavy traffic) where the M/M/1 density is almost flat.

This conclusion is emphasized by the distribution functions which are also given in the subsequent diagrams. A close look at Figures 1-4 indicates that a small percentage of customers, whose waiting time in M/M/1 is rather short, do have to wait slightly longer in the correlated system. This stems from the fact that when the arrival rate is low the server does in fact serve in a very leisurely manner.

We are now in a position to compare means and variances of waiting time in the two systems. In M/M/1

$$E(w) = \frac{\lambda}{1-\lambda} \quad \text{and} \quad \mathrm{Var}(w) = \left(\frac{\lambda}{1-\lambda}\right)^2 .$$

In the correlated system

$$E_c(w) = \lambda \sum_{i\geq 0} \frac{1}{s_i} \quad \text{and} \quad \mathrm{Var}_c(w) = \lambda^2 \sum_{i\geq 0} \frac{1}{s_i^2} .$$

It is now evident that

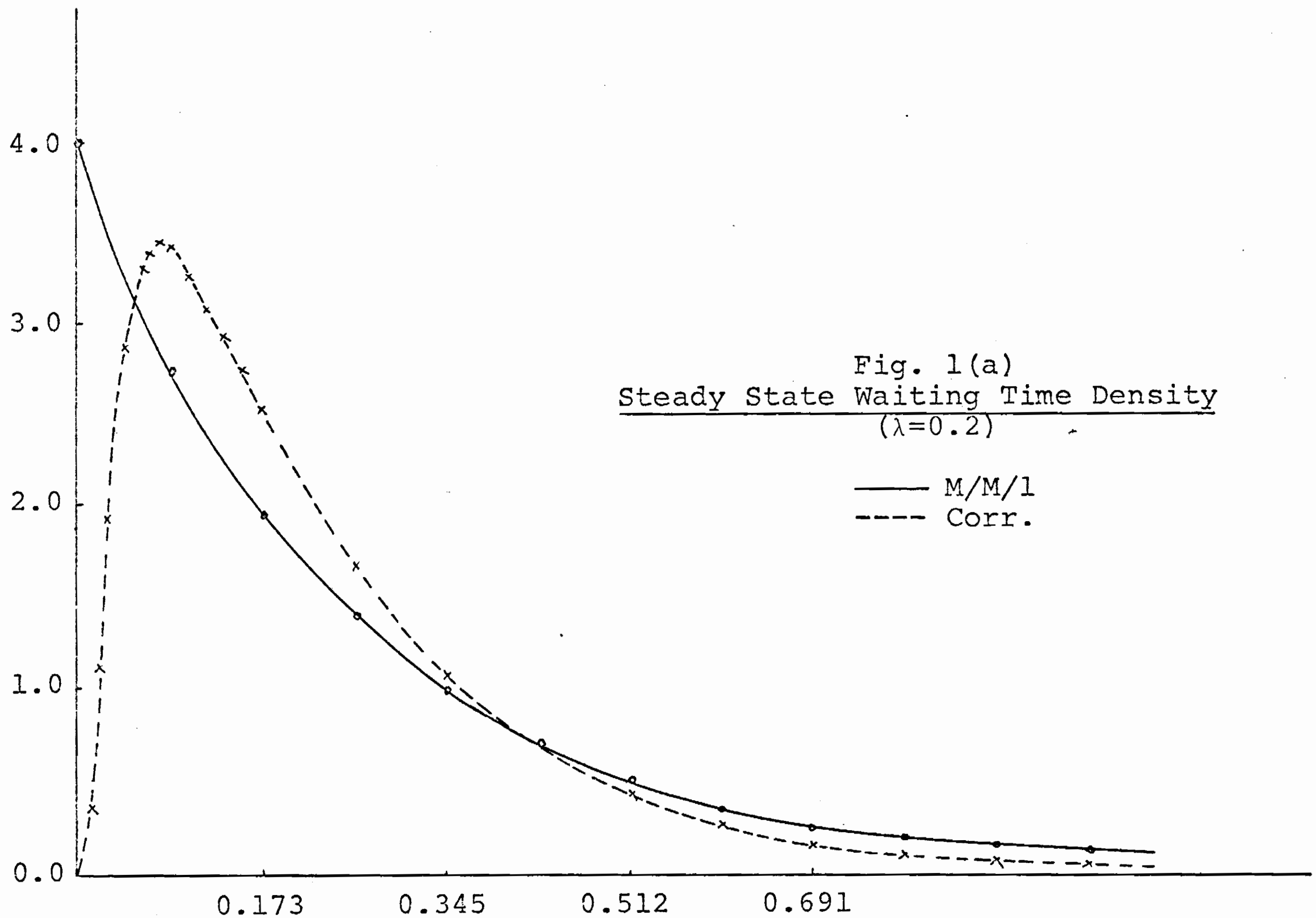


Fig. 1(a)
Steady State Waiting Time Density
(λ=0.2)

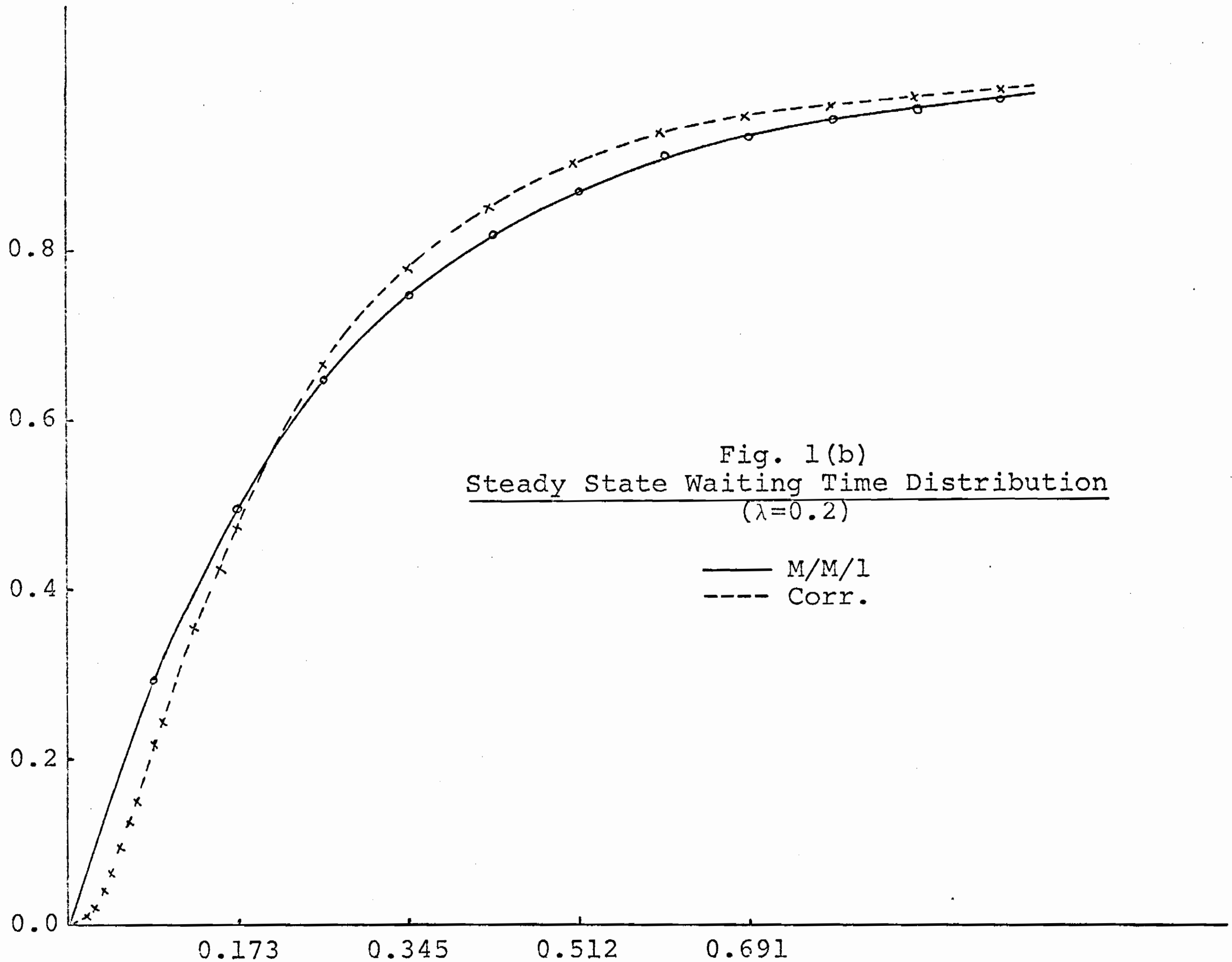


Fig. 1(b)
Steady State Waiting Time Distribution
(λ=0.2)

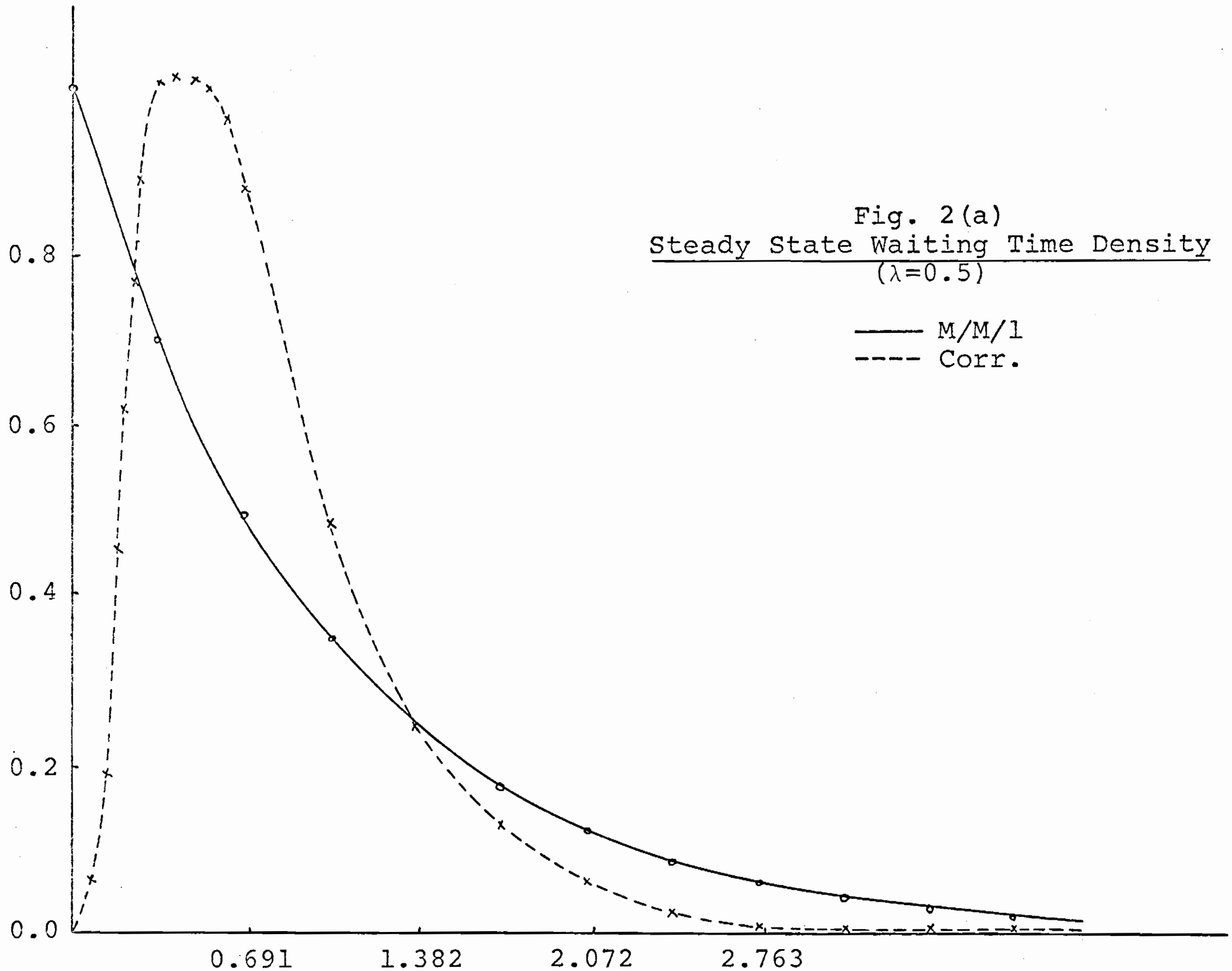


Fig. 2(a)
Steady State Waiting Time Density
(λ=0.5)

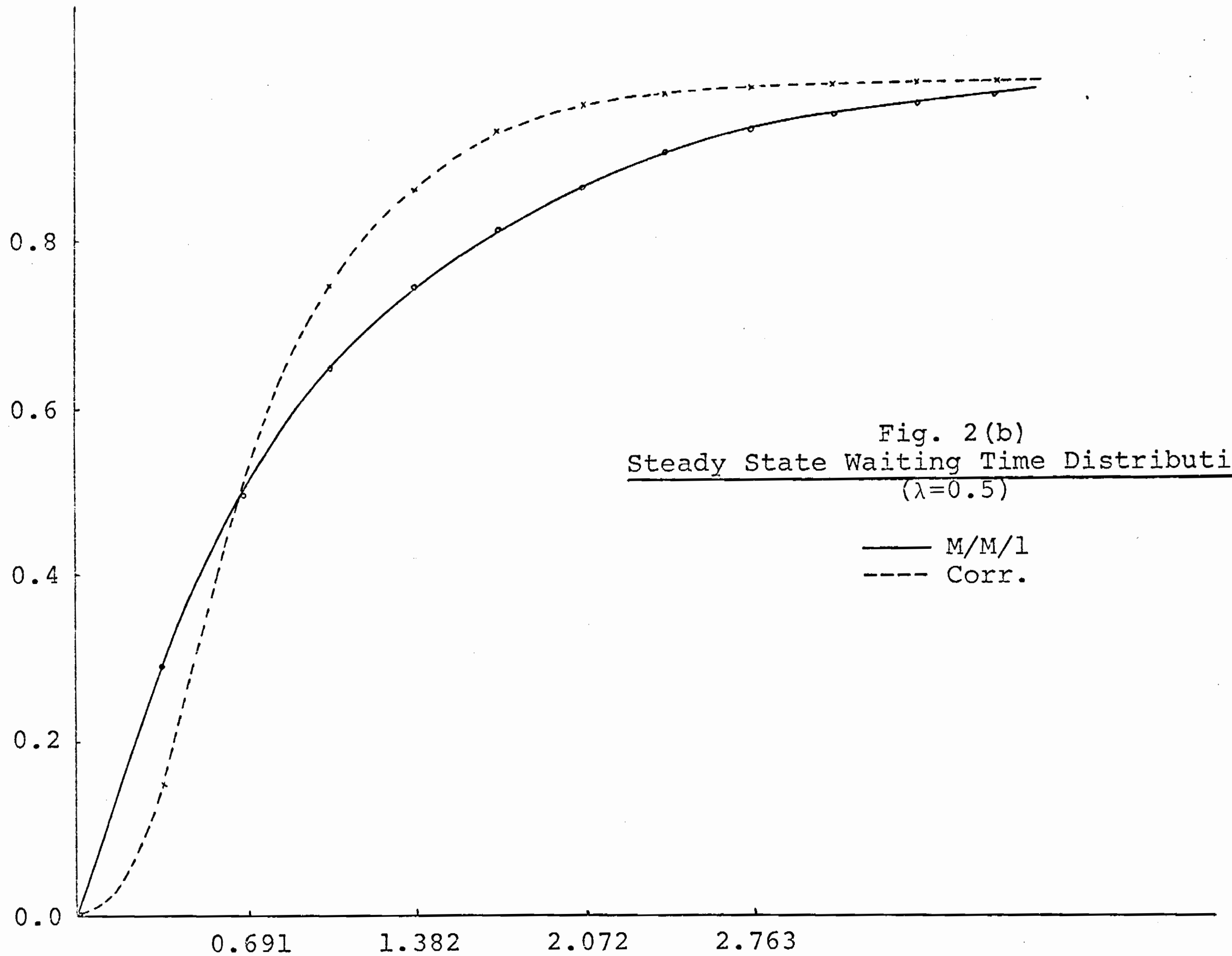


Fig. 2(b)
Steady State Waiting Time Distribution
(λ=0.5)

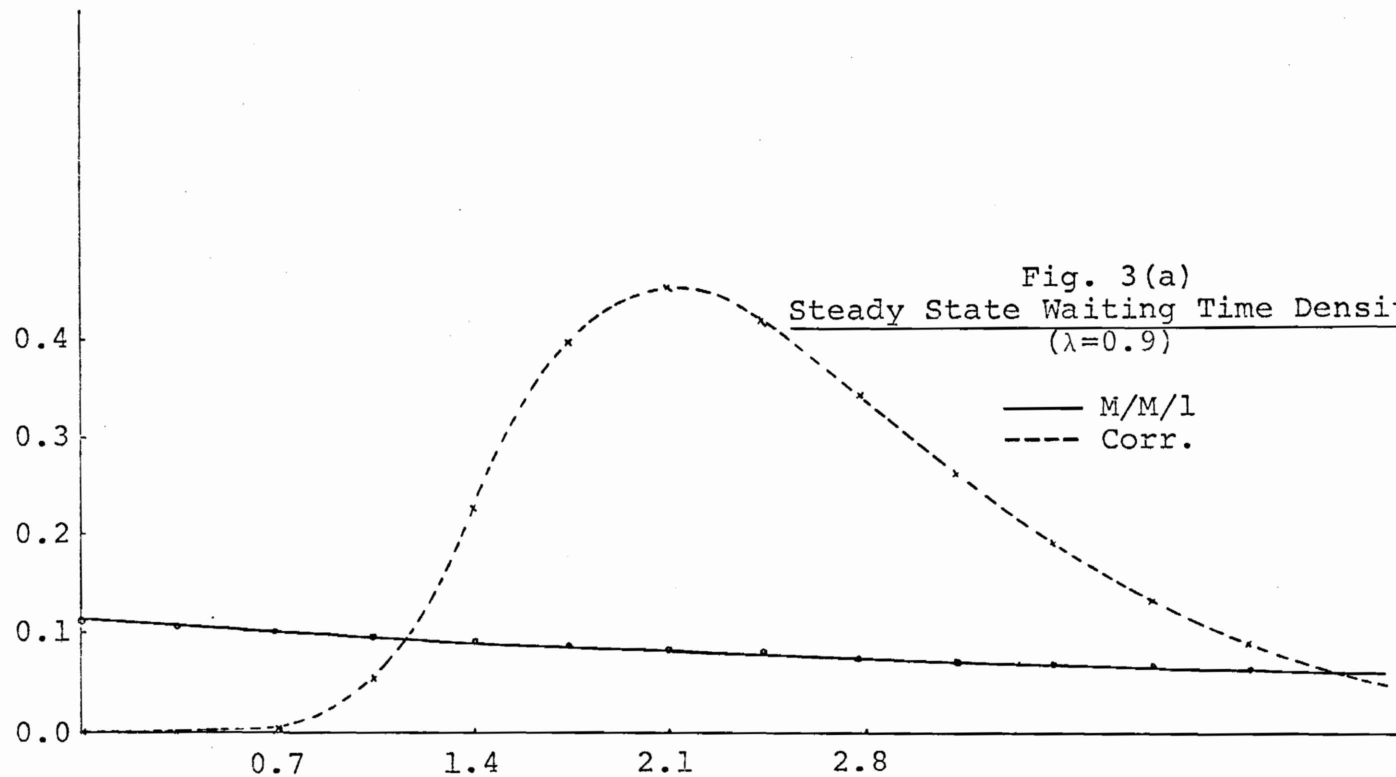


Fig. 3(a)
Steady State Waiting Time Density
(λ=0.9)

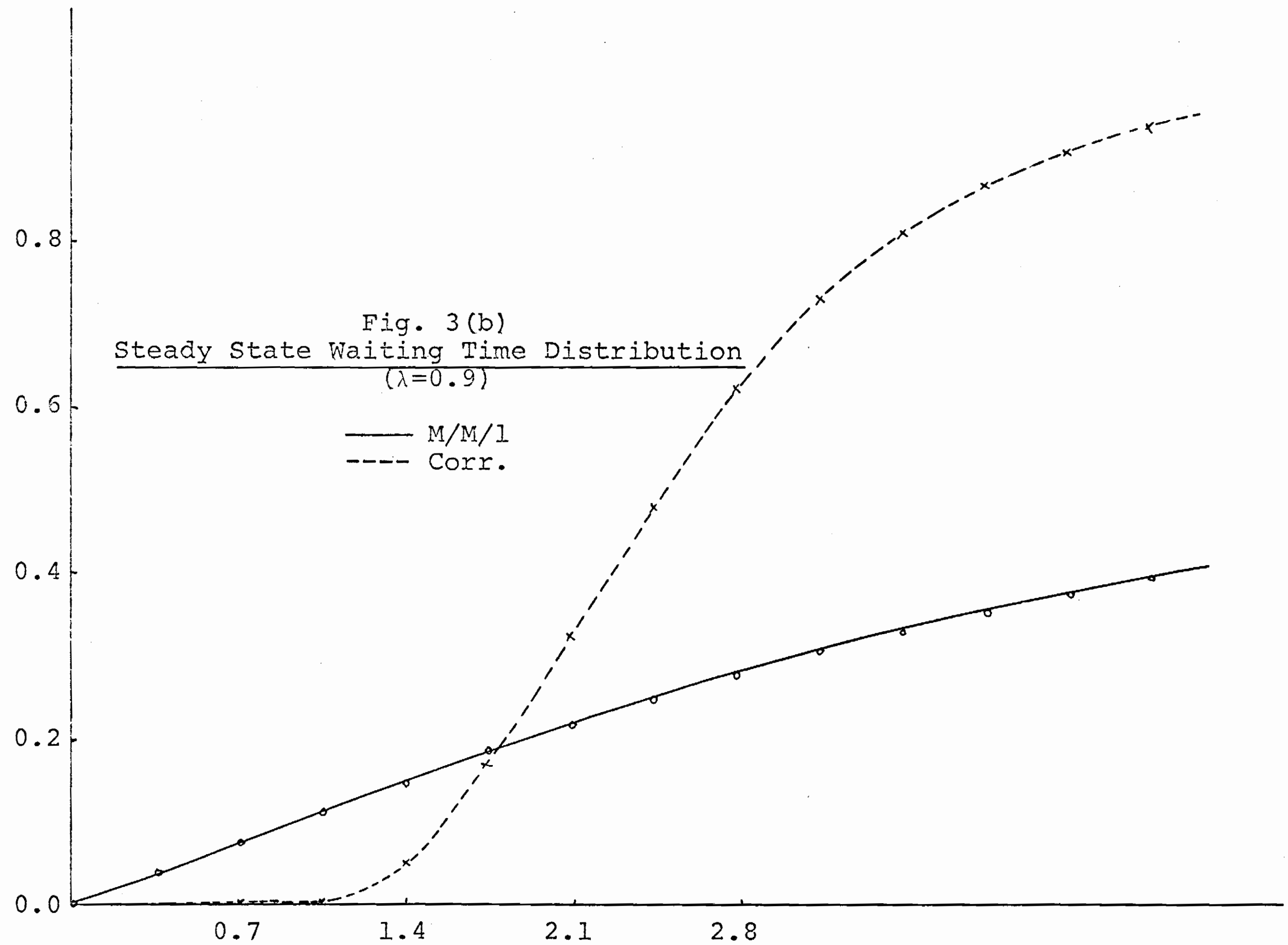


Fig. 3(b)
Steady State Waiting Time Distribution
(λ=0.9)
M/M/1
Corr.
0.8
0.6
0.4
0.2
0.0
0.7
1.4
2.1
2.8

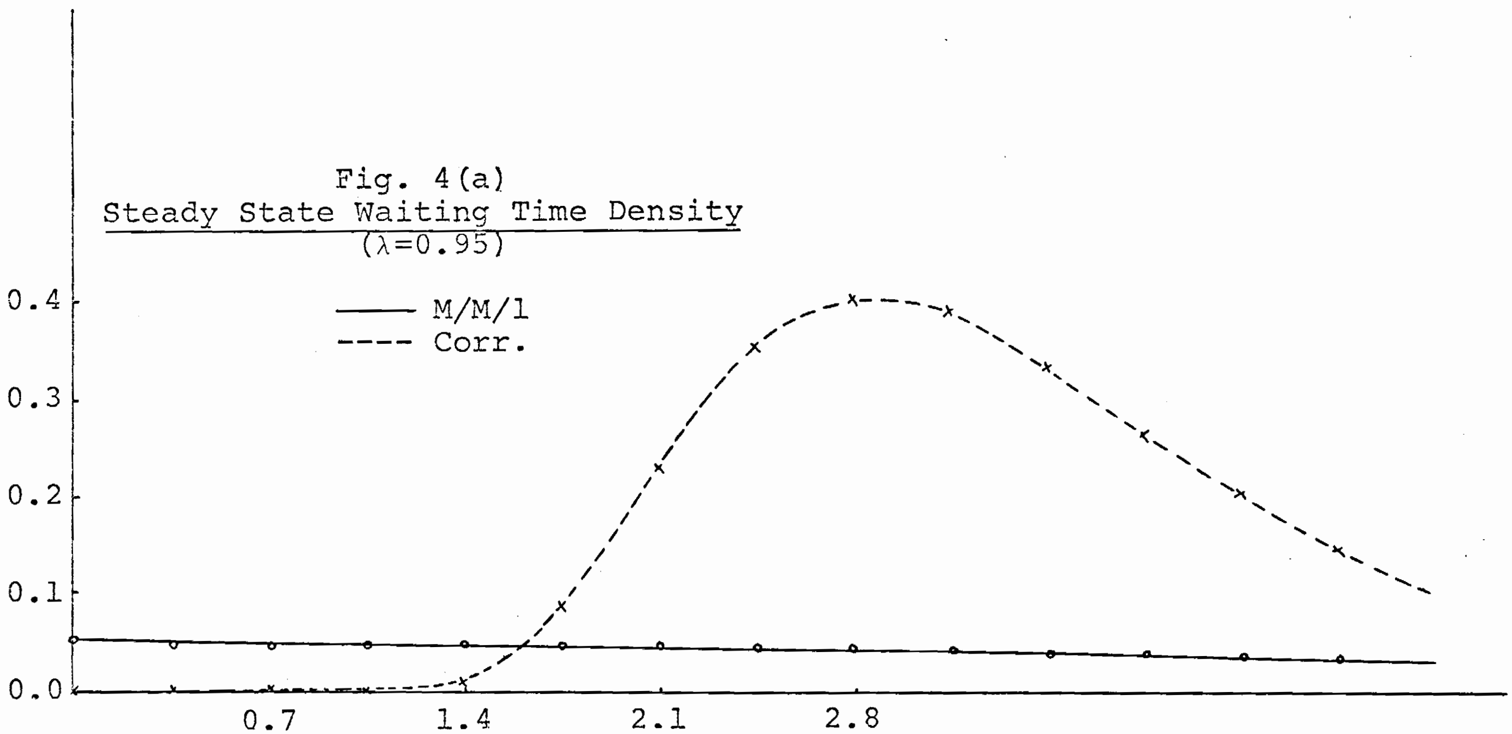


Fig. 4(a)
Steady State Waiting Time Density
(λ=0.95)

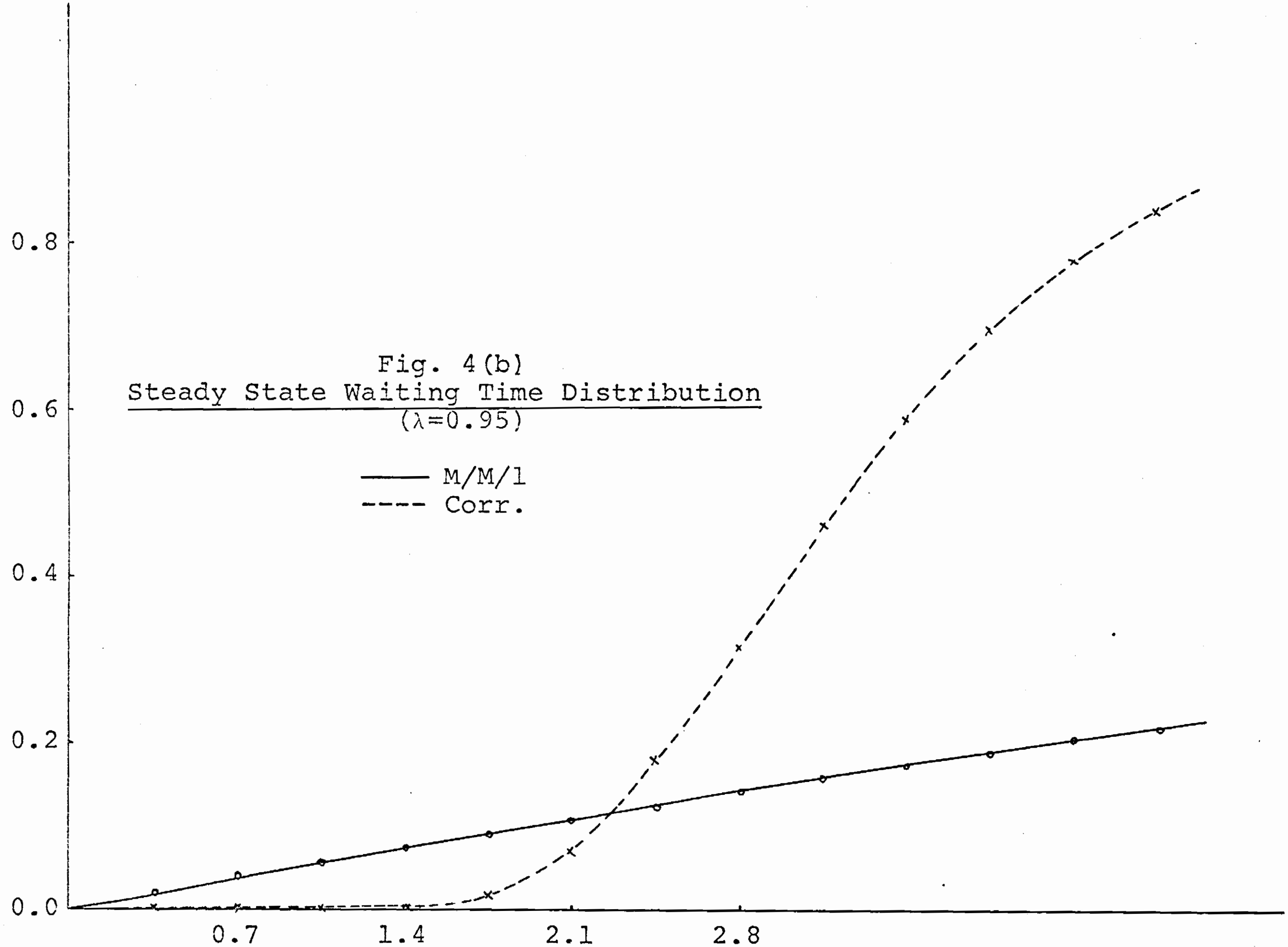


Fig. 4(b)
Steady State Waiting Time Distribution
(λ=0.95)

$$\lambda\left(\frac{1}{1-\lambda} - \sum_{i\geq 0} \frac{1}{s_i}\right) = \lambda \sum_{i\geq 1} \frac{\lambda^i s_{i-1}}{s_i} > 0 \quad ,$$

and, in fact, $\lambda \sum_{i\geq 1} \frac{\lambda^i s_{i-1}}{s_i}$ is a monotonically increasing function of λ since

$$\frac{\partial \frac{s_{i-1}}{s_i}}{\partial \lambda} = \left(\frac{i}{\lambda^{i+1}} + \frac{i-1}{\lambda^{i+2}} + \ldots + \frac{1}{\lambda^{2i}}\right) \Big/ s_i^2 > 0 \; .$$

We can produce a similar argument for the variances, however, Table X which gives the means and the variances and their ratios in the steady state is believed to bring out the point much more readily.

Table X
Mean and Variance of Waiting Time and Their Ratios in Steady State

λ	$E(w)$	$E_c(w)$	$\frac{E_c(w)}{E(w)}$	$Var(w)$	$Var_c(w)$	$\frac{Var_c(w)}{Var(w)}$
0.2	0.25000	0.24139	0.96556	0.06250	0.04115	0.65840
0.5	1.00000	0.80334	0.80334	1.00000	0.28433	0.28433
0.7	2.33333	1.42685	0.61151	5.44443	0.61324	0.11264
0.9	9.00000	2.70856	0.30095	81.00000	1.16391	0.01437
0.95	19.00000	3.47027	0.18265	361.00000	1.36524	0.00378
0.97	32.33333	4.01441	0.12416	1045.44423	1.46053	0.00140
0.99	99.00000	5.15293	0.05205	9801.00000	1.57198	0.00016

It should be pointed out here that B. W. Conolly, in his original paper where he introduced the problem [11], did essentially produce the above table, a glance at which implies the following conclusion:

(ix) The correlation mechanism drastically reduces the expected value and variance of the waiting time of the customers. This reduction is startling when traffic becomes heavy.

This conclusion along with conclusion (i) implies that correlation, or self regulation, provides twin benefits in the form of improved service and better server utilization. It is in this respect that this method of queue regulation, leading to the improvement of operational characteristics, is different from the more obvious methods (such as increasing the number of service channels) which suggest themselves.

It is evident that non-steady state results can also easily be compared, but it is believed they will not contribute further conclusions.

9.3 Comparison of Busy Period Processes

The p.d.f. of the busy period in M/M/1 with unit mean arrivals and traffic intensity λ can be shown to be

$$b(t) = \sqrt{\frac{1}{\lambda}}\, e^{-s_1 t}\, I_1\left(2t\sqrt{\frac{1}{\lambda}}\right)\Big/ t$$

where $I_1(x)$ is the modified Bessel function of the first kind and order 1. (See, for example, B. W. Conolly [7].) The mean and variance are respectively

$$E(t) = \frac{\lambda}{1-\lambda} \quad \text{and} \quad Var(t) = \left(\frac{\lambda}{1-\lambda}\right)^2 \left(\frac{1+\lambda}{1-\lambda}\right) .$$

The corresponding density in the correlated system was shown in Chapter 4 to be

$$b(t) = \sum_{n\geq 1} \sum_{j=0}^{n-1} (-)^j \lambda^{\frac{1}{2}n(n-3)-\frac{1}{2}j(j-1)} \frac{(s_{n-j-1})^{n-2}(s_{n-j-1}s_{n-j}\cdots s_{n-2})}{s_0 s_1 \cdots s_{j-1}} e^{-\frac{s_{n-j}}{s_{n-j-1}}t}$$

with mean $E_c(t) = \sum_{i\geq 1} \frac{1}{s_i}$ and variance $Var_c(t) =$

$$= \sum_{i\geq 1} \frac{s_{i-1}}{s_i} \sum_{j\geq i} \frac{1}{s_j} - \left(\sum_{i\geq 1} \frac{1}{s_i}\right)^2 .$$

The densities and distributions are plotted in Figures 5(a)-8(b). We may conclude that:

(x) More of the weight of the correlated b.p. density is concentrated on the lower range of time scale than its M/M/1 counterpart; the distribution function reaches unity sooner in the correlated system.

This then has the effect of more busy periods of smaller length over a given interval of time.

One notices from Figures 5-8 that at low values of λ the difference between the two systems is not very sharp, but as λ approaches unity it becomes more and more significant following, apparently, the common trend.

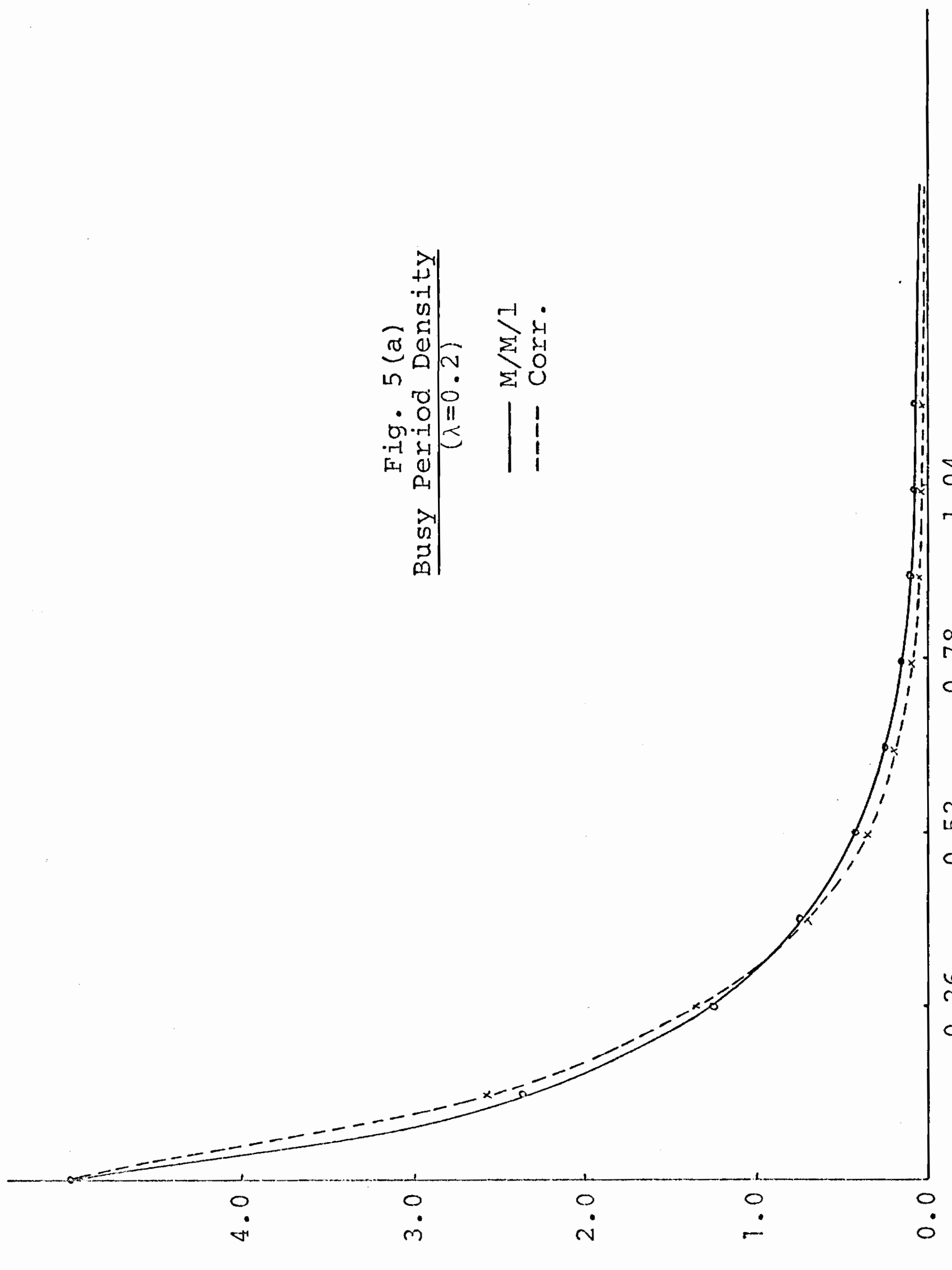


Fig. 5(a)
Busy Period Density
(λ=0.2)

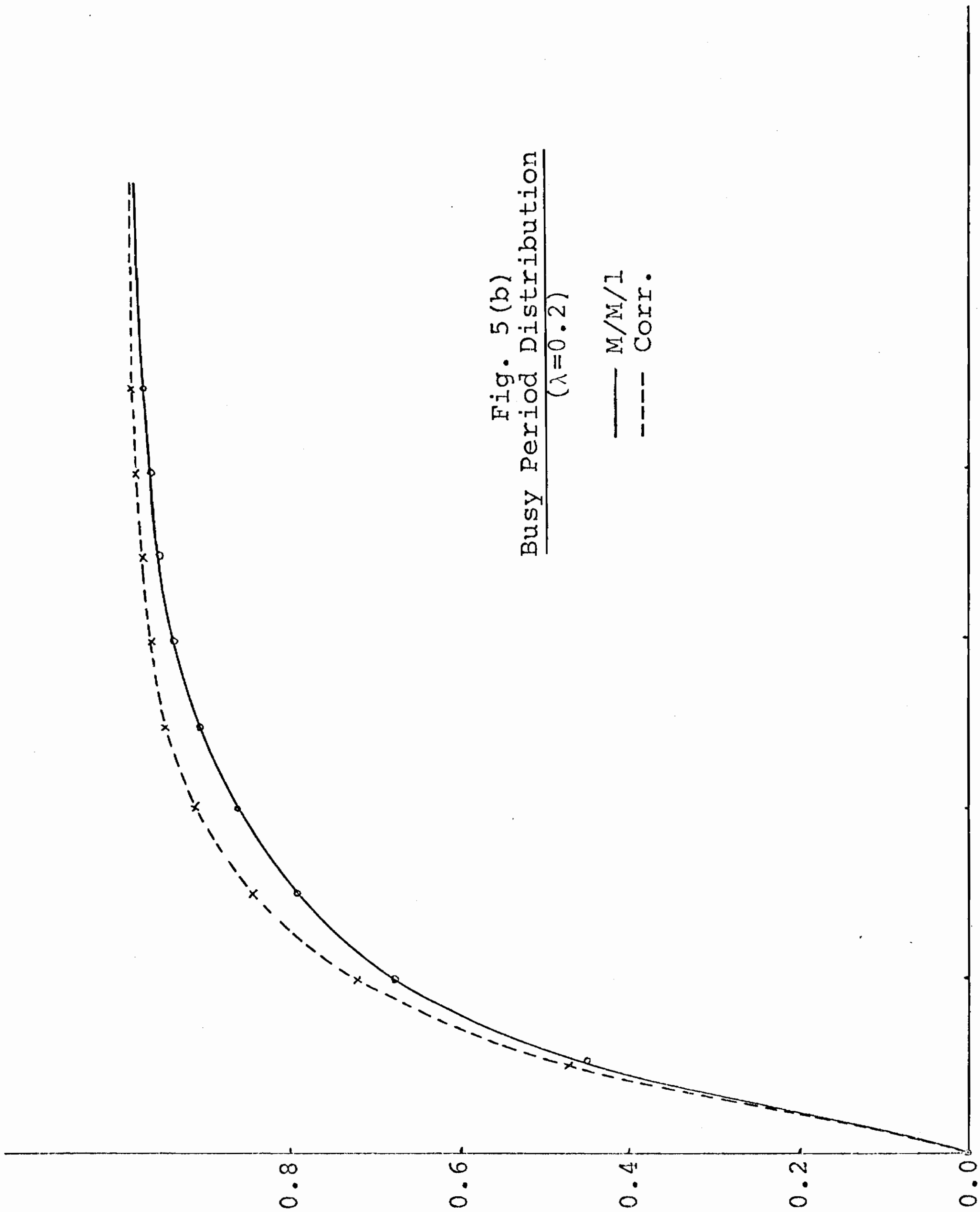


Fig. 5(b)
Busy Period Distribution
(λ=0.2)

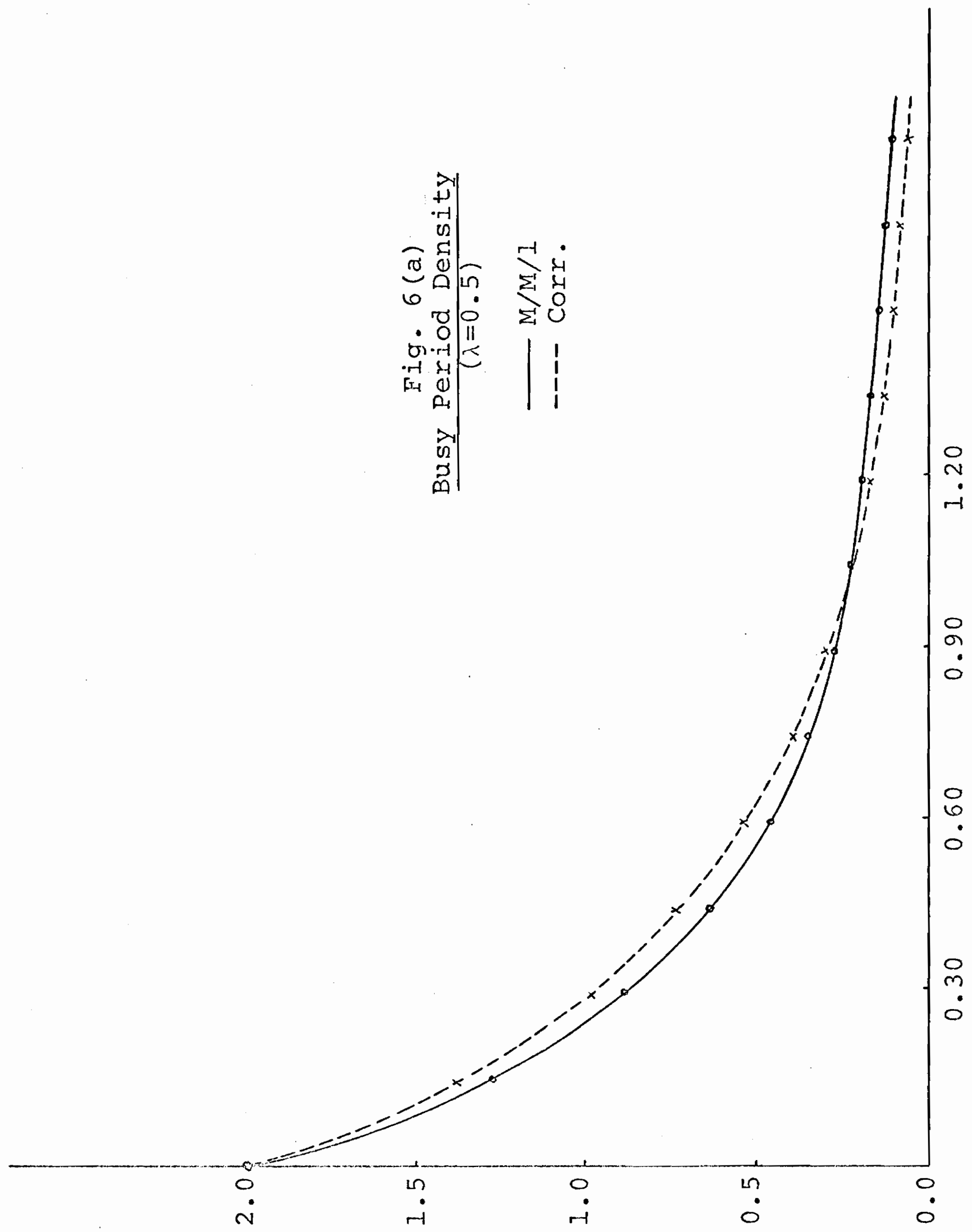


Fig. 6(a)
Busy Period Density
(λ=0.5)

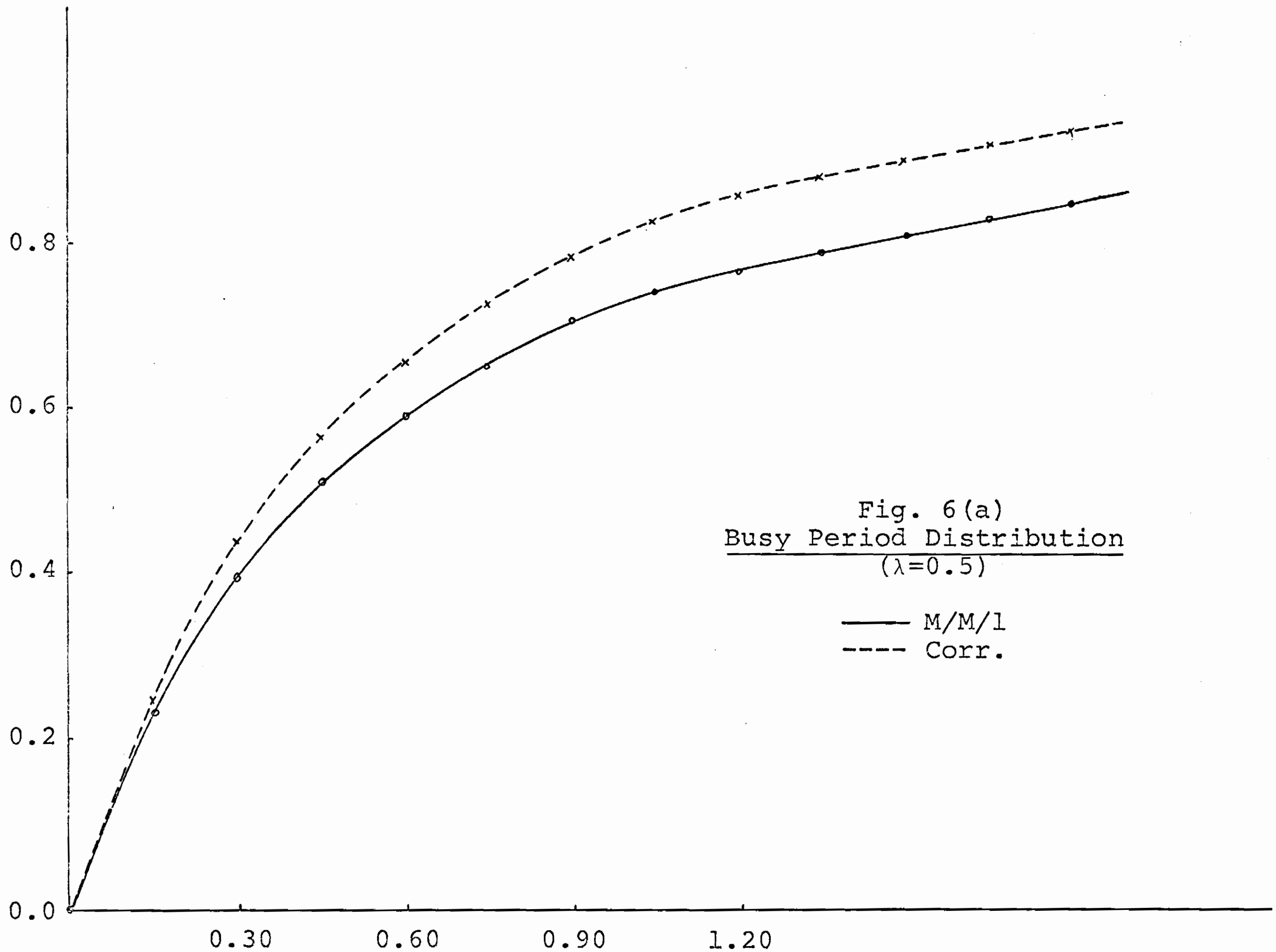


Fig. 6(a)
Busy Period Distribution
(λ=0.5)

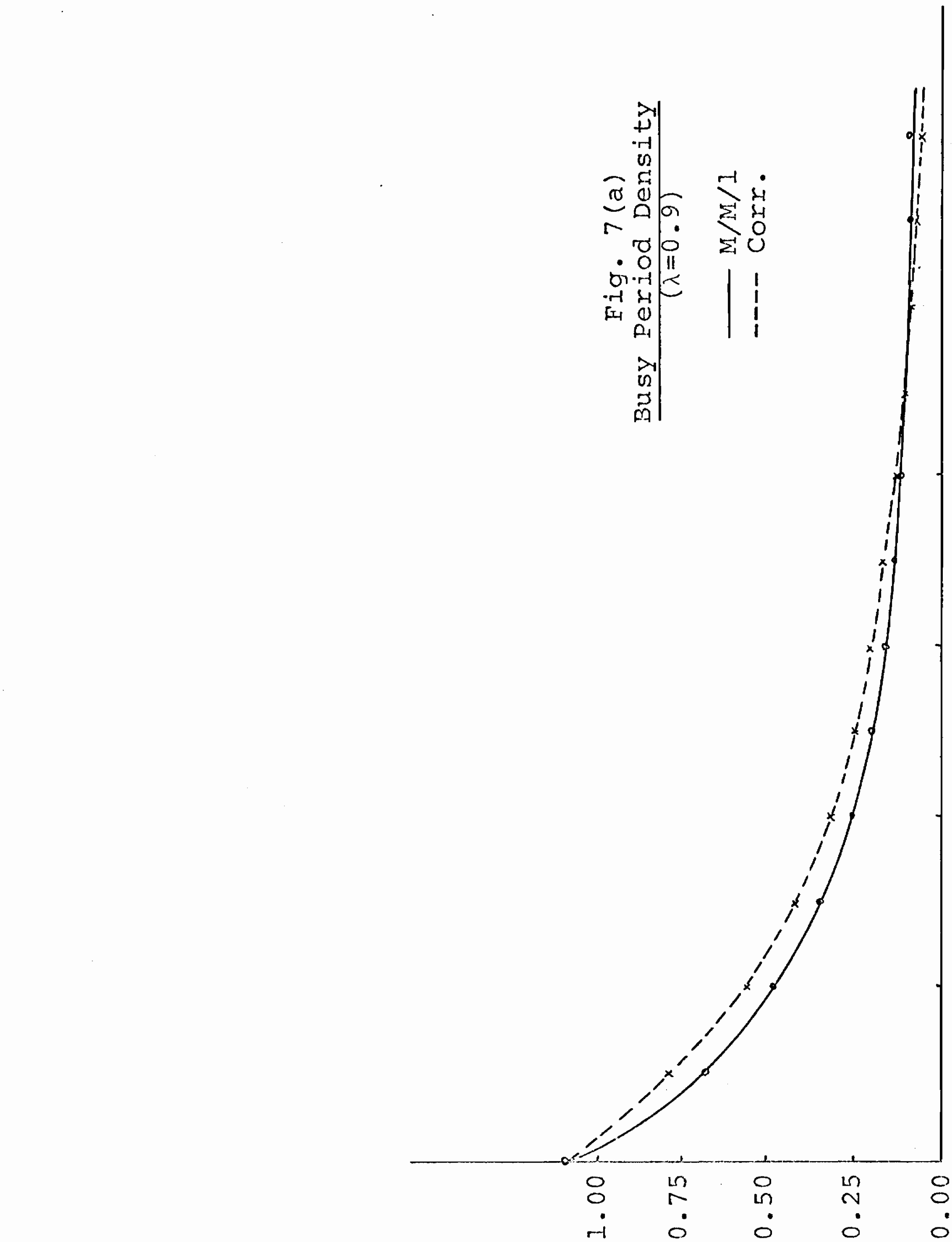


Fig. 7(a)
Busy Period Density
(λ=0.9)

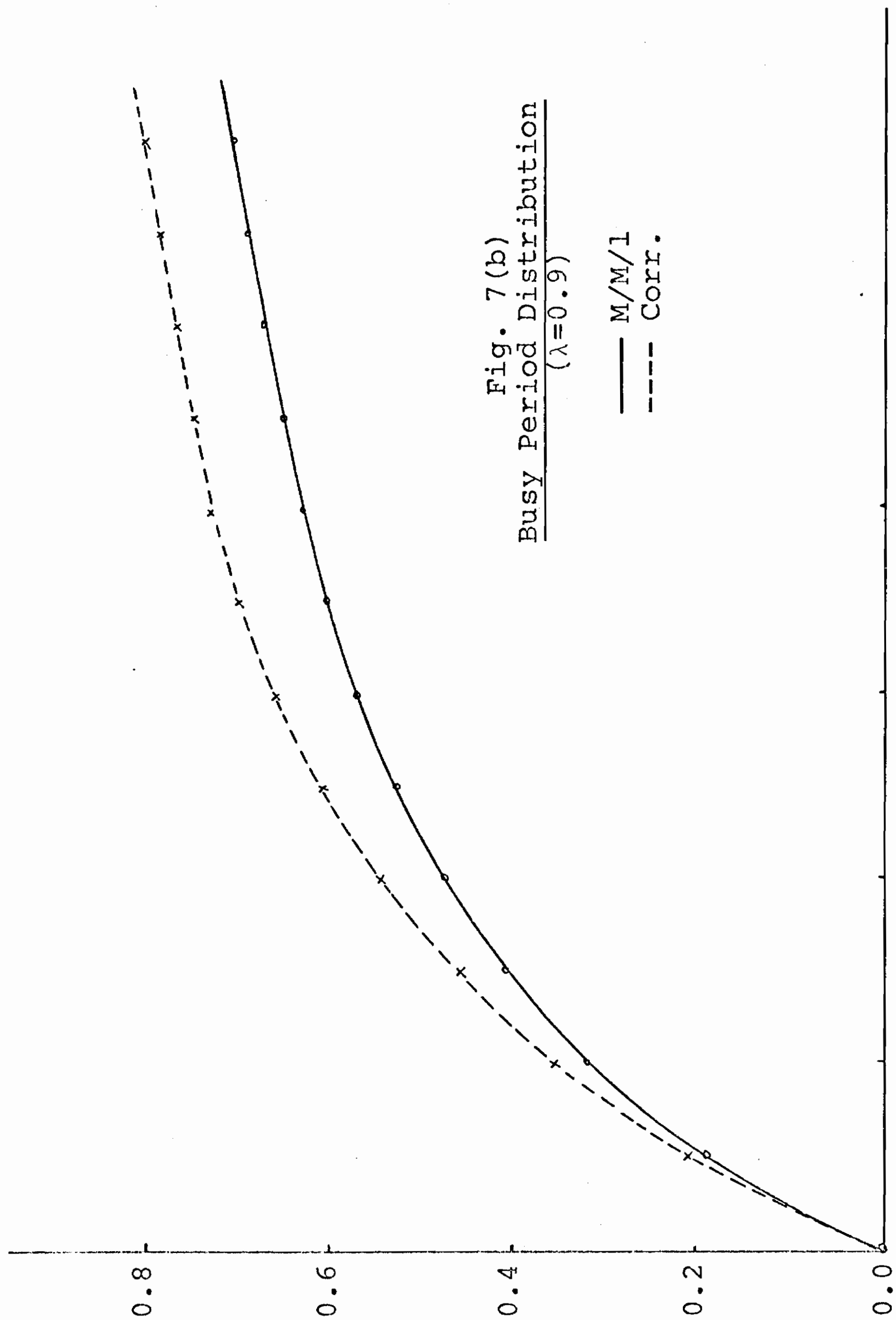


Fig. 7(b)
Busy Period Distribution
(λ=0.9)

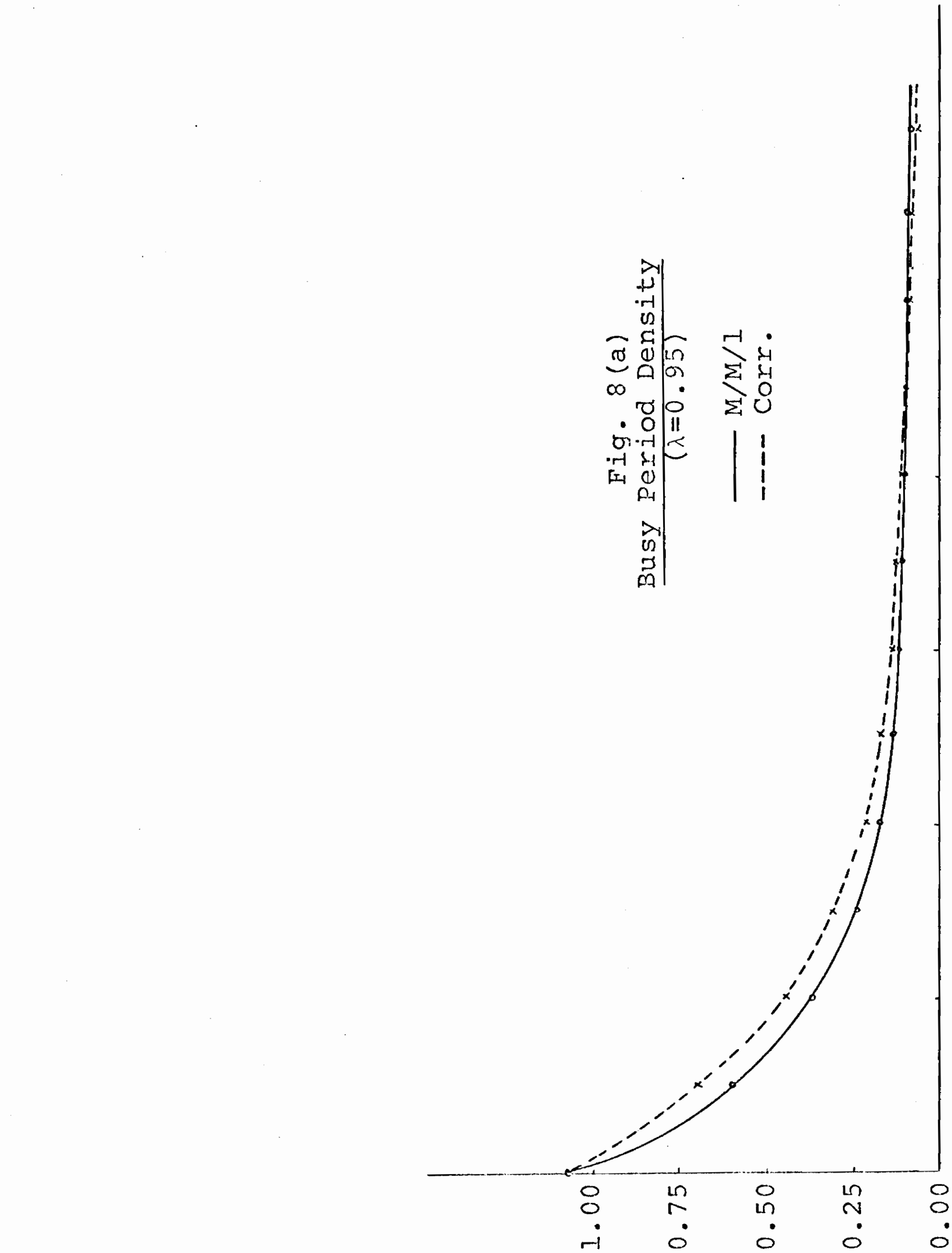


Fig. 8(a)
Busy Period Density
(λ=0.95)

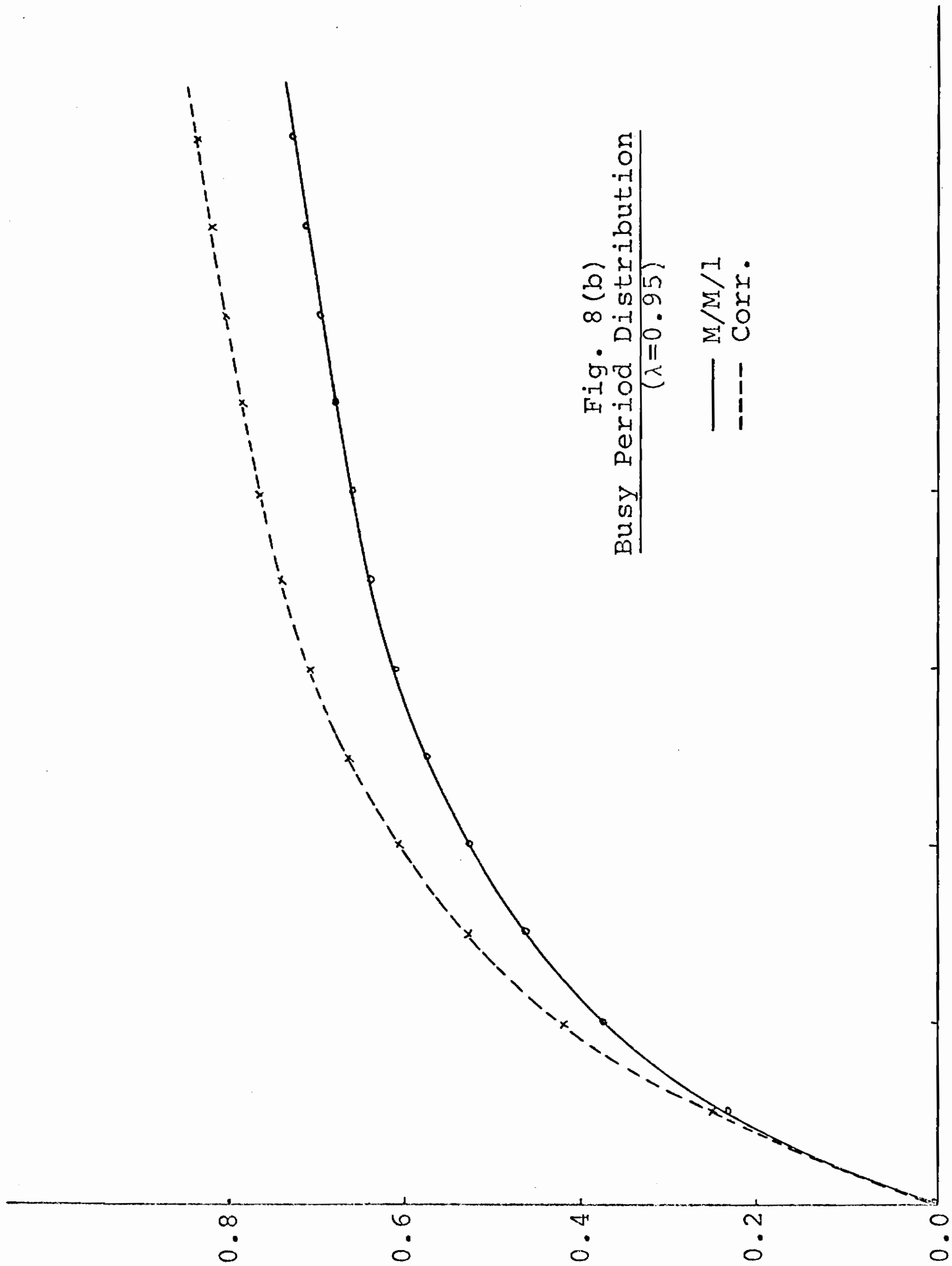


Fig. 8(b)
Busy Period Distribution
(λ=0.95)

An argument similar to that employed for the waiting time will easily show that $E_c(t) < E(t)$, and $E(t)-E_c(t)$ is a monotonic increasing function of λ . Further

$$\frac{s_{i-1}}{s_i}\left(\frac{1}{s_i} + \frac{1}{s_{i-1}} + \ldots\right) < \lambda^{i+1}(1+\lambda+\lambda^2+\ldots) ,$$

which implies that

$$\mathrm{Var}_c(t) < \sum_{i\geq 1} \frac{s_{i-1}}{s_i} \sum_{j\geq i} \frac{1}{s_j} < \sum_{i\geq 1} \lambda^{i+1}(1+\lambda+\lambda^2+\ldots) =$$

$$= \lambda^2(1+\lambda+\lambda^2+\ldots)^2 < \mathrm{Var}(t) ,$$

and again the difference is a monotonically increasing function of λ , but the point is brought up more clearly by the following table.

Table XI
Mean and Variance of Busy Period and Their Ratios

λ	$E(t)$	$E_c(t)$	$\frac{E_c(t)}{E(t)}$	$\mathrm{Var}(t)$	$\mathrm{Var}_c(t)$	$\frac{\mathrm{Var}_c(t)}{\mathrm{Var}(t)}$
0.2	0.25000	0.20693	0.82772	0.09375	0.00145	0.01547
0.5	1.00000	0.60668	0.60668	3.00000	0.07413	0.02471
0.7	2.33333	1.03836	0.44501	30.85185	0.49984	0.01620
0.9	9.00000	2.00951	0.22328	1539.00000	5.57891	0.00363
0.95	19.00000	2.65291	0.13963	14079.00000	16.64661	0.00118
0.97	32.33333	3.13857	0.09707	68650.85180	33.69309	0.00049
0.99	99.00000	4.20498	0.04247	1950399.00000	129.68977	0.00007

We now conclude that:

(xi) The regularity induced by correlation reduces the mean and variance of busy period as well. The closer the value of traffic intensity to unity, the more significant is this reduction.

9.4 Comparison of the Output Processes

The interval u between successive departures from the system in the steady state is purely negative exponential in M/M/1 , and is composed of negative exponential components in the correlated model. The steady state M/M/1 density is

$$h(u) = e^{-u}$$

with unit mean arrival interval (see, for example, P.J.Burke [3]). The corresponding density for the correlated system was shown in Chapter 5 to be

$$h_c(u) = \frac{1}{\lambda}\left[\sum_{r\geq 0} \frac{g_r s_{r+1}}{s_r(s_1 s_r+1)} e^{-\frac{s_{r+1}}{\lambda}u} + e^{-\frac{u}{\lambda s_1}} \sum_{r\geq 0} \frac{g_r}{(s_1 s_r+1)}\right] .$$

Figures 9-12 show the steady state output densities and distributions for a number of values of λ . It will be noticed that the probability of a very large output interval is greater in the correlated queue than in M/M/1 ; this is logical, for if an interarrival interval is very large, with probability one, the corresponding service interval is also going to be very large in the correlated queue, while in M/M/1 , regardless of the interarrival interval length, the probability of a very large service is small. In spite of this fact, we see that:

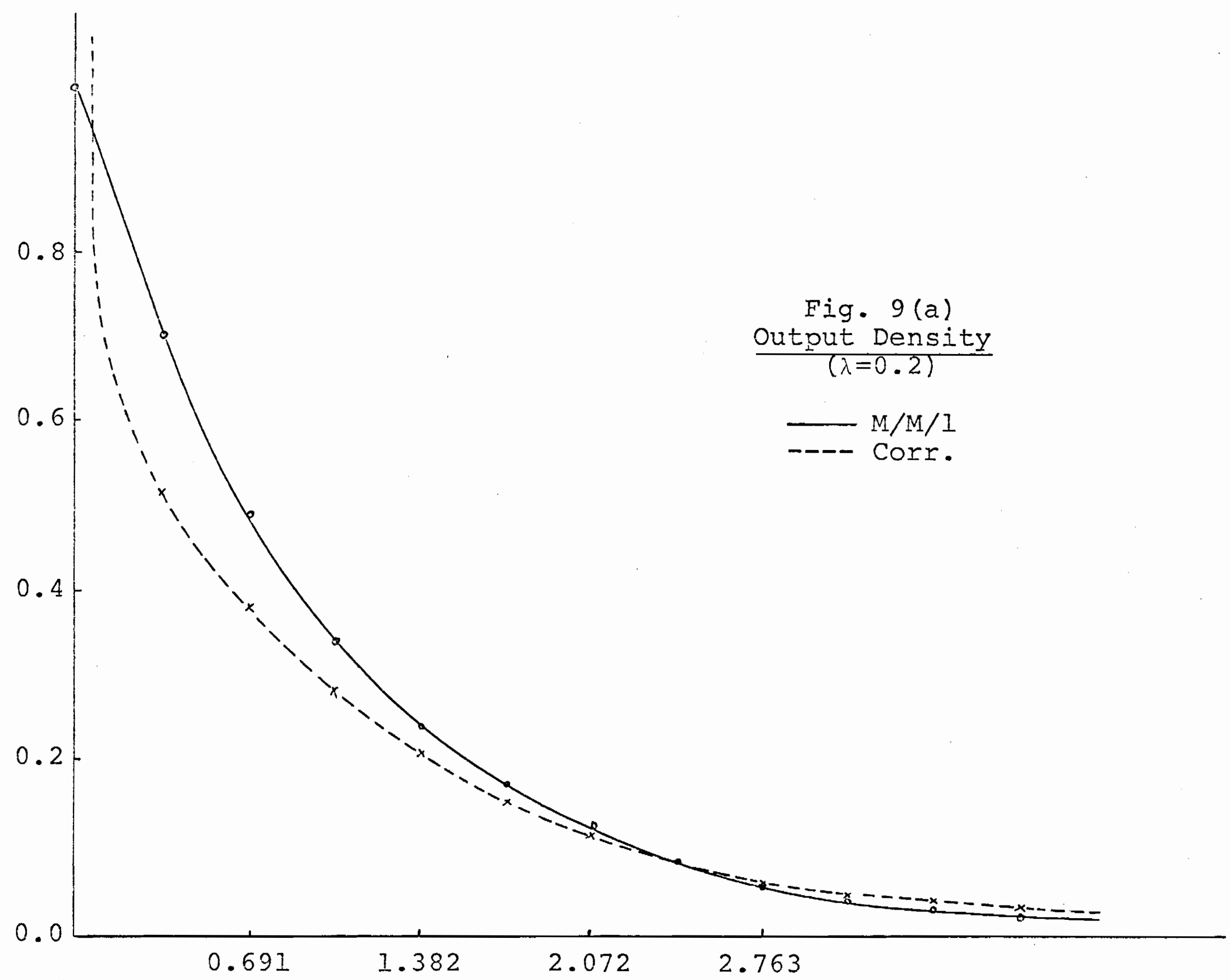


Fig. 9(a)
Output Density
(λ=0.2)

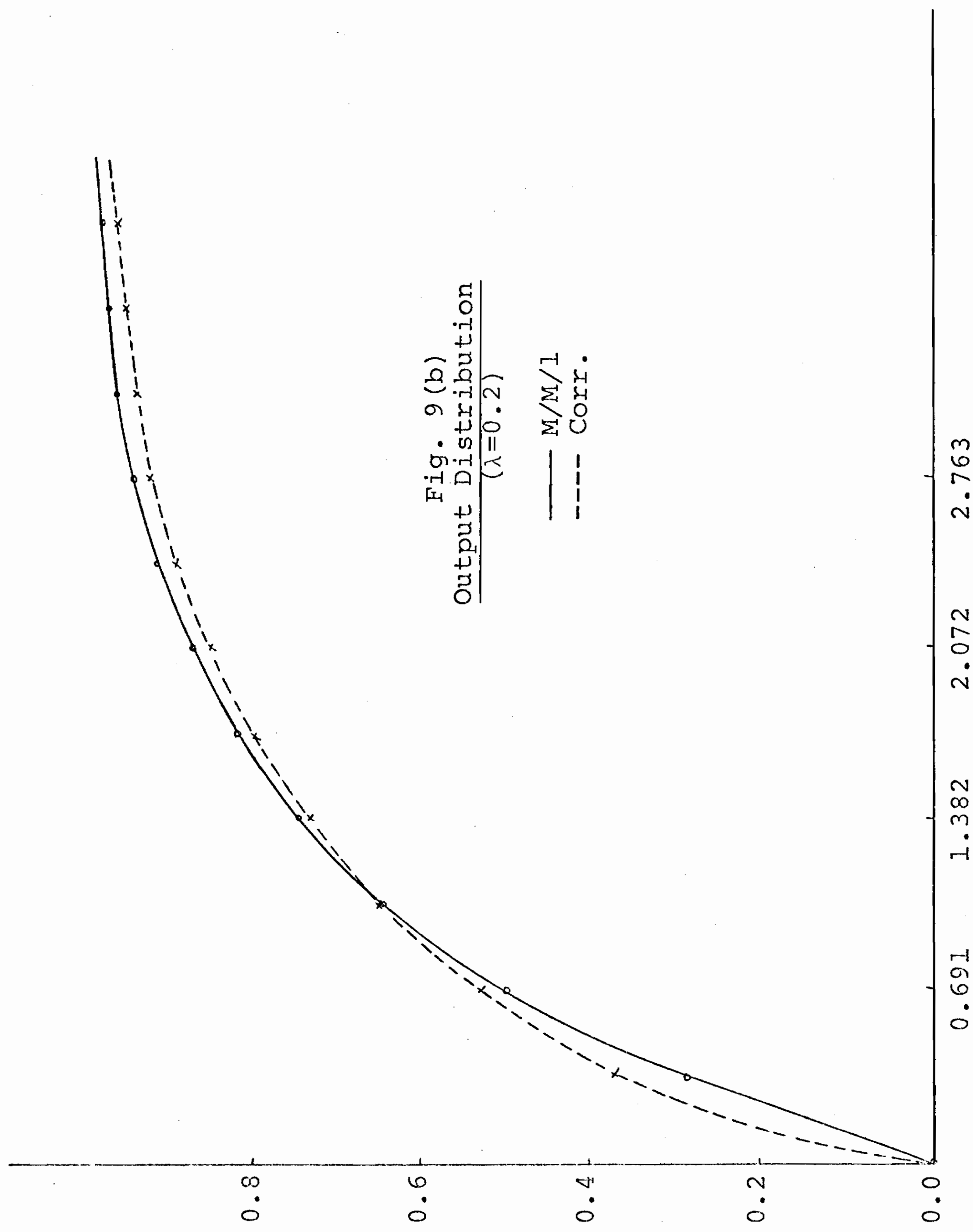


Fig. 9(b)
Output Distribution
(λ=0.2)

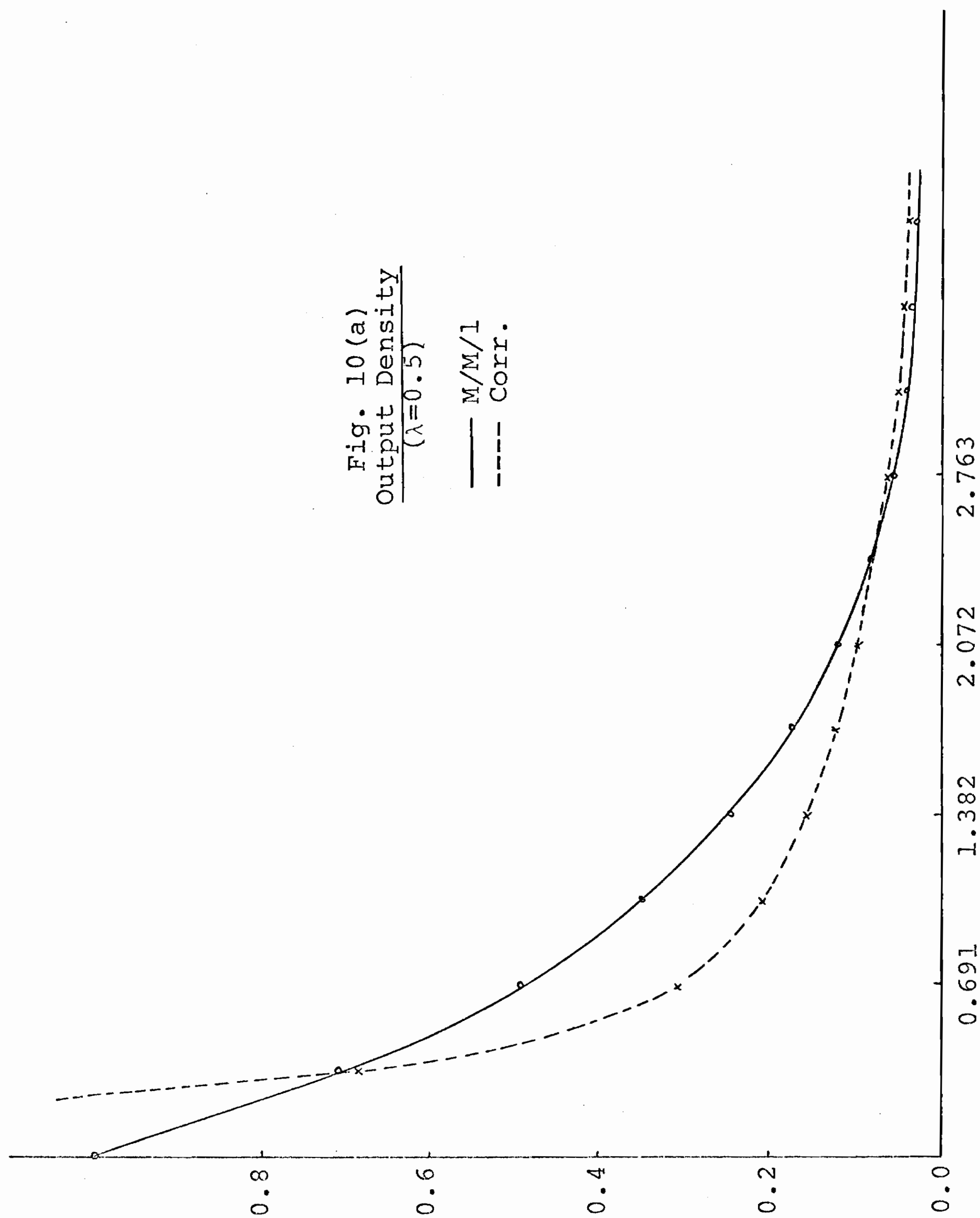


Fig. 10(a)
Output Density
(λ=0.5)

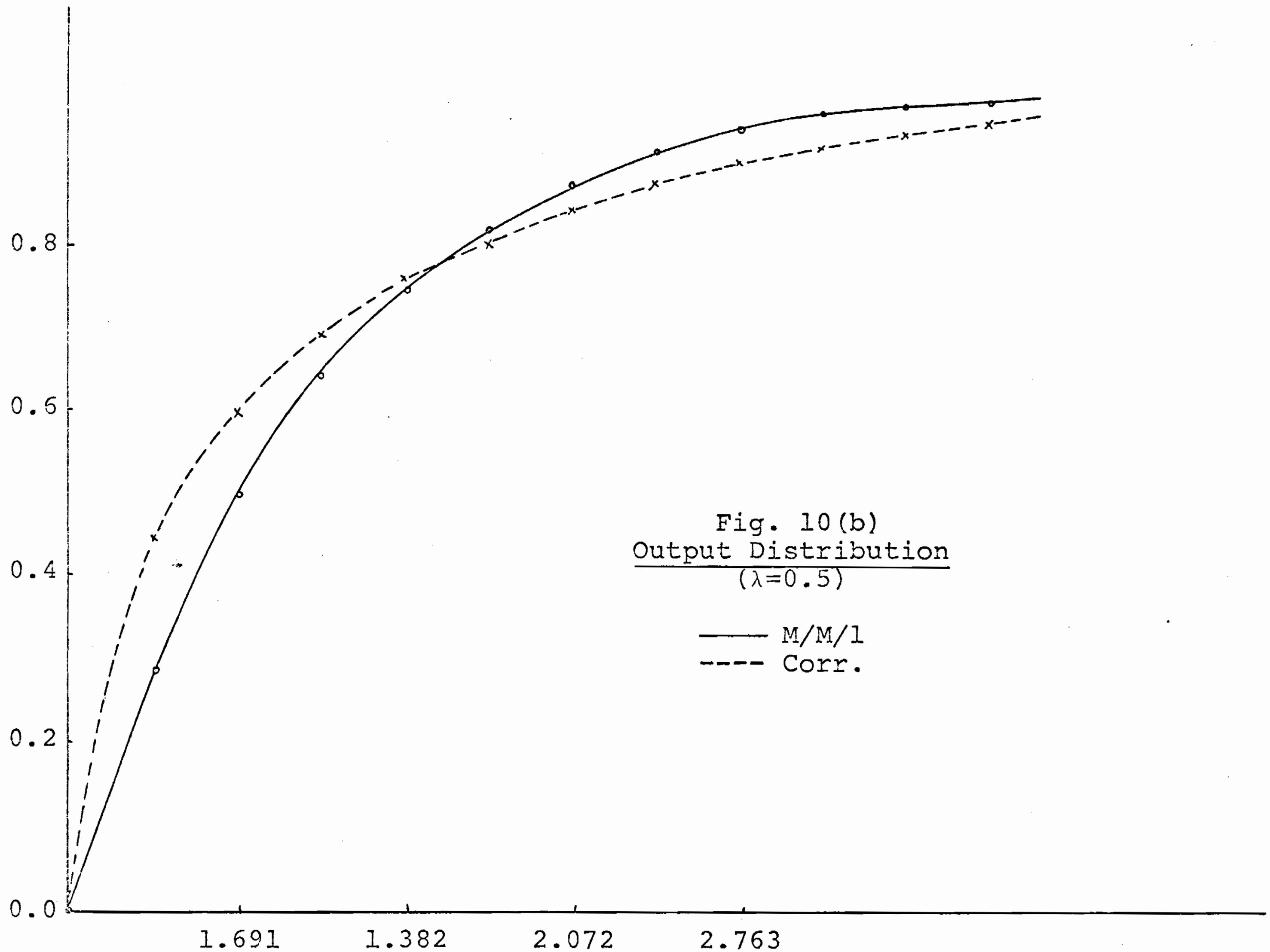


Fig. 10(b)
Output Distribution
(λ=0.5)

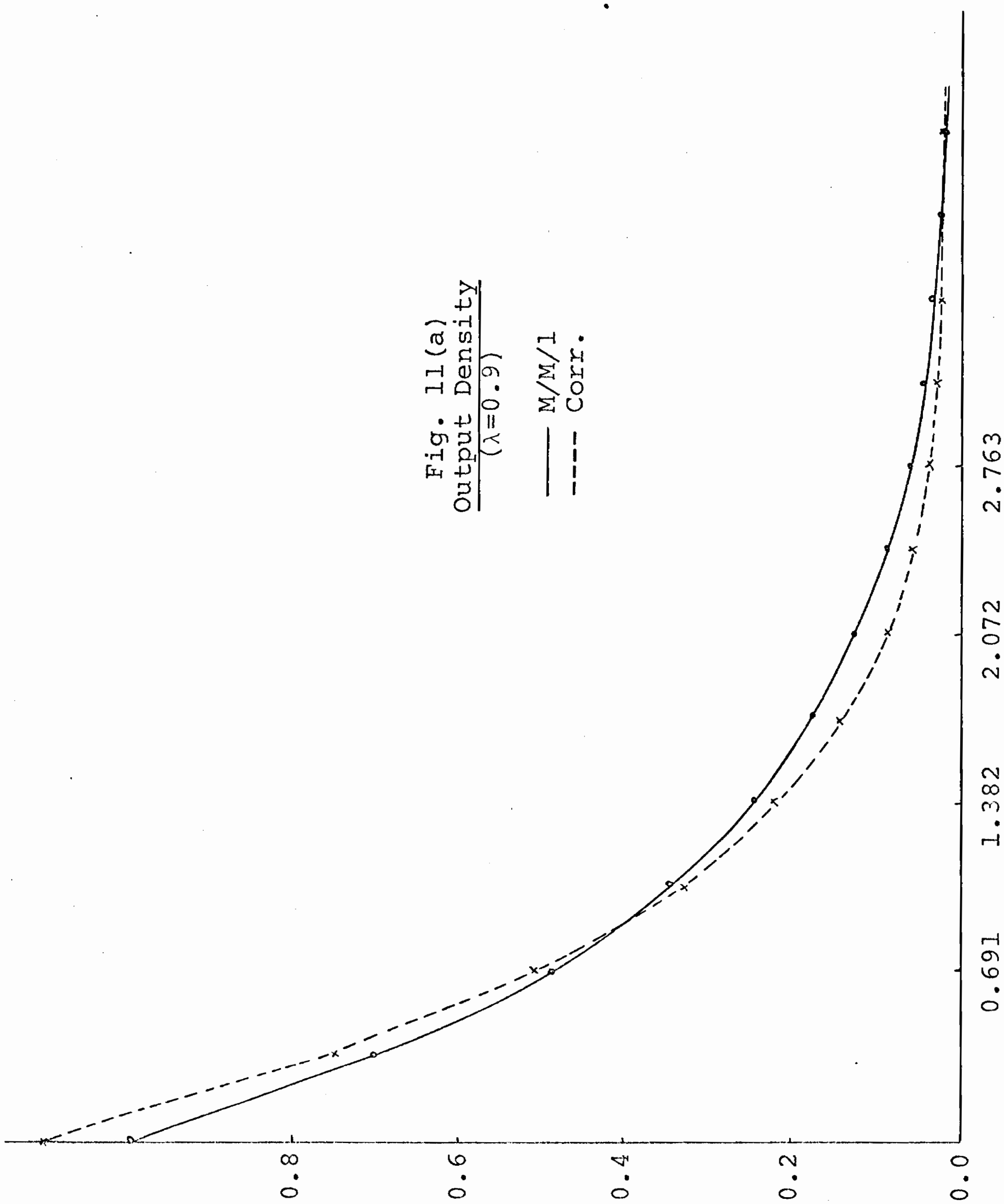


Fig. 11(a)
Output Density
(λ=0.9)

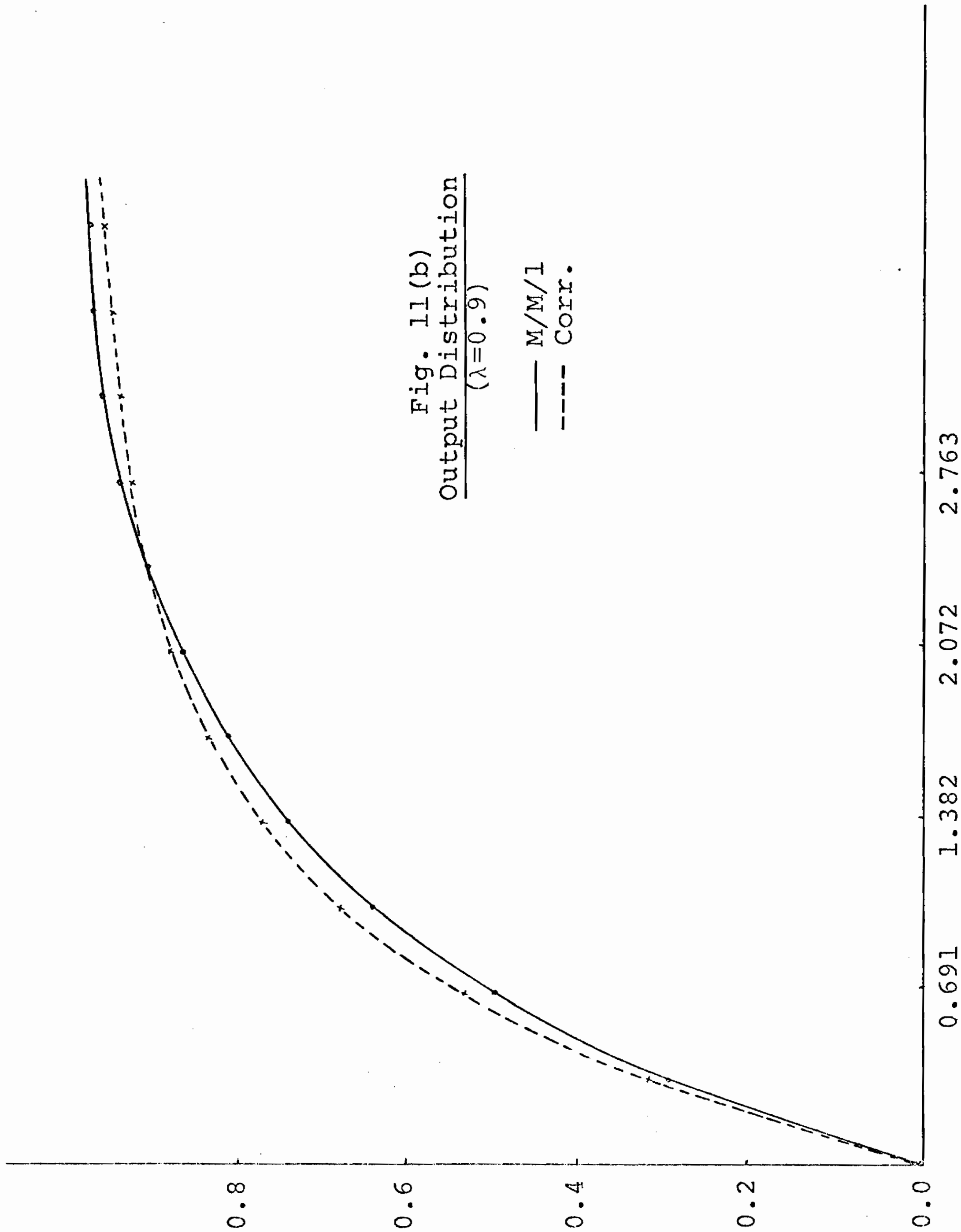


Fig. 11(b)
Output Distribution
(λ=0.9)

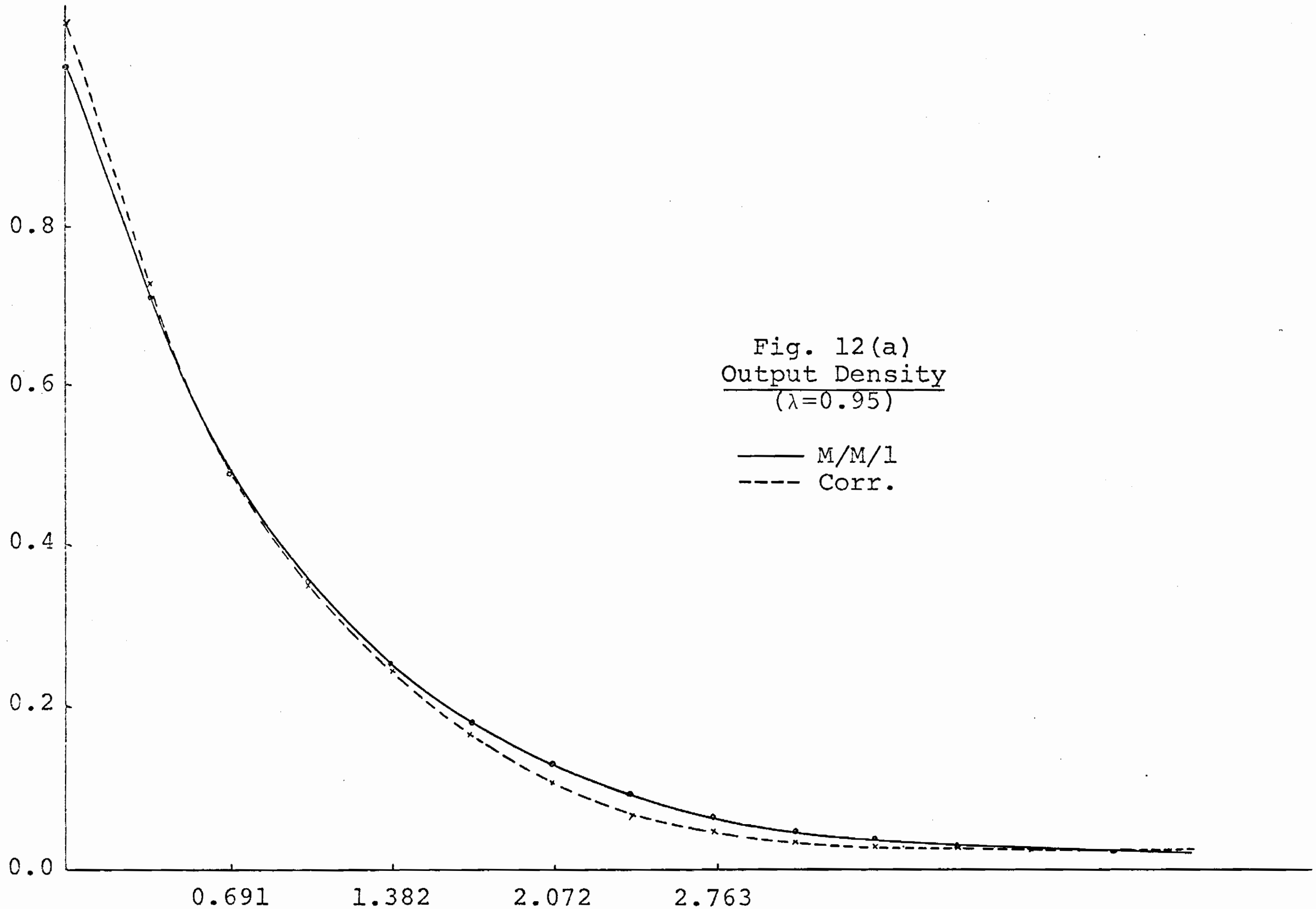


Fig. 12(a)
Output Density
(λ=0.95)

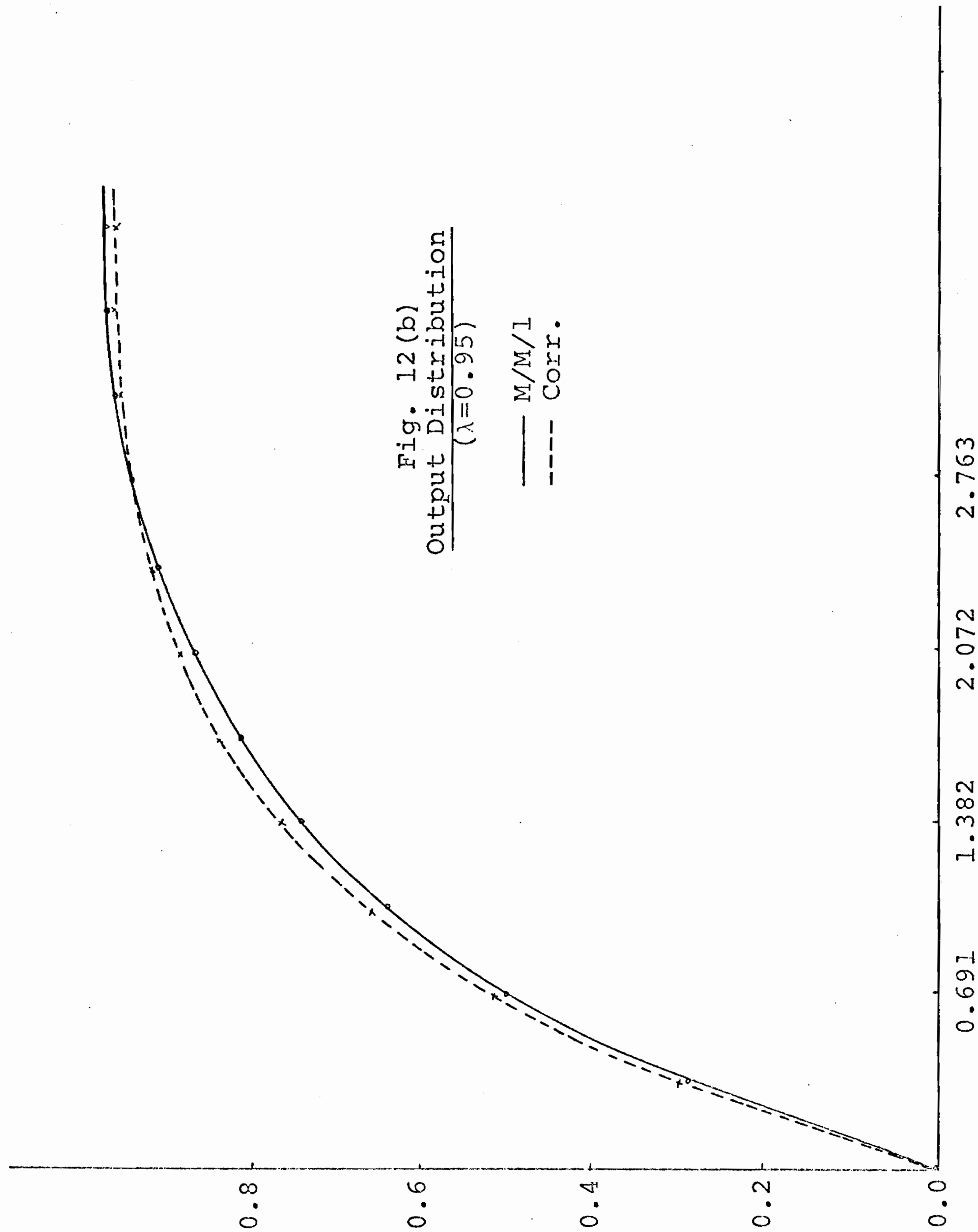


Fig. 12(b)
Output Distribution
(λ=0.95)

(xii) The output densities and distributions become almost the same in both cases and the <u>similarity becomes stronger</u> as traffic intensity (λ) approaches one.

The steady state mean output interval, as it was argued in Chapter 5, is the same in both cases and equal to the mean arrival interval-unity. The variances, however, vary slightly; Table XII shows the values, from which we conclude that:

(xiii) The variance of the output interval is slightly larger in the correlated system.

Table XII
Variance of the Steady State Output Interval

λ	M/M/1	Corr.
0.2	1.00000	1.38622
0.5	1.00000	1.80334
0.7	1.00000	1.85611
0.9	1.00000	1.54173
0.95	1.00000	1.34703
0.97	1.00000	1.24086
0.99	1.00000	1.10306

This, then, concludes the comparison of the correlated system with its most elementary uncorrelated counterpart, M/M/1. It might, however, be argued that M/M/1 is essentially a chaotic system which can be improved upon operationally by the device of regularizing arrivals or service. The question then arises as to how the correlated system compares with such systems as D/M/1 and M/D/1. This is the subject of the next section.

9.5 Comparison of Waiting Time of the Correlated Model with D/M/1 and M/D/1.

Setting up appointments for the arrival of customers, or fixed intervals for the services, essentially has the effect of reducing waiting times as compared with M/M/1. For this reason, we shall here consider it sufficient to compare only the mean and variance of waiting time.

In D/M/1, that is, if customers are given appointments for their arrivals, it can be shown that in the steady state, when arrivals have unit mean and traffic intensity is λ , the mean and variance of waiting time, excluding service, is given by

$$E^D(w) = \frac{\lambda^2}{1-\xi_0} \qquad \text{and}$$

$$Var^D(w) = \frac{\lambda^3(2-\lambda)}{(1-\xi_0)^2} ,$$

where ξ_0 is the smallest (less than unity) root of

$$x = e^{-(1-x)/\lambda} .$$

(See, for example, D. V. Lindley [28].) Superscript D indicates D/M/1. Using the values of ξ_0 given by B. W. Conolly [9] and adding λ and λ^2 respectively to the mean and variance (to include service), along with the corresponding results for the correlated system in Chapter 2, we set up Table XIII from which we conclude that:

(xiv) The mean and variance of customers' waiting time with the correlation mechanism are, on the whole, very much smaller even than their corresponding D/M/1 counterparts.

Table XIII
Mean and Variance of Waiting Time (in the Steady State) in the Correlated Queue and D/M/1 with Their Ratios

λ	$E^D(w)$	$E_c(w)$	$\frac{E_c(w)}{E^D(w)}$	$Var^D(w)$	$Var_c(w)$	$\frac{Var_c(w)}{Var^D(w)}$
0.2	0.24028	0.24139	1.00462	0.05460	0.04115	0.75366
0.5	0.81375	0.80334	0.98721	0.54533	0.28433	0.52139
0.7	1.61932	1.42685	0.88114	2.05958	0.61324	0.29775
0.9	5.09472	2.70856	0.53164	22.31580	1.16391	0.05216
0.95	9.98403	3.47027	0.34758	91.10719	1.36524	0.01498
0.97	16.36934	4.01441	0.24524	252.74918	1.46053	0.00578
0.99	37.97491	5.15293	0.13569	1396.49735	1.57198	0.00112

One would imagine that, as far as waiting time is concerned, the best one can do is to set up an appointment system for the arrival of customers, but as Table XIII indicates, in the correlated system, if for example $\lambda = 0.95$, the mean waiting time is as little as 35 per cent of the corresponding appointment system value, with a variance of only 1 per cent. It should not be overlooked, however, that at low values of traffic intensity the gain is very small, and in fact at $\lambda = 0.2$ the correlated mean is only slightly higher.

We shall finally compare the waiting time results with M/D/1; namely, the system in which services are completed at regular time intervals. The mean and variance of waiting time excluding service can easily be derived from M/G/1 results given, for example, by W. L. Smith [34] to be

$$E_D(w) = \frac{\lambda^2}{2(1-\lambda)} \quad \text{and}$$

$$\mathrm{Var}_D(w) = \frac{\lambda^3(4-\lambda)}{12(1-\lambda)^2} \quad ,$$

when arrivals have unit mean and traffic intensity is λ. Subscript D indicates M/D/1. After adding λ and λ^2 to the mean and variance respectively to allow for the inclusion of service time, we compile the following table.

Table XIV
Mean and Variance of Waiting Time (in the Steady State) in the Correlated Queue and M/D/1 with Their Ratios

λ	$E_D(w)$	$E_C(w)$	$\frac{E_C(w)}{E_D(w)}$	$\mathrm{Var}_D(w)$	$\mathrm{Var}_C(w)$	$\frac{\mathrm{Var}_C(w)}{\mathrm{Var}_D(w)}$
0.2	0.22500	0.24139	1.07284	0.04396	0.04115	0.93608
0.5	0.75000	0.80334	1.07112	0.39583	0.28433	0.71831
0.7	1.51667	1.42685	0.94078	1.23250	0.61324	0.49756
0.9	4.95000	2.70856	0.54718	19.64250	1.16391	0.05925
0.95	9.97500	3.47027	0.34790	88.06897	1.36524	0.01550
0.97	16.65167	4.01441	0.24108	256.99636	1.46053	0.00568
0.99	49.99500	5.15293	0.10307	2434.81343	1.57198	0.00064

We here notice that at low traffic, at least as far as the mean waiting time is concerned, M/D/1 is slightly superior, but with a larger variance. However,

(xv) in moderate and heavy traffic, the mean and variance of customer's waiting time under the correlation mechanism is very much smaller even than their M/D/1 counterpart.

9.6 Comments

In spite of all the advantages associated with the correlation mechanism, it is not suggested that it is necessarily a universally applicable technique. It could be objected, for instance, that in the case of a human server a lower limit must be set on service time by virtue of the fact that a human being is physically limited in the number of functions he can perform in a given time. Further, to extend service longer than perhaps strictly necessary is to ignore the nature of the task, and therefore, could be interpreted as nothing more than a disguised form of idleness. Nevertheless, it should be pointed out that when the server is not a human being the amount of service is not necessarily represented by the service time, since the same amount of work could be obtained in only a portion of time it might usually take by operating a machine, for instance, at a higher speed. On the other hand, further models, such as ones which have lower and upper bounds on service time but incorporating correlation, could be analyzed, as was suggested in Chapter 8.

It is therefore suggested that the research reported in this dissertation is indicative of the type of improvement on conventional systems which could be obtained where it is practical to implement the kind of mechanism which has been examined. It is further suggested that much remains to be done in connection with the investigation of more realistic specific models which take into account the physical limitations of the systems involved, and the payoffs to management and clients of the economies effected. It seems clear, nevertheless, that the results and findings which have been achieved will form a solid basis for such further investigation.

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VITA

Nasser Hadidi, the second of the five sons of Mohammad Baqher and Neshat Hadidi, was born on October 21, 1942 at Shiraz in Iran, of which country he is proudly a citizen. Upon graduation from Shahpour High School in 1961, he enrolled as a freshman in the American University of Beirut, where he received intensive training in Mathematics, Statistics, as well as in the Theory and Practice of Education. He was awarded the degree of Bachelor of Science in Statistics and the Normal Diploma in Education in June 1965. In September of the same year he joined the Statistics Department of Virginia Polytechnic Institute, where he commenced working towards his Ph.D. degree. Upon completion he will return to Iran; he has accepted the position of assistant professor of mathematics in the Department of Mathematics of Pahlavi University at Shiraz.

Nasser Hadidi

Nasser Hadidi

CORRELATION BETWEEN ARRIVAL AND SERVICE PATTERNS AS A MEANS OF QUEUE REGULATION

by

Nasser Hadidi

ABSTRACT

A major cause of congestion in queueing situations, that is of immoderate waits and lengthening queues, is often the assumed independence of the arrival and service mechanisms. This dissertation is concerned with single server "correlated" models, defined to be such that either the service mechanism is somehow tailored to the arrival pattern, or vice versa. The greatest attention is given to a particular model in which the service time allotted to the nth arrival is λT_n, where λ is a non-time dependent constant and numerically has the value of congestion index, and T_n is the interval between the (n-1)th and the nth arrivals which, it is important to note, could be observed by the server before service is initiated. It is shown that the effect of the correlation mechanism is to reduce congestion under a given level of traffic intensity, as compared with single server systems in which arrivals and service are independent. This result is achieved without inflicting on the service facility the penalty of increased periods of

idleness. The particular model is a queueing interpretation of a stochastic-kinematic situation studied by B. W. Conolly in connection with a military tactical analysis.

The dissertation is divided into two parts. Part I develops the theory of the main model with particular reference to state probabilities, waiting time, busy period, and output. Some consideration is also given to a related model where service depends on the arrival pattern, and to what is referred to as the "dual" problem in which the arrival mechanism is geared to service capability. Further, the state probabilities at arrival epochs for a conventional M/M/1 queue are obtained by employing a simple probabilistic argument. This is needed for Part II.

Part II applies the theory to give a practical comparison of the correlation mechanism with the elementary "independent" single server queues M/M/1, M/D/1 and D/M/1; and it is shown in detail that the practical result referred to above is achieved. The superiority of the correlation mechanism increases with traffic intensity. State probability, busy period and output comparisons are made only with the M/M/1 system. The main conclusions are found to extend also to these processes.

It is concluded that, where its application is practicable, a mechanism of correlation can achieve important gains in efficiency.