SOME ASPECTS OF TIME-DEPENDENT

ONE-DIMENSIONAL RANDOM WALKS

by

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CHAPTER I

INTRODUCTION

The notion of random walk is a convenient descriptive framework in which to study the fluctuations of sums of random variables. In the one-dimensional case considered in this dissertation one imagines a particle constrained to move along the x-axis at discrete time intervals by amounts selected on a chance-dependent basis. We suppose that initially the particle has coordinate S_0 and that the step lengths X have a common probability distribution while successive steps X_1, X_2, ... are statistically independent. The coordinate of the particle immediately after the n^{th} step is

$$S_n = X_1 + X_2 + \ldots + X_n \ . \tag{1.1}$$

The random variable X may be either discrete or continuous. In this dissertation, however, we assume that X is discrete with possible values ± 1.

In this formulation no attention is given to the time required for the particle to accomplish its steps and consequently the probabilistic properties of S_n depend only on the fact that n steps have been taken. Thus, they are completely independent of the amount of time which has elapsed during the course of the n steps. Consequently, we refer to these walks as "discrete-time" or "time-independent" random walks.

For many practical situations which can be viewed in the random walk framework, however, it is appropriate to require that the time intervals between steps are themselves also subject to chance. For example, in collective-risk theory, claims are assumed to occur at random intervals. In Queuing Theory applications, customer arrivals (steps of +1) and departures (steps of -1) occur at random time intervals. In population models, the intervals of time between births (steps of +1) and deaths (steps of -1) are also subject to chance. In these time-dependent models we are interested in the behavior of the random variable $S(t)$, the coordinate of the particle at time t after the process was initiated. Under these assumptions, the value of $S(t)$ may result from any finite number of steps which can occur during the interval $(0,t)$.

Random walks in which the time between steps is also chance dependent are called "randomized random walks" by Feller [19]. In this dissertation we prefer the expressions "time-dependent random walks" or "continuous-time random walks", and it is with these walks that we shall be exclusively concerned.

Most of our work, in fact, will be further specialized to walks such that in any small time interval $(t,t+dt)$ the particle has probability $\lambda dt + 0(dt^2)$ of making instantaneously a positive step, or step to the right ($X = +1$), and probability $\mu dt + O(dt^2)$ of making instantaneously a negative step, or step to the left ($X = -1$), where λ, μ are positive time-independent constants. The consequence of this assumption is that successive steps of the same kind are separated by negative exponentially distributed time intervals with parameters λ and μ for steps to the right and left respectively. Moreover, the walk itself can be regarded as the interaction of two independent Poisson streams of events. For if $S(0) = 0$ and $S(t) = n(\geq 0)$, the particle must have taken $n + r$ positive steps ($r \geq 0$), and r negative steps during $(0,t)$ so that

$$p_n(t) = \Pr\{S(t) = n \mid S(0) = 0\} = e^{-(\lambda+\mu)t} \sum_{r=0}^{\infty} \frac{(\lambda t)^{n+r}}{(n+r)!} \frac{(\mu t)^r}{r!}, \qquad n = 0,1,2,\ldots. \tag{1.2}$$

Similarly,

$$p_{-n}(t) = e^{-(\lambda+\mu)t} \sum_{r=0}^{\infty} \frac{(\lambda t)^r}{r!} \frac{(\mu t)^{n+r}}{(n+r)!}, \qquad n = 0,1,2,\ldots. \tag{1.3}$$

The probability $p_n(t)$ will often be called a "state probability." As we shall see, $p_n(t)$ can be expressed in terms of a modified Bessel function of the first kind for all values of n. We shall comment on the repeated occurrence of this Bessel function in probabilistic and allied physical studies.

The most obvious practical application of this type of walk is perhaps to the problem of the "state probabilities" in Queuing Theory where, however, there is a barrier at the origin in the sense that if the particle reaches the origin the next step must be a positive one. In this case (1.2) and (1.3) are not applicable and more complicated formulae arise. In the queuing situation $\lambda dt + O(dt^2)$ is to be interpreted as the probability that an arrival occurs during $(t,t+dt)$ and $\mu dt + O(dt^2)$ as the probability that a departure (or service completion) occurs during $(t,t+dt)$ provided, of course, that $S(t) \geq 1$. In this case $S(t)$ represents the number of customers in the system at time t. The usual convention in the literature of Queuing Theory is to use M to denote the negative exponential distribution. Thus, M/M/1 denotes the single-server queuing process in which both time intervals between arrivals, and service times, have negative exponential distributions with different parameters in general. The letter G is used to indicate the general distribution. Thus, M/G/1 denotes the single-server queuing process in which the intervals between arrivals have a negative exponential distribution and the service times have a general distribution. The GI/M/1 system is analogously defined, the letters GI denoting "general independent."

In order to distinguish the queuing process from the random walk without barriers, we shall refer to the random walk which describes the queuing process as the "queuing random walk." In this case, the position of the particle at time t represents the number in, or "state of," the system. On the other hand, the random walk without barriers will be referred to as the "unrestricted random walk," or "doubly-infinite random walk."

A perhaps flippant, but instructive, application of the unrestricted random walk is provided by the score difference between two teams engaged in a soccer game. We assume that the two teams have probability differentials λdt and μdt, respectively, of scoring a goal. We shall refer to this hypothetical soccer game from time to time throughout the dissertation for the purpose of illustration.

The initial motivation for the research embodied in this dissertation arose from a study of busy time in the single-server queuing system. Busy time is a measure of that part of a fixed time interval during which the server is occupied. The remaining time during which the server is idle is called the idle time. The reader should be careful to note the difference between the busy time and a busy period. A busy period is the length of time which elapses from the instant the server becomes occupied until the moment when he becomes idle again for the first time. Thus, the busy time during a fixed time interval $(0,t)$ may be composed of several busy periods. Analogous considerations apply to the idle time and idle period. Busy time is of considerable practical interest, particularly under conditions of heavy traffic when the mean arrival and service rates tend to equality. Alternatively expressed in terms of a randomly moving particle, busy time is the so-called sojourn time of the particle in a non-zero state. Since sojourn time problems possess some interest in their own right, attention was first directed towards these in a rather general framework which nevertheless permits application to the queuing and other situations. Chapter IV contains a development of the existing theory on the two-state sojourn time problem as well as a development of new results concerning the more difficult three-state sojourn time problem in which it is assumed that at any given time t, the particle may be in any one of three possible states. In addition, Chapter IV contains new applications of the theory of sojourn times to the M/M/1 queuing process and the unrestricted random walk with negative exponentially distributed intervals between steps.

It is also of interest in queuing problems to be able to make probability statements about the time when the queue size first reaches a given level and the maximum queue size during a given time interval. This motivated the material in Chapter III, again placed in the more general setting of an unrestricted random walk. Unfortunately, it seems difficult to make much impact on the queuing application although Chapter III does contain new additions to the theory. In the case of the unrestricted random walk, however, it is shown that the relationship known to exist between the maximum and sojourn time problems for discrete-time random walks can be extended to the continuous-time random walk considered in this dissertation.

This relationship, not obvious at first glance, led to a search for other time-dependent results possessing time-independent analogues. This work is described in Chapter V.

Finally, some new results are given in Chapter VI for unrestricted random walks in which it is assumed that either the distribution of time between positive steps or that between negative steps is general, instead of negative exponential.

The dissertation thus contains topics, treated to a large extent independently, which are in fact linked by a common interest and have a useful application to practical situations. In this respect both the precise results given in Chapter III and IV and the asymptotic formulae developed from them are interesting and important.

To avoid duplication the basic mathematical tools necessary for the development of the main topics are presented in Chapter II. References to related work in the literature are given in the relevant sections.

CHAPTER II

MATHEMATICAL REQUISITES

2.1 Introduction

The theoretical developments carried out in this dissertation require repeated use of a limited range of unconnected mathematical techniques and results. For self-sufficiency and convenience of reference these are all collected and described in this Chapter. Thus, we deal summarily in turn with the Laplace transformation, some properties of modified Bessel and of hypergeometric functions, and with the so-called stable distributions. The description is limited to those properties referred to in the rest of the work and, in general, no proofs are given. Authoritative references are included.

2.2 The Laplace Transformation

An important tool used throughout this dissertation is the Laplace transformation. If $f(t)$ is a real-valued function of, in general, a real variable t defined on $-\infty < t < \infty$ such that $f(t) = 0$ for $t < 0$, the function $f^*(z)$ defined by the equation

$$f^*(z) = \int_0^\infty e^{-zt} f(t)dt \tag{2.2.1}$$

is called the Laplace transform of $f(t)$. The operation of producing $f^*(z)$ from $f(t)$ is called the Laplace transformation. We frequently write

$$f^*(z) = L\{f(t)\} \ . \tag{2.2.2}$$

The variable z is in general complex and it can be shown that the integral in (2.2.1) converges for all z such that $Re(z) > c$ where c is a positive constant (see [41] for example).

The Laplace transformation has been used frequently in mathematical physics because of the convenient property that

$$\int_0^\infty e^{-zt} \frac{df(t)}{dt} dt = -f(0) + zf^*(z), \tag{2.2.3}$$

with similar results for the higher derivatives. This property permits certain ordinary differential equations to be transformed into algebraic equations which can be solved for $f^*(z)$, which is then "inverted," or transformed back, into the t domain. In the same way certain partial differential equations involving a time and a space variable can be reduced to ordinary differential equations.

The process of inversion may be carried out directly by using the complex integral inversion formula

$$f(t) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{c-iT}^{c+iT} e^{zt} f^*(z) dz \tag{2.2.4}$$

or as we shall henceforth write it,

$$f(t) = \frac{1}{2\pi i} \int_C e^{zt} f^*(z)dz . \tag{2.2.5}$$

It is understood that C, the contour of integration, is a large semicircle described in the positive sense with diameter parallel to and displaced a distance c to the right of the imaginary axis and containing all the singularities of the integrand. An alternative method of inversion is to consult a table of Laplace transforms and their inverses. Currently [17], Vol. I, contains the most extensive set of such tables.

Many time-dependent stochastic processes are formulated in terms of differential-difference equations. By use of equation (2.2.3), the Laplace transformation then permits the "derivative to be removed" so that one is left to deal with a pure difference equation. This explains the convenience of the Laplace transformation in modern probability theory.

Another important feature of the Laplace transform is the convolution property. The convolution of two functions f(t) and g(t) is written $f(t)\circledast g(t)$ which is shorthand for

$$f(t)\circledast g(t) = \int_0^t f(x)g(t-x)dx \tag{2.2.6}$$

and may be extended arbitrarily many times. Thus

$$h(t)\circledast[f(t)\circledast g(t)] = \int_0^t h(x) \int_0^{t-x} f(y)g(t-x-y)dydx .$$

The convolution operation denoted by $\circledast$ is commutative and associative. The effect of applying the Laplace transformation is to transform a convolution into the product of the Laplace transforms of its components. Thus, for example,

$$f^*(z)g^*(z)h^*(z) = L\{f(t)\circledast g(t)\circledast h(t)\} \quad . \qquad (2.2.7)$$

Even when there are difficulties in inverting $f^*(z)$ which, unhappily, is often the case, there may be useful information which can be derived directly from the Laplace transform. For example, if $f(t)$ is a probability density function such that

$$f(t)dt = Pr\{t < X < t+dt\}$$

where X is a real-valued random variable defined for $X \geq 0$, then if $Pr\{X = \infty\} = 0$,

$$(-1)^n\left[\frac{d^n f^*(z)}{dz^n}\right]_{z=0} = \int_0^\infty t^n\, f(t)dt \ , \qquad (2.2.8)$$

so that the moments of X can be obtained in principle from the Laplace transform without having to evaluate $f(t)$ explicitly.

The following limiting results also provide useful information about the asymptotic behavior of a function $f(t)$ and its Laplace transform $f^*(z)$.

First of all we define the two symbols $\rightarrow$ and $\sim$ which will be used repeatedly throughout the dissertation. By the expression $\psi(x) \to \theta(x)$, $x \to \delta$, where ψ, θ are arbitrary functions of x, and δ is in general an arbitrary number, we shall mean that $\psi(x)$ and $\theta(x)$ approach or tend to the same value as x approaches δ. We shall sometimes use the expression $\psi(x) \sim \theta(x)$ as $x \to \delta$ which is equivalent to the statement $\psi(x) \to \theta(x)$, $x \to \delta$. We now summarize the limiting results referred to above.

We denote by L(t) a function defined for $0 < t < \infty$ such that for each $x > 0$,

$$\frac{L(tx)}{L(t)} \to 1, \quad t \to \infty \ . \tag{2.2.9}$$

The function L is then said to vary slowly at infinity. A function f(t) is said to be "ultimately monotone" if it is monotone on some interval (a,∞) where a is a non-negative constant. Under this assumption it can be shown that for some ε such that $0 < \varepsilon < \infty$, as $t \to \infty$, then

$$f(t) \sim \frac{1}{\Gamma(\varepsilon)} t^{\varepsilon-1} \, L(t) \ . \tag{2.2.10}$$

If and only if the ultimately monotone function f(t) satisfies (2.2.10) as $t \to \infty$, then its Laplace transform $f^*(z)$ as $|z| \to 0$ is such that

$$f^*(z) \sim z^{-\varepsilon} \, L(\frac{1}{z}) \ . \tag{2.2.11}$$

For proofs of (2.2.10) and (2.2.11), see [19].

Equations (2.2.10) and (2.2.11) will be useful in this dissertation since most, if not all, of the probability distribution functions and probability density functions encountered will satisfy the assumptions of the preceding paragraph. In particular, we note that if $\varepsilon = 1$ and $L(x)=K$, a constant, then from (2.2.11) it follows that

$$\lim_{z \to 0} z f^{*}(z) = \lim_{t \to \infty} f(t) \ . \qquad (2.2.12)$$

We remark that the roles of z and t may be interchanged in (2.2.10) - (2.2.12) to obtain corresponding results for the case when $|z| \to \infty$ and $t \to 0$. For a complete discussion of results of the type given by (2.2.10) and (2.2.11), known as Tauberian theorems, see [19], for example.

In conclusion, it is worth noting that the Laplace transform itself sometimes provides important probabilistic information and leads to a better understanding of the structure of the problem from which it was derived. For example, it can be shown (see [36], for example) that the Laplace transform of the probability density function corresponding to the distribution of the length of the busy period in the M/G/1 queue is given by the root with smallest modulus of the functional equation

$$\eta(z) = b^{*}(z+\lambda-\lambda\eta(z)) \ , \qquad (2.2.13)$$

where $b(t)$ is the probability density function corresponding to the general service time distribution and λ is the parameter of the negative exponential interarrival-interval distribution. As stated previously, the busy period is the length of time which elapses from the instant the server becomes occupied until the moment when the server becomes idle again for the first time.

A well known theorem of Lagrange (see [40], for example) states that if $\phi(w)$ is regular on and inside a closed contour C surrounding a point Z, and if X is such that the inequality

$$|X\phi(w)| < |w-Z| \tag{2.2.14}$$

is satisfied at all points w on the perimeter of C, then the equation

$$U = Z - X\phi(U) \ , \tag{2.2.15}$$

regarded as an equation in U, has one root in the interior of C; and furthermore, any function f(U) regular on and inside C can be expanded as a power series in X by the formula

$$f(U) = f(Z) + \sum_{m=1}^{\infty} \frac{(-X)^m}{m!} \frac{d^{m-1}}{dZ^{m-1}} \left[\{\phi(Z)\}^m f'(Z) \right] \ . \tag{2.2.16}$$

Upon applying Lagrange's result to (2.2.13), we find that the Laplace transform $g^*(z)$ of the busy period density is given by

$$g^*(z) = \frac{1}{\lambda} \sum_{m=1}^{\infty} (-1)^{m-1} \frac{\lambda^m}{m!} \frac{d^{m-1}}{dz^{m-1}} \left[\{b^*(z+\lambda)\}^m \right] \ . \tag{2.2.17}$$

Upon inverting (2.2.17) term by term, we find that $g(t)$, the busy period density, is given by

$$g(t) = e^{-\lambda t} \sum_{m=1}^{\infty} \frac{(\lambda t)^{m-1}}{m!} b^{(m)}(t) \qquad (2.2.18)$$

where $b^{(m)}(t)$ is the m-fold convolution of $b(t)$ with itself. It can be shown that the individual terms in the sum given by

$$g_m(t) = e^{-\lambda t} \frac{(\lambda t)^{m-1}}{m!} b^{(m)}(t) \qquad (2.2.19)$$

represent the joint probability density function and probability that the busy period is of length t and m customers are served, respectively. We note that $g_m(t)$ is also equal to $\frac{1}{m}$ times the probability that at time t, the m^{th} negative step occurs resulting in a passage to -1; at the same time we assume an unrestricted random walk in which $S(0) = 0$ and the time between positive steps has negative exponential distribution, while the time between negative steps has a general distribution with probability density function $b(t)$. This suggests a probabilistic link between the unrestricted random walk and the queuing random walk.

For a more thorough account of the theory of the Laplace transformation, see [15], [38], or [41]. For a basic development of the theory with special emphasis on applications, [6] is recommended.

2.3 Modified Bessel Functions and Their Properties

A function which recurs often in widely diverse applications is the modified Bessel function of the first kind of order n and argument x which can be defined by the series

$$I_n(x) = \sum_{m=0}^{\infty} \frac{(x/2)^{2m+n}}{m!\,\Gamma(m+n+1)} \quad . \qquad (2.3.1)$$

Both x and n are in general complex. As $|x| \to \infty$, we have for fixed n,

$$I_n(x) \sim \frac{e^x}{(2\pi x)^{1/2}} \sum_{m=0}^{\infty} \frac{a_m^{(n)}}{(2x)^m} \qquad (2.3.2)$$

where

$$a_m^{(n)} = \frac{(-1)^m}{m!} \frac{\Gamma(n+m+\frac{1}{2})}{\Gamma(n-m+\frac{1}{2})} \quad . \qquad (2.3.3)$$

The function $I_n(x)$ is called "modified" to distinguish it from its more familiar counterpart $J_n(x)$ which can be defined by the series

$$J_n(x) = (-i)^n \sum_{m=0}^{\infty} \frac{(ix/2)^{2m+n}}{m!\,\Gamma(m+n+1)} \quad . \qquad (2.3.4)$$

It is evident that $I_n(x)$ and $J_n(x)$ are related by the equation

$$I_n(x) = (-i)^n J_n(ix) \quad . \qquad (2.3.5)$$

The function $I_n(x)$ occurs in mathematical physics as the solution of the differential equation

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - (x^2+n^2)y = 0. \tag{2.3.6}$$

In our work we shall often be required to recognize $I_n(x)$ as a contour integral, namely,

$$I_n(x) = \frac{1}{2\pi i} \int_C e^{zx} \frac{dz}{\sqrt{z^2-1}\left[z+\sqrt{z^2-1}\right]^n} \tag{2.3.7}$$

or

$$\frac{nI_n(x)}{x} = \frac{1}{2\pi i} \int_C e^{zx} \frac{dz}{\left[z+\sqrt{z^2-1}\right]^n} \tag{2.3.8}$$

where C is the usual Laplace inversion contour defined in Section 2.2.

In order to manipulate various equations involving $I_n(x)$, it will be useful to note the relations

$$\frac{2nI_n(x)}{x} = I_{n-1}(x) - I_{n+1}(x) \ , \tag{2.3.9}$$

and

$$2I_n'(x) = I_{n-1}(x) + I_{n+1}(x) \ . \tag{2.3.10}$$

A relationship which, among other uses, will enable us to show that certain probabilities add to unity is Schlomilch's identity given by

$$e^{\frac{1}{2}x(s+s^{-1})} = \sum_{m=-\infty}^{\infty} s^m I_m(x) \quad . \tag{2.3.11}$$

A definitive study of Bessel functions and their properties is contained in [39].

This is not the place to enumerate all the areas in which these functions naturally arise. They do appear quite often, however, in probability problems involving negative exponential distributions which will be of greatest concern to us in this dissertation. For example, from equation (1.2) we obtain for the "state probabilities"

$$p_n(t) = \rho^{n/2} e^{-(\lambda+\mu)t} I_n(2t\sqrt{\lambda\mu}) \quad , \; n=0,1,2,..., \tag{2.3.12}$$

where

$$\rho = \lambda/\mu \quad . \tag{2.3.13}$$

Feller [19] has obtained (2.3.12) by extending a discrete-time random walk result to continuous time. His technique will be illustrated in Chapter V. Since

$$I_n(x) = I_{-n}(x) \quad , \; n=0,1,2,..., \tag{2.3.14}$$

we see, with the aid of (1.3) and (2.3.1), that (2.3.12) holds for any positive or negative integer n.

As an example of the use of (2.3.11), we see immediately that the state probabilities $p_n(t)$ given in (2.3.12) add to unity, as they must since $S(t)$ must have some integer value for every t. Thus,

$$\sum_{n=-\infty}^{\infty} p_n(t) = 1 . \tag{2.3.15}$$

The usefulness of Schlomilch's identity in this connection has been pointed out by Feller [20].

2.4 The J-function

A function related to the modified Bessel function of the first kind which recurs throughout this dissertation will be denoted by $J_n(x,y)$ and defined by the series

$$J_n(x,y) = e^{-(x+y)} \sum_{m=0}^{\infty} \eta^{m+n} I_{m+n}(\xi) \tag{2.4.1}$$

where

$$\eta = \sqrt{y/x} , \xi = 2\sqrt{xy} . \tag{2.4.2}$$

We assume that x and y are functions of some parameter t. Then, for example, from equation (2.3.12) we see that for the unrestricted random walk,

$$\Pr\{S(t)\geq n\} = J_n(\mu t,\lambda t) , n=0,1,2,.... \tag{2.4.3}$$

Writing $J_0(x,y) = J(x,y)$, we find from Schlomilch's identity given in (2.3.11) that

$$J_0(x,y) = 1 - J_1(x,y) \ . \tag{2.4.4}$$

$J_0(x,y)$ is the so-called J-function discussed in detail by Goldstein [22] in connection with a diffusion problem, again illustrating the link with random walks which are frequently used as probabilistic models for diffusion processes. A thorough summary of results concerning $J(x,y)$ is given in Luke [27].

From the work of Goldstein we have as $|\xi| \to \infty$,

$$\begin{aligned} J(x,y) &\sim \frac{e^{-(x+y-\xi)}}{\sqrt{2\pi\xi}\,(1-\eta)} \sum_{m=0}^{\infty} \frac{A_m \lambda_m}{(2\xi)^m} \ , \quad \eta < 1, \\ &\sim 1 - \frac{e^{-(x+y-\xi)}}{\sqrt{2\pi\xi}\,(\eta-1)} \sum_{m=0}^{\infty} \frac{A_m \lambda_m}{(2\xi)^m} \ , \quad \eta > 1 \ , \end{aligned} \tag{2.4.5}$$

where

$$A_m = \frac{\{\Gamma(m+\frac{1}{2})\}^2}{m!} \ , \tag{2.4.6}$$

$$\lambda_m = (1-\eta)/2 + (1+\eta)\beta_m/2 \ , \tag{2.4.7}$$

$$\beta_m = \sum_{j=0}^{m} \frac{(m)_j}{(m-\frac{1}{2})_j} \left[\frac{-4\eta}{(1-\eta)^2}\right]^m \ , \tag{2.4.8}$$

and

$$(N)_j = N(N-1)\cdots(N-j+1);\ N_0 = 1 . \qquad (2.4.9)$$

For η near unity (2.4.5) will not provide a good approximation. To obtain expressions valid for values of η near unity, see [22].

It should be noted that the asymptotic behavior of $J_n(x,y)$, $n=1,2,\ldots$, can be obtained from (2.4.5) and (2.3.2) by observing that

$$J_n(x,y) = J(x,y) - e^{-(x+y)} \sum_{j=0}^{n-1} \eta^j I_j(\xi) . \qquad (2.4.10)$$

It may be convenient, for certain applications, to express $J_n(x,y)$ in forms different from the one given in (2.4.1). A simple integral form is given by

$$J_n(x,y) = \int e^{-(x+y)} [y'\eta^{n-1}I_{n-1}(\xi) - x'\eta^n I_n(\xi)]dt , \qquad (2.4.11)$$

where the dash denotes differentiation with respect to the parameter t. Equation (2.4.11) is a direct consequence of a result due to Maximon [28]. As an example of the use of (2.4.11), we have from (2.4.3)

$$Pr\{S(t) \geq n\} = \rho^{n/2} \int_0^t e^{-(\lambda+\mu)} \{\sqrt{\lambda\mu}I_{n-1}[2w\sqrt{\lambda\mu}] -$$

$$- \mu I_n[2w\sqrt{\lambda\mu}]\}dw \ , \quad n=1,2,\ldots, \qquad (2.4.12)$$

and

$$\Pr\{S(t) \geq 0\} = 1 + \int_0^t e^{-(\lambda+\mu)w} \{\sqrt{\lambda\mu}I_1[2w\sqrt{\lambda\mu}] - \qquad (2.4.13)$$

$$- \mu I_0[2w\sqrt{\lambda\mu}]\}dw \ ,$$

where

$$\rho = \lambda/\mu \ . \qquad (2.4.14)$$

Another quantity which is related to $J_n(x,y)$ is the function $f(c,X,T)$ defined by

$$f(c,X,T) = \int_X^T e^{-cw} \frac{I_1[\sqrt{w^2-X^2}]}{\sqrt{w^2-X^2}} dw \qquad (2.4.15)$$

where c,X,T are positive constants. The distribution functions which will be derived in connection with sojourn times in Chapter IV are expressible in terms of $f(c,X,T)$. In order to use the asymptotic properties of $J_n(x,y)$, it will be convenient to express $f(c,X,T)$ in terms of $J_0(x,y)$ and $J_1(x,y)$. We proceed to do this by using Maximon's result given in (2.4.11). By considering $f(c,X,T)$ as a function of the parameter T, it is not difficult to show, with the aid of (2.4.11), that

$$J_0[b(T+X),a(T-X)] = e^{X\sqrt{c^2-1}} \int_X^T e^{-cw} \left[\frac{1}{2}\sqrt{\frac{w+X}{w-X}}\, I_1[\sqrt{w^2-X^2}] - \right. \tag{2.4.16}$$

$$\left. - bI_0[\sqrt{w^2-X^2}]\right] dw + e^{-2bX} ,$$

and

$$J_1[a(T+X),b(T-X)] = e^{-X\sqrt{c^2-1}} \int_X^T e^{-cw} \left[bI_0[\sqrt{w^2-X^2}] - \right. \tag{2.4.17}$$

$$\left. - \frac{1}{2}\sqrt{\frac{w-X}{w+X}}\, I_1[\sqrt{w^2-X^2}]\right] dw ,$$

where

$$\begin{matrix} a \\ b \end{matrix} = \frac{c \pm \sqrt{c^2-1}}{2} . \tag{2.4.18}$$

Upon combining (2.4.15) - (2.4.18), it easily follows that

$$f(c,X,T) = - \frac{e^{-cX}}{X} + \frac{1}{X}\left[e^{-X\sqrt{c^2-1}}\, J[b(T+X),a(T-X)] + \right. \tag{2.4.19}$$

$$\left. + e^{X\sqrt{c^2-1}}\, J_1[a(T+X),b(T-X)]\right] ,$$

a result obtained by Ford [21] by a different method. Equation (2.4.19) will enable us to go directly from integral expressions of the form given in (2.4.15) to expressions involving the J-function.

2.5 The Generalized Hypergeometric Function

In Chapter III we shall encounter a probability density function expressed in terms of a fractional integral (for a discussion of fractional integrals, see [17], Vol. II) which in turn can be expressed in terms of the generalized hypergeometric function. To determine the asymptotic behavior of this density function, we can then invoke the well documented asymptotic properties of the generalized hypergeometric function. We summarize here the basic properties of the generalized hypergeometric function which will be necessary in Chapter III.

Following Luke [27], we denote the generalized hypergeometric function by

$$ {}_pF_q\left[\begin{matrix} a_1,a_2,\ldots,a_p \\ 1+b_1,1+b_2,\ldots,1+b_q \end{matrix}\middle| z\right] = {}_pF_q(a_p;1+b_q;z) \quad ,(2.5.1)$$

which is defined by the equation

$$ {}_pF_q[a_p;1+b_q;z] = \sum_{k=0}^{\infty} \frac{(a_1)_k\cdots(a_p)_k}{(1+b_1)_k\cdots(1+b_q)_k}\frac{z^k}{k!} \quad , \quad (2.5.2)$$

where

$$ (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} \quad . \qquad (2.5.3)$$

We shall be concerned specifically with the case p = 2,

$q = 3$. For $0 \leq p \leq q-1$, we have from [27] the result, as $z \to \infty$,

$$(M_{p,q})\,{}_pF_q[a_p;1+b_q;z] \sim K_{p,q}(z) \tag{2.5.4}$$

where

$$M_{p,q} = \frac{\prod_{j=1}^{p} \Gamma(a_j)}{\prod_{j=1}^{q} \Gamma(1+b_j)}, \tag{2.5.5}$$

$$K_{p,q}(z) = [\exp(u^{-1}z^u)]z^v(2\pi)^{(p-q)/2}\sqrt{u}\sum_{k=0}^{\infty} N_k z^{-ku}, \tag{2.5.6}$$

and

$$N_0 = 1;\ u = (q-p+1)^{-1};\ v = u\left[\frac{(q-p)}{2} + \sum_{k=1}^{p} a_k - \sum_{k=1}^{q} (1+b_k)\right]. \tag{2.5.7}$$

We give the value of N_k only for $k = 0$ since the N_k vary with p and q and their expressions are very cumbersome for $k = 1,2,\ldots$. For our purposes $N_0 = 1$ will suffice. For a complete discussion, see [27]. For a broader discussion of the generalized hypergeometric function, see [4].

2.6 Stable Distributions

Of great importance in the theory of sums of random variables are the so-called stable distributions, which are, in fact, the only ones occuring as the limiting distributions of a sum S_n of independent random variables $X_1, X_2, \ldots, X_n$.

We let $X, X_1, X_2, \ldots, X_n$ denote mutually independent random variables with common distribution R and let $S_n = X_1 + X_2 + \ldots + X_n$. We say that R is stable (broad sense) if, for all n, there exist constants $c_n > 0$ and γ_n such that

$$S_n \overset{d}{=} c_n X + \gamma_n \qquad (2.6.1)$$

where R is not concentrated at zero and the symbol "$\overset{d}{=}$" means "has the same distribution as." R is said to be stable in the strict sense if (2.6.1) holds with $\gamma_n = 0$.

A comprehensive account of stable distributions and their properties is given in [19] with reference to earlier work by Levy, Doblin and others.

The connection between time-independent or discrete-time random walks and sums of random variables is apparent since the sum $S_n = X_1 + X_2 + \ldots + X_n$ represents the position of the particle at the end of the n^{th} step as described in Chapter I. The connection with time-dependent random walks is not as obvious. One example connected with time-dependent random walks where we may profit from the theory of sums of random variables is the time of the r^{th} return to the origin. We say that a return to the origin occurs during $(t,t+dt)$ if $S(t) \neq 0$ and $S(t+dt) = 0$. If at each return to the origin the initial conditions are reproduced, the time up to the r^{th} return to the origin can be thought of as the sum of r recurrence times, that is, the total time elapsed during the course of r successive returns to the origin. Other examples where the theory of sums of random variables may apply to time-dependent random walks will be indicated as they arise.

We give here a summary of important definitions, examples, and results pertaining to the theory of stable distributions.

i) The only possible values of c_n in (2.6.1) are

$$c_n = n^{1/\alpha} \tag{2.6.2}$$

where $0 \leq \alpha \leq 2$; α is called the characteristic exponent or index of the stable distribution R.

For example, assume X_i to be independent and normally distributed with mean μ and variance σ^2. Then $S_n = X_1 + X_2 + \ldots + X_n$ is also normally distributed with mean $n\mu$ and variance $n\sigma^2$, i.e., $S_n \stackrel{d}{=} \sqrt{n}X + (n-\sqrt{n})\mu$. Therefore, for the normal distribution we have $c_n = n^{1/2}$, $\gamma_n = n - \sqrt{n}$, and $\alpha = 2$.

ii) All (broad sense) stable distributions are continuous.

iii) If R is stable with exponent $\alpha \neq 1$, then there exists a constant b such that R(X+b) is strictly stable, i.e., (2.6.1) holds with $\gamma_n = 0$.

For example, if X is normal with mean μ and variance σ^2, then $X - \mu$ is normal with mean zero and variance σ^2. Hence, $S_n' \overset{d}{=} \sqrt{n}X$ where $S_n' = \sum_{i=1}^{n} (X_i - \mu)$.

A basic notion in the theory of the limiting behavior of sums of random variables is that of the domain of attraction. The distribution F of the independent random variables X_1, X_2, ... is said to belong to the domain of attraction of a distribution R if there exist norming constants $a_n > 0$, b_n such that the distribution of $(S_n - b_n)/a_n$ tends to R with increasing n. See [19] for further discussion of this topic.

The connection between stable distributions and domains of attraction is revealed by the fact that a distribution R possesses a domain of attraction if and only if it is stable. It is known that a stable distribution with characteristic exponent α has absolute moments of all orders less than α. It follows from the Central Limit Theorem that a distribution with a finite variance belongs to the domain of attraction of the normal distribution. Together the two preceding statements imply that the exponents α are necessarily less than or equal to 2.

We shall not present here all the results which enable us to classify distributions according to their domains of attraction. Instead we confine attention to distribution functions F(x) which possess a probability density function f(x) since all the distribution functions we shall encounter in this dissertation will be of this type. The results summarized here depend on a function L that varies slowly at infinity (see(2.2.9) and subsequent remarks).

The two results most relevant to our work are as follows:

1) For fixed α, $0 < \alpha < 1$, the function $g_\alpha^*(z) = e^{-z^\alpha}$ is the Laplace transform of a probability density function $g_\alpha(x)$ with corresponding distribution function $G_\alpha(x)$ possessing the following properties: $G_\alpha(x)$ is stable; more precisely, if $X_1, X_2, \ldots, X_n$ are independent random variables with distribution function $G_\alpha(x)$, then $(X_1 + \ldots + X_n)/n^{1/\alpha}$ has the distribution function $G_\alpha(x)$. Moreover,

$$x^\alpha[1 - G_\alpha(x)] \to \frac{1}{\Gamma(1-\alpha)}, \quad x \to \infty, \tag{2.6.3}$$

and

$$e^{x^{-\alpha}} G_\alpha(x) \to 0, \quad x \to 0. \tag{2.6.4}$$

2) Suppose that F is a probability distribution concentrated on $(0,\infty)$ such that

$$F^{(n)}(a_n x) \to G(x), \quad \text{as } n \to \infty, \tag{2.6.5}$$

(at points of continuity) where $G(x)$ is a proper distribution function not concentrated at a single point, and $F^{(n)}(x)$ is defined by the equations

$$F^{(0)}(x) = 1$$

and

$$F^{(n)}(x) = \int_0^x F^{(n-1)}(x-s)dF(s) \quad .$$

Then

(a) there exists a function L that varies slowly at infinity such that

$$1 - F(x) \sim \frac{x^{-\alpha}L(x)}{\Gamma(1-\alpha)} \quad . \qquad (2.6.6)$$

(b) Conversely, if F is of the form (2.6.7), it is possible to choose constants a_n such that

$$\frac{nL(a_n)}{a_n^{\alpha}} \to 1 \ , \ n \to \infty, \qquad (2.6.7)$$

and in this case (2.6.5) holds with $G = G_\alpha$.

Result (2) implies that the possible limits G in (2.6.5) differ only by scale factors from some G_α.

An example of the stable distribution mentioned in result (1) is the stable distribution with characteristic exponent $\alpha = 1/2$ with parameter γ defined by

$$F_\gamma(x) = 2[1 - \Phi(\alpha/\sqrt{x})]$$

$$= \sqrt{\frac{2}{\pi}} \int_{\gamma/\sqrt{x}}^{\infty} e^{-s^2/2} ds \quad , \; x > 0. \qquad (2.6.8)$$

$F_\gamma(x)$ has probability density function $f_\gamma(x)$ given by

$$f_\gamma(x) = \frac{\gamma}{\sqrt{2\pi}} x^{-3/2} \; e^{-\gamma^2/(2x)}, \; x > 0. \qquad (2.6.9)$$

The Laplace transform of $f_\gamma(x)$ denoted by $f_\gamma^*(z)$ is given by

$$f_\gamma^*(z) = e^{-\gamma\sqrt{2z}} \; . \qquad (2.6.10)$$

Unfortunately, explicit expressions for the stable distributions $F_\alpha(x)$ have been obtained only for the two cases with characteristic exponents $\alpha = 1/2$ and $\alpha = 2$, respectively. The characteristic function $\phi_\alpha(z)$ of the distribution function $F_\alpha(x)$ can be obtained for general α, however, (see [19], for example) and is given by

$$\phi_\alpha(z) = \exp\{-|z|^\alpha(\cos\frac{\pi\alpha}{2} - i\sin\frac{\pi\alpha}{2}\text{sgn } z)\Gamma(1-\alpha)\} \quad (2.6.11)$$

where $\alpha \neq 1$ and $\phi_\alpha(z)$ is defined by

$$\phi_\alpha(z) = \int_0^\infty e^{izx} dF_\alpha(x) \; . \qquad (2.6.12)$$

Although the asymptotic behavior of certain random variables will be of concern in this dissertation, we have found it more convenient to develop the analysis from the exact formulae which are obtainable. Wherever applicable, however, we shall point out the connection between our work and the results of this Section.

CHAPTER III

FIRST-PASSAGE TIMES AND MAXIMA FOR THE RANDOM WALK WITH NEGATIVE EXPONENTIALLY DISTRIBUTED INTERVALS BETWEEN STEPS

3.1 Introduction

In this chapter we shall be concerned with the general class of problems connected with first-passage times. We say that the particle makes a first passage from state m to state n during the small time interval $(t, t+dt)$ if $S(0) = m$, $S(s) \neq n$, $0 < s \leq t$, and $S(t+dt) = n$ where $S(t)$ represents the position of the particle at time t. The probability of this event will be denoted by $f_{mn}(t)dt$. We define $F_{mn}(t)$ by the equation

$$F_{mn}(t) = 1 - \int_0^t f_{mn}(x)dx . \tag{3.1.1}$$

$F_{mn}(t)$ is the probability that given $S(0) = m$, no first passage to n occurs prior to or at time t. The corresponding Laplace transforms are defined by

$$\phi_{mn}(z) = \int_0^\infty e^{-zt} f_{mn}(t)dt; \quad \Phi_{mn}(z) = \int_0^\infty e^{-zt} F_{mn}(t)dt . \tag{3.1.2}$$

Intimately connected with the first-passage-time problem is the problem of the maximum value of $S(t)$ during the interval of time $(0,t)$. The expression "first maximum," extensively used by Feller [18], is perhaps ambiguous. In deference to usage we continue to employ it with the following meaning. Suppose that there exists a time t' in (0,t) such that:

(i) $S(t') > 0$;

(ii) $S(t') > S(x)$ for $0 \leq x < t'$; and

(iii) $S(t') \geq S(x)$ for $t' \leq x \leq t$.

Then we say that the particle attains its first maximum at time t' in the interval (0,t). If no first passage to a positive value (i.e. no first passage to +1) occurs during (0,t), we say that $t' = 0$.

In the subsequent sections we shall discuss in turn the topics:

(i) $f_{mn}(t)$, $F_{mn}(t)$;

(ii) the time of the r^{th} return to zero;

(iii) the number of returns to zero during (0,t); and

(iv) t', the time of the first maximum, and S(t').

Sections 3.2--3.5 deal with topics (i)--(iv) in connection with the time-dependent unrestricted random walk with negative exponentially distributed intervals between steps. In Sections 3.6--3.8 we shall treat topics (i)--(iii) for the M/M/1 queuing process described in Chapter I.

The results obtained in this Chapter are of interest by themselves. In addition, we shall refer to them often in Chapter IV. For example, it has been demonstrated by E. Sparre Andersen [1] and [2] in connection with non-time-dependent random walks that a strong relationship exists between the problem of the maximum partial sum S_ν among S_1, $S_2, \ldots, S_n$ and that of the particle's sojourn in a given state. It will be shown in Chapter IV that an analogous relationship holds in the case of the time dependent walks under study here.

We turn now to the discussion of $f_{mn}(t)$.

3.2 First-Passage Time from m to n for the Unrestricted Random Walk with Negative Exponentially Distributed Intervals between Steps

We now proceed to derive $f_{mn}(t)$ for the case $m \neq n$. The case $m = n$ will be considered in Section 3.3. Because of the assumption that steps to the right and to the left belong to two independent Poisson streams with parameters λ and μ, respectively, the probability that a positive or negative step occurs at any time is independent of the value of $S(t)$, the displacement of the particle at time t. Moreover, a first passage from state m to state n is equivalent to a first passage to a point $|n-m|$ units from the starting point in either a positive or negative direction, according to whether or not $n - m$ is positive or negative. Because the particle may move an unlimited number of steps in either direction, whatever its starting point, the probability of moving $|n-m|$ steps in a positive or negative direction is independent of the fact that $S(0) = m$. Therefore, for the unrestricted walk we have

$$f_{mn}(t) = f_{0,n-m}(t) \tag{3.2.1}$$

which implies that

$$f_{n0}(t) = f_{0,-n}(t) \ . \tag{3.2.2}$$

From considerations of symmetry it is evident that $f_{0,-n}(t) = f_{0,n}(t)$ except for an interchange of the parameters, λ and μ, of the underlying Poisson distributions. The foregoing considerations imply that it suffices to consider only $f_{0m}(t)$ in order to derive the density function $f_{mn}(t)$.

We begin by deriving $f_{01}(t)$. In order that a first passage to +1 occur during $(t,t+dt)$, the particle must either remain at zero throughout $(0,t)$ and take a positive step during $(t,t+dt)$, the probability of this event being $e^{-(\lambda+\mu)t}\lambda dt$; or it may remain at zero until a negative step occurs during the small interval $(s,s+ds)$, the probability of this event being $e^{-(\lambda+\mu)s}\mu ds$, and then make a first passage from -1 to +1 in the remaining time $t - s$ when $0 < s < t$. This leads to the equation

$$f_{01}(t) = \lambda e^{-(\lambda+\mu)t} + \mu\int_0^t e^{-(\lambda+\mu)s} f_{-1,1}(t-s)ds \ . \tag{3.2.3}$$

By (3.2.1) we have

$$f_{01}(t) = \lambda e^{-(\lambda+\mu)t} + \mu\int_0^t e^{-(\lambda+\mu)s} f_{0,2}(t-s)ds \ . \tag{3.2.4}$$

For n = 2,3,..., a first passage to n must be preceded by a first passage to n - 1. Thus we can write

$$f_{01}(t) = \int_0^t f_{0,n-1}(t-s)\, f_{01}(s)ds \ , \ n = 2,3,.... \quad (3.2.5)$$

From (3.2.5) it then follows by induction that

$$f_{0n}(t) = f_{01}^{(n)}(t) \ , \ n = 1,2,... \ . \quad (3.2.6)$$

where

$$f_{01}^{(n)}(t) = \int_0^t f_{01}^{(n-1)}(t-s)\, f_{01}(s)ds \ , \ n = 2,3,..., \quad (3.2.7)$$

and

$$f_{01}^{(1)}(t) = f_{01}(t) \ . \quad (3.2.8)$$

Therefore $f_{02}(t) = f_{01}^{(2)}(t)$ and using the convolution property of the Laplace transform given by (2.2.7), it follows from (3.2.4) that

$$\mu\, [\phi_{01}(z)]^2 - Z\phi_{01}(z) + \lambda = 0 \quad (3.2.9)$$

where

$$Z = \lambda + \mu + z \ . \quad (3.2.10)$$

Since $\int_0^\infty f(t)dt = \phi_{01}(0) \leq 1$, $\phi_{01}(z)$ must be equal to the smaller root of (3.2.9) and hence

$$\phi_{01}(z) = \frac{Z-R}{2\mu} = \frac{2\lambda}{Z+R} \quad (3.2.11)$$

where

$$R^2 = Z^2 - 4\lambda\mu \ . \quad (3.2.12)$$

From (3.2.6) it follows that

$$\phi_{0n}(z) = \left(\phi_{01}(z)\right)^n = \left(\frac{2\lambda}{Z+R}\right)^n, \quad n = 1,2,3,\ldots. \tag{3.2.13}$$

Upon inversion of (3.2.13) using [17], Vol. I, (28), p. 240, it follows that

$$f_{0n}(t) = \rho^{n/2} \frac{nI_n(2t\sqrt{\lambda\mu})}{t} e^{-(\lambda+\mu)t}, \quad n = 1,2,\ldots, \tag{3.2.14}$$

where

$$\rho = \frac{\lambda}{\mu}. \tag{3.2.15}$$

By symmetry we have

$$\begin{aligned} f_{n0}(t) &= \rho^{-n/2} \frac{nI_n(2t\sqrt{\lambda\mu})}{t} e^{-(\lambda+\mu)t} \\ &= \rho^{-n} f_{0n}(t). \end{aligned} \tag{3.2.16}$$

Equation (3.2.14) has been derived by Feller [19] by extending a discrete-time result to continuous time with the aid of conditional probabilities. This technique will be illustrated in Chapter V.

Expressions for $F_{n0}(t)$ and $F_{0n}(t)$ can be obtained from (3.1.1), (3.2.14) and (3.2.16) and are given by

$$\begin{aligned} F_{n0}(t) &= 1 - n\rho^{-n/2} \int_0^t e^{-(\lambda+\mu)x} I_n(2x\sqrt{\lambda\mu}) \frac{dx}{x} \\ &= 1 - \rho^{-n} + \rho^{-n} F_{0n}(t). \end{aligned} \tag{3.2.17}$$

From (3.1.1), (3.1.2) and (3.2.13) we obtain

$$\Phi_{0n}(z) = \frac{1 - \phi_{0n}(z)}{z} = \frac{1}{z}\left[1 - \left(\frac{2\lambda}{Z+R}\right)\right]^n \quad (3.2.18)$$

and

$$\Phi_{0,n+1}(z) - \Phi_{0,n}(z) = \phi_{0n}(z)\ \Phi_{01}(z)\ , \quad (3.2.19)$$

a result which will be used in Chapter IV. Corresponding expressions for $\phi_{n0}(z)$ and $\Phi_{n0}(z)$ follow by symmetry.

In order to make use of the asymptotic properties of the J-function and the modified Bessel function we now proceed to derive alternate expressions for $F_{01}(t)$ and $F_{10}(t)$. We do this by appealing again to Maximon's result given in (2.4.11). Letting $x = \lambda t$ and $y = \mu t$ in (2.4.11), we obtain

$$J(\lambda t,\mu t) = 1 + \int_0^t e^{-(\lambda+\mu)x}\,[\sqrt{\lambda\mu}I_1(2x\sqrt{\lambda\mu}) - \lambda I_0(2x\sqrt{\lambda\mu})]dx\ , \quad (3.2.20)$$

and

$$J_2(\lambda t,\mu t) = \int_0^t e^{-(\lambda+\mu)x}\,[\mu\rho^{-1/2}I_1(2x\sqrt{\lambda\mu}) - \lambda\rho^{-1}I_2(2x\sqrt{\lambda\mu})]dx\ , \quad (3.2.21)$$

where $J_n(x,y)$ is defined in (2.4.1). Upon combining (3.2.20) and (3.2.21) with (2.3.9), we find that

$$J(\lambda t,\mu t) - \rho J_2(\lambda t,\mu t) = 1 - \rho^{1/2}\int_0^t e^{-(\lambda+\mu)x}I_1(2x\sqrt{\lambda\mu})\frac{dx}{x}$$
$$= F_{01}(t)\ . \quad (3.2.22)$$

By (3.2.17) and (3.2.22) we have

$$F_{10}(t) = 1 - \rho^{-1} + \rho^{-1}J(\lambda t, \mu t) - J_2(\lambda t, \mu t) \ . \quad (3.2.23)$$

It is clear from (3.2.22) and (3.2.23) that the asymptotic properties of the probabilities $F_{01}(t)$ and $F_{10}(t)$ can be deduced from the asymptotic behavior of $J_n(x,y)$. We can deduce the limiting behavior of $J_n(x,y)$ by the use of equations (2.4.10), (2.3.2) and (2.4.5). For completeness and for later use, these results can be summarized as follows:

$$F_{10}(t) \sim \frac{4e^{-\mu(1-\sqrt{\rho})^2 t}}{(1-\sqrt{\rho})^2 \ (2\pi\xi)^{1/2}} \left[\frac{1}{(2\xi)} + 0\left(\frac{1}{4\xi^2}\right)\right], \ \rho < 1,$$

$$(3.2.24)$$

$$\sim (1-\rho)^{-1} + \frac{4e^{-\mu(1-\sqrt{\rho})^2 t}}{(1-\sqrt{\rho})^2 \ (2\pi\xi)^{1/2}} \left[\frac{1}{(2\xi)} + 0\left(\frac{1}{4\xi^2}\right)\right], \ \rho > 1,$$

where

$$\xi = 2t\sqrt{\lambda\mu}, \ \rho = \frac{\lambda}{\mu} \ . \quad (3.2.25)$$

The corresponding result for $F_{0,1}(t)$ follows from (3.2.17).

To obtain asymptotic results for $F_{0,n}(t)$ and $F_{n,0}(t)$, it is more convenient to proceed by direct methods rather than using the results of Chapter II. From (3.2.13) it follows that

$$\int_0^\infty f_{0n}(x)\,dx = \rho^n \ , \ \rho < 1, \quad (3.2.26)$$
$$= 1, \ \rho \geq 1,$$

and hence

$$F_{0n}(t) = 1 - \rho^n + \int_t^\infty f_{0n}(x)dx \ , \quad \rho < 1 \ ,$$

$$= \int_t^\infty f_{0n}(x)dx \ , \quad \rho \geq 1 \ . \tag{3.2.27}$$

We need then to determine the asymptotic behavior of $\int_t^\infty f_{0n}(x)dx$. From (3.2.14) and the asymptotic properties of $I_n(x)$ given in (2.3.2), we have

$$\int_t^\infty f_{0n}(x)dx = n\rho^{n/2} \int_t^\infty e^{-(\lambda+\mu)x} \, I_n(2x\sqrt{\lambda\mu}) \frac{dx}{x}$$

$$\sim \frac{n\rho^{n/2}}{2\sqrt{\pi}(\lambda\mu)^{1/4}} \int_t^\infty e^{-\mu(1-\sqrt{\rho})^2 x} \left[x^{-3/2} + 0(x^{-5/2})\right] dx, \tag{3.2.28}$$

as $t \to \infty$.

Upon substituting $y = x - t$ and expanding the denominator of (3.2.28) in powers of (x/t) by means of the binomial expansion, we find that

$$\int_t^\infty f_{0n}(x)dx \sim \frac{4n\rho^{\frac{n+1}{2}} e^{-\mu(1-\sqrt{\rho})^2 t}}{(1-\sqrt{\rho})^2 \ (2\pi\xi)^{1/2}} \left[\frac{1}{(2\xi)} + 0\left(\frac{1}{4\xi^2}\right)\right] \tag{3.2.29}$$

where ξ and ρ are defined in (3.2.25).

Substituting (3.2.29) into (3.3.37) we obtain

$$F_{0n}(t) \sim 1 - \rho^n + \frac{4n\rho^{(n+1)/2} e^{-\mu(1-\sqrt{\rho})^2 t}}{(1-\sqrt{\rho})^2 \ (2\pi\xi)^{1/2}} \left[\frac{1}{(2\xi)} + 0\left(\frac{1}{4\xi^2}\right)\right], \ \rho < 1,$$

$$\frac{4n\rho^{(n+1)/2} e^{-\mu(1-\sqrt{\rho})^2 t}}{(1-\sqrt{\rho})^2 \ (2\pi\xi)^{1/2}} \left[\frac{1}{(2\xi)} + 0\left(\frac{1}{4\xi^2}\right)\right], \ \rho > 1 . \tag{3.2.30}$$

The corresponding results for $F_{n0}(t)$ follow from (3.2.17).

We note from (3.2.30) that $F_{0n}(t)$ tends to $1 - \rho^n$ for $\rho < 1$ but tends to zero as $t \to \infty$ when $\rho > 1$. Since $F_{0n}(t)$ represents the probability that first passage to n occurs after time t, we would expect that it tend to zero with increasing t when $\rho > 1$, i.e., when there is a drift of the random walk toward $+\infty$. The result given in (3.2.30) supports our intuition. On the other hand, when $\rho < 1$, there is a drift of the random walk toward $-\infty$; and the fact that $F_{0n}(t)$ tends to a non-zero limit in this case, i.e., there is a non-zero probability that the particle never passes to +n, is not surprising. It is also interesting to note that the probability $1 - \rho^n$ that passage to +n never occurs tends to unity as n increases for $\rho < 1$. The case $\lambda = \mu$, or $\rho = 1$, is particularly interesting, since we would expect no tendency to drift in either direction. We examine this case in detail next.

When $\lambda = \mu$, our results take on a pleasing form. From (3.2.16) we see that $f_{0n}(t) = f_{n0}(t)$ and hence, $F_{0n}(t) = F_{n0}(t)$ when $\rho = 1$. In particular,

$$F_{0n}(t) = 1 - n\int_0^t e^{-2\mu x}\, I_n(2\mu x)\frac{dx}{x} \tag{3.2.31}$$

$$= e^{-2\mu t}\left[I_0(2\mu t) + 2\sum_{j=1}^{n-1} I_j(2\mu t) + I_n(2\mu t)\right], \tag{3.2.32}$$

$$n = 2,3,4,\ldots,$$

and

$$F_{01}(t) = e^{-\mu t}\left[I_0(2\mu t) + I_1(2\mu t)\right] . \tag{3.2.33}$$

Equations (3.2.32) and (3.2.33) follow from standard results for integrals of Bessel functions, given, among other places, in [37], Section 11,3. From (3.2.32) and (3.2.33) it follows that

$$F_{0,n+1}(t) - F_{0n}(t) = e^{-2\mu t}\left[I_n(2\mu t) + I_{n+1}(2\mu t)\right] \text{ , } n = 1,2,\ldots. \tag{3.2.34}$$

From (3.2.32) and the asymptotic properties of the modified Bessel function of order n given by (2.3.2), we find that for $\rho = 1$, as $t \to \infty$,

$$F_{0n}(t) = F_{n0}(t) \sim \left[\frac{n}{(\pi\mu t)^{1/2}} + 0\left(\frac{1}{(\mu t)^{3/2}}\right)\right]. \tag{3.2.35}$$

From (3.2.35) it is evident that both $F_{0n}(t)$ and $F_{n0}(t)$ tend to zero as $t \to \infty$. Since $F_{0n}(t)$ represents the probability that a first passage to +n occurs after time t and $F_{n0}(t)$ is equal to the probability that a first passage to -n occurs after time t, we expect, based on (3.2.35), that the particle eventually makes a first passage to all finite states $\pm n$, $n = 1,2,\ldots$. Therefore, in a soccer game between two evenly matched teams, as described in Chapter I, we would expect that if the two teams played long enough, we might eventually witness any given score difference $\pm n$, $n = 1,2,\ldots$. This result is perhaps not intuitively obvious although consistent with the results for the so-called coin-tossing random walk discussed by Feller [18], Chapter III.

In order that first passage to both $\pm n$ eventually occur, it is evident that the particle must cross the origin at least once. This consideration raises the question as to how fast the path of the random walk oscillates between positive and negative values. At least a partial answer to this question can be obtained by studying the time between returns to the origin, which we shall discuss in the next Section.

3.3 The Time of Occurrence of the r^{th} Return to Zero for the Unrestricted Random Walk with Negative Exponentially Distributed Intervals Between Steps

One measure of the nature of the oscillation of the particle is the amount of time which elapses between returns to zero. We say that a first return to zero occurs at time t if $S(0) = S(t) = 0$ and $S(s) \neq 0$, $0 < s < t$. We let T_r denote the time up to the r^{th} return to the origin and define $f_r(t)$ by

$$f_r(t)dt = \Pr\{t < T_r < t + dt \mid S(0) = 0\} . \quad (3.3.1)$$

In this Section we shall discuss the distributional properties of T_r which will yield information concerning the time of occurrence of the r^{th} tie in a soccer game where $S(t)$, in this case, represents the score difference between the two teams at time t. We proceed now to derive $f_r(t)$.

In order that a first return to zero take place at time t, the particle must take either a positive or negative step at some time s, $0 < s < t$, and then make a first return to zero from +1 or -1 at time $t - s$. This fact is expressed by the equation

$$f_1(t) = \int_0^t \{\lambda e^{-(\lambda+\mu)s} f_{10}(t-s) + \mu e^{-(\lambda+\mu)s} f_{01}(t-s)\}ds \; . \tag{3.3.2}$$

From (3.2.11) and symmetry considerations, it follows that

$$\phi_1(z) = \frac{Z-R}{Z} \tag{3.3.3}$$

where

$$\phi_r(z) = \int_0^\infty e^{-zt} f_r(t)dt \; . \tag{3.3.4}$$

and Z,R are defined by (3.2.10) and (3.2.12), respectively. For integer values of r greater than unity, it is evident that an r^{th} return to zero must be preceded by $r - 1$ returns to zero. Therefore we can write

$$f_r(t) = \int_0^t f_{r-1}(t-s) \; f_1(s)ds \tag{3.3.5}$$

which, by the convolution property of the Laplace transform, implies that

$$\phi_r(z) = \phi_{r-1}(z)\phi_1(z) \quad , \; r = 2,3,4,\ldots. \tag{3.3.6}$$

By induction, starting from (3.3.3), we have

$$\phi_r(z) = \left(\frac{Z-R}{Z}\right)^r = \left(\frac{4\lambda\mu}{Z(Z+R)}\right)^r \quad , \; r = 1,2,3,\ldots. \tag{3.3.7}$$

From [17], Vol. I, (28), p. 240, it follows that

$$f_r(t) = \frac{r(2\sqrt{\lambda\mu})^r}{(r-1)!} e^{-(\lambda+\mu)t} \int_0^t (t-s)^{r-1} I_r(2s\sqrt{\lambda\mu})\frac{ds}{s} \quad , \tag{3.3.8}$$

$$r = 1,2,3,\ldots.$$

From (3.3.7) we have

$$\frac{d\phi_r(z)}{dz} = r\left(\frac{Z-R}{Z}\right)^{r-1}\left(\frac{-4\lambda\mu}{Z^2R}\right). \tag{3.3.9}$$

From (3.3.7) and (3.3.9) we see that

$$\begin{aligned} \phi_r(0) = \int_0^\infty f_r(t)dt &= (2p)^r \quad , \; \rho < 1 \\ &= 1 \qquad\quad , \; \rho = 1 \\ &= (2q)^r \quad , \; \rho > 1 \end{aligned} \tag{3.3.10}$$

and

$$\begin{aligned} \frac{E(T_r)}{\phi_r(0)} = \int_0^\infty tf_r(t)dt &= \frac{2qr}{\mu(1-\rho)} \quad , \; \rho < 1 \\ &= \infty \qquad\qquad , \; \rho = \\ &= \frac{2pr}{\mu(\rho-1)} \quad , \; \rho > 1, \end{aligned} \tag{3.3.11}$$

where

$$p = \frac{\lambda}{\lambda+\mu}\ ,\ q = \frac{\mu}{\lambda+\mu}\ ,\ \rho = \frac{\lambda}{\mu}\ . \tag{3.3.12}$$

By (3.3.10) a return to zero is certain only when $\rho = 1$, but the mean recurrence time in this case is infinite as shown in (3.3.11). This result is identical to the corresponding result for the coin-tossing random walk described in [18], Chapter III. This similarity is not surprising, for it can be shown that

$$\int_0^\infty t^k f_r(t)\,dt = \frac{1}{(\lambda+\mu)^k} \sum_{n=0}^{\infty} \frac{\Gamma(n+k)}{\Gamma(n)} \Pr\{T_r = n\}$$

where $\Pr\{T_r = n\}$ represents the probability that the r^{th} return to zero occurs on the n^{th} step.

This completes the derivation of $f_r(t)$ and some of the important properties connected with it. The result in (3.3.8) does not seem to have been noted previously. In order to make some deductions about T_r for large values of time, we deal next with the asymptotic behavior of $f_r(t)$.

In order to determine the limiting behavior of $f_r(t)$ as $t \to \infty$, we shall express $f_r(t)$ in terms of the generalized hypergeometric function defined by equation (2.5.2). Then we can utilize the asymptotic properties of this function summarized in equations (2.5.4)--(2.5.7). First we need a result from [17], Vol. II, which states that

$$\frac{1}{\Gamma(m)} \int_0^y x^{n-1}(y-x)^{m-1} J_\nu(\alpha x)\,dx = \frac{\Gamma(\nu+n)}{\Gamma(\nu+1)\Gamma(m+n+\nu)}\left(\frac{\alpha}{2}\right)^\nu \cdot \tag{3.3.13}$$

$$\cdot\; y^{m+n+\nu-1}\; {}_2F_3\left[\frac{n+\nu}{2}, \frac{n+\nu+1}{2}; \nu+1, \frac{m+n+\nu}{2}, \frac{m+n+\nu+1}{2}; -\frac{\alpha^2 y^2}{4}\right],$$

where $\mathrm{Re}(m) > 0$, $\mathrm{Re}(n+\nu) > 0$, and $J_\nu(x)$ is defined by equation (2.3.4). Equation (3.2.5) implies that (3.3.13) is equivalent to

$$\frac{1}{\Gamma(m)} \int_0^y x^{n-1}(y-x)^{m-1} I_\nu(\alpha x)\,dx = \frac{\Gamma(\nu+n)}{\Gamma(\nu+1)\Gamma(m+n+\nu)}\left(\frac{\alpha}{2}\right)^\nu y^{m+n+\nu-1} \cdot$$

$$\cdot\; {}_2F_3\left[\frac{n+\nu}{2}, \frac{n+\nu+1}{2}; \nu+1, \frac{m+n+\nu}{2}, \frac{m+n+\nu+1}{2}; \frac{\alpha^2 y^2}{4}\right]. \tag{3.3.14}$$

From (3.3.8) and (3.3.14) we then have

$$f_r(t) = \frac{(2\lambda\mu)^r t^{2r-1}}{\Gamma(2r)}\; e^{-(\lambda+\mu)t}\; {}_2F_3\left[\frac{r}{2}, \frac{r+1}{2}; r+1, r, r+\frac{1}{2}; \lambda\mu t^2\right]. \tag{3.3.15}$$

This expression for $f_r(t)$ is new. Using the asymptotic properties of the generalized hypergeometric function given in (2.5.4) - (2.5.7), it follows from (3.3.15) that as $t \to \infty$,

$$f_r(t) \sim \frac{2^{r-1}\Gamma(r+1)\Gamma(r)\Gamma\left(r+\frac{1}{2}\right)}{\Gamma(2r)\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{r+1}{2}\right)}\;\frac{e^{-\mu(1-\sqrt{\rho})^2 t}}{\sqrt{\pi}(\lambda\mu)^{1/4}t^{3/2}} + O(rt^{-5/2}) . \tag{3.3.16}$$

Using induction, or directly using Gamma function formulae, it can be shown that

$$\frac{\Gamma(r+1)\Gamma(r)\Gamma\left(r+\frac{1}{2}\right)}{\Gamma(2r)\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{r+1}{2}\right)} = \frac{r}{2^r}\ , \quad r = 1,2,\ldots, \tag{3.3.17}$$

and, therefore, as $t \to \infty$, we have

$$f_r(t) \sim \frac{re^{-\mu(1-\sqrt{\rho})^2 t}}{2\sqrt{\pi}(\lambda\mu)^{1/4}t^{3/2}} + 0(rt^{-5/2})\ . \tag{3.3.18}$$

To obtain an approximate value for $\Pr\{T_r > t\}$ for large t, we use (3.3.18) and the relation which follows from (3.3.10), namely,

$$\begin{aligned}
\Pr\{T_r > t\} &= 1 - \int_0^t f_r(x)\,dx \\
&= 1 - (2p)^r + \int_t^\infty f_r(x)\,dx\ , \quad \rho < 1 \\
&= \int_t^\infty f_r(x)\,dx \qquad\qquad\qquad , \quad \rho = 1 \\
&= 1 - (2q)^r + \int_t^\infty f_r(x)\,dx\ , \quad \rho > 1,
\end{aligned} \tag{3.3.19}$$

where p and q are given in (3.3.12). After integrating (3.3.18) by parts and combining with (3.3.19), we obtain the result

$$\Pr\{T_r > t\} \sim 1 - (2p)^r + \frac{re^{-\mu(1-\sqrt{\rho})^2 t}}{(\pi t)^{1/2}(\lambda\mu)^{1/4}} -$$

$$- r\rho^{-1/4}\ (1-\sqrt{\rho})\left(\frac{2}{\pi}\right)^{1/2} \int_{\sqrt{2\mu t}(1-\sqrt{\rho})}^{\infty} e^{-y^2/2}dy +$$

$$+ 0\left(\frac{r}{t^{3/2}}\right), \ \rho < 1,$$

$$\sim \frac{r}{(\pi\mu t)^{1/2}} + 0(rt^{-3/2}) \ , \ \rho = 1,$$

$$\sim 1 - (2q)^r + \frac{re^{-\mu(1-\sqrt{\rho})^2 t}}{(\pi t)^{1/2}(\lambda\mu)^{1/4}} -$$

$$- r\rho^{-1/4} \ (\sqrt{\rho}-1) \ \left(\frac{2}{\pi}\right)^{1/2} \int_{\sqrt{2\mu t}(1-\sqrt{\rho})}^{\infty} e^{-y^2/2}dy +$$

$$+ 0(rt^{-3/2}) \ , \ \rho > 1 \ . \tag{3.3.20}$$

In terms of a soccer game like that described in Chapter I, $Pr\{T_r > t\}$ represents the probability that the time up to the r^{th} tie is greater than t. We see from (3.3.20) that for $\rho \neq 1$, as $t \to \infty$, $Pr\{T_r > t\}$ tends to the probability that the r^{th} tie never occurs. When $\rho = 1$, however, $Pr\{T_r > t\}$ tends to zero as $t \to \infty$. Moreover, (3.3.20) shows that during a long game of length t between two evenly matched teams, the probability that the r^{th} tie does not occur is proportional to $rt^{-1/2}$.

From the form of the Laplace transform of $f_r(t)$ given in (3.3.7), we conclude that T_r is the sum of r independent random variables, each having the distribution of T_1 with probability density function $f_1(t)$. In order to apply the results concerning stable distributions described in Section 2.6, the asymptotic form of $1 - F_1(t) = Pr\{T_1 > t\}$ must be expressible in terms of a slowly varying function $L(t)$ defined in (2.2.9). From (3.3.20) we see that $\Pr\{T_1 > t\}$ cannot be expressed in this way when $\rho \neq 1$, since the negative exponential function does not vary slowly at ∞. On the other hand, when $\rho = 1$, we have

$$1 - F_1(t) = \Pr\{T_1 > t\} \sim \frac{t^{-1/2}}{\Gamma\left(\frac{1}{2}\right)} L(t) \qquad (3.3.21)$$

where

$$L(t) = \frac{1}{\sqrt{\mu}} \; ; \; F_1(t) = \int_0^t f_1(x)\,dx \; . \qquad (3.3.22)$$

Obviously $L(t) = 1/\sqrt{\mu}$ is slowly varying and, therefore, we can apply result (2) of Section 2.6 and conclude that when $\rho = 1$, there exist norming constants a_r such that

$$F_r(a_r t) \to G_{\frac{1}{2}}(t) \text{ as } r \to \infty, \qquad (3.3.23)$$

where $G_{\frac{1}{2}}(t)$ is the stable distribution of order $\frac{1}{2}$ given by (2.6.8) and

$$F_r(t) = \int_0^t f_r(x)\,dx \; . \qquad (3.3.24)$$

Although the same results do not apply when $\rho \neq 1$, it is interesting to note that the result for $\rho \neq 1$ given in (3.3.20) does contain a term which has the form of the stable distribution of order $\frac{1}{2}$ given in (2.6.8).

To indicate the usefulness of (3.3.18) as an approximation to $f_r(t)$, we present Tables I and II, giving true values and approximate values of $f_r(t)$ based on (3.3.18). In each Table we let $\mu = 1$ and give the exact and approximate values of $f_r(t)$ for $t = 10$, 20, and 30. Under each value of t the exact and approximate values are listed side by side for the values of ρ given in the first column and the values of r given in the second column. The numbers are believed to be correct to one unit of the final decimal place.

We would expect, based on (3.3.18), that the approximation of $f_r(t)$ should be most accurate when the quantity $rt^{-5/2}$ is small. Tables I and II bear this out. We note that for fixed r, the approximation of $f_r(t)$ tends to the exact value as t increases. We also note that convergence to the exact value as t increases is slower for larger values of r. This is consistent with the fact that the accuracy depends on $rt^{-5/2}$ being small.

TABLE I

EXACT AND APPROXIMATE VALUES OF $f_r(t)$: $\rho = .6, .8$

		t = 10		t = 20		t = 30	
ρ	r	Exact	Approx.	Exact	Approx.	Exact	Approx.
0.6	1	0.00663	0.00609	0.00134	0.00129	0.00043	0.00042
0.6	2	0.01326	0.01219	0.00269	0.00259	0.00087	0.00084
0.6	3	0.01794	0.01829	0.00389	0.00389	0.00127	0.00127
0.6	4	0.01884	0.02439	0.00477	0.00518	0.00161	0.00169
0.6	5	0.01572	0.03049	0.00523	0.00648	0.00187	0.00212
0.6	6	0.01047	0.03658	0.00521	0.00778	0.00201	0.00254
0.6	7	0.00561	0.04268	0.00474	0.00908	0.00204	0.00297
0.6	8	0.00243	0.04878	0.00394	0.01037	0.00195	0.00339
0.6	9	0.00086	0.05488	0.00300	0.01167	0.00177	0.00382
0.6	10	0.00025	0.06098	0.00208	0.01297	0.00153	0.00424
0.8	1	0.00905	0.00843	0.00275	0.00266	0.00132	0.00129
0.8	2	0.01811	0.01687	0.00551	0.00533	0.00265	0.00259
0.8	3	0.02505	0.02531	0.00800	0.00800	0.00390	0.00389
0.8	4	0.02777	0.03375	0.00996	0.01067	0.00498	0.00519
0.8	5	0.02544	0.04218	0.01117	0.01334	0.00583	0.00649
0.8	6	0.01934	0.05069	0.01151	0.01601	0.00693	0.00779
0.8	7	0.01220	0.05906	0.01098	0.01867	0.00663	0.00909
0.8	8	0.00641	0.06750	0.00973	0.02134	0.00655	0.01039
0.8	9	0.00281	0.07593	0.00801	0.02401	0.00619	0.01169
0.8	10	0.00104	0.08437	0.00611	0.02668	0.00559	0.01299

TABLE II

EXACT AND APPROXIMATE VALUES OF $f_r(t)$: ρ = 1.0, 1.2, 1.4

		t = 10		t = 20		t = 30	
ρ	r	Exact	Approx.	Exact	Approx.	Exact	Approx.
1.0	1	0.00949	0.00892	0.00324	0.00315	0.00175	0.00171
1.0	2	0.01899	0.01784	0.00646	0.00630	0.00350	0.00343
1.0	3	0.02659	0.02676	0.00946	0.00946	0.00515	0.00515
1.0	4	0.03041	0.03568	0.01189	0.01261	0.00662	0.00686
1.0	5	0.02943	0.04460	0.01352	0.01576	0.00781	0.00858
1.0	6	0.02426	0.05352	0.01423	0.01892	0.00866	0.01030
1.0	7	0.01702	0.06244	0.01397	0.02207	0.00912	0.01201
1.0	8	0.01016	0.07136	0.01286	0.02523	0.00919	0.01373
1.0	9	0.00516	0.08028	0.01111	0.02838	0.00888	0.01545
1.0	10	0.00224	0.08920	0.00899	0.03153	0.00826	0.01716
1.2	1	0.00823	0.00778	0.00257	0.00251	0.00127	0.00124
1.2	2	0.01646	0.01556	0.00515	0.00502	0.00254	0.00249
1.2	3	0.02324	0.02334	0.00754	0.00753	0.00374	0.00374
1.2	4	0.02712	0.03112	0.00952	0.01004	0.00483	0.00499
1.2	5	0.02721	0.03890	0.01095	0.01255	0.00573	0.00624
1.2	6	0.02368	0.04668	0.01169	0.01506	0.00640	0.00748
1.2	7	0.01787	0.05446	0.01171	0.01758	0.00682	0.00873
1.2	8	0.01168	0.06224	0.01106	0.02009	0.00696	0.00998
1.2	9	0.00661	0.07002	0.00987	0.02660	0.00684	0.01123
1.2	10	0.00324	0.07781	0.00832	0.02511	0.00649	0.01248
1.4	1	0.00617	0.00586	0.00151	0.00148	0.00058	0.00057
1.4	2	0.01234	0.01172	0.00303	0.00296	0.00117	0.00115
1.4	3	0.01753	0.01758	0.00444	0.00444	0.00173	0.00172
1.4	4	0.02076	0.02344	0.00564	0.00592	0.00223	0.00230
1.4	5	0.02135	0.02931	0.00654	0.00740	0.00266	0.00288
1.4	6	0.01931	0.03517	0.00705	0.00888	0.00299	0.00345
1.4	7	0.01537	0.04103	0.00717	0.01037	0.00321	0.00403
1.4	8	0.01075	0.04689	0.00690	0.01185	0.00332	0.00461
1.4	9	0.00659	0.05276	0.00631	0.01333	0.00330	0.00518
1.4	10	0.00354	0.05862	0.00547	0.01481	0.00318	0.00576

3.4 The Number of Returns to Zero During (0,t) for the Unrestricted Random Walk with Negative Exponentially Distributed Intervals Between Steps

In the previous Section we were concerned with the distributional properties of T_r, the time of occurrence of the r^{th} return to zero. In this section we shall discuss the random variable $N(t)$, which will denote the number of returns to zero during an arbitrary time interval $(0,t)$. As an application of our results, we shall be able to give probabilistic information about the number of ties that occur during a soccer game of length t. A tie at a given time s is, of course, equivalent to the event $S(s) = 0$.

First of all, we shall derive an expression for

$$k_n(t) = \Pr\{N(t) = n\} . \tag{3.4.1}$$

We do this by using a well known and obvious relationship between T_r and $N(t)$, namely,

$$\Pr\{T_n \leq t\} = \Pr\{N(t) \geq n\} , \quad n = 1,2,3,.... \tag{3.4.2}$$

Let $K(x,z)$ denote the Laplace transform of the generating function of $k_n(t)$ defined by

$$K(x,z) = \sum_{n=0}^{\infty} x^n k_n^*(z) \tag{3.4.3}$$

where

$$k_n^*(z) = \int_0^\infty e^{-zt} k_n(t)dt . \tag{3.4.4}$$

As a consequence of (3.4.2), we have

$$\frac{1}{z} + \sum_{n=1}^{\infty} x^n \int_0^\infty e^{-zt} \Pr\{T_n \le t\}dt = \frac{\frac{1}{z} - xK(x,z)}{1 - x} . \tag{3.4.5}$$

From (3.3.7) it follows that

$$\left[z\{1 - x\left(1 - \frac{R}{Z}\right)\}\right]^{-1} = \frac{\frac{1}{z} - xK(x,z)}{1 - x} , \tag{3.4.6}$$

and hence,

$$K(x,z) = \frac{R}{z[Z - x(Z-R)]} \tag{3.4.7}$$

where

$$Z = \lambda + \mu + z , \quad R^2 = Z^2 - 4\lambda\mu . \tag{3.4.8}$$

Combining (3.4.3) and (3.4.7) we obtain

$$k_n^*(z) = \frac{R(Z-R)^n}{zZ^{n+1}} , \quad n = 0,1,2,.... \tag{3.4.9}$$

By (3.4.8) it follows that $k_n^*(z)$ can be written as

$$k_n^*(z) = \frac{(4\lambda\mu)^n}{zRZ^{n-1}(Z+R)^n} - \frac{(4\lambda\mu)^{n+1}}{zRZ^{n+1}(Z+R)^n} , \tag{3.4.10}$$

and upon inversion, we obtain

$$k_n(t) = \frac{(2\sqrt{\lambda\mu})^n}{(\lambda+\mu)^{n-1}} \int_0^t e^{-(\lambda+\mu)x}\, I_n(2x\sqrt{\lambda\mu}) \times$$

$$\times\left(\frac{\gamma[(n-1),(\lambda+\mu)(t-x)]}{\Gamma(n-1)} - \frac{4\lambda\mu}{(\lambda+\mu)^2}\,\frac{\gamma[(n+1),(\lambda+\mu)(t-x)]}{\Gamma(n+1)}\right)dx\ ,$$

$$n = 2,3,\ldots; \qquad (3.4.11)$$

$$k_1(t) = 2\sqrt{\lambda\mu} \int_0^t e^{-(\lambda+\mu)x}\, I_1(2x\sqrt{\lambda\mu}) \times$$

$$\times \{1 - \frac{4\lambda\mu}{(\lambda+\mu)^2}\,\gamma[2,(\lambda+\mu)(t-x)]\}dx; \qquad (3.4.12)$$

and

$$k_0(t) = \Pr\{T_1 > t\} = 1 - \int_0^t f_1(x)dx \qquad (3.4.13)$$

where $f_1(t)$ is given by (3.3.8) and

$$\gamma[n,t] = \int_0^t x^{n-1}\, e^{-x} dx \qquad (3.4.14)$$

is the incomplete Gamma function.

For the symmetric case, i.e., when $\lambda = \mu$, the fact that

$$\frac{\gamma[n-1,t]}{\Gamma(n-1)} - \frac{\gamma[n+1,t]}{\Gamma(n+1)} = e^{-t}\left(\frac{t^{n-1}}{\Gamma(n)} + \frac{t^n}{\Gamma(n+1)}\right) (3.4.15)$$

implies that

$$k_n(t) = 2\mu e^{-2\mu t}\int_0^t I_n(2\mu s)\left(\frac{[2\mu(t-s)]^{n-1}}{\Gamma(n)} + \frac{[2\mu(t-s)]^n}{\Gamma(n+1)}\right)ds\ ,$$

$$n = 1,2,3,\ldots. \qquad (3.4.16)$$

To provide a feeling for the probabilities themselves, we give a numerical table of $k_n(t)$ for selected values of n and t for the case $\lambda = \mu = 1$. Each entry in Table III is the exact value of $k_n(t)$ corresponding to the value of t given at the top of the column and the value of n given at the beginning of the row. The values are believed to be correct to one unit of the final decimal place.

It is interesting to observe from Table III that $k_0(t)$ and $k_1(t)$ appear to be equal, or nearly equal, for all values of t. This can be explained by noting from (3.4.9) that for $\lambda = \mu$,

$$k_1^*(z) = k_0^*(z) - \frac{R^2}{zZ^2} . \qquad (3.4.17)$$

Upon using (3.4.8) and inverting, we obtain

$$k_1(t) = k_0(t) - \varepsilon(t) \qquad (3.4.18)$$

where

$$\varepsilon(t) = e^{-2\mu t} (1+2\mu t) . \qquad (3.4.19)$$

The rapid decrease of $\varepsilon(t)$ as t increases accounts for the two probabilities $k_0(t)$ and $k_1(t)$ being so nearly equal. For example, for the case $\mu = 1$, we have $\varepsilon(5) = 4.9 \times 10^{-4}$, and $\varepsilon(10) = 4.3 \times 10^{-10}$.

TABLE III

EXACT VALUES OF $k_n(t)$: $\lambda = \mu = 1$

n \ t	5	10	15	20	25	30
0	0.2637	0.1820	0.1475	0.1273	0.1137	0.1036
1	0.2632	0.1820	0.1475	0.1273	0.1137	0.1036
2	0.2203	0.1711	0.1420	0.1239	0.1112	0.1018
3	0.1445	0.1494	0.1310	0.1170	0.1064	0.0982
4	0.0716	0.1192	0.1153	0.1069	0.0993	0.0928
5	0.0267	0.0857	0.0961	0.0943	0.0903	0.0859
6	0.0076	0.0547	0.0754	0.0801	0.0798	0.0778
7	0.0017	0.0307	0.0554	0.0652	0.0684	0.0689
8	0.0003	0.0150	0.0378	0.0508	0.0569	0.0595
9	0.0004	0.0063	0.0239	0.0376	0.0457	0.0500
10	0.0000	0.0023	0.0138	0.0265	0.0354	0.0410

To find the moments of N(t), i.e., the quantities defined by

$$E\{[N(t)]^k\} = \sum_{n=1}^{\infty} n^k \Pr\{N(t) = n\} , \tag{3.4.20}$$

we use the relation

$$\int_0^{\infty} e^{-zt} E\{N(t)[N(t) - 1]\ldots[N(t) - (k+1)]\}dt = \left.\frac{d^k K(x,z)}{dx^k}\right|_{x=1} . \tag{3.4.21}$$

For example, we find by differentiating (3.4.7) that

$$\int_0^{\infty} e^{-zt} E\{N(t)\}dt = \frac{4\lambda\mu}{zR(Z+R)} . \tag{3.4.22}$$

Upon inversion we obtain

$$\begin{aligned} E[N(t)] &= 2\sqrt{\lambda\mu} \int_0^t e^{-(\lambda+\mu)x} I_1(2x\sqrt{\lambda\mu})dx \\ &= \frac{(1+\rho)}{|1-\rho|} - 1 + 2\sqrt{\lambda\mu} \int_t^{\infty} e^{-(\lambda+\mu)x} I_1(2x\sqrt{\lambda\mu})dx , \end{aligned} \tag{3.4.23}$$

provided that $\rho = \frac{\lambda}{\mu} \neq 1$. The latter part of (3.4.23) follows from [17], Vol. I, (30), p. 240. Using the asymptotic properties of the modified Bessel function given in (2.3.2), we find that, as $t \to \infty$,

$$E[N(t)] \sim \frac{(1+\rho)}{|1-\rho|} - 1 - \frac{2\rho^{1/4}}{|1-\sqrt{\rho}|}\{1 - \Phi[\sqrt{2\mu t}\;|1-\sqrt{\rho}|]\} + + 0\left(\frac{1}{\mu t}\right) \qquad (3.4.24)$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy , \qquad (3.4.25)$$

and $\rho = \frac{\lambda}{\mu} \neq 1$.

When $\lambda = \mu$, (3.4.23) becomes

$$\begin{aligned} E[N(t)] &= 2\mu t\, e^{-2\mu t}\{I_1(2\mu t) + I_2(2\mu t)\} + \\ &+ e^{-2\mu t}\{I_0(2\mu t) + 2I_1(2\mu t)\} - 1, \end{aligned} \qquad (3.4.26)$$

which follows from [27], (8), p. 122. Therefore, we have from (2.3.2) that for $\lambda = \mu$,

$$E[N(t)] \sim \sqrt{\frac{\mu t}{\pi}} \text{ as } t \to \infty \qquad (3.4.27)$$

In terms of a soccer game between two evenly matched teams, (3.4.27) means that over a long period of time t, the expected number of ties is proportional to $\sqrt{t}$ which increases without bound. We note from (3.4.24), however, that the expected number of ties between two teams of unequal ability remains bounded above by $\frac{(1+\rho)}{|1-\rho|} - 1$ regardless of how long they play.

The asymptotic behavior of $k_n(t)$ and the conclusions about a long soccer game drawn from this behavior could have been predicted from the discrete-time random walk results. For example, it is shown in [18], Chapter III, that in the symmetric case, the mean number of returns to zero during the course of n steps is proportional to $\sqrt{n}$ for large values of n. Since the number of steps that occur in $(0,t)$ increases with t, we would expect that the average number of returns to zero during $(0,t)$ be proportional to $\sqrt{t}$ for large values of t. Equation (3.4.27) shows that this is the case. On the other hand, to obtain probabilistic information about the number of returns to zero during $(0,t)$ for small values of t, it is convenient to use the explicit formulae given in this Section.

We note that $\bar{k}_n = \lim_{t\to\infty} k_n(t)$ can be interpreted as the probability that exactly n returns to zero occur if the random walk is allowed to continue forever. Since $\int_0^\infty f_1(x)dx$ represents the probability that a return to the origin ever occurs and $k_1(t) = \int_0^t f_1(x)dx$, where $f_1(t)$ is given by (3.3.8), we should find, based on (3.3.10), that

$$\begin{aligned} \bar{k}_n &= (2p)^n(1-2p) \quad , \; p < q \; , \\ &= 0 \quad , \; p = q \; , \\ &= (2q)^n(1-2q) \quad , \; p > q \; , \end{aligned}$$

where $p = \frac{\lambda}{\lambda+\mu}$, and $q = 1 - p$. An application of (2.2.12) to equation (3.4.9) does yield (3.4.28). For $\lambda \neq \mu$, (3.4.28) may be useful for approximating the probability that over a long period of time exactly n returns to zero occur.

3.5 The Time of Occurrence and Magnitude of the First Maximum for the Unrestricted Random Walk with Negative Exponentially Distributed Intervals Between Steps

In this Section we use the results of Section 3.2 to determine the distributional properties of the first maximum of the unrestricted random walk with negative exponentially distributed intervals between steps during an arbitrary interval of time $(0,t)$. We let t' denote the time of occurrence of the first maximum during $(0,t)$ as defined in Section 3.1. $S(t')$ then represents the magnitude of the first maximum. If no first passage to a positive value, i.e., no first passage to +1, occurs during $(0,t)$, then we say $t' = 0$. The probability of this event is given by

$$\Pr\{t' = 0\} = F_{01}(t) \ , \qquad (3.5.1)$$

where $F_{01}(t)$ is defined by equation (3.1.1). Analogous definitions and results hold for the first minimum.

Baxter and Donsker [5] have studied the probability $\Pr\{S(t') < \alpha\}$ for a general process in continuous time with discrete steps. They have derived the double Laplace transform of $\Pr\{S(t') < \alpha\}$ with respect to t and α and expressed it in terms of a function $\psi(\xi)$ defined by the equation

$$E\{e^{i\xi S(t)}\} = e^{t\psi(\xi)} \quad . \tag{3.5.2}$$

Unfortunately, application of this result is difficult, and it seems more convenient to proceed directly in our case. Thus we can also obtain probabilistic information about t', the time of occurrence of the first maximum, as well as the value of S(t'). We can then use these explicit results to obtain asymptotic approximations for large values of t. We turn next to the derivations of the joint probability and probability density function that t' = s and S(t') = k.

We define $m_k(s,t)ds$ as the probability that

(i) $S(t') = k$, $k = 1,2,3,\ldots$; and

(ii) $s < t' < s + ds$, $0 < s \leq t$.

In order that $S(t') = k \geq 1$, and that the maximum occur for the first time at time s, $0 < s < t$, a first passage from 0 to k must occur at time s followed by no first passage from k to k + 1 during (s,t). Therefore, because of the Markov property of the Poisson process, we can write

$$m_k(s,t) = f_{0k}(s)\ F_{01}(t-s) \quad , \ 0 < s < t, \tag{3.5.3}$$

where $f_{0k}(t)$ and $F_{01}(t)$ are given in (3.2.14) and (3.2.27), respectively. Evidently,

$$m_k(t,t) = f_{0k}(t) \tag{3.5.4}$$

and therefore (3.5.3) holds for $0 < s \leq t$.

We define $m(s,t)$ by

$$m(s,t)ds = \Pr\{s < t' < s + ds\}\ , \quad 0 < s \leq t, \tag{3.5.5}$$

and it follows from the definition of $m_k(s,t)$ that

$$m(s,t) = \sum_{k=1}^{\infty} m_k(s,t) \ . \tag{3.5.6}$$

To determine $m(s,t)$ completely, it is necessary to find $\sum_{k=1}^{\infty} f_{0k}(s)$. By (3.2.13) we have

$$\sum_{k=1}^{\infty} f_{0k}(s) = \frac{1}{2\pi i} \int_C e^{zs} \sum_{k=1}^{\infty} \phi_{0k}(z)dz \tag{3.5.7}$$

$$= \frac{1}{2\pi i} \int_C e^{zs} \frac{\lambda}{z}\left[1 - \frac{2\mu}{Z+R}\right]dz \ ,$$

interchange of summation and integration being justified by the absolute convergence of the right-hand side of (3.5.7). Upon comparing (3.5.7) with (3.2.18), we obtain by symmetry the result

$$\sum_{k=1}^{\infty} f_{0k}(s) = \lambda F_{10}(s) \ . \tag{3.5.8}$$

On combining (3.5.6) and (3.5.8), we obtain the elegant formula

$$m(s,t) = F_{10}(s)\ F_{01}(t-s) \ . \tag{3.5.9}$$

The similarity of this result to the work of E. Sparre

Andersen [1] and [2] for discrete-time random walks, of which the coin-tossing game described in [18], Chapter III, is an example, is most striking. Further observations on this subject will be made in Chapter V.

From (3.5.3) it is easy to obtain the probability

$$M_k(t) = \Pr\{S(t') = k\} , \qquad (3.5.10)$$

for, in order that $S(t') = k \geq 1$, the maximum attained by the particle during $(0,t)$ must occur for the first time at some point s, $0 < s \leq t$. Hence, we have

$$M_k(t) = \int_0^t m_k(s,t)\,ds . \qquad (3.5.11)$$

From (3.5.3), (3.5.11) and the convolution property of the Laplace transform, we have

$$M_k(t) = \frac{1}{2\pi i} \int_C e^{zt} \phi_{0k}(z)\ \Phi_{01}(z)\,dz , \qquad (3.5.12)$$

and from (3.2.19) it follows that

$$M_k(t) = F_{0,k+1}(t) - F_{0,k}(t) . \qquad (3.5.13)$$

As a consequence of (3.5.13) and the fact that

$$F_{0,k+1}(t) - F_{0,k}(t) = \int_0^t [f_{0k}(x) - f_{0k+1}(x)]\,dx ,$$

we have

$$\Pr\{S(t') \geq k\} = \sum_{j=k}^{\infty} M_j(t) = \int_0^t f_{0k}(x)\,dx \qquad (3.5.14)$$

where $f_{0k}(t)$ is given in (3.2.14). The result given in (3.5.14) was obtained by Baxter and Donsker [5] using the Fourier transform method mentioned earlier. The other finite-time results given in this Section are original as far as we have been able to ascertain.

In addition to the asymptotic behavior of $M_k(t)$, which can be deduced from equations (3.2.17) and (3.2.30), it is interesting to note that $M_k(t)$ has the following stationary distribution which follows from (2.2.12) and (3.5.12):

$$\begin{aligned} \overline{M}_k &= \rho^k(1-\rho) \quad , \; \rho < 1 \\ &= 0 \qquad\qquad\quad , \; \rho \geq 1 \end{aligned} \tag{3.5.15}$$

where $\overline{M}_k = \lim\limits_{t\to\infty} M_k(t)$, $\rho = \frac{\lambda}{\mu}$.

The asymptotic behavior of $m(s,t)$ can be deduced from (3.2.24) for the case $\lambda \neq \mu$ and from (3.2.35) for the case $\lambda = \mu$. From (3.2.24) we have as $s,(t-s) \to \infty$,

$$m(s,t) \sim \frac{2\lambda(1+\sqrt{\rho})e^{-\mu(1-\sqrt{\rho})^2 s}}{[\pi s\sqrt{\lambda\mu}]^{1/2}(1-\sqrt{\rho})}\left[\frac{1}{[4s\sqrt{\lambda\mu}]} + 0\left(\frac{1}{16s^2\lambda\mu}\right)\right] \quad , \; \rho < 1,$$

$$\sim \frac{2\lambda(1+\sqrt{\rho})e^{-\mu(1-\sqrt{\rho})^2(t-s)}}{[\pi(t-s)\sqrt{\lambda\mu}]^{1/2}(\sqrt{\rho}-1)}\left[\frac{1}{[4(t-s)\sqrt{\lambda\mu}]} + 0\left(\frac{1}{16\lambda\mu(t-s)^2}\right)\right],$$

$$\rho > 1, \tag{3.5.16}$$

and from (3.2.35) we have

$$m(s,t) \sim \frac{1}{\pi}\frac{1}{\sqrt{s(t-s)}} \quad , \; \rho = 1. \tag{3.5.17}$$

Upon integrating by parts, we obtain from (3.5.16) and (3.5.17) the result

$$\int_{\alpha t}^{t} m(s,t)ds \sim \frac{\rho^{1/4}(1+\sqrt{\rho})}{\sqrt{\pi}}\left[\frac{e^{-\Lambda\alpha t}}{\sqrt{\Lambda\alpha t}} - \frac{e^{-\Lambda t}}{\sqrt{\Lambda t}} + \int_{\sqrt{2\Lambda\alpha t}}^{\sqrt{2\Lambda t}} e^{-y^2/2}\,dy\right], \quad \rho<1,$$

$$\sim 1 - \frac{2}{\pi}\text{ arc sin }\sqrt{\alpha}\ , \quad \rho = 1, \tag{3.5.18}$$

$$\sim \frac{2\rho^{1/4}(1+\sqrt{\rho})}{\sqrt{\pi}(\sqrt{\rho}-1)}\left[1 - \frac{e^{-\Lambda(1-\alpha)t}}{\sqrt{t(1-\alpha)}} - \sqrt{\Lambda}\int_{0}^{\sqrt{2\Lambda t(1-\alpha)}} e^{-y^2/2}dy\right],$$

$$\rho > 1,$$

where

$$\Lambda = \mu(1-\sqrt{\rho})^2 . \tag{3.5.19}$$

From (3.5.18) we observe that for $\rho < 1$, $0 < \alpha < 1$, $\Pr\{t' > \alpha t\} \to 0$ as $t \to \infty$, i.e., the first maximum has a strong tendency to occur during $(0,\alpha t)$ after which the particle has a tendency to drift toward $-\infty$. On the other hand, when $\rho > 1$, there is a drift toward $+\infty$ and in this case, $\Pr\{t' > \alpha t\}$, $0 < \alpha < 1$, approaches a non-zero limit as $t \to \infty$. The case $\rho = 1$ is especially interesting. We note from (3.5.17) that the first maximum is more likely to occur at a time near zero or t as t increases, a result which is not intuitively obvious. This same phenomenon has been shown to occur in the coin-tossing random walk discussed in [18], Chapter III. This similarity to the coin-tossing random walk will be discussed in more detail in Chapter V.

In terms of a soccer game between two evenly matched teams, we conclude that the largest score difference during a very long game is more likely to occur near the beginning or the end of the game rather than at any time in between.

The results pertaining to stable distributions are not directly applicable here. It is interesting to note, however, the appearance in (3.5.18) of a term which has the form of the stable distribution of index $\frac{1}{2}$ defined by (2.6.8).

The usefulness of (3.5.18) as an approximation will be commented upon in Section 4.6.

We turn now to the consideration of first-passage times for the M/M/1 queuing process described in Chapter I.

3.6 First-Passage Time from m to n for the M/M/1 Queuing Process

A complete analysis of first-passage times and the number of returns to zero for the queuing random walk is complicated because of the barrier at zero. To derive $f_{0n}(t)$, for example, one must deal with a more complicated two-barrier problem rather than the one-barrier problem appropriate to the case of the doubly-infinite random walk. Bailey [3] has given the Laplace transforms of $f_{mn}(t)$ for the cases $m < n$ and $m > n$ and obtained explicit results for the case $m > n$. Apparently, no explicit formulae for $f_{mn}(t)$ have been published for the case $m \leq n$. A step in this direction will be made in this Section. Heathcote [25] has obtained systems of equations relating $f_{m,m+1}(t)$ for the GI/M/1 queue (defined in Chapter I), but no explicit formulae for M/M/1 seem to be obtainable from Heathcote's work.

There are also no published results concerning the time of occurrence of the first maximum, or maximum number of customers present during an interval of time of arbitrary length t. Cohen [8], however, has obtained expressions for the distribution of the maximum number of customers present during a busy period for the M/G/1 and GI/M/1 queues.

We now proceed to derive $f_{mn}(t)$, the probability density function of first passage to n at time t given $S(0) = m$. In the sequel the most useful results will be the expressions for $f_{m0}(t)$.

As before, we let

$$p_{mn}(t) = \Pr\{S(t) = n \mid S(0) = m\} \qquad (3.6.1)$$

where $S(t)$, as related to the queuing process, represents the number of customers in the system at time t. The Laplace transform of $p_{mn}(t)$ is defined by

$$\pi_{mn}(z) = \int_0^\infty e^{-zt}\, p_{mn}(t)dt\ . \qquad (3.6.2)$$

Explicit expressions for $p_{mn}(t)$ were first given by A.B. Clarke [7]. Using a difference equation technique, Conolly [9] found that

$$\pi_{m,n}(z) = A_m\alpha^n + B_m\beta^n \quad , \; n \leq m,$$
$$= C_m\beta^n \qquad\qquad , \; n \geq m, \tag{3.6.3}$$

where

$$A_m = \frac{1}{\alpha^m R} \; , \; B_m = \frac{(1-\beta)}{(\alpha-1)\alpha^m R} \; , \; C_m = A_m\left(\frac{\alpha}{\beta}\right)^m + B_m \; . \tag{3.6.4}$$

The letters α,β denote the larger and smaller roots, respectively, of the quadratic equation

$$\mu x^2 - Zx + \lambda = 0 \; , \tag{3.6.5}$$

where

$$Z = \lambda + \mu + z \; , \; R^2 = Z^2 - 4\lambda\mu \; . \tag{3.6.6}$$

Let $u_{mn}(t)dt$ denote the probability that during $(t,t+dt)$ the system passes to state n given that $S(0) = m$, and let

$$u^*_{mn}(z) = \int_0^\infty e^{-zt}\, u_{mn}(t)dt \; . \tag{3.6.7}$$

The passage to n during $(t,t+dt)$ may be the first or else a first passage to n has occurred at some time s, $0 < s < t$, followed by a passage to n at time $t - s$ given $S(0) = n$. Accordingly, it follows that

$$u_{mn}(t) = f_{mn}(t) + \int_0^t f_{mn}(s)\, u_{nn}(t-s)ds \; . \tag{3.6.8}$$

Using (3.6.8) and the convolution property of the Laplace transform, we find that

$$\phi_{mn}(z) = \frac{u^*_{mn}(z)}{1 + u^*_{nn}(z)} \,. \tag{3.6.9}$$

Taking a different point of view one can think of the probability of a passage to n during (t,t+dt) as the probability that at time t there are n - 1 or n + 1 customers in the system and during the following small interval (t,t+dt) an arrival or departure occurs. Hence, we can write

$$u^*_{mn}(z) = \lambda\pi_{m,n-1}(z) + \mu\lambda_{m,n+1}(z) \;, \; n = 1,2,\ldots, \tag{3.6.10}$$

and

$$u^*_{m0}(z) = \mu\pi_{m,1}(z) \;, \; m = 0,1,2,\ldots. \tag{3.6.11}$$

Using the quadratic relation given in (3.6.5), together with equations (3.6.3) and (3.6.10), it follows that

$$\begin{aligned} u^*_{mn}(z) &= Z[A_m\alpha^n + B_m\beta^n] \;, && n < m, \\ &= 2\lambda A_n\alpha^{n-1} + ZB_n\beta^n \;, && n = m, \\ &= ZC_m\beta^n \;, && n > m. \end{aligned} \tag{3.6.12}$$

In the same manner we obtain from (3.6.11) the equation

$$\begin{aligned} u^*_{m0}(z) &= \mu[A_m\alpha + B_m\beta] \;, && m = 1,2,\ldots, \\ &= \mu C_0\beta \;, && m = 0. \end{aligned} \tag{3.6.13}$$

From (3.6.9) we have

$$\phi_{mn}(z) = \frac{1}{\alpha^{m-n}} \qquad , \; m > n \geq 1,$$

$$= \rho^{n-m} \frac{\{\alpha^m(\alpha-1) + \beta^m(1-\beta)\}}{\alpha^n(\alpha-1) + \beta^n(1-\beta)} \qquad , \; m < n, \tag{3.6.14}$$

$$= \frac{\lambda}{Z}\left[\frac{1}{\alpha} + \frac{\alpha^{n-1}(\alpha-1) + \beta^{n-1}(1-\beta)}{\alpha^n(\alpha-1) + \beta^n(1-\beta)}\right] \qquad , \; m = n.$$

Also,

$$\phi_{m0}(z) = \alpha^{-m} \; , \; m = 1,2,3,\ldots,$$
$$\phi_{00}(z) = \frac{\lambda\alpha^{-1}}{\lambda+z} . \tag{3.6.15}$$

Using [17], Vol. I, and equations (3.6.14) and (3.6.15), we have

$$f_{mn}(t) = \rho^{-(m-n)/2} \frac{e^{-(\lambda+\mu)t}}{t}(m-n) I_{(m-n)}(2t\sqrt{\lambda\mu}) \quad , \; m > n \geq 0. \tag{3.6.16}$$

The function $f_{10}(t)$ is the classical busy period density for the M/M/1 queue. Equations (3.6.14) - (3.6.16) have been given by Bailey [3].

When $m < n$, $\phi_{mn}(z)$ does not seem to be invertible in terms of known tabulated functions although it possesses an elegant form. For the case $\lambda = \mu$, however, an inverted form does exist although it is somewhat formidable. To see this, consider $\phi_{mn}(z)$, $m < n$, as given by (3.6.14). In this case,

$\alpha\beta = 1$, and we have

$$\phi_{mn}(z) = \frac{\beta^{n-m} + \beta^{n+m+1}}{1 + \beta^{2n+1}} . \tag{3.6.17}$$

Upon inversion of (3.6.17) we find that

$$f_{mn}(t) = \frac{e^{-2\mu t}}{t} \sum_{k=0}^{\infty} (-1)^k \{a_k I_{a_k}(2\mu t) + (a_k+2m+1) I_{(a_k+2m+1)}(2\mu t)\} \tag{3.6.18}$$

where

$$a_k = (2n+1)k + n - m . \tag{3.6.19}$$

Equation (3.6.18) has not been published previously.

Further light is shed on the problem if one considers the equation relating $f_{mn}(t)$ and $f_{n-1,n}(t)$ for $n > m$, namely,

$$f_{mn}(t) = \int_0^t f_{m,n-1}(s)\, f_{n-1,n}(t-s)ds . \tag{3.6.20}$$

Therefore, we could derive $f_{mn}(t)$ recursively if we knew the form of $f_{n-1,n}(t)$. The functions $f_{n-1,n}(t)$ and $f_{n,n+1}(t)$ are related by the equation

$$f_{n,n+1}(t) = \lambda e^{-(\lambda+\mu)t} + \mu\int_0^t e^{-(\lambda+\mu)s} f_{n-1,n}(t-s) * f_{n,n+1}(t-s)ds, \tag{3.6.21}$$

and hence,

$$\phi_{n,n+1}(z) = \frac{\lambda/Z}{1 - \frac{\mu}{Z}\phi_{n-1,n}(z)} = \frac{\lambda}{Z}\sum_{j=0}^{\infty}\left[\frac{\mu\phi_{n-1,n}(z)}{Z}\right]^j \tag{3.6.22}$$

which implies that

$$f_{n,n+1}(t) = \lambda e^{-(\lambda+\mu)t} + \lambda \sum_{j=0}^{\infty} \int_0^t \frac{(\mu s)^j}{j!} e^{-(\lambda+\mu)s} f_{n-1,n}^{(j)}(t-s)ds \, . \tag{3.6.23}$$

Therefore, since we know $f_{01}(t) = \lambda e^{-\lambda t}$, we could derive $f_{n,n+1}(t)$ recursively using (3.6.23), albeit a cumbersome process. For example, $f_{12}(t)$ is given by

$$f_{12}(t) = \lambda e^{-(\lambda+\mu)t} + \lambda\sqrt{\lambda\mu}\ e^{-\lambda t} \int_0^t e^{-\mu s} \sqrt{\frac{s}{t-s}}\, I_1[2\sqrt{\lambda\mu s(t-s)}]ds \, . \tag{3.6.24}$$

Equation (3.6.23) appears to be new.

An explicit solution in terms of known functions can be obtained from the integral equation given by (3.6.8). In fact, it is easily verified that

$$f_{mn}(t) = u_{mn}(t) + \sum_{k=1}^{\infty} (-1)^k u_{mn}(t) * u_{nn}^{(k)}(t) \tag{3.6.25}$$

satisfies (3.6.8). From (3.6.10) and (3.6.11) we see that $u_{mn}(t)$ can be expressed in terms of the state probabilities $p_{mn}(t)$ which are known (see [30], Chapter I, for example). Explicit expressions for $f_{mn}(t)$ are not easily obtained from (3.6.25), however.

3.7 The Time of Occurrence of the r^{th} Return to Zero for the M/M/1 Queuing Process

We now consider $f_r(t)$ for the M/M/1 queue where

$$f_r(t)dt = \Pr\{t < T_r < t + dt \mid S(0) = 0\} \qquad (3.7.1)$$

and T_r denotes the time of occurrence of the r^{th} return to zero in the queuing random walk. In this case the random variable T_r represents the amount of time which elapses before the server becomes idle for the r^{th} time. Throughout this Section we assume that $S(0) = 0$, i.e., the server is idle at time $t = 0$.

From (3.6.15) we have

$$\phi_1(z) = \frac{\lambda}{\alpha(\lambda+\mu)} \qquad (3.7.2)$$

where

$$\phi_r(z) = \int_0^\infty e^{-zt} f_r(t)dt \ , \qquad (3.7.3)$$

and α is the root of (3.6.5) with greater absolute value. As in the unrestricted random walk, we have

$$f_r(t) = \int_0^t f_{r-1}(t-s) \ f_1(s)ds \ , \qquad (3.7.4)$$

and hence,

$$\phi_r(z) = \left[\frac{\lambda}{(\lambda+z)\alpha}\right]^r \ , \ r = 1,2,3,\ldots. \qquad (3.7.5)$$

Upon inversion, equation (3.7.5) yields the result

$$f_r(t) = \frac{r(2\lambda\mu)^{r/2}}{(r-1)!} e^{-\lambda t} \int_0^t e^{-\mu s} \ [t-s]^{r-1} I_r(2s\sqrt{\lambda\mu})\frac{ds}{s} \ . \qquad (3.7.6)$$

We note that

$$\begin{aligned} \phi_1(0) &= 1 \ , \ \rho \le 1, \\ &= \frac{1}{\rho} \ , \ \rho > 1, \end{aligned} \tag{3.7.7}$$

and, therefore, a return to zero is certain only when $\rho \le 1$. Differentiating $\phi_1(z)$ we find that the expected value of T_1 is given by

$$\begin{aligned} E(t) &= \frac{2}{\mu(1-\rho)} \ , \ \rho < 1, \\ &= \infty \qquad\quad , \ \rho \ge 1, \end{aligned} \tag{3.7.8}$$

where $\rho = \lambda/\mu$. We note that in the case $\rho = 1$, a return to zero is certain; but, according to (3.7.8), the mean recurrence time is infinite, a phenomenon which appeared in connection with the unrestricted random walk as well.

Since T_r represents the sum of r mutually independent random variables each distributed as T_1, we can examine the behavior of $1 - \int_0^t f_1(x)dx$ as $t \to \infty$ to determine whether or not the theory of stable distributions given in Section 2.6 is applicable. From (3.7.6) it is not difficult to show that

$$\begin{aligned} F_1(t) &= \int_0^t f_1(x)dx \\ &= F_{11}(t) - F_{12}(t) \end{aligned} \tag{3.7.9}$$

where

$$F_{11}(t) = \rho^{-1/2} \int_0^t e^{-(\lambda+\mu)s} I_1(2s\sqrt{\lambda\mu}) \frac{ds}{s} , \qquad (3.7.10)$$

and

$$F_{12}(t) = \rho^{-1/2} e^{-\lambda t} \int_0^t e^{-\mu s} I_1(2s\sqrt{\lambda\mu}) \frac{ds}{s} . \qquad (3.7.11)$$

By (3.7.7) and [17], Vol. I, (4), p. 195, we have

$$F_1(\infty) = F_{11}(\infty) = 1 \text{ for } \rho \leq 1. \qquad (3.7.12)$$

Therefore we conclude that

$$\lim_{t\to\infty} F_{12}(t) = 0 , \qquad (3.7.13)$$

and that

$$1 - F_1(t) \sim 1 - F_{11}(t) . \qquad (3.7.14)$$

From (3.7.10), (3.7.12), and (2.3.2) it then follows that

$$1 - F_1(t) \sim \frac{\rho^{-3/4} e^{-\mu(1-\sqrt{\rho})^2 t}}{\sqrt{\pi\mu t}} - \rho^{-3/4}(1-\sqrt{\rho})\sqrt{\frac{2}{\pi}} \int_{\sqrt{2\mu t}(1-\sqrt{\rho})}^{\infty} e^{-y^2/2} dy , \qquad \rho < 1,$$

$$\sim \frac{1}{\sqrt{\pi\mu t}} , \ \rho = 1. \qquad (3.7.15)$$

We note from (3.7.15) that for $\rho < 1$, the limiting form of

$1 - F_1(t)$ is not expressible in terms of a slowly varying function $L(t)$ defined in (2.2.9). Therefore the results of Section 2.6 do not apply. We note, however, that the limiting form of $1 - F_1(t)$ for the case of $\rho < 1$ is expressible in terms of the stable distribution of index $\frac{1}{2}$ given by (2.6.8). For $\rho = 1$, however, we see from (3.7.15) that

$$1 - F_1(t) \sim \frac{L(t)}{\sqrt{t}\,\Gamma(1/2)} \tag{3.7.16}$$

where

$$L(t) = \frac{1}{\sqrt{\mu}}\,, \tag{3.7.17}$$

and is slowly varying. Therefore we can apply result (2) of Section 2.6 and conclude that for $\rho = 1$, there exist norming constants a_r such that

$$F_r(a_r t) \to G_{\frac{1}{2}}(t) \text{ as } r \to \infty\,, \tag{3.7.18}$$

where $G_{\frac{1}{2}}(t)$ is the stable distribution of index $\frac{1}{2}$ given by (2.6.8) and

$$F_r(t) = \int_0^t f_r(x)\,dx\,. \tag{3.7.19}$$

3.8 The Number of Returns to Zero During (0,t) for the M/M/1 Queuing Process

In the queuing process, the number of returns to zero during an arbitrary interval of time (0,t) can be interpreted as the number of times the system becomes empty, i.e., the number of times the server becomes idle.

As before, we let N(t) denote the number of returns to zero during (0,t) and define the probability

$$k_n(t) = \Pr\{N(t) = n \mid S(0) = 0\} . \qquad (3.8.1)$$

Evidently,

$$k_0(t) = \Pr\{T_1 > t\} = 1 - \int_0^t f_1(s)ds . \qquad (3.8.2)$$

To derive $k_n(t)$, $n = 1,2,3,\ldots$, we follow the same steps as we did in Section 3.4 for the unrestricted random walk utilizing the relation

$$\Pr\{T_n \leq t\} = \Pr\{N(t) \geq n\} . \qquad (3.8.3)$$

We define K(x,z), the Laplace transform of the generating function of $k_n(t)$, by equation (3.4.3). We can then use equations (3.4.5) and (3.7.5) to find that

$$K(x,z) = \frac{\frac{1}{z}\left[1 - \frac{2\lambda\mu}{(\lambda+z)(Z+R)}\right]}{\left[1 - \frac{2\lambda\mu x}{(\lambda+z)(Z+R)}\right]} . \qquad (3.8.4)$$

Upon expanding (3.8.4) in powers of x, we observe that

$$k_n^*(z) = \frac{1}{z}\left[\left(\frac{\lambda}{(\lambda+z)\alpha}\right)^n - \left(\frac{\lambda}{(\lambda+z)\alpha}\right)^{n+1}\right] . \qquad (3.8.5)$$

Upon inversion we obtain

$$k_n(t) = \Lambda_n(t) - \Lambda_{n+1}(t) \quad , \; n = 1,2,\ldots, \tag{3.8.6}$$

where

$$\Lambda_n(t) = \frac{n\rho^{-n/2}}{(n-1)!} \int_0^t e^{-(\lambda+\mu)y} \, \gamma[n,\lambda(t-y)] \, I_n(2y\sqrt{\lambda\mu})dy \; , \tag{3.8.7}$$

and

$$\gamma[n,t] = \int_0^t e^{-x} \, x^{n-1} dx \; . \tag{3.8.8}$$

From (2.2.11), (2.2.12), and (3.8.5), we conclude that $k_n(t)$ has the stationary distribution given by

$$\begin{aligned} \bar{k}_n &= 0 \qquad\qquad\quad , \; \rho \le 1, \\ &= \rho^{-n}(1-\rho^{-1}) \; , \; \rho > 1, \end{aligned} \tag{3.8.9}$$

where $\rho = \frac{\lambda}{\mu}$ and $\bar{k}_n = \lim_{t\to\infty} k_n(t)$.

For the queuing process we refer to ρ = (mean arrival rate) / (mean service rate) = $\frac{\lambda}{\mu}$ as the traffic intensity. From (3.8.9) we see that as $t \to \infty$, the probability that the server becomes idle a finite number of times tends to zero when the traffic intensity ρ does not exceed unity.

Upon application of (3.4.21) to (3.8.4), we find that

$$E[N(t)] = \int_0^t e^{-(\lambda+\mu)x} \sum_{n=2}^{\infty} n\rho^{-n/2} \, I_n(2x\sqrt{\lambda\mu}) \frac{dx}{x} \; . \tag{3.8.10}$$

For $\rho = 1$, (3.8.10) becomes

$$E[N(t)] = \int_0^{2\mu t} e^{-x} [I_1(x) + I_2(x)]dx . \qquad (3.8.11)$$

The integral in (3.8.11) can be evaluated (see [27], p. 122, for example,) to obtain for $\rho = 1$,

$$E[N(t)] = (2\mu t)e^{-2\mu t} [I_o(2\mu t) + 2I_1(2\mu t) + I_2(2\mu t)] +$$

$$+ 4e^{-2\mu t} I_1(2\mu t) + 3e^{-2\mu t} [I_0(2\mu t) - 1] . \qquad (3.8.12)$$

By using the asymptotic properties of the modified Bessel function given in (2.3.2), we find that as $t \to \infty$,

$$E[N(t)] \sim 4\sqrt{\frac{\mu t}{\pi}} , \ \rho = 1. \qquad (3.8.13)$$

We conclude that when the traffic intensity is equal to unity, the expected number of returns to zero is proportional to $t^{1/2}$ as $t \to \infty$ although the expected length of time between returns is infinite. This result suggests that it may be possible even when $\rho = 1$ to find an asymptotic approximation to the probability that the fraction of an interval during which the server is busy is equal to α, $0 < \alpha < 1$. Such a result does exist and will be derived in Chapter IV.

This concludes our discussion of the number of returns to zero during an arbitrary time interval for the M/M/1 queuing process.

CHAPTER IV

SOJOURN TIME PROBLEMS

4.1 Introduction

In Chapter III we were concerned with the amount of time required for the particle to pass from one state to another. In this Chapter we shall be concerned with the proportion of an arbitrary time interval of length t that the particle spends in a given state. For example, in the queuing process one may wish to know what proportion of time the system is in state zero, that is, what proportion of time the server is idle. A state T will consist of a set of integral valued real numbers since we assume in the random walks under consideration that the particle moves in unit steps. We say that the particle is in state T at time t if $S(t) \in T$ where $S(t)$ denotes the position of the particle at time t. For any state T, let $\chi_T(t)$ be a random variable such that

$$\begin{aligned} \chi_T(t) &= 1, \; S(t) \in T \\ &= 0, \; S(t) \notin T. \end{aligned} \tag{4.1.1}$$

The sojourn time $\sigma_T(t)$ in state T is then defined by the equation

$$\sigma_T(t) = \int_0^t \chi_T(x)\,dx \tag{4.1.2}$$

and represents the amount of time the particle spends in state T during the time interval (0,t).

From the structure of the random walks described in Chapter I, it is evident that an infinite number of states could be defined. Good [23] has studied the general problem where n possible states are considered. Takács [34] has derived some general results for the two-state sojourn problem. The stochastic process which alternates between two possible states, A and B, is sometimes called an alternating renewal process. Such processes are discussed in [13], Chapter VII. In this Chapter we shall deal only with two and three state problems, that is, we will be concerned only with time spent in two or three possible disjoint states. We shall obtain general and specific results for cases of particular interest in the unrestricted and queuing random walks. Further references to specific results by Good, Takács and others will be made in the appropriate Sections.

In the next Section we shall derive a general expression for the two-state sojourn distribution and density functions. This will be followed, in Sections 4.3 - 4.5, by applications of this general result to both the unrestricted and queuing random walks. The three-state sojourn problem will be discussed in Section 4.7.

In Section 4.6 we shall give numerical results with the objective of assessing the effectiveness of the asymptotic approximations derived in Sections 4.3 - 4.5 and to compare our results with those of Takács where possible. We note that the asymptotic formulae given in this Chapter will frequently be expressed in terms of the stable distributions discussed in Section 2.6. This is perhaps not surprising since the sojourn time in a given state is the sum of disjoint intervals of time spent in that state. Therefore we might expect that the stable laws which govern the asymptotic behavior of sums of random variables would influence the asymptotic behavior of the sojourn time distribution. In fact, this is the point of view taken by Takács [35] and the asymptotic distributions for sojourn times obtained in his work are based on the theory of stable distributions. We shall discuss in Section 4.6 the usefulness of Takács approach to asymptotic results, contrasting it with the exact asymptotic approximations obtainable from our precise solutions to the problems investigated.

4.2 General Solution for the Two-State Sojourn Time Problem

In this Section we assume two possible states A and B, i.e. at any time t, either $S(t) \varepsilon A$ or $S(t) \varepsilon B$ and there is no other possibility. We let

$$p_B^{(k)}(s,t)ds = \Pr\{s < \sigma_B(t) < s+ds | S(0) = k\} \quad (4.2.1)$$

and

$$P_B^{(k)}(s,t) = Pr\{\sigma_B(t) \leq s | S(0) = k\}, \tag{4.2.2}$$

the range of k being the set of all integers for the unrestricted walk while for the queuing walk, k must be a non-negative integer. The probabilities $P_A^{(k)}(s,t)$ and $p_A^{(k)}(s,t)ds$ are defined analogously. The absence of any superscript k in (4.2.1) and (4.2.2) will be understood to mean $S(0) = 0$.

We let τ_{AB} denote the first-passage time from A to B, i.e. $\tau_{AB} = \tau$ if $S(t_0-dt) \in B$; $S(t_0+x) \in A$, $(0 \leq x < \tau)$; and $S(t_0+\tau) \in B$. Then let

$$G(t) = Pr\{\tau_{AB} \leq t\} \tag{4.2.3}$$

and

$$g(t)dt = Pr\{t < \tau_{AB} < t+dt\}. \tag{4.2.4}$$

Analogous expressions for state B will be denoted by τ_{BA}, $H(t)$ and $h(t)$, respectively.

We shall assume that the distributions of τ_{AB} and τ_{BA} depend only on the fact that a transition is made from B to A or vice-versa, i.e. the Markov property holds. Under this assumption general expressions for $p_B(s,t)$ and $P_B(s,t)$ will be derived. It should be remarked that these results could be obtained as a special case of the three-state sojourn problem to be considered in Section 4.7. It is more convenient and more instructive, however, to derive the two-state result first.

We begin by obtaining expressions for $P_B^{(k)}(s,t)$ and

$p_B^{(k)}(s,t)$. The corresponding expressions, $P_A^{(k)}(s,t)$ and $p_A^{(k)}(s,t)$, for state A can then be obtained by use of the relationships

$$P_A^{(k)}(s,t) = 1-P_B^{(k)}(t-s,t) \tag{4.2.5}$$

and

$$p_A^{(k)}(s,t) = p_B^{(k)}(t-s,t). \tag{4.2.6}$$

Equation (4.2.5) follows from the fact that at any time t, either $S(t) \varepsilon A$ or else $S(t) \varepsilon B$, and (4.2.6) follows directly from (4.2.5) by differentiation.

For convenience we assume that $S(0) = 0$ and that a transition from state B to state A has just occurred. To avoid having to enumerate all the possible cases, the general results for $S(0) = k$ will be derived in subsequent sections for the particular cases considered.

Since the particle begins its walk in state A, it cannot remain in state B throughout the entire interval (0,t). Consequently,

$$\Pr\{\sigma_B(t) = t|S(0) = 0\} = 0. \tag{4.2.7}$$

In order that $\sigma_B(t) = 0$, we must have $\tau_{AB} > t$ and hence

$$\Pr\{\sigma_B(t) = 0|S(0) = 0\} = 1-G(t). \tag{4.2.8}$$

We now proceed to derive an expression for $p_B(s,t)$ assuming $0 < s < t$. At time t, either $S(t) \varepsilon A$ or $S(t) \varepsilon B$. Since the particle may leave and re-enter state B more than once during the time interval $(0,t)$, $\sigma_B(t)$ represents the sum of disjoint intervals of time during which the particle remained in state B. Accordingly, let us define $u_n(s,t)ds$ as the probability that

(i) $S(t) \varepsilon A$;

(ii) $\sigma_B(t)$ is composed of n disjoint intervals of time during which the particle remained in state B; and

(iii) $s < \sigma_B(t) < s+ds$.

Similarly, we define $v_n(s,t)ds$ as the probability that

(i) $S(t) \varepsilon B$;

(ii) $\sigma_B(t)$ is composed of n disjoint intervals of time during which the particle remained in state B; and

(iii) $s < \sigma_B(t) < s+ds$.

We begin by considering $u_1(s,t)$. In order that $S(t) \varepsilon A$ and $\sigma_B(t)$ consist of one interval of time of duration s, there must be an initial period during which the particle remains in state A followed by an interval of length s during which the particle remains in state B while in the remainder of the interval $(0,t)$ the particle is in state A. Consequently,

$$\begin{aligned} u_1(s,t) &= \int_0^{t-s} g(x)h(s)[1 - G(t-s-x)]dx \\ &= h(s)\{G(t-s) - G^{(2)}(t-s)\} \end{aligned} \tag{4.2.9}$$

where $G^{(n)}(t)$ is defined by the set of equations

$$G^{(0)}(t) = 1,$$

$$G^{(1)}(t) = G(t), \tag{4.2.10}$$

$$G^{(n)}(t) = \int_0^t G^{(n-1)}(t-s)g(s)ds, \; n = 2,3,4,\ldots.$$

Since the distribution of τ_{AB} depends only on the fact that a transition from B to A has just occurred, we see that whenever a transition from B to A occurs, the initial conditions assumed above are duplicated. Consequently, we can write $u_2(s,t)$ as

$$\begin{aligned} u_2(s,t) &= \int_{x=0}^{t-s} g(x)dx \int_{y=0}^{s} h(y)u_1(s-y,t-y-x)dy \\ &= h^{(2)}(s)\{G^{(2)}(t-s) - G^{(3)}(t-s)\} \end{aligned} \tag{4.2.11}$$

where $h^{(n)}(t)$ is the n-fold convolution of h(t) with itself and is defined by the set of equations

$$h^{(0)}(t) = 0,$$

$$h^{(1)}(t) = h(t), \tag{4.2.12}$$

$$h^{(n)}(t) = \int_0^t h(s)h^{(n-1)}(t-s)ds, \; n = 2,3,4,\ldots.$$

In general it follows that

$$u_n(s,t) = \int_{x=0}^{t-s} g(x)dx \int_{y=0}^{s} h(y)u_{n-1}(s-y,t-y-x)dy. \tag{4.2.13}$$

If we assume that

$$u_j(s,t) = h^{(j)}(s)\{G^{(j)}(t-s) - G^{(j+1)}(t-s)\},\ j = 1,2,\ldots,n-1,$$

then we obtain from (4.2.9) and (4.2.13) by mathematical induction that

$$u_n(s,t) = h^{(n)}(s)\{G^{(n)}(t-s) - G^{(n+1)}(t-s)\},$$

$$n = 1,2,3,\ldots. \qquad (4.2.14)$$

The expression for $v_n(s,t)$ is derived in a similar manner. Upon following the steps analogous to those followed in the derivation of (4.2.14) we find that

$$v_1(s,t) = g(t-s)[1-H(s)] \qquad (4.2.15)$$

$$v_n(s,t) = \int_0^{t-s} g(x)dx \int_0^s h(y)v_{n-1}(s-y,t-y-x)dy,$$

$$n = 2,3,\ldots, \qquad (4.2.16)$$

and

$$v_n(s,t) = g^{(n)}(t-s)\{H^{(n-1)}(s) - H^{(n)}(s)\},$$

$$n = 1,2,3,\ldots, \qquad (4.2.17)$$

Since the probabilities expressed by $u_n(s,t)ds$ and $v_n(s,t)ds$ account for all possible time spent in state B, provided $0 < s < t$, we conclude that

$$p_B(s,t) = \sum_{n=1}^{\infty} \{u_n(s,t) + v_n(s,t)\}. \qquad (4.2.18)$$

From equations (4.2.9), (4.2.13), (4.2.15), and (4.2.16), it then follows that

$$p_B(s,t) = h(s)\{G(t-s) - G^{(2)}(t-s)\} + g(t-s)\{1 - H(s)\} +$$

$$+ \int_0^{t-s} g(x)\,dx \int_0^s h(y)\,p_B(s-y),t-y-x)\,dy \qquad (4.2.19)$$

and from (4.2.14), (4.2.17) and (4.2.18) we have

$$p_B(s,t) = \sum_{n=1}^{\infty} \{h^{(n)}(s)[G^{(n)}(t-s) - G^{(n+1)}(t-s)] +$$

$$+ g^{(n)}(t-s)[H^{(n-1)}(s) - H^{(n)}(s)]\},$$

$$0 < s < t. \qquad (4.2.20)$$

Upon rearranging the terms of (4.2.20) it is not difficult to show that $p_B(s,t)$ is the derivative of

$$P_B(s,t) = \sum_{n=0}^{\infty} H^{(n)}(s)[G^{(n)}(t-s) - G^{(n+1)}(t-s)] \qquad (4.2.21)$$

and that $P_B(s,t)$ satisfies the equation

$$P_B(s,t) = 1 - G(t-s) +$$

$$+ \int_0^{t-s} g(x)\,dx \int_0^s h(y)\,P_B(s-y,t-y-x)\,dy. \qquad (4.2.22)$$

Good [23] has given an expression corresponding to (4.4.20) for the general n-state problem in terms of a multiple contour integral. Good assumes, however, that the infinitesimal transition probabilities, $\Pr\{S(t+dt) \in B|S(t) \in A\}$ and $\Pr\{S(t+dt) \in A|S(t) \in B\}$, are constant, i.e. independent of time. Our assumptions do not eliminate the possibility that these transition probabilities are time dependent and in this sense the results given here are more general than Good's results. Equation (4.2.21) has been obtained by Takács [34] using an argument similar to, but more obscure than, the arguments used to derive (4.2.20). The expressions given in (4.2.20) and (4.2.21) could be obtained directly from the integral equations (4.2.19) and (4.2.22) using an iterative procedure. The underlying probabilistic aspects of the sojourn distributions are more clearly illustrated, however, by the derivation given in this Section. We now turn to some particular cases.

4.3 Strictly Positive Sojourn Time for the Unrestricted Random Walk with Negative Exponentially Distributed Intervals Between Steps

In this Section we shall say that the particle is in state A at time t if $S(t) \leq 0$ and in state B if $S(t) > 0$. The particle passes from state A to state B when a transition from 0 to +1 is made and from B to A when a passage from +1 to 0 occurs. Therefore, the time required for the random walk to pass from state A to state B after entering state A is, in the terminology of Chapter III, the first-passage time from 0 to +1. Similarly, the time required to pass from state B to state A is equal to the first-passage time from +1 to 0. In the notation of Chapter III, we have then

$$g(t) = f_{01}(t) \tag{4.3.1}$$

and

$$h(t) = f_{10}(t) \tag{4.3.2}$$

where $f_{01}(t)$ and $f_{10}(t)$ are given in (3.2.14) and (3.2.16), respectively.

Thus, from the results of Section 3.2, we have

$$g^{(n)}(t) = \rho^{n/2} e^{-(\lambda+\mu)t} \frac{nI_n(2t\sqrt{\lambda\mu})}{t}, \tag{4.3.3}$$

and

$$h^{(n)}(t) = \rho^{-n/2} e^{-(\lambda+\mu)t} \frac{nI_n(2t\sqrt{\lambda\mu})}{t}, \tag{4.3.4}$$

where $\rho = \lambda/\mu$.

From equation (3.2.11) and symmetry considerations we have

$$g^*(z) = \phi_{01}(z) = \frac{Z - R}{2\mu} \tag{4.3.5}$$

and

$$h^*(z) = \phi_{10}(z) = \frac{Z - R}{2\lambda} \tag{4.3.6}$$

where

$$Z = \lambda + \mu + z, \quad R^2 = Z^2 - 4\lambda\mu, \tag{4.3.7}$$

$\phi_{mn}(z)$ is defined by equation (3.1.2), and $g^*(z)$ and $h^*(z)$ denote the Laplace transforms of $g(t)$ and $h(t)$, respectively.

Upon combining the above equations and using the convolution property of the Laplace transformation together with equation (4.2.20), it follows that

$$p_B(s,t) = \frac{1}{(2\pi i)^2} \int_{C_2} e^{vs} \sum_{n=1}^{\infty} \{[\phi_{10}(v)]^n dv \cdot \int_{C_1} e^{u(t-s)} [\phi_{10}(u)]^n \phi_{01}(u)\,du\} + \tag{4.3.8}$$

$$+ \frac{1}{(2\pi i)^2} \int_{C_1} e^{u(t-s)} \sum_{n=1}^{\infty} \{[\phi_{01}(u)]^n du \cdot \int_{C_2} e^{vs} [\phi_{10}(v)]^{n-1} \phi_{10}(v)\,dv\}$$

where C_1, C_2 are Laplace inversion contours in the u,v planes, respectively, and $\phi_{mn}(z)$ is defined by (3.1.2). From (4.3.5) and (4.3.6) it is not difficult to show that $|\phi_{01}(z)| < 1$ and $|\phi_{10}(z)| < 1$ for $Re(z) > 0$. Therefore it follows from (4.3.8) that

$$p_B(s,t) = \frac{1}{(2\pi i)^2} \int_{C_1} e^{u(t-s)} \phi_{01}(u)\,du \int_{C_2} e^{vs} \cdot \frac{[\phi_{10}(v)\phi_{01}(u) + \phi_{10}(v)]}{1 - \phi_{10}(v)\phi_{01}(u)} dv. \tag{4.3.9}$$

Upon making the substitutions $u + \lambda + \mu = (\lambda\mu)^{1/2}(e^{x} + e^{-x})$ and $v + \lambda + \mu = (\lambda\mu)^{1/2}(e^{y} + e^{-y})$, we obtain

$$\phi_{01}(u) = \rho^{1/2}e^{-x}, \quad \phi_{10}(v) = \rho^{-1/2}e^{-y}, \tag{4.3.10}$$

and

$$\Phi_{01}(u) = \frac{e^{-x}}{(\lambda\mu)^{1/2}(1-\rho^{-1/2}e^{-x})}, \quad \Phi_{10}(v) = \frac{e^{-y}}{(\lambda\mu)^{1/2}(1-\rho^{1/2}e^{-y})}. \tag{4.3.11}$$

When these expressions are substituted in the integrand of (4.3.9), it follows immediately that

$$\frac{\phi_{01}(u)[\phi_{10}(v)\Phi_{01}(u)+\Phi_{10}(v)]}{1-\phi_{01}(u)\,\phi_{10}(v)} = \lambda\Phi_{10}(v)\Phi_{01}(u). \tag{4.3.12}$$

Using the inversion properties of the Laplace transform, it then follows from (4.3.9) and (4.3.12) that

$$p_B(s,t) = \lambda F_{01}(t-s)F_{10}(s) \tag{4.3.13}$$

where $F_{01}(t)$ and $F_{10}(t)$ are given by (3.2.17).

Various forms of (4.3.13) may be obtained by making use of the expressions for $F_{01}(t)$ and $F_{10}(t)$ given by (3.2.17), (3.2.22), and (3.2.23).

We note that the result for $p_B(s,t)$ given by (4.3.13) coincides with the expression for $m(s,t)$ given in (3.5.9). This suggests an analogy to the "coin-tossing" results in

[18], Chapter III. Further reference to this analogy will be made in Chapter V.

We note that in the symmetric case, i.e., when $\lambda = \mu$, we have

$$\Phi_{01}(z) = \Phi_{10}(z) = \frac{1}{z} - \frac{[Z-R]}{2\mu z} \qquad (4.3.14)$$

which, after appropriate algebraic manipulation, becomes

$$\Phi_{01}(z) = \Phi_{10}(z) = \frac{1}{R} + \frac{2\mu}{R(Z+R)}. \qquad (4.3.15)$$

Hence, with the aid of [17], Vol. I, (30), p. 240, we find that

$$F_{01}(t) = F_{10}(t) = e^{-2\mu t}\{I_o(2\mu t) + I_1(2\mu t)\}, \quad (4.3.16)$$

and therefore, when $\lambda = \mu$, we have from (4.3.13) and (4.3.16) that

$$p_B(s,t) = \mu e^{-2\mu t}\{I_0(2\mu s) + I_1(2\mu s)\}\{I_0(2\mu[t-s]) + I_1(2\mu[t-s])\},$$
$$0 < s < t. \qquad (4.3.17)$$

Corresponding results for state A follow from (4.2.5) and (4.2.6).

In the foregoing we have assumed that $S(0) = 0$. Suppose now $S(0) = k$ where $k = 1,2,3,\ldots$. To find $p_B^{(k)}(s,t)$ we observe that

$$p_B^{(k)}(s,t) = f_{k0}(s)F_{01}(t-s) + \int_0^s f_{k0}(x)p_B^{(0)}(s-x,t-x)dx. \quad (4.3.18)$$

Equation (4.3.18) follows from the fact that after accounting for the first-passage time from k to 0, the process renews itself as if it were beginning again in state zero. Using (4.3.13), (4.3.18) and the convolution property of the Laplace transform, it follows that

$$p_B^{(k)}(s,t) = f_{k0}(s)F_{01}(t-s)+\lambda F_{01}(t-s)\frac{1}{2\pi i}\int_C e^{zs}\Phi_{10}(z)[\Phi_{10}(z)]^k dz$$

$$= F_{01}(t-s)\{f_{k0}(s)+\lambda[F_{k+1,0}(s)-F_{k,0}(s)]\}, \qquad (4.3.19)$$

the latter expression having been obtained with the aid of equation (3.2.19). Now since $\phi_{10}(z)$ satisfies the quadratic equation given by (3.2.9), it follows that

$$f_{k0}(s) + \lambda\{F_{k+1,0}(s) - F_{k,0}(s)\} = \mu\{F_{k,0}(s) - F_{k-1,0}(s)\},$$

$$k = 1,2,\dots. \qquad (4.3.20)$$

Equation (4.3.19) can then be rewritten as

$$p_B^{(k)}(s,t) = \lambda F_{10}(s)F_{01}(t-s),\ k = 0,$$

$$= \mu F_{10}(s)F_{01}(t-s),\ k = 1, \qquad (4.3.21)$$

$$= \mu F_{01}(t-s)[F_{k,0}(s) - F_{k-1,0}(s)],$$

$$0 < s < t,\ k = 2,3,\dots,$$

where $F_{n,0}(t)$ is given by equation (3.2.17). The exact expressions for $p_B^{(k)}(s,t)$ given in (4.3.21) are apparently new.

As in the case for S(0) = 0, it follows that

$$\Pr\{\sigma_B(t) = 0|S(0) = k\} = F_{01}(t) \quad , \; k = 1, \qquad (4.3.22)$$
$$= 0 \qquad , \; k = 1,2,...,$$

$$\Pr\{\sigma_B(t) = t|S(0) = k\} = 0 \qquad , \; k = 0, \qquad (4.3.23)$$
$$= F_{k0}(t) \quad , \; k = 1,2,....$$

Analogous results hold for the cases k = -1, -2,....

During the interval (0,t) the particle must spend some, none, or all of the time in state B. Therefore we expect that

$$\Pr\{0\leq\sigma_B(t)\leq t|S(0) = k\} = \Pr\{\sigma_B(t) = 0|S(0) = k\} + \qquad (4.3.24)$$
$$+ \Pr\{\sigma_B(t) = t|S(0) = k\} + \int_0^t p_B^{(k)}(s,t)ds = 1 \; .$$

Combining equations (4.3.21) - (4.4.23), (3.2.18), and (3.2.9), we see that for k = 0,

$$\int_0^\infty e^{-zt} \Pr\{0\leq\sigma_B(t)\leq t\}dt = \Phi_{01}(z)[1+\lambda\Phi_{10}(z)] = \frac{1}{z} \; , \qquad (4.3.25)$$

which implies that (4.3.24) holds. That equation (4.3.24) remains true for k = 1,2,3,... is verified in a similar manner.

From the foregoing results we can also deduce the result for the more general situation where the random walk is said to be in state A_n at time t if $S(t) \leq n$, and in state B_n otherwise. We assume for convenience that $S(0) = 0$ and $n = 0,1,2,\ldots.$ Then $p_{B_0}(s,t)$ is given in (4.3.21). For $n = 1,2,3,\ldots$, we observe that after a first passage to n is made, the problem reduces to that of the positive sojourn time considered in the beginning of this section. That is, $p_{B_n}(s,t)$ satisfies the equation

$$p_{B_n}(s,t) = \int_0^{t-s} f_{0n}(x)p_{B_0}(s,t-x)dx \tag{4.3.26}$$

where $p_{B_0}^{(0)}(s,t)$ is given in (4.3.21). Substituting the result for $p_{B_0}(s,t)$ into (4.3.26) yields

$$p_{B_n}(s,t) = F_{10}(s)\{F_{0,n+1}(t-s) - F_{0,n}(t-s)\}. \tag{4.3.27}$$

This completes the derivation of $p_B^{(k)}(s,t)$ and the related probability density function $p_{B_n}(s,t)$. These results give probabilistic information about the proportion of time during a soccer game of length t that one team remains in the lead. To obtain similar information about a very long soccer game, we may examine the asymptotic behavior of $p_B^{(k)}(s,t)$. Because of the similarity of the discrete-time random walk discussed in [18], Chapter III, to the continuous-time walk considered here, we would expect the asymptotic results to be similar to those for the discrete-time walk. In particular, for $\lambda = \mu$, we expect the distribution of

$\sigma_B(t)$ to follow a so-called arc sine law. This is the case as will be shown in the following.

Since $m(s,t) = p_B^{(0)}(s,t)$, the asymptotic results of Section 3.5 apply. To obtain asymptotic results for $p_B^{(k)}(s,t)$, we can apply the results obtained in Section 3.2. From equations (3.2.30), (3.5.16), (3.5.17), and (4.3.21), it follows that for fixed λ, μ, as t-s, $s\to\infty$,

$$p_B^{(0)}(s,t) \sim \frac{2\lambda(1+\sqrt{\rho})e^{-\mu(1-\sqrt{\rho})^2 s}}{\sqrt{\pi}(1-\sqrt{\rho})(s\sqrt{\lambda\mu})^{1/2}} \left[\frac{1}{(4s\sqrt{\lambda\mu})} + \right.$$

$$\left. + 0\left(\frac{1}{16\lambda\mu s^2}\right)\right], \quad \rho < 1,$$

$$\sim \frac{1}{\pi[s(t-s)]^{1/2}}, \quad \rho = 1. \tag{4.3.28}$$

$$\sim \frac{2\ (1+\sqrt{\rho})e^{-\mu(1-\sqrt{\rho})^2(t-s)}}{\sqrt{\pi}[(t-s)\sqrt{\lambda\mu}]^{1/2}(\sqrt{\rho}-1)} \left[\frac{1}{[4(t-s)\sqrt{\lambda\mu}]} + \right.$$

$$\left. + 0\left(\frac{1}{16\lambda\mu(t-s)^2}\right)\right], \qquad \rho > 1.$$

By (4.3.21), $p_B^{(1)}(s,t) = \rho^{-1}p_B^{(0)}(s,t)$ and therefore its asymptotic behavior can be deduced from (4.3.28). For $k = 2,3,4,\ldots$, we have

$$p_B^{(k)}(s,t) \sim \frac{2\lambda(1+\sqrt{\rho})\rho^{-(k-1)/2}e^{-\mu(1-\sqrt{\rho})^2 s}}{[s\sqrt{\lambda\mu}]^{1/2}} \times$$

$$\times\left[\frac{1}{(4s\sqrt{\lambda\mu})} + 0\left(\frac{1}{16\lambda\mu s^2}\right)\right], \qquad \rho < 1,$$

$$\sim \frac{1}{\pi\sqrt{s(t-s)}}, \qquad \rho = 1,$$

$$(4.3.29)$$

$$\sim \frac{2\lambda(1+\sqrt{\rho})\rho^{-k}e^{-\mu(1-\sqrt{\rho})^2(t-s)}}{\sqrt{\pi}[(t-s)\sqrt{\lambda\mu}]^{1/2}} \times$$

$$\times\left[\frac{1}{[4(t-s)\sqrt{\lambda\mu}]} + 0\left(\frac{1}{16\lambda\mu(t-s)^2}\right)\right], \quad \rho > 1.$$

Upon integrating by parts and applying the binomial expansion, we find that, provided $0 < \alpha < 1$, as $t \to \infty$,

$$\Pr\{\alpha t<\sigma_B(t)<t\,|\,S(0) = 0\} = \int_{\alpha t}^{t} p_B^{(0)}(s,t)ds \sim \frac{\rho^{1/4}(1+\sqrt{\rho})}{\sqrt{\pi}} \times$$

$$\times\left[\frac{e^{-\Lambda\alpha t}}{\sqrt{\Lambda\alpha t}} - \frac{e^{-\Lambda t}}{\sqrt{\Lambda t}} - \int_{\sqrt{2\Lambda\alpha t}}^{\sqrt{2\Lambda t}} e^{-y^2/2}dy\right], \ \rho < 1,$$

$$\sim 1 - \frac{2}{\pi}\ \text{arc sin}\sqrt{\alpha}\ , \ \rho = 1,$$

$$(4.3.30)$$

where

$$\Lambda = \mu(1-\rho^{1/2})^2 \tag{4.3.31}$$

For k = 1, we have

$$\Pr\{\alpha t<\sigma_B(t)<t|S(0) = 1\} = \rho^{-1}\int_{\alpha t}^{t} p_B^{(0)}(s,t)ds, \tag{4.3.32}$$

and for k = 2,3,..., we have

$$\Pr\{\alpha t<\sigma_B(t)<t|S(0) = k\} \sim \frac{\rho^{1/4}(1+\rho^{1/2})\rho^{-(k-1)/2}}{\pi^{1/2}} \cdot$$

$$\cdot \left[\frac{e^{-\Lambda\alpha t}}{(\Lambda\alpha t)^{1/2}} - \frac{e^{-\Lambda t}}{(\Lambda t)^{1/2}} - \int_{(2\Lambda\alpha t)^{1/2}}^{(2\Lambda t)^{1/2}} e^{-y^2/2}dy\right],$$

$$\rho < 1,$$

$$\sim 1 - \frac{2}{\pi} \text{ arc sin } \sqrt{\alpha}\ , \qquad \rho = 1, \tag{4.3.33}$$

$$\sim \frac{2\rho^{1/4}(1+\sqrt{\rho})\rho^{-k}}{\sqrt{\pi}(\sqrt{\rho} - 1)}\left[1 - \frac{e^{-\Lambda(1-\alpha)t}}{[t(1-\alpha)]^{1/2}} - \right.$$

$$\left. - \Lambda^{1/2}\int_0^{[2\Lambda t(1-\alpha)]^{1/2}} e^{-y^2/2}\ dy\right], \quad \rho > 1.$$

The result contained in (4.3.33) for $\rho = 1$ has also been suggested by the work of Takács [35]. For the case of $\rho \neq 1$, however, Takacs' work does not apply.

In terms of the soccer game described in Chapter I, (4.3.33) can be interpreted to mean that during a long game between two evenly matched teams, the most likely event is that one team or the other leads most of the time. As predicted, this result corresponds exactly to that for the coin-tossing game discussed in [18], Chapter III; that is, Peter's chances of winning are not improved by tossing the coin at random instants in time. More will be said about the interpretation of this result and its validity as an approximation to true values in Section 4.6. In Chapter V, we shall refer again to the arc-sine distribution given in (4.3.33) in the course of discussing the connection of our work with that of E. Sparre Andersen.

Here again, we note the appearance of the stable distribution of index 1/2 defined in (2.6.8). As mentioned in Section 4.1, its presence is perhaps not surprising since the positive sojourn time is the sum of disjoint intervals of time during which the particle remains in the positive state. Therefore, we might expect that the stable laws which govern the asymptotic behavior of sums of random variables also influence the asymptotic behavior of the positive sojourn time.

4.4 Zero and Non-zero Sojourn Time for the Unrestricted Random Walk with Negative Exponentially Distributed Intervals Between Steps

In this Section we say that the particle is in state A at time t if $S(t) = 0$ and in state B if $S(t) \neq 0$. As before we shall concern ourselves with the probability $\Pr\{s<\sigma_B(t)<s+ds\} = p_B(s,t)ds$. In terms of a soccer game, we wish to determine the distributional properties of the amount of time during the time interval $(0,t)$ that the score remains tied or untied.

After the particle enters state A, it passes from the zero state as soon as a positive or negative step occurs. Since these steps occur according to two independent Poisson streams with parameters λ,μ, we can write, in our usual notation,

$$g(t) = (\lambda+\mu)e^{-(\lambda+\mu)t}, \quad 0 < t < \infty, \qquad (4.4.1)$$

and

$$G(t) = 1 - e^{-(\lambda+\mu)t}, \quad 0 < t < \infty. \qquad (4.4.2)$$

To obtain $h(t)$ we consider the two possible cases, that of a first passage from state A to +1 and first passage from state A to -1. Given that the former occurs, we have $\Pr\{t<\tau_{BA}<t+dt|\text{passage to }+1\} = f_{10}(t)dt$, and if the latter occurs, we have $\Pr\{t<\tau_{BA}<t+dt|\text{passage to }-1\} = f_{01}(t)dt$. Because of the properties of the negative exponential distribution, it follows by elementary probability argument

that given a step occurs, either positive or negative, the probability that this step is positive is equal to $\lambda/(\lambda+\mu)$ and that it is negative is equal to $\mu/(\lambda+\mu)$. Therefore, we conclude that

$$h(t)dt = \frac{\mu}{(\lambda+\mu)} \Pr\{t<\tau_{BA}<t + dt|\text{ passage to } -1\}+$$

$$+ \frac{\lambda}{(\lambda+\mu)} \Pr\{t<\tau_{BA}<t + dt|\text{ passage to } +1\} \qquad (4.4.3)$$

$$= \frac{[\mu f_{01}(t) + \lambda f_{10}(t)]dt}{(\lambda+\mu)}$$

We are now in a position to derive $P_B(s,t)$ and $p_B(s,t)$.

We assume first of all that $S(0) = 0$. From (4.2.21) we have

$$P_B(s,t) = e^{-(\lambda+\mu)(t-s)} \sum_{n=0}^{\infty} \frac{[(\lambda+\mu)(t-s)]^n}{n!} H^{(n)}(s) \qquad (4.4.4)$$

where

$$H^{(n)}(s) = \int_0^s h^{(n)}(x)dx = \int_0^s H^{(n-1)}(x)h(s-x)dx. \qquad (4.4.5)$$

We define, as usual,

$$h^*(z) = \int_0^\infty e^{-zt}h(t)dt, \qquad (4.4.6)$$

and upon applying the convolution property of the Laplace transform to (4.4.5), it follows that

$$H^{(n)}(s) = \frac{1}{2\pi i}\int_C e^{zs} \frac{[h^*(z)]^n}{z} dz \qquad (4.4.7)$$

where C is the usual Laplace inversion contour. Combining (4.4.3), (4.4.4) and (4.4.7), and using the fact that $\phi_{01}(z) = (Z-R)/2\mu$ and $\phi_{10}(z) = (Z-R)/2\lambda$, we have

$$P_B(s,t) = e^{-(\lambda+\mu)(t-s)} \sum_{n=0}^{\infty} \frac{[(\lambda+\mu)(t-s)]^n}{n!} \frac{1}{2\pi i}\int_C \frac{e^{zs}}{z} \left[\frac{Z-R}{(\lambda+\mu)}\right]^n dz$$

$$= e^{-(\lambda+\mu)(t-s)} \frac{1}{2\pi i} \int_C \frac{e^{zs}}{z} e^{(t-s)(Z-R)} dz \qquad (4.4.8)$$

where

$$Z = \lambda + \mu + z, \ R^2 = Z^2 - 4\lambda\mu. \qquad (4.4.9)$$

Interchanging summation and integration in (4.4.8) is justified by the absolute convergence of the right-hand side. With the aid of [17], Vol. I, (41), p. 250, we obtain

$$P_B(s,t) = e^{-(\lambda+\mu)(t-s)}\left[1 + 2\sqrt{\lambda\mu}\ (t-s)\int_0^s e^{-(\lambda+\mu)x} \cdot \frac{I_1[2\sqrt{\lambda\mu x[x+2(t-s)]}}{\sqrt{x^2 + 2x(t-s)}}\right]dx, \qquad (4.4.10)$$

and upon making the substitution $w = 2\sqrt{\lambda\mu}\ [x+t-s]$, (4.4.10) becomes

$$P_B(s,t) = e^{-(\lambda+\mu)(t-s)} + 2\sqrt{\lambda\mu}(t-s) \int_{2\sqrt{\lambda\mu}\ (t-s)}^{2\sqrt{\lambda\mu}\ t} e^{-\frac{(\lambda+\mu)w}{2\sqrt{\lambda\mu}}} \cdot \frac{I_1[\sqrt{w^2-4\lambda\mu(t-s)^2}]}{\sqrt{w^2-4\lambda\mu(t-s)^2}}\, dw \qquad (4.4.11)$$

We observe that the integral expression in (4.4.11) is in the form of $f(c,X,T)$ given by (2.4.15) with

$$c = \frac{(\lambda+\mu)}{2\sqrt{\lambda\mu}}, \quad X = 2\sqrt{\lambda\mu}\,(t-s), \quad T = 2\sqrt{\lambda\mu}\,t. \tag{4.4.12}$$

By equation (2.4.19) it then follows that

$$P_B(s,t) = e^{(\mu-\lambda)(t-s)}J_1[\mu(2t-s),\lambda s]+e^{(\lambda-\mu)(t-s)}J_0[\lambda(2t-s),\mu s] \tag{4.4.13}$$

where $J_n(x,y)$ is defined by (2.4.1).

Upon differentiating the expression for $P_B(s,t)$ given in (4.4.13) and using the identities for modified Bessel functions of the first kind given in (2.3.9) and (2.3.10), we obtain

$$\begin{aligned} p_B(s,t) = {} & e^{-(\lambda+\mu)t}\left[(\lambda+\mu)I_0[\ell(s)] + 2t\left(\frac{\lambda\mu}{s(2t-s)}\right)^{1/2}I_1[\ell(s)]\right]+ \\ & + (\lambda-\mu)e^{(\mu-\lambda)(t-s)}J_1[\mu(2t-s),\lambda s] + \\ & + (\mu-\lambda)e^{(\lambda-\mu)(t-s)}J_1[\lambda(2t-s),\mu s] , \end{aligned} \tag{4.4.14}$$

where $\ell(x)$ is defined by

$$\ell(x) = 2\sqrt{\lambda\mu x(2t-x)} . \tag{4.4.15}$$

Alternate forms of $p_B(s,t)$ may be obtained by using the results for J-functions given in Section 2.4.

It follows from (4.4.14) that for $\lambda = \mu$,

$$p_B(s,t) = 2\mu e^{-2\mu t}\left[I_0[\ell(s)] + t\,\frac{I_1[\ell(s)]}{\sqrt{s(2t-s)}}\right], \qquad (4.4.16)$$

and hence

$$P_B(s,t) = 2\mu e^{-2\mu t}\int_0^s \left(I_0[\ell(x)] + t\,\frac{I_1[\ell(x)]}{\sqrt{x(2t-x)}}\right)dx \qquad (4.4.17)$$

where $\ell(x)$ is given by (4.4.15).

Expressions for $P_A(s,t)$ and $p_A(s,t)$ may be obtained from equation (4.2.5) and (4.2.6), respectively.

If we assume that $S(0) = k > 0$, then we can derive expressions for $P_B^{(k)}(s,t)$ and $p_B^{(k)}(s,t)$ by observing that

$$p_B^{(k)}(s,t) = \int_0^s f_{k0}(x)p_B^{(0)}(s-x,t-x)\,dx \qquad (4.4.18)$$

where $f_{k0}(t)$ is given in (3.2.16). From (4.4.13) and (4.4.18) it follows that

$$p_B^{(k)}(s,t) = e^{-(\lambda+\mu)t}\,\rho^{-k/2}\left[\sum_{n=0}^{\infty} \rho^{-n/2}\,L_{k,n}(s,2t-s) + \right.$$

$$\left. + \sum_{n=1}^{\infty} \rho^{n/2}\,L_{k,n}(s,2t-s)\right] \qquad (4.4.19)$$

where

$$L_{k,n}(v,w) = k\int_0^v \frac{I_k(2x\sqrt{\lambda\mu})}{x}\left(\frac{v-x}{w-x}\right)^{n/2} I_n[2\sqrt{\lambda\mu(v-x)(w-x)}]\,dx$$

$$(4.4.20)$$

$$= k\int_0^v \frac{I_k(2[v-x]\sqrt{\lambda\mu})}{v-x}\left(\frac{x}{w-v+x}\right)^{n/2} I_n[2\sqrt{\lambda\mu x(w-v+x)}]\,dx$$

Making use of [17], Vol. I, (44), p. 250, and the convolution property of the Laplace transform, we obtain

$$L_{k,n}(v,w) = \frac{1}{2\pi i} \int_C e^{zs} \frac{(2\sqrt{\lambda\mu})^{n+k}}{R(Z+R)^{n+k}} e^{(t-s)(Z-R)} dz \qquad (4.4.21)$$

where

$$Z = z + \lambda + \mu, \quad R^2 = z^2 - 4\lambda\mu, \qquad (4.4.22)$$

and C is the usual Laplace inversion contour. Upon inversion we find that

$$L_{k,n}(v,w) = \left(\frac{v}{w}\right)^{(n+k)/2} I_{n+k}[2\sqrt{\lambda\mu v w}\,] . \qquad (4.4.23)$$

Therefore by (4.4.23) and (2.4.1) we have

$$P_B^{(k)}(s,t) = e^{(\mu-\lambda)(t-s)} \rho^{-k} J_{k+1}[\mu(2t-s),\lambda s] +$$

$$+ e^{(\lambda-\mu)(t-s)} J_k [\lambda(2t-s),\mu s]. \qquad (4.4.24)$$

We verify that $P_B^{(k)}(t,t) = 1$ by noting that putting $s = t$ in (4.4.24) and using (2.3.14) yields Schlomilch's identity (see equation 2.3.11).

The $\Pr\{\sigma_B(t) = 0|S(0) = k\}$ can be obtained from (4.4.24) by observing that

$$\Pr\{\sigma_B(t) = 0|S(0) = k\} = P_B^{(k)}(0,t). \qquad (4.4.25)$$

On the other hand, $\sigma_B(t) = t$ only if $S(0) = k$, $k = 1,2,...,$

and no first passage to zero occurs during $(0,t)$. Therefore we have

$$\Pr\{\sigma_B(t) = t|S(0) = k\} = 0, \; k = 0,$$
$$= F_{k0}(t), \; k = 1,2,..., \tag{4.4.26}$$

where $F_{k0}(t)$ is given by (3.2.17).

Upon differentiating $P_B^{(k)}(s,t)$ as given in (4.4.24) and using (2.3.9) and (2.3.10), we find that the probability density function $p_B^{(k)}(s,t)$ is given by

$$p_B^{(k)}(s,t) = e^{-(\lambda+\mu)t}\left[\mu\left(\frac{\mu s}{\lambda(2t-s)}\right)^{\frac{k-1}{2}} I_{k-1}[\ell(s)] + \right.$$

$$\left. + (\lambda+\mu)\left(\frac{\mu s}{\lambda(2t-s)}\right)^{k/2} I_k[\ell(s)] + \lambda\left(\frac{\mu s}{\lambda(2t-s)}\right)^{\frac{k+1}{2}} I_{k+1}[\ell(s)]\right]$$

$$+ (\lambda-\mu)\rho^{-k}e^{(\mu-\lambda)(t-s)}J_{k+1}[\mu(2t-s), \lambda s] + \tag{4.4.27}$$

$$+ (\mu-\lambda)e^{(\lambda-\mu)(t-s)}J_{k+1}[\lambda(2t-s),\mu s], \; k = 0,1,2,...,$$

where $\ell(x)$ is given by (4.4.15). The expressions for $P_B^{(k)}(s,t)$ and $p_B^{(k)}(s,t)$ given in (4.424) and (4.4.27) respectively, are apparently new.

For the symmetric case, $\lambda = \mu$, (4.4.24) and (4.4.27) become

$$p_B^{(k)}(s,t) = J_{k+1}[\mu(2t-s),\mu s] + J_k[\mu(2t-s),\mu s] \quad ,(4.4.28)$$

and

$$p_B^{(k)}(s,t) = \mu e^{-2\mu t}\left[\left(\frac{s}{2t-s}\right)^{\frac{(k-1)}{2}} I_{k-1}[\ell(s)] + 2\left(\frac{s}{2t-s}\right)^{\frac{k}{2}} I_k[\ell(s)] + \right.$$

$$\left. + \left(\frac{s}{2t-s}\right)^{\frac{(k+1)}{2}} I_{k+1}[\ell(s)]\right] \quad , \; k = 0,1,2,\ldots. \qquad (4.4.29)$$

From (4.4.29) we can obtain a pleasing asymptotic formula for $p_B^{(k)}(s,t)$ in the case $\lambda = \mu$. Using the asymptotic properties of the modified Bessel function and the J-function given in (2.3.2) and (2.4.2), respectively, we find that as $t \to \infty$,

$$p_B^{(k)}(s,t) \sim \sqrt{\frac{\mu}{\pi}} \; e^{-2\mu\{t-\sqrt{s(2t-s)}\}}\left(\frac{s}{2t-s}\right)^{k/2} \times$$

$$\times\left[\frac{t}{[s(2t-s)]^{3/4}} + \frac{t}{[s(2t-s)]^{1/4}}\right]. \qquad (4.4.30)$$

By making the substitution $y^2/2 = 2\mu\{t-\sqrt{s(2t-s)}\}$ and applying the binomial expansion to the denominator of the integrand, we find for fixed α, $0 < \alpha < 1$, as $t \to \infty$,

$$\int_{\alpha t}^{t} p_B^{(k)}(s,t)\,ds \sim \sqrt{\frac{2}{\pi}}\int_0^{2[\mu t(1-\sqrt{2\alpha-\alpha^2}]^{1/2}} e^{-y^2/2} \times$$

$$\times \left[1 - \frac{ky}{\sqrt{2\mu t}} + 0\left(\frac{1}{\mu t}\right)\right]dy \; . \qquad (4.4.31)$$

We note that as $t\to\infty$, $\Pr\{\alpha t<\sigma_B(t)<t\}\to 1$. In terms of the soccer application, this result can be interpreted to mean that if two evenly matched teams play for a long time, they are likely to remain untied most of the time. This is consistent with the results of Section 4.3 but may be contrary to intuition. We remark that the larger the value of k, the more slowly $\Pr\{\alpha t<\sigma_B(t)<t\}$ tends to unity.

We observe that the limiting form of $\Pr\{\alpha t<\sigma_B(t)<t\}$ given in (4.4.31) is expressed in terms of the stable distribution of index 1/2. Takács [35] has given a similar, but slightly different, asymptotic result, which will be discussed in Section 4.6.

Unfortunately, no meaningful asymptotic results have been obtained for the case $\lambda \neq \mu$ as yet.

Numerical complements based on (4.4.31) will be given in Section 4.6.

4.5 The Busy Time for the M/M/1 Queuing Process

In this Section we apply the results of Section 4.2 to a queuing situation, i.e., we shall derive the distribution of the busy time and the idle time (as described in Chapter I) of the server in the M/M/1 queuing process. We assume that $S(0) = 0$, i.e., the server is idle at time $t = 0$. We say that the system is in state A if the server is idle and in state B if the server is busy. In the notation of this Chapter, $\sigma_A(t)$ represents the amount of time during the interval $(0,t)$ that the server is idle and $\sigma_B(t)$ the amount of time during $(0,t)$ that the server is busy. Accordingly, $g(t)$ and $h(t)$ correspond to the idle and busy period density functions for the M/M/1 queue, respectively, where idle and busy period have been defined in Chapter I. We write then

$$g(t) = \lambda e^{-\lambda t} \tag{4.5.1}$$

and

$$h(t) = f_{10}(t) = \rho^{-1/2} e^{-(\lambda+\mu)t} I_1 \frac{(2t\sqrt{\lambda\mu})}{t}, \tag{4.5.2}$$

the latter having been obtained from equation (3.6.16).

Noting that $f_{10}(t)$ is the same for both the queuing random walk and the doubly infinite walk, it follows from (4.2.21) that

$$P_B(s,t) = e^{-\lambda(t-s)} \sum_{n=0}^{\infty} \frac{[\lambda(t-s)]^n}{n!} B^{(n)}(s)$$

$$= e^{-\lambda(t-s)} \sum_{n=0}^{\infty} \frac{[\lambda(t-s)]^n}{n!} \frac{1}{2\pi i} \int_C \frac{e^{zs}}{z} [\phi_{10}(z)]^n dz$$

$$= e^{-\lambda(t-s)} \frac{1}{2\pi i} \int_C \frac{e^{zs}}{z} e^{\frac{(t-s)}{2}(Z-R)} dz, \tag{4.5.3}$$

the interchange of integration and summation having been justified because of the absolute convergence of the right-hand side of (4.5.3). By comparison of (4.5.3) and (4.4.8) it follows from (4.4.10) that

$$p_B(s,t) = e^{-\lambda(t-s)}\{1+2(t-s)(\lambda\mu)\int_0^s e^{-(\lambda+\mu)s}\frac{I_1[\ell(x)]dx\}}{\ell(x)} \tag{4.5.4}$$

where

$$\ell(x) = 2\{\lambda\mu[x^2+x(t-s)]\}^{1/2} \tag{4.5.5}$$

By substituting $w = 2\sqrt{\lambda\mu}\ [x + (t-s)/2]$ in the integral in (4.5.4), we obtain

$$p_B(s,t) = e^{-\lambda(t-s)} + \sqrt{\lambda\mu}(t-s)e^{(\mu-\lambda)(t-s)/2} \times$$
$$\times \int_{\sqrt{\lambda\mu}(t-s)}^{\sqrt{\lambda\mu}(t+s)} e^{-\frac{(\lambda+\mu)w}{2\sqrt{\lambda\mu}}} \frac{I_1\left[\sqrt{w^2-\lambda\mu(t-s)^2}\right]}{\sqrt{w^2-\lambda\mu(t-s)^2}}dw , \tag{4.5.6}$$

which corresponds to equation (4.4.11) with obvious modification. Following the same procedure as in Section 4.4, we obtain

$$p_B(s,t) = J_1(\lambda t,\mu s) + e^{(\mu-\lambda)(t-s)}J(\mu t,\lambda s) , \tag{4.5.7}$$

where $J_n(x,y)$ is defined in (2.4.1).

Upon differentiating (4.5.7), we obtain

$$p_B(s,t) = e^{-(\lambda t+\mu s)}\left[\mu I_0[2\sqrt{\lambda\mu st}] + \left(\frac{\lambda\mu t}{s}\right)^{1/2} I_1[2\sqrt{\lambda\mu st}]\right] - \tag{4.5.8}$$

$$- (\mu-\lambda)e^{(\mu-\lambda)(t-s)}J(\mu t,\lambda s) \ .$$

Alternate forms of $p_B(s,t)$ and $P_B(s,t)$ may be obtained with the use of the results of Section 2.4. An expression for the sojourn density function $p_B(s,t)$ under steady state or equilibrium conditions has been obtained by Greenberg [24], who uses a result given by Good [23]. Linhart [26] has given the double Laplace transform of $p_B(s,t)$ with respect to the variables s and t as well as the equilibrium distribution of $\sigma_B(t)$ for the M/G/1 queuing process, which may be specialized to M/M/1.

We can generalize from $p_B(s,t)$ to $p_B^{(k)}(s,t)$ by noting that

$$p_B^{(k)}(s,t) = \int_0^s f_{k0}(x)p_B^{(0)}(s-x,t-x)dx \ , \tag{4.5.9}$$

where $f_{k0}(t)$ is given by equation (3.6.16). From (4.5.7) and (4.5.9), it follows that

$$p_B^{(k)}(s,t) = e^{-(\lambda t+\mu s)}\rho^{-k/2}\left[\sum_{n=0}^{\infty} \rho^{n/2} L_{k,n}(s,t) + \right. \tag{4.5.10}$$

$$\left. + \sum_{n=1}^{\infty} \rho^{-n/2} L_{k,n}(s,t)\right] ,$$

where $L_{k,n}(s,t)$ is defined by (4.4.20). Therefore, by (4.4.23) and (4.5.10) we have

$$P_B^{(k)}(s,t) = e^{(\mu-\lambda)(t-s)}\rho^{-k}J_k(\mu t,\lambda s) + J_{k+1}(\lambda t,\mu s). \quad (4.5.11)$$

We note from (4.5.11) that the identity $P_B(t,t) = 1$ follows directly from (2.3.11) and (2.3.14). Upon differentiating (4.5.11) with respect to s and using the recurrence formulae given in (2.3.9) and (2.3.10), we obtain

$$\begin{aligned} p_B^{(k)}(s,t) = \mu e^{-(\lambda t+\mu s)}\Bigg[\left(\frac{\mu s}{\lambda t}\right)^{k/2} I_k[2\sqrt{\lambda\mu st}] \; + \\ + \left(\frac{\mu s}{\lambda t}\right)^{(k-1)/2} I_{(k-1)}[2\sqrt{\lambda\mu st}]\Bigg] - \quad (4.5.12) \\ - (\mu-\lambda)\rho^{-k}e^{(\mu-\lambda)(t-s)}J_k(\mu t,\lambda s). \end{aligned}$$

The expressions for $P_B^{(k)}(s,t)$ and $p_B^{(k)}(s,t)$ given in (4.5.11) and (4.5.12), respectively, appear to be new.

We observe that $\Pr\{\sigma_B(t) = 0|S(0) = k\}$ can be obtained from (4.5.11) since

$$\Pr\{\sigma_B(t) = 0|S(0) = k\} = P_B^{(k)}(0,t). \quad (4.5.13)$$

On the other hand, $\sigma_B(t) = t$ only if $S(0) = k$ where $k = 1, 2,3,\ldots,$ and no first passage to zero occurs during $(0,t)$, i.e. the initial busy period does not end. Therefore we have

$$\Pr\{\sigma_B(t) = t\} = 0, \ k = 0, \qquad\qquad (4.5.14)$$

$$= F_{k0}(t), \ k = 1,2,\ldots,$$

where

$$F_{k0}(t) = 1 - \int_0^t f_{k0}(x)dx, \qquad (4.5.15)$$

and $f_{k0}(t)$ is given by (3.6.16).

For $\lambda = \mu$, (4.5.11) and (4.5.12) reduce to

$$P_B^{(k)}(s,t) = J_k(\mu t, \mu s) + J_{k+1}(\mu t, \mu s) \qquad (4.5.16)$$

and

$$p_B^{(k)}(s,t) = \mu e^{-\mu(t+s)}\left[\left(\frac{s}{t}\right)^{k/2} I_k[2\mu\sqrt{st}] + \right. \qquad (4.5.17)$$

$$\left. +\left(\frac{s}{t}\right)^{\frac{(k-1)}{2}} I_{(k-1)}[2\mu\sqrt{st}]\right], \quad k = 0,1,2,\ldots.$$

Asymptotic representations of $p_B^{(k)}(s,t)$ can be obtained from the asymptotic result for the modified Bessel function and the J-function given by equations (2.3.2) and (2.4.5), respectively. Upon applying these results to (4.5.12) we find that as $t\to\infty$,

$$p_B^{(k)}(s,t) \sim \frac{e^{-(\sqrt{\lambda t} - \sqrt{\mu s})^2}}{2\sqrt{\pi}\,(\lambda\mu st)^{1/4}}\left(\frac{\mu s}{\lambda t}\right)^{\frac{(k-1)}{2}} \frac{\mu(1-s/t)}{\left[1 - \sqrt{\frac{\mu s}{\lambda t}}\right]} + O\left(\frac{1}{\mu t}\right). \qquad (4.5.18)$$

Using (4.5.18), the substitution $y = \sqrt{2}\,(\sqrt{\lambda t} - \sqrt{\mu s}\,)$, and the binomial expansion, we obtain for fixed α, $0 < \alpha < 1$, as $t\to\infty$,

$$\int_{\alpha t}^{t} p_B^{(k)}(s,t)\,ds \sim \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{2\mu t}(1-\sqrt{\rho})}^{\sqrt{2\mu t}(\sqrt{\rho}\,-\,\sqrt{\alpha}\,)} e^{-y^2/2}dy \; + \tag{4.5.19}$$

$$+\frac{\left[\frac{\rho+1}{\rho-1} - k\right]}{2\sqrt{\pi\lambda t}}\left[e^{-\mu t(1-\sqrt{\rho})^2} - e^{-\mu t(\sqrt{\rho}-\sqrt{\alpha})^2}\right] + O\left(\frac{1}{\lambda t}\right),$$

where $\rho = \lambda/\mu < 1$.

We note that the result given in (4.5.19) depends on the quantity $\mu t(1-\sqrt{\rho})^2$ being large so that if ρ is near unity, for example, this approximation will require a large value of t in order to be very accurate. A numerical illustration of this point will be provided in the next section.

It is interesting to observe that (4.5.19) implies that for fixed α, $0 < \alpha < 1$, and $\rho < 1$, as $t\to\infty$,

$$\begin{aligned} \Pr\{\alpha t \le \sigma_B(t) < t\} &\to 0, \quad \rho < \alpha, \\ &\to 1/2, \quad \rho = \alpha, \\ &\; 1, \quad \rho < \alpha. \end{aligned} \tag{4.5.20}$$

In terms of the single-server queuing process, equation

(4.5.40) implies that over a long period of time, the fraction of time during which the server is busy is not likely to exceed ρ. This leaves the question as to the behavior of $Pr\{\alpha t<\sigma_B(t)<t\}$ when $\rho = 1$. It is not possible, of course, for the server to be occupied for a fraction of time which exceeds unity.

To answer this question we use equation (4.5.17) and follow the same steps as in the non-symmetric case. This procedure leads to the result that for $\rho = 1$ and fixed α, $0 < \alpha < 1$, then as $t\to\infty$,

$$\int_{\alpha t}^{t} p_B^{(k)}(s,t)ds \sim \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{2\mu t}\,(1-\sqrt{\alpha})} e^{-y^2/2}dy + \frac{k\,[1 - e^{-\mu t(1-\sqrt{\alpha})^2}]}{\sqrt{\pi\mu t}} + O\left(\frac{1}{\mu t}\right),\quad k = 0,1,2,... \tag{4.5.21}$$

Evidently when $\rho = 1$, $Pr\{\alpha t<\sigma_B(t)<t\}\to 1$ as $t\to\infty$, i.e., over a long period of time, for fixed α, the server is likely to be busy for a fraction of time which exceeds α. Therefore, over a long period of time the server is likely to be busy most of the time. It is possible, however, to approximate $Pr\{\alpha t<\sigma_B(t)<t\}$ to a high degree of accuracy by using (4.5.21) as will be illustrated in the next Section.

The limiting expression for $Pr\{\alpha t<\sigma_B(t)<t\}$ given by (4.5.19) and (4.5.21) are similar to, but differ slightly

from, results obtainable from the work of Takács [3.5]. In the next Section, we shall compare Takács' asymptotic results with our own.

4.6 Numerical Complements

The object of this Section is to illustrate the degree of accuracy of the asymptotic formulae derived in Sections 4.3 - 4.5 and to compare these asymptotic approximations with those suggested by the work of Takács [35] wherever applicable. As noted in Section 4.1, Takács' work is based on the theory of stable distributions discussed in Section 2.6. Although we have used entirely different methods from those used by Takács, our limiting forms are similar to his, but there are slight differences which appear to affect the accuracy of the approximations as we shall point out.

To this end we shall discuss in order the asymptotic results pertaining to the following:

(i) the strictly positive sojourn time for the unrestricted random walk;

(ii) the zero and non-zero sojourn time for the unrestricted walk; and

(iii) the busy time for the M/M/1 queuing process.

We assume throughout this Section that $S(0) = 0$. All entries in the Tables are believed to be correct to one unit of the final decimal place.

We proceed now to discuss topics (i) - (iii) in turn.

i) <u>The Strictly Positive Sojourn Time for the Unrestricted Random Walk with Negative Exponentially Distributed Intervals Between Steps</u>

Because of the difficulties of integrating the sojourn density $p_B(s,t)$ given in (4.3.13), we shall focus attention on the density function itself instead of on the distribution function.

For the case $\lambda \neq \mu$, Takács' results do not seem to apply. The accuracy of our own approximation to $p_B(s,t)$ given in (4.3.28) is illustrated in Table IV. It is assumed that $S(0) = 0$.

In Table IV we give the exact and approximate values of $p_B(\alpha t,t)$ for $t = 50, 100, 150, 200$; $\alpha = .2, .4, .6, .8$; and $\rho = .6, .8, 1.2, 1.4$. Under each value of t the exact and approximate values are listed side by side for the different values of ρ given in the first column and the values of α given in the second column. We have assumed $\mu = 1$ in each case.

From Table IV it is evident that equation (4.3.28) does not provide a good approximation for the positive sojourn density function. Nevertheless, it is clear that the relative error in the approximation decreases as t increases. Unfortunately, the convergence of the approximation to the true value with increasing t is extremely slow. Equation (4.3.28) was derived using only the leading terms of the relevant asymptotic formulae, however, and greater accuracy could be obtained by including higher order terms, which would not be difficult.

TABLE IV

EXACT AND APPROXIMATE VALUES OF $p_B(\alpha t, t)$

		t = 50		t = 100		t = 150		t = 200	
ρ	α	Exact	Approx.	Exact	Approx.	Exact	Approx.	Exact	Approx.
0.6	0.2	0.012766	0.037189	0.003824	0.007910	0.001475	0.002590	0.000636	0.001012
0.6	0.4	0.003859	0.007910	0.000637	0.001012	0.000141	0.000199	0.000035	0.000046
0.6	0.6	0.001511	0.002590	0.000141	0.000199	0.000018	0.000023	0.000002	0.000003
0.6	0.8	0.000687	0.001012	0.000036	0.000046	0.000002	0.000003	0.000000	2×10^{-7}
0.8	0.2	0.018970	0.135421	0.009253	0.042829	0.005739	0.020854	0.003924	0.012116
0.8	0.4	0.010240	0.042829	0.004107	0.012116	0.002164	0.005277	0.001276	0.002742
0.8	0.6	0.006935	0.020854	0.002317	0.005277	0.001047	0.002056	0.000537	0.000956
0.8	0.8	0.005586	0.012116	0.001542	0.002742	0.000594	0.000956	0.000263	0.000397
1.2	0.2	0.007719	0.017797	0.002331	0.004370	0.000979	0.001652	0.000472	0.000745
1.2	0.4	0.009185	0.030015	0.003283	0.008074	0.001585	0.003344	0.000867	0.001652
1.2	0.6	0.013035	0.060400	0.005471	0.017797	0.003019	0.008074	0.001863	0.004370
1.2	0.8	0.023163	0.187132	0.011552	0.060400	0.007346	0.030015	0.005152	0.017797
1.4	0.2	0.002344	0.003774	0.000251	0.000348	0.000037	0.000049	0.000004	0.000008
1.4	0.4	0.004138	0.008128	0.000668	0.001049	0.000147	0.000208	0.000034	0.000049
1.4	0.6	0.008555	0.020890	0.002064	0.003774	0.000660	0.001049	0.000236	0.000348
1.4	0.8	0.022675	0.082656	0.008363	0.020890	0.003933	0.008128	0.002053	0.003774

For the case $\lambda = \mu$, the asymptotic result suggested by the work of Takács and our approximation given in (4.3.28) are the same. Table V compares the exact and approximate values of $p_B(s,t)$ assuming $\lambda = \mu = .1$ and $t = 100$. As applied to a soccer game in which goals are scored at random according to a Poisson process, these assumptions imply that each team scores a goal on the average of one every ten minutes during a 100 minute game. In Table V the values of s are arranged in the left hand column while the exact and approximate values are arranged in columns headed by the words "exact" and "approximate", respectively.

We observe from Table V that away from the ends of the range, $[\pi\sqrt{s(t-s)}]^{-1}$ provides a good approximation for the density $p_B(s,t)$. The approximate values do tend to overestimate the true values, however, and for this reason we expect that the arc sine approximation given in (4.3.30) will have this same tendency to overestimate the distribution function.

TABLE V

VALUES OF $p_B(s,t)$ FOR $\rho = 1$, $t = 100$

s	Exact	Approximate
0	0.1773	∞
20	0.0763	0.0796
40	0.0633	0.0650
60	0.0633	0.0650
80	0.0763	0.0796
100	0.1773	∞

We turn now to

ii) The Zero and Non-Zero Sojourn Distribution for the Unrestricted Random Walk with Negative Exponentially Distributed Intervals Between Steps

Unfortunately, no meaningful approximations have been derived for the case $\lambda \neq \mu$ as yet. Moreover, it is not clear that the work of Takács applies in this case since the distribution of τ_{BA} does not satisfy (2.6.6).

For the case $\lambda = \mu$, however, accurate asymptotic approximations can be obtained. Our approximation for $\Pr\{\alpha t < \sigma_B(t) < t\}$ is given in equation (4.4.31). The corresponding formula obtained from Takács' results ([35], Theorem 4, p. 284) is given by

$$\int_{\alpha t}^{t} p_B(s,t)ds = \Pr\{\alpha t < \sigma_B(t) < t\} \sim \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{2\mu t}(1-\alpha)} e^{-x^2/2}dx \ , \tag{4.6.1}$$

which is the form of the stable distribution of order 1/2 given in (2.6.8). A comparison of (4.4.31) and (4.6.1) shows that the two approximations are the same except for the difference in the upper limit of the integral. We point out that (4.6.1) implies a dependence on α while our result given in (4.4.31) depends upon $\sqrt{\alpha}$. We are unable to explain this difference at present. The practical effect of this difference is illustrated in Table VI below.

Table VI gives exact and approximate values of $\Pr\{\alpha t < \sigma_B(t) < t\}$ for $t = 10, 20, 30$, and $\alpha = .5, .6, .7, .8, .9$. In this case $\sigma_B(t)$ represents the amount of time during $(0,t)$ that the random walk remains in the non-zero state. Under each value of t, the exact and approximate values are given, with Takács' approximation in the column headed "Takács" and the approximation based on (4.4.31) given in the column headed "(4.4.31)." The corresponding values of α are listed at the beginning of each row. Again we have assumed $S(0) = 0$ and $\mu = 1$.

It is evident from Table VI that (4.4.31) is a uniformly better approximation than Takács' result given by (4.6.1). Moreover, (4.4.31) has the desirable property of being very accurate even for relatively small values of t.

TABLE VI

VALUES OF $\Pr\{\alpha t < \sigma_B(t) < t\}$ FOR $\rho = 1$

	t = 10			t = 20			t = 30		
α	Exact	(4.4.31)	Takács	Exact	(4.4.31)	Takács	Exact	(4.4.31)	Takács
0.5	0.9784	0.9793	0.9984	0.9988	0.9989	0.9999	0.9999	0.9999	0.9999
0.6	0.9304	0.9323	0.9885	0.9899	0.9902	0.9996	0.9984	0.9984	0.9999
0.7	0.8224	0.8253	0.9422	0.9442	0.9450	0.9927	0.9810	0.9812	0.9989
0.8	0.6282	0.6313	0.7940	0.7949	0.7963	0.9263	0.8797	0.8805	0.9715
0.9	0.3436	0.3456	0.4729	0.4721	0.4734	0.6289	0.5610	0.5619	0.7266

iii) The Busy Time for the M/M/1 Queuing Process

Consider first of all the case $\lambda < \mu$ or $\rho < 1$. Assuming $S(0) = 0$, our approximation for $\Pr\{\alpha t < \sigma_B(t) < t\}$ is given by equation (4.5.19) where $\sigma_B(t)$ represents the amount of time during $(0,t)$ that the server is occupied. The work of Takács ([35], Theorem 2, p. 281) leads to the result

$$\Pr\{\alpha t < \sigma_B(t) < t\} \sim \frac{1}{\sqrt{2\pi}} \int_{\frac{\sqrt{\mu t}(\alpha-\rho)}{\sqrt{2\rho}}}^{\frac{\sqrt{\mu t}(1-\rho)}{\sqrt{2\rho}}} e^{-y^2/2} dy \;, \quad 0 < \alpha < 1. \tag{4.6.2}$$

Again we note the appearance of $\sqrt{\alpha}$ in (4.5.19) while the lower limit of (4.6.2) contains α. Both, however, have the same limiting behavior as $t \to \infty$ for the cases $\alpha < \rho$, $\alpha = \rho$, and $\alpha > \rho$ as given in (4.5.20). We compare the two approximations to the exact values in Tables VII - IX, which follow.

Tables VII - IX below give exact and approximate values of $\Pr\{\alpha t < \sigma_B(t) < t\}$ for $t = 15, 20, 25$; $\alpha = .2, .4, .6, .8$; and $\rho = .2, .4, .6, .8$. We have assumed $\mu = 1$ and $S(0) = 0$. The exact values of the probability are given in the third column corresponding to the values of ρ listed in the first column and the values of α listed in the second column. The fourth and fifth columns contain respectively the dominant and correction terms in (4.5.19), and the sixth column their sum. The last column in each Table contains the values calculated from Takács' result given in (4.6.2).

The values given in Tables VII - IX illustrate that for smaller values of ρ, (4.5.19) gives a better approximation than that of Takács. For values of ρ near unity, however, the accuracy of (4.5.19) diminishes as predicted because of the dependence on $(1-\rho)$. For ρ near unity we notice that as α increases, the correction term decreases the accuracy of the approximation. The dominant term is then seen to be of the same order of magnitude as the value of Takács' approximation.

TABLE VII

VALUES OF $\Pr\{\alpha t < \sigma_B(t) < t\}$ FOR $t = 15$

(4.5.19)

ρ	α	Exact	Dominant	Cor.	Approx.	Takács
0.2	0.2	0.38095	0.49876	-0.12090	0.37786	0.49999
0.2	0.4	0.08861	0.15391	-0.07175	0.08206	0.11033
0.2	0.6	0.01404	0.03524	-0.02322	0.01202	0.00715
0.2	0.8	0.00140	0.00592	-0.00483	0.00109	0.00011
0.4	0.2	0.77107	0.82280	-0.06259	0.76021	0.80207
0.4	0.4	0.37446	0.47794	-0.11664	0.36130	0.49531
0.4	0.6	0.11794	0.19607	-0.08152	0.11455	0.18855
0.4	0.8	0.02216	0.05361	-0.03028	0.02333	0.03694
0.6	0.2	0.93764	0.85503	0.05008	0.90511	0.84270
0.6	0.4	0.67774	0.67338	-0.05112	0.62226	0.68160
0.6	0.6	0.33710	0.39150	-0.10029	0.29121	0.42135
0.6	0.8	0.10031	0.14731	-0.06385	0.08346	0.16110
0.8	0.2	0.99833	0.71129	0.29178	1.00300	0.69675
0.8	0.4	0.86781	0.64279	0.17913	0.82192	0.61951
0.8	0.6	0.58614	0.46264	0.01459	0.47723	0.45970
0.8	0.8	0.24870	0.21845	-0.05641	0.16204	0.22985

TABLE VIII

VALUES OF $\Pr\{\alpha t < \sigma_B(t) < t\}$ FOR t = 20

(4.5.19)

ρ	α	Exact	Dominant	Cor.	Approx.	Takács
0.2	0.2	0.39633	0.49976	-0.10555	0.39420	0.49999
0.2	0.4	0.07164	0.12044	-0.05302	0.06742	0.07864
0.2	0.6	0.00767	0.01896	-0.01216	0.00679	0.00233
0.2	0.8	0.00047	0.00210	-0.00170	0.00040	0.00001
0.4	0.2	0.82502	0.86926	-0.05077	0.81849	0.83999
0.4	0.4	0.38965	0.48995	-0.10855	0.38140	0.49865
0.4	0.6	0.10428	0.17428	-0.06987	0.10441	0.15730
0.4	0.8	0.01502	0.03072	-0.02168	0.01704	0.02140
0.6	0.2	0.96753	0.90380	0.03986	0.94366	0.89752
0.6	0.4	0.73254	0.73867	-0.04977	0.68890	0.74165
0.6	0.6	0.54301	0.58825	-0.08972	0.49852	0.44876
0.6	0.8	0.09378	0.14726	-0.06325	0.08401	0.15587
0.8	0.2	0.99548	0.74549	0.24813	0.99363	0.74330
0.8	0.4	0.91561	0.69906	0.17350	0.87257	0.68160
0.8	0.6	0.64320	0.52357	0.01580	0.53938	0.52049
0.8	0.8	0.26890	0.24783	-0.06341	0.18442	0.26024

TABLE IX

VALUES OF $\Pr\{\alpha t < \sigma_B(t) < t\}$ FOR $t = 25$

(4.5.19)

ρ	α	Exact	Dominant	Cor.	Approx.	Takács
0.2	0.2	0.40694	0.49953	-0.09457	0.40496	0.50000
0.2	0.4	0.05793	0.09507	-0.04007	0.05500	0.05692
0.2	0.6	0.00422	0.01026	-0.00644	0.00382	0.00078
0.2	0.8	0.00016	0.00073	-0.00059	0.00014	1×10^{-6}
0.4	0.2	0.86374	0.90020	-0.04058	0.85962	0.86782
0.4	0.4	0.40036	0.49532	-0.10052	0.39480	0.49960
0.4	0.6	0.09185	0.15275	-0.05924	0.09351	0.13137
0.4	0.8	0.01018	0.02730	-0.00498	0.00301	0.00219
0.6	0.2	0.98270	0.93420	0.03090	0.96510	0.93211
0.6	0.4	0.77451	0.78708	-0.04700	0.74008	0.78540
0.6	0.6	0.36764	0.44451	-0.10477	0.33974	0.46605
0.6	0.8	0.08680	0.14291	-0.06083	0.08208	0.16671
0.8	0.2	0.99849	0.77153	0.21291	0.98444	0.77654
0.8	0.4	0.94504	0.74033	0.16377	0.90410	0.72847
0.8	0.6	0.68828	0.57391	0.01658	0.59049	0.57080
0.8	0.8	0.28444	0.27232	-0.06902	0.20330	0.28540

For the case $\lambda = \mu$, sometimes referred to as the "heavy traffic" situation, our result is given in (4.5.21). Takács' analysis ([35], Theorem 4, p. 284) gives a similar result, namely,

$$\Pr\{\alpha t < \sigma_B(t) < t\} \sim \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{\frac{\mu t}{2}}(1-\alpha)} e^{-y^2/2} dy \; . \qquad (4.6.3)$$

Again both (4.5.21) and (4.6.3) are given in terms of the stable distribution of index 1/2 while (4.5.21) depends on $\sqrt{\alpha}$ and (4.6.3) on α. As an approximation (4.5.21) seems to be preferable to (4.6.3) as illustrated in Table X below.

Table X gives exact and approximate values of $\Pr\{\alpha t < \sigma_B(t) < t\}$ for t = 15, 20, 25, 30; α = .5, .6, .7, .8, .9. It has been assumed that $S(0) = 0$ and $\lambda = \mu = 1$. Under each value of t, the second column contains the correct value corresponding to the value of α given in the first column. The third and fourth columns under each value of t are the values of equations (4.5.21) and (4.6.3), respectively, and are labeled "(4.5.21)" and "Takács".

TABLE X

VALUES OF Pr{$\alpha t < \sigma_B(t) < t$} FOR $\lambda = \mu = 1$

	t = 15			t = 20		
α	Exact	(4.5.21)	Takács	Exact	(4.5.21)	Takács
0.5	0.8892	0.8913	0.8290	0.9348	0.9360	0.8861
0.6	0.7805	0.7830	0.7266	0.8443	0.8460	0.7940
0.7	0.6266	0.6290	0.5886	0.6966	0.6984	0.6572
0.8	0.4350	0.4369	0.4161	0.4941	0.4956	0.4729
0.9	0.2203	0.2213	0.2158	0.2536	0.2544	0.2481

	t = 25			t = 30		
α	Exact	(4.5.21)	Takács	Exact	(4.5.21)	Takács
0.5	0.9609	0.9616	0.9229	0.9763	0.9767	0.9471
0.6	0.8878	0.8890	0.8427	0.9183	0.9191	0.8786
0.7	0.7504	0.7519	0.7111	0.7930	0.7942	0.7547
0.8	0.5433	0.5446	0.5204	0.5854	0.5865	0.5614
0.9	0.2825	0.2832	0.2763	0.3083	0.3089	0.3014

4.7 The Three-State Sojourn Time Problem

In this Section we consider the more difficult sojourn time problem in which it is assumed that the particle may be in any one of three possible states at a given time t. We assume that $S(0) = 0$ and that a transition to state A has occurred just prior to $t = 0$. The alternative states are denoted by B_1, B_2. We define $g(t)$ by the equation

$$g(t)dt = \Pr\{t < \min[\tau_{AB_1}, \tau_{AB_2}] < t + dt\}, \qquad (4.7.1)$$

and $h_1(t)$, $h_2(t)$ are similarly defined in a manner consistent with the definitions of Section 4.2.

We shall assume that from states B_1, B_2 the random walk can only make a transition to state A. We shall also assume that, given that a transition from A to B_1 or B_2 occurs, the probability of transition to B_1 (or B_2) is a constant, i.e. independent of time. Because of this assumption we define

$$p_1 = \Pr\{S(t+dt) \in B_1 | S(t) \in A; S(t+dt) \in (B_1 \cup B_2)\}, \qquad (4.7.2)$$

and

$$p_2 = \Pr\{S(t+dt) \in B_2 | S(t) \in A; S(t+dt) \in (B_1 \cup B_2)\}. \qquad (4.7.3)$$

Using these assumptions we proceed to derive an expression for $p(r,s,t)$ which is defined by

$$p(r,s,t)drds = \Pr\{r<\sigma_A(t)<r+dr;\ s<\sigma_{B_1}(t)<s+ds \mid S(0) = 0\},$$

$$0 < r,\ s < t. \qquad (4.7.4)$$

The derivation of $p(r,s,t)$ follows the same general pattern as that given in Section 4.2 for the two-state sojourn density function. As mentioned previously, Good [23] has obtained the counterpart of $p(r,s,t)$ for n states and expressed it in terms of a multiple contour integral. Good has obtained his results, however, under assumptions which differ slightly from ours. In Good's analysis he has assumed that the infinitesimal transition probabilities, $\Pr\{S(t+dt) \in A \mid S(t) \in B_i\}$ and $\Pr\{S(t+dt) \in B_i \mid S(t)\} \in A$, are constant, $i = 1, 2$. In this Section we shall use a direct approach to derive $p(r,s,t)$ which is instructive and leads to an explicit result.

The general expression obtained will then be applied to the unrestricted random walk in which it is assumed that the particle is in state A at time t if $S(t) = 0$, in state B_1 if $S(t) > 0$, and in state B_2 if $S(t) < 0$.

Both the general result and the application given in this Section have not been discussed in the literature previously.

At a given time t the particle must be in one of the three states A, B_1, or B_2 and there is no other possibility. We begin our analysis by defining $u_{mn}(r,s,t)drds$ as the probability that

(i) $S(t) \in A$;

(ii) $\sigma_A(t)$ is composed of m disjoint intervals of time and $r < \sigma_A(t) < r + dr$; and

(iii) $\sigma_{B_1}(t)$ is composed of n disjoint intervals of time and $s < \sigma_{B_1}(t) < s + ds$.

Obviously the definition of $u_{mn}(r,s,t)$ implies that $t-r-s-ds-dr < \sigma_{B_2}(t) < t-r-s$. We define $v_{mn}(r,s,t)$ and $w_{mn}(r,s,t)$ in the same manner except for the substitution of $S(t) \varepsilon B_1$ and $S(t) \varepsilon B_2$ in (i) above, respectively.

For convenience of notation we shall let $u_{mn}(r,0,t)dr$ be defined according to (i) - (iii) above with the exception that in (iii) we replace $s < \sigma_{B_1}(t) < s + ds$ by $\sigma_{B_1}(t) = 0$. Similarly, the probability $u_{mn}(r,t-r,t)dr$ is also defined according to (i) - (iii) with the exception that in (iii) the phrase, $t-r-dr < \sigma_{B_1}(t) < t - r$, is substituted for $s < \sigma_{B_1}(t) < s + ds$. An analogous convention will be used to define $v_{mn}(r,0,t)dr$, $v_{mn}(r,t-r,t)dr$, $w_{mn}(r,0,t)dr$, and $w_{mn}(r,t-r,t)dr$. Although not usually done, it seems logical and most convenient to extend the usual probability density function notation to apply to the mixed probability and probability density function in this manner.

We note that since $S(0) \varepsilon A$, we have

$$\Pr\{\sigma_A(t) = 0;\ s < \sigma_{B_1}(t) < s + ds\} = 0,\quad 0 \le s \le t, \tag{4.7.5}$$

and since $\sigma_A(t) = t$ implies that no transition from state A to B_1 or B_2 occurs during the interval $(0,t)$, we have

$$\Pr\{\sigma_A(t) = t\} = 1 - G(t). \tag{4.7.6}$$

We first derive $u_{mn}(r,s,t)$ and then indicate the modification necessary to obtain expressions for $v_{mn}(r,s,t)$ and $w_{mn}(r,s,t)$. Consider then

<u>Case I. $0 < r < t,\ s = 0$</u>

The assumption $s = 0$ implies that no passage to state B_1 occurs during $(0,t)$ and consequently $u_{mn}(r,0,t)$ is zero if $n \neq 0$. Since $0 < r < t$, the particle must make a passage to state B_2 at some time x followed by an interval of time of length y, $0 < y \leq t-r$, during which the particle remains in state B_2. In order to have $S(t) \varepsilon A$, the particle must return to state A at some time and hence $\sigma_A(t)$ must be composed of at least two disjoint intervals of time, i.e. $m \geq 2$. If $m = 2$, we see from the foregoing considerations that

$$\begin{aligned} u_{20}(r,0,t) &= \int_0^r g(x)p_2h_2(t-r)[1-G(r-x)]dx \\ &= p_2h_2(t-r)[G(r)-G^{(2)}(r)]. \end{aligned} \tag{4.7.7}$$

For $m \geq 2$, it is evident that after the initial return to state A from state B_2, the process begins again with the remaining portion of the interval $(0,t)$ consisting of $m - 1$ disjoint intervals during which the particle remains in state A and no time spent in state B_1. Consequently, we can write

$$u_{m0}(r,0,t) = \int_0^r g(x)dx \int_0^{t-r} p_2h_2(y)u_{m-1,0}(r-x,0,t-y-x)dy,$$

$$m = 3,\ 4,\ \ldots. \qquad (4.7.8)$$

By application of mathematical induction to (4.7.7) and (4.7.8), we find that

$$u_{m0}(r,0,t) = p_2^{(m-1)}(t-r)[G^{(m-1)}(r) - G^{(m)}(r)],$$

$$m = 2,\ 3,\ \ldots. \qquad (4.7.9)$$

We turn then to

<u>Case II. $0 < r < t,\ s = t - r$</u>

Because of the assumption that $s = t - r$, Case II corresponds exactly to Case I except that now $\sigma_{B_2}(t) = 0$ rather than $\sigma_{B_1}(t) = 0$. Since $S(t) \varepsilon A$, $\sigma_{B_2}(t) = 0$ implies that $m = n+1$, $n = 1,2,3,\ldots.$ By argument analogous to that used for Case I, it is evident that

$$u_{21}(r,t-r,t) = \int_0^r g(x)p_1h_1(t-r)[1-G(r-x)]dx \qquad (4.7.10)$$

$$= p_1h(t-r)[G(r)-G^{(2)}(r)]$$

and

$$u_{m,m-1}(r,t-r,t) = \int_0^r g(x)dx \int_0^{t-r} h_1(y) \times$$

$$\times\, u_{m-1,m-2}(r-x,t-r-y,t-x-y)dy,$$

$$m = 3,\ 4,\ 5,\ \ldots . \qquad (4.7.11)$$

Again by mathematical induction we find that

$$u_{m,m-1}(r,t-r,t) = p_1^{(m-1)} h_1^{(m-1)}(t-r)\,[G^{(m-1)}(r)-G^{(m)}(r)],$$

$$m = 2,\ 3,\ \ldots . \qquad (4.7.12)$$

Finally, we consider

<u>Case III. $0 < r < t,\ 0 < s < t - r$</u>

In this case we have $0 < \sigma_A(t),\ \sigma_{B_1}(t),\ \sigma_{B_2}(t) < t$, i.e. the particle spends some time in all three states. Therefore we conclude that $n \geq 1$ and $m \geq n + 2$.

Consider first of all $u_{31}(r,s,t)drds$. In this case $\sigma_{B_1}(t)$ and $\sigma_{B_2}(t)$ each consist of one interval of time and satisfy the inequalities $s < \sigma_{B_1}(t) < s + ds$ and $t-s-r-ds-dr < \sigma_{B_2}(t) < t-s-r$, respectively. There are two possibilities: either state B_1 is visited first or state B_2 is visited first.

Consequently, we write

$$u_{31}(r,s,t) = \int_0^r g(x)p_1h_1(s)dx \int_0^{r-x} p_2g(y)h_2(t-r-s)[1-G(r-x-y)]dy +$$

$$+ \int_0^r g(x)p_2h_2(t-r-s)dx \int_0^{r-x} g(y)p_1h_1(s)[1-G(r-x-y)]dy$$

$$= 2p_1p_2h_1(s)h_2(t-r-s)[G^{(2)}(r)-G^{(3)}(r)]. \qquad (4.7.13)$$

For higher values of m we use the fact that the process renews itself whenever a transition to state A occurs to write

$$u_{m1}(r,s,t) = \int_0^r g(x)dxp_1h_1(s)u_{m-1,0}(r-x,0,t-s-x)$$

$$+ \int_0^r g(x)dx \int_0^{t-r-s} p_2h_2(y)u_{m-1,1}(r-x,s,t-x-y)dy,$$

$$m = 4, 5, \ldots . \qquad (4.7.14)$$

Substituting (4.7.9) in (4.7.14) and applying induction, we obtain

$$u_{m1}(r,s,t) = \binom{m-1}{1} p_1p_2^{(m-2)}h_1(s)h_2^{(m-2)}(t-s-r)[G^{(m-1)}(r)-G^{(m)}(r)],$$

$$m = 3, 4, \ldots . \qquad (4.7.15)$$

In a similar manner by induction on n and using the relation

$$u_{mn}(r,s,t) = \int_0^r g(x)dx \int_0^s p_1 h_1(y) u_{m-1,n-1}(r-x,s-y,t-y-x)dy +$$

$$+ \int_0^r g(x)dx \int_0^s p_2 h_2(y) u_{m-1,n}(r-x,s,t-x-y)dy, \tag{4.7.16}$$

we find that

$$u_{mn}(r,s,t) = \binom{m-1}{n} p_1^n \, p_2^{m-n-1} h_1^{(n)}(s) h_2^{(m-n-1)}(t-s-r) \times$$

$$\times \, [G^{(m-1)}(r) - G^{(m)}(r)], \; m \geq n + 2, \; n = 1,2,3,\ldots,$$

$$0 < r, \; s < t. \tag{4.7.17}$$

Since the arguments used to obtain expressions for $v_{mn}(r,s,t)$ and $w_{mn}(r,s,t)$ are essentially the same as those used to obtain $u_{mn}(r,s,t)$, with obvious modification, we simply summarize these results as follows:

$$v_{mn}(r,0,t) = 0, \; m \geq n, \; n = 0,1,2,\ldots; \tag{4.7.18}$$

$$v_{m,m}(r,t-r,t) = g^{(m)}(r) p_1^m \; [H^{(m-1)}(t-r) - H^{(m)}(t-r)]; \tag{4.7.19}$$

$$v_{m,n}(r,s,t) = \binom{m-1}{n-1} p_1^n p_2^{m-n} g^{(m)}(r) h_2^{(m-n)}(t-r-s) [H_1^{(n-1)}(s) - H^{(n)}(s)],$$

$$m \geq n+1, \; n = 1,2,\ldots, \; 0 < r, \; s < t. \tag{4.7.20}$$

$$w_{mn}(r,t-r,t) = 0, \; m, \; n = 1, \; 2, \; \ldots; \tag{4.7.21}$$

$$w_{m0}(r,0,t) = p_2^m g^{(m)}(r)[H_2^{(m-1)}(t-r)-H_2^{(m)}(t-r)], \; m = 1,2,\ldots; \tag{4.7.22}$$

and

$$w_{mn}(r,s,t) = \binom{m-1}{n} g^{(m)}(r) p_1^n p_2^{m-n} \times \tag{4.7.23}$$

$$\times [H_2^{(m-n-1)}(t-r-s)-H_2^{(m-n)}(t-r-s)]h_1^{(n)}(s),$$

$$m \geq n+1, \; n = 1,2,\ldots, \; 0 < r, \; s < t.$$

Combining the above we have

$$p(r,s,t) = \sum_{n=1}^{\infty} \sum_{m=n+2}^{\infty} u_{mn}(r,s,t) + \tag{4.7.24}$$

$$+ \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} [v_{mn}(r,s,t) + w_{mn}(r,s,t)], \; 0 < r, \; s < t;$$

$$\Pr\{\sigma_A(t) = t\} = 1 - G(t); \tag{4.7.25}$$

$$\Pr\{r<\sigma_A(t)<r+dr; \; \sigma_{B_1}(t) = 0\} = \sum_{m=2}^{\infty} u_{m0}(r,0,t)dr + \tag{4.7.26}$$

$$+ \sum_{m=1}^{\infty} w_{m0}(r,0,t)dr, \; 0 < r < t;$$

and

$$\Pr\{r<\sigma_A(t)<r+dr; \; \sigma_{B_2}(t) = 0\} = \sum_{m=2}^{\infty} u_{m,m-1}(r,t-r,t)dr + \tag{4.7.27}$$

$$+ \sum_{m=1}^{\infty} v_{m,m}(r,t-r,t)dr, \; 0 < r < t.$$

This completes the derivation of the general sojourn density function for the three-state sojourn time problem.

As a partial check on our analysis we should find that

$$\int_0^t \int_0^{t-r} p(r,s,t)\,ds\,dr + \int_0^t \Pr\{r<\sigma_A(t)<r+dr;\ \sigma_{B_1}(t) = 0\}dr +$$

$$+ \int_0^t \Pr\{r<\sigma_A(t)<r+dr;\ \sigma_{B_2}(t) = 0\}dr + \Pr\{\sigma_A(t) = t\} = 1. \tag{4.7.28}$$

Taking Laplace transforms with respect to the variable t, we find that

$$L\left[\int_0^t \int_0^{t-r} p(r,s,t)\,ds\,dr\right] = p_1 g^*(z) G_c^*(z) h_1^*(z) \times$$

$$\times \left[\frac{1}{[1-p_2 g^*(z) h_2^*(z)][1-p_1 g^*(z) h_1^*(z) - p_2 g^*(z) h_2^*(z)]} - \right.$$

$$\left. - \frac{1}{[1-p_1 g^*(x) h_1^*(z)]}\right] ; \tag{4.7.29}$$

$$L\left[\int_0^t \Pr\{r<\sigma_A(t)<r+dr;\ \sigma_{B_1}(t) = 0\}dr\right] = \frac{p_2 g^*(z)[1-g^*(z) h_2^*(z)]}{z[1-p_2 g^*(z) h_2^*(z)]}; \tag{4.7.30}$$

$$L\left[\int_0^t \Pr\{r<\sigma_A(t)<r+dr;\ \sigma_{B_2}(t) = 0\}dr\right] = \frac{p_1 g^*(z)[1-g^*(z) h_1^*(z)]}{z[1-p_1 g^*(z) h_1^*(z)]}; \tag{4.7.31}$$

and

$$L[\Pr\{\sigma_A(t) = t\}] = \frac{[1-g^*(z)]}{z} ; \tag{4.7.32}$$

where

$$g^*(z) = \int_0^\infty e^{-zt} g(t)dt, \tag{4.7.33}$$

$$h_i^*(z) = \int_0^\infty e^{-zt} h_i(t)dt, \quad i = 1, 2, \tag{4.7.34}$$

and

$$G_c^*(z) = \frac{1}{z} [1-g^*(z)]. \tag{4.7.35}$$

Equation (4.7.29) follows from (4.7.24) with the aid of the identity

$$\sum_{m=0}^{\infty} \binom{m+n}{n} x^m = (1-x)^{-(n+1)}, \quad |x| < 1.$$

The derivations of (4.7.30) - (4.7.32) follow in a straightforward manner from (4.7.25) - (4.7.27).

Adding (4.7.29) - (4.7.32) yields a sum equal to 1/z which implies that (4.7.28) holds.

We now apply the results given in (4.7.24) - (4.7.27) to the unrestricted random walk in which it is assumed that steps of +1 and -1 occur according to a Poisson stream with parameters λ, μ, respectively. We say that the particle is in state A at time t if $S(t) = 0$, in state B_1 if $S(t) > 0$ and in state B_2 if $S(t) < 0$. In order to obtain an expression for $p(r,s,t)$, we proceed to determine the quantities contained in equations (4.7.24) - (4.7.27). A transition from state A to states B_1 or B_2 takes place if a positive or negative step occurs. Therefore, we have

$$\begin{aligned} p_1 &= \Pr\{S(t+dt) \in B_1 | S(t) \in A;\ S(t+dt) \in (B_1 \cup B_2)\} \\ &= \Pr\{S(t+dt) > 0 | S(t) = 0;\ S(t+dt) \neq 0\} \\ &= \Pr\{\text{positive step occurs during } (t,t+dt) \text{ a step occurs during } (t,t+dt)\} \\ &= \frac{\lambda dt}{(\lambda+\mu)dt} = \frac{\lambda}{\lambda+\mu} . \end{aligned} \tag{4.7.36}$$

Similarly,

$$p_2 = \frac{\mu}{\lambda+\mu} . \tag{4.7.37}$$

It is evident that $\Pr\{t<\min[\tau_{AB_1},\tau_{AB_2}] < t + dt\}$ is equal to the probability that a positive or negative step occurs during (t,t+dt), and hence,

$$g(t) = (\lambda+\mu)e^{-(\lambda+\mu)t} . \tag{4.7.38}$$

Also, $\Pr\{t<\sigma_{B_1A}<t+dt\}$ is the probability that a first passage to zero occurs during (t,t+dt) given that a passage from

0 to +1 occurred at time t = 0. Hence,

$$h_1(t) = f_{10}(t). \tag{4.7.39}$$

Similarly,

$$h_2(t) = f_{01}(t). \tag{4.7.40}$$

We can now summarize the results corresponding to equations (4.7.24) - (4.7.27) as follows:

$$\Pr\{\sigma_A(t) = t\} = e^{-(\lambda+\mu)t} ; \tag{4.7.41}$$

$$\Pr\{r<\sigma_A(t)<r+dr;\ \sigma_{B_1}(t) = 0\} = e^{-(\lambda+\mu)r} \sum_{m=1}^{\infty} \left[\frac{(\mu r)^m}{m!} f_{01}^{(m)}(t-r) + \right. \tag{4.7.42}$$

$$\left. + \frac{\mu(\mu r)^{m-1}}{(m-1)!} [F_{0,m}(t-r)-F_{0,m-1}(t-r)]\right] dr;$$

$$\Pr\{r<\sigma_A(t)<r+dr;\ \sigma_{B_2}(t) = 0\} = e^{-(\lambda+\mu)r} \sum_{m=1}^{\infty} \left[\frac{(\lambda r)^m}{m!} f_{10}^{(m)}(t-r) + \right. \tag{4.7.43}$$

$$\left. + \frac{\lambda(\lambda r)^{m-1}}{(m-1)!} [F_{m,0}(t-r)-F_{m-1,0}(t-r)]\right] dr;$$

and

$$p(r,s,t) = e^{-(\lambda+\mu)r} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\frac{(\lambda r)^m}{m!} \frac{(\mu r)^n}{n!} f_{10}^{(n)}(s) f_{01}^{(m)}(t-r-s) + \right.$$

$$+ \frac{(\lambda r)^n}{n!} \frac{\mu(\mu r)^{m-1}}{(m-1)!} f_{10}^{(n)}(s) [F_{0,m}(t-r-s)-F_{0,m-1}(t-r-s)] +$$

$$+ \frac{\lambda(\lambda r)^{n-1}}{(n-1)!} \frac{(\mu r)^m}{m!} f_{01}^{(m)}(t-r-s)[F_{n,0}(s)-F_{n-1,0}(s)]\Bigg],$$

$$0 < r,\ s < t. \qquad (4.7.44)$$

Adopting the Laplace transform technique used in the previous Sections in this Chapter, we obtain after tedious calculation the following explicit expressions:

$$\Pr\{r<\sigma_A(t)<r+dr;\ \sigma_{B_1}(t) = 0\} = e^{-(\lambda+\mu)t}\{\lambda I_0[\ell(r)] +$$

$$+ \left(\frac{\lambda\mu(t-r)}{t}\right)^{1/2} I_1[\ell(r)]\}dr +$$

$$+ (\mu-\lambda)e^{-\mu r} J[\lambda t, \mu(t-r)]dr; \qquad (4.7.45)$$

$$\Pr\{r<\sigma_A(t)<r+dr;\ \sigma_{B_2}(t) = 0\} = e^{-(\lambda+\mu)t}\{\mu I_0[\ell(r)] +$$

$$+ \left(\frac{\lambda\mu(t-r)}{t}\right)^{1/2} I_1[\ell(r)]\}dr +$$

$$+ (\lambda-\mu)e^{-\lambda r} J[\mu t, \lambda(t-r)]dr; \qquad (4.7.46)$$

where

$$\ell(r) = 2\sqrt{\lambda\mu t(t-r)} \qquad (4.7.47)$$

and

$$p(r,s,t) = e^{-(\lambda+\mu)t}M(r,s)N(r,s,t) +$$

$$+ e^{-(\lambda+\mu)s}M(r,s)\Pr\{r<\sigma_A(t-s)<r+dr;\ \sigma_{B_2}(t-s) = 0\} +$$

$$+ e^{-(\lambda+\mu)(t-r-s)}\Pr\{r<\sigma_A(r+s)<r+dr;\ \sigma_{B_2}(r+s) = 0\} \times$$

$$\times N(r,s,t), \quad 0 < r,\ s < t, \tag{4.7.48}$$

where

$$M(r,s) = \left(\frac{\lambda\mu}{s(r+s)}\right)^{1/2} rI_1[\ell_1(r,s)], \tag{4.7.49}$$

$$N(r,s,t) = \left(\frac{\lambda\mu}{(t-s)(t-s-r)}\right)^{1/2} rI_1[\ell_2(r,s,t)], \tag{4.7.50}$$

and

$$\ell_1(r,s) = 2\sqrt{\lambda\mu s(r+s)}\ ;\ \ell_2(r,s,t) = 2\sqrt{\lambda\mu(t-s)(t-s-r)}\ . \tag{4.7.51}$$

We note that $p(r,s,t)$ is the product of functions of arguments $r(r+s)$ and $(t-r)(t-r-s)$ which is analogous to the two-state result.

As a partial check on our analysis we should find that

$$\Pr\{r<\sigma_A(t)<r+dr;\ \sigma_{B_1}(t) = 0\} + \Pr\{r<\sigma_A(t)<r+dr;\ \sigma_{B_2}(t) = 0\} +$$

$$+ \int_0^{t-r} p(r,s,t)\,ds = p_B(t-r,t) \tag{4.7.52}$$

where $p_B(s,t)$ is the non-zero sojourn density given by equation (4.4.14). With the aid of [17], Vol. I, we find, after much algebraic manipulation, that

$$\Pr\{r<\sigma_A(t)<r+dr;\ \sigma_{B_1}(t) = 0\}= \frac{1}{2\pi i}\int_C e^{zt-Zr}\{e^{\lambda r\phi_{10}(z)}-1 + \tag{4.7.53}$$

$$+ \lambda\Phi_{10}(z)e^{\lambda r\phi_{10}(z)}\}dzdr;$$

$$\Pr\{r<\sigma_A(t)<r+dr;\ \sigma_{B_2}(t) = 0\}= \frac{1}{2\pi i}\int_C e^{zt-Zr}\{e^{\mu r\phi_{01}(z)}-1 + \tag{4.7.54}$$

$$+ \mu\Phi_{01}(z)e^{\mu r\phi_{01}(z)}\}dzdr;$$

and

$$\int_0^{t-r} p(r,s,t)\,ds = \frac{1}{2\pi i}\int_C e^{zt-Zr}\{[e^{\lambda r\phi_{10}(z)}-1][e^{\mu r\phi_{01}(z)}-1] +$$

$$+ \mu\Phi_{01}(z)e^{\mu\phi_{01}(z)r}[e^{\lambda\phi_{10}(z)}-1] +$$

$$+\lambda\Phi_{10}(z)e^{\lambda\phi_{10}(z)r}[e^{\mu\phi_{01}(z)}-1]\}dz; \tag{4.7.55}$$

where $Z = \lambda + \mu + z$, C is the usual Laplace inversion contour and $\phi_{mn}(z)$ and $\Phi_{mn}(z)$ are defined in (3.1.2). Adding, we obtain, with the aid of (3.2.13) and (3.2.18), the result

$$\Pr\{r<\sigma_A(t)<r+dr\} = \frac{1}{2\pi i}\int_C e^{zt}\left[\frac{Z}{R}e^{-Rr} + \right. \tag{4.7.56}$$

$$\left. + e^{-Rr}\left(\frac{(\lambda+\mu)}{R} + \frac{(\lambda-\mu)^2}{ZR}\right)\right]dz$$

where $R^2 = Z^2 - 4\lambda\mu$. Upon inversion using [17], Vol. I, (36), (37), p. 249, (4.7.56) becomes

$$\Pr\{r<\sigma_A(t)<r+dr\} = e^{-(\lambda+\mu)t}\left[\frac{2t\sqrt{\lambda\mu}}{\sqrt{t^2-r^2}} I_1[\ell_3(t)] + \right.$$

$$\left. + (\lambda+\mu)I_0[\ell_3(r)]\right]dr + (\lambda-\mu)^2\int_r^t e^{-(\lambda+\mu)x} \times$$

$$\times I_0[\ell_3(x)]dxdr , \tag{4.7.57}$$

where

$$\ell_3(x) = 2\sqrt{\lambda\mu(x^2-r^2)} . \tag{4.7.58}$$

Then using (2.4.11), it is not difficult to show that

$$(\lambda-\mu)^2\int_r^t e^{-(\lambda+\mu)x}I_0[\ell_3(x)]dx = (\lambda-\mu)e^{(\mu-\lambda)r}J_1[\mu(t+r),\lambda(t-r)] +$$

$$+ (\mu-\lambda)e^{(\lambda-\mu)r}J_1[\lambda(t+r),\mu(t-r)]. \tag{4.7.59}$$

Upon combining (4.7.57) and (4.7.59), we see that (4.7.57) agrees with (4.4.14).

No asymptotic results have been obtained so far.

CHAPTER V

CONNECTION OF THE RESULTS FOR THE UNRESTRICTED RANDOM WALK IN DISCRETE TIME AND THE WORK OF E. SPARRE ANDERSEN WITH THE UNRESTRICTED RANDOM WALK IN CONTINUOUS TIME

5.1 Introduction

By discrete-time random walk we mean the random walk in which the particle moves to the right or left in steps of unit length at epochs which may be numbered consecutively 1, 2, 3, ..., but where we are unconcerned with the amount of time between these epochs. The work contained in [18], Chapter III, dealing with the discrete-time random walk, demonstrates a connection between our results and those for the unrestricted random walk in discrete time. It is possible to derive at least some of the results for the unrestricted random walk in continuous time from the discrete-time results. These derivations utilize probabilities conditioned on the number of steps during a given time t and the probability that a certain number of steps take place in time t. This procedure will be used in Section 2 to show how first-passage time densities and state probabilities for the continuous-time random walk can be obtained from the corresponding results in discrete time. In Section 3, we shall show the connection of several of our results with analogous results for discrete time discovered by E. Sparre Andersen. The connection does not, at first sight, seem obvious.

We summarize now the basic results for discrete-time random walks needed in Section 2 and 3. For a detailed discussion, see [18]. It is assumed that the particle moves only at fixed epochs numbered 1, 2, 3, ... We let $X_i = \pm 1$ according to whether or not the i^{th} step is positive or negative, $i = 1, 2, 3, \ldots$. It is assumed that

$$\Pr\{X_i = +1\} = p; \ \Pr\{X_i = -1\} = q; \ i=1,2,3,\ldots, \tag{5.1.1}$$

where

$$p + q = 1 \ , \ 0 < p < 1. \tag{5.1.2}$$

The position of the particle at the end of the r^{th} step is then equal to

$$S_r = X_1 + X_2 + \ldots + X_r \ , \ r=1,2,3,\ldots, \tag{5.1.3}$$

assuming that

$$S_0 = 0 \ , \tag{5.1.4}$$

where S_0 denotes the position of the particle initially. The state probabilities are given by the equations

$$p_n(r) = \Pr\{S_r = n\} = \binom{r}{\frac{r+n}{2}} p^{\frac{r+n}{2}} q^{\frac{r-n}{2}} \tag{5.1.5}$$

and

$$p_{-n}(r) = \Pr\{S_r = -n\} = \binom{r}{\frac{r+n}{2}} p^{\frac{r-n}{2}} q^{\frac{r+n}{2}} \quad ,n\leq r, \qquad r=0,1,2,..., \tag{5.1.6}$$

where r, n are both odd or both even.

The probability of first passage from 0 to n at time 2r-n is given by the equation

$$f_{0n}(2r-n) = \frac{n}{2r-n}\binom{2r-n}{r} p^r q^{r-n} \quad , r=n,n+1,..., \qquad n=1,2,.... \tag{5.1.7}$$

We now turn to the application of these results to the unrestricted random walk in continuous time with negative exponentially distributed intervals between steps.

5.2 Extension of Discrete-Time Results to Continuous Time for the Random Walk with Negative Exponentially Distributed Intervals Between Steps

We assume, as before, that positive and negative steps occur according to two independent Poisson streams with parameters λ and μ, respectively. The probability that a positive step occurs during the small interval (t,t+dt) is then λdt and, similarly, the probability that a negative step occurs during (t,t+dt) is equal to μdt. Consequently, the probability that a step (either positive or negative) occurs during $(t,t+dt)$ is equal to $(\lambda+\mu)dt$. To extend the discrete-time results to continuous time we need the probabilities p and q that a positive or negative step occurs given that a step of some kind has occurred. From the preceding considerations, it follows by elementary probability argument that

$$p = \frac{\lambda}{\lambda+\mu}, \quad q = \frac{\mu}{\lambda+\mu}. \tag{5.2.1}$$

Now let $c(t)$ denote the number of steps that the particle takes during $(0,t)$, and let T_r be the time of the r^{th} step. From the properties of the negative exponential distribution, it follows that

$$\Pr\{c(t) = r\} = \frac{[(\lambda+\mu)t]^r}{r!} e^{-(\lambda+\mu)t}, \qquad r=0,1,2,\ldots, \tag{5.2.2}$$

and

$$\Pr\{t < T_r < t+dt\} = (\lambda+\mu)\frac{[(\lambda+\mu)t]^{r-1}}{(r-1)!} e^{-(\lambda+\mu)t}dt, \qquad r=1,2,3,\ldots. \tag{5.2.3}$$

We now proceed by elementary probability argument to obtain $p_n(t)$ and $f_{0n}(t)$ from the discrete-time results, where $p_n(t)$ and $f_{0n}(t)$ are defined by equations (1.2) and (3.2.1), respectively.

First of all, from (5.1.5) and (5.2.2) we find that

$$p_n(t) = \Pr\{S(t) = n \mid S(0) = 0\}$$

$$= \sum_{r=n}^{\infty} \Pr\{S_r = n\}\Pr\{c(t) = r\}$$

$$= \sum_{r=n}^{\infty} \binom{r}{\frac{r+n}{2}} p^{\frac{r+n}{2}} q^{\frac{r-n}{2}} \frac{[(\lambda+\mu)t]^r}{r!} e^{-(\lambda+\mu)t} \qquad (5.2.4)$$

where it is understood that the terms in (5.2.4) vanish if $r+n$ is odd. By use of (5.2.1) and upon simplification, (5.2.4) becomes

$$p_n(t) = \left(\frac{\lambda}{\mu}\right)^{n/2} e^{-(\lambda+\mu)t} \sum_{r=0}^{\infty} \frac{(\lambda\mu t^2)^{r+n/2}}{(r+n)!r!}$$

$$= \rho^{n/2} e^{-(\lambda+\mu)t} I_n(2t\sqrt{\lambda\mu}) \quad , n=0,1,2,..., \qquad (5.2.5)$$

where

$$\rho = \frac{\lambda}{\mu} . \qquad (5.2.6)$$

The latter part of (5.2.5) follows directly from (2.3.1). We note that (5.2.5) agrees with equation (2.3.12).

Similarly, by use of equation (5.1.7) and (5.2.3), it follows that

$$f_{0n}(t) = \sum_{m=n}^{\infty} f_{0n}(m)\Pr\{t < T_m < t+dt\}$$

$$= e^{-(\lambda+\mu)t}\, n \sum_{m=0}^{\infty} \frac{\lambda^{n+m}\mu^{m}t^{2m+n-1}}{(m+n)!m!} \qquad (5.2.7)$$

$$= e^{-(\lambda+\mu)t}\, n\rho^{n/2}\, I_n(2t\sqrt{\lambda\mu})/t \ , \ n=1,2,\ldots.$$

Equations (5.2.5) and (5.2.7) are given in [19] where these same methods were used to obtain them. It is possible that other results for the continuous-time case can be derived using discrete-time results. It seems simpler, however, to follow the methods used in Chapters III and IV.

5.3 Connection with the Work of E. Sparre Andersen

E. Sparre Andersen's fundamental research [1] and [2] concerning the fluctuations of sums of random variables is summarized, perhaps more accessibly, in [19] and [33]. We concentrate attention on four of Andersen's results which bear a striking resemblance to our results for continuous time:

i) Consider a discrete-time unrestricted random walk for which the size of each step X_i has a common distribution and let $S_0 = 0$, S_1, S_2, S_3, ..., S_n be the sequence of first n partial sums of the X_i. If N_n is the number of positive terms among S_1, S_2, ..., S_n, then

$$\Pr\{N_n = k\} = \Pr\{N_k = k\}\Pr\{N_{n-k} = 0\} \ . \quad (5.3.1)$$

ii) Let ν be the number of steps (or time) required to reach the first maximal term S_ν among $S_1, S_2, \ldots, S_n$ so that $S_\nu > 0$, $S_\nu > S_1, \ldots, S_\nu > S_{\nu-1}$, $S_\nu \geq S_{\nu+1}$, $S_\nu \geq S_{\nu+2}$, $S_\nu \geq S_{\nu+3}, \ldots, S_\nu \geq S_n$. Then

$$\Pr\{\nu = k\} = \Pr\{N_n = k\} \ . \qquad (5.3.2)$$

iii) For fixed α, $0 < \alpha < 1$, as $n \to \infty$,

$$\Pr\{N_n < n\alpha\} \to \frac{2}{\pi}\text{arc sin } \sqrt{\alpha} \ . \qquad (5.3.3)$$

iv) For a walk with steps of ± 1, let τ_n be the probability of first passage from 0 to +1 at the n^{th} step (time n). Moreover, let $\tau(s)$ be the generating function of τ_n, viz.,

$$\tau(s) = \sum_{n=1}^{\infty} s^n \tau_n \ .$$

Then

$$\log[1 - \tau(s)]^{-1} = \sum_{n=1}^{\infty} \frac{s^n}{n} \Pr\{S_n > 0\} \ . \qquad (5.3.4)$$

Equation (5.3.1) corresponds to $p_B(s,t)$ for the positive sojourn time defined in (4.2.1) and given by (4.3.13), because $F_{01}(t-s)$ is the probability that a particle, beginning its walk at the origin, does not pass into the positive state throughout the time interval (0,t-s). In other words,

$$F_{01}(t-s) = \Pr\{\sigma_B(t-s) = 0 \mid S(0) = 0\} , \qquad (5.3.5)$$

where $\sigma_B(t)$ denotes the amount of time during (0,t) that the particle spends in state B, the positive state. Similarly,

$$F_{10}(s) = \Pr\{\sigma_B(s) = s \mid S(0) = 1\} . \qquad (5.3.6)$$

The factor λ in (4.3.13) can be explained on the grounds that equation (4.3.13) was derived assuming S(0) = 0 so that λds is the probability of its immediate passage into the strictly positive state.

The result stated in paragraph (ii) is simply the analogue of the result

$$m(s,t) = p_B(s,t) , \qquad (5.3.7)$$

where m(s,t) is defined by (3.5.5) and given in (3.5.9).

Equation (5.3.3) corresponds directly to equation (4.3.30), which expresses the asymptotic behavior of the random variable $\sigma_B(t)/t$, i.e., the distribution of $\sigma_B(t)/t$ and the distribution of N_n/n have the same asymptotic behavior.

Concerning (iv) we note from (5.2.5) and (5.2.7) that

$$\frac{1}{n} f_{0n}(t) = \frac{1}{t} p_n(t) , \qquad (5.3.8)$$

and hence,

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} e^{-zt} f_{0n}(t)dt = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-zt} p_n(t)\frac{dt}{t}$$

(5.3.9)

$$= \int_0^{\infty} e^{-zt} \Pr\{S(t) > 0\}\frac{dt}{t} .$$

From equation (3.2.13), which states that

$$\int_0^{\infty} e^{-zt} f_{0n}(t)dt = [\phi_{01}(z)]^n \qquad (5.3.10)$$

where

$$\phi_{0n}(z) = \int_0^{\infty} e^{-zt} f_{0n}(t)dt , \qquad (5.3.11)$$

we see that

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} e^{-zt} f_{0n}(t)dt = \log[1 - \phi_{01}(z)]^{-1} .$$

(5.3.12)

Combining (5.3.9) and (5.3.12), we have

$$\int_0^{\infty} e^{-zt} \Pr\{S(t) > 0\}\frac{dt}{t} = \log[1 - \phi_{01}(z)]^{-1} , \qquad (5.3.13)$$

a result which corresponds to equation (5.3.4).

Although we have given no new results here, with the exception of (5.3.13), it is worthwhile to note the obvious comparisons between our work and that of E. Sparre Andersen. An elegant exposition of the results given in this Section is included in an unpublished paper by B. W. Conolly [11].

CHAPTER VI

SOME GENERAL RESULTS FOR THE UNRESTRICTED RANDOM WALK AND THE SINGLE-SERVER QUEUING PROCESS

6.1 Introduction

Until now we have considered only cases in which positive and negative steps occur according to a Poisson stream. In this Chapter we shall summarize certain results pertaining to the unrestricted and queuing random walks under more general assumptions. Since most of our work is concerned with the unrestricted random walk, we shall postpone any further remarks about the queuing process until Section 6.5. Therefore, the reader may assume that, with the exception of Section 6.5, the discussion in this Chapter applies only to the unrestricted random walk.

The extension that will be made in Sections 6.2 - 6.4 is a natural consequence of removing the barrier at zero in the queuing random walk. We shall be concerned with the development of results in the unrestricted case which correspond to the GI/M/1 and M/G/1 queuing systems described in Chapter I.

First of all, let $A(t)$ denote the distribution function of the length of time between positive steps, and let $a(t)$ denote its probability density function. Similarly, denote by $B(t)$ the distribution function of the time between negative steps with its corresponding probability density function denoted by $b(t)$. We define $A_c(t)$ and $B_c(t)$ by the equations

$$A_c(t) = 1 - A(t) \tag{6.1.1}$$

and

$$B_c(t) = 1 - B(t) \; . \tag{6.1.2}$$

We continue to assume throughout Sections 6.2 - 6.4 that positive and negative steps occur independently of each other. As a result of this assumption, we have, for the unrestricted random walk,

$$\begin{aligned} p_n(t) &= \Pr\{S(t) = n \mid S(0) = 0\} \\ &= \sum_{m=0}^{\infty} \{A^{(m+n)}(t) - A^{(m+n+1)}(t)\}\{B^{(m)}(t) - B^{(m+1)}(t)\} \; , \end{aligned} \tag{6.1.3}$$

and

$$\begin{aligned} p_{-n}(t) &= \Pr\{S(t) = -n \mid S(0) = 0\} \\ &= \sum_{m=0}^{\infty} \{A^{(m)}(t) - A^{(m+1)}(t)\}\{B^{(m+n)}(t) - B^{(m+n+1)}(t)\} \; , \\ & \qquad n = 0,1,2,\ldots, \end{aligned} \tag{6.1.4}$$

where

$$A^{(m)}(t) = \int_0^t A^{(m-1)}(s)a(t-s)ds \quad , \; m = 1,2,3,\ldots,$$

and

$$A^{(0)}(t) = 1 \; .$$

The function $B^{(m)}(t)$ is similarly defined.

Borrowing from the notation of queuing theory, we shall denote by "∞^2/M/G" the doubly-infinite or unrestricted random walk for which

$$a(t) = \lambda e^{-\lambda t} \tag{6.1.5}$$

and b(t) is general. Similarly, we let "∞^2/GI/M" designate the unrestricted random walk for which

$$b(t) = \mu e^{-\mu t} \tag{6.1.6}$$

and a(t) is general.

Under the more general assumptions of ∞^2/M/G and ∞^2/GI/M, we lose the convenient Markov property associated with the Poisson process. For example, in the ∞^2/GI/M walk, if one considers an arbitrary point t_0 in time, the probability that the next positive step occurs at time $t_0 + \tau$ depends on the length of time which has elapsed since the previous positive step. Consequently, the probability that the next positive step occurs at time $t_0 + \tau$ is independent of the time elapsed since the previous positive step only if that step has just occurred, i.e., it occurred during the interval (t_0-dt,t_0). For this reason we say that the process renews itself at the time of a positive step, and the instants at which positive steps occur we call renewal epochs. Analogous considerations apply to the walk $\infty^2/M/G$. To avoid difficulties in analysis, we shall always assume that time $t = 0$ is a renewal epoch. The fact that the process begins again, so to speak, at a renewal epoch will be very important in the analysis to follow.

In Sections 6.2 and 6.3 we shall discuss first-passage times and the time of occurrence and magnitude of the first maximum for the $\infty^2/M/G$ random walk. The corresponding topics for $\infty^2/GI/M$ are discussed in Section 6.4. As mentioned previously, we shall discuss the sojourn time problems for the M/G/1 and GI/M/1 queuing processes in Section 6.5. Although the techniques used are not new, the results given here have apparently not been published before.

6.2 First-Passage Time from m to n for $\infty^2/M/G$

As in Chapter III we define $f_{mn}(t)$ by the equation

$$f_{mn}(t)dt = \Pr\{S(t+dt) = n \; ; \; S(x) \neq n \; , \; 0 \leq x \leq t \mid S(0) = m\} \tag{6.2.1}$$

given that $t = 0$ is a renewal epoch, i.e., a transition from state $m + 1$ to state m takes place. Since we shall discuss only the unrestricted random walk in this Section, it suffices to consider the case where $S(0) = 0$ and a transition from $+1$ to 0 occurs at $t = 0$.

We shall derive a system of equations which relates the probabilities $f_{0n}(t)dt$, $n = 1,2,\ldots,$ by using the fact that

$$f_{mn}(t) = f_{0,m-n}(t) \quad , \; m = n+1, n+2, \ldots, \tag{6.2.2}$$

and analyzing the possible behavior of the particle until the time when the first negative step occurs, that is, until the first renewal epoch.

During the interval $(0,t)$ either no negative step occurs or at least one negative step occurs. If no negative step occurs during $(0,t)$, then the n^{th} positive step must occur at time t in order that a first passage to n take place at time t. On the other hand, suppose the first negative step occurs at time s, $0 < s < t$, and j positive steps occur during $(0,s)$, $0 \leq j < n$. Since s is a renewal epoch, a first passage to n takes place at time t only if a first passage from $j - 1$ to n occurs at the end of a further time $t - s$. The foregoing considerations are summarized by the equation

$$\begin{aligned} f_{0n}(t) &= \frac{\lambda(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t} B_c(t) + \\ &+ \int_0^t e^{-\lambda s} b(s) \sum_{r=0}^{n-1} \frac{(\lambda s)^r}{r!} f_{0,n+1-r}(t-s)ds \, . \end{aligned} \tag{6.2.3}$$

Now define the generating function $F(x,t)$ and its Laplace transform $\Phi(x,z)$ by

$$F(x,t) = \sum_{n=1}^{\infty} x^n f_{0n}(t) \tag{6.2.4}$$

and

$$\Phi(x,z) = \int_0^{\infty} e^{-zt} F(x,t)\,dt . \tag{6.2.5}$$

From (6.2.3) it follows that

$$\begin{aligned} F(x,t) &= \lambda x e^{-\lambda t(1-x)} B_c(t) + \\ &+ \int_0^t b(s) e^{-\lambda s(1-x)} \left\{\frac{F(x,t-s)}{x} - f_{01}(t-s)\right\} ds \end{aligned} \tag{6.2.6}$$

which implies that

$$\Phi(x,z) = \frac{\lambda x^2 B_c^*(z+\lambda-\lambda x) - x b^*(z+\lambda-\lambda x)\phi_{01}(z)}{x - b^*(z+\lambda-\lambda x)} . \tag{6.2.7}$$

To determine $\phi_{01}(z)$ we use an argument familiar in Queuing Theory beginning with consideration of the analyticity of $\Phi(x,z)$. First of all, it is evident that

$$|\phi_{0n}(z)| < 1 \tag{6.2.8}$$

for $\mathrm{Re}(z) > 0$ since

$$\int_0^{\infty} f_{0n}(t)\,dt \leq 1 . \tag{6.2.9}$$

By comparing the series

$$\Phi(x,z) = \sum_{n=1}^{\infty} x^n \phi_{0n}(z) \tag{6.2.10}$$

with the geometric series having common ratio x, it follows that $\Phi(x,z)$ converges at least for values of x such that $|x| < 1$. Therefore, $\Phi(x,z)$ must be analytic for all x,z such that $|x| < 1$ and $\mathrm{Re}(z) > 0$. The analyticity of $\Phi(x,z)$ implies that if the denominator of (6.2.7) vanishes for values of x,z such that $|x| < 1$ and $\mathrm{Re}(z) > 0$, then, simultaneously, the numerator must also vanish. Upon consideration of the roots of the equation

$$x = b^*(z+\lambda-\lambda x) \quad , \tag{6.2.11}$$

it follows from Rouche's theorem (see [29], for example) that there exists exactly one root w of equation (6.2.11) which lies within the unit circle. (For a thorough treatment of equations like (6.2.11) and their roots, see [35].) From analyticity considerations it follows that the numerator of (6.2.7) must vanish when x = w. Thus,

$$\phi_{01}(z) = B_c^*(z+\lambda-\lambda w) \tag{6.2.12}$$

and hence, upon combining (6.2.7) and (6.2.12), we have

$$\Phi(x,z) = \frac{\lambda x[xB_c^*(z+\lambda-\lambda x) - b^*(z+\lambda-\lambda x)B_c^*(z+\lambda-\lambda w)]}{[x - b^*(z+\lambda-\lambda x)]} \tag{6.2.13}$$

The quantities $\phi_{0n}(z)$, n = 2,3,..., may be obtained from (6.2.13) by using the relation

$$\phi_{0n}(z) = \frac{1}{n!} \left. \frac{d^n[\Phi(x,z)]}{dx^n} \right|_{x=0} . \qquad (6.2.14)$$

We can invert (6.2.12) and obtain an expansion for $f_{01}(t)$ by using the theorem of Lagrange given in Section 2.2. We find that

$$f_{01}(t) = \lambda e^{-\lambda t} B_c(t) + e^{-\lambda t} \sum_{m=1}^{\infty} \frac{\lambda^{m+1} t^{m-1}}{m!} \int_0^t s B_c(s) b^{(m)}(t-s)ds . \qquad (6.2.15)$$

We note that if B(t) represents the interarrival time distribution and $A(t) = 1 - e^{-\lambda t}$ represents the service time distribution, then (6.2.15) is identical to the expression for the busy period density for the GI/M/1 queuing process (see [30], for example). This similarity is not surprising because a first passage to +1 at time t in the doubly-infinite walk implies $S(x) \leq 0$, $0 \leq x < t$, i.e., the particle is restricted to non-positive states just as the system state in the queuing process must remain non-negative.

6.3 The Time of Occurrence and Magnitude of the First Maximum for ∞^2/M/G

We suppose, as in the preceding section, that a transition from +1 to 0 occurs at t = 0 and thus S(0) = 0. As in Chapter III we use the term "first maximum" with the following meaning. Suppose that there exists an epoch t' in (0,t) such that

(i) $S(t') > 0$;

(ii) $S(t') > S(x)$ for $0 \leq x < t'$; and

(iii) $S(t') \geq S(x)$ for $t' \leq x \leq t$.

Then we say that the particle attains its first maximum at the epoch t' in the interval $(0,t)$. In this section we shall be primarily concerned with the quantities $m_k(s,t)$ and $m(s,t)$ where $m_k(s,t)ds$ is the joint probability that

(1) $S(t') = k$, $k = 1,2,3,\ldots$; and

(2) $s < t' < s+ds$; (6.3.1)

and

$$m(s,t)ds = Pr\{s < t' < s+ds\}$$
$$= \sum_{k=1}^{\infty} m_k(s,t)ds \ . \tag{6.3.2}$$

Clearly,

$$m_k(t,t) = f_{0k}(t) \ . \tag{6.3.3}$$

In order to derive $m_k(s,t)$, $0 < s < t$, we again use the fact that the times at which negative steps occur constitute renewal epochs. We consider the three possible mutually exclusive cases:

(a) no negative step occurs during $(0,t)$;

(b) the first negative step occurs during (s,t);

(c) the first negative step occurs during $(0,s)$.

(6.3.4)

To illustrate the procedure we first derive $m_1(s,t)$ and then generalize to $m_k(s,t)$, $k = 1,2,3....$

If case (a), then (1) and (2) in (6.3.1) are satisfied for $k = 1$ if the first positive step takes place at time s and no positive step takes place during (s,t). The contribution to $m_1(s,t)$ is then given by

$$B_c(s)\lambda e^{-\lambda s} \frac{B_c(t)}{B_c(s)} e^{-\lambda(t-s)} = B_c(t)e^{-\lambda t} . \qquad (6.3.5)$$

If case (b) in (6.3.4) occurs, (1) and (2) in (6.3.1) are satisfied if the first positive step occurs at time s, the first negative step takes place at time s+v, $0 < v < t-s$, and no first passage from 0 to +2 occurs during (s+v,t). The contribution to $m_1(s,t)$ is then

$$B_c(s)\lambda e^{-\lambda s} \int_{v=0}^{t-s} e^{-\lambda v} \frac{b(s+v)}{B_c(s)} F_{02}(t-s-v)dv = \qquad (6.3.6)$$

$$= \lambda e^{-\lambda s} \int_{v=0}^{t-s} e^{-\lambda v} b(s+v) F_{02}(t-s-v)dv$$

where

$$F_{02}(t) = 1 - \int_0^t f_{02}(x)dx . \qquad (6.3.7)$$

Finally, for case (c) in (6.3.4), the first negative step must take place at some time v, $0 < v < s$, prior to the first positive step. Then to satisfy (1) and (2) in (6.3.1), a first maximum of magnitude two must occur at time s - v of the remaining time interval (0,t-v). Consequently, the contribution to $m_1(s,t)$ is

$$\int_0^s b(v)e^{-\lambda v}\, m_2(s-v,t-v)\,dv\ . \tag{6.3.8}$$

Summing up, we have from (6.3.5), (6.3.6), and (6.3.8),

$$\begin{aligned} m_1(s,t) = \lambda e^{-\lambda t}\, B_c(t) &+ \lambda e^{-\lambda s}\int_{v=0}^{t-s} e^{-\lambda v} b(s+v)\, F_{02}(t-s-v)\,dv\ + \\ &+ \int_{v=0}^{s} b(v)e^{-\lambda v}\, m_2(s-v,t-v)\,dv\ . \end{aligned} \tag{6.3.9}$$

By an analogous procedure we find, in general, that

$$\begin{aligned} m_k(s,t) &= \frac{\lambda(\lambda s)^{k-1}}{(k-1)!}\, e^{-\lambda t}\, B_c(t)\ + \\ &+ \frac{\lambda(\lambda s)^{k-1}}{(k-1)!}\, e^{-\lambda s}\int_{v=0}^{t-s} b(v+s)\, e^{-\lambda v}\, F_{02}(t-s-v)\,dv\ + \\ &+ \int_{v=0}^{s} b(v)e^{-\lambda v}\sum_{r=0}^{k-1}\frac{(\lambda v)^r}{r!}\, m_{k+1-r}(s-v,t-v)\,dv\ , \end{aligned}$$

$$k = 1,2,3,\ldots. \tag{6.3.10}$$

The summation in the last term takes account of the possibility that the particle may take up to k - 1 positive steps prior to the first negative step at time v.

We now define the generating function m(x,s,t) by

$$m(x,s,t) = \sum_{k=1}^{\infty} x^k m_k(s,t) . \qquad (6.3.11)$$

From (6.3.10) it follows that

$$m(x,s,t) = \lambda x B_c(t) e^{-\lambda(t-sx)} + $$
$$+ \lambda x e^{-\lambda s(1-x)} \int_{v=0}^{t-s} b(v+s) e^{-\lambda v} F_{02}(t-s-v)dv +$$
$$+ \int_{v=0}^{s} b(v) e^{-\lambda v(1-x)} \left\{\frac{m(x,s-v,t-v)}{x} - m_1(s-v,t-v)\right\}dv .$$
$$(6.3.12)$$

By (6.3.11) and (6.3.12) it is clear that

$$m(s,t) = m(1,s,t) = \lambda e^{-\lambda(t-s)} B_c(t) +$$
$$+ \lambda\int_{v=0}^{t-s} b(v+s)e^{-\lambda v} F_{02}(t-s-v)dv + \qquad (6.3.13)$$
$$+ \int_{v=0}^{s} b(v)\{m(s-v,t-v) - m_1(s-v,t-v)\}dv$$

where m(s,t)ds is the probability that $s < t' < s+ds$, $0 < s < t$.

$\phi_{02}(z)$ can be obtained from (6.2.13) and (6.2.14) and is given by

$$\phi_{02}(z) = \frac{\lambda B_c^*(z+\lambda-\lambda w) - \lambda B_c^*(z+\lambda)}{b^*(z+\lambda)} \tag{6.3.14}$$

where w is the root of (6.2.11) with smallest absolute value. The Laplace transform of $F_{02}(t)$ is then given by

$$\Phi_{02}(z) = \frac{1}{z}\,[1 - \phi_{02}(z)] \;. \tag{6.3.15}$$

Using an iterative procedure we can now obtain an expression for $m(s,t)$ in terms of $m_1(s,t)$. We will then proceed to derive an expression for $m_1(s,t)$ which will complete the derivation. To this end we define, first of all, the quantities

$$\ell_1(s,t) = \int_{v=0}^{s} b(v)m(s-v,t-v)dv \;,$$

$$f_1(s,t) = \int_{v=0}^{s} b(v)m_1(s-v,t-v)dv \;,$$

$$(6.3.16)$$

$$g_1(s,t) = B_c(t)e^{-\lambda(t-s)} \;,$$

and

$$h_1(s,t) = \int_{v=0}^{t-s} e^{-\lambda v}\, b(s+v)\, F_{02}(t-s-v)dv \;.$$

From (6.3.13) we have that

$$m(s,t) = \ell_1(s,t) - f_1(s,t) + \lambda\{g_1(s,t) + h_1(s,t)\} \; . \quad (6.3.17)$$

Now define

$$\ell_n(s,t) = \int_0^s b(v)\,\ell_{n-1}(s-v,t-v)dv \; , \; n = 2,3,4,\ldots, \quad (6.3.18)$$

and analogously define $f_n(s,t)$, $g_n(s,t)$ and $h_n(s,t)$. From (6.3.13) it then follows that

$$\ell_1(s,t) = \ell_2(s,t) - f_2(s,t) + \lambda[g_2(s,t) + h_2(s,t)] \quad (6.3.19)$$

and in general,

$$\ell_n(s,t) = \ell_{n+1}(s,t) - f_{n+1}(s,t) + \lambda[g_{n+1}(s,t) + h_{n+1}(s,t)] \; . \quad (6.3.20)$$

Adding, we obtain from (6.3.19) and (6.3.20) that

$$\ell_1(s,t) = \lim_{n\to\infty} \ell_n(s,t) - \sum_{n=2}^{\infty} f_n(s,t) + \lambda\sum_{n=2}^{\infty} [g_n(s,t) + h_n(s,t)] \; . \quad (6.3.21)$$

Hence, we have from (6.3.17) that

$$m(s,t) = \lim_{n\to\infty} \ell_n(s,t) - \sum_{n=1}^{\infty} f_n(s,t) + \lambda\sum_{n=1}^{\infty} [g_n(s,t) + h_n(s,t)] \; . \quad (6.3.22)$$

We can obtain expressions for $\ell_n(s,t)$, $f_n(s,t)$, $g_n(s,t)$, and $h_n(s,t)$ from their definitions given in (6.3.18) by using mathematical induction. These results are summarized as follows:

$$\ell_n(s,t) = \int_{v=0}^{s} b^{(n)}(v)m(s-v,t-v)dv$$

$$f_n(s,t) = \int_{v=0}^{s} b^{(n)}(v)m_1(s-v,t-v)dv$$

$$g_n(s,t) = e^{-\lambda(t-s)} \int_{v=0}^{s} b^{(n-1)}(v)B_c(t-v)dv$$

and

$$h_n(s,t) = \int_{v=0}^{s} b^{(n-1)}(v)h_1(s-v,t-v)dv \ , \qquad n = 2,3,4,\ldots. \tag{6.3.23}$$

To determine $\lim_{n\to\infty} \ell_n(s,t)$, we must find $\lim_{n\to\infty} b^{(n)}(v)$ for fixed v such that $0 \leq v \leq s$. We know that

$$\int_0^\infty b(t)dt \leq 1$$

and hence,

$$|b^*(z)| = \left|\int_0^\infty e^{-zt}\, b(t)dt\right| < 1 \tag{6.3.24}$$

for $\mathrm{Re}(z) > 0$. From the convolution property of the Laplace transform, we have that

$$[b^*(z)]^n = \int_0^\infty e^{-zt}\, b^{(n)}(t)dt \ . \tag{6.3.25}$$

From (6.3.24) we see that

$$\lim_{n\to\infty} [b^*(z)]^n = 0 \tag{6.3.26}$$

which implies by (6.3.25) and the uniqueness property of the Laplace transform that

$$\lim_{n\to\infty} b^{(n)}(v) = 0 \text{ , a.e.} \tag{6.3.27}$$

Therefore,

$$\lim_{n\to\infty} \ell_n(s,t) = 0 \text{ , a.e.} \tag{6.3.28}$$

Combining the above results we see that

$$\begin{aligned} m(s,t) = {} & e^{-\lambda(t-s)} B_c(t) + \lambda\int_{v=0}^{t-s} e^{-\lambda v} b(s+v) F_{02}(t-s-v)dv + \\ & + \lambda \sum_{n=1}^{\infty} \int_{v=0}^{s} b^{(n)}(v)dv \int_{u=0}^{t-s} e^{-\lambda u} b(s+u-v) F_{02}(t-s-u)du + \\ & + \lambda e^{-\lambda(t-s)} \sum_{n=1}^{\infty} \int_{v=0}^{s} b^{(n)}(v)B_c(t-v)dv - \\ & - \sum_{n=1}^{\infty} \int_{v=0}^{s} b^{(n)}(v)m_1(s-v,t-v)dv \text{ .} \end{aligned} \tag{6.3.29}$$

It is tedious but not difficult to show that (6.3.29) satisfies (6.3.13).

We now direct our attention to the task of obtaining an expression for $m_1(s,t)$ which will complete the analysis of

$m(s,t)$. To do this we shall need the auxiliary quantity $f_{01}(s,t)ds\,dt$ defined as the probability that

(i) first passage to +1 occurs during $(t,t+dt)$ given $S(0) = 0$; and

(ii) the last negative step prior to time t occurs during $(t-s-ds,ts)$;

given that a transition from +1 to 0 takes place at time $t = 0$.

The contributions of cases (a) and (b) in (6.3.4) to $m_1(s,t)$ are given in (6.3.5) and (6.3.6), respectively, in terms of known quantities. We shall obtain the contribution of case (c) in (6.3.4), however, in terms of $f_{01}(s,t)$ and subsequently derive $f_{01}(s,t)$ in terms of known quantities. Case (c) implies a first passage to +1 at time s and a last negative step at some time $s - v$, $0 < v < s$. To satisfy (1) and (2) in the definition of $m_1(s,t)$, given by (6.3.1), no first passage to +2 can take place during (s,t). Therefore, the first step taken during (s,t) must be a negative one unless, of course, no steps occur during (s,t). Therefore the contribution of case (c) to $m_1(s,t)$ can be expressed as

$$\int_{v=0}^{s} f_{01}(v,s)\ \{\frac{B_c(v+t-s)}{B_c(v)}\ e^{-\lambda(t-s)} + \int_{u=0}^{t-s} \frac{b(v+u)}{B_c(v)}\ e^{-\lambda u} F_{02}(t-s-u)du\}dv. \tag{6.3.30}$$

Combining (6.3.30) with (6.3.5) and (6.3.6), we have

$$m_1(s,t) = e^{-\lambda t} B_c(t) + \lambda e^{-\lambda s} \int_{v=0}^{t-s} b(v+s) e^{-\lambda v} F_{02}(t-s-v)\,dv +$$

$$+ \int_{v=0}^{s} f_{01}(v,s) \{ \frac{B_c(v+t-s)}{B_c(v)} e^{-\lambda(t-s)} + \qquad (6.3.31)$$

$$+ \int_{u=0}^{t-s} \frac{b(v+u)}{B_c(v)} e^{-\lambda u} F_{02}(t-s-u)\,du\}dv .$$

To complete the analysis of $m_1(s,t)$, we need to determine $f_{01}(s,t)$. To do this it will be convenient to use the quantity $q_{-r}(s,t)ds$, $r = 1,2,3,\ldots$, defined as the probability that, given a transition from +1 to 0 at time $t = 0$,

(i) $S(t) = -r$;

(ii) no first passage to +1 takes place during $(0,t)$; and

(iii) the last negative step prior to time t occurs during $(t-s-ds, t-s)$.

We define

$$q_{-r}(0,t) = \lim_{s \to 0} q_{-r}(s,t) \qquad (6.3.32)$$

and interpret $q_{-r}(0,t)dt$ to mean the probability of events (i) and (ii) above and that the last negative step prior to time t occurs during $(t-dt,t)$, i.e., just prior to time t.

We can now express $f_{01}(s,t)$ in terms of $q_{-r}(0,t)$ and then we shall derive the Laplace transform of $q_{-r}(0,t)$. To see the connection between $f_{01}(s,t)$ and $q_{-r}(0,t)$, we note the position of the particle at the time of the last negative step prior to first passage to +1 at time t. It is then clear that

$$f_{01}(s,t) = \lambda e^{-\lambda s} B_c(s) \sum_{r=1}^{\infty} \frac{(\lambda s)^r}{r!} q_{-r}(0,t-s) , \qquad (6.3.33)$$

for all s such that $0 < s < t$. We note that if the particle takes its last negative step prior to time t at time $t - s$ and $S(t-s) = -(n+r-1)$, $r = 1,2,\ldots$, then in order that a negative step occur at time t and $S(t+dt) = -r$, n positive steps must occur during the remaining interval $(t-s,t)$. Upon consideration of all possible positions of the particle at the time of the last negative step prior to time t, we obtain

$$q_{-1}(0,t) = b(t)e^{-\lambda t} + \int_0^t \sum_{n=1}^{\infty} q_{-n}(0,t-s)b(s) \frac{(\lambda s)^n}{n!} e^{-\lambda s} ds , \qquad (6.3.34)$$

and

$$q_{-r}(0,t) = \int_0^t \sum_{n=0}^{\infty} q_{-(n+r-1)}(0,t-s)b(s)\frac{(\lambda s)^n}{n!} e^{-\lambda s} ds , \quad r = 2,3,4,\ldots . \qquad (6.3.35)$$

Defining

$$q^*_{-r}(0,z) = \int_0^{\infty} e^{-zt} q_{-r}(0,t)dt , \qquad (6.3.36)$$

it follows that

$$q_{-1}^{*}(0,z) = b^{*}(\lambda+z) + \sum_{n=1}^{\infty} q_{-n}^{*}(0,z)\lambda_n(z) \tag{6.3.37}$$

and

$$q_{-r}^{*}(0,z) = \sum_{n=0}^{\infty} q_{-(n+r-1)}^{*}(0,z)\lambda_n(z) \tag{6.3.38}$$

where

$$\lambda_n(z) = \int_0^{\infty} e^{-(z+\lambda)t} \frac{(\lambda t)^n}{n!} b(t)dt \ . \tag{6.3.39}$$

We now apply the method of difference equations to (6.3.37) and (6.3.38) to obtain the Laplace transform of $q_{-r}(0,t)$ which will complete the derivation of $f_{01}(s,t)$. We seek a solution of the form

$$q_{-r}^{*}(0,z) = \sum_{i=1}^{\infty} A_i {y_i}^r \ . \tag{6.3.40}$$

In order for $q_{-r}^{*}(0,z) = y^r$ to be a particular solution of (6.3.38), it follows that y must satisfy the equation

$$y = \sum_{n=0}^{\infty} \lambda_n(z) y^n = b^{*}(z+\lambda-\lambda y) \ . \tag{6.3.41}$$

For fixed t, $\sum_{r=1}^{\infty} q_{-r}(0,t)$ is bounded and hence $q_{-r}^{*}(0,z)$ must be bounded for all possible values of r. In order that $q_{-r}^{*}(0,z)$ be of the form given in (6.3.40) and remain bounded, we must have $|y_i| < 1$ for all values of i. Upon application of Rouché's theorem, we can show that there exists a unique root y which lies within the unit circle and satisfies (6.3.41). Therefore, the general solution of (6.3.34) and (6.3.35) is given by

$$q^*_{-r}(0,z) = Ay^r \ , \ r = 1,2,3,\ldots . \tag{6.3.42}$$

From (6.3.34) it follows that

$$A = 1 \tag{6.3.43}$$

and hence,

$$q^*_{-r}(0,z) = y^r \ , \ r = 1,2,3,\ldots, \tag{6.3.44}$$

where y is the root of equation (6.3.41) with smallest absolute value. Noting that equation (6.3.41) is identical to (2.2.13), it follows that $q_{-1}(0,t)$ is equal to the probability density function for the length of the busy period in the M/G/1 queuing process given in (2.2.17). We can write, therefore,

$$q_{-1}(0,t) = e^{-\lambda t} \sum_{m=1}^{\infty} \frac{(\lambda t)^{m-1}}{m!} b^{(m)}(t) . \tag{6.3.45}$$

From (6.3.44) and the convolution property of the Laplace transform, we obtain

$$q_{-r}(0,t) = q_{-1}{}^{(r)}(0,t) . \tag{6.3.46}$$

This completes the derivation of $q_{-r}(0,t)$ and the analysis of $m(s,t)$.

Although it can be shown that $f_{10}(t)$ for $\infty^2/M/G$ is identical to the busy period density for M/G/1, given in (2.2.17), nevertheless, it is not obvious that the function $q_{-1}(0,t)$ and the busy period density function for M/G/1 should be equal. The event for which $q_{-1}(0,t)$ is the corresponding probability density function and the event first passage to 0 at time t given $S(0) = 1$ appear to be quite dissimilar, and no explanation for the common distribution of these two events has been found. A similar phenomenon occurs in the random walk $\infty^2/GI/M$, as will be shown in the next Section.

As a partial check on our analysis, we now show that the expressions for $m_1(s,t)$ and $m(s,t)$ given in (6.3.31) and (6.3.29), respectively, agree with the results given in Chapter III for the case in which positive and negative steps occur according to two independent Poisson streams with means λ and μ, respectively.

First of all, by letting $b(t) = \mu e^{-\mu t}$ in (6.3.45), we find, with the aid of (2.3.1), that

$$q_{-1}(0,t) = f_{10}(t) \tag{6.3.47}$$

and hence,

$$q_{-r}(0,t) = f_{r0}(t) \tag{6.3.48}$$

where $f_{r0}(t)$ is given in (3.2.16). By (6.3.33) we then have

$$f_{01}(s,t) = e^{-(\lambda+\mu)s} \sum_{r=1}^{\infty} \frac{\lambda(\lambda s)^r}{r} f_{r0}(t-s) \ . \qquad (6.3.49)$$

By substitution in (6.3.31) of the quantities $f_{01}(s,t)$ given by (6.3.49) and $b(t) = \mu e^{-\mu t}$, we find that

$$m_1(s,t) = \lambda e^{-(\lambda+\mu)t} + \lambda\mu e^{-(\lambda+\mu)s} \int_{v=0}^{t-s} e^{-(\lambda+\mu)v} F_{02}(t-s-v)dv +$$

$$+ \lambda e^{-(\lambda+\mu)(t-s)} \int_{v=0}^{s} e^{-(\lambda+\mu)v} \sum_{r=1}^{\infty} f_{r0}(s-v)dv +$$

$$+ \lambda\mu \int_{v=0}^{s} e^{-(\lambda+\mu)v} \sum_{r=1}^{\infty} f_{r0}(s-v)dv \times \qquad (6.3.50)$$

$$\times \int_{u=0}^{t-s} e^{-(\lambda+\mu)u} F_{02}(t-s-u)du \ .$$

We recall from Chapter III that the Laplace transforms of $f_{r0}(t)$ and $F_{02}(t)$ are denoted by $\phi_{r0}(z)$ and $\Phi_{02}(z)$, respectively. An expression for $\phi_{r0}(z)$ can be obtained from (3.2.13) by symmetry, and $\Phi_{02}(z)$ is given by (3.2.18). Using the quadratic relationship given in (3.2.9) together with (3.2.18), we obtain

$$\frac{\mu\Phi_{02}(z)}{z} = \Phi_{01}(z) - \frac{1}{z} \qquad (6.3.51)$$

and hence, upon inversion, we see that

$$\mu \int_{v=0}^{t-s} e^{-(\lambda+\mu)v} F_{02}(t-s-v)dv = F_{01}(t-s) - e^{-(\lambda+\mu)(t-s)} \qquad (6.3.52)$$

where

$$Z = \lambda + \mu + z \, . \qquad (6.3.53)$$

In the same manner we find that

$$\frac{\lambda}{Z} \sum_{r=1}^{\infty} \frac{\lambda^r}{Z^r} \phi_{r0}(z) = \phi_{01}(z) - \frac{\lambda}{Z} \qquad (6.3.54)$$

where $\phi_{01}(z)$ is given in (3.2.11). Upon inversion of (6.3.54), it follows that

$$\lambda \int_{v=0}^{s} e^{-(\lambda+\mu)v} \sum_{r=1}^{\infty} \frac{(\lambda v)^r}{r!} f_{r0}(s-v)dv = f_{01}(s) - \lambda e^{-(\lambda+\mu)s} \, . \qquad (6.3.55)$$

Now by substituting (6.3.52) and (6.3.55) into (6.3.50) and combining terms, we find that

$$m_1(s,t) = f_{01}(s) \, F_{01}(t-s) \qquad (6.3.56)$$

which agrees with (3.5.4)

We are now in a position to check our result for m(s,t) given in (6.3.29). Upon substitution of $b(t) = \mu e^{-\mu t}$ in (6.3.29), we should find that the resulting expression agrees with the known result for this case given by equation (3.5.8).

After making this substitution, using (6.3.56) and summing, we find immediately that

$$m(s,t) = \lambda e^{-\lambda(t-s)} e^{-\mu t} + \lambda\mu e^{-\mu s} \int_0^{t-s} e^{-(\lambda+\mu)v} F_{02}(t-s-v)dv +$$

$$+ \lambda\mu[1 - e^{-\mu s}] \int_{u=0}^{t-s} e^{-(\lambda+\mu)u} F_{02}(t-s-u)du +$$

$$+ \lambda[e^{-(\lambda+\mu)(t-s)} - e^{-\lambda(t-s)} e^{-\mu t}] -$$

$$- \mu F_{01}(t-s) \int_{v=0}^{s} f_{01}(s-v)dv \ . \tag{6.3.57}$$

Now using equation (6.3.52) and the fact that $\mu f_{01}(t) = \lambda f_{10}(t)$, it follows from (6.3.57) that

$$m(s,t) = \lambda F_{01}(t-s)\ F_{10}(s) \tag{6.3.58}$$

which agrees with (3.5.8). We have succeeded in showing that the general results derived in this section agree with the known particular cases given in Chapter III.

6.4 First-Passage Time from m to n and the Time of Occurrence and Magnitude of the First Maximum for $\infty^2/GI/M$

We assume in this Section that the intervals between positive steps are independently and identically distributed with distribution function $A(t)$ and probability density function $a(t)$ while negative steps occur in a Poisson stream with parameter μ.

As before, we let $f_{mn}(t)dt$ denote the probability that first passage from m to n occurs during $(t,t+dt)$ given that $t = 0$ is a renewal epoch, i.e., at $t = 0$ a transition from $m - 1$ to m takes place. Again, we consider only $f_{0n}(t)$, $n = 1,2,3,\ldots$, since

$$f_{mn}(t) = f_{0,n-m}(t) \ , \ n = m+1, m+2 \ldots . \tag{6.4.1}$$

We shall derive an expression for $f_{0n}(t)$ using the fact that the times at which positive steps occur are renewal epochs. In order that a first passage to n may take place at time t, $n = 1,2,\ldots$, at least n positive steps must occur during the time interval $(0,t)$. If r negative steps occur prior to the first positive step, then the particle must make a first passage from $-(r-1)$ to n during the remainder of the interval $(0,t)$. Consequently, we can write

$$f_{01}(t) = a(t)e^{-\mu t} + \int_{x=0}^{t} e^{-\mu x} \ a(x) \sum_{r=1}^{\infty} \frac{(\mu x)^r}{r!} \ f_{0r}(t-x)dx \ . \tag{6.4.2}$$

and

$$f_{0n}(t) = \int_{x=0}^{t} a(x)e^{-\mu x} \sum_{r=0}^{\infty} \frac{(\mu x)^r}{r!} \ f_{0,r+n-1}(t-x)dx \ , \quad n = 2,3,4,\ldots . \tag{6.4.3}$$

Recalling that

$$\phi_{0n}(z) = \int_0^\infty e^{-zt} f_{0n}(t)dt \ , \qquad (6.4.4)$$

we see from (6.4.2) and (6.4.3) that

$$\phi_{01}(z) = \mu_0(z) + \sum_{r=1}^{\infty} \mu_r(z)\phi_{0r}(z) \ , \qquad (6.4.5)$$

and

$$\phi_{0n}(z) = \sum_{r=0}^{\infty} \mu_r(z)\phi_{0,r+n-1}(z) \qquad (6.4.6)$$

where

$$\mu_r(z) = \int_0^\infty e^{-(z+\mu)t} a(t) \frac{(\mu t)^r}{r!} dt \ . \qquad (6.4.7)$$

We observe that the problem of solving (6.4.5) and (6.4.6) to obtain an expression for $\phi_{0n}(z)$ is identical to the problem of solving equations (6.3.37) and (6.3.38) for $q^*_{-r}(0,z)$ with obvious modification. Upon following the steps used in Section 6.3 to solve (6.3.37) and (6.3.38), we find that

$$\phi_{0n}(z) = \eta^n \ , \ n = 1,2,3,\ldots, \qquad (6.4.8)$$

where η is the root with smallest absolute value of the equation

$$x = a^*(z+\mu-\mu x) \ . \qquad (6.4.9)$$

As before, an application of Lagrange's theorem given in Section 2.2 yields the result

$$f_{01}(t) = e^{-\mu t} \sum_{m=1}^{\infty} \frac{(\mu t)^{m-1}}{m!} a^{(m)}(t) . \qquad (6.4.10)$$

By (6.4.8) and the convolution property of the Laplace transform, we have

$$f_{0n}(t) = f_{01}^{(n)}(t) . \qquad (6.4.11)$$

We note that if $A(t)$ represents the distribution of service times and $B(t) = 1 - e^{-\mu t}$ is the distribution of the length of time between successive arrivals, then the expression for the busy period density for the M/G/1 queuing process given in (2.2.17) is identical to (6.4.10). In Section 6.2 we noted that for the walk ∞^2/M/G, the expression for $f_{01}(t)$ given in (6.2.15) is identical to the busy period density for the GI/M/1 queuing process with obvious modification. For the random walks ∞^2/M/G and ∞^2/GI/M, the first-passage time density $f_{01}(t)$ is identical to the busy period density for M/G/1 and GI/M/1, respectively. Therefore, we see from the preceding remarks that for ∞^2/M/G we can obtain an expression for $f_{01}(t)$ from the expression for $f_{10}(t)$ for ∞^2/GI/M and vice-versa. The same is true with regard to obtaining $f_{10}(t)$ for ∞^2/M/G from $f_{01}(t)$ for ∞^2/GI/M and vice-versa.

If $q_1'(0,t)dt$ is defined as the joint probability that a positive step occurs during $(t-dt,t)$, $S(t) = 1$, and no first passage to -1 has occurred during $(0,t)$, then it is easily shown that for ∞^2/GI/M $q_1'(0,t) = f_{01}(t)$ given by (6.4.10). This result corresponds to the equality of $q_{-1}(0,t)$ and $f_{10}(t)$ for ∞^2/M/G as discussed in Section 6.3. As remarked in Section 6.3, no satisfactory probabilistic explanation of these identities has been found.

To derive $m_k(s,t)$ and $m(s,t)$, as defined at the beginning of Section 6.3, we note that the first maximum during $(0,t)$ must occur at the time of a positive step, i.e., at a renewal epoch. Moreover, if the first maximum is to occur at time s and be of magnitude k, then by the definition of first maximum given in Section 6.3, a first passage to k must occur at time s and no first passage from k to k + 1 may occur during the remaining time interval (s,t). Consequently, we can write

$$m_k(s,t) = f_{0k}(s)\ F_{01}(t-s) \tag{6.4.12}$$

and

$$m(s,t) = \sum_{k=1}^{\infty} m_k(s,t) \ , \tag{6.4.13}$$

where

$$F_{01}(t) = 1 - \int_0^t f_{01}(x)dx \ . \tag{6.4.14}$$

Evidently, for the case $a(t) = \lambda e^{-\lambda t}$, equations (6.4.12) and (6.4.13) will agree with (3.5.4) and (3.5.8), respectively, if the expression for $f_{01}(t)$ given in (6.4.10) agrees with (3.2.14). By letting $a(t) = \lambda e^{-\lambda t}$ in (6.4.10) and applying the series expansion for the modified Bessel function of the first kind given by (2.3.1), we find that (6.4.10) becomes

$$f_{01}(t) = \rho^{1/2} e^{-(\lambda+\mu)t} \frac{I_1(2t\sqrt{\lambda\mu})}{t} \qquad (6.4.15)$$

which agrees with (3.2.14).

This completes the analysis of first-passage times and first maxima for ∞^2/M/G and ∞^2/GI/M. Unfortunately, the corresponding results for sojourn times and returns to the origin for these partial generalizations of ∞^2/M/M are not as easy to obtain, and there is no progress to report in this area as yet. We can make an extension of the sojourn time results for the M/M/1 queuing system to the M/G/1 and GI/M/1 systems, however, as will be shown in the next section.

6.5 The Two-State Sojourn Time Problem for the M/G/1 and GI/M/1 Queuing Processes

Consider, first of all, the M/G/1 queue where it is assumed that arrivals occur in a Poisson stream with parameter λ, and service times are independently and identically distributed with distribution function $B(t)$ and probability density function $b(t)$. We assume, as usual, that $t = 0$ is a renewal epoch, i.e., a transition from +1 to 0 occurs at $t = 0$ and thus $S(0) = 0$. As in the walk $\infty^2/M/G$, we shall use the fact that the departure times are renewal epochs in order to derive the probability

$$p_B(s,t)ds = \Pr\{s < \sigma_B(t) < s+ds \mid S(0) = 0\} \quad (6.5.1)$$

given that $t = 0$ is a renewal epoch. The notation used here has been discussed in detail in Sections 4.1 and 4.2. As defined in Section 4.5 for the M/M/1 queue, we say that the system is in state B at time t if $S(t) > 0$, i.e., if the server is occupied.

There is one important difference between $\infty^2/M/G$ and the M/G/1 queuing process which simplifies the analysis of the sojourn density defined in (6.5.1). In the queuing process, the probability that a departure occurs while no customers are present in the system is equal to zero. Consequently, when a customer arrives and finds no one in the queue, his service time is independent of the length of time which has elapsed since the previous customer completed service, i.e., his service time is independent of the length of the preceding idle period. As a result, the length of the busy period, or first-passage time from +1 to 0, is also independent of the length of the preceding idle period. In effect, therefore, the process renews itself not only when a departure occurs but each time a busy period begins as well.

Making use of the fact that epochs of service completion and the times when a busy period is initiated are renewal epochs, we see that $p_B(s,t)$ satisfies the equation

$$p_B(s,t) = \lambda e^{-\lambda(t-s)} F_{10}(s) + \lambda\int_0^{t-s} e^{-\lambda x} f_{10}(s) e^{-\lambda(t-s-x)} dx +$$

$$+ \int_0^{t-s} \lambda e^{-\lambda x} dx \int_0^{s} f_{10}(y)\, p_B(s-y,t-x-y)\, dy =$$

$$= \lambda e^{-\lambda(t-s)} F_{10}(s) + \lambda(t-s) e^{-\lambda(t-s)} f_{10}(s) + \tag{6.5.2}$$

$$+ \lambda\int_0^{t-s} e^{-\lambda x} dx \int_0^{s} f_{10}(y)\, p_B(s-y,t-x-y)\, dy$$

provided that $0 < s < t$. The first-passage time density $f_{10}(t)$ is, of course, the same as the busy period density function for M/G/1 given in equation (2.2.17)

We note that (6.5.2) coincides with (4.2.19) if we let $g(t) = \lambda e^{-\lambda t}$ and $h(t) = f_{10}(t)$. From (4.2.20) we see that the expression

$$p_B(s,t) = e^{-\lambda(t-s)} \sum_{n=1}^{\infty} \left[\frac{\lambda[\lambda(t-s)]^{n-1}}{(n-1)!} [F_{n0}(s) - F_{n-1,0}(s)] + \right.$$

$$\left. + \frac{[\lambda(t-s)]^n}{n!} f_{10}^{(n)}(s) \right] \tag{6.5.3}$$

satisfies (6.5.2), where

$$F_{n0}(t) = 1 - \int_0^t f_{10}^{(n)}(x)\,dx \tag{6.5.4}$$

and

$$F_{00}(t) = 0 \ . \tag{6.5.5}$$

Since $S(0) = 0$, the server cannot remain busy throughout the entire interval $(0,t)$ and hence,

$$\Pr\{\sigma_B(t) = t\} = 0 \ . \tag{6.5.6}$$

On the other hand, if no customers arrive during $(0,t)$, the server will not be busy at all. Therefore,

$$\Pr\{\sigma_B(t) = 0\} = e^{-\lambda t} \ . \tag{6.5.7}$$

From (4.2.21) we find that the distribution function $P_B(s,t)$ is given by

$$\begin{aligned} P_B(s,t) &= e^{-\lambda(t-s)} \sum_{n=0}^{\infty} \frac{[\lambda(t-s)]^n}{n!} \int_0^s f_{10}^{(n)}(x)\,dx \\ &= \Pr\{0 \leq \sigma_B(t) \leq s\} \ . \end{aligned} \tag{6.5.8}$$

We can also express $P_B(s,t)$ and $p_B(s,t)$ in terms of the Laplace transforms of $f_{10}(t)$ and $F_{10}(t)$ denoted by $\phi_{10}(z)$ and $\Phi_{10}(z)$, respectively. The resulting expressions are given by

$$p_B(s,t) = e^{-\lambda(t-s)} \frac{1}{2\pi i} \int_{C_1} \frac{e^{zs}}{z} e^{\lambda(t-s)\phi_{10}(z)} dz \qquad (6.5.9)$$

and

$$p_B(s,t) = e^{-\lambda(t-s)} \frac{1}{2\pi i} \int_{C_2} e^{zs} \{e^{\lambda(t-s)\phi_{10}(z)} [1 + \frac{\lambda}{z} \{1 - \phi_{10}(z)\}] - 1\} dz , \qquad (6.5.10)$$

where C_1, C_2 are the usual Laplace inversion contours.

The analysis for the GI/M/1 queue is more difficult. Although the times at which arrivals occur constitute renewal epochs, the probability that an arrival occurs at any given time always depends on the amount of time which has elapsed since the previous arrival. We can, however, derive an integral equation satisfied by $p_B^{(1)}(s,t)$ where

$$p_B^{(1)}(s,t)ds = \Pr\{s < \sigma_B(t) < s+ds \mid S(0) = 1\} , \qquad (6.5.11)$$

given that a transition from 0 to +1 occurs at time t = 0 so that t = 0 is a renewal epoch. We assume that interarrival intervals, or lengths of time between arrivals, are independently distributed with a general distribution function A(t) and corresponding probability density function a(t). We define $A_c(t)$ by

$$A_c(t) = 1 - A(t) . \qquad (6.5.12)$$

We shall give an expression for $p_B^{(1)}(s,t)$ in terms of $f_{10}(s,t)dsdt$ defined as the joint probability that

(i) first passage from +1 to 0 occurs during $(t,t+dt)$ given that $S(0) = 1$ and an arrival occurs at $t = 0$; and

(ii) the last previous arrival occurred during $(t-s-ds,t-s)$, $0 < s < t$.

By considering all possible combinations of times when first passage from 1 to 0 and the last previous arrival could occur in order that $s < \sigma_B(t) < s+ds$, we see that $p_B^{(1)}(s,t)$ satisfies the equation

$$\begin{aligned} p_B^{(1)}(s,t) = {} & \mu e^{-\mu s} A_c(t) + \int_{u=0}^{s} f_{10}(u,s) \frac{A_c(t-s+u)}{A_c(u)} du + \\ & + \mu \int_{v=0}^{s} e^{-\mu v} \{a(t-s+v) F_{10}(s-v) + \\ & + \int_{0}^{t-s} a(u+v) p_B^{(1)}(s-v,t-u-v) du\} dv + \\ & + \int_{v=0}^{s} \int_{u=0}^{v} f_{10}(u,v) \{\frac{a(t-s+u)}{A_c(u)} F_{10}(s-v) \\ & + \int_{y=0}^{t-s} \frac{a(y+u)}{A_c(u)} p_B^{(1)}(s-v,t-y-v) dy\} du dv, \quad 0 < s < t . \end{aligned} \tag{6.5.13}$$

Equation (6.5.13) takes into consideration the following mutually exclusive possibilities:

(i) first passage from +1 to 0 occurs at time s and last arrival occurred at time $t = 0$;

(ii) first passage from +1 to 0 occurs at time s and the last arrival occurred at time $s - u$, $0 < u < s$;

(iii) first passage from +1 to 0 occurs at time v, $0 < v < s$, and the last previous arrival occurred at time $t = 0$; and

(iv) first passage from +1 to 0 occurs at time v, $0 < v < s$, and the last previous arrival occurred at time $v - u$, $0 < u < v$.

Since $S(0) = 1$, the server cannot remain idle throughout the interval of time $(0,t)$ and hence,

$$\Pr\{\sigma_B(t) = 0\} = 0 \ . \tag{6.5.14}$$

If no first passage from 1 to 0 occurs, however, the server remains busy throughout $(0,t)$. Therefore,

$$\begin{aligned} \Pr\{\sigma_B(t) = t\} &= F_{10}(t) \\ &= 1 - \int_0^t f_{10}(x)dx \ . \end{aligned} \tag{6.5.15}$$

For completeness we give an expression for $f_{10}(s,t)$. The derivation follows steps analogous to those used in Section 6.3 to derive $f_{01}(s,t)$ for $\infty^2/M/G$ and therefore we omit the details. In fact, the desired expression for $f_{10}(s,t)$

can be obtained directly from the expression for $f_{01}(s,t)$ given in (6.3.33) with obvious change in notation and interchange of the roles of the positive and negative steps. Therefore, we simply state this result as

$$f_{10}(s,t) = \mu e^{-\mu s} A_c(s) \sum_{n=1}^{\infty} \frac{(\mu s)^n}{n!} r_{n+1}(0,t-s) , \qquad (6.5.16)$$

where

$$r_2(0,t) = e^{-\mu t} \sum_{m=1}^{\infty} \frac{(\mu t)^{m-1}}{m!} a^{(m)}(t) , \qquad (6.5.17)$$

and

$$r_n(0,t) = r_2^{(n-1)}(0,t) , \; n = 2,3,4,\ldots. \qquad (6.5.18)$$

As a partial check on our analysis, we shall show that (6.5.13) yields the known results in Chapter IV for the case of independent, negative exponentially distributed interarrival and service times. Using the fact that for the M/M/1 queuing process we have

$$p_B^{(1)}(s,t) = e^{-\lambda(t-s)} f_{10}(s) + \int_0^s f_{10}(x) p_B^{(0)}(s-x,t-x)\,dx \qquad (6.5.19)$$

together with equation (4.2.20), it can be shown that

$$p_B^{(1)}(s,t) = e^{-\lambda(t-s)} f_{10}(s) + \lambda e^{-\lambda(t-s)} [F_{20}(s) - F_{10}(s)] +$$

$$+ \lambda \int_{y=0}^{t-s} e^{-\lambda y} dy \int_{v=0}^{s} f_{10}(v) p_B^{(1)}(s-v, t-v-y) dv \,, \tag{6.5.20}$$

where $F_{n0}(t)$ is defined by (6.5.4) and (6.5.5), $n = 0,1,2,\ldots.$ Therefore, (6.5.13) should agree with (6.5.20) for the case when interarrival and service times are negative exponentially distributed with parameters λ and μ, respectively.

Letting $a(t) = \lambda e^{-\lambda t}$ in (6.5.17), we find that

$$\begin{aligned} r_2(0,t) &= \sqrt{\rho}\, e^{-(\lambda+\mu)t} \frac{I_1(2t\sqrt{\lambda\mu})}{t} \\ &= \rho f_{10}(t) \,. \end{aligned} \tag{6.5.21}$$

Therefore, by (6.5.16) and (6.5.18) we have

$$f_{10}(s,t) = e^{-(\lambda+\mu)s} \sum_{r=1}^{\infty} \frac{(\lambda s)^r}{r!} f_{10}^{(r)}(t-s) \,. \tag{6.5.22}$$

Upon substitution of (6.5.21) and (6.5.22) into (6.5.13), we obtain

$$p_B^{(1)}(s,t) = \mu e^{-(\lambda+\mu)s} e^{-\lambda(t-s)} +$$

$$+ \mu e^{-\lambda(t-s)} \int_{u=0}^{s} e^{-(\lambda+\mu)u} \sum_{r=1}^{\infty} \frac{(\lambda u)^r}{r!} f_{10}^{(r)}(s-u) du +$$

$$+ \lambda\mu \int_{v=0}^{s} e^{-(\lambda+\mu)v} \{ e^{-\lambda(t-s)} F_{10}(s-v) +$$

$$+ \int_0^{t-s} e^{-\lambda u} p_B^{(1)}(s-v,t-u-v)\,du\}dv \; +$$

$$+ \lambda\mu \int_{v=0}^{s} \int_{u=0}^{v} e^{-(\lambda+\mu)u} \sum_{r=1}^{\infty} \frac{(\lambda u)^r}{r!} f_{10}^{(r)}(v-u) \; \times$$

$$\times \{e^{-\lambda(t-s)} F_{10}(s-v) \; + \tag{6.5.23}$$

$$+ \int_{y=0}^{t-s} e^{-\lambda y} p_B^{(1)}(s-v,t-y-v)\,dy\}dudv \; .$$

Since $f_{10}(t)$ is the same for both the M/M/1 queuing process and the ∞^2/M/M random walk, we can apply (6.3.52) and (6.3.55) to (6.5.23) and combine terms to find that

$$p_B^{(1)}(s,t) = e^{-\lambda(t-s)} f_{10}(s) + \lambda e^{-\lambda(t-s)}[F_{20}(s) - F_{10}(s)] \; +$$

$$+ \lambda \int_{y=0}^{t-s} e^{-\lambda y} dy \int_{v=0}^{s} f_{10}(v) p_B^{(1)}(s-v,t-v-y)\,dv \; , \tag{6.5.24}$$

which is identical with (6.5.20).

CHAPTER VII

SUMMARY AND CONCLUDING REMARKS

This dissertation contains a study of certain topics related to the random walk which proceeds by steps of ± 1 occurring at random time intervals. In general it is assumed that the intervals of time between steps of the same kind are independently and identically distributed. This model may be specialized to the queuing process by placing a reflecting barrier at the origin so that the displacement $S(t)$ of the random walk at any time t is non-negative.

Throughout most of the dissertation it is assumed that the steps of ± 1 occur according to two independent Poisson streams with parameters λ, μ, respectively. Under this assumption we designate the single-server queuing process by the usual notation M/M/1, and we denote by ∞^2/M/M the unrestricted random walk in which $S(t)$ may range over the entire set of positive and negative integers including zero. The letter M is used here to denote a negative exponential distribution. Where it is assumed that the lengths of time between steps of +1 or between steps of -1 are independently distributed according to a general distribution, the notations ∞^2/GI/M and ∞^2/M/G are used.

The study begins necessarily, because of its fundamental importance, with a discussion of the first passage time to an arbitrary state n given that the initial displacement S(0) is equal to m. Explicit expressions for the probability density function of the first-passage time from m to n are given for $\infty^2/M/M$. In Chapter VI, this result is extended to the more general random walks $\infty^2/M/G$ and $\infty^2/GI/M$, an extension which appears to be new. For the M/M/1 queuing process, Bailey [3] gives the distribution of the first-passage time from m to n for the case $m > n$. In this dissertation new results are given for the case $m < n$.

Intimately connected with the first-passage time problem is the problem of the time of occurrence and magnitude of the first maximum during (0,t). For the walk $\infty^2/M/M$ expressions are given in Chapter III for the joint probability and probability density function of t', the epoch of the first maximum, and S(t'), the maximum displacement during (0,t). The marginal distributions of t' and S(t') are obtained from the joint distribution. In Chapter VI these results are extended to the walks $\infty^2/M/G$ and $\infty^2/GI/M$. Apparently, this extension is new. For the walk $\infty^2/M/M$, it is found that the asymptotic distribution of t' can be expressed in terms of the stable distribution of index 1/2 for the case $\lambda \neq \mu$ and in terms of the arc sine distribution for the case $\lambda = \mu$. The asymptotic results for the case $\lambda \neq \mu$ have not been published before. Unfortunately, an investigation of the distributional properties of t' and S(t') for the M/M/1 queuing process has so far proved unsuccessful and must therefore remain an area for future research.

In addition to the topics of first-passage time and first maximum, Chapter III contains a study of two other random variables, viz. the number of returns to zero during an arbitrary time interval (0,t) and the time up to the r^{th} return to zero. For both ∞^2/M/M and the M/M/1 queuing process, explicit expressions are developed for the probability of n returns to zero during (0,t) and the probability that the r^{th} return to zero occurs during the small interval (t,t+dt). Moments and moment generating functions are given as well as asymptotic approximations valid for large values of t. Many of the results obtained do not appear to have been published previously.

A major portion of the dissertation deals with the so-called two-state sojourn time problem in which it is assumed that at any time t the particle may be in either one of two possible states, A and B. General expressions are derived for the probability density and distribution functions of the sojourn time $\sigma_B(t)$ in a given state B during an arbitrary time interval (0,t). In Chapter IV the general two-state sojourn time result has been applied to the M/M/1 queuing process in order to obtain the distribution of the busy time. Little attention has been given previously to the application of the sojourn time results to single-server queues with the exception of papers by Greenberg [24] and Linhart [26]. A similar application is made to ∞^2/M/M to obtain the distribution of $\sigma_B(t)$ for the two cases:

(i) B is the set of all non-zero integers; and

(ii) B is the set of all positive integers.

In each of the three particular applications considered, the distribution function and probability density function of the sojourn time $\sigma_B(t)$ are expressible in terms of the modified Bessel function of the first kind and the J-function studied by Goldstein [22] in connection with certain diffusion processes in a non-probabilistic context. These distribution and density functions have apparently not been expressed in this form previously. Such expressions are more easily adapted for numerical computation. Also, asymptotic results for large values of time can be obtained directly from these expressions. Consequently, asymptotic formulae for the probability $\Pr\{\alpha t<\sigma_B(t)<t\}$, $0<\alpha<1$, valid for large t, are obtained from the explicit results for finite values of t. In most cases these limiting forms are expressible in terms of the stable distribution of index 1/2. Section 4.6 compares numerically the asymptotic approximations of $\Pr\{\alpha t<\sigma_B(t)<t\}$ developed from our work to the approximations obtained from Takács' results [35]. In contrast to our direct approach, Takács' asymptotic formulae are based on the general theory of the asymptotic behavior of sums of random variables. Although Takács' results are also expressible in terms of the stable distribution of index 1/2, our approximations are generally more accurate, especially for relatively small values

of time.

Chapter V contains a discussion of several results by E. Sparre Andersen concerning fluctuations of sums of random variables and their time-dependent analogues. These analogues have not been mentioned in the literature with the exception of an unpublished paper by Conolly [11]. For example, it is found that in the $\infty^2/M/M$ walk beginning at zero that t', the epoch of the first maximum during (0,t), and $\sigma_B(t)$, where B is the set of positive integers, are identically distributed. This result has a counterpart for non-time-dependent random walks given by Andersen [1] and [2].

Although the volume of literature on random walks is great, few explicit results have been published concerning the time-dependent random walk considered in this dissertation. In addition to the theoretical interest generated by such a random walk, its close link with certain time-dependent random walk models of practical interest makes it worthy of further study. For example, the application most often cited in this dissertation is the single-server queuing process. Other queuing systems, however, such as multi-server systems, inventory models characterized by a last-come first-served queue discipline, and systems with arrival and service mechanisms dependent on queue length are also examples of models in which the time-dependent random walk is basic. In addition to queuing situations, the continuous-time random walk is applicable to models used to describe population changes over a given period of time. In these models, a step of +1 corresponds to a birth or any other addition to the population while a step of -1 corresponds to a death or a migration from the population.

Although many related areas for possible future research could be suggested, the following are two of the most important ones.

1) Instead of restricting consideration to step lengths of ± 1, one could study the time-dependent random walk in which the step length X is a continuous random variable. This type of model is applicable, for example, to insurance models in which it is assumed that claims occur at random time intervals and that the amount X of an individual claim has a continuous distribution. In principle these claims may be positive or negative. (For example, a death may free the company of an obligation and increase the reserves.) Some aspects of such a random walk in discrete time have been discussed by Conolly [10]. An analysis of various aspects of the continuous-time case remains to be done.

2) In this dissertation we have assumed, except in Chapter VI, that during the small time interval $(t,t+dt)$, a step of +1 or -1 occurs with instantaneous probability λdt, μdt, respectively. For many applications, however, it is convenient to assume that these in-

stantaneous probabilities vary with time, or in the so-called birth and death models, these probabilities are assumed to depend on the number in the population at time t. For a discussion of birth and death processes see [18], Chapter XVII. An attempt should be made to extend the sojourn time results of Chapter IV to these more general models. Such an analysis would yield probabilistic information about the proportion of time that the number in the population remains above or below a given level.

The above is a sample of the problems remaining to be solved in this field. The results obtained in this dissertation should form the basis for further investigation and generalization.

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In September, 1960, Mr. Gibson entered Denison University, Granville, Ohio. While at Denison he majored in mathematics and received the Bachelor of Arts degree in June, 1964. In September, 1964, Mr. Gibson began study for the Master of Science Degree in Mathematics at Case Institute of Technology, Cleveland, Ohio. He received this degree in June, 1966.

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SOME ASPECTS OF TIME-DEPENDENT

ONE-DIMENSIONAL RANDOM WALKS

Allen E. Gibson

Abstract

This dissertation contains a study of related topics connected with the one-dimensional random walk which proceeds by steps of ± 1 occurring at random time intervals. In general it is assumed that these intervals are identically and independently distributed. This model may be specialized to the queuing process by inserting a reflecting barrier at the origin so that the displacement $S(t)$ of the random walk at any time t is non-negative.

Throughout most of the dissertation it is assumed that the time intervals between steps of the same kind are independently and negative exponentially distributed with non-time-dependent parameter λ for positive steps, and μ for negative steps. Under this assumption we designate the single-server queuing process by the usual notation M/M/1. Using an obvious extension to the queuing notation, we denote by ∞^2/M/M the unrestricted walk in which $S(t)$ may range over the entire set of positive and negative integers including zero.

Topics of classical interest are discussed such as first-passage times, first maxima, the time of occurrence of the

r^{th} return to zero, and the number of returns to zero during an arbitrary time interval $(0,t)$. In addition to the discussion of these topics for $\infty^2/M/M$ and M/M/1, probability density functions are obtained for the first-passage times and the epoch of the first maximum on the assumption that time intervals between steps of +1 have a general distribution and steps of -1 occur in a Poisson stream and vice-versa. These more general expressions are new.

Special emphasis is placed on the two-state sojourn problem in which it is assumed that at any time t, $S(t)$ belongs to one of two possible states, A and B. The distribution of the sojourn time $\sigma_B(t)$ in a given state B during the arbitrary time interval $(0,t)$ is given. The general result for the distribution of $\sigma_B(t)$ is applied to the M/M/1 queuing process to obtain the distribution of the busy time. A similar application is made to the walk $\infty^2/M/M$ to obtain the distribution of $\sigma_B(t)$ for the two cases:

(i) B is the set of all non-zero integers; and

(ii) B is the set of all positive integers.

New expressions are given for the distribution function of $\sigma_B(t)$ in all three cases. New asymptotic formulae for these cases are derived and compared numerically with those obtained by Takacs using different methods.

For the more difficult sojourn time problem assuming three possible states, A, B_1, and B_2, the joint probability density function of $\sigma_A(t)$ and $\sigma_{B_1}(t)$ is derived. This result, not published before, is applied to $\infty^2/M/M$ assuming that A contains zero only and that B_1 and B_2 consist of the sets of positive and negative integers, respectively.

The dissertation also includes a discussion of several results by E. Sparre Andersen concerning fluctuations of sums of random variables and their time-dependent analogues.