UNITARY EQUIVALENCE OF SPECTRAL MEASURES

ON A BAER *-SEMIGROUP

by

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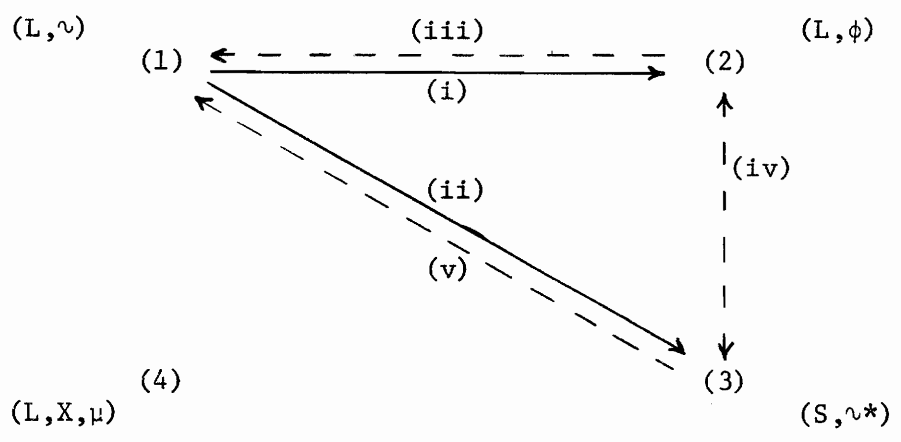
CHAPTER I

INTRODUCTION

It has been observed in reference [1] that there is a connection among the following four distinct mathematical structures: (1) Loomis dimension lattices $(L, \sim)$, (2) orthomodular lattices with a center valued quantifier (L, ϕ), (3) Baer *-semigroups with some form of *-equivalence $(S, \sim^*)$, and (4) abstract spectral measures (L, X, μ). The connections between these structures are illustrated in the figure below. The solid lines indicate complete results, and the dashed lines indicate partial results.



The Quadripartite Diagram

Connection (i) is due to L. H. Loomis [10]. Connection (ii) is due to M. F. Janowitz [9]. Connection (iii) is given by J. H. Bevis [1]. David Foulis [2] has proven a relation that exists between orthomodular lattices and Baer *-semigroups, and Janowitz [9] has shown how the properties of a center-valued quantifier on an orthomodular lattice are carried over into the coordinatizing Baer *-semigroup. In (v), there have been several partial results, including work by Foulis [3]. However, there has been very little done concerning abstract spectral measures, except in the case of the lattice of closed projections on a Hilbert space. For this case, P. R. Halmos [6] has established a complete set of invariants for the unitary equivalence of normal operators on a Hilbert space, H. That is, he has proven the existence of a multiplicity function, u_A, which is associated with every normal operator, A, such that A and B are unitarily equivalent normal operators on H if and only if $u_A = u_B$. Since two normal operators are unitarily equivalent if and only if their respective spectral measures are unitarily equivalent, then all analysis dealt with this later equivalence.

The purpose of this research was to generalize in some form the notion of unitary equivalence of spectral measures so as to make a connection in the Quadripartite diagram between abstract spectral measures and the other three mathematical systems. The major portion of this paper replaces the space of bounded linear operators on a Hilbert space by a general Baer *-semigroup, S. The remainder of this chapter will be devoted to the consideration of properties of certain elements in a Baer *-semigroup.

By an <u>involution semigroup</u>, we mean a semigroup, S, equipped with a mapping $* : S \to S$ called the <u>involution</u>, such that for $a, b \in S$ we have that:

(i) $(a^*)^* = a$ (ii) $(ab)^* = b^*a^*$

An element $e \in S$ having the properties that:

(i) $e^* = e$ (e is <u>self adjoint</u>)

(ii) $e^2 = e$ (e is <u>idempotent</u>)

is called a <u>projection</u>.

A <u>Baer *-semigroup</u> is then defined as an involution semigroup with a two sided zero such that for $a \in S$, $\{x \mid x \in S \text{ and } ax = 0\} = eS$ where e is a projection in S, i.e., the right anihilating ideal of any element a is a principal right ideal generated by a projection.

Let P(S) = {projections in S}. Note that P(S) is partially ordered by the following rule: For $e, f \in P(S)$, then $e \leq f$ whenever $ef = e$ (or equivalently whenever $fe = e$).

Now for $a \in S$, let $' : S \to P(S)$ represent the mapping of S into P(S) given by $' : a \to a'$, where a' is the projection which generates the right anihilating ideal of a. Note: In a Baer *-semigroup, $0'$ is denoted by I and is the multiplicative identity in S, and $xx^* = 0$ if and only if $x = 0$.

The following well known lemmas (1 - 3) prove several important properties of the mapping, '.

<u>Lemma 1 (i)</u>: For $a \in S$, a Baer *-semigroup, the mapping ' is uniquely defined.

Proof: If $'$ were not uniquely defined, then there would exist $e \neq f$, with $e, f \in P(S)$ and $eS = fS$, the right anihilating ideal of a. But $e = e \cdot e \in eS = f \cdot S$ so there exists an $s_1 \in S$ such that $e = fs_1$. Then $fe = f \cdot fs_1 = fs_1 = e$, or $e \leq f$. Similarly $f = f \cdot f \in fS = eS$ so there exists an $s_2 \in S$ such that $f = es_2$. Then $ef = e \cdot es_2 = es_2 = f$, or $f \leq e$. Thus $f \leq e \leq f$, or $e = f$.

Lemma 1 (ii): For $a, b \in S$ with $ab = 0$, then $b = a'b$.

Proof: $ab = 0$ implies that there exists an $s_1 \in S$ such that $b = a's_1$. Since $a' \in P(S)$, then $a'b = a'a's_1 = a's_1 = b$.

Lemma 2: For $e, f \in P(S)$ and $a, b \in S$ then the following properties are true:

(i) $e \leq e''$

(ii) $e \leq f$ implies that $f' \leq e'$

(iii) $e' = e'''$

(iv) $ab = a$ implies that $b' \leq a'$

(v) $a = aa''$

Proof: (i) $ee' = 0$ so taking adjoints gives: $e'e = 0$ which implies that $e = (e')'e = e''e$, or $e \leq e''$.

(ii) Given that $ef = e$, then $ef' = ef \cdot f' = e \cdot 0 = 0$ which implies that $f' = e'f'$, or $f' \leq e'$.

(iii) By (i), $e \leq e''$ and so by (ii), $e''' \leq e'$. By (i) applied to e', then $e' \leq e'''$.

(iv) $a \cdot b' = ab \cdot b' = a \cdot 0 = 0$ so that $b' = a'b'$.

(v) Taking adjoints of $aa' = 0$ yields $a'a^* = 0$, so that $a^* = a''a^*$. Taking adjoints again yields $a = a^{**}a'' = aa''$.

We now single out certain elements of P(S) by saying that $e \in P(S)$ is <u>closed</u> if $e = e''$. The property which separates these closed projections from P(S) is given by the following lemma.

<u>Lemma 3</u>: A projection $e \in P(S)$ is closed if and only if e is in the range of the mapping, $' : S \to P(S)$.

<u>Proof</u>: If e is closed, then $e'' = e$, so that e is the image of e' under the mapping, $'$.

Conversely, if e is in the range of $'$, then for some $a \in S$ we must have that $e = a'$, or $e'' = a''' = a'$ by Lemma 2 (iii), and $a' = e$. Thus $e'' = a' = e$, i.e., e is closed.

Now let P'(S) denote the range of $'$ on S.

Below, we shall see that the elements of P'(S) satisfy the properties of a mathematical system known as a lattice.

A <u>lattice</u> is a partially ordered set in which every pair of elements a, b has a least upper bound and a greatest lower bound. These are denoted by $a \vee b$ and $a \wedge b$ respectively and are also referred to as the join and meet of a and b. A <u>complete</u> lattice is a lattice in which every subset has a least upper bound and a greatest lower bound in the lattice.

By an <u>involution poset</u>, we mean a partially ordered set equipped with a mapping, ', such that for elements a, b in the set, we have that: (i) $a \leq b$ implies $b' \leq a'$, and (ii) $a'' = (a')' = a$.

Now let L be a lattice with a least element and a greatest element denoted by 0 and I respectively. If $a \wedge b = 0$ and $a \vee b = I$, then we say that a and b are <u>complements</u>. If $' : L \to L$ is an involution mapping and a' is a complement for a for all $a \in L$, then ' is called an <u>ortho-complementation</u>. If the lattice has an orthocomplementation and if $e \leq f$ implies that $e \vee (e' \wedge f) = f$, then the lattice is called an <u>ortho-modular lattice</u>.

If for any elements a, b, c in the lattice, L, we have that $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, then the lattice is said to be <u>distributive</u>. Let L be an involution lattice. For a, $b \in L$, we say that a <u>commutes</u> with b and write a C b whenever $(a \vee b') \wedge b = a \wedge b$.

Reference [2], pg. 650 gives the relationship between the elements of P'(S) and lattice theoretic concepts. This is if S is a Baer *-semi-group then P'(S) is an orthomodular lattice in which orthocomplementation is given by the mapping ' defined above.

Furthermore, in P'(S) we have $e \wedge f = (f'e)'e$ and $e \vee f = [(fe')']$ for $e, f \in P'(S)$.

Proof: Reference [2], pg. 650.

Another important result which is useful in making lattice computations is the Foulis-Holland Theorem [5] which states that if L is an orthomodular lattice, and if any two of the following relations hold; $a \, C \, b$, $a \, C \, d$, $b \, C \, d$, then $a \wedge (b \vee d) = (a \wedge b) \vee (a \wedge d)$ and $a \vee (b \wedge d) = (a \vee b) \wedge (a \vee d)$.

Lemma 4: For $e, f \in P'(S)$, then $ef = fe$ if and only if $e \, C \, f$ and in this case $ef = e \wedge f$.

Proof: Theorem 2 (ii) of [4].

CHAPTER II

UNITARY EQUIVALENCE OF SPECTRAL MEASURES

This chapter will be devoted to developing the major results of the research. That is, to define spectral measures on a Baer*-semigroup and to determine necessary and sufficient conditions for which two spectral measures will be unitarily equivalent.

Let $\mathcal{S}$ be a σ-algebra of subsets of a set X. Let μ be a mapping from sets $A \in \mathcal{S}$ into P'(S) for some Baer*-semigroup S. Let M represent the range of μ, so that $M \subseteq P'(S)$. If μ satisfies the following properties, then we say that μ is a <u>spectral measure</u>.

(i) $\mu(X) = I$, the least upper bound of all elements in P'(S)

(ii) $\mu(\phi) = 0$, the greatest lower bound of all elements in P'(S)

(iii) $\mu(\bigcup_n A_n) = \bigvee_n \mu(A_n)$ for $\{A_n\}$ a countable collection of disjoint subsets of $\mathcal{S}$.

(iv) $A \cap B = \phi$ implies $\mu(A) \wedge \mu(B) = 0$, for $A, B \in \mathcal{S}$.

(v) $\mu(A)\mu(B) = \mu(B)\mu(A)$ for $A, B \in \mathcal{S}$.

<u>Lemma 5</u>: If μ is a spectral measure, then

(i) $\mu(\bigcup_n A_n) = \bigvee_n \mu(A_n)$ for a countable collection of subsets of $\mathcal{S}$.

(ii) $A \subseteq B$ implies that $\mu(A) \leq \mu(B)$.

(iii) $\mu(A \cap B) = \mu(A) \cdot \mu(B) = \mu(A) \wedge \mu(B)$.

Proof: (i) $\mu(A \cup B) = \mu[(A-B) \cup (A \cap B) \cup (B-A)] =$ (by iii above)
$= \mu(A-B) \vee \mu(A \cap B) \vee \mu(B-A) = \mu(A-B) \vee \mu(A \cap B) \vee \mu(B-A) \vee \mu(A \cap B)$.
But $\mu(A-B) \vee \mu(A \cap B) = \mu[(A-B) \cup (A \cap B)] = \mu(A)$ and
$\mu(B-A) \vee \mu(A \cap B) = \mu[(B-A) \cup (A \cap B)] = \mu(B)$.
Thus the above equation is equal to $\mu(A) \vee \mu(B)$.
By applying the above result n times, we get that $\mu(\bigcup_1^n A_i) = \bigvee_1^n \mu(A_i)$.
Now $\bigvee_1^n \mu(A_i) \leq \bigvee_1^n \mu(A_i) \vee \bigvee_{n+1}^{\infty} \mu(A_i) = \bigvee_1^{\infty} \mu(A_i)$. Since the right hand side is independent of n, then $\mu(\bigcup_1^{\infty} A_i) \leq \bigvee_1^{\infty} \mu(A_i)$.

Consider now $\mu(\bigcup_1^{\infty} A_i) = \mu[(\bigcup_1^n A_i) \cup (\bigcup_{n+1}^{\infty} A_i)]$. Let $A \doteq \bigcup_1^n A_i$ and $B \doteq \bigcup_{n+1}^{\infty} A_i$. Then $\mu(\bigcup_1^{\infty} A_i) = \mu(A \cup B) = \mu(A) \vee \mu(B) \geq \mu(A) = \bigvee_1^n \mu(A_i)$. The left hand side is independent of n, so $\mu(\bigcup_1^{\infty} A_i) \geq \bigvee_1^{\infty} \mu(A_i)$.

(ii) $A \subseteq B$ implies that $\mu(B) = \mu[A \cup (B-A)] = \mu(A) \vee \mu(B-A) \geq \mu(A)$.

(iii) By Lemma 4, since $\mu(A)\mu(B) = \mu(B)\mu(A)$ and $M \subseteq P'(S)$, then $\mu(A)\ C\ \mu(B)$. Thus $\mu(A) \wedge \mu(B) = \mu(A)\mu(B)$. Hence
$\mu(A)\mu(B) = [\mu(A-B) \vee \mu(A \cap B)] \wedge [\mu(B-A) \vee \mu(A \cap B)] =$
$[\mu(A-B) \wedge \mu(B-A)] \vee \mu(A \cap B) = \mu(A \cap B)$.

Let $L \doteq P'(S)$ for S a Baer*-semigroup. For $\overline{M} \subseteq L$, we define
$Z(\overline{M}) = \{s \mid s \in S \text{ and } xs = sx \text{ for every } x \in \overline{M}\}$ and denote by:
$C(\overline{M}) = \{e \mid e \in L \text{ and } e\ C\ x \text{ for every } x \in \overline{M}\}$

The relationship between $Z(\overline{M})$ and $C(\overline{M})$ is given by the following well known lemma.

Lemma 6: $C(\overline{M}) = P'(S) \cap Z(\overline{M})$ for $\overline{M} \subseteq P'(S)$

Proof: By Lemma 4, $ex = xe$ if and only if $e \, C \, x$, where $e, x \in L$. If $e \in C(\overline{M})$, then $e \in P'(S)$ and $e \, C \, x$, for every $x \in \overline{M}$. Thus $ex = xe$ for every $x \in \overline{M}$, and $e \in Z(\overline{M})$.

Conversley if $e \in P'(S) \cap Z(\overline{M})$, then $e \in L$ and $ex = xe$ for every $x \in \overline{M}$, and hence $e \, C \, x$ for every $x \in \overline{M}$.

Corollary 7: $C(C(\overline{M})) = Z(C(\overline{M})) \cap P'(S)$ for $\overline{M} \subseteq P'(S)$.

Lemma 8: If $e_\alpha \in C(\overline{M})$, then

(i) $e_\alpha' \in C(\overline{M})$

(ii) $\bigvee_\alpha e_\alpha \in C(\overline{M})$

(iii) $\bigwedge_\alpha e_\alpha \in C(\overline{M})$

whenever $\bigvee_\alpha e_\alpha$ and $\bigwedge_\alpha e_\alpha$ exist.

Proof: This is corollary (iv) of [4].

As a result of Lemma 8, we are now able to classify the structure of $C(\overline{M})$ by the following well known result.

Theorem 9: $C(\overline{M})$ is a suborthomodular sublattice of L which is complete if L is complete.

Proof: By (ii) and (iii) of Lemma 8, $C(\overline{M})$ is complete if L is complete.

By (i) of Lemma 8, $C(\overline{M})$ is orthocomplemented. $C(\overline{M})$ satisfies the "Orthomodular Identity"; $e, f \in C(\overline{M})$ with $e \leq f$ implies $e \vee (e' \wedge f) = f$, since L does.

Several properties of $Z(\ )$ are given by the following four lemmas which are also well known.

Lemma 10: If $N_1 \subseteq N_2$ with $N_1, N_2 \subseteq S$, then $Z(N_2) \subseteq Z(N_1)$.

Proof: If $e \in Z(N_2)$, then $en = ne$ for every $n \in N_2$. Hence for any $m \in N_1$, then $m \in N_1 \subseteq N_2$, so $em = me$, i.e., $e \in Z(N_1)$.

Lemma 11: For $N \subseteq S$, we have $N \subseteq Z(Z(N))$.

Proof: If $n \in N$ and $e \in Z(N)$, then $en = ne$, i.e., $n \in Z(Z(N))$.

Lemma 12: For $N \subseteq S$, then $Z(N) = Z(Z(Z(N)))$.

Proof: Since from Lemma 11, $N \subseteq Z(Z(N))$, then by Lemma 10, $Z(Z(Z(N))) \subseteq Z(N)$.

Conversely, applying Lemma 11 to $Z(N)$, instead of N, yields $Z(N) \subseteq Z(Z(Z(N)))$.

Lemma 13: If N is a commutative subset of S, then $Z(Z(N)) \subseteq Z(N)$.

Proof: Since N is commutative, then $N \subseteq Z(N)$. Applying Lemma 10 yields $Z(Z(N)) \subseteq Z(N)$.

We now apply these results to our particular case in which the subset N of S is the range of a spectral measure.

Theorem 14: For M the range of spectral measure μ, we have that $M \subseteq Z(Z(M)) \subseteq Z(M)$.

Proof: We have that M is commutative, so that the conclusion follows from Lemmas 11 and 13.

Theorem 15: For M the range of μ, we have $M \subseteq C(C(M)) \subseteq C(M)$.

Proof: $M \subseteq P'(S)$ and M commutative implies that $M \subseteq P'(S) \cap Z(M) = C(M)$ by Corollary 6. Now by Lemma 10 then $Z(C(M)) \subseteq Z(M)$. But, $C(C(M)) = P'(S) \cap Z(C(M)) \subseteq P'(S) \cap Z(M) = C(M)$, i.e., $C(C(M)) \subseteq C(M)$. Also, $C(M) = P'(S) \cap Z(M)$ implies $C(M) \subseteq Z(M)$, so by Lemma 10 $Z(Z(M)) \subseteq Z(C(M))$. But $M \subseteq Z(Z(M))$ by Theorem 14, so $M \subseteq Z(Z(M)) \subseteq Z(C(M))$. Thus $M \subseteq Z(C(M)) \cap P'(S) = C(C(M))$.

At this point we begin the development of results which are required to implement the unitary equivalence of spectral measures.

Lemma 16: Let M and N be two subsets of P'(S), S a Baer*-semigroup, for which there exists an $x \varepsilon X$ such that $x^*Mx = N$ and $xx^* = x^*x = I$, i.e., for every $m \varepsilon M$, there exists an $n \varepsilon N$ such that $x^*mx = n$, and conversley. Then $x^* C(M)x = C(N)$.

Proof: Let $e \varepsilon C(N)$, so as n ranges over N, with $en = ne$, then $n = x^*mx$ and m ranges over M. Then $(x^*mx)e = e(x^*mx)$. Premultiplying by x yields $xx^*mxe = xex^*mx$ and $mxe = xex^*mx$. Postmultiplying this last equation by x^* yields $mxex^* = xex^*mxx^* = (xex^*)m$. Thus, if $e \varepsilon C(N)$, then $xex^* \varepsilon Z(M)$. It will now be shown that $xex^* \varepsilon P'(S)$. Since $(xex^*)^2 = (xex^*)(xex^*) = xe \cdot ex^* = xex^*$ and $(xex^*)^* = x^{**}e^*x^* = xex^*$, then $xex^* \varepsilon P(S)$.

By Lemma 2(v), $x = xx''$, so that $x^* = (xx'')^* = x''x^*$. Thus $I = xx^* = (xx^*)'' = (x''x^*)''$, by Theorem 1 (xii) of [4], and $(x''x^*)'' = (x^*)''$. By symmetry of arguement, $(x)'' = I$, also. Let $f = xex^*$. Now $e = x^*xex^*x = x^*fx = (x^*fx)'' = ((x^*)''fx)'' = (fx)'' = (f''x)''$ $= ((x^*)''\, f''\, x)'' = (x^*f''x)''$. Thus $x^*fx = (x^*f''x)''$. Again by Lemma 2(v), $x^*f''x = (x^*f''x)\ (x^*f''x)'' = (x^*f''x)(x^*fx) = x^*f''fx = x^*fx$, since $f = ff''$; and $f = f^* = (ff'')^* = f''f^*$. Then $f'' = xx^*f''xx^* = x(x^*f''x)x^* = x(x^*fx)x^*$ $= xx^*fxx^* = f$, so $f'' = f = xex^* \in P'(S)$. Therefore, $e \in C(N)$ implies $xex^* \in P'(S) \cap X(M) = C(M)$. Thus, $xC(N)x^* \subseteq C(M)$.

Now let $y = x^*$, so that as n ranges over N, m ranges over M, with $m = y^*ny$. Repeating the above proof interchanging m and n, and using $y = x^*$ yields: $e \in C(M)$ implies $yey^* \in P'(S) \cap Z(N) = C(N)$. Premultiplication by x and postmultiplication by x^* yields $C(M) \subseteq xC(N)x^*$; hence equality.

<u>Corollary 17</u>: For hypothesis identical to those of Lemma 16, $x^*C^2(M)x = C^2(N)$, where $C^2(\ ) = C(C(\ ))$.

<u>Proof</u>: Replace M and N by C(M) and C(N) respectively in the proof of Lemma 16.

We define a pair of mappings from L onto L which serve to implement the two previous results.

Lemma 18: Let $(\ )E$ and $(\ )F$ be mappings defined on S for which $(\ )E = x^*(\ )x$; $(\ )F = x(\ )x^*$; $x^*x = xx^* = I$; $(M)E = N$ and $M, N \subseteq P'(S)$. Then $(\ )E$ is an orthocomplementation preserving lattice isomorphism from L onto L such that $(C(M))E = C(N)$ and $(C^2(M))E = C^2(N)$, and the inverse mapping of $(\ )E$ is $(\ )F$.

Proof: In Theorem 7.5 of [9], letting $e = f = I$, then E is an orthocomplementation preserving lattice isomorphism whose inverse is F. By hypothesis $(M)E = N$. By Lemma 16, E may be extended to C(M), and when it is, the range of E becomes C(N). Similarly by Corollary 17, when the domain of E is restricted from C(M) to $C^2(M)$, the range becomes $C^2(N)$.

We now consider mappings from L to L which will be of particular importance in our study of unitary equivalence of spectral measures.

Let L be a complete orthomodular lattice possessing both a greatest element I, and a least element, 0. A mapping, $q : L \to L$ is called a quantifier on L if it satisfies the following conditions:

(i) $(0)q = 0$

(ii) $e \leq (e)q$ for every $e \in L$

(iii) $(e \wedge (f)q)q = (e)q \wedge (f)q$

In general a mapping $\phi : L \to L$ is said to be monotone in case $e, f \in L$ with $e \leq f$ implies $e\phi \leq f\phi$. M(L) will denote the semigroup (under function composition) of all monotone maps on L. Given two elements ϕ, ϕ^* of M(L), we say that ϕ and ϕ^* are <u>mutually adjoint</u> if $(e'\phi)'\phi^* \leq e$, and $(e'\phi^*)'\phi \leq e$ hold for every $e \in L$. (ϕ^* is unique if it exists.)

S(L) is the subset of M(L) consisting of those monotone mappings, ϕ, for which there exists $\phi^* \in M(L)$ such that ϕ and ϕ^* are mutually adjoint. Also, a mapping $\phi : L \to L$ is called a <u>hemimorphism</u> if $(e \vee f)\phi = e\phi \vee f\phi$ for $e, f \in L$, and $0\phi = 0$. $\phi \in S(L)$ is a hemimorphism.

In the following lemmas, we will prove some important results for particular types of elements in S(L).

<u>Lemma 19</u>: (i) A quantifier on a complete orthomodular lattice, L, determines and is determined by its invariant elements, i.e., $(e)q = \bigwedge \{f \mid e \leq f \text{ and } (f)q = f\}$.

(ii) If Q is a subcomplete suborthomodular lattice of the orthomodular lattice, L, with $Q \subseteq C(L)$, and if $\bigwedge \{f \mid f \in Q \text{ and } e \leq f\}$ exists for every $e \in L$, then Q determines a quantifier in the sense of (i).

<u>Proof</u>: This is lemma 1.3 of [1].

(iii) Let Q be a subcomplete, suborthomodular lattice of the orthomodular lattice, L. For each $e \in L$, define: $(e)q = \bigwedge \{f \mid f \geq e \text{ and } f \in Q\}$. Then $q \in S(L)$ and $q = q^2 = q^*$; and q restricted to C(Q) is a centervalued quantifier if $Q \subseteq C^2(Q)$.

<u>Proof</u>: By Lemma 1.3 (iii) of [1].

Lemma 20: Let q be defined as in Lemma 19 (iii). Then the following are true:

(i) $(I)q = I$

(ii) $e \leq f$ implies $(e)q \leq (f)q$ for any $e, f \in L$.

(iii) $((e)q)'q = ((e)q)'$ for $e \in L$.

Proof: Let $f \in Q$. Then clearly $(f)q = f$, and $(e)q \geq e$ for any $e \in L$.

(i) Also, since Q is a suborthomodular lattice of L, then $f' \in Q$. Since Q is a sublattice, then $e \vee e' = I \in Q$. Now $f \geq I$ and $f \in Q$ implies $f = I$, so $\{f \mid f \in Q \text{ and } f \geq I\} = \{I\}$. Therefore, $(I)q = \bigwedge \{f \mid f \in Q \text{ and } f \geq I\} = I$.

(ii) If $e_1 \leq e_2$; $e_1, e_2 \in L$, then $e_1 \leq e_2 \leq (e_2)q \in Q$. Thus, $(e_2)q \in$ {elements of Q which are $\geq e_1$}; hence $(e_1)q = \bigwedge \{f \mid f \in Q \text{ and } f \geq e_1\} \leq (e_2)q$, so $(e_1)q \leq (e_2)q$.

(iii) $(e)q \in Q$ for $e \in L$, and Q is suborthomodular implies that $((e)q)' \in Q$. Thus $((e)q)'q = ((e)q)'$.

Let M be the range of a spectral measure, μ. Define a mapping $q_\mu : L \to C^2(M)$ by $(e)q_\mu = \bigwedge \{f \mid e \leq f \text{ and } f \in C^2(M)\}$, and $e \in L$.

Corollary 21: Let q_μ be defined as above. Then $q_\mu \in S(L)$, $q_\mu = q_\mu{}^* = q_\mu{}^2$ and q_μ restricted to C(M) is a centervalued quantifier. Furthermore the conclusions of Lemma 20 are true for q_μ.

Proof: By Theorem 9 and Lemma 19, and Lemma 20.

Given e, $f \in P(S)$ with S a Baer *-semigroup, we say that e is *-equivalent to f and write $e \sim^* f$ whenever there exists an $x \in S$ for which $xx^* = e$ and $x^*x = f$. x is said to implement the * equivalence of e and f, written $x : e \sim^* f$, and x is called a partially unitary element.

Lemma 22: If $x : e \sim^* f$, then $e = xfx^*$ and $x^*ex = f$.

Proof: $e = e^2 = (xx^*)(xx^*) = x(x^*x)x^* = xfx^*$. Also $f = f^2 = (x^*x)(x^*x) = x^*(xx^*)x = x^*ex$.

The following two lemmas give a necessary and sufficient condition for which certain projections are *-equivalent in $S(L)$.

Lemma 23: For $i = 1, 2$; let $q_i : L \to Q_i$ defined by $(e)q_i = \bigwedge \{f \mid f \geq e \text{ and } f \in Q_i\}$ for all $e \in L$, where Q_i is a subcomplete suborthomodular lattice, and $Q_i \subseteq C^2(Q_i)$. Let T be an orthocomplementation preserving lattice isomorphism between Q_1 and Q_2. Then $x : q_1 \sim^* q_2$ in $S(L)$, where $x = q_1T$.

Proof: Since Tq_2T^{-1} and $T^{-1}q_1T$ are the identity maps on Q_1 and Q_2 respectively and since Q_i is closed under ', (Q_i is suborthomodular), for any $e \in L$ we have $(eq_1T)'q_2T^{-1} = (eq_1)' \leq e'$. Also, $(eq_2T^{-1})'\,q_1T = (eq_2)'T^{-1}q_1T = (eq_2)' \leq e'$. Hence $x = q_1T \in S(L)$ with $x^* = q_2T^{-1}$. Then $xx^* = q_1Tq_2T^{-1} = q_1$ and $x^*x = q_2T^{-1}q_1T = q_2$, so that $x : q_1 \sim^* q_2$.

The converse to Lemma 23 is given by the following lemma, Lemma 24.

Lemma 24: Let q_1 and q_2 be defined as in Lemma 23 with $q_1 \sim^* q_2$ in $S(L)$. Then there exists an orthocomplementation preserving lattice isomorphism between their respective ranges.

Proof: By hypothesis, there exists an $x \in S(L)$ for which $xx^* = q_1$, $x^*x = q_2$ and $q_1 = xq_2x^*$. Define $T : \text{Range }(q_1) \to L$ by $eT = ex$ for all $e = eq_1$. If $e = eq_1$, then $eT = exq_2x^*x = exq_2$. Thus $T : \text{Range }(q_1) \to \text{Range }(q_2)$. Similarly if $f = fq_2$, then $fx^* = fx^*q_1xx^* = fx^*q_1$. Now if $e_1, e_2 \in \text{Range }(q_1)$ and $e_1T = e_2T$, then $e_1 = e_1q_1 = e_1xq_2x^* = e_2xq_2x^* = e_2q_1 = e_2$. If $f \in \text{Range }(q_2)$, then $fx^*T = fx^*q_1x = fq_2 = f$. Thus $T : \text{Range }(q_1) \to \text{Range }(q_2)$ is a 1:1 onto map whose inverse is given by $fT^{-1} = fx^*$ for all $f = fq_2$. Since $x \in S(L)$, then T preserves joins on the Range (q_1). Now suppose that $e = eq_1$ and $f = eT \wedge e'T$. Then $fT^{-1} \leq e$, and $fT^{-1} \leq e'$, i.e., $fT^{-1} = 0$. Hence, $eT \wedge e'T = 0$. Also $eT \vee e'T = (e \vee e')T = I$ and $eT\, C\, e'T$ by hypothesis. Hence $e'T = (eT)'$ and T preserves orthocomplementation.

We now apply the preceeding results to the particular case concerning the range of a spectral measure.

Lemma 25: Let μ and ν be spectral measures with $\mu : (X,\mathcal{S}) \to M \subseteq P'(S)$, with $q_\mu : P'(S) \to C^2(M)$. Also, $\nu : (X,\mathcal{S}) \to N \subseteq P'(S)$, with $q_\nu : P'(S) \to C^2(N)$.

(i) If there exists an $x \in S$ such that $xx^* = x^*x = I$ and $x^*Mx = N$, then there exists an $E \in S(P'(S))$ such that $(M)E = N$, $E : C^2(M) \to C^2(N)$ where E is an orthocomplementation preserving lattice isomorphism, and $y : q_\mu \sim^* q_\nu$ with $y = q_\mu E \in S(P'(S))$.

(ii) If $q_\mu \sim^* q_\nu$, then there exists an orthocomplementation preserving lattice isomorphism between $C^2(M)$ and $C^2(N)$.

Proof: (i) By Lemma 18 and Lemma 23.

(ii) By Lemma 24.

Corollary 26: There exists an orthocomplementation preserving lattice isomorphism between $C^2(M)$ and $C^2(N)$ if and only if $q_\mu \sim^* q_\nu$.

In the following definitions, let q be a center valued quantifier on a complete orthomodular lattice, L. We say that:

(i) e is q-invariant if $(e)q = e$.

(ii) e is q-simple if $f \leq e$ implies $f = e \wedge (f)q$.

(iii) e is type I with respect to q if every subelement of e is an orthogonal supremum of q-simple elements.

(iv) L is type I with respect to q if and only if every element of L is type I with respect to q.

(v) e is q-unrelated to f if $(e)q$ is orthogonal to $(f)q$ (or e is very orthogonal to f). e is q-related to f if it is not orthogonal to $(f)q$.

The pair $(C(M), q_\mu)$ denotes that q_μ is a center valued quantifier on C(M). In keeping with the definitions of reference [1], an element a of $(C(M), q_\mu)$ is said to have uniform multiplicity c provided that c is the smallest cardinal number such that ae can be expressed as the union of c nonzero, orthogonal, simple elements, where $e \in C^2(M)$ and $ae \neq 0$. An element is said to have uniform multiplicity if there is some cardinal number, c , such that a has uniform multiplicity c.

$(C(M), q_\mu)$ is said to be homogeneous if for every element a of uniform multiplicity, there is some cardinal number, c, such that if a is a union of a family $\{a_\gamma\}$ of nonzero, orthogonal, simple, subelements of a with $(a_\gamma)q_\mu = (a)q_\mu$, then the cardinality of the family $\{a_\gamma\}$ is c.

Let $\mathbb{C}$ denote the set of cardinal numbers less than or equal to the cardinality of L. In reference [1] it is shown that if $(C(M), q_\mu)$ is type I and homogeneous, we may associate with each a in C(M) a function written $\{(e_\gamma, \gamma)\}$ from $\mathbb{C}$ into $C^2(M)$ where $\gamma \in \mathbb{C}$ and e_γ is the value of the function at γ. $\{(e_\gamma, \gamma)$ is called the multiplicity family of a and is uniquely characterized by the properties

(i) $e_\gamma \in C^2(M)$

(ii) $e_\lambda \perp e_\gamma$ for $\lambda \neq \gamma$

(iii) $\bigvee_\gamma e_\gamma = aq_\mu$

(iv) $a \wedge e_\gamma$ is the union of γ, orthogonal, simple elements $\{a_{\gamma j}\}$ of $(C(M), q_\mu)$ such that $a_{\gamma j}\, q_\mu = e_\gamma$ for all j.

We use the notation $MF(a) = \{(e_\gamma, \gamma)\}$ to denote that $\{(e_\gamma, \gamma)\}$ is the multiplicity family of a.

We now give a precise meaning to the statement that two spectral measures on a Baer*-semigroup are unitarily equivalent.

Let μ and ν be spectral measures with ranges M and N, respectively, in P'(S). We say that μ and ν are <u>unitarily equivalent</u> ($\mu \sim \nu$) if there exists an $x \in X$ such that $x^*\mu(A)x = \nu(A)$ for every $A \in \mathcal{S}$. (Note: $xx^* = x^*x = I$, and also; $y = \mu(A)x : \mu(A) \sim^* \nu(A)$)

<u>Lemma 27</u>: The unitary equivalence between spectral measures is an equivalence relation.

<u>Proof</u>: Let μ_1, μ_2 and μ_3 be spectral measures with ranges in L of M_1, M_2 and M_3 respectively.

(i) $\mu_1 \sim \mu_1$ by letting $x = I$.

(ii) If $\mu_1 \sim \mu_2$, then there exists an $x \in S$ such that $x^*\mu_1(A)x = \mu_2(A)$ for every $A \in \mathcal{S}$. Then since $x^*x = xx^* = I$, $\mu_1(A) = x\mu_2(A)x^*$, so that $\mu_2 \sim \mu_1$ by x^*.

(iii) If there exists x and $y \in S$ such that $x^*\mu_1(A)x = \mu_2(A)$ and $y^*\mu_2(A)y = \mu_3(A)$, then $y^*x^*\mu_1(A)xy = \mu_3(A) = (xy)^*\mu_1(A)(xy)$, so $\mu_1 \sim \mu_2$ and $\mu_2 \sim \mu_3$ imply that $\mu_1 \sim \mu_3$.

We are now ready to begin the proof of the major result summarized below in Theorem 31, which is a necessary and sufficient condition for which two spectral measures will be unitarily equivalent.

<u>Theorem 28</u>: Let μ and ν be unitarily equivalent spectral measures with ranges M and N respectively; q_μ and q_ν the associated quantifiers. Let $a \in C(M)$ with $MF(a) = \{(e_\gamma, \gamma)\}$. Assume that $(C(M), q_\mu)$ and $(C(M), q_\nu)$

are both type I homogeneous. If $(\ )E = x^*(\ )x$ is as in Lemma 18, then $MF(aE) = \{(e_\gamma E, \gamma)\}$.

Proof: Since $E : C^2(M) \to C^2(N)$ and $E^{-1} = F : C^2(N) \to C^2(M)$, we derive $e \leq eq_\mu$ implies $eE \leq eq_\mu E \in C^2(N)$, and thus $eEq_\nu \leq eq_\mu E$; also, $eE \leq (eE)q_\nu$ implies that $e \leq eEq_\nu F \in C^2(M)$ and thus $eq_\mu \leq eEq_\nu F$. Finally $eq_\mu E \leq eEq_\nu FE = eEq_\nu$, and $eq_\mu E = eEq_\nu$. Thus for all $e \in L$, $q_\mu E = Eq_\nu$.

If $e_{\gamma i} \leq e'_{\gamma j}$, then by Lemma 18, $(e_{\gamma i})E \leq (e'_{\gamma j})E = (e_{\gamma j}E)'$. Also $\bigvee_\gamma(e_\gamma E) = (\bigvee_\gamma e_\gamma)E = (aq_\mu)E = (aE)q_\nu$ by the above.

Let $\{a_{\gamma j}\}$ be the family of orthogonal, simple elements of $C(M)$ such that $a_{\gamma j}q_\mu = e_\gamma$ for all j and $\bigvee_j a_{\gamma j} = e_\gamma \wedge a$. Then $\bigvee_j(a_{\gamma j}E) = (\bigvee_j a_{\gamma j})E = (ae_\gamma)E = (a \wedge e_\gamma)E = aE \wedge e_\gamma E = (aE)(e_\gamma E)$ since $E : C^2(M) \to C^2(N)$. As shown above, since E preserves orthogonality, then $\{a_{\gamma j}E\}$ are mutually orthogonal. Since $(a_{\gamma j}E)q_\nu = (a_{\gamma j}q_\mu)E$, and since $a_{\gamma j}E$ are equal for all j, then the $(a_{\gamma j}E)q_\nu$ are equal for all j.

If $dE \leq a_{\gamma j}E$ with $a_{\gamma j}$ simple in $C(M)$ with respect to q_μ, then $d = dEF \leq a_{\gamma j}EF = a_{\gamma j}$, and hence $d = a_{\gamma j} \wedge dq_\mu$. Then $dE = (a_{\gamma j} \wedge dq_\mu)E = a_{\gamma j}E \wedge dq_\mu E = a_{\gamma j}E \wedge (dE)q_\nu$. Thus $dE \leq a_{\gamma j}E$ implies $dE = a_{\gamma j}E \wedge (dE)q_\nu$ so that if $a_{\gamma j}$ is simple with respect to q_μ, then $a_{\gamma j}E$ is simple with respect to q_ν.

Lemma 29: Let μ and ν be spectral measures with ranges M and N; q_μ and q_ν the associated quantifiers. Let $A \in \mathcal{S}$ with $MF(\mu(A)) = \{(e_\gamma, \gamma)\}$ and $MF(\nu(A)) = \{(f_\gamma, \gamma)\}$. Let T be an orthocomplementation preserving lattice isomorphism defined on L for which $e_\gamma T = f_\gamma$, for every value of γ. Then $\mu(A)T = \nu(A)$.

Proof: Since $e_\gamma = (\mu(A))q_\mu$ and $\mu(A) \varepsilon C^2(M)$ implies that $(\mu(A))q_\nu = \mu(A)$, then $\mu(A) = \bigvee_\gamma e_\gamma$. Similarly, we have $\nu(A) = \bigvee_\gamma f_\gamma$. Thus $\mu(A)T = (\bigvee_\gamma e_\gamma)T = \bigvee_\gamma (e_\gamma T) = \bigvee_\gamma f_\gamma = \nu(A)$, then $\mu(A)T = \nu(A)$.

Corollary 30: Let (i) $(C(M), q_\mu)$ and $(C(N), q_\nu)$ be type I homogeneous.

(ii) S is the common domain of spectral measures μ and ν.

(iii) For every $A \varepsilon S$, with $MF(\mu(A)) = \{(e_\gamma, \gamma)\}$ and $MF(\nu(A)) = \{(f_\gamma, \gamma)\}$ we have $e_\gamma E = f_\gamma$, for every value of γ, where $(\ )E = x^*(\ )x$ with $x^*x = xx^* = I$.

Then $x : \mu \sim \nu$, with $\mu(A)x : \mu(A) \sim^* \nu(A)$ for every $A \varepsilon S$.

Proof: By Lemma 18, E is an orthocomplementation preserving lattice isomorphism, then by Lemma 29, $x^*\mu(A)x = \nu(A)$ for every $A \varepsilon S$, i.e., $x : \mu \sim \nu$.

Let $y = \mu(A)x$. Then $yy^* = \mu(A)xx^*\mu(A) = \mu(A)^2 = \mu(A)$, and $y^*y = x^*\mu(A)\mu(A)x = x^*\mu(A)x = \nu(A)$. Thus $\mu(A)x : \mu(A) \sim^* \nu(A)$.

As a result of Theorem 28 and Corollary 30, we may state the promised result.

Theorem 31: Let μ and ν be spectral measures with a common domain, S. Let $(C(M), q_\mu)$ and $(C(n), q_\nu)$ be type I homogeneous. Then $\mu \sim \nu$ if and only if there exists an orthocomplementation preserving lattice isomorphism that preserves the multiplicity families between $\mu(A)$ and $\nu(A)$ for every $A \varepsilon \mathcal{S}$.

In addition, if a stronger assumption is made concerning the element x of S for which $x : \mu \sim \nu$, then equality between the spectral measures is achieved. The following lemmas terminate in this result in Theorem 35.

Lemma 32: If $y : \mu(A) \sim^* \nu(A)$ and $y \varepsilon Z(M)$, then

(i) $\mu(A) \geq \nu(A)$

(ii) $y\mu(A)y^* = \mu(A)$

Proof: If $y\mu(A) = \mu(A)y$, then premultiplication by y^* yields $y^*y\mu(A) = y^*\mu(A)y = \nu(A)$. But $y^*y = \nu(A)$, so $\nu(A)\mu(A) = \nu(A)$. Since $\nu(A) = \nu(A)^* = (\nu(A)\mu(A))^* = \mu(A)^*\nu(A)^* = \mu(A)\nu(A)$, then $\mu(A)$ and $\nu(A)$ commute so that $\nu(A)\mu(A) = \nu(A) \wedge \mu(A) = \nu(A)$.
Thus $\mu(A) \geq \nu(A)$. For (ii), $\mu(A) = \mu(A)^2 = \mu(A)yy^* = y\mu(A)y^*$.

Corollary 33: If $y : \mu(A) \sim^* \nu(A)$ and $y \varepsilon Z(M) \cap Z(N)$, then $\mu(A) = \nu(A)$.

Proof: Note that $y \varepsilon Z(M) \cap Z(N)$ implies that $y^* \varepsilon Z(M) \cap Z(N)$. Thus, since $y^* : \nu(A) \sim^* \mu(A)$, then by Lemma 32, $\mu(A) \leq \nu(A)$. Lemma 32 applied to $y : \mu(A) \sim^* \nu(A)$ yields $\mu(A) \geq \nu(A)$, thus equality.

Lemma 34: If $x \varepsilon Z(M)$ and $x^*x = xx^* = I$ with $x^*\mu(A)x = \nu(A)$, then $\mu(A) = \nu(A)$.

Proof: This is immediate.

Theorem 35: If $x : \mu \sim \nu$ and $x \in Z(M)$ (or $Z(N)$), then $\mu = \nu$.

Proof: Taking $X = A$, yields $xx^* = x^*x = I$. Then apply Lemma 34.

CHAPTER III

CYCLES AND SIMPLE ELEMENTS

In the analysis of Chapter II it was assumed, in part, that each $(C(M), q_\mu)$ was type I. Chapter III will develop sufficient conditions for which each $(C(M), q_\mu)$ will be type I, where S is a Baer*-semigroup, and $L = P'(S)$ is type I with respect to its associated quantifier. This will be accomplished by proving that every element in C(M) can be expressed as an orthogonal supremum of certain elements in C(M) called cycles, and hence that $(C(M), q_\mu)$ is type I if every cycle is simple with respect to q_μ.

However before producing these general results, we consider a special case in which we may prove directly that each $(C(M), q_\mu)$ is type I.

Theorem 36: (i) $(C(M), q_\mu)$ is type I if C(M) is atomic, where μ is a spectral measure with range, $M \subseteq P'(S)$.

(ii) If C(M) is a complete, atomic, M-symmetric orthomodular lattice, then $(C(M), q_\mu)$ is type I homogeneous.

Proof: By Lemma 3.3 (ii) and Theorem 3.5, respectively of [1].

For $g \in L$, define $\phi_g : L \to L$ for every $e \in L$ by $e\phi_g = (e \vee g') \wedge g$. These mappings are called Sasaki Projections. (Also $\phi_g^2 = \phi_g{}^* = \phi_g \in S(L)$)

The following theorem shows the relation between S and S(L) as well as the structure of L.

Theorem 37: If L is an orthomodular lattice, then S(L) is a Baer*-semigroup and the correspondence between L and P'(S(L)) is an orthocomplementation preserving lattice isomorphism. (reference [2])

Now let S be a Baer*-semigroup with its lattice of closed projections, L = P'(S). For each $x \in S$, define a mapping $\phi_x : L \to L$ by $e\phi_x = (ex)''$ for every $e \in L$.

Theorem 38: For $x, y \in S$; $\phi_{xy} = \phi_x \phi_y$; hence the mapping $\phi : S \to S(L)$ defined by $\phi(x) = \phi_x$ for $x \in S$ is an involution preserving semigroup homomorphism from S into S(L). Moreover for $e, g \in L$, $e\phi_g = (eg)'' = (e \vee g') \wedge g$. (reference [2])

We now pick out those elements of C(M) which serve as the basic building blocks for all of C(M). These are the cycles.

Let e be an element in L = P'(S). Then define $(e)Z_\mu = \bigvee \{(e\mu(A))'' \mid A \in \mathcal{S}\} \doteq \bigvee_{\mathcal{S}} (e\mu(A))''$, the cycle generated by e, where μ is a spectral measure defined on measurable sets, $A \in \mathcal{S}$.

In Lemmas 39 through Lemma 44 below, we prove several important facts about these cycles, the first being that each cycle is in C(M).

<u>Lemma 39</u>: $(e)Z_\mu \in C(M)$.

<u>Proof</u>: Let $B \in \mathcal{S}$. Consider $\{[\bigvee_{\mathcal{S}} (e\mu(A))'']\mu(B)\}'' = [\bigvee_{\mathcal{S}} (e\mu(A))'']\phi_{\mu(B)}$, by Theorem 38, $= \bigvee_{\mathcal{S}} [(e\mu(A))''\phi_{\mu(B)}]$, since $\phi_{\mu(B)}$ is a hemimorphism $= \bigvee_{\mathcal{S}} [(e\mu(A))''\mu(B)]'' = \bigvee_{\mathcal{S}} [e\mu(A)\mu(B)]''$, by Theorem 2 (xii) of [4], $= \bigvee_{\mathcal{S}} [e\mu(A \cap B)]'' \leq \bigvee_{\mathcal{S}} (e\mu(A))''$; since $A \cap B \in \mathcal{S}$, for every A, and hence $\{\mu(A \cap B) \mid A \in \mathcal{S}\} \subseteq \{\mu(A) \mid A \in \mathcal{S}\}$. Thus $\{[\bigvee_{\mathcal{S}} (e\mu(A))'']\mu(B)\}'' \leq \bigvee_{\mathcal{S}} (e\mu(A))''$. By Theorem 2 (i), of [4], then $\mu(B)[\bigvee_{\mathcal{S}} (e\mu(A))''] = [\bigvee_{\mathcal{S}} (e\mu(A))'']\mu(B)$. But $B \in \mathcal{S}$ was arbitrary, so $(e)Z_\mu \in Z(M)$. Also $\bigvee_{\mathcal{S}} (e\mu(A))'' \in P'(S)$. so that $(e)Z_\mu \in C(M)$.

<u>Lemma 40</u>: $(e)Z_\mu \geq e$

<u>Proof</u>: Since $X \in \mathcal{S}$, and $\mu(X) = I$, then $(e)Z_\mu = \bigvee_{\mathcal{S}} (e\mu(A))'' \geq (e\mu(X))'' = (e)'' = e$.

<u>Lemma 41</u>: If $f \in C(M)$ and $f \leq (e)Z_\mu$, then $f = (e\phi_f)Z_\mu$.

<u>Proof</u>: $(e\phi_f)Z_\mu = \bigvee_{\mathcal{S}} [(ef)''\mu(A)]'' = \bigvee_{\mathcal{S}} [ef\mu(A)]'' = \bigvee_{\mathcal{S}} [e\mu(A)f]''$, since $f \in C(M)$, $= \bigvee_{\mathcal{S}} [(e\mu(A))''f]'' = \bigvee_{\mathcal{S}} [(e\mu(A))''\phi_f] = [\bigvee_{\mathcal{S}} (e\mu(A))'']\phi_f = [(e)Z_\mu]\phi_f$. But $f \leq (e)Z_\mu$ and L an orthomodular lattice implies that $f \; C \; (e)Z_\mu$ and hence $[(e)Z_\mu]\phi_f = (e)Z_\mu \wedge f$. Since $f \leq (e)Z_\mu$, then $(e)Z_\mu \wedge f = f$.

<u>Corollary</u> (to proof) <u>42</u>: If $f \in C^2(M)$, then $(e)Z_\mu \cdot f = ((e)\phi_f)Z_\mu$.

<u>Proof</u>: Use first two lines in proof in Lemma 41. Since $f \in C^2(M)$, then $f \; C \; (e)Z_\mu$ and hence $((e)Z_\mu)\phi_f = (e)Z_\mu \cdot f$.

Lemma 43: If $f \varepsilon C(M)$, then $e \leq f'$ and $e \leq f$ are equivalent to $(e)Z_\mu \leq f'$ and $(e)Z_\mu \leq f$ respectively.

Proof: If $e \leq f'$, then $0 = e\phi_f = (ef)''$. Then $((e)Z_\mu)\phi_f = [\bigvee_s (e\mu(A))'']\phi_f$ $= \bigvee_s [e\mu(A)f]'' = \bigvee_s [ef\mu(A)]'' = \bigvee_s [(ef)''\mu(A)]'' = \bigvee_s [0\cdot\mu(A)]'' = 0$. Conversley if $(e)Z_\mu \leq f'$, then $e \leq (e)Z_\mu \leq f'$, so $e \leq f'$. Now if $e \leq f$, then $e = e\phi_f = (ef)''$, so as above $((e)Z_\mu)\phi_f = \bigvee_s [(ef)''\mu(A)]'' = \bigvee_s [e\mu(A)]''$ $= (e)Z_\mu$ so that $(e)Z_\mu \leq f$. Conversley if $(e)Z_\mu \leq f$, then $e \leq (e)Z_\mu \leq f$ or $e \leq f$.

Lemma 44: If $f \varepsilon C^2(M)$ and $0 \neq f \leq ((e)Z_\mu)q_\mu$, then $e\phi_f \neq 0$.

Proof: If $e\phi_f = 0$, then by Corollary 42, $(e)Z_\mu \cdot f = 0$, so that $f = ((e)Z_\mu)q_\mu \wedge f = ((e)Z_\mu \wedge f)q_\mu = ((e)Z_\mu \wedge f)q_\mu = ((e)Z_\mu \cdot f)q_\mu$ since $f \varepsilon C^2(M)$, and hence $f = ((e)Z_\mu \cdot f)q_\mu = (0)q_\mu = 0$, a contradiction since $f \neq 0$.

It will now be shown that if L is a type I orthomodular (complete) lattice with respect to a center valued quantifier, then every element of C(M) can be written as a supremum of orthogonal cycles.

Theorem 45: If $p \varepsilon C(M)$, then there exists an orthogonal family $\{(e_j)Z_\mu\}$ of cycles such that $p = \bigvee_j (e_j)Z_\mu$, where $\{e_j\}$ is a family of simple elements in L.

Proof: Let $\{e_j\}$ be a maximal family of simple elements in L such that $e_j \leq p$ and $(e_j)Z_\mu \leq (e_i)Z_\mu'$ if $i \neq j$. Note that $(e_j)Z_\mu \leq p$ for every j by

Lemma 43. If $p \wedge [\bigvee_j (e_j)Z_\mu]' \neq 0$, then let a be a simple element in L such that $a \leq p \wedge [\bigvee_j (e_j)Z_\mu]' = p \wedge [\bigwedge_j (e_j)Z_\mu']$, so $a \leq (e_j)Z_\mu'$ for every value of j. Again by Lemma 43, then $(a)Z_\mu \leq (e_j)Z_\mu'$ for every j, and $(a)Z_\mu \leq p$ contradicting the maximality of $\{e_j\}$. Thus $p \wedge [\bigvee_j (e_j)Z_\mu]' = 0$. Now since $\bigvee_j (e_j)Z_\mu \leq p$ and since L satisfies the orthomodular identity (so that C(M) does), then $p = (p \wedge [\bigvee_j (e_j)Z_\mu]) \vee (p \wedge [\bigvee_j (e_j)Z_\mu]') = p \wedge [\bigvee_j (e_j)Z_\mu] \leq \bigvee_j (e_j)Z_\mu$ Thus $p \leq \bigvee_j (e_j)Z_\mu \leq p$, and hence equality.

Hence if it can be shown that for every simple element, e, in an orthomodular lattice, L, which is type I with respect to its associated quanitfier every $(e)Z_\mu$ is simple for all spectral measures, μ, with range $M \subseteq L$, then every $(C(M), q_\mu)$ will be a type I sublattice of L.

The following theorem states a necessary and sufficient condition, in terms of cycles, for which $(C(M), q_\mu)$ will be type I.

<u>Theorem 46</u>: $(C(M), q_\mu)$ is type I if and only if for every $p \in C(M)$, there exists an orthogonal family of simple cycles, $\{(e_\alpha)Z_\mu\}$ such that $p = \bigvee_\alpha (e_\alpha)Z_\mu$

<u>Proof</u>: Clearly, if for every $p \in C(M)$, there exists an orthogonal family of simple cycles $\{(e_\alpha)Z_\mu\}$ such that $p = \bigvee_\alpha (e_\alpha)Z_\mu$, then $(C(M), q_\mu)$ is type I.

Conversley, if $(C(M), q_\mu)$ is type I, then for every $p \in C(M)$, there exists an orthogonal family of simple elements, $\{a_i\}$ such that $p = \bigvee_i a_i$.

By Theorem 45, since each $a_i \in C(M)$, then there exists an orthogonal family of cycles $\{(e_{ij})Z_\mu\}$ such that $a_i = \bigvee_j (e_{ij})Z_\mu$. If there exists a $b \in C(M)$ such that $b \leq (e_{ij})Z_\mu \leq a_i$, then since a_i is simple, $b = a_i \wedge (b)q_\mu$, so that $b = b \wedge (e_{ij})Z_\mu = (a_i \wedge (b)q_\mu) \wedge (e_{ij})Z_\mu =$ $(a_i \wedge (e_{ij})Z_\mu) \wedge (b)q_\mu = (e_{ij})Z_\mu \wedge (b)q_\mu$, and hence each $(e_{ij})Z_\mu$ is simple. Thus for fixed value of i, $\{(e_{ij})Z_\mu\}$ are an orthogonal simple family with $a_i = \bigvee_j (e_{ij})Z_\mu$

For $i \neq k$, since $a_i \leq a_k{}'$, then for every value of j, $(e_{ij})Z_\mu \leq a_i \leq a_k{}' \leq (e_{k\iota})Z_\mu'$ for every value of ι. Thus $\{(e_{ij})Z_\mu\}_{i,j}$ is an orthogonal family of simple elements with $p = \bigvee_i a_i = \bigvee_i (\bigvee_j (e_{ij})Z_\mu) =$ $\bigvee_{i,j} (e_{ij})Z_\mu$

CHAPTER IV

LOOMIS *-SEMIGROUPS

In proving the major results of Chapter II, it was also assumed that each $(C(M), q_\mu)$ was homogeneous. Chapter IV will be primarily devoted to proving that this criterion will be satisfied for every $(C(M), q_\mu)$ if, instead of taking S to be a Baer *-semigroup, we require that S be a Loomis *-semigroup.

After reference [5], we call S a Loomis *-semigroup if S satisfies the following postulates:

(i) S is an involution semigroup with 0.

(ii) all projections in P(S) are closed.

(iii) if $x, y \in S$ with $xx^* = yx^* = yy^*$ implies that $x = y$. (the *-cancellation law)

(iv) if $\{a_\alpha\}$ is an orthogonal family of partially unitary elements of S, then there exists an element $a \in S$ such that $aa_\alpha^* = a_\alpha a_\alpha^*$ for all α, and for $b, c \in S$ with $a_\alpha b = a_\alpha c$ for all α, then $ab = ac$ (We denote that (iv) is satisfied by writing $a \in \mathrm{sum}_\alpha a_\alpha$.).

(v) for every non-zero element $x \in S$, there exists an element $y \in S$ such that $y = y^* \in ZZ(x^*x)$ and x^*xy^2 is a non-zero closed projection (S has the property (EP).).

Note: if S is a Loomis *-semigroup, then S is a Baer *-semigroup.

Let L be a complete orthomodular lattice. We call L a Loomis dimension lattice if it is equipped with an equivalence relation $\sim$ satisfying the following:

(i) If $e \sim 0$, then $e = 0$.

(ii) If $\{e_\alpha\}$ is an orthogonal family in L, and if $f \sim \bigvee_\alpha e_\alpha$, then there exists an orthogonal family $\{f_\alpha\}$ in L such that $f = \bigvee_\alpha f_\alpha$ and such that $f_\alpha \sim e_\alpha$ for every α.

(iii) If $\{e_\alpha\}$ is an orthogonal family in L and if $\{f_\alpha\}$ is a second orthogonal family in L with the same indices such that $e_\alpha \sim f_\alpha$ for every α, then $\bigvee_\alpha e_\alpha \sim \bigvee_\alpha f_\alpha$.

(iv) If e and f have a common complement, then $e \sim f$.

D. J. Foulis [3] has related these two concepts by the following theorem.

Theorem 47: If S is a Loomis *-semigroup, then P(S) is a Loomis dimension lattice under the relation of *-equivalence. (Proof: Theorem 24 of [3].)

The connection between Loomis *-semigroup and type I homogeneous lattices is then completed by the relation proved by J. H. Bevis [1].

Theorem 48: If $(L, \sim)$ is a type I Loomis dimension lattice, then (L, ϕ) is type I homogeneous for any center valued quantifier on L.

Proof: Theorem 4.10 (i) of [1].)

We will now determine the structure of $Z(M)$ in the case that S is a Loomis *-semigroup.

Lemma 49: Let S be a Loomis *-semigroup and $M \subseteq P(S)$, with M the range of spectral measure, μ. Then $Z(M)$ is a Loomis *-semigroup.

Proof: (i) By Lemma 2 (iv) [3], since $M = M^*$, then $Z(M)$ is a subsemigroup of S containing 0 and closed under the involution *.

(ii) Now $P(S) = P'(S)$ and clearly $P(Z(M)) = Z(M) \cap P(S)$. By Lemma 4 of [4] we have $P'(Z(M)) = Z(M) \cap P'(S)$. Thus $P(Z(M)) = P'(Z(M))$.

(iii) Let $\{a_\alpha\}$ be an orthogonal family of partially unitary elements of $Z(M)$. We show that the $\text{sum}_\alpha\, a_\alpha$ is an element of $Z(M)$. Since $\mu(A) \varepsilon M$ for $A \varepsilon \mathcal{S}$, then $a_\alpha \mu(A) = \mu(A) a_\alpha$ for all α since $a_\alpha \varepsilon Z(M)$. Then where a is the unique element in the $\text{sum}_\alpha \{a_\alpha\}$, Corollary 10 (ii) of [3], implies that $a\mu(A) = \mu(A)a$ for every $A \varepsilon \mathcal{S}$. Hence $a \varepsilon Z(M)$.

(iv) If $x, y \varepsilon Z(M)$, then $x^*, y^* \varepsilon Z(M)$ by (i). Thus $xx^* = yx^* = yy$ implies that $x = y$.

(v) Let $0 \neq x \varepsilon Z(M)$. Then there exists an element $y \varepsilon S$ such that $y = y^* \varepsilon ZZ(x^*x)$ and $x^*xy^2 \neq 0$ is a closed projection. We show that $y \varepsilon Z(M)$. Since $M \subseteq Z(x^*x)$, then by Lemma 10, $ZZ(x^*x) \subseteq Z(M)$, but $y \varepsilon ZZ(x^*x)$ so $y \varepsilon Z(M)$.

We now apply these theorems to prove our promised result.

Theorem 50: If S ia a Loomis *-semigroup with each $(C(M), q_\mu)$ type I, then each $(C(M), q_\mu)$ is type I homogeneous.

Proof: By Lemma 49 each Z(M) is a Loomis *-semigroup, and $P(Z(M) = Z(M) \cap P'(S) = C(M)$ by Lemma 6. Thus $P(Z(M))$ is a Loomis dimension lattice by Theorem 47, and since it is also type I, then $(C(M), q_\mu)$ is type I homogeneous by Theorem 48.

The remainder of this chapter will be devoted to relation between *-equivalence of two orthogonal families of elements in the ranges of two different spectral measures on S, and the *-equivalence of their respective sums.

Lemma 51: Let $x_\alpha : \mu(A_\alpha) \sim^* \nu(A_\alpha)$ for orthogonal families $\{\mu(A_\alpha)\}$ and $\{\nu(A_\alpha)\}$. Then $\{x_\alpha\}$ is a family of mutually orthogonal elements of S.

Proof: Let $\alpha \neq \beta$. Then $\mu(A_\alpha)\mu(A_\beta)=0$, or in terms of the partially unitary elements, $x_\alpha x_\alpha^* x_\beta x_\beta^* = 0$. We show this implies $x_\beta x_\alpha^* = x_\beta^* x_\alpha = 0$.

Premultiplication by x_β^* and postmultiplication by x_α gives $0 = x_\beta^* x_\alpha x_\alpha^* x_\beta x_\beta^* x_\alpha = (x_\beta^* x_\alpha)(x_\beta^* x_\alpha)^*(x_\beta^* x_\alpha)$. Multiplication by $(x_\beta^* x_\alpha)^*$ yields $(x_\beta^* x_\alpha)^*(x_\beta^* x_\alpha)(x_\beta^* x_\alpha)^*(x_\beta^* x_\alpha) = 0$. Let $a = (x_\beta^* x_\alpha)^*(x_\beta^* x_\alpha)$. Note that $a^* = a$, so the above is $a^* a = 0$, and hence by *-cancellation, $a = 0$ (from [4], pg. 75). Thus $(x_\beta^* x_\alpha)^*(x_\beta^* x_\alpha) = 0$; hence $x_\beta^* x_\alpha = 0$. Similarly, by considering $x_\alpha^* x_\alpha x_\beta^* x_\beta = 0$, premultiplying and postmultiplying by x_β and x_α^*, respectively, yields $x_\beta x_\alpha^* = 0$ for $\alpha \neq \beta$.

Lemma 52: Let the hypothesis of Lemma 51 be satisfied and define $x = \sum_\alpha x_\alpha$. Then $x : \bigvee_\alpha \mu(A_\alpha) \sim^* \bigvee_\alpha \nu(A_\alpha)$.

Proof: By Theorem 8,[5], $x^* = \sum_\alpha x_\alpha{}^*$. Then by Corollary 11 [3] $xx^* = \sum_\alpha x_\alpha x_\alpha{}^* = \bigvee_\alpha x_\alpha x_\alpha{}^*$ (by Lemma 13 [3]) $= \bigvee_\alpha \mu(A_\alpha)$. Also $x^*x = x^*x^{**} = \sum_\alpha x_\alpha{}^* x_\alpha{}^{**} = \sum_\alpha x_\alpha{}^* x_\alpha = \bigvee_\alpha \nu(A_\alpha)$.

The fact that this same x relates the individual elements of these families is proven in Lemma 54 below.

Lemma 53: Let the hypothesis of Lemma 51 be satisfied. Then: $x^*x_\alpha = x_\alpha{}^*x = x_\alpha{}^*x_\alpha = \nu(A_\alpha)$ and $x_\alpha x^* = xx_\alpha{}^* = x_\alpha x_\alpha{}^* = \mu(A_\alpha)$.

Proof: Since $x_\beta{}^*x_\alpha = 0$ except for $\beta = \alpha$, then by Lemma 6 [3], $x_\beta{}^*x = x_\beta{}^*x_\beta = \nu(A_\beta)$. Taking adjoints of both sides of the last expression yield $x^*x_\beta = \nu(A_\beta)$.

Similarly, $x_\beta x^* = x_\beta x_\beta{}^* = \mu(A_\beta)$, and taking adjoints yields $x\, x_\beta{}^* = \mu(A_\beta)$.

Lemma 54: Let the hypothesis of Lemma 51 be satisfied. Then $x^*\mu(A_\alpha)x = \nu(A_\alpha)$ for every α.

Proof: Let $t = \mu(A_\beta)x$ where $x = \sum_\alpha x_\alpha$. Then $tt^* = \mu(A_\beta)xx^*\mu(A_\beta) = \mu(A_\beta)\cdot \bigvee_\alpha \mu(A_\alpha) \cdot \mu(A_\beta) = \mu(A_\beta)$. Now consider:
$x^*\mu(A_\beta) = x^*x_\beta x_\beta{}^* = x^*x_\beta x^* = x_\beta{}^*xx^*$ by Lemma 53 applied twice. Now $(x_\beta{}^*)(xx^*) = x_\beta{}^*$ if and only if $(x_\beta{}^*)'' \leq xx^*$ by Theorem 1 (iv) [4].

But $(x_\beta{}^*)'' = \mu(A_\beta)$ by Lemma 7.2 [9], so $(x_\beta{}^*)'' = \mu(A_\beta) \leq \bigvee_\alpha \mu(A_\alpha) = xx^*$. Thus $t^*t = x^*\mu(A_\beta)x = (x_\beta{}^*)(xx^*)(x) = x_\beta{}^*x = \nu(A_\beta)$ by Lemma 53.

The following theorem shows that even non-orthogonal pairs which are each *-equivalent, can both have their *-equivalences related by a single element, x, of S for which $xx^* = xx^* = I$.

<u>Theorem 55</u>: Suppose that for every $A_\alpha \in \mathcal{S}$, there exists an element $x_\alpha \in S$ for which $x_\alpha : \mu(A) \sim^* \nu(A)$. Then for every pair of elements $\mu(A)$, $\mu(B)$ (and not necessarily orthogonal), there exists an $x \in S$ for which $x^*x = xx^* = I$ and $x^*\mu(A)x = \nu(A)$, $x^*\mu(B)x = \nu(B)$.

<u>Proof</u>: Take $x_1 : \mu(A \cap \tilde{B}) \sim^* \nu(A \cap \tilde{B})$

$x_2 : \mu(A \cap B) \sim^* \nu(A \cap B)$

$x_3 : \mu(\tilde{A} \cap B) \sim^* \nu(\tilde{A} \cap B)$

$x_4 : \mu(\widetilde{A \cup B}) \sim^* \nu(\widetilde{A \cup B})$

Let $x = \Sigma x_i$. Then $x^*x = xx^* = I$ by Lemma 52. Also $x^*\mu(A)x = x^*(\mu(A \cap \tilde{B}) \vee \mu(A \cap B))x =$, (by Lemma 18), $x^*\mu(A \cap \tilde{B})x \vee x^*\mu(A \cap B)x = \nu(A \cap \tilde{B}) \vee \nu(A \cap B)$, (by Lemma 54), $= \nu(A)$. Similarly for $x^*\mu(B)x = \nu(B)$.

<u>Theorem 56</u>: Let $\{A_\alpha\}$ be a collection of mutually disjoint elements of $\mathcal{S}$. If $x_\alpha : \mu(A_\alpha) \sim^* \nu(A_\alpha)$ for every value of α, $x_\alpha \in S$, then there exists an $x \in S$ such that $xx^* = x^*x = I$ and

(i) $x^*\mu(A_\alpha)x = \nu(A_\alpha)$ for every α.

(ii) $x^*\mu(\bigcup_\alpha A_\alpha)x = \nu(\bigcup_\alpha A_\alpha)$.

<u>Proof</u>: By Lemma 5, and Lemmas 51-54.

CHAPTER V

FINITE DIMENSIONAL BAER*-ALGEBRAS

This chapter will relate the studies of unitary equivalence of spectral measures and unitary equivalence of normal elements in a Finite Dimensional Baer*-Algebra.

Let F be the field of complex numbers. An <u>involution algebra</u> S over F is an algebra over F with a mapping $* : S \to S$, the involution, such that :

(i) S is an involution semigroup under mulitplication.

(ii) For every a, b ε S, and λ_1, λ_2 ε F, we have that $(\lambda_1 a + \lambda_2 b)^* = \overline{\lambda}_1 a^* + \overline{\lambda}_2 b^*$.

A <u>*-algebra</u> over F is an involution algebra over F having a unity element and such that $a^*a = 0$ if and only if $a = 0$. A Baer*-algebra is a *-algebra which is a Baer*-semigroup under multiplication.

Let a be an element of a Baer*-algebra for which $aa^* = a^*a$. For a complex number, λ, we define the <u>λth eigenprojection</u> of a, denoted by a_λ, by $a_\lambda = (a-\lambda I)'$. A complex number is called an <u>eigenvalue</u> of a if $a_\lambda \neq 0$. The set of eigenvalues of a is called the <u>spectrum of a</u> and is denoted by Λ_a.

The following theorem gives the relation between a normal element of S and its associated eigenvalues and eigenprojections.

Theorem 57: (Spectral Decomposition Theorem). Let $a \in S$, a finite dimensional Baer *-algebra, with $aa^* = a^*a$. Then:

(i) The spectrum of a, Λ_a, is a finite non-empty set of F.

(ii) $\{a_\lambda \mid \lambda \in \Lambda_a\}$ is an orthogonal family of nonzero projections.

(iii) $\Sigma a_\lambda = I$ for $\lambda \in \Lambda_a$.

(iv) $a = \Sigma \lambda a_\lambda$, for $\lambda \in \Lambda_a$.

Proof: Theorem 2.27 of [7].

The next lemma shows that the *-equivalence of the eigenprojections of two normal elements, a and b in S, implies that a and b are unitarily equivalent.

Lemma 58: Let $a, b \in S$ with $a^*a = aa^*$, $b^*b = bb^*$, $\Lambda_a = \Lambda_b = \{\lambda_i\}_1^n$. Let $a_i \doteq a_{\lambda_i}$, $b_i \doteq b_{\lambda_i}$ and let x_i be the element of S for which $x_i : a_i \sim^* b_i$ for each $\lambda_i \in \Lambda_a$. Then $x^*ax = b$ where $x^*x = xx^* = I$, $x = \sum_1^n x_i$.

Proof: Since $d^*d = 0$ implies $d = 0$ for any $d \in S$, then a duplication of the proof of Lemma 51 implies that $\{x_i\}_1^n$ is an orthogonal family of elements of S.

Define $x = \sum_1^n x_i$. Then $x^* = \sum_1^n x_i^*$ and $xx^* = \sum_1^n x_i x_i^*$ by the orthogonality of the $\{x_i\}$, and $\sum_1^n x_i x_i^* = \sum_1^n a_i = I$ by Theorem 57 (iii). Similarly we get that $x^*x = I$.

By a repetition of the proof of Lemmas 53 and 54, we prove that $x^*a_i x = b_i$. Thus $x^*ax = x^*(\sum_1^n \lambda_i a_i)x = \sum_1^n \lambda_i (x^*a_i x) = \sum_1^n \lambda_i b_i = b$.

Theorem 59: Let Λ be a non-empty finite set of complex numbers and $\{b_\lambda \mid \lambda \in \Lambda\}$ an orthogonal family of projections such that $\Sigma b_\lambda = I$. Then if $a = \Sigma\lambda b_\lambda$, then $aa^* = a^*a$, $b_\lambda = a_\lambda$ for all $\lambda \in \Lambda$, and $\Lambda = \Lambda_a$ if and only if $b_\lambda \neq 0$ for each nonzero $\lambda \in \Lambda$.

Proof: Theorem 2.28 of [7].

The following gives the converse to Theorem 58.

Lemma 60: Let $a, b \in S$ with $a^*a = aa^*$, $b^*b = bb^*$, and $x^*ax = b$, where $xx^* = x^*x = I$. Then $a_\lambda \sim^* b_\lambda$ for every λ, and $\Lambda_a = \Lambda_b$.

Proof: By Theorem 57 (iv), for $\lambda \in \Lambda_a$, then $a = \Sigma\lambda a_\lambda$ with $\Sigma a_\lambda = I$. Thus, $I = x^*x = x^*Ix = x^*(\Sigma a_\lambda)x = \Sigma x^*a_\lambda x$. By Lemma 18, $x^*a_\lambda x \in P'(S)$ for every λ, and $\{a_\lambda\}$ mutually orthogonal implies that $\{x^*a_\lambda x\}$ are mutually orthogonal, and that $a_\lambda \neq 0$ implies $x^*a_\lambda x \neq 0$ for every λ.

Then $b = x^*ax = x^*(\Sigma\lambda a_\lambda)x = \Sigma\lambda(x^*a_\lambda x)$, so by Theorem 59, $b_\lambda = x^*a_\lambda x$, and $\Lambda_a = \Lambda_b$. Now define $x_\lambda = a_\lambda x$. Then $x_\lambda : a_\lambda \sim^* b_\lambda$.

The preceeding results are summarized below.

Theorem 61: Let $a, b \in S$ with $a^*a = aa^*$, $bb^* = b^*b$. Then $x^*ax = b$ with $x^*x = xx^* = I$ if and only if $\Lambda_a = \Lambda_b$ and $a_\lambda \sim^* b_\lambda$ for every $\lambda \in \Lambda_a$.

CHAPTER VI

CONCLUDING REMARKS

The results of this research have been to establish a connection between the abstract spectral measure, (L, X, μ), and the other three mathematical systems in the Quadripartite Diagram. One of the assumptions made on the $L = C(M)$ of the abstract spectral measures was that it was type I. Chapter III shows the relation between cycles and the type I condition.

(1) What conditions are needed on S (or on L) so that each $(C(M), q_\mu)$ will be type I?

Another unsolved problem is given in the following discussion. Let $\Gamma = \{f : \mathbb{C} \to L \mid f(\lambda) \perp f(\gamma) \text{ for } \lambda \neq \gamma\}$. Define an equivalence relation on Γ by $f_1 \sim f_2$ if there exists an $x \in S$ such that $xx^* = x^*x = I$ and $x^* f_1(\gamma) x = f_2(\gamma)$ for all $\gamma \in \mathbb{C}$. For $f \in \Gamma$, let $\overline{f}$ be the equivalence class of Γ to which f belongs. Let $\overline{\Gamma} = \{\overline{f} \mid f \in \Gamma\}$. For each spectral measure $\mu : (x, \mathcal{S}) \to L$ such that $C(M)$ is type I homogeneous with respect q_μ, we associate a mapping $\overline{\mu} : (x, \mathcal{S}) \to \overline{\Gamma}$ by $\overline{\mu}(A) = \overline{MF(\mu(A))}$. Let μ and ν be spectral measures from $(X, \mathcal{S})$ to L such that $C(M)$ and $C(N)$ are type I and homogeneous with respect to q_μ and q_ν respectively. It is clear from Theorem 28 that if $\mu \sim \nu$, then $\overline{\mu} = \overline{\nu}$.

(2) What conditions are required so that if $\overline{\mu} = \overline{\nu}$, then $\mu \sim \nu$?

Theorems 55 and 56 give related results for a Loomis *-semigroup. But under what conditions will the x of Theorem 56 be such that $x^*\mu(A)x = \nu(A)$ for every $A \in \mathcal{S}$?

In the case of the Finite Dimensional Baer *-Algebras discussed in Chapter V, the necessary conditions for (2) are already satisfied. Here, for normal elements, a and b, we take for X the spectrum of a, $\Lambda_a = \Lambda_b$. The ranges of spectral measures μ and ν are the sets of eigenprojections of a and b respectively. Lemma 58 then shows that if $\overline{\mu} = \overline{\nu}$, then $\mu \sim \nu$. The key to the general solution of (2) may well turn out to be the requirement of having a general "Spectral Decomposition Theorem" for the space S under consideration.

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VITA

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Kenneth Ross Garren

UNITARY EQUIVALENCE OF SPECTRAL MEASURES ON A BAER *-SEMIGROUP

Kenneth Ross Garren

ABSTRACT

This paper is concerned with a generalization of the concept of unitary equivalence of spectral measures on a Baer *-semigroup. A connection is made between abstract spectral measures, and three other distinct mathematical systems.

Chapter II is devoted specifically to generalizing the concept of a spectral measure and to determining necessary and sufficient conditions for which two spectral measures will be unitarily equivalent.

Chapter III discusses the problem of each $(C(M), q_\mu)$ being type I in terms of cycles, the basic elements of $C(M)$.

In Chapter IV it is shown that in a Loomis *-semigroup each type I $(C(M), q_\mu)$ will be type I homogeneous.

Chapter V relates the study of unitary equivalence of spectral measures and the unitary equivalence of normal elements in a Finite Dimensional Baer *-algebra.