A RACE TOWARD THE ORIGIN BETWEEN N RANDOM WALKS

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I. INTRODUCTION

The concept of random walks is familiar and possesses a rich literature. Important sources for reference on random walk theory are Bailey [4], Feller [12], Parzen [13] and Spitzer [17]. There exist many diverse applications for random walk theory. For example, Beightler and Shamblin [7] and Terrell et.al. [19] utilize this theory to solve some practical problems in the areas of process and quality control. On the other hand, Brown [8] applies random walk concepts to obtain win probabilities for each side engaging in a stochastic battle.

In this dissertation we shall be concerned with a particular class of practical problems whose theoretical models can all be analyzed in terms of random walks and certain of their associated stochastic processes. It is the purpose of this dissertation to develop the requisite theory - not to be found in the standard works, and to describe some of the applications.

The class of walks with which we shall be concerned is one-dimensional and discrete. That is to say we consider an imaginary particle which moves on a line (parallel to the x-axis in Cartesian space, say) according to certain probabilistic laws. The motive consists of jumps of discrete amounts (possibly zero) at time epochs which may be regularly spaced (discrete time, discrete walk), or which possess a continuous probability density function (continuous time, discrete walk).

The classical problem of the Gambler's Ruin (vide Feller [12]) provides an illustration of the type of situation we have in mind, though as will be seen, our model possesses an important difference which necessitates a special treatment. Two players, Peter and Paul, possessing initial fortunes of A and B dollars, repeatedly toss a coin. The rules are such that if the outcome is a head, Peter wins one of Paul's dollars. The fortunes of Peter and Paul at the n^{th} toss may then be regarded as the coordinates, A(n) and B(n), of two particles moving randomly on two lines parallel to the x-axis with origins on the y-axis. We may without confusion speak of the "A particle" and the "B particle", or of the "A walk" and the "B walk". The initial coordinates, A(0) and B(0), of the particles are A and B.

The motions of the particles are in this case correlated. For if Peter (A) wins a toss, then the A particle moves one unit to the right and *simultaneously* the B particle moves one unit to the left. Thus we can say that the motion is determined by the outcomes of a series of Bernoulli trials which, with probability p (heads), cause the A particle to move one unit to the right, and the B particle to move one unit to the left.

The Gambler's Ruin situation can thus be likened to a race between two particles, or more picturesquely, as a race between two random walks governed by the above rules. The origin in a sense corresponds to a winning post, but in this case it is such that if the A particle gets there first, then Peter has lost his initial fortune and Paul (B) has won the game and vice versa.

Items with which the theory concerns itself are for example:

(i) The probability that Peter wins in n trials or tosses.

(ii) The probability that Paul wins, however many trials or tosses were required.

It is sufficient for many applications to ignore the time element, but if the coin were tossed at random time intervals possessing a common proper distribution function it could be relevant to ask

(iii) What is the probability that Peter has won the game by time t?

This would require us to find the probability that, given that n steps were necessary to move the B particle to the origin, the corresponding time is less than t. We should then have to multiply by the probability that A wins in n steps, and add over all possible n. In this situation it is appropriate to denote the coordinates of the particles at time t by A(t) and B(t).

We have dwelt at some length on the Gambler's Ruin because it provides a useful description in familiar terms of the ideas we wish to introduce. In fact, as is well known, it is sufficient to associate this particular problem with the motion of a single particle (vide Feller [12]).

We turn now to the random walks with which this dissertation will be concerned. As in the Gambler's Ruin, we consider particles which move on lines parallel to the x-axis, but now there may be any number of them. Also, the motion of each particle is completely independent of that of the others. In the discrete time situation the coordinates of the particles immediately after the n^{th} trials can be denoted by A(n), B(n), C(n), ..., and without confusion the initial positions A(0), B(0), C(0), ..., can for notational convenience be written simply A, B, C, ... If continuous time is in question, then the interpretation of coordinates A(t), B(t), C(t), ... is obvious. The phrase "A walk" etc., is also without ambiguity.

Returning to discrete time, if there are N particles, or competitors, their motion is determined by N independent streams of Bernoulli trials with success probabilities $p_1, p_2, \ldots, p_n$ respectively. However, success will now be associated with a step to the left, and a failure with a zero step, or the particle remaining stationary. Thus in the case of two particles, the first trial may result in the following four possibilities:

Coordinate of Particle A	A	A-1	A	A-1
Coordinate of Particle B	B	B	B-1	B-1
Probability	$(1-p_1)(1-p_2)$	$p_1(1-p_2)$	$(1-p_1)p_2$	p_1p_2

The difference between this model and that of the Gambler's Ruin is now clear. The spirit of competition is still central, however, and it is now rather the particle which first reaches the origin that is declared the winner. It should be noted, moreover, that as long as we are dealing with discrete steps, the probability of a draw is non-zero and, in some applications to be discussed later, is of considerable interest.

The study of this type of random walk originated in an attempt to study duels which portray the microscopic aspects of combat as opposed to the macroscopic viewpoint of the Lanchester-type models (see the works by Ancker [3], Brown [8] and Springall [18]). For example, the outcomes of an encounter between a submarine and an antisubmarine unit may be mutual destruction, the destruction of one, or of neither. This fits into our framework.

It was soon realized that the same framework (extended to any number of walks) has a number of other applications, all of which embody the spirit of competition in an un-warlike context. These will be discussed in some detail in the text. Here it suffices to mention the areas of inventory control, reliability and queueing theory as providing examples to which a common theory has application.

This dissertation can be considered to be divided into four major portions. The second chapter deals with a race between random walks in discrete time as described above whereas the third chapter concerns itself with these walks racing in continuous time. Pertinent questions that will be answered in these chapters deal with the time and non-time dependent state, win and draw probabilities. Another topic of interest will be the duration of the race.

The fourth chapter will deal with approximations to many of the previously obtained results. The purpose of these approximations, as will become evident, is not so much for numerical convenience as for enlightenment and insight into the operation of the system.

Chapter five deals with practical applications. Here we will show how the theory of random walks racing towards the origin can be utilized as a basic framework for explaining the operation of, and answering pertinent questions concerning, several apparently diverse systems.

II. A RACE BETWEEN RANDOM WALKS IN DISCRETE TIME

In this chapter we shall consider "competing" random walks whose motion along lines parallel to the x-axis is governed by Bernoulli trials with, in general, different "success" probabilities for each walk. As a result, the number of trials that have occurred since the start of the race will take on the role of time in the discrete sense.

2.1 Two Walks

Reviewing what was briefly described in the introduction, we shall consider two particles independently performing discrete random walks on lines parallel to the real or x-axis. These walks will be performed in such a manner that each particle may only move to the left or remain stationary according to the outcome of each trial. Denote by $A(k)$ and $B(k)$ the respective co-ordinates of the two particles immediately after the k^{th} trials.

We shall assume not only that each particle has a different starting point or co-ordinate, but also a different probability of moving to the left or remaining stationary. In other words, each walk is governed by trials with different "success" probabilities. The initial co-ordinates, $A(0)$ and $B(0)$, will have the numerical values A and B.

Let us represent by α and β the probabilities that the A and B particles respectively move one unit to the left at any trial. Then $(1-\alpha)$ and $(1-\beta)$ are the respective probabilities of remaining stationary at any trial. For a graphic representation see Figure 2.1.1 .

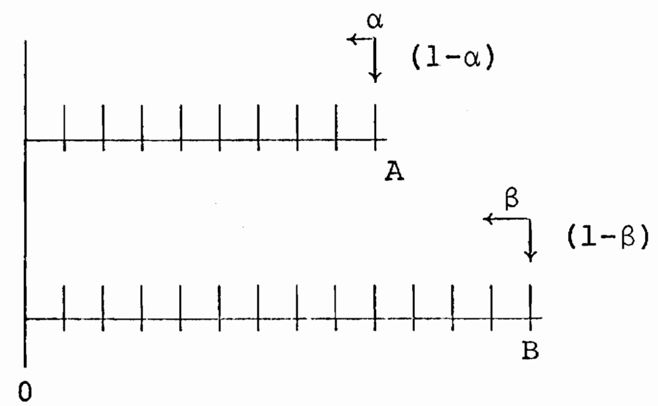

Figure 2.1.1

Graphical Representation of the System

For ease in analysis and without loss of generality, we will always assume in the sequel that $A \leq B$. No restrictions will be placed on α or β except, of course, that they be less than or equal to one, and be nonnegative.

The purpose of the following analysis will be to make probability statements concerning which of the two particles reaches the origin first. Since both particles may move at each trial, a tie is also possible. Thus, as soon as one or both of the competing particles is absorbed at the origin, the process terminates and we may say that the race is over.

As a preliminary we shall derive formulae for "state probabilities," i.e. for

$$p[i,j;k] = \Pr[A(k)=i,B(k)=j] \quad .$$

Other expressions will be derived for win and draw probabilities and also probabilities for the duration of the race.

2.1.1 State Probabilities

The event $\varepsilon \equiv [A(k)=i,B(k)=j]$ means that in k trials, the A particle has advanced $(A-i)$ units and the B particle, $(B-j)$ units.

Since each walk is acting independently of the other, then $p[i,j;k]$, for $i > 0$, $j > 0$, may be thought of as being derived from two independent binomial distributions. Hence, we have had $(A-i)$ successes or steps from one binomial distribution with parameters k and α , and $(B-j)$ successes or steps from another binomial distribution with parameters k and β . Thus, for $i,j>0$, we have,

$$(2.1.1) \qquad p[i,j;k] = \binom{k}{A-i}\alpha^{A-i}(1-\alpha)^{k-A+i} \cdot \binom{k}{B-j}\beta^{B-j}(1-\beta)^{k-B+j} \; , \; \begin{pmatrix} A\geq i>0 \\ B\geq j>0 \end{pmatrix} \quad .$$

The events to which p[0,j;k] and p[i,0;k] refer are respectively victory for A and B at the k^{th} trial; and the p's are victory probabilities. p[0,0;k] refers to a draw at the k^{th} trial. In all three cases the race terminates upon the occurrence of any of these events. Now [0,j,k] comes about as a result of [1,j,k-1] or [1,j+1, k-1] and so

$$p[0,j;k] = p[1,j;k-1]\cdot\alpha\cdot(1-\beta)+p[1,j+1,;k-1]\cdot\alpha\cdot\beta \quad .$$

If we now substitute equation (2.1.1) we obtain

$$\begin{aligned} p[0,j;k] &= \binom{k-1}{A-1}\alpha^A(1-\alpha)^{k-A}\binom{k-1}{B-j}\beta^{B-j}(1-\beta)^{k-B+j} + \\ &\quad + \binom{k-1}{A-1}\alpha^A(1-\alpha)^{k-A}\binom{k-1}{B-j-1}\beta^{B-j}(1-\beta)^{k-B+j} \\ &= \binom{k-1}{A-1}\alpha^A(1-\alpha)^{k-A}\cdot\beta^{B-j}\cdot(1-\beta)^{k-B+j}\cdot\left[\binom{k-1}{B-j}+\binom{k-1}{B-j-1}\right] . \end{aligned}$$

It can be easily verified that

$$\binom{k-1}{B-j}+\binom{k-1}{B-j-1} = \binom{k}{B-j} \quad .$$

Hence, we may write

$$\text{(2.1.2)} \quad p[0,j;k] = \binom{k-1}{A-1}\alpha^A(1-\alpha)^{k-A}\binom{k}{B-j}\beta^{B-j}(1-\beta)^{k-B+j} \quad , \; B\geq j>0 \; .$$

Similarly, by symmetry,

$$(2.1.3)\quad p[i,0;k] = \binom{k-1}{B-1}\beta^B(1-\beta)^{k-B}\binom{k}{A-i}\alpha^{A-i}(1-\alpha)^{k-A+i}\ , \quad A\geq i>0\ .$$

For the case of a draw at the k^{th} trial we must have $A(k-1) = B(k-1) = 1$. Thus

$$p[0,0;k] = p[1,1;k-1]\cdot\alpha\beta\ .$$

Employing equation (2.1.1) again yields

$$p[0,0;k] = \binom{k-1}{A-1}\alpha^{A-1}(1-\alpha)^{k-A}\binom{k-1}{B-1}\beta^{B-1}(1-\beta)^{k-B}\alpha\beta\ ,$$

or

$$(2.1.4)\quad p[0,0;k] = \binom{k-1}{A-1}\alpha^{A}(1-\alpha)^{k-A}\binom{k-1}{B-1}\beta^{B}(1-\beta)^{k-B}\ .$$

Equations (2.1.2) and (2.1.3) are the respective probabilities that the A and B particles win <u>at</u> the k^{th} trial, the co-ordinate of the B particle at that time being j , or that of the A particle being i . Equation (2.1.4) is the probability of a tie or draw occurring <u>at</u> the k^{th} trial. It should be pointed out again that as soon as one of the particles is absorbed at the origin, the process stops and we say the race is over. We also note here that if either α or β equals 1 , then the number of trials which have occurred, k, can not possibly be larger than A or B respectively.

Displaying all of the state probabilities as a set we have:

$$
(2.1.5)\quad \begin{cases}
p[i,j;k] = \begin{pmatrix} k \\ A-i \end{pmatrix} \alpha^{A-i}(1-\alpha)^{k-A+i} \cdot \\
\qquad \cdot \begin{pmatrix} k \\ B-j \end{pmatrix} \beta^{B-j}(1-\beta)^{k-B+j} \;, \quad \begin{pmatrix} A \geq i > 0 \\ B \geq j > 0 \end{pmatrix} \\[2ex]
p[0,j;k] = \begin{pmatrix} k-1 \\ A-1 \end{pmatrix} \alpha^{A}(1-\alpha)^{k-A} \begin{pmatrix} k \\ B-j \end{pmatrix} \beta^{B-j}(1-\beta)^{k-B+j} \;, \quad B \geq j > 0 \\[2ex]
p[i,0;k] = \begin{pmatrix} k-1 \\ B-1 \end{pmatrix} \beta^{B}(1-\beta)^{k-B} \begin{pmatrix} k \\ A-i \end{pmatrix} \alpha^{A-i}(1-\alpha)^{k-A+i} \;, \quad A \geq i > 0 \\[2ex]
p[0,0;k] = \begin{pmatrix} k-1 \\ A-1 \end{pmatrix} \alpha^{A}(1-\alpha)^{k-A} \begin{pmatrix} k-1 \\ B-1 \end{pmatrix} \beta^{B}(1-\beta)^{k-B} \;.
\end{cases}
$$

For completeness we now demonstrate that the functions defined by equations (2.1.5) do indeed form a proper probability distribution. This is also an instructive exercise in algebraic manipulation coupled with probabilistic argument. It is required, then, to show that

$$\sum_{i,j} p[i,j;k] = 1 \;,$$

for all possible non-negative integers k .

2.1.2 Proof of Being a Proper Probability Distribution

As stated earlier, without loss of generality we shall assume in the sequel that $A \leq B$. With this in mind, we break our proof into three cases, namely $k<A,B$; $A\leq k<B$;

$A \le B \le k$.

Case I - $k<A,B$

Here, since k is less than both A and B , it is obvious that $p[0,j;k] = p[i,0;k] = p[0,0;k] = 0$. We are left with $p[i,j;k]$ to be summed over all i and j . Since each particle can advance at most k units, we see that the total range on i is $(A-k)$ to A and the total range on j is $(B-k)$ to B .

Hence

$$\sum_i \sum_j p[i,j;k] = \sum_{i=A-k}^{A} \binom{k}{A-i} \alpha^{A-i}(1-\alpha)^{k-A+i} \cdot$$

$$\cdot \sum_{j=B-k}^{B} \binom{k}{B-j} \beta^{B-j}(1-\beta)^{k-B+j} .$$

By letting $x = A-i$ and $y = B-j$ we obtain

$$\sum_i \sum_j p[i,j;k] = \sum_{x=0}^{k} \binom{k}{x} \alpha^{x}(1-\alpha)^{k-x} \sum_{y=0}^{k} \binom{k}{y} \beta^{y}(1-\beta)^{k-y} = 1 .$$

Case II - $A \le k < B$

Here since k is less than B , it is again obvious that $p[i,0;k] = p[0,0;k] = 0$. Thus, we need only sum $\{p[i,j;k]+p[0,j;k]\}$ over all possible i and j where they appear. Again, since each particle can advance at most k units we see that the total range on i is 0 to A while the total range on j is $(B-k)$ to B. From equation (2.1.1) we obtain

$$\sum_i \sum_j p[i,j;k] = \sum_{i=1}^{A} \binom{k}{A-i} \alpha^{A-i} (1-\alpha)^{k-A+i} \cdot$$

$$\cdot \sum_{j=B-k}^{B} \binom{k}{B-j} \beta^{B-j} (1-\beta)^{k-B+j} .$$

By letting $x = A-i$ and $y = B-j$ we obtain

$$\sum_i \sum_j p[i,j;k] = \sum_{x=0}^{A-1} \binom{k}{x} \alpha^x (1-\alpha)^{k-x} \cdot$$

$$\cdot \sum_{y=0}^{k} \binom{k}{y} \beta^y (1-\beta)^{k-y} ,$$

or

$$\text{(2.1.6)} \qquad \sum_i \sum_j p[i,j;k] = \sum_{x=0}^{A-1} \binom{k}{x} \alpha^x (1-\alpha)^{k-x} .$$

From equation (2.1.2) we also obtain

$$\sum_j p[0,j;k] = \binom{k-1}{A-1} \alpha^A (1-\alpha)^{k-A} \sum_{j=B-k}^{B} \binom{k}{B-j} \beta^{B-j} (1-\beta)^{k-B+j} .$$

Upon letting $y = B-j$ again and performing the indicated sum, we are left with

$$(2.1.7) \qquad \sum_j p[0,j;k] = \binom{k-1}{A-1}\alpha^A(1-\alpha)^{k-A} \quad .$$

Now equation (2.1.6) is the probability of neither particle winning by the k^{th} trial, whereas equation (2.1.7) is the probability of the A particle winning at the k^{th} trial. In order that all possible situations be taken into account, we must consider the probabilities of the A particle winning before the k^{th} trial. We must, therefore, sum equation (2.1.7) over all trials. Hence, since the minimum number of trials required for the A particle to win is A, we may write

$$(2.1.8) \qquad \sum_{s=A}^{k} \sum_j p[0,j;s] = \sum_{s=A}^{k} \binom{s-1}{A-1}\alpha^A(1-\alpha)^{s-A} \quad .$$

Since we have taken into account all possible situations, equations (2.1.6) and (2.1.8) should sum to unity. Calling their sum S, we write

$$(2.1.9) \quad S = \sum_{x=0}^{A-1} \binom{k}{x}\alpha^x(1-\alpha)^{k-x} + \sum_{s=A}^{k} \binom{s-1}{A-1}\alpha^A(1-\alpha)^{s-A} \quad .$$

We may show that S equals one by any of several methods, but the simplest appears to be one involving the well known relationship between the number of successes in n Bernoulli trials and the waiting time required for r

successes to occur. See, for example, Feller [12] and Parzen [13]. The identity we are about to develop is extremely important and one which we will have occasion to use many times throughout this work.

Consider a sequence of Bernoulli trials with probability p for success.

Let N_n = the number of successes in n trials ,
T_r = the number of trials required to produce r successes.

Then obviously, (no matter what the distribution)

$$\Pr[N_n \geq r] = \Pr[T_r \leq n] \quad . \tag{2.1.10}$$

In our case

$$\Pr[N_n = s] = \binom{n}{s} p^s (1-p)^{n-s}$$

and

$$\Pr[T_r = t] = \binom{t-1}{r-1} p^r (1-p)^{t-r} \quad .$$

Hence, by the relationship (2.1.10) we have

$$\sum_{s=r}^{n} \binom{n}{s} p^s (1-p)^{n-s} = \sum_{t=r}^{n} \binom{t-1}{r-1} p^r (1-p)^{t-r}$$

$$\therefore 1 - \sum_{s=0}^{r-1} \binom{n}{s} p^s (1-p)^{n-s} = \sum_{t=r}^{n} \binom{t-1}{r-1} p^r (1-p)^{t-r}$$

$$\therefore \sum_{s=0}^{r-1} \binom{n}{s} p^s (1-p)^{n-s} + \sum_{t=r}^{n} \binom{t-1}{r-1} p^r (1-p)^{t-r} = 1 \quad . \tag{2.1.11}$$

Equation (2.1.11) expresses a fundamental relationship between the binomial and negative binomial distributions.

Noting that equations (2.1.9) and (2.1.11) are exactly of the same form, it immediately follows that $S = 1$. Hence, under Case II we also have a proper probability distribution.

Case III - $A \leq B \leq k$

Now the total ranges of i and j are respectively, 0 to A and 0 to B . Summing equation (2.1.1) we obtain

$$\sum_i \sum_j p[i,j;k] = \sum_{i=1}^{A} \binom{k}{A-i} \alpha^{A-i} (1-\alpha)^{k-A+i} \sum_{j=1}^{B} \binom{k}{B-j} \beta^{B-j} (1-\beta)^{k-B+j} .$$

Letting $x = A-i$ and $y = B-j$ yields

$$(2.1.12) \quad \sum_i \sum_j p[i,j;k] = \sum_{x=0}^{A-1} \binom{k}{x} \alpha^x (1-\alpha)^{k-x} \sum_{y=0}^{B-1} \binom{k}{y} \beta^y (1-\beta)^{k-y} .$$

Equation (2.1.12) may be interpreted as the probability that the race is not over after k trials.

Summing equation (2.1.2) we obtain

$$\sum_j p[0,j;k] = \binom{k-1}{A-1} \alpha^A (1-\alpha)^{k-A} \sum_{j=1}^{B} \binom{k}{B-j} \beta^{B-j} (1-\beta)^{k-B+j} .$$

Letting $y = B-j$ again yields

$$\sum_j p[0,j;k] = \binom{k-1}{A-1} \alpha^A (1-\alpha)^{k-A} \sum_{y=0}^{B-1} \binom{k}{y} \beta^y (1-\beta)^{k-y}$$

which is to be interpreted as the probability of the A particle winning the race at the k trial. By summing this equation over all possible trials previous to the k^{th}, we take into account the total probability of the A particle winning by the k^{th} trial. Performing this sum yields

$$(2.1.13)\quad \sum_{s=A}^{k} \sum_{j} p[0,j;s] = \sum_{s=A}^{k} \binom{s-1}{A-1} \alpha^{A}(1-\alpha)^{s-A} \sum_{y=0}^{B-1} \binom{s}{y} \beta^{y}(1-\beta)^{s-y} \quad .$$

By symmetry we may immediately write from equation 2.1.13 that

$$(2.1.14)\quad \sum_{s=B}^{k} \sum_{j} p[i,0;s] = \sum_{s=B}^{k} \binom{s-1}{B-1} \beta^{B}(1-\beta)^{s-B} \sum_{x=0}^{A-1} \binom{s}{x} \alpha^{x}(1-\alpha)^{s-x} \quad .$$

Similarly, from equation (2.1.14), after realizing that a tie or draw can occur anywhere between the B^{th} and k^{th} trials inclusive (remember $A \leq B$), we may write

$$(2.1.15)\quad \sum_{s=B}^{k} p[0,0;s] = \sum_{s=B}^{k} \binom{s-1}{A-1} \alpha^{A}(1-\alpha)^{s-A} \binom{s-1}{B-1} \beta^{B}(1-\beta)^{s-B} \quad .$$

The task at hand is to now show that equations (2.1.12) through (2.1.15) sum to unity. To do this we will have occasion to utilize the relationship between the binomial and negative binomial distributions, (2.1.11). We will use this relationship in equations (2.1.12) through (2.1.14) by writing each binomial distribution in terms of its negative binomial complement. Hence, performing this substitution in (2.1.12) yields

$$\sum_{x=0}^{A-1}\binom{k}{x}\alpha^{x}(1-\alpha)^{k-x}\sum_{y=0}^{B-1}\binom{k}{y}\beta^{y}(1-\beta)^{k-y} = \left[1-\sum_{x=A}^{k}\binom{x-1}{A-1}\alpha^{A}(1-\alpha)^{x-A}\right]\cdot$$

$$\cdot\left[1-\sum_{y=B}^{k}\binom{y-1}{B-1}\beta^{B}(1-\beta)^{y-B}\right]$$

or

$$(2.1.16)\quad = 1+\sum_{x=A}^{k}\binom{x-1}{A-1}\alpha^{A}(1-\alpha)^{x-A}\sum_{y=B}^{k}\binom{y-1}{B-1}\beta^{B}(1-\beta)^{y-B} -$$

$$-\sum_{x=A}^{k}\binom{x-1}{A-1}\alpha^{A}(1-\alpha)^{x-A}-\sum_{y=B}^{k}\binom{y-1}{B-1}\beta^{B}(1-\beta)^{y-B}\quad .$$

Equation (2.1.13) may be written as

$$\sum_{s=A}^{k}\binom{a-1}{A-1}\alpha^{A}(1-\alpha)^{s-A}\sum_{y=0}^{B-1}\binom{s}{y}\beta^{y}(1-\beta)^{s-y} = \sum_{s=A}^{k}\binom{s-1}{A-1}\alpha^{A}(1-\alpha)^{s-A}\cdot$$

$$\cdot\left[1-\sum_{y=B}^{s}\binom{y-1}{B-1}\beta^{B}(1-\beta)^{y-B}\right]$$

or

$$(2.1.17) = \sum_{s=A}^{k}\binom{s-1}{A-1}\alpha^{A}(1-\alpha)^{s-A}-\sum_{s=A}^{k}\binom{s-1}{A-1}\alpha^{A}(1-\alpha)^{s-A}\sum_{y=B}^{s}\binom{y-1}{B-1}\beta^{B}(1-\beta)^{y-B}.$$

Similarly, equation (2.1.14) can be written as

$$(2.1.18)\quad \sum_{s=B}^{k}\binom{s-1}{B-1}\beta^{B}(1-\beta)^{s-B}\sum_{x=0}^{A-1}\binom{s}{x}\alpha^{x}(1-\alpha)^{s-x} = \sum_{s=B}^{k}\binom{s-1}{B-1}\beta^{B}(1-\beta)^{s-B} -$$

$$- \sum_{s=B}^{k}\binom{s-1}{B-1}\beta^{B}(1-\beta)^{s-B}\sum_{x=A}^{s}\binom{x-1}{A-1}\alpha^{A}(1-\alpha)^{x-A} \quad .$$

Equation (2.1.15) will be left alone since it already is in the negative binomial form.

Since $\binom{n}{r} = 0$ by definition for all $r > n$, nothing will be changed in the above equations if we replace B by A whenever it occurs as a lower index of summation over s. By doing this, we shall be able to write the sum of these equations in a compact form. Calling the sum of equations (2.1.15) through (2.1.18) S, we write

$$S = 1+\sum_{x=A}^{k}\binom{x-1}{A-1}\alpha^{A}(1-\alpha)^{x-A}\sum_{x=B}^{k}\binom{y-1}{B-1}\beta^{B}(1-\beta)^{y-B} - \sum_{x=A}^{k}\binom{x-1}{A-1}\alpha^{A}(1-\alpha)^{x-A} -$$

$$- \sum_{y=B}^{k}\binom{y-1}{B-1}\beta^{B}(1-\beta)^{y-B} + \sum_{s=A}^{k}\left[\binom{s-1}{A-1}\alpha^{A}(1-\alpha)^{s-A} - \binom{s-1}{A-1}\alpha^{A}(1-\alpha)^{s-A}\cdot\right.$$

$$\cdot\sum_{y=B}^{s}\binom{y-1}{B-1}\beta^{B}(1-\beta)^{y-B} + \binom{s-1}{B-1}\beta^{B}(1-\beta)^{s-B} - \binom{s-1}{B-1}\beta^{B}(1-\beta)^{s-B}\cdot$$

$$\left.\cdot\sum_{x=A}^{s}\binom{x-1}{A-1}\alpha^{A}(1-\alpha)^{x-A} + \binom{s-1}{A-1}\alpha^{A}(1-\alpha)^{s-A}\binom{s-1}{B-1}\beta^{B}(1-\beta)^{s-B}\right] \quad .$$

Noting that the third and fourth summations in S are the negatives of the first and third terms within the accolades simplifies S to

$$S = 1+\sum_{x=A}^{k}\binom{x-1}{A-1}\alpha^A(1-\alpha)^{x-A}\sum_{y=B}^{k}\binom{y-1}{B-1}\beta^B(1-\beta)^{y-B}+\sum_{s=A}^{k}\left[-\binom{s-1}{A-1}\alpha^A(1-\alpha)^{s-A}\cdot\right.$$

$$\cdot\sum_{y=B}^{s}\binom{y-1}{B-1}\beta^B(1-\beta)^{y-B}-\binom{s-1}{B-1}\beta^B(1-\beta)^{s-B}\sum_{x=A}^{s}\binom{x-1}{A-1}\alpha^A(1-\alpha)^{x-A}+$$

$$\left.+\binom{s-1}{A-1}\alpha^A(1-\alpha)^{s-A}\binom{s-1}{B-1}\beta^B(1-\beta)^{s-B}\right] .$$

For brevity in what is to follow, let us simplify notation by setting

$$a_x = \binom{x-1}{A-1}\alpha^A(1-\alpha)^{x-A} \quad \text{and} \quad b_x = \binom{x-1}{B-1}\beta^B(1-\beta)^{x-B} .$$

We may now rewrite S as

$$S = 1+\sum_{x=A}^{k}a_x\sum_{y=B}^{k}b_y+\sum_{s=A}^{k}\left[-a_s\sum_{y=B}^{s}b_y-b_s\sum_{x=A}^{s}a_x+a_sb_s\right] .$$

By adding and subtracting the factor $\sum\limits_{x=A}^{s} a_x \sum\limits_{y=B}^{s} b_y$ within the accolades, S becomes after factoring,

$$S = 1+\sum_{x=A}^{k}a_x\sum_{y=B}^{k}b_y+\sum_{s=A}^{k}\left\{\left[\sum_{x=A}^{s}a_x-a_s\right]\left[\sum_{y=B}^{s}b_y-b_s\right]-\sum_{x=A}^{s}a_x\sum_{y=B}^{s}b_y\right\}$$

or

$$S = 1+\sum_{x=A}^{k}a_x\sum_{y=B}^{k}b_y+\sum_{s=A}^{k}\left[\sum_{x=A}^{s-1}a_x\sum_{y=B}^{s-1}b_y-\sum_{x=A}^{s}a_x\sum_{y=B}^{s}b_y\right] . \tag{2.1.19}$$

Now, by taking the upper limit of the summation over s (i.e. the terms within the accolades by setting $s = k$), we notice that the product of the last two sums within accolades will cancel the product of the two sums immediately following the 1. After this cancelation we are left with

$$S = 1+\sum_{x=A}^{k-1} a_x \sum_{y=B}^{k-1} b_y + \sum_{s=A}^{k-1}\left[\sum_{x=A}^{s-1} a_x \sum_{y=B}^{s-1} b_y - \sum_{x=A}^{s} a_x \sum_{y=B}^{s} b_y\right] .$$

Again, by taking the upper limit of summation over s within the accolades, the same two sums will cancel the ones immediately following the 1. We continue this process of cancelation in the same manner until the upper limit of summation over s becomes B. When this occurs, S will become

$$S = 1+b_B \sum_{x=A}^{B} a_x + \sum_{s=A}^{B}\left[\sum_{x=A}^{s-1} a_x \cdot 0 - \sum_{x=A}^{s} a_x \sum_{y=B}^{s} b_y\right] .$$

Hence

$$S = 1+b_B \sum_{x=A}^{B} a_x - \sum_{s=A}^{B}\left[\sum_{x=A}^{s} a_x \sum_{y=B}^{s} b_y\right] .$$

Upon taking the upper limit of summation over s again we obtain,

$$S = 1+b_B \sum_{x=A}^{B} a_x - b_B \sum_{x=A}^{B} a_x - \sum_{s=A}^{B-1}\left[\sum_{x=A}^{s} a_x \cdot 0\right] .$$

$\therefore$ S = 1 as was to be shown, and thus under Case III we also have a proper probability distribution.

We have now shown that equations (2.1.5) do indeed form a proper probability distribution for <u>all</u> choices of k , the number of trials.

2.1.3 Determining Win and Draw Probabilities

In the situations considered so far, we have been concerned only with the cases of: the race not being over; the A particle winning; the B particle winning; or a draw occurring; but all at a specified number of trials, k . It is also of considerable interest to sum over all possible trials thereby obtaining the probabilities of the A or B particles eventually winning or of a tie occurring however many trials are involved.

Equation (2.1.2) gives $p(0,j;k)$, the probability of the A particle winning the race <u>at</u> the k^{th} trial, the co-ordinate of the B walk then being j . Thus, by summing equation (2.1.2) over all possible k we have

$$(2.1.20)\quad p[0,j] = \sum_{k=A}^{\infty} \binom{k-1}{A-1} \alpha^{A} (1-\alpha)^{k-A} \binom{k}{B-j} \beta^{B-j} (1-\beta)^{k-B+j} \quad (B \geq j > 0)\ ,$$

which is the probability that the A particle eventually wins and that the co-ordinate of the B particle at that time is j . From equation (2.1.3) we obtain

$$(2.1.21)\quad p[i,0] = \sum_{k=B}^{\infty} \binom{k-1}{B-1} \beta^{B}(1-\beta)^{k-B} \binom{k}{A-i} \alpha^{a-i}(1-\alpha)^{k-A+i} \qquad (A\geq i>0)$$

which has a similar interpretation. Summing equation (2.1.4) over all k yields

$$(2.1.22)\quad P_{AB} = \sum_{k=B}^{\infty} \binom{k-1}{A-1} \alpha^{A}(1-\alpha)^{k-A} \binom{k-1}{B-1} \beta^{B}(1-\beta)^{k-B} \quad .$$

which is the probability of a draw or tie eventually occurring.

Expressions for P_A and P_B, the unconditional probabilities of the A or B particles winning, can be obtained directly by summing equation (2.1.20) over all j and equation (2.1.21) over all i. Hence, by summing (2.1.19) from $j = 1$ to B we obtain

$$P_A = \sum_{j=1}^{B} p[0,j] = \sum_{k=A}^{\infty} \binom{k-1}{A-1} \alpha^{A}(1-\alpha)^{k-A} \sum_{j=1}^{B} \binom{k}{B-j} \beta^{B-j}(1-\beta)^{k-B+j} \quad .$$

Letting $y = B-j$ yields

$$(2.1.23)\quad P_A = \sum_{k=A}^{\infty} \binom{k-1}{A-1} \alpha^{A}(1-\alpha)^{k-A} \sum_{y=0}^{B-1} \binom{k}{y} \beta^{y}(1-\beta)^{k-y} \quad .$$

Likewise, by symmetry we also obtain

$$(2.1.24)\quad P_B = \sum_{k=B}^{\infty} \binom{k-1}{B-1} \beta^{B}(1-\beta)^{k-B} \sum_{x=0}^{A-1} \binom{k}{x} \alpha^{x}(1-\alpha)^{k-x} \quad .$$

Equations (2.1.22) through (2.1.24) are the ones with which we will be primarily concerned.

Before proceeding any farther, we must take note that equation (2.1.12), which is interpreted as the probability of the race not being over in k trials, clearly tends to zero for an infinite number of trials. This implies that an infinitely prolonged race has zero probability of occurring.

2.1.4 Alternative Forms for the Win and Draw Probabilities

It is possible to express neatly equations (2.1.22) through (2.1.24) in terms of hypergeometric functions of the first type, viz

$$ {}_2F_1(\alpha,\beta;\gamma:z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n z^n}{(\gamma)_n n!} \quad , \qquad |z| < 1 \ , $$

where

$$ (2.1.25) \quad (\lambda)_n = \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1) = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \ . $$

For a complete discussion of hypergeometric functions see Abramowitz [1], Bailey [5], Bateman [6] and Rainville [14].

Let us now interchange the order of summation in (2.1.23) yielding

$$P_A = \alpha^A(1-\alpha)^{-A}\sum_{y=0}^{B-1}\beta^y(1-\beta)^{-y}\sum_{k=A}^{\infty}\binom{k-1}{A-1}\binom{k}{y}[(1-\alpha)(1-\beta)]^k \quad .$$

Rewriting the binomial coefficients in terms of Gamma functions yields

$$P_A = \frac{\alpha^A(1-\alpha)^{-A}}{\Gamma(A)}\sum_{y=0}^{B-1}\frac{\beta^y(1-\beta)^{-y}}{\Gamma(y+1)}\sum_{k=A}^{\infty}\frac{\Gamma(k)\Gamma(k+1)[(1-\alpha)(1-\beta)]^k}{\Gamma(k-A+1)\Gamma(k-y+1)} \quad .$$

Making the change of variables $n = k-A$ yields

$$P_A = \frac{\alpha^A(1-\alpha)^{-A}}{\Gamma(A)}\sum_{y=0}^{B-1}\frac{\beta^y(1-\beta)^{-y}}{\Gamma(y+1)}\sum_{n=0}^{\infty}\frac{\Gamma(A+n)\Gamma(A+n+1)[(1-\alpha)(1-\beta)]^{A+n}}{\Gamma(n+1)\Gamma(A-x+n+1)} \quad .$$

Utilizing the notation (2.1.25) we obtain

$$P_A = \alpha^A\sum_{y=0}^{B-1}\frac{\Gamma(A+1)\beta^y(1-\beta)^{A-y}}{\Gamma(y+1)\Gamma(A-x+1)}\sum_{n=0}^{\infty}\frac{(A)_n(A+1)_n[(1-\alpha)(1-\beta)]^n}{(A-x+1)_n\ n!} \quad .$$

Hence,

$$(2.1.26)\quad P_A = \alpha^A\sum_{y=0}^{B-1}\binom{A}{y}\beta^y(1-\beta)^{A-y}\ {}_2F_1[A,A+1;A-x+1:(1-\alpha)(1-\beta)] \quad .$$

Similarly,

$$(2.1.27)\quad P_B = \beta^B\sum_{x=0}^{A-1}\binom{B}{x}\alpha^x(1-\alpha)^{B-x}\ {}_2F_1[B,B+1;B-x+1:(1-\alpha)(1-\beta)] \quad .$$

By following the same procedure outlined above we may also obtain P_{AB} in the following pleasing form,

$$(2.1.28)\quad P_{AB} = \binom{B-1}{A-1}\alpha^{A}(1-\alpha)^{B-A}\beta^{B}\ {}_{2}F_{1}[B,B;B-A+1:(1-\alpha)(1-\beta)] \quad .$$

Apart from their aesthetic appeal, these forms give us the opportunity of invoking, if necessary, the known properties of the hypergeometric functions.

2.1.5 Duration of the Race

The purpose of this section is to obtain the probability distribution for the duration of the race, i.e. the number of trials until the race is over. We recall that this will occur if any of three mutually exclusive events occur namely, either the A or B particle wins or a tie occurs.

At this point we have at our disposal expressions for $p[0,j;k]$ and $p[i,0;k]$ (viz (2.1.2) and (2.1.3)), the respective probabilities of the A and B particles winning at the k^{th} trial, the other particle then having a specified co-ordinate. By summing these probabilities over all i and j greater than zero, we will obtain the unconditional probabilities of each particle winning at the k^{th} trial. Hence, summing (2.1.2) over all $j > 0$, and (2.1.3) over all $i > 0$, yields

$$P_A^{(k)} = \sum_{j>0} p[0,j;k] = \binom{k-1}{A-1}\alpha^{A}(1-\alpha)^{k-A}\sum_{y=0}^{B-1}\binom{k}{y}\beta^{y}(1-\beta)^{k-y} \quad ;$$

$$P_B^{(k)} = \sum_{i>0} p[i,0;k] = \binom{k-1}{B-1} \beta^B (1-\beta)^{k-B} \sum_{x=0}^{A-1} \binom{k}{x} \alpha^x (1-\alpha)^{k-x} \quad .$$

We merely rewrite (2.1.4) as

$$P_{AB}^{(k)} = p[0,0;k] = \binom{k-1}{A-1} \alpha^A (1-\alpha)^{k-A} \binom{k-1}{B-1} \beta^B (1-\beta)^{k-B} \quad ,$$

which we recall is the probability of a tie occurring at the k^{th} trial.

Since $P_A^{(k)}$, $P_B^{(k)}$ and $P_{AB}^{(k)}$ represent the probabilities of mutually exclusive events, the probability that the race terminates at the k^{th} trial can easily be written as

$$P^{(k)} = P_A^{(k)} + P_B^{(k)} + P_{AB}^{(k)} \quad .$$

Hence

$$\begin{aligned} (2.1.29) \quad P^{(k)} = & \binom{k-1}{A-1} \alpha^A (1-\alpha)^{k-A} \sum_{y=0}^{B-1} \binom{k}{y} \beta^y (1-\beta)^{k-y} + \\ & + \binom{k-1}{B-1} \beta^B (1-\beta)^{k-B} \sum_{x=0}^{A-1} \binom{k}{x} \alpha^x (1-\alpha)^{k-x} + \\ & + \binom{k-1}{A-1} \alpha^A (1-\alpha)^{k-A} \binom{k-1}{B-1} \beta^B (1-\beta)^{k-B} \quad . \end{aligned}$$

To obtain this probability in a more compact form, we shall write the summation of the binomial probabilities in (2.1.29) in terms of their negative binomial complements as given in the relationship (2.1.11). Hence, by applying this relationship to (2.1.29) we obtain after utilizing the shorthand notation for the negative binomial introduced in Section 2.1.2,

$$P^{(k)} = a_k[1-\sum_{y=B}^{k} b_y]+b_k[1-\sum_{x=A}^{k} a_x]+a_k b_k \quad .$$

By adding and subtracting out the product

$$[1-\sum_{x=A}^{k} a_x][1-\sum_{y=B}^{k} b_y] \quad ,$$

we obtain, after factoring,

$$P^{(k)} = \left[a_k+\left(1-\sum_{x=A}^{k} a_x\right)\right]\left[b_k+\left(1-\sum_{y=B}^{k} b_y\right)\right]-\left(1-\sum_{x=A}^{k} a_x\right)\left(1-\sum_{y=B}^{k} b_y\right) \quad .$$

This may be further simplified to

$$P^{(k)} = \left(1-\sum_{x=A}^{k-1} a_x\right)\left(1-\sum_{y=B}^{k-1} b_y\right)-\left(1-\sum_{x=A}^{k} a_x\right)\left(1-\sum_{y=B}^{k} b_y\right) \quad .$$

Changing these negative binomial sums back to binomials by (2.1.11) we obtain, after returning to standard notation,

$$(2.1.30)\quad P^{(k)} = \left[\sum_{x=0}^{A-1}\binom{k-1}{x}\alpha^x(1-\alpha)^{k-1-x}\right]\left[\sum_{y=0}^{B-1}\binom{k-1}{y}\beta^y(1-\beta)^{k-1-y}\right] -$$

$$-\left[\sum_{x=0}^{A-1}\binom{k}{x}\alpha^x(1-\alpha)^{k-x}\right]\left[\sum_{y=0}^{B-1}\binom{k}{y}\beta^y(1-\beta)^{k-y}\right] \quad .$$

This appears to be in a more useful form for calculations than (2.1.29).

It should be pointed out that the product of the first two brackets and the product of the second two brackets in (2.1.30) are merely the probabilities of the race not being over in $(k-1)$ and k trials respectively [vide equation (2.1.12)]. From this viewpoint it seems that we could have written (2.1.30) down directly. For notice that

$$P^{(k)} = \Pr[\text{race }\underline{\text{over}}\text{ by } k \text{ trials}] - \\ -\Pr[\text{race }\underline{\text{over}}\text{ by } (k-1) \text{ trials}] \quad ,$$

or

$$P^{(k)} = 1-\Pr[\text{race }\underline{\text{not over}}\text{ by } k \text{ trials}]- \\ -1+\Pr[\text{race }\underline{\text{not over}}\text{ by } (k-1) \text{ trials}] \quad .$$

Hence,

$$(2.1.31)\quad P^{(k)} = \Pr[\text{race }\underline{\text{not over}}\text{ by } (k-1) \text{ trials}]- \\ - \Pr[\text{race }\underline{\text{not over}}\text{ by } k \text{ trials}] \quad .$$

Thus, if we substitute equation (2.1.12) for these probabilities, we will obtain (2.1.30) exactly.

We expect to find that (2.1.29), or (2.1.30), do indeed form a proper probability distribution. That is to say, their summation over all possible k must equal unity. To demonstrate that this is true, let us consider (2.1.30) and introduce a shorthand notation. Set

$$(2.1.32)\quad \alpha(n) = \sum_{x=0}^{A-1} \binom{n}{x} \alpha^x (1-\alpha)^{n-x} \quad \text{and} \quad \beta(n) = \sum_{x=0}^{B-1} \binom{n}{x} \beta^x (1-\beta)^{n-x} \quad ,$$

and notice that

$$\alpha(n) = 1 \quad , \quad n \leq A-1 \quad ,$$

$$\beta(n) = 1 \quad , \quad n \leq B-1 \quad .$$

Thus, utilizing this notation, (2.1.30) becomes

$$(2.1.33)\quad P^{(k)} = \alpha(k-1)\beta(k-1) - \alpha(k)\beta(k) \quad .$$

Let us now sum (2.1.33) over all possible k . Since the race cannot possibly terminate in fewer than A trials, we see the range on k is A , ∞ , hence

$$\sum_{k=A}^{\infty} P^{(k)} = \sum_{k=A}^{\infty} \alpha(k-1)\beta(k-1) - \sum_{k=A}^{\infty} \alpha(k)\beta(k) \quad ,$$

or

$$\sum_{k=A}^{\infty} P^{(k)} = \alpha(A-1)\beta(A-1) + \sum_{k=A+1}^{\infty} \alpha(k-1)\beta(k-1) - \sum_{k=A}^{\infty} \alpha(k)\beta(k) \quad .$$

We now notice that the two summations cancel each other. Thus,

$$\sum_{k=A}^{\infty} P^{(k)} = \alpha(A-1)\beta(A-1) = 1 \quad ,$$

and we have demonstrated (2.1.30) or equivalently, (2.1.29), to be proper distributions.

Besides having the probability distribution for the duration of the race at his disposal, one might also desire its expected duration, E(k). This is most easily obtained by multiplying (2.1.33) by k and then summing over all possible k. Performing this operation yields,

$$\sum_{k=A}^{\infty} kP^{(k)} = \sum_{k=A}^{\infty} k\alpha(k-1)\beta(k-1) - \sum_{k=A}^{\infty} k\alpha(k)\beta(k) \quad ,$$

or

$$E(k) = A\alpha(A-1)\beta(A-1) + \sum_{k=A+1}^{\infty} k\alpha(k-1)\beta(k-1) - \sum_{k=A}^{\infty} k\alpha(k)\beta(k) \quad ,$$

or

$$E(k) = A + \sum_{k=A}^{\infty} \alpha(k)\beta(k) \quad .$$

Returning to the original notation we obtain,

$$(2.1.34) \quad E(k) = A + \sum_{k=A}^{\infty} \left[\sum_{x=0}^{A-1} \binom{k}{x} \alpha^{x}(1-\alpha)^{k-x}\right]\left[\sum_{x=0}^{B-1} \binom{k}{x} \beta^{x}(1-\beta)^{k-x}\right] \quad .$$

The leading constant, A, should not come as a surprise, since we know that the race cannot possibly terminate before A trials have occurred. The rest of the formula reflects the probabilistic behavior of the process.

The second moment, $E(k^2)$, is obtained in a similar manner. We merely multiply (2.1.33) by k^2 and sum over all k. Performing the same operations as above yields,

$$E(k^2) = A^2 + \sum_{k=A+1}^{\infty} k^2\alpha(k-1)\beta(k-1) - \sum_{k=A}^{\infty} k^2\alpha(k)\beta(k) \quad .$$

Further simplification results in,

$$E(k^2) = A^2 + \sum_{k=A}^{\infty} (2k+1)\alpha(k)\beta(k) \quad .$$

Returning to the original notation we obtain,

$$(2.1.35) \quad E(k^2) = A^2 + \sum_{k=A}^{\infty} \left\{ (2k+1) \left[\sum_{x=0}^{A-1} \binom{k}{x} \alpha^x (1-\alpha)^{k-x} \right] \cdot \left[\sum_{x=0}^{B-1} \binom{k}{x} \beta^x (1-\beta)^{k-x} \right] \right\} \quad .$$

2.2 Three or More Walks

In this section, all of the assumptions and methodology utilized in the first half of this chapter will be used again but in a more general sense. Analogous formulas for the win and, in this case, multiple draw probabilities, will be developed. Proof of forming proper probability distributions will be limited to the three walk case because of the complexity of the algebra. However, it will be obvious to the reader that it could be extended to any number of walks given enough time and paper. The method used will be general enough.

Consider the situation of n walks performed on the real axis as described in Section 2.1 , and as graphically represented in Figure 2.2.1 . As before, we will assume

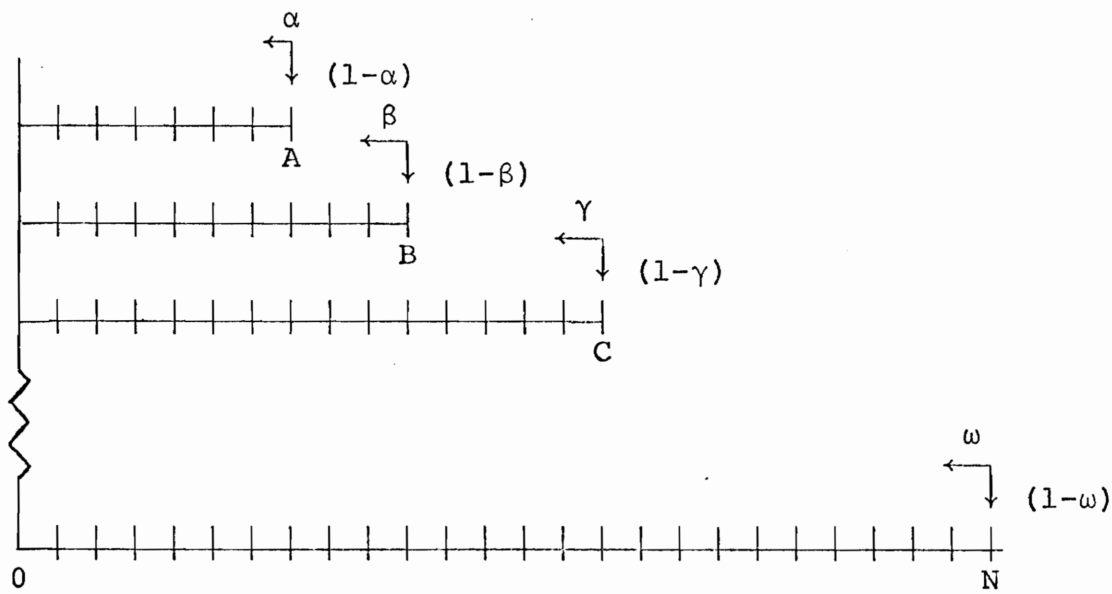

Figure 2.2.1

Graphical Representation of the System

without loss of generality that $A \le B \le C \le \ldots \le N$. Again we place no restrictions on α, β, γ, ..., ω, the respective probabilities of each particle moving one unit to the left at each trial. For brevity in notation in what is to follow, we shall now denote $(1-\alpha)$, $(1-\beta)$, ..., $(1-\omega)$ by α', β', ..., ω'.

Since most of the following analysis will be carried out with three walks, the general extension to n walks will be inserted wherever it appears that the result is not obvious. It should be pointed out that since we are still considering these walks in discrete time, ties between two, three or all of the walks are possible. Numerically speaking however, these probabilities become very small as the number of walks considered increases.

By extension of our previous notation we shall denote by $p[i,j,\ell,\ldots,n;k]$, the probability that immediately after the k^{th} trial, the A, B, C, ... and N particles have co-ordinates i, j, ℓ, ... and n respectively. Thus

$$p[i,j,\ell,\ldots n;k] = \Pr[A(k)=i,B(k)=j,C(k)=\ell,\ldots,N(k)=n] \quad ,$$

where, as before, $A(k)$ is the random variable defining the co-ordinate of the A particle immediately after the k^{th} trial, similarly for $B(k)$, $C(k)$, ... etc.

2.2.1 State Probabilities

As discussed in section 2.1.1 , $p[i,j,\ell,\ldots,n;k]$ is the product of n independent binomial distributions just as long as none of the co-ordinates of the walks is equal to zero. Hence, in the case of three walks we have

$$(2.2.1)\quad p[i,j,\ell;k] = \binom{k}{A-i}\alpha^{A-i}(1-\alpha)^{k-A+i}\binom{k}{B-j}\beta^{B-j}(1-\beta)^{k-B+j} \cdot$$

$$\cdot\binom{k}{C-\ell}\gamma^{C-\ell}(1-\gamma)^{k-C+\ell} \qquad \begin{array}{l} 0<i\leq A \\ 0<j\leq B \\ 0<\ell\leq C \ . \end{array}$$

The extension to n walks is obvious here. The probability of any one particular particle's victory (that it reaches the origin before the competitors) is easily obtained as in the two walk case. Thus, for example,

$$p[0,j,\ell;k] = p[1,j,\ell;k]\alpha\beta'\gamma' + p[1,j+1,\ell;k]\alpha\beta\gamma' +$$

$$+ p[1,j,\ell+1;k]\alpha\beta'\gamma + p[1,j+1,\ell+1;k]\alpha\beta\gamma$$

$$= \binom{k-1}{A-1}\alpha^{A}\alpha'^{\,k-A}\beta^{B-j}\beta'^{\,k-B+j}\gamma^{C-\ell}\gamma'^{\,k-C+\ell}\left[\binom{k-1}{B-j}\binom{k-1}{C-\ell} + \right.$$

$$\left. + \binom{k-1}{B-j-1}\binom{k-1}{C-\ell} + \binom{k-1}{B-j}\binom{k-1}{C-\ell-1} + \binom{k-1}{B-j-1}\binom{k-1}{C-\ell-1}\right] .$$

By twice using the fact that $\binom{r}{s-1}+\binom{r}{s} = \binom{r+1}{s}$ after factoring the binomial coefficients in the brackets, we have

$$\text{(2.2.2)} \quad p[0,j,\ell;k] = \binom{k-1}{A-1}\alpha^{A}\alpha'^{\,k-A}\binom{k}{B-j}\beta^{B-j}\beta'^{\,k-B+j}\binom{k}{C-\ell}\cdot$$

$$\cdot\gamma^{C-\ell}\gamma'^{\,k-C+\ell} \quad , \quad \begin{pmatrix} 0<j\le B \\ 0<\ell\le C \end{pmatrix} .$$

Similarly by symmetry

$$\text{(2.2.3)} \quad p[i,0,\ell;k] = \binom{k}{A-i}\alpha^{A-i}\alpha'^{\,k-A+i}\binom{k-1}{B-1}\beta^{B}\beta'^{\,k-B}\binom{k}{C-\ell}\cdot$$

$$\cdot\gamma^{C-\ell}\gamma'^{\,k-C+\ell} \quad , \quad \begin{pmatrix} 0<i\le A \\ 0<\ell\le C \end{pmatrix} ,$$

$$\text{(2.2.4)} \quad p[i,j,0;k] = \binom{k}{A-i}\alpha^{A-i}\alpha'^{\,k-A+i}\binom{k}{B-j}\beta^{B-j}\beta'^{\,k-B+j}\binom{k-1}{C-1}\cdot$$

$$\cdot\gamma^{C}\gamma'^{\,k-C} \quad , \quad \begin{pmatrix} 0<i<A \\ 0<j\le B \end{pmatrix} .$$

A similar argument yields the probabilities of a tie between two or all three of the walks at the k^{th} trial. Thus,

$$(2.2.5)\quad p[0,0,\ell;k] = \binom{k-1}{A-1}\alpha^{A}\alpha'^{k-A}\binom{k-1}{B-1}\beta^{B}\beta'^{k-B}\binom{k}{C-\ell}\cdot$$

$$\cdot\,\gamma^{C-\ell}\gamma'^{k-C+\ell} \qquad 0<\ell\leq C$$

$$(2.2.6)\quad p[0,j,0;k] = \binom{k-1}{A-1}\alpha^{A}\alpha'^{k-A}\binom{k}{B-j}\beta^{B-j}\beta'^{k-B+j}\binom{k-1}{C-1}\cdot$$

$$\cdot\,\gamma^{C}\gamma'^{k-C} \qquad 0<j\leq B$$

$$(2.2.7)\quad p[i,0,0;k] = \binom{k}{A-i}\alpha^{A-i}\alpha'^{k-A+i}\binom{k-1}{B-1}\beta^{B}\beta'^{k-B}\binom{k-1}{C-1}\cdot$$

$$\cdot\,\gamma^{C}\gamma'^{k-C} \qquad 0<i\leq A$$

$$(2.2.8)\quad p[0,0,0;k] = \binom{k-1}{A-1}\alpha^{A}\alpha'^{k-A}\binom{k-1}{B-1}\beta^{B}\beta'^{k-B}\binom{k-1}{C-1}\gamma^{C}\gamma'^{k-C}\quad .$$

The general extension of these formulas is obvious for the situation of more than three walks. Each formula is the product of binomial and negative binomial probabilities. If the co-ordinate of any particle is zero, then that part of the formula is represented negative binomially while if the co-ordinate of any other particle is not zero, its part of the formula is represented binomially. For example, the probability of the A particle winning at the k^{th} trial when five particles are racing and the co-ordinates of the other four are specified is

$$p[0,j,\ell,d,e;k] = \binom{k-1}{A-1}\alpha^{A}\alpha'^{k-A}\binom{k}{B-j}\beta^{B-j}\beta'^{k-B+j}\binom{k}{C-\ell}\gamma^{C-\ell}\gamma'^{k-C+\ell}\cdot$$

$$\cdot\binom{k}{D-d}\delta^{D-d}\delta'^{k-D+d}\cdot\binom{k}{E-e}\varepsilon^{E-e}\varepsilon'^{k-E+e}$$

where the extra parameter sets for the two additional walks are D , δ and E , ε respectively.

We now show that the sum of equations (2.2.1) through (2.2.8) over all i , j and ℓ equals unity for any k , thus confirming that these equations define a proper probability distribution.

2.2.2 Proof of Being a Proper Probability Distribution

Due to the complexity of the algebra and notation, this analysis and proof will be done for three walks only. The method developed however, will indicate the obvious extensions. We assume without loss of generality that $A\leq B\leq C$ and so have to deal with four cases, namely $k<A\leq B\leq C$; $A\leq k<B\leq C$; $A\leq B\leq k<C$; $A\leq B\leq C\leq k$. It will be shown that the first three of these cases reduce to the three cases of Section 2.1.2 . The fourth case will be dealt with in a manner which is simply a general extension of the methods in Section 2.1.2 .

Case I - $k<A\leq B\leq C$

Here the maximum possible number of steps is k . Hence $p[0,j,\ell;k] = p[i,0,\ell;k] = p[i,j,0;k] = p[0,0,\ell;k] = p[o,j,0;k] = p[i,0,0;k] = p[0,0,0;k] = 0$. All of these probabilities are zero except the ones given by equation (2.2.1) because none of the particles can possibly reach zero in k trials. Hence, we need only show that the summation over all i , j and ℓ in (2.2.1) is unity to confirm that it defines a probability distribution. Performing this summation yields

$$\sum_{i,j,\ell} p[i,j,\ell;k] = \sum_{i=A-k}^{A} \binom{k}{A-i} \alpha^{A-i}\alpha'^{k-A+i} \sum_{j=B-k}^{B} \binom{k}{B-j} \beta^{B-j}\beta'^{k-B+j} \cdot \sum_{\ell=C-k}^{C} \binom{k}{C-\ell} \gamma^{C-\ell}\gamma'^{k-C+\ell} .$$

Letting $x = A-i$; $y = B-j$ and $z = C-\ell$ yields

$$\sum_{i,j,\ell} p[i,j,\ell;k] = \sum_{x=0}^{k} \binom{k}{x} \alpha^x\alpha'^{k-x} \sum_{y=0}^{k} \binom{k}{y} \beta^y\beta'^{k-y} \sum_{z=0}^{k} \binom{k}{z} \gamma^z\gamma'^{k-z}$$

which is obviously 1 .

Case II - $A\leq k<B\leq C$

Here since $k<B,C$, then only the A particle could possibly win by the k^{th} trial, or the race would not be

over.

Hence, summing equation (2.2.1) over all possible i , j and ℓ and making the usual change of variables yields

$$(2.2.9) \qquad \sum_{i,j,\ell} p[i,j,\ell;k] = \sum_{x=0}^{A-1} \binom{k}{x} \alpha^x \alpha'^{k-x}$$

which is interpreted as the race not being over in k trials.

Similarly, summing (2.2.2) over all j and ℓ and changing variables again yields

$$\sum_{j,\ell} p[0,j,\ell;k] = \binom{k-1}{A-1} \alpha^A \alpha'^{k-A}$$

which is interpreted as the A particle winning at the k^{th} trial. Taking into account the possibility of A's winning prior to the k^{th} trial by summing up to k yields

$$(2.2.10) \qquad \sum_{x=A}^{k} \sum_{j,\ell} p[0,j,\ell;x] = \sum_{x=A}^{k} \binom{x-1}{A-1} \alpha^A \alpha'^{x-A} \quad .$$

We see immediately that equations (2.2.9) and (2.2.10) sum to unity because this is exactly Case II in Section 2.1.2. (*Vide* equation 2.1.9 which was shown to be 1.)

Case III - $A \le B \le k < C$

Here only the A and B particles could possibly win the race in k trials. As a result, all state probabilities will be zero in which the C particle is represented as having reached the origin (i.e. if $\ell = 0$). Hence, performing the same summations and transformations as before on (2.2.1) yields

$$(2.2.11) \qquad \sum_{i,j,\ell} p[i,j,\ell;k] = \sum_{x=0}^{A-1} \binom{k}{x} \alpha^x \alpha'^{k-x} \sum_{y=0}^{B-1} \binom{k}{y} \beta^y \beta'^{k-y} \quad .$$

Similarly, by performing the appropriate summations over i, j and ℓ and by also summing over trials in equations (2.2.2), (2.2.3) and (2.2.5) we obtain

$$(2.2.12) \begin{cases} \displaystyle\sum_{s=A}^{k} \sum_{j,\ell} p[0,j,\ell;s] = \sum_{s=A}^{k} \binom{s-1}{A-1} \alpha^A \alpha'^{s-A} \sum_{x=0}^{B-1} \binom{s}{x} \beta^x \beta'^{s-x} \\ \displaystyle\sum_{s=B}^{k} \sum_{i,\ell} p[i,0,\ell;s] = \sum_{s=B}^{k} \binom{s-1}{B-1} \beta^B \beta'^{s-B} \sum_{x=0}^{A-1} \binom{s}{x} \alpha^x \alpha'^{s-x} \\ \displaystyle\sum_{s=B}^{k} \sum_{\ell} p[0,0,\ell;s] = \sum_{s=B}^{k} \binom{s-1}{A-1} \alpha^A \alpha'^{s-A} \binom{s-1}{B-1} \beta^B \beta'^{s-B} \quad . \end{cases}$$

These equations, (2.2.11) and (2.2.12), represent the probabilities of the only possible outcomes after k trials. We again see immediately that they sum to unity because they are exactly equations (2.1.12) through (2.1.15) under Case III in Section 2.1.2.

Case IV - $A \le B \le C < k$

Here all events are now possible. Performing appropriate summations on (2.2.1) through (2.2.8) yields:

$$(2.2.13) \qquad \sum_{i,j,\ell} p[i,j,\ell;k] = \sum_{x=0}^{A-1} \binom{k}{x} \alpha^x \alpha'^{k-x} \sum_{y=0}^{B-1} \binom{k}{y} \beta^y \beta'^{k-y} \cdot \sum_{z=0}^{C-1} \binom{k}{z} \gamma^z \gamma'^{k-z} .$$

This is interpreted as the probability of no one winning by the k^{th} trial or in other words, of the race not being over by the k^{th} trial.

$$(2.2.14) \begin{cases} \displaystyle\sum_{s=A}^{k} \sum_{j,\ell} p[0,j,\ell;s] = \sum_{s=A}^{k} \binom{s-1}{A-1} \alpha^A \alpha'^{s-A} \sum_{y=0}^{B-1} \binom{s}{y} \beta^y \beta'^{s-y} \cdot \sum_{z=0}^{C-1} \binom{s}{z} \gamma^z \gamma'^{s-z} \\ \displaystyle\sum_{s=B}^{k} \sum_{i,\ell} p[i,0,\ell;s] = \sum_{s=B}^{k} \binom{s-1}{B-1} \beta^B \beta'^{s-B} \sum_{x=0}^{A-1} \binom{s}{x} \alpha^x \alpha'^{s-x} \cdot \sum_{z=0}^{C-1} \binom{s}{z} \gamma^z \gamma'^{s-z} \\ \displaystyle\sum_{s=C}^{k} \sum_{i,j} p[i,j,0;s] = \sum_{s=C}^{k} \binom{s-1}{C-1} \gamma^C \gamma'^{s-C} \sum_{x=0}^{A-1} \binom{s}{x} \alpha^x \alpha'^{s-x} \cdot \sum_{y=0}^{B-1} \binom{s}{y} \beta^y \beta'^{s-y} . \end{cases}$$

Equations (2.2.14) are interpreted as the respective probabilities of each particle winning by the k^{th} trial.

$$
(2.2.15)\quad \begin{cases}
\displaystyle\sum_{s=B}^{k} \sum_{\ell} p[0,0,\ell;s] = \sum_{s=B}^{k} \binom{s-1}{A-1} \alpha^{A} \alpha'^{\,s-A} \binom{s-1}{B-1} \beta^{B} \beta'^{\,s-B} \cdot \sum_{z=0}^{C-1} \binom{s}{z} \gamma^{z} \gamma'^{\,s-z} \\[2ex]
\displaystyle\sum_{s=C}^{k} \sum_{j} p[0,j,0;s] = \sum_{s=C}^{k} \binom{s-1}{A-1} \alpha^{A} \alpha'^{\,s-A} \binom{s-1}{C-1} \gamma^{C} \gamma'^{\,s-c} \cdot \sum_{y=0}^{B-1} \binom{s}{y} \beta^{y} \beta'^{\,s-y} \\[2ex]
\displaystyle\sum_{s=C}^{k} \sum_{i} p[i,0,0;s] = \sum_{s=C}^{k} \binom{s-1}{B-1} \beta^{B} \beta'^{\,s-B} \binom{s-1}{C-1} \gamma^{C} \gamma'^{\,s-C} \cdot \sum_{x=0}^{A-1} \binom{s}{x} \alpha^{x} \alpha'^{\,s-x} .
\end{cases}
$$

Equations (2.2.15) are interpreted as the probabilities of two way ties occurring <u>by</u> the k^{th} trial.

$$
(2.2.16)\quad \sum_{s=C}^{k} p[0,0,0;s] = \sum_{s=C}^{k} \binom{s-1}{A-1} \alpha^{A} \alpha'^{\,s-A} \binom{s-1}{B-1} \beta^{B} \beta'^{\,s-B} \cdot \binom{s-1}{C-1} \gamma^{C} \gamma'^{\,s-C}
$$

is interpreted as the probability of a three tie occurring <u>by</u> the k^{th} trial.

It is now desirable to show that equations (2.2.13)

through (2.2.16) sum to unity, as they should. Again we write each binomial sum in terms of its negative binomial complement as in the relationship (2.1.11). Also, again for brevity we will employ the simplified notation for the negative binomial distribution, $\left(\text{viz. } c_s = \binom{s-1}{C-1}\gamma^C\gamma'^{s-C}\right)$.
Equation (2.2.13) becomes

$$\sum_{i,j,\ell} p[i,j,\ell;k] = 1+\sum^k a_x \sum^k b_x + \sum^k a_x \sum^k c_x + \sum^k b_x \sum^k c_x - \sum^k a_x - \sum^k b_x - \sum^k c_x - \sum^k a_x \sum^k b_x \sum^k c_x \quad . \tag{2.2.17}$$

Equations (2.2.14) become

$$\left\{\begin{aligned} \sum_{s=A}^{k} \sum_{j,\ell} p[0,j,\ell;s] &= \sum_{s=A}^{k}\left[a_s - a_s\sum^s b_x - a_s\sum^s c_x + a_s\sum^s b_x \sum^s c_x\right] \\ \sum_{s=B}^{k} \sum_{i,\ell} p[i,0,\ell;s] &= \sum_{s=B}^{k}\left[b_s - b_s\sum^s a_x - b_s\sum^s c_x + b_s\sum^s a_x \sum^s c_x\right] \\ \sum_{s=C}^{k} \sum_{i,j} p[i,j,0;s] &= \sum_{s=C}^{k}\left[c_s - c_s\sum^s a_x - c_s\sum^s b_x + c_s\sum^s a_x \sum^s b_x\right] \end{aligned}\right. \quad . \tag{2.2.18}$$

Equations (2.2.15) become

$$\left\{\begin{aligned} \sum_{s=B}^{k} \sum_{\ell} p[0,0,\ell;s] &= \sum_{s=B}^{k}\left[a_s b_s - a_s b_s \sum^s c_x\right] \\ \sum_{s=C}^{k} \sum_{j} p[0,j,0;s] &= \sum_{s=C}^{k}\left[a_s c_s - a_s c_s \sum^s b_x\right] \end{aligned}\right. \tag{2.2.19}$$

$$\left|\sum_{s=C}^{k}\sum_{i} p[i,0,0;s] = \sum_{s=C}^{k}\left(b_s c_s - b_s c_s \overset{s}{\Sigma} a_x\right)\right. .$$

Rewriting (2.2.16) using the simplified notation yields

$$\text{(2.2.20)} \qquad \sum_{s=C}^{k} p[0,0,0;s] = \sum_{s=C}^{k}\left(a_s b_s c_s\right) .$$

When we set the lower index of summation over s to A in each of the above equations, nothing is changed since $\binom{n}{r} = 0$ for all $r > n$ by definition. Setting this lower index of summation over s to A will enable us to write all of the equations in a compact form later on. Let us call the sum of equations (2.2.17) through (2.2.20), S. We note immediately then that the summation over s of the first terms within accolades in (2.2.18) cancels out with $-\overset{k}{\Sigma} a_x - \overset{k}{\Sigma} b_x - \overset{k}{\Sigma} c_x$ in (2.2.17).

After this cancelation is performed and after setting the lower index of summation over s to A we may write equations (2.2.18) through (2.2.20) in the following compact form. [neglecting (2.2.17) temporarily].

$$\sum_{s=A}^{k}\Bigg[\left(-a_s \overset{s}{\Sigma} b_x - a_s \overset{s}{\Sigma} c_x + a_s \overset{s}{\Sigma} b_x \overset{s}{\Sigma} c_x\right) + \left(-b_s \overset{s}{\Sigma} a_x - b_s \overset{s}{\Sigma} c_x + b_s \overset{s}{\Sigma} a_x \overset{s}{\Sigma} c_x\right) +$$

$$+ \left(-c_s \overset{s}{\Sigma} a_x - c_s \overset{s}{\Sigma} b_x + c_s \overset{s}{\Sigma} a_x \overset{s}{\Sigma} b_x\right) + \left(a_s b_s - a_s b_s \overset{s}{\Sigma} c_x\right) + \left(a_s c_s - a_s c_s \overset{s}{\Sigma} b_x\right) +$$

$$+\left(b_s c_s - b_s c_s \sum^{s} a_x\right) + \left(a_s b_s c_s\right)\Big]\Big] \quad .$$

Now by adding and subtracting the terms

$$\sum^{s} a_x \sum^{s} b_x \ ; \ \sum^{s} a_x \sum^{s} c_x \ ; \ \sum^{s} b_x \sum^{s} c_x \ ; \ \sum^{s} a_x \sum^{s} b_x \sum^{s} c_x \quad ,$$

this may be written as

$$\sum_{s=A}^{k}\Bigg[\left[\left(a_s - \sum^{s} a_x\right)\left(b_s - \sum^{s} b_x\right)\left(c_s - \sum^{s} b_x\right) + \left(a_s - \sum^{s} a_x\right)\left(b_s - \sum^{s} b_x\right) + \right.$$

$$+\left(a_s - \sum^{s} a_x\right)\left(c_s - \sum^{s} c_x\right) + \left(b_s - \sum^{s} b_x\right)\left(c_s - \sum^{s} c_x\right) +$$

$$\left. + \sum^{s} a_x \sum^{s} b_x \sum^{s} c_x - \sum^{s} a_x \sum^{s} b_x - \sum^{s} a_x \sum^{s} c_x - \sum^{s} b_x \sum^{s} c_x\right] \quad .$$

And this can easily be seen to simplify to

$$\sum_{s=A}^{k}\Bigg[\left(- \sum^{s-1} a_x \sum^{s-1} b_x \sum^{s-1} c_x + \sum^{s-1} a_x \sum^{s-1} b_x + \sum^{s-1} a_x \sum^{s-1} c_x + \sum^{s-1} b_x \sum^{s-1} c_x\right) +$$

$$+\left(\sum^{s} a_x \sum^{s} b_x \sum^{s} c_x - \sum^{s} a_x \sum^{s} b_x - \sum^{s} a_x \sum^{s} c_x - \sum^{s} b_x \sum^{s} c_x\right)\Bigg] \quad .$$

Now by adding what remains of (2.2.17) to the above we have

$$S = 1 - \sum^{k} a_x \sum^{k} b_x \sum^{k} c_x + \sum^{k} a_x \sum^{k} b_x + \sum^{k} a_x \sum^{k} c_x + \sum^{k} b_x \sum^{k} c_x + \sum_{s=A}^{k}\Bigg[\left(- \sum^{s-1} a_x \sum^{s-1} b_x \sum^{s-1} c_x + \right.$$

$$+\sum^{s-1} a_x \sum^{s-1} b_x + \sum^{s-1} a_x \sum^{s-1} c_x + \sum^{s-1} b_x \sum^{s-1} c_x \Bigg\} +$$

$$+\left\{\sum^{s} a_x \sum^{s} b_x \sum^{s} c_x - \sum^{s} a_x \sum^{s} b_x - \sum^{s} a_x \sum^{s} c_x - \sum^{s} b_x \sum^{s} c_x\right\}\Bigg] \,.$$

By taking the upper limit of the summation over s (i.e. the terms within the accolades by setting $s = k$), we notice that all the terms within the second set of brackets cancel everything in the first line of S immediately following the 1 and we are left with

$$S = 1 - \sum^{k-1} a_x \sum^{k-1} b_x \sum^{k-1} c_x + \sum^{k-1} a_x \sum^{k-1} b_x + \sum^{k-1} a_x \sum^{k-1} c_x + \sum^{k-1} b_x \sum^{k-1} c_x +$$

$$+\sum_{s=A}^{k-1}\Bigg[\left\{-\sum^{s-1} a_x \sum^{s-1} b_x \sum^{s-1} c_x + \sum^{s-1} a_x \sum^{s-1} b_x + \sum^{s-1} a_x \sum^{s-1} c_x + \sum^{s-1} b_x \sum^{s-1} c_x\right\} +$$

$$+\left\{\sum^{s} a_x \sum^{s} b_x \sum^{s} c_x - \sum^{s} a_x \sum^{s} b_x - \sum^{s} a_x \sum^{s} c_x - \sum^{s} b_x \sum^{s} c_x\right\}\Bigg] \,.$$

We do exactly the same thing, letting s equal its upper index of summation (in this case $s = k-1$) and note that again all the terms within the second set of brackets cancel everything in the first line of S immediately following the 1. We continue this process of reduction until the upper index of summation becomes c. When this happens we are left with,

$$S = 1-\gamma^{c}\sum^{c} a_x \sum^{c} b_x + \sum^{c} a_x \sum^{c} b_x + \gamma^{c}\sum^{c} a_x + \gamma^{c}\sum^{c} b_x + \sum_{s=A}^{c}\left[\left(-\sum^{s-1} a_x \sum^{s-1} b_x \cdot 0 + \sum^{s-1} a_x \sum^{s-1} b_x + \sum^{s-1} a_x \cdot 0 + \sum^{s-1} b_x \cdot 0\right) + \left(\sum^{s} a_x \sum^{s} b_x \sum^{s} c_x - \sum^{s} a_x \sum^{s} b_x - \sum^{s} a_x \sum^{s} c_x - \sum^{s} b_x \sum^{s} c_x\right)\right] .$$

This reduces to

$$S = 1-\gamma^{c}\sum^{c} a_x \sum^{c} b_x + \sum^{c} a_x \sum^{c} b_x + \gamma^{c}\sum^{c} a_x + \gamma^{c}\sum^{c} b_x + \sum_{s=A}^{c}\left[\left(\sum^{s-1} a_x \sum^{s-1} b_x\right) + \left(\sum^{s} a_x \sum^{s} b_x \sum^{s} c_x - \sum^{s} a_x \sum^{s} b_x - \sum^{s} a_x \sum^{s} c_x - \sum^{s} b_x \sum^{s} c_x\right)\right] .$$

Letting s equal its upper limit of summation again yields

$$S = 1+ \sum^{c-1} a_x \sum^{c-1} b_x + \sum_{s=A}^{c-1}\left[\left(\sum^{s-1} a_x \sum^{s-1} b_x\right) + \left(\sum^{s} a_x \sum^{s} b_x \cdot 0 - \sum^{s} a_x \sum^{s} b_x - \sum^{s} a_x \cdot 0 - \sum^{s} b_x \cdot 0\right)\right] .$$

Hence

$$S = 1+ \sum_{x=A}^{c-1} a_x \sum_{x=B}^{c-1} b_x + \sum_{s=A}^{c-1}\left(\sum_{x=A}^{s-1} a_x \sum_{x=B}^{s-1} b_x - \sum_{x=A}^{s} a_x \sum_{x=B}^{s} b_x\right) .$$

We immediately note however that this is in exactly the same form as the S in Section 2.1.2 under Case III (refer to equation (2.1.19)), and we already showed that this was equal to one. Thus, we have reduced our present Case IV through simple straightforward algebraic manipulations to that of a previous case. This will always be the situation no matter how many walks are considered. However, the work becomes prohibitive.

We have thus shown that equations (2.2.1) through (2.2.8) do indeed form a proper probability distribution for all choices of k , the number of trials performed.

2.2.3 Determining Win and Draw Probabilities

Analogously to Section 2.1.3 we will consider first the situation of one particle winning or a tie occurring, when the remaining particles have specified co-ordinates. This is easily accomplished by summing equations (2.2.2) through (2.2.8) over all possible trials, k . Hence

$$
(2.2.21)\begin{cases}
p[0,j,\ell] = \sum\limits_{k=A}^{\infty} \binom{k-1}{A-1} \alpha^A \alpha'^{k-A} \binom{k}{B-j} \beta^{B-j} \beta'^{k-B+j} \cdot \\
\qquad \cdot \binom{k}{C-\ell} \gamma^{C-\ell} \gamma'^{k-C+\ell} \ , \ \begin{bmatrix} 0<j\le B \\ 0<\ell\le C \end{bmatrix} , \\
p[i,0,\ell] = \sum\limits_{k=B}^{\infty} \binom{k-1}{B-1} \beta^B \beta'^{k-B} \binom{k}{A-i} \alpha^{A-i} \alpha'^{k-A+i} \cdot \\
\qquad \cdot \binom{k}{C-\ell} \gamma^{C-\ell} \gamma'^{k-C+\ell} \ , \begin{bmatrix} 0<i\le A \\ 0<\ell\le C \end{bmatrix} ,
\end{cases}
$$

$$\left|\begin{aligned} p[i,j,0] &= \sum_{k=C}^{\infty} \binom{k-1}{C-1} \gamma^{C} \gamma'^{k-C} \binom{k}{A-i} \alpha^{A-i} \alpha'^{k-A+i} \cdot \\ &\quad \cdot \binom{k}{B-j} \beta^{B-j} \beta'^{k-B+j} , \quad \begin{pmatrix} 0<i\le A \\ 0<j\le B \end{pmatrix} . \end{aligned}\right.$$

Equations (2.2.21) are interpreted as the probabilities of the A , B and C particles eventually winning, while the other two are at specified co-ordinates. Similarly

$$(2.2.22)\quad \left\{\begin{aligned} p[0,0,\ell] &= \sum_{k=B}^{\infty} \binom{k-1}{A-1} \alpha^{A} \alpha'^{k-A} \binom{k-1}{B-1} \beta^{B} \beta'^{k-B} \cdot \\ &\quad \cdot \binom{k}{C-\ell} \gamma^{C-\ell} \gamma'^{k-C+\ell} , \quad 0<\ell\le C , \\ p[0,j,0] &= \sum_{k=C}^{\infty} \binom{k-1}{A-1} \alpha^{A} \alpha'^{k-A} \binom{k-1}{C-1} \gamma^{C} \gamma'^{k-C} \cdot \\ &\quad \cdot \binom{k}{B-j} \beta^{B-j} \beta'^{k-B+j} , \quad 0<j\le B , \\ p[i,0,0] &= \sum_{k=C}^{\infty} \binom{k-1}{B-1} \beta^{B} \beta'^{k-B} \binom{k-1}{C-1} \gamma^{C} \gamma'^{k-C} \cdot \\ &\quad \cdot \binom{k}{A-i} \alpha^{A-i} \alpha'^{k-A+i} , \quad 0<i\le A . \end{aligned}\right.$$

Equations (2.2.22) are interpreted as the probabilities of two-way ties eventually occurring while the remaining particle is at a specified co-ordinate. Likewise, by summing (2.2.8) over all k we obtain

$$(2.2.23)\quad p[0,0,0] = \sum_{k=C}^{\infty} \binom{k-1}{A-1} \alpha^{A} \alpha'^{k-A} \binom{k-1}{B-1} \beta^{B} \beta'^{k-B} \binom{k-1}{C-1} \gamma^{C} \gamma'^{k-C}$$

which is the probability of a three-way tie occurring.

The above equations are of limited interest. More useful are the time independent probabilities of each particle winning, and drawing. These are easily obtained by summing the above equations over all i , j and $\ell > 0$. Thus, performing the indicated sums and making the usual change of variables yields

$$(2.2.24)\quad \begin{cases} P_A = \sum_{k=A}^{\infty} \binom{k-1}{A-1} \alpha^{A} \alpha'^{k-A} \sum_{y=0}^{B-1} \binom{k}{y} \beta^{y} \beta'^{k-y} \cdot \\ \qquad \cdot \sum_{z=0}^{C-1} \binom{k}{z} \gamma^{z} \gamma'^{k-z} \quad ; \\ P_B = \sum_{k=B}^{\infty} \binom{k-1}{B-1} \beta^{B} \beta'^{k-B} \sum_{x=0}^{A-1} \binom{k}{x} \alpha^{x} \alpha'^{k-x} \cdot \\ \qquad \cdot \sum_{z=0}^{C-1} \binom{k}{z} \gamma^{z} \gamma'^{k-z} \quad ; \\ P_C = \sum_{k=C}^{\infty} \binom{k-1}{C-1} \gamma^{C} \gamma'^{k-C} \sum_{x=0}^{A-1} \binom{k}{x} \alpha^{x} \alpha'^{k-x} \cdot \\ \qquad \cdot \sum_{y=0}^{B-1} \binom{k}{y} \beta^{y} \beta'^{k-y} \quad ; \\ P_{AB} = \sum_{k=B}^{\infty} \binom{k-1}{A-1} \alpha^{A} \alpha'^{k-A} \binom{k-1}{B-1} \beta^{B} \beta'^{k-B} \cdot \\ \qquad \cdot \sum_{z=0}^{C-1} \binom{k}{z} \gamma^{z} \gamma'^{k-z} \quad ; \end{cases}$$

$$\left\{\begin{array}{l}
P_{AC} = \sum_{k=C}^{\infty} \binom{k-1}{A-1} \alpha^{A} \alpha'^{k-A} \binom{k-1}{C-1} \gamma^{C} \gamma'^{k-C} \cdot \\
\qquad\qquad\qquad\qquad \cdot \sum_{y=0}^{B-1} \binom{k}{y} \beta^{y} \beta'^{k-y} \; ; \\
P_{BC} = \sum_{k=C}^{\infty} \binom{k-1}{B-1} \beta^{B} \beta'^{k-B} \binom{k-1}{C-1} \gamma^{C} \gamma'^{k-C} \cdot \\
\qquad\qquad\qquad\qquad \cdot \sum_{x=0}^{A-1} \binom{k}{x} \alpha^{x} \alpha'^{k-x} \; ; \\
P_{ABC} = \sum_{k=C}^{\infty} \binom{k-1}{A-1} \alpha^{A} \alpha'^{k-A} \binom{k-1}{B-1} \beta^{B} \beta'^{k-B} \binom{k-1}{C-1} \gamma^{C} \gamma'^{k-C} \; .
\end{array}\right.$$

The general extension of these formulas is quite obvious for the situation of more than three walks. In comparing the above formulas with the two-walk case, the pattern is easily discerned.

Each formula contains binomial and negative binomial probabilities. If the co-ordinate of any particle is to be zero, (i.e. either it has won or tied) then its portion of the formula is represented negative binomially. The remaining particles whose co-ordinates are not zero are represented in the formula by the binomial distribution. This is an extension of what was observed in the two-walk case (cf Section 2.1.3).

Probabilistically speaking, we interpret the formulas for one or more of the particles winning as the product of:

the probabilities of one or more of the particles reaching the origin exactly at the k^{th} trial (i.e. the negative binomial distribution); and the probability of there being less than the required number of successes or steps for the other walks to reach the origin in k trials (i.e. the distribution function of the binomial distribution). This product is then summed over all trials, k .

It should be pointed out again that the probability that there is less than the required number of steps for a particular particle to reach the origin in k trials, is equivalent to saying that the waiting time required to produce the required number of steps is greater than k . This is by virtue of the relationship (2.1.11) which connects the binomial and negative binomial distributions.

Also before proceeding any further we must take note that equation (2.2.13), which is interpreted as the probability of no one reaching the origin in k trials, clearly goes to zero as k goes to infinity. This is a result of the fact that the probability is zero that only finitely many successes occur in an infinite number of Bernoulli trials. Hence, the probability of an infinitely prolonged race is zero.

2.2.4 Alternative Forms For the Win and Draw Probabilities

Analogously to Section 2.1.4, we will express the win and draw probabilities obtained in the preceding section in a different form. In Section 2.1.4 we found that the probabilities could be written in terms of Hypergeometric Functions of the first type. In this section we will see that our probabilities can be written in terms of the Generalized Hypergeometric Function. See, for example, Abramowitz [1], Bailey [5], Bateman [6] and Rainville [13].

In the three-walk case the win and draw probabilities will be written in terms of the function ${}_3F_2$ where

$${}_3F_2\left[\begin{matrix}\alpha,\beta,\gamma \\ \delta,\varepsilon\end{matrix};z\right] = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n(\gamma)_n z^n}{(\delta)_n(\varepsilon)_n\, n!} \qquad |z| < 1$$

and the notation $(\lambda)_n$ is given by (2.1.25). Similarly, it will also be seen in this section that the win and draw probabilities for the case of n walks can be written in terms of the function ${}_nF_{(n-1)}$ whose definition is the obvious extension of ${}_2F_1$ and ${}_3F_2$.

Considering the win and draw probabilities, (2.2.24), let us interchange the order of summation as in Section 2.1.4. P_A yields, after reversing the summation,

$$P_A = \alpha^A \alpha'^{-A} \sum_{y=0}^{B-1} \beta^y \beta'^{-y} \sum_{z=0}^{C-1} \gamma^z \gamma'^{-z} \sum_{k=A}^{\infty} \binom{k-1}{A-1}\binom{k}{y}\binom{k}{z}[\alpha'\beta'\gamma']^k \quad .$$

Rewriting the binomial coefficients in terms of gamma functions and making the change of variables $n = k-A$ yields

$$P_A = \frac{\alpha^A \alpha'^{-A}}{\Gamma(A)} \sum_{y=0}^{B-1} \frac{\beta^y \beta'^{-y}}{\Gamma(y+1)} \sum_{z=0}^{C-1} \frac{\gamma^z \gamma'^{-z}}{\Gamma(z+1)} \cdot$$

$$\cdot \sum_{n=0}^{\infty} \frac{\Gamma(A+n)\Gamma(A+n+1)\Gamma(A+n+1)[\alpha'\beta'\gamma']^{A+n}}{\Gamma(n+1)\Gamma(A-y+n+1)\Gamma(A-z+n+1)} .$$

Utilizing the notation in (2.1.25) we obtain

$$P_A = \alpha^A \sum_{y=0}^{B-1} \frac{\Gamma(A+1)\beta^y \beta'^{A-y}}{\Gamma(y+1)\Gamma(A-y+1)} \cdot$$

$$\cdot \sum_{z=0}^{C-1} \frac{\Gamma(A+1)\gamma^z \gamma'^{A-z}}{\Gamma(z+1)\Gamma(A-z+1)} \sum_{n=0}^{\infty} \frac{(A)_n (A+1)_n (A+1)_n [\alpha'\beta'\gamma']^n}{(A-y+1)_n (A-z+1)_n \ n!} .$$

Hence

$$P_A = \alpha^A \sum_{y=0}^{B-1} \binom{A}{y} \beta^y \beta'^{A-y} \cdot$$

$$\cdot \sum_{z=0}^{C-1} \binom{A}{z} \gamma^z \gamma'^{A-z} \ {}_3F_2 \left[\begin{matrix} A, A+1, A+1 & ; \\ A-y+1, A-z+1 & ; \end{matrix} \ (\alpha'\beta'\gamma') \right] .$$

Similarly, by symmetry, and by following the same procedure outlined above, we may also obtain all the remaining probabilities in (2.2.24) in the following forms:

$$P_B = \beta^B \sum_{x=0}^{A-1} \binom{B}{x} \alpha^x \alpha'^{B-x} \cdot$$

$$\cdot \sum_{z=0}^{C-1} \binom{B}{z} \gamma^z \gamma'^{B-z} \ {}_3F_2 \left[\begin{matrix} B, B+1, B+1 & ; \\ B-x+1, B-z+1 & ; \end{matrix} \ (\alpha'\beta'\gamma') \right]$$

$$P_C = \gamma^C \sum_{x=0}^{A-1} \binom{C}{x} \alpha^x \alpha'^{C-x} \cdot$$

$$\cdot \sum_{y=0}^{B-1} \binom{C}{y} \beta^y \beta'^{C-y} \; {}_3F_2 \left[\begin{matrix} C, C+1, C+1 & ; \\ C-x+1, C-y+1 & ; \end{matrix} \; (\alpha'\beta'\gamma')\right]$$

$$P_{AB} = \beta^B \binom{B-1}{A-1} \alpha^A \alpha'^{B-A} \cdot$$

$$\cdot \sum_{z=0}^{C-1} \binom{B}{z} \gamma^z \gamma'^{B-z} \; {}_3F_2 \left[\begin{matrix} B, B, B+1 & ; \\ B-A+1, B-z+1 & ; \end{matrix} \; (\alpha'\beta'\gamma')\right]$$

$$P_{AC} = \gamma^C \binom{C-1}{A-1} \alpha^A \alpha'^{C-A} \cdot$$

$$\cdot \sum_{y=0}^{B-1} \binom{C}{y} \beta^y \beta'^{C-y} \; {}_3F_2 \left[\begin{matrix} C, C, C+1 & ; \\ C-A+1, C-y+1 & ; \end{matrix} \; (\alpha'\beta'\gamma')\right]$$

$$P_{BC} = \gamma^C \binom{C-1}{B-1} \beta^B \beta'^{C-B} \cdot$$

$$\cdot \sum_{x=0}^{A-1} \binom{C}{x} \alpha^x \alpha'^{C-x} \; {}_3F_2 \left[\begin{matrix} C, C, C+1 & ; \\ C-B+1, C-X+1 & ; \end{matrix} \; (\alpha'\beta'\gamma')\right]$$

$$P_{ABC} = \gamma^C \binom{C-1}{A-1} \alpha^A \alpha'^{C-A} \binom{C-1}{B-1} \beta^B \beta'^{C-B} \; {}_3F_2 \left[\begin{matrix} C, C, C & ; \\ C-A+1; C-B+1; & \end{matrix} \; (\alpha'\beta'\gamma')\right].$$

The extensions to the case of n walks can be easily seen from Section 2.1.4 and from the above. For example, if we are considering four walks and we seek the probability that the A particle wins, we could immediately write:

$$P_A = \alpha^A \sum_{y=0}^{B-1} \binom{A}{y} \beta^y \beta'^{A-y} \sum_{z=0}^{C-1} \binom{A}{z} \gamma^z \gamma'^{A-z} \cdot$$

$$\cdot \sum_{x=0}^{D-1} \binom{A}{x} \delta^x \delta'^{A-x} \; {}_4F_3 \left[\begin{matrix} A, A+1, A+1, A+1 & ; \\ A-y+1, A-z+1, A-x+1 & ; \end{matrix} \; (\alpha'\beta'\gamma'\delta') \right] .$$

Similarly, a three-way tie between A , B and D has probability

$$P_{ABD} = \delta^D \begin{pmatrix} D-1 \\ A-1 \end{pmatrix} \alpha^A \alpha'^{D-A} \begin{pmatrix} D-1 \\ B-1 \end{pmatrix} \beta^B \beta'^{D-B} \cdot$$

$$\cdot \sum_{z=0}^{C-1} \binom{D}{z} \gamma^z \gamma'^{D-z} \; {}_4F_3 \left[\begin{matrix} D, D, D, D+1 & ; \\ D-A+1, D-B+1, D-z+1 & ; \end{matrix} \; (\alpha'\beta'\gamma'\delta') \right] .$$

All other probabilities may be obtained in the same fashion. This holds true for an arbitrary number of walks.

2.2.5 Duration of the Race

As in Section 2.1.5 we will now determine the probability distribution of the duration of the race. By employing the reasoning utilized in the latter part of that section, we are able to write this distribution down directly. Equation (2.1.31) enables us to write $P^{(k)}$, the probability that the race is terminated at the k^{th} trial, down immediately, no matter how many walks are involved. Thus, by substituting the probabilities that the race is not over by $(k-1)$ and k

trials respectively [viz. (2.1.13)] in (2.1.31) yields for the case of three walks,

$$(2.2.25) \quad P^{(k)} = \left[\sum_{x=0}^{A-1}\binom{k-1}{x}\alpha^{x}\alpha'^{\,k-1-x} \sum_{x=0}^{B-1}\binom{k-1}{x}\beta^{x}\beta'^{\,k-1-x} \cdot \sum_{x=0}^{C-1}\binom{k-1}{x}\gamma^{x}\gamma'^{\,k-1-x}\right] - \left[\sum_{x=0}^{A-1}\binom{k}{x}\alpha^{x}\alpha'^{\,k-x} \cdot \sum_{x=0}^{B-1}\binom{k}{x}\beta^{x}\beta'^{\,k-x} \sum_{x=0}^{C-1}\binom{k}{x}\gamma^{x}\gamma'^{\,k-x}\right] .$$

Upon comparing (2.1.30) and (2.2.25), the generalization to the case of more than three walks is obvious.

We should now demonstrate that (2.2.25) defines a proper probability distribution by showing that its summation over all k is unity. Utilizing the shorthand notation, (2.1.32), we may write (2.2.25) as,

$$(2.2.26) \quad P^{(k)} = \alpha(k-1)\beta(k-1)\gamma(k-1) - \alpha(k)\beta(k)\gamma(k) .$$

Summing this over all k yields as before,

$$\sum_{k=A}^{\infty} P^{(k)} = \alpha(A-1)\beta(A-1)\gamma(A-1) + \sum_{k=A+1}^{\infty} \alpha(k-1)\beta(k-1)\gamma(k-1) - \sum_{k=A}^{\infty} \alpha(k)\beta(k)\gamma(k) .$$

We again notice that the two summations cancel each other and also that the leading product is unity. Hence,

$$\sum_{k=A}^{\infty} P^{(k)} = 1 \quad .$$

It is not difficult to see that this will always be the situation, no matter how many walks are involved.

Let us now obtain the expected duration of the race, E(k) . Multiplying (2.2.26) by k , summing and then collecting terms as in Section 2.1.5 we obtain

$$E(k) = A + \sum_{k=A}^{\infty} \alpha(k)\beta(k)\gamma(k) \quad .$$

After returning to our original notation we obtain

$$(2.2.27) \quad E(k) = A + \sum_{k=A}^{\infty} \left(\left[\sum_{x=0}^{A-1} \binom{k}{x} \alpha^{x} \alpha'^{\,k-x} \right] \cdot \left[\sum_{x=0}^{B-1} \binom{k}{x} \beta^{x} \beta'^{\,k-x} \right] \left[\sum_{x=0}^{C-1} \binom{k}{x} \gamma^{x} \gamma'^{\,k-x} \right] \right) \quad .$$

The expected duration of the race for more than three walks is obvious. Again we should point out that the leading constant, A , is a result of the race having to last <u>at least</u> A trials for one of the competitors to win. The rest of the formula reflects the probabilistic behavior of the process.

The second moment, $E(k^2)$, is obtained exactly as in Section 2.1.5. The only difference is the factor $\gamma(k)$ which does not affect the analysis. Hence, analogously to (2.1.35) we have immediately

$$E(k^2) = A^2 + \sum_{k=A}^{\infty} \left\{ (2k+1) \left[\sum_{x=0}^{A-1} \binom{k}{x} \alpha^x \alpha'^{k-x} \right] \left[\sum_{x=0}^{B-1} \binom{k}{x} \alpha^x \alpha'^{k-x} \right] \cdot \right.$$

$$\left. \cdot \left[\sum_{x=0}^{C-1} \binom{k}{x} \gamma^x \gamma'^{k-x} \right] \right\}.$$

III. A RACE BETWEEN RANDOM WALKS IN CONTINUOUS TIME

In this chapter we are again going to consider a race between discrete random walks toward the origin, but now we shall take account of the time interval between steps. We will find that most of the formulas obtained will contain continuous probability functions that are analogous to the discrete ones obtained in the last chapter. For example, just as the negative binomial distribution describes the probability law associated with the waiting time for the r^{th} success in discrete time, so does the gamma distribution in continuous time.

3.0 Preliminary Description and Analysis of a Single Walk

Before undertaking the analysis of a system of two or more random walks, it will be convenient for us to establish some basic notation in terms of a single random walk. Let this walk be called the A walk and again let the letter A perform double duty by also representing the starting co-ordinate of the A particle, A(0) .

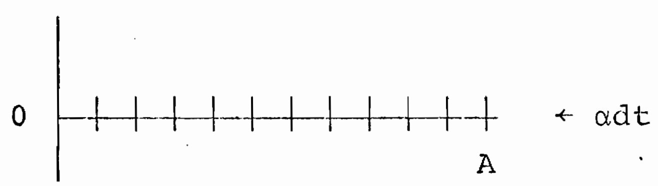

Figure 3.0.1

Graphical Representation of One Walk

Instead of α being the probability of the A particle moving one unit to the left at each trial as in the last chapter, let it now represent the average rate at which the particle moves one unit to the left. This is equivalent to saying that the probability of the A walk making a transition to the left in a small increment of time dt is αdt.

Starting with this premise leads us to the well known result that the inter-movement time or the time required for the walk to move one step to the left is distributed negative exponentially with parameter α. This in turn leads to another well known result concerning the Poisson distribution. See Feller [12] and Parzen [13]. If A were infinite, then we could immediately say that the probability of n successes (that is, moving n steps to the left) in a time interval t has a Poisson distribution with parameter αt.

Since we will not consider A as being infinite, it becomes obvious that we are dealing with a truncated Poisson distribution. Looking at the model from a different viewpoint reveals that this is in reality a pure death process which is homogeneous in time and whose transition probabilities are independent of the state of the system.

We will now give a rigorous derivation for the state probabilities of the system which will serve as a sound foundation for the analysis to be performed in the next section. Letting $A(t)$ represent the co-ordinate of the A particle at time t after the process begins, we may define the state probabilities as

$$p[n;t] = \Pr[A(t)=n] \qquad n=0,1,2,\ldots,A \ .$$

Then

$$p[A;t+dt] = p[A;t]\cdot(1-\alpha dt) \ .$$

After multiplying, collecting terms, dividing through by dt and then utilizing the fundamental definition of a derivative we obtain

$$\text{(3.0.1)} \qquad \frac{dp[A;t]}{dt} = -\alpha p[A;t] \ .$$

Similarly,

$$p[n;t+dt] = p[n;t](1-\alpha dt)+p[n+1;t]\alpha dt \quad ; \quad n=1,2,\ldots,A-1$$

leads to

$$\text{(3.0.2)} \quad \frac{dp[n;t]}{dt} = -\alpha p[n;t]+\alpha p[n+1;t] \quad ; \quad n=1,2,\ldots,A-1 \ .$$

Likewise,

$$p[0\ ;\ t+dt] = p[0;t]\cdot 1+p[1;t]\alpha dt$$

leads to,

$$\text{(3.0.3)} \qquad \frac{dp[0;t]}{dt} = \alpha p[1;t] \ .$$

We will now be able to obtain the desired state probabilities from these three differential equations. From (3.0.1) we may write

$$\frac{dp[A;t]}{p[A;t]} = -\alpha dt \; .$$

Integrating directly yields

$$p[A;t] = Ce^{-\alpha t}$$

where C is the constant of integration. Utilizing the boundary condition that $p[A;0] = 1$ yields

$$p[A;t] = e^{-\alpha t} \; .$$

After setting $n = A-1$ in (3.0.2) and then substituting for $p[A;t]$, we obtain

$$\frac{dp[A-1;t]}{dt} = p[A-1;t]+\alpha e^{-\alpha t} \; .$$

Solving this equation using $e^{\alpha t}$ as an integrating factor, and then utilizing the additional boundary condition, $p[A-1;0] = 0$, results in

$$p[A-1;t] = \alpha t e^{-\alpha t} \; .$$

We can continue this process in the same fashion by solving each equation successively for all n in (3.0.2). Then it is easy to show inductively that

$$(3.0.4) \quad p[A-n;t] = \frac{e^{-\alpha t}(\alpha t)^n}{n!} \quad \text{for } n=0,1,\ldots,A-1 \; .$$

The above development parallels the derivation of a Poisson process as described in Feller [12] and Parzen [13]. Because our process is defined for only a finite sample space, we shall see that the remaining probabilities not contained in (3.0.4) will be in $p[0;t]$. We need now only evaluate $p[0;t]$. This could be obtained by substituting (3.0.4) for $p[1;t]$ in (3.0.3) and then integrating directly by parts. However, it will be to our advantage to evaluate $p[0;t]$ by another method which we will have occasion to use many times in what is to follow.

We are about to prove a direct analogy to the relationship (2.1.11) but in continuous time. It will associate the probabilities between the number of successes, n , occurring in time t and the waiting time required for the n^{th} success to occur. Let us first begin by defining two random variables, $T(n)$ and $N(t)$, such that

- $T(n)$ is the waiting time up to the occurrence of the n^{th} success according to some probability law,
- $N(t)$ is the number of successes that have occurred by time t .

Then, analogously to relationship (2.1.10), we may write immediately that

$$\Pr[N(t)<n] = \Pr[T(n)>t] \quad . \tag{3.0.5}$$

Since our model specifies αdt as the probability differential of a unit transition, we know then that the number of events occurring in an interval t follows a Poisson distribution. See, for example, Bailey [4], Feller [12] and Parzen [13]. In a Poisson process, it is a well known result that the time between events is distributed negative exponentially. Thus, since the gamma distribution is simply a convolution of negative exponential distributions, we may assert that the waiting time for the n^{th} event follows the gamma distribution. Hence

$$\Pr[t \leq T(n) \leq t+dt] = \frac{\alpha e^{-\alpha t}(\alpha t)^{n-1}}{(n-1)!}\,dt \ ,$$

while,

$$\Pr[N(t)=n] = \frac{e^{-\alpha t}(\alpha t)^n}{n!} \qquad \alpha>0 \ ; \ n=0,1,\ldots \ .$$

Applying the relationship (3.0.5) to these two probability statements yields

$$\sum_{x=0}^{n-1} \frac{e^{-\alpha t}(\alpha t)^x}{x!} = \int_t^\infty \frac{e^{-\alpha x}(\alpha x)^{n-1}}{(n-1)!}\,dx \ . \tag{3.0.6}$$

This is an extremely important and fundamental relationship. See Feller [12], Parzen [13] and Bailey [4]. It is directly analogous to (2.1.11) which relates the binomial and negative binomial distributions. It should be pointed out that (3.0.6) could also have been obtained by simply integrating its right hand side directly by parts n times.

We now return to the problem at hand, namely, evaluating $p[0;t]$. Substituting (3.0.4) for $p[1;t]$ in (3.0.3) yields

$$\frac{dp[0;t]}{dt} = \frac{\alpha e^{-\alpha t}(\alpha t)^{A-1}}{(A-1)!} \quad . \tag{3.0.7}$$

We may now write $p[0;t]$ as

$$p[0;t] = \int_0^t \frac{\alpha e^{-\alpha x}(\alpha x)^{A-1}}{(A-1)!}\, dx \ . \tag{3.0.8}$$

Since the gamma density function integrates to unity, this may be rewritten as

$$p[0;t] = 1 - \int_t^\infty \frac{\alpha e^{-\alpha x}(\alpha x)^{A-1}}{(A-1)!}\, dx \ .$$

Hence in view of the relationship (3.0.6), this may be written as

$$p[0;t] = 1 - \sum_{x=0}^{A-1} \frac{e^{-\alpha t}(\alpha t)^x}{x!} \ . \tag{3.0.9}$$

Thus, equations (3.0.4) and (3.0.9) form the complete set of state probabilities for the system. It is quite obvious that these sum to unity by virtue of the fact that the Poisson distribution is a proper probability distribution. We also note that $p[0;t] \to 1$ as $t \to \infty$ which is certainly to be expected.

It is very interesting to note that (3.0.9) may be interpreted as the probability of A or more events occurring in a Poisson process with parameter αt. In other words, all of the probabilities in a Poisson distribution not contained in (3.0.4) are contained in $p[0;t]$, thus making this a truncated Poisson distribution.

Since we shall have occasion to use the alternative forms (3.0.8) and (3.0.9) several times later on, it is convenient to introduce more compact notation. Thus, we may write

$$(3.0.10) \qquad I = \int_0^t e^{-ax} x^m \, dx \ .$$

We have, by virtue of (3.0.9),

$$(3.0.11) \qquad I = \frac{m!}{a^{m+1}} \left[1 - \sum_{x=0}^{m} \frac{e^{-at}(at)^x}{x!} \right] \ .$$

3.1 Two Walks

Consider two independent discrete random walks performed in continuous time, each as described in the last section. Let these walks be called A and B and as before let these letters represent the starting co-ordinates of each particle. Let α and β represent the average rate at which each particle moves one step or unit to the left.

We are now going to consider a race between these two particles toward the origin. Each particle will act independently of the other until one of them reaches the origin, in which case the whole process stops. Graphically the system is represented in Figure 3.1.1.

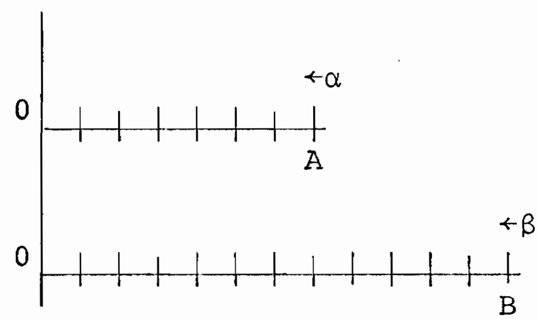

Figure 3.1.1
Graphical Representation of the System

Since this system is operating in continuous time, the probability of a tie or draw between the two walks is zero and need not be considered in the development of the theory. However, in some of the applications to be discussed later, we will want to maximize the probability that both walks finish very close together in terms of distance. It should be noted that in this case the probability that both the A and B particles move together in the same small time interval is by our assumptions vanishingly small.

3.1.1 State Probabilities

Represent by p[i,j;t] the "state" probability of the event: "the A particle has co-ordinate i, the B particle has co-ordinate j at time t." Utilizing previous notation, it is defined more precisely as

$$p[i,j;t] = \Pr[A(t)=i, B(t)=j] \quad .$$

This is the obvious generalization of $p[n;t]$ in the last section and the analogous representation in continuous time of $p[i,j;k]$ from Chapter II.

Since each particle proceeds independently until one or the other is absorbed at the origin, we may write immediately

$$(3.1.1) \quad p[i,j;t] = \frac{e^{-\alpha t}(\alpha t)^{A-i}}{(A-i)!} \frac{e^{-\beta t}(\beta t)^{B-j}}{(B-j)!} \quad , \quad \begin{bmatrix} i=1,2,\ldots,A \\ j=1,2,\ldots,B \end{bmatrix} .$$

Here we have merely used equation (3.0.4) by making the simple change of variables $k = A-n$. Hence $i=1,2,\ldots,A$ and $j=1,2,\ldots,B$ as above.

Now, the probability that the A particle reaches the origin in the small interval $(t,t+dt)$ at which time the B particle has co-ordinate j is clearly

$$p[0,j;t+dt] = p[0,j;t]\cdot 1 + p[1,j;t]\cdot \alpha dt(1-\beta dt) \quad .$$

Notice that a term $p[1,j+1;t]$ does not occur because the probability of <u>both</u> particles advancing in the small increment of time, dt, is of order of magnitude $(dt)^2$ and hence goes to zero in the limit. The usual operations result in

$$\frac{dp[0,j;t]}{dt} = \alpha p[1,j;t] \quad .$$

Substituting (3.1.1) for $p[1,j;t]$ yields

$$\text{(3.1.2)} \quad \frac{dp[0,j;t]}{dt} = \frac{\alpha^A \beta^{B-j} t^{A-1+B-j} e^{-(\alpha+\beta)t}}{(A-1)!(B-j)!} \qquad j=1,2,\ldots,B \ .$$

The integral of the above with respect to t is exactly in the same format of the relationship (3.0.10), when constants are neglected. Hence

$$p[0,j;t] = \frac{\alpha^A \beta^{B-j}}{(A-1)!(B-j)!} \int_0^t x^{A-1+B-j} \, e^{-(\alpha+\beta)x} \, dx \quad .$$

And thus by virtue of (3.0.11) we obtain

$$p[0,j;t] = \frac{(A-1+B-j)!\,\alpha^A \beta^{B-j}}{(A-1)!(B-j)!(\alpha+\beta)^{A+B-j}} \cdot$$

$$\cdot \left[1 - \sum_{x=0}^{A-1+B-j} \frac{e^{-(\alpha+\beta)t}[(\alpha+\beta)t]^x}{x!}\right] \qquad 1 \le j \le B \ .$$

Collecting terms and rewriting the factorials as a Binomial coefficient yields

$$\text{(3.1.3)} \quad p[0,j;t] = \binom{A-1+B-j}{A-1} \left(\frac{\alpha}{\alpha+\beta}\right)^A \left(\frac{\beta}{\alpha+\beta}\right)^{B-j} \cdot$$

$$\cdot \left[1 - \sum_{x=0}^{A-1+B-j} \frac{e^{-(\alpha+\beta)t}[(\alpha+\beta)t]^x}{x!}\right] , \quad 1 \le j < B \quad .$$

Similarly by symmetry we may write immediately

$$(3.1.4)\quad p[i,0;t] = \binom{A-i+B-1}{B-1}\left(\frac{\beta}{\alpha+\beta}\right)^{B}\left(\frac{\alpha}{\alpha+\beta}\right)^{A-i} \cdot$$
$$\cdot\left[1-\sum_{x=0}^{A-i+B-1}\frac{e^{-(\alpha+\beta)t}[(\alpha+\beta)t]^{x}}{x!}\right],\quad 1\le i\le A \quad .$$

Equation (3.1.3) is interpreted as the state probability of the A particle winning by time t when the co-ordinate of the B walk is at that time, j. A similar interpretation exists for (3.1.4).

Now equations (3.1.1), (3.1.3) and (3.1.4) summed over all i and j should equal unity for all t. We will show that this is indeed the case in the next section.

In order to do this however, we must obtain the unconditional state probabilities of either of the walks winning or of the race not being over by time t. As is to be expected, we obtain these by summing each of the equations (3.1.1), (3.1.3) and (3.1.4) over all i and j. Summing (3.1.1) over i and j yields

$$\sum_{i,j} p[i,j;t] = \sum_{i=1}^{A}\frac{e^{-\alpha t}(\alpha t)^{A-i}}{(A-i)!}\sum_{j=1}^{B}\frac{e^{-\beta t}(\beta t)^{B-j}}{(B-j)!} \quad .$$

Setting $x = A-i$ and $y = B-j$ results in

$$(3.1.5)\quad \sum_{i,j} p[i,j;t] = \sum_{x=0}^{A-1}\frac{e^{-\alpha t}(\alpha t)^{x}}{x!}\sum_{y=0}^{B-1}\frac{e^{-\beta t}(\beta t)^{y}}{y!} \quad .$$

This is interpreted as the probability of the race not being over by time t. We note that as $t \to \infty$, (3.1.5) goes to zero as is to be expected. Thus, the probability of an infinitely prolonged race is zero.

Now summing (3.1.3) over $j>0$ yields

$$\sum_j p[0,j;t] = \sum_{j=1}^{B} \binom{A-1+B-j}{A-1} \left(\frac{\alpha}{\alpha+\beta}\right)^A \left(\frac{\beta}{\alpha+\beta}\right)^{B-j} \cdot \left[1-\sum_{x=0}^{A-1+B-j} \frac{e^{-(\alpha+\beta)t}[(\alpha+\beta)t]^x}{x!}\right] .$$

By making the change of variables $z = A+B-j$ we obtain

$$(3.1.6) \quad P_A(t) = \sum_{z=A}^{A+B-1} \binom{z-1}{A-1} \left(\frac{\alpha}{\alpha+\beta}\right)^A \left(\frac{\beta}{\alpha+\beta}\right)^{z-A} \cdot \left[1-\sum_{x=0}^{z-1} \frac{e^{-(\alpha+\beta)t}[(\alpha+\beta)t]^x}{x!}\right] .$$

Summing (3.1.4) over $i>0$ and making a similar change of variables yeilds

$$(3.1.7) \quad P_B(t) = \sum_{z=B}^{A+B-1} \binom{z-1}{B-1} \left(\frac{\beta}{\alpha+\beta}\right)^B \left(\frac{\alpha}{\alpha+\beta}\right)^{z-B} \cdot \left[1-\sum_{x=0}^{z-1} \frac{e^{-(\alpha+\beta)t}[(\alpha+\beta)t]^x}{x!}\right] .$$

Equations (3.1.6) and (3.1.7) are the unconditional state probabilities of each particle winning the race by time t, whereas equation (3.1.5) is the probability of the race not being over by that time.

We note that both of the win probabilities, $P_A(t)$ and $P_B(t)$, involve negative binomial and Poisson distributions. Probabilistically speaking in $P_A(t)$, we may interpret the negative binomial portion by saying that, in at most $A+B-1$ steps, there have been A "successes" and less than B "failures" with the A^{th} "success" occurring at the last step. Here a "success" is considered to be the A particle advancing one unit and a failure, the B particle advancing one unit. We see that the probability of a success at each step turns out to be $[\alpha/(\alpha+\beta)]$ and of a failure $[\beta/(\alpha+\beta)]$.

By virtue of the relationship (3.0.6) between the Poisson and gamma distributions, we can interpret the Poisson portion of $P_A(t)$ as being the probability that there have been between A and $A+B-1$ steps taken during time t when they are occurring at a rate $(\alpha+\beta)$.

As already mentioned, taking the limit as $t\to\infty$ in (3.1.5) demonstrates that the probability of an infinitely prolonged race is zero. However, taking this limit in equations (3.1.6) and (3.1.7) yields the unconditional non-time dependent win probabilities for each walk. These probabilities, which we shall represent by P_A and P_B, are the probabilities of each walk eventually winning. Thus, taking the limit as $t \to \infty$ in (3.1.6) and (3.1.7) yields

$$(3.1.8) \qquad P_A = \sum_{z=A}^{A+B-1} \binom{z-1}{A-1} \left(\frac{\alpha}{\alpha+\beta}\right)^A \left(\frac{\beta}{\alpha+\beta}\right)^{z-A} ,$$

$$(3.1.9) \qquad P_B = \sum_{z=B}^{A+B-1} \binom{z-1}{B-1} \left(\frac{\beta}{\alpha+\beta}\right)^B \left(\frac{\alpha}{\alpha+\beta}\right)^{z-B} .$$

One might now ask why these non-time dependent win probabilities are not of the same form as those obtained in Chapter II when a race was being performed in discrete time. It should be pointed out that even though both walks are acting independently of the other, when one particle moves the other can not at the same instant since we are considering their movement in continuous time. This is because, by our basic assumptions, the probability of both walks advancing in a small increment of time is negligible. Thus, if we were to consider a step as being the movement of any one of the particles, there could be at most $A+B-1$ steps. In the discrete time situation discussed in Chapter II, it was possible for both particles to advance simultaneously at each trial, and there could be an undetermined number of trials before one of the particles was absorbed at the origin, thus ending the process.

Let us now demonstrate that both the time dependent and non-time dependent state probabilities form proper probability distributions.

3.1.2 Proof of Being a Proper Probability Distribution

To show that the time dependent state probabilities, namely (3.1.5), (3.1.6) and (3.1.7), and the non-time dependent win probabilities, (3.1.8) and (3.1.9), form proper probability distributions, we must show that each set sums to unity. Since the former set contains the latter, we shall deal first with the non-time dependent win probabilities. Calling the sum of (3.1.8) and (3.1.9) U , we may write

$$U = \sum_{z=A}^{A+B-1} \binom{z-1}{A-1} p^A q^{z-A} + \sum_{z=B}^{A+B-1} \binom{z-1}{B-1} q^B p^{z-B}$$

where for brevity in notation we have put

$$p = \frac{\alpha}{\alpha+\beta} \quad \text{and} \quad q = \frac{\beta}{\alpha+\beta} \quad .$$

Now writing each of the above negative binomials in terms of its binomial complement by the relationship (2.1.11), we obtain

$$U = \left[1-\sum_{s=0}^{A-1} \binom{A+B-1}{s} p^s q^{A+B-1-s}\right] + \left[1-\sum_{s=0}^{B-1} \binom{A+B-1}{s} q^s p^{A+B-1-s}\right] \quad .$$

Hence

$$U = \sum_{s=A}^{A+B-1} \binom{A+B-1}{s} p^s q^{A+B-1-s} + \sum_{s=B}^{A+B-1} \binom{A+B-1}{s} q^s p^{A+B-1-s} \quad .$$

And now by using the well known (see Feller [12] and Abramowitz [1]) identity

$$\sum_{s=a}^{n} \binom{n}{s} p^s q^{n-s} = I_p(a,n-a+1)$$

where $I_p(a,n-a+1)$ is the Incomplete Beta function defined by

$$I_p(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^p t^{a-1}(1-t)^{b-1}\,dt \quad , \quad \begin{pmatrix} 0 \leq p \leq 1 \\ a,b>0 \end{pmatrix} \quad ,$$

we can write

$$U = I_p(A,B)+I_q(B,A) \quad .$$

But since $I_q(B,A) = 1-I_p(A,B)$ by virtue of its definition, we have immediately that

$$U = 1 \quad .$$

Thus, we have demonstrated that the non-time dependent win probabilities given by (3.1.8) and (3.1.9) form a proper distribution.

Utilizing the fact that U equals 1, we will now call the sum of equations (3.1.5), (3.1.6) and (3.1.7) S, and write

$$S = \sum_{x=0}^{A-1} \frac{e^{-\alpha t}(\alpha t)^x}{x!} \cdot \sum_{x=0}^{B-1} \frac{e^{-\beta t}(\beta t)^x}{x!} + 1 -$$

$$- \sum_{z=A}^{A+B-1} \binom{z-1}{A-1} p^A q^{z-A} \sum_{x=0}^{z-1} \frac{e^{-(\alpha+\beta)t}[(\alpha+\beta)t]^x}{x!} -$$

$$- \sum_{z=B}^{A+B-1} \binom{z-1}{B-1} q^B p^{z-B} \sum_{x=0}^{z-1} \frac{e^{-(\alpha+\beta)t}[(\alpha+\beta)t]^x}{x!} .$$

Hence, in order to show that S equals 1, it will suffice to show that

$$(3.1.10) \quad \sum_{x=0}^{A-1} \frac{e^{-\alpha t}(\alpha t)^x}{x!} \sum_{x=0}^{B-1} \frac{e^{-\beta t}(\beta t)^x}{x!} = \sum_{z=A}^{A+B-1} \binom{z-1}{A-1} p^A q^{z-A} \cdot \sum_{x=0}^{z-1} \frac{e^{-(\alpha+\beta)t}[(\alpha+\beta)t]^x}{x!} + \sum_{z=B}^{A+B-1} \binom{z-1}{B-1} q^B p^{z-B} \cdot \sum_{x=0}^{z-1} \frac{e^{-(\alpha+\beta)t}[(\alpha+\beta)t]^x}{x!} .$$

Calling the right hand side of this equation R, after making a simple change of variable we have

$$(3.1.11) \quad R = \sum_{z=0}^{B-1} \binom{A+z-1}{A-1} p^A q^z \sum_{x=0}^{A+z-1} \frac{e^{-(\alpha+\beta)t}[(\alpha+\beta)t]^x}{x!} + \sum_{z=0}^{A-1} \binom{B+z-1}{B-1} q^B p^z \sum_{x=0}^{B+z-1} \frac{e^{-(\alpha+\beta)t}[(\alpha+\beta)t]^x}{x!} .$$

Calling the left hand side of (3.1.10) L, we write

$$(3.1.12) \quad L = \sum_{x=0}^{A-1} \frac{e^{-\alpha t}(\alpha t)^x}{x!} \sum_{x=0}^{B-1} \frac{e^{-\beta t}(\beta t)^x}{x!} .$$

The task at hand is to now show $L = R$.

For brevity in notation in what is to follow, let us write the Poisson distribution function as

$$\sum_{x=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^x}{x!} = P(\lambda,n,t) \quad ,$$

and the Gamma density function as

$$\frac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{(n-1)!} = \Gamma(\lambda,n,x) \quad .$$

Then from the relationship (3.0.6) developed earlier, we may write

$$(3.1.13) \qquad P(\lambda,n,t) = \int_t^\infty \Gamma(\lambda,n,x)dx \; .$$

Now, by employing the identity (3.0.6) on equation (3.1.11) we obtain, after substituting back for p and q ,

$$R = \sum_{z=0}^{B-1} \binom{A+z-1}{A-1} \left(\frac{\alpha}{\alpha+\beta}\right)^A \left(\frac{\beta}{\alpha+\beta}\right)^z \int_t^\infty \frac{e^{-(\alpha+\beta)x}(\alpha+\beta)^{A+z}x^{A+z-1}}{(A+z-1)!} dx \; +$$

$$+ \sum_{z=0}^{A-1} \binom{B+z-1}{B-1} \left(\frac{\beta}{\alpha+\beta}\right)^B \left(\frac{\alpha}{\alpha+\beta}\right)^z \int_t^\infty \frac{e^{-(\alpha+\beta)x}(\alpha+\beta)^{B+z}x^{B+z-1}}{(B+z-1)!} dx \; .$$

Rewriting the Binomial coefficients as factorials and then canceling constants yields,

$$R = \sum_{z=0}^{B-1} \frac{\alpha^A \beta^z}{(A-1)!z!} \int_t^\infty e^{-(\alpha+\beta)x} x^{A+z-1} dx \; +$$

$$+ \sum_{z=0}^{A-1} \frac{\alpha^B \beta^z}{(B-1)!z!} \int_t^\infty e^{-(\alpha+\beta)x} x^{B+z-1} dx \quad .$$

After interchanging the order of summation and integration we obtain

$$R = \int_t^\infty \frac{\alpha^A e^{-\alpha x} x^{A-1}}{(A-1)!} \sum_{z=0}^{B-1} \frac{e^{-\beta x}(\beta x)^z}{z!}\, dx +$$

$$+ \int_t^\infty \frac{\beta^B e^{-\beta x} x^{B-1}}{(B-1)!} \sum_{z=0}^{A-1} \frac{e^{-\alpha x}(\alpha x)^z}{z!}\, dx \quad .$$

Applying the simplified notation in (3.1.13) we have

$$(3.1.14) \quad R = \int_t^\infty \Gamma(\alpha,A,x)\cdot P(\beta,B,x)dx +$$

$$+ \int_t^\infty \Gamma(\beta,B,x)\cdot P(\alpha,A,x)dx \ .$$

Let us consider the first integration in (3.1.14) and integrate this by parts setting

$$u = P(\beta,B,x) \quad \text{and} \quad dv = \Gamma(\alpha,A,x)dx \ .$$

We obtain du immediately by differentiating the integral in (3.1.13). Hence

$$du = -\Gamma(\beta,B,x)dx \ .$$

We obtain v by writing the indefinite integral of dv as

$$v = \int_{C}^{x} \Gamma(\alpha,A,s)ds = -\int_{x}^{C} \Gamma(\alpha,A,s)ds \ .$$

Since C can be considered to be any arbitrary constant, setting it to infinity yields

$$v = -\int_{x}^{\infty} \Gamma(\alpha,A,s)ds \ .$$

Upon application of (3.1.13) we obtain

$$v = -P(\alpha,A,x) \quad .$$

Thus, by calling the first integration in (3.1.14) I, we obtain, after performing the integration by parts,

$$I = uv \Big|_{t}^{\infty} - \int_{t}^{\infty} vdu \ .$$

Hence

$$I = -P(\beta,B,x)P(\alpha,A,x) \Big|_{t}^{\infty} - \int_{t}^{\infty} P(\alpha,A,x)\Gamma(\beta,B,x)dx \ .$$

After evaluating I at its upper and lower limits we are left with

$$I = P(\alpha,A,t)P(\beta,B,t) - \int_{t}^{\infty} \Gamma(\beta,B,x)P(\alpha,A,x)dx \ .$$

Combining this with the second integration in (3.1.14) results in the immediate cancellation of both integrals. Hence (3.1.14) becomes

$$R = P(\alpha,A,t)P(\beta,B,t) \ .$$

Reverting to the original notation we have

$$R = \sum_{x=0}^{A-1} \frac{e^{-\alpha t}(\alpha t)^x}{x!} \sum_{x=0}^{B-1} \frac{e^{-\beta t}(\beta t)^x}{x!} \ .$$

This is the exact form of L in (3.1.12) and thus we have proved that $R = L$, implying that $S = 1$.

Recapitulating, we see that we have just shown that equations (3.1.5), (3.1.6) and (3.1.7) do indeed form a proper probability distribution.

3.1.3 Duration of the Race

In this section we are going to derive the probability distribution for the duration of the race. As mentioned earlier, our assumptions do not permit a tie or draw in this continuous time walk. Hence the only way the race can terminate is by A or B winning it. We are therefore interested in determining a probability distribution on when the race will end.

First, let us define a probability distribution function on t, $P(t)$, such that

(3.1.15) $$P(t) = P_A(t) + P_B(t) .$$

We note that $P_A(t)$ and $P_B(t)$ are both probability distribution functions on t and represent the mutually exclusive events of the A and B walks winning by time t. We noticed earlier that by letting $t\to\infty$ in $P_A(t)$ and $P_B(t)$ we obtain the non-time-dependent solutions, P_A and P_B, for the A and B walks winning the race. We also have shown, by virtue of the fact that $U = 1$ in the last section, that P_A plus P_B, (viz. (3.1.8) and (3.1.9)), form a proper distribution. Thus, we may say that

$$\lim_{t\to\infty} P(t) = 1 .$$

Let us now define a probability density function $f(t)$ for the duration of the race. Thus, $f(t)dt$ will denote the probability that the race terminates in the interval $(t,t+dt)$. Hence, we may write

(3.1.16) $$f(t) = f_A(t) + f_B(t)$$

where $f_A(t)$ and $f_B(t)$ are the probability density functions for the mutually exclusive events of A and B winning the race at time t.

We may obtain (3.1.16) by differentiating (3.1.15) directly. However, because of the complicated nature of $P_A(t)$ and $P_B(t)$ [viz. (3.1.6) and (3.1.7)] this would be rather involved. It is much simpler to notice how $P_A(t)$

and $P_B(t)$ were obtained from the basic differential equations. For example, $P_A(t)$ was obtained by integrating equation (3.1.2) up to t and then summing over all j. Thus,

$$P_A(t) = \sum_{j=1}^{B} \int_0^t \frac{dp[0,j;t]}{dt}\, dt \; .$$

Hence, we may obtain $f_A(t)$ directly by summing $\frac{dp[0,j;t]}{dt}$ over all j. Therefore, by summing equation (3.1.2) over all j we obtain

$$f_A(t) = \sum_{j=1}^{B} \frac{\alpha^A \beta^{B-j} t^{A-1+B-j} e^{-(\alpha+\beta)t}}{(A-1)!\,(B-j)!} \; .$$

Making the change of variables $x = B-j$ yields

$$f_A(t) = \frac{\alpha^A t^{A-1} e^{-\alpha t}}{(A-1)!} \sum_{x=0}^{B-1} \frac{e^{-\beta t}(\beta t)^x}{x!} \; .$$

By symmetry we can also obtain $f_B(t)$. Thus, with expressions for $f_A(t)$ and $f_B(t)$ at our disposal, we may write down the probability density function for the duration of the race namely,

$$(3.1.17) \qquad f(t) = \frac{\alpha^A t^{A-1} e^{-\alpha t}}{(A-1)!} \sum_{x=0}^{B-1} \frac{e^{-\beta t}(\beta t)^x}{x!} + \\ + \frac{\beta^B t^{B-1} e^{-\beta t}}{(B-1)!} \sum_{x=0}^{A-1} \frac{e^{-\alpha t}(\alpha t)^x}{x!} \; .$$

In order for $f(t)$ to be a proper density, its integral over all t must equal 1. By the very nature of the relationship between $f(t)$ and $P(t)$, it turns out that

$$\int_0^\infty f(t)dt = \lim_{t\to\infty} P(t) .$$

Since we showed earlier in this section that $\lim_{t\to\infty} P(t) = 1$, we have shown $f(t)$ to be a proper density.

Besides having the density and distribution functions for the duration of the race, one may also desire its expected duration. This is most easily obtained from $f(t)$ in (3.1.17). Multiplying by t and integrating over all t yields,

$$E(t) = \int_0^\infty \frac{(\alpha t)^A e^{-\alpha t}}{(A-1)!} \sum_{x=0}^{B-1} \frac{e^{-\beta t}(\beta t)^x}{x!} dt +$$

$$+ \int_0^\infty \frac{(\beta t)^B e^{-\beta t}}{(B-1)!} \sum_{x=0}^{A-1} \frac{e^{-\alpha t}(\alpha t)^x}{x!} dt .$$

By interchanging the order of integration and summation, collecting terms and then integrating we obtain,

$$E(t) = \sum_{x=0}^{B-1} \frac{(A+x)!\alpha^A\beta^x}{x!(A-1)!(\alpha+\beta)^{A+x+1}} + \sum_{x=0}^{A-1} \frac{(B+x)!\beta^B\alpha^x}{x!(B-1)!(\alpha+\beta)^{B+x+1}} .$$

Making the usual change of variables yields

$$(3.1.18) \quad E(t) = \frac{1}{(\alpha+\beta)}\left[\left(\frac{\alpha}{\alpha+\beta}\right)^{A} \sum_{z=A}^{A+B-1} \binom{z-1}{A-1} \left(\frac{\beta}{\alpha+\beta}\right)^{z-A} \cdot z + \right.$$

$$\left. + \left(\frac{\beta}{\alpha+\beta}\right)^{B} \sum_{z=B}^{A+B-1} \binom{z-1}{B-1} \left(\frac{\alpha}{\alpha+\beta}\right)^{z-B} \cdot z \right] .$$

We note that except for the constant term, $1/(\alpha+\beta)$, this is simply the expected value of the number of <u>trials</u> required for the race to terminate, when the trials are distributed according to the non-time dependent win probabilities, P_A and P_B [<u>vide</u> (3.1.8) and (3.1.9)]. Unfortunately no pleasing closed form exists for this moment. However, its value is most easily calculated by utilizing tables of the negative binomial distribution or with the use of a high speed computor.

The second moment; $E(t^2)$, is also obtained from $f(t)$ in (3.1.17). After multiplying by t^2 , integrating and then performing operations similar to those utilized in deriving $E(t)$, we obtain

$$(3.1.19) \quad E(t^2) = \frac{1}{(\alpha+\beta)^2}\left[\left(\frac{\alpha}{\alpha+\beta}\right)^{A} \sum_{z=A}^{A+B-1} \binom{z-1}{A-1} \left(\frac{\beta}{\alpha+\beta}\right)^{z-A} \cdot z(z+1) + \right.$$

$$\left. + \left(\frac{\beta}{\alpha+\beta}\right)^{B} \sum_{z=B}^{A+B-1} \binom{z-1}{B-1} \left(\frac{\alpha}{\alpha+\beta}\right)^{z-B} \cdot z(z+1) \right] .$$

3.2 Three or More Walks

In this section we will merely generalize the methodology and assumptions utilized in the first half of this chapter. As mentioned earlier, because of the nature of the system operating in continuous time, we need not consider the possibilities of ties between two or more of the competing particles. This simplifies the analysis and the number of formulas involved. As a result, this section will be more of an outline and general description of how to determine win probabilities when more than two particles are racing. As in the preceding chapter however, the actual analysis will still be performed on three walks. When the required probabilities are calculated for three walks, it will be shown how the most general results are obtained.

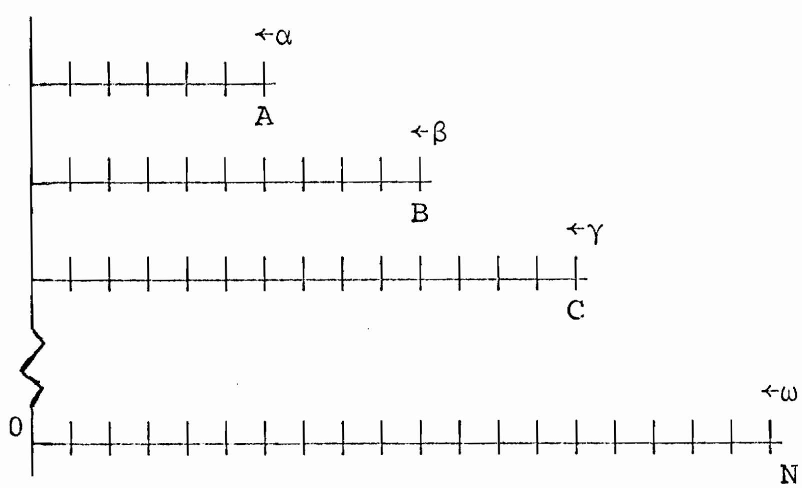

Figure 3.2.1
Graphical Representation of the System

Consider the situation of n random walks performed on the real axis, but in continuous time as described in Section 3.0 . We may graphically picture the system as in Figure 3.2.1 . Each particle is racing toward the origin independently of the others. However, as before, as soon as one of these particles reaches the origin, the whole process terminates.

By extending our previous notation we will denote by $p[i,j,\ldots,n;t]$, the "state" probability that the A , B , C , ... and N particles are at co-ordinates i , j , k , ... and n respectively at time t . More precisely, we define $p[i,j,k,\ldots,n;t]$ by

$$p[i,j,k,\ldots,n;t] = \Pr[A(t)=i,B(t)=j;C(t)=k;\ldots,N(t)=n]$$

where $A(t)$, $B(t)$, ... are, as before, the random variables representing the co-ordinates of each particle at time t .

3.2.1 State Probabilities

As discussed in Section 3.1.1, $p[i,j,k,\ldots,n;t]$ is most easily seen to be the product of n independent Poisson distributions just as long as none of the co-ordinates of each particle is zero. Hence, in the case of three walks and analogously to equation (3.1.1) we have

$$p[i,j,k;t] = \frac{e^{-\alpha t}(\alpha t)^{A-i}}{(A-i)!}\,\frac{e^{-\beta t}(\beta t)^{B-j}}{(B-j)!} \quad . \tag{3.2.1}$$

$$\cdot \frac{e^{-\gamma t}(\gamma t)^{C-k}}{(C-k)!} \quad , \quad \begin{pmatrix} i=1,2,\ldots,A \\ j=1,2,\ldots,B \\ k=1,2,\ldots,C \end{pmatrix} \quad .$$

Writing now the probability that the A particle reaches the origin sometime between time t and t+dt while the B and C particles are at that time at co-ordinates j and k (both greater than zero) in terms of state and transition probabilities we have

$$p[0,j,k;t+dt] = p[0,j,k;t]\cdot 1+p[1,j,k;t]\alpha dt[1-\beta dt][1-\gamma dt] \quad .$$

The usual operations result in

$$(3.2.2) \quad \frac{dp[0,j,k;t]}{dt} =$$

$$= \frac{\alpha^A\beta^{B-j}\gamma^{C-k}e^{-(\alpha+\beta+\gamma)t}t^{A-1+B-j+C-k}}{(A-1)!(B-j)!(C-k)!} \quad , \quad \begin{pmatrix} j=1,2,\ldots,B \\ k=1,2,\ldots,C \end{pmatrix} \quad .$$

The integral of this with respect to t is in exactly the same format as the relationship (3.0.10) when constants are neglected and hence by virtue of (3.0.11) we may write immediately,

$$(3.2.3) \quad p[0,j,k;t] = \frac{(A-1+B-j+C-k)!\,\alpha^A\beta^{B-j}\gamma^{C-k}}{(A-1)!(B-j)!(C-k)!(\alpha+\beta+\gamma)^{A+B-j+C-k}} \quad .$$

$$\cdot\left[1-\sum_{x=0}^{A-1+B-j+C-k}\frac{e^{-(\alpha+\beta+\gamma)t}[(\alpha+\beta+\gamma)t]^x}{x!}\right] ,$$

which is an obvious extension of (3.1.3). By symmetry we could also obtain expressions for $p[i,0,k;t]$ and $p[i,j,0;t]$. These are interpreted as the probabilities of one of the particles winning by time t while the other particles have non-zero co-ordinates.

By summing (3.2.3) over all j and k we will obtain the unconditional probability that the A particle wins the race. After making the usual change of variables we obtain

$$(3.2.4)\qquad P_A(t)=\sum_{y=0}^{B-1}\sum_{z=0}^{C-1}\frac{(A-1+y+z)!\alpha^A\beta^y\gamma^z}{(A-1)!y!z!(\alpha+\beta+\gamma)^{A+y+z}}\cdot$$

$$\cdot\left[1-\sum_{\nu=0}^{A-1+y+z}\frac{e^{-(\alpha+\beta+\gamma)t}[(\alpha+\beta+\gamma)t]^\nu}{\nu!}\right] .$$

By symmetry we also obtain

$$(3.2.5)\qquad P_B(t)=\sum_{x=0}^{A-1}\sum_{z=0}^{C-1}\frac{(x+b-1+z)!\alpha^x\beta^B\gamma^z}{x!(B-1)!z!(\alpha+\beta+\gamma)^{x+B+z}}\cdot$$

$$\cdot\left[1-\sum_{\nu=0}^{x+B-1+z}\frac{e^{-(\alpha+\beta+\gamma)t}[(\alpha+\beta+\gamma)t]^\nu}{\nu!}\right]$$

and

$$(3.2.6) \quad P_C(t) = \sum_{x=0}^{A-1} \sum_{y=0}^{B-1} \frac{(x+y+C-1)!\alpha^x\beta^y\gamma^C}{x!y!(C-1)!(\alpha+\beta+\gamma)^{x+y+C}} \cdot$$

$$\cdot\left[1- \sum_{\nu=0}^{x+y+C-1} \frac{e^{-(\alpha+\beta+\gamma)t}[(\alpha+\beta+\gamma)t]^\nu}{\nu!}\right] .$$

Similarly, by summing equation (3.2.1) over all i, j and k, we obtain the probability that the race is not over by time t. Performing this summation yields, after the usual change of variables,

$$(3.2.7) \quad \sum_{i,j,k} p[i,j,k;t] = \sum_{x=0}^{A-1} \frac{e^{-\alpha t}(\alpha t)^x}{x!} \cdot$$

$$\cdot \sum_{y=0}^{B-1} \frac{e^{-\beta t}(\beta t)^y}{y!} \sum_{z=0}^{C-1} \frac{e^{-\gamma t}(\gamma t)^z}{z!} .$$

Just as in the two-walk situation, these four equations must sum to unity in order that they define a proper probability distribution. The fact that this is true will be demonstrated in the next section.

Just from the nature of equations (3.2.4) through (3.2.6) and the way they were obtained we could write down immediately the time dependent unconditional win probability for any walk no matter how many walks were involved. On comparing the two-walk situation with the three-walk situation above [i.e. viz. (3.1.6) and (3.1.7) compared with (3.2.4), (3.2.5) and (3.2.6)] we see that they are of exactly the same format except with extra parameters and a summation involved. Keeping this in mind we could write down the time-dependent unconditional win probability for the A walk, for example, when there are five walks involved. Let the parameters for these two extra walks be D and δ and E and ε respectively. Then

$$(3.2.8)\quad P_A(t) = \sum_{b=0}^{B-1} \sum_{c=0}^{C-1} \sum_{d=0}^{D-1} \sum_{e=0}^{E-1} \cdot \frac{(A-1+b+c+d+e)!\,\alpha^A \beta^b \gamma^c \delta^d \varepsilon^e}{(A-1)!\,b!\,c!\,d!\,e!\,(\alpha+\beta+\gamma+\delta+\varepsilon)^{A+b+c+d+e}} \cdot \left[1 - \sum_{\nu=0}^{A-1+b+c+d+e} \frac{e^{-(\alpha+\beta+\gamma+\delta+\varepsilon)t}[(\alpha+\beta+\gamma+\delta+\varepsilon)t]^{\nu}}{\nu!}\right].$$

Before proceeding further we must obtain the non-time-dependent win probabilities for each of the walks. These are obtained by simply letting $t\to\infty$ in equations (3.2.4) through (3.2.6). Taking this limit yields

$$(3.2.9)\quad \begin{cases} P_A = \displaystyle\sum_{y=0}^{B-1} \sum_{z=0}^{C-1} \frac{(A-1+y+z)!\,\alpha^A \beta^y \gamma^z}{(A-1)!\,y!\,z!\,(\alpha+\beta+\gamma)^{A+y+z}} ; \\[2ex] P_B = \displaystyle\sum_{x=0}^{A-1} \sum_{z=0}^{C-1} \frac{(x+B-1+z)!\,\alpha^x \beta^B \gamma^z}{x!\,(B-1)!\,z!\,(\alpha+\beta+\gamma)^{x+B+z}} ; \\[2ex] P_C = \displaystyle\sum_{x=0}^{A-1} \sum_{y=0}^{B-1} \frac{(x+y+C-1)!\,\alpha^x \beta^y \gamma^C}{x!\,y!\,(C-1)!\,(\alpha+\beta+\gamma)^{x+y+C}} . \end{cases}$$

These are interpreted as the probabilities that each walk eventually wins the race. Naturally these too must sum to unity as will be shown in the next section. Just as in the two-walk situation, equation (3.2.7), which is the probability of the race not being over by time t, tends to zero as t approaches infinity. Recall this implies that the probability of an infinitely prolonged race is zero.

In a similar manner we can also obtain the non-time-dependent win probability in the five-walk situation. Taking the limit as $t \to \infty$ in (3.2.8) we obtain

$$P_A = \sum_{b=0}^{B-1} \sum_{c=0}^{C-1} \sum_{d=0}^{D-1} \sum_{e=0}^{E-1} \frac{(A-1+b+c+d+e)!}{(A-1)!b!c!d!e!} \frac{\alpha^A \beta^b \gamma^c \delta^d \varepsilon^e}{(\alpha+\beta+\gamma+\delta+\varepsilon)^{A+b+c+d+e}} .$$

Again we see immediately how this generalizes to the N walk situation.

Probabilistically speaking, all of the equations derived above have the same interpretations as those given in Section 3.1.1 .

3.2.2 Proof of Being a Proper Probability Distribution

In this section we are going to show that equations (3.2.4) through (3.2.7), the time dependent state probabilities, and also equations (3.2.9), the non-time-dependent win probabilities, form proper distributions by showing that each system of equations sums to unity. Since the former system contains the latter, we will deal with these equations first. In order to show that (3.2.9) sums to unity in a simple manner, we will utilize a most straightforward argument.

Let us define a new random walk system which again involves a race toward the origin between the particles. This race will be performed in discrete time as in Chapter II but in a different manner. In this system, only one of the particles will be able to advance at each trial. We recall that in the model discussed in Chapter II, at each trial any one, or all or none, or any combination of the particles could advance at each trial. The system can be represented graphically as in Figure 3.2.2 .

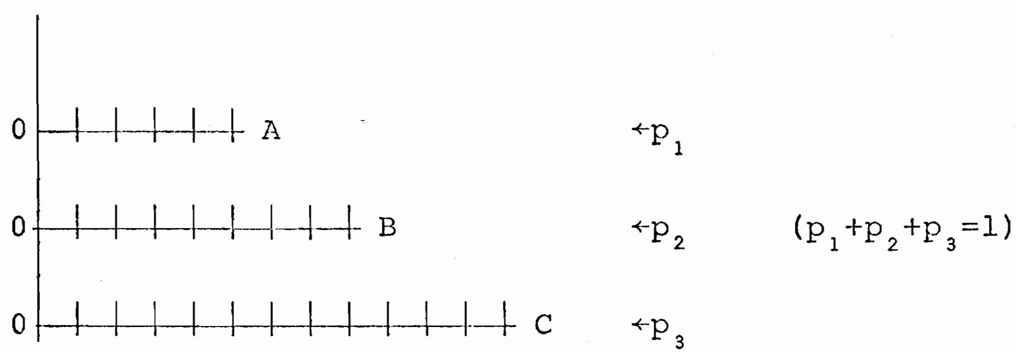

Figure 3.2.2
Graphical Representation of the System

As already mentioned, in this situation only one of the particles will be allowed to advance at each trial. Also, one of the particles must advance. Thus, at each trial either walk A , B , or C is chosen with probabilities p_1 , p_2 or p_3 respectively and then the one chosen advances one unit toward the origin. This continues until one of the particles reaches the origin at which time the whole process stops.

It is important to notice the difference here between this system and the one described in Chapter II. In the earlier system, the process could continue for an indefinite number of trials depending on the starting points of each particle and the probabilities of moving to the left or remaining stationary. In this system it is obvious that the maximum number of trials possible before one of the walks wins is $(A+B+C-2)$. This is to say, it is certain that one of the three particles will have reached the origin in $(A+B+C-2)$ trials.

We may also note that even though we are operating in discrete time again, a tie or draw is impossible here since only one of the walks may advance at each trial.

We will now develop the formulas for P_A , P_B and P_C , the respective probabilities of A , B or C winning. Since the only possible situation after at most $(A+B+C-2)$ trials is that one of these walks has won, then it is trivially true that $P_A+P_B+P_C = 1$. Let us first consider the A walk and develop P_A . The others will be obtained by symmetry.

The A particle can win only if there have been at least A trials. Let the number of trials required for the A particle to win be represented by $T = A+r+s$, where r is the number of steps the B walk has taken and s is the number of steps the C walk has taken. Obviously then, in order for A to win, $(0 \leq r \leq B-1)$ and $(0 \leq s \leq C-1)$. Let us represent by $P_A^{(T)}$ the probability that the A particle wins at the T'th trial. This means that on the last trial, A must have advanced one unit. Hence, in the trials preceeding the T'th , A must have taken $(A-1)$ steps, B must have taken r steps $(0 \leq r \leq B-1)$, and C must have taken s steps $(0 \leq s \leq C-1)$.

The total number of ways the choice of these steps could be arranged is simply

$$\frac{(A-1+r+s)!}{(A-1)!r!s!} ,$$

and the probability of any one of these combinations is $p_1^{A-1}\ p_2^r\ p_3^s$. Upon multiplying these two factors together with another p_1 , which is the probability of an A step on the last trial we have

$$P_A^{(T)} = \frac{(A-1+r+s)!}{(A-1)!r!s!}\ p_1^A\ p_2^r\ p_3^s \quad , \begin{pmatrix} 0 \leq r \leq B-1 \\ \\ 0 \leq s \leq C-1 \end{pmatrix} .$$

This is the probability of the A particle winning on the T'th trial. Hence, by summing over r and s we obtain the unconditional probability, P_A, of A eventually winning the race. Thus

$$P_A = \sum_{r=0}^{B-1} \sum_{s=0}^{C-1} \frac{(A-1+r+s)!}{(A-1)!r!s!} p_1^A p_2^r p_3^s \quad .$$

We may obtain P_B and P_C by symmetry. Now, since it is trivially true that $P_A+P_B+P_C = 1$ we are able to write immediately that

$$(3.2.10) \qquad \sum_{b=0}^{B-1} \sum_{c=0}^{C-1} \frac{(A-1+b+c)!}{(A-1)!b!c!} p_1^A p_2^b p_3^c \; + \\ + \sum_{a=0}^{A-1} \sum_{c=0}^{C-1} \frac{(a+B-1+c)!}{a!(B-1)!c!} p_1^a p_2^B p_3^c \; + \\ + \sum_{a=0}^{A-1} \sum_{b=0}^{B-1} \frac{(a+b+C-1)!}{a!b!(C-1)!} p_1^a p_2^b p_3^C \; = \; 1 \; .$$

Before showing how this identity is to be used in proving that the earlier mentioned systems in this chapter are proper probability distributions, we should note how general the analysis performed above was. The same reasoning could have been applied to any number of walks. As a matter of fact, in the case of two walks, this is an alternate way of showing that $U = 1$ in Section 3.1.2 . Recall that this was accomplished by use of several operations involving Incomplete Beta functions.

Returning to the problem at hand, we must show that equations (3.2.9) sum to unity. This is now a trivial operation in view of the identity (3.2.10). If we set

$$p_1 = \left(\frac{\alpha}{\alpha+\beta+\gamma}\right) \quad , \quad p_2 = \left(\frac{\beta}{\alpha+\beta+\gamma}\right) \quad \text{and} \quad p_3 = \left(\frac{\gamma}{\alpha+\beta+\gamma}\right)$$

we see that (3.2.9) is exactly in the format of (3.2.10) and hence sums to unity. In the case of more walks, the same procedure holds.

Now that we have shown that the non-time-dependent win probabilities form a proper distribution, let us now show that this is also true for the time dependent state probabilities, equations (3.2.4) through (3.2.7). As in Section 3.1.2, noting that the time dependent probabilities contain the non-time dependent ones (which we have just shown sum to unity), we see that showing the sum of equations (3.2.4) through (3.2.7) is unity is equivalent to showing that

$$(3.2.11) \quad \sum_{x=0}^{A-1} \frac{e^{-\alpha t}(\alpha t)^x}{x!} \sum_{x=0}^{B-1} \frac{e^{-\beta t}(\beta t)^x}{x!} \sum_{x=0}^{C-1} \frac{e^{-\gamma t}(\gamma t)^x}{x!} =$$

$$\sum_{b=0}^{B-1} \sum_{c=0}^{C-1} \frac{(A-1+b+c)!}{(A-1)!b!c!} \frac{\alpha^A \beta^b \gamma^c}{(\alpha+\beta+\gamma)^{A+b+c}} \cdot$$

$$\cdot \sum_{\nu=0}^{A-1+b+c} \frac{e^{-(\alpha+\beta+\gamma)t}[(\alpha+\beta+\gamma)t]^\nu}{\nu!} +$$

$$+ \sum_{a=0}^{A-1} \sum_{c=0}^{C-1} \frac{(a+B-1+c)!}{a!(B-1)!c!} \frac{\alpha^a \beta^B \gamma^c}{(\alpha+\beta+\gamma)^{a+B+c}} \cdot$$

$$\cdot \sum_{\nu=0}^{a+B-1+c} \frac{e^{-(\alpha+\beta+\gamma)t}[(\alpha+\beta+\gamma)t]^{\nu}}{\nu!} +$$

$$+ \sum_{a=0}^{A-1} \sum_{b=0}^{B-1} \frac{(a+b+C-1)!}{a!b!(C-1)!} \frac{\alpha^a \beta^b \gamma^C}{(\alpha+\beta+\gamma)^{a+b+C}} \cdot$$

$$\cdot \sum_{\nu=0}^{a+b+C-1} \frac{e^{-(\alpha+\beta+\gamma)t}[(\alpha+\beta+\gamma)t]^{\nu}}{\nu!} .$$

Calling the left hand side of (3.2.11) L, and the right hand side R, we will proceed to show that $L = R$. Working first with R, let us employ the fundamental relationship (3.0.6) derived earlier connecting the Poisson and Gamma distributions. Substituting for the Poisson portion in R, we obtain after interchanging the order of summation and integration

$$R = \int_t^{\infty} \frac{\alpha^A e^{-\alpha x} x^{A-1}}{(A-1)!} \sum_{b=0}^{B-1} \frac{e^{-\beta x}(\beta x)^b}{b!} \sum_{c=0}^{C-1} \frac{e^{-\gamma x}(\gamma x)^c}{c!} dx +$$

$$+ \int_t^{\infty} \frac{\beta^B e^{-\beta x} x^{B-1}}{(B-1)!} \sum_{a=0}^{A-1} \frac{e^{-\alpha x}(\alpha x)^a}{a!} \sum_{c=0}^{C-1} \frac{e^{-\gamma x}(\gamma x)^c}{c!} dx +$$

$$+ \int_t^{\infty} \frac{\gamma^C e^{-\gamma x} x^{C-1}}{(C-1)!} \sum_{a=0}^{A-1} \frac{e^{-\alpha x}(\alpha x)^a}{a!} \sum_{b=0}^{B-1} \frac{e^{-\beta x}(\beta x)^b}{b!} dx .$$

After applying the simplified notation in (3.1.13) we obtain

$$(3.2.12) \qquad R = \int_t^\infty \Gamma(\alpha,A,x)P(\beta,B,x)P(\gamma,C,x)dx + $$
$$+ \int_t^\infty \Gamma(\beta,B,x)P(\alpha,A,x)P(\gamma,C,x)dx +$$
$$+ \int_t^\infty \Gamma(\gamma,C,x)P(\alpha,A,x)P(\beta,B,x)dx \quad .$$

We are now going to proceed just as we did in Section 3.1.2. We will consider only the first integral in (3.2.12) and integrate it by parts but this time setting

$$u = P(\beta,B,x)P(\gamma,C,x) \quad \text{and} \quad dv = \Gamma(\alpha,A,x)dx \; .$$

By differentiating the product we will obtain

$$du = [-P(\beta,B,x)\Gamma(\gamma,C,x)-\Gamma(\beta,B,x)P(\gamma,C,x)]dx \quad .$$

We note that dv here is the same as it was in Section 3.1.2 and so using that result we obtain

$$v = -P(\alpha,A,x) \; .$$

Now by calling the first integral in R i.e. equation (3.2.12), I, we obtain after integrating by parts and evaluating

$$I = P(\alpha,A,t)P(\beta,B,t)P(\gamma,C,t) - \int_t^\infty \Gamma(\gamma,C,x)P(\alpha,A,x)P(\beta,B,x)dx -$$

$$- \int_t^{\infty} \Gamma(\beta,B,x)P(\alpha,A,x)P(\gamma,C,x)dx \quad .$$

Combining I with the remainder of R in (3.2.12) results in the cancellation of all the integrals. Hence R becomes

$$R = P(\alpha,A,t)P(\beta,B,t)P(\gamma,C,t) \quad .$$

After changing back to the standard notation we obtain

$$R = \sum_{x=0}^{A-1} \frac{e^{-\alpha t}(\alpha t)^x}{x!} \sum_{x=0}^{B-1} \frac{e^{-\beta t}(\beta t)^x}{x!} \sum_{x=0}^{C-1} \frac{e^{-\gamma t}(\gamma t)^x}{x!} \quad ,$$

which is exactly L in (3.2.11). Thus, we have proved the identity (3.2.11), which was equivalent to proving that the time dependent state probabilities, i.e. equations (3.2.4) through (3.2.7), form a proper probability distribution.

The most pleasing feature of the method described above, of integrating by parts, is that it is easily carried out no matter how many walks are involved. Because of the symmetry of the equations, it is quite easy to see how the extension would be made.

3.2.2 Duration of the Race

Since all of the theory and methods used to determine the probability density function were discussed quite rigorously in Section 3.1.3, we will only attempt to show here the general results for three or more walks. Analogously to equation (3.1.15) we may define the distribution function for the duration of the race to be

$$(3.2.13) \qquad P(t) = P_A(t) + P_B(t) + P_C(t) \quad .$$

Similarly we define the density function for the duration of the race to be

$$(3.2.14) \qquad f(t) = f_A(t) + f_B(t) + f_C(t) \quad .$$

Both of these equations are direct extensions of (3.1.15) and (3.1.16). We recall that (3.2.14) is the derivative of (3.2.13) and hence may either be obtained in that way or in a simpler manner by summing the basic differential equations. Hence

$$f_A(t) = \sum_{j=1}^{B} \sum_{k=1}^{C} \frac{dp[0,j,k;t]}{dt} \quad .$$

Therefore, by performing the indicated summation on equation (3.2.2) we obtain

$$f_A(t) = \frac{\alpha^A t^{A-1} e^{-\alpha t}}{(A-1)!} \sum_{x=0}^{B-1} \frac{e^{-\beta t}(\beta t)^x}{x!} \sum_{x=0}^{C-1} \frac{e^{-\gamma t}(\gamma t)^x}{x!} \quad .$$

Similar expressions also exist for $f_B(t)$ and $f_C(t)$.

It is quite obvious what these densities would look like if more than three walks were involved. For example, if we were to consider five walks in the system with the extra parameter sets being D, δ and E , ε as before, then $f_A(t)$ would become

$$f_A(t) = \frac{\alpha^A t^{A-1} e^{-\alpha t}}{(A-1)!} \sum_{x=0}^{B-1} \frac{e^{-\beta t}(\beta t)^x}{x!} \sum_{x=0}^{C-1} \frac{e^{-\gamma t}(\gamma t)^x}{x!} \cdot$$

$$\cdot \sum_{x=0}^{D-1} \frac{e^{-\delta t}(\delta t)^x}{x!} \sum_{x=0}^{E-1} \frac{e^{-\varepsilon t}(\varepsilon t)^x}{x!} \; .$$

Thus, returning to the three-walk situation, we find that the probability density function for the duration of the race is

$$(3.2.15) \quad f(t) = \frac{\alpha^A t^{A-1} e^{-\alpha t}}{(A-1)!} \sum_{x=0}^{B-1} \frac{e^{-\beta t}(\beta t)^x}{x!} \sum_{x=0}^{C-1} \frac{e^{-\gamma t}(\gamma t)^x}{x!} +$$

$$+ \frac{\beta^B t^{B-1} e^{-\beta t}}{(B-1)!} \sum_{x=0}^{A-1} \frac{e^{-\alpha t}(\alpha t)^x}{x!} \sum_{x=0}^{C-1} \frac{e^{-\gamma t}(\gamma t)^x}{x!} +$$

$$+ \frac{\gamma^C t^{C-1} e^{-\gamma t}}{(C-1)!} \sum_{x=0}^{A-1} \frac{e^{-\alpha t}(\alpha t)^x}{x!} \sum_{x=0}^{B-1} \frac{e^{-\beta t}(\beta t)^x}{x!} \; .$$

Again it must be repeated that by merely observing the structure of (3.2.15), the extension to any number of walks becomes obvious.

If $f(t)$ is a proper density function, its integral over all t must equal 1 . We must point out again just as in the latter part of Section 3.1.3, that due to the very nature of the relationship between $P(t)$ and $f(t)$,

$$\lim_{t\to\infty} P(t) = \int_0^\infty f(t)dt \ .$$

Thus, since this limit is merely the sum of the non-time dependent win probabilities defined in (3.2.9), and since this sum was shown to be one in Section 3.2.2, we have shown $f(t)$ to be a proper probability density function.

Just as in the two-walk situation, we may be interested in obtaining the expected duration of the race, $E(t)$. As before, we merely multiply $f(t)$ by t in (3.2.15) and integrate over all t . We obtain

$$(3.2.16) \quad E(t) = \sum_{y=0}^{B-1} \sum_{z=0}^{C-1} \frac{(A+y+z)!\alpha^A \beta^y \gamma^z}{(A-1)!y!z!(\alpha+\beta+\gamma)^{A+y+z+1}} + \sum_{x=0}^{A-1} \sum_{z=0}^{C-1} \frac{(x+B+z)!\alpha^x \beta^B \gamma^z}{x!(B-1)!z!(\alpha+\beta+\gamma)^{x+B+z+1}} + \sum_{x=0}^{A-1} \sum_{y=0}^{B-1} \frac{(x+y+C)!\alpha^x \beta^y \gamma^C}{x!y!(C-1)!(\alpha+\beta+\gamma)^{x+y+C+1}} \ .$$

Again, we should notice the similarity between this expected value and the sum of the non-time dependent win probabilities (3.2.9).

Similarly, the second moment, $E(t^2)$ is found to be

$$(3.2.17) \quad E(t^2) = \sum_{y=0}^{B-1} \sum_{z=0}^{C-1} \frac{(A+y+z+1)!\alpha^A \beta^y \gamma^z}{(A-1)!y!z!(\alpha+\beta+\gamma)^{A+y+z+2}} +$$

$$+ \sum_{x=0}^{A-1} \sum_{z=0}^{C-1} \frac{(x+B+z+1)!\alpha^x \beta^B \gamma^z}{x!(B-1)!z!(\alpha+\beta+\gamma)^{x+B+z+2}} +$$

$$+ \sum_{x=0}^{A-1} \sum_{y=0}^{B-1} \frac{(x+y+C+1)!\alpha^x \beta^y \gamma^C}{x!y!(C-1)!(\alpha+\beta+\gamma)^{x+y+C+2}} .$$

IV. NORMAL APPROXIMATIONS

4.0 Preliminary Discussion

So far in our analysis, we have considered several models and have developed formulas for win and draw probabilities and also for the duration of the race. The probabilities that we have obtained are not difficult to calculate, expecially with the aid of a computer. One may then ask, "if this is indeed the case, why should we concern ourselves with approximations"?

Quite often, as will be indicated by this chapter, these approximations give a descriptive insight into the operation of a system, besides being excellent numerically. For example, one may wish to know how the function

$$f(n) = \binom{2n}{n} 2^{-2n}$$

behaves. Merely examining this function for different values of n is not only extremely tedious, but also unrewarding in terms of the results desired. However, if Stirling's approximation were used on $f(n)$, we would find quite simply that, as $n \to \infty$,

$$f(n) \sim (\pi n)^{-\frac{1}{2}} .$$

In other words $f(n)$ approaches zero like $n^{-\frac{1}{2}}$. This is not obvious from the functional form of $f(n)$. See Feller [12].

Our situation is similar in that the functional forms of the probabilities do not lend themselves especially well to simple manipulations even though they are relatively easy to calculate. Thus, one is tempted to seek approximations in order to obtain insight into the effect of changes in one or more of the parameters involved.

To begin this investigation, numerical calculations were carried out in the hope that the results would yield inspiration and insight into the operation of the system. A program was written for the IBM 7040 computer to calculate P_A, P_B and P_{AB} for all values of α and β ranging from .1 to .9 in increments of .1. The values of A and B were allowed to range from 10 to 100 both in increments of 10.

The numerical results suggested that $P_A = P_B$ approximately when $A/\alpha = B/\beta$. They also suggested that P_{AB} was approximately maximum when $A/\alpha = B/\beta$. However, as is to be expected, even with $A/\alpha = B/\beta$ for a fixed α and β, P_{AB} decreases with increasing A and B. In fact, when $A/\alpha = B/\beta$ it was found that the relationship $P_A = P_B = \frac{1}{2}$ became more and more true as $A, B \to \infty$.

Several graphs of P_A, P_B and P_{AB} were drawn against B/β for fixed A/α. Figure 4.0.1 illustrates the kind of result observed.

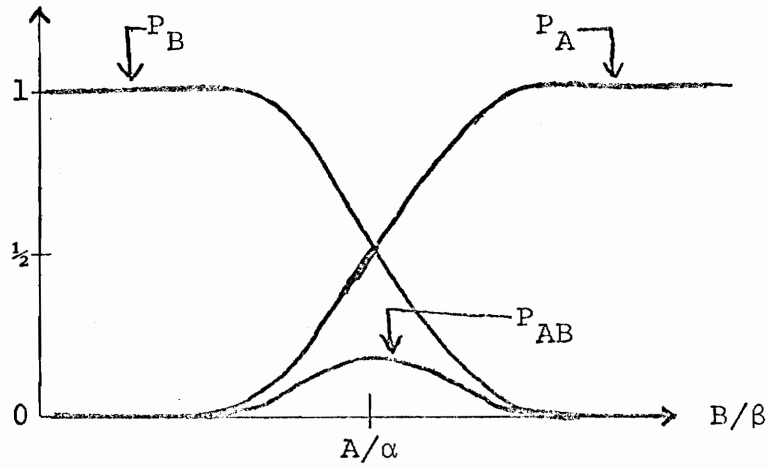

Figure 4.0.1

Typical Behavior of P_A , P_B and P_{AB}

The above diagram merely illustrates the typical behavior of the probabilities which rise and fall extremely fast close to the point where $B/\beta = A/\alpha$. On the basis of these numerical results and graphs, an attempt was made to show that in the limit (i.e. for large values of A and B , but still preserving their ratio such that $A/\alpha = B/\beta$, keeping α and β fixed) that $P_A = P_B = \frac{1}{2}$.

After many analytic and unsuccessful approaches to determine an exact relationship between A , α , B and β such that $P_A = P_B$, the argument that was adopted is contained in Section 4.1 .

4.1 An Approximation Utilizing the Multivariate Normal Distribution

In this section we shall develop general theory which will enable us to analyze in subsequent sections the behavior of a race between two or more particles both in discrete and continuous time. We shall give a general description here, and then inject particular restrictions in the sections to follow.

The most important property, common to both the race in discrete and in continuous time, is that each particle is acting completely independently of the others. If we were to represent by T_A, the time required for the A particle to reach the origin, and T_B the time it would take the B particle also to reach the origin, then it is quite obvious that A wins the race if and only if T_A is less than T_B. Now T_A and T_B are both random variables and are also independent of each other.

This type of approach, involving the time required for each walk to reach the origin independently of the others, will be used to obtain the desired approximations. It should be pointed out that in the discrete walk situation, the random variables involved will be discrete whereas in the continuous situation, they will naturally be continuous. However, this will not affect the following analysis.

Consider as before, a system of n discrete random walks whose particles race toward the origin either in discrete or continuous time. Let x_i be the random variable representing the total amount of time required for the i^{th} random walk to reach the origin independently of the other walks.

Then

$$x_1 = \sum_{j=1}^{A} t_j$$

in the case of the first or A walk. Here x_1 is merely the sum of A , independent, identically distributed random variables, t_j , which represent the time required for the A walk to make a transition from co-ordinate $(A-j+1)$ to $(A-j)$. Since both in the discrete and continuous situation, the transition probabilities from one co-ordinate to the next are independent of both time and co-ordinate, we see that x_1 is indeed the sum of A , independent, identically distributed random variables, t_j . In the case of the discrete time situation, we will see later that each t_j is distributed Geometrically whereas in the continuous time situation, it will be distributed Negative Exponentially. However, as mentioned earlier, we shall not differentiate between these two distributions until later.

Since each x_i is the sum of independent, identically distributed random variables, each possessing finite mean and variance, we know that the mean of x_i will simply be the sum of means and its variance the sum of the variances of the t_j involved. Call the mean of x_i, μ_i and its variance σ_i^2.

Now if we consider the random variables x_i for sufficiently large values of $A, B, C, \ldots, N$ (the starting points of each walk), we know then that by the Central Limit Theorem for sums of random variables, that each x_i will tend to be normally distributed with mean μ_i and variance σ_i^2. The only condition is that μ_i and σ_i^2 must exist, which they do in our situation. Symbolically we may write

$$x_i \cap N(\mu_i, \sigma_i^2) \ . \tag{4.1.1}$$

Again we must emphasize not only the independence of the t_j whose sum make up the total time for each walk to reach the origin (viz. $x_i, x_j, \ldots$) but also the mutual independence among these total times.

Let us now consider the vector of random variables $\underline{x}$, such that

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} .$$

Thus, assuming that we have chosen $A, B, \ldots, N$ sufficiently large for the Central Limit Theorem to hold, we may write

$$(4.1.2) \qquad f(\underline{x}) = \frac{|\Sigma|^{-\frac{1}{2}}}{(2\pi)^{n/2}} \exp[-\tfrac{1}{2}(\underline{x}-\underline{\mu})'\Sigma^{-1}(\underline{x}-\underline{\mu})]$$

where

$$\underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & 0 \\ & 0 & \ddots \; \sigma_n^2 \end{bmatrix}, \text{ and } |\Sigma|^{-\frac{1}{2}} = (\sigma_1^2 \cdot \sigma_2^2 \cdots \sigma_n^2)^{-\frac{1}{2}},$$

where (4.1.2) is the multivariate normal density function. See Anderson [2]. Since the x_i are mutually independent, we see then that the variance-covariance matrix, Σ , is diagonal.

We shall now make a transformation on the normal vector, $\underline{x}$. The purpose of this transformation will become obvious in the following sections. We shall see that the transformed vector, which shall be called $\underline{y}$, will enable us to determine approximate win and draw probabilities and also discover relationships between the parameters involved. It will also give excellent numerical approximations in the case of two walks, even with relatively small A and B .

Let us now consider a change of variables on the normal vector, $\underline{x}$. Let us make the linear transformation $\underline{y} = T\underline{x}$ where

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_1 - x_3 \\ \vdots \\ x_1 - x_n \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 1 & -1 & 0 & 0 & \ldots & 0 \\ 1 & 0 & -1 & & & 0 \\ 1 & 0 & 0 & -1 & & \\ \vdots & \vdots & & & & \\ 1 & 0 & & 0 & & -1 \end{bmatrix}_{(n-1)\times n} .$$

It can easily be verified that T is of rank $(n-1)$ implying that the transformation is a non-singular one. Hence, we may write symbolically that

$$(4.1.3) \qquad \underline{y} \cap N_{n-1}[T\underline{\mu}, T\ddagger T']$$

where it can be verified that

$$T\mu = \begin{bmatrix} \mu_1 - \mu_2 \\ \mu_1 - \mu_3 \\ \vdots \\ \mu_1 - \mu_n \end{bmatrix}$$

and

$$T\ddagger T' = \begin{bmatrix} (\sigma_1^2+\sigma_2^2) & \sigma_1^2 & \sigma_1^2 & \ldots & \sigma_1^2 \\ \sigma_1^2 & (\sigma_1^2+\sigma_3^2) & \sigma_1^2 & \ldots & \sigma_1^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \sigma_1^2 & \sigma_1^2 & \sigma_1^2 & & (\sigma_1^2+\sigma_n^2) \end{bmatrix}_{(n-1)\times(n-1)} .$$

The above variance-covariance matrix, $T\ddagger T'$, shows immediately that the covariance between any two of the variables in the vector $\underline{y}$ is σ_1^2 which is simply the variance of x_1 or of the time required for the A particle to finish independently of the others.

4.2 Two Walks

In this section we shall utilize the general theory developed in Section 4.1 for the special case of two walks racing toward the origin in both discrete and continuous time. In this case we shall be dealing with the uni-variate normal distribution and hence will be able to obtain numerical values quite easily. We shall also prove some theorems which will describe these random walks in the limit.

As mentioned several times previously, a tie or draw, when the walks are operating in continuous time, has probability zero. However, we know that this is not the case in discrete time. Intuition does tell us though, that as A and B grow very large, it would seem that P_{AB} should grow very small and ultimately approach zero. This is borne out by numerical calculations to be presented in Table 4.1 and also will be proven in Section 4.2.1 . For now, let us assume this to be correct and prove a theorem in the case of two walks.

The theorem we shall prove deals with the equality of P_A and P_B . We shall prove it in general in terms of μ_1 and μ_2 , the mean times required for each walk to reach the origin and in terms of the variances of these times, σ_1^2 and σ_2^2. Later we will see what these μ's and σ^2's actually are in terms of the parameters α, A, β, and B for both the discrete and continuous time situation. In other words, we will use the following general theorem to show the underlying relationships between the parameters of the walks, and under what conditions the win probabilities of each walk are equal.

<u>Theorem 4.1</u>

Assuming μ_1, μ_2, σ_1^2 and σ_2^2 all exist and that sufficiently large values of A and B are chosen such that the Central Limit Theorem for sums of random variables holds, and if $\mu_1 = \mu_2$, then $P_A \doteqdot P_B$ tends to $\frac{1}{2}$ despite the values of σ_1^2 and σ_2^2.

<u>Proof</u>: Here in the case of only two walks we are dealing with the univariate normal situation. From (4.1.3) we may symbolically write

$$(4.2.1) \quad y_1 = (x_1 - x_2) \cap N[(\mu_1 - \mu_2), (\sigma_1^2 + \sigma_2^2)] \quad .$$

Now the A walk will win the race if and only if $x_1 < x_2$, that is, if and only if the time required for the A walk to cover the distance A, is less than the time required for the B walk to cover the distance B to the origin. Hence, we may write P_A as

$$P_A = \Pr[x_1<x_2] = \Pr[x_1-x_2<0] = \Pr[y_1<0] \quad .$$

Now, if $\mu_1 = \mu_2$ as asserted, then from (4.2.1) we have

$$y_1 \cap N[0,(\sigma_1^2+\sigma_2^2)] \quad .$$

Thus

$$P_A = \Pr[y_1<0] = ½ \quad ,$$

no matter what the variances are, as the area to the left of the mean under a normal curve is always ½ . Hence, we have proven if $\mu_1 = \mu_2$, then $P_A \doteqdot P_B \doteqdot ½$, in the limit. (Note P_B is simply $\Pr[y_1>0] = ½$).

4.2.1 Discrete Time

As before we may write x_i , the time required for the i^{th} walk to reach the origin, as

(4.2.2) $$x_i = \sum_{j=1}^{s_i} t_j$$

where s_i represents the starting co-ordinate of the i^{th} walk (i.e. viz. A , B , C , ... etc.) and t_j is the random variable representing the time required for the i^{th} walk to make a transition from co-ordinates (s_i-j+1) to (s_i-j) . We recall that for each walk, the t_j are independent, identically distributed random variables.

In the discrete time situation, each t_j is distributed geometrically, i.e. with probability mass function

$$(4.2.3) \quad P(t_j) = \lambda(1-\lambda)^{t_j-1} \quad , \quad t_j=1,2,3, \ldots \quad .$$

Here λ is the probability of the particular particle under consideration moving one unit toward the origin at each trial. Naturally $(0\le\lambda\le1)$ and since we are considering the number of trials required, t_j is a discrete random variable with a countably infinite sample space.

As a result of (4.2.3) we may write

$$(4.2.4) \quad E(t_j) = 1/\lambda \quad \text{and} \quad Var(t_j) = (1-\lambda)/\lambda^2 \quad .$$

Thus, since the t_j are independent, we may write by virtue of the relationship (4.2.2) that

$$(4.2.5) \quad E(x_i) = s_i/\lambda \quad \text{and} \quad Var(x_i) = s_i(1-\lambda)/\lambda^2 \quad .$$

Hence, by substituting the values of α , β , A and B in the above we have that

$$(4.2.6) \begin{cases} \mu_1 = A/\alpha \quad ; \quad \mu_2 = B/\beta \\ \text{and} \\ \sigma_1^2 = \dfrac{A(1-\alpha)}{\alpha^2} \quad ; \quad \sigma_2^2 = \dfrac{B(1-\beta)}{\beta^2} \quad . \end{cases}$$

Stating the results of Theorem 4.1 in terms of the present situation we may say that if $A/\alpha = B/\beta$, then for very large values of A and B , $P_A \doteqdot P_B \doteqdot \frac{1}{2}$. This confirms what the numerical results mentioned earlier seemed to indicate.

So far in this analysis, we have not mentioned how this normal approximation may be used to determine the probability of a tie or draw occurring, P_{AB}. In terms of the times required for each walk to reach the origin, a tie occurs when $x_1 = x_2$ or equivalently, when $y_1 = 0$ in (4.2.1). Recapitulating we may say that

$$P_A = Pr[y_1<0] \quad ,$$

$$P_B = Pr[y_1>0] \quad ,$$

$$P_{AB} = Pr[y_1=0] \quad .$$

Since we are dealing with a continuous distribution the last of these three relationships is zero. However, for finite A and B, we know that the probability of a tie is not zero. As is the common practice (see Feller [12]) in utilizing a continuous distribution (i.e. the normal) for an approximation to a discrete distribution (i.e. the binomial) we alter the above relationships to

$$\text{(4.2.7)} \qquad \begin{cases} P_A = Pr[y_1<-\tfrac{1}{2}] & , \\ P_B = Pr[y_1>+\tfrac{1}{2}] & , \\ P_{AB} = Pr[-\tfrac{1}{2}<y_1<+\tfrac{1}{2}] & . \end{cases}$$

Let us now write (4.2.1) in terms of the parameters α, β, A and B. We have from (4.2.1) and (4.2.6) that

$$(4.2.8)\quad y_1 = (x_1 - x_2) \cap N\left[\left(\frac{A}{\alpha}\right) - \left(\frac{B}{\beta}\right), \left(\frac{A(1-\alpha)}{\alpha^2}\right) + \left(\frac{B(1-\beta)}{\beta^2}\right)\right] .$$

We can see now from P_{AB} in (4.2.7) and also the distribution of y_1 in (4.2.8), why $P_{AB} \to 0$ as A and $B \to \infty$. This is because the variance of the normal distribution is becoming infinite and hence the area in the interval $(-\frac{1}{2}, +\frac{1}{2})$ is going to zero because the curve is becoming flatter. Recall that we utilized the fact that $P_{AB} = 0$ in the limit in the proof of Theorem 4.1. There we said that $P_A = \Pr[y_1 < 0]$ instead of $P_A = \Pr[y_1 < -\frac{1}{2}]$ because the area in the interval $[-\frac{1}{2}, +\frac{1}{2}]$ was zero anyway.

We may use (4.2.7) and (4.2.8) to prove another fact which the numerical results seemed to indicate. This theorem deals with the conditions under which the draw probability is a maximum. We see from Figure 4.0.1 that P_{AB} appears to be a maximum when $B/\beta = A/\alpha$.

That P_{AB} is a maximum when $B/\beta = A/\alpha$ may be shown quite simply through a geometric argument. From (4.2.7) we know that $P_{AB} = \Pr[-\frac{1}{2} < y_1 < +\frac{1}{2}]$. Suppose B/β did not equal A/α and in fact $\left(\frac{A}{\alpha} - \frac{B}{\beta}\right)$ in (4.2.8) was some positive number, say M. Then, P_{AB} would be the shaded area in Figure 4.2.1.

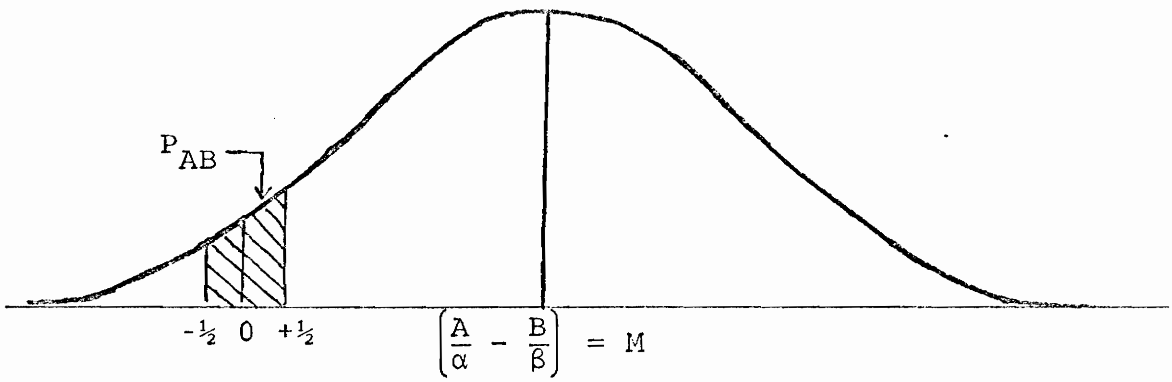

Figure 4.2.1

Draw Probability When Mean Times Are Unequal

On the other hand, if $A/\alpha = B/\beta$ implying $\left(\frac{A}{\alpha} - \frac{B}{\beta}\right) = 0$ in (4.2.8), then P_{AB} would be the shaded area in Figure 4.2.2 .

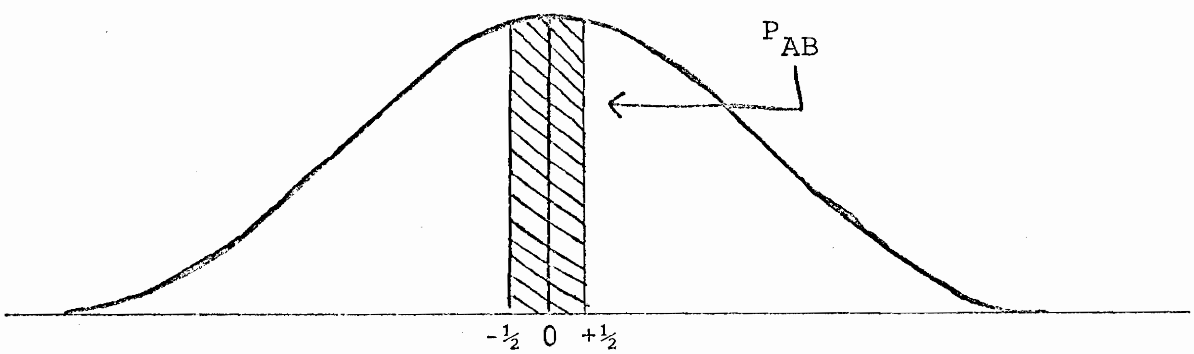

Figure 4.2.2

Draw Probability When Mean Times Are Equal

It is quite obvious upon comparing these two figures and in view of the fact that $P_{AB} = Pr[-\frac{1}{2}<0<+\frac{1}{2}]$ that P_{AB} will be a maximum only when the mean of the distribution in (4.2.8) is zero. This is equivalent to saying P_{AB} is a maximum when $A/\alpha = B/\beta$ for given values of A and B .

Let us now see what kind of numerical results these approximations give. For simplicity in calculations, we will standardize our Normal variable y_1 in (4.2.8) to the standard Normal with mean zero and variance 1. Setting, for brevity in notation,

$$\mu = \frac{A}{\alpha} - \frac{B}{\beta} \quad \text{and} \quad \sigma^2 = \frac{A(1-\alpha)}{\alpha^2} + \frac{B(1-\beta)}{\beta^2}$$

in (4.2.8), we have

$$(4.2.9) \qquad y_1 \frown N(\mu, \sigma^2) \ .$$

Hence, putting

$$z = \frac{y_1 - \mu}{\sigma^2}$$

will result in

$$z \frown N[0,1] \ .$$

Thus, if we set

$$z_1 = \frac{-\tfrac{1}{2}-\mu}{\sigma} \quad \text{and} \quad z_2 = \frac{+\tfrac{1}{2}-\mu}{\sigma} \ ,$$

we may write the win and draw probabilities given in (4.2.7) in an equivalent form namely,

$$(4.2.10) \qquad \begin{cases} P_A = \Pr[z \leq z_1] , \\ P_B = \Pr[z \geq z_2] , \\ P_{AB} = \Pr[z_1 \leq z \leq z_2] . \end{cases}$$

Since we now have the desired probabilities in terms of percentage points of the standard normal distribution, we may perform approximate calculations quite rapidly.

The most satisfying feature which arises from considering our model from this viewpoint is that, use of the Central Limit Theorem gives excellent approximations to the actual values of P_A , P_B and P_{AB} even for very small values of A and B . Table 4.1, which follows, will give an indication of how good an approximation we have developed. It compares the actual and approximate values for P_A , P_B and P_{AB} for a few choices of the parameters.

It is obvious from Table 4.1, that this approximation is excellent even for values of A and B as small as ten. Naturally, the larger A and B , the better the approximation.

Another advantage of using this Normal approximation is that it lends itself expecially well to solving approximately for one of the parameters in terms of the other three when a particular result is desired. For example, suppose

		Actual	Approximate
$A = 50$ $B = 80$	$P_A =$	.1307	.1357
$\alpha = .5$ $\beta = .9$	$P_B =$	.8457	.8438
	$P_{AB} =$	.0236	.0205
$A = 30$ $B = 40$	$P_A =$	.6330	.6255
$\alpha = .4$ $\beta = .5$	$P_B =$	.3402	.3446
	$P_{AB} =$	.0268	.0299
$A = 40$ $B = 80$	$P_A =$	1	1
$\alpha = .9$ $\beta = .9$	$P_B =$	0	0
	$P_{AB} =$	0	0
$A = 10$ $B = 30$	$P_A =$	.5216	.4920
$\alpha = .1$ $\beta = .3$	$P_B =$	.4664	.4920
	$P_{AB} =$	.0120	.0160
$A = 10$ $B = 10$	$P_A =$	.3568	.3745
$\alpha = .9$ $\beta = .9$	$P_B =$	.3568	.3745
	$P_{AB} =$	.2865	.2510

Table 4.1
Actual and Approximate Values for
Some Win and Draw Probabilities

we wish to determine approximately what the value of A should be so that $P_A \geq P_B$. We know that $P_A \doteq P_B$ when $A/\alpha = B/\beta$. Hence, we may write immediately that $P_A \geq P_B$ when $A \leq \alpha B/\beta$.

4.2.2 Continuous Time

In this section we will show what the parameters of the general theory developed earlier actually are in terms of two random walks operating in continuous time. Most of the work in this section will directly parallel that which has already been done in Section 4.2.1 .

In the continuous time situation, each t_j , described earlier, is distributed negative exponentially with probability density function

$$(4.2.11) \qquad f(t_j) = \lambda e^{-\lambda t_j} \quad , \quad \begin{bmatrix} \lambda > 0 \\ 0 \leq t_j \leq \infty \end{bmatrix} .$$

Here λ is the average rate per unit time that it takes the particular walk under consideration to move one unit toward the origin. We note that t_j is now a continuous random variable.

It follows from (4.2.11) that

$$(4.2.12) \qquad E(t_j) = 1/\lambda \quad \text{and} \quad Var(t_j) = 1/\lambda^2 .$$

Again, since the t_j are independent, we may write by virtue of the relationship (4.2.2),

$$(4.2.13) \qquad E(x_i) = s_i/\lambda \quad \text{and} \quad Var(x_i) = s_i/\lambda^2 .$$

Hence, substituting values for the particular parameters in the two-walk situation we obtain

$$(4.2.14)\quad \begin{cases} \mu_1 = A/\alpha \quad , & \mu_2 = B/\beta \\ \sigma_1^2 = A/\alpha^2 \quad , & \sigma_2^2 = B/\beta^2 \quad . \end{cases}$$

Restating the results of Theorem 4.1 in terms of the present situation, we may say that as $A\to\infty$, $B\to\infty$ in such a way that $A/\alpha = B/\beta$, then both P_A and $P_B\to\frac{1}{2}$.

Numerical calculations are carried out in exactly the same fashion as in Section 4.2.1 . The only difference is that now a tie or draw is <u>not</u> possible and so analogously to (4.2.10) we have

$$(4.2.15)\quad \begin{cases} \qquad P_A = \Pr[z \leq z_1] & , \\ \qquad P_B = \Pr[z \geq z_2] & , \\ \text{where} & \\ \qquad z_1 = z_2 = -\mu/\sigma & , \\ \text{and} & \\ \qquad \mu = \dfrac{A}{\alpha} - \dfrac{B}{\beta} \; ; \; \sigma^2 = \dfrac{A}{\alpha^2} + \dfrac{B}{\beta^2} & . \end{cases}$$

It should be pointed out here that even though a tie is <u>not</u> possible, in some applications to be discussed later, it is desirable to maximize the probability of the two walks finishing very close together, say within some specified interval of time. With this end in mind, it is quite easy to see that the development in Section 4.2.1 showing where

P_{AB} is an approximate maximum, carries over to this situation. Figures 4.2.1 and 4.2.2 indicate despite the size of the interval around zero considered, the probability that the two walks finish within this interval is a maximum when the difference between the mean times to reach the origin of each walk is zero. Translated in terms of the present parameters, we may say that the probability that the A and B walks finish very close together or within some specified time interval, is a maximum approximately when $A/\alpha = B/\beta$.

4.3 Three or More Walks

In this section we will not be concerned so much with numerical approximations as with the application of the approximations obtained to describe the operation of a system of more than two walks. Theorem 4.1 states that in the limit, or for sufficiently large values of the starting co-ordinates of the two walks involved, the probabilities of each walk winning are equal if the mean or expected times required for each walk to reach the origin are equal, despite the variances of these times. Unfortunately this property does not carry over into the case of three or more walks racing toward the origin, as might have been expected.

This may be illustrated most vividly by resorting to a graphical representation of the situation. For simplicity let us only consider three walks. As before, let μ_i and

σ_i^2 represent the mean and variance of the time, x_i, required for the i^{th} walk to reach the origin independently of the other two. Just as in Theorem 4.1, let us set the means of these times equal so that $\mu_1=\mu_2=\mu_3=\mu$, but place no restrictions on the variances, σ_i^2.

Let us now draw the distributions of x_1, x_2 and x_3 in the same figure with their means equal. Since we are going to give a counter-example to the proposed theorem, and since no restrictions are necessary on the variances of x_i, without loss of generality let us consider σ_1^2 and σ_2^2 as being extremely small while σ_3^2 is extremely large. This situation is represented in Figure 4.3.1.

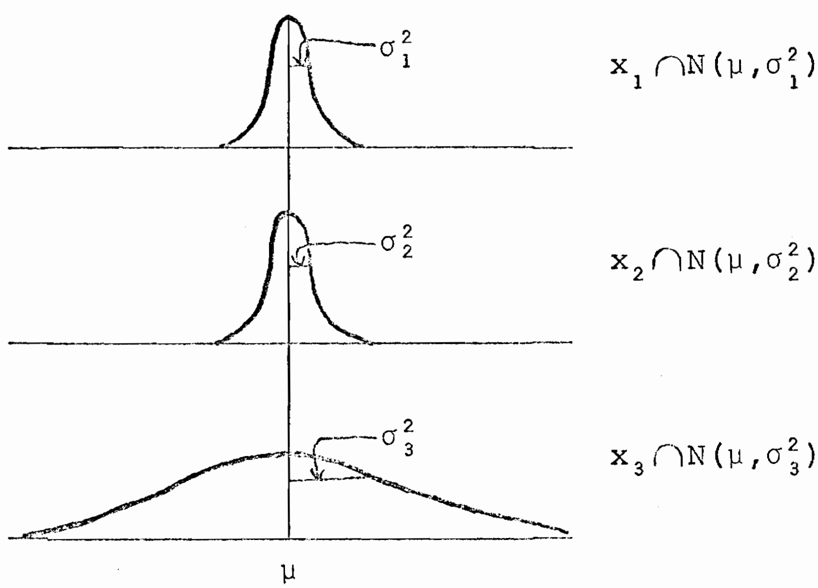

Figure 4.3.1
Time Distributions for Each Walk Finishing the Race

We know that the third or C walk can win the race only if its time to reach the origin is less than that of the other two walks. That is to say, C wins the race iff $x_3 < x_1$ and $x_3 < x_2$. Now we can see from Figure 4.3.1 that if x_3 is less than μ it will almost certainly win the race, and in fact, the probability that x_3 is less than μ is $\frac{1}{2}$. A similar argument holds also for C losing the race. Hence, even though all three means are equal, the probability that the C walk wins the race is close to $\frac{1}{2}$ because its variance is so large in relation to the variances of the other two walks. Thus, we have found a counter-example to the proposed generalization of Theorem 4.1 . In fact, it can be shown that the only way for the victory probabilities P_A , P_B , ... to be equal is for $\mu_1=\mu_2= \ldots =\mu_n$ <u>and</u> $\sigma_1^2=\sigma_2^2= \ldots =\sigma_n^2$ where n is greater than two.

Since we showed in Section 4.2 how to express μ_i and σ_i^2 in terms of the parameters for walks in discrete and continuous time, we will not repeat the argument since nothing essential changes when three or more walks are involved. One point which does deserve our attention is under what conditions are the probabilities of a tie or draw maximized. Fortunately, as we shall soon see, the results obtained in the last section do generalize to the multi-walk situation. As pointed out in the latter part of Section 4.2.2, even though the probability that a tie actually occurs in continuous time is zero, the analysis used to determine under what conditions the probability of a tie is a maximum carries over into the continuous time situation. This is because quite often we are interested in maximizing the probability that the walks finish within some specified interval of time. Thus, we may consider the following analysis to hold for both discrete and continuous time walks. For the moment, we shall restrict our attention to ties between all of the walks involved in the race. Later we shall consider partial ties, or ties between any subset of the walks.

For purposes of graphing distributions later, consider again only three walks. The generalization to n walks will be immediate and obvious. From (4.1.3) we have

$$\underline{y} \frown N_2[T\underline{\mu}, T\ddagger T']$$

where

$$T\underline{\mu} = \begin{bmatrix} \mu_1 - \mu_2 \\ \\ \mu_2 - \mu_3 \end{bmatrix} \quad \text{and} \quad T\ddagger T' = \begin{bmatrix} (\sigma_1^2 + \sigma_2^2) & \sigma_1^2 \\ \\ \sigma_1^2 & (\sigma_1^2 + \sigma_3^2) \end{bmatrix} \quad \text{and} \quad \underline{y} = \begin{bmatrix} x_1 - x_2 \\ \\ x_1 - x_3 \end{bmatrix} .$$

The bivariate normal distribution of $\underline{y}$ may be represented pictorially as in Figure 4.3.2 .

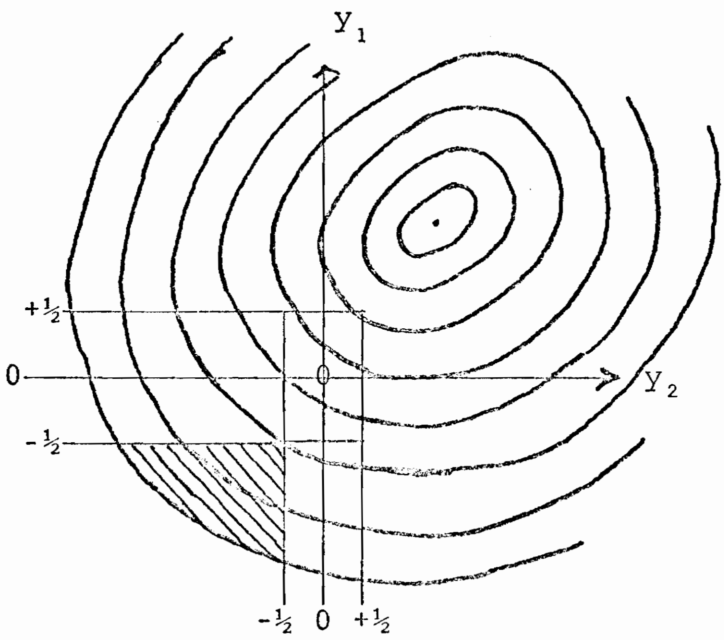

Figure 4.3.2

Bivariate Normal Distribution of $\underline{y}$

Now the A walk will win the race if and only if x_1 is jointly less than x_2 and x_3, i.e. if and only if

$$(4.3.1) \qquad \underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_1 - x_3 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

If a tie is possible as in the discrete time situation, the zero vector in (4.3.1) above will be replaced by the vector $[-\tfrac{1}{2},-\tfrac{1}{2}]'$, as is usually the situation when one uses a continuous distribution to approximate a discrete distribution, see Feller [12]. Mathematically we may now write,

$$P_A = \int_{-\infty}^{-\frac{1}{2}} \int_{-\infty}^{-\frac{1}{2}} f(\underline{y})\,d\underline{y} \quad ,$$

which is the shaded area in Figure 4.3.2 to the left of $y_2 = -\frac{1}{2}$ and below $y_1 = -\frac{1}{2}$. Similar expressions can be written for P_B and P_C .

The probability of a two-way tie between A and B , P_{AB} , is represented by the infinite strip in Figure 4.3.2 to the left of $y_2 = -\frac{1}{2}$ and between $y_1 = -\frac{1}{2}$ and $y_1 = +\frac{1}{2}$. Thus,

$$P_{AB} = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\infty}^{-\frac{1}{2}} f(\underline{y})\,d\underline{y} \quad .$$

The probability of a three-way tie between A , B , and C , is P_{ABC} , represented by the square surrounding the origin formed by the boundries $[-\frac{1}{2} \le y_1 \le +\frac{1}{2}]$ and $[-\frac{1}{2} \le y_2 \le +\frac{1}{2}]$. Thus,

$$P_{ABC} = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} f(\underline{y})\,dy \quad .$$

It is quite easy to see from Figure 4.3.2 that the probability of a three-way tie will be a maximum when the center of the distribution is coincident with the origin of the two axes. When this is the case, the given square encompasses the maximum amount of volume under the surface for its size.

Thus, since P_{ABC} is a maximum when the center of the bivariate normal distribution is coincident with the origin, and since this center is actually the mean of each of the random variables involved, we see then that P_{ABC} is a maximum when

$$\begin{bmatrix} \mu_1-\mu_2 \\ \mu_1-\mu_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mu_1=\mu_2=\mu_3 \quad .$$

We can easily express these μ_i in terms of the α's , A's (etc.) as was done in Sections 4.2.1 and 4.2.2 . Again we note here that the variances, σ_i^2 , do not play a role in determining when P_{ABC} is a maximum under fixed conditions. However, we can see from Figure 4.3.2 that if besides setting $\mu_1=\mu_2=\mu_3$, we could also make the variances and covariances of y_1 and y_2 very small so that more of the volume of the surface came within the boundaries of the square, this too would increase P_{ABC}. For example, in the discrete time situation we have

$$\mu_1 = A/\alpha \quad , \quad \mu_2 = B/\beta \quad , \quad \mu_3 = C/\gamma \quad ,$$

and

$$\sigma_1^2 = \frac{(1-\alpha)A}{\alpha^2} \quad , \quad \sigma_2^2 = \frac{(1-\beta)B}{\beta^2} \quad , \quad \sigma_3^2 = \frac{(1-\gamma)C}{\gamma^2} \quad .$$

We now see how we can reduce the σ_i^2 while keeping the μ_i equal to each other. This is because the σ_i^2 are constant multiples of the μ_i . Thus, if we reduce A , B and C or increase α , β , and γ in such a way that the equality of the means still holds, then we will increase P_{ABC} . A similar result holds for the continuous time situation. Contrary to Intuition, in the case of partial or two-way ties, say between A and B , equality of the mean times for each walk involved is not a sufficient condition for the probability of this tie to be a maximum over other two-way ties. As a matter of fact, if $\mu_1 = \mu_2$ only, it is quite possible for P_{BC} and P_{AC} both to be greater than P_{AB} , even though A and B's mean times to reach the origin are equal. This is borne out by numerical calculations, and also by the nature of the Bivariate Normal Density function. This situation usually occurs when $\mu_1 = \mu_2$ but $\mu_3 < \mu_1$ or μ_2 . Here, not only is P_C greater than P_A and P_B , but also $P_{AC} > P_{AB}$ and $P_{BC} > P_{AB}$ which is not what intuition might tell us.

We can see on the basis of what has been presented in this chapter, how important and useful approximations can be. Besides giving excellent numerical results in the case of two walks, they enabled us to get a simplified yet meaningful representation of the entire system. This is an advantage which could not have been obtained from the exact win and draw probabilities due to their complicated structure. In addition to giving a very broad insight into the operation of the system, these approximations enabled us to prove some useful theorems whose results we shall have occasion to use in the following chapter on applications.

V. APPLICATIONS

This chapter will be devoted to describing applications. We shall show how the theory of random walks racing towards the origin (both in discrete and continuous time) can be utilized as a basic framework in explaining the operation of several diverse "systems". We shall also find that many pertinent questions will be answered concerning certain objectives to be achieved, and the relationship of the parameters involved to these objectives.

5.1 A Stochastic Battle Between Two Opposing Forces

As mentioned in the Introduction, the analysis of a race between random walks originated in an attempt to study duels which portray the microscopic aspects of combat as opposed to the macroscopic viewpoint of the Lanchester type models. (See Ancker [3], Brown [8] and Springall [18].) It was soon realized that the same framework had a number of other applications, all of which embody the spirit of competition in an un-warlike context. As a result, instead of the analysis of a stochastic battle being an end, it became a means to a more general and useful end.

Here, we shall show how a battle between two opposing forces or opponents may be represented in terms of a race between two random walks toward the origin in discrete time (vide Section 2.1). Actually, we need not even limit ourselves to a battle in the usual sense. Keeping our terminology completely general, we will merely consider a struggle between two opposing forces. Thus, with ecological applications in mind, these opposing forces would be two opposing species of animals.

Denote the opposing forces by A and B. Let these letters also represent the initial number of units on each side. Let us emphasize the word units here as opposed to the word individuals. These units besides being individuals (i.e. men) in an army, or animals belonging to a particular species, could also be submarines and anti-submarine units.

Suppose the opposing forces A and B are mixing homogeneously in some unspecified area, such that units of opposing sides are bound to come in contact with one another. We will call the contact of two units on opposing sides an encounter. Thus, in the case of submarines versus anti-submarine units, we will call an encounter the actual contact, or the detection of each vessel by the other.

At each encounter, a fight or struggle between the two units will begin, and we shall assume in this model that each encounter may have any one of four mutually exclusive outcomes:

1. The A unit kills the B unit.
2. The B unit kills the A unit.
3. Each kills the other (i.e. Perhaps one unit is killed immediately while the other dies shortly thereafter from a fatal wound. This could also be the situation in rutting season when two bull moose lock antlers hopelessly, thus resulting in mutual starvation.)
4. Neither kills the other. (i.e. here both units have missed. In this situation, the units involved can either remain and engage in another "encounter" or they may escape until they meet each other or another unit of the opposite side at a later time.)

Let us now introduce the notion of kill probabilities. We shall assume that all the members on each side are equally effective in killing a unit on the opposite side at every encounter. We will call this measure of effectiveness the kill probability. Represent the kill probability of the A side by β and of the B side by α. Hence at each encounter, the probabilities of the four mutually exclusive outcomes described above are:

1. $\beta(1-\alpha)$
2. $(1-\beta)\alpha$
3. $\beta\alpha$
4. $(1-\beta)(1-\alpha)$.

With the above description and definitions at our disposal, we now see that this model fits exactly into the framework of a race to the origin between two random walks in discrete time described in Section 2.1. We are not really concerned with time here. Actually, time or the number of trials, as described in Section 2.1, corresponds to encounters as defined above.

Thus using the notation of Section 2.1 we say that $p[i,j;k]$ now represents the state probability of the A side having i units and the B side having j units remaining after k encounters or contacts. There is one fundamental difference that should be pointed out between the model in Section 2.1, and the application of the model in this section.

In this section, the winner of the overall battle or struggle is <u>not</u> the side who reaches zero first, but the one who does not reach zero first. In other words, previously we were considering a race towards the origin between two particles and the one which reached zero first was the winner. In the present situation, if a particular side reaches zero, all of his units have been annihilated and hence he loses the struggle. Thus to determine win probabilities in this section, we merely interchange the subscripts A and B in the formulas for P_A and P_B in equations (2.1.23) and (2.1.24). The tie probability, P_{AB}, remains unchanged.

The theory developed earlier also enables us to obtain the probability distribution for the number of men remaining on the side of the victor after the losing side is annihilated. These distributions are given by equations (2.1.19) and (2.1.20). All of the analysis given in Section 2.1.5 for the duration of the race carries directly over for the duration of the battle or struggle.

Given that a battle or struggle fits the model described above, we now see how the approximations obtained in Chapter IV may be used to obtain quick and useful answers. For example, suppose the parameters α, β, and B were known to the A side and he wishes to know how many A units are necessary such that at least $P_A \geq P_B$. This is not easily obtained from the functional equations for P_A and P_B. [vide (2.1.22) and (2.1.23)]. However, in view of the discussion at the end of Section 4.2.1, we see that since $P_A = P_B$ approximately when $A/\alpha = B/\beta$, then it is obvious that $P_A \geq P_B$ when $A \geq \alpha B/\beta$.

This result is quite fundamental and exactly parallels Lanchester's Linear Law.

Lanchester's equations are derived on the assumption that multiple-kills (i.e. two units dying simultaneously) are not possible. In our analysis, this assumption was dropped and we still obtained a similar result, namely that the two opposing sides or forces have approximately equal "strength" if

$$A/\alpha = B/\beta.$$

5.2 An Inventory Control Model

There is a rich literature on inventory control and this area alone forms quite an important portion of the field operations research. See Fabrycky and Banks [11] and Sasieni et.al.[16]. Most of the work has been directed towards devising optimum reorder levels, reorder points and the amount to order at these points. Quite sophisticated methods and algorithms have been developed to enable the decision maker to effect a "trade-off" in various costs so that a long range minimum cost policy can be put to use.

Very often however, certain aspects of inventory control are overlooked. For example, the government is known to tax businesses for its inventory on the basis of what a firm has on hand on December 31$^{\underline{st}}$. (Naturally, there are other factors taken into account, but for purposes of formulating a mathematical model, let us consider this one only.) Obviously, in order to save on taxes, it would benefit a company to adopt, in addition to its usual "optimum-policies", a policy near the end of the year that would minimize the stock on hand on December 31. This new policy should also offer protection against running out previously to that date because a great deal could be lost due to missed sales, good will, etc. (For the same reason, if the objective in mind is to have a zero or near zero inventory on a certain date, it would pay to be sure that a new shipment was due the next day.)

Before showing how the analysis of a race between random walks fits the above situation, let us discuss another one which also fits. Consider a rather large restaurant which sells the same brand of beer in many forms (i.e. kegs, bottles, cans, etc.). Suppose that all of these forms are delivered by the same delivery man, say once a week. We see here that the manager of the restaurant is interested in all of the types of beer running out at approximately the same time, exactly when the delivery truck is scheduled to arrive. Just as in the previous example, running out prior to this time results in lost sales, and having too much on hand yields increased inventory and handling costs.

In both of the above examples, there are some parameters which are considered to be under management's control and some which are not. Except for the influence of advertising, it will be assumed that the average selling rate per unit time for each item under consideration is not under management's control. However, from past sales records and business experience, it would be a simple matter to estimate these rates which we shall denote by α, β, ... and ω.

The parameters which are under management's control are the levels at which each item should be stocked. These levels will be represented by A, B, ... and N. The question to be answered in the first example is "At which level should each item be stocked so that they will all run out on December 31st?" In the beer situation, we desire the optimum amount of each type of beer to stock so they will all run out when the delivery man arrives.

The assumption of Poisson-distributed demands for a particular item is quite realistic and one which is common in many inventory control problems. With this assumption, we can see immediately how the above situations fit exactly into the framework of a race between N random walks in continuous time, and we are interested in maximizing the probability of a draw or "near-draw". Graphically, we may represent both situations as in Figure 5.2.1.

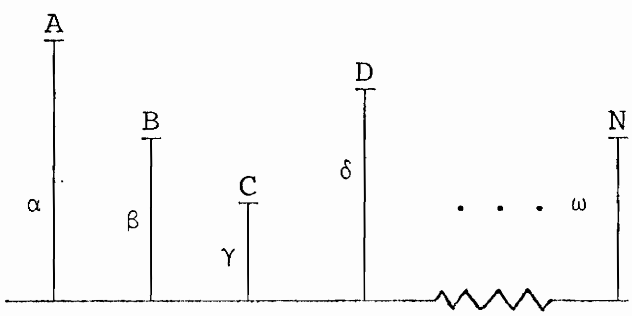

Figure 5.2.1
Graphical Representation of Inventory Levels and Sales Parameters.

Given that we know the time until the items are to run out (i.e. December 31st in the first example or one week in the second), we will rely on the analysis developed in Chapter III for a race between walks in continuous time, and the discussion and approximations given in Chapter IV. From Chapter IV we recall that the probability of a tie or draw occurring between all of the walks involved is approximately maximized when $\mu_1 = \mu_2 = \ldots = \mu_N$. Hence in the present situation, we will maximize the probability of all the items running out together by setting $A/\alpha = B/\beta = \ldots = N/\omega$.

We need now only to determine one of the item levels and then the rest will be automatically determined by the above equality. Calling the time until all of the items are to run out, T, we may determine the A level say, by simply setting $A/\alpha = T$ and solving for A. Thus, knowing α, β, ..., ω and A, the other item levels can be obtained. We now see on the basis of the discussion in Chapter IV that the probability of a draw between all of the walks at time T is maximized.

5.3 Optimum Redundancy in a Reliability Model

Let us consider a complex assembly composed, for example, of many electronic components, all necessary for the assembly to function properly. Quite often, especially if the proposed function of the assembly is of paramount importance or if perhaps several lives depend on its functioning properly, then a concept known as redundancy is utilized in building the assembly.

Redundancy can most simply be defined as incorporating extra components in the assembly so that if one of the original components ceases to function, one of the "spares" takes over. For example, if an assembly needs one transistor and one capacitor, then in order to achieve a high degree of reliability for the complete system, four transistors and three capacitors might be built in in such a way that if one of the components were to fail, then the next one available would take over. We shall be concerned to find the optimum number of redundant components necessary to achieve a desired reliability level.

In most reliability models, particularly electronic circuits, it is reasonable to assume that component lives follow the negative exponential distribution. The parameter of this distribution, called the failure rate in reliability applications, is usually given in failures per hour and is quite often obtained from manufacturers' specifications. However, sometimes this parameter is estimated by testing similar components until they burn out or fail. The many methods utilized in the estimation of failure rates is a self-contained field of statistics and is called Life Testing (see for example, Roberts [15]).

Suppose that we are interested in obtaining a desired reliability at minimum cost. Thus, given that redundancy is necessary to achieve this reliability, how many redundant items for each component should be incorporated into the system so that the desired reliability is obtained but without being "over-redundant" (ie. putting too many redundant items in, hence increasing costs)? Since negative exponential lives are being assumed for each component, we shall now see how the theory for a race between random walks in continuous time (Chapter III) can help answer this question.

Let each type of component involved be labeled A, B, ..., N and as before, let these letters also represent the number of each type of component in the system (i.e. the level of redundancy for each component, which is at this point still undetermined). Let α, β, ... ω represent the respective failure rates for each of the above mentioned components. Since the life time of each component is distributed negative exponentially, we know then that the life of each string of similar components is Gamma distributed, the parameters being the failure rate for that type of component and the number of components involved.

It will be assumed that we know how long the assembly is required to function, noting that perhaps after that time, it will not ever be used again. (For example, in the case of a space module required for a two week orbit around the Earth with astronauts, this module would never be used again for a flight after its recovery.) Thus, with the objective in mind being to maximize reliability without being over-redundant, and given that we know how long the system is to operate, we see that we are interested in maximizing the probability of each string of redundant components failing together or very close together.

The analysis of this situation follows exactly the same lines as that of the inventory control model given in the last section. There we were interested in maximizing the probability of all the items running out together, whereas here we want all the components simultaneously to fail after a certain time, but as close after it as feasible (given of course, that the system were allowed to operate after its function was satisfied). Naturally this "target-time" would probably have a safety margin already added to it.

5.4 A Multi-Channel Queueing System

Consider a multi-channel queueing system, that is, n service facilities in parallel all performing the same service but perhaps at different rates [viz. turnpike booths, barber shops, ships unloading at different piers, airplanes landing at different airports, etc.]. Let us say that a customer (i.e. someone or something that requires service) enters the system and a decision has to be made as to which channel should be chosen to perform the required service. Let us assume that once the service facility is chosen, the customer may not change his mind for some reason or another, perhaps because of set-up costs, etc.

If no information at all is available concerning the service rates of each channel per unit time, then the obvious and most natural decision is to choose that service facility with the least number of customers waiting in the queue. Suppose however that both the service rates for each facility and the number in each queue is available to the arriving customer, what should the decision procedure be now?

In order to answer this question intelligently, the objective of the arriving customer should be clearly defined. For example, if the arriving customer merely wants to minimize his waiting time (ie. the time in the queue and in service), then the best decision would be to choose that facility which offers the smallest expected waiting time including service. This is a quite logical and obvious procedure, and one which is almost always used, given that the service rates and the number in each queue is known and the objective is to minimize waiting time.

Suppose however, that minimum waiting time per se is not an objective, but being served before a certain time is. For example, consider a tanker entering a port with several unloading facilities. The tanker has to be emptied within three days because after that date, it is under contract to a new client. Hence, if the tanker is emptied before the third day then nothing is lost, but if it is not emptied by that time, then a large penalty or fine must be paid to the new contractor for failure to comply with the terms of the contract. Before showing why the theory developed in Chapter III of this dissertation should be used instead of the minimum expected time argument described above, let us discuss another example.

Consider an airplane with a limited amount of fuel remaining in its tanks, say one hour's worth. Suppose that it is equidistant between two airports or aircraft carriers each 45 minutes away from the plane and an hour from each other. Suppose further that each airport or carrier is already committed to several other planes and if our plane chooses to go to the first, the expected waiting time before it can land is 15 minutes, whereas the expected waiting time in the second is 14 minutes. Which airport or carrier should be chosen?

This is quite similar to the tanker example described above. If the plane can land within 15 minutes after reaching either airport, then nothing is lost. However, if he is not able to land within this time period, then a great deal is lost, namely the plane and perhaps the pilot. The whole purpose of these two examples is to demonstrate that expected time to be serviced is not always the important objective of the customer. Sometimes, as in these examples, the customer is indifferent as to his service time just as long as the service is accomplished within a certain time.

Our initial reaction in both of these examples is to still choose that facility with the smallest expected service time. However, what is really desired in this situation is the choice of that facility which has the highest probability of finishing first or "winning the race" to serve its customers. Just because a facility has the lowest expected time to service its customers, this does not always imply that its probability of finishing first is greater than the others. Naturally this depends on the particular distributions and on the number and kind of parameters involved.

Let us now formulate our examples mathematically in terms of the theory developed earlier. Let each service facility available be represented by A, B, ..., N. We shall also denote the number of customers at each facility by these letters (as before). The mean service rates per unit time of each facility will be represented by α, β, ..., ω. Then, in the case of the plane we have only two facilities, A and B, with mean rates α and β.

If we assume that the service time at each facility is distributed negative exponentially, then the theory developed in Chapter III gives us values for $P_A(t)$, $P_B(t)$, ..., $P_N(t)$, each representing the probabilities for each walk winning by a certain time. Hence, if we were to set t equal to three days in the case of the tanker or fifteen minutes in the case of the airplane, we should choose that facility whose win probability at that time is largest. By doing this, we will obtain the smallest expected loss in terms of money in the case of the tanker, or the plane and pilot in the airplane example.

To be more explicit as to why we should use the win probabilities instead of the expected waiting times in these examples, we must note that each waiting time distribution is Gamma. This is to say that if there are N customers in a facility and the mean service rate per unit time is ω, then the <u>total</u> waiting time before all of its customers are served is Gamma distributed with parameters ω and N. Unlike a one-parameter distribution [vide the negative exponential or the Poisson] where the smallest expected waiting time <u>does</u> imply which service facility should finish first, this does not always hold true for a two-parameter distribution. Here, distributions can be extremely skewed depending on the types and values of each parameter involved. Because of this skewness, it is quite possible for the facility with the smallest expected waiting time to not have the highest win probability. Thus, if there is a great deal at stake, it is most definitely to the decision maker's advantage to be exact in his procedure by using the win probability criterion as opposed to selecting the smallest expected waiting time.

5.5 Miscellaneous Examples

There exist numerous examples, perhaps not as important as the ones given earlier, but such that the theory already developed could be utilized to answer pertinent questions. For example, an Immigration process (vide Feller [12] and Bailey [4]) is often regarded as a Poisson process where individuals immigrate into an environment (or country) independently of the number of individuals already there (assuming of course there is room or that no barriers are created to hinder them). Suppose that we can estimate the immigration rates of certain people into this country, Indians and Japanese for example. When our immigration bureau sets yearly quotas for each race, the theory developed in this dissertation could help answer questions by obtaining probabilities for:

(i) which quota will be reached first;

(ii) how many more individuals are needed to fill a quota, given that one of the other quotas has already been filled;

(iii) how long will it be before the first quota is filled?

Along these same lines, parking lots filling up can be considered to be an immigration process and hence the same questions can be answered as to which of several lots will fill up first, etc. Here we must assume that no cars are leaving. This is a realistic assumption if we limit ourselves to parking lots at say a sporting event like a football or basketball game. There, everyone is arriving and no one is leaving until after the game. Thus here again, we could now answer questions about which lot will empty first, etc. The answer to these questions could play a strategic role in traffic control, design of stadiums, and so on.

There has been some work done in applying the theory of random walks and a race between random walks to quality and process control problems. However, all of it was under different assumptions with different objectives in mind than the models we have developed. See for example, the articles by Beightler and Shamblin [7] and Terrell et.al. [19].

For a quality control example that fits one of our models, let us consider a particular item which can have several types of defects. For instance, a desk could have finish defects, the drawers might not fit properly, the legs might not be level, etc. We can represent the process average, or the proportion of defects, for each type by α, β, ..., ω. Thus if we are concerned with the number of "non-defects" of each type discovered after examining several desks, we could write $p[i, j, \ldots, n; k]$ as the probability of finding i non-defects of the first type, j non-defects of the second type, etc., after having examined k desks. In calculating this probability we use equation (2.2.1) where here the roles of α and $(1-\alpha)$, β and $(1-\beta)$, etc. are interchanged. The letters A, B, ..., N, might now represent some quota or goal set up by management for each department in a quality contest. Hence, we may now make probabilistic statements as to which department reaches its quota first or how many desks will be manufactured before a department is declared the winner and so on.

Other examples could be quoted. The main objective is to emphasize that whenever an operating system can be modeled mathematically in terms of a race between discrete random walks, in either discrete or continuous time, our theory can be applied.

VI. BIBLIOGRAPHY

[1] Abramowitz, M. and Stegun, I. A., editors, Handbook of Mathematical Functions. National Bureau of Standards, Washington D.C., 1966.

[2] Anderson, T. W., An Introduction to Multivariate Analysis. John Wiley and Sons, Inc., New York, 1965.

[3] Ancker, C. J., Jr., "The Status of Developments in the Theory of Stochastic Duels-II." Operations Research, 15, 388-406.

[4] Bailey, Norman T. J., The Elements of Stochastic Processes With Applications to the Natural Sciences. John Wiley and Sons, Inc., New York, 1964.

[5] Bailey,W. Norman , Generalized Hypergeometric Series. Cambridge University Press, London, 1935.

[6] Bateman, Harry, "Higher Transcendental Functions." Bateman Manuscript Project, McGraw-Hill, New York, 1953-1955.

[7] Beightler, Charles S. and Shamblin, James E., "Sequential Process Control." The Journal of Industrial Engineering, March-April, 1965, 101-108.

[8] Brown, Richard H., "A Stochastic Analysis of Lanchesters Theory of Combat." Operations Research Office of Johns Hopkins University, Publication ORO-T-323, 1955.

[9] Copson, E. T., "The Asymptotic Expansion of a Function Defined by a Definite Integral or a Contour Integral." Admiralty Computing Service, London.

[10] Erdélyi, A., Asymptotic Expansions. Dover Publications, Inc., New York, 1956.

[11] Fabrycky, W. J.and Banks, Jerry, Procurement and Inventory Systems. Reinhold Publishing Corporation, New York, 1967.

[12] Feller, William, An Introduction to Probability Theory and Its Applications. John Wiley and Sons, Inc., New York, 1965.

[13] Parzen, Emanuel, Stochastic Processes. Holden-Day, Inc., San Francisco, 1964.

[14] Rainville, Earl D., Special Functions. The Macmillan Co., New York, 1960.

[15] Roberts, Norman H., Mathematical Methods in Reliability Engineering. McGraw-Hill, New York, 1964.

[16] Sasieni, M., Yaspan, A. and Friedman, L., Operations Research. John Wiley and Sons, Inc., New York, 1965.

[17] Spitzer, Frank L., Principles of Random Walk. Van Nostrand, Princeton, N.J., 1964.

[18] Springall, Anthony, "Lanchester Combat Models." Ph.D. Dissertation in Statistics, Virginia Polytechnic Institute, 1968.

[19] Terrell, M. Palmer, Beightler, Charles S., and Reis, Irvin L., "Process Control by Means of Three Dimensional Random Walks." The Journal Of Industrial Engineering, August, 1967, pp.474-481.

[20] Watson, G. N., Theory of Bessel Functions. Cambridge University Press, London, 1944.

VII. VITA

Daniel Caleb Denby, the oldest of ten children, was born June 9, 1942, in Washington, D. C. Since his father was in the U. S. Navy, his family moved frequently and as a result, his early education was obtained at several different schools. He graduated from Fluvanna County High School in Carysbrook, Virginia, in June, 1960. In September of the same year he entered the College of William and Mary, from which he received the Bachelor of Science degree in mathematics in June, 1964.

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Daniel C. Denby

A RACE TOWARD THE ORIGIN BETWEEN N RANDOM WALKS

by

Daniel Caleb Denby

ABSTRACT

This dissertation studies systems of "competing" discrete random walks as discrete and continuous time processes. A system is thought of as containing n imaginary particles performing random walks on lines parallel to the x-axis in Cartesian space. The particles act completely independently of each other and have, in general, different starting coordinates.

In the discrete time situation, the motion of the n particles is governed by n independent streams of Bernoulli trials with success probabilities p_1, p_2, ..., and p_n respectively. A success for any particle at a trial causes that particle to move one unit toward the origin, and a failure causes it to take a "zero-step" (i.e. remain stationary). A probabilistic description is first given of the positions of the particles at arbitrary points in time, and this is extended to provide time dependent and independent probabilities of which particle is the winner, that is to say, of which particle

first reaches the origin. In this case "draws" are possible and the relevant probabilities are derived. The results are expressed, in particular, in terms of Generalized Hypergeometric Functions. In addition, formulae are given for the duration of what may now be regarded as a race with winning post at the origin.

In the continuous time situation, the motion of the n particles is governed by n independent Poisson streams, in general, having different parameters. A treatment similar to that for the discrete time situation is given with the exception of draw probabilities which in this case are not possible.

Approximations are obtained in many cases. Apart from their practical utility, these give insight into the operation of the systems in that they reveal how changes in one or more of the parameters may affect the win and draw probabilities and also the duration of the race.

A chapter is devoted to practical applications. Here it is shown how the theory of random walks racing toward the origin can be utilized as a basic framework for explaining the operation of, and answering pertinent questions concerning several apparently diverse situations. Examples are Lanchester Combat theory, inventory control, reliability and queueing theory.