ON VIBRATION AND STABILITY PROBLEMS OF LAMINATED PLATES AND SHELLS USING SHEAR DEFORMATION THEORIES

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(ABSTRACT)

This study deals with the vibration and stability analyses of laminated plates and shells, using classical, first–order and third–order equivalent single–layer theories and the layer–wise theory of Reddy. Analytical solutions of these theories for natural frequencies and critical buckling loads of plates and shells under various boundary conditions are developed using an improved analytical procedure. A solution for the transient response of viscously damped cross–ply laminated plates, subjected to a sonic–boom type loading, is developed using the third–order shear deformation plate theory of Reddy and the first–order shear deformation plate theory.

The nonlinear dynamic equations of the first–order shear deformation plate theory and the third–order shear deformation plate theory of Reddy are reformulated in terms of a pair of equations describing the interior and the edge–zone problems of rectangular plates laminated of transversely isotropic layers. The pure–shear frequencies of the plate in linear and nonlinear problems are identified from the edge–zone equation. For certain boundary conditions the original system of equations are reduced to three in number, as in the classical plate theory.

The frequency and buckling equations of symmetric plates laminated of transversely isotropic layers are obtained using the Levinson's third–order shear deformation plate theory. Using the interior and the edge–zone equations, the

frequency and buckling equations are also obtained according to the first–order shear deformation plate theory. The solution contribution of the edge–zone equation is analyzed. By introducing a mixed approach, the bending problem of laminated plates with various boundary conditions is studied according to the first–order and Reddy's third–order shear deformation plate theories.

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CHAPTER I

INTRODUCTION

1.1 Motivation

The expanded use of fibrous composite materials in aircraft, automotive, shipbuilding, and other industries has stimulated interest in the accurate prediction of response characteristics of laminated plates and shells.

While composite materials offer many desirable structural properties over conventional materials, they also present challenging technical problems in the understanding of their structural behavior. Most of the advanced composites in use to date have a low ratio of the transverse shear modulus to the in–plane modulus, and, therefore, the transverse shear deformation plays a much more important role in reducing the effective flexural stiffness of laminated plates and shells made of these composites than in the corresponding metallic ones.

One of the important steps to develop accurate analysis of composite structures is to select a proper structural theory for the problem. It has long been recognized that classical two–dimensional (single–layer) plate and shell theories yield accurate results only when these structures are thin, the dynamic excitations are within the low–frequency range, and material anisotropy is not severe.

Laminated plates and shells are modelled according to one of the following three classes of theories:

1. Equivalent single–layer, 2–D shear deformation theories
2. Layer–wise, 2–D shear deformation theories
3. Continuum–based, 3–D and 2–D theories

In the past couple of years, increased interest is seen in the use of single–layer and layer–wise shear deformation theories for the study of bending, vibration and stability problems of laminated plates and shells.

Although the two–dimensional single–layer and layer–wise shear deformation theories are adequate for predicting the gross response characteristics of laminated plates and shells, only in a relatively few cases analytical results are obtained. In most cases, the studies were limited to special boundary conditions and the classical theory. Often, approximate procedures such as the Ritz, Galerkin, finite difference, and finite element methods are used to develop numerical results. These procedures, if properly used, will approach the exact solutions as closely as desired when sufficient terms or degrees of freedom are retained in the solution, although the roots of very large determinants may be required. The standard procedure for determining theoretical values of natural frequency and critical buckling load is to solve a mathematical eigenvalue problem, that is, a problem governed by differential equations and boundary conditions, all of which are homogeneous. The difficulties in finding exact closed–form solutions in vibration and stability problems of laminated plates and shells are: i) bending–stretching coupling due to material anisotropy, ii) the order and number of governing equations involved in the transverse shear deformation theories, and iii) in–plane and rotatory intertia terms present in the refined theories.

In the next section, the existing single–layer and layer–wise shear deformation plate and shell theories are reviewed, which will provide a background for the present study.

1.2. Various Theories–A Review

It is well known by now that the classical theory of plates introduced by Kirchhoff and extended to composite laminates in [1–8], is inadequate in modelling, especially dynamic aspects of, laminated composite plates. The Kirchhoff assumptions amount to treating plates to be infinitely rigid in the transverse direction by neglecting transverse strains. This theory underestimates deflections and overestimates natural frequencies and buckling loads. In addition, the classical plate theory is plagued with the inconsistency between the order of the governing equation and the number of boundary conditions (see Stoker [9]). Since the transverse shear moduli of modern composite materials are usually very low compared to the in–plane moduli, the transverse shearing strains must be taken into account if an accurate representation of the behaviour of the laminated plate is to be achieved.

Numerous plate theories which include transverse shear deformations are documented in the literature. The shear deformation plate theories known in the literature can be grouped into two classes; (1) stress–based theories, and (2) displacement–based theories. The first stress–based shear deformation plate theory is due to Reissner [10–12]. The theory is based on a linear distribution of the in–plane normal and shear stresses through the thickness of the plate as

$$\sigma_x = \frac{M_x}{(h^2/6)}\frac{2}{h}z,\ \sigma_y = \frac{M_y}{(h^2/6)}\frac{2}{h}z,\ \sigma_{xy} = \frac{M_{xy}}{(h^2/6)}\frac{2}{h}z. \tag{1–1}$$

Here (σ_x,σ_y) and σ_{xy} are the normal and shear stresses, (M_x,M_y) and M_{xy} are the associated bending moments (which are functions of the in–plane coordinates x and y), $|z| \le h/2$, and h is the total thickness of the plate. The transverse stresses σ_{xz},

σ_{yz}, and σ_{zz} are then obtained by integrating the equilibrium equations of the 3–D elasticity theory:

$$\sigma_{xz} = \frac{Q_x}{(2h/3)}[1 - (\frac{2}{h}z)^2],\ \sigma_{yz} = \frac{Q_y}{(2h/3)}[1 - (\frac{2}{h}z)^2]$$

$$\sigma_{zz} = -\frac{3}{4}(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y})[\frac{2z}{h} - \frac{1}{3}(\frac{2}{3}z)^3], \tag{1–2}$$

where the shear stress resultants Q_x and Q_y are related to the bending moments M_x, M_y, and M_{xy} by the relations

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y},\ Q_y = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y}. \tag{1–3}$$

The governing equations and the associated boundary conditions of the theory are then obtained by using the principle of virtual forces. Gol'denveizer [13,14] generalized Reissner's theory by replacing the linear variation of stresses (1–1) through thickness by a general function of the thickness coordinate. Recently, Voyiadjis and Baluch [15,16] presented a stress–based theory that is a modification and extension of Reissner's theory to dynamics of isotropic plates and bending of composite plates. Kromm [17,18] developed a shear deformation theory that is a special case of Gol'denveizer's extension of Reissner's theory. The displacement field in Kromm's theory is of the form;

$$u_1 = u - z\frac{\partial w}{\partial x} + \frac{3}{2}z(1 - \frac{4}{3h^2}z^2)\varphi_x$$

$$u_2 = v - z\,\frac{\partial w}{\partial y} + \frac{3}{2}\,z(1 - \frac{4}{3h^2}\,z^2)\varphi_y$$

$$u_3 = w \tag{1–4}$$

where u, v, w, φ_x, and φ_y are displacement functions which are functions of in–plane coordinates x and y. The displacement field (1–4) is determined so that the stress–free boundary conditions on the bounding planes of the plate are satisfied. A mixed approach is also presented by Reissner [19,20] where both assumed–displacement and assumed–stress expansions are used. The governing equations are obtained by means of the mixed or Reissner's variational principle.

The origin of displacement–based theories is attributed to Basset [21], who proposed that the displacement components be expanded in series of powers of the thickness coordinate z. For example, the displacement component u_1 along the x–direction in the N–th order theory is written in the form

$$u_1(x,y,z) = u(x,y) + \sum_{n=1}^{N} z^n \varphi_x^{(n)}(x,y) \tag{1–5}$$

where the functions $\varphi_x^{(n)}$ have the meaning

$$\varphi_x^{(n)}(x,y) = \frac{1}{n!}\left.\frac{d^n u_1}{dz^n}\right|_{z=0},\ n = 0,1,2,... \tag{1–6}$$

Basset's work has not received as much attention as it deserves. Hildebrand, Reissner, and Thomas [22] presented, following the work of Basset [21], a first–order shear deformation theory for shells with the following displacement field [a special case of Eq. (1–5) when N = 1]:

$$u_1(x,y,z) = u(x,y) + z\varphi_x(x,y)$$
$$u_2(x,y,z) = v(x,y) + z\varphi_y(x,y)$$
$$u_3(x,y,z) = w(x,y). \tag{1–7}$$

The governing equations of the theory are then derived by using the principle of minimum total potential energy. This gives five equilibrium equations in the five generalized displacement variables u, v, w, φ_x, and φ_y.

The *first–order shear deformation plate theory* (the phrase was first introduced by Reddy [23]) is based on the displacement field (1–7) and is often known as the Mindlin Plate Theory. Mindlin [25] extended Hencky's theory [24] of isotropic plates to the dynamic case. Historical evidence (from the review of the literature) points out that the basic idea of the displacement–based shear deformation theory came from Basset [21] and Hencky [24]. Mindlin should however be credited with the extension to the dynamic case. The first–order shear deformation theory yields a constant value of transverse shear strain through the thickness of the plate, and thus requires shear correction factors. The shear correction factors are dimensionless quantities introduced to account for the discrepancy between the constant state of shear strains in the first–order theory and the quadratic distribution of shear strains in the elasticity theory. For composite laminates, the shear correction factors, in general, depend on the constituent ply properties, lamination scheme, and type of structure (i.e., geometry and boundary conditions) [26–28].

Stavsky [29] extended the first–order shear deformation theory of isotropic plates to plates laminated of isotropic layers. The work was extended to anisotropic laminated plates by Yang, Norris, and Stavsky [30]. Following these works, higher–order displacement–based theories have also been documented in the literature [31–52]. In these theories, the three components of the displacement vector are expanded in power series of thickness coordinate and in terms of unknown generalized displacement functions:

$$u_1(x,y,z,t) = u^o + z\phi_1^1 + z^2\phi_1^2 + z^3\phi_1^3 + \dots$$

$$u_2(x,,y,z,t) = v^o + z\phi_2^1 + z^2\phi_2^2 + z^3\phi_2^3 + \dots$$

$$u_3(x,y,z,t) = w + z\phi_3^1 + z^2\phi_3^2 + \dots \qquad (1\text{–}8)$$

where u^o, v^o, w, and ϕ_α^i are functions of time and in–plane coordinates x and y. Generally, these higher–order theories are cumbersome and computationally demanding because with each additional power of the thickness coordinate an additional dependent unknown is introduced (per displacement component) into the theory.

In the third–order theories [38–47], the series (1–8) are truncated such that the displacement field is:

$$u_\alpha(x,y,z,t) = u_\alpha^o + z\phi_\alpha^1 + z^2\phi_\alpha^2 + z^3\phi_\alpha^3$$

$$u_3(x,y,z,t) = u_3^o \quad (\alpha = 1,2) \qquad (1\text{–}9)$$

By satisfying the transverse stress–free boundary conditions on the bounding planes of plate, the number of the unknown functions in (1–9) is then reduced to five, and the displacement field becomes [38–47],

$$u_\alpha(x,y,z,t) = u_\alpha^o + f(z)\phi_\alpha - zu_{3,\alpha}^o$$

$$u_3(x,y,,z,t) = u_3^o \qquad (1\text{–}10a)$$

where

$$f(z) = [1 - \frac{4}{3}(\frac{z}{h})^2]z. \tag{1–10b}$$

In the displacement–based theories, two different approaches are generally adopted to derive the governing equations of a plate [48]: one is called the variational approach, which is based on the principle of virtual displacement (see [38,41,44,48]); the other one is referred to as the vectorial approach (or the method of moments), which involves integration of 3–D elasticity equations. In principle, the two approaches can yield the same governing equations if one has the foresight to select the weight functions properly using the vectorial approach. This has been demonstrated in [48] for the displacement field (1–10) and a general class of weight function, f(z). Equations of motion of the first–order shear deformation theory provide an example where the two approaches are known to yield the same result. However, the third–order theories developed (based on (1–10)) by Jemielita [39], Schmidt [40]. Levinson [42], Murthy [43], and Librescu and Khdeir [46,47] are inconsistent in the sense that the equilibrium equations of the first–order shear deformation theory are used as the governing equations of a third–order theory. As a consequence, the higher–order terms of displacement field in (1–10) are accounted for only in the calculation of the strains but not in the governing differential equations or in the boundary conditions.

Recently, Reddy [49] showed that most of the third–order theories [38–47] that satisfy vanishing of transverse shear stresses on the bounding planes of a plate are not new but duplicates of other theories. More explicitly, it is shown in [49] that all third–order theories [38,40–44], which are based on the virtual displacement principle, are identical and the ones based on the equations of motion of first–order shear deformation theory are the same. In addition, the latter ones can be obtained as a special case of the former ones. Reddy [38,50,51] is the first one (see [49]) to develop the equilibrium equations of a third–order shear deformation theory for composite laminates, based on the displacement field similar to (1–10) and using the principle of virtual displacements. The displacement field adopted by Reddy in [38,50,51] is:

$$u_1(x,y,z,t) = u^o + f(z)\varphi_x - \frac{4}{3h^2} z^3 \frac{\partial w}{\partial x}$$

$$u_2(x,y,z,t) = v^o + f(z)\varphi_y - \frac{4}{3h^2} z^3 \frac{\partial w}{\partial y}$$

$$u_3(x,y,z,t) = w \qquad (1\text{–}11)$$

where u^o, v^o, w, φ_x, and φ_y are functions of time and in–plane coordinates x and y.

The theories presented in [38,50] are for small–deflection and large–deflection theories, respectively, while that in [51] accounts for moderately large rotations but is limited to orthotropic plates (see also [52]). Vlasov [53] is the first one to develop the third–order displacement field (1–11) that satisfies the stress–free boundary conditions on the top and bottom planes of a plate. Jemielita [39] considered the more general case of satisfying non–zero transverse stress state on the bounding planes of the plate [the displacement field (1–10) is a special case of that considered by Jemielita [39]]. The technical note of Krishna Murty [54] suggested a general, higher–order, displacement field of the type (1–11). In a recent paper, Krishna Murty [55] introduced a modification to the displacement field in (1–11) by splitting the transverse deflection into two parts: $w = w_b + w_s$, one due to bending and another due to shear. Such a representation was suggested by

Huffington [56] in a 1963 paper on beams. All of the above first–order and higher–order shear deformation plate theories are referred to as equivalent single–layer 2–D theories. These theories provide improved global response estimates for deflections, vibration frequencies, and buckling loads of moderately thick composites when compared to the classical laminate theory. Although the three–dimensional theories give more accurate results than the 2–D classical and shear–deformation theories, they are intractable. For example, the local theory of Pagano [57] results in a mathematical model consisting of 23N partial differential equations in the midplane coordinates of the laminate and 7N edge boundary conditions, where N is the number of layers in the laminate. Because the refined shear–deformation theories give as accurate a global response as the three–dimensional theory but is computationally less demanding, they are considered for problems not involving regions of acute discontinuities.

Recently, Reddy [58] developed a displacement–based theory in which the three–dimensional elasticity theory is reduced to a layer–wise two–dimensional laminate theory. In this theory [58,59,60], the displacements (u_1,u_2,u_3) at a point (x,y,z) in the laminate are assumed to be of the form

$$u_1(x,y,z,t) = u(x,y,t) + \sum_{j=1}^{N} U_j(x,y,t)\phi_j(z)$$

$$u_2(x,y,z,t) = v(x,y,t) + \sum_{j=1}^{N} V_j(x,y,t)\phi_j(z)$$

$$u_3(x,y,z,t) = w(x,y,t) + \sum_{j=1}^{N} W_j(x,y,t)\varphi_j(z) \tag{1–12}$$

where N is the number of layers in the laminate and ϕ_j and φ_j are known functions of thickness coordinate z. The functions ϕ_j and φ_j are piecewise continuous functions, defined only on two adjacent layers, and can be viewed as the global Lagrange interpolation functions associated with the Jth interface of the layers through the laminate thickness. Because of this local nature of ϕ_j and φ_j, the displacements are continuous through the thickness but their derivatives with respect to z are not. This implies that all transverse strains will be discontinuous at layer interfaces, leaving the possibility that the interlaminar transverse stresses computed from constitutive equations can be continuous. The inplane strains will be continuous but the inplane stresses will be discontinuous at layer interfaces because of the difference in material properties of adjacent layers. In Eqs. (1–12), (u,v,w) are the displacements of a point (x,y,0) on the reference plane of the laminate, and U, V, and W are functions that vanish on the reference plane:

$$U(x,y,0,t) = V(x,y,0,y) = W(x,y,0,t) = 0 \tag{1–13}$$

The principle of virtual displacements is used to derive a consistent set of differential equations governing the equilibrium of a laminate composed of N constant thickness orthotropic laminae; the material axes of each lamina are arbitrarily oriented with respect to the laminate coordinates. The theory results in (3 + 3N) unknown variables and as many differential equations in two dimensions.

Surveys of various shell theories can be found in the works of Naghdi [61] and Bert [62], and a detailed study of thin ordinary (i.e., not laminated) shells can be found in the monographs by Kraus [63], Ambartsumyan [62], and Vlasov [65]. Many of the classical shell theories are developed originally for thin elastic shells, and are based on the Love–Kirchhoff assumptions (or the first approximation theory): (1) plane sections normal to the undeformed middle surface remain plane and normal to the deformed middle surface, (2) the normal stresses perpendicular to the middle surface can be neglected in the stress–strain relations, and (3) the transverse displacement is independent of the thickness coordinate. The first assumption leads to the neglect of the transverse shear strains. These theories, known as the Love's first approximation theories (Love [66]) are expected to yield sufficiently accurate results when: (i) the lateral dimension–to–thickness ratio (a/h) is large, (ii) the dynamic excitations are within the low–frequency range, and (iii) the material anisotropy is not severe. However, application of such theories to layered anisotropic composite shells could lead to as much as 30% or more errors in deflections, stresses, and frequencies [67].

Ambartsumyan [64,68] was considered to be the first to analyze laminates that incorporated the bending–stretching coupling due to material anisotropy. The laminates that Ambartsumyan analyzed are now known as laminated orthotropic shells because the individual orthotropic layers were oriented such that the principal axes of material symmetry coincided with the principal coordinates of the shell reference surface. In 1962, Dong, Pister, and Taylor [69] formulated a theory of thin shells laminated of anisotropic material. Cheng and Ho [70] presented an analysis of laminated anisotropic cylindrical shells using Flugge's shell theory [71]. A first approximation theory for the unsymmetric deformation of nonhomogeneous, anisotropic, elastic cylindrical shells was derived by Widera and his colleagues [72,73] by means of asymptotic integration of the elasticity equation. For a homogeneous, isotropic material, the theory reduced to Donnell's shallow shell theory [74]. All of the aforementioned works are based on Love–Kirchhoff's hypotheses in which the transverse shear deformation is neglected. An exposition of various shell theories can be found in the articles of Bert [62] and Librescu [75].

The effect of transverse shear deformation and transverse isotropy as well as thermal expansion through the shell thickness were considered by Gulati and Essenberg [76] and Zukas and Vinson [77]. Dong and Tso [78] presented a theory applicable to layered, orthotropic cylindrical shells. Whitney and Sun [78] developed a higher–order shear deformation theory. This theory is based on a displacement field in which the displacements in the surface of the shell are expanded as a quadratic function of the thickness coordinate. Reddy [80] presented a shear–deformation version of the Sanders shell theory for laminated composite shells. Such theories account for constant transverse shear stresses through thickness, and therefore require a correction to the transverse shear stiffnesses.

Third–order shear deformation shell theories are also developed by Reddy and Liu [81], Bhimaraddi [82], and Librescu et al. [83] all based on a displacement field similar to (1–11). In these works, the stress–free boundary conditions on the bounding planes of the shell are satisfied and, therefore, the number of governing equations are five in total (the same as in first–order shear deformation shell theory). In [83], the equations of motion of first–order shear deformation shell theory are used to derive the governing equations of the third–order theory, whereas in [81] and [82], the governing equations are developed from the principle of virtual displacements. Recently, the layer–wise 2–D plate theory developed in [58–60] is extended by Barbero and Reddy [84] to a laminated cylindrical shells.

Applications of first–order and higher–order shear deformation plate and shell theories in solving static, stability, and vibration problems are numerous. Salerno and Goldberg [85] reformulated the Reissner Theory [10–12] in terms of a fourth–order equation (known as interior equation) governing the transverse deflection and a second–order equation (known as edge–zone equation) describing the edge effect. They applied the equations to static bending problems of a homogeneous isotropic plate with simply–supported edges and a plate having two opposite edges simply supported. The bending of homogeneous transversely isotropic plate with simply–supported edges are also considered by Ambartsumyan [1]. The Lévy–type problem (where two opposite edges of a plate are assumed to be invariably simply–supported) of the same plate is also considered in [1]. The remaining two edges were assumed to be either simply–supported or clamped (fixed). The Navier solution where all of the edges are assumed to be simply–supported, and Lévy–type solutions of thick homogeneous isotropic plates were obtained by Levinson and Cooke [86] and Cooke and Levinson [87], respectively. In [86,87], the first–order theory and the third–order theory of Levinson [42] were used.

The Navier solutions of the first–order shear deformation plate theory applied to laminated plates is due to Whitney and Pagano [88], who presented solutions for bending and vibration of symmetric and antisymmetric cross–ply and antisymmetric angle–ply rectangular plates. Fortier and Rosettos [89] analyzed the free–vibration of thick rectangular plates of unsymmetric cross–ply construction. Sinha and Rath [90] considered both vibration and buckling for the same type of plates. Following Whitney and Pagano [88], Bert and Chen [91] and Reddy and Chao [92] presented closed–form solutions for the free–vibration of simply–supported rectangular plates of antisymmetric angle–ply laminates. Reddy and Phan [93] developed the Navier solution for buckling and free–vibration problems of unsymmetric cross–ply and antisymmetric angle–ply plates for Reddy's third–order theory [38]. Recently, Reddy et al. [94] developed a Lévy–type solution for symmetric laminated cross–ply plates using the first–order theory. The method used is based on the state space concept, allowing one to generate analytical solutions for the critical buckling load and natural frequencies of rectangular laminated plates with various boundary conditions. This technique is used in a series of papers [94–102,46,47,83]. Khdeir, Reddy, and Librescu [95] analyzed the state of stress of symmetric cross–ply laminated plates using Reddy's third–order theory [38], while Khdeir and Librescu [47] investigated the free–vibration and buckling of symmetric cross–ply laminated plates using a third–order theory developed in [46]. Analysis of free–vibration and buckling problems of shallow spherical and cylindrical panels with various boundary conditions is considered in [83], whereas, in [102], the same problem is considered using the third–order theory developed in [81]. The Navier solution is also developed in [84] for the layer–wise cylindrical shell theory for eigenfrequency problem and results were compared with those of three–dimensional elasticity problems.

Papers dealing with the damped and undamped response of plates and shells are in less evidence. Librescu [103,2] obtained some conclusions concerning the influence of rotatory inertia terms of moderately thick rectangular plates built up of a transversely isotropic material. Forced–motion of impulsively loaded homogeneous isotropic rectangular plate was considered by Reismann and Lee [104] using the first–order theory (also known as Mindlin's theory). Dobyns [105] presented an analysis of simply–supported symmetric cross–ply plates subjected to static and dynamic loads using the first–order shear deformation theory. Sun and Whitney [106,107] analyzed the response of anisotropic plates in cylindrical bending using the first–order theory. Birman and Bert [108] considered the response of a simply–supported antisymmetrically laminated angle–ply plates to explosive blast loading using the first–order theory. In the analysis, the in–plane and rotatory inertia terms were omitted. Khdeir and Reddy [109] considered the response of symmetric cross–ply plates using the third–order theory of Reddy [38]. By assuming the exact form of the spatial variation of the solution, the governing equations of the plate were reduced to a system of ordinary differential equations in time which were analytically solved by the state space concept utilized in [94–102,46,67,83]. The same technique was used by Khdeir and Reddy in [110] to analyze the forced–motion of antisymmetric cross–ply laminated plates. Dynamic response of a homogeneous orthotropic shell due to line and patch loads is considered by Cederbaum and Heller [111] using the first–order shear deformation shell theory. There exist numerous other works (e.g. see [112,113]) pertinent to undamped transient analysis of plates and shells where shear deformation theories are used.

The review of the literature clearly reveals that the following topics related to the stability and vibration problems of laminated plates and shells have not been considered or closely examined:

- The study of damped response of laminated plates and shells using the first–order and third–order shear deformation theories with in–plane and rotatory inertia terms included
- Stability and eigenfrequency problems of laminated shells with various boundary conditions using the layer–wise theory of Reddy.
- Uncoupling of linear and nonlinear dynamic equations of variationally consistent third–order shear deformation plate and shell theories into

interior and edge–zone equations and determination of pure–shear frequencies

- Analytical solutions of buckling and eigenfrequency equations of laminated plates with various boundary conditions using the first–order and third–order theories
- Analytical solutions of forced–vibration problems of laminated plates with various boundary conditions using the first–order and third–order theories
- A general analytical technique for solving stability and eigenfrequency problems of laminated shells and plates having Lévy–type boundary conditions
- Lévy–type solutions of bending problems of laminated plates using a variationally consistent third–order shear deformation theory

1.3. Objectives of the Research

The major objectives of the present study are identified as follows:

1. Development of an analytical technique for the solution of eigenfrequency and buckling problems of thick and thin laminated shells and plates with various boundary conditions using single–layer and layer–wise shear deformation theories.
2. Development of a solution methodology for damped vibration analysis of composite laminates using higher–order shear deformation theories. The procedure will be applicable to both self–adjoint and non–self–adjoint equations, and in–plane inertia terms, rotatory

inertia terms, and sonic boom and blast type loading will also be included in the analysis.

3. Reformulation and uncoupling of linear and nonlinear dynamic equations of shear deformation theories of plates laminated of transversely isotropic layers into interior and edge–zone equations. The pure–shear frequencies of plates will be identified and the five governing equations of the theories will be reduced to three (as in the classical theory) for certain boundary conditions.
4. Determination of frequency and buckling determinants and equations of plates laminated of transversely isotropic layers and with various boundary conditions. The results will be obtained within the third–order and first–order shear deformation theories by using the full system of equations and interior and edge–zone equations.
5. Solution of bending problems of transversely isotropic plates with various boundary conditions will be obtained with the help of interior and edge–zone equations of a variationally consistent third–order shear deformation theory.

These objectives constituted a topic for this dissertation. The details of this study are included in the forthcoming chapters. While analytical solutions are presented for a number of cases, actual numerical results are presented only for some typical cases.

CHAPTER II

FORMULATION OF EQUIVALENT SINGLE–LAYER PLATE AND SHELL THEORIES

Throughout this study the following theories are considered: i) the nonlinear equations of the third–order shear deformation theories of plates and shells developed by Reddy [50] and Reddy and Liu [67], ii) the first–order shear deformation theories of plates and shells, iii) the third–order shear deformation plate theory of Levinson [42], and iv) the generalized layer–wise shell theory developed in [84]. The equations of the first–order theory can be obtained as a special case from those of the third–order shell theory of Reddy and Liu [67]. We start our study by presenting the geometrically nonlinear third–order theory of laminated shells developed in [67].

2.1 Kinematics of a Shell

The displacement field, for a doubly–curved shell with principal radii of curvature R_1 and R_2, in the third–order shear deformation shell theory (TSDST) of Reddy and Liu [67] is assumed to be

$$u_1(x_1,x_2,\xi,t) = (1 + \frac{\xi}{R_1})u + \xi\phi_1 + \lambda\xi^2\varphi_1 + \lambda\xi^3\theta_1$$

$$u_2(x_1,x_2,\xi,t) = (1 + \frac{\xi}{R_2})v + \xi\phi_2 + \lambda\xi^2\varphi_2 + \lambda\xi^3\theta_2$$

$$u_3(x_1,x_2,\xi,t) = w(x_1,x_2,t) \qquad (2\text{–}1)$$

where u, v, ϕ_1, ϕ_2, φ_1, φ_2, θ_1, and θ_2 are functions of x_1, x_2, and time t only; x_1 and x_2 are the coordinates lying on the middle surface of shell, ξ is the coordinate normal to the middle surface of the shell, and λ is a tracer equal to 1 in TSDST and 0 in the first–order shear deformation shell theory (FSDST). In TSDST, by satisfying the transverse stress–free boundary conditions on the bounding surfaces of the shell, the displacement field (2–1) is reduced to

$$u_1 = (1 + \frac{\xi}{R_1})u + \xi\phi_1 - \lambda \frac{4}{3h^2} \xi^3(\phi_1 + \frac{\partial w}{\partial x_1})$$

$$u_2 = (1 + \frac{\xi}{R_2})v + \xi\phi_2 - \lambda \frac{4}{3h^2} \xi^3(\phi_2 + \frac{\partial w}{\partial x_2})$$

$$u_3 = w \tag{2–2}$$

where h denotes the total thickness of the shell.

2.2 Equations of Motion

Using the principle of virtual displacements and assuming that

1. the thickness of the shell is small compared to the principal radii of the curvature (i.e., h/R_1 and $h/R_2 << 1$), and
2. the transverse normal stress is negligible,

the equations of motion of shallow elastic shells developed in [67] can be expressed as,

$$\frac{\partial N_1}{\partial x_1} + \frac{\partial N_6}{\partial x_2} = I_1\ddot{u} + \bar{I}_2\ddot{\phi}_1 - \bar{I}_4\ddot{w}_{,x_1} \tag{2–3a}$$

$$\frac{\partial N_2}{\partial x_2} + \frac{\partial N_6}{\partial x_1} = I_1\ddot{v} + \overline{I}_2\ddot{\phi}_2 - \overline{I}_4\ddot{w}_{,x_2} \tag{2–3b}$$

$$\frac{\partial M_1}{\partial x_1} + \frac{\partial M_6}{\partial x_2} - Q_1 + \lambda\frac{4}{h^2}K_1 - \lambda\frac{4}{3h^2}\left(\frac{\partial P_1}{\partial x_1} + \frac{\partial P_6}{\partial x_2}\right)$$

$$= \overline{I}_2\ddot{u} + \overline{I}_3\ddot{\phi}_1 - \overline{I}_5\ddot{w}_{,x_1} \tag{2–3c}$$

$$\frac{\partial M_2}{\partial x_2} + \frac{\partial M_6}{\partial x_1} - Q_2 + \lambda\frac{4}{h^2}K_2 - \lambda\frac{4}{3h^2}\left(\frac{\partial P_2}{\partial x_2} + \frac{\partial P_6}{\partial x_1}\right)$$

$$= \overline{I}_2\ddot{v} + \overline{I}_3\ddot{\phi}_2 - \overline{I}_5\ddot{w}_{,x_2} \tag{2–3d}$$

$$\frac{\partial Q_1}{\partial x_1} + \frac{\partial Q_2}{\partial x_2} - \lambda\frac{4}{h^2}\left(\frac{\partial K_1}{\partial x_1} + \frac{\partial K_2}{\partial x_2}\right)$$

$$+ \lambda\frac{4}{3h^2}\left(\frac{\partial^2 P_1}{\partial x_1^2} + 2\frac{\partial^2 P_6}{\partial x_1 \partial x_2} + \frac{\partial^2 P_2}{\partial x_2^2}\right) - \frac{N_1}{R_1} - \frac{N_2}{R_2} + P_z + N(u,v,w)$$

$$= \overline{I}_4(\ddot{u}_{,x_1} + \ddot{v}_{,x_2}) + \overline{I}_5(\ddot{\phi}_{1,x_1} + \ddot{\phi}_{,x_2}) + I_1\ddot{w} - \overline{I}_7(\ddot{w}_{,x_1x_1} + \ddot{w}_{,x_2x_2}) \tag{2–3e}$$

where

$$N(u,v,w) = \frac{\partial}{\partial x_1}\left(N_1\frac{\partial w}{\partial x_1} + N_6\frac{\partial w}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(N_6\frac{\partial w}{\partial x_1} + N_2\frac{\partial w}{\partial x_2}\right). \tag{2–4}$$

To be more consistent with assumption 1, the mass terms in Eqs. (2–3) are slightly modified from those given in [67]. The stress resultants in Eqs. (2–3) and (2–4) are expressed in terms of the strain and curvature components as

$$N_i = A_{ij}\epsilon_j^o + B_{ij}\kappa_j^o + E_{ij}\kappa_j^2$$

$$M_i = B_{ij}\epsilon_j^o + D_{ij}\kappa_j^o + F_{ij}\kappa_j^2$$

$$P_i = E_{ij}\epsilon_j^o + F_{ij}\kappa_j^o + H_{ij}\kappa_j^2 \quad (i,j = 1,2,6) \tag{2–5}$$

$$Q_2 = A_{4j}\epsilon_j^o + D_{4j}\kappa_j^1$$

$$Q_1 = A_{5j}\epsilon_j^o + D_{5j}\kappa_j^1$$

$$K_2 = D_{4j}\epsilon_j^o + F_{4j}\kappa_j^1$$

$$K_1 = D_{5j}\,\epsilon_j^o + F_{5j}\kappa_j^1 \quad (j = 4,5) \tag{2–6}$$

where A_{ij}, B_{ij}, etc. are the laminate stiffnesses,

$$(A_{ij},B_{ij},D_{ij},E_{ij},F_{ij},H_{ij})$$

$$= \sum_{K=1}^{N} \int_{\xi_{k+1}}^{\xi_k} \overline{Q}_{ij}^{(k)}(1,\zeta,\zeta^2,\zeta^3,\zeta^4,\zeta^6)d\xi \quad (i,j = 1,2,4,5,6) \tag{2–7}$$

In Eqs. (2–5) and (2–6) a repeated index indicates summation of terms over all values of that index. In the first–order shear deformation shell (and plate) theories the rigidity terms A_{ij} for i,j = 4,5 are replaced by $K_i K_j A_{ij}$ where K_i are the shear correction factors. Also the strain and curvature components in Eqs. (2–5) and (2–6) are

$$\epsilon_1^o = \frac{\partial u}{\partial x_1} + \frac{w}{R_1} + \frac{1}{2}(\frac{\partial w}{\partial x_1})^2,\ \kappa_1^o = \frac{\partial \phi_1}{\partial x_1},\ \kappa_1^2 = -\lambda\frac{4}{3h^2}(\frac{\partial \phi_1}{\partial x_1} + \frac{\partial^2 w}{\partial x_1^2})$$

$$\epsilon_2^o = \frac{\partial v}{\partial x_2} + \frac{w}{R_2} + \frac{1}{2}(\frac{\partial w}{\partial x_2})^2,\ \kappa_2^o = \frac{\partial \phi_2}{\partial x_2},\ \kappa_2^2 = -\lambda\frac{4}{3h^2}(\frac{\partial \phi_2}{\partial x_2} + \frac{\partial^2 w}{\partial x_2^2})$$

$$\epsilon_4^o = \phi_2 + \frac{\partial w}{\partial x_2},\ \kappa_4^1 = -\lambda\frac{4}{h^2}(\phi_2 + \frac{\partial w}{\partial x_2})$$

$$\epsilon_5^o = \phi_1 + \frac{\partial w}{\partial x_1},\ \kappa_5^1 = -\lambda\frac{4}{h^2}(\phi_1 + \frac{\partial w}{\partial x_1})$$

$$\epsilon_6^o = \frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2} + \frac{\partial w}{\partial x_1}\frac{\partial w}{\partial x_2},\ \kappa_6^o = \frac{\partial \phi_2}{\partial x_1} + \frac{\partial \phi_1}{\partial x_2}$$

$$\kappa_6^2 = -\lambda\frac{4}{3h^2}(\frac{\partial \phi_2}{\partial x_1} + \frac{\partial \phi_1}{\partial x_2} + 2\frac{\partial^2 w}{\partial x_1 \partial x_2}). \qquad (2\text{–}8)$$

The mass terms in Eqs. (2–3) are defined as

$$\overline{I}_2 = I_2 - \lambda\frac{4}{3h^2}I_4,\ \overline{I}_3 = I_3 - \lambda\frac{4}{3h^2}I_5 + \lambda(\frac{4}{3h^2})^2 I_7$$

$$\overline{I}_4 = \lambda \frac{4}{3h^2} I_4, \ \overline{I}_5 = \lambda \frac{4}{3h^2}\left(I_5 - \frac{4}{3h^2} I_7\right), \ \overline{I}_7 = \lambda \left(\frac{4}{3h^2}\right)^2 I_7 \tag{2–9}$$

where

$$(I_1, I_2, I_3, I_4, I_5, I_7) = \sum_{k=1}^{N} \int_{\xi_{k+1}}^{\xi_k} \rho^{(k)}(1, \xi, \xi^2, \xi^3, \xi^4, \xi^6) d\xi. \tag{2–10}$$

In Eqs. (2–7) and (2–10), N denotes the total number of layers and $\overline{Q}_{ij}^{(k)}$ and $\rho^{(k)}$ are the reduced stiffnesses and mass density of kth layer of the shell (see [67]).

2.3 Boundary Conditions

The boundary conditions of a shell with rectangular planform involve the specification of

$$u \quad \text{or} \quad N_1 \tag{2–11a}$$

$$v \quad \text{or} \quad N_6 \tag{2–11b}$$

$$\phi_1 \quad \text{or} \quad M_1 - \lambda \frac{4}{3h^2} P_1 \tag{2–11c}$$

$$\phi_2 \quad \text{or} \quad M_6 - \lambda \frac{4}{3h^2} P_6 \tag{2–11d}$$

$$w \qquad \text{or} \qquad Q_1 - \lambda \frac{4}{h^2} K_1 + \lambda \frac{4}{3h^2} \left(\frac{\partial P_1}{\partial x_1} + 2 \frac{\partial P_6}{\partial x_2}\right)$$

$$- \overline{I}_4 \ddot{u} - \overline{I}_5 \ddot{\phi}_1 + \overline{I}_7 \ddot{w}_{,x_1} \tag{2–11e}$$

$$w_{,x_1} \qquad \text{or} \qquad P_1 \tag{2–11f}$$

at $x_1 = 0$ and a and

$$u \qquad \text{or} \qquad N_6$$

$$v \qquad \text{or} \qquad N_2$$

$$\phi_1 \qquad \text{or} \qquad M_6 - \lambda \frac{4}{3h^2} P_6$$

$$\phi_2 \qquad \text{or} \qquad M_2 - \lambda \frac{4}{3h^2} P_2$$

$$w \qquad \text{or} \qquad Q_2 - \lambda \frac{4}{h^2} K_2 + \lambda \frac{4}{3h^2} \left(\frac{\partial P_2}{\partial x_2} + 2 \frac{\partial P_6}{\partial x_1}\right)$$

$$- \overline{I}_4 \ddot{v} - \overline{I}_5 \ddot{\phi}_2 + \overline{I}_7 \ddot{w}_{,x_2}$$

$$w_{,x_2} \qquad \text{or} \qquad P_2 \tag{2–12}$$

at $x_2 = 0$ and b.

The equations of motion and boundary conditions of the first–order shear deformation shell theory (FSDST) can be obtained from Eqs. (2–3) through (2–12) by letting $\lambda = 0$. The equations of motion of the classical shell theory (CLST) can be obtained from those of FSDST by letting $\phi_1 = -\frac{\partial w}{\partial x_1}$ and $\phi_2 = -\frac{\partial w}{\partial x_2}$. Also in FSDST only the first five boundary conditions in Eqs. (2–11) and (2–12) are imposed. Equations (2–3) can be specialized to spherical shells, cylindrical shells, and plates by setting $R_1 = R_2 = R$, $R_2 = R$ and $\frac{1}{R_1} = 0$ (the x_1–axis is taken along the generator of the cylinder), and $\frac{1}{R_1} = \frac{1}{R_2} = 0$, respectively. In the next chapter we will formulate the layer–wise and the generalized layer–wise shell theories [84]. The third–order shear deformation plate theory of Levinson will be considered in Chapter VIII.

CHAPTER III

FORMULATION OF LAYER–WISE AND GENERALIZED LAYER–WISE SHELL THEORIES

In this chapter we present the layer–wise shell theory and the generalized layer–wise shell theory developed in [84] with a slight modification in the assumed displacement field. The formulation will be developed for circular cylindrical panels and shells with radius R.

3.1 Displacements and Strain–Displacement Relations

In the layer–wise shell theory (LST) we assume that the displacement components in a laminated shell have the form

$$u_1(x_1,x_2,z) = \sum_{j=1}^{N+1} u^j(x_1,x_2)\phi^j(z)$$

$$u_2(x_1,x_2,z) = \sum_{j=1}^{N+1} v^j(x_1,x_2)\phi^j(z)$$

$$u_3(x_1,x_2,z) = w(x_1,x_2) \tag{3–1}$$

where N denotes the number of layers and ϕ^j are known shape functions defined in Fig. 3.1. In LST we assumed that the radial displacement is constant throughout the thickness of the shell. In the generalized layer–wise shell theory (GLST) we remove this assumption and let

$$u_1(x_1,x_2,z) = \sum_{j=1}^{N+1} u^j(x_1,x_2)\phi^j(z)$$

$$u_2(x_1,x_2,z) = \sum_{j=1}^{N+1} v^j(x_1,x_2)\phi^j(z)$$

$$u_3(x_1,x_2,z) = \sum_{j=1}^{N+1} w^j(x_1,x_2)\phi^j(z) \tag{3-2}$$

Assuming that the normal coordinate $z << R$, the strain components at a generic point of the shell are related to u_1, u_2, and u_3 as

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1},\ \epsilon_{22} = \frac{\partial u_2}{\partial x_2} + \frac{u_3}{R},\ \epsilon_{33} = \frac{\partial u_3}{\partial z}$$

$$\gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1},\ \gamma_{23} = \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial x_2} - \frac{u_2}{R},\ \gamma_{13} = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial z} \tag{3-3}$$

The strain–displacement relations in LST and GLST are obtained, respectively, by substituting Eqs. (3–1) and (3–2) into Eqs. (3–3):

LST:

$$\epsilon_{11} = \sum_{j=1}^{N+1} u^j_{,x_1}\phi^j,\ \epsilon_{22} = \sum_{j=1}^{N+1} v^j_{,x_2}\phi^j + \frac{1}{R}w$$

$$\gamma_{12} = \sum_{j=1}^{N+1} u^j_{,x_2}\phi^j + \sum_{j=1}^{N+1} v^j_{,x_1}\phi^j,\ \epsilon_{33} = 0$$

$$\gamma_{23} = \sum_{j=1}^{N+1} v^j\frac{d\phi^j}{dz} + w_{,x_2} - \frac{1}{R}\sum_{j=1}^{N+1} v^j\phi^j$$

$$\gamma_{13} = w_{,x_1} + \sum_{j=1}^{N+1} u^j \frac{d\phi^j}{dz} \qquad (3\text{–}4)$$

GLST:

$$\epsilon_{11} = \sum_{j=1}^{N+1} u^j_{,x_1} \phi^j, \; \epsilon_{22} = \sum_{j=1}^{N+1} v^j_{,x_2} \phi^j + \sum_{j=1}^{N+1} \frac{1}{R} w^j \phi^j$$

$$\gamma_{12} = \sum_{j=1}^{N+1} u^j_{,x_2} \phi^j + \sum_{j=1}^{N+1} v^j_{,x_1} \phi^j, \; \epsilon_{33} = \sum_{j=1}^{N+1} w^j \frac{d\phi^j}{dz}$$

$$\gamma_{23} = \sum_{j=1}^{N+1} v^j \frac{d\phi^j}{dz} + \sum_{j=1}^{N+1} w^j_{,x_2} \phi^j - \sum_{j=1}^{N+1} \frac{1}{R} v^j \phi^j$$

$$\gamma_{13} = \sum_{j=1}^{N+1} u^j \frac{d\phi^j}{dz} + \sum_{j=1}^{N+1} w^j_{,x_1} \phi^j \qquad (3\text{–}4)$$

3.2 Equilibrium Equations and Boundary Conditions

The principle of minimum total potential energy will be used to derive the equilibrium equations of circular cylindrical panels and shells laminated of N constant–thickness orthotropic laminae whose principal material directions are arbitrarily oriented with respect to the laminate coordinates. With a distributed load $p_z(x_1,x_2)$ acting in the outward (positive) direction, the principle can be stated as:

$$0 = \int_V (\sigma_{11}\delta\epsilon_{11} + \sigma_{22}\delta\epsilon_{22} + \sigma_{33}\delta\epsilon_{33}$$

$$+ \sigma_{13}\delta\gamma_{13} + \sigma_{23}\delta\gamma_{23} + \sigma_{12}\delta\gamma_{12})dV - \int_\Omega p_z \delta u_3 d\Omega \tag{3–5}$$

where σ_{11}, σ_{22}, σ_{33}, σ_{13}, σ_{23}, and σ_{12} are the stress components, V is the total volume of the laminate, Ω is the reference surface of the laminate (assumed to be the middle surface of the shell), and δ denotes the variational symbol.

The equilibrium equations in LST and GLST and their corresponding boundary conditions for a shell with rectangular planform are obtained by substituting the strain–displacement relations (3–3) and (3–4), respectively, into Eqs. (3–5), integrating through the thickness, and integrating the derivatives of the varied quantities by parts and collecting the coefficients of δu^j, δv^j, and δw in LST and δu^j, δv^j, and δw^j in GLST.

<u>LST</u>

$$\delta u^j: \quad M^j_{1,x_1} + M^j_{6,x_2} - Q^j_1 = 0$$

$$\delta v^j: \quad M^j_{2,x_2} + M^j_{6,x_1} + \frac{1}{R} K^j_2 - Q^j_2 = 0$$

$$\delta w: \quad Q_{1,x_1} + Q_{2,x_2} - \frac{1}{R} N_2 + P_z = 0 \tag{3–6}$$

with the geometric and force boundary conditions, at x_1 = constant:

Geometric (essential)	Force (natural)	
u^j	M_1^j	
v^j	M_6^j	
w	Q_1	(3–7)

and

at x_2 = constant:

Geometric (essential)	Force (natural)	
u^j	M_6^j	
v^j	M_2^j	
w	Q_2	(3–8)

GLST

$$\delta u^j:\quad M^j_{1,x_1} + M^j_{6,x_2} - Q^j_1 = 0$$

$$\delta v^j:\quad M^j_{2,x_2} + M^j_{6,x_1} + \frac{1}{R}K^j_2 - Q^j_2 = 0 \quad (j=1,2,\ldots,N+1)$$

$$\delta w^j:\quad K^j_{1,x_1} + K^j_{2,x_2} - \frac{1}{R}M^j_2 - Q^j_3 + \delta_{\bar{k}j}P_z = 0 \tag{3–9a}$$

with $\bar{k} = N + 1$ and

$$\delta_{\bar{k}j} = \begin{cases} 1 \text{ if } j = \bar{k} \\ 0 \text{ if } j \neq \bar{k} \end{cases} \tag{3–9b}$$

and the boundary conditions

at x_1 = constant:

Geometric (essential)	Force (natural)	
u^j	M_1^j	
v^j	M_6^j	
w^j	K_1^j	(3–10)

and

at x_2 = constant:

Geometric (essential)	Force (natural)	
u^j	M_6^j	
v^j	M_2^j	
w^j	K_2^j	(3–11)

3.3 Stress Resultants

In Eqs. (3–6) through (3–11) the stress resultants are defined as

$$M_1^j = \int_{-h/2}^{h/2} \sigma_{11}\phi^j dz \, , \; M_2^j = \int_{-h/2}^{h/2} \sigma_{22}\phi^j dz$$

$$M_6^j = \int_{-h/2}^{h/2} \sigma_{12}\phi^j dz \, , \; K_1^j = \int_{-h/2}^{h/2} \sigma_{13}\phi^j dz$$

$$K_2^j = \int_{-h/2}^{h/2} \sigma_{23}\phi^j dz \, , \; Q_1^j = \int_{-h/2}^{h/2} \sigma_{13}\frac{d\phi^j}{dz} dz$$

$$Q_2^j = \int_{-h/2}^{h/2} \sigma_{23} \frac{d\phi^j}{dz} dz \, , \; Q_3^j = \int_{-h/2}^{h/2} \sigma_{33} \frac{d\phi^j}{dz} dz$$

$$Q_1 = \int_{-h/2}^{h/2} \sigma_{13} dz \, , \; Q_2 = \int_{-h/2}^{h/2} \sigma_{23} dz$$

and

$$N_2 = \int_{-h/2}^{h/2} \sigma_{22} dz. \tag{3–12}$$

3.4 Constitutive Law

Like in the equivalent single–layer theories of Chapter II, in LST we will use the plane–stress constitutive law

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}^k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^k \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{Bmatrix}^k$$

$$\begin{Bmatrix} \sigma_{23} \\ \sigma_{13} \end{Bmatrix}^k = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix}^k \begin{Bmatrix} \gamma_{23} \\ \gamma_{13} \end{Bmatrix}^k \tag{3–13}$$

where $\bar{Q}_{ij}$ are the plane–stress transformed reduced material stiffnesses of the k^{th} layer. However, in GLST we remove the plane–stress assumption and use the three–dimensional constitutive law

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}^k = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & 0 & 0 & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{23} & 0 & 0 & \bar{C}_{26} \\ \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{33} & 0 & 0 & \bar{C}_{36} \\ 0 & 0 & 0 & \bar{C}_{44} & \bar{C}_{45} & 0 \\ 0 & 0 & 0 & \bar{C}_{45} & \bar{C}_{55} & 0 \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{36} & 0 & 0 & \bar{C}_{66} \end{bmatrix}^k \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}^k \qquad (3\text{–}14)$$

where $\bar{C}_{ij}$ are the transformed material stiffnesses of the k^{th} layer (see [3]).

3.5 Stress Resultants in Terms of Generalized Displacements

For simplicity we have defined the shape functions ϕ^j to be linear functions of z in Fig. 3.1. More explicitly, we have

$$\phi^j = \begin{cases} 0 & z > z_{j-1} \\ \psi_1^j & z_{j-1} > z > z_j \\ \psi_2^j & z_j > z > z_{j+1} \\ 0 & z < z_j \end{cases} \qquad (3\text{–}15a)$$

with

$$\psi_1^j = \frac{1}{t_{j-1}} (z_{j-1} - z)$$

$$\psi_2^j = \frac{1}{t_j}(z - z_{j+1}) \tag{3–15b}$$

where t_j is the thickness of the j^{th} layer (see Fig. 3.1). That is, we are assuming that u^j, v^j, and w^j are the displacement components at the j^{th} interface in the laminate and (u^1, v^1, w^1) and $(u^{N+1}, v^{N+1}, w^{N+1})$ are the displacement components at the top and bottom surfaces, respectively, of the laminated shell.

With ϕ^j given by Eq. (3–15) we substitute Eqs. (3–4) into (3–13) and the subsequent results into Eqs. (3–12) to obtain (in LST):

$$M_1^j = \sum_{i=1}^{N+1} D_{11}^{ij} u^i_{,x_1} + \sum_{i=1}^{N+1} D_{12}^{ij} v^i_{,x_2} + B_{12}^j \frac{w}{R}$$

$$+ \sum_{i=1}^{N+1} D_{16}^{ij} u^i_{,x_2} + \sum_{i=1}^{N+1} D_{16}^{ij} v^i_{,x_1}$$

$$M_2^j = \sum_{i=1}^{N+1} D_{12}^{ij} u^i_{,x_1} + \sum_{i=1}^{N+1} D_{22}^{ij} v^j_{,x_2} + B_{22}^j \frac{w}{R}$$

$$+ \sum_{i=1}^{N+1} D_{26}^{ij} u^i_{,x_2} + \sum_{i=1}^{N+1} D_{26}^{ij} v^i_{,x_2}$$

$$M_6^j = \sum_{i=1}^{N+1} D_{16}^{ij} u^i_{,x_1} + \sum_{i=1}^{N+1} D_{26}^{ij} v^i_{,x_2} + B_{26}^j \frac{w}{R}$$

$$+ \sum_{i=1}^{N+1} D_{66}^{ij} u^i_{,x_2} + \sum_{i=1}^{N+1} D_{66}^{ij} v^i_{,x_1}$$

$$K_2^j = \sum_{i=1}^{N+1} (\bar{D}_{44}^{ij} - \frac{1}{R} D_{44}^{ij}) v^i + \sum_{i=1}^{N+1} \bar{D}_{45}^{ij} u^i$$

$$+ B_{45}^j w_{,x_1} + B_{44}^j w_{,x_2}$$

$$Q_1^j = \sum_{i=1}^{N+1} (\bar{\bar{D}}_{45}^{ij} - \frac{1}{R} \bar{D}_{45}^{ji}) v^i + \sum_{i=1}^{N+1} \bar{\bar{D}}_{55}^{ij} u^i$$

$$+ \bar{B}_{55}^j w_{,x_1} + \bar{B}_{45}^j w_{,x_2}$$

$$Q_2^j = \sum_{i=1}^{N+1} (\bar{\bar{D}}_{44}^{ij} - \frac{1}{R} \bar{D}_{44}^{ji}) v^i + \sum_{i=1}^{N+1} \bar{\bar{D}}_{45}^{ij} u^i$$

$$+ \bar{B}_{45}^j w_{,x_1} + \bar{B}_{44}^j w_{,x_2}$$

$$Q_1 = \sum_{i=1}^{N+1} (\bar{B}_{45}^{i} - \frac{1}{R} B_{45}^{i}) v^i + \sum_{i=1}^{N+1} \bar{B}_{55}^{i} u^i$$

$$+ A_{55} w_{,x_1} + A_{45} w_{,x_2}$$

$$Q_2 = \sum_{i=1}^{N+1} (\bar{B}_{44}^{i} - \frac{1}{R} B_{44}^{i}) v^i + \sum_{i=1}^{N+1} \bar{B}_{45}^{i} u^i$$

$$+ A_{45} w_{,x_1} + A_{44} w_{,x_2}$$

$$N_2 = \sum_{i=1}^{N+1} B_{12}^{i} u^{i}_{,x_1} + \sum_{i=1}^{N+1} B_{26}^{i} u^{i}_{,x_2} + \sum_{i=1}^{N+1} B_{26}^{i} v^{i}_{,x_1}$$

$$+ \sum_{i=1}^{N+1} B_{22}^{i} v^{i}_{,x_2} + \frac{1}{R} A_{22} w \tag{3–16a}$$

with j = 1,...,N + 1 and

$$D_{pq}^{ij} = D_{pq}^{ji} = \sum_{k=1}^{N} \int_{z_{k+1}}^{z_k} \bar{Q}_{pq}^{k} \phi^i \phi^j dz$$

$$\bar{D}_{pq}^{ij} = \sum_{k=1}^{N} \int_{z_{k+1}}^{z_k} \bar{Q}_{pq}^{k} \frac{d\phi^i}{dz} \phi^j dz \neq \bar{D}_{pq}^{ji}$$

$$\bar{\bar{D}}_{pq}^{ij} = \bar{\bar{D}}_{pq}^{ji} = \sum_{k=1}^{N} \int_{z_{k+1}}^{z_k} \bar{Q}_{pq}^{k} \frac{d\phi^i}{dz} \frac{d\phi^j}{dz} dz$$

$$B_{pq}^{j} = \sum_{k=1}^{N} \int_{z_{k+1}}^{z_k} \bar{Q}_{pq}^{k} \phi^j dz \,, \quad \bar{B}_{pq}^{j} = \sum_{k=1}^{N} \int_{z_{k+1}}^{z_k} \bar{Q}_{pq}^{k} \frac{d\phi^j}{dz} dz$$

$$A_{pq} = \sum_{k=1}^{N} \int_{z_{k+1}}^{z_k} \bar{Q}_{pq}^{k} dz \;, \quad p,q = (1,2,4,5,6) \tag{3–16b}$$

By carrying out the integrations in Eqs. (3–16b) the rigidity terms are found to be

$$D^{ij}_{pq} = \begin{cases} \frac{1}{3}\, t_i \bar{Q}^{i}_{pq} + \frac{1}{3}\, t_{i-1} \bar{Q}^{i-1}_{pq} & \text{if } i = j \\ \frac{1}{6}\, \bar{Q}^{i}_{pq}\, t_i & \text{if } j = i + 1 \\ 0 & \text{if } j > i + 1 \end{cases}$$

$$\bar{D}^{ij}_{pq} = \begin{cases} \frac{1}{2}\, \bar{Q}^{i}_{pq} - \frac{1}{2}\, \bar{Q}^{i-1}_{pq} & \text{if } i = j \\ \frac{1}{2}\, \bar{Q}^{i}_{pq} & \text{if } j = i + 1 \\ -\frac{1}{2}\, \bar{Q}^{i-1}_{pq} & \text{if } j = i - 1 \\ 0 & \text{if } j > i + 1 \text{ and } j < i - 1 \end{cases}$$

$$\bar{\bar{D}}^{ij}_{pq} = \begin{cases} \frac{1}{t_i}\, \bar{Q}^{i}_{pq} + \frac{1}{t_{i-1}}\, \bar{Q}^{i-1}_{pq} & \text{if } i = j \\ -\frac{1}{t_i}\, \bar{Q}^{i}_{pq} & \text{if } j = i + 1 \\ 0 & \text{if } j > i + 1 \end{cases}$$

$$B^{i}_{pq} = \frac{1}{2}\, t_i \bar{Q}^{i}_{pq} + \frac{1}{2}\, t_{i-1} \bar{Q}^{i-1}_{pq}$$

$$\bar{B}^{i}_{pq} = \bar{Q}^{i}_{pq} - \bar{Q}^{i-1}_{pq} \text{ and } A_{pq} = \sum_{k=1}^{N} \bar{Q}^{k}_{pq} t_k. \tag{3–17}$$

In GLST the stress resultants are expressed in terms of the generalized displacement components by substituting Eqs. (3–4) into (3–14) and the subsequent results into Eqs. (3–12):

$$M_1^j = \sum_{i=1}^{N+1} D_{11}^{ij} u^i_{,x_1} + \sum_{i=1}^{N+1} D_{12}^{ij} v^i_{,x_2} + \sum_{i=1}^{N+1} \frac{1}{R} D_{12}^{ij} w^i$$

$$+ \sum_{i=1}^{N+1} \bar{D}_{13}^{ij} w^i + \sum_{i=1}^{N+1} D_{16}^{ij} u^i_{,x_2} + \sum_{i=1}^{N+1} D_{16}^{ij} v^i_{,x_1}$$

$$M_2^j = \sum_{i=1}^{N+1} D_{12}^{ij} u^i_{,x_1} + \sum_{i=1}^{N+1} D_{22}^{ij} v^i_{,x_2} + \sum_{i=1}^{N+1} \frac{1}{R} D_{22}^{ij} w^i$$

$$+ \sum_{i=1}^{N+1} \bar{D}_{23}^{ij} w^i + \sum_{i=1}^{N+1} D_{26}^{ij} u^i_{,x_2} + \sum_{i=1}^{N+1} D_{26}^{ij} v^i_{,x_1}$$

$$M_6^j = \sum_{i=1}^{N+1} D_{16}^{ij} u^i_{,x_1} + \sum_{i=1}^{N+1} D_{66}^{ij} v^i_{,x_2} + \sum_{i=1}^{N+1} \frac{1}{R} D_{26}^{ij} w^i$$

$$+ \sum_{i=1}^{N+1} \bar{D}_{36}^{ij} w^i + \sum_{i=1}^{N+1} D_{66}^{ij} u^i_{,x_2} + \sum_{i=1}^{N+1} D_{66}^{ij} v^i_{,x_1}$$

$$K_1^j = \sum_{i=1}^{N+1} (\bar{D}_{45}^{ij} - \frac{1}{R} D_{45}^{ij}) v^i + \sum_{i=1}^{N+1} \bar{D}_{55}^{ij} u^i$$

$$+ \sum_{i=1}^{N+1} D_{55}^{ij} w^i_{,x_1} + \sum_{i=1}^{N+1} D_{45}^{ij} w^i_{,x_2}$$

$$K_2^j = \sum_{i=1}^{N+1} (\bar{D}_{44}^{ij} - \frac{1}{R} D_{44}^{ij}) v^i + \sum_{i=1}^{N+1} \bar{D}_{45}^{ij} u^i$$

$$+ \sum_{i=1}^{N+1} D_{45}^{ij} w^i_{,x_1} + \sum_{i=1}^{N+1} D_{44}^{ij} w^i_{,x_2}$$

$$Q_1^j = \sum_{i=1}^{N+1} (\bar{\bar{D}}_{45}^{ij} - \frac{1}{R} \bar{D}_{45}^{ji}) v^i + \sum_{i=1}^{N+1} \bar{\bar{D}}_{55}^{ij} u^i$$

$$+ \sum_{i=1}^{N+1} \bar{D}_{55}^{ji} w^i_{,x_1} + \sum_{i=1}^{N+1} \bar{D}_{45}^{ji} w^i_{,x_2}$$

$$Q_2^j = \sum_{i=1}^{N+1} (\bar{\bar{D}}_{44}^{ij} - \frac{1}{R} \bar{D}_{44}^{ji}) v^i + \sum_{i=1}^{N+1} \bar{\bar{D}}_{45}^{ij} u^i$$

$$+ \sum_{i=1}^{N+1} \bar{D}_{45}^{ji} w^i_{,x_1} + \sum_{i=1}^{N+1} \bar{D}_{44}^{ji} w^i_{,x_2}$$

$$Q_3^j = \sum_{i=1}^{N+1} \bar{D}_{13}^{ji} u^i_{,x_1} + \sum_{i=1}^{N+1} \bar{D}_{23}^{ji} v^i_{,x_2} + \sum_{i=1}^{N+1} \bar{D}_{36}^{ji} u^i_{,x_2}$$

$$+ \sum_{i=1}^{N+1} \bar{D}_{36}^{ji} v^i_{,x_1} + \sum_{i=1}^{N+1} (\bar{\bar{D}}_{33}^{ij} + \frac{1}{R} \bar{D}_{23}^{ji}) w^i, \quad j = 1, \ldots, N+1$$

(3–18a)

where

$$D_{pq}^{ij} = D_{pq}^{ji} = \sum_{k=1}^{N} \int_{z_{k+1}}^{z_k} \bar{C}_{pq}^k \phi^i \phi^j dz \quad , \quad p,q = 1,2,4,5,6$$

$$\bar{D}^{ij}_{pq} = \sum_{k=1}^{N} \int_{z_{k+1}}^{z_k} \bar{C}^{k}_{pq} \frac{d\phi^i}{dz} \phi^j dz \neq \bar{D}^{ji}_{pq} \quad , \quad p,q = \overline{1,6}$$

$$\bar{\bar{D}}^{ij}_{pq} = \bar{\bar{D}}^{ji}_{pq} = \sum_{k=1}^{N} \int_{z_{k+1}}^{z_k} \bar{C}^{k}_{pq} \frac{d\phi^i}{dz} \frac{d\phi^j}{dz} dz \quad , \quad p,q = 4,5 \tag{3–18b}$$

By evaluating the integrals in Eqs. (3–18b), results similar to those in Eqs. (3–17) can readily be obtained (in short, $\bar{Q}_{pq}$ in Eqs. (3–17) are simply replaced by $\bar{C}_{pq}$).

With the help of the expressions in Eqs. (3–16a) and (3–18a), the governing equations in LST and GLST can be expressed explicitly in terms of the generalized displacement components. This, however, will be done in Chapter V where we consider the stability problem of anisotropic shells.

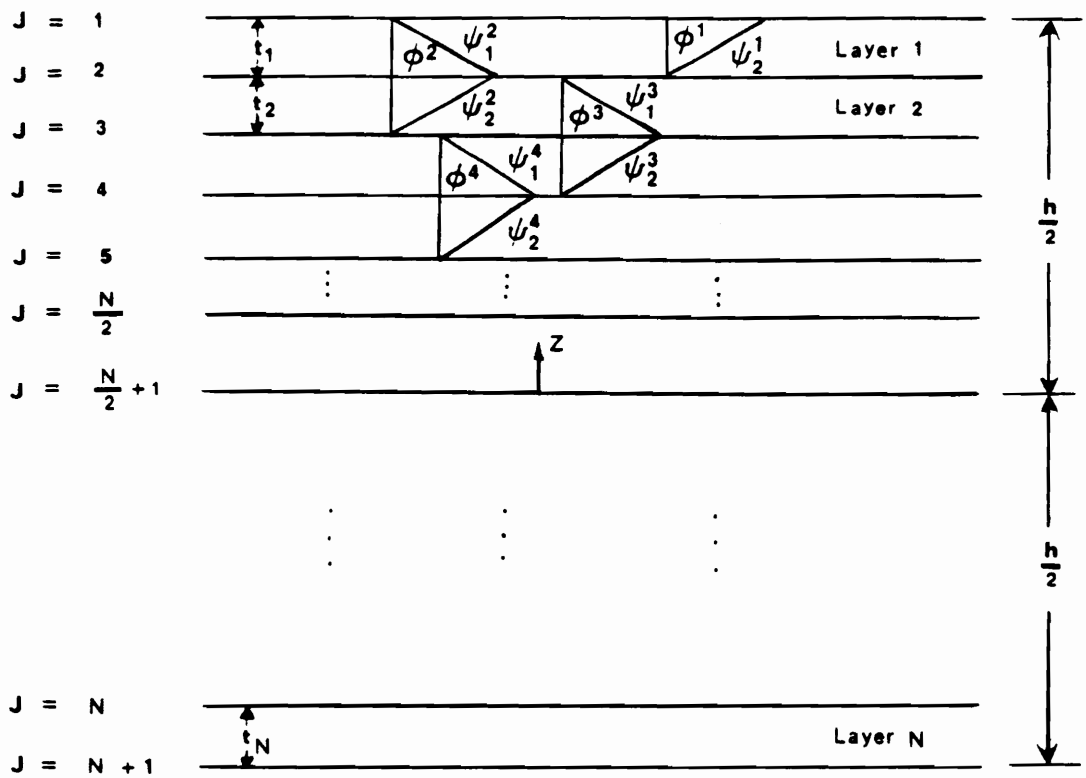

Figure 3.1 The thickness functions used in the displacement expansions (they represent the global linear interpolation functions)

CHAPTER IV

AN ANALYTICAL PROCEDURE FOR FREE VIBRATION AND STABILITY ANALYSES OF CROSS–PLY LAMINATED CIRCULAR CYLINDRICAL SHELLS

Laminated circular cylindrical shells are increasingly used as basic elements in lightweight aerospace structures. Therefore, it is of practical importance to have an analytical technique for the free–vibration and stability analyses of such elements. It has long been known that the classical Love–Kirchhoff laminated shell theories, in which transverse shear deformations are neglected, overpredict natural frequencies and critical buckling loads.

Recently, a general analytical method has been used by Khdeir and Reddy [102] for the study of free–vibration and buckling problems of laminated circular cylindrical shells with various boundary conditions and using the third–order shear deformation shell theory of Reddy and Liu [81]. Numerical results were compared with the ones obtained from the first–order shear deformation theory and the classical theory. The same technique was previously used by Khdeir et al. [95–101], to generate Lévy–type solutions of various shear deformation theories. In [95–102], the resulting system of ordinary differential equations are replaced by a new system of first–order equations by introducing state variables, similar to the approach used in the modern control theory. Subsequently, a formal solution is adopted for the state variables. The method is general and is particularly useful for the shear deformation theories where the number of governing equations is generally greater than that of the classical theory. The advantage of the method is that no assumption is made concerning the nature of the roots of the auxiliary equation. The roots can have complex values and need not be monitored during computation.

The major drawback of the method, however, is that the characteristic determinant, obtained by imposing the boundary conditions, becomes ill–conditioned when the ratio of the characteristic length of the structure to its thickness (ℓ/h) is near or larger than 20. Indeed, no numerical results for natural frequencies and critical buckling loads are presented in [95–102] for ℓ/h larger than 20.

The objective of the present chapter is to present a new analytical procedure for the analysis of free–vibration and buckling problems of circular cylindrical shells with various boundary conditions. The proposed method will remove the aforementioned drawback of the method used in [95–102]. The method has a simpler form and is, therefore, computationally more efficient for generating Lévy–type solutions for frequency and buckling problems of plates and shell panels.

4.1.1 Equations of Motion

The linear equations of motion of the first–order shear deformation shell theory (FSDST) of a laminated circular cylindrical shell are readily obtained from Eqs. (2–3) by letting $\lambda = 0$, $\frac{1}{R_2} = 0$, $R_1 = R$, and dropping the nonlinear term:

$$\frac{\partial N_1}{\partial x_1} + \frac{\partial N_6}{\partial x_2} = I_1\ddot{u} + I_2\ddot{\phi}_1$$

$$\frac{\partial N_6}{\partial x_1} + \frac{\partial N_2}{\partial x_2} = I_1\ddot{v} + I_2\ddot{\phi}_2$$

$$\frac{\partial M_1}{\partial x_1} + \frac{\partial M_6}{\partial x_2} - Q_1 = I_2\ddot{u} + I_3\ddot{\phi}_1$$

$$\frac{\partial M_6}{\partial x_1} + \frac{\partial M_2}{\partial x_2} - Q_2 = I_2\ddot{v} + I_3\ddot{\phi}_2$$

$$\frac{\partial Q_1}{\partial x_1} + \frac{\partial Q_2}{\partial x_2} - \frac{N_2}{R} - \bar{N}\,\frac{\partial^2 w}{\partial x_1^2} = I_1\ddot{w} \tag{4–1}$$

where R is the radius of the cylinder; $\bar{N}$ is the axial compressive load (positive in compression); u, v, and w are the displacement components along the x_1 – ,x_2 –, and ξ–axes (see Fig. 4.1); ϕ_1 and ϕ_2 are the rotation functions; and a superposed dot indicates differentiation with respect to time t. For a general cross–ply laminate (i.e., laminate with stiffnesses $A_{16} = A_{26} = A_{45} = B_{16} = B_{26} = D_{16} = D_{26} = 0$), the stress resultants are obtained from Eqs. (2–6):

$$N_1 = A_{11}\epsilon_1^o + A_{12}\epsilon_2^o + B_{11}\kappa_1^o + B_{12}\kappa_2^o$$

$$N_2 = A_{21}\epsilon_1^o + A_{22}\epsilon_2^o + B_{21}\kappa_1^o + B_{22}\kappa_2^o$$

$$N_6 = A_{66}\epsilon_6^o + B_{66}\kappa_6^o$$

$$M_1 = B_{11}\epsilon_1^o + B_{12}\epsilon_2^o + D_{11}\kappa_1^o + D_{12}\kappa_2^o$$

$$M_2 = B_{21}\epsilon_1^o + B_{22}\epsilon_2^o + D_{21}\kappa_1^o + D_{22}\kappa_2^o$$

$$M_6 = B_{66}\epsilon_6^o + D_{66}\kappa_6^o$$

$$Q_1 = K_{55}^2 A_{55} \epsilon_5^o$$

$$Q_2 = K_{44}^2 A_{44} \epsilon_4^o \tag{4–2}$$

where K_{44}^2 and K_{55}^2 are the shear correction factors. In Eqs. (4–1) and (4–2), the mass terms and strain and curvature measures are defined as:

$$(I_1, I_2, I_3) = \sum_{k=1}^{N} \int_{\xi_{k+1}}^{\xi_k} \rho^{(k)}(1, \xi, \xi^2) d\xi \tag{4–3}$$

where N denotes the total number of layers and $\rho^{(k)}$ is the material mass density of k^{th} layer (see Fig. 4.1) and

$$\epsilon_1^o = \frac{\partial u}{\partial x_1}, \; \epsilon_2^o = \frac{\partial v}{\partial x_2} + \frac{w}{R}, \; \epsilon_4^o = \phi_2 + \frac{\partial w}{\partial x_2}$$

$$\epsilon_5^o = \phi_1 + \frac{\partial w}{\partial x_1}, \; \epsilon_6^o = \frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2}, \; \kappa_1^o = \frac{\partial \phi_1}{\partial x_1}$$

$$\kappa_2^o = \frac{\partial \phi_2}{\partial x_2}, \; \kappa_6^o = \frac{\partial \phi_2}{\partial x_1} + \frac{\partial \phi_1}{\partial x_2}. \tag{4–4}$$

The equations of motion of the classical shell theory (CLST) are obtained from Eqs. (4–1) by setting

$$\phi_1 = -\frac{\partial w}{\partial x_1} \text{ and } \phi_2 = -\frac{\partial w}{\partial x_2}. \tag{4–5}$$

Finally, the equations of motion of FSDST and CLST can be expressed in terms of generalized displacement components and presented as

$$[L]\{\Delta\} = \{0\} \tag{4–6}$$

by substituting Eqs. (4–4) and (4–5) into Eqs. (4–2) and the subsequent results into Eqs. (4–1). The linear operators L_{ij} for FSDST and CLST are displayed in Appendix 4.1. Moreover

$$\{\Delta\} = \{u, v, \phi_1, \phi_2, w\}^T \tag{4–7}$$

for FSDST and

$$\{\Delta\} = \{u, v, w\}^T \tag{4–8}$$

for CLST.

4.1.2 Solution of the Governing Equations

For the circular cylindrical shell with arbitrary boundary conditions at $x_1 = \pm L/2$, we assume the following representations for the generalized displacement components:

$$\begin{Bmatrix} u \\ v \\ \phi_1 \\ \phi_2 \\ w \end{Bmatrix} = \begin{Bmatrix} U_m(x_1)\cdot\cos\beta_m x_2 \\ V_m(x_1)\cdot\sin\beta_m x_2 \\ X_m(x_1)\cdot\cos\beta_m x_2 \\ Y_m(x_1)\cdot\sin\beta_m x_2 \\ W_m(x_1)\cdot\cos\beta_m x_2 \end{Bmatrix}\cdot T_m(t) \tag{4–9}$$

where $T_m = e^{i\omega_m t}$ in eigenfrequency analysis and $T_m(t) = 1$ in stability analysis and ω_m is the natural frequency corresponding to m^{th} mode (we keep in mind that there are denumerable infinite frequencies for each value of m); $\beta_m = \frac{m}{R}$ with m = 0,1,2,... and $i = \sqrt{-1}$. Substitution of Eqs. (4–9) into Eqs. (4–6) results in five and three coupled ordinary differential equations for FSDST and CLST, respectively. Since the proposed solution technique of these equations will be general, we only present the equations of FSDST and include the numerical results of CLST for the sake of comparison:

$$U_m'' = C_1U_m + C_2V_m' + C_3X_m + C_4Y_m' + C_5W_m'$$

$$V_m'' = C_6U_m' + C_7V_m + C_8X_m' + C_9Y_m + C_{10}W_m$$

$$X_m'' = C_{11}U_m + C_{12}V_m' + C_{13}X_m + C_{14}Y_m' + C_{15}W_m'$$

$$Y_m'' = C_{16}U_m' + C_{17}V_m + C_{18}X_m' + C_{19}Y_m + C_{20}W_m$$

$$W_m'' = C_{21}U_m' + C_{22}V_m + C_{23}X_m' + C_{24}Y_m + C_{25}W_m \qquad (4\text{–}10)$$

where a prime indicates a derivative with respect to x_1. The coefficients C_j ($j = \overline{1,25}$) in Eqs. (4–10) are given in Appendix 4.2. With some simple algebraic operations (addition and subtraction) it is made sure that only one unknown variable with highest derivative appears in Eqs. (4–10). This will save the computational time required in the method that we will introduce for solving Eqs. (4–10).

There exist a number of ways to solve a system of ordinary differential equations. However, when there are more than three governing equations, as in Eqs. (4–10), it is more practical to introduce new unknown variables and replace the original system of equations by an equivalent system of first–order equations. We introduce the following variables:

$$Z_{1m}(x_1) = U_m(x_1) \ , \ Z_{2m}(x_1) = U'_m(x_1) = Z'_{1m}(x_1)$$

$$Z_{3m}(x_1) = V_m(x_1) \ , \ Z_{4m}(x_1) = V'_m(x_1) = Z'_{3m}(x_1)$$

$$Z_{5m}(x_1) = X_m(x_1) \ , \ Z_{6m}(x_1) = X'_m(x_1) = Z'_{5m}(x_1)$$

$$Z_{7m}(x_1) = Y_m(x_1) \ , \ Z_{8m}(x_1) = Y'_m(x_1) = Z'_{7m}(x_1)$$

$$Z_{9m}(x_1) = W_m(x_1) \ , \ Z_{10m}(x_1) = W'_m(x_1) = Z'_{9m}(x_1) \qquad (4\text{–}11)$$

for $m = 1,2,...$ Substituting Eqs. (4–11) into Eqs. (4–10), the resulting equations along with Eqs. (4–11) can be expressed in the form:

$$\{Z'\} = [A]\{Z\} \qquad (4\text{–}12)$$

where

$$\{Z'\} = \{Z'_{1m}, Z'_{2m}, ..., Z'_{10m}\}^T, \qquad (4\text{–}13)$$

$$\{Z\} = \{Z_{1m}, Z_{2m}, ..., Z_{10m}\}^T, \qquad (4\text{–}14)$$

and the coefficient matrix [A] is

$$[A] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_1 & 0 & 0 & C_2 & C_3 & 0 & 0 & C_4 & 0 & C_5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_6 & C_7 & 0 & 0 & C_8 & C_9 & 0 & C_{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ C_{11} & 0 & 0 & C_{12} & C_{13} & 0 & 0 & C_{14} & 0 & C_{15} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & C_{16} & C_{17} & 0 & 0 & C_{18} & C_{19} & 0 & C_{20} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & C_{21} & C_{22} & 0 & 0 & C_{23} & C_{24} & 0 & C_{25} & 0 \end{bmatrix} \qquad (4\text{–}15)$$

A formal solution of Eqs. (4–13) (see [114,115]) is given as

$$\{Z(x_1)\} = \Psi(x_1)\{D\} \qquad (4\text{–}16)$$

where $\Psi(x_1)$ is a fundamental matrix whose columns consist of ten linearly independent solutions of Eqs. (4–12) and $\{D\}$ is an unknown constant vector. Some or all components of this vector, as will be seen later, are in general complex. The nonsingular fundamental matrix $\Psi(x_1)$ is not unique. All fundamental matrices, however, differ from each other by a multiplicative constant matrix. Since Eqs. (4–16) are the solutions of Eqs. (4–12) and $\Psi(0)$ is a nonsingular constant matrix, from Eqs. (4–16) a special fundamental matrix $\Phi(x_1)$ (known as the state transition matrix) for Eqs. (4–12) can be defined such that

$$\{Z(x_1\} = \Phi(x_1)\{Z(0)\} \qquad (4\text{–}17)$$

are also the solutions of Eqs. (4–12), with

$$\Phi(x_1) = \Psi(x_1)\Psi^{-1}(0) \qquad (4\text{–}18a)$$

and

$$\{D\} = \Psi^{-1}(0)\{Z(0)\}. \tag{4–18b}$$

Since [A] is a constant matrix, the state transition matrix is given by a matrix exponential function as

$$\Phi(x_1) = e^{[A]x_1} \tag{4–19}$$

By imposing the ten boundary conditions at $x_1 = \pm L/2$ on the solution given by Eqs. (4–17), a homogeneous system of algebraic equations can be found:

$$[M]\{Z(0)\} = \{0\}. \tag{4–20}$$

For nontrivial solution of natural frequency or critical buckling load, the determinant of the coefficient matrix [M] must be set to zero:

$$|M| = 0. \tag{4–21}$$

Since the constant vector {Z(0)} is real, the determinant of [M] is also real. Hence, in a trial and error procedure, one can easily find the correct value of natural frequency (in a free–vibration problem) or of the critical buckling load (in a stability problem) which would make $|M| = 0$. A nonzero compressive (or tensile) edge load can also be included in the free–vibration analysis.

Numerous methods are available (e.g., see [114,115]) for determining the matrix exponential $e^{[A]x_1}$ appearing in Eq. (4–19). However, regardless of any method used, it is found that $|M|$ becomes ill–conditioned when the ratio of the characteristic length of the structure to its thickness is near or larger than 20. This is also the case in the Lévy–type eigenfrequency and stability problems of laminated plates and shell panels when shear deformation theories are used.

When the eigenvalues of the coefficient matrix [A] are distinct, the fundamental matrix $\Psi(x_1)$ is given by:

$$\Psi(x_1) = [U][Q(x_1)] \tag{4–22}$$

where $[Q(x_1)]$ is another fundamental matrix defined as

$$[Q(x_1)] = \begin{bmatrix} e^{\lambda_1 x_1} & & & 0 \\ & e^{\lambda_2 x_1} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_{10} x_1} \end{bmatrix} \tag{4-23}$$

and [U] is a modal matrix that transforms [A] into a diagonal form (i.e., the j^{th} column of [U] constitutes the eigenvectors of [A] corresponding to the j^{th} eigenvalue of [A]). In Eq. (4–23), $\lambda_j (j = \overline{1,10})$ are the distinct eigenvalues of [A], which in general can be real and complex. We note that the eigenvalues of [A] are the same as the roots of the auxiliary equation of Eqs. (4–10). These eigenvalues in most eigenfrequency and stability problems of plates and shells are distinct. The axisymmetric buckling problem and axisymmetric eigenfrequency problem (when all inertia forces, except the radial inertia force, are neglected) of a cylindrical shell are two examples where two of the eigenvalues, as will be seen, are identical. When the eigenvalues are repeated, Eq. (4–23) is no longer valid and a Jordan canonical form (see [114,115]) of [A] must be used.

Substituting Eqs. (4–22) into Eqs. (4–18a) and (4–18b) yields

$$\Phi(x_1) = [U][Q(x_1)][U]^{-1} \tag{4-24}$$

and

$$\{D\} = [U]^{-1}\{Z(0)\}. \tag{4-25}$$

Equations (4–17) and (4–24) were used in previous works [94–102] related to stability and vibration problems of laminated plates and shells. However, due to occurrence of an ill–conditioned determinant $|M|$ in Eq. (4–21), we propose the following approach. Instead of imposing the boundary conditions on Eqs. (4–17), we can impose the boundary conditions on Eqs. (4–16) which have a simpler form. This way we come up with a set of homogeneous algebraic equations of the form

$$[K]\{D\} = \{0\}. \tag{4–26}$$

For a nontrivial solution of Eqs. (4–26) to exist, the determinant of the generally complex coefficient matrix [K] must vanish. Since $|K|$ in general can be complex, it may be computationally more convenient to substitute Eqs. (4–25) into Eqs. (4–26) to get

$$[K][U]^{-1}\{Z(0)\} = \{0\} \tag{4–27a}$$

and set the coefficient matrix in Eqs. (4–27a) to zero

$$|([K][U]^{-1})| = 0 \tag{4–27b}$$

This way the determinant in Eq. (4–27b) will always be a real number. However, it will have the same computational problem as $|M|$ in Eq. (4–21). This is not a surprising result since

$$|M| = |([K][U]^{-1})|. \tag{4–28}$$

The key point in overcoming this difficulty is to rewrite Eq. (4–27b) as

$$\frac{|K|}{|U|} = 0 \tag{4–29}$$

that is, to evaluate the determinants of [K] and [U] separately, rather than evaluating the determinant of $([K][U]^{-1})$. It should also be noted that, this way, the inverse of [U] is never needed. For very thin shells (or long shells), computer overflow and underflow may occur when we evaluate the elements of the coefficient matrix [K]. This problem is addressed in detail in Appendix 4.3. It should, however, be kept in mind that the determinant of [K] never becomes ill–conditioned. In summary, after assuming a trial value for natural frequency (in free–vibration analysis) or for buckling load (in stability analysis) for a particular m, we will impose the ten boundary conditions at $x_1 = \pm L/2$ on Eqs. (4–16) and come up with Eqs. (4–26) and check if Eq. (4–29) is satisfied. It should be reminded that $|U|$ appearing in Eq. (4–29) is never zero since the eigenvectors in [U] are independent of each other.

A remark must be made concerning the computation of eigenvalues and eigenvectors of the coefficient matrix [A]. Since the diagonal elements of [A] are all zero, during the computation of eigenvalues and eigenvectors computer overflow or underflow may occur. To resolve this problem, we can subtract a nonzero constant number from the diagonal elements of [A] and compute the eigenvalues and eigenvectors of the new matrix. The eigenvalues of [A] can then be obtained by adding the same number to each eigenvalue of the new matrix. The eigenvectors of [A] will be identical to those of the new matrix (see page 52 of [116], for example).

4.1.3 Boundary Conditions

In theory, a combination of boundary conditions can be assumed to exist at the edges of the shell. We classify these boundary conditions for FSDST according to:

S1: $\quad w = M_1 = \phi_2 = N_1 = N_6 = 0$

S2: $\quad w = M_1 = \phi_2 = u = N_6 = 0$

S3: $\quad w = M_1 = \phi_2 = N_1 = v = 0$

$$\text{S4:} \quad w = M_1 = \phi_2 = u = v = 0 \tag{4–30}$$

for simply–supported edges and

$$\text{C1:} \quad w = \phi_1 = \phi_2 = N_1 = N_6 = 0$$

$$\text{C2:} \quad w = \phi_1 = \phi_2 = u = N_6 = 0$$

$$\text{C3:} \quad w = \phi_1 = \phi_2 = N_1 = v = 0$$

$$\text{C4:} \quad w = \phi_1 = \phi_2 = u = v = 0 \tag{4–31}$$

for clamped edges. For a free edge we assume

$$\text{F:} \quad N_1 = N_6 = M_1 = M_6 = Q_1 - \bar{N}\frac{\partial w}{\partial x_1} = 0. \tag{4–32}$$

Similar boundary conditions are also classified for CLST and displayed in Appendix 4.1. The boundary type S3 is referred to as shear diaphragm by Leissa [117].

Up to this point, it was assumed that $m \neq 0$. Next we will consider the axisymmetric mode (i.e., when $m = 0$).

4.2.1. Axisymmetric Problem

When $m = 0$, Eqs. (4–9) and (4–10) will reduce to

$$v = \phi_2 = 0 \tag{4–33a}$$

$$(u,\phi_1,w) = (U_o,X_o,W_o)\cdot T_o(t) \tag{4–33b}$$

and

$$U_o'' = \bar{C}_1U_o + \bar{C}_2X_o + \bar{C}_3W_o'$$

$$X_o'' = \bar{C}_4U_o + \bar{C}_5X_o + \bar{C}_6W_o'$$

$$W_o'' = \bar{C}_7U_o' + \bar{C}_8X_o' + \bar{C}_9W_o \tag{4–34}$$

where $T_o(t) = e^{i\omega_o t}$ and 1, respectively, in free–vibration and stability problems, and the coefficients $\bar{C}_j(j = \overline{1,9})$ are given in Appendix 4.4. The stress resultants reduce to

$$N_1 = A_{11}\frac{\partial u}{\partial x_1} + \frac{A_{12}}{R}w + B_{11}\frac{\partial \phi_1}{\partial x_1} \tag{4–35a}$$

$$M_1 = B_{11}\frac{\partial u}{\partial x_1} + \frac{B_{12}}{R}w + D_{11}\frac{\partial \phi_1}{\partial x_1} \tag{4–35b}$$

$$Q_1 = K_{55}^2A_{55}(\phi_1 + \frac{\partial w}{\partial x_1}) \tag{4–35c}$$

$$N_6 = M_6 = Q_2 = 0. \tag{4–35d}$$

In stability analysis, the partial derivatives in Eqs. (4–35) should be replaced by the total derivatives. Also the boundary conditions for FSDST are reduced to:

S1: $w = M_1 = N_1 = 0$, S2: $w = M_1 = u = 0$

$$\text{C1:} \quad w = \phi_1 = N_1 = 0 \;\; , \;\; \text{C2:} \;\; w = \phi_1 = u = 0$$

$$\text{F:} \quad N_1 = M_1 = Q_1 - \bar{N}\,\frac{\partial w}{\partial x_1} = 0, \tag{4–36}$$

with S3, S4, C3, and C4 cases being identical to S1, S2, C1, and C2 cases, respectively. The boundary conditions for CLST are given in Appendix 4.1.

In the eigenfrequency analysis, with all the inertia forces preserved, the eigenvalues will again be distinct and, therefore, a solution procedure similar to the one suggested for Eqs. (4–10) can be used. For the sake of completeness we will present the numerical results for this case. However, in the stability problem and eigenfrequency problem (with all the inertia forces, except the radial inertia force, neglected) the eigenvalues will no longer be distinct. Here, for such problems, we adopt two different approaches which will, of course, yield identical results.

4.2.2 Approach I

When only the radial inertia force is retained in Eqs. (4–34), we have (see Appendix 4.4)

$$\bar{C}_1 = \bar{C}_4 = 0 \tag{4–37}$$

as in the stability problem. Here, without altering the form of Eqs. (4–34) and assuming

$$(U_o, X_o, W_o) = (A, B, C)e^{\lambda x_1} \tag{4–38}$$

where A, B, and C are unknown complex constants, Eqs. (4–34) yield

$$\begin{bmatrix} \lambda^2 & -\bar{C}_2 & -\bar{C}_3\lambda \\ 0 & \lambda^2-\bar{C}_5 & -\bar{C}_6\lambda \\ -\bar{C}_7\lambda & -\bar{C}_8\lambda & \lambda^2-\bar{C}_9 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{4–39}$$

Expansion of the determinant of the coefficient matrix in Eqs. (4–39) yields a sixth–order auxiliary equation of the form

$$\lambda^2[(\lambda^2)^2 + \theta_1(\lambda^2) + \theta_2] = 0 \tag{4–40}$$

whose roots are given by

$$\lambda_1 = \lambda_2 = 0,$$

$$\lambda_3 = \tfrac{1}{2}[\theta_1 - (\theta_1^2 - 4\theta_2)^{1/2}]^{1/2},\ \lambda_4 = -\lambda_3,$$

$$\lambda_5 = \tfrac{1}{2}[\theta_1 + (\theta_1^2 - 4\theta_2)^{1/2}]^{1/2},\ \text{and } \lambda_6 = -\lambda_5. \tag{4–41}$$

The coefficients $\theta_j(j = \overline{1,2})$ are displayed in Appendix 4.4. Based on Eqs. (4–38) and (4–41), a general solution for Eqs. (4–34) can be written with eighteen unknown constants. The relationship among the constants is established by substituting the general solution back into Eqs. (4–34). The general solution may then be presented as:

$$U_o(x_1) = A_1 + A_2x_1 + \sum_{j=3}^{6} A_j e^{\lambda_j x_1}$$

$$X_o(x_1) = \sum_{j=1}^{6} A_j u_{2j} e^{\lambda_j x_1}$$

$$W_o(x_1) = A_2 u_{32} + \sum_{j=3}^{6} A_j u_{3j} e^{\lambda_j x_1} \tag{4–42}$$

where

$$u_{32} = -\frac{\bar{C}_7}{\bar{C}_9},\ u_{2j} = \frac{\bar{C}_6 \lambda_j^2}{\bar{C}_3 \lambda_j^2 + (\bar{C}_2 \bar{C}_6 - \bar{C}_3 \bar{C}_5)},$$

and

$$u_{3j} = \frac{\lambda_j (\lambda_j^2 - \bar{C}_5)}{\bar{C}_3 \lambda_j^2 + (\bar{C}_2 \bar{C}_6 - \bar{C}_3 \bar{C}_5)}\ (j = \overline{3,6}), \tag{4–43}$$

with

$$u_{23} = u_{24}\ ,\ u_{25} = u_{26},$$

$$u_{33} = u_{34}\ ,\ u_{35} = u_{36}. \tag{4–44}$$

Equations (4–42) can be presented in a matrix form as:

$$\begin{Bmatrix} U_o \\ U_o' \\ X_o \\ X_o' \\ W_o \\ W_o' \end{Bmatrix} = [U] \begin{bmatrix} 1 & x_1 & & & & \\ & 1 & & & & \\ & & e^{\lambda_3 x_1} & & \underset{\sim}{0} & \\ & & & \ddots & & \\ & \underset{\sim}{0} & & & & \\ & & & & & e^{\lambda_6 x_1} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_6 \end{Bmatrix} \tag{4–45}$$

where

$$[U] = \begin{bmatrix} 1 & 0 & 1 & \dots & 1 \\ 0 & 1 & \lambda_3 & \dots & \lambda_6 \\ 0 & 0 & u_{23} & \dots & u_{26} \\ 0 & 0 & u_{23}\lambda_3 & \dots & u_{26}\lambda_{26} \\ 0 & u_{32} & u_{33} & \dots & u_{36} \\ 0 & 0 & u_{33}\lambda_3 & \dots & u_{36}\lambda_6 \end{bmatrix}. \tag{4–46}$$

From Eqs. (4–45) we have

$$\begin{Bmatrix} A_1 \\ \vdots \\ A_6 \end{Bmatrix} = [U]^{-1}\{U_o(0),U_o'(0),X_o(0),X_o'(0),W_o(0),W_o'(0)\}^T. \tag{4–47}$$

When the boundary conditions (see Eqs. (4–36)) involve the specification of the axial displacement u at least at one end of the shell, a system of six homogeneous algebraic equations of the form

$$[K]\{A\} = \{0\} \tag{4–48}$$

is obtained. Since $A_j(j = \overline{1,6})$ have, in general, complex values, the determinant of [K] may also be complex. However, as we mentioned before, it may be computationally more convenient to find the zeros of a real–valued determinant. For this reason, we substitute Eqs. (4–47) into Eqs. (4–48) and set the determinant of the new coefficient matrix to zero:

$$\frac{|K|}{|U|} = 0 \tag{4–49}$$

As in the non–axisymmetric problems, Eq. (4–49) is the condition which must be satisfied for nontrivial solution.

When $N_1 = 0$ is imposed at both ends of the shell (this will be the case when any combination of S1, C1, and F boundary conditions are imposed), then a system of five homogeneous algebraic equations of the form

$$[\bar{K}]\begin{Bmatrix} A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_6 \end{Bmatrix} = \{0\} \tag{4–50}$$

are obtained. This happens because both the boundary conditions

$$N_1\Big|_{x_1=L/2} = 0 \text{ and } N_1\Big|_{x_1=-L/2} = 0 \tag{4–51}$$

yield the same result

$$A_2 = 0. \tag{4–52}$$

This can readily be seen by substituting Eqs. (4–33b), (4–42), (4–43), and the expressions of $\bar{C}_j(j = \overline{1,9})$ into Eqs. (4–35a). When this is done, we obtain:

$$N_1 = [(A_{11} - \frac{A_{12}^2}{A_{22}})T_o(t)]A_2 \tag{4–53}$$

Equation (4–52) will therefore be the outcome when the boundary conditions in Eq. (4–51) are imposed on Eq. (4–53).

For a nontrivial solution we must set the determinant of the coefficient matrix in Eqs. (4–50) to zero. However, from Eqs. (4–45) we first write

$$\begin{Bmatrix} A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_6 \end{Bmatrix} = [\bar{U}]^{-1}\{U_o'(0),X_o(0),X_o'(0),W_o(0),W_o'(0)\}^T \tag{4–54}$$

where

$$[\bar{U}] = \begin{bmatrix} 1 & \lambda_3 & \cdots & \lambda_6 \\ 0 & u_{23} & \cdots & u_{26} \\ 0 & u_{23}\lambda_3 & \cdots & u_{26}\lambda_6 \\ u_{32} & u_{33} & \cdots & u_{36} \\ 0 & u_{33}\lambda_3 & \cdots & u_{36}\lambda_6 \end{bmatrix}. \tag{4–55}$$

Again, by substituting Eqs. (4–55) into (4–50) we obtain the condition for nontrivial solution

$$\frac{|\bar{K}|}{|\bar{U}|} = 0. \tag{4–56}$$

It should be noted that $|U| = |\bar{U}|$. Indeed, in theory, from Eqs. (4–42) an infinite number of transformation matrices similar to [U] and $[\bar{U}]$ can be developed whose determinants can be used in place of $|U|$ and $|\bar{U}|$ in Eqs. (4–49) and (4–56). A similar argument can also be made as far as $|U|$ appearing in Eq. (4–29) is concerned.

4.2.3 Approach II

Equations (4–29), (4–49), and (4–56) are known as the analytic frequency (or buckling) determinants. It is relatively cumbersome to expand these determinants to find the corresponding frequency (or buckling) equations. However, in axisymmetric free–vibration and stability problems, it is possible to come up with the explicit expressions of frequency and buckling equations by rewriting Eqs. (4–42) as:

$$U_o(x_1) = E_1 + E_2x_1 + E_3 \cosh\lambda_3x_1 + E_4 \sinh \lambda_3x_1 + E_5 \cosh\lambda_5x_1 + E_6 \sinh\lambda_5x_1$$

$$X_o(x_1) = E_3u_{23} \cosh\lambda_3x_1 + E_4u_{24} \sinh\lambda_3x_1 + E_5u_{25} \cosh\lambda_5x_1 + E_6u_{26} \sinh\lambda_5x_1$$

$$W_o(x_1) = E_2u_{32} + E_3u_{33} \sinh\lambda_3x_1 + E_4u_{34} \cosh\lambda_3x_1 + E_5u_{35} \sinh\lambda_5x_1$$

$$+ E_6u_{36} \cosh\lambda_5x_1 \tag{4.57}$$

where $E_j(j = \overline{1,6})$ are six unknown arbitrary constants.

When the boundary conditions at both ends of the shell are the same, the radial displacement of shell is either symmetric or antisymmetric with respect to the cross section of the shell at $x_1 = 0$. For antisymmetric and symmetric deformations we set $E_2 = E_4 = E_6 = 0$ and $E_1 = E_3 = E_5 = 0$, respectively, and impose the boundary conditions only at one end of the shell. In summary, the frequency and the buckling equations that are obtained for various boundary conditions are as follows:

(a) <u>Symmetric Mode</u>

S1–S1:

$$f_1 \equiv [u_{34}(B_{11} + D_{11}u_{26})\lambda_5 - u_{36}(B_{11} + D_{11}u_{24})\lambda_3]\cdot\cosh\alpha_3 \cosh\alpha_5 = 0 \tag{4–58a}$$

S2–S2:

$$u_{32}(B_{11} + D_{11}u_{24})\lambda_3 \cosh\alpha_3 \sinh\alpha_5 - u_{32}(B_{11} + D_{11}u_{26})\lambda_5 \sinh\alpha_3 \cosh\alpha_5$$

$$+ \frac{L}{2}[u_{34}(B_{11} + D_{11}u_{26})\lambda_5 - u_{36}(B_{11} + D_{11}u_{24})\lambda_3]\cosh\alpha_3 \cosh\alpha_5 = 0 \tag{4–58b}$$

C1–C1:

$$g_1 \equiv u_{34}u_{26} \cosh\alpha_3 \sinh\alpha_5 - u_{36}u_{24} \sinh\alpha_3 \cosh\alpha_5 = 0 \tag{4–58c}$$

C2–C2:

$$u_{34}u_{26} \cosh\alpha_3 \sinh\alpha_5 - u_{36}u_{24} \sinh\alpha_3 \cosh\alpha_5$$

$$-\frac{2}{L} u_{32}(u_{26} - u_{24}) \sinh\alpha_3 \sinh\alpha_5 = 0 \tag{4–58d}$$

F–F:

$$(B_{11}\lambda_3 + \frac{B_{12}}{R} u_{34} + D_{11}u_{24}\lambda_3)[K_{55}^2A_{55}u_{26} + (K_{55}^2A_{55} - \bar{N})u_{36}\lambda_5]\cosh\alpha_3 \sinh\alpha_5$$

$$- (B_{11}\lambda_5 + \frac{B_{12}}{R} u_{36} + D_{11}u_{26}\lambda_5)[K_{55}^2A_{55}u_{24} + (K_{55}^2A_{55} - \bar{N})u_{34}\lambda_3]\sinh\alpha_3 \cosh\alpha_5 = 0 \tag{4–58e}$$

(b) <u>Antisymmetric Mode</u>

S1–S1 and S2–S2:

$$f_2 \equiv [u_{33}(B_{11} + D_{11}u_{25})\lambda_5 - u_{35}(B_{11} + D_{11}u_{23})\lambda_3]\sinh\alpha_3 \sinh\alpha_5 = 0 \tag{4–59a}$$

C1–C1 and C2–C2:

$$g_2 \equiv u_{33}u_{25} \sinh\alpha_3 \cosh\alpha_5 - u_{35}u_{23} \cosh\alpha_3 \sinh\alpha_5 = 0 \tag{4–59b}$$

F–F:

$$(B_{11}\lambda_3 + \frac{B_{12}}{R} u_{33} + D_{11}u_{23}\lambda_3)[K_{55}^2 A_{55}u_{25} + (K_{55}^2 A_{55} - \bar{N})u_{35}\lambda_5]\sinh\alpha_3 \cosh\alpha_5$$

$$-(B_{11}\lambda_5 + \frac{B_{12}}{R} u_{35} + D_{11}u_{25}\lambda_5)[K_{55}^2 A_{55}u_{23} + (K_{55}^2 A_{55}$$

$$- \bar{N})u_{33}\lambda_3] \sinh\alpha_5 \cosh\alpha_3 = 0 \qquad (4\text{–}59c)$$

where $\alpha_3 = \lambda_3\, L/2$, $\alpha_5 = \lambda_5\, L/2$, and the symbol S1–S1, for example, indicates that the boundary conditions imposed at $x_1 = \pm\, L/2$ are both of the type S1.

In general, Eqs. (4–58) and (4–59) have complex values. However, we can introduce, as before, a transformation matrix whose determinant renders a real value to these equations. For a symmetric mode ($E_1 = E_3 = E_5 = 0$) from Eqs. (4–57) we can write:

$$\begin{Bmatrix} X_o'(0) \\ \\ W_o''(0) \end{Bmatrix} = [U_s] \begin{Bmatrix} A_4 \\ \\ A_6 \end{Bmatrix} \equiv \begin{bmatrix} u_{24}\lambda_3 & u_{26}\lambda_5 \\ \\ u_{34}\lambda_3^2 & u_{36}\lambda_5^2 \end{bmatrix} \begin{Bmatrix} A_4 \\ \\ A_6 \end{Bmatrix} \qquad (4\text{–}60)$$

with

$$|U_s| = \lambda_3\lambda_5(u_{24}u_{36}\lambda_5 - u_{34}u_{26}\lambda_3). \qquad (4\text{–}61)$$

Similarly, for an antisymmetric mode ($E_2 = E_4 = E_6 = 0$) we have

$$\begin{Bmatrix} X_o(0) \\ \\ W_o'(0) \end{Bmatrix} = [U_a] \begin{Bmatrix} A_3 \\ \\ A_5 \end{Bmatrix} \equiv \begin{bmatrix} u_{23} & u_{25} \\ \\ u_{33}\lambda_3 & u_{35}\lambda_5 \end{bmatrix} \begin{Bmatrix} A_3 \\ \\ A_5 \end{Bmatrix} \tag{4–62}$$

with

$$|U_a| = u_{23}u_{35}\lambda_5 - u_{25}u_{33}\lambda_3. \tag{4–63}$$

Finally, for the reason discussed above, we divide Eqs. (4–58a)–(4–58e) and Eqs. (4–59a)–(4–59c) by $|U_s|$ and $|U_a|$, respectively. Instead of hyperbolic functions in Eqs. (4–57) we could have used trigonometric functions had we replaced $e^{\lambda x_1}$ by $e^{i\lambda x_1}$ in Eqs. (4–38). Indeed, this is done by Yamaki [118] who developed buckling equations similar to Eqs. (4–58a)–(4–58d) and Eqs. (4–59a)–(4–58b) for a homogeneous isotropic circular cylindrical shell using Donnell's classical equations of shells. However, in [118], only simply supported and clamped boundary conditions were considered and it was assumed that both ends of the shell have identical boundary conditions. Further, no discussion was presented concerning the transformation matrices that are introduced throughout the present paper. Indeed, in [118] a different approach is adopted for transforming the complex determinants into real ones.

When the boundary conditions at $x_1 = \pm L/2$ are not identical, it is found that the frequency (and buckling) equations for the cases (i) S1–C1, S1–C2, and S2–C1 are identical and different from S2–C2 (ii) S1–F and S2–F are identical and (iii) C1–F and C2–F are identical. For the sake of brevity we will not present these equations. For the remaining cases S1–S2 and C1–C2, it is found that the frequency (and buckling) equations are

$$\frac{f_1 \cdot f_2}{|U_s| \cdot |U_a|} = 0 \tag{4–64}$$

for S1–S2 and

$$\frac{g_1 \cdot g_2}{|U_s| \cdot |U_a|} = 0 \tag{4–65}$$

for C1–C2. Therefore, the natural frequencies and buckling loads for the boundary types S1–S2 and C1–C2 are identical to those of S1–S1 and C1–C1, respectively. All of the results developed in this section are numerically verified by the method introduced in Section 4.2.2.

4.3.1 Numerical Results and Discussions

Numerical results are developed for an orthotropic material with the following properties:

$$E_1/E_2 = 10,\ G_{13} = G_{12} = 0.6E_2,\ G_{23} = 0.5E_2,\ \nu_{12} = 0.25,$$

and for an isotropic material with Poisson's ratio $\nu = 0.25$. It is assumed that $K_{44}^2 = K_{55}^2 = 5/6$ and the total thickness ($\equiv$h) of the shell is equal to 1 inch in all the numerical examples. Further, all layers are assumed to be of equal thickness.

The effect of altering the lamination scheme on the fundamental frequency of a cross–ply shell with various boundary conditions is demonstrated in Table 4.1 (a number in the parentheses denotes the circumferential mode number m). It is observed that, except for the S4–F case, the fundamental frequency for a (90/0) laminated shell is slightly smaller than that of a (0/90) laminated shell. However, an analysis based on the generalized layer–wise shell theory (GLST) indicates that this exception for boundary type S4–F does not occur. The (90/0) designation refers to the case in which the fibers of the outside layer are in the circumferential direction (90°) and those of the inside layer are along the longitudinal axis (0°) of the shell.

The numerical results indicate, unless the shell is extremely short, that the minimum axisymmetric frequency is always quite larger than the fundamental frequency of cross–ply and isotropic shells. This is particularly true for cross–ply shells as can be seen from Table 4.2, where the results are tabulated for cases C1–C1 through C4–C4. It should be noted that the effect of imposing various in–plane boundary conditions is more severe for isotropic shells than cross–ply shells. In the two minimum axisymmetric frequencies presented in Table 4.2, the second number corresponds to the case when only the radial force is preserved. In such a problem it was found that two of the eigenvalues of Eq. (4–40) were equal to zero. Hence it was assumed that, for all trial values of frequency ω_0, Eqs. (4–42) were the solution of Eqs. (4–34). However, it can readily be verified that the coefficient θ_2 appearing in Eq. (4–40) will become zero when the trial value of the frequency ω_0 becomes identical to

$$\omega_0 = \left[\frac{1}{I_1R^2}(A_{22} - A_{12}^2/A_{11})\right]^{1/2}. \tag{4–66}$$

Therefore, Eqs. (4–42) will no longer be valid since two more eigenvalues of Eq. (4–40) will become identically zero. Numerical results indicate that the frequency in Eq. (4–66) is very close to the actual minimum axisymmetric frequency. This is, in fact, demonstrated in Table 4.3 where we tabulated the roots of Eq. (4–40) for several trial values of the frequency ω_0. It should be noted that the last two trial values of the frequency ω_0 in Table 4.3 are actually the minimum axisymmetric frequencies of a (0/90) laminated shell with boundary conditions C1–C1 (and C3–C3) and C2–C2 (and C4–C4), respectively, as can be seen from Table 4.2.

The influence of various simply–supported boundary conditions on the critical buckling loads of laminated and isotropic shells can be studied with the help of Table 4.4. As in the frequency problem, it is seen that various in–plane boundary conditions have a more severe influence on the critical buckling load of isotropic shells. Also the ratio of the minimum axisymmetric buckling load to the critical buckling load is larger in cross–ply shells than in isotropic shells. Indeed, the actual computations indicate that only for extremely short cross–ply shells will the axisymmetric buckling load be the actual critical load. It should be noted that the bending–extension coupling induced by the lamination asymmetry substantially decreases the buckling loads. However, for antisymmetrically laminated shells, the effect of the coupling dies out rapidly as the number of layers is increased as can be seen from the results of Table 4.5. While analytical solutions developed herein are valid for unsymmetric cross–ply shells, no numerical results are included here. For antisymmetric cross–ply shells, we further have $B_{12} = B_{66} = 0$, $B_{22} = -B_{11}$, $A_{22} = A_{11}$, and $D_{22} = D_{11}$.

4.3.2 Conclusions

The main objective of this chapter was to introduce a modified procedure in solving the eigenfrequency and buckling problems of laminated shells. The new procedure will specially be useful when the order and the number of governing equations are higher than eight and three, respectively. To demonstrate the new technique, a simple shear–deformation shell theory is used, which can be considered as a Donnell–type theory. Numerical results are also developed using the Classical Donnell Shell theory. The present technique, can also be used in solving other related shell and plate problems (e.g., Lévy–type problems).

As far as the accuracy of the theories is concerned, it is commonly accepted (from the study of isotropic shells) that Donnell–type theories are fairly accurate for shallow shells when the circumferential mode number m is large and the shell is not very long. With the advent of high–precision eigenvalue routines in computers, however, the simplicity of the governing equations is not a consideration and therefore, more general theories can be used. From Table 4.5, it can be seen that for most antisymmetrically (90/0/90/...) laminated shells and most boundary conditions the critical buckling loads are larger than those of (0/90/0,...) laminated shells (i.e., interchange 0° and 90° layers). Some exceptions occurred for the boundary types C4–F and S3–S3. However, an analysis (presented in Chapter V) based on a more accurate theory indicates that indeed these exceptions do not occur.

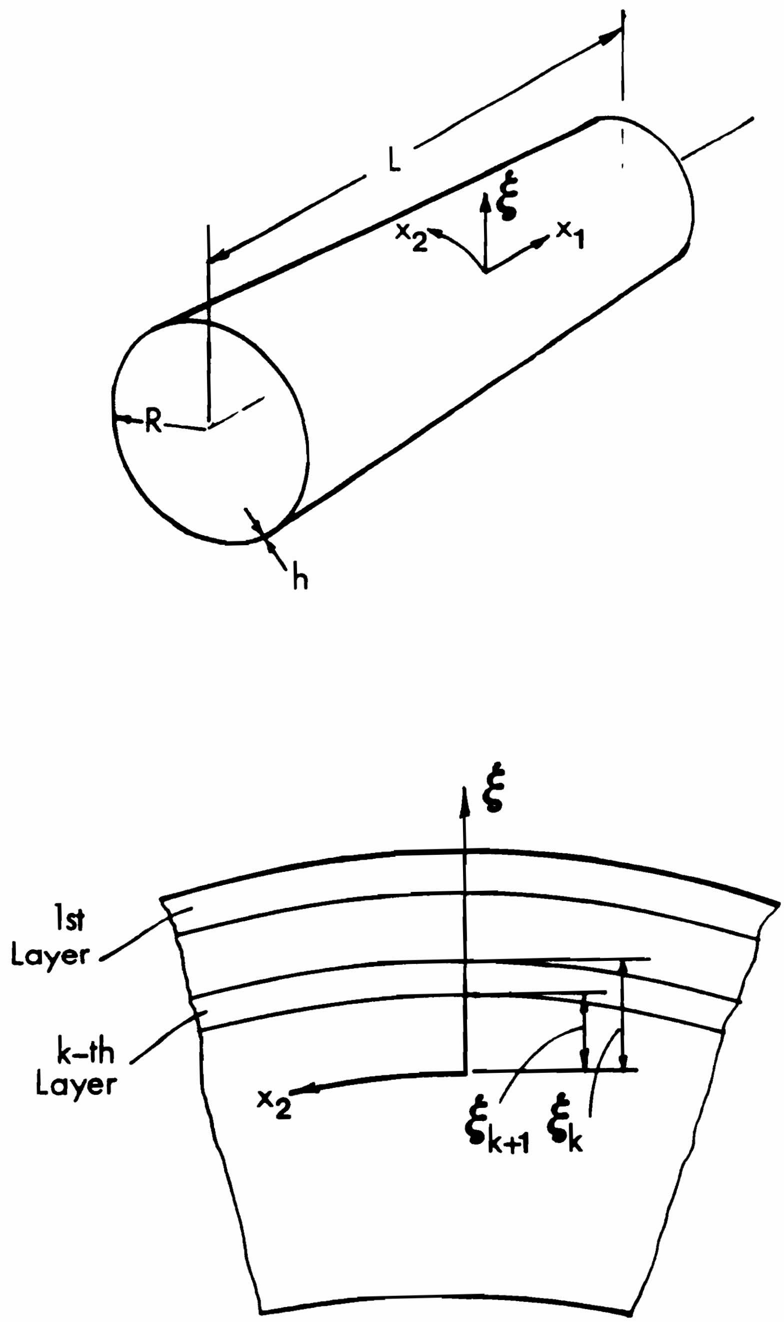

Figure 4.1 Laminated shell coordinate system and geometry

Table 4.1

The effects of lamination and various boundary conditions on the dimensionless fundamental –frequency $\bar{\omega}_m$ of a shell; $R/h = 60$, $L/R = 1$, $\bar{N} = 0$, and $\bar{\omega}_m = \frac{\omega_m L^2}{10h}\sqrt{\frac{\rho}{E_2}}$.

Lamination	Theory	F–F	S3–F	C4–F	S3–S3	S3–C4	C4–C4
(0/90)	FSDST	0.4096(3)	0.4579(3)	1.7158(5)	2.8497(6)	3.0291(6)	3.2659(6)
	CLST	0.4098(3)	0.4585(3)	1.7193(5)	2.8535(6)	3.0358(6)	3.2762(6)
(90/0)	FSDST	0.4071(3)	0.4542(3)	1.7200(5)	2.7747(6)	2.9745(6)	3.2424(6)
	CLST	0.4076(3)	0.4545(3)	1.7233(5)	2.7788(6)	2.9805(6)	3.2508(6)

Table 4.2

Comparison of the dimensionless fundamental frequency with dimensionless minimum axisymmetric frequency of a shell according to FSDST; R/h = 60, L/R = 1, $\bar{N} = 0$, and $\bar{\omega}_m = \frac{\omega_m L^2}{10h} \sqrt{\frac{\rho}{E_2}}$.

Lamination	C1–C1	C2–C2	C3–C3	C4–C4
(0/90)	3.0836(6) 14.1068(0) 14.1140(0)	3.2007(6) 14.1198(0) 14.1239(0)	3.1133(6) 14.1068(0) 14.1140(0)	3.2659(6) 14.1198(0) 14.1239(0)
Homogeneous Isotropic	1.9928(6) 6.0155(0) 6.0368(0)	2.1882(5) 6.1657(0) 6.1685(0)	2.0196(6) 6.0155(0) 6.0368(0)	1.2090(6) 6.1657(0) 6.1685(0)

Table 4.3

The roots $\lambda_j (j = \overline{1,6})$ of Eq. (4.40) with various trial values for frequencies of a (0/90) laminated shell according to FSDST;

$$i = \sqrt{-1} \text{ and } \bar{\omega}_o = \frac{\omega_o L^2}{10h}\sqrt{\frac{\rho}{E_2}}, \frac{R}{h} = 60, \frac{L}{R} = 1, \bar{N} = 0.$$

$\bar{\omega}_o$	$\lambda_1 = \lambda_2$	$\lambda_3 = -\lambda_4$	$\lambda_5 = -\lambda_6$
14.100000	0	i0.010378	i0.085836
14.100834	0	0	i0.086463
14.114034	0	0.037514	i0.094284
14.123885	0	0.047372	i0.098645

Table 4.4

The effects of various simply–supported boundary conditions on the dimensionless critical buckling load

T_{11} of cross–ply shells ($T_{11} = \frac{\bar{N}L^2}{100 h^3 E_2}$) and an isotropic

shell ($T_{11} = \frac{\bar{N}L^2}{10 h^3 E}$); R/h = 40 and L/R = 2.

Lamination	Theory	S1–S1	S2–S2	S3–S3	S4–S4
(90/0)	FSDST	1.5451(4) 3.6512(0)	1.5793(4) 3.6637(0)	1.8479(6) 3.6512(0)	1.8849(6) 3.6637(0)
	CLST	1.5705(4) 3.7693(0)	1.6081(4) 3.7839(0)	1.8663(6) 3.7693(0)	1.9044(6) 3.7839(0)
(0/90/0)	FSDST	1.8234(4) 5.5233(0)	1.8266(5) 5.5233(0)	2.0372(6) 5.5233(0)	2.0959(6) 5.5233(0)
	CLST	1.8396(5) 5.7739(0)	1.8430(5) 5.7739(0)	2.0507(6) 5.7739(0)	2.1095(6) 5.7739(0)
Homogeneous Isotropic	FSDST	4.7535(1) 9.5074(0)	4.7560(1) 9.5128(0)	9.4428(3) 9.5074(0)	9.4440(3) 9.5128(0)
	CLST	4.8062(1) 9.5923(0)	4.8090(1) 9.5977(0)	9.5448(4) 9.5923(0)	9.5492(3) 9.5977(0)

Table 4.5

The influences of lamination scheme and boundary conditions on the dimensionless critical buckling load T_{11} of a shell according to

FSDST; $R/h = 80, L/R = 1$, and $T_{11} = \dfrac{\bar{N}L^2}{10h^3E_2}$.

Lamination	F–F	S3–F	C4–F	S3–S3	C4–C4
(90/0)	1.6372(5)	2.1895(4)	5.2273(8)	9.3966(9)	9.8950(8)
(0/90)	1.6329(5)	2.1753(4)	5.2542(8)	9.3325(8)	9.8394(8)
(90/0/90/0)	2.4524(5)	3.4898(4)	6.2835(7)	11.6085(7)	12.5659(7)
(0/90/0/90)	2.4486(5)	3.4775(4)	6.2953(7)	11.6417(7)	12.5078(7)
(90/0/90/0/90/0)	2.5988(5)	3.7177(4)	6.4610(7)	11.9466(7)	12.9768(7)
(0/90/0/90/0/90)	2.5961(5)	3.7089(4)	6.4691(7)	11.9688(7)	12.9381(7)
(90/0/,... 100 lyers)	2.7138(5)	3.8941(4)	6.6030(7)	12.2219(7)	13.2932(7)
(0/90/,... 100 lyers)	2.7136(5)	3.8936(4)	6.6035(7)	12.2233(7)	13.2909(7)

Appendix 4.1

The operators L_{ij} for FSDST appearing in Eqs. (4–6) are:

$$L_{11} = A_{11}\frac{\partial^2}{\partial x_1^2} + A_{66}\frac{\partial^2}{\partial x_2^2} - I_1\frac{\partial^2}{\partial t^2}\ ,\ L_{12} = (A_{12} + A_{66})\frac{\partial^2}{\partial x_1 \partial x_2},$$

$$L_{13} = B_{11}\frac{\partial^2}{\partial x_1^2} + B_{66}\frac{\partial^2}{\partial x_2^2} - I_2\frac{\partial^2}{\partial t^2}\ ,\ L_{14} = (B_{12} + B_{66})\frac{\partial^2}{\partial x_1 \partial x_2},$$

$$L_{15} = \frac{A_{12}}{R}\frac{\partial}{\partial x_1},\ L_{22} = A_{66}\frac{\partial^2}{\partial x_1^2} + A_{22}\frac{\partial^2}{\partial x_2^2} - I_1\frac{\partial^2}{\partial t^2},\ L_{33} = L_{14},$$

$$L_{24} = B_{66}\frac{\partial^2}{\partial x_1^2} + B_{22}\frac{\partial^2}{\partial x_2^2} - I_2\frac{\partial^2}{\partial t^2},\ L_{25} = \frac{A_{22}}{R}\frac{\partial}{\partial x_2},$$

$$L_{33} = -K_{55}^2 A_{55} + D_{11}\frac{\partial^2}{\partial x_1^2} + D_{66}\frac{\partial^2}{\partial x_2^2} - I_3\frac{\partial^2}{\partial t^2},\ L_{34} = (D_{12} + D_{66})\frac{\partial^2}{\partial x_1 \partial x_2},$$

$$L_{35} = \left(\frac{B_{12}}{R} - K_{55}^2 A_{55}\right)\frac{\partial}{\partial x_1},\ L_{44} = -K_{44}^2 A_{44} + D_{66}\frac{\partial^2}{\partial x_1^2} + D_{22}\frac{\partial^2}{\partial x_2^2} - I_3\frac{\partial^2}{\partial t^2},$$

$$L_{45} = \left(\frac{B_{22}}{R} - K_{44}^2 A_{44}\right)\frac{\partial}{\partial x_2},$$

$$L_{55} = \frac{A_{22}}{R^2} + (\bar{N} - K_{55}^2 A_{55}) \frac{\partial^2}{\partial x_1^2} - K_{44}^2 A_{44} \frac{\partial^2}{\partial x_2^2} + I_1 \frac{\partial^2}{\partial t^2},$$

and $L_{ij} \equiv L_{ji}$.

The operators L_{ij} for CLST appearing in Eqs. (4–6) are:

$$L_{11} = A_{11} \frac{\partial^2}{\partial x_1^2} + A_{66} \frac{\partial^2}{\partial x_2^2} - I_1 \frac{\partial^2}{\partial t^2}, \; L_{12} = (A_{12} + A_{66}) \frac{\partial^2}{\partial x_1 \partial x_2},$$

$$L_{13} = - B_{11} \frac{\partial^2}{\partial x_1^3} - (B_{12} + 2B_{66}) \frac{\partial^3}{\partial x_1 \partial x_2^2} + \frac{A_{12}}{R} \frac{\partial}{\partial x_1} + I_2 \frac{\partial^3}{\partial x_1 \partial t^2},$$

$$L_{22} = A_{66} \frac{\partial^2}{\partial x_1^2} + A_{22} \frac{\partial^2}{\partial x_2^2} - I_1 \frac{\partial^2}{\partial t^2},$$

$$L_{23} = - (B_{12} + 2B_{66}) \frac{\partial^3}{\partial x_1^2 \partial x_2} - B_{22} \frac{\partial^3}{\partial x_2^3} + \frac{A_{22}}{R} \frac{\partial}{\partial x_2} + I_2 \frac{\partial^3}{\partial x_2 \partial t^2}$$

$$L_{33} = D_{11} \frac{\partial^4}{\partial x_1^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4}{\partial x_1^2 \partial x_2^2} + D_{22} \frac{\partial^4}{\partial x_2^4} + \frac{A_{22}}{R^2}$$

$$+ (\bar{N} - 2 \frac{B_{12}}{R}) \frac{\partial^2}{\partial x_1^2} - 2 \frac{B_{22}}{R} \frac{\partial^2}{\partial x_2^2} + I_1 \frac{\partial^2}{\partial t^2}$$

$$- I_3 \frac{\partial^2}{\partial t^2} (\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}),$$

and $L_{ij} = L_{ji}$. The longitudinal, circumferential, and rotatory inertia terms are not neglected in CLST.

When m = 1,2,..., the boundary conditions for CLST are classified as:

S1: $w = M_1 = N_1 = N_6 = 0,$ C1: $w = \frac{\partial w}{\partial x_1} = N_1 = N_6 = 0,$

S2: $w = M_1 = u = N_6 = 0,$ C2: $w = \frac{\partial w}{\partial x_1} = u = N_6 = 0,$

S3: $w = M_1 = N_1 = v = 0,$ C3: $w = \frac{\partial w}{\partial x_1} = N_1 = v = 0,$

S4: $w = M_1 = u = v = 0,$ C4: $w = \frac{\partial w}{\partial x_1} = u = v = 0,$

and

$$\text{F:}\ N_1 = N_6 = M_1 = \frac{\partial M_1}{\partial x_1} + 2\,\frac{\partial M_6}{\partial x_2} - I_2\,\frac{\partial^2 u}{\partial t^2} + I_3\,\frac{\partial^3 w}{\partial x_1\,\partial t^2} - \bar{N}\,\frac{\partial w}{\partial x_1} = 0$$

where N_1, N_6, M_1, M_6, and Q_1 are obtained by substituting Eqs. (4–4) and (4–5) into Eqs. (4–2). When m = 0, the above boundary conditions are reduced to:

S1: $w = M_1 = N_1 = 0,$ C1: $w = \frac{\partial w}{\partial x_1} = N_1 = 0,$

S2: $w = M_1 = u = 0,$ C2: $w = \frac{\partial w}{\partial x_1} = u = 0,$

$$\text{F:}\qquad \frac{\partial M_1}{\partial x_1} - I_2 \frac{\partial^2 u}{\partial t^2} + I_3 \frac{\partial^3 w}{\partial x_1 \partial t^2} - \bar{N} \frac{\partial w}{\partial x_1} = 0,$$

with S3, S4, C3, and C4 cases being identical to S1, S2, C1, and C2 cases, respectively.

Appendix 4.2

In free–vibration analysis, the coefficients $C_j (j = \overline{1,25})$ for FSDST appearing in Eqs. (4–10) are:

$$C_1 = (e_2 - e_1 e_{14})/e_{26},\ C_2 = (e_3 - e_1 e_{17})/e_{26},\ C_3 = (e_4 - e_1 e_{16})/e_{26},$$

$$C_4 = (e_5 - e_1 e_{17})/e_{26},\ C_5 = (e_6 - e_1 e_{18})/e_{26},\ C_6 = (e_8 - e_7 e_{20})/e_{27},$$

$$C_7 = (e_3 - e_7 e_{21})/e_{27},\ C_8 = (e_{10} - e_7 e_{22})/e_{27},\ C_9 = (e_{11} - e_7 e_{23})/e_{27},$$

$$C_{10} = (e_{12} - e_7 e_{24})/e_{27},\ C_{11} = (e_{14} - e_{13} e_2)/e_{26},$$

$$C_{12} = (e_{15} - e_{13} e_3)/e_{26},$$

$$C_{13} = (e_{16} - e_{13} e_4)/e_{26},\ C_{14} = (e_{17} - e_{13} e_5)/e_{26},$$

$$C_{15} = (e_{18} - e_{13} e_6)/e_{26},$$

$$C_{16} = (e_{20} - e_{19} e_8)/e_{27},\ C_{17} = (e_{21} - e_{19} e_9)/e_{27},$$

$$C_{18} = (e_{22} - e_{19} e_{10})/e_{27},$$

$$C_{19} = (e_{23} - e_{19} e_{11})/e_{27},\ C_{20} = (e_{24} - e_{19} e_{12})/e_{27},\ C_{21} = e_{28}/e_{25},$$

$$C_{22} = e_{29}/e_{25},\ C_{23} = e_{30}/e_{25},\ C_{24} = e_{31}/e_{25},\ C_{25} = e_{32}/e_{25},$$

where

$$e_1 = B_{11}/A_{11},\ e_2 = (A_{66}\beta_m^2 - \omega_m^2 I_1)/A_{11},\ e_3 = -(A_{12} + A_{66})\beta_m/A_{11},$$

$$e_4 = (B_{66}\beta_m^2 - \omega_m^2 I_2)/A_{11},\ e_5 = -(B_{12} + B_{66})\beta_m/A_{11},$$

$$e_6 = -A_{12}/(A_{11}R),$$

$$e_7 = B_{66}/A_{66},\ e_8 = (A_{12} + A_{66})\beta_m/A_{66},\ e_9 = (A_{22}\beta_m^2 - \omega_m^2 I_1)/A_{66},$$

$$e_{10} = (B_{12} + B_{66})\beta_m/A_{66},\ e_{11} = (B_{22}\beta_m^2 - \omega_m^2 I_2)/A_{66},$$

$$e_{12} = A_{22}\beta_m/(A_{66}R),$$

$$e_{13} = B_{11}/D_{11},\ e_{14} = (B_{66}\beta_m^2 - \omega_m^2 I_2)/D_{11},\ e_{15} = -(B_{12} + B_{66})\beta_m/D_{11},$$

$$e_{16} = (K_{55}^2 A_{55} + D_{66}\beta_m^2 - \omega_m^2 I_3)/D_{11},\ e_{17} = -(D_{12} + D_{66})\beta_m/D_{11},$$

$$e_{18} = -(B_{12}/R - K_{55}^2 A_{55})/D_{11},\ e_{19} = B_{66}/D_{66},$$

$$e_{20} = (B_{12} + B_{66})\beta_m/D_{66},$$

$$e_{21} = (B_{22}\beta_m^2 - \omega_m^2 I_2)/D_{66},\ e_{22} = (D_{12} + D_{66})\beta_m/D_{66},$$

$$e_{23} = (K_{44}^2 A_{44} + D_{22}\beta_m^2 - \omega_m^2 I_3)/D_{66},\ e_{24} = (B_{22}/R - K_{44}^2 A_{44})\beta_m/D_{66},$$

$$e_{25} = K_{55}^2 A_{55} - \bar{N},\ e_{26} = 1 - e_1 e_{13},\ e_{27} = 1 - e_7 e_{19},\ e_{28} = A_{12}/R,$$

$$e_{29} = A_{22}\beta_m/R,\ e_{30} = -\,K_{55}^2 A_{55} + B_{12}/R,\ e_{31} = (-K_{44}^2 A_{44} + B_{22}/R)\beta_m,$$

$$e_{32} = K_{44}^2 A_{44}\beta_m^2 - \omega_m^2 I_1 + A_{22}/(R^2).$$

In the stability analysis we let $\omega_m \to 0$ in $e_j (j = \overline{1,32})$.

Appendix 4.3

For very thin shells (or long shells) some elements of the coefficient matrix [K] appearing in Eq. (4–26) may become extremely large or small. To see this, we note that Eq. (4–26) has the form:

$$\begin{bmatrix} \bar{K}_{11}e^{\lambda_1 L/2} & \bar{K}_{12}e^{\lambda_2 L/2} & \cdots & \bar{K}_{110}e^{\lambda_{10} L/2} \\ \bar{K}_{21}e^{\lambda_1 L/2} & \bar{K}_{22}e^{\lambda_2 L/2} & \cdots & \bar{K}_{210}e^{\lambda_{10} L/2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \bar{K}_{51}e^{\lambda_1 L/2} & \bar{K}_{52}e^{\lambda_2 L/2} & \cdots & \bar{K}_{510}e^{\lambda_{10} L/2} \\ \bar{K}_{61}e^{-\lambda_1 L/2} & \bar{K}_{62}e^{-\lambda_2 L/2} & \cdots & \bar{K}_{610}e^{-\lambda_{10} L/2} \\ \bar{K}_{71}e^{-\lambda_1 L/2} & \bar{K}_{72}e^{-\lambda_2 L/2} & \cdots & \bar{K}_{710}e^{-\lambda_{10} L/2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \bar{K}_{101}e^{-\lambda_1 L/2} & \bar{K}_{102}e^{-\lambda_2 L/2} & \cdots & \bar{K}_{1010}e^{-\lambda_{10} L/2} \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ D_{10} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} \tag{a}$$

That is, each column in [K] involves only one of the eigenvalues of the coefficient matrix [A]. Because of exponential functions in [K], computer overflow (or underflow) may occur whenever the real part of $\pm\, \lambda_j L/2$ ($j = \overline{1,10}$) becomes a relatively large positive (or negative) number. To resolve this problem we assume, for example, that $a_1 L/2$ is a large positive number where a_1 is the real part of $\lambda_1 =$

$a_1 + ib_1$ $(i = \sqrt{-1})$. Now by introducing the new unknown constant

$$\bar{D}_1 = e^{a_1 L/2} D_1,$$

we can write

$$\bar{K}_{j1} e^{\lambda_1 L/2} D_1 = \bar{K}_{j1} e^{ib_1 L/2} \bar{D}_1 \quad (j = \overline{1,5})$$

and

$$\bar{K}_{j1} e^{-\lambda_1 L/2} D_1 = \frac{\bar{K}_{j1} e^{-ib_1 L/2}}{e^{a_1 L}} \cdot \bar{D}_1 \to 0 \cdot \bar{D}_1 \quad (j = \overline{6,10})$$

That is, we can replace the first column in [K] by

$$\{\bar{K}_{11} e^{ib_1 L/2} \cdots \bar{K}_{51} e^{ib_1 L/2} \; 0 \cdots 0\}^T$$

and replace D_1 by $\bar{D}_1$ in Eqs. (a). Similarly, if $a_2 L/2$ is a large negative number (with $\lambda_2 = a_2 + ib_2$), then the second column in [K] and D_2 in Eqs. (a) can be replaced by

$$\{0 \cdots 0 \; \bar{K}_{62} e^{-ib_2 L/2} \cdots \bar{K}_{102} e^{-ib_2 L/2}\}^T$$

and $\bar{D}_2$, respectively, with $\bar{D}_2 = e^{-a_2 L/2} D_2$. Similar arguments can be made concerning the remaining columns in [K].

Appendix 4.4

The coefficients $\bar{C}_j(j = \overline{1,9})$ appearing in Eqs. (4–34) are:

$$\bar{C}_1 = (e_2 - e_1e_6)/e_{13},\ \bar{C}_2 = (e_3 - e_1e_7)/e_{13},\ \bar{C}_3 = (e_4 - e_1e_8)/e_{13},$$

$$\bar{C}_4 = (e_6 - e_5e_2)/e_{13},\ \bar{C}_5 = (e_7 - e_5e_3)/e_{13},\ \bar{C}_6 = (e_8 - e_5e_4)/e_{13},$$

$$\bar{C}_7 = e_9/e_{12},\ \bar{C}_8 = e_{10}/e_{12},\ \text{and}\ \bar{C}_9 = e_{11}/e_{12},$$

where

$$e_1 = B_{11}/A_{11},\ e_2 = -I_1\omega_o^2/A_{11},\ e_3 = -I_2\omega_o^2/A_{11},\ e_4 = -A_{12}/(RA_{11}),$$

$$e_5 = B_{11}/D_{11},\ e_6 = -I_2\omega_o^2/D_{11},\ e_7 = (K_{55}^2A_{55} - I_3\omega_o^2)/D_{11},$$

$$e_8 = (K_{55}^2A_{55} - B_{12}/R)/D_{11},\ e_9 = A_{12}/R,\ e_{10} = B_{12}/R - K_{55}^2A_{55},$$

$$e_{11} = A_{22}/R^2 - I_1\omega_o^2,\ e_{12} = K_{55}^2A_{55} - \bar{N},\ \text{and}\ e_{13} = 1 - e_1e_5.$$

In the stability problem we let $\omega_o \to 0$ in $e_j(j = \overline{1,13})$. In the vibration problem, when only the radial inertia force is preserved, we will have

$$e_2 = e_3 = e_6 = 0,\ \text{and}\ e_7 = K_{55}^2A_{55}/D_{11}.$$

The coefficients $\theta_j (j = 1,2)$ appearing in Eq. (4–40) are

$$\theta_1 = -(\bar{C}_3\bar{C}_7 + \bar{C}_6\bar{C}_8 + \bar{C}_5 + \bar{C}_9) \text{ and}$$

$$\theta_2 = \bar{C}_5\bar{C}_9 + \bar{C}_3\bar{C}_5\bar{C}_7 - \bar{C}_2\bar{C}_6\bar{C}_7,$$

where $\bar{C}_j (j = \overline{1,9})$ are given in above.

CHAPTER V

STABILITY OF ANISOTROPIC CIRCULAR CYLINDRICAL SHELLS ACCORDING TO VARIOUS THEORIES

We have studied the free–vibration and stability problems of cross–ply laminated circular cylindrical shells using the first–order shear deformation shell theory (FSDST) and the classical shell theory (CLST). In the present chapter we develop solutions for the stability of generally laminated circular cylindrical shells under compressive loads.

Buckling of anisotropic shells under axial compression has been investigated experimentally by Card [119], Tasi et al. [120], and Holston et al. [121]. The experimental results lie within 65 to 85% of the theoretically predicted loads for perfect shells using linear anisotropic shell theories. Theoretical investigations of anisotropic shells has been made by Cheng and Ho [122], Tasi [123], Holston [124], Khot [125], and Wu [126]. In their analysis, Cheng and Ho (see also [127]) assumed a particular solution to Flugge's classical differential equations of equilibrium. Their solution was originally used by Donnell in 1934 to solve the buckling problem of a thin tube under torsion [74]. However, it is argued in [120] that when the coupling between stretching and shear is large, the nonhomogeneity of the coupling stiffnesses B_{ij} will make the analysis of Cheng and Ho inapplicable. Khot [125] investigated the effect of fiber orientation on the buckling and postbuckling behavior of anisotropic cylindrical shells under axial compression using the von Karman–Donnell large–displacement equations. The investigation also includes some data on the effect of initial imperfections on the buckling behavior of the anisotropic shells. Anisotropic cylindrical shells having axisymmetric shape imperfections are also investigated by Tennyson et al. [128, 129] using the nonlinear

Donnell–type shell equations. It is concluded in [128] that since the axisymmetric shape imperfection has a dominant effect on the buckling of composite cylindrical shells under axial compression, there is no significant advantage from a design point of view in selecting the optimum fiber orientations that would give maximum buckling strength for a perfect shell. Using Cheng and Ho's solution, Wu [126] studied the buckling of anisotropic circular cylindrical shells under the combined radial pressure, axial compression, and torsion. In his analysis, Wu utilized Flügge's and Donnell's equations and developed general solutions for asymmetric (non–axisymmetric) and axisymmetric buckling modes. The numerical results are, however, presented only for asymmetric modes.

Here we develop a general solution to the buckling of anisotropic circular cylindrical shells under axial compression within the frameworks of FSDST, the generalized layer–wise shell theory (GLST), and the layer–wise shell theory (LST). Our analysis will include both the asymmetric and axisymmetric modes and it can be used for buckling analysis with combined loadings. For completeness, we will also compare the results with those obtained according to Donnell's classical shell theory (CLST).

5.1.1 Stability Analysis According to FSDST

We start our study of generally laminated shells using the first–order shear deformation shell theory (FSDST). For general anisotropic shells, the stiffnesses A_{16}, A_{26}, A_{45}, B_{16}, B_{26}, D_{16}, and D_{26} are no longer equal to zero. The stability equations are (see Chapters II and IV):

$$A_{11}u_{,x_1x_1} + 2A_{16}u_{,x_1x_2} + A_{66}u_{,x_2x_2} + A_{16}v_{,x_1x_1} + (A_{12} + A_{66})v_{,x_1x_2}$$

$$+ A_{26}v_{,x_2x_2} + B_{11}\phi_{1,x_1x_1} + 2B_{16}\phi_{1,x_1x_2} + B_{66}\phi_{1,x_2x_2} + B_{16}\phi_{2,x_1x_1}$$

$$+ (B_{12} + B_{66})\phi_{2,x_1x_2} + B_{26}\phi_{2,x_2x_2} + \frac{A_{12}}{R}w_{,x_1} + \frac{A_{26}}{R}w_{,x_2} = 0$$

$$A_{16}u_{,x_1x_1} + (A_{12} + A_{66})u_{,x_1x_2} + A_{26}u_{,x_2x_2} + A_{66}v_{,x_1x_1} + 2A_{26}v_{,x_1x_2}$$

$$+ A_{22}v_{,x_2x_2} + B_{16}\phi_{1,x_1x_1} + (B_{12} + B_{66})\phi_{1,x_1x_2} + B_{26}\phi_{1,x_2x_2}$$

$$+ B_{66}\phi_{2,x_1x_1} + 2B_{26}\phi_{2,x_1x_2} + B_{22}\phi_{2,x_2x_2} + \frac{A_{26}}{R}w_{,x_1} + \frac{A_{22}}{R}w_{,x_2} = 0$$

$$B_{11}u_{,x_1x_1} + 2B_{16}u_{,x_1x_2} + B_{66}u_{,x_2x_2} + B_{16}v_{,x_1x_1} + (B_{12} + B_{66})v_{,x_1x_2}$$

$$+ B_{26}v_{,x_2x_2} - K_{55}^2A_{55}\phi_1 + D_{11}\phi_{1,x_1x_1} + 2D_{16}\phi_{1,x_1x_2} + D_{66}\phi_{1,x_2x_2}$$

$$- K_{45}^2A_{45}\phi_2 + D_{16}\phi_{2,x_1x_1} + (D_{12} + D_{66})\phi_{2,x_1x_2} + D_{26}\phi_{2,x_2x_2}$$

$$+ (\frac{B_{12}}{R} - K_{55}^2A_{55})w_{,x_1} + (\frac{B_{26}}{R} - K_{45}^2A_{45})w_{,x_2} = 0$$

$$B_{16}u_{,x_1x_1} + (B_{12} + B_{66})u_{,x_1x_2} + B_{26}u_{,x_2x_2} + B_{66}v_{,x_1x_1} + 2B_{26}v_{,x_1x_2}$$

$$+ B_{22}v_{,x_2x_2} - K_{45}^2A_{45}\phi_1 + D_{16}\phi_{1,x_1x_1} + (D_{12} + D_{66})\phi_{1,x_1x_2}$$

$$+ D_{26}\phi_{1,x_2x_2} - K_{44}^2 A_{44}\phi_2 + D_{66}\phi_{2,x_1x_1} + 2D_{26}\phi_{2,x_1x_2} + D_{22}\phi_{2,x_2x_2}$$

$$+ (\frac{B_{26}}{R} - K_{45}^2 A_{45}) w_{,x_1} + (\frac{B_{22}}{R} - K_{44}^2 A_{44}) w_{,x_2} = 0$$

$$\frac{A_{12}}{R} u_{,x_1} + \frac{A_{26}}{R} u_{,x_2} + \frac{A_{26}}{R} v_{,x_1} + \frac{A_{22}}{R} v_{,x_2} + (\frac{B_{12}}{R} - K_{55}^2 A_{55})\phi_{1,x_1}$$

$$+ (\frac{B_{26}}{R} - K_{45}^2 A_{45})\phi_{1,x_2} + (\frac{B_{26}}{R} - K_{45}^2 A_{45})\phi_{2,x_1} + (\frac{B_{22}}{R} - K_{44}^2 A_{44})\phi_{2,x_2}$$

$$+ \frac{A_{22}}{R^2} w + (\bar{N} - K_{55}^2 A_{55}) w_{,x_1x_1} - 2K_{45}^2 A_{45} w_{,x_1x_2} - K_{44}^2 A_{44} w_{,x_2x_2} = 0 \tag{5–1}$$

where $\bar{N}$ is the distributed axial compressive (positive) load acting at the ends of the shell.

5.1.2 Boundary Conditions

Various boundary conditions can be assumed to exist at the edges of a shell. We have classified these conditions in Eqs. (4–30), (4–31), and (4–32) of the previous chapter. For anisotropic shells the stress resultants appearing in these equations are given (see Chapter II) as

$$M_1 = B_{11} u_{,x_1} + B_{16} u_{,x_2} + B_{12} v_{,x_2} + B_{16} v_{,x_1} + D_{11}\phi_{1,x_1}$$

$$+ D_{16}\phi_{1,x_2} + D_{16}\phi_{2,x_1} + D_{12}\phi_{2,x_2} + \frac{B_{12}}{R} w$$

$$M_6 = B_{16}u_{,x_1} + B_{66}u_{,x_2} + B_{26}v_{,x_2} + B_{66}v_{,x_1} + D_{16}\phi_{1,x_1}$$

$$+ D_{66}\phi_{1,x_2} + D_{66}\phi_{2,x_1} + D_{26}\phi_{2,x_2} + \frac{B_{26}}{R} w$$

$$N_1 = A_{11}u_{,x_1} + A_{16}u_{,x_2} + A_{12}v_{,x_2} + A_{16}v_{,x_1} + B_{11}\phi_{1,x_1}$$

$$+ B_{16}\phi_{1,x_2} + B_{16}\phi_{2,x_1} + B_{12}\phi_{2,x_2} + \frac{A_{12}}{R} w$$

$$N_6 = A_{16}u_{,x_1} + A_{66}u_{,x_2} + A_{26}v_{,x_2} + A_{66}v_{,x_1} + B_{16}\phi_{1,x_1}$$

$$+ B_{66}\phi_{1,x_2} + B_{66}\phi_{2,x_1} + B_{26}\phi_{2,x_2} + \frac{A_{26}}{R} w$$

$$Q_1 = K_{55}^2 A_{55}\phi_1 + K_{45}^2 A_{45}\phi_2 + K_{55}^2 A_{55}w_{,x_1} + K_{45}^2 A_{45}w_{,x_2}. \qquad (5\text{–}2)$$

5.1.3 Solution Procedure (Asymmetric Case)

Assuming that the shell buckles in m (with m = 1,2,...) circumferential waves, we seek a solution in the form

$$(u,v,\phi_1,\phi_2,w) = \mathrm{Re}\{[U_m(x_1),V_m(x_1),X_m(x_1),Y_m(x_1),W_m(x_1)]\cdot e^{i\beta_m x_2}\}$$
(5–3)

where Re(·) denotes the real part, $i = \sqrt{-1}$, and $\beta_m = \frac{m}{R}$. Upon introduction of this solution and the new variables,

$$Z_{1m}(x_1) = U_m(x_1)\ ,\ Z_{2m}(x_1) = U'_m(x_1) = Z'_{1m}(x_1)$$

$$Z_{3m}(x_1) = V_m(x_1)\ ,\ Z_{4m}(x_1) = V'_m(x_1) = Z'_{3m}(x_1)$$

$$Z_{5m}(x_1) = X_m(x_1)\ ,\ Z_{6m}(x_1) = X'_m(x_1) = Z'_{5m}(x_1)$$

$$Z_{7m}(x_1) = Y_m(x_1)\ ,\ Z_{8m}(x_1) = Y'_m(x_1) = Z'_{7m}(x_1)$$

$$Z_{9m}(x_1) = W_m(x_1)\ ,\ Z_{10m}(x_1) = W'_m(x_1) = Z'_{9m}(x_1) \tag{5–4}$$

into Eqs. (5–1), we obtain

$$[B]\{Z'\} = [C]\{Z\} \tag{5–5a}$$

where

$$\{Z'\} = \{Z'_{1m},Z'_{2m},\ldots,Z'_{10m}\}^T$$

$$\{Z\} = \{Z_{1m},Z_{2m},\ldots,Z_{10m}\}^T \tag{5–5b}$$

and the coefficient matrices [B] and [C] are

$$[B] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & a_2 & 0 & a_3 & 0 & a_4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_1 & 0 & b_2 & 0 & b_3 & 0 & b_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 & 0 & c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & d_1 & 0 & d_2 & 0 & d_3 & 0 & d_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_1 \end{bmatrix} \quad (5\text{–}6a)$$

and

$$[C] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_5 & b_6 & b_7 & b_8 & b_9 & b_{10} & b_{11} & b_{12} & b_{13} & b_{14} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} & c_{12} & c_{13} & c_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ d_5 & d_6 & d_7 & d_8 & d_9 & d_{10} & d_{11} & d_{12} & d_{13} & d_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} \end{bmatrix} \quad (5\text{–}6b)$$

In Eqs. (5–6) we have

$$a_1 = A_{11},\ a_2 = b_1 = A_{16},\ a_3 = c_1 = B_{11},\ a_4 = b_3 = c_2 = d_1 = B_{16}$$

$$a_5 = \beta_m^2 A_{66},\ a_6 = -2i\beta_m A_{16},\ a_7 = b_5 = \beta_m^2 A_{26},$$

$$a_8 = b_6 = -i\beta_m(A_{12} + A_{66}),\ a_9 = c_5 = \beta_m^2 B_{66},\ a_{10} = c_6 = -2i\beta_m B_{16}$$

$$a_{11} = b_9 = c_7 = d_5 = \beta_m^2 B_{26},\ a_{12} = b_{10} = c_8 = d_6 = -i\beta_m(B_{12} + B_{66})$$

$$a_{13} = e_2 = -i\beta_m \frac{A_{26}}{R},\ a_{14} = e_3 = -\frac{A_{12}}{R},\ b_2 = A_{66},\ b_4 = d_2 = B_{66}$$

$$b_7 = \beta_m^2 A_{22},\ b_8 = -2i\beta_m A_{26},\ b_{11} = d_7 = \beta_m^2 B_{22},\ b_{12} = d_8 = -2i\beta_m B_{26}$$

$$b_{13} = -i\beta_m \frac{A_{22}}{R} = e_4,\ b_{14} = e_5 = -\frac{A_{26}}{R},\ c_3 = D_{11},\ c_4 = d_3 = D_{16}$$

$$c_9 = \beta_m^2 D_{66} + K_{55}^2 A_{55},\ c_{10} = -2i\beta_m D_{16},\ c_{11} = d_9 = \beta_m^2 D_{26} + K_{45}^2 A_{45}$$

$$c_{12} = d_{10} = -i\beta_m(D_{12} + D_{66}),\ c_{13} = e_6 = -i\beta_m\left(\frac{B_{26}}{R} - K_{45}^2 A_{45}\right)$$

$$c_{14} = e_7 = K_{55}^2 A_{55} - \frac{B_{12}}{R},\ d_4 = D_{66},\ d_{11} = \beta_m^2 D_{22} + K_{44}^2 A_{44}$$

$$d_{12} = -2i\beta_m D_{26},\ d_{13} = e_8 = -i\beta_m\left(\frac{B_{22}}{R} - K_{44}^2 A_{44}\right),$$

$$d_{14} = e_9 = K_{45}^2 A_{45} - \frac{B_{26}}{R}, \; e_1 = \bar{N} - K_{55}^2 A_{55}, \; e_{10} = -\beta_m^2 K_{44}^2 A_{44} - \frac{A_{22}}{R^2}$$

$$e_{11} = 2i\beta_m K_{45}^2 A_{45}. \tag{5–7}$$

It should be noted that [B] is a general real matrix and [C] is a general complex matrix. Equation (5–5a) can alternatively be written as

$$\{Z'\} = [A]\{Z\} \tag{5–8a}$$

where

$$[A] = [B]^{-1}[C]. \tag{5–8b}$$

A formal solution of Eq. (5–8a) can be shown to be (see [114,115])

$$\{Z\} = [U][Q(x_1)]\{D\} \tag{5–9a}$$

where

$$[Q(x_1)] = \begin{bmatrix} e^{\lambda_1 x_1} & & & \\ & e^{\lambda_2 x_1} & \underset{\sim}{0} & \\ & & \ddots & \\ & \underset{\sim}{0} & & e^{\lambda_{10} x_1} \end{bmatrix} \tag{5–9b}$$

where [U] is a modal matrix that transforms [A] into a diagonal form (i.e., the jth column of [U] constitutes the eigenvectors of [A] corresponding to the jth eigenvalue of [A]). It is found that for m ≠ 0 the eigenvalues λ_j ($j = \overline{1,10}$) of [A] are always distinct. In Eq. (5–9a) {D} is an unknown constant vector whose components are in general complex.

Equation (5–9a) can alternatively be rewritten as

$$Z_{im} = \sum_{j=1}^{10} u_{ij} e^{\lambda_j x_1} D_j \qquad i = (\overline{1,10}) \tag{5–10}$$

Hence, from Eqs. (5–3), (5–4), and (5–10) we have

$$\begin{Bmatrix} u \\ v \\ \phi_1 \\ \phi_2 \\ w \end{Bmatrix} = \mathrm{Re}\left[\sum_{j=1}^{10} \begin{Bmatrix} u_{1j} \\ u_{3j} \\ u_{5j} \\ u_{7j} \\ u_{9j} \end{Bmatrix} (\cosh \lambda_j x_1 + \sinh \lambda_j x_1) D_j \cdot e^{i\beta_m x_2}\right] \tag{5–11}$$

Throughout our analysis we will assume that the boundary conditions at $x_1 = \pm L/2$ are identical. At this point we assume, for the sake of illustration, that both edges of the shell are clamped (of the type C4, see Chapter IV). Upon imposing the boundary conditions on Eq. (5–11), we obtain

$$[K]\{D\} = \{0\} \tag{5–12}$$

where

$$
[K] = \begin{bmatrix}
u_{11}\cosh\lambda_1 L/2 & u_{12}\cosh\lambda_2 L/2 & \cdots & u_{110}\cosh\lambda_{10} L/2 \\
u_{31}\cosh\lambda_1 L/2 & u_{32}\cosh\lambda_2 L/2 & \cdots & u_{310}\cosh\lambda_{10} L/2 \\
u_{51}\cosh\lambda_1 L/2 & u_{52}\cosh\lambda_2 L/2 & \cdots & u_{510}\cosh\lambda_{10} L/2 \\
u_{71}\cosh\lambda_1 L/2 & u_{72}\cosh\lambda_2 L/2 & \cdots & u_{710}\cosh\lambda_{10} L/2 \\
u_{91}\cosh\lambda_1 L/2 & u_{92}\cosh\lambda_2 L/2 & \cdots & u_{910}\cosh\lambda_{10} L/2 \\
u_{11}\sinh\lambda_1 L/2 & u_{12}\sinh\lambda_2 L/2 & \cdots & u_{110}\sinh\lambda_{10} L/2 \\
u_{31}\sinh\lambda_1 L/2 & u_{32}\sinh\lambda_2 L/2 & \cdots & u_{310}\sinh\lambda_{10} L/2 \\
u_{51}\sinh\lambda_1 L/2 & u_{52}\sinh\lambda_2 L/2 & \cdots & u_{510}\sinh\lambda_{10} L/2 \\
u_{71}\sinh\lambda_1 L/2 & u_{72}\sinh\lambda_2 L/2 & \cdots & u_{710}\sinh\lambda_{10} L/2 \\
u_{91}\sinh\lambda_1 L/2 & u_{92}\sinh\lambda_2 L/2 & \cdots & u_{910}\sinh\lambda_{10} L/2
\end{bmatrix}
\tag{5–13}
$$

For a nontrivial solution the determinant of [K] should vanish:

$$
|K| = 0. \tag{5–14}
$$

Generally, some of the eigenvalues λ_j ($j = \overline{1,10}$) are complex. However, it is found that if one of these eigenvalues is complex and has the form $a_1 + ib_1$, there always exists another eigenvalue of the form $-a_1 + ib_1$. That is, the eigenvalues appear as $\pm a_1 + ib_1$. For this reason, the columns in Eq. (5–13) corresponding to these eigenvalues will also be complex numbers whose real parts will be the negative of one another but will have identical imaginary parts. Therefore, the determinant $|K|$, when it is complex, can easily be transformed into a real one. Indeed, it turns out that $|K|$ is either a real number or a pure imaginary number when it is complex. Hence we can simply divide $|K|$ by $|U|$ to obtain a real number as we discussed in the previous chapter.

During computation of the elements in the coefficient matrix [K] for thin shells (or long shells) overflow or underflow may occur in computer calculations. As explained in Chapter IV, this problem is primarily due to the fact that the product of L/2 and the real part of some of eigenvalues may result in a very large positive (or negative) number. To resolve this problem, we assume that $a_1L/2$ is a large positive number with a_1 being the real part of, say, $\lambda_1 = a_1 + ib_1$. With this in mind, we note that

$$D_1 \cosh \lambda_1 L/2 = \frac{1}{2} D_1 \left[e^{a_1 L/2} e^{ib_1 L/2} + \frac{e^{-ib_1 L/2}}{e^{a_1 L/2}} \right] \to \frac{1}{2} e^{a_1 L/2} D_1 e^{ib_1 L/2}$$

$$= \bar{D}_1 e^{ib_1 L/2} = \bar{\bar{D}}_1 \tag{5–15a}$$

and

$$D_1 \sinh \lambda_1 L/2 = \frac{1}{2} D_1 \left[e^{a_1 L/2} e^{ib_1 L/2} - \frac{e^{-ib_1 L/2}}{e^{a_1 L/2}} \right] \to \frac{1}{2} e^{a_1 L/2} D_1 e^{ib_1 L/2}$$

$$= \bar{D}_1 e^{ib_1 L/2} = \bar{\bar{D}}_1 \tag{5–15b}$$

where $\bar{D}_1 = (\frac{1}{2} e^{a_1 L/2})D_1$ and $\bar{\bar{D}}_1 = (\frac{1}{2} e^{\lambda_1 L/2})D_1$. That is, we can replace D_1 in Eq. (5–12) by a new unknown constant $\bar{D}_1$ and replace $\cosh \lambda_1 L/2$ and $\sinh \lambda_1 L/2$ by $e^{ib_1 L/2}$. Alternatively, we can replace D_1 by $\bar{\bar{D}}_1$ and replace $\cosh \lambda_1 L/2$ and $\sinh \lambda_1 L/2$ by 1 as can be seen from Eqs. (5–15). Similarly, if $\lambda_2 = -a_1 + ib_1$ (with $-a_1 L/2$ being a large negative number), we note that

$$D_2 \cosh \lambda_2 L/_2 = \frac{1}{2} D_2 \left[\frac{e^{ib_1 L/2}}{e^{a_1 L/2}} + e^{a_1 L/2} e^{-ib_1 L/2} \right] \to \frac{1}{2} e^{a_1 L/2} D_2 e^{-ib_1 L/2}$$

$$= \bar{D}_2 e^{-ib_1 L/2} = \bar{\bar{D}}_2 \qquad (5\text{–}16a)$$

and

$$D_2 \sinh \lambda_2 L/2 = \frac{1}{2} D_2 \left[\frac{e^{ib_1 L/2}}{e^{a_1 L/2}} - e^{a_1 L/2} e^{-ib_1 L/2} \right] \to -\frac{1}{2} e^{a_1 L/2} D_2 e^{-ib_1 L/2}$$

$$= -\bar{D}_2 e^{-ib_1 L/2} = -\bar{\bar{D}}_2 \qquad (5\text{–}16b)$$

where $\bar{D}_2 = (\frac{1}{2} e^{a_1 L/2})D_2$ and $\bar{\bar{D}}_2 = (\frac{1}{2} e^{-\lambda_2 L/2})D_2$. That is, we can replace D_2 by $\bar{D}_2$ and replace $\cosh \lambda_2 L/2$ and $\sinh \lambda_2 L/2$ by $e^{-ib_1 L/2}$ and $-e^{-ib_1 L/2}$, respectively. Also, alternatively, we can replace D_2, $\cosh \lambda_2 L/2$, and $\sinh \lambda_2 L/2$, respectively, by $\bar{\bar{D}}_2$, 1, and –1. Similar arguments can be made for the remaining columns in [K].

5.1.5 <u>Axisymmetric Buckling (m = 0)</u>

In the special case when the buckling takes place with the axisymmetric mode, the generalized displacement components will be functions of x_1 only, and Eqs. (5–1) reduce to

$$A_{11} U_o'' + A_{16} V_o'' + B_{11} X_o'' + B_{16} Y_o'' + \frac{A_{12}}{R} W_o' = 0$$

$$A_{16}U_o'' + A_{66}V_o'' + B_{16}X_o'' + B_{66}Y_o'' + \frac{A_{26}}{R} W_o' = 0$$

$$B_{11}U_o'' + B_{16}V_o'' + D_{11}X_o'' - K_{55}^2 A_{55} X_o + D_{16}Y_o'' - K_{45}^2 A_{45} Y_o$$

$$+ \left[\frac{B_{12}}{R} - K_{55}^2 A_{55}\right] W_o' = 0$$

$$B_{16}U_o'' + B_{66}V_o'' + D_{16}X_o'' - K_{45}^2 A_{45} X_o + D_{66}Y_o'' - K_{44}^2 A_{44} Y_o$$

$$+ \left[\frac{B_{26}}{R} - K_{45}^2 A_{45}\right] W_o' = 0$$

$$\frac{A_{12}}{R} U_o' + \frac{A_{26}}{R} V_o' + \left[\frac{B_{12}}{R} - K_{55}^2 A_{55}\right] X_o' + \left[\frac{B_{26}}{R} - K_{45}^2 A_{45}\right] Y_o'$$

$$+ (\bar{N} - K_{55}^2 A_{55}) W_o'' + \frac{A_{22}}{R^2} W_o = 0 \qquad (5\text{–}17)$$

where a derivative with respect to x_1 is denoted by a prime. By assuming that

$$(U_o, V_o, X_o, Y_o, W_o) = (A, B, C, D, E)e^{\lambda x_1} \qquad (5\text{–}18)$$

the general solution of Eqs. (5–17) can be shown to be

$$\begin{Bmatrix} U_o \\ V_o \\ X_o \\ Y_o \\ W_o \end{Bmatrix} = \sum_{j=1}^{6} \begin{Bmatrix} A_j \\ B_j \\ C_j \\ D_j \\ E_j \end{Bmatrix} e^{\lambda_j x_1} + \begin{Bmatrix} E_7 + E_8 x_1 + B_7 x_1^2 + B_8 x_1^3 \\ E_9 + E_{10} x_1 + B_9 x_1^2 + B_{10} x_1^3 \\ C_7 + C_8 x_1 + C_9 x_1^2 + C_{10} x_1^3 \\ D_7 + D_8 x_1 + D_9 x_1^2 + D_{10} x_1^3 \\ A_7 + A_8 x_1 + A_9 x_1^2 + A_{10} x_1^3 \end{Bmatrix} \tag{5-19}$$

where λ_j ($j = \overline{1,6}$) are the roots of

$$\theta_1(\lambda^2)^3 + \theta_2(\lambda^2)^2 + \theta_3(\lambda^2) + \theta_4 = 0. \tag{5-20}$$

The expressions of θ_j ($j = \overline{1,4}$) are displayed in Appendix 5.1. The relationship among the unknown constants appearing in Eq. (5–19) are obtained with the help of Eqs. (5–17) and the general solution finally can be presented as

$$\begin{Bmatrix} U_o \\ V_o \\ X_o \\ Y_o \\ W_o \end{Bmatrix} = \sum_{j=1}^{6} \begin{Bmatrix} u_{1j} \\ u_{2j} \\ u_{3j} \\ u_{4j} \\ u_{5j} \end{Bmatrix} \cdot e^{\lambda_j x_1} E_j + \begin{Bmatrix} E_7 + E_8 x_1 \\ E_9 + E_{10} x_1 \\ 0 \\ 0 \\ u_{58} E_8 + u_{510} E_{10} \end{Bmatrix} \tag{5-21}$$

where the expressions of u_{ij} ($i = \overline{1,5}$ and $j = \overline{1,6}$), u_{58}, and u_{510} are given in Appendix 5.1. For an axisymmetric problem the stress resultant in Eqs. (5–2) reduce to

$$M_1 = B_{11}U_o' + B_{16}V_o' + D_{11}X_o' + D_{16}Y_o' + \frac{B_{12}}{R} W_o \tag{5–22a}$$

$$M_6 = B_{16}U_o' + B_{66}V_o' + D_{16}X_o' + D_{66}Y_o' + \frac{B_{26}}{R} W_o \tag{5–22b}$$

$$N_1 = A_{11}U_o' + A_{16}V_o' + B_{11}X_o' + B_{16}Y_o' + \frac{A_{12}}{R} W_o \tag{5–22c}$$

$$N_6 = A_{16}U_o' + A_{66}V_o' + B_{16}X_o' + B_{66}Y_o' + \frac{A_{26}}{R} W_o \tag{5–22d}$$

$$Q_1 = K_{55}^2 A_{55} X_o + K_{45}^2 A_{45} Y_o + K_{55}^2 A_{55} W_o'. \tag{5–22e}$$

Further, by substituting Eqs. (5–21) and the expressions for u_{ij} into Eqs. (5–22c) and (5–22d), we obtain

$$N_1 = \left[A_{11} + \frac{A_{12}}{R} u_{58}\right] E_8 + \left[A_{16} + \frac{A_{12}}{R} u_{510}\right] E_{10}$$

$$N_6 = \left[A_{16} + \frac{A_{26}}{R} u_{58}\right] E_8 + \left[A_{66} + \frac{A_{26}}{R} u_{510}\right] E_{10} \tag{5–23}$$

It is to be noted that the unknown constants E_7 and E_9 appear only in the expressions of U_o and V_o, respectively. Also only E_8 and E_{10} appear in the expressions of N_1 and N_6. Therefore when we impose the boundary conditions (see Eqs. (4–30)–(4–32)), we will have the following four cases:

CASE 1

For the boundary types S4–S4 and C4–C4 we obtain a system of ten homogeneous algebraic equations of the form

$$[K]\{E\} = \{0\} \tag{5–24}$$

which involve all the ten unknown constants E_1 through E_{10}.

CASE 2

For the boundary types S3–S3 and C3–C3 we obtain a system of nine homogeneous algebraic equations of the form

$$[K] = \begin{Bmatrix} E_1 \\ \vdots \\ E_6 \\ E_8 \\ E_9 \\ E_{10} \end{Bmatrix} = \{0\} \tag{5–25}$$

not involving the unknown constant E_7.

Case 3

For the boundary types S2–S2 and C2–C2 we obtain a system of nine homogeneous algebraic equations of the form

$$[K]\begin{Bmatrix} E_1 \\ \vdots \\ E_8 \\ E_{10} \end{Bmatrix} = \{0\} \tag{5–26}$$

not involving E_9.

Case 4

For the boundary types S1–S1, C1–C1, and F–F we obtain a system of six homogeneous algebraic equations of the form

$$[K]\begin{Bmatrix} E_1 \\ \vdots \\ E_6 \end{Bmatrix} = \{0\} \tag{5–27}$$

not involving E_j $(j = \overline{7,10})$. In Case 4 we will have $E_8 = E_{10} = 0$. For a nontrivial solution, the determinant of the coefficient matrix in Eqs. (5–24)–(5–27) must be set to zero. Finally, by the procedure that we developed in Section 4.2.2 the complex determinant $|K|$ can readily be transformed to a real one.

It should be recalled that here we assumed the shell is anisotropic. For a cross–ply shell, however, V_0 and Y_0 will be equal to zero, as we noted in Chapter IV, and, therefore, the general solution of the axisymmetric buckling problem developed in Chapter IV should be used.

5.2.1 Analysis Based on GLST

The small–displacement equilibrium equations of circular cylindrical shells according to the generalized layer–wise shell theory (GLST) were presented in Chapter III. Here, without presenting all the details of the generally known statements and definitions on which the stability of elastic shells is based (see [119], for example), we present the approximate stability equations of circular cylindrical shells according to GLST as

$$\delta u^j:\quad M^j_{1,x} + M^j_{6,y} - Q^j_1 = 0$$

$$\delta v^j:\quad M^j_{2,y} + M^j_{6,x} + \frac{1}{R}K^j_2 - Q^j_2 = 0$$

$$\delta w^j:\quad K^j_{1,x} + K^j_{2,y} - \frac{1}{R}M^j_2 - Q^j_3 - \bar{N}w^{\bar{k}}_{,xx}\cdot\delta_{\bar{k}j} = 0 \quad (j = 1,\ldots,N+1) \tag{5–28a}$$

where

$$\delta_{\bar{k}j} = \begin{cases} 0, \text{ when } j \neq \bar{k}(\equiv \frac{1}{2}N + 1) \\ 1, \text{ when } j = \bar{k}(\equiv \frac{1}{2}N + 1) \end{cases} \tag{5–28b}$$

and it is assumed that $\bar{N}$ is acting at the midsurface of the shell and that there are even number of mathematical layers in the shell. However it is possible to remove this restriction by distributing the $\bar{N}$ over the shell thickness. Here, for convenience, we have replaced x_1 and x_2 appearing in Eqs. (3–9a) by x and y, respectively. It should be recalled that u^j, v^j, and w^j appearing in Eqs. (5–28) are small increments to the equilibrium position. Generally, the coefficient $\bar{N}$ should be obtained from the axisymmetric prebuckling deformation of cylindrical shells entailing both bending and stretching. When this is done, this coefficient will be a function of x. However, for simplicity, this coefficient is usually determined from the linear axisymmetric membrane equations. The major analytical advantage of this assumption is that the stability equations will have a constant coefficient rather than a variable coefficient.

In stability analysis, the specification of boundary conditions at $x = \pm L/2$ are slightly different from those given by Eqs. (3–10). Here, at $x = \pm L/2$, we specify

$$M_1^j \quad \text{or} \quad u^j$$

$$M_6^j \quad \text{or} \quad v^j$$

$$K_1^j - \bar{N} w^{\bar{k}}_{,x} \quad \text{or} \quad w^j \tag{5–29}$$

5.2.2 Stability Equations in Terms of Generalized Displacements

Substituting the stress resultants given by Eqs. (3–18a) into Eqs. (5–28) results in:

$$\sum_{i=1}^{N+1} D_{11}^{ij} u^i_{,xx} + \sum_{i=1}^{N+1} D_{66}^{ij} u^i_{,yy} - \sum_{i=1}^{N+1} \bar{\bar{D}}_{55}^{ij} u^i + \sum_{i=1}^{N+1} 2D_{16}^{ij} u^i_{,xy}$$

$$+ \sum_{i=1}^{N+1} D_{16}^{ij} v^i_{,xx} + \sum_{i=1}^{N+1} D_{26}^{ij} v^i_{,yy} + \sum_{i=1}^{N+1} \left(\frac{1}{R} \bar{D}_{45}^{ji} - \bar{\bar{D}}_{45}^{ji}\right) v^i$$

$$+ \sum_{i=1}^{N+1} (D_{12}^{ij} + D_{66}^{ij}) v^i_{,xy} + \sum_{i=1}^{N+1} \left(\bar{D}_{36}^{ij} + \frac{1}{R} D_{26}^{ij} - \bar{D}_{45}^{ji}\right) w^i_{,y}$$

$$+ \sum_{i=1}^{N+1} \left(\frac{1}{R} D_{12}^{ij} + \bar{D}_{13}^{ij} - \bar{D}_{55}^{ji}\right) w^i_{,x} = 0 \qquad (5\text{–}30a)$$

$$\sum_{i=1}^{N+1} D_{16}^{ij} u^i_{,xx} + \sum_{i=1}^{N+1} D_{26}^{ij} u^i_{,yy} + \sum_{i=1}^{N+1} \left(\frac{1}{R} \bar{D}_{45}^{ij} - \bar{\bar{D}}_{45}^{ij}\right) u^i$$

$$+ \sum_{i=1}^{N+1} (D_{12}^{ij} + D_{66}^{ij}) u^i_{,xy} + \sum_{i=1}^{N+1} D_{66}^{ij} v^i_{,xx} + \sum_{i=1}^{N+1} D_{22}^{ij} v^i_{,yy}$$

$$+ \sum_{i=1}^{N+1} \left(\frac{1}{R} \bar{\bar{D}}_{44}^{ij} - \frac{1}{R^2} D_{44}^{ij} - \bar{\bar{D}}_{44}^{ij} + \frac{1}{R} \bar{\bar{D}}_{44}^{ji}\right) v^i + \sum_{i=1}^{N+1} 2D_{26}^{ij} v^i_{,xy}$$

$$+ \sum_{i=1}^{N+1} \left(\bar{D}_{23}^{ij} + \frac{1}{R} D_{22}^{ij} + \frac{1}{R} D_{44}^{ij} - \bar{D}_{44}^{ji}\right) w^i_{,y}$$

$$+ \sum_{i=1}^{N+1} \left(\bar{D}_{36}^{ij} + \frac{1}{R} D_{26}^{ij} + \frac{1}{R} D_{45}^{ij} - \bar{D}_{45}^{ji}\right) w^i_{,x} = 0 \qquad (5\text{–}30b)$$

$$\sum_{i=1}^{N+1} \left(\bar{D}_{45}^{ij} - \bar{D}_{36}^{ji} - \frac{1}{R} D_{26}^{ij}\right) u^i_{,y} - \sum_{i=1}^{N+1} \left(\bar{D}_{13}^{ji} + \frac{1}{R} D_{12}^{ij} - \bar{D}_{55}^{ij}\right) u^i_{,x}$$

$$+\sum_{i=1}^{N+1}(\bar{D}_{44}^{ij}-\frac{1}{R}D_{44}^{ij}-\frac{1}{R}D_{22}^{ij}-\bar{D}_{23}^{ji})v^{i}_{,y}$$

$$+\sum_{i=1}^{N+1}(\bar{D}_{45}^{ij}-\frac{1}{R}D_{45}^{ij}-\frac{1}{R}D_{26}^{ij}-\bar{D}_{36}^{ji})v^{i}_{,x}+\sum_{i=1}^{N+1}D_{55}^{ij}w^{i}_{,xx}$$

$$-\sum_{i=1}^{N+1}\bar{N}w^{i}_{,xx}\delta_{\bar{k}i}\delta_{\bar{k}j}+\sum_{i=1}^{N+1}D_{44}^{ij}w^{i}_{,yy}+\sum_{i=1}^{N+1}2D_{45}^{ij}w^{i}_{,xy}$$

$$-\sum_{i=1}^{N+1}(\frac{1}{R^2}D_{22}^{ij}+\frac{1}{R}\bar{D}_{23}^{ij}+\frac{1}{R}\bar{D}_{23}^{ji}+\bar{\bar{D}}_{33}^{ij})w^{i}=0 \quad (j=1,\ldots,N+1)$$

(5–30c)

Following the procedure introduced in Section 5.1.3 we assume

$$u^{i}(x,y)=\mathrm{Re}\{U_{m}^{i}(x)e^{\bar{i}\beta_m y}\}$$

$$v^{i}(x,y)=\mathrm{Re}\{V_{m}^{i}(x)e^{\bar{i}\beta_m y}\}$$

$$w^{i}(x,y)=\mathrm{Re}\{W_{m}^{i}(x)e^{\bar{i}\beta_m y}\} \qquad (i=1,\ldots,N+1) \quad (5\text{–}31)$$

where $\beta_m=\frac{m}{R}$, Re indicates the real part, and $\bar{i}=\sqrt{-1}$. We substitute Eqs. (5–31) into Eqs. (5–30) and obtain:

$$\sum_{i=1}^{N+1}a_1^{ij}U_m^{i\,\prime\prime}+\sum_{i=1}^{N+1}a_2^{ij}V_m^{i\,\prime\prime}=\sum_{i=1}^{N+1}a_3^{ij}U_m^{i}+\sum_{i=1}^{N+1}a_4^{ij}U_m^{i\,\prime}$$

$$+\sum_{i=1}^{N+1} a_5^{ij}V_m^i + \sum_{i=1}^{N+1} a_6^{ij}V_m^{i\,\prime} + \sum_{i=1}^{N+1} a_7^{ij}W_m^i + \sum_{i=1}^{N+1} a_8^{ij}W_m^{i\,\prime}$$

(5–32a)

$$\sum_{i=1}^{N+1} b_1^{ij}U_m^{i\,\prime\prime} + \sum_{i=1}^{N+1} b_2^{ij}V_m^{i\,\prime\prime} = \sum_{i=1}^{N+1} b_3^{ij}U_m^i + \sum_{i=1}^{N+1} b_4^{ij}U_m^{i\,\prime}$$

$$+\sum_{i=1}^{N+1} b_5^{ij}V_m^i + \sum_{i=1}^{N+1} b_6^{ij}V_m^{i\,\prime} + \sum_{i=1}^{N+1} b_7^{ij}W_m^i + \sum_{i=1}^{N+1} b_8^{ij}W_m^{i\,\prime}$$

(5–32b)

$$\sum_{i=1}^{N+1} c_1^{ij}W_m^{i\,\prime\prime} = \sum_{i=1}^{N+1} c_2^{ij}U_m^i + \sum_{i=1}^{N+1} c_3^{ij}U_m^{i\,\prime} + \sum_{i=1}^{N+1} c_4^{ij}V_m^i$$

$$+\sum_{i=1}^{N+1} c_5^{ij}V_m^{i\,\prime} + \sum_{i=1}^{N+1} c_6^{ij}W_m^i + \sum_{i=1}^{N+1} c_7^{ij}W_m^{i\,\prime} \quad j = 1,\ldots,N+1$$

(5–32c)

where a prime denotes differentiation with respect to x. In Eqs. (5–32) we have

$$a_1^{ij} = D_{11}^{ij},\ a_2^{ij} = D_{16}^{ij},\ a_3^{ij} = \beta_m^2 D_{66}^{ij} + \bar{\bar{D}}_{55}^{ij}$$

$$a_4^{ij} = -2\bar{i}\beta_m D_{16}^{ij},\ a_5^{ij} = \beta_m^2 D_{26}^{ij} - (\frac{1}{R}\bar{D}_{45}^{ji} - \bar{\bar{D}}_{45}^{ij})$$

$$a_6^{ij} = -\bar{i}\beta_m(D_{12}^{ij} + D_{66}^{ij}),\ a_7^{ij} = -\bar{i}\beta_m(\bar{D}_{36}^{ij} + \frac{1}{R}D_{26}^{ij} - \bar{D}_{45}^{ji})$$

$$a_8^{ij} = -\left(\frac{1}{R} D_{12}^{ij} + \bar{D}_{13}^{ij} - \bar{D}_{55}^{ji}\right),\ b_1^{ij} = D_{16}^{ij},\ b_2^{ij} = D_{66}^{ij}$$

$$b_3^{ij} = \beta_m^2 D_{26}^{ij} - \left(\frac{1}{R} \bar{D}_{45}^{ij} - \bar{\bar{D}}_{45}^{ij}\right),\ b_4^{ij} = -\bar{i}\beta_m\left(D_{12}^{ij} + D_{66}^{ij}\right)$$

$$b_5^{ij} = \beta_m^2 D_{22}^{ij} - \left(\frac{1}{R} \bar{D}_{44}^{ij} - \frac{1}{R^2} D_{44}^{ij} - \bar{\bar{D}}_{44}^{ij} + \frac{1}{R} \bar{D}_{44}^{ji}\right)$$

$$b_6^{ij} = -2\bar{i}\beta_m D_{26}^{ij},\ b_7^{ij} = -\bar{i}\beta_m\left(\bar{D}_{23}^{ij} + \frac{1}{R} D_{22}^{ij} + \frac{1}{R} D_{44}^{ij} - \bar{D}_{44}^{ji}\right)$$

$$b_8^{ij} = -\left(\bar{D}_{36}^{ij} + \frac{1}{R} D_{26}^{ij} + \frac{1}{R} D_{45}^{ij} - \bar{D}_{45}^{ji}\right),\ c_1^{ij} = D_{55}^{ij} - \bar{N}\delta_{\bar{k}i}\delta_{\bar{k}j}$$

$$c_2^{ij} = -\bar{i}\beta_m\left(\bar{D}_{45}^{ij} - \bar{D}_{36}^{ji} - \frac{1}{R} D_{26}^{ij}\right),\ c_3^{ij} = \bar{D}_{13}^{ji} + \frac{1}{R} D_{12}^{ij} - \bar{D}_{55}^{ij}$$

$$c_4^{ij} = -\bar{i}\beta_m\left(\bar{D}_{44}^{ij} - \frac{1}{R} D_{44}^{ij} - \frac{1}{R} D_{22}^{ij} - \bar{D}_{23}^{ji}\right)$$

$$c_5^{ij} = -\left(\bar{D}_{45}^{ij} - \frac{1}{R} D_{45}^{ij} - \frac{1}{R} D_{26}^{ij} - \bar{D}_{36}^{ji}\right)$$

$$c_6^{ij} = \beta_m^2 D_{44}^{ij} + \left(\frac{1}{R^2} D_{22}^{ij} + \frac{1}{R} \bar{D}_{23}^{ij} + \frac{1}{R} \bar{D}_{23}^{ji} + \bar{\bar{D}}_{23}^{ij}\right),\ c_7^{ij} = -2\bar{i}\beta_m D_{45}^{ij} \tag{5–33}$$

Equations (5–32) are (3N + 3) coupled ordinary differential equations with a total order of (6N + 6). To solve these equations, we first replace them with a new system of first–order equations by introducing the following variables:

$$
\begin{array}{lcl}
Z_1 = U_m^1 & ; & Z_{N+2} = U_m^{1\,\prime} \rightarrow U_m^{1\,\prime\prime} = Z'_{N+2} \\
Z_2 = U_m^2 & ; & Z_{N+3} = U_m^{2\,\prime} \rightarrow U_m^{2\,\prime\prime} = Z'_{N+3} \\
\vdots & \vdots & \vdots \\
Z_{N+1} = U_m^{N+1} & ; & Z_{2N+2} = U_m^{N+1\,\prime} \rightarrow U_m^{N+1\,\prime\prime} = Z'_{2N+2} \\
Z_{2N+3} = V_m^1 & ; & Z_{3N+4} = V_m^{1\,\prime} \rightarrow V_m^{1\,\prime\prime} = Z'_{3N+4} \\
Z_{2N+4} = V_m^2 & ; & Z_{3N+5} = V_m^{2\,\prime} \rightarrow V_m^{2\,\prime\prime} = Z'_{3N+5} \\
\vdots & & \vdots \\
Z_{3N+3} = V_m^{N+1} & ; & Z_{4N+4} = V_m^{N+1\,\prime} \rightarrow V_m^{N+1\,\prime\prime} = Z'_{4N+4} \\
Z_{4N+5} = W_m^1 & ; & Z_{5N+6} = W_m^{1\,\prime} \rightarrow W_m^{1\,\prime\prime} = Z'_{5N+6} \\
Z_{4N+6} = W_m^2 & ; & Z_{5N+7} = W_m^{2\,\prime} \rightarrow W_m^{2\,\prime\prime} = Z'_{5N+7} \\
\vdots & & \vdots \\
Z_{5N+5} = W_m^{N+1} & ; & Z_{6N+6} = W_m^{N+1\,\prime} \rightarrow W_m^{N+1\,\prime\prime} = Z'_{6N+6}.
\end{array}
\tag{5–34}
$$

Introducing Eqs. (5–34) into Eqs. (5–32) yields:

$$[C]\{Z'\} = [B]\{Z\} \tag{5–35}$$

where

$$\{Z'\} = \{Z'_1, Z'_2, \ldots, Z'_{6N+6}\}^T \tag{5–36a}$$

and

$$\{Z\} = \{Z_1, Z_2, \ldots, Z_{6N+6}\}^T. \tag{5–36b}$$

Equations (5–35) can alternatively be presented as

$$\{Z'\} = [A]\{Z\} \tag{5–37a}$$

where

$$[A] = [C]^{-1}[B]. \tag{5–37b}$$

The expressions of the elements of the matrices [B] and [C] are given in Appendix 5.2. A formal solution for Eqs. (5–37a) can be shown (see [114,115]) to be

$$\{Z\} = [U]\begin{bmatrix} e^{\lambda_1 x} & & & \mathcal{Q} \\ & e^{\lambda_2 x} & & \\ & & \ddots & \\ \mathcal{Q} & & & e^{\lambda_{6N+6} x} \end{bmatrix}\{D\} \tag{5–38}$$

where $\lambda_1 \ldots \lambda_{6N+6}$ are the distinct eigenvalues of the general complex matrix [A] and [U] is the modal matrix. That is, the columns of [U] constitute the eigenvectors of [A] corresponding to $\lambda_1 \ldots \lambda_{6N+6}$. Also {D} is an unknown vector. Substituting Eq. (5–38) into Eqs. (5–34) yields:

$$\begin{Bmatrix} U_m^i \\ V_m^i \\ W_m^i \end{Bmatrix} = \sum_{j=1}^{6N+6} \begin{Bmatrix} u_{i\,j} \\ u_{\ell+i\,j} \\ u_{2\ell+ij} \end{Bmatrix} (\cosh \lambda_j x + \sinh \lambda_j x) D_j \tag{5–39a}$$

where $i = 1,\ldots, N + 1$, u_{ij} are the elements of [U], and

$$\ell = 2N + 2. \tag{5–39b}$$

Based on the boundary types given by Eqs. (4–30)–(4–32) in the first–order shear deformation shell theory (FSDST) we classify the following boundary types in GLST (at $x = \pm L/2$):

S1:

$$W_m^j = M_1^j = 0;\ j = 1,\ldots,N + 1$$

$$M_6^{\bar{k}} = 0,\ V_m^j = V_m^{j+1};\ j = 1,\ldots,N$$

S2:

$$W_m^j = 0;\ j = 1,\ldots,N + 1$$

$$M_1^j = 0;\ j = 1,\ldots, \frac{N}{2} \text{ and } \frac{N}{2} + 2,\ldots,N + 1$$

$$U_m^{\bar{k}} = 0,\ M_6^{\bar{k}} = 0,\ V_m^j = V_m^{j+1};\ j = 1,\ldots,N$$

S3:

$$W_m^j = V_m^j = M_1^j = 0;\ j = 1,\ldots,N + 1$$

S4:

$$W_m^j = V_m^j = 0;\ j = 1,\ldots,N + 1$$

$$M_1^j = 0;\ j = 1,\ldots, \frac{N}{2} \text{ and } \frac{N}{2} + 2,\ldots,N + 1$$

$$U_m^{\bar{k}} = 0$$

<u>C1</u>:

$$W_m^j = 0; \ j = 1,\ldots,N+1$$

$$M_1^{\bar{k}} = M_6^{\bar{k}} = 0, \ U_m^j = U_m^{j+1}; \ j = 1,\ldots,N$$

$$V_m^j = V_m^{j+1}; \ j = 1,\ldots,N$$

<u>C2</u>:

$$W_m^j = U_m^j = 0; \ j = 1,\ldots,N+1$$

$$M_6^{\bar{k}} = 0, \ V_m^j = V_m^{j+1}; \ j = 1,\ldots,N$$

<u>C3</u>:

$$W_m^j = V_m^j = 0; \ j = 1,\ldots,N+1$$

$$M_1^{\bar{k}} = 0, \ U_m^j = U_m^{j+1}; \ j = 1,\ldots,N$$

<u>C4</u>:

$$W_m^j = U_m^j = V_m^j = 0, \ j = 1,\ldots,N+1$$

F:

$$M_1^j = M_6^j = K_1^j - \bar{N}w^{\bar{k}}_{,x}\,\delta_{\bar{k}j} = 0,\ j = 1,\ldots,N + 1 \qquad (5\text{–}40)$$

with $\bar{k} = \frac{1}{2}N + 1$. Assuming that the edges of the shell at $x = \pm L/2$ have identical boundary conditions, imposing Eqs. (5–40) on Eqs. (5–39a) results in a system of (6N + 6) homogeneous algebraic equations of the form

$$[K]\{D\} = \{0\} \qquad (5\text{–}41)$$

For a nontrivial solution we set $|K| = 0$.

5.3 Analysis Based on LST

The stability analysis of anisotropic circular cylindrical shells based on the layer–wise shell theory (LST) is very similar to the one based on GLST. Here, again, without going through the linearization of the nonlinear equations, we present the approximate stability equations of LST (see Eqs. (3–6)) as:

$$\delta u^j:\quad M^j_{1,x} + M^j_{6,y} - Q^j_1 = 0$$

$$\delta v^j:\quad M^j_{2,y} + M^j_{6,x} + \frac{1}{R}K^j_2 - Q^j_2 = 0 \qquad (j = 1, 2, \ldots, N+1)$$

$$\delta w:\quad Q_{1,x} + Q_{2,y} - \frac{1}{R}N_2 - \bar{N}w_{,xx} = 0 \qquad (5\text{–}42)$$

where the stress resultants are defined in Eqs. (3–16a). With the help of Eqs. (3–16a), Eqs. (5–42) can be expressed in terms of u^i, v^i, and w. As we mentioned in Chapter III, in LST we assume that the radial displacement w is constant through the thickness of the shell. By assuming

$$u^i(x,y) = \mathrm{Re}\{U_m^i(x)e^{\bar{i}\beta_m y}\}$$

$$v^i(x,y) = \mathrm{Re}\{V_m^i(x)e^{\bar{i}\beta_m y}\}$$

$$w(x,y) = \mathrm{Re}\{W_m(x)e^{\bar{i}\beta_m y}\} \tag{5–43}$$

and introducing the new variables $Z_1 = U_m^1$,... up to $Z_{4N+4} = V_m^{N+1\prime}$ as in Eqs. (5–34) and $Z_{4N+5} = W_m$ and $Z_{4N+6} = W_m'$, the stability equations of LST can be presented as a system of first–order equations similar to Eq. (5–35). The solution procedure of these equations and the imposition of boundary conditions closely follow our developments for FSDST and GLST. For completeness, however, here we classify the boundary types at $x = \pm L/2$ in LST as:

<u>S1</u>:

$$W_m = 0,\ M_1^j = 0;\ j = 1,...,N + 1$$

$$M_6^{\bar{k}} = 0,\ V_m^j = V_m^{j+1};\ j = 1,...,N$$

<u>S2</u>:

$$W_m = 0,\ M_1^j = 0;\ j = 1,...,\ \frac{N}{2} \text{ and } \frac{N}{2} + 2,...,\ N + 1$$

$$U_m^{\bar{k}} = M_6^{\bar{k}} = 0,\ V_m^j = V_m^{j+1};\ j = 1,...,\ N$$

S3:

$$W_m = V_m^j = M_1^j = 0;\ j = 1, \ldots, N+1$$

S4:

$$W_m = U_m^{\bar{k}} = 0,\ V_m^j = 0;\ j = 1, \ldots, N+1$$

$$M_1^j = 0;\ j = 1, \ldots, \frac{N}{2} \text{ and } \frac{N}{2} + 2, \ldots, N+1$$

C1:

$$W_m = M_1^{\bar{k}} = M_6^{\bar{k}} = 0$$

$$U_m^j = U_m^{j+1},\ V_m^j = V_m^{j+1};\ j = 1, \ldots, N$$

C2:

$$W_m = M_6^{\bar{k}} = 0,\ U_m^j = 0;\ j = 1, \ldots, N+1$$

$$V_m^j = V_m^{j+1};\ j = 1, \ldots, N$$

C3:

$$W_m = M_1^{\bar{k}} = 0,\ V_m^j = 0;\ j = 1,\ldots,N + 1$$

$$U_m^j = U_m^{j+1};\ j = 1,\ldots,N$$

C4:

$$W_m = U_m^j = V_m^j = 0;\ j = 1,\ldots,N + 1$$

F:

$$M_1^j = M_6^j = 0;\ j = 1,\ldots,N + 1$$

$$Q_1 - \bar{N}w_{,x} = 0 \tag{5–44}$$

where $\bar{k} = \frac{1}{2}N + 1$. At this point it should be pointed out that no separate analysis is required for the axisymmetric buckling mode ($m = 0$) since the eigenvalues will not be repeated. This is also the case as far as GLST is concerned. However, in axisymmetric buckling analysis the magnitude of some of the eigenvalues will be very close to zero and a quadruple precision must be used in the computer programming.

5.4 Numerical Results and Discussions

For simplicity we assume that the thicknesses and the material properties of all laminae are the same. The total thickness of the shells is assumed to be 1 inch and the material properties of a lamina are taken to be

$$E_1 = 10\ E_2 = 10\ E_3\ ,\ G_{12} = G_{13} = 0.6\ E_2,$$

$$G_{23} = 0.5\ E_2, \text{ and } \nu_{12} = \nu_{13} = \nu_{23} = 0.25.$$

Also the shear correction factor $K^2(\equiv K_{44}^2 = K_{45}^2 = K_{55}^2)$ in FSDST is assumed to be 5/6.

The dimensionless critical buckling loads of (0/45/–45/90) laminated shells having various clamped boundary types are presented in Table 5.1. A number in parentheses following the buckling load denotes the circumferential mode number m. It is to be recalled that the 0–degree layer is the outside layer of the shell. In a 0–degree layer and a 90–degree layer the fibers are oriented, respectively, in the longitudinal and circumferential directions of the shell. It is noted from Table 5.1 that the critical buckling loads of the shell with various clamped boundary types are only slightly changed. Further, in some cases the buckling loads obtained within GLST and LST are slightly larger than those predicted by FSDST. The results of FSDST depend, of course, on the assumed value for the shear correction factors. On the other hand, from the assumed displacement fields in GLST and LST it is noted that each layer in the shell can be assumed to be composed of many layers in these theories. In Table 5.1 we have assumed that a (0/45/–45/90) shell is composed of four layers in GLST and LST. However, if more accurate results are sought, each layer should be modeled as if it were composed of many layers. This is demonstrated in Table 5.2 where a (0/45/–45/90) laminated shell is once modeled as a four–layer shell and then as an eight–layer shell. In an eight–layer shell, each layer in the (0/45/–45/90) shell is considered as two layers. It is noted that, when this is done, the critical buckling loads obtained within GLST and LST are smaller than those obtained within FSDST. This is also demonstrated in Table 5.3 where the buckling loads of a homogeneous isotropic shell with boundary type S3–S3 are determined according to various theories. It is to be recalled that in FSDST and CLST a laminated shell is modeled as an equivalent single–layer shell.

In order to put into evidence the importance of fiber orientation, we have plotted in Figs. 5.1 and 5.2, the variation of dimensionless critical buckling load according to FSDST vs. θ for the following three–layer shells with S3–S3 boundary type:

Shell 1: $(0/-\theta/\theta)$

Shell 2: $(\theta/-\theta/0)$

Shell 3: $(\theta/0/-\theta)$

Shell 4: $(\theta/90/-\theta)$

The numerical results are generated for values of θ ranging from 0° to 90° with 5° increments. In the $(\theta/-\theta/0)$ laminated shell the maximum critical buckling load occurs when θ is approximately equal to 41°. Also for $0 \le \theta \le 90$ the circumferential mode number assumes values between 8 and 13. The minimum critical buckling load occurs when $\theta = 85°$. The ratio of the maximum load to the minimum load is equal to 1.34. For the $(0/-\theta/\theta)$ shell we have also plotted the minimum axisymmetric ($m = 0$) buckling loads in Fig. 5.1. It is seen that for $55° < \theta < 65°$ the minimum axisymmetric loads are slightly smaller than the minimum asymmetric loads. For $0^\circ \leq \theta < 40^\circ$ and $70^\circ < \theta \leq 90^\circ$ the circumferential mode number m is larger than 2 whereas for $40 < \theta < 70$ m is smaller than 2. For this shell the ratio of maximum critical load to minimum critical load is 1.37 (see Fig. 5.1). For shells 3 and 4 this ratio is equal to 1.29 and 1.33, respectively. The critical buckling loads of Shells 1 through 4 are displayed in Table 5.4 for various fiber orientation angle θ. The values of θ approximately represent the fiber orientations at which the minimum and maximum critical buckling loads occur for Shells 1 through 4 (see Figs. 5.1 and 5.2). For the $(\theta/90/-\theta)$ shell the critical buckling loads are predicted by asymmetric buckling modes with m being larger than 4 and smaller than 10. For the $(\theta/0/-\theta)$ shell the minimum axisymmetric buckling loads are slightly smaller than the minimum asymmetric buckling loads for $36^\circ < \theta < 44^\circ$ (see Fig. 5.2).

5.5 Conclusions

In this chapter we have developed a general solution for the stability analysis of generally laminated shells. It is found by Wu [126] that when m = 2 and L/R = 100 there is a considerable difference between the results obtained from Donnell's classical theory and Flugge's theory. This is expected since, as we mentioned in Chapter V, Donnell–type theories are accurate so long as m is larger than 2 and the shell is not very long.

It is found that changing the lamination scheme and fiber winding angle can produce a considerable decrease or increase in the critical axial buckling load of a laminated shell. It should be pointed out that the general solution procedure presented here can also be used for the free–vibration analysis and the stability analysis of anisotropic shells under combined loadings.

Table 5.1

The effects of various clamped boundary types on the dimensionless critical buckling load T_{11} $(\equiv \frac{\bar{N}L^2}{100E_2h^3})$ of a (0/45/–45/90) laminated shell; $\frac{R}{h} = 40$ and $\frac{L}{R} = 2$.

Theory	C1–C1	C2–C2	C3–C3	C4–C4
GLST	2.2401(6)	2.2500(6)	2.2542(6)	2.2586(6)
LST	2.2413(6)	2.2511(6)	2.2554(6)	2.2597(6)
FSDST	2.2483(6)	2.2500(6)	2.2573(6)	2.2584(6)
CLST	2.2915(6)	2.2934(6)	2.3016(6)	2.3029(6)

Table 5.2

Effects of number of layers on the dimensionless critical buckling load T_{11} $(\equiv \frac{\bar{N}L^2}{100E_2h^3})$ of a (0/45/–45/90) laminated shell, $\frac{R}{h} = 40$, $\frac{L}{R} = 2$.

Boundary	Theory	T_{11}	Number of Layers
C4–C4	GLST	2.2586(6)	4
		2.2548(6)	8
	LST	2.2597(6)	4
		2.2579(6)	8
	FSDST	2.2558(6)	4
	CLST	2.3029(6)	4
S3–S3	GLST	2.2337(6)	4
		2.2295(6)	8
	LST	2.2348(6)	4
		2.2327(6)	8
	FSDST	2.2339(6)	4
	CLST	2.2755(6)	4

Table 5.3

Dimensionless critical buckling load T_{11} $(\equiv \frac{\bar{N}L^2}{Eh^3})$ of an isotropic shell according to various theories; boundary type is S3–S3, $\frac{R}{h} = 30$, $\frac{L}{R} = 1$, and $\nu = 0.25$.

Theory	T_{11}	Number of Layers
	17.7701(2)	2
GLST	17.5587(2)	4
	17.5048(2)	6
	17.6677(3)	2
LST	17.6363(3)	4
	17.6292(3)	6
FSDST	17.6605(3)	1
CLST	17.9005(3)	1

Table 5.4

The dimensionless critical buckling load T_{11} $(\equiv \frac{\bar{N}L^2}{100E_2h^3})$ of Shells 1 through 4 with various fiber orientations; $\frac{L}{R} = 2$, $\frac{R}{h} = 100$.

Lamination	T_{11}
(0/–65/65)	5.850(1)
(0/–45/45)	4.267(1)
(40/–40/0)	5.806(11)
(85/–85/0)	4.329(9)
(60/0/–60)	5.491(5)
(15/0/–15)	4.252(8)
(40/90/–40)	5.677(7)
(80/90/–80)	4.284(8)

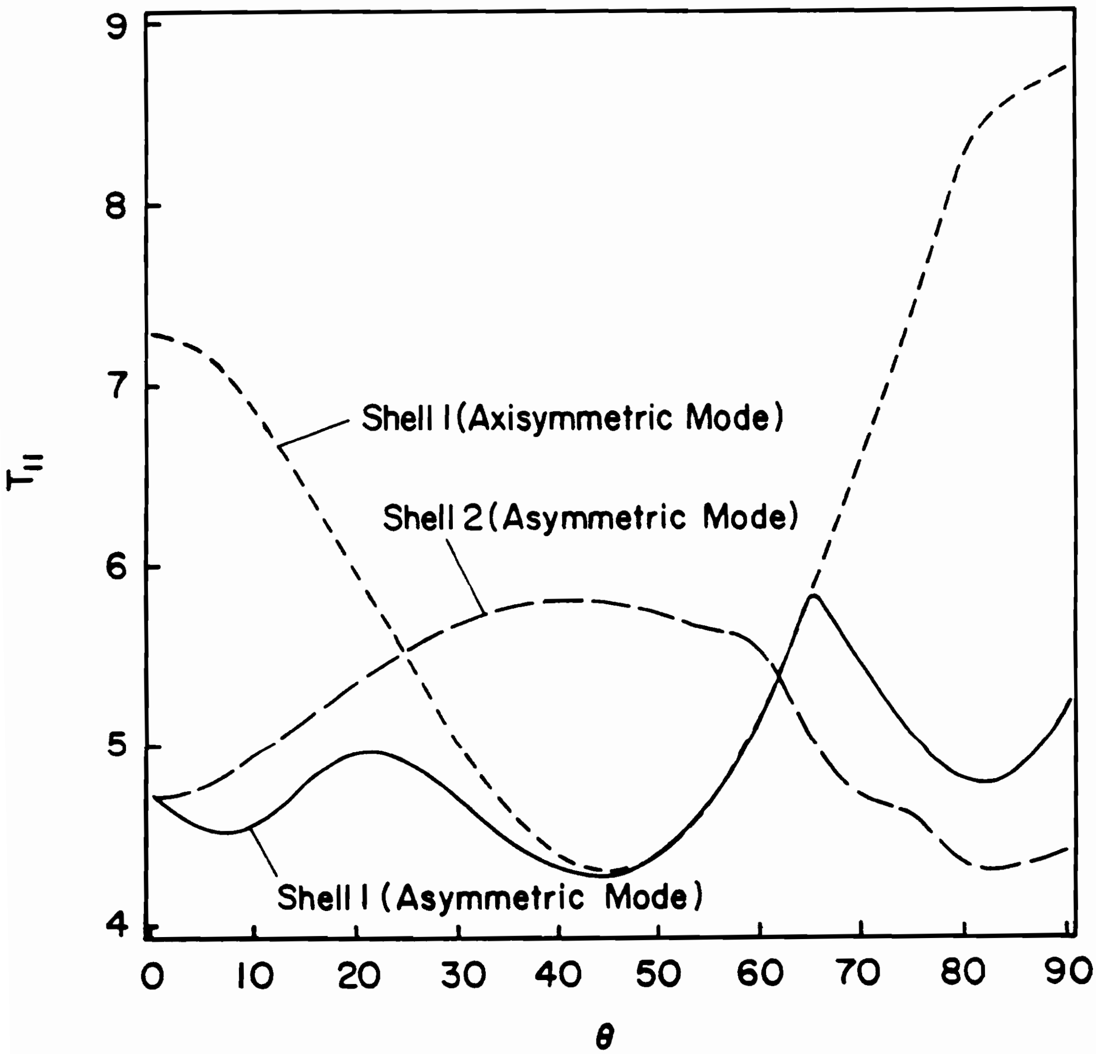

Figure 5.1 Variation of dimensionless critical buckling load $T_{11}(\equiv \frac{\bar{N}L^2}{100 E_2 h^3})$ vs. θ according to FSDST; $\frac{R}{h}$ = 100, L/R = 2, S3–S3 boundary type.

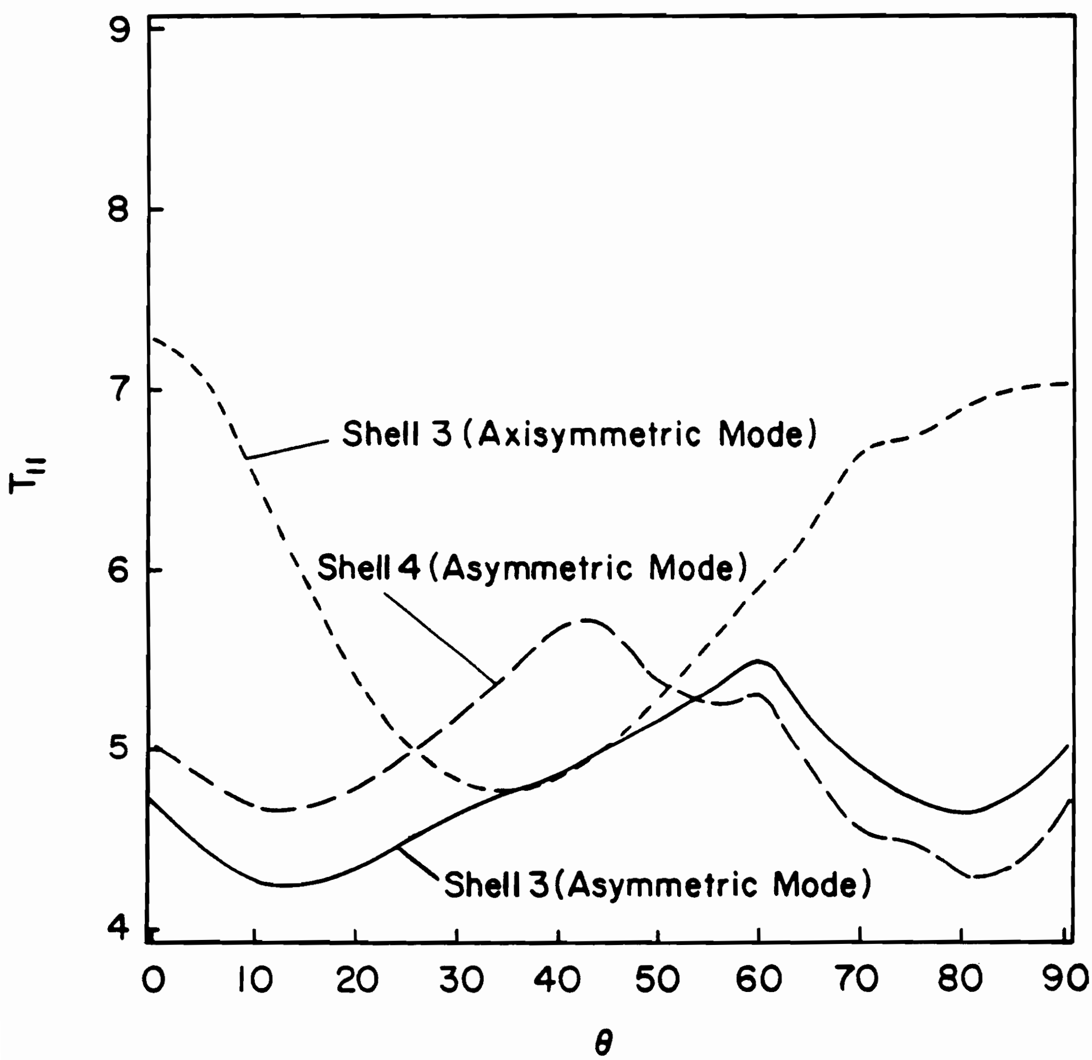

Figure 5.2 Variation of dimensionless critical buckling load $T_{11}(\equiv \frac{\bar{N}L^2}{100E_2h^3})$ vs. θ according to FSDST; $\frac{R}{h}$ = 100, L/R = 2, S3–S3 boundary type.

Appendix 5.1

The coefficients $\theta_j (j = \overline{1,4})$ appearing in Eq. (5–20) are

$$\theta_1 = f_5 f_7,\ \theta_2 = f_1 + f_3 + f_5 f_8 + f_6 f_7$$

$$\theta_3 = f_2 + f_4 + f_5 f_9 + f_6 f_8,\ \theta_4 = f_6 f_9$$

where

$$f_1 = \bar{e}_1(\bar{d}_5 \bar{c}_2 - \bar{c}_5 \bar{d}_2),\ f_2 = \bar{e}_1(\bar{d}_5 \bar{c}_4 - \bar{c}_5 \bar{d}_4)$$

$$f_3 = -\bar{e}_2(\bar{d}_5 \bar{c}_1 - \bar{c}_5 \bar{d}_1),\ f_4 = -\bar{e}_2(\bar{d}_5 \bar{c}_3 - \bar{c}_5 \bar{d}_3)$$

$$f_5 = \bar{e}_3,\ f_6 = \bar{e}_4,\ f_7 = \bar{d}_2 \bar{c}_1 - \bar{c}_2 \bar{d}_1$$

$$f_8 = \bar{d}_4 \bar{c}_1 + \bar{d}_2 \bar{c}_3 - \bar{c}_2 \bar{d}_3 - \bar{c}_4 \bar{d}_1,\ f_9 = \bar{d}_4 \bar{c}_3 - \bar{c}_4 \bar{d}_3$$

$$\bar{c}_1 = c_3 - c_1 a_2/a_1 - c_2 b_2/b_1,\ \bar{c}_2 = c_5 - c_1 a_3/a_1 - c_2 b_3/b_1$$

$$\bar{c}_3 = c_4,\ \bar{c}_4 = c_6,\ \bar{c}_5 = c_7 - c_1 a_4/a_1 - c_2 b_4/b_1$$

$$\bar{d}_1 = d_3 - d_1 a_2/a_1 - d_2 b_2/b_1,\ \bar{d}_2 = d_5 - d_1 a_3/a_1 - d_2 b_3/b_1$$

$$\bar{d}_3 = d_4,\ \bar{d}_4 = d_6,\ \bar{d}_5 = d_7 - d_1 a_4/a_1 - d_2 b_4/b_1$$

$$\bar{e}_1 = e_3 - e_1 a_2/a_1 - e_2 b_2/b_1,\ \bar{e}_2 = e_4 - e_1 a_3/a_1 - e_2 b_3/b_1$$

$$\bar{e}_3 = e_5,\ \bar{e}_4 = e_6 - e_1 a_4/a_1 - e_2 b_4/b_1$$

and

$$a_1 = A_{11}A_{66} - A_{16}^2,\ a_2 = B_{11}A_{66} - A_{16}B_{16},\ a_3 = B_{16}A_{66} - A_{16}B_{66}$$

$$a_4 = A_{66}A_{12}/R - A_{16}A_{26}/R,\ b_1 = A_{16}^2 - A_{11}A_{66},$$

$$b_2 = A_{16}B_{11} - A_{11}B_{16}$$

$$b_3 = A_{16}B_{16} - A_{11}B_{66},\ b_4 = A_{16}A_{12}/R - A_{11}A_{26}/R,\ c_1 = B_{11}$$

$$c_2 = B_{16},\ c_3 = D_{11},\ c_4 = -K_{55}^2 A_{55},\ c_5 = D_{16},\ c_6 = -K_{45}^2 A_{45}$$

$$c_7 = B_{12}/R - K_{55}^2 A_{55},\ d_1 = B_{16},\ d_2 = B_{66},\ d_3 = D_{16},\ d_4 = -K_{45}^2 A_{45}$$

$$d_5 = D_{66},\ d_6 = -K_{44}^2 A_{44},\ d_7 = B_{26}/R - K_{45}^2 A_{45},\ e_1 = A_{12}/R$$

$$e_2 = A_{26}/R,\ e_3 = B_{12}/R - K_{55}^2 A_{55},\ e_4 = B_{26/R} - K_{45}^2 A_{45}$$

$$e_5 = \bar{N} - K_{55}^2 A_{55},\ e_6 = A_{22}/R^2.$$

Expressions of $u_{ij}(i = \overline{1,5}$ and $j = \overline{1,6})$, u_{58}, and u_{510} appearing in Eqs. (5–21):

$$u_{ij} = \frac{-a_2\lambda_j u_{3j} - a_3 u_{4j}\lambda_j - a_4}{a_1\lambda_j}, \; u_{2j} = \frac{-b_2\lambda_j u_{3j} - b_3 u_{4j}\lambda_j - b_4}{b_1\lambda_j}$$

where

$$u_{3j} = \frac{g_{6j}g_{2j} - g_{3j}g_{5j}}{g_{1j}g_{5j} - g_{2j}g_{4j}} \;, \; u_{4j} = \frac{g_{6j}g_{1j} - g_{3j}g_{4j}}{g_{4j}g_{2j} - g_{1j}g_{5j}} \;, \; u_{5j} = 1$$

with

$$g_{ij} = \bar{c}_1\lambda_j^2 + \bar{c}_3 \;, \; g_{2j} = \bar{c}_2\lambda_j^2 + \bar{c}_4 \;, \; g_{3j} = \bar{c}_5\lambda_j$$

$$g_{4j} = \bar{d}_1\lambda_j^2 + \bar{d}_3 \;, \; g_{5j} = \bar{d}_2\lambda_j^2 + \bar{d}_4 \;, \; g_{6j} = \bar{d}_5\lambda_j.$$

Appendix 5.2

Expressions of [C] and [B] appearing in Eq. (5–35):

$$
[C] = \begin{bmatrix}
[I] & 0 & 0 & 0 & 0 & 0 \\
0 & [A_1] & 0 & [A_2] & 0 & 0 \\
0 & 0 & [I] & 0 & 0 & 0 \\
0 & [B_1] & 0 & [B_2] & 0 & 0 \\
0 & 0 & 0 & 0 & [I] & 0 \\
0 & 0 & 0 & 0 & 0 & [C_1]
\end{bmatrix}
$$

$$
[B] = \begin{bmatrix}
0 & [I] & 0 & 0 & 0 & 0 \\
[A_3] & [A_4] & [A_5] & [A_6] & [A_7] & [A_8] \\
0 & 0 & 0 & [I] & 0 & 0 \\
[E_3] & [B_4] & [B_5] & [B_6] & [B_7] & [B_8] \\
0 & 0 & 0 & 0 & 0 & [I] \\
[C_2] & [C_3] & [C_4] & [C_5] & [C_6] & [C_7]
\end{bmatrix}
$$

where [I] is an (N + 1) by (N + 1) identity matrix and

$$
[A_i] = \begin{bmatrix}
a_i^{11} & a_i^{21} & \cdots & a_i^{(N+1)1} \\
a_i^{12} & a_i^{22} & \cdots & a_i^{(N+1)2} \\
\vdots & \vdots & & \vdots \\
a_i^{1(N+1)} & a_i^{2(N+1)} & \cdots & a_i^{(N+1)(N+1)}
\end{bmatrix} \qquad i = \overline{1,8}
$$

$$[B_i] = \begin{bmatrix} b_i^{11} & b_i^{21} & \dots & b_i^{(N+1)1} \\ b_i^{12} & b_i^{22} & \dots & b_i^{(N+1)2} \\ \vdots & \vdots & & \vdots \\ b_i^{1(N+1)} & b_i^{2(N+1)} & \dots & b_i^{(N+1)(N+1)} \end{bmatrix} \quad i = \overline{1,8}$$

$$[C_i] = \begin{bmatrix} c_i^{11} & c_i^{21} & \dots & c_i^{(N+1)1} \\ c_i^{12} & c_i^{22} & \dots & c_i^{(N+1)2} \\ \vdots & \vdots & & \vdots \\ c_i^{1(N+1)} & c_i^{2(N+1)} & \dots & c_i^{(N+1)(N+1)} \end{bmatrix} \quad i = \overline{1,7}$$

CHAPTER VI

DAMPED FORCED–VIBRATION ANALYSIS OF LAMINATES SUBJECTED TO SONIC BOOM AND BLAST LOADINGS

In this chapter we will introduce an analytic procedure for obtaining the damped response of laminates with simple supports. Special attention will be given to the investigation of the dynamic response to shock waves produced by sonic booms.

In modern warfare, naval ships could be subjected to considerable air blast load which is mainly due to the overpressure in a shock wave produced by a supersonic aircraft in flight. Typical pressure–time histories of sonic booms (shock waves) measured at ground level are seen to have the shape of the capital letter N [1]. For this reason sonic booms are referred to as N–waves. The research studies devoted to the damped response of structures have been mainly confined to metallic type structures [2–5] using the classical plate theory. As we mentioned in Chapter I, Birman and Bert [108] considered the response of antisymmetrically laminated angle–ply plates subjected to explosive blast loading using the first–order shear deformation plate theory (FSDPT). In their analysis the in–plane and rotatory inertia forces were omitted. This way, by expanding the loads and the generalized displacement functions in double Fourier series, the problem was reduced to a second–order ordinary differential equation in time t.

Here in our analysis we will not neglect the inplane and rotatory inertia terms. Also we will add viscous damping terms (proportional to mass terms) to the equations of motion for the sake of generality. Our analysis will involve the applications of Laplace transformation and double finite Fourier sine and cosine transformations of the governing equations of the laminate. As will be seen, the analysis will be general in the sense that it is applicable to moving loads and to those theories which result in nonself–adjoint governing equations.

We demonstrate the solution procedure by considering the damped response of a symmetric cross–ply plate according to the third–order shear deformation plate theory (TSDPT) of Reddy and the first–order shear deformation plate theory (FSDPT). For the sake of completeness, the numerical results based on the classical plate theory (CLPT) will also be included.

6.1 Governing Equations of TSDPT and FSDPT

For symmetric plates in linear theory, the bending and stretching problems are no longer coupled. In bending problems, the equations of motion of TSDPT and FSDPT can be obtained from Eqs. (2–3c), (2–3d), and (2–3e) by letting $\lambda = 1$ and $\lambda = 0$, respectively, and $\frac{1}{R_1} = \frac{1}{R_2} = 0$. Further, for a symmetric cross–ply plate we have

$$B_{ij} = E_{ij} = 0 \qquad (i,j = 1,2,6)$$

$$A_{16} = A_{26} = A_{45} = D_{16} = D_{26} = D_{45} = 0$$

$$F_{16} = F_{26} = F_{45} = H_{16} = H_{26} = 0$$

$$\bar{I}_2 = \bar{I}_4 = 0 \tag{6–1}$$

By substituting Eqs. (6–1) and (2–5) through (2–8) into Eqs. (2–3c) through (2–3e), the equations of motion of TSDPT and FSDPT can be expressed and presented in a compact form as

$$a_1\psi_{x,xx} + a_2\psi_{x,yy} + a_3\psi_x + a_4\psi_{y,xy} + a_5 w_{,xxx} + a_6 w_{,xyy}$$

$$a_7 w_{,x} = m_3\ddot{\psi}_x - m_5\ddot{w}_{,x} + d_3\dot{\psi}_x - d_5\dot{w}_{,x} \tag{6–2a}$$

$$b_1\psi_{y,yy} + b_2\psi_{y,xx} + b_3\psi_y + b_4\psi_{x,xy} + b_5 w_{,yyy} + b_6 w_{,yxx}$$

$$+ b_7 w_{,y} = m_3\ddot{\psi}_y - m_5\ddot{w}_{,y} + d_3\dot{\psi}_y - d_5\dot{w}_{,y} \tag{6–2b}$$

$$c_1\psi_{x,xxx} + c_2\psi_{x,xyy} + c_3\psi_{x,x} + c_4\psi_{y,yyy} + c_5\psi_{y,xxy} + c_6\psi_{y,y}$$

$$+ c_7 w_{,xxxx} + c_8 w_{,xxyy} + c_9 w_{,yyyy} + c_{10} w_{,xx} + c_{11} w_{,yy} + P_z$$

$$= m_1\ddot{w} + m_5(\ddot{\psi}_{x,x} + \ddot{\psi}_{y,y}) - m_7\nabla^2\ddot{w} + d_1\dot{w} + d_5(\dot{\psi}_{x,x} + \dot{\psi}_{y,y}) - d_7\nabla^2\dot{w} \tag{6–2c}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{6–3}$$

and d_1, d_3, d_5, and and d_7 are the proportional viscous damping coefficients. Also the coefficients a_i, b_i ($i = \overline{1,7}$), and c_i ($i = \overline{1,11}$) are

$$a_1 = D_{11} + \lambda\left[-\frac{8}{3h^2}F_{11} + \left[\frac{4}{3h^2}\right]^2 H_{11}\right]$$

$$a_2 = D_{66} + \lambda\left[-\frac{8}{3h^2}F_{66} + \left[\frac{4}{3h^2}\right]^2 H_{66}\right]$$

$$a_3 = -\ A_{55} + \lambda\left[\frac{8}{h^2}D_{55} - \left[\frac{4}{h^2}\right]^2 F_{55}\right]$$

$$a_4 = D_{12} + D_{66} + \lambda\left[-\frac{8}{3h^2}(F_{12} + F_{66}) + \left[\frac{4}{3h^2}\right]^2 (H_{12} + H_{66})\right]$$

$$a_5 = \lambda\left[-\frac{4}{3h^2}F_{11} + \left[\frac{4}{3h^2}\right]^2 H_{11}\right]$$

$$a_6 = \lambda\left[-\frac{4}{3h^2}(F_{12} + 2F_{66}) + \left[\frac{4}{3h^2}\right]^2 (H_{12} + 2H_{66})\right]$$

$$a_7 = a_3\ ,\ m_3 = \bar{I}_3\ ,\ m_5 = \lambda\frac{4}{3h^2}\bar{I}_5$$

$$b_1 = D_{22} + \lambda\left[-\frac{8}{3h^2}F_{22} + \left[\frac{4}{3h^2}\right]^2 H_{22}\right]$$

$$b_2 = a_2\ ,\ b_3 = -\ A_{44} + \lambda\left[\frac{8}{h^2}D_{44} - \left[\frac{4}{h^2}\right]^2 F_{44}\right]$$

$$b_4 = a_4\ ,\ b_5 = \lambda\left[-\frac{4}{3h^2}F_{22} + \left[\frac{4}{3h^2}\right]^2 H_{22}\right]$$

$$b_6 = a_6\ ,\ b_7 = b_3\ ,\ c_1 = -\ a_5\ ,\ c_2 = -a_6,\ c_3 = -\ a_3$$

$$c_4 = -b_5 \,,\; c_5 = c_2 \,,\; c_6 = -b_3 \,,\; c_7 = -\lambda\left[\frac{4}{3h^2}\right]^2 H_{11}$$

$$c_8 = -\lambda\left[\frac{4}{3h^2}\right]^2 (2H_{12} + 4H_{66}) \,,\; c_9 = -\lambda\left[\frac{4}{3h^2}\right]^2 H_{22}$$

$$c_{10} = T_{11} - a_3 \,,\; c_{11} = T_{22} - b_3 \,,\; m_1 = I_1 \,,\; m_7 = \lambda\left[\frac{4}{3h^2}\right]^2 I_7 \tag{6–4}$$

where T_{11} and T_{22} are the tensile edge loads in the x and y directions, respectively. In Eqs. (6–4) we let $\lambda = 1$ and $\lambda = 0$ in TSDPT and FSDPT, respectively. Also in FSDPT the rigidity terms A_{44} and A_{55} are replaced, respectively, by $K_{44}^2 A_{44}$ and $K_{55}^2 A_{55}$ with K_{44}^2 and K_{55}^2 being the shear correction coefficients. For convenience we replaced the rotation functions ϕ_1 and ϕ_2 and (x_1,x_2) appearing in Eqs. (2–3c)–(2–3e) by ψ_x and ψ_y and (x,y), respectively.

6.2 Initial Conditions

For the sake of generality we assume the following nonhomogeneous initial conditions:

$$\psi_x(x,y,0) \equiv \bar{\psi}_x(x,y) \,,\; \dot{\psi}_x(x,y,0) \equiv \bar{\bar{\psi}}_x(x,y)$$

$$\psi_y(x,y,0) \equiv \bar{\psi}_y(x,y) \,,\; \dot{\psi}_y(x,y,0) \equiv \bar{\bar{\psi}}_y(x,y)$$

$$w(x,y,0) \equiv w(x,y) \,,\; \dot{w}(x,y,0) \equiv \bar{\bar{w}}(x,y) \tag{6–5}$$

6.3 Boundary Conditions

From Eqs. (2–11) and (2–12), the boundary conditions in TSDPT for a simply–supported plate are given by:

At $x = 0,a$;

$$w = \psi_y = P_1 = M_1 = 0 \tag{6–6}$$

At $y = 0,b$;

$$w = \psi_x = P_2 = M_2 = 0 \tag{6–7}$$

where from Eqs. (2–5) and (2–8) and using Eqs. (6–1) and (6–4) we have;

$$P_1 = \left[F_{11} - \left[\frac{4}{3h^2}\right]H_{11}\right]\psi_{x,x} + \left[F_{12} - \left[\frac{4}{3h^2}\right]H_{12}\right]\psi_{y,y}$$

$$- \left[\frac{4}{3h^2}\right]\left[H_{11}w_{,xx} + H_{12}w_{,yy}\right]$$

$$M_1 = \left[D_{11} - \left[\frac{4}{3h^2}\right]F_{11}\right]\psi_{x,x} + \left[D_{12} - \left[\frac{4}{3h^2}\right]F_{12}\right]\psi_{y,y}$$

$$- \left[\frac{4}{3h^2}\right]\left[F_{11}w_{,xx} + F_{12}w_{,yy}\right] \tag{6–8}$$

and

$$P_2 = \left[F_{12} - \left[\frac{4}{3h^2}\right]H_{12}\right]\psi_{x,x} + \left[F_{22} - \left[\frac{4}{3h^2}\right]H_{22}\right]\psi_{y,y}$$

$$- \left[\frac{4}{3h^2}\right]\left[H_{12}w_{,xx} + H_{22}w_{,yy}\right]$$

$$M_2 = \left[D_{12} - \left[\frac{4}{3h^2}\right]F_{12}\right]\psi_{x,x} + \left[D_{22} - \left[\frac{4}{3h^2}\right]F_{22}\right]\psi_{y,y}$$

$$-\left[\frac{4}{3h^2}\right]\left[F_{12}w_{,xx} + F_{22}w_{,yy}\right] \tag{6–9}$$

Also in FSDPT we have:

At x = 0,a;

$$w = \psi_y = M_1 = 0 \tag{6–10}$$

At y = 0,b;

$$w = \psi_x = M_2 = 0 \tag{6–11}$$

where

$$M_1 = D_{11}\psi_{x,x} + D_{12}\psi_{y,y} \tag{6–12}$$

and

$$M_2 = D_{12}\psi_{x,x} + D_{22}\psi_{y,y} \tag{6–13}$$

It is to be noted that the total order of the governing equations in TSDPT is eight and that in FSDPT is six.

6.4 Solution Procedure

By taking into account the initial conditions in Eqs. (6–5), the Laplace transformation of Eqs. (6–2) yields

$$a_1\bar{\psi}_{x,xx} + a_2\bar{\psi}_{x,yy} + a_3\bar{\psi}_x + a_4\bar{\psi}_{y,xy} + a_5\bar{w}_{,xxx} + a_6\bar{w}_{,xyy}$$

$$+ a_7\bar{w}_{,x} = m_3s^2\bar{\psi}_x - m_3s\tilde{\psi}_x - m_3\tilde{\tilde{\psi}}_x - m_5s^2\bar{w}_{,x} + m_5s\tilde{w}_{,x}$$

$$+ m_5\tilde{\tilde{w}}_{,x} + d_3 s\bar{\psi}_x - d_3\tilde{\psi}_x - d_5 s\bar{w}_{,x} + d_5\tilde{w}_{,x} \tag{6–14a}$$

$$b_1\bar{\psi}_{y,yy} + b_2\bar{\psi}_{y,xx} + b_3\bar{\psi}_y + b_4\bar{\psi}_{x,xy} + b_5\bar{w}_{,yyy} + b_6\bar{w}_{,yxx}$$

$$+ b_7\bar{w}_{,y} = m_3 s^2\bar{\psi}_y - m_3 s\tilde{\psi}_y - m_3\tilde{\tilde{\psi}} - m_5 s^2\bar{w}_{,y} + m_5 s\tilde{w}_{,y}$$

$$+ m_5\tilde{\tilde{w}}_{,y} + d_3 s\bar{\psi}_y - d_3\tilde{\psi}_y - d_5 s\bar{w}_{,y} + d_5\tilde{w}_{,y} \tag{6–14b}$$

$$c_1\bar{\psi}_{x,xxx} + c_2\bar{\psi}_{x,xyy} + c_3\bar{\psi}_{x,x} + c_4\bar{\psi}_{y,yyy} + c_5\bar{\psi}_{y,xxy} + c_6\bar{\psi}_{y,y}$$

$$+ c_7\bar{w}_{,xxxx} + c_8\bar{w}_{,xxyy} + c_9\bar{w}_{,yyyy} + c_{10}\bar{w}_{,xx} + c_{11}\bar{w}_{,yy} + \bar{P}_z$$

$$= m_1 s^2\bar{w} - m_1 s\tilde{w} - m_1\tilde{\tilde{w}} + m_5 s^2(\bar{\psi}_{x,x} + \bar{\psi}_{y,y}) - m_5 s(\tilde{\psi}_{x,x} + \tilde{\psi}_{y,y})$$

$$- m_5(\tilde{\tilde{\psi}}_{x,x} + \tilde{\tilde{\psi}}_{y,y}) - m_7 s^2\nabla^2\bar{w} + m_7 s\nabla^2\tilde{w} + m_7\nabla^2\tilde{\tilde{w}}$$

$$+ d_1 s\bar{w} - d_1\tilde{w} + d_5 s(\bar{\psi}_{x,x} + \bar{\psi}_{y,y}) - d_5(\tilde{\psi}_{x,x} + \tilde{\psi}_{y,y})$$

$$- d_7 s\nabla^2\bar{w} + d_7\nabla^2\tilde{w} \tag{6–14c}$$

where s denotes the Laplace variable and a bar over a variable represents the variable in the transformed domain, i.e.,

$$\bar{Z}(s) = L\{(t)\} \tag{6–15}$$

with L being the Laplace transformation operator.

Next, we will use the finite Fourier transformation technique to transform the partial differential equations (6–14) into a system of linear algebraic equations. To this end we define

$$\bar{\psi}_x^{(cs)}(m,n,s) = \int_0^b \int_0^a \bar{\psi}_x(x,y,s) \cos \alpha_m x \sin \beta_n y \, dxdy$$

$$\bar{\psi}_y^{(sc)}(m,n,s) = \int_0^b \int_0^a \bar{\psi}_y(x,y,s) \sin \alpha_m x \cos \beta_n y \, dxdy$$

$$\bar{w}^{(ss)}(m,n,s) = \int_0^b \int_0^a \bar{w}(x,y,s) \sin \alpha_m x \sin \beta_n y \, dxdy \tag{6–16}$$

and apply the mixed cosine–sine, sine–cosine, and double sine–sine Fourier transformations to Eqs. (6–14a), (6–14b), and (6–14c), respectively. Because of the boundary conditions in Eqs. (6–6) and (6–7) for TSDPT and in Eqs. (6–10) and (6–11) for FSDPT, the terms resulted from integration by parts will vanish identically and we will have (for now we assume that $m \neq 0$ and $n \neq 0$):

$$\begin{bmatrix} (M_{11}s^2 + C_{11}s + K_{11}) & K_{12} & (M_{13}s^2 + C_{13}s + K_{13}) \\ K_{21} & (M_{22}s^2 + C_{22}s + K_{22}) & (M_{23}s^2 + C_{23}s + K_{23}) \\ (M_{31}s^2 + C_{31}s + K_{31}) & (M_{32}s^2 + C_{32}s + K_{32}) & (M_{33}s^2 + C_{33}s + K_{33}) \end{bmatrix}$$

$$\cdot\begin{Bmatrix} \bar{\psi}_x^{(cs)}(m,n,s) \\ \bar{\psi}_y^{(sc)}(m,n,s) \\ \bar{w}^{(ss)}(m,n,s) \end{Bmatrix} = \{\bar{T}_1 \quad \bar{T}_2 \quad \bar{T}_3 + \bar{P}_z^{(ss)}\}^T \tag{6–17}$$

In Eqs. (6–16) we have

$$\alpha_m = \frac{m\pi}{a} \text{ and } \beta_n = \frac{n\pi}{b} \tag{6–18}$$

with m and n being the integer Fourier variables. Also in Eqs. (6–17) we have

$$M_{ij} = M_{ji}\,,\ C_{ij} = C_{ji}\,,\ K_{ij} = K_{ji},$$

$$M_{11} = m_3\,,\ M_{13} = -\,m_5\alpha_m\,,\ M_{22} = m_3,$$

$$M_{23} = -\,m_5\beta_n\,,\ M_{33} = m_1 + m_7(\alpha_m^2 + \beta_n^2),$$

$$C_{11} = d_3\,,\ C_{13} = -\,d_5\alpha_m\,,\ C_{22} = d_3,$$

$$C_{23} = -\,d_5\beta_n\,,\ C_{33} = d_1 + d_7(\alpha_m^2 + \beta_n^2),$$

$$K_{11} = a_1\alpha_m^2 + a_2\beta_n^2 - a_3\,,\ \ K_{12} = a_4\alpha_m\beta_n,$$

$$K_{13} = a_5\alpha_m^3 + a_6\alpha_m\beta_n^2 - a_7\alpha_m,$$

$$K_{22} = b_1\beta_n^2 + b_2\alpha_m^2 - b_3,$$

$$K_{23} = b_5\beta_n^3 + b_6\beta_n\alpha_m^2 - b_7\beta_n,$$

$$K_{33} = -c_7\alpha_m^4 - c_8\alpha_m^2\beta_n^2 - c_9\beta_n^4 + c_{10}\alpha_m^2 + c_{11}\beta_n^2, \qquad (6\text{–}19)$$

and

$$\bar{P}_z^{(ss)}(m,n,s) \equiv \int_0^b\int_0^a \bar{P}_z(x,y,s)\sin\alpha_m x\,\sin\beta_n y\;dxdy \qquad (6\text{–}20)$$

$$\bar{T}_i = sC_i + S_i \quad (i = \overline{1,3}) \qquad (6\text{–}21)$$

with

$$C_1 = m_3\tilde{\psi}_x^{(cs)} - m_5\tilde{w}^{(ss)}\alpha_m \;,\; C_2 = m_3\tilde{\psi}_y^{(sc)} - m_5\tilde{w}^{(ss)}\beta_n$$

$$C_3 = m_1\tilde{w}^{(ss)} + m_7(\alpha_m^2 + \beta_n^2)\tilde{w}^{(ss)} - m_5(\alpha_m\tilde{\psi}_x^{(cs)} + \beta_n\tilde{\psi}_y^{(sc)}),$$

$$S_1 = m_3\dot{\tilde{\psi}}_x^{(cs)} - m_5\dot{\tilde{w}}^{(ss)}\alpha_m + d_3\tilde{\psi}_x^{(cs)} - d_5\tilde{w}^{(ss)}\alpha_m,$$

$$S_2 = m_3\dot{\tilde{\psi}}_y^{(sc)} - m_5\dot{\tilde{w}}^{(ss)}\beta_n + d_3\tilde{\psi}_y^{(sc)} - d_5\tilde{w}^{(ss)}\beta_n$$

$$S_3 = m_1\tilde{\tilde{w}}^{(ss)} + m_7(\alpha_m^2 + \beta_n^2)\tilde{\tilde{w}}^{(ss)} - m_5(\alpha_m\tilde{\tilde{\psi}}_x^{(cs)} + \beta_n\tilde{\tilde{\psi}}_y^{(sc)})$$

$$+ d_1\tilde{w}^{(ss)} + d_7(\alpha_m^2 + \beta_n^2)\tilde{w}^{(ss)} - d_5(\alpha_m\tilde{\psi}_x^{(cs)} + \beta_n\tilde{\psi}_y^{(sc)}), \tag{6–22}$$

where $\tilde{\psi}_x^{(cs)}$, $\tilde{\tilde{\psi}}_x^{(cs)}$,..., $\tilde{w}^{(ss)}$, and $\tilde{\tilde{w}}^{(ss)}$ are defined similar to $\bar{\psi}_x^{(cs)}$, $\bar{\psi}_y^{(sc)}$, and $\bar{w}^{(ss)}$ in Eqs. (6–16). It is noted that C_i and S_i $(i = \overline{1,3})$ are not functions of the Laplace variable s.

From Eqs. (6–17), by using Cramer's rule, we find and present $\bar{\psi}_x^{(cs)}$, $\bar{\psi}_y^{(sc)}$, and $\bar{w}^{(ss)}$ as

$$\begin{Bmatrix} \bar{\psi}_x^{(cs)} \\ \bar{\psi}_y^{(sc)} \\ \bar{w}^{(ss)} \end{Bmatrix} = \sum_{j=1}^{3} s\frac{C_j}{\bar{D}} \begin{Bmatrix} \bar{F}_{1j} \\ \bar{F}_{2j} \\ \bar{F}_{3j} \end{Bmatrix} + \sum_{j=1}^{3} \frac{S_j}{\bar{D}} \begin{Bmatrix} \bar{F}_{1j} \\ \bar{F}_{2j} \\ \bar{F}_{3j} \end{Bmatrix} + \frac{\bar{P}_z^{(ss)}}{\bar{D}} \begin{Bmatrix} \bar{F}_{13} \\ \bar{F}_{23} \\ \bar{F}_{33} \end{Bmatrix} \tag{6–23}$$

where $\bar{D}$ is the determinant of the coefficient matrix in Eqs. (6–17) which upon expansion can be presented as

$$\bar{D} = \lambda_1 s^6 + \lambda_2 s^5 + \lambda_3 s^4 + \lambda_4 s^3 + \lambda_5 s^2 + \lambda_6 s + \lambda_7 \tag{6–24}$$

for any nonzero pair of m and n. Also $\bar{F}_{ij}$ in Eqs. (6–23) can be expressed as

$$\bar{F}_{ij}(m,n,s) \equiv \sum_{r=1}^{5} I_{ijr} \cdot s^{5-r} \tag{6–25}$$

where λ_k ($k = \overline{1,7}$) and I_{ijr} are given in Appendix 6.1.

For light damping, by denoting the six roots of $\bar{D} = 0$ as s_k ($k = \overline{1,3}$) and their complex conjugates s_k^* ($k = (\overline{1,3})$, Eq. (6–24) can be presented as

$$\bar{D} = \lambda_1 \prod_{k=1}^{3} (s - s_k)(s - s_k^*) \tag{6–26}$$

for any nonzero value of m and n. In Eq. (6–26) Π denotes the product sign. Also by expressing s_k as

$$s_k = -\beta_k + J \cdot \omega_k \tag{6–27}$$

with $J = \sqrt{-1}$, Eq. (6–26) can be rewritten as

$$\bar{D} = \lambda_1 \prod_{k=1}^{3} [(s + \beta_k)^2 + \omega_k^2] \tag{6–28}$$

where β_k is not to be considered the same as β_n ($= \frac{n\pi}{b}$) introduced earlier.

From Eqs. (6–25) and (6–28) we have

$$\frac{\bar{F}_{ij}}{\bar{D}} \equiv \frac{\sum_{r=1}^{5} I_{ijr} \cdot s^{5-r}}{\lambda_1 \prod_{k=1}^{3} [(s + \beta_k)^2 + \omega_k^2]}. \tag{6–29}$$

Now we can use the partial fraction expansion technique and express $\bar{F}_{ij}/\bar{D}$

equivalently as:

$$\frac{\bar{F}_{ij}}{\bar{D}} = \sum_{k=1}^{3} \frac{A_{ijk}(s + \beta_k) + B_{ijk}\omega_k}{(s + \beta_k)^2 + \omega_k^2} \tag{6–30}$$

where A_{ijk} and B_{ijk} are <u>real</u> coefficients which are determined with the help of Heaviside–expansion theorem as follows. We multiply Eq. (6–30) by $(s - s_k)$ and let $s \to s_k$ and then by $(s - s_k^*)$ and let $s \to s_k^*$ and use Heaviside–expansion theorem to obtain

$$A_{ijk}\cdot(J\omega_k) + B_{ijk}\cdot\omega_k = 2J\omega_k\cdot\left.\left[\frac{\bar{F}_{ij}}{\frac{d\bar{D}}{ds}}\right]\right|_{s=s_k} \tag{6–31a}$$

$$A_{ijk}\cdot(-J\omega_k) + B_{ijk}\cdot\omega_k = -2J\omega_k\cdot\left.\left[\frac{\bar{F}_{ij}}{\frac{d\bar{D}}{ds}}\right]\right|_{s=s_k^*} \tag{6–31b}$$

where from Eq. (6–24) we have

$$\frac{d\bar{D}}{ds} = 6\lambda_1 s^5 + 5\lambda_2 s^4 + 4\lambda_3 s^3 + 3\lambda_4 s^2 + 2\lambda_5 s + \lambda_6. \tag{6–32}$$

With the help of Eqs. (6–25) and (6–32), it can readily be shown that

$$\left[\frac{\bar{F}_{ij}}{\frac{d\bar{D}}{ds}}\right]\Bigg|_{s=s_k^*} = \text{conjugate of} \left[\frac{\bar{F}_{ij}}{\frac{d\bar{D}}{ds}}\right]\Bigg|_{s=s_k} \equiv \overline{\left[\frac{\bar{F}_{ij}}{\frac{d\bar{D}}{ds}}\right]\Bigg|_{s=s_k}} \tag{6–33}$$

By solving Eqs. (6–31) for A_{ijk} and B_{ijk} and using Eq. (6–33), we obtain

$$A_{ijk} = 2\ \mathrm{Re}\left\{\left[\frac{\bar{F}_{ij}}{\frac{d\bar{D}}{ds}}\right]\Bigg|_{s=s_k}\right\} \tag{6–34a}$$

$$B_{ijk} = -2\ \mathrm{Im}\left\{\left[\frac{\bar{F}_{ij}}{\frac{d\bar{D}}{ds}}\right]\Bigg|_{s=s_k}\right\} \tag{6–34b}$$

where by Re and Im we mean the real part and imaginary part, respectively.

Now by direct substitution of $s_k(\equiv -\beta_k + J\omega_k)$ into Eqs. (6–25) and (6–32), we find

$$[\bar{F}_{ij}]\big|_{s=s_k} = \breve{F}_{ijk} + J\breve{\breve{F}}_{ijk} \tag{6–35a}$$

$$\left[\frac{d\bar{D}}{ds}\right]\Bigg|_{s=s_k} = \tilde{D}_k + J\tilde{\tilde{D}}_k. \tag{6–35b}$$

Finally, substituting Eqs. (6–35) into Eqs. (6–34) will result in

$$A_{ijk} = 2\,\frac{\tilde{F}_{ijk}\ \tilde{D}_k + \tilde{\tilde{F}}_{ijk}\ \tilde{\tilde{D}}_k}{\tilde{D}_k^2 - \tilde{\tilde{D}}_k^2} \tag{6–36a}$$

$$B_{ijk} = -2\,\frac{\tilde{\tilde{F}}_{ijk}\ \tilde{D}_k - \tilde{F}_{ijk}\ \tilde{\tilde{D}}_k}{\tilde{D}_k^2 - \tilde{\tilde{D}}_k^2} \tag{6–36b}$$

for any nonzero pair of m and n. Also $\tilde{F}_{ijk}$, $\tilde{\tilde{F}}_{ijk}$, $\tilde{D}_k$, and $\tilde{\tilde{D}}_k$ appearing in Eqs. (6–35) are

$$\tilde{F}_{ijk} = \sum_{\ell=1}^{5} I_{ij\ell}\tilde{\alpha}_{\ell k}$$

$$\tilde{\tilde{F}}_{ijk} = \sum_{\ell=1}^{5} I_{ij\ell}\tilde{\tilde{\alpha}}_{\ell k}$$

$$\tilde{D}_k = \sum_{\ell=1}^{6} \lambda_\ell\tilde{\gamma}_{\ell k}$$

$$\tilde{\tilde{D}}_k = \sum_{\ell=1}^{6} \lambda_\ell \tilde{\tilde{\gamma}}_{\ell k} \tag{6–37}$$

where

$$\tilde{\alpha}_{1k} = \beta_k^4 - 6\beta_k^2\omega_k^2 + \omega_k^4 \;,\; \tilde{\alpha}_{2k} = -\beta_k^3 + 3\beta_k\omega_k^2,$$

$$\tilde{\alpha}_{3k} = \beta_k^2 - \omega_k^2 \;,\; \tilde{\alpha}_{4k} = -\beta_k \;,\; \tilde{\alpha}_{5k} = 1,$$

$$\tilde{\tilde{\alpha}}_{1k} = -4\beta_k^3\omega_k + 4\beta_k\omega_k^3 \;,\; \tilde{\tilde{\alpha}}_{2k} = 3\beta_k^2\omega_k - \omega_k^3,$$

$$\tilde{\tilde{\alpha}}_{3k} = -2\beta_k\omega_k \;,\; \tilde{\tilde{\alpha}}_{4k} = \omega_k \;,\; \tilde{\tilde{\alpha}}_{5k} = 0$$

$$\tilde{\gamma}_{1k} = -6(\beta_k^5 - 10\beta_k^3\omega_k^2 + 5\beta_k\omega_k^4) \;,\; \tilde{\gamma}_{2k} = 5\tilde{\alpha}_{1k},$$

$$\tilde{\gamma}_{3k} = 4\tilde{\alpha}_{2k} \;,\; \tilde{\gamma}_{4k} = 3\tilde{\alpha}_{3k} \;,\; \tilde{\gamma}_{5k} = 2\tilde{\alpha}_{4k} \;,\; \tilde{\gamma}_{6k} = 1,$$

$$\tilde{\tilde{\gamma}}_{1k} = 6(5\beta_k^4\omega_k - 10\beta_k^2\omega_k^3 + \omega_k^5) \;,\; \tilde{\tilde{\gamma}}_{2k} = 5\tilde{\tilde{\alpha}}_{1k},$$

$$\tilde{\tilde{\gamma}}_{3k} = 4\tilde{\tilde{\alpha}}_{2k} \;,\; \tilde{\tilde{\gamma}}_{4k} = 3\tilde{\tilde{\alpha}}_{3k} \;,\; \tilde{\tilde{\gamma}}_{5k} = 2\tilde{\tilde{\alpha}}_{4k} \;,\; \tilde{\tilde{\gamma}}_{6k} = 1. \tag{6–38}$$

Now the application of the inverse Laplace transformation to Eq. (6–30) results in

$$\mathcal{L}^{-1}\left\{\frac{\bar{F}_{ij}}{\bar{D}}\right\} \equiv H_{ij}(m,n,t)$$

$$= \sum_{k=1}^{3} e^{-\beta_k t}(A_{ijk} \cos \omega_k t + B_{ijk} \sin \omega_k t) \tag{6–39}$$

With the result in Eq. (6–39), we can write

$$\mathcal{L}^{-1}\left\{s\,\frac{\bar{F}_{ij}}{\bar{D}}\right\} \equiv L_{ij}(m,n,t)$$

$$= \frac{dH_{ij}}{dt} + H_{ij}(m,n,0)\delta(t) \tag{6–40}$$

where the generalized function $\delta(t)$ is the so–called Dirac's delta function. From Eq. (6–39) we have

$$H_{ij}(m,n,0) = \sum_{k=1}^{3} A_{ijk}. \tag{6–41}$$

However, by comparing Eqs. (6–29) and (6–30) we conclude that

$$\sum_{k=1}^{3} A_{ijk} = 0 \tag{6–42}$$

and therefore

$$L_{ij}(m,n,t) = \dot{H}_{ij}(m,n,t) \tag{6–43}$$

$$= \sum_{k=1}^{3} e^{-\beta_k t}[(\omega_k B_{ijk} - \beta_k A_{ijk})\cos \omega_k t$$

$$- (\beta_k B_{ijk} + \omega_k A_{ijk})\sin \omega_k t] \tag{6–44}$$

With the help of Eqs. (6–39), (6–40), and (6–43), application of the inverse Laplace transformation to Eqs. (6–23) yields

$$\begin{Bmatrix} \psi_x^{(cs)}(m,n,t) \\ \psi_y^{(sc)}(m,n,t) \\ w^{(ss)}(m,n,t) \end{Bmatrix} = \sum_{j=1}^{3} C_j \begin{Bmatrix} L_{1j}(m,n,t) \\ L_{2j}(m,n,t) \\ L_{3j}(m,n,t) \end{Bmatrix} + \sum_{j=1}^{3} S_j \begin{Bmatrix} H_{1j}(m,n,t) \\ H_{2j}(m,n,t) \\ H_{3j}(m,n,t) \end{Bmatrix}$$

$$+ \int_0^t P_z^{(ss)}(m,n,\tau) \begin{Bmatrix} H_{13}(m,n,t-\tau) \\ H_{23}(m,n,t-\tau) \\ H_{33}(m,n,t-\tau) \end{Bmatrix} d\tau \tag{6–45}$$

where the first two terms are the response due to nonzero initial conditions and the last term is the response due to the transverse load P_z. With zero initial conditions we will have $C_j = S_j = 0$ in Eqs. (6–45) and substituting Eq. (6–39) into Eqs.

(6–45) results in:

$$\psi_x^{(cs)}(m,n,t) = \sum_{k=1}^{3} \int_0^t P_z^{(ss)}(m,n,\tau) e^{-\beta_k(t-\tau)} [A_{13k} \cos \omega_k(t-\tau)$$

$$+ B_{13k} \sin \omega_k(t-\tau)] d\tau,$$

$$\psi_y^{(sc)}(m,n,t) = \sum_{k=1}^{3} \int_0^t P_z^{(ss)}(m,n,\tau) e^{-\beta_k(t-\tau)} [A_{23k} \cos \omega_k(t-\tau)$$

$$+ B_{23k} \sin \omega_k(t-\tau)] d\tau,$$

$$w^{(ss)}(m,n,t) = \sum_{k=1}^{3} \int_0^t P_z^{(ss)}(m,n,\tau) e^{-\beta_k(t-\tau)} [A_{33k} \cos \omega_k(t-\tau)$$

$$+ B_{33k} \sin \omega_k(t-\tau)] d\tau, \tag{6–46}$$

for all nonzero values of m and n. The primary response quantities ψ_x, ψ_y, and w are related to the transformed quantities $\psi_x^{(cs)}$, $\psi_y^{(sc)}$, and $w^{(ss)}$, respectively, as

$$\psi_x(x,y,t) = \frac{2}{ab} \sum_{n=1}^{\infty} \psi_x^{(cs)}(0,n,t) \sin \beta_n y$$

$$+ \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \psi_x^{(cs)}(m,n,t) \cos \alpha_m x \sin \beta_n y$$

$$\psi_y(x,y,t) = \frac{2}{ab} \sum_{m=1}^{\infty} \psi_y^{(sc)}(m,0,t) \sin \alpha_m x$$

$$+\frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\psi_y^{(sc)}(m,n,t)\sin\alpha_m x\cos\beta_n y$$

$$w(x,y,t)=\frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}w^{(ss)}(m,n,t)\sin\alpha_m x\sin\beta_n y \tag{6–47}$$

where $\psi_x^{(cs)}(m,n,t)$, $\psi_y^{(sc)}(m,n,t)$, and $w^{(ss)}(m,n,t)$ are given by Eqs. (6–45) for all nonzero values of m and n. In the next sections we will find $\psi_x^{(cs)}(0,n,t)$ (i.e., when $m = 0$ and $n \geq 1$) and $\psi_y^{(sc)}(m,0,t)$ (i.e., when $m \geq 1$ and $n = 0$) appearing in Eqs. (6–47).

6.5 Determination of $\psi_x^{(cs)}(0,n,t)$

It is clear from Eqs. (6–16) that when $m = n = 0$ we have $\bar{\psi}_x^{(cs)}(0,0,s) = \bar{\psi}_y^{(sc)}(0,0,s) = \bar{w}^{(ss)}(0,0,s) = 0$. However, when $m = 0$ and $n \geq 1$ we have

$$\bar{\psi}_y^{(sc)}(0,n,s) = \bar{w}^{(ss)}(0,n,s) = 0 \tag{6–48a}$$

and

$$\bar{\psi}_x^{(cs)}(0,n,s) = \int_0^b\int_0^a \bar{\psi}_x(y,s)\sin\beta_n y\,dx\,dy \tag{6–48b}$$

i.e., $\psi_y = w = 0$ and $\psi_x \equiv \psi_x(y,t)$. This case is known as the pure–shear deformation mode of a plate. For such a deformation, Eqs. (6–14b) and (6–14c) identically vanish and Eq. (6–14a) reduces to

$$a_2\bar{\psi}_{x,yy} + a_3\bar{\psi}_x = m_3 s^2\bar{\psi}_x - m_3 s\tilde{\psi}_x - m_3\dot{\tilde{\psi}}_x + d_3 s\bar{\psi}_x - d_3\tilde{\psi}_x. \tag{6–49a}$$

Also the boundary conditions at y = 0 and b given by Eqs. (6–7) and (6–11) reduce to

$$\psi_x = 0. \tag{6–49b}$$

Using Eq. (6–49b), the finite Fourier transformation of Eq. (6–49a) according to Eq. (6–48b) yields:

$$(M_{11}s^2 + C_{11}s + K_{11})\bar{\psi}_x^{(cs)}(0,n,s) = \bar{T}_1 \tag{6–50}$$

where

$$M_{11} = m_3 \;,\; C_{11} = d_3 \;,\; K_{11} = a_2\beta_n^2 - a_3, \tag{6–41a}$$

and

$$\bar{T}_1 = sC_1 + S_2 \tag{6–51b}$$

with

$$C_1 = m_3\tilde{\psi}_x^{(cs)}(0,n)$$

$$S_1 = m_3\dot{\tilde{\psi}}_x^{(cs)}(0,n) + d_3\tilde{\psi}_x^{(cs)}(0,n). \tag{6–51c}$$

Substituting Eq. (6–51b) into Eq. (6–50) results in

$$\bar{\psi}_x^{(cs)}(0,n,s) = \frac{sC_1 + S_1}{M_{11}s^2 + C_{11}s + K_{11}} \tag{6–52}$$

and the application of inverse Laplace transformation to Eq. (6–52) yields:

$$\psi_x^{(cs)}(0,n,t) = L_1C_1 + H_1S_1 \quad (n \geq 1) \tag{6–53}$$

where

$$H_1 = \frac{1}{M_{11}} e^{-\beta_k \omega_k t} \frac{1}{\bar{\omega}_k} \sin \bar{\omega}_k t \tag{6–54a}$$

and

$$L_1 = \frac{dH_1}{dt} = \frac{1}{M_{11}} e^{-\beta_k \omega_k t} \left[\cos \bar{\omega}_k t - \frac{\beta_k}{(1 - \beta_k^2)^{1/2}} \sin \bar{\omega}_k t \right]. \tag{6–54b}$$

In Eqs. (6–54), ω_k is the natural pure–shear frequency of the plate

$$\omega_k^2 = \frac{K_{11}}{M_{11}}. \tag{6–55}$$

Also $\bar{\omega}_k = \omega_k(1 - \beta_k^2)^{1/2}$ can be regarded as the natural pure–shear frequency of the damped plate with β_k being the damping ratio given by

$$2\beta_k\omega_k = \frac{C_{11}}{M_{11}}. \tag{6–56}$$

It should be noted that when $\bar{\psi}_x^{(cs)}(0,n) = \tilde{\bar{\psi}}^{(cs)}(0,n) = 0$, from Eq. (6–53) we have

$$\psi_x^{(cs)}(0,n,t) = 0 \tag{6–57}$$

In Eqs. (6–54) we assumed that $\beta_k^2 < 1$ which amounts to light damping.

6.6 Determination of $\psi_y^{(sc)}(m,0,t)$

When $n = 0$ and $m \geq 1$, we have $\psi_x = w = 0$ and $\psi_y \equiv \psi_y(x,t)$ and from Eqs. (6–16) we obtain

$$\bar{\psi}_x^{(cs)}(m,0,s) = \bar{w}^{(ss)}(m,0,s) = 0 \tag{6–58a}$$

and

$$\bar{\psi}_y^{(sc)}(m,0,s) = \int_0^b \int_0^a \bar{\psi}_y(x,s)\sin \alpha_m x \, dxdy \tag{6–58b}$$

which correspond to another pure–shear deformation mode of the plate. For this case, Eqs. (6–14a) and (6–14c) identically vanish and Eq. (6–14b) reduces to

$$b_2\bar{\psi}_{y,xx} + b_3\bar{\psi}_y = m_3 s^2\bar{\psi}_y - m_3 s\tilde{\psi}_y - m_3\tilde{\dot{\psi}}_y + d_3 s\bar{\psi}_y - d_3\tilde{\psi}_y. \tag{6–59a}$$

Also the boundary conditions at x = 0 and a given by Eqs. (6–6) and (6–10) reduce to

$$\psi_y = 0. \tag{6–59b}$$

Following the same procedure as in Section 6.5, we find

$$\psi_y^{(sc)}(m,0,t) = L_2C_2 + H_2S_2 \quad (m \geq 1) \tag{6–60}$$

where for light damping

$$H_2 = \frac{1}{M_{22}} e^{-\beta_k\omega_k t} \frac{1}{\bar{\omega}_k} \sin \bar{\omega}_k t \tag{6–61a}$$

and

$$L_2 = \frac{dH_2}{dt} = \frac{1}{M_{22}} e^{-\beta_k\omega_k t}\left[\cos \bar{\omega}_k t - \frac{\beta_k}{(1 - \beta_k^2)^{1/2}} \sin \bar{\omega}_k t\right]. \tag{6–61b}$$

In Eqs. (6–60) and (6–61) we have

$$C_2 = m_3\tilde{\psi}_y^{(sc)}(m,0),$$

$$S_2 = m_3\tilde{\dot{\psi}}_y^{(sc)}(m,0) + d_3\tilde{\psi}_y^{(sc)}(m,0),$$

$$M_{22} = m_3,$$

$$\omega_k^2 = \frac{K_{22}}{M_{22}} \ , \ 2\beta_k\omega_k = \frac{C_{22}}{M_{22}} \ , \ \bar{\omega}_k = \omega_k(1-\beta_k^2)^{1/2},$$

$$C_{22} = d_3 \ , \ K_{22} = b_2\alpha_m^2 - b_3. \tag{6–62}$$

When the initial conditions are homogeneous i.e., when $\bar{\psi}_y^{(sc)}(m,0) = \dot{\bar{\psi}}_y^{(sc)}(m,0) =$ 0, from Eq. (6–60) we will have

$$\psi_y^{(sc)}(m,0,t) = 0. \tag{6–63}$$

6.7 Response to Sonic Boom and Blast–Type Loadings

As we discussed earlier, a sonic boom type loading can be described by an N–shaped normal pressure pulse as

$$P_z(x,y,t) \equiv P(t) = \begin{cases} P_o(1 - t/t_p) \ \text{for} \ 0 \ < t \ < rt_p \\ 0 \ \text{for} \ t < 0 \ \text{and} \ t > \ rt_p \end{cases} \tag{6–64}$$

where P_o is the peak reflected pressure (when $r \leq 2$) uniformly distributed over the plate, t_p denotes the positive phase duration of the pulse (see Fig. 6.1) and r is a pulse length parameter. It is to be noted that this pulse (referred to as an N–wave) becomes a triangular one, a symmetric N–shaped pulse, and an asymmetric N–shaped pulse when $r = 1$, $r = 2$, and $r > 1$ $(r \neq 2)$, respectively. Further, a step load is also obtained by letting the positive phase time of the pressure $t_p \to \infty$ with r = 1.

We have

$$P_z^{(ss)}(m,n,t) = \int_0^b \int_0^a P_z(x,y,t) \sin \alpha_m x \sin \beta_n y \, dxdy. \tag{6–65}$$

Substituting Eq. (6–64) into Eq. (6–65) results in

$$P_z^{(ss)}(m,n,t) = \frac{4\,ab}{mn\pi^2} \begin{cases} P_o(1 - t/t_p) & \text{for } 0 < t < rt_p \\ 0 & \text{for } t < 0 \text{ and } t > rt_p \end{cases} \quad (m,n = 1,3\ldots) \tag{6–66}$$

With zero initial conditions we substitute Eq. (6–66) into Eq. (6–46) and obtain:

$$w^{(ss)}(m,n,t) = \frac{4\,ab}{mn\pi^2} \frac{P_o}{t_p} \sum_{k=1}^{3} e^{-\beta_k t} \Big\{ (A_{33k} \cos \omega_k t + B_{33k} \sin \omega_k t)$$

$$\cdot \left[t_p \int_0^t e^{\beta_k \tau} \cos \omega_k \tau d\tau - \int_o^t \tau e^{\beta_k \tau} \cos \omega_k \tau d\tau \right] + (A_{33k} \sin \omega_k t$$

$$- B_{33k} \cos \omega_k t) \left[t_p \int_0^t e^{\beta_k \tau} \sin \omega_k \tau d\tau - \int_0^t \tau e^{\beta_k \tau} \sin \omega_k \tau d\tau \right] \Big\}$$

$$\tag{6–67}$$

where for $0 < t \leq rt_p$ we have

$$\int_0^t e^{\beta_k \tau} \cos \omega_k \tau d\tau = \frac{1}{\beta_k^2 + \omega_k^2} [e^{\beta_k t} (\beta_k \cos \omega_k t + \omega_k \sin \omega_k t) - \beta_k]$$

$$\int_0^t e^{\beta_k\tau}\sin\omega_k\tau d\tau = \frac{1}{\beta_k^2+\omega_k^2}[e^{\beta_k t}(\beta_k\sin\omega_k t-\omega_k\cos\omega_k t)+\omega_k]$$

$$\int_0^t \tau e^{\beta_k\tau}\cos\omega_k\tau d\tau = \frac{te^{\beta_k t}}{\beta_k^2+\omega_k^2}(\beta_k\cos\omega_k t+\omega_k\sin\ \omega_k t)$$

$$-\frac{e^{\beta_k t}}{(\beta_k^2+\omega_k^2)^2}[(\beta_k^2-\omega_k^2)\cos\omega_k t+2\beta_k\omega_k\sin\omega_k t]+\frac{(\beta_k^2-\omega_k^2)}{(\beta_k^2+\omega_k^2)^2}$$

$$\int_0^t \tau e^{\beta_k\tau}\sin\ \omega_k\tau d\tau = \frac{te^{\beta_k t}}{\beta_k^2+\omega_k^2}(\beta_k\sin\omega_k t-\omega_k\cos\omega_k t)$$

$$-\frac{e^{\beta_k t}}{(\beta_k^2+\omega_k^2)^2}[(\beta_k^2-\omega_k^2)\sin\omega_k t-2\beta_k\omega_k\cos\omega_k t]-\frac{2\beta_k\omega_k}{(\beta_k^2+\omega_k^2)^2}$$

(6–68)

For $t > rt_p$ we replace t by rt_p in the expressions appearing at the right side of equal signs in Eqs. (6–68). Expressions for $\psi_x^{(cs)}(m,n,t)$ and $\psi_y^{(sc)}(m,n,t)$ are obtained by simply replacing (A_{33k}, B_{33k}) in Eqs. (6–67) and (6–68) by (A_{13k}, B_{13k}) and (A_{23k}, B_{23k}), respectively.

6.8 Numerical Results and Discussions

For numerical illustrations a (0/90/90/0) laminated square plate with layers of equal thickness will be considered. The material properties of all layers are also assumed to be the same and given by

$$E_1 = 10\,E_2\ ,\ G_{13} = G_{12} = 0.6\,E_2\ ,\ G_{23} = 0.5\,E_2,$$

$$\nu_{12} = 0.25\ ,\ \text{and}\ \rho = 0.00013\ \text{lb. sec}^2/\text{in}^4.$$

In order to study the influence of viscous damping

$$\frac{\Delta_j}{\pi} = \frac{d_j}{m_j\omega} \quad j = 1,3,5,7 \tag{6–69}$$

on the time history of the transverse displacement, we consider the following cases:

Case 1: $\frac{\Delta_j}{\pi} = 0$ (i.e., no damping)

Case 2: $\frac{\Delta_j}{\pi} = 0.1$ $(j = 1,3,5,7)$

Case 3: $\frac{\Delta_1}{\pi} = 0.1$ and $\frac{\Delta_j}{\pi} = 0$ with $j = 3,5,7$.

In Eq. (6–69) m_j and d_j are the mass terms and the viscous damping coefficients appearing in Eqs. (6–2). Also ω is the fundamental undamped frequency of the plate according to CLPT. It is to be noted that in Case 3 we are neglecting the damping coefficients corresponding to rotatory inertia forces. However, all the rotatory inertia forces are preserved in Cases 1 through 3. In Fig. 6.2 we have plotted the time history of the dimensionless center deflection of the plate subjected to an asymmetric N–wave ($r = 1.8$) according to TSDPT. It is seen that the responses of the plate for Cases 2 and 3 are practically identical. That is, the reduction in deflection is mainly due to damping term corresponding to the transverse inertia force. This point can further be clarified by comparing the frequencies of the plate for Cases 1 through 3. In Table 6.1 we have presented the real and imaginary parts of s_k ($= -\beta_k + J\omega_k$), where the real part ($-\beta_k$) is proportional to the damping while the imaginary part (ω_k) represents the damped frequency. It is noted that for each pair of m and n only the lowest frequency is affected by the addition of damping terms. Also for each pair of m and n the real part of s_1 is much larger than the real parts of s_2 and s_3 in Case 3. On the other hand, the real part of s_k will remain the same for all the frequencies of the plate in Case 2. The fact that the transverse displacements in Cases 2 and 3 are practically the same is an indication that only the first few vibration modes corresponding to the lowest frequencies furnish a significant contribution to the displacement.

The undamped response of the plate subjected to a symmetric N–wave ($r = 2$) is plotted in Fig. 6.3 according to various theories. It is seen that for $t > rt_p$ ($\equiv 0.0016$ sec) CLPT provides unreliable results in the sense that the deflection is quite underpredicted. In fact, depending on the values of t_p and r, it is found that in the free–vibration region (i.e., when $t > rt_p$) CLPT will sometimes overpredict the deflection. Also from Fig. 6.3 it is seen that, depending on the assumed value of the shear correction factor $K^2(\equiv K_{44}^2 = K_{55}^2)$, FSDPT can provide results in good agreement with TSDPT.

Figure 6.4 displays the variation of the dynamic magnification factor (DMF) according to TSDPT vs. ωt_p where ω is the fundamental undamped frequency of the plate according to CLPT. DMF is defined as the ratio of the largest (in the absolute sense) dynamic deflection to static deflection. In determining the static deflection, it is assumed that the plate is subjected to a uniformly distributed load of magnitude P_0. It is seen that there exists three distinct branches in the DMF curve. This is due to the fact that the largest dynamic deflection can occur during the positive ($0 < t < t_p$) and negative ($t_p < t < rt_p$) phases of the N–wave and during the free–vibration period ($t > rt_p$). The most significant amplitude attenuation due to the damping effect occurs during the free–vibration period (i.e., when $t > rt_p$). More on this subject and the effect of rotatory inertia forces will be said in the next chapter.

6.9 Conclusions

In this chapter we have developed a closed–form solution for the damped forced–vibration problem of a simply–supported symmetric cross–ply laminated plate. The analysis is general and can be applied to vibration problems of, for example, unsymmetric cross–ply plates, antisymmetric angle–ply plates, and cross–ply shells and shell panels. The equations of motion of many existing shell and plate theories in the literature are not self–adjoint (e.g., see [40,42,43,46,83]) and therefore it is necessary to use the biorthogonality relations of the modes if modal analysis is used. However, as we noted in this chapter, the response quantities are more directly obtained by using the Laplace transformation technique without the consideration of self–adjointness (or non–self–adjointness) of the equations. It should be recalled that the equations of motion of FSDPT and Reddy's TSDPT are self–adjoint (see [49]). Also the combined use of Fourier and Laplace transformations makes our analysis more complete since it could also be used when the structure is subjected to moving loads.

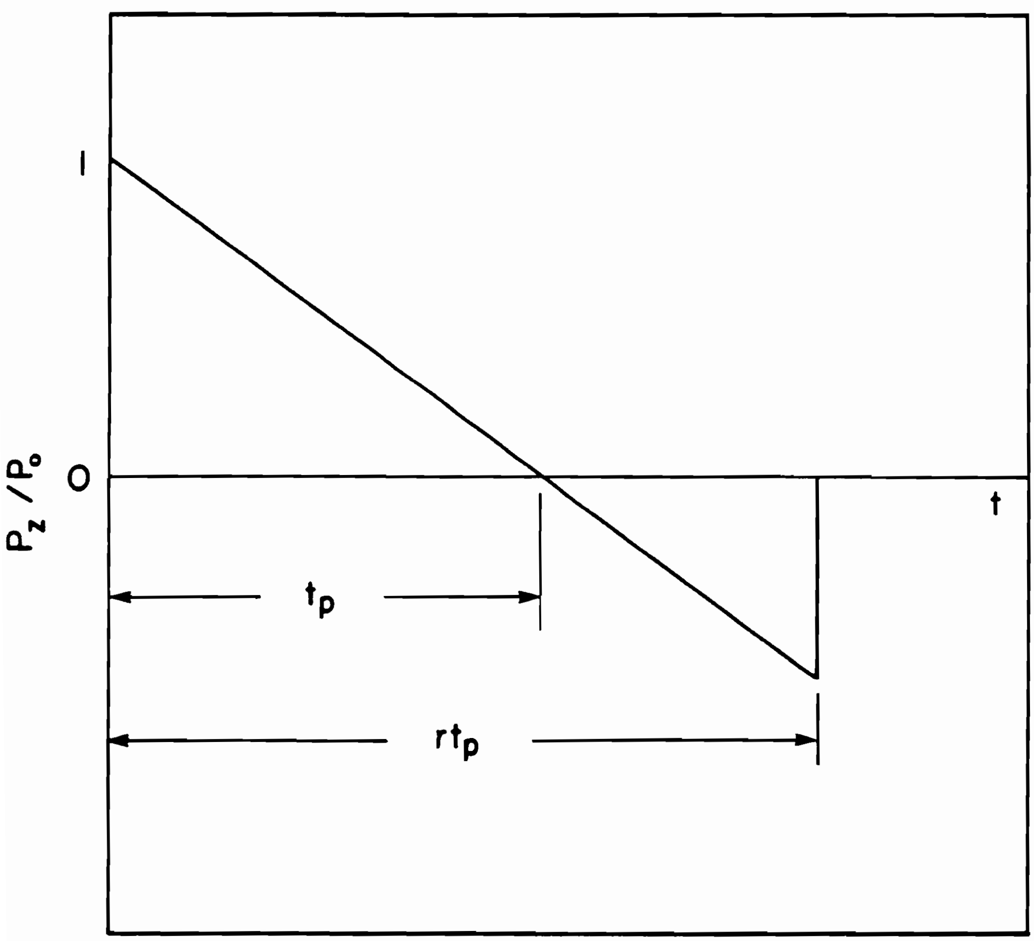

Figure 6.1 A typical asymmetric N–shaped normal pressure pulse.

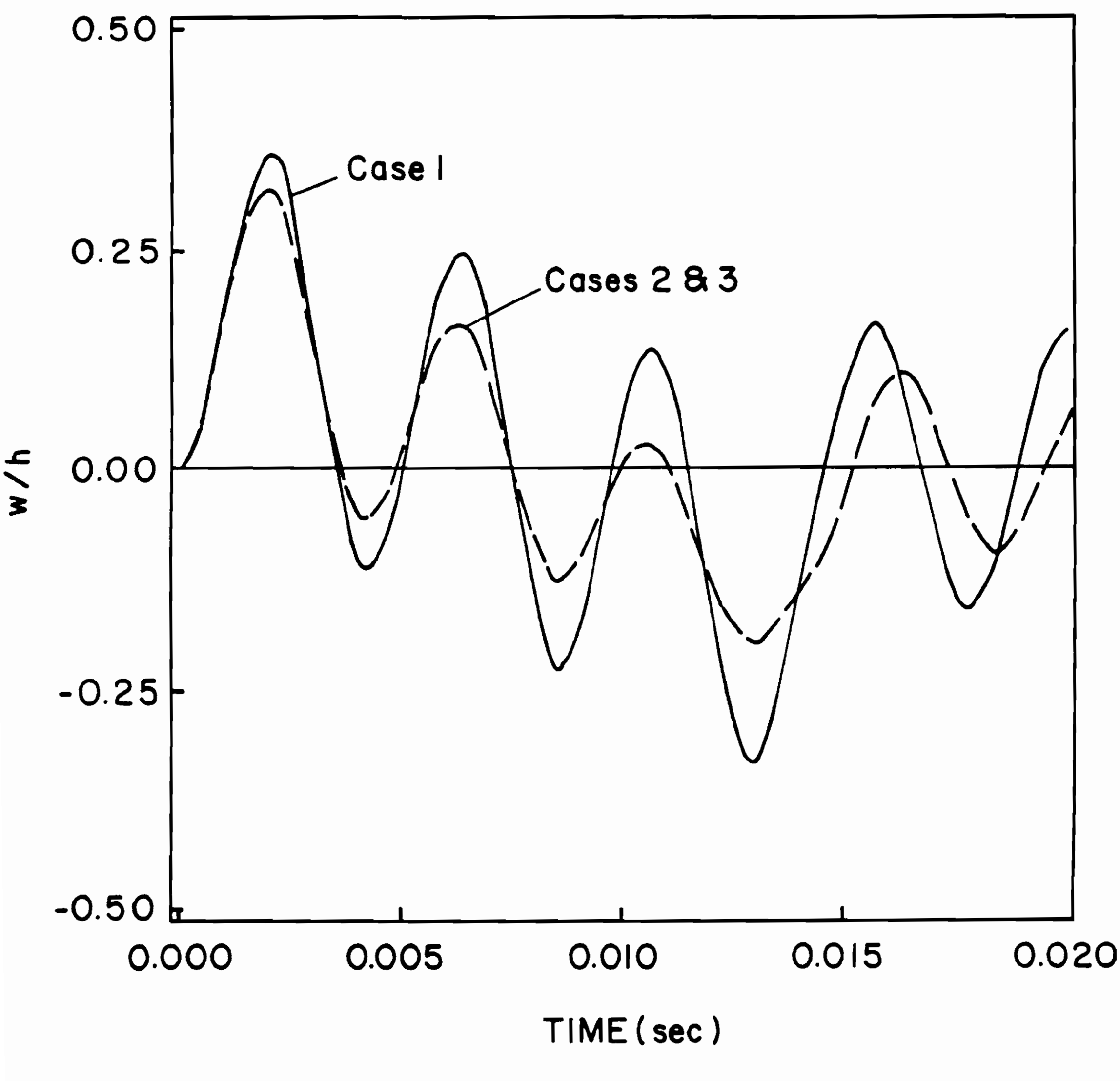

Figure 6.2 Time history of the dimensionless center deflection of a (0/90/90/0) laminated plate subjected to an asymmetric N–wave ($r = 1.8$); $t_p = 0.008$ sec, $P_o = 35$ psi, $h = 1$ in, $a = b = 30$ in.

Table 6.1

The real and imaginary parts of $s_k(= -\beta_k + J\omega_k)$ for Cases 1 through 3; a = b = 30 in., h = 1 in.

(m,n)	$s_k(k = \overline{1,3})$	Case 1	Case 2	Case 3
(1,1)	s_1	0 + J(1453.74)	−73.39 + J(1451.89)	−73.26 + J(1451.89)
	s_2	0 + J(284187.11)	−73.39 + J(284187.10)	−0.06 + J(284187.11)
	s_3	0 + J(297819.40)	−73.39 + J(297819.39)	−0.07 + J(297819.40)
(1,2)	s_1	0 + J(2864.51)	−73.39 + J(2863.57)	−73.07 + J(2863.58)
	s_2	0 + J(284694.21)	−73.39 + J(284694.20)	−0.05 + J(284694.21)
	s_3	0 + J(300420.47)	−73.39 + J(300420.46)	−0.26 + J(300420.47)
(2,2)	s_1	0 + J(5657.60)	−73.39 + J(5657.12)	−72.91 + J(5657.13)
	s_2	0 + J(292257.41)	−73.39 + J(292257.40)	−0.15 + J(292257.41)
	s_3	0 + J(301127.39)	−73.39 + J(301127.38)	−0.33 + J(301127.39)

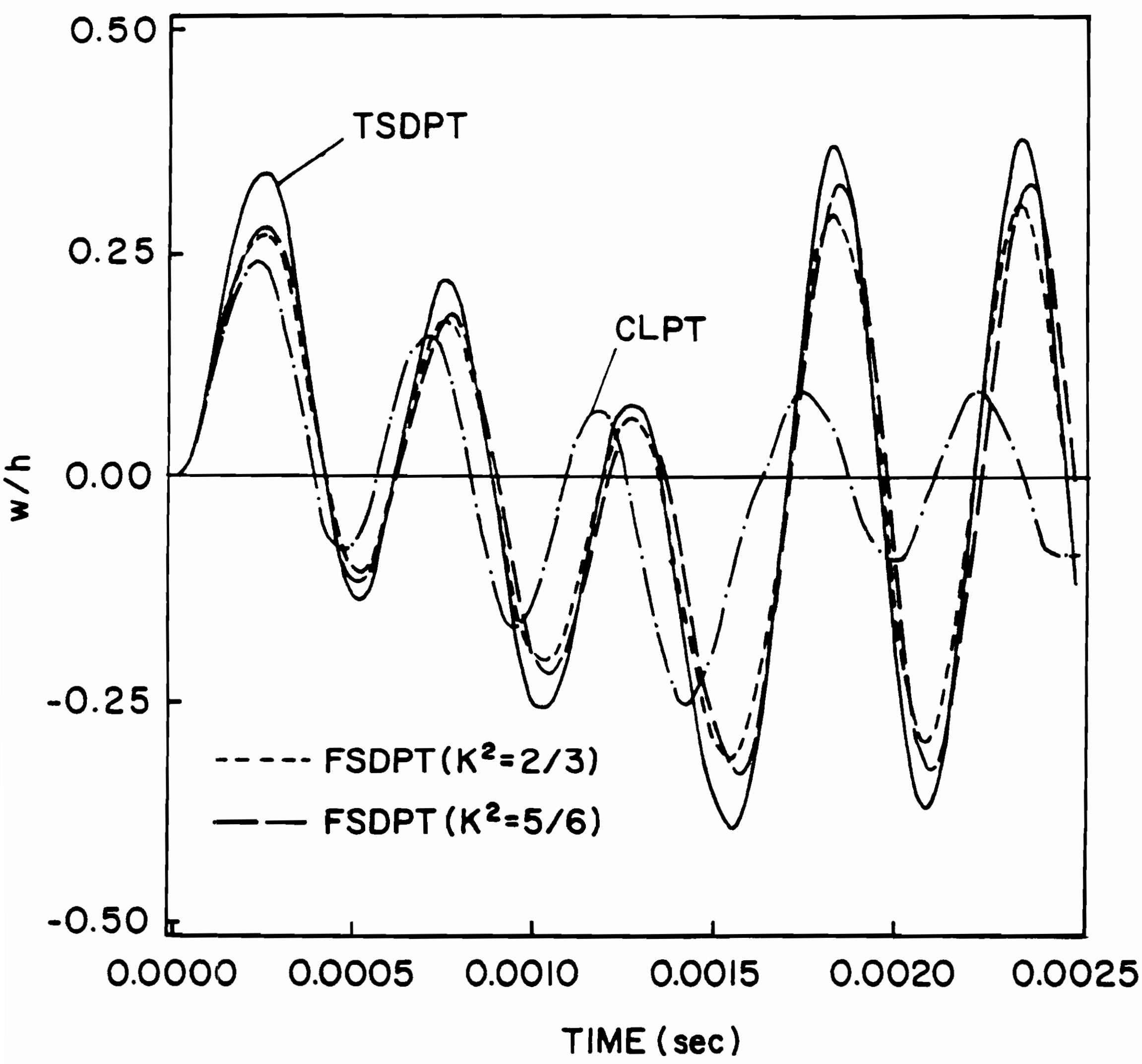

Figure 6.3 Time history of the dimesionless center deflection of a (0/90/90/0) laminated plate subjected to a symmetric N–wave (r = 2) according to various theories; $t_p = 0.0008$ sec, h = 1 in., a = b = 10 in, and $P_o = 2500$ psi.

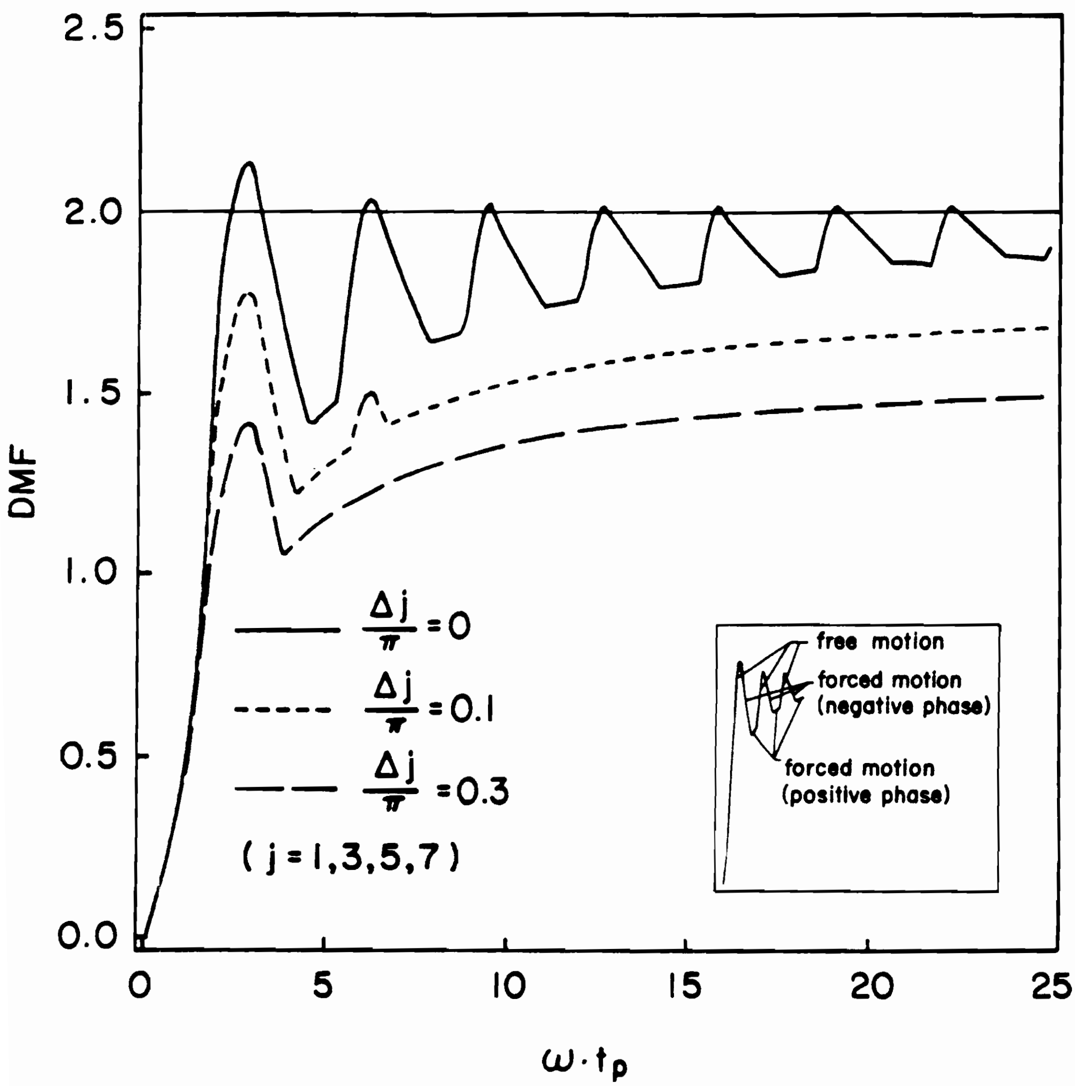

Figure 6.4 Variation of dynamic magnification factor (DMF) vs. ωt_p of a (0/90/90/0) laminated plate subjected to a symmetric N–wave (r = 2); h = 1 in., a = b = 30 in.

Appendix 6.1

The coefficients λ_k appearing in (6–24) are:

$$\lambda_1 = M_{11}(M_{22}M_{33} - M_{23}M_{32}) - M_{31}M_{13}M_{22}$$

$$\lambda_2 = M_{11}(M_{22}C_{33} + C_{22}M_{33} - C_{23}M_{32} - M_{23}C_{32}) + C_{11}(M_{22}M_{33} - M_{23}M_{32})$$
$$- M_{31}(M_{13}C_{22} + C_{13}M_{22}) - C_{31}(M_{13}M_{22})$$

$$\lambda_3 = M_{11}(M_{22}K_{33} + C_{22}C_{33} + K_{22}M_{33} - M_{23}K_{32} - C_{23}C_{32} - K_{23}M_{32})$$
$$+ C_{11}(M_{22}C_{33} + C_{22}M_{33} - C_{23}M_{32} - M_{23}C_{32}) + K_{11}(M_{22}M_{33}$$
$$- M_{23}M_{32}) + K_{21}M_{13}M_{32} + M_{31}(K_{12}M_{23} - M_{13}K_{22} - C_{13}C_{22}$$
$$- K_{13}M_{22}) - C_{31}(M_{13}C_{22} + C_{13}M_{22}) - K_{31}M_{13}M_{22}$$

$$\lambda_4 = M_{11}(C_{22}K_{33} + K_{22}C_{33} - C_{23}K_{32} - K_{23}C_{32}) + C_{11}(M_{22}K_{33} + C_{22}C_{33}$$
$$+ K_{22}M_{33} - M_{23}K_{32} - C_{23}C_{32} - K_{23}M_{32}) + K_{11}(M_{22}C_{33} + C_{22}M_{33}$$
$$- C_{23}M_{32} - M_{23}C_{32}) + K_{21}(M_{13}C_{32} + C_{13}M_{32}) + M_{31}(K_{12}C_{23}$$
$$- C_{13}K_{22} - K_{13}C_{22}) + C_{31}(K_{12}M_{23} - M_{13}K_{22} - C_{13}C_{22} - K_{13}M_{22})$$

$$- K_{31}(M_{13}C_{22} + C_{13}M_{22})$$

$$\lambda_5 = M_{11}(K_{22}K_{33} - K_{23}K_{32}) + C_{11}(C_{22}K_{33} + K_{22}C_{33} - C_{23}K_{32} - K_{23}C_{32})$$

$$+ K_{11}(M_{22}K_{33} + C_{22}C_{33} + K_{22}M_{33} - M_{23}K_{32} - C_{23}C_{32} - K_{23}M_{32})$$

$$- K_{21}(K_{12}M_{33} - M_{13}K_{32} - C_{13}C_{32} - K_{13}M_{32}) + M_{31}(K_{12}K_{23}$$

$$- K_{13}K_{22}) + C_{31}(K_{12}C_{23} - C_{13}K_{22} - K_{13}C_{22}) + K_{31}(K_{12}M_{23}$$

$$- M_{13}K_{22} - C_{13}C_{22} - K_{13}M_{22})$$

$$\lambda_6 = C_{11}(K_{22}K_{33} - K_{23}K_{32}) + K_{11}(C_{22}K_{33} + K_{22}C_{33} - C_{23}K_{32}$$

$$- K_{23}C_{32}) - K_{21}(K_{12}C_{33} - C_{13}K_{32} - K_{13}C_{32}) + C_{31}(K_{12}K_{23}$$

$$- K_{13}K_{22}) + K_{31}(K_{12}C_{23} - C_{13}K_{22} - K_{13}C_{22})$$

$$\lambda_7 = K_{11}(K_{22}K_{33} - K_{23}K_{32}) - K_{21}(K_{12}K_{33} - K_{13}K_{32}) + K_{31}(K_{12}K_{23}$$

$$- K_{13}K_{22})$$

The coefficients I_{ijr} appearing in (6–25) are:

$$I_{111} = M_{22}M_{33} - M_{23}M_{32}$$

$$I_{112} = M_{22}C_{33} + C_{22}M_{33} - C_{23}M_{32} - M_{23}C_{32}$$

$$I_{113} = M_{22}K_{33} + C_{22}C_{33} + K_{22}M_{33} - M_{23}K_{32} - C_{23}C_{32} - K_{23}M_{32}$$

$$I_{114} = C_{22}K_{33} + K_{22}C_{33} - C_{23}K_{32} - K_{23}C_{32}$$

$$I_{115} = K_{22}K_{33} - K_{23}K_{32}$$

$$I_{121} = - M_{13}M_{32} \ , \ I_{122} = -(M_{13}C_{32} + C_{13}M_{32})$$

$$I_{123} = K_{12}M_{33} - M_{13}K_{32} - C_{13}C_{32} - K_{13}M_{32}$$

$$I_{124} = K_{12}C_{33} - C_{13}K_{32} - K_{13}C_{32}$$

$$I_{125} = K_{12}K_{33} - K_{13}K_{32}$$

$$I_{131} = - M_{13}M_{22} \ , \ I_{132} = -(M_{13}C_{22} + C_{13}M_{22})$$

$$I_{133} = K_{12}M_{23} - M_{13}K_{22} - C_{13}C_{22} - K_{13}M_{22}$$

$$I_{134} = K_{12}C_{23} - C_{13}K_{22} - K_{13}C_{22}$$

$$I_{135} = K_{12}K_{23} - K_{13}K_{22}$$

$$I_{211} = -M_{31}M_{23}$$

$$I_{212} = -\ (M_{23}C_{31} + C_{23}M_{31})$$

$$I_{213} = K_{21}M_{33} - M_{23}K_{31} - C_{23}C_{31} - K_{23}M_{31}$$

$$I_{214} = K_{21}C_{33} - C_{23}K_{31} - K_{23}C_{31}$$

$$I_{215} = K_{21}K_{33} - K_{23}K_{31}$$

$$I_{221} = M_{11}M_{33} - M_{13}M_{31}$$

$$I_{222} = M_{11}C_{33} + C_{11}M_{33} - M_{13}C_{31} - C_{13}M_{31}$$

$$I_{223} = M_{11}K_{33} + C_{11}C_{33} + K_{11}M_{33} - M_{13}K_{31} - C_{13}C_{31} - K_{13}M_{31}$$

$$I_{224} = C_{11}K_{33} + K_{11}C_{33} - C_{13}K_{31} - K_{13}C_{31}$$

$$I_{225} = K_{11}K_{33} - K_{13}K_{31}$$

$$I_{231} = M_{11}M_{23}\ ,\ I_{232} = M_{11}C_{23} + C_{11}M_{23}$$

$$I_{233} = M_{11}K_{23} + C_{11}C_{23} + K_{11}M_{23} - K_{21}M_{13}$$

$$I_{234} = C_{11}K_{23} + K_{11}C_{23} - C_{13}K_{21}$$

$$I_{235} = K_{11}K_{23} - K_{13}K_{21}$$

$$I_{311} = -M_{22}M_{31} \ , \ I_{312} = -(M_{22}C_{31} + C_{22}M_{31})$$

$$I_{313} = (K_{21}M_{32} - M_{22}K_{31} - C_{22}C_{31} - K_{22}M_{31})$$

$$I_{314} = K_{21}C_{32} - C_{22}K_{31} - K_{22}C_{31}$$

$$I_{315} = K_{21}K_{32} - K_{22}K_{31}$$

$$I_{321} = M_{11}M_{32} \ , \ I_{322} = M_{11}C_{32} + C_{11}M_{32}$$

$$I_{323} = M_{11}K_{32} + C_{11}C_{32} + K_{11}M_{32} - K_{12}M_{31}$$

$$I_{324} = C_{11}K_{32} + K_{11}C_{32} - K_{12}C_{31}$$

$$I_{325} = K_{11}K_{32} - K_{12}K_{31}$$

$$I_{331} = M_{11}M_{22} \ , \ I_{332} = M_{11}C_{22} + C_{11}M_{22}$$

$$I_{333} = M_{11}K_{22} + C_{11}C_{22} + K_{11}M_{22} \;\; , \;\; I_{334} = C_{11}K_{22} + K_{11}C_{22}$$

$$I_{335} = K_{11}K_{22} - K_{12}K_{21}$$

CHAPTER VII

A STUDY OF NONLINEAR DYNAMIC EQUATIONS OF HIGHER–ORDER SHEAR DEFORMATION PLATE THEORIES

In this chapter the nonlinear dynamic equations of the first–order shear deformation plate theory (FSDPT) and the third–order shear deformation plate theory (HSDPT) of Reddy will be reformulated into equations describing the interior and edge–zone problems of rectangular plates. More explicitly, the system of five coupled equations of these theories for symmetric plates laminated of transversely isotropic layers will be recast into four equations, whose total order will be the same as that of the original equations. Three of these equations are nonlinear and coupled, and they are expressed in terms of in–plane displacement components u and v and transverse displacement w; the fourth one is a linear, second–order equation, defining the edge–zone problem of a plate. It will be shown that this latter equation will have zero solution contribution for a simply–supported plate with arbitrary in–plane boundary conditions and in the absence of rotatory inertia terms. Further, by ignoring the in–plane inertia terms, the remaining three equations will be reduced to two by introducing a force function. Viscous damping terms are also included in the governing equations. Two problems related to static large–deflection and dynamic small–deflection problems of rectangular plates are then considered. Numerical results will be presented to demonstrate the effects of nonlinearity, shear deformation, rotatory inertia, damping, and sonic boom type loadings.

7.1 Governing Equation

The nonlinear equations of motion of a symmetric laminated plate, according to the third–order shear deformation plate theory (HSDPT) of Reddy [50,52] are obtiained as a special case from Eqs. (2–3) by letting $\frac{1}{R_1} = \frac{1}{R_2} = 0$:

$$\frac{\partial N_1}{\partial x} + \frac{\partial N_6}{\partial y} = m_1\ddot{u} + c_1\dot{u} \tag{7–1a}$$

$$\frac{\partial N_6}{\partial x} + \frac{\partial N_2}{\partial y} = m_1\ddot{v} + c_1\dot{v} \tag{7–1b}$$

$$\frac{\partial M_1}{\partial x} + \frac{\partial M_6}{\partial y} - Q_1 + \lambda\left[\frac{4}{h^2}K_1 - \frac{4}{3h^2}\left[\frac{\partial P_1}{\partial x} + \frac{\partial P_6}{\partial y}\right]\right]$$

$$= m_3\ddot{\psi}_x + c_3\dot{\psi}_x - \lambda[m_5\ddot{w}_{,x} + c_5\dot{w}_{,x}] \tag{7–1c}$$

$$\frac{\partial M_6}{\partial x} + \frac{\partial M_2}{\partial y} - Q_2 + \lambda\left[\frac{4}{h^2}K_2 - \frac{4}{3h^2}\left[\frac{\partial P_6}{\partial x} + \frac{\partial P_2}{\partial y}\right]\right]$$

$$= m_3\ddot{\psi}_y + c_3\dot{\psi}_y - \lambda[m_5\ddot{w}_{,y} + c_5\dot{w}_{,y}] \tag{7–1d}$$

$$\frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial y} + P_z + \lambda\left[\frac{4}{3h^2}\left[\frac{\partial^2 P_1}{\partial x^2} + 2\frac{\partial^2 P_6}{\partial x\partial y} + \frac{\partial^2 P_2}{\partial y^2}\right] - \frac{4}{h^2}\left[\frac{\partial K_1}{\partial x} + \frac{\partial K_2}{\partial y}\right]\right]$$

$$= m_1\ddot{w} + c_1\dot{w} + \lambda[m_5(\ddot{\psi}_{x,x} + \ddot{\psi}_{y,y}) + c_5(\dot{\psi}_{x,x} + \dot{\psi}_{y,y})$$

$$- m_7\nabla^{w}\ddot{w} - c_7\nabla^2\dot{w}] - \mathcal{N}(u,v,w) \tag{7–1e}$$

where a superposed dot on the variables indicates differentiation with respect to time, and a comma followed by a variable denotes differentiation, e.g., $w_{,x} \equiv \partial w/\partial x$, and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{7–2}$$

$$\mathcal{N} = \frac{\partial}{\partial x}(N_1 \frac{\partial w}{\partial x} + N_6 \frac{\partial w}{\partial y}) + \frac{\partial}{\partial y}(N_6 \frac{\partial w}{\partial x} + N_2 \frac{\partial w}{\partial y}). \tag{7–3}$$

In Eqs. (1), c_1, c_3, c_5, and c_7 are the viscous damping coefficients which are assumed, for simplicity, to be proportional to mass terms m_1, m_3, m_5, and m_7, respectively. Also λ is a tracer which takes on the values of 1 and 0 to give theories TSDPT and FSDPT, respectively. In Eq. (7–1e), $P_z \equiv P_z(x,y,t)$ denotes the transverse load. The mass terms $m_1,...,m_7$ are defined in Appendix 7.1. For convenience, we have replaced x_1, x_2, ϕ_1, and ϕ_2 by x, y, ψ_x, and ψ_y, respectively (see Eqs. (2–3)). It should be noted that Eqs. (7–1a) and (7–1b) remain the same for TSDPT, FSDPT, and classical plate theory (CLPT), and Eqs. (7–1c) and (7–1d) are linear. Thus, Eqs. (7–1a), (7–1b) and (7–1e) are the only nonlinear equations. This is due to the fact that the nonlinear terms involving the rotation functions ψ_x and ψ_y are ignored in the derivation of these equations (see [50]), and only nonlinear terms involving the derivatives of the transverse displacement w with respect to x and y are retained (like in von Kármán's plate theory). Also the nonlinear expression $\mathcal{N}$ given by Eq. (7–3) remains the same in all three theories.

Equations (7–1a) and (7–1b) can be expressed in terms of the displacement components u, v, and w for a symmetric plate with transversely isotropic layers (the plane of isotropy is assumed to be parallel to the mid–plane of the plate) by introducing the new rigidity terms $\bar{\bar{M}}$ and $\bar{\bar{B}}$ (see Appendix 7.1). When this is done, Eqs. (2–5) can be expressed as

$$N_1 = \bar{\bar{M}}\epsilon_1^o + (\bar{\bar{M}} - \bar{\bar{B}})\epsilon_2^o$$

$$N_2 = (\bar{\bar{M}} - \bar{\bar{B}})\epsilon_1^o + \bar{\bar{M}}\epsilon_2^o$$

$$N_6 = \frac{\bar{\bar{B}}}{2}\,\epsilon_6^o \tag{7–4}$$

where the extensional strain measures are given (see Eqs. (2–8)) by

$$\epsilon_1^o = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$\epsilon_2^o = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

$$\epsilon_6^o = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \tag{7–5}$$

Substituting Eqs. (7–5) into Eqs. (7–4) and the result into Eqs. (7–1a) and (7–1b), we obtain two nonlinear equations in terms of u, v, and w:

$$\bar{\bar{M}}(u_{,xx} + w_{,x}w_{,xx}) + \frac{\bar{\bar{B}}}{2}(u_{,yy} + w_{,x}w_{,yy})$$

$$+ (\bar{\bar{M}} - \frac{\bar{\bar{B}}}{2})(v_{,xy} + w_{,y}w_{,xy}) = m_1\ddot{u} + c_1\dot{u}$$

$$\bar{\bar{M}}(v_{,yy} + w_{,y}w_{,yy}) + \frac{\bar{\bar{B}}}{2}(v_{,xx} + w_{,y}w_{,xx})$$

$$+ (\bar{\bar{M}} - \frac{\bar{\bar{B}}}{2})(u_{,xy} + w_{,x}w_{,xy}) = m_1\ddot{v} + c_1\dot{v} \qquad (7\text{–}6)$$

These equations remain the same for the three theories, as stated earlier. With the help of Eqs. (7–4) and (7–5), the nonlinear expression N in Eq. (7–1e) (which is defined by Eq. (7–3)) can also be expressed in terms of u, v, and w.

For a symmetric plate with transversely isotropic layers the remaining stress resultants in Eqs. (2–5) and (2–6) can be expressed in terms of the generalized displacement components as

$$M_1 = \bar{D}\psi_{x,x} + (\bar{D} - 2\bar{C})\psi_{y,y} - \bar{F}w_{,xx} - (\bar{F} - 2\bar{L})w_{,yy}$$

$$M_2 = \bar{D}\psi_{y,y} + (\bar{D} - 2\bar{C})\psi_{x,x} - \bar{F}w_{,yy} - (\bar{F} - 2\bar{L})w_{,xx}$$

$$M_6 = \bar{C}(\psi_{x,y} + \psi_{y,x}) - 2\bar{L}w_{,xy}$$

$$P_1 = \frac{3h^2}{4}\{F\psi_{x,x} + (F - 2L)\psi_{y,y} - Hw_{,xx} - [H - 2(\bar{L} - L)]w_{,yy}\}$$

$$P_2 = \frac{3h^2}{4}\{F\psi_{y,y} + (F - 2L)\psi_{x,x} - Hw_{,yy} - [H - 2(\bar{L} - L)]w_{,xx}\}$$

$$P_6 = \frac{3h^2}{4}[L(\psi_{x,y} + \psi_{y,x}) - 2(\bar{L} - L)w_{,xy}]$$

$$Q_1 = \bar{A}(\psi_x + w_{,x}) \ , \ Q_2 = \bar{A}(\psi_y + w_{,y})$$

$$K_1 = \frac{h^2}{4}(\bar{A} - A)(\psi_x + w_{,x}) \ , \ K_2 = \frac{h^2}{4}(\bar{A} - A)(\psi_y + w_{,y}) \tag{7–7a}$$

in HSDPT and

$$M_1 = \bar{\bar{D}}\psi_{x,x} + (\bar{\bar{D}} - 2\bar{\bar{C}})\psi_{y,y}$$

$$M_2 = \bar{\bar{D}}\psi_{y,y} + (\bar{\bar{D}} - 2\bar{\bar{C}})\psi_{x,x}$$

$$M_6 = \bar{\bar{C}}(\psi_{x,y} + \psi_{y,x})$$

$$Q_1 = K^2\bar{\bar{A}}(\psi_x + w_{,x}) \ , \ Q_2 = K^2\bar{\bar{A}}(\psi_y + w_{,y}) \tag{7–7b}$$

in FSDPT. The new rigidity terms that are introduced here are defined in Appendix 7.1. Also K^2 in Eqs. (7–7b) is a shear correction factor and h in Eqs. (7–7a) is the total thickness of the plate.

With the help of Eqs. (7–7a) and letting $\lambda = 1$, the remaining equations of motion of TSDPT [i.e., Eqs. (7–1c) through (7–1e)] are expressed in terms of u, v, ψ_x, ψ_y, and w and presented as

$$D\psi_{x,xx} + C\psi_{x,yy} + (D - C)\psi_{y,xy} - A(\psi_x + w_{,xx})$$

$$- F\nabla^2 w_{,x} = m_3\ddot{\psi}_x + c_3\dot{\psi}_x - m_5\ddot{w}_{,x} - c_5\dot{w}_{,x} \tag{7–8a}$$

$$D\psi_{y,yy} + C\psi_{y,xx} + (D - C)\psi_{x,xy} - A(\psi_y + w_{,y})$$

$$- F\nabla^2 w_{,y} = m_3\ddot{\psi}_y + c_3\dot{\psi}_y - m_5\ddot{w}_{,y} - c_5\dot{w}_{,y} \tag{7–8b}$$

$$F\nabla^2(\psi_{x,x} + \psi_{y,y}) + A(\psi_{x,x} + \psi_{y,y}) - H\nabla^2\nabla^2 w$$

$$+ A\nabla^2 w + P_z = m_1\ddot{w} + c_1\dot{w} + m_5(\ddot{\psi}_{x,x} + \ddot{\psi}_{y,y})$$

$$+ c_5(\dot{\psi}_{x,x} + \dot{\psi}_{y,y}) - m_7\nabla^2\ddot{w} - c_7\nabla^2\dot{w} - \mathcal{N}(u,v,w) \tag{7–8c}$$

Similarly, with the help of Eqs. (7–7b) and setting $\lambda = 0$ the remaining governing equations of FSDPT are obtained from Eqs. (7–1c) through (7–1e):

$$\bar{\bar{D}}\psi_{x,xx} + \bar{\bar{C}}\psi_{x,yy} + (\bar{\bar{D}} - \bar{\bar{C}})\psi_{y,xy} - K^2\bar{\bar{A}}(\psi_x + w_{,x})$$

$$= \bar{\bar{m}}_3\ddot{\psi}_x + c_3\dot{\psi}_x \tag{7–9a}$$

$$\bar{\bar{D}}\psi_{y,yy} + \bar{\bar{C}}\psi_{y,xx} + (\bar{\bar{D}} - \bar{\bar{C}})\psi_{x,xy} - K^2\bar{\bar{A}}(\psi_y + w_{,y})$$

$$= \bar{\bar{m}}_3\ddot{\psi}_y + c_3\dot{\psi}_y \tag{7–9b}$$

$$K^2\bar{\bar{A}}(\psi_{x,x} + \psi_{y,y}) + K^2\bar{\bar{A}}\nabla^2 w + P_z$$

$$= m_1\ddot{w} + c_1\dot{w} - \mathcal{N}(u,v,w) \tag{7–9c}$$

Next, it is shown that by introducing a potential function Φ, Eqs. (7–8) can be uncoupled into two equations, one in terms of u, v, and w, and the other one in terms of Φ only.

7.2 Uncoupling of Equations

By differentiating Eqs. (7–8a) and (7–8b) with respect to y and x, respectively, and then subtracting one from the other, we obtain the following second–order linear equation:

$$C\nabla^2\Phi - A\Phi = m_3\ddot{\Phi} + c_3\dot{\Phi} \tag{7–10a}$$

where the potential function Φ is introduced as

$$\Phi = \psi_{x,y} - \psi_{y,x}. \tag{7–10b}$$

Differentiating Eqs. (7–8a) and (7–8b) with respect to x and y, respectively, and adding the results, the following relation is found:

$$L_1(\psi_{x,x} + \psi_{y,y}) = L_2(w) \tag{7–11a}$$

where

$$L_1 = D\nabla^2 - A - m_3 \frac{\partial^2}{\partial t^2} - c_3 \frac{\partial}{\partial t}$$

$$L_2 = F\nabla^2\nabla^2 + A\nabla^2 - m_5\nabla^2(\ )_{,tt} - c_5\nabla^2(\ )_{,t}. \tag{7–11b}$$

Equation (7–8c) can also be presented as:

$$L_3(\psi_{x,x} + \psi_{y,y}) = L_4(w) - P_z - \mathcal{N}(u,v,w) \tag{7–12a}$$

where

$$L_3 = F\nabla^2 + A - m_5 \frac{\partial^2}{\partial t^2} - c_5 \frac{\partial}{\partial t}$$

$$L_4 = H\nabla^2\nabla^2 - A\nabla^2 + m_1(\ )_{,tt} + c_1(\ )_{,t}$$

$$- m_7\nabla^2(\ )_{,tt} - c_7\nabla^2(\ )_{,t} \tag{7–12b}$$

The equation describing the interior problem of a plate is found by operating L_3 and L_1 on Eqs. (7–11a) and (7–12a), respectively, and subtracting the subsequent results. Realizing that $L_3L_1 = L_1L_3$, the result is:

$$(L_1L_4 - L_3L_2)(w) = L_1(P_z) + L_1(\mathcal{N}) \tag{7–13}$$

Substitution of the operators L_j $(j = \overline{1,4})$ into (7–13) results in an equation in terms of u, v, and w as

$$\frac{1}{A}(\bar{\bar{D}}H - \bar{F}^2)\nabla^2\nabla^2\nabla^2 w - \bar{\bar{D}}\nabla^2\nabla^2 w - \frac{D}{A}\nabla^2 P_z + \frac{m_3}{A}\ddot{P}_z$$

$$+ \frac{c_3}{A}\dot{P}_z = \frac{1}{A}(\bar{m}_5^2 - \bar{\bar{m}}_3 m_7)\nabla^2\ddddot{w} + \frac{1}{A}(2m_5c_5 - m_3c_7 - m_7c_3)\nabla^2\dddot{w}$$

$$- (\bar{\bar{m}}_3 + \frac{D}{A}m_1 + \frac{c_3c_7}{A} - \frac{c_5^2}{A})\nabla^2\ddot{w} - (2c_5 + c_3 + c_7 + \frac{D}{A}c_1)\nabla^2\dot{w}$$

$$+ \frac{m_1m_3}{A}\ddddot{w} + \frac{1}{A}(m_1c_3 + m_3c_1)\dddot{w} + (m_1 + \frac{c_1c_3}{A})\ddot{w} + c_1\dot{w}$$

$$+ \frac{1}{A}(H\bar{\bar{m}}_3 + m_7\bar{\bar{D}} - 2\bar{m}_5\bar{F})\nabla^2\nabla^2\ddot{w}$$

$$+ \frac{1}{A}(Dc_7 + Hc_3 - 2Fc_5)\nabla^2\nabla^2\dot{w} + \frac{1}{A}L_1(\mathcal{N}(u,v,w)), \qquad (7\text{–}14)$$

where L_1 is given by Eq. (7–11b). Similarly, by following the same procedure as above, the governing equations of the first–order shear deformation theory (i.e., Eqs. (7–9)) are recast as:

$$\bar{\bar{C}}\nabla^2\Phi - K^2\bar{\bar{A}}\Phi = \bar{\bar{m}}_3\ddot{\Phi} + c_3\dot{\Phi} \qquad (7\text{–}15)$$

$$- \bar{\bar{D}}\nabla^2\nabla^2 w - \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\nabla^2 P_z + P_z + \frac{\bar{\bar{m}}_3}{K^2\bar{\bar{A}}}\ddot{P}_z + \frac{c_3}{K^2\bar{\bar{A}}}\dot{P}_z$$

$$= -(\bar{\bar{m}}_3 + \frac{m_1\bar{\bar{D}}}{K^2\bar{A}})\nabla^2\ddot{w} - (c_3 + \frac{c_1\bar{\bar{D}}}{K^2\bar{A}})\nabla^2\dot{w} + \frac{m_1\bar{\bar{m}}_3}{K^2\bar{A}}\ddddot{w}$$

$$+ \frac{1}{K^2\bar{A}}(m_1c_3 + c_1\bar{\bar{m}}_3)\dddot{w} + (m_1 + \frac{c_1c_3}{K^2\bar{A}})\ddot{w}$$

$$+ c_1\dot{w} + \frac{1}{K^2\bar{A}} L_1(N(u,v,w)) \tag{7–16}$$

where the operator L_1 is

$$L_1 = \bar{\bar{D}}\nabla^2 - K^2\bar{A} - c_3(\ )_{,t} - \bar{\bar{m}}_3(\ )_{,tt}. \tag{7–17}$$

In summary, Eqs. (7–6), (7–10a) and (7–14) comprise the four governing equations of TSDPT in terms of u, v, w, and Φ, and Eqs. (7–6), (7–15), and (7–16) are those of FSDPT, in terms of u, v, w, and Φ. Equation (7–10a), which looks like the equation of a freely–vibrating membrane on an elastic foundation, is known as the edge–zone or boundary layer equation of the plate.

Since the boundary conditions in these theories are expressed in terms of u, v, ψ_x, ψ_y, and w, it is also necessary to express ψ_x and ψ_y in terms of u, v, w, and Φ. This will be illustrated for TSDPT, and the results will be outlined for FSDPT. For the sake of simplicity, the rotatory inertia terms and their corresponding damping terms will be ignored, and with such simplification, it will also be shown that the contribution of the edge–zone equation (i.e., Eq. (7–10a) will be identically zero for a simply supported plate.

If the rotatory inertia terms and their corresponding damping terms are neglected, Eqs. (7–10a) and (7–14) are reduced to:

$$C\nabla^2\Phi - A\Phi = 0 \tag{7–18}$$

and

$$\frac{1}{A}(\bar{\bar{D}}H - \bar{F}^2)\nabla^2\nabla^2\nabla^2 w - \bar{\bar{D}}\nabla^2\nabla^2 w - \frac{D}{A}\nabla^2 P_z + P_z$$

$$= -\frac{D}{A} m_1 \nabla^2 \ddot{w} - \frac{D}{A} c_1 \nabla^2 \dot{w} + m_1 \ddot{w} + c_1 \dot{w} + \frac{1}{A} L_1(N) \tag{7–19a}$$

with

$$L_1 = D\nabla^2 - A. \tag{7–19b}$$

Further, with the help of relation (7–10b), Eqs. (7–8a)–(7–8c) can be rewritten as:

$$\psi_x = \frac{D}{A}(\psi_{x,x} + \psi_{y,y})_{,x} + \frac{C}{A}\Phi_{,y} - \frac{F}{A}\nabla^2 w_{,x} - w_{,x} \tag{7–20a}$$

$$\psi_y = \frac{D}{A}(\psi_{x,x} + \psi_{y,y})_{,y} - \frac{C}{A}\Phi_{,x} - \frac{F}{A}\nabla^2 w_{,y} - w_{,y} \tag{7–20b}$$

and

$$\nabla^2(\psi_{x,x} + \psi_{y,y}) = -\frac{A}{F}(\psi_{x,x} + \psi_{y,y}) + \frac{H}{F}\nabla^2\nabla^2 w$$

$$-\frac{A}{F}\nabla^2 w - \frac{1}{F}p_z + \frac{m_1}{F}\ddot{w} + \frac{c_1}{F}\dot{w} - \frac{1}{F}\mathcal{N}(u,v,w). \qquad (7\text{–}20c)$$

Differentiating Eqs. (7–20a) and (7–20b) with respect to x and y, respectively, and adding the results we obtain

$$\nabla^2(\psi_{x,x} + \psi_{y,y}) = \frac{A}{D}(\psi_{x,x} + \psi_{y,y}) + \frac{F}{D}\nabla^2\nabla^2 w + \frac{A}{D}\nabla^2 w. \qquad (7\text{–}21)$$

Subtracting Eq. (7–20c) from Eq. (7–21) results in

$$(\psi_{x,x} + \psi_{y,y}) = \frac{(\bar{\bar{D}}H - \bar{F}^2)}{A\bar{D}}\nabla^2\nabla^2 w - \nabla^2 w - \frac{D}{A\bar{D}}P_z$$

$$+ \frac{Dm_1}{A\bar{D}}\ddot{w} + \frac{D}{A\bar{D}}c_1\dot{w} - \frac{D}{A\bar{D}}N(u,v,w) \qquad (7\text{–}22)$$

where $\bar{D} \equiv F + D = \bar{\bar{D}} - \bar{F}$ (see Appendix 7.1) is introduced. Substitution of Eq. (7–22) into Eqs. (7–20a) and (7–20b) yields the explicit expressions for ψ_x and ψ_y in terms of u, v, w, and Φ as follows:

$$\psi_x = \frac{D}{A^2\bar{D}}(\bar{\bar{D}}H - F^2)\nabla^2\nabla^2 w_{,x} - \frac{\bar{D}}{A}\nabla^2 w_{,x} - w_{,x} + \frac{C}{A}\Phi_{,y}$$

$$+ \frac{D^2}{A^2\bar{D}}(m_1\ddot{w}_{,x} + c_1\dot{w}_{,x} - P_{z,x} - \frac{\partial}{\partial x}\mathcal{N}(u,v,w)) \qquad (7\text{–}23a)$$

$$\psi_y = \frac{D}{A^2\bar{D}}(\bar{\bar{D}}H - \bar{F}^2)\nabla^2\nabla^2 w_{,y} - \frac{\bar{D}}{A}\nabla^2 w_{,y} - w_{,y} - \frac{C}{A}\Phi_{,x}$$

$$+ \frac{D^2}{A^2\bar{D}}(m_1\ddot{w}_{,y} + c_1\dot{w}_{,y} - P_{z,y} - \frac{\partial}{\partial y}\mathcal{N}(u,v,w)). \qquad (7\text{–}23b)$$

Results similar to Eqs. (7–18), (7–19), and (7–23) can also be developed for FSDPT; they are summarized as

$$\bar{\bar{C}}\nabla^2\Phi - K^2\bar{\bar{A}}\Phi = 0 \qquad (7\text{–}24a)$$

$$-\bar{\bar{D}}\nabla^2\nabla^2 w - \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\nabla^2 P_z + P_z = -\frac{\bar{\bar{D}}m_1}{K^2\bar{\bar{A}}}\nabla^2\ddot{w} + m_1\ddot{w} + c_1\dot{w} + \frac{1}{K^2\bar{\bar{A}}}L_1(\mathcal{N})$$

$$(7\text{–}24b)$$

with

$$L_1 = \bar{\bar{D}}\nabla^2 - K^2\bar{\bar{A}} \qquad (7\text{–}24c)$$

and

$$\psi_x = -\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\nabla^2 w_{,x} - w_{,x} + \frac{\bar{\bar{C}}}{K^2\bar{\bar{A}}}\Phi_{,y} + \frac{\bar{\bar{D}}}{(K^2\bar{\bar{A}})^2}(m_1\ddot{w}_{,x}$$

$$+ c_1\dot{w}_{,x} - P_{z,x} - \frac{\partial}{\partial x} N(u,v,w)) \tag{7–25a}$$

$$\psi_y = -\frac{\bar{\bar{D}}}{K^2\bar{A}} \nabla^2 w_{,y} - w_{,y} - \frac{\bar{\bar{C}}}{K^2\bar{A}} \Phi_{,x} + \frac{\bar{\bar{D}}}{(K^2\bar{A})^2} (m_1\ddot{w}_{,y}$$

$$+ c_1\dot{w}_{,y} - P_{z,y} - \frac{\partial}{\partial y} N(u,v,w)). \tag{7–25b}$$

7.3 Application to Simply–Supported Plates

We show that, for a simply–supported plate, the solution contribution of the edge–zone equation is identically zero in the absence of rotatory inertia terms and their corresponding damping terms. To this end, it will suffice to work with TSDPT. It should be noded that the total order of equations in TSDPT is twelve and, therefore, six boundary conditions are specified at each edge of a rectangular plate with lengths a and b in the x– and y–directions, respectively. At $x = 0$ and a, for example, we can classify four different types of simple–support boundary conditions as follows (see Eqs. (2–11)):

S1;

$$w = \psi_y = P_1 = M_1 = 0 \text{ and } N_1 = N_6 = 0 \tag{7–26a}$$

S2;

$$w = \psi_y = P_1 = M_1 = 0 \text{ and } u = N_6 = 0 \tag{7–26b}$$

S3;

$$w = \psi_y = P_1 = M_1 = 0 \text{ and } N_1 = v = 0 \tag{7–26c}$$

S4;

$$w = \psi_y = P_1 = M_1 = 0 \text{ and } u = v = 0 \tag{7–26d}$$

It should be noted that the conditions

$$w = 0 \tag{7–27a}$$

$$\psi_y = 0 \tag{7–27b}$$

$$P_1 = M_1 = 0 \tag{7–27c}$$

are common in boundary types S1 through S4. The remaining two boundary conditions, referred to as the in–plane boundary conditions, involve the specification of one element of each part: (u or N_1) and (v or N_6). The explicit specifications of in–plane boundary conditions will not be needed in the remaining discussions. For x = 0 or a (see Eq. (7–27a,b), since the derivatives (of any order) of w and ψ_y with respect to y and time are also zero, the boundary conditions (7–27c) are equivalent to (see Eqs. (7–7a)):

$$\psi_{x,x} = 0 \tag{7–28a}$$

and

$$w_{,xx} = 0 \tag{7–28b}$$

Substitution of Eqs. (7–27a), (7–27b), and (7–28) into (7–20b), on the other hand, results in

$$\Phi_{,x} = 0 \text{ at } x = 0 \text{ or a.} \tag{7–29a}$$

Also substituting Eqs. (7–27a), (7–27b), and (7–28) into Eq. (7–22) yields

$$\left[\frac{\bar{D}H - \bar{F}^2}{D}\right] w_{,xxxx} - P_z - \mathcal{N}(u,v,w) = 0 \text{ at } x = 0 \text{ or a,} \tag{7–29b}$$

where the nonlinear expression $\mathcal{N}$ is evaluated at the edge of the plate with the help of Eq. (7–3) and consideration of in–plane boundary conditions. For example, $\mathcal{N}$

vanishes if the edge is considered to be completely movable and not subjected to any in–plane loads.

In summary, Eqs. (7–27a), (7–28b), and (7–29) comprise the four boundary conditions at x = 0 or a in terms of w and Φ and, in general, u and v. Similarly, it can be verified that the boundary conditions at y = 0 or b are

$$w = w_{,yy} = 0 \tag{7–30a}$$

$$\Phi_{,y} = 0 \tag{7–30b}$$

and

$$\left[\frac{\bar{\bar{D}}H - \bar{F}^2}{D}\right] w_{,yyyy} - P_z - \mathcal{N}(u,v,w) = 0. \tag{7–30c}$$

The boundary conditions developed above can be classified into two groups; one group involving w, u and v, and the other group involving Φ only. Further, it can easily be verified that the edge–zone equation (7–18), with the boundary conditions (7–29a) and (7–30b), admits only the trivial solution $\Phi = 0$. Hence, when the rotatory inertia terms are omitted, the five governing equations of TSDPT are reduced to three in number, namely Eqs. (7–6) and (7–19a), with a total order of ten. This conclusion is also valid in static–flexural, buckling, and postbuckling problems since $\Phi = 0$ will again be the only solution of the edge–zone equation (7–18). Based on similar arguments, the five equations of FSDPT are also reduced to three, namely Eqs. (7–6) and (7–24b), with a total order of eight. This result is also obtained in [130] in the context of postbuckling analysis of plates.

The boundary conditions (7–29a) and (7–30b) will also be obtained if the rotatory inertia terms are not omitted in Eqs. (7–20a) through (7–23b). Moreover, again the edge–zone equation (7–10a) with its corresponding boundary conditions

(7–29a) and (7–30b) can be considered and solved as a separate linear problem. Therefore, the vibration mode corresponding to this problem is independent of those corresponding to mid–plane displacements u, v, and w. More on this will be said in the subsequent chapters. The boundary conditions (7–29a) and (7–30b) are identically satisfied by assuming

$$\Phi(x,y,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cos \alpha_m x \cos \beta_n y \, T_{mn}(t) \tag{7–31}$$

where $\alpha_m = \frac{m\pi}{a}$, $\beta_n = \frac{n\pi}{b}$, and $T_{mn}(t)$ are the generalized coordinates.

Substitution of Eq. (7–31) into Eq. (7–10a) results in

$$\ddot{T}_{mn}(t) + \frac{c_3}{m_3} \dot{T}_{mn}(t) + \omega_{mn}^2 T_{mn}(t) = 0 \tag{7–32}$$

where

$$\omega_{mn} = \frac{1}{m_3^{1/2}} \left[C\left[\alpha_m^2 + \beta_n^2\right] + A \right]^{1/2} \tag{7–33}$$

are the pure–shear frequencies of the undamped system corresponding to zero mid–plane displacement components u, v, and w. The pure–shear frequencies corresponding to the damped system, as can be seen from Eq. (7–32), are

$$\omega_{dmn} = \left[\omega_{mn}^2 - \frac{1}{4} \left[\frac{c_3}{m_3}\right]^2 \right]^{1/2} \tag{7–34}$$

where ω_{mn} are given by Eq. (7–33). Similar results are readily obtained for FSDPT by merely replacing m_3, C, and A in Eqs. (7–32) through (7–34) by $\bar{\bar{m}}_3$, $\bar{\bar{C}}$, and $K^2\bar{\bar{A}}$, respectively (see Eq. (7–15)). It can readily be shown that FSDPT with $K^2 = \frac{14}{17}$ and TSDPT predict identical pure–shear frequency for a homogeneous transversely isotropic (and isotropic) plate.

7.4 Equations in Terms of w and a Force Function

In certain problems, it is more convenient to introduce a force function Ψ such that:

$$N_1 = \Psi_{,yy},\ N_2 = \Psi_{,xx},\ \text{and}\ N_6 = -\Psi_{,xy}. \tag{7–35}$$

It can readily be verified that this function satisfies the equations of motion (7–1a) and (7–1b) identically in the absence of in–plane inertia and their corresponding damping terms. Further, with the help of Eqs. (7–1a) and (7–1b) with zero right–hand side, and Eq. (7–35), Eq. (7–3) is reduced to

$$\mathcal{N}(w,\Psi) = \Psi_{,yy}w_{,xx} - 2\Psi_{,xy}w_{,xy} + \Psi_{,xx}w_{,yy} \tag{7–36}$$

Hence, the interior equations of TSDPT and FSDPT can be expressed in terms of the transverse displacement w and the potential function Ψ. These equations are given by (7–14) and (7–16), respectively, or by (7–19a) and (7–24b) when the rotatory inertia terms are also neglected.

A second equation, which relates w to Ψ, is derived from the condition of compatibility:

$$\epsilon_1^o + \epsilon_2^o - \epsilon_6^o = w_{,xy}^2 - w_{,xx}w_{,yy} \tag{7–37}$$

which is satisfied by Eqs. (7–5). To this end, the inverse of Eqs. (7–4) is written as

$$\begin{Bmatrix} \epsilon_1^o \\ \\ \epsilon_2^o \end{Bmatrix} = \frac{1}{(2\bar{\bar{M}}\bar{\bar{B}} - \bar{\bar{B}}^2)} \begin{bmatrix} \bar{\bar{M}} & -(\bar{\bar{M}} - \bar{\bar{B}}) \\ \\ -(\bar{\bar{M}} - \bar{\bar{B}}) & \bar{\bar{M}} \end{bmatrix} \begin{Bmatrix} N_1 \\ \\ N_2 \end{Bmatrix} \tag{7–38a}$$

and

$$\epsilon_6^o = \frac{2}{\bar{\bar{B}}} N_6. \tag{7–38b}$$

By making use of (7–35) and (7–38a), the condition of compatibility (7–37) is, therefore, expressed in terms of w and Ψ as:

$$\frac{\bar{\bar{M}}}{(2\bar{\bar{M}}\bar{\bar{B}} - \bar{\bar{B}}^2)} \nabla^2\nabla^2\Psi = w_{,xx}^2 - w_{,xx}w_{,yy}. \tag{7–39}$$

It should be recalled that Eq. (7–39) remains the same for FSDPT and also the classical plate theory (CLPT).

In what follows, two simple problems will be considered in order to study the effects of nonlinearity, transverse shear strain, rotatory inertia, and damping on the response behavior of a simply–supported plate. The analysis will be shown for

TSDPT and relevant results will be summarized for FSDPT and the classical plate theory.

7.5 Static Large–Deflection of a Rectangular Plate

In static problems, Eq. (7–6) and (7–14) are reduced to

$$\bar{\bar{M}}(u_{,xx} + w_{,x}w_{,xx}) + \frac{\bar{\bar{B}}}{2}(u_{,yy} + w_{,x}w_{,yy}) + (\bar{\bar{M}} - \frac{\bar{\bar{B}}}{2})(v_{,xy} + w_{,y}w_{,xy}) = 0 \tag{7–40a}$$

$$\bar{\bar{M}}(v_{,yy} + w_{,y}w_{,yy}) + \frac{\bar{\bar{B}}}{2}(v_{,xx} + w_{,y}w_{,xx}) + (\bar{\bar{M}} - \frac{\bar{\bar{B}}}{2})(u_{,xy} + w_{,x}w_{,xy}) = 0 \tag{7–40b}$$

$$\frac{1}{A}(\bar{D}H - \bar{F}^2)\nabla^2\nabla^2\nabla^2 w - \bar{D}\nabla^2\nabla^2 w - \frac{D}{A}\nabla^2 P_z + P_z - \frac{1}{A}L_1(\mathcal{N}(u,v,w)) = 0 \tag{7–40c}$$

where

$$\mathcal{N}(u,v,w) = N_1 w_{,xx} + 2N_6 w_{,xy} + N_2 w_{,yy} \tag{7–41}$$

and L_1 is given by (7–19b). Expression for $\mathcal{N}$ can be recast in terms of u, v, and w by simply substituting (7–5) into (7–4) and the result into Eq. (7–41). Equations (7–40) will be used to study the nonlinear behavior of a simply–supported rectangular plate subjected to a transverse load $P_z(x,y)$. Here we assume the edges of the plate are immovable (see [131]):

$$u = N_6 = 0 \text{ at } x = 0,a \tag{7–42a}$$

and

$$v = N_6 = 0 \text{ at } y = 0,b \tag{7–42b}$$

The remaining boundary conditions are given by Eqs. (7–27a), (7–28b), (7–29b), (7–30a), and (7–30c). The boundary conditions (7–29b) and (7–30c) can be simplified with the help of Eqs. (7–41) and (7–42) to:

$$\left[\frac{\bar{\bar{D}}H - \bar{F}^2}{\bar{D}}\right] w_{,xxxx} - P_z - N_1 w_{,xx} = 0, \text{ at } x = 0,a \tag{7–43a}$$

and

$$\left[\frac{\bar{\bar{D}}H - \bar{F}^2}{\bar{D}}\right] w_{,yyyy} - P_z - N_2 w_{,yy} = 0, \text{ at } y = 0,b. \tag{7–43b}$$

The transverse load $P_z(x,y)$ is assumed to be of the form

$$P_z(x,y) = P_o \sin \frac{\pi}{a} x \sin \frac{\pi}{b} y. \tag{7–44}$$

The Galerkin method (see [131]) is used to obtain an approximate solution. It can easily be verified, by direct substitution, that by assuming the expansions

$$u(x,y) = \frac{w_o^2 \pi}{16a} \left[\left[\frac{a}{b}\right]^2 \left[\frac{\bar{M} - \bar{\bar{B}}}{\bar{\bar{M}}}\right] - 2\sin^2 \frac{\pi}{b} y\right] \sin \frac{2\pi}{a} x$$

$$v(x,y) = \frac{w_o^2 \pi}{16b} \left[\left[\frac{b}{a}\right]^2 \left[\frac{\bar{M} - \bar{\bar{B}}}{\bar{\bar{M}}}\right] - 2\sin^2 \frac{\pi}{a} x\right] \sin \frac{2\pi}{b} y \tag{7–45}$$

$$w(x,y) = w_o \sin \frac{\pi}{a} x \sin \frac{\pi}{b} y,$$

all the boundary conditions and Eqs.(7–40a) and (7–40b) are identically satisfied. In Eqs. (7–45), w_o denotes the transverse deflection at the center of the plate. The Galerkin method requires that Eq. (7–40c) be satisfied in the weighted–integral sense:

$$\int_0^b \int_0^a R(u,v,w) \sin \frac{\pi}{a} x \sin \frac{\pi}{b} y \, dxdy = 0, \tag{7–46}$$

where R(u,v,w) is the (residual) expression on the left side of Eq. (7–40c).

By substituting Eqs. (7–45) into Eq. (7–40c) and the result into Eq. (7–46) and evaluating all the integrals, the following relation is obtained:

$$\gamma_1 \left[\frac{w_o}{h}\right] + \gamma_2 \left[\frac{w_o}{h}\right]^3 = \frac{P_o a^4}{\bar{\bar{D}} h} \tag{7–47}$$

where the constant coefficients γ_1 and γ_2 are displaced in Appendix 7.2. An equation similar to Eq. (7–47) is also obtained for FSDPT and the classical plate theory by using the solution representation (7–45). The coefficients γ_1 and γ_2 for these two theories are also given in Appendix 7.2. If the plate is subjected to a uniformly–distributed load, then the right side of Eq. (7–47) will be replaced by $\frac{16}{\pi^2} P_o a^4/(\bar{\bar{D}}h)$ for a one–term approximation of the load. This way, with γ_1 and γ_2

corresponding to the classical plate theory (see Appendix 7.2), the result (7–47) will become identical to that found by Niyogi [132] for a homogeneous isotropic plate.

The rotation functions ψ_x and ψ_y can be determined by substituting the solution (7–45) into the static counterpart of Eqs. (7–23). It should be remarked that the terms involving Φ in Eqs. (7–23) should be ignored since, as it was argued earlier, the solution contribution of the edge–zone equation (Eq. (7–18)) is identically zero.

7.6 Transient Small–Deflection of a Rectangular Plate

In the linear theory, in the absence of rotatory inertia and their corresponding damping terms, the interior equation of TSDPT is obtained from Eq. (7–14) (or Eq. (7–19a)) by simply considering N_1, N_2, and N_6 as edge–loads in the nonlinear expression N. The resulting equation is

$$\frac{1}{A}(\bar{D}H - \bar{F}^2)\nabla^2\nabla^2\nabla^2 w - \bar{D}\nabla^2\nabla^2 w - \frac{D}{A}\nabla^2 q_z + q_z$$

$$= -\frac{D}{A} m_1 \nabla^2 \ddot{w} - \frac{D}{A} c_1 \nabla^2 \dot{w} + m_1 \ddot{w} + c_1 \dot{w} \tag{7–48}$$

with

$$q_z = P_z + N_x w_{,xx} + N_y w_{,yy}. \tag{7–49}$$

Solution of Eq. (7–48) will result in the transverse displacement w, and thereafter ψ_x and ψ_y can be found from

$$\psi_x = \frac{D}{A^2\bar{D}}(\bar{\bar{D}}H - \bar{F}^2)\nabla^2\nabla^2 w_{,x} - \frac{\bar{D}}{A}\nabla^2 w_{,x} - w_{,x}$$

$$+ \frac{D^2}{A^2\bar{D}}(m_1\ddot{w}_{,x} + c_1\dot{w}_{,x} - q_{z,x}) \tag{7–50a}$$

$$\psi_y = \frac{D}{A^2\bar{D}}(\bar{\bar{D}}H - \bar{F}^2)\nabla^2\nabla^2 w_{,y} - \frac{\bar{D}}{A}\nabla^2 w_{,y} - w_{,y}$$

$$+ \frac{D^2}{A^2\bar{D}}(m_1\ddot{w}_{,y} + c_1\dot{w}_{,y} - q_{z,y}), \tag{7–50b}$$

which are the linear counterparts of Eqs. (7–23). In Eqs. (7–48)–(7–50), the tensile edge–loads N_x and N_y are introduced to replace N_1 and N_2, respectively, with $N_6 = 0$. Once again, the terms involving Φ are not included in Eqs. (7–50) because they are identically zero in this problem.

It can readily be seen that the representations

$$w(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\alpha_m x \sin\beta_n y\ \eta_{mn}(t) \tag{7–51a}$$

and

$$P_z(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m_1 \sin\alpha_m x \sin\beta_n y\ Q_{mn}(t) \tag{7–51b}$$

satisfy all the boundary conditions of a simply–supported plate. In Eqs. (7–51), $\eta_{mn}(t)$ and $Q_{mn}(t)$ can be considered as the generalized displacements and forces, respectively.

Upon substituting (7–51) into (7–48), the following equation, which is similar to the equation of a simple damped oscillator, is found;

$$\ddot{\eta}_{mn}(t) + 2\xi_{mn}\omega_{mn}\dot{\eta}_{mn}(t) + \omega_{mn}^2\eta_{mn}(t) = Q_{mn}(t) \tag{7–52a}$$

with

$$2\xi_{mn}\omega_{mn} = \frac{c_1}{m_1}. \tag{7–52b}$$

In Eqs. (7–52), ξ_{mn} can be considered as damping ratios and ω_{mn} are the natural frequencies of the plate. The explicit expressions of ω_{mn}^2 obtained within TSDPT, FSDPT, and the classical plate theory are given in Appendix 7.2.

For light damping, the solution of Eq. (7–52a) can be shown (with the help of the Laplace transformation technique) to be

$$\begin{aligned}\eta_{mn}(t) = {} & e^{-\xi_{mn}\omega_{mn}t}\left[\cos \omega_{mn}t + \frac{\xi_{mn}}{(1 - \xi_{mn}^2)^{1/2}} \sin \omega_{dmn}t\right]\eta_{mn}(0) \\ & + \left[\frac{1}{\omega_{dmn}}e^{-\xi_{mn}\omega_{mn}t} \sin \omega_{dmn}t\right]\dot{\eta}_{mn}(0) \\ & + \frac{1}{\omega_{dmn}}\int_0^t Q_{mn}(\tau)e^{-\xi_{mn}\omega_{mn}(t-\tau)} \sin \omega_{dmn}(t-\tau)d\tau\end{aligned} \tag{7–53}$$

where $\omega_{dmn} = \omega_{mn}(1 - \xi_{mn}^2)^{1/2}$ can be regarded as the natural frequencies associated with the damped system. In Eq. (7–53), $\eta_{mn}(0)$ and $\dot{\eta}_{mn}(0)$ are the initial values of the generalized displacements and velocities, respectively. By assuming the following initial conditions:

$$w(x,y,0) \equiv \tilde{w}(x,y)$$
$$\dot{w}(x,y,0) \equiv \tilde{\tilde{w}}(x,y) \tag{7–54}$$

and using Eq. (7–51a) and the orthogonality properties of sine functions (e.g., see [131], page 39), $\eta_{mn}(0)$ and $\dot{\eta}_{mn}(0)$ are found as

$$\eta_{mn}(0) = \frac{4}{ab} \int_0^b \int_0^a \tilde{w}(x,y) \sin \alpha_m x \sin \beta_n y \, dxdy \tag{7–55a}$$

$$\dot{\eta}_{mn}(0) = \frac{4}{ab} \int_0^b \int_0^a \tilde{\tilde{w}}(x,y) \sin \alpha_m x \sin \beta_n y \, dxdy \tag{7–55b}$$

Similarly, from Eq. (7–51b), the generalized forces $Q_{mn}(t)$ are also seen to be

$$Q_{mn}(t) = \frac{4}{abm_1} \int_0^b \int_0^a P_z(x,y,t) \sin \alpha_m x \sin \beta_n y \, dxdy \tag{7–56}$$

When the initial conditions (7–54) are homogeneous, the solution (7–53) is simplified to

$$\eta_{mn}(t) = \frac{1}{\omega_{dmn}} \int_0^t Q_{mn}(\tau) e^{-\xi_{mn}\omega_{mn}(t-\tau)} \sin \omega_{dmn}(t-\tau) d\tau. \tag{7–57}$$

7.7 Numerical Results and Discussions

To study the effects of nonlinearity, shear deformation, rotatory inertia, and damping, the nondimensionalized center deflection of a single–layer plate (referred to as Structure I) and a three–layer symmetric plate (Structure II), with total thickness h, is evaluated with the help of Eqs. (7–47), (7–51a), and (7–57). The material properties of Structure I are taken to be

$$E = 19.2 \times 10^6 \text{ psi}, \ \nu = 0.24,$$

$$G_z = 0.82 \times 10^6 \text{ psi, and } \rho = 0.00013 \text{ lb sec}^2/\text{in}^4,$$

where E, ν, ..., and ρ are defined in Appendix 7.1. The material properties of outside layers (whose total thickness is equal to that of the middle layer) of Structure II are assumed to be

$$E = 20.83 \times 10^6 \text{ psi}, \ \nu = 0.44,$$

$$G_z = 3.71 \times 10^6 \text{ psi, and } \rho = 0.00013 \text{ lb sec}^2/\text{in}^4,$$

with the middle layer having the same properties as those of structure I.

For the transient problem, the loading $P_z(x,y,t)$ in Eq. (7–56) is assumed to be of sonic boom type described by an N–shaped normal pressure pulse (see Chapter VI) as follows:

$$P_z(x,y,t) \equiv P(t) = \begin{cases} P_o(1 - t/t_p) \text{ for } 0 < t < st_p \\ 0 \text{ for } t < 0 \text{ and } t > st_p \end{cases} \tag{7–58}$$

where P_o is the peak reflected pressure (when $s \leq 2$) uniformly distributed over the plate; t_p denotes the positive phase duration of the pulse; s denotes the pulse length parameter.

Substituting Eq. (7–58) into Eq. (7–56) results in

$$Q_{mn}(t) = \frac{16P_o}{mn\pi^2 m_1}\begin{cases}(1 - t/t_p) & \text{for } 0 < t < st_p \\ 0 & \text{for } t < 0 \text{ and } t > st_p\end{cases} \qquad (m,n = 1,3,\ldots)$$

(7–59)

With the help of Eq. (7–59), the generalized displacements $\eta_{mn}(t)$ are found from Eq. (7–57) by evaluating all the necessary integrals.

The variation of the dimensionless center deflection $\frac{w_o}{h}$ vs. $P_o a^4/(10\bar{D}h)$ of Structure II is shown in Fig. 7.1. It is seen that, for this particular structure, the results predicted by FSDPT with K^2 being equal to $\frac{\pi^2}{12}$, $\frac{14}{17}$, and $\frac{5}{6}$ are almost identical. The slight differences in the results cannot be seen in the plot. Moreover, FSDPT with $K^2 = \frac{2}{3}$ and TSDPT yield practically identical results. The effect of transverse shear strain and geometric nonlinearity on the amplitude of the center deflection parameter $\frac{w_o}{h}$ is also apparent from Fig. 7.1; the effect of transverse shear strain is to increase the center deflection, whereas the effect of nonlinearity is to decrease the deflection.

To study the effects of rotatory inertia, transverse shear strain, and damping in the linear theory, the dimensionless center deflection of Structures I and II subjected to various transverse loadings vs. time are plotted in Figs. 7.2 through 7.5 with the help of Eqs. (7–51a) and (7–57). In particular, Fig. 7.2 demonstrates the negligible effect of rotatory inertia terms on the deflection of Structure II when subjected to a step load and a triangular pulse. It should be noted that the analysis of full system of equations, which include all the inertia terms, were presented in

Chapter VI. The time–history of center deflection of Structures I and II subjected to a symmetric N–wave (s=2) is shown in Figs. 7.3 and 7.4. These figures clearly demonstrate that the results predicted by CLPT can be completely unreliable particularly in the free–vibration region (i.e., when $t > st_p$) where the amplitude of deflection is extremely underpredicted. Indeed, depending on the value of t_p, CLPT can yield a higher amplitude that TSDPT and FSDPT in this region (this is not shown here). It is also seen from Fig. 7.3 that the response of Structure I predicted by FSDPT with $K^2 = \pi^2/12$, 14/17, and 5/6 is almost identical with that predicted by TSDPT (the slight difference is not shown in Fig. 7.3). This is due to the fact that the fundamental frequency predicted by FSDPT (with such choices for K^2) is very close to that predicted by TSDPT as shown in Table 7.1. It is to be noted that although FSDPT with $K^2 = 14/17$ and TSDPT yield an identical pure–shear frequency for Structure I, the numerical value of the fundamental frequency obtained within FSDPT with $K^2 = 5/6$ is closer to that determined within TSDPT (see Table 7.1). Figure 7.4 and Table 7.1 indicate, on the other hand, that the response of Structure II is more accurately predicted within FSDPT when $K^2 = 2/3$ is used.

The effect of viscous damping $\frac{\Delta}{\pi}$ $(\equiv \frac{c_1}{m_1 \omega})$ on the response of Structure II subjected to a symmetric N–wave (s = 2) is studied with TSDPT and the results are included in Figs. 7.5 and 7.6. Here ω denotes the fundamental undamped frequency of Structure II, as predicted by CLPT. Figure 7.5 shows the role played by damping in reducing the deflection amplitudes. It is also seen, from Fig. 7.5, that the most significant amplitude attenuation due to the damping effect occurs during the free–motion range. This conclusion was also reached in the case of metallic type structures [133,134]. Figure 7.6 displays the variation of the dynamic magnification factor (DMF) vs. ωt_p where ω is the fundamental undamped frequency of Structure II (determined with CLPT) and DMF is defined as the ratio of the largest (in the absolute sense) dynamic deflection to static deflection. In determining the static deflection, it is assumed that the plate is subjected to a uniformly distributed load of magnitude P_o. The largest dynamic deflection is seen to occur during positive and negative phases of the N–wave and during the free–vibration period (i.e., after the negative phase of pressure pulse). This can be seen from the three distinct branches appearing in the DMF curves of Fig. 7.6. This phenomenon was also observed in [133] where the three branches were shown to exist by actually differentiating an equation similar to Eq. (7–57) with respect to time to obtain the times at which the maximum and minimum deflections occurred in a mass–spring system. It is further to be noted that when damping is included, the maximum deflection most often occurs during forced motion when the pressure pulse is positive (see Figs. 7.5 and 7.6).

7.8 Conclusions

It is demonstrated that the nonlinear dynamic equations of the first–order and third–order shear deformation plate theories can be recast into interior and edge–zone equations. It is further shown that, for a plate with simply–supported boundary types S1 through S4, the contribution of the edge–zone equation is merely due to the presence of rotatory inertia terms. In the absence of rotatory inertia terms, the number of governing equations of these theories is reduced to three, as in the classical plate theory. Moreover, in the absence of in–plane inertia terms, the three equations are reduced to two by introducing a force function. Therefore, the solution methodologies that are used in the classical plate theory are also applicable to the shear deformation plate theories with the aforementioned boundary conditions. This is demonstrated by an example where the static large–deflection problem of a rectangular plate was considered.

It is noted that, because of neglecting the nonlinear terms involving the rotation functions ψ_x and ψ_y, the edge–zone equations of the linear and nonlinear theories are the same. Consequently, the pure–shear frequencies predicted by both the linear and nonlinear theories are the same.

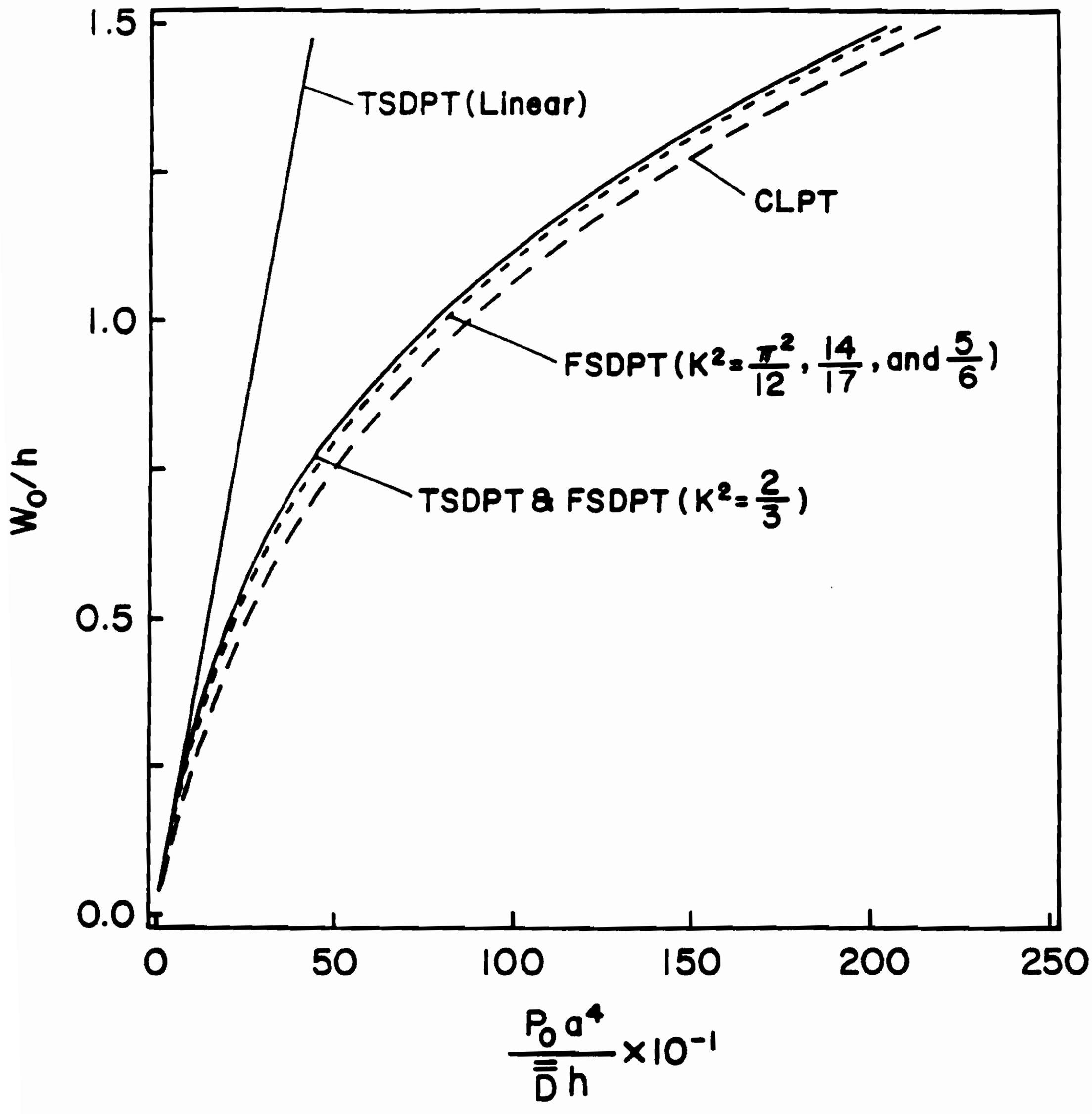

Figure 7.1 Variation of dimensionless center deflection of a square plate (Structure II) vs. $P_o a^4/(10\bar{\bar{D}}h)$; $\frac{a}{h} = 10$; h = 1 in.

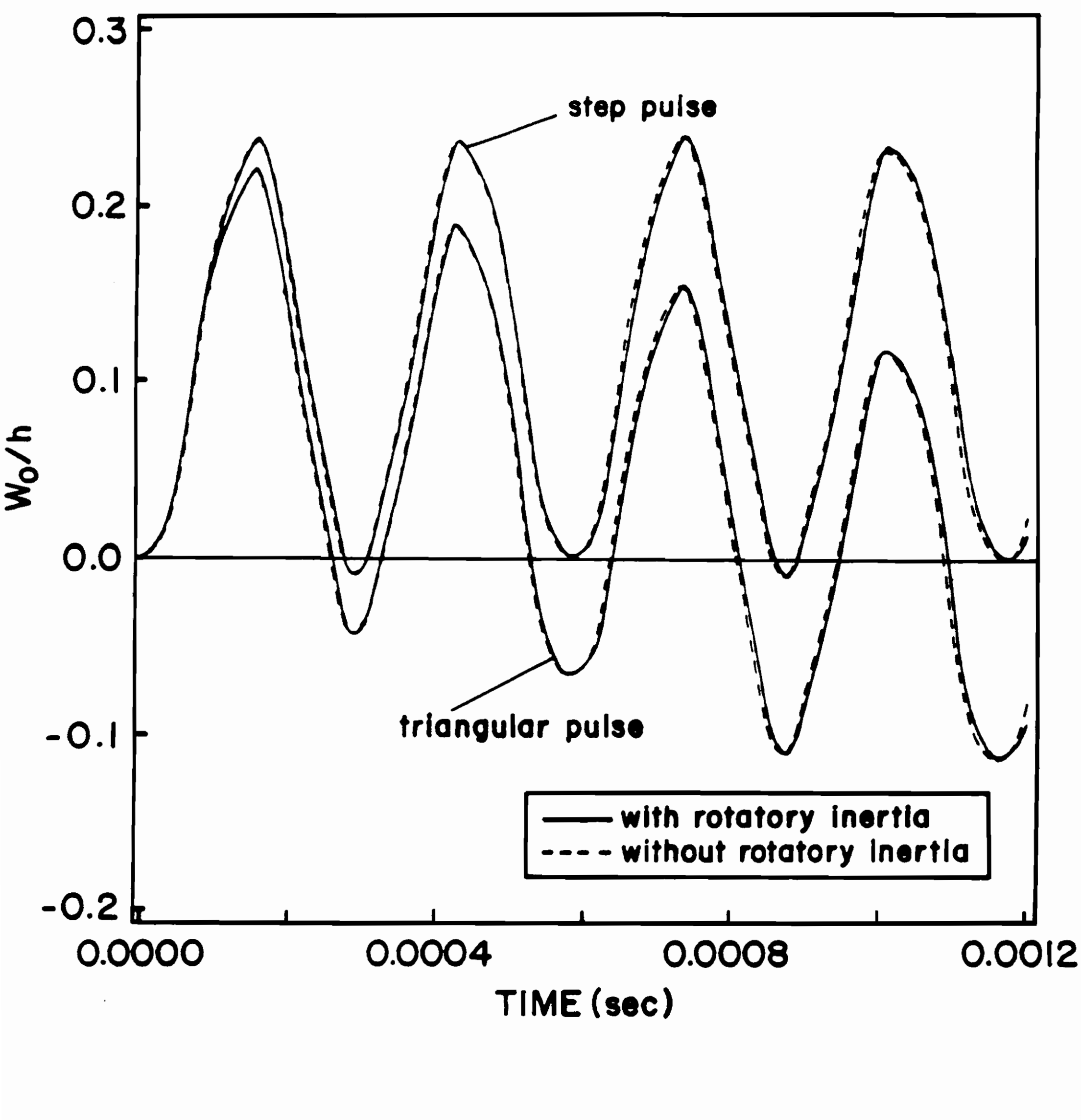

Figure 7.2 Time–history of dimensionless center deflection of a square plate (Structure II) subjected to step and triangular (t_p = 0.001 sec) pulses; P_o = 4500 psi; a/h = 10; h = 1 in.

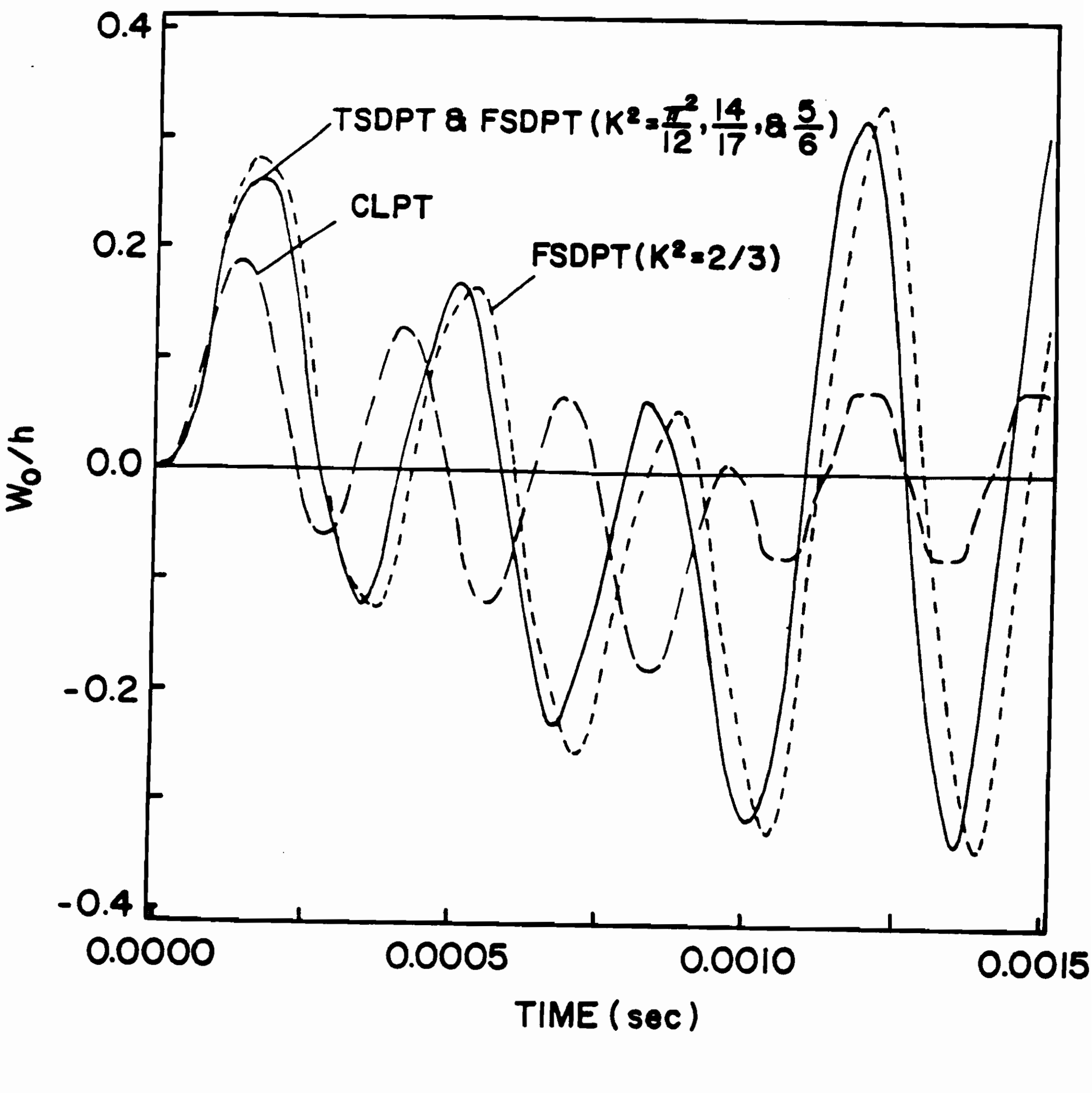

Figure 7.3 Time–history of dimensionless center deflection of a square plate (Structure I) subjected to a symmetric N–wave (s = 2); P_o = 4500 psi; t_p = 0.0005 sec; a/h = 10; h = 1 in.

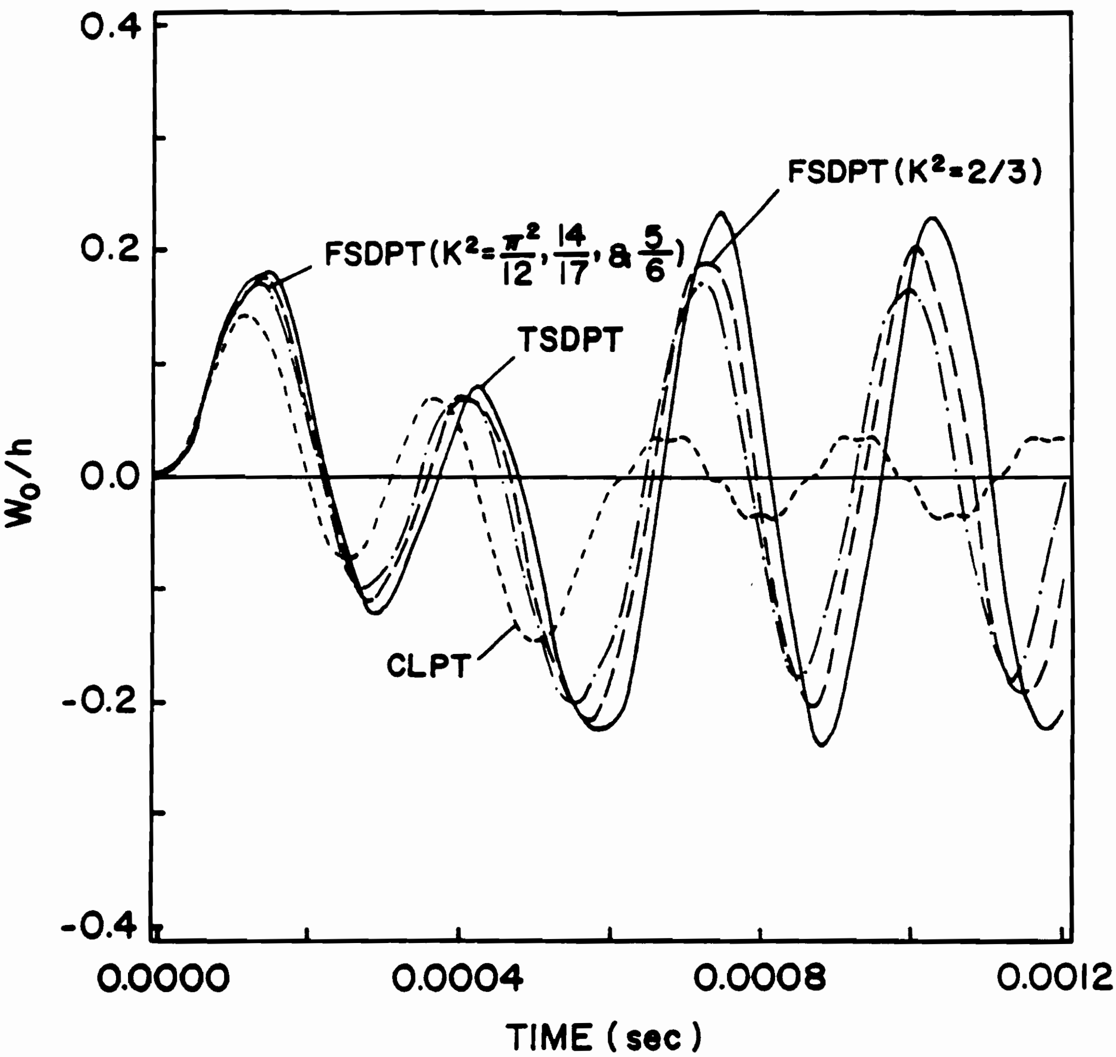

Figure 7.4 Time–history of dimensionless center deflection of a square plate (Structure II) subjected to a symmetric N–wave ($s = 2$); P_o = 4500 psi; t_p = 0.0003 sec; $a/h = 10$; $h = 1$ in.

Table 7.1

Nondimensional Fundamental Freqnency $\bar{\omega} = \omega a^2\sqrt{\rho h/\bar{\bar{D}}}$ of Square Plates Predicted by Different Theories (a = 10 in., h = 1 in.).

Structure	TSDPT	FSDPT with K^2 Equal to				CLPT
		$\pi^2/12$	14/17	5/6	2/3	
1	16.184	16.134	16.137	16.169	15.542	19.739
2	17.020	17.856	17.858	17.877	17.488	19.739

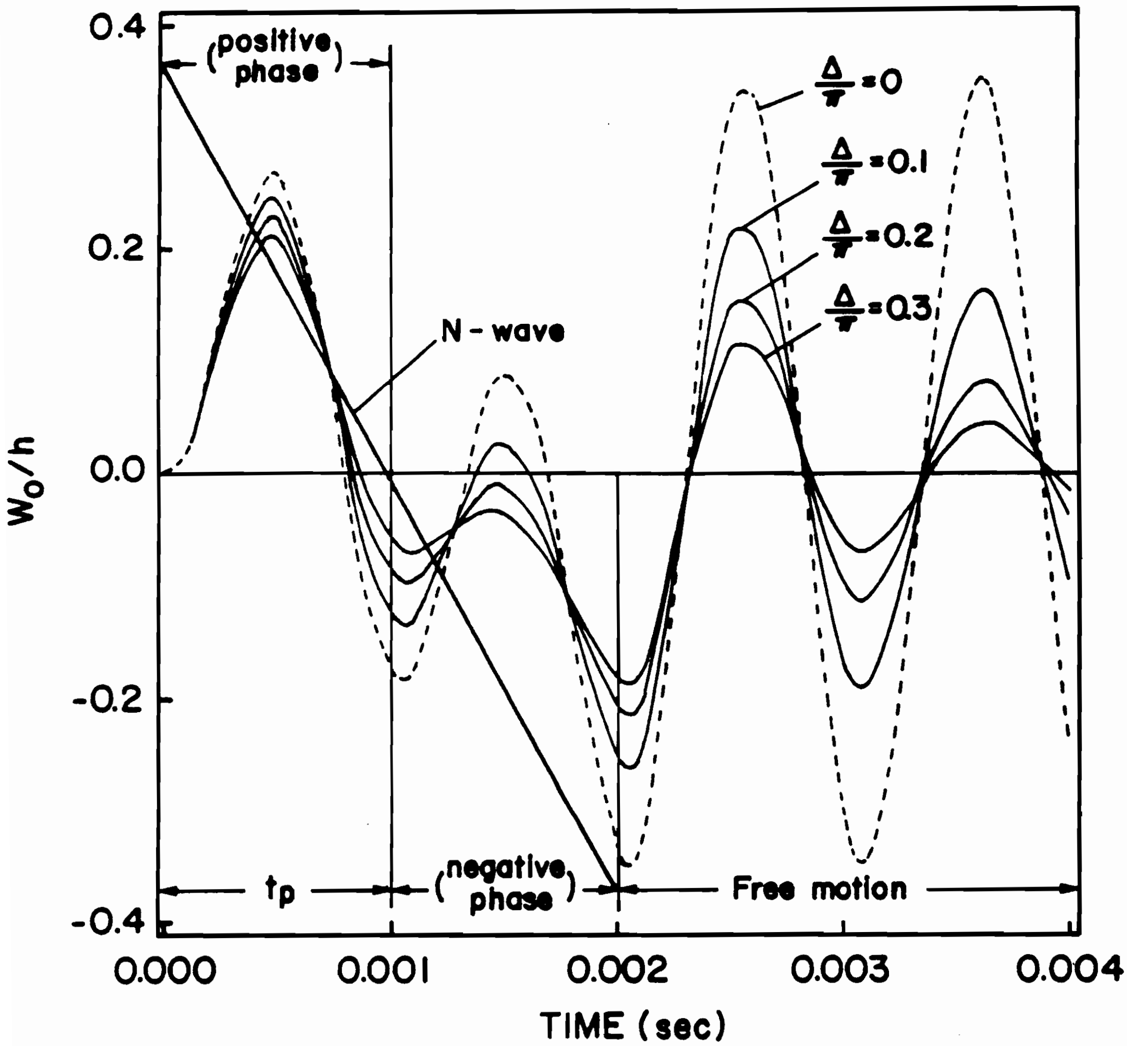

Figure 7.5 Time–history of dimensionless center deflection of a square plate (Structure II) subjected to a symmetric N–wave (s = 2), determined within TSDPT; $P_o = 500$ psi; $t_p = 0.001$ sec; a/h = 20; h = 1 in.

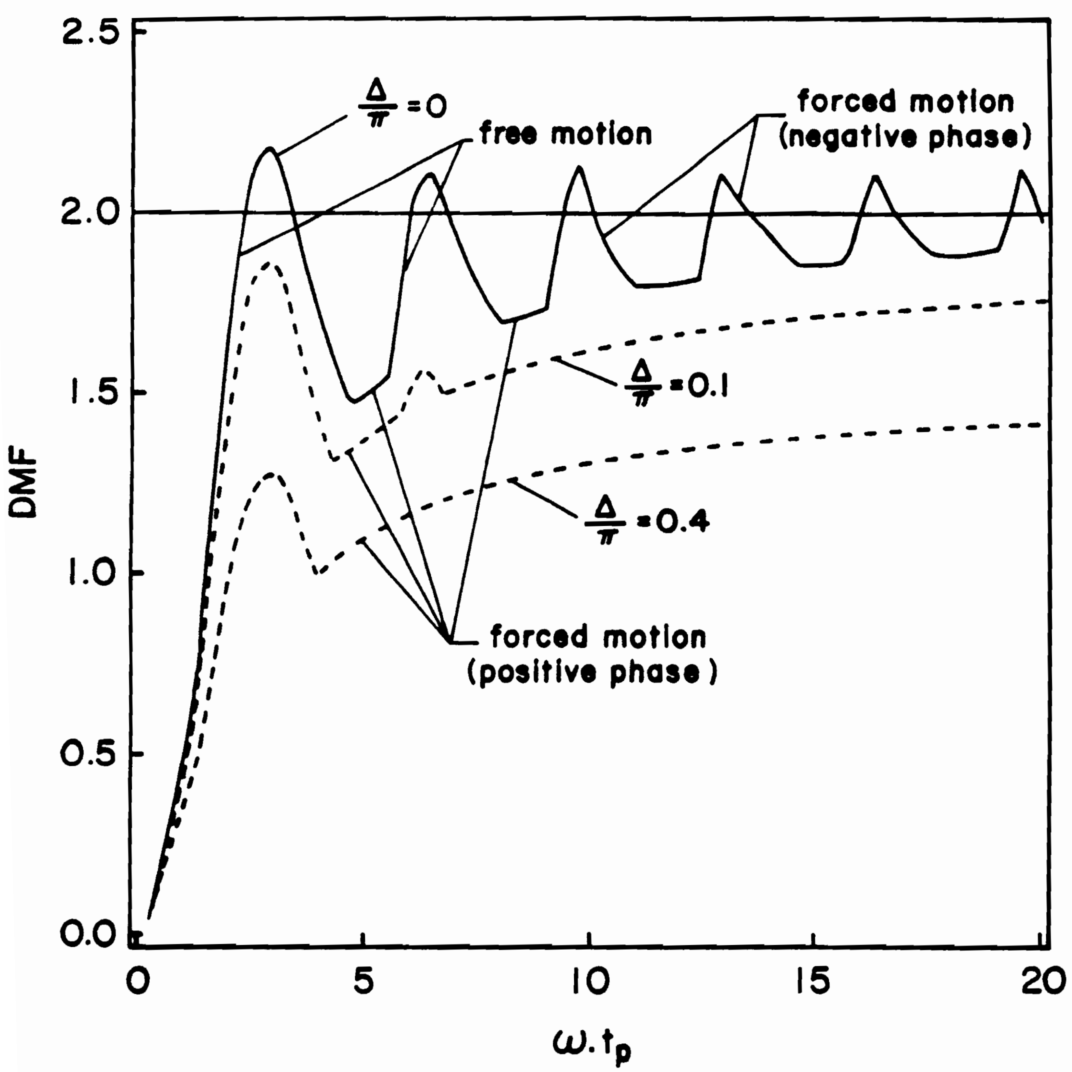

Figure 7.6 Variation of Dynamic Magnification Factor (DMF) of a square plate (Structure II) vs. ωt_p, determined within TSDPT. The plate is subjected to a symmetric N–wave ($s = 2$); $a/h = 20$; $h = 1$ in.

APPENDIX 7.1

Definitions of constant coefficients appearing in Eqs. (7–4), (7–7), (7–8), and (7–9):

$$\bar{\bar{M}} = A_{11} = A_{22} = \sum_{k=1}^{N} \left[\frac{E}{1 - \nu^2}\right]_k (z_k - z_{k+1})$$

$$\bar{\bar{B}} = 2A_{66} = \sum_{k=1}^{N} \left[\frac{E}{1 + \nu}\right]_k (z_k - z_{k+1})$$

$$A = A_{44} - \frac{8}{h^2} D_{44} + \left[\frac{4}{h^2}\right]^2 F_{44} = A_{55} - \frac{8}{h^2} D_{55} + \left[\frac{4}{h^2}\right]^4 F_{55}$$

$$= \sum_{k=1}^{N} (G_z)_k \left[(z_k - z_{k+1}) - \frac{8}{3h^2}(z_k^3 - z_{k+1}^3) + \frac{1}{5}\left[\frac{4}{h^2}\right]^2 (z_k^5 - z_{k+1}^5)\right]$$

$$\bar{A} = A_{44} - \frac{4}{h^2} D_{44} = A_{55} - \frac{4}{h^2} D_{55} = \sum_{k=1}^{N} (G_z)_k \Big[(z_k - z_{k+1})$$

$$- \frac{4}{3h^2}(z_k^3 - z_{k+1}^3)\Big]$$

$$\bar{\bar{A}} = A_{44} = A_{55} = \sum_{k=1}^{N} (G_z)_k (z_k - z_{k+1})$$

$$C = D_{66} - \frac{8}{3h^2} F_{66} + \left[\frac{4}{3h^2}\right]^2 H_{66} = \sum_{k=1}^{N} \frac{1}{2}\left[\frac{E}{1 + \nu}\right]_k \Big[\frac{1}{3}(z_k^3 - z_{k+1}^3)$$

$$- \frac{1}{5}\left[\frac{8}{3h^2}\right](z_k^5 - z_{k+1}^5) + \frac{1}{7}\left[\frac{4}{3h^2}\right]^2 (z_k^7 - z_{k+1}^7)\Big]$$

$$\bar{C} = D_{66} - \frac{4}{3h^2} F_{66} = \sum_{k=1}^{N} \frac{1}{2} \left[\frac{E}{1 + \nu}\right]_k \left[\frac{1}{3} (z_k^3 - z_{k+1}^3)\right.$$

$$\left. - \frac{1}{5}\left[\frac{4}{3h^2}\right](z_k^5 - z_{k+1}^5)\right]$$

$$\bar{\bar{C}} = D_{66} = \sum_{k=1}^{N} \frac{1}{6} \left[\frac{E}{1 + \nu}\right]_k (z_k^3 - z_{k+1}^3)$$

$$D = D_{11} - \frac{8}{3h^2} F_{11} + \left[\frac{4}{3h^2}\right]^2 H_{11} = D_{22} - \frac{8}{3h^2} F_{22} + \left[\frac{4}{3h^2}\right]^2 H_{22}$$

$$= \sum_{k=1}^{N} \left[\frac{E}{1 - \nu^2}\right]_k \left[\frac{1}{3}(z_k^3 - {}_{k+1}^3) - \frac{1}{5}\left[\frac{8}{3h^2}\right](z_k^5 - {}_{k+1}^5)\right.$$

$$\left. + \frac{1}{7} \left[\frac{4}{3h^2}\right]^2 (z_k^7 - z_{k+1}^7)\right]$$

$$\bar{D} = D_{11} - \frac{4}{3h^2} F_{11} = D_{22} - \frac{4}{3h^2} F_{22} = \sum_{k=1}^{N} \left[\frac{E}{1 - \nu^2}\right]_k \left[\frac{1}{3} (z_k^3 - z_{k+1}^3)\right.$$

$$\left. - \frac{1}{5}\left[\frac{4}{3h^2}\right](z_k^5 - z_{k+1}^5)\right]$$

$$\bar{\bar{D}} = D_{11} = D_{22} = \sum_{k=1}^{N} \frac{1}{3} \left[\frac{E}{1 - \nu^2}\right]_k (z_k^3 - z_{k+1}^3)$$

$$F = \frac{4}{3h^2} F_{11} - \left[\frac{4}{3h^2}\right]^2 H_{11} = \frac{4}{3h^2} F_{22} - \left[\frac{4}{3h^2}\right]^2 H_{22}$$

$$= \sum_{k=1}^{N} \left[\frac{E}{1 - \nu^2}\right]_k \left[\frac{1}{5}\left[\frac{4}{3h^2}\right](z_k^5 - z_{k+1}^5) - \frac{1}{7}\left[\frac{4}{3h^2}\right]^2 (z_k^7 - z_{k+1}^7)\right]$$

$$\bar{F} = \frac{4}{3h^2} F_{11} = \frac{4}{3h^2} F_{22} = \sum_{k=1}^{N} \frac{1}{5} \left[\frac{E}{1 - \nu^2}\right]_k \left[\frac{4}{3h^2}\right](z_k^5 - z_{k+1}^5)$$

$$H = \left[\frac{4}{3h^2}\right]^2 H_{11} = \left[\frac{4}{3h^2}\right]^2 H_{22} = \sum_{k=1}^{N} \frac{1}{7} \left[\frac{4}{3h^2}\right]^2 \left[\frac{E}{1 - \nu^2}\right]_k (z_k^7 - z_{k+1}^7)$$

$\bar{L} = \bar{\bar{C}} - \bar{C}$ and $L = \bar{C} - C$

$$m_1 = I_1 = \sum_{k=1}^{N} (\rho)_k (z_k - z_{k+1})$$

$$m_3 = I_3 - \frac{8}{3h^2} I_5 + \left[\frac{4}{3h^2}\right]^2 I_7$$

$$= \sum_{k=1}^{N} (\rho)_k \left[\frac{1}{3}(z_k^3 - z_{k+1}^3) - \frac{1}{5}\left[\frac{8}{3h^2}\right](z_k^5 - z_{k+1}^5)\right.$$

$$\left. + \frac{1}{7}\left[\frac{4}{3h^2}\right]^2 (z_k^7 - z_{k+1}^7)\right]$$

$$\bar{m}_3 = I_3 - \frac{4}{3h^2} I_5 = \sum_{k=1}^{N} (\rho)_k \left[\frac{1}{3}(z_k^3 - z_{k+1}^3) - \frac{1}{5}\left[\frac{4}{3h^2}\right](z_k^5 - z_{k+1}^5)\right]$$

$$\bar{\bar{m}}_3 = I_3 = \sum_{k=1}^{N} \frac{1}{3} (\rho)_k (z_k^3 - z_{k+1}^3)$$

$$m_5 = \frac{4}{3h^2} I_5 - \left[\frac{4}{3h^2}\right]^2 I_7 = \frac{4}{3h^2} \sum_{k=1}^{N} (\rho)_k \left[\frac{1}{5}(z_k^5 - z_{k+1}^5)\right.$$

$$\left. - \frac{1}{7}\left[\frac{4}{3h^2}\right](z_k^7 - z_{k+1}^7)\right]$$

$$\bar{m}_5 = \frac{4}{3h^2} I_5 = \frac{4}{3h^2} \sum_{k=1}^{N} \frac{1}{5} (\rho)_k (z_k^5 - z_{k+1}^5)$$

$$m_7 = \left[\frac{4}{3h^2}\right]^2 I_7 = \left[\frac{4}{3h^2}\right]^2 \sum_{k=1}^{N} \frac{1}{7} (\rho)_k (z_k^7 - z_{k+1}^7)$$

In the above expressions h denotes the total thickness of the plate; N is the total number of layers; ρ is the density; E and ν are the Young's modulus and Poisson's ratio in the plane of isotropy (x–y plane) and G_z is the shear modulus in the plane normal to the plane of isotropy. The rigidity terms A_{ij}, D_{ij}, F_{ij}, and H_{ij} and the mass terms I_1, I_3, I_5, and I_7 are also defined in Chapter II.

APPENDIX 7.2

Definitions of γ_1 and γ_2 appearing in Eq. (7–47) for different theories:

TSDPT:

$$\gamma_1 = \frac{a^4}{\alpha}\left\{\frac{1}{A}\left[\frac{\bar{D}H - \bar{F}^2}{\bar{\bar{D}}}\right]\left[\left[\frac{\pi}{a}\right]^2 + \left[\frac{\pi}{b}\right]^2\right]^3 + \left[\left[\frac{\pi}{a}\right]^2 + \left[\frac{\pi}{b}\right]^2\right]^2\right\}$$

where

$$\alpha = \frac{D}{A}\left[\left[\frac{\pi}{a}\right]^2 + \left[\frac{\pi}{b}\right]^2\right] + 1.$$

FSDPT:

$$\gamma_1 = \frac{a^4}{\alpha}\left[\left[\frac{\pi}{a}\right]^2 + \left[\frac{\pi}{b}\right]^2\right]^2$$

where

$$\alpha = \frac{\bar{\bar{D}}}{k^2\bar{\bar{A}}}\left[\left[\frac{\pi}{a}\right]^2 + \left[\frac{\pi}{b}\right]^2\right] + 1.$$

Classical Plate Theory:

$$\gamma_1 = a^4\left[\left[\frac{\pi}{a}\right]^2 + \left[\frac{\pi}{b}\right]^2\right]^2$$

In addition, for all of the above theories, γ_2 is given by

$$\gamma_2 = \frac{a^4h^2}{16\bar{\bar{D}}}\left\{4(\bar{\bar{M}} - \bar{\bar{D}})\left[\frac{\pi}{a}\right]^2\left[\frac{\pi}{b}\right]^2 + \left[2(\bar{\bar{M}} + \bar{\bar{B}}) - \frac{\bar{\bar{B}}^2}{\bar{\bar{M}}}\right]\left[\left[\frac{\pi}{a}\right]^4 + \left[\frac{\pi}{b}\right]^4\right]\right\}.$$

Definitions of ω_{mn}^2 appearing in Eqs. (7–52a) for different theories:

<u>TSDPT</u>:

$$\omega_{mn}^2 = \frac{1}{\beta}\left[\frac{1}{A}(\bar{\bar{D}}H - \bar{\bar{F}}^2)(\alpha_m^2 + \beta_n^2)^3 + \bar{\bar{D}}(\alpha_m^2 + \beta_n^2)^2\right.$$

$$\left. + \frac{D}{A}(N_x\alpha_m^2 + N_y\beta_n^2)(\alpha_m^2 + \beta_n^2 + \frac{A}{D})\right]$$

where

$$\beta = m_1\left[\frac{D}{A}(\alpha_m^2 + \beta_n^2) + 1\right].$$

<u>FSDPT</u>:

$$\omega_{mn}^2 = \frac{1}{\beta}\left[\bar{\bar{D}}(\alpha_m^2 + \beta_n^2)^2 + \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}(N_x\alpha_m^2 + N_y\beta_n^2)\left[\alpha_m^2 + \beta_n^2 + \frac{K^2\bar{\bar{A}}}{\bar{\bar{D}}}\right]\right]$$

where

$$\beta = m_1\left[\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}(\alpha_m^2 + \beta_n^2) + 1\right].$$

Classical Plate Theory:

$$\omega_{mn}^2 = \frac{1}{m_1}\left[\bar{\bar{D}}(\alpha_m^2 + \beta_n^2)^2 + (N_x\alpha_m^2 + N_y\beta_n^2)\right].$$

CHAPTER VIII

FREQUENCY AND BUCKLING EQUATIONS OF A SYMMETRIC PLATE ACCORDING TO LEVINSON'S THIRD–ORDER THEORY

The third–order shear deformation plate theory of Levinson is developed in [42] for homogeneous isotropic plates. As we mentioned in Chapter I, the Navier solution and Lévy–type solutions for homogeneous isotropic plates in bending according to this theory are subsequently devleoped in [86] and [87], respectively. Here we extend this theory to symmetric laminated plates and then obtain the governing equations of this theory for symmetric plates laminated of transversely isotropic layers. Following the procedure developed in Chapter VII, the interior and edge–zone equations of this theory will be obtained. The remainder of this chapter will be devoted to developing analytic frequency and buckling equations for a plate with Lévy–type boundary conditions using the original form of the governing equations. These analyses will be extended in the next chapter within the framework of the first–order shear deformation plate theory using exclusively the interior and edge–zone equations of a plate.

8.1 Governing Equations of a Symmetric Plate

The equations of motion of a symmetric plate according to the third–order shear deformation plate theory (TSDPT) of Levinson [42] can be presented as:

$$M_{1,x} + M_{6,y} - Q_1 = \bar{m}_3\ddot{\psi}_x - \bar{m}_5\ddot{w}_{,x}$$

$$M_{6,x} + M_{2,y} - Q_2 = \bar{m}_3\ddot{\psi}_y - \bar{m}_5\ddot{w}_{,y}$$

$$Q_{1,x} + Q_{2,y} + N_x w_{,xx} + N_y w_{,yy} + P_z = m_1 \ddot{w} \tag{8–1}$$

where N_x and N_y are the tensile edge loads in the x– and y–directions, respectively, and

$$m_1 = I_1 \ , \ \bar{m}_3 = I_3 - \frac{4}{3h^2} I_5 \ , \ \bar{m}_5 = \frac{4}{3h^2} I_5 \tag{8–2}$$

with I_1, I_3, and I_5 given in Eq. (2–10). In Levinson's TSDPT the displacement field is assumed to be

$$u_1(x,y,z,t) = f(z)\psi_x - \frac{4}{3h^2} z^3 w_{,x}$$

$$u_2(x,y,z,t) = f(z)\psi_y - \frac{4}{3h^2} z^3 w_{,y}$$

$$u_3(x,y,z,t) = w \tag{8–3a}$$

with

$$f(z) = z - \frac{4}{3h^2} z^3 \tag{8–3b}$$

and the rotation functions ψ_x and ψ_y and the transverse displacement w being functions of x, y, and time t. It is to be noted that the displacement field in Eqs. (8–3) satisfies the transverse shear stress–free boundary conditions on the boundary planes of a plate. This displacement field is identical to the one used in Reddy's TSDPT (see Chapter II and [38]) and in Librescu's TSDPT [46]. However, the equations of motion in Levinson's and Librescu's theories are obtained by the method of moments (also known as the vectorial approach) which involves the integration of 3–D elasticity equations whereas Reddy's theory is based on the virtual work principle. For this reason, the total order of the equations in Levinson's and Librescu's theories is six whereas in Reddy's TSDPT (see Chapters II and VII) the total order of the equations is eight. It should be pointed out that, in principle, the vectorial approach and variational approach can yield identical equations if the former approach is carefully applied. In general, however, the vectorial approach does not yield the exact form of the boundary conditions. In passing, it should be added that, in contrast to Levinson's TSDPT, the effect of transverse normal stress is also incorporated in Librescu's TSDPT. That is, when the transverse normal stress effect is ignored, the two theories become identical. Based on Eqs. (8–3), the strain–displacement relations are

$$\epsilon_1 = f(z)\psi_{x,x} - \frac{4}{3h^2} z^3 w_{,xx}$$

$$\epsilon_2 = f(z)\psi_{y,y} - \frac{4}{3h^2} z^3 w_{,yy}$$

$$\epsilon_4 = f'(z)(\psi_y + w_{,y})$$

$$\epsilon_5 = f'(z)(\psi_x + w_{,x})$$

$$\epsilon_6 = f'(z)(\psi_{x,y} + \psi_{y,x}) - \frac{8}{3h^2} z^3 w_{,xy} \tag{8–4a}$$

where

$$f'(z) = 1 - \frac{4}{h^2} z^2. \tag{8–4b}$$

In Eqs. (8–1), the stress resultants are defined as

$$M_i = \int_{-h/2}^{h/2} \sigma_1 \cdot z dz, \quad (i = 1,2,6)$$

$$(Q_1,Q_2) = \int_{-h/2}^{h/2} (\sigma_5,\sigma_4) dz, \tag{8–5}$$

where the stress components σ_i in a symmetric plate with orthotropic layers are related to strain components according to

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}^k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & 0 \\ 0 & 0 & \bar{Q}_{66} \end{bmatrix}^k \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{Bmatrix}^k$$

$$\begin{Bmatrix} \sigma_4 \\ \sigma_5 \end{Bmatrix}^k = \begin{bmatrix} \bar{Q}_{44} & 0 \\ 0 & \bar{Q}_{55} \end{bmatrix}^k \begin{Bmatrix} \epsilon_4 \\ \epsilon_5 \end{Bmatrix}^k \tag{8–6}$$

By substituting Eqs. (8–4) into (8–6) and the subsequent results into Eqs. (8–5), we obtain

$$M_1 = (D_{11} - \frac{4}{3h^2} F_{11})\psi_{x,x} + (D_{12} - \frac{4}{3h^2} F_{12})\psi_{y,y}$$

$$- \frac{4}{3h^2} F_{11} w_{,xx} - \frac{4}{3h^2} F_{12} w_{,yy}$$

$$M_2 = (D_{12} - \frac{4}{3h^2} F_{12})\psi_{x,x} + (D_{22} - \frac{4}{3h^2} F_{22})\psi_{y,y}$$

$$- \frac{4}{3h^2} F_{12} w_{,xx} - \frac{4}{3h^2} F_{22} w_{,yy}$$

$$M_6 = (D_{66} - \frac{4}{3h^2} F_{66})(\psi_{x,y} + \psi_{y,x}) - \frac{8}{3h^2} F_{66} w_{,xy}$$

$$Q_1 = (A_{55} - \frac{4}{h^2} D_{55})(\psi_x + w_{,x})$$

$$Q_2 = (A_{44} - \frac{4}{h^2} D_{44})(\psi_y + w_{,y}) \tag{8–7}$$

For a symmetric plate with transversely isotropic layers (the plane of isotropy is assumed to be parallel to the x–y plane) it is more convenient to introduce (see Chapter VII)

$$\bar{A} = A_{44} - \frac{4}{h^2} D_{44} = A_{55} - \frac{4}{h^2} D_{55}$$

$$\bar{C} = D_{66} - \frac{4}{3h^2} F_{66}$$

$$\bar{D} = D_{11} - \frac{4}{3h^2} F_{11} = D_{22} - \frac{4}{3h^2} F_{22}$$

$$\bar{F} = \frac{4}{3h^2} F_{11} = \frac{4}{3h^2} F_{22}. \tag{8–8}$$

Substituting Eqs. (8–8) into Eqs. (8–7), we obtain

$$M_1 = \bar{D}\psi_{x,x} + (\bar{D} - 2\bar{C})\psi_{y,y} - \bar{F}w_{,xx} - (\bar{F} - 2\bar{L})w_{,yy}$$

$$M_2 = (\bar{D} - 2\bar{C})\psi_{x,x} + \bar{D}\psi_{y,y} - (\bar{F} - 2\bar{L})w_{,xx} - \bar{F}w_{,yy}$$

$$M_6 = \bar{C}(\psi_{x,y} + \psi_{y,x}) - 2\bar{L}w_{,xy}$$

$$Q_1 = \bar{A}(\psi_x + w_{,x}) \ , \ Q_2 = \bar{A}(\psi_y + w_{,y}) \tag{8–9}$$

where $\bar{L} \equiv \bar{\bar{C}} - \bar{C}$ is introduced for convenience (see Appendix 7.1 in Chapter VII). Substitution of Eqs. (8–7) and Eqs. (8–9) into the equations of motion (Eqs. (8–1)) yields the governing equations of orthotropic and transversely isotropic plates, respectively, in terms of ψ_x, ψ_y, and w. However, here we only present the governing equations of a symmetric multilayered transversely isotropic plate:

$$\bar{D}\psi_{x,xx} + \bar{C}\psi_{x,yy} - \bar{A}(\psi_x + w_{,x}) + (\bar{D} - \bar{C})\psi_{y,xy} - \bar{F}\nabla^2 w_{,x} = \bar{m}_3\ddot{\psi}_x - \bar{m}_5\ddot{w}_{,x} \tag{8-10a}$$

$$\bar{D}\psi_{y,yy} + \bar{C}\psi_{y,xx} - \bar{A}(\psi_y + w_{,y}) + (\bar{D} - \bar{C})\psi_{x,xy} - \bar{F}\nabla^2 w_{,y} = \bar{m}_3\ddot{\psi}_y - \bar{m}_5\ddot{w}_{,y} \tag{8-10b}$$

$$\bar{A}(\psi_{x,x} + \psi_{y,y}) + \bar{A}\nabla^2 w + N_x w_{,xx} + N_y w_{,yy} + P_z = m_1\ddot{w} \tag{8-10c}$$

where ∇^2 is the Laplace operator.

8.2 Interior and Edge–Zone Equations

Following the procedure developed in Chapter VII, by introducing

$$\Phi = \psi_{x,y} - \psi_{y,x} \tag{8-11}$$

the interior and the edge–zone equations of Levinson's TSDPT can readily be obtained from Eqs. (8–10) and presented as:

$$-\bar{\bar{D}}\nabla^2\nabla^2 w - \frac{\bar{D}}{\bar{A}}\nabla^2 q_z + q_z + \frac{\bar{m}_3}{\bar{A}}\ddot{q}_z = \frac{m_1\bar{m}_3}{\bar{A}}\ddddot{w} - (\bar{m}_3 + \frac{\bar{D}}{\bar{A}}m_1)\nabla^2\ddot{w} + m_1\ddot{w} \tag{8-12a}$$

$$\bar{C}\nabla^2\Phi - \bar{A}\Phi = \bar{m}_3\ddot{\Phi} \tag{8-12b}$$

where (see Chapter VII)

$$\bar{\bar{D}} = \bar{D} + \bar{F} \tag{8-12c}$$

$$q_z = P_z + N_x w_{,xx} + N_y w_{,yy} \tag{8–12d}$$

Equation (8–12a) was obtained for the first time by Levinson [42] for a homogeneous isotropic plate in the absence of tensile edge loads N_x and N_y. Equations (8–10) and (8–12) constitute two different (and equivalent) formulations of Levinson's TSDPT. However, when Eqs. (8–12) are being used, it is necessary to express the stress resultants in Eqs. (8–9) in terms of w and Φ. To this end, we introduce Eq. (8–11) into Eqs. (8–10a) and (8–10b) and get

$$\psi_x + \frac{\bar{m}_3}{\bar{A}}\ddot{\psi}_x = -w_{,x} - \frac{\bar{F}}{\bar{A}}\nabla^2 w_{,x} + \frac{\bar{D}}{\bar{A}}(\psi_{x,x} + \psi_{y,y})_{,x} + \frac{\bar{C}}{\bar{A}}\Phi_{,y} \tag{8–13a}$$

$$\psi_y + \frac{\bar{m}_3}{\bar{A}}\ddot{\psi}_y = -w_{,y} - \frac{\bar{F}}{\bar{A}}\nabla^2 w_{,y} + \frac{\bar{D}}{\bar{A}}(\psi_{x,x} + \psi_{y,y})_{,y} - \frac{\bar{C}}{\bar{A}}\Phi_{,x} \tag{8–13b}$$

Also Eq. (8–10c) can be rewritten as

$$(\psi_{x,x} + \psi_{y,y}) = -\nabla^2 w - \frac{q_z}{\bar{A}} + \frac{m_1 \ddot{w}}{\bar{A}} \tag{8–14}$$

Substitution of Eq. (8–14) into Eqs. (8–13) results in

$$\psi_x + \frac{\bar{m}_3}{\bar{A}}\ddot{\psi}_x = -w_{,x} - \frac{\bar{\bar{D}}}{\bar{A}}\nabla^2 w_{,x} + \frac{\bar{D}}{\bar{A}}\frac{m_1}{\bar{A}}\ddot{w}_{,x} + \frac{\bar{C}}{\bar{A}}\Phi_{,y} - \frac{\bar{D}}{\bar{A}^2}q_{z,x} \quad (8\text{–}15a)$$

$$\psi_y + \frac{\bar{m}_3}{\bar{A}}\ddot{\psi}_y = -w_{,y} - \frac{\bar{\bar{D}}}{\bar{A}}\nabla^2 w_{,y} + \frac{\bar{D}}{\bar{A}}\frac{m_1}{\bar{A}}\ddot{w}_{,y} - \frac{\bar{C}}{\bar{A}}\Phi_{,x} - \frac{\bar{D}}{\bar{A}^2}q_{z,y} \quad (8\text{–}15b)$$

Equations (8–15) can be rewritten as

$$\psi_x = \frac{1}{L_1}\left[-w_{,x} - \frac{\bar{\bar{D}}}{\bar{A}}\nabla^2 w_{,x} + \frac{\bar{D}}{\bar{A}}\frac{m_1}{\bar{A}}\ddot{w}_{,x} + \frac{\bar{C}}{\bar{A}}\Phi_{,y} - \frac{\bar{D}}{\bar{A}^2}q_{z,x}\right] \quad (8\text{–}16a)$$

$$\psi_y = \frac{1}{L_1}\left[-w_{,y} - \frac{\bar{\bar{D}}}{\bar{A}}\nabla^2 w_{,y} + \frac{\bar{D}}{\bar{A}}\frac{m_1}{\bar{A}}\ddot{w}_{,y} - \frac{\bar{C}}{\bar{A}}\Phi_{,x} - \frac{\bar{D}}{\bar{A}^2}q_{z,y}\right] \quad (8\text{–}16b)$$

where

$$L_1 = (1 + \frac{\bar{m}_3}{\bar{A}}\frac{\partial^2}{\partial t^2}). \quad (8\text{–}16c)$$

Clearly by substituting Eqs. (8–16) into Eqs. (8–9), the stress–resultants will be obtained explicitly in terms of w and Φ. This will be done in the next chapter where the interior and the edge–zone equations of the first–order shear deformation plate theory (FSDPT) will exclusively be used for developing Lévy–type frequency and buckling equations. In this chapter, however, we will use the original form of the governing equations (i.e., Eqs. (8–10)) to study the plate buckling and vibration problems.

8.3 Boundary Conditions

Throughout our study in this chapter we will use the capital letters S, C, and F to indicate that an edge of the plate is simply–supported, clamped, and free, respectively. Further, we will assume that two opposite edges of the plate at $x = 0$ and a are invariably simply supported (i.e., $\psi_y = w = M_1 = 0$). At the other two edges of the plate (i.e., at $y = 0$ and b) we can have a combination of the following cases:

S (simply–supported edge):

$$\psi_x = w = 0$$

$$M_2 = 0 \rightarrow \bar{D}\psi_{y,y} - \bar{F}w_{,yy} = 0 \tag{8–17}$$

C (clamped edge):

$$\psi_x = w = 0 \tag{8–18a}$$

$$4\psi_y - w_{,y} = 0 \tag{8–18b}$$

F (free edge):

$$M_2 = M_6 = Q_2 + N_y w_{,y} = 0 \tag{8–19}$$

These boundary conditions are known as the relaxed boundary conditions as opposed to the exact boundary conditions in 3–D elasticity theory (see [1]). For a clamped edge, instead of $4\psi_y - w_{,y} = 0$ (see Eq (8–18b)) we can alternatively impose $\psi_y = 0$ which was originally suggested by Levinson in [42]. The matter will be further examined later in Section 8.7 and in the next chapter. In the next two sections we will develop the frequency and buckling equations of a plate having various boundary conditions at y = 0,b. Later in this chapter, for the purpose of verifying our results, we will develop an alternative solution to these problems.

8.4.1 Frequency Equations

In Chapter VII we have shown that when the plate is completely simply supported, the boundary conditions accompanying the edge–zone equation are

$$\Phi_{,x} = 0 \text{ at } x = 0 \text{ and } a,$$
$$\Phi_{,y} = 0 \text{ at } y = 0 \text{ and } b, \tag{8–20}$$

in FSDPT and Reddy's TSDPT. Further, we have shown that the solution of the edge–zone equation with the boundary conditions in Eqs. (8–20) resulted in an infinite number of pure–shear frequencies. That is, no transversal displacement occurs when the plate is vibrating with these frequencies. By following the procedure of Chapter VII, it can readily be verified that the same boundary conditions as in Eqs. (8–20) and the same results can be obtained in Levinson's TSDPT. Here, however, we choose an alternative approach to obtain the same results.

When the remaining edges of the plate (i.e., at y = 0,b) are also simply supported, it can be verified that

$$\psi_x(x,y,t) = \sum_m^\infty \sum_n^\infty A_{mn} \cdot \cos \alpha_m x \cdot \sin \beta_n y \cdot T_{mn}(t) \tag{8–21a}$$

$$\psi_y(x,y,t) = \sum_m^\infty \sum_n^\infty B_{mn} \cdot \sin \alpha_m x \cdot \cos \beta_n y \cdot T_{mn}(t) \tag{8–21b}$$

$$w(x,y,t) = \sum_{m}^{\infty} \sum_{n}^{\infty} C_{mn} \cdot \sin \alpha_m x \cdot \sin \beta_n y \cdot T_{mn}(t) \tag{8–21c}$$

with $T_{mn} = e^{i\omega_{mn}t}$, $i = \sqrt{-1}$, $\alpha_m = \frac{m\pi}{a}$, and $\beta_n = \frac{n\pi}{b}$ satisfy all boundary conditions in Eqs. (8–17) and their corresponding counterparts at x = 0 and a. With nonzero values of m and n and $P_z = 0$, substitution of Eqs. (8–21) into Eqs. (8–10) results in the following system of homogeneous algebraic equations:

$$\begin{bmatrix} -\bar{D}\alpha_m^2 - \bar{C}\beta_n^2 - \bar{A} + \bar{m}_3\omega_{mn}^2 & -\alpha_m\beta_n(\bar{D}-\bar{C}) & [\bar{F}(\alpha_m^2+\beta_n^2) - \bar{A} - \bar{m}_5\omega_{mn}^2]\alpha_m \\ -\alpha_m\beta_n(\bar{D}-\bar{C}) & -\bar{C}\alpha_m^2 - \bar{D}\beta_n^2 - \bar{A} + \bar{m}_3\omega_{mn}^2 & [\bar{F}(\alpha_m^2+\beta_n^2) - \bar{A} - \bar{m}_5\omega_{mn}^2]\beta_n \\ -\bar{A}\alpha_m & -\bar{A}\beta_n & -\bar{A}(\alpha_m^2+\beta_n^2) + m_1\omega_{mn}^2 - (N_x\alpha_m^2 + N_y\beta_n^2) \end{bmatrix}$$

$$\begin{Bmatrix} A_{mn} \\ B_{mn} \\ C_{mn} \end{Bmatrix} = \{0 \quad 0 \quad 0\}^T \tag{8–22}$$

For a nontrivial solution, the determinant of the coefficient matrix in Eqs. (8–22) must be set equal to zero. It can be verified that the expansion of this determinant yields

$$D_I \cdot D_E = 0 \tag{8–23}$$

where

$$\begin{aligned} D_I = {} & m_1 \bar{m}_3 (\omega_{mn}^2)^2 - [\bar{A}(\bar{m}_3 + \frac{\bar{D}}{\bar{A}} m_1)(\alpha_m^2 + \beta_n^2) + m_1 \bar{A} \\ & + \bar{m}_3 (N_x \alpha_m^2 + N_y \beta_n^2)] \omega_{mn}^2 + [\bar{D}(N_x \alpha_m^2 + N_y \beta_n^2)(\alpha_m^2 + \beta_n^2) \\ & + \bar{A}(N_x \alpha_m^2 + N_y \beta_n^2) + \bar{A}\bar{\bar{D}}(\alpha_m^2 + \beta_n^2)^2] \end{aligned} \tag{8–24a}$$

and

$$D_E = \bar{m}_3 \omega_{mn}^2 - [\bar{A} + \bar{C}(\alpha_m^2 + \beta_n^2)] \tag{8–24b}$$

Equation (8–23) is a cubic polynomial in terms of ω_{mn}^2. From Eq. (2–23) we can further write

$$D_I = 0 \tag{8–25a}$$

$$D_E = 0 \tag{8–25b}$$

which are quadratic and linear functions of ω_{mn}^2. Hence, we have the explicit expressions for the three natural frequencies for all nonzero values of m and n. From equation (8–25b) we have

$$\omega_{mn}^2 = \frac{1}{\bar{m}_3}[\bar{C}(\alpha_m^2 + \beta_n^2) + \bar{A}]. \tag{8–26}$$

To find the relations among A_{mn}, B_{mn}, and C_{mn} for these frequencies, we substitute Eq. (8–26) into Eq. (8–22) and obtain

$$C_{mn} = 0 \tag{8–27a}$$

and

$$A_{mn}\alpha_m = -B_{mn}\beta_n \tag{8–27b}$$

Hence for the frequencies in Eq. (8–26) we have

$$w(x,y,t) = 0 \tag{8–28}$$

i.e., no transversal displacement occurs when the plate is vibrating with any of the frequencies given by Eq. (8–26). For this reason, we refer to these frequencies as the pure–shear frequencies. This result was obtained for the first time by Mindlin et al. [135] in a different manner for homogeneous isotropic plates using the first–order shear deformation plate theory (also known as Mindlin's plate theory [25]).

When m = 0 and n ≥ 1, we have from Eqs. (8–21)

$$\psi_y = w = 0$$

$$\psi_x(y,t) = \sum_{n=1}^{\infty} A_{on} \sin \beta_n y \cdot e^{i\omega_{on}t} \tag{8–29}$$

and from the characteristic determinant of Eqs. (8–22) we get

$$\omega_{on}^2 = \frac{1}{\bar{m}_3}[\bar{A} + \bar{C}\beta_n^2] \tag{8–30}$$

which is also predicted by Eq. (8–26). Similarly, when n = 0 and m ≥ 1 we have

$$\psi_x = w = 0$$

$$\psi_y(x,t) = \sum_{m=1}^{\infty} B_{mo} \sin \alpha_m x \cdot e^{i\omega_{mo}t} \tag{8–31}$$

and

$$\omega_{mo}^2 = \frac{1}{\bar{m}_3}[\bar{A} + \bar{C}\alpha_m^2] \tag{8–31}$$

which again is also predicted by Eq. (8–26). At this point, it should be emphasized that Eq. (8–25a) can also be obtained from the interior equation by simply substituting Eq. (8–21c) into Eq. (8–12a). Also it is readily seen that the solution representation

$$\Phi(x,y,t) = \sum_m^{\infty} \sum_n^{\infty} E_{mn} \cos \alpha_m x \cos \beta_n y e^{i\omega_{mn}t} \tag{8–32}$$

satisfies the boundary conditions given in Eqs. (8–20). Further, substitution of Eq. (8–32) into the edge–zone equation (i.e., Eq. (8–12b)) results in Eq. (8–25b). That is, the pure–shear frequencies of the plate are predicted by the edge–zone equation. It is noted that when m = n = 0, the solution will be a trivial one according to Eqs. (8–21) and a nonzero constant (with respect to space) according to Eq. (8–32) with

$$\omega_{oo} = \left[\frac{\bar{A}}{\bar{m}_3}\right]^{1/2} \tag{8–33}$$

according to Eq. (8–25b). This frequency is the pure–shear frequency of a simply–supported beam of any length with a rectangular cross section. For this frequency we have

$$\Phi(t) = E_{oo}e^{i\omega_{oo}t} \tag{8–34}$$

from Eq. (8–32). A close examination of Eqs. (8–16) reveals that this solution does not affect the states of strain and stress since ψ_x and ψ_y are dependent only on the derivatives of Φ with respect to y and x, respectively.

The above solutions are known as Navier's solutions. In the next section we will develop the Lévy–type solutions corresponding to the cases where the remaining edges of the plate (i.e., at y = 0,b) are not simply–supported.

8.4.2 <u>Lévy–Type Solutions</u>

The solution representations

$$\psi_x(x,y,t) = (Ae^{py})\cos \lambda_m x \cdot T_m(t) \tag{8–35a}$$

$$\psi_y(x,y,t) = (Be^{py})\sin \lambda_m x \cdot T_m(t) \tag{8–35b}$$

$$w(x,y,t) = (Ce^{py})\sin \lambda_m x \cdot T_m(t) \tag{8-35c}$$

with $T_m(t) = e^{i\omega_m t}$, $i = \sqrt{-1}$, and $\lambda_m = \frac{m\pi}{a}$ are seen to identically satisfy the simply-supported boundary conditions $\psi_y = w = M_1 = 0$ at $x = 0$ and a. Formally, a subscript m should be added to the unknown parameters A, B, C, and p in Eqs. (8-35) since for different values of m these parameters change. For convenience, however, this is not done here. With $P_z = 0$, substitution of Eqs. (8-35) into the governing equations (Eqs. (8-10)) yields the following system of homogeneous algebraic equations:

$$\begin{bmatrix} -\bar{D}\lambda_m^2 + \bar{C}p^2 - \bar{A} + \bar{m}_3\omega_m^2 & \lambda_m(\bar{D} - \bar{C})p & [-\bar{A} + \bar{F}(\lambda_m^2 - p^2) - \bar{m}_5\omega_m^2]\lambda_m \\ -\lambda_m(\bar{D} - \bar{C})p & \bar{D}p^2 - \bar{C}\lambda_m^2 - \bar{A} + \bar{m}_3\omega_m^2 & [-\bar{A} + \bar{F}(\lambda_m^2 - p^2) - \bar{m}_5\omega_m^2]p \\ -\bar{A}\lambda_m & \bar{A}p & \bar{A}(p^2 - \lambda_m^2) + m_1\omega_m^2 - N_x\lambda_m^2 + N_y p^2 \end{bmatrix} \cdot$$

$$\begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{8-36}$$

The expansion of the coefficient matrix in Eqs. (8–36) yields the auxiliary equation (also known as the characteristic equation) which can be presented as

$$D_I \cdot D_E = 0 \tag{8–37}$$

where

$$D_I = p^4 + \theta_1 p^2 + \theta_2 \equiv (p^2 - p_1^2)(p^2 - p_2^2) \tag{8–38a}$$

and

$$D_E = (p^2 - p_3^2) \tag{8–38b}$$

with

$$\theta_1 = \left[2\bar{\bar{D}}\lambda_m^2 + \frac{\bar{D}}{\bar{A}}\lambda_m^2 N_x + N_y\left[\frac{\bar{D}}{\bar{A}}\lambda_m^2 - \frac{\bar{m}_3}{\bar{A}}\omega_m^2 + 1\right]\right.$$

$$\left. - \left[\bar{m}_3 + \frac{\bar{D}}{\bar{A}} m_1\right]\omega_m^2\right] / \left[-\bar{\bar{D}} - \frac{\bar{D}}{\bar{A}} N_y\right]$$

$$\theta_2 = \left[-\bar{\bar{D}}\lambda_m^4 + N_x\left[-\frac{\bar{D}}{\bar{A}}\lambda_m^4 - \lambda_m^2 + \frac{\bar{m}_3}{\bar{A}}\omega_m^2\lambda_m^2\right]\right.$$

$$\left. - \frac{m_1\bar{m}_3}{\bar{A}}\omega_m^4 + \left[\bar{m}_3 + \frac{\bar{D}}{\bar{A}} m_1\right]\omega_m^2\lambda_m^2 + m_1\omega_m^2\right] / \left[-\bar{\bar{D}} - \frac{\bar{D}}{\bar{A}} N_y\right] \tag{8–39}$$

and

$$p_1^2 = -\frac{\theta_1}{2} + \frac{1}{2}(\theta_1^2 - 4\theta_2)^{1/2}$$

$$p_2^2 = -\frac{\theta_1}{2} - \frac{1}{2}(\theta_1^2 - 4\theta_2)^{1/2}$$

$$p_3^2 = \lambda_m^2 + \frac{\bar{A}}{\bar{C}} - \frac{\bar{m}_3}{\bar{C}}\,\omega_m^2. \tag{8–40}$$

Based on Eqs. (8–35), the general solution of Eqs. (8–10) can be presented as

$$\psi_x = [A_1 e^{p_1 y} + A_2 e^{-p_1 y} + A_3 e^{p_2 y} + A_4 e^{-p_2 y} + A_5 e^{p_3 y} + A_6 e^{-p_3 y}]\cos \lambda_m x \cdot T_m(t) \tag{8–41a}$$

$$\psi_y = [B_1 e^{p_1 y} + B_2 e^{-p_1 y} + B_3 e^{p_2 y} + B_4 e^{-p_2 y} + B_5 e^{p_3 y} + B_6 e^{-p_3 y}]\sin \lambda_m x \cdot T_m(t) \tag{8–41b}$$

$$w = [C_1 e^{p_1 y} + C_2 e^{-p_1 y} + C_3 e^{p_2 y} + C_4 e^{-p_2 y}]\sin \lambda_m x \cdot T_m(t) \tag{8–41c}$$

The relations among A_i, $B_i(i = \overline{1,6})$ and $C_i(i = \overline{1,4})$ can be found by solving any two equations in Eqs. (8–36). It should be reminded that we purposefully did not include the remaining terms

$$C_5 e^{p_3 y} + C_6 e^{-p_3 y} \tag{8–42}$$

to the expression of w in Eq. (8–41c). Had we done otherwise, solving Eqs. (8–36) for A_i, B_i, and $C_i(i = \overline{1,6})$ would yield

$$C_5 = C_6 = 0. \tag{8–43}$$

This will always be the case since substitution of Eq. (8–35c) into the interior equation (Eq. (8–12a)) will result in

$$D_I = 0 \tag{8–44}$$

which, on the other hand, indicates that the transversal displacement w is dependent on p_1 and p_2 only. Indeed, substituting

$$\Phi(x,y,t) = (Ee^{py})\cos \lambda_m x \cdot e^{i\omega_m t} \tag{8–45}$$

into the edge–zone equation (Eq. (8–12b)) will also yield

$$D_E = 0. \tag{8–46}$$

These results merely reflect the fact that the auxiliary equation of the problem is invariant.

At this point we simplify the problem by neglecting the rotatory inertia forces and by assuming that $N_x = N_y = 0$. This way from Eqs. (8–40) we get

$$p_1^2 \equiv \alpha_m^2 = \left\{ \lambda_m^2 - \frac{1}{2} \frac{m_1}{\bar{\bar{D}}} \frac{\bar{D}}{\bar{A}} \omega_m^2 + \left[\frac{m_1}{\bar{\bar{D}}} \omega_m^2 + \left[\frac{m_1 \bar{D} \omega_m^2}{2 \bar{\bar{D}} \bar{A}} \right]^2 \right]^{1/2} \right\}, \tag{8–47a}$$

$$p_3^2 \equiv \gamma_m^2 = \left[\lambda_m^2 + \frac{\bar{A}}{\bar{C}} \right] \tag{8–47b}$$

which are always positive. As far as p_2^2 is concerned, we can have two different cases:

CASE I

If

$$\left\{\frac{1}{2}\frac{m_1}{\bar{\bar{D}}}\frac{\bar{D}}{\bar{A}}\omega_m^2 + \left[\frac{m_1}{\bar{\bar{D}}}\omega_m^2 + \left[\frac{m_1\bar{D}\omega_m^2}{2\bar{\bar{D}}\bar{A}}\right]^2\right]^{1/2}\right\} > \lambda_m^2,$$

then

$$p_2^2 \equiv -\beta_m^2 = -\left\{-\lambda_m^2 + \frac{1}{2}\frac{m_1}{\bar{\bar{D}}}\frac{\bar{D}}{\bar{A}}\omega_m^2 + \left[\frac{m_1}{\bar{\bar{D}}}\omega_m^2 + \left[\frac{m_1\bar{D}\omega_m^2}{2\bar{\bar{D}}\bar{A}}\right]^2\right]^{1/2}\right\} \quad (8\text{–}48)$$

with β_m^2 being a positive number.

CASE II

If

$$\left\{\frac{1}{2}\frac{m_1}{\bar{\bar{D}}}\frac{\bar{D}}{\bar{A}}\omega_m^2 + \left[\frac{m_1}{\bar{\bar{D}}}\omega_m^2 + \left[\frac{m_1\bar{D}\omega_m^2}{2\bar{\bar{D}}\bar{A}}\right]^2\right]^{1/2}\right\} < \lambda_m^2,$$

then

$$p_2^2 \equiv \bar{\beta}_m^2 = \left\{\lambda_m^2 - \frac{1}{2}\frac{m_1}{\bar{\bar{D}}}\omega_m^2 - \left[\frac{m_1}{\bar{\bar{D}}}\omega_m^2 + \left[\frac{m_1\bar{D}\omega_m^2}{2\bar{\bar{D}}\bar{A}}\right]^2\right]^{1/2}\right\} \quad (8\text{–}49)$$

with $\bar{\beta}_m^2$ being a positive number. The general solution for CASE I can now be presented as

$$\psi_x(x,y,t) = [Aa_1 \sinh \alpha_m y + Bb_1 \cosh \alpha_m y + Cc_1 \sin \beta_m y$$

$$+ Dd_1 \cos \beta_m y + Ee_1 \cosh \gamma_m y + Ff_1 \sinh \gamma_m y]\cos \lambda_m x \cdot T_m(t)$$

$$\psi_y(x,y,t) = [Aa_2 \cosh \alpha_m y + Bb_2 \sinh \alpha_m y + Cc_2 \cos \beta_m y$$

$$+ Dd_2 \sin \beta_m y + Ee_2 \sinh \gamma_m y + Ff_2 \cosh \gamma_m y]\sin \lambda_m x \cdot T_m(t)$$

$$w(x,y,t) = [A \sinh \alpha_m y + B \cosh \alpha_m y + C \sin \beta_m y$$

$$+ D \cos \beta_m y]\sin \lambda_m x \cdot T_m(t) \qquad (8\text{–}50)$$

where A, B,...,F are the new unknown constants replacing A_i, B_i and C_i appearing in Eqs. (8–41). It is to be noted that Eqs. (8–50) could be obtained more directly by substituting Eq. (8–41c) and Eq. (8–45) into Eq. (8–16a) and Eq. (8–16b), keeping in mind that we are neglecting the rotatory inertia forces and $N_x = N_y = P_z = 0$. In Eqs. (8–50) we have

$$a_1 = b_1 = -\lambda_m - \frac{\bar{\bar{D}}}{\bar{A}} \lambda_m(\beta_m^2 + \lambda_m^2)$$

$$c_1 = d_1 = -\lambda_m + \frac{\bar{\bar{D}}}{\bar{A}} \lambda_m(\alpha_m^2 - \lambda_m^2)$$

$$a_2 = b_2 = -\alpha_m - \frac{\bar{\bar{D}}}{\bar{A}}\alpha_m(\beta_m^2 + \lambda_m^2)$$

$$c_2 = -d_2 = -\beta_m + \frac{\bar{\bar{D}}}{\bar{A}}\beta_m(\alpha_m^2 - \lambda_m^2)$$

$$e_1 = f_1 = \frac{\bar{C}}{\bar{A}}\gamma_m \ , \ e_2 = f_2 = \frac{\bar{C}}{\bar{A}}\lambda_m \tag{8–51}$$

The above expressions are slightly simplified in appearance by using

$$\frac{m_1}{\bar{\bar{D}}}\frac{\bar{D}}{\bar{A}}\omega_m^2 = \beta_m^2 - \alpha_m^2 + 2\lambda_m^2 \tag{8–52}$$

which is readily obtained by adding Eq. (8–47a) to Eq. (8–48).

Next, based on Eqs. (8–50), we will find the frequency equations of the plate having various boundary conditions at the remaining edges. Assuming that the remaining edges are located at y = b/2 and –b/2, the vibration modes will separate into ones which are either symmetric or antisymmetric (with respect to the x–axis) when the edges at y = ± b/2 are both clamped (CC) or both free (FF). For a symmetric mode and antisymmetric mode we assume

$$A = C = F = 0 \tag{8–53}$$

and

$$B = D = E = 0 \tag{8–54}$$

in Eqs. (8–50), respectively, and impose the boundary conditions given by Eqs.

(8–18) for a clamped plate and by Eqs. (8–19) for a free plate (with $N_y = 0$) at (say) $y = b/2$. This way a system of three homogeneous algebraic equations in terms of B, D, and E (for a symmetric mode) or in terms of A, C, and F (for an antisymmetric mode) will be obtained. The frequency equation is then developed by expanding the determinant of the coefficient matrix. In summary, we have for

<u>CC</u>:

1) Symmetric mode

$$\beta_m\left[1-\frac{4}{5}\frac{\bar{\bar{D}}}{\bar{A}}(\alpha_m^2-\lambda_m^2)\right]\tan\beta_m b/2+\alpha_m\left[1+\frac{4}{5}\frac{\bar{\bar{D}}}{\bar{A}}(\beta_m^2+\lambda_m^2)\right]\tanh\alpha_m b/2$$

$$-\delta_A\cdot\frac{4}{5}\frac{\bar{\bar{D}}}{\bar{A}}\frac{\lambda_m^2}{\gamma_m}(\alpha_m^2+\beta_m^2)\tanh\gamma_m b/2=0 \qquad (8\text{–}55a)$$

2) Antisymmetric Mode

$$\beta_m\left[1-\frac{4}{5}\frac{\bar{\bar{D}}}{\bar{A}}(\alpha_m^2-\lambda_m^2)\right]\tanh\alpha_m b/2-\alpha_m\left[1+\frac{4}{5}\frac{\bar{\bar{D}}}{\bar{A}}(\beta_m^2+\lambda_m^2)\right]\tan\beta_m b/2$$

$$+\delta_A\cdot\frac{4}{5}\frac{\bar{\bar{D}}}{\bar{A}}\frac{\lambda_m^2}{\gamma_m}(\alpha_m^2+\beta_m^2)\tanh\alpha_m b/2\cdot\tan\beta_m b/2\cdot\coth\gamma_m b/2=0 \qquad (8\text{–}55b)$$

and for

<u>FF</u>:

1) Symmetric Mode

$$\beta_m\left[\alpha_m^2 + \frac{2\bar{\bar{C}} - \bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2 + \left[1 + \frac{2\bar{C}}{\bar{A}}\lambda_m^2\right]\left[\beta_m^2 + \lambda_m^2\right]\right]\tan\beta_m b/2$$

$$-\alpha_m\left[\beta_m^2 - \frac{2\bar{\bar{C}} - \bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2 - \left[1 + \frac{2\bar{C}}{\bar{A}}\lambda_m^2\right]\left[\alpha_m^2 - \lambda_m^2\right]\right]\tanh\alpha_m b/2$$

$$-\delta_A \cdot 4\frac{\bar{C}\bar{\bar{C}}}{\bar{\bar{D}}\bar{A}}\lambda_m^2\alpha_m\beta_m\gamma_m(\alpha_m^2 + \beta_m^2)\tanh\alpha_m b/2 \cdot \tan\beta_m b/2 \cdot \coth\gamma_m b/2 = 0$$

(8–56a)

2) Antisymmetric Mode

$$\alpha_m\left[\beta_m^2 - \frac{2\bar{\bar{C}} - \bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2 - \left[1 + \frac{2\bar{C}}{\bar{A}}\lambda_m^2\right]\left[\alpha_m^2 - \lambda_m^2\right]\right]\tan\beta_m b/2$$

$$+\beta_m\left[\alpha_m^2 + \frac{2\bar{\bar{C}} - \bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2 + \left[1 + \frac{2\bar{C}}{\bar{A}}\lambda_m^2\right]\left[\beta_m^2 + \lambda_m^2\right]\right]\tanh\alpha_m b/2$$

$$-\delta_A \cdot 4\frac{\bar{C}\bar{\bar{C}}}{\bar{\bar{D}}\bar{A}}\lambda_m^2\alpha_m\beta_m\gamma_m(\alpha_m^2 + \beta_m^2)\tanh\gamma_m b/2 = 0 \qquad (8\text{–}56\text{b})$$

where $\bar{\bar{C}} = \bar{C} + \bar{L}$ is introduced (also see Chapter VII). The frequency equations for the SC (simply–supported at y = 0, and clamped at y = b) and the SF (simply–supported at y = 0 and free at y = b) cases can be obtained simply by replacing ($\alpha_m b/2$, $\beta_m b/2$, $\gamma_m b/2$) by ($\alpha_m b$, $\beta_m b$, $\gamma_m b$) in the trigonometric and hyperbolic functions appearing in Eq. (8–55b) and Eq. (8–56), respectively. That is, the antisymmetric vibration modes (and the corresponding frequencies) of a CC plate of width b are the same as those of a SC plate of width b/2. This is due to the fact that the conditions along the x–axis of a CC plate are the same as those of a simple support. The same arguments apply as far as FF and SF cases are concerned. Hence, we have for

SC:

$$\beta_m\left[1 - \frac{4}{5}\frac{\bar{\bar{D}}}{\bar{A}}\left[\alpha_m^2 - \lambda_m^2\right]\right]\tanh \alpha_m b - \alpha_m\left[1 + \frac{4}{5}\frac{\bar{\bar{D}}}{\bar{A}}(\beta_m^2 + \lambda_m^2)\right]\tan \beta_m b$$

$$+ \delta_A\cdot\frac{4}{5}\frac{\bar{\bar{D}}}{\bar{A}}\frac{\lambda_m^2}{\gamma_m}(\alpha_m^2 + \beta_m^2)\tanh \alpha_m b\cdot\tan \beta_m b\cdot\coth \gamma_m b = 0 \quad (8\text{–}57)$$

and for

SF:

$$\alpha_m\left[\beta_m^2 - \frac{2\bar{C} - \bar{D}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{C}}{\bar{\bar{D}}}\lambda_m^2 - \left[1 + \frac{2\bar{C}}{\bar{A}}\lambda_m^2\right]\left[\alpha_m^2 - \lambda_m^2\right]\right]\tan \beta_m b$$

$$+ \beta_m\left[\alpha_m^2 + \frac{2\bar{\bar{C}} - \bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2 + \left[1 + \frac{2\bar{C}}{\bar{A}}\lambda_m^2\right](\beta_m^2 + \lambda_m^2)\right]\tanh \alpha_m b$$

$$- \delta_A \cdot 4\,\frac{\bar{C}\bar{\bar{C}}}{\bar{\bar{D}}\bar{A}}\,\lambda_m^2\alpha_m\beta_m\gamma_m(\alpha_m^2 + \beta_m^2)\tanh \gamma_m b = 0 \qquad (8\text{–}58)$$

The frequency equation for CF (clamped at y = 0 and free at y = b) case can not as easily be found as for the cases that we considered so far. This is because the frequency equation of CF case must be found by expanding and simplifying the determinant of a six by six coefficient matrix. This case will, however, be considered along with all of the above cases later in this chapter when we introduce a more general method for determining the frequencies and buckling load of a plate.

In Eqs. (8–55) through (8–58) we have added the tracer $\delta_A(\equiv 1)$ in front of the last terms which involve either a trigonometric or a hyperbolic function of $\gamma_m b$ or $\gamma_m b/2$. These last terms may be viewed as the direct contribution of the boundary layer (edge–zone) equation. It will be shown in the next chapter that by ignoring this contribution (i.e., by setting $\delta_A = 0$) the numerical values of frequencies and buckling loads will only change slightly.

Equations (8–55) through (8–58) are the solutions for CASE I (see Eq. (8–48)). The frequency equations for CASE II are easily obtained by replacing β_m^2 in Eqs. (8–55) through (8–58) by $-\bar{\beta}_m^2$ (see Eqs. (8–48) and (8–49)). In summary, we have

CC

1) Symmetric Mode

$$-\bar{\beta}_m\left[1-\frac{4}{5}\frac{\bar{\bar{D}}}{\bar{A}}(\alpha_m^2-\lambda_m^2)\right]\tanh\bar{\beta}_m b/2+\alpha_m\left[1-\frac{4}{5}\frac{\bar{\bar{D}}}{\bar{A}}(\bar{\beta}_m^2-\lambda_m^2)\right]\tanh\alpha_m b/2$$

$$-\delta_A\cdot\frac{4}{5}\frac{\bar{\bar{D}}}{\bar{A}}\frac{\lambda_m^2}{\gamma_m}(\alpha_m^2-\bar{\beta}_m^2)\tanh\gamma_m b/2=0 \qquad (8\text{–}59a)$$

2) Antisymmetric Mode

$$-\bar{\beta}_m\left[1-\frac{4}{5}\frac{\bar{\bar{D}}}{\bar{A}}(\alpha_m^2-\lambda_m^2)\right]\tanh\alpha_m b/2+\alpha_m\left[1-\frac{4}{5}\frac{\bar{\bar{D}}}{\bar{A}}(\bar{\beta}_m^2-\lambda_m^2)\right]\tanh\bar{\beta}_m b/2$$

$$-\delta_A\cdot\frac{4}{5}\frac{\bar{\bar{D}}}{\bar{A}}\frac{\lambda_m^2}{\gamma_m}(\alpha_m^2-\bar{\beta}_m^2)\tanh\alpha_m b/2\cdot\tanh\bar{\beta}_m b/2\cdot\coth\gamma_m b/2=0$$

(8–59b)

FF

1) Symmetric Mode

$$-\bar{\beta}_m\left[\alpha_m^2+\frac{2\bar{\bar{C}}-\bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2-\left[1+\frac{2\bar{C}}{\bar{A}}\lambda_m^2\right]\left[\bar{\beta}_m^2-\lambda_m^2\right]\right]\tanh\bar{\beta}_m b/2$$

$$+\alpha_m\left[\bar{\beta}_m^2+\frac{2\bar{\bar{C}}-\bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2-\left[1+\frac{2\bar{C}}{\bar{A}}\lambda_m^2\right]\left[\alpha_m^2-\lambda_m^2\right]\right]\tanh\alpha_m b/2$$

$$+\ \delta_A \cdot 4\,\frac{\bar{C}\bar{\bar{C}}}{\bar{\bar{D}}\bar{A}}\,\lambda_m^2 \alpha_m \bar{\beta}_m \gamma_m (\alpha_m^2 - \bar{\beta}_m^2) \tanh \alpha_m b/2 \cdot \tanh \bar{\beta}_m b/2 \cdot \coth \gamma_m b/2 = 0$$

(8–60a)

2) Antisymmetric Mode

$$\alpha_m\left[\bar{\beta}_m^2 + \frac{2\bar{\bar{C}} - \bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2 - \left[1 + \frac{2\bar{C}}{\bar{A}}\lambda_m^2\right]\left[\alpha_m^2 - \lambda_m^2\right]\right]\tanh \bar{\beta}_m b/2$$

$$-\bar{\beta}_m\left[\alpha_m^2 + \frac{2\bar{\bar{C}} - \bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2 - \left[1 + \frac{2\bar{C}}{\bar{A}}\lambda_m^2\right]\left[\bar{\beta}_m^2 - \lambda_m^2\right]\right]\tanh \alpha_m b/2$$

$$+\ \delta_A \cdot 4\,\frac{\bar{C}\bar{\bar{C}}}{\bar{\bar{D}}\bar{A}}\,\lambda_m^2 \alpha_m \bar{\beta}_m \gamma_m (\alpha_m^2 - \bar{\beta}_m^2) \tanh \gamma_m b/2 = 0 \qquad (8\text{–}60b)$$

<u>SC</u>

$$-\bar{\beta}_m\left[1 - \tfrac{4}{5}\frac{\bar{\bar{D}}}{\bar{A}}(\alpha_m^2 - \lambda_m^2)\right]\tanh \alpha_m b + \alpha_m\left[1 - \tfrac{4}{5}\frac{\bar{\bar{D}}}{\bar{A}}(\bar{\beta}_m^2 - \lambda_m^2)\right]\tanh \bar{\beta}_m b$$

$$-\ \delta_A \cdot \tfrac{4}{5}\frac{\bar{\bar{D}}}{\bar{A}}\frac{\lambda_m^2}{\gamma_m}(\alpha_m^2 - \bar{\beta}_m^2)\tanh \alpha_m b \cdot \tanh \bar{\beta}_m b \cdot \coth \gamma_m b = 0 \qquad (8\text{–}61)$$

SF

$$\alpha_m\left[\bar{\beta}_m^2 + \frac{2\bar{C} - \bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{D}}{\bar{\bar{D}}}\lambda_m^2 - \left[1 + \frac{2\bar{C}}{\bar{A}}\lambda_m^2\right]\left[\alpha_m^2 - \lambda_m^2\right]\right]\tanh \bar{\beta}_m b$$

$$-\bar{\beta}_m\left[\alpha_m^2 + \frac{2\bar{C} - \bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2 - \left[1 + \frac{2\bar{C}}{\bar{A}}\lambda_m^2\right]\left[\bar{\beta}_m^2 - \lambda_m^2\right]\right]\tanh \alpha_m b$$

$$+ \delta_A \cdot 4\frac{\bar{C}\bar{\bar{C}}}{\bar{\bar{D}}\bar{A}}\lambda_m^2\alpha_m\bar{\beta}_m\gamma_m(\alpha_m^2 - \bar{\beta}_m^2)\tanh \gamma_m b = 0 \qquad (8\text{–}62)$$

8.5.1 Buckling Equations

The stability equations in Levinson's TSDPT are obtained from Eqs. (8–10) by simply dropping all the inertia forces, letting $P_z = 0$, $\bar{N}_x = -N_x$, and $\bar{N}_y = -N_y$ where $\bar{N}_x$ and $\bar{N}_y$ are the compressive edge loads. For a completely simply–supported plate, substitution of Eqs. (8–21), with $T_{mn} = 1$, into the stability equations and following the same procedure as in Section 8.4.1 yields the buckling equation

$$(\bar{N}_x\alpha_m^2 + \bar{N}_y\beta_n^2) = \frac{\bar{A}\bar{\bar{D}}(\alpha_m^2 + \beta_n^2)^2}{\bar{D}(\alpha_m^2 + \beta_n^2) + \bar{A}} \qquad (8\text{–}63)$$

where $\alpha_m = \frac{m\pi}{a}$ and $\beta_n = \frac{n\pi}{b}$ as in Section 8.4.1.

8.5.2 Lévy–Type Solutions

The Lévy–type stability analysis is similar to the analysis presented in Section 8.4.2. By assuming $T_m = 1$ in Eqs. (8–35), results similar to those in Eqs. (8–36) through (8–41) can readily be obtained. However, here we simplify the analysis by letting $\bar{N}_y = 0$ and obtain

$$P_1^2 \equiv \alpha_m^2 = \left\{\lambda_m^2 - \frac{1}{2}\frac{\bar{N}_x}{\bar{\bar{D}}}\frac{\bar{D}}{\bar{A}}\lambda_m^2 + \left[\frac{\bar{N}_x}{\bar{\bar{D}}}\lambda_m^2 + \left[\frac{1}{2}\frac{\bar{N}_x}{\bar{\bar{D}}}\frac{\bar{D}}{\bar{A}}\lambda_m^2\right]^2\right]^{1/2}\right\} \tag{8–64a}$$

$$P_3^2 \equiv \gamma_m^2 = \left[\lambda_m^2 + \frac{\bar{A}}{\bar{C}}\right] \tag{8–64b}$$

which are the counterparts of Eqs. (8–47). Again, as far as P_2^2 is concerned, we can have two different cases:

CASE I

If

$$\left\{\frac{1}{2}\frac{\bar{N}_x}{\bar{\bar{D}}}\frac{\bar{D}}{\bar{A}}\lambda_m^2 + \left[\frac{\bar{N}_x}{\bar{\bar{D}}}\lambda_m^2 + \left[\frac{1}{2}\frac{\bar{N}_x}{\bar{\bar{D}}}\frac{\bar{D}}{\bar{A}}\lambda_m^2\right]^2\right]^{1/2}\right\} > \lambda_m^2,$$

then

$$P_2^2 \equiv -\beta_m^2 = -\left\{-\lambda_m^2 + \frac{1}{2}\frac{\bar{N}_x}{\bar{\bar{D}}}\frac{\bar{D}}{\bar{A}}\lambda_m^2 + \left[\frac{\bar{N}_x}{\bar{\bar{D}}}\lambda_m^2 + \left[\frac{1}{2}\frac{\bar{N}_x}{\bar{\bar{D}}}\frac{\bar{D}}{\bar{A}}\lambda_m^2\right]^2\right]^{1/2}\right\} \tag{8–65}$$

where β_m^2 is a positive number.

CASE II

If

$$\left\{\frac{1}{2}\frac{\bar{N}_x}{\bar{\bar{D}}}\frac{\bar{D}}{\bar{A}}\lambda_m^2 + \left[\frac{\bar{N}_x}{\bar{\bar{D}}}\lambda_m^2 + \left(\frac{1}{2}\frac{\bar{N}_x}{\bar{\bar{D}}}\frac{\bar{D}}{\bar{A}}\lambda_m^2\right)^2\right]^{1/2}\right\} < \lambda_m^2,$$

then

$$P_2^2 \equiv \bar{\beta}_m^2 = \left\{\lambda_m^2 - \frac{1}{2}\frac{\bar{N}_x}{\bar{\bar{D}}}\frac{\bar{D}}{\bar{A}}\lambda_m^2 - \left[\frac{\bar{N}_x}{\bar{\bar{D}}}\lambda_m^2 + \left(\frac{1}{2}\frac{\bar{N}_x}{\bar{\bar{D}}}\frac{\bar{D}}{\bar{A}}\lambda_m^2\right)^2\right]^{1/2}\right\} \qquad (8\text{–}66)$$

with $\bar{\beta}_m^2$ being a positive number. The general solution for CASE I is the same as that given by Eqs. (8–50) and (8–51) with $T_m = 1$ and α_m^2 and β_m^2 given by Eqs. (8–64a) and (8–65). The buckling equations for CASE I and CASE II are also the same as those given by Eq. (8–55) through Eq. (8–62) with α_m^2, β_m^2, and $\bar{\beta}_m^2$ given by Eq. (8–64a), (8–65), and (8–66). In the stability analysis, it is unikely for the solutions of CASE II, except for the FF case, to give the lowest buckling load. This is expected merely from the analysis based on the classical plate theory (CLPT). Also in free–vibration problems, some of frequency equations corresponding to CASE II may not have any roots (specifically those of the CC and SC cases). Melkonyan and Khachatryan [136] obtained the buckling equation of a homogeneous transversely isotropic plate for the SF case using Ambartsumyan's shear–deformation plate theory [1]. Their result is also presented in [1].

8.6 An Alternative Solution Procedure

In the previous sections, in order to find the frequency equations, we made the simplifying assumption that the rotatory inertia forces could be neglected. Also we obtained the buckling equations of a plate under uniaxial load in the x–direction. In this section we develop an alternative analysis in which the aforementioend assumptions will be removed.

We have seen in Section 8.4.2 that the general solution of Eqs. (8–10) is given by Eqs. (8–41). As we mentioned before, the relationships between the unknown constants A_i, B_i ($i = \overline{1,6}$), and C_i ($i = \overline{1,4}$) are found by substituting Eqs. (8–41) back into Eqs. (8–10). This way Eqs. (8–41) will change to

$$\psi_x = \left[\sum_{j=1}^{6} A_j e^{r_j y}\right] \cos \lambda_m x \cdot T_m(t)$$

$$\psi_y = \left[\sum_{j=1}^{6} A_j u_{2j} e^{r_j y}\right] \sin \lambda_m x \cdot T_m(t)$$

$$w = \left[\sum_{j=1}^{4} A_j u_{3j} e^{r_j y}\right] \sin \lambda_m x \cdot T_m(t) \tag{8–67}$$

where

$$r_1 = P_1,\ r_2 = -P_1,\ r_3 = P_2,\ r_4 = -P_2,\ r_5 = P_3,\ r_6 = -P_3 \tag{8–68}$$

and P_j ($j = \overline{1,3}$) are given in Eqs. (8–40). Also

$$u_{2j} = \frac{r_j}{\lambda_m}\ ,\ u_{3j} = \frac{-[\bar{D}(r_j^2 - \lambda_m^2) - \bar{A} + \bar{m}_3 \omega_m^2]}{\lambda_m [\bar{F}(\lambda_m^2 - r_j^2) - \bar{A} - \bar{m}_5 \omega_m^2]} \quad \text{for } j = \overline{1,4} \tag{8–69a}$$

and

$$u_{2j} = \frac{\lambda_m}{r_j} \qquad \text{for } j = 5,6. \tag{8–69b}$$

It is to be recalled that, in Eqs. (8–67), $\lambda_m = \frac{m\pi}{a}$ and $T_m = e^{i\omega_m t}$ and 1 in free–vibration and stability problems, respectively. Also in stability problems we replace (N_x,N_y) by $(-\bar{N}_x,-\bar{N}_y)$ where $\bar{N}_x$ and $\bar{N}_y$ are the compressive edge loads acting in the x– and y–directions, respectively. It should be noted that r_j $(j = \overline{1,6})$ can be real or complex. With m = 1,2,..., when the boundary conditions in Eqs. (17)–(19) are imposed at $y = \pm\, b/2$ (or alternatively at y = 0 and b) on Eqs. (8–67), we obtain a system of homogeneous algebraic equations of the form

$$[K]\{A\} = \{0\}. \tag{8–70a}$$

For a nontrivial solution to exist, the determinant of [K] must be set equal to zero. That is

$$|K| = 0 \tag{8–70b}$$

It is to be noted that $|K|$ can in general be complex since the unknown constants A_j $(j = \overline{1,6})$ are in general complex. However, the determinant of [K] can readily be transformed into a real number by the procedure that we introduced in Chapter IV.

In free–vibration problems, it is possible for m to be equal to 0. With m = 0 we will have

$$\psi_y = w = 0 \tag{8–71a}$$

and

$$\psi_x \equiv \psi_x(y,t). \tag{8–71b}$$

Also the governing equations (i.e., Eqs. (8–10)) will reduce to

$$\bar{C}\psi_{x,yy} - \bar{A}\psi_x = \bar{m}_3\ddot{\psi}_x \tag{8–72}$$

and from Eqs. (8–9) the stress resultants will become

$$M_2 = Q_2 = 0 \tag{8–73a}$$

and

$$M_6 = \bar{C}\psi_{x,y}. \tag{8–73b}$$

With the help of Eqs. (8–71) and Eqs. (8–73) the boundary conditions in Eq. (8–17) through Eq. (8–19) will also reduce to

$$\begin{aligned} &\text{S:} \quad \psi_x = 0 \\ &\text{C:} \quad \psi_x = 0 \\ &\text{F:} \quad \psi_{x,y} = 0 \end{aligned} \tag{8–74}$$

We will refer to Eq. (8–72) as the <u>shear</u> equation since when m = 0 the transverse displacement of the plate is identically zero. Also it is seen from Eqs. (8–74) that the pure–shear frequencies of a plate having any combination of S type and C type boundary conditions at y = 0 and b will be identical. These frequencies are readily seen to be the same as those given by Eq. (7–30). We demonstrate this alternatively by assuming

$$\psi_x(y,t) = (A \cdot e^{py})e^{i\omega_o t} \tag{8–75}$$

in Eq. (8–72). This way we find

$$P_1^2 = \frac{1}{\bar{C}}\,[\bar{A} - \bar{m}_3\omega_o^2] \tag{8–76}$$

and, therefore, the general solution of Eq. (8–72) (when $\bar{m}_3\omega_o^2 \neq \bar{A}$) is

$$\psi_x(y,t) = [A_1 e^{P_1 y} + A_2 e^{-P_1 y}]\cdot e^{i\omega_o t} \tag{8-77}$$

Now by assuming various boundary conditions for the plate at y = 0 and b we obtain the following results:

<u>**SS, SC, CC, FF**</u>

$$\omega_o^2 = \frac{1}{\bar{m}_3}[\bar{A} + \bar{C}\beta_n^2] \tag{8-78}$$

with $\beta_n = \frac{n\pi}{b}$ (n = 1,2,3,...) and

<u>**SF, CF**</u>

$$\omega_o^2 = \frac{1}{\bar{m}_3}[\bar{A} + \bar{C}\bar{\beta}_n^2] \tag{8-79}$$

with $\bar{\beta}_n = \frac{n\pi}{2b}$ (n = 1,3,5,...). It is readily seen that no new results will be obtained when $\bar{m}_3\omega_o^2 = \bar{A}$ (i.e., when $P_1^2 = 0$ in Eq. (8–76)).

8.7 <u>Discussions</u>

In this chapter we have studied the free–vibration and buckling problems of a symmetric plate with transversely isotropic layers using the original form of the governing equations (i.e., Eqs. (8–10)) in Levinson's TSDPT. Throughout our analyses we have adopted two different approaches. With some simplifying assumptions concerning the edge loads and the rotatory inertia forces, the first approach resulted in analytic frequency and buckling equations. By removing these assumptions, however, in the second approach we came up with the analytic frequency and buckling determinants. We postpone the development of numerical results to the next chapter where we will develop the frequency and buckling equations of a plate within the framework of the first–order shear deformation plate theory (FSDPT), by using the interior and edge–zone equations exclusively.

In section 8.3 we have pointed out that for a clamped edge, instead of the boundary conditions in Eq. (8–18), we may alternatively impose (see [1])

$$\psi_x = w = 0 \tag{8–80a}$$

$$\psi_y = 0. \tag{8–80b}$$

If this is done, the frequency equations for CC and SC cases can be obtained by simply replacing the constant factor $\frac{4}{5}$ by 1 in Eqs. (8–55), (8–57), (8–59), and (8–61). This conclusion also holds as far as the buckling equations are concerned. It will be seen, in the next chapter, that for a single layer isotropic (and transversely isotropic) plate the frequency and buckling equations according to Levinson's TSDPT and FSDPT become identical if the conditions in Eqs. (8–18) are imposed at a clamped edge. However, if the conditions given in Eqs. (8–80) are used, the results of these theories will be identical only for FF, SF, and SS cases. In the free–vibration analysis these conclusions hold only when the rotatory inertia forces are neglected.

CHAPTER IX

FREQUENCY AND BUCKLING EQUATIONS OF A SYMMETRIC PLATE ACCORDING TO THE FIRST–ORDER SHEAR DEFORMATION PLATE THEORY

In this chapter we will develop the frequency and buckling equations of a symmetric plate with transversely isotropic layers according to the first–order shear deformation plate theory (FSDPT). However, for the sake of illustration, we will alternatively use the interior and edge–zone equations of the plate throughout our analysis. Later in this chapter the frequency and buckling equations of the plate according to the classical plate theory (CLPT) will be obtained as a special case from the results of FSDPT.

For the sake of comparison, analytical frequency and buckling determinants according to Reddy's TSDPT will be developed by using the procedure presented in Section 8.6. Finally, numerical results will be generated to demonstrate the influence of the edge–zone equation and the effects of rotatory inertia forces, transverse shear deformation, shear correction factor in FSDPT, and various clamped–type boundary conditions in Levinson's TSDPT on the natural frequencies and buckling loads of plates with various edge conditions.

9.1 Governing Equations

The nonlinear governing equations of a symmetric plate laminated of transversely isotropic layers were developed in Chapter VII and are given by Eqs. (7–6) and (7–9). As we mentioned in Chapter VI, in the linear theory the stretching and bending equations will be uncoupled for a symmetric plate. The bending equations in FSDPT are obtained from Eqs. (7–9) by simply dropping the nonlinear term appearing in Eq. (7–9c):

$$\bar{\bar{D}}\psi_{x,xx} + \bar{\bar{D}}\psi_{x,yy} - K^2\bar{\bar{A}}(\psi_x + w_{,x}) + (\bar{\bar{D}} - \bar{\bar{C}})\psi_{y,xy} = \bar{\bar{m}}_3\ddot{\psi}_x \tag{9–1a}$$

$$\bar{\bar{D}}\psi_{y,yy} + \bar{\bar{C}}\psi_{y,xx} - K^2\bar{\bar{A}}(\psi_y + w_{,y}) + (\bar{\bar{D}} - \bar{\bar{C}})\psi_{x,xy} = \bar{\bar{m}}_3\ddot{\psi}_y \tag{9–1b}$$

$$K^2\bar{\bar{A}}(\psi_{x,x} + \psi_{y,y}) + K^2\bar{\bar{A}}\nabla^2 w + N_x w_{,xx} + N_y w_{,yy} + P_z = m_1\ddot{w} \tag{9–1c}$$

with N_x and N_y being the tensile edge loads in the x– and y–directions.

9.2 An Alternative Formulation

In Chapter VII we have also shown how alternatively these equations can be reformulated to yield the interior and edge–zone equations of a plate (see Eqs. (7–15) and (7–16). In the linear theory these equations are

$$-\bar{\bar{D}}\nabla^2\nabla^2 w - \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\nabla^2 q_z + q_z + \frac{\bar{\bar{m}}_3}{K^2\bar{\bar{A}}}\ddot{q}_z = \frac{1}{K^2\bar{\bar{A}}}(m_1\bar{\bar{m}}_3)\ddot{\ddot{w}}$$

$$-(\bar{\bar{m}}_3 + \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}m_1)\nabla^2\ddot{w} + m_1\ddot{w} \tag{9–2a}$$

$$\bar{\bar{C}}\nabla^2\Phi - K^2\bar{\bar{A}}\Phi = \bar{\bar{m}}_3\ddot{\Phi} \tag{9–2b}$$

where

$$q_z = N_x w_{,xx} + N_y w_{,yy} + P_z \tag{9–2c}$$

and

$$\Phi = \psi_{x,y} - \psi_{y,x}. \tag{9–3}$$

Equations (9–2) are also obtained in [137] by using tensorial notations and a different approach. We will use these equations in this chapter to develop the frequency and buckling equations of a plate having various boundary conditions.

9.3 Stress Resultants in Terms of Φ and w

The stress resultants for a symmetric plate laminated of transversely isotropic layers in FSDPT are given by Eqs. (7–7b). For convenience we present them here:

$$M_1 = \bar{\bar{D}}(\psi_{x,x} + \psi_{y,y}) - 2\bar{\bar{C}}\psi_{y,y}$$

$$M_2 = \bar{\bar{D}}(\psi_{x,x} + \psi_{y,y}) - 2\bar{\bar{C}}\psi_{x,x}$$

$$M_6 = \bar{\bar{C}}(\psi_{x,y} + \psi_{y,x})$$

$$Q_1 = K^2\bar{\bar{A}}(\psi_x + w_{,x}) \ , \ Q_2 = K^2\bar{\bar{A}}(\psi_y + w_{,y}) \tag{9–4}$$

The next step in our analysis is to express these stress resultants in terms of w and Φ only, i.e., to eliminate ψ_x and ψ_y in Eqs. (9–4). By following the procedure developed in Chapter VII, we first rewrite Eqs. (9–1) as

$$L_1\psi_x = \left[-w_{,x} + \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}(\psi_{x,x} + \psi_{y,y})_{,x} + \frac{\bar{\bar{C}}}{K^2\bar{\bar{A}}}\Phi_{,y}\right] \tag{9–5a}$$

$$L_1\psi_y = \left[-w_{,y} + \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}(\psi_{x,x} + \psi_{y,y})_{,y} - \frac{\bar{\bar{C}}}{K^2\bar{\bar{A}}}\Phi_{,x}\right] \tag{9–5b}$$

$$(\psi_{x,x} + \psi_{y,y}) = -\nabla^2 w - \frac{q_z}{K^2\bar{\bar{A}}} + \frac{m_1}{K^2\bar{\bar{A}}}\ddot{w} \tag{9–5c}$$

where

$$L_1 = \left[1 + \frac{\bar{\bar{m}}_3}{K^2\bar{\bar{A}}}\frac{\partial^2}{\partial t^2}\right] \tag{9–5d}$$

and q_z and Φ are given by Eqs. (9–2c) and (9–3), respectively. Now substitution of Eq. (9–5c) into Eqs. (9–5a) and (9–5b) results in

$$\psi_x = \frac{1}{L_1}\left[-w_{,x} - \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\nabla^2 w_{,x} + \frac{\bar{\bar{D}}m_1}{(K^2\bar{\bar{A}})^2}\ddot{w}_{,x} + \frac{\bar{\bar{C}}}{K^2\bar{\bar{A}}}\Phi_{,y} - \frac{\bar{\bar{D}}}{(K^2\bar{\bar{A}})^2}q_{z,x}\right]$$

$$\psi_y = \frac{1}{L_1}\left[-w_{,y} - \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\nabla^2 w_{,y} + \frac{\bar{\bar{D}}m_1}{(K^2\bar{\bar{A}})^2}\ddot{w}_{,y} - \frac{\bar{\bar{C}}}{K^2\bar{\bar{A}}}\Phi_{,x} - \frac{\bar{\bar{D}}}{(K^2\bar{\bar{A}})^2}q_{z,y}\right]$$

(9–6)

which can be used in Eqs. (9–4) to express the stress resultants explicitly in terms of w and Φ. This will, however, be done later in this chapter.

9.4.1 Vibration Analysis

In this section we consider the free–vibration problems of the plate with various boundary conditions. The stability analysis of the plate will be developed in Section 9.5. It should be kept in mind that in free–vibration and stability analyses we let $P_z = 0$.

9.4.2 Simply–Supported Plate

When a plate is simply supported at x = 0 and a, we impose the boundary conditions (see Eqs. (2–11c)–(2–11e) in Chapter II)

$$\psi_y = w = 0 \tag{9–7a}$$

and

$$M_1 = 0. \tag{9–7b}$$

From Eqs. (9–4), the boundary condition (9–7b) can alternatively be stated as

$$\psi_{x,x} = 0. \tag{9–7c}$$

At this point we need to express the boundary conditions in Eqs. (9–7a) and (9–7c) in terms of w and Φ only. This is easily done by evaluating Eqs. (9–5c) and (9–5b) at x = 0 and a and using the information given by Eqs. (9–7a) and (9–7c). This way the boundary conditions at x = 0 and a will turn out to be

$$w = w_{,xx} + \frac{1}{K^2\bar{A}}(P_z + N_x w_{,xx}) = 0 \tag{9–8a}$$

will

$$\Phi_{,x} = 0. \tag{9–8b}$$

Similarly we can show that the boundary conditions for a plate simply supported at $y = 0$ and b are

$$w = w_{,yy} + \frac{1}{K^2\bar{A}}(P_z + N_y w_{,yy}) = 0 \tag{9–9a}$$

and

$$\Phi_{,y} = 0. \tag{9–9b}$$

with $P_z = 0$ in free–vibration and stability problems. For a completely simply–supported plate, it is readily seen that the representations

$$w = A_{mn} \sin \alpha_m x \sin \beta_n y \; T_{mn}(t) \tag{9–10a}$$

$$\Phi = B_{mn} \cos \alpha_m x \cos \beta_n y \; T_{mn}(t) \tag{9–10b}$$

with $\alpha_m = \frac{m\pi}{a}$ and $\beta_n = \frac{n\pi}{b}$ satisfy all the boundary conditions given in Eqs. (9–8) and (9–9). Here $T_{mn} = e^{i\omega_{mn}t}$ with $i = \sqrt{-1}$ and ω_{mn} are the natural frequencies of the plate.

At this point it should be reminded that Eqs. (9–8) and (9–9) indicate that since w and Φ are not coupled through the boundary conditions, the interior and edge–zone equations can be solved independently. Physically, on the other hand, this is an indication that these equations are predicting two completely different motions of the plate, as will soon be seen.

Substituting Eq. (9–10a) into Eq. (9–2a) results in

$$D_I = 0 \tag{9–11a}$$

where

$$D_I = \frac{m_1\bar{\bar{m}}_3}{K^2\bar{\bar{A}}}(\omega_{mn}^2)^2 - \left[m_1 + \left[\bar{\bar{m}}_3 + \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}m_1\right]\left[\alpha_m^2 + \beta_n^2\right]\right.$$

$$\left. + \frac{\bar{\bar{m}}_3}{K^2\bar{\bar{A}}}\left[N_x\alpha_m^2 + N_y\beta_n^2\right]\right]\omega_{mn}^2$$

$$+ \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}[N_x\alpha_m^2 + N_y\beta_n^2)(\alpha_m^2 + \beta_n^2)] + \bar{\bar{D}}(\alpha_m^2 + \beta_n^2)^2 + (N_x\alpha_m^2 + N_y\beta_n^2) \tag{9–11b}$$

Equation (9–11a) is a quadratic polynomial in terms of ω_{mn}^2 which results in two natural frequencies for each pair of m and n. The vibration modes of these frequencies correspond to $\Phi = 0$. When $\Phi = 0$, on the other hand, we have

$$\psi_{x,y} = \psi_{y,x} \tag{9–12}$$

which is obtained from Eq. (9–3). Also substitution of Eq. (9–10b) into Eq. (9–2b) yields

$$D_E = 0 \tag{9–13a}$$

where

$$D_E = \bar{\bar{m}}_3\omega_{mn}^2 - K^2\bar{\bar{A}} - \bar{\bar{C}}(\alpha_m^2 + \beta_n^2) \tag{9–13b}$$

The frequencies obtained from Eq. (9–13a) are

$$\omega_{mn}^2 = \frac{1}{\bar{\bar{m}}_3}[\bar{\bar{C}}(\alpha_m^2 + \beta_n^2) + \bar{\bar{A}}]. \tag{9-14}$$

The vibration modes for these frequencies correspond to w = 0. For this reason, these frequencies were referred to as the pure–shear frequencies of the plate in previous chapters.

Next we assume that the edges of the plate at x = 0 and a are invariably simply supported and develop the frequency equations of a plate having various boundary conditions at y = 0 and b.

9.4.3 Lévy–Type Solutions

When the edges of the plate at x = 0 and a are simply supported, the solution representations

$$w = (Ae^{py})\sin \lambda_m x \cdot T_m(t) \tag{9-15a}$$

and

$$\Phi = (Be^{py})\cos \lambda_m x \cdot T_m(t) \tag{9-15b}$$

with $\lambda_m = \frac{m\pi}{a}$ are seen to satisfy the boundary conditions in Eqs. (9–8). In Eqs. (9–15) $T_m = e^{i\omega_m t}$ where $i = \sqrt{-1}$. Substituting Eq. (9–15a) into Eq. (9–2a) results in the auxiliary equation

$$D_I = 0 \tag{9-16a}$$

where

$$D_I = P^4 + \theta_1 P^2 + \theta_2 \equiv (P^2 - P_1^2)(P^2 - P_2^2) \tag{9-16b}$$

and

$$\theta_1 = \left[2\bar{\bar{D}}\lambda_m^2 + \bar{\bar{D}}\,\frac{N_x}{K^2\bar{\bar{A}}}\,\lambda_m^2 + N_y\left[\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\,\lambda_m^2 - \frac{\bar{\bar{m}}_3}{K^2\bar{\bar{A}}}\,\omega_m^2 + 1\right]\right.$$

$$\left. - \left[\bar{\bar{m}}_3 + \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\,m_1\right]\omega_m^2\right] / \left[-\bar{\bar{D}}\left[1 + \frac{N_y}{K^2\bar{\bar{A}}}\right]\right]$$

$$\theta_2 = \left[-\bar{\bar{D}}\lambda_m^2 - N_x\lambda_m^2\left[\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\,\lambda_m^2 - \frac{\bar{\bar{m}}_3}{K^2\bar{\bar{A}}}\,\omega_m^2 + 1\right] - \frac{m_1\bar{\bar{m}}_3}{K^2\bar{\bar{A}}}\,(\omega_m^2)^2\right.$$

$$\left. + m_1\omega_m^2 + \left[\bar{\bar{m}}_3 + \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\,m_1\right]\lambda_m^2\omega_m^2\right] / \left[-\bar{\bar{D}}\left[1 + \frac{N_y}{K^2\bar{\bar{A}}}\right]\right] \tag{9–16c}$$

with

$$P_1^2 = -\frac{\theta_1}{2} + \frac{1}{2}\,(\theta_1^2 - 4\theta_2)^{1/2}$$

$$P_2^2 = -\frac{\theta_1}{2} - \frac{1}{2}\,(\theta_1^2 - 4\theta_2)^{1/2} \tag{9–16d}$$

Also substitution of Eq. (9–15b) into Eq. (9–2b) yields

$$D_E = 0 \tag{9–17a}$$

where

$$D_E = (P^2 - P_3^2) \tag{9–17b}$$

and

$$P_3^2 = \lambda_m^2 + \frac{K^2\bar{A}}{\bar{\bar{C}}} - \frac{\bar{\bar{m}}_3}{\bar{\bar{C}}}\,\omega_m^2. \tag{9–17c}$$

On the basis of Eqs. (9–15), (9–16d), and (9–17c) the general solutions of Eqs. (9–2) can be presented as

$$w = \left[A_1 e^{P_1 y} + A_2 e^{-P_1 y} + A_3 e^{P_2 y} + A_4 e^{-P_2 y}\right] \sin \lambda_m x \cdot T_m(t) \tag{9–18a}$$

$$\Phi = \left[B_1 e^{P_3 y} + B_2 e^{-P_3 y}\right] \cos \lambda_m x \cdot T_m(t) \tag{9–18b}$$

It is to be noted that at this point P_1, P_2, and P_3 can be real or complex. We can simplify the problem by assuming $N_x = N_y = 0$ and neglecting the rotatory inertia forces. When this is done, we will have

$$P_1^2 \equiv \alpha_m^2 = \left\{\lambda_m^2 - \frac{1}{2}\frac{m_1}{K^2\bar{A}}\,\omega_m^2 + \left[\frac{m_1}{\bar{\bar{D}}}\,\omega_m^2 + \left[\frac{m_1}{2K^2\bar{A}}\right]^2 \omega_m^4\right]^{1/2}\right\} \tag{9–19a}$$

$$P_3^2 \equiv \gamma_m^2 = \left[\lambda_m^2 + \frac{K^2\bar{A}}{\bar{\bar{C}}}\right] \tag{9–19b}$$

which are always positive. Also as far as P_2^2 is concerned we can have the following two cases:

CASE I;

If

$$\left\{\frac{1}{2}\frac{m_1}{K^2\bar{\bar{A}}}\,\omega_m^2 + \left[\frac{m_1}{\bar{\bar{D}}}\,\omega_m^2 + \left[\frac{m_1}{2K^2\bar{\bar{A}}}\right]^2\omega_m^4\right]^{1/2}\right\} > \lambda_m^2,$$

then

$$P_2^2 \equiv -\beta_m^2 = -\left\{-\lambda_m^2 + \frac{1}{2}\frac{m_1}{K^2\bar{\bar{A}}}\,\omega_m^2 + \left[\frac{m_1}{\bar{\bar{D}}}\,\omega_m^2 + \left[\frac{m_1}{2K^2\bar{\bar{A}}}\right]^2\omega_m^4\right]^{1/2}\right\} \tag{9–20}$$

where β_m^2 is always positive.

CASE II;

If

$$\left\{\frac{1}{2}\frac{m_1}{K^2\bar{\bar{A}}}\,\omega_m^2 + \left[\frac{m_1}{\bar{\bar{D}}}\,\omega_m^2 + \left[\frac{m_1}{2K^2\bar{\bar{A}}}\right]^2\omega_m^4\right]^{1/2}\right\} < \lambda_m^2,$$

then

$$P_2^2 \equiv \bar{\beta}_m^2 = \left\{\lambda_m^2 - \frac{1}{2}\frac{m_1}{K^2\bar{\bar{A}}}\,\omega_m^2 - \left[\frac{m_1}{\bar{\bar{D}}}\,\omega_m^2 + \left[\frac{m_1}{2K^2\bar{\bar{A}}}\right]^2\omega_m^4\right]^{1/2}\right\} \tag{9–21}$$

with $\bar{\beta}_m^2$ being a positive number.

Based on Eqs. (9–19) and (9–20), the general solutions of Eqs. (9–2) for CASE I will be

$$w = [A \sinh \alpha_m y + B \cosh \alpha_m y$$

$$+ C \sin \beta_m y + D \cos \beta_m y] \sin \lambda_m x \cdot T_m(t) \tag{9–22a}$$

$$\Phi = [E \sinh \gamma_m y + F \cosh \gamma_m y] \cos \lambda_m x \cdot T_m(t) \tag{9–22b}$$

where A, B,..., and F are six arbitrary constants.

9.4.4 Boundary Conditions at y = ± b/2 (or at y = 0 and b)

As we mentioned earlier, we need to express the stress resultants in terms of w and Φ. To this end we keep in mind that we have asumed $N_x = N_y = 0$ and neglected the rotatory inertia forces. With these assumptions, we substitute Eqs. (9–6), (9–2b), and (9–5c) into Eqs. (9–4) and obtain

$$M_2 = -\bar{\bar{D}}\nabla^2 w + \frac{m_1 \bar{\bar{D}}}{K^2 \bar{\bar{A}}} \ddot{w} + 2\bar{\bar{C}} w_{,xx} - \frac{2\bar{\bar{C}}^2}{K^2 \bar{\bar{A}}} \Phi_{,xy}$$

$$- 2 \frac{\bar{\bar{C}}\bar{\bar{D}}}{K^2 \bar{\bar{A}}} \left[\frac{m_1}{K^2 \bar{\bar{A}}} \ddot{w}_{,xx} - \nabla^2 w_{,xx} \right], \tag{9–23a}$$

$$M_6 = -2\bar{\bar{C}} w_{,xy} + \bar{\bar{C}}\Phi - 2 \frac{\bar{\bar{C}}^2}{K^2 \bar{\bar{A}}} \Phi_{,xx}$$

$$+ 2\,\frac{\bar{\bar{C}}\bar{\bar{D}}}{K^2\bar{\bar{A}}}\left[\frac{m_1}{K^2\bar{\bar{A}}}\,\ddot{w}_{,xy} - \nabla^2 w_{,xy}\right], \tag{9–23b}$$

and

$$Q_2 = \bar{\bar{D}}\left[\frac{m_1}{K^2\bar{\bar{A}}}\,\ddot{w}_{,y} - \nabla^2 w_{,y}\right] - \bar{\bar{C}}\Phi_{,x}. \tag{9–23c}$$

For the remaining edges of the plate at $y = \pm\, b/2$ we now classify the following boundary types:

S (simply supported):

We have already shown in Section 9.4.2 that the boundary conditions for a simply–supported edge at $y = +\, b/2$ or $-b/2$ are given by Eqs. (9–9) with $P_z = N_y = 0$.

C (clamped):

When the edge at $y = +b/2$ or $-b/2$ is clamped, we impose

$$w = 0 \tag{9–24a}$$

and

$$\psi_x = \psi_y = 0 \tag{9–24b}$$

These boundary conditions can be expressed in terms of w and Φ by simply substituting Eqs. (9–24) into Eqs. (9–6). This way we get

$$w = 0$$

$$-\bar{\bar{D}}w_{,xyy} + \bar{\bar{C}}\Phi_{,y} = 0$$

$$\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\left[\frac{m_1}{K^2\bar{\bar{A}}}\ddot{w}_{,y} - \nabla^2 w_{,y}\right] - w_{,y} - \frac{\bar{\bar{C}}}{K^2\bar{\bar{A}}}\Phi_{,x} = 0. \tag{9–25}$$

F (free):

At a free edge we impose

$$M_2 = M_6 = Q_2 = 0 \tag{9–26}$$

where M_2, M_6, and Q_2 are given by Eqs. (9–23).

When the remaining edges of the plate at $y = \pm b/2$ are either both clamped (denoted by CC) or both free (denoted by FF), w and ψ_x will be either symmetric or antisymmetric with respect to the x–axis. For symmetric modes and antisymmetric modes we let

$$A = C = F = 0 \tag{9–27a}$$

and

$$B = D = E = 0 \tag{9–27b}$$

in Eqs. (9–22), respectively. By imposing the boundary conditions (9–25) for the CC case or (9–26) for the FF case at (say) $y = b/2$ we obtain a system of three homogeneous algebraic equations. These equations will be in terms of B, D, and E for symmetric modes and in terms of A, C, and F for antisymmetric modes. The frequency equations are readily obtained by setting the determinants of the coefficient matrices in these equations equal to zero:

CC

1) Symmetric Mode

$$\beta_m\left[1-\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}(\alpha_m^2-\lambda_m^2)\right]\tan\beta_m b/2$$

$$+\ \alpha_m\left[1+\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}(\beta_m^2+\lambda_m^2)\right]\tanh\alpha_m b/2$$

$$-\ \delta_A\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\frac{\lambda_m^2}{\gamma_m}(\alpha_m^2+\beta_m^2)\tanh\gamma_m b/2=0 \tag{9–28a}$$

2) Antisymmetric Mode

$$\beta_m\left[1-\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}(\alpha_m^2-\lambda_m^2)\right]\tanh\alpha_m b/2$$

$$-\ \alpha_m\left[1+\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}(\beta_m^2+\lambda_m^2\right]\tan\beta_m b/2$$

$$+\ \delta_A\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\frac{\lambda_m^2}{\gamma_m}(\alpha_m^2+\beta_m^2)\tanh\alpha_m b/2\cdot\tan\beta_m b/2\cdot\coth\gamma_m b/2=0 \tag{9–28b}$$

FF

1) Symmetric Mode

$$\beta_m\left[\alpha_m^2 + \frac{2\bar{\bar{C}} - \bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2 + \left[1 + \frac{2\bar{\bar{C}}}{K^2\bar{\bar{A}}}\lambda_m^2\right]\left[\beta_m^2 + \lambda_m^2\right]\right]\tan\beta_m b/2$$

$$-\,\alpha_m\left[\beta_m^2 - \frac{2\bar{\bar{C}} - \bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2 - \left[1 + \frac{2\bar{\bar{C}}}{K^2\bar{\bar{A}}}\lambda_m^2\right]\left[\alpha_m^2 - \lambda_m^2\right]\right]\tanh\alpha_m b/2$$

$$-\,\delta_A 4\,\frac{\bar{\bar{C}}^2}{\bar{\bar{D}}}\,\frac{\lambda_m^2}{K^2\bar{\bar{A}}}\,\alpha_m\beta_m\gamma_m(\alpha_m^2 + \beta_m^2)\cdot\tanh\alpha_m b/2\cdot\tan\beta_m b/2\cdot\coth\gamma_m b/2 = 0$$

(9–29a)

2) Antisymmetric Mode

$$\alpha_m\left[\beta_m^2 - \frac{2\bar{\bar{C}} - \bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2 - \left[1 + \frac{2\bar{\bar{C}}}{K^2\bar{\bar{A}}}\lambda_m^2\right]\left[\alpha_m^2 - \lambda_m^2\right]\right]\tan\beta_m b/2$$

$$+\,\beta_m\left[\alpha_m^2 + \frac{2\bar{\bar{C}} - \bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2 + \left[1 + \frac{2\bar{\bar{C}}}{K^2\bar{\bar{A}}}\lambda_m^2\right]\left[\beta_m^2 + \lambda_n^2\right]\right]\tanh\alpha_m b/2$$

$$-\,\delta_A 4\,\frac{\bar{\bar{C}}^2}{\bar{\bar{D}}}\,\frac{\lambda_m^2}{K^2\bar{\bar{A}}}\,\alpha_m\beta_m\gamma_m(\alpha_m^2 + \beta_m^2)\tanh\gamma_m b/2 = 0 \qquad (9\text{–}29b)$$

Following our discussions in Chapter VIII, the frequency equations for the SC case (simply supported at, say, y = 0 and clamped at y = b) and the SF case (simply supported at y = 0 and free y = b) are easily obtained from Eqs. (9–28b) and (9–29b), respectively:

SC

$$\beta_m\left[1 - \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}(\alpha_m^2 - \lambda_m^2)\right]\tanh \alpha_m b$$

$$- \alpha_m\left[1 + \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}(\beta_m^2 + \lambda_m^2)\right]\tan \beta_m b$$

$$+ \delta_A \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\frac{\lambda_m^2}{\gamma_m}(\alpha_m^2 + \beta_m^2)\tanh \alpha_m b\cdot\tan \beta_m b\cdot\coth \gamma_m b = 0 \qquad (9\text{–}30)$$

SF

$$\alpha_m\left[\beta_m^2 - \frac{2\bar{\bar{C}} - \bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2 - \left[1 + \frac{2\bar{\bar{C}}}{K^2\bar{\bar{A}}}\lambda_m^2\right](\alpha_m^2 - \lambda_m^2)\right]\tan \beta_m b$$

$$+ \beta_m\left[\alpha_m^2 + \frac{2\bar{\bar{C}} - \bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2 + \left[1 + \frac{2\bar{\bar{C}}}{K^2\bar{\bar{A}}}\lambda_m^2\right](\beta_m^2 + \lambda_m^2)\right]\tanh \alpha_m b$$

$$- \delta_A 4 \frac{\bar{\bar{C}}^2}{\bar{\bar{D}}} \frac{\lambda_m^2}{K^2 \bar{\bar{A}}} \alpha_m \beta_m \gamma_m (\alpha_m^2 + \beta_m^2) \tanh \gamma_m b = 0 \qquad (9\text{–}31)$$

In obtaining Eqs. (9–28) through (9–31) we occasionally used

$$\frac{m_1}{K^2 \bar{\bar{A}}} \omega_m^2 = \beta_m^2 - \alpha_m^2 + 2\lambda_m^2 \qquad (9\text{–}32)$$

which is readily obtained from Eqs. (9–19a) and (9–20), to simplify the final results. As far as the CF case is concerned, similar arguments can be made as in Chapter VIII. In Eqs. (9–28) through (9–31) we have added the tracer $\delta_A (\equiv 1)$ in front of the terms which are the direct contribution of the boundary layer (edge–zone) equation. The effect of the boundary layer on the natural frequencies and also the critical buckling loads of a plate with various boundary conditions can be studied by simply setting $\delta_A = 0$.

The general solutions of interior and edge–zone equations for CASE II are obtained by replacing β_m in Eqs. (9–22) by $i\bar{\beta}_n$ ($i = \sqrt{-1}$). Also the frequency equations for CASE II are obtained by replacing β_m in Eqs. (9–28) through (9–31) by $i\bar{\beta}_m$:

CC

1) Symmetric Mode

$$-\bar{\beta}_m\left[1-\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}(\alpha_m^2-\lambda_m^2)\right]\tanh\bar{\beta}_m b/2$$

$$+\alpha_m\left[1-\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}(\bar{\beta}_m^2-\lambda_m^2)\right]\tanh\alpha_m b/2$$

$$-\delta_A\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\frac{\lambda_m^2}{\gamma_m}(\alpha_m^2-\bar{\beta}_m^2)\tanh\gamma_m b/2=0 \tag{9–33a}$$

2) Antisymmetric Mode

$$-\bar{\beta}_m\left[1-\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}(\alpha_m^2-\lambda_m^2)\right]\tanh\alpha_m b/2$$

$$+\alpha_m\left[1-\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}(\bar{\beta}_m^2-\lambda_m^2)\right]\tanh\bar{\beta}_m b/2$$

$$-\delta_A\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\frac{\lambda_m^2}{\gamma_m}(\alpha_m^2-\bar{\beta}_m^2)\tanh\alpha_m b/2\cdot\tanh\bar{\beta}_m b/2\cdot\coth\gamma_m b/2=0 \tag{9–33b}$$

FF

1) Symmetric Mode

$$-\bar{\beta}_m\left[\alpha_m^2 + \frac{2\bar{\bar{C}} - \bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2 - \left[1 + \frac{2\bar{\bar{C}}}{K^2\bar{\bar{A}}}\lambda_m^2\right](\bar{\beta}_m^2 - \lambda_m^2)\right]\tanh \bar{\beta}_m b/2$$

$$+ \alpha_m\left[\bar{\beta}_m^2 + \frac{2\bar{\bar{C}} - \bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2 - \left[1 + \frac{2\bar{\bar{C}}}{K^2\bar{\bar{A}}}\lambda_m^2\right](\alpha_m^2 - \lambda_m^2)\right]\tanh \alpha_m b/2$$

$$+ \delta_A 4\frac{\bar{\bar{C}}^2}{\bar{\bar{D}}}\frac{\lambda_m^2}{K^2\bar{\bar{A}}}\alpha_m\bar{\beta}_m\gamma_m(\alpha_m^2 - \bar{\beta}_m^2)\tanh \alpha_m b/2\cdot\tanh \bar{\beta}_m b/2\cdot\coth \gamma_m b/2 = 0$$

(9–34a)

2) Antisymmetric Mode

$$\alpha_m\left[\bar{\beta}_m^2 + \frac{2\bar{\bar{C}} - \bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2 - \left[1 + \frac{2\bar{\bar{C}}}{K^2\bar{\bar{A}}}\lambda_m^2\right](\alpha_m^2 - \lambda_m^2)\right]\tanh \bar{\beta}_m b/2$$

$$-\bar{\beta}_m\left[\alpha_m^2 + \frac{2\bar{\bar{C}} - \bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2 - \left[1 + \frac{2\bar{\bar{C}}}{K^2\bar{\bar{A}}}\lambda_m^2\right](\bar{\beta}_m^2 - \lambda_m^2)\right]\tanh \alpha_m b/2$$

$$+ \delta_A 4\frac{\bar{\bar{C}}^2}{\bar{\bar{D}}}\frac{\lambda_m^2}{K^2\bar{\bar{A}}}\alpha_m\bar{\beta}_m\gamma_m(\alpha_m^2 - \bar{\beta}_m^2)\tanh \gamma_m b/2 = 0 \qquad (9\text{–}34b)$$

SC

$$-\bar{\beta}_m\left[1-\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}(\alpha_m^2-\lambda_m^2)\right]\tanh\alpha_m b$$

$$+\alpha_m\left[1-\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}(\bar{\beta}_m^2-\lambda_m^2)\right]\tanh\bar{\beta}_m b$$

$$-\delta_A\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\frac{\lambda_m^2}{\gamma_m}(\alpha_m^2-\bar{\beta}_m^2)\tanh\alpha_m b\cdot\tanh\bar{\beta}_m b\cdot\coth\gamma_m b=0 \qquad (9\text{–}35)$$

SF

$$\alpha_m\left[\bar{\beta}_m^2+\frac{2\bar{\bar{C}}-\bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2-\left[1+\frac{2\bar{\bar{C}}}{K^2\bar{\bar{A}}}\lambda_m^2\right](\alpha_m^2-\lambda_m^2)\right]\tanh\bar{\beta}_m b$$

$$-\bar{\beta}_m\left[\alpha_m^2+\frac{2\bar{\bar{C}}-\bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2-\left[1+\frac{2\bar{\bar{C}}}{K^2\bar{\bar{A}}}\lambda_m^2\right](\bar{\beta}_m^2-\lambda_m^2)\right]\tanh\alpha_m b$$

$$+\delta_A 4\frac{\bar{\bar{C}}^2}{\bar{\bar{D}}}\frac{\lambda_m^2}{K^2\bar{\bar{A}}}\alpha_m\bar{\beta}_m\gamma_m(\alpha_m^2-\bar{\beta}_m^2)\tanh\gamma_m b=0 \qquad (9\text{–}36)$$

When the rotatory inertia forces are not neglected, the full system of equations (i.e., Eqs. (9–1) or eqs. (9–2)) can alternatively be solved by using the procedure developed in Section 8.6. This is in fact done in Section 9.7 for numerical comparisons.

9.5.1 Stability Analysis

The stability equations in FSDPT are obtained from Eqs. (9–2) by dropping the inertia forces and assuming $P_z = 0$:

$$-\bar{\bar{D}}\nabla^2\nabla^2 w - \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\nabla^2 q_z + q_z = 0 \tag{9–37a}$$

$$\bar{\bar{C}}\nabla^2\Phi - K^2\bar{\bar{A}}\Phi = 0 \tag{9–37b}$$

where

$$q_z = -\bar{N}_x w_{,xx} - \bar{N}_y w_{,yy}. \tag{9–37c}$$

It is to be noted that we have replaced (N_x, N_y) by $(-\bar{N}_x, -\bar{N}_y)$ where $\bar{N}_x$ and $\bar{N}_y$ are the axial compressive edge loads acting in the x– and y–directions, respectively.

9.5.2 Simply–Supported Plate

The boundary conditions for a simply–supported plate are given by Eqs. (9–8) and (9–9). With $T_m = 1$, substitution of Eq. (9–10a) into Eq. (9–37a) yields

$$(\bar{N}_x\alpha_m^2 + \bar{N}_y\beta_n^2) = \frac{\bar{\bar{D}}(\alpha_m^2 + \beta_n^2)^2}{\dfrac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}(\alpha_m^2 + \beta_n^2) + 1} \tag{9–38}$$

where $\alpha_m = \frac{m\pi}{a}$ and $\beta_n = \frac{n\pi}{b}$ as in Section 9.4.2. Also with $T_m = 1$, substitution of Eq. (9–10b) into Eq. (9–37b) results in

$$\Phi = 0. \tag{9–39}$$

That is, the solution contribution of the edge–zone equation is identically zero as we mentioned in Chapter VII.

9.5.3 Lévy–Type Stability Analysis

In the stability analysis Eqs. (9–5c) and (9–6) reduce to

$$(\psi_{x,x} + \psi_{y,y}) = -\nabla^2 w - \frac{q_z}{K^2 \bar{A}} \tag{9–40a}$$

and

$$\psi_x = \left[-w_{,x} - \frac{\bar{\bar{D}}}{K^2 \bar{A}} \nabla^2 w_{,x} + \frac{\bar{\bar{C}}}{K^2 \bar{A}} \Phi_{,y} - \frac{\bar{\bar{D}}}{(K^2 \bar{A})^2} q_{z,x}\right] \tag{9–40b}$$

$$\psi_y = \left[-w_{,y} - \frac{\bar{\bar{D}}}{K^2 \bar{A}} \nabla^2 w_{,y} - \frac{\bar{\bar{C}}}{K^2 \bar{A}} \Phi_{,x} - \frac{\bar{\bar{D}}}{(K^2 \bar{A})^2} q_{z,y}\right] \tag{9–40c}$$

respectively. Substitution of Eqs. (9–40) into Eqs. (9–4) results in the stress resultants

$$M_2 = -\bar{\bar{D}}\nabla^2 w - \frac{\bar{\bar{D}}}{K^2 \bar{A}} q_z + 2\bar{\bar{C}} w_{,xx} - 2\frac{\bar{\bar{C}}^2}{K^2 \bar{A}} \Phi_{,xy}$$

$$+ 2\frac{\bar{\bar{C}}\bar{\bar{D}}}{K^2\bar{\bar{A}}}\left[\frac{1}{K^2\bar{\bar{A}}}q_{z,xx} + \nabla^2 w_{,xx}\right]$$

$$M_6 = -2\bar{\bar{C}}w_{,xy} + \bar{\bar{C}}\Phi - 2\frac{\bar{\bar{C}}^2}{K^2\bar{\bar{A}}}\Phi_{,xx}$$

$$- 2\frac{\bar{\bar{C}}\bar{\bar{D}}}{K^2\bar{\bar{A}}}\left[\frac{1}{K^2\bar{\bar{A}}}q_{z,xy} + \nabla^2 w_{,xy}\right]$$

$$Q_2 = -\bar{\bar{D}}\left[\frac{1}{K^2\bar{\bar{A}}}q_{z,y} + \nabla^2 w_{,y}\right] - \bar{\bar{C}}\Phi_{,x} \tag{9–41}$$

which are the counterparts of Eqs. (9–23). Similar to the frequency analysis we assume that the edges of the plate at $x = 0$ and a are invariably simply supported. We have already mentioned that boundary conditions for a simply–supported edge at $y = b/2$ or $-b/2$ are given by Eqs. (9–9). By a similar procedure to that presented in Section 9.4.4 we can show that for a clamped edge at any of the remaining edges we must impose

$$w = 0$$

$$-\bar{\bar{D}}w_{,xyy} + \bar{\bar{C}}\Phi_{,y} + \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\bar{N}_y w_{,xyy} = 0$$

$$-\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\left[\frac{1}{K^2\bar{\bar{A}}}q_{z,y}+\nabla^2 w_{,y}\right]-w_{,y}-\frac{\bar{\bar{C}}}{K^2\bar{\bar{A}}}\Phi_{,x}=0 \tag{9–42}$$

which are the counterparts of Eqs. (9–25). Also at a free edge we impose

$$M_2 = M_6 = Q_2 - \bar{N}_y w_{,y} = 0 \tag{9–43}$$

where M_2, M_6, and Q_2 are given by Eqs. (9–41).

Lévy–type solutions of Eqs. (9–37) can be developed by assuming $T_m = 1$ in Eqs. (9–15). Alternatively, the general solutions of Eqs. (9–37) can be obtained by letting $\omega_m^2 \to 0$ in Eqs. (9–16a) through (9–17c). This way we will obtain Eqs. (9–18) with $T_m = 1$. However, we simplify the analysis by assuming $\bar{N}_y = 0$ and obtain

$$P_1^2 \equiv \alpha_m^2 = \left\{\lambda_m^2 - \frac{1}{2}\frac{\bar{N}_x}{K^2\bar{\bar{A}}}\lambda_m^2 + \left[\frac{\bar{N}_x}{\bar{\bar{D}}}\lambda_m^2 + \left[\frac{\bar{N}_x\lambda_m^2}{2K^2\bar{\bar{A}}}\right]^2\right]^{1/2}\right\} \tag{9–44a}$$

$$P_3^2 \equiv \gamma_m^2 = \left[\lambda_m^2 + \frac{K^2\bar{\bar{A}}}{\bar{\bar{C}}}\right] \tag{9–44b}$$

which are positive numbers for all values of $\bar{N}_x$. Also for P_3^2 we can have the following two cases:

CASE I;

If

$$\left\{\frac{1}{2}\frac{\bar{N}_x}{K^2\bar{A}}\lambda_m^2 + \left[\frac{\bar{N}_x}{\bar{\bar{D}}}\lambda_m^2 + \left[\frac{\bar{N}_x\lambda_m^2}{2K^2\bar{A}}\right]^2\right]^{1/2}\right\} > \lambda_m^2,$$

then

$$P_2^2 \equiv -\beta_m^2 = -\left\{-\lambda_m^2 + \frac{1}{2}\frac{\bar{N}_x}{K^2\bar{A}}\lambda_m^2 + \left[\frac{\bar{N}_x}{\bar{\bar{D}}}\lambda_m^2 + \left[\frac{\bar{N}_x\lambda_m^2}{2K^2\bar{A}}\right]^2\right]^{1/2}\right\}$$

(9–45)

CASE II;

If

$$\left\{\frac{1}{2}\frac{\bar{N}_x}{K^2\bar{A}}\lambda_m^2 + \left[\frac{\bar{N}_x}{\bar{\bar{D}}}\lambda_m^2 + \left[\frac{\bar{N}_x\lambda_m^2}{2K^2\bar{A}}\right]^2\right]^{1/2}\right\} < \lambda_m^2$$

then

$$P_2^2 \equiv \bar{\beta}_m^2 = \left\{\lambda_m^2 - \frac{1}{2}\frac{\bar{N}_x}{K^2\bar{A}}\lambda_m^2 - \left[\frac{\bar{N}_x}{\bar{\bar{D}}}\lambda_m^2 + \left[\frac{\bar{N}_x\lambda_m^2}{2K^2\bar{A}}\right]^2\right]^{1/2}\right\} \qquad (9\text{–}46)$$

This way, β_m^2 and $\bar{\beta}_m^2$ will always be positive for all values of $\bar{N}_x$. The general solutions of Eqs. (9–37) for CASE I will be identical to those given by Eqs. (9–22) if we set $T_m = 1$ and use α_m^2 and β_m^2 given by Eqs. (9–44a) and (9–45). Similarly, to obtain the solutions for CASE II, we use α_m^2 and $\bar{\beta}_m^2$ given here by Eqs. (9–44a) and (9–46) in Eqs. (9–22). The buckling equations for CASE I and CASE II are also identical to those given by Eqs. (9–28) through (9–31) and Eqs. (9–33) through (9–36) with α_m^2, β_m^2, and $\bar{\beta}_m^2$ given by Eqs. (9–44a), (9–45), and (9–46).

9.6.1 The Classical Plate Theory (CLPT)

The governing equation of a plate according to the classical plate theory (CLPT) can be obtained by letting $K^2\bar{\bar{A}} \to \infty$, $\psi_x = -\frac{\partial w}{\partial x}$, and $\psi_y = -\frac{\partial w}{\partial y}$ in Eqs. (9–2a). It is to be noted that this way from Eq. (9–3) we obtain $\Phi = 0$ in CLPT.

The frequency and buckling equations of the plate according to the classical plate theory can also be obtained from the results of FSDPT by letting $K^2\bar{\bar{A}} \to \infty$. For a completely simply–supported plate, the frequency equation and buckling equation are obtained from Eqs. (9–11) and (9–38), respectively:

$$\omega_{mn}^2 = \frac{\bar{\bar{D}}(\alpha_m^2 + \beta_n^2)^2 + (N_x\alpha_m^2 + N_y\beta_n^2)}{m_1 + \bar{\bar{m}}_3(\alpha_m^2 + \beta_n^2)} \tag{9–47a}$$

$$(\bar{N}_x\alpha_m^2 + \bar{N}_y\beta_n^2) = \bar{\bar{D}}(\alpha_m^2 + \beta_n^2)^2 \tag{9–47b}$$

with $\alpha_m = \frac{m\pi}{a}$ and $\beta_n = \frac{n\pi}{b}$. It is to be noted that in Eq. (9–47a) the rotatory inertia terms are not neglected.

9.6.2 Lévy–Type Frequency Equations

The Lévy–type frequency equations for CASE I according to CLPT are obtained by letting $K^2\bar{\bar{A}} \to \infty$ in Eqs. (9–28) through (9–31):

CC

1) Symmetric Mode

$$\beta_m \tan \beta_m \frac{b}{2} + \alpha_m \tanh \alpha_m \frac{b}{2} = 0 \tag{9–48a}$$

2) Antisymmetric Mode

$$\beta_m \tanh \alpha_m \frac{b}{2} - \alpha_m \tan \beta_m \frac{b}{2} = 0 \tag{9–48b}$$

FF

1) Symmetric Mode

$$\beta_m\left[\alpha_m^2 + \frac{2\bar{\bar{C}} - \bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2 + \beta_m^2 + \lambda_m^2\right]\tan \beta_m \frac{b}{2}$$

$$- \alpha_m\left[\beta_m^2 - \frac{2\bar{\bar{C}} - \bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2 - \alpha_m^2 + \lambda_m^2\right]\tanh \alpha_m \frac{b}{2} = 0$$

(9–49a)

2) Antisymmetric Mode

$$\alpha_m\left[\beta_m^2-\frac{2\bar{\bar{C}}-\bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2-\alpha_m^2+\lambda_m^2\right]\tan\beta_m\frac{b}{2}$$

$$+\beta_m\left[\alpha_m^2+\frac{2\bar{\bar{C}}-\bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2+\beta_m^2+\lambda_m^2\right]\tanh\alpha_m\frac{b}{2}=0$$

(9–49b)

<u>SC</u>

$$\beta_m\tanh\alpha_m b-\alpha_m\tan\beta_m b=0 \tag{9–50}$$

<u>SF</u>

$$\alpha_m\left[\beta_m^2-\frac{2\bar{\bar{C}}-\bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2-\alpha_m^2+\lambda_m^2\right]\tan\beta_m b$$

$$+\beta_m\left[\alpha_m^2+\frac{2\bar{\bar{C}}-\bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}}\lambda_m^2+\beta_m^2+\lambda_m^2\right]\tanh\alpha_m b=0$$

(9–51)

where

$$\alpha_m^2=\left[\frac{m_1}{\bar{\bar{D}}}\omega_m^2\right]^{1/2}+\lambda_m^2 \tag{9–52a}$$

and

$$\beta_m^2=\left[\frac{m_1}{\bar{\bar{D}}}\omega_m^2\right]^{1/2}-\lambda_m^2 \tag{9–52b}$$

which are obtained form Eqs. (9–19a) and (9–20). Similarly, the frequency equations for CASE II are obtained from Eqs. (9–33) through (9–36):

CC

1) Symmetric Mode

$$\bar{\beta}_m \tanh \bar{\beta}_m \frac{b}{2} - \alpha_m \tanh \alpha_m \frac{b}{2} = 0 \tag{9–53a}$$

2) Antisymmetric Mode

$$\bar{\beta}_m \tanh \alpha_m \frac{b}{2} - \alpha_m \tanh \bar{\beta}_m \frac{b}{2} = 0 \tag{9–53b}$$

FF

1) Symmetric Mode

$$\bar{\beta}_m \left[\alpha_m^2 + \frac{2\bar{\bar{C}} - \bar{\bar{D}}}{\bar{\bar{D}}} \lambda_m^2\right] \left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}} \lambda_m^2 - \bar{\beta}_m^2 + \lambda_m^2\right] \tanh \bar{\beta}_m b/2$$

$$- \alpha_m \left[\bar{\beta}_m^2 + \frac{2\bar{\bar{C}} - \bar{\bar{D}}}{\bar{\bar{D}}} \lambda_m^2\right] \left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}} \lambda_m^2 - \alpha_m^2 + \lambda_m^2\right] \tanh \alpha_m b/2 = 0$$

(9–54a)

2) Antisymmetric Mode

$$\alpha_m \left[\bar{\beta}_m^2 + \frac{2\bar{\bar{C}} - \bar{\bar{D}}}{\bar{\bar{D}}} \lambda_m^2\right] \left[\frac{2\bar{\bar{C}}}{\bar{\bar{D}}} \lambda_m^2 - \alpha_m^2 + \lambda_m^2\right] \tanh \bar{\beta}_m b/2$$

$$-\bar{\beta}_m\left[\alpha_m^2+\frac{2\bar{C}-\bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{C}}{\bar{\bar{D}}}\lambda_m^2-\bar{\beta}_m^2+\lambda_m^2\right]\tanh\alpha_m b/2=0$$

(9–54b)

SC

$$\bar{\beta}_m \tanh \alpha_m b - \alpha_m \tanh \bar{\beta}_m b = 0 \tag{9–55}$$

SF

$$\alpha_m\left[\bar{\beta}_m^2+\frac{2\bar{C}-\bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{C}}{\bar{\bar{D}}}\lambda_m^2-\alpha_m^2+\lambda_m^2\right]\tanh\bar{\beta}_m b$$

$$-\bar{\beta}_m\left[\alpha_m^2+\frac{2\bar{C}-\bar{\bar{D}}}{\bar{\bar{D}}}\lambda_m^2\right]\left[\frac{2\bar{C}}{\bar{\bar{D}}}\lambda_m^2-\bar{\beta}_m^2+\lambda_m^2\right]\tanh\alpha_m b=0$$

(9–56)

where

$$\alpha_m^2 = \lambda_m^2 + \left[\frac{m_1}{\bar{\bar{D}}}\omega_m^2\right]^{1/2} \tag{9–57a}$$

and

$$\bar{\beta}_m^2 = \lambda_m^2 - \left[\frac{m_1}{\bar{\bar{D}}}\omega_m^2\right]^{1/2} \tag{9–57b}$$

which are obtained from Eqs. (9–19a) and (9–21). Throughout this chapter we have assumed that the symmetric plate is laminated of transversely isotropic layers (with the x–y plane being the plane of isotropy). On the other hand, we know that transverse shear deformation is neglected in the classical plate theory (CLPT).

Hence the above results (i.e., Eqs. (9–47)–(9–57)) will remain the same for symmetric laminated plates with isotropic layers. Further, for a homogeneous isotropic plate we have (see Appendix 7.1 in Chapter VII)

$$\bar{\bar{C}} = \frac{1}{12}\,Gh^3 \text{ and } \bar{\bar{D}} = \frac{Eh^3}{12(1 - \nu^2)} \tag{9–58}$$

where h is the total thickness of the plate and $G(= \frac{E}{2(1 + \nu)})$ and E are the shear and Young's moduli with ν being the Poisson ratio. Also the mass terms are given by

$$m_1 = \rho h \text{ and } \bar{\bar{m}}_3 = \frac{1}{12}\,\rho h^3 \tag{9–59}$$

where ρ is the mass density. With the help of Eqs. (9–58) and (9–59), it can readily be shown that the frequency equations of CLPT reduce to the well–known frequency equations of a homogeneous isotropic plate (e.g., see [138]).

9.6.3 Lévy–Type Buckling Equations

The Lévy–type buckling equations according to CLPT can be obtained from Eqs. (9–48) through (9–51) for CASE I by merely replacing α_m and β_m appearing in these equations by

$$\alpha_m = \left[\left[\frac{\bar{N}_x}{\bar{\bar{D}}}\,\lambda_m^2\right]^{1/2} + \lambda_m^2\right]^{1/2} \tag{9–60a}$$

and

$$\beta_m = \left[\left[\frac{\bar{N}_x}{\bar{\bar{D}}}\lambda_m^2\right]^{1/2} - \lambda_m^2\right]^{1/2} \tag{9–60b}$$

which are obtained from Eqs. (9–44a) and (9–45) by letting $K^2\bar{\bar{A}} \to \infty$. Also for CASE II we replace α_m and $\bar{\beta}_m$ appearing in Eqs. (9–53)–(9–57) by

$$\alpha_m = \left[\lambda_m^2 + \left[\frac{\bar{N}_x}{\bar{\bar{D}}}\lambda_m^2\right]^{1/2}\right]^{1/2}, \tag{9–61a}$$

and

$$\bar{\beta}_m = \left[\lambda_m^2 - \left[\frac{\bar{N}_x}{\bar{\bar{D}}}\lambda_m^2\right]^{1/2}\right]^{1/2} \tag{9–61b}$$

which are obtained from Eqs. (9–44a) and (9–46) by letting $K^2\bar{\bar{A}} \to \infty$. It is expected that, except for the FF case, the critical buckling load of a plate will be predicted from the buckling equations corresponding to CASE I. This conclusion is merely based on the well–known results of the classical plate theory of homogeneous isotropic plates (e.g., see [139]).

9.7 Numerical Results and Discussions

To study the effects of shear deformation, rotatory inertia forces, and boundary layer equation, the natural frequencies and the critical buckling loads of a single–layer transversely isotropic plate (referred to as Structure I) and a three–layer transversely isotropic plate (Structure II), with total thickness h (= 1 in.), are determined according to Reddy's TSDPT, Levinson's TSDPT, FSDPT, and

CLPT. The material properties of Structure I are assumed to be

$$E = 19 \cdot 2 \times 10^6 \text{ psi}, \ \nu = 0.24$$

$$G_z = 0.82 \times 10^6 \text{ psi}, \ \rho = 0.00013 \text{ lb sec}^2/\text{in}^4.$$

The material properties of the outside layers (whose total thickness is equal to that of the middle layer) of Structure II are assumed to be

$$E = 20.83 \times 10^6 \text{ psi}, \ \nu = 0.44$$

$$G_z = 3.71 \times 10^6 \text{ psi}, \ \rho = 0.00013 \text{ lb sec}^2/\text{in}^4.$$

The material properties of the middle layer of Structure II are assumed to be the same as those of Structure I.

It is to be noted that we have not developed the frequency and buckling equations of the plate according to Reddy's TSDPT. The frequencies and buckling loads according to Reddy's TSDPT (see Chapters VI and VII) are, however, determined by using a procedure similar to the one developed in Section 8.6. As we mentioned in Chapter VIII, compared to Levinson's TSDPT, Reddy's TSDPT is believed to be more consistent and accurate since it is derived by using the principle of virtual displacements.

For a homogeneous transversely isotropic plate we have (see Appendix 7.1 of Chapter VII)

$$\bar{A} = \frac{2}{3} G_z h,\ \bar{\bar{A}} = G_z h,\ \bar{C} = \frac{1}{15} Gh^3,\ \bar{\bar{C}} = \frac{1}{12} Gh^3$$

$$\bar{D} = \frac{1}{15} \frac{Eh^3}{1 - \nu^2},\ \bar{\bar{D}} = \frac{1}{12} \frac{Eh^3}{1 - \nu^2},\ \bar{F} = \frac{1}{60} \frac{Eh^3}{1 - \nu^2}$$

$$m_1 = \rho h,\ \bar{m}_3 = \frac{1}{15} \rho h^3,\ \bar{\bar{m}}_3 = \frac{1}{12} \rho h^3,\ \bar{m}_5 = \frac{1}{60} \rho h^3. \tag{9–62}$$

For a homogeneous isotropic plate we simply replace G_z by G in Eq. (9–62). In Chapter VIII we have considered two different clamped boundary conditions in Levinson's TSDPT. These conditions are defined in Eqs. (8–18) and (8–80). For convenience we will refer to the boundary conditions in Eq. (8–18) as C1–type conditions and to those in Eq. (8–80) as C2–type conditions. It is readily seen, with the help of Eq. (9–62), that the frequency and buckling equations according to Levinson's TSDPT and FSDPT are identical for a homogeneous transversely isotropic (and isotropic) plate with various boundary conditions if C1–type conditions are imposed in Levinson's theory and $K^2 = \frac{5}{6}$ is used in FSDPT. However, when C2–type conditions are imposed, this conclusion holds only for the SF, SS, and FF cases. It should also be reminded that, in the free–vibration analysis, these conclusions are valid only when the rotatory inertia forces are neglected. The pure–shear frequencies of a completely simply–supported homogeneous transversely isotropic (and isotropic) plate according to Levinson's TSDPT and FSDPT are also seen to be identical (see Eqs. (8–26) and (9–14)) when $K^2 = 5/6$ is used in FSDPT.

The nondimensional fundamental frequencies of Structures I and II according to various theories are displayed in Table 9.1. The results indicate that for the single–layer plate (Structure I) Levinson's TSDPT with C1–type conditions for a clamped edge and FSDPT with $K^2 = 5/6$ yield accurate results as compared with Reddy's TSDPT. On the other hand, for the three–layer plate (Structure II) Levinson's TSDPT with C2–type conditions and FSDPT with $K^2 = 2/3$ yield more accurate results. It is to be noted that the rotatory inertia forces are not neglected in Table 9.1. The effects of rotatory inertia forces on the fundamental frequencies of Structure II can be studied with the help of the numerical results in Table 9.2. It is seen that by ignoring the rotatory inertia forces the frequencies are only slightly increased. Also the results of the classical plate theory (CLPT) can be completely erroneous by overpredicting the frequencies. In Table 9.2 the shear correction factor K^2 is assumed to be 2/3 in FSDPT and C2–type conditions are used for a clamped edge in Levinson's TSDPT.

As we have mentioned earlier (and also in Chapter VIII) the influence of the edge–zone (boundary layer) equation on the natural frequency and the critical buckling load can be studied by setting the tracer δ_A equal to zero in the frequency and buckling equations. The numerical results in Table 9.3 indicate that, when δ_A is set to zero, the fundamental frequencies of Structures I and II are slightly decreased in the SC and CC cases and decreased in the SF and FF cases. In Fig. 9.1 we have plotted the dimensionless fundamental frequency of Structure I vs. a/h (= b/h) ratio for the CC case. It is observed that the boundary layer contribution diminishes as a/h increases. Also by varying the value of G_z of Structure I, it is seen from Fig. 9.2 that the boundary layer contribution is slightly increased as

E/G_z ratio is increased (that is, as G_z is decreased). Indeed the error introduced, by ignoring the boundary layer contribution, is less than 2.2% for $E/G_z \leq 50$.

The critical buckling loads of Structures I and II according to various theories are displayed in Table 9.4. It is again observed that C2–type conditions yield more accurate results in Levinson's theory as far as Structure II is concerned. The numerical results in Table 9.5 indicate that by ignoring the boundary layer contribution the critical buckling loads are slightly decreased in the SC and CC cases and increased in the FF and SF cases. The edge–zone equation is seen to be most influential in the SF case. In Fig. 9.3 we have plotted the dimensionless critical buckling load of Structure II vs. the E/G_z ratio for the SF case by varying the transverse shear modulus G_z of the middle layer of Structure II. Here E also is the Young modulus of the middle layer. It is seen that the error introduced by ignoring the boundary layer effect is less than 4.7% for $E/G_z \leq 60$. It should be pointed out that the results concerning the boundary layer effects are qualitatively similar for Structures I and II.

9.8 Conclusions

The frequency and buckling equations of symmetric plates laminated of transversely isotropic layers according to Levinson's third–order shear deformation plate theory (TSDPT) are developed in Chapter VIII. Similar equations are developed according to the first–order shear deformation plate theory (FSDPT) and the classical plate theory (CLPT) in the present chapter. For the sake of completeness, numerical results according to these theories are compared with the ones obtained within the framework of Reddy's TSDPT. The governing equations of Reddy's TSDPT were solved by following the procedure developed in Section 8.6.

The roots of the auxiliary equation in Reddy's TSDPT are obtained with the help of a computer. Since the roots of the auxiliary equation could not be explicitly obtained, the frequency and buckling equations according to Reddy's TSDPT are not obtained. This is primarily due to the fact that the auxiliary equation in Reddy's TSDPT is an eighth–order equation. It will, however, be shown in the next chapter that the roots of this equation can be explicitly obtained in bending problems.

The numerical results indicate that, compared to FSDPT, Levinson's TSDPT is a more reliable theory. It should, however, be pointed out that the choice of a particular set of boundary conditions for a clamped edge may be very significant in Levinson's TSDPT.

It is demonstrated, for a moderately thick plate, that the boundary layer effect only slightly alters the numerical values of the natural frequencies and the critical buckling loads. The boundary layer effect is seen to diminish for thin plates and also when the transverse shear modulus G_z is not very small compared to Young's modulus E.

Table 9.1

The effects of different clamped–type boundary conditions in Levinson's TSDPT and shear correction factors in FSDPT on the dimensionless fundamental frequencies of Structures I and II; $\bar{\omega} = \omega a^2 \left[\frac{\rho h}{\bar{\bar{D}}}\right]^{1/2}$, a = b = 10h., and rotatory inertia forces are not neglected.

Structure	Theory		SC	CC	CF
I	Reddy's TSDPT		17.680	19.430	10.810
	Levinson's TSDPT	C1–type	17.558	19.124	10.824
		C2–type	17.296	18.529	10.761
	FSDPT	$K^2 = 5/6$	17.540	19.107	10.818
		$K^2 = 2/3$	16.688	17.980	10.484
II	Reddy's TSDPT		18.837	21.007	10.619
	Levinson's TSDPT	C1–type	19.343	21.631	10.842
		C2–type	19.101	21.047	10.798
	FSDPT	$K^2 = 5/6$	20.073	22.862	11.064
		$K^2 = 2/3$	19.457	21.818	10.880

Table 9.2

The effects of rotatory inertia forces on the dimensionless fundamental frequency of Structure II; $\bar{\omega} = \omega a^2 \left[\frac{\rho h}{\bar{\bar{D}}}\right]^{1/2}$, a = b = 10h.

Theory		SS	SC	CC	FF	SF	CF
Reddy's TSDPT		16.943†	18.837	21.007	8.545	10.081	10.619
		17.020‡	18.915	21.083	8.573	10.123	10.664
Levinson's TSDPT		17.338†	19.101	21.047	8.689	10.276	10.798
		17.402‡	19.164	21.105	8.714	10.315	10.841
FSDPT		17.400†	19.457	21.818	8.709	10.304	10.879
		17.488‡	19.548	21.906	8.739	10.349	10.929
CLPT		19.739‡	23.646	28.951	9.354	11.258	12.157

†Rotatory inertia forces are not neglected.

‡Rotatory inertia forces are neglected.

Table 9.3

Boundary layer effects on the dimensionless fundamental frequencies of Structures I and II; $\bar{\omega} = \omega a^2 \left[\frac{\rho h}{\bar{\bar{D}}}\right]^{1/2}$ and a = b = 10h.

Structure	Theory		SC	CC	FF	SF
I	Levinson's TSDPT	$\delta_A=1$	17.598†	19.160†	8.702	10.334
		$\delta_A=0$	17.447†	18.829†	8.751	10.497
	FSDPT ($K^2=5/6$)	$\delta_A=1$	17.598	19.160	8.702	10.334
		$\delta_A=0$	17.447	18.829	8.751	10.497
II	Levinson's TSDPT	$\delta_A=1$	19.164‡	21.105‡	8.714	10.315
		$\delta_A=0$	19.059‡	20.871‡	8.825	10.546
	FSDPT ($K^2=2/3$)	$\delta_A=1$	19.548	21.906	8.739	10.349
		$\delta_A=0$	19.450	21.678	8.847	10.575

†For a clamped edge C1–type conditions are assumed.

‡For a clamped edge C2–type conditions are assumed.

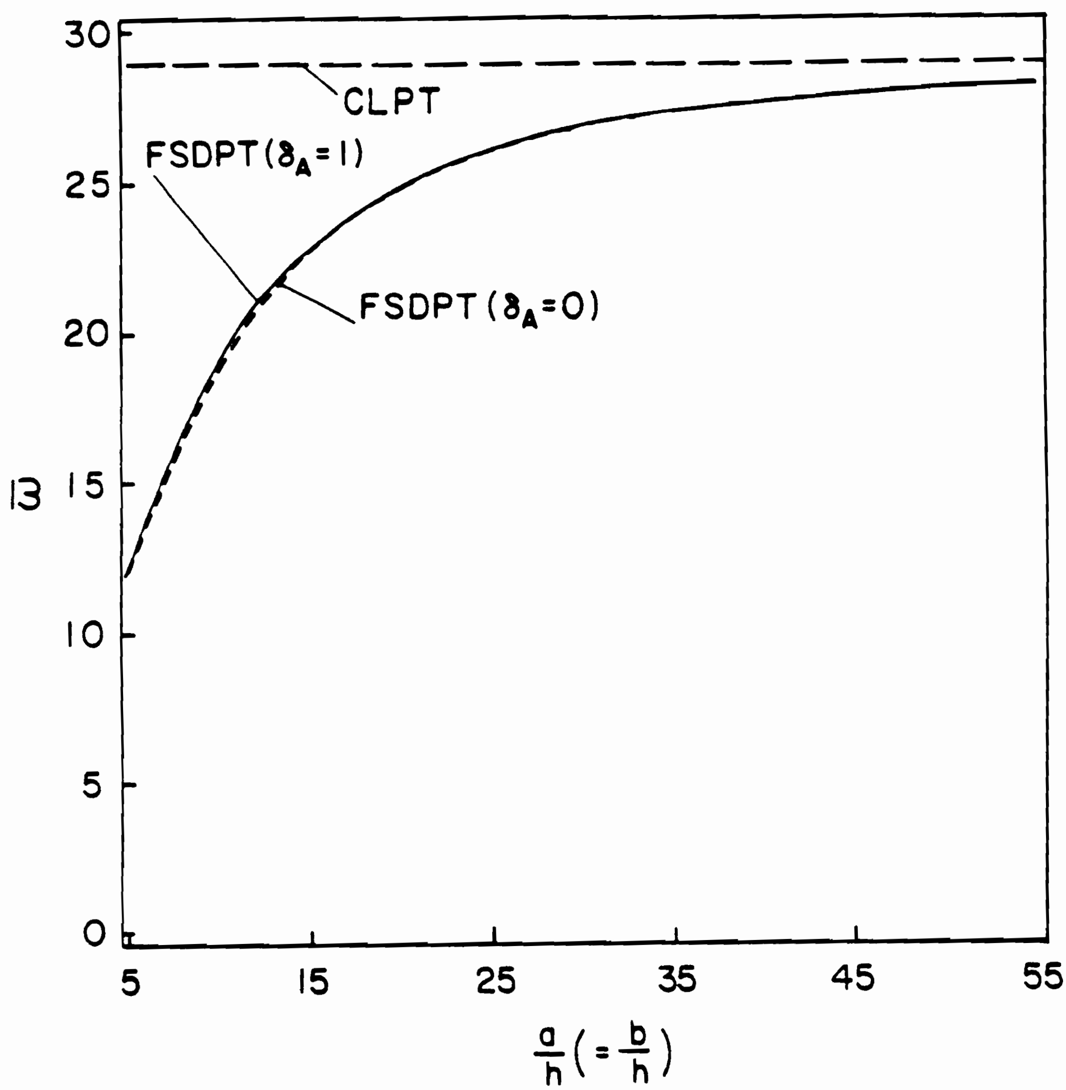

Figure 9.1 Variation of the nondimensional fundamental frequency $\bar{\omega}$ ($= \omega a^2 \sqrt{\rho h/\bar{\bar{D}}}$) of Structure I versus a/h for CC case; a = b and h = 1 in.

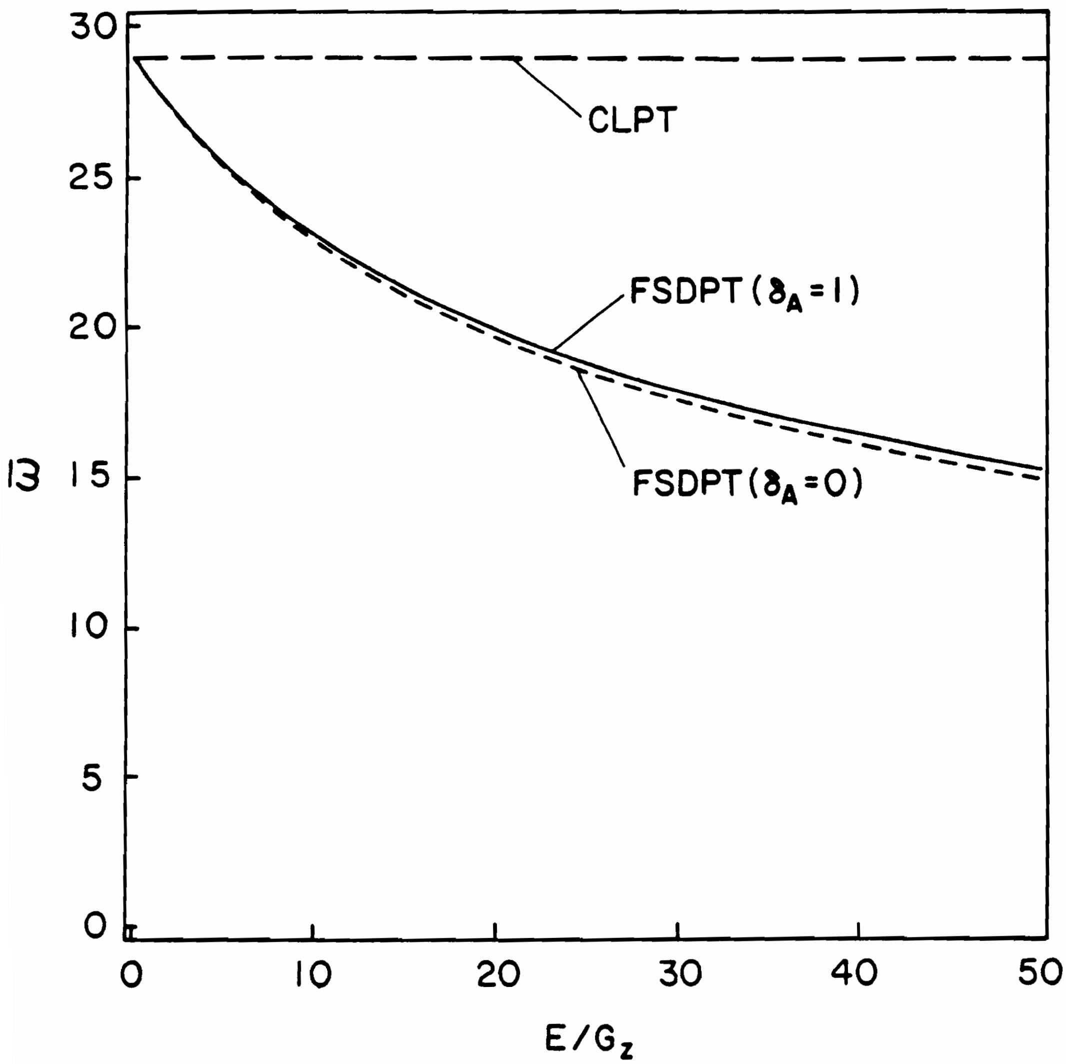

Figure 9.2 Variation of the nondimensional fundamental frequency $\bar{\omega}$ (= $\omega a^2 \sqrt{\rho h/\bar{\bar{D}}}$) of Structure I versus E/G_z ratio for CC case; a/h = b/h = 10.

Table 9.4

The effects of different clamped–type boundary conditions in Levinson's TSDPT and shear correction factors in FSDPT on the dimensionless critical buckling loads of Structures I and II; $T_{11} = \bar{N}_x a^2/\bar{\bar{D}}$ and $a = b = 10h$.

Structure	Theory		SC	CC	CF
I	Reddy's TSDPT		28.778	29.734	11.921
I	Levinson's TSDPT	C1–type	28.353	29.052	11.937
I	Levinson's TSDPT	C2–type	28.115	28.538	11.798
I	FSDPT	$K^2=5/6$	28.353	29.052	11.937
I	FSDPT	$K^2=2/3$	24.823	25.323	11.203
II	Reddy's TSDPT		34.372	35.706	11.524
II	Levinson's TSDPT	C1–type	37.146	38.510	12.005
II	Levinson's TSDPT	C2–type	36.848	37.839	11.907
II	FSDPT	$K^2=5/6$	41.249	43.326	12.521
II	FSDPT	$K^2=2/3$	37.890	39.338	12.103
I	CLPT		56.652	75.910	16.845
II	CLPT		56.652	75.910	14.976

Table 9.5

Boundary layer effects on the dimensionless critical buckling load of Structure II; $T_{11} = \bar{N}_x a^2/\bar{\bar{D}}$, a = b = 10h and C2–type conditions are assumed at a clamped edge in Levinson's TSDPT.

Theory		SC	CC	FF	SF
Levinson's TSDPT	$\delta_A=1$	36.848	37.839	7.694	10.779
	$\delta_A=0$	36.645	37.402	7.892	11.269
FSDPT ($K^2=2/3$)	$\delta_A=1$	37.890	39.338	7.737	10.851
	$\delta_A=0$	36.647	38.793	7.931	11.331
CLPT		56.652	75.910	8.865	12.842

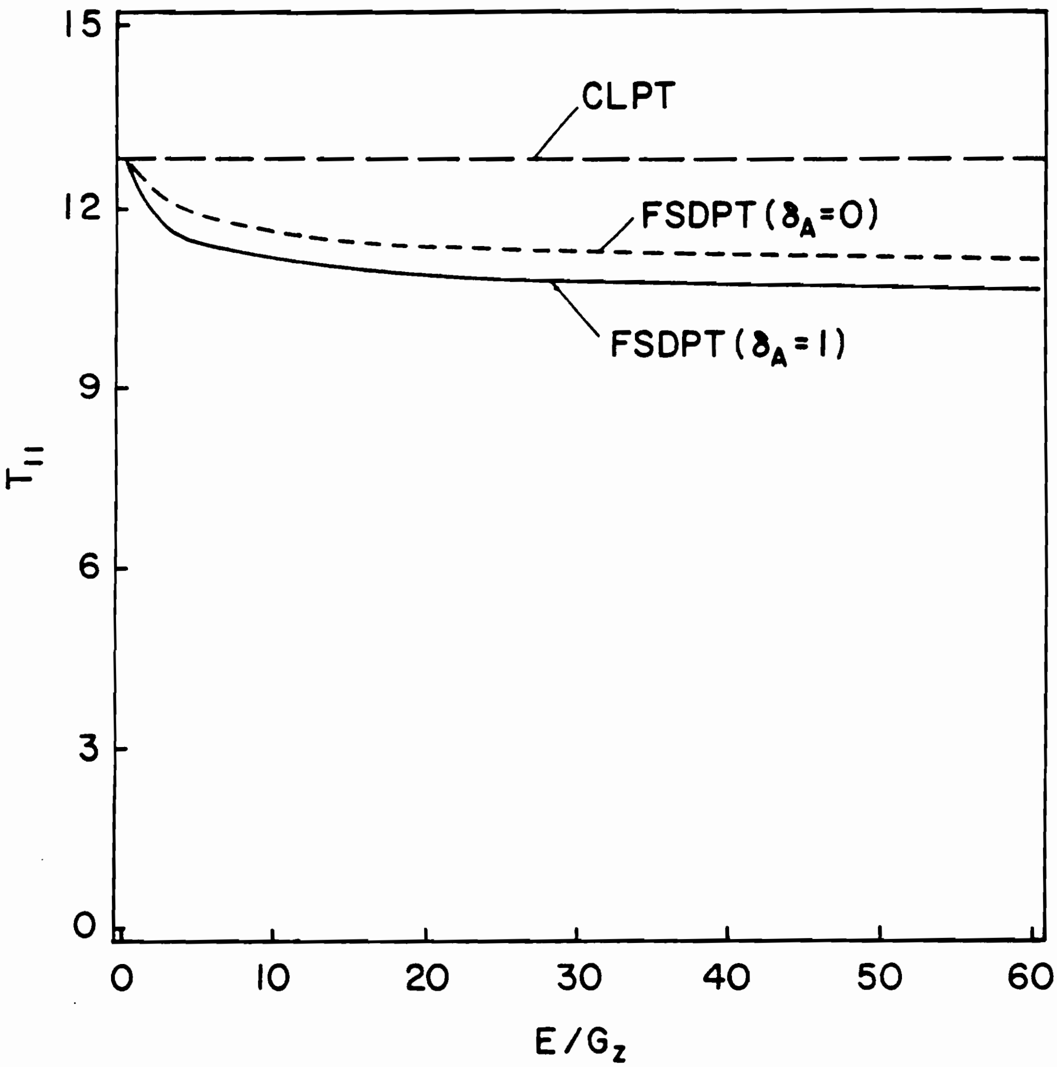

Figure 9.3 Variation of the dimesionless critical buckling load $T_{11}(= \bar{N}_x a^2/\bar{\bar{D}})$ of Structure II versus E/G_z ratio for SF case; $\frac{a}{h} = \frac{b}{h} = 10$.

CHAPTER X

ON THE BENDING PROBLEMS OF LAMINATED PLATES

We mentioned in Chapters I and VIII that the Navier and Lévy–type solutions for bending problems of a homogeneous isotropic plate are developed respectively in [86] and [87] using both FSDPT and Levinson's TSDPT. However, since the analysis presented in [87] is incorrect, we will briefly consider here the bending problem using FSDPT. It should be noted that a similar analysis can readily be developed for Levinson's TSDPT since, as we noted in Chapters VIII and IX, the total order of the governing equations in both theories is six. For the sake of generality, we will assume that the plate is laminated of transversely isotropic layers. Later in this chapter we will consider the same problem by using a more consistent and yet more complicated third–order shear deformation plate theory (i.e., Reddy's TSDPT).

10.1.1 Analysis Based on FSDPT

For bending problems the simplest approach may be to solve for w and Φ using the interior and edge–zone equations

$$\bar{\bar{D}}\nabla^2\nabla^2 w = P_z - \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\nabla^2 P_z \tag{10–1a}$$

$$\bar{\bar{C}}\nabla^2\Phi - K^2\bar{\bar{A}}\Phi = 0 \tag{10–1b}$$

and the corresponding boundary conditions that are explicitly in terms of w and Φ.

Equations (10–1) are the static counterparts of Eqs. (9–2). It is to be noted that we have already expressed the boundary conditions explicitly in terms of w and Φ in the previous chapter. Here, however, we use a mixed approach. That is, after obtaining the general solutions of Eqs. (10–1) we find ψ_x and ψ_y from the relations

$$\psi_x = -w_{,x} - \frac{\bar{\bar{D}}}{K^2\bar{A}} \nabla^2 w_{,x} + \frac{\bar{\bar{C}}}{K^2\bar{A}} \Phi_{,y} - \frac{\bar{\bar{D}}}{(K^2\bar{A})^2} P_{z,x} \tag{10–2a}$$

and

$$\psi_y = -w_{,y} - \frac{\bar{\bar{D}}}{K^2\bar{A}} \nabla^2 w_{,y} - \frac{\bar{\bar{C}}}{K^2\bar{A}} \Phi_{,x} - \frac{\bar{\bar{D}}}{(K^2\bar{A})^2} P_{z,y} \tag{10–2b}$$

which also are the static counterparts of Eqs. (9–6). This way, to obtain the final solution, we can impose those boundary conditions which are in terms of w, ψ_x, and ψ_y.

10.1.2 Navier Solution

The boundary conditions (at x = 0,a and y = 0,b) for a completely simply–supported plate are obtained from Eqs. (9–8) and (9–9) by setting $N_x = N_y = 0$. It can be seen that the solution representations

$$w = \sum_m^\infty \sum_n^\infty A_{mn} \sin \alpha_m x \sin \beta_n y \tag{10–3a}$$

$$\Phi = \sum_m^{\infty} \sum_n^{\infty} E_{mn} \cos \alpha_m x \cos \beta_n y \tag{10–3b}$$

with $\alpha_m = \frac{m\pi}{a}$ and $\beta_n = \frac{n\pi}{b}$ satisfy all the boundary conditions. Substitution of Eq. (10–3a) and

$$P_z = \sum_m^{\infty} \sum_n^{\infty} P_{mn} \sin \alpha_m x \sin \beta_n y \tag{10–4}$$

into Eq. (10–1a) results in

$$A_{mn} = \left[1 + \frac{\bar{\bar{D}}}{K^2 \bar{\bar{A}}} (\alpha_m^2 + \beta_n^2)\right] \frac{P_{mn}}{\bar{\bar{D}}(\alpha_m^2 + \beta_n^2)^2} \tag{10–5}$$

where P_{mn} is given by

$$P_{mn} = \frac{4}{ab} \int_0^b \int_0^a P_z(x,y) \sin \alpha_m x \sin \beta_n y \, dxdy \tag{10–6}$$

It is also seen that substitution of Eq. (10–3b) into Eq. (10–1b) results in $E_{mn} = 0$. That is,

$$\Phi(x,y) = 0. \tag{10–7}$$

The expressions for ψ_x and ψ_y are now obtained by substituting Eqs. (10–3a), (10–4), (10–5), and (10–7) into Eqs. (10–2):

$$\psi_x = \sum_m^\infty \sum_n^\infty \frac{-\alpha_m P_{mn}}{\bar{\bar{D}}(\alpha_m^2 + \beta_n^2)^2} \cos \alpha_m x \sin \beta_n y \tag{10–8a}$$

$$\psi_y = \sum_m^\infty \sum_n^\infty \frac{-\beta_n P_{mn}}{\bar{\bar{D}}(\alpha_m^2 + \beta_n^2)^2} \sin \alpha_m x \cos \beta_n y \tag{10–8b}$$

where P_{mn} is given by Eq. (10–6).

10.1.3 Lévy–Type Solutions

When the edges of a plate at x = 0 and a are simply supported, we assume that the total solution for w consists of a homogeneous solution w_h and a particular solution w_p. That is,

$$w = w_h + w_p \tag{10–9}$$

where w_h is the solution of the biharmonic equation

$$\bar{\bar{D}}\nabla^2\nabla^2 w_h = 0 \tag{10–10}$$

It can easily be shown with $\lambda_m = \frac{m\pi}{a}$ that

$$\begin{aligned} w_h = \sum_m^\infty (A_m \sinh \lambda_m y + B_m \cosh \lambda_m y + C_m y \sinh \lambda_m y \\ + D_m y \cosh \lambda_m y) \sin \lambda_m x \end{aligned} \tag{10–11a}$$

and

$$\Phi = \sum_{m}^{\infty} (E_m \sinh \gamma_m y + F_m \cosh \gamma_m y)\cos \lambda_m x \tag{10–11b}$$

are the solutions of Eqs. (10–10) and (10–1b), respectively. In Eq. (10–11b) we have

$$\gamma_m^2 = \lambda_m^2 + \frac{K^2 \bar{\bar{A}}}{\bar{\bar{C}}}. \tag{10–12}$$

It should be noted that the solutions in Eqs. (10–11) identically satisfy the boundary conditions of the plate at x = 0 and x = a which are given in Eqs. (9–8). These boundary conditions are also satisfied if the particular solution w_p is expressed as

$$w_p = \sum_{m}^{\infty} w_m(y) \sin \lambda_m x. \tag{10–13}$$

Also by expanding $P_z(x,y)$ in terms of a single Fourier series

$$P_z(x,y) = \sum_{m}^{\infty} P_m(y) \sin \lambda_m x. \tag{10–14a}$$

with

$$P_m(y) = \frac{2}{a}\int_0^a P_z(x,y) \sin \lambda_m x \, dx \tag{10–14b}$$

and substituting Eqs. (10–13) and (10–14a) into Eq. (10–1a) we obtain

$$\bar{\bar{D}}\left[w_m'''' - 2\lambda_m^2 w'' + \lambda_m^4 w_m\right] = Q_m(y). \tag{10–15}$$

A prime in Eq. (10–15) denotes differentiation with respect to y. Also $Q_m(y)$ in Eq. (10–15) is given by

$$Q_m(y) = -\frac{\bar{\bar{D}}}{K^2 \bar{\bar{A}}} P_m'' + \left[1 + \frac{\bar{\bar{D}}}{K^2 \bar{\bar{A}}} \lambda_m^2\right] P_m. \tag{10–16}$$

Upon determination of the particular solution w_m of Eq. (10–15), we will then have the total solution w from Eq. (10–9). At this point we can impose the boundary conditions in terms of Φ and w at $y = \pm b/2$ to obtain the unknown arbitrary constants A_m, B_m,..., F_m. However, as we pointed out earlier, we wish to impose the boundary conditions which are in terms of ψ_x, ψ_y, and w. For this reason, we first substitute Eqs. (10–9), (10–11), and (10–14a) into Eqs. (10–2) to obtain the expressions for ψ_x and ψ_y:

$$\psi_x = \sum_m^{\infty} \left[\left[-\lambda_m A_m - 2\lambda_m^2 \frac{\bar{\bar{D}}}{K^2 \bar{\bar{A}}} D_m\right] \sinh \lambda_m y + \left[-\lambda_m B_m\right.\right.$$

$$- 2\lambda_m^2 \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}} C_m\Big] \cosh \lambda_m y - \lambda_m C_m y \sinh \lambda_m y - \lambda_m D_m y \cosh \lambda_m y$$

$$+ \gamma_m \frac{\bar{\bar{C}}}{K^2\bar{\bar{A}}} E_m \cosh \lambda_m y + \gamma_m \frac{\bar{\bar{C}}}{K^2\bar{\bar{A}}} F_m \sinh \gamma_m y\Big] \cos \lambda_m x + \psi_{xp}$$

(10–17a)

and

$$\psi_y = \sum_m^{\infty} \left\{ -\left[\lambda_m A_m + \left[1 + 2\lambda_m^2 \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\right] D_m\right] \cosh \lambda_m y - \left[\lambda_m B_m\right.\right.$$

$$\left.+ \left[1 + 2\lambda_m^2 \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\right] C_m\right] \sinh \lambda_m y - \lambda_m D_m y \sinh \lambda_m y - \lambda_m C_m y \cosh \lambda_m y$$

$$\left. + \lambda_m \frac{\bar{\bar{C}}}{K^2\bar{\bar{A}}} E_m \sinh \gamma_m y + \lambda_m \frac{\bar{\bar{C}}}{K^2\bar{\bar{A}}} F_m \cosh \lambda_m y \right\} \sin \lambda_m x + \psi_{yp}$$

(10–17b)

where

$$\psi_{xp} = \sum_m^{\infty} \left[\left[-\lambda_m + \lambda_m^2 \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\right] w_m - \lambda_m \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}} w_m'' - \lambda_m \frac{\bar{\bar{D}}}{(K^2\bar{\bar{A}})^2} P_m\right] \cos \lambda_m x$$

(10–17c)

and

$$\psi_{yp} = \sum_m^{\infty} \left[\left[-1 + \lambda_m^2 \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\right] w_m' - \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}} w_m''' - \frac{\bar{\bar{D}}}{(K^2\bar{\bar{A}})^2} P_m'\right] \sin \lambda_m x$$

(10–17d)

10.1.4 A Plate Under Uniform Loading

To illustrate our method we assume for now that the boundary conditions at the remaining edges at $y = \pm b/2$ are identical and the plate is subjected to a uniformly distributed load. That is:

$$P_z(x,y) = P_o \tag{10–18a}$$

and therefore

$$P_m = \frac{4P_o}{m\pi} \quad m = 1,3,... \tag{10–18b}$$

which is evaluated with the help of Eq. (10–14b). Because of the symmetry of the boundary conditions and the loading the deflection w and the rotation ψ_x will be symmetric with respect to the x–axis. Hence we let

$$A_m = D_m = F_m = 0. \tag{10–19}$$

The particular solution of Eq. (10–15) is also seen to be

$$w_m = \frac{4a^4 P_o}{\pi^5 m^5 \bar{\bar{D}}}\left[1 + \frac{\bar{\bar{D}}}{K^2 \bar{\bar{A}}}\lambda_m^2\right]. \qquad m=1,3,... \tag{10–20}$$

Now with the help of Eqs. (10–19) and (10–20), substitution of Eqs. (10–13) and (10–11a) into Eq. (10–9) results in

$$w(x,y) = \sum_{m=1,3,...}^{\infty} (B_m \cosh \lambda_m y + C_m y \sinh \lambda_m y)\sin \lambda_m x$$

$$+\frac{4a^4P_o}{\pi^5\bar{\bar{D}}}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^5}\left[1+\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\lambda_m^2\right]\sin\lambda_m x \qquad (10\text{–}21)$$

Also substituting Eqs. (10–18b), (10–19), and (10–20) into Eqs. (10–17) yields

$$\psi_x(x,y)=\sum_{m=1,3,\ldots}^{\infty}\left[\left[-\lambda_m B_m-2\lambda_m^2\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}C_m\right]\cosh\lambda_m y-\lambda_m C_m y\sinh\lambda_m y\right.$$

$$\left.+\gamma_m\frac{\bar{\bar{C}}}{K^2\bar{\bar{A}}}E_m\cosh\gamma_m y\right]\cos\lambda_m x-\frac{4a^3P_o}{\pi^4\bar{\bar{D}}}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^4}\cos\lambda_m x$$

(10–22a)

and

$$\psi_y(x,y)=\sum_{m=1,3,\ldots}^{\infty}\left\{-\left[\lambda_m B_m+\left[1+2\lambda_m^2\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\right]C_m\right]\sinh\lambda_m y\right.$$

$$\left.-\lambda_m C_m y\cosh\lambda_m y+\lambda_m\frac{\bar{\bar{C}}}{K^2\bar{\bar{A}}}E_m\sinh\lambda_m y\right\}\sin\lambda_m x \qquad (10\text{–}22b)$$

10.1.5 A Clamped Plate Under Uniform Loading

Assuming that the remaining edges at $y = \pm\, b/2$ are clamped, we impose

$$w=\psi_x=\psi_y=0 \qquad (10\text{–}23)$$

at, say, $y = b/2$. This way we obtain three nonhomogeneous algebraic equations in terms of B_m, C_m, and E_m. Solving these equations yields

$$B_m = \frac{U_{m1}}{U_m}\ ,\ C_m = \frac{U_{m2}}{U_m}\ ,\ \text{and } E_m = \frac{U_{m3}}{U_m} \tag{10–24a}$$

where

$$U_{m1} = \frac{4P_o}{m\pi\bar{\bar{D}}}\left\{-\frac{1}{\lambda_m^4}\left[1 + \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\lambda_m^2\right]\left[\sinh\alpha_m + \alpha_m\cosh\alpha_m\right]\cosh\beta_m\right.$$

$$+ 2\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\left[\frac{1}{\lambda_m^2}\left[1 + \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\lambda_m^2\right]\left[\frac{\lambda_m}{\gamma_m}\cosh\alpha_m\sinh\beta_m - \sinh\alpha_m\cosh\beta_m\right]\right.$$

$$\left.\left. + \frac{1}{2}\frac{\alpha_m}{\lambda_m\gamma_m}\sinh\alpha_m\sinh\beta_m\right]\right\} \tag{10–24b}$$

$$U_{m2} = \frac{4P_o}{m\pi\bar{\bar{D}}}\left[\frac{1}{\lambda_m^3}\left[1 + \frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\lambda_m^2\right]\sinh\alpha_m\cosh\beta_m - \frac{1}{\gamma_m}\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\cosh\alpha_m\sinh\beta_m\right] \tag{10–24c}$$

$$U_{m3} = \frac{4P_o}{m\pi\bar{\bar{C}}}\frac{1}{\lambda_m\gamma_m}(\cosh\alpha_m\sinh\alpha_m - \alpha_m) \tag{10–24d}$$

$$U_m = (\sinh\alpha_m\cosh\alpha_m + \alpha_m)\cosh\beta_m + 2\frac{\bar{\bar{D}}}{K^2\bar{\bar{A}}}\lambda_m^2\left[\sinh\alpha_m\cosh\beta_m\right.$$

$$-\frac{\lambda_m}{\gamma_m}\cosh\alpha_m\sinh\beta_m\Big]\cosh\alpha_m \tag{10–24e}$$

with

$$\alpha_m = \frac{b}{2}\lambda_m \text{ and } \beta_m = \frac{b}{2}\gamma_m. \tag{10–24f}$$

We can obtain similar results within the classical plate theory (CLPT) by simply letting $K^2\bar{A} \to \infty$ in the above expressions:

$$B_m = \frac{4P_o}{m\pi\bar{\bar{D}}}\left[-\frac{1}{\lambda_m^4}(\sinh\alpha_m + \alpha_m\cosh\alpha_m)\right]/\bar{U}_m$$

$$C_m = \frac{4P_o}{m\pi\bar{\bar{D}}}\left[\frac{1}{\lambda_m^3}\right]\sinh\alpha_m/\bar{U}_m$$

$$E_m = 0 \tag{10–25a}$$

where

$$\bar{U}_m = \sinh\alpha_m\cosh\alpha_m + \alpha_m \text{ and } \alpha_m = \frac{b}{2}\lambda_m. \tag{10–25b}$$

10.2.1 Analysis Based on Reddy's TSDPT

In Chapter VII we have developed and studied the nonlinear dynamic equations of the third–order shear deformation plate theory (TSDPT) of Reddy for symmetric plates with transversely isotropic layers. As we mentioned before, the governing equations in stretching and bending theories of symmetric plates will be uncoupled in the linear theory. For linear bending problems, the governing equations in Reddy's TSDPT are obtained from Eqs. (7–8):

$$D\psi_{x,xx} + C\psi_{x,yy} - A\psi_x + (D - C)\psi_{y,xy} - Aw_{,x} - F\nabla^2 w_{,x} = 0 \tag{10–26a}$$

$$D\psi_{y,yy} + C\psi_{y,xx} - A\psi_y + (D - C)\psi_{x,xy} - Aw_{,y} - F\nabla^2 w_{,x} = 0 \tag{10–26b}$$

$$F\nabla^2(\psi_{x,x} + \psi_{y,y}) + A(\psi_{x,x} + \psi_{y,y}) - H\nabla^2\nabla^2 w + A\nabla^2 w + P_z = 0 \tag{10–26c}$$

Also the interior and the edge–zone equations corresponding to Eqs. (10–26) are

$$\frac{1}{A}(\bar{\bar{D}}H - \bar{F}^2)\nabla^2\nabla^2\nabla^2 w - \bar{\bar{D}}\nabla^2\nabla^2 w = -P_z + \frac{D}{A}\nabla^2 P_z \tag{10–27a}$$

and

$$C\nabla^2\Phi - A\Phi = 0 \tag{10–27b}$$

which are the static (and linear) counterparts of Eqs. (7–14) and (7–10a). It should be recalled that the edge–zone equation was a linear equation also in the nonlinear theory (see Chapter VII). Also in all the existing equivalent single–layer plate theories this equation is seen to be a second–order equation with respect to both space and time. On the other hand, unlike most of the existing theories which have a fourth–order interior equation, Reddy's interior equation is a sixth–order equation with respect to spatial coordinates. This is expected since the total order of Eqs. (10–26) is eight.

10.2.2 Navier Solution

When the edges of the plate are simply supported we have shown in Chapter VII that the boundary conditions are those given by Eqs. (7–27a), (7–28b), (7–29), and (7–30). In the linear theory, these conditions reduce to

$$w = w_{,xx} = \frac{\bar{\bar{D}}H - \bar{F}^2}{D} w_{,xxxx} - P_z = 0 \tag{10–28a}$$

$$\Phi_{,x} = 0 \tag{10–28b}$$

at x = 0 and a and

$$w = w_{,yy} = \frac{\bar{\bar{D}}H - \bar{F}^2}{D} w_{,yyyy} - P_z = 0 \tag{10–29a}$$

$$\Phi_{,y} = 0 \tag{10–29b}$$

at y = 0 and b. It is seen that, as in all the theories that we have considered so far, these boundary conditions are completely uncoupled. This, of course, is the case only for a completely simply–supported plate as we have noticed in the previous chapters. From the analysis that we have developed in Section 10.1.2, it is readily seen that $\Phi = 0$ is the only solution of the edge–zone equation. Therefore, for a simply–supported plate, the bending theory is reduced to a sixth–order equation in terms of w and the specification of three boundary conditions also in terms of w at each edge of the plate. This conclusion also holds in the stability problem and vibration problem, with all the rotatory inertia forces neglected, since $\Phi = 0$ will again be the only solution of the edge–zone equation.

Following the analysis of section 10.1.2, we expand w and P_z in double Fourier series (see Eqs. (10–3a) and (10–4)) and obtain

$$A_{mn} = \left[1 + \frac{D}{A}(\alpha_m^2 + \beta_n^2)\right] \frac{P_{mn}}{\frac{1}{A}(\bar{\bar{D}}H - \bar{F}^2)(\alpha_m^2 + \beta_n^2)^3 + \bar{\bar{D}}(\alpha_m^2 + \beta_n^2)^2}$$

(10–30)

where P_{mn} is defined in Eq. (10–6) and

$$\alpha_m = \frac{m\pi}{a} \text{ and } \beta_n = \frac{n\pi}{b}. \qquad (10\text{–}31)$$

The expressions for ψ_x and ψ_y can be determined from

$$\psi_x = \frac{D}{A^2\bar{D}}(\bar{\bar{D}}H - \bar{F}^2)\nabla^2\nabla^2 w_{,x} - \frac{\bar{D}}{A}\nabla^2 w_{,x} - w_{,x} + \frac{C}{A}\Phi_{,y} - \frac{D^2}{A^2\bar{D}}P_{z,x}$$

(10–32a)

and

$$\psi_y = \frac{D}{A^2\bar{D}}(\bar{\bar{D}}H - \bar{F}^2)\nabla^2\nabla^2 w_{,y} - \frac{\bar{D}}{A}\nabla^2 w_{,y} - w_{,y} - \frac{C}{A}\Phi_{,x} - \frac{D^2}{A^2\bar{D}}P_{z,y}$$

(10–32b)

which are the linear (and static) counterparts of Eqs. (7–23):

$$\psi_x = \sum_m^{\infty} \sum_n^{\infty} \frac{[\frac{F}{A}(\alpha_m^2 + \beta_n^2) - 1]\alpha_m P_{mn}}{\frac{1}{A}(\bar{\bar{D}}H - \bar{F}^2)(\alpha_m^2 + \beta_m^2)^3 + \bar{\bar{D}}(\alpha_m^2 + \beta_n^2)^2} \cos \alpha_m x \sin \beta_n y$$

(10–33a)

$$\psi_y = \sum_m^{\infty} \sum_n^{\infty} \frac{[\frac{F}{A}(\alpha_m^2 + \beta_n^2) - 1]\beta_n P_{mn}}{\frac{1}{A}(\bar{\bar{D}}H - \bar{F}^2)(\alpha_m^2 + \beta_n^2)^3 + \bar{\bar{D}}(\alpha_m^2 + \beta_n^2)^2} \sin \alpha_m x \cos \beta_n y$$

(10–33b)

In obtaining these results the relations $D\bar{\bar{D}} - \bar{D}^2 = \bar{\bar{D}}H - \bar{F}^2$ and $\bar{D} = D + F$ are used (see Appendix 7.1 in Chapter VII).

10.2.3 Lévy–Type Solutions

Assuming that the edges of a plate at $x = 0$ and $x = a$ are invariably simply supported, the homogeneous solution w_h of the interior equation

$$\frac{1}{A}(\bar{\bar{D}}H - \bar{F}^2)\nabla^2\nabla^2\nabla^2 w_h - \bar{\bar{D}}\nabla^2\nabla^2 w_h = 0 \qquad (10\text{–}34)$$

is seen to be

$$w_h = \sum_m^{\infty} (A_m \sinh \lambda_m y + B_m \cosh \lambda_m y + C_m y \sinh \lambda_m y + D_m y \cosh \lambda_m y$$

$$+ E_m \sinh \delta_m y + F_m \cosh \delta_m y)\sin \lambda_m x \tag{10–35a}$$

where

$$\lambda_m = \frac{m\pi}{a} \text{ and } \delta_m = \left[\lambda_m^2 + \frac{\bar{\bar{D}}A}{\bar{\bar{D}}H - \bar{F}^2}\right]^{1/2}. \tag{10–35b}$$

Also the general solution of the edge–zone equation, Eq. (10–27b), is

$$\Phi = \sum_m^\infty (G_m \sinh \gamma_m y + H_m \cosh \gamma_m y)\cos \lambda_m y \tag{10–36a}$$

where

$$\gamma_m = (\lambda_m^2 + \frac{A}{C})^{1/2}. \tag{10–36b}$$

As in Eqs. (10–13) and (10–14a), by expressing the particular solution w_p and the loading P_z in single Fourier series we obtain

$$\bar{K}w_m'''''' - (3\lambda_m^2\bar{K} + \bar{\bar{D}})w_m'''' + (3\lambda_m^4\bar{K} + 2\lambda_m^2\bar{\bar{D}})w_m''$$

$$- (\lambda_m^6\bar{K} + \lambda_m^4\bar{\bar{D}})w_m = \frac{D}{A}P_m'' - (1 + \frac{D}{A}\lambda_m^2)P_m \tag{10–37a}$$

where

$$\bar{K} \equiv \frac{\bar{D}H - \bar{F}^2}{A} \tag{10–37b}$$

and P_m is defined in Eq. (10–14b). After determining the particular solution w_m of Eq. (10–37a), the total deflection w is obtained from Eq. (10–9). The expressions for ψ_x and ψ_y are subsequently determined by substituting Eqs. (10–14a), (10–9), and (10–36a) into Eqs. (10–32):

$$\psi_x = \sum_m^\infty \Bigg[\left[-\lambda_m A_m - 2\lambda_m^2 \frac{\bar{D}}{A} D_m\right] \sinh \lambda_m y + \left[-\lambda_m B_m - 2\lambda_m^2 \frac{\bar{D}}{A} C_m\right] \cosh \lambda_m y$$

$$- \lambda_m C_m y \sinh \lambda_m y - \lambda_m D_m y \cosh \lambda_m y + \lambda_m \frac{\bar{F}}{\bar{D}} E_m \sinh \delta_m y$$

$$+ \lambda_m \frac{\bar{F}}{\bar{D}} F_m \sinh \delta_m y + \gamma_m \frac{C}{A} G_m \cosh \gamma_m y$$

$$+ \gamma_m \frac{C}{A} H_m \sinh \gamma_m y\Bigg] \cos \lambda_m x + \psi_{xp} \tag{10–38a}$$

and

$$\psi_y = \sum_m^\infty \Bigg\{-\left[\lambda_m A_m + \left[1 + 2\lambda_m^2 \frac{\bar{D}}{A}\right] D_m\right] \cosh \lambda_m y - \Bigg[\lambda_m B_m$$

$$+ \left[1 + 2\lambda_m^2 \frac{\bar{D}}{A}\right] C_m\Bigg] \sinh \lambda_m y - \lambda_m C_m y \cosh \lambda_m y - \lambda_m D_m y \sinh \lambda_m y$$

$$+ \delta_m \frac{\bar{F}}{\bar{D}} E_m \cosh \delta_m y + \delta_m \frac{\bar{F}}{\bar{D}} F_m \sinh \delta_m y + \lambda_m \frac{C}{A} G_m \sinh \gamma_m y$$

$$+ \lambda_m \frac{C}{A} H_m \cosh \gamma_m y \Big\} \sin \lambda_m x + \psi_{xp} \tag{10–38b}$$

where

$$\psi_{xp} = \sum_m^\infty \left[\lambda_m \frac{D\bar{K}}{A\bar{D}} w_m'''' - (2\lambda_m^3 \frac{D\bar{K}}{A\bar{D}} + \frac{\bar{D}}{A} \lambda_m) w_m'' + \left[\lambda_m^5 \frac{D\bar{K}}{A\bar{D}} \right.\right.$$

$$\left.\left. + \lambda_m^3 \frac{\bar{D}}{A} - \lambda_m \right] w_m - \frac{D^2}{\bar{D}A^2} \lambda_m P_m \right] \cos \lambda_m x \tag{10–38c}$$

and

$$\psi_{yp} = \sum_m^\infty \left[\frac{D\bar{K}}{A\bar{D}} w_m''''' - \left[2\lambda_m^2 \frac{D\bar{K}}{A\bar{D}} + \frac{\bar{D}}{A} \right] w_m''' + \left[\lambda_m^4 \frac{D\bar{K}}{A\bar{D}} + \lambda_m^2 \frac{\bar{D}}{A} - 1 \right] w_m' \right.$$

$$\left. - \frac{D^2}{\bar{D}A^2} P_m' \right] \sin \lambda_m x. \tag{10–38d}$$

For a plate subjected to a uniformly distributed load the particular solution w_m of Eq. (10–37a) is seen to be

$$w_m = \frac{4P_o}{m\pi} \frac{\lambda_m^2 D + A}{A(\lambda_m^6 \bar{K} + \lambda_m^4 \bar{\bar{D}})} \tag{10–39}$$

Substituting Eq. (10–39) into Eqs. (10–38c) and (10–38d) results in

$$\psi_{xp} = \sum_{m=1,3,\ldots}^{\infty} \frac{4P_o}{m\pi} \frac{\lambda_m^2 F - A}{A(\lambda_m^5 \bar{K} + \lambda_m^3 \bar{\bar{D}})} \cos \lambda_m x \quad \text{and} \quad \psi_{yp} = 0 \tag{10–40}$$

where $F = \bar{D} - D$ (see Chapter VII). When the edges of the plate at $y = \pm b/2$ are clamped we impose, according to Eqs. (2–12) of Chapter II, the boundary conditions

$$w = w_{,y} = \psi_x = \psi_y = 0. \tag{10–41}$$

Because of the symmetry we also let

$$A_m = D_m = E_m = H_m = 0 \tag{10–42}$$

in Eqs. (10–35), (10–38a), and (10–38b). By imposing the boundary conditions in Eq. (10–41) at, say, $y = b/2$ we will obtain a system of four nonhomogeneous algebraic equations in terms of B_m, C_m, F_m, and G_m. With the help of a computer these unknown constants can readily be obtained. Perhaps the simplest case in the Lévy–type problems is when the edges at $y = \pm b/2$ are also simply supported. For such a plate, with the help of the boundary condition in Eq. (10–29b), we will have $G_m = H_m = 0$. That is

$$\Phi = 0 \tag{10–43}$$

which was also obtained in the development of the Navier solution. By imposing the remaining boundary conditions given by Eq. (10–29a), a system of three algebraic equations in terms of B_m, C_m, and F_m will then be obtained which can readily be solved. It should be reminded that the result in Eq. (10–43) will always be obtained for a completely simply–supported plate subjected to any arbitrary transverse load P_z since the edge–zone equation is a homogeneous equation.

10.3 Discussions

In this chapter we have outlined a simple approach for solving plate bending problems using FSDPT and Reddy's TSDPT. It is seen that the transverse displacement w is not directly dependent on the fundamental solutions of the edge–zone equation. However, since Φ and w are (except for a simply–supported plate) coupled through the boundary conditions, it can be argued that the total solution for w is indirectly dependent on the fundamental solutions of the edge–zone equation. On the other hand, by noting that the edge–zone equation looks like the equation of a membrane on an elastic foundation (with no transverse load), we expect that the solution contribution of the edge–zone equation will diminish as we move towards the interior region of the plate. We also note that in, for example, Reddy's TSDPT A is the only rigidity term that is dependent on the transverse shear modulus G_z (see Appendix 7.1 in Chapter VII). Now by rewriting Eq. (10–27b) as

$$\frac{C}{A}\nabla^2\Phi - \Phi = 0, \tag{10–44}$$

it is readily seen that we obtain $\Phi = 0$ when $G_z \to \infty$. Also when the plate is extremely thin we will again have $\Phi = 0$ since $\frac{C}{A} \to 0$ (see Chapter VII).

It is to be noted that when $G_z \to \infty$ (or when the plate is extremely thin) from the interior equation, Eq. (10–27a), we obtain

$$\bar{\bar{D}}\nabla^2\nabla^2 w = P_z \tag{10–45}$$

which is the classical plate equation. Also from Eqs. (10–32) we automatically obtain

$$\psi_x = -w_{,x} \tag{10–46a}$$

and

$$\psi_y = -w_{,y} \tag{10–46b}$$

which are the well–known Kirchhoff constraints in the classical plate theory.

CHAPTER XI

CONCLUDING REMARKS

In this study analytical solutions for vibration and stability of the classical, shear deformation and layer–wise theories of laminated plates and shells are presented. An improved analytical technique is presented for the free–vibration and buckling problems of cross–ply laminated circular cylindrical shells with various boundary conditions. Special attention is also given to the axisymmetric vibration and buckling modes. A general solution procedure is also developed for the stability analysis of anisotropic circular cylindrical shells, with various boundary conditions, within the frameworks of Reddy's layer–wise and generalized layer–wise shell theories, a first–order shear deformation shell theory, and Donnell's classical shell theory.

A closed–form solution is obtained for the damped forced–vibration problem of symmetric cross–ply laminated plates, subjected to sonic–boom type loading, using the third–order shear deformation plate theory of Reddy and the first–order shear deformation plate theory. The numerical results are compared with those obtained with the classical laminate plate theory.

The nonlinear dynamic equations of the third–order shear deformation plate theory of Reddy and the first–order shear deformation plate theory are closely examined. The equations are uncoupled into those defining the interior and edge–zone problems of the plates. It is shown that the pure–shear frequencies according to the linear and nonlinear equations are identical. Further, for certain boundary conditions, these equations are reduced to three in number, as in the classical plate theory. By introducing a force function, these equations are further reduced to two in number. Therefore, the solution methodologies used in the classical plate theory for nonlinear oscillation and postbuckling problems can readily be extended to the shear deformation plate theories. This has been demonstrated by considering a simple large–deflection bending problem.

Analytic Lévy–type solutions for frequency and buckling equations for symmetric plates laminated of transversely isotropic layers are developed using Levinson's third–order shear deformation plate theory and the first–order shear deformation plate theory. The solutions are developed by using the original form of the governing equations as well as the pair of interior and edge–zone (boundary layer) equations. The effects of the edge–zone equation on the natural frequencies and the critical buckling loads of the plates are thoroughly studied. The numerical results are compared with those obtained according to the third–order shear deformation plate theory of Reddy and the classical plate theory. Finally, by proposing a mixed approach, Lévy–type solution for bending problems of laminated plates are developed according to the first–order shear deformation plate theory and the third–order shear deformation plate theory of Reddy.

By no means can the present work be considered exhaustive. Each of the later chapters is amenable to expansion. Specifically, the developments in Chapters VI and VII could readily be extended to the analysis of laminated circular cylindrical and spherical shells and shell panels. Other effects, such as thermal loads, the effect of stiffeners, etc. can also be readily included.

The governing equations of several shear deformation plate theories are reformulated in the present study into the interior and the edge–zone equations. A significant aspect of these equations is that all differential operators in them appear in terms of the Laplace operators. Therefore, the stability, vibration, and bending problems of circular plates can more directly be analyzed using these equations.

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