NUMERICAL SOLUTIONS FOR A CLASS OF SINGULAR INTEGRODIFFERENTIAL EQUATIONS

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(ABSTRACT)

In this study, we consider numerical schemes for a class of singular integrodifferential equations with a nonatomic difference operator. Our interest in this particular class has been motivated by an application in aeroelasticity. By applying nonconforming finite element methods, we are able to establish convergence for a semi-discrete scheme. We use an ordinary differential equation solver for the semi-discrete scheme and then improve the result by using a fully discretized scheme. We report our numerical findings and comment on the rates of convergence.

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Chapter 1

Introduction

1.1 Background

In recent years the feasibility and advantages of active control surfaces to reduce maneuver gust, fatigue loads and damp vibration that contributes to flutter have been extensively studied. A systematic procedure for control design requires the development of a "realistic" mathematical model that predicts the dynamic behavior of the physical system. The development of state space models for aeroelastic systems, including unsteady aerodynamics, is potentially important for the design and development of highly maneuverable aircraft.

In [5], a complete dynamic model for the elastic motions of a three-degree-of-freedom airfoil section, with flap, in a two dimensional, incompressible flow (Theodorsen's problem) was formulated. An evolution equation for the circulation on the airfoil was derived and coupled to the rigid-body dynamics of the airfoil to obtain a complete set of functional differential equations that describes the composite system. The resulting model for the aeroelastic system including a forcing term f has the form

$$\begin{aligned}\frac{d}{dt}[Ax(t) \quad + \quad \int_{-\infty}^{0} A(s)x(t+s)ds] = \\ Bx(t) + \int_{-\infty}^{0} B(s)x(t+s)ds + f(t),\end{aligned} \tag{1.1}$$

for $t > 0$, where

$$x(t) = (h(t), \theta(t), \beta(t), \dot{h}(t), \dot{\theta}(t), \dot{\beta}(t), \Omega(t), \dot{\Omega}(t))^T.$$

The functions h, θ, β denote the plunge, pitch angle and flap angle, respectively. The 8×8 matrix A is singular (each entry of the last row is zero) while the 8×8

matrix function $A(s)$ is weakly singular ($A_{88}(s) = ((Us-2)/Us)^{1/2}$, U denotes the undisturbed stream velocity). The function Ω denotes the total airfoil circulation and has representation

$$\Omega(t) = \begin{cases} \int_{-\infty}^{t} \psi(s)ds, & t < 0, \\ \eta + \int_0^t \frac{\partial}{\partial s} \int_{-1}^{1} \gamma_a(s,\sigma)d\sigma ds, & t \geq 0, \end{cases}$$

for the given initial condition $\psi \in L_1(-\infty, 0)$, $\eta \in R$. Here $\gamma_a(t,x)$, $-1 < x < 1$, represents the circulation per unit distance on the airfoil. The state of the system (1.1) includes the past history of $\dot{\Omega}$ which may not be observed for the entire past $(-\infty, 0]$. However, one could possibly observe this past history over a finite time interval, say $[-b, 0]$. It is to be noted that the kernel function $A(s)$ in (1.1) is not integrable on $(-\infty, 0]$, therefore it is not possible to address the infinite delay problem as a finite delay problem by mapping the interval $(-\infty, 0]$ to some finite interval $[-T, 0]$.

A "finite delay" version of (1.1),

$$\begin{aligned} \frac{d}{dt}[Ax(t) \quad + \quad & \int_{-b}^{0} A(s)x(t+s)ds] = \\ & Bx(t) + \int_{-b}^{0} B(s)x(t+s)ds + f(t), \end{aligned} \tag{1.2}$$

has been studied extensively. We view the system (1.2) as having a singular integral component and concentrate on this particular component. In [15], approximation techniques for the finite delay system have been developed. We wish to follow the similar techniques for a specific finite delay system (1.2) and devote our attention to the scalar equation case.

1.2 Introduction

In this study, we consider numerical schemes for the scalar case with a singular kernel as mentioned above. It can be extended to a class of singular functional differential equations with a nonatomic difference operator D. The operator D is nonatomic since we can change the scalar equation into the special form

$$\frac{d}{dt}Dx_t = Lx_t + f(t),$$

with $x_t(\cdot) = x(t + \cdot)$ and the difference operator D defined as

$$D\phi = \int_{-b}^{0} |s|^{-p}\phi(s)ds,$$

where

$$b > 0, \quad p \in (0, 1).$$

The functional differential equation is singular because D contains a weakly singular kernel, $|s|^{-p}$.

In the literature, one finds the use of product spaces to analyze retarded functional differential equations ([1], [2], [9]) and to prove the well-posedness of atomic neutral functional differential equations ([6], [7], [17], [19]). The assumption that the difference operator D is nonatomic plays a crucial role in establishing well-posedness for the associated abstract Cauchy problems ([8], [15]). However, it has been established that the difference operator D being nonatomic is not necessary for the generation of a C_0 semigroup on product spaces ([6]). Weighted product spaces are recommended in [11], [12] and [13]; there it is assumed that the circulation history of the airfoil problem belongs to a weighted L_2 space. In other words, the weighted product spaces are used to establish well-posedness for the model.

In [8] and [15], the authors show that the finite delay version of the model equation of the aeroelastic system generates a C_0-semigroup on the weighted space $L_{2,g}$ (g is the weight function) and the infinitesimal generator of that semigroup satisfies a dissipative estimate, which is important for the stability of approximating schemes ([14], [15]). We follow this idea and present an approximation scheme with several examples to indicate the feasibility of the approach. We discuss the rates of convergence of the approximation scheme.

The study is organized as follows. In the second chapter, we state several important results from [8] and [15]. We formulate a scalar singular integrodifferential equation as a first order hyperbolic partial differential equation with a nonlocal boundary condition. Then we characterize the solutions on a weighted $L_{2,g}$ space by establishing well-posedness of the abstract evolution problem associated with this neutral system. One is then able to develop a computational algorithm and establish convergence of the numerical scheme ([18]). Chapter three contains exact solutions for a class of singular integral equations. The method of steps is introduced. Chapter four contains a description of the specific numerical scheme. By applying the nonconforming finite element methods to the abstract evolution problem, we are able to establish convergence for a semi-discrete scheme ([15]). We use an ordinary differential equation solver for the semi-discrete scheme and then improve the result by using a fully discretized scheme. We report our numerical findings and comment on the rates of convergence. Chapter five contains a summary and list of future work.

1.3 Notation

AC	:	absolutely continuous function on $[-b, 0]$, b fixed , $b > 0$.
C	:	continuous function on $[-b, 0]$, b fixed , $b > 0$.
$C(0, T)$	:	continuous function on $(0, T)$, T fixed , $T > 0$.
$C^1(0, T)$	:	$\phi \in C^1(0, T)$ if and only if $\phi \in C(0, T)$, $\dot{\phi} \in C(0, T)$, $T > 0$.
$Dom(D)$	:	domain of the operator D.
$L_{2,g}$	:	weighted L_2 space with a weight function g.
Z	:	$L_{2,g}$.
L_r	:	Banach space on $[-b, 0]$, b fixed , $b > 0$, $r \geq 1$.
BV	:	function of bounded variation on $[0, T]$, $T > 0$.
NBV	:	function of normalized bounded variation on $[0, T]$, $T > 0$, *i.e.*, $F \in NBV$ if and only if $F(0) = 0$ and $F \in BV$.
R	:	set of real numbers.
R^+	:	set of positive real numbers.
$\rho(A)$	:	spectra of A.

Chapter 2

A Semigroup on $L_{2,g}$

2.1 Introduction

In this chapter, we describe the associated singular integrodifferential equation for the finite delay version of the aeroelastic system. We show that a certain weighted L_2 space can be used as a state space for such system. We transform the system into a Cauchy problem in the weighted space and use linear semigroup theory to establish the well-posedness of the problem.

2.2 Problem description

We consider the well-posedness of the singular integrodifferential equation

$$\frac{d}{dt}\int_{-b}^{0} g(s)x(t+s)ds \quad = \quad f(t), \tag{2.1}$$

with initial data

$$x(s) \quad = \quad \phi(s), \quad -b \leq s \leq 0, \tag{2.2}$$

where $f(t)$ is locally integrable on $t \geq 0$, the function g belongs to L_1 and is positive, nondecreasing, and weakly singular at $s = 0$. Specifically, we study the case where $g(s) = |s|^{-p}$, $0 < p < 1$, on $[-b, 0]$, i.e., Abel's kernel. In particular, for the aeroelasticity model, $p = 0.5$.

Note that the initial value problem (2.1)–(2.2) can be written as

$$\int_{-b}^{0} g(s)x(t+s)ds = \int_{-b}^{0} g(s)\phi(s)ds + \int_{0}^{t} f(\tau)d\tau, \tag{2.3}$$

provided that the function $t \longrightarrow \int_{-b}^{0} g(s)x(t+s)ds$ is absolutely continuous and the function $g(\cdot)\phi(\cdot)$ belongs to L_1.

By defining

$$x_t(s) = x(t+s), \tag{2.4}$$

$$D\zeta = \int_{-b}^{0} g(s)\zeta(s)ds, \tag{2.5}$$

$$L\zeta = 0, \tag{2.6}$$

the associated singular functional differential equation with a nonatomic difference operator D for (2.1) is

$$\frac{d}{dt}Dx_t = Lx_t + f(t), \quad t \geq 0, \tag{2.7}$$

with the initial condition

$$x_0 = \phi, \tag{2.8}$$

and equation (2.3) can be expressed as

$$Dx_t = D\phi + \int_0^t f(\tau)d\tau, \quad t \geq 0. \tag{2.9}$$

Since $g(\cdot)$ belongs to L_1, it follows that $D : L_{2,g} \longrightarrow R$ is a bounded linear operator satisfying

$$|D\phi| \leq [\int_{-b}^{0} g(s)ds]^{\frac{1}{2}}[\int_{-b}^{0} |\phi(s)|^2 g(s)ds]^{\frac{1}{2}}.$$

2.3 Semigroup formulation

The well-posedness of (2.7) has been studied by several authors. In [7], it was shown that under rather general conditions on D and L, (2.7) is well-posed in the sense that the solution exists and depends continuously on the initial data $(\eta, \phi(\cdot))$ in $R \times L_p$ and the forcing function f in $L_p(0,T)$ for $T > 0$. Most of these studies have been based on an explicit representation of the solution to Abel's equation. Moreover, when these results were applied to certain aeroelastic control problems it was shown that (2.7) is well-posed in $R \times L_p$ only if $p > 2$ (see [6], [7], [12], [16] for details). We shall investigate the well-posedness of (2.7) in the weighted Hilbert space $L_{2,g}$.

In order to introduce the basic idea, we indicate how equation (2.1) can be (formally) written as an evolution equation on the weighted state space $L_{2,g}$. Given x: $[-b, \infty) \longrightarrow R$, we define

$$\xi : [0, \infty) \times [-b, 0) \longrightarrow R$$

by

$$\xi(t,\theta) = x(t+\theta).$$

If $x(\cdot)$ is locally absolutely continuous and the function $t \longrightarrow \int_{-b}^{0} g(s)x(t+s)ds$ is locally absolutely continuous, then equation (2.1) can be reformulated as an initial-boundary value problem for the first order hyperbolic equation

$$\frac{\partial}{\partial t}\xi(t,\theta) = \frac{\partial}{\partial \theta}\xi(t,\theta), \tag{2.10}$$

with the constraint

$$\int_{-b}^{0} g(\theta)\frac{\partial}{\partial \theta}\xi(t,\theta)d\theta = f(t). \tag{2.11}$$

The constraint (2.11) is obtained by interchanging the order of integration and the differentiation d/dt in equation (2.1) and then using equation (2.10). We use the weighted space $L_{2,g}$ as the function space for the history function ξ as in [8] and [15].

Let Z be the weighted space $L_{2,g}$ and $\langle \cdot, \cdot \rangle$ be the scalar product defined on Z:

$$\langle u, v \rangle = \int_{-b}^{0} u(\zeta)v(\zeta)g(\zeta)d\zeta, \quad u, v \in Z.$$

Then we can write (2.10)–(2.11) with $f \equiv 0$ as a Cauchy problem for $z(t) = \xi(t,\cdot) \in Z$:

$$\begin{aligned} \frac{d}{dt}z(t) &= Az(t), \\ z(0) &= \phi, \end{aligned} \tag{2.12}$$

where the linear operator A on Z is defined by

$$Au = u' = \frac{d}{d\theta}u,$$

with the domain, $Dom(A)$, dense in Z:

$$Dom(A) = \{u \in Z : u \in AC, \quad u' \in Z, \quad Du' = 0\}.$$

We shall use the following definition of solutions to (2.1)–(2.2). This definition is analogous to the definition of a generalized solution given in [7].

Definition 2.1 *A solution to the initial value problem (2.1)–(2.2) is a measurable function $x : [-b, \infty) \to R$ satisfying*

1. $x(s) = \phi(s)$ *on* $[-b, 0]$,

2. $x_t \in L_{2,g}$ *for* $t \geq 0$,

3. x *satisfies (2.1)–(2.2) almost everywhere on* $[0, \infty)$.

The well-posedness of (2.1)–(2.2) is established by the following result ([8]):

Theorem 2.1 *Let* $x(t)$, $t \geq -b$, *be the solution to (2.1)–(2.2) with* $f \equiv 0$ *and define the solution semigroup* $S(t)$, $t \geq 0$, *on* $L_{2,g}$ *by*

$$(S(t)\phi)(s) = x(t+s) \quad on \quad [-b, 0].$$

Then, $S(t)$, $t \geq 0$, *forms a* C_0 *semigroup on* $L_{2,g}$ *whose infinitesimal generator* A *is given by*

$$Dom(A) = \{u \in L_{2,g} : u \text{ is locally absolutely continuous with } u' \in L_{2,g} \text{ and } Du' = 0\}$$

and $Au = u'$ *for* $u \in Dom(A)$. *Moreover,* $\lambda \in \rho(A)$ *if and only if* $\Delta(\lambda) \neq 0$, *where the characteristic function* $\Delta(\lambda)$ *is given by*

$$\Delta(\lambda) = \lambda \int_{-b}^{0} g(s) e^{\lambda s} ds.$$

If $\lambda \in \rho(A) \cap R^+$ *and* $\psi_\lambda = \Delta(\lambda)^{-1} e^{\lambda s} \in L_{2,g}$, *then the solution* $x(t+\cdot) \in L_{2,g}$ *to (2.1)–(2.2) can be represented as*

$$x(t+\cdot) = S(t)\phi - (A - \lambda I) \int_0^t S(t-s) \psi_\lambda f(s) ds.$$

2.4 Summary

In this chapter, we transformed the finite delay version (2.1)–(2.2) into a neutral functional differential equation (2.7)–(2.8). Theorem 2.1 established the basic equivalence between the solutions to (2.1)–(2.2) and the semigroup generated by A. Moreover, the space $L_{2,g}$ can be used in the development of approximating systems. Numerical results based on this framework will be introduced in Chapter 4.

Chapter 3

Exact Solutions

3.1 Introduction

In this chapter, we discuss the exact solution to the singular equation

$$\int_{-b}^{0} |s|^{-p} x(t+s)ds = F(t), \quad t > 0, \tag{3.1}$$

with the initial condition

$$x(s) = \phi(s), \quad -b \leq s \leq 0, \tag{3.2}$$

where $F(0) = \int_{-b}^{0} |s|^{-p}\phi(s)ds$ and $F(t)$ satisfies certain properties. Without loss of generality, for the rest of this presentation, we assume that $b = 1$. In order to find the exact solution to (3.1)−(3.2), we integrate both sides of equation (2.1) from 0 to t with $g(\cdot) = |\cdot|^{-p}$ to obtain

$$\int_{-1}^{0} |s|^{-p} x(t+s)ds = \int_{-1}^{0} |s|^{-p}\phi(s)ds + \int_{0}^{t} f(\tau)d\tau, \tag{3.3}$$

provided that the function $t \longrightarrow \int_{-1}^{0} |s|^{-p} x(t+s)ds$ is absolutely continuous and $|\cdot|^{-p}\phi(\cdot)$ belongs to L_1.

Defining $x_t(s) = x(t+s)$ and

$$D\zeta = \int_{-1}^{0} |s|^{-p}\zeta(s)ds,$$

equation (3.3) can be written as

$$Dx_t = D\phi + \int_{0}^{t} f(\tau)d\tau \equiv F(t). \tag{3.4}$$

Our definition of an exact solution to (3.1)−(3.2) is the following:

Definition 3.1 *If $u(t)$ is an exact solution to (3.1)–(3.2) on an interval $J = \{t \in R : 0 \le t \le T\}$, $T > 0$, then $u(t)$ is a real valued, measurable function defined on J and satisfies (3.1)–(3.2) almost everywhere.*

In this chapter, we consider the case with $F(t) \in C^1(0,1)$ and solve for the exact solution $x(t)$ to (3.1)–(3.2), $t \in (0,1]$, then extend the solution to $(1, T_1)$, $T_1 > 1$ by the method of steps. Solutions for specific $F(t)$ are discussed. We also discuss the solutions for special case $p = 1/2$.

3.2 $F(t) \in C^1(0,1)$

We start with the problem in [6] (p.111):

$$\begin{aligned} \int_{-1}^{0} |s|^{-p} x(t+s)ds &= \eta, \quad t > 0, & (3.5)\\ x(s) &= \phi(s), \quad s \in [-1,0], \quad p \in (0,1), & (3.6) \end{aligned}$$

and using the definition of the operator D, we have

$$D\phi = \int_{-1}^{0} |s|^{-p}\phi(s)ds.$$

The exact solution to (3.5)–(3.6) for $\phi \in C$, $\eta \in R$ is stated below:

Theorem 3.1 *Let $\phi \in C$, $\eta \in R$. Then the unique integrable solution to (3.5)–(3.6) is given by*

$$\begin{aligned} x(t) &= \frac{\sin(p\pi)}{\pi}\left[\int_{-1}^{0} \frac{1}{t-s}\left|\frac{t}{s}\right|^{p} \phi(s)ds + \int_{0}^{t} \frac{(t-s)^{p-1}}{(t-s)+1}\phi(s-1)ds\right] & (3.7)\\ &+ \frac{\sin(p\pi)}{\pi}\left[(\eta - D\phi)t^{p-1}\right], \quad 0 < t \le 1. & (3.8) \end{aligned}$$

We introduce an useful result from [10] (p.41):

Lemma 3.1 *Let $h \in C(0,1)$ and $h(0) = 0$. Then $x(\cdot)$ satisfies*

$$\int_{0}^{t} |t-s|^{-p} x(s)ds = h(t) \quad a.e. \ on \ (0,1) \qquad (3.9)$$

if and only if

$$\int_{0}^{t} x(s)ds = \frac{\sin(p\pi)}{\pi}\int_{0}^{t} |t-s|^{p-1} h(s)ds \ on \ [0,1]. \qquad (3.10)$$

By applying Lemma 3.1, and following the proof in [6], we are able to find the solution to (3.1)–(3.2) with $F(t) \in C^1(0,1)$.

Lemma 3.2 *Let $F \in C^1(0,1)$. Then (3.1)–(3.2) has the unique integrable solution*

$$\begin{aligned} x(t) &= \frac{\sin(p\pi)}{\pi}\left[\int_{-1}^{0} \frac{1}{t-s}\left|\frac{t}{s}\right|^{p} \phi(s)ds + \int_{0}^{t} \frac{(t-s)^{p-1}}{(t-s)+1}\phi(s-1)ds\right] \\ &\quad + \frac{\sin(p\pi)}{\pi}\left[\int_{0}^{t} F'(s)(t-s)^{p-1}ds\right], \quad 0 < t \leq 1. \end{aligned}$$

Proof The proof is constructed in steps:

1. We transform equation (3.1) into the form of equation (3.9),
2. Identify $h(t)$ in equation (3.9),
3. Show that the right-hand side of equation (3.10) is differentiable where h is as defined in step 2.

First, by changing variables, equation (3.1) becomes

$$\begin{aligned} & \int_{-1}^{0} |s|^{-p} x(t+s)ds \\ =\ & \int_{t-1}^{t} |s-t|^{-p} x(s)ds \\ =\ & \int_{t-1}^{0} |s-t|^{-p} x(s)ds + \int_{0}^{t} |s-t|^{-p} x(s)ds \\ =\ & \int_{t-1}^{0} |s-t|^{-p} \phi(s)ds + \int_{0}^{t} |s-t|^{-p} x(s)ds \\ =\ & F(t). \end{aligned}$$

Therefore,

$$\int_{0}^{t} |s-t|^{-p} x(s)ds = F(t) - \int_{t-1}^{0} |s-t|^{-p}\phi(s)ds. \tag{3.11}$$

Comparing (3.11) with (3.9), it follows that

$$\begin{aligned} h(t) &= F(t) - \int_{t-1}^{0} |s-t|^{-p}\phi(s)ds \\ &= F(t) - \int_{-1}^{0} |s-t|^{-p}\phi(s)ds + \int_{-1}^{t-1} |s-t|^{-p}\phi(s)ds. \end{aligned}$$

Note that the assumption $F(0) = D\phi = \int_{-1}^{0} |s|^{-p}\phi(s)ds$ implies

$$\begin{aligned} h(0) &= F(0) - \int_{-1}^{0} |s|^{-p}\phi(s)ds \\ &= \int_{-1}^{0} |s|^{-p}\phi(s)ds - \int_{-1}^{0} |s|^{-p}\phi(s)ds \\ &= 0. \end{aligned}$$

By changing variables, $h(t)$ can be written as

$$\begin{aligned} h(t) &= F(t) - \int_{-1}^{0} |s-t|^{-p}\phi(s)ds + \int_{0}^{t} |u-1-t|^{-p}\phi(u-1)du \\ &= F(t) - \int_{-1}^{0} |t-s|^{-p}\phi(s)ds + \int_{0}^{t} (t+1-u)^{-p}\phi(u-1)du \\ &= \int_{-1}^{0} \left[|s|^{-p} - |t-s|^{-p}\right] \phi(s)ds + \int_{0}^{t} (t+1-s)^{-p}\phi(s-1)ds \\ &\quad + \left[F(t) - \int_{-1}^{0} |s|^{-p}\phi(s)ds\right] \\ &= \int_{-1}^{0} \left[|s|^{-p} - |t-s|^{-p}\right] \phi(s)ds + \int_{0}^{t} (t+1-s)^{-p}\phi(s-1)ds \\ &\quad + \left[F(t) - D\phi\right]. \end{aligned} \tag{3.12}$$

Next we show that $\int_0^t h(s)|t-s|^{p-1}ds$ is differentiable where h is as defined above. By substituting $h(t)$ in (3.12) into equation (3.10), we define

$$\begin{aligned} \frac{d}{dt}\int_0^t h(s)|t-s|^{p-1}ds &= \frac{d}{dt}\int_0^t \int_{-1}^{0} \left[|u|^{-p} - |s-u|^{-p}\right] \phi(u)|t-s|^{p-1}duds \\ &\quad + \frac{d}{dt}\int_0^t \left[\int_0^s (s+1-u)^{-p}\phi(u-1)du\right] |t-s|^{p-1}ds \\ &\quad + \frac{d}{dt}\int_0^t \left[F(s) - D\phi\right] |t-s|^{p-1}ds \\ &\equiv I_1 + I_2 + I_3. \end{aligned}$$

By changing the order of integration and making several changes of variables, one has

$$\begin{aligned} I_1 &= \frac{d}{dt}\int_{-1}^{0} \left[\int_0^t \left(|u|^{-p} - |s-u|^{-p}\right) |t-s|^{p-1}ds\right] \phi(u)du \\ &= \frac{d}{dt}\int_{-1}^{0} \left[\int_0^t \left(\frac{1}{|u|^p} - \frac{1}{|t-v-u|^p}\right) v^{p-1}dv\right] \phi(u)du \\ &= \int_{-1}^{0} \left[\int_0^t p(t-v-u)^{-p-1}v^{p-1}dv\right] \phi(u)du \\ &= \int_{-1}^{0} \frac{p}{t-u} \left[\int_0^{\frac{t}{t-u}} (1-s)^{-p-1}\, s^{p-1}ds\right] \phi(u)du \end{aligned}$$

$$
\begin{aligned}
&= \int_{-1}^{0} \frac{p}{t-u}\left[\int_{0}^{-\frac{t}{u}} v^{p-1}dv\right]\phi(u)du \\
&= \int_{-1}^{0} \frac{1}{(t-u)}\left|\frac{t}{u}\right|^{p}\phi(u)du
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= \frac{d}{dt}\int_0^t \left[\int_0^s (s+1-u)^{-p}\phi(u-1)du\right]|t-s|^{p-1}ds \\
&= \frac{d}{dt}\int_0^t \left[\int_u^t (s+1-u)^{-p}(t-s)^{p-1}ds\right]\phi(u-1)du \\
&= \frac{d}{dt}\int_0^t \left[\int_0^{t-u} (\tau+1)^{-p}(t-\tau-u)^{p-1}d\tau\right]\phi(u-1)du \\
&= \frac{d}{dt}\int_0^t \left[\int_0^{t-u} \frac{1}{(t-u-s)^{1-p}}\,\frac{1}{(s+1)^p}ds\right]\phi(u-1)du.
\end{aligned}
$$

By the property of convolution, we have

$$
\begin{aligned}
& \int_0^{t-u} \frac{1}{(t-u-s)^{1-p}}\,\frac{1}{(s+1)^p}ds \\
= & \int_0^{t-u} \frac{1}{s^{1-p}}\,\frac{1}{(t-u-s+1)^p}ds \\
= & \int_0^{\frac{t-u}{1+t-u}} \frac{1}{(1+t-u)^{1-p}w^{1-p}}\,\frac{1}{(t-u-(1+t-u)w+1)^p}\,(1+t-u)dw \\
= & \int_0^{\frac{t-u}{1+t-u}} \frac{1}{(1+t-u)^{1-p}w^{1-p}}\,\frac{1}{(1+t-u)^p(1-w)^p}\,(1+t-u)dw \\
= & \int_0^{\frac{t-u}{1+t-u}} \frac{1}{w^{1-p}}\,\frac{1}{(1-w)^p}dw.
\end{aligned}
$$

Therefore, we have the identity

$$
\begin{aligned}
I_2 &= \frac{d}{dt}\int_0^t \left[\int_0^{\frac{t-u}{1+t-u}} \frac{1}{w^{1-p}}\,\frac{1}{(1-w)^p}dw\right]\phi(u-1)du \\
&= \int_0^t \left(\frac{t-u}{1+t-u}\right)^{p-1}\left(\frac{1}{1+t-u}\right)^{-p}\frac{1}{(1+t-u)^2}\,\phi(u-1)du \\
&= \int_0^t \frac{(t-u)^{p-1}}{(1+t-u)}\;\phi(u-1)du.
\end{aligned}
$$

Since $F \in C^1(0,1)$, F' exists and $F(0) = D\phi$, we use integration by parts to obtain

$$
\begin{aligned}
I_3 &= \frac{d}{dt}\int_0^t [F(s)-D\phi]\,(t-s)^{p-1}ds \\
&= \frac{d}{dt}\left[\int_0^t F(s)(t-s)^{p-1}ds - \int_0^t (D\phi)(t-s)^{p-1}ds\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{d}{dt}\left[F(s)\ \frac{-1}{p}\ (t-s)^p\ |_0^t + \int_0^t F'(s)\ \frac{1}{p}\ (t-s)^p ds + (D\phi)\ \frac{1}{p}(t-s)^p\ |_0^t\right] \\
&= \frac{d}{dt}\left[F(0)\ \frac{1}{p}\ t^p + \int_0^t F'(s)\ \frac{1}{p}\ (t-s)^p ds - (D\phi)\ \frac{1}{p}\ t^p\right] \\
&= F(0)t^{p-1} + F'(t)\ \frac{1}{p}\ (t-t)^p + \int_0^t F'(s)(t-s)^{p-1} ds - (D\phi)\ t^{p-1} \\
&= \int_0^t F'(s)(t-s)^{p-1} ds.
\end{aligned}
$$

□

Lemma 3.2 shows the solution $x(t)$ to (3.1)–(3.2) for $t \in (0,1]$. In order to find the solution for $t \in (1, T_1)$, $T_1 > 1$, we shall use the method of steps.

3.3 Method of steps

The idea behind the method of steps is that once the exact solution to (3.1)–(3.2) is known for $t \in (0,1]$, we assume this part of solution as a new initial condition, then we apply Lemma 3.1 to obtain the solution for $t \in (1,2]$. We repeat the same procedure to find the solution on $(2,3]$, $(3,4]$, and so forth.

The following is an example illustrates how the method of steps can be used to find the solution on $(1,2]$.

Example 3.1

Consider the problem

$$
\begin{aligned}
\int_{-1}^{0} |s|^{-0.5} x(t+s) ds &= 1, && (3.13) \\
\phi(s) &= 0, \quad s \in [-1,0]. && (3.14)
\end{aligned}
$$

By [16], the exact solution is

$$
x(t) = \begin{cases} \frac{1}{\pi} t^{-0.5} & \textit{for} \quad 0 < t \le 1, \\ \frac{2}{\pi} t^{-0.5} & \textit{for} \quad 1 < t \le 2. \end{cases}
$$

Proof The method of steps applied to this example proceeds as follows:

1. We assume $x(t) = t^{-0.5}/\pi$ for $0 < t \le 1$ as an initial condition,

2. Identify the function $h(t)$ in Lemma 3.1,

3. Apply Lemma 3.1.

Suppose $1 < t \leq 2$, equation (3.13) can be written as

$$\begin{aligned} 1 &= \int_{-1}^{0} x(t+s)|s|^{-0.5} ds \\ &= \int_{-1}^{0} x(t+s)(-s)^{-0.5} ds \\ &= \int_{t-1}^{t} x(u)(t-u)^{-0.5} du \\ &= \int_{t-1}^{1} x(u)(t-u)^{-0.5} du + \int_{1}^{t} x(u)(t-u)^{-0.5} du. \end{aligned} \tag{3.15}$$

Since $0 < t-1 \leq 1$, by applying the "new" initial condition to (3.15), we obtain

$$\begin{aligned} \int_{1}^{t} x(u)(t-u)^{-0.5} du &= 1 - \int_{t-1}^{1} x(u)(t-u)^{-0.5} du \\ &= 1 - \int_{t-1}^{1} \frac{1}{\pi} u^{-0.5}(t-u)^{-0.5} du \\ &= 1 - \int_{1-\frac{1}{t}}^{\frac{1}{t}} \frac{1}{\pi} t^{-0.5}\tau^{-0.5} t^{-0.5}(1-\tau)^{-0.5} t\, d\tau \\ &= 1 - \int_{1-\frac{1}{t}}^{\frac{1}{t}} \frac{1}{\pi} \tau^{-0.5}(1-\tau)^{-0.5} d\tau. \end{aligned} \tag{3.16}$$

Using a change of variables, the left-hand side of (3.16) becomes

$$\int_{1}^{t} x(u)(t-u)^{-0.5} du = \int_{0}^{t-1} x(\tau+1)(t-\tau-1)^{-0.5} d\tau. \tag{3.17}$$

Defining the right-hand side of (3.16) to be

$$\nu(t) = 1 - \int_{1-\frac{1}{t}}^{\frac{1}{t}} \frac{1}{\pi} \tau^{-0.5}(1-\tau)^{-0.5} d\tau, \tag{3.18}$$

and assuming $s = t-1$ (then $s \in (0, 1]$) and $y(\tau) = x(\tau+1)$, (3.16), (3.17) and (3.18) imply

$$\begin{aligned} \int_{0}^{s} y(\tau)(s-\tau)^{-0.5} d\tau &= \nu(s+1) \\ &= 1 - \int_{1-\frac{1}{s+1}}^{\frac{1}{s+1}} \frac{1}{\pi} \tau^{-0.5}(1-\tau)^{-0.5} d\tau. \end{aligned} \tag{3.19}$$

Comparing (3.19) with (3.9), we obtain

$$h(s) = 1 - \int_{1-\frac{1}{s+1}}^{\frac{1}{s+1}} \frac{1}{\pi} \tau^{-0.5}(1-\tau)^{-0.5} d\tau.$$

Note that

$$h(0) = 1 - \int_{0}^{1} \frac{1}{\pi} \tau^{-0.5}(1-\tau)^{-0.5} d\tau$$

$$
\begin{aligned}
&= 1-\frac{1}{\pi}\int_0^1 \tau^{-0.5}(1-\tau)^{-0.5}d\tau \\
&= 1-\frac{1}{\pi}\frac{\Gamma(0.5)\Gamma(0.5)}{\Gamma(1)} \\
&= 1-1 \\
&= 0.
\end{aligned}
$$

After identifying $h(s)$, we show that $\int_0^s |s-\tau|^{-0.5}h(\tau)d\tau$ is differentiable. By equation (3.10), we have

$$
\begin{aligned}
y(s) &= \frac{\sin(0.5\pi)}{\pi}\frac{d}{ds}\int_0^s h(\tau)(s-\tau)^{-0.5}d\tau \\
&= \frac{1}{\pi}\frac{d}{ds}\left[\int_0^s\left[1-\int_{\frac{\tau}{\tau+1}}^{\frac{1}{\tau+1}}\frac{1}{\pi}u^{-0.5}(1-u)^{-0.5}du\right](s-\tau)^{-0.5}d\tau\right] \\
&= \frac{1}{\pi}\frac{d}{ds}\left[\int_0^s(s-\tau)^{-0.5}d\tau-\int_0^s\int_{\frac{\tau}{\tau+1}}^{\frac{1}{\tau+1}}\frac{1}{\pi}u^{-0.5}(1-u)^{-0.5}(s-\tau)^{-0.5}dud\tau\right] \\
&= \frac{1}{\pi}\frac{d}{ds}\left[\int_0^s(s-\tau)^{-0.5}d\tau\right]-\frac{1}{\pi}\frac{d}{ds}\left[\int_0^s\int_{\frac{\tau}{\tau+1}}^{\frac{1}{\tau+1}}\frac{1}{\pi}u^{-0.5}(1-u)^{-0.5}(s-\tau)^{-0.5}dud\tau\right] \\
&= I_4+I_5.
\end{aligned}
$$

The terms, I_4 and I_5, can be simplified to

$$
\begin{aligned}
I_4 &= \frac{1}{\pi}\frac{d}{ds}\left[-2(s-\tau)^{0.5}\,|_0^s\right] \\
&= \frac{2}{\pi}\frac{d}{ds}s^{0.5} \\
&= \frac{1}{\pi}s^{-0.5},
\end{aligned}
$$

and

$$
\begin{aligned}
I_5 &= -\frac{1}{\pi}\frac{d}{ds}\left[\int_0^s\int_{\frac{\tau}{\tau+1}}^{\frac{1}{\tau+1}}\frac{1}{\pi}u^{-0.5}(1-u)^{-0.5}(s-\tau)^{-0.5}dud\tau\right] \\
&= -\frac{1}{\pi^2}\frac{d}{ds}\left[\int_0^s\left[\int_{\frac{\tau}{\tau+1}}^{\frac{1}{\tau+1}}u^{-0.5}(1-u)^{-0.5}du\right](s-\tau)^{-0.5}d\tau\right],
\end{aligned}
$$

respectively. By [3] (p.229), we obtain

$$
\begin{aligned}
I_5 &= \frac{1}{\pi^2}\frac{d}{ds}\left[\int_0^s\left[\sin^{-1}(-2u+1)\left|_{\frac{\tau}{\tau+1}}^{\frac{1}{\tau+1}}\right.\right](s-\tau)^{-0.5}d\tau\right] \\
&= \frac{1}{\pi^2}\frac{d}{ds}\left[\int_0^s\left[\sin^{-1}\left(\frac{-2}{\tau+1}+1\right)-\sin^{-1}\left(\frac{-2\tau}{\tau+1}+1\right)\right](s-\tau)^{-0.5}d\tau\right] \\
&= \frac{1}{\pi^2}\frac{d}{ds}\left[\int_0^s\left[\sin^{-1}\left(\frac{\tau-1}{\tau+1}\right)-\sin^{-1}\left(\frac{1-\tau}{\tau+1}\right)\right](s-\tau)^{-0.5}d\tau\right].
\end{aligned}
$$

Since $\sin^{-1}(\cdot)$ is an odd function, it follows that

$$
\begin{aligned}
I_5 &= \frac{1}{\pi^2}\frac{d}{ds}\left[\int_0^s 2\sin^{-1}\left(\frac{\tau-1}{\tau+1}\right)(s-\tau)^{-0.5}d\tau\right] \\
&= \frac{2}{\pi^2}\frac{d}{ds}\left[\sin^{-1}\left(\frac{\tau-1}{\tau+1}\right)(-2)(s-\tau)^{0.5}\,|_0^s + \int_0^s 2(s-\tau)^{0.5}\frac{1}{\sqrt{\tau}(\tau+1)}d\tau\right] \\
&= \frac{2}{\pi^2}\frac{d}{ds}\left[2\sin^{-1}(-1)s^{0.5} + \int_0^s 2(s-\tau)^{0.5}\frac{1}{\sqrt{\tau}(\tau+1)}d\tau\right] \\
&= \frac{2}{\pi^2}\frac{d}{ds}\left[2\,\frac{-\pi}{2}s^{0.5} + \int_0^s 2(s-\tau)^{0.5}\frac{1}{\sqrt{\tau}(\tau+1)}d\tau\right] \\
&= \frac{2}{\pi^2}\frac{d}{ds}\left[-\pi s^{0.5} + \int_0^s 2(s-\tau)^{0.5}\frac{1}{\sqrt{\tau}(\tau+1)}d\tau\right] \\
&= \frac{2}{\pi^2}\left[-\frac{\pi}{2}s^{-0.5} + \int_0^s (s-\tau)^{-0.5}\frac{1}{\sqrt{\tau}(\tau+1)}d\tau\right] \\
&= -\frac{s^{-0.5}}{\pi} + \frac{2}{\pi^2}\int_0^s (s-\tau)^{-0.5}\frac{1}{\sqrt{\tau}(\tau+1)}d\tau.
\end{aligned}
$$

Our argument is complete if

$$y(s) = \frac{2}{\pi}(s+1)^{-0.5}.$$

Namely, compared to the exact solution $x(t)$ for $t \in (1,2]$, we need to show

$$\frac{2}{\pi^2}\int_0^s (s-\tau)^{-0.5}\frac{1}{\sqrt{\tau}(\tau+1)}d\tau = \frac{2}{\pi}(s+1)^{-0.5}. \tag{3.20}$$

By a change of variables and following [3] (p.236), we have

$$
\begin{aligned}
&\frac{2}{\pi^2}\int_0^s (s-\tau)^{-0.5}\frac{1}{\sqrt{\tau}(\tau+1)}d\tau \\
=\ &\frac{2}{\pi^2}\int_0^{\sqrt{s}} (s-u^2)^{-0.5}\frac{2u}{u(u^2+1)}du \\
=\ &\frac{4}{\pi^2}\int_0^{\sqrt{s}} (s-u^2)^{-0.5}\frac{1}{u^2+1}du \\
=\ &\frac{4}{\pi^2}\;\frac{1}{\sqrt{s+1}}\;\tan^{-1}\frac{u\sqrt{s+1}}{\sqrt{s-u^2}}\,|_0^{\sqrt{s}} \\
=\ &\frac{4}{\pi^2}\;\frac{1}{\sqrt{s+1}}\;(\tan^{-1}\frac{\sqrt{s}\sqrt{s+1}}{\sqrt{s-s}} - \tan^{-1}0) \\
=\ &\frac{4}{\pi^2}\;\frac{1}{\sqrt{s+1}}\;\frac{\pi}{2} \\
=\ &\frac{2}{\pi}(s+1)^{-0.5}.
\end{aligned}
$$

$\square$

We therefore claim that the method of steps is a feasible approach to obtain the solution to (3.1)–(3.2) for $t \in (0, \infty)$.

3.4 Special cases

We find solutions to two special cases of (3.1)–(3.2) which are of interest to this study.

3.4.1 Initial condition $\phi = 0$ and $F(t) = t^a$

In this section, we compute the solution to (3.1)–(3.2) with

$$\begin{aligned} F(t) &= t^a, \quad a > 0, \\ \phi(s) &= 0, \quad -1 \leq s \leq 0. \end{aligned}$$

We have the following result:

Lemma 3.3 *Let $F(t) = t^a$, $a > 0$, $\phi(s) = 0$, $-1 \leq s \leq 0$. Then the exact solution to (3.1)–(3.2) is*

$$x(t) = \frac{\Gamma(a+1)}{\Gamma(1-p)\Gamma(p+a)} \; t^{p+a-1}, \quad 0 < t \leq 1. \tag{3.21}$$

Proof It is simply an application of Lemma 3.2. Since $F(t) = t^a$, we have

$$F'(t) = at^{a-1}.$$

Therefore, using the initial condition, applying a change of variables and using the definition of gamma functions, the solution is given by

$$\begin{aligned} x(t) &= \frac{\sin(p\pi)}{\pi} \cdot \int_0^t a \cdot s^{a-1}(t-s)^{p-1} ds \\ &= \frac{\sin(p\pi)}{\pi} \cdot a \int_0^1 t^{a-1} u^{a-1} t^{p-1} (1-u)^{p-1} t du \\ &= \frac{\sin(p\pi)}{\pi} \cdot a \cdot t^{p+a-1} \cdot \int_0^1 u^{a-1}(1-u)^{p-1} t du \\ &= \frac{\sin(p\pi)}{\pi} \cdot a \cdot t^{p+a-1} \cdot \frac{\Gamma(a)\Gamma(p)}{\Gamma(p+a)} \\ &= \frac{1}{\Gamma(1-p)\Gamma(p)} \cdot t^{p+a-1} \cdot \frac{\Gamma(a+1)\Gamma(p)}{\Gamma(p+a)} \end{aligned}$$

$$= \frac{\Gamma(a+1)}{\Gamma(1-p)\Gamma(p+a)} t^{p+a-1}.$$

□

By Lemma 3.3 and the linearity of the equation (3.1), we extend the exact solution for the case with $F(t) = \sum_{i=1}^{l} c_i t^{a_i}$, $a_i > 0$. Namely, we are able to find the solution for $t \in (0, 1]$:

$$\int_{-1}^{0} |s|^{-p} x(t+s) ds = \text{ a linear combination of terms like } t^a, \quad a > 0.$$

with initial function

$$\phi(s) = 0, \quad s \in [-1, 0].$$

Four such examples together with exact solutions are:

1. $F(t) = t + t^2/2!$, the solution is

$$x(t) = \sum_{i=1}^{2} \frac{\Gamma(i+1)}{i\Gamma(1-p)\Gamma(p+i)} t^{p+i-1},$$

2. For $F(t) = \sin(t) = t - t^3/3! + t^5/5! - \cdots$, the solution $x(t)$ is

$$x(t) = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{\Gamma(2i)}{(2i-1)!\Gamma(1-p)\Gamma(p+2i-1)} t^{p+2i-2},$$

3. For $F(t) = \cos(t) - 1 = -t^2/2! + t^4/4! - \cdots$, $x(t)$ can be written as

$$x(t) = \sum_{i=1}^{\infty} (-1)^{i} \frac{\Gamma(2i+1)}{(2i)!\Gamma(1-p)\Gamma(p+2i)} t^{p+2i-1},$$

   and

4. For $F(t) = e^t - 1 = t + t^2/2! + t^3/3! + \cdots$, the solution is

$$x(t) = \sum_{i=1}^{\infty} \frac{\Gamma(i+1)}{(i)!\Gamma(1-p)\Gamma(p+i)} t^{p+i-1}.$$

3.4.2 Initial condition $\phi = 0$ and $F(t) \in NBV$

Another special case of (3.1)–(3.2) is $\phi(s) = 0$, $-1 \le s \le 0$, and $F(t) \in NBV$, $t > 0$. We recall the following useful result from [16] (p.342):

Theorem 3.2 *Assume $\phi = 0$ and let $F(t) : [0,\infty) \to R$ be a normalized function of bounded variation on $[0,T]$ for any $T > 0$ (here normalized means $F(0) = 0$, i.e., $F \in NBV$). Then (3.1)–(3.2) has an unique locally integrable solution given by*

$$x(t) = \int_0^t Y(t-u)dF(u), \quad t \geq 0,$$

where $Y(t)$ is the solution of

$$\begin{aligned} \int_{-1}^{0} |s|^{-p} Y(t+s)ds &= 1, \quad t > 0, && (3.22) \\ Y(t) &= 0, \quad t < 0. && (3.23) \end{aligned}$$

i.e., $Y(t)$ is the fundamental solution.

Theorem 3.2 implies that once the fundamental solution is known for $t \in (0,\infty)$, we can solve for $x(t)$, $t \in (0,\infty)$, provided $F(t)$ is a normalized function of bounded variation. However, by complicated calculations, the fundamental solution is solved explicitly on the interval $(-1,3]$. In [15], we have the following lemma:

Lemma 3.4 *For $p = 0.5$, the solution to (3.22)–(3.23) for $t \in (0,3]$ is*

$$x(t) = \begin{cases} \frac{1}{\pi} t^{-0.5} & \text{for} \quad 0 < t \leq 1, \\ \frac{2}{\pi} t^{-0.5} & \text{for} \quad 1 < t \leq 2, \\ \frac{2}{\pi} t^{-0.5} \left[1 + \frac{2}{\pi} \arctan \left(\frac{t-2}{t} \right)^{0.5} \right] & \text{for} \quad 2 < t \leq 3. \end{cases}$$

Application of Theorem 3.2 to the four examples in Section 3.4.1, solutions for $t \in (0,3]$ are:

1. For $F(t) = t + t^2/2!$, the solution is

$$x(t) = \begin{cases} \frac{2}{\pi} t^{0.5} (1 + \frac{2t}{3}) & \text{for} \quad 0 < t \leq 1, \\ -\frac{4}{3\pi} + \frac{4t^{0.5}}{\pi} - \frac{2t}{\pi} + \frac{8t^{1.5}}{3\pi} & \text{for} \quad 1 < t \leq 2, \\ -\frac{4}{3\pi} + \frac{4t^{0.5}}{\pi} - \frac{2t}{\pi} + \frac{8t^{1.5}}{3\pi} & \\ + \frac{4}{\pi^2} [(1+t) \int_2^t s^{-0.5} \arctan \left(\frac{s-2}{s} \right)^{0.5} ds & \\ - \int_2^t s^{0.5} \arctan \left(\frac{s-2}{s} \right)^{0.5} ds] & \text{for} \quad 2 < t \leq 3. \end{cases}$$

2. For $F(t) = \sin(t)$,

$$x(t) = \begin{cases} \int_0^t \frac{1}{\pi} s^{-0.5} \cos(t-s)ds & \text{for} \quad 0 < t \leq 1, \\ \\ \int_0^1 \frac{1}{\pi} s^{-0.5} \cos(t-s)ds + \int_1^t \frac{2}{\pi} s^{-0.5} \cos(t-s)ds & \text{for} \quad 1 < t \leq 2, \\ \\ \int_0^1 \frac{1}{\pi} s^{-0.5} \cos(t-s)ds + \int_1^2 \frac{2}{\pi} s^{-0.5} \cos(t-s)ds & \\ + \int_2^t \frac{2}{\pi} s^{-0.5} \cos(t-s)[1 + \frac{2}{\pi} \arctan\left(\frac{s-2}{s}\right)^{0.5}]ds & \text{for} \quad 2 < t \leq 3. \end{cases}$$

3. For $F(t) = \cos(t) - 1$,

$$x(t) = \begin{cases} - \int_0^t \frac{1}{\pi} s^{-0.5} \sin(t-s)ds & \text{for} \quad 0 < t \leq 1, \\ \\ - \int_0^1 \frac{1}{\pi} s^{-0.5} \sin(t-s)ds - \int_1^t \frac{2}{\pi} s^{-0.5} \sin(t-s)ds & \text{for} \quad 1 < t \leq 2, \\ \\ - \int_0^1 \frac{1}{\pi} s^{-0.5} \sin(t-s)ds - \int_1^2 \frac{2}{\pi} s^{-0.5} \sin(t-s)ds & \\ - \int_2^t \frac{2}{\pi} s^{-0.5} \sin(t-s)[1 + \frac{2}{\pi} \arctan\left(\frac{s-2}{s}\right)^{0.5}]ds & \text{for} \quad 2 < t \leq 3. \end{cases}$$

4. For $F(t) = e^t - 1$,

$$x(t) = \begin{cases} \int_0^t \frac{1}{\pi} s^{-0.5} e^{t-s} ds & \text{for} \quad 0 < t \leq 1, \\ \\ \int_0^1 \frac{1}{\pi} s^{-0.5} e^{t-s} ds + \int_1^t \frac{2}{\pi} s^{-0.5} e^{t-s} ds & \text{for} \quad 1 < t \leq 2, \\ \\ \int_0^1 \frac{1}{\pi} s^{-0.5} e^{t-s} ds + \int_1^2 \frac{2}{\pi} s^{-0.5} e^{t-s} ds & \\ + \int_2^t \frac{2}{\pi} s^{-0.5} e^{t-s}[1 + \frac{2}{\pi} \arctan\left(\frac{s-2}{s}\right)^{0.5}]ds & \text{for} \quad 2 < t \leq 3. \end{cases}$$

3.5 Summary

In this chapter, we considered the exact solution for

$$\begin{aligned} Dx_t &= F(t), \quad t > 0, \\ x_0 &= \phi. \end{aligned}$$

We showed the exact solution on $(0, 1]$ for $F(t) \in C^1(0, 1)$, $\phi \in C$ and discussed the special cases such as $\phi = 0$, $F(t)$ is a linear combination of t^a and the singularity $p = 0.5$. However, the exact solutions are given only on either $(0, 1]$ or $(0, 3]$. For any $T > 0$, the method of steps is an ideal way to find the solution on $(0, T]$.

Chapter 4

Numerical Approximation

4.1 Introduction

In Chapter 2, we discussed the semigroup formulation for the integrodifferential equation (2.1) with initial data (2.2) and the well-posedness of the associated Cauchy problem. In Chapter 3, we solved for the exact solution for different initial conditions and some special $F(t)$. We claimed that by obtaining a solution for $t \in (0,1]$, we can use the method of steps to extend the solution to $t \in (1,\infty)$.

In this chapter, we develop numerical methods to approximate the solution of the equation (i.e., (2.1)–(2.2))

$$\frac{d}{dt}\int_{-1}^{0} |s|^{-p} x(t+s)ds = f(t), \quad t>0, \quad p\in(0,1), \tag{4.1}$$

with initial data

$$x(s) = \phi(s), \qquad -1 \le s \le 0, \tag{4.2}$$

where $f(t)$ is a locally integrable function for $t > 0$.

By the definitions of a solution to (2.1)–(2.2) in Chapter 2 and exact solution to (3.1)–(3.2) in Chapter 3, we argue that a solution of (4.1)–(4.2) (or (2.1)–(2.2)) is also a solution to (3.1)–(3.2), i.e.,

$$\int_{-1}^{0} |s|^{-p} x(t+s)ds = F(t), \quad t>0, \quad p\in(0,1), \tag{4.3}$$

$$x(s) = \phi(s), \qquad -1 \le s \le 0, \tag{4.4}$$

provided that

$$F(t) = D\phi + \int_0^t f(\tau)d\tau, \tag{4.5}$$

where

$$D\phi = \int_{-1}^{0} |s|^{-p}\phi(s)ds.$$

Since $f(t)$ is locally integrable, $d/dt(F(t))$ exists. Therefore, equation (4.1) can be written as

$$\frac{d}{dt}\int_{-1}^{0} |s|^{-p}x(t+s)ds = \frac{d}{dt}F(t).$$

In this chapter, we assume

$$f(t) = \frac{d}{dt}F(t).$$

4.2 The numerical approach

We begin with defining a new variable ξ:

$$\xi(t,s) = x(t+s), \qquad -1 \le s \le 0, \qquad t > 0. \tag{4.6}$$

Equation (4.1) can be reformulated as a first order hyperbolic equation

$$\frac{\partial}{\partial t}\xi(t,s) = \frac{\partial}{\partial s}\xi(t,s), \qquad -1 \le s \le 0, \tag{4.7}$$

with the condition

$$\int_{-1}^{0} |s|^{-p}\frac{\partial}{\partial s}\xi(t,s)ds = f(t). \tag{4.8}$$

Next, we assume that the solution to (4.7)–(4.8) has the form

$$\xi(t,s) = \sum_{i=0}^{n} \alpha_i(t)B_i(s), \tag{4.9}$$

where the basis, $B_i(s)$, $i = 0, \ldots, n$, are given by

$$B_i(s) = \begin{cases} \frac{1}{\delta_{i+1}}(s - \tau_{i+1}) & \text{on } [\tau_{i+1}, \tau_i], \\ \frac{1}{\delta_i}(\tau_{i-1} - s) & \text{on } [\tau_i, \tau_{i-1}], \\ 0 & \text{otherwise.} \end{cases}$$

Namely, $B_i(s)$, $i = 0, \cdots, n$, are piecewise continuous linear functions or hat functions or linear splines. The mesh points, $\tau_0, \tau_1, \ldots, \tau_n$, are defined by $-1 = \tau_n < \tau_{n-1} < \cdots < \tau_1 < \tau_0 = 0$ and $\delta_i = \tau_{i-1} - \tau_i > 0$, for $i = 1, \ldots, n$. Here we choose two distributions of the mesh points, $\tau_0, \ldots, \tau_n$:

1. Uniform mesh points (i.e., δ_i = constant)

$$\tau_i = -\frac{i}{n}, \quad i = 0, \quad \ldots, \quad n$$

2. Nonuniform mesh points according to the equal area, i.e., τ_i is chosen so that

$$\int_{\tau_i}^{\tau_{i-1}} |s|^{-p} ds = \frac{1}{n} \int_{-1}^{0} |s|^{-p} ds, \quad i = 0, \ \ldots, \ n.$$

The procedure to find nonuniform mesh points, $\tau_0, \ldots, \tau_n$, is (note that $\tau_0 = 0$, $\tau_n = -1$): solve τ_1 first, then τ_2, then $\tau_3, \ldots$, then τ_{n-1}. As an example, we show how to obtain τ_1:

By the way we define $\tau_1, \ldots, \tau_n$, we have

$$\begin{aligned}
& \int_{\tau_1}^{\tau_0} |s|^{-p} ds = \frac{1}{n} \int_{-1}^{0} |s|^{-p} ds \\
\Rightarrow \quad & \int_{\tau_1}^{0} |s|^{-p} ds = \frac{1}{n} \int_{-1}^{0} |s|^{-p} ds \\
\Rightarrow \quad & \int_{\tau_1}^{0} (-s)^{-p} ds = \frac{1}{n} \int_{-1}^{0} (-s)^{-p} ds \\
\Rightarrow \quad & \frac{-1}{1-p} (-s)^{1-p} |_{\tau_1}^{0} = \frac{1}{n} \frac{-1}{1-p} (-s)^{1-p} |_{-1}^{0} \\
\Rightarrow \quad & \frac{1}{1-p} (-\tau_1)^{1-p} = \frac{1}{n} \frac{1}{1-p} \\
\Rightarrow \quad & (-\tau_1)^{1-p} = \frac{1}{n} \\
\Rightarrow \quad & -\tau_1 = \frac{1}{n^{\frac{1}{1-p}}} \\
\Rightarrow \quad & \tau_1 = \frac{-1}{n^{\frac{1}{1-p}}},
\end{aligned}$$

then for τ_2: we have

$$\int_{\tau_2}^{\tau_1} |s|^{-p} ds = \frac{1}{n} \int_{-1}^{0} |s|^{-p} ds.$$

From the previous step, we have τ_1, so we can find τ_2. The same routine can find τ_3, $\tau_4, \ldots, \tau_{n-1}$. A general form for nonuniform mesh points τ_i, $i = 0, \ldots, n$, is

$$\tau_i = -\left(\frac{i}{n}\right)^{\frac{1}{1-p}}.$$

We approach the numerical solution of equation (4.8) in two ways: semi-discretization and full discretization.

4.3 Semi-discretization

In this section, we substitute the special form of ξ in equation (4.9) into (4.7)–(4.8), then the governing equations for $\alpha_i(t)$, $i = 0, \ldots, n$, are as follows ([15]):

$$\begin{aligned} \frac{d}{dt}\alpha_i(t) &= \frac{1}{\delta_i}(\alpha_{i-1}(t) - \alpha_i(t)), \quad i = 1, \ \ldots, \ n, & (4.10) \\ \int_{-1}^{0} |s|^{-p} \sum_{i=0}^{n} \alpha_i(t) \frac{d}{ds} B_i(s) ds &= f(t). & (4.11) \end{aligned}$$

By defining

$$g_i = \int_{\tau_i}^{\tau_{i-1}} |s|^{-p} ds, \quad i = 1, \ \ldots, \ n,$$

equations (4.10)–(4.11) can be written as

$$\begin{aligned} \frac{d}{dt}\alpha_i(t) &= \frac{1}{\delta_i}(\alpha_{i-1}(t) - \alpha_i(t)), \quad i = 1, \ \ldots, \ n, & (4.12) \\ \sum_{i=1}^{n} \frac{g_i}{\delta_i}(\alpha_{i-1}(t) - \alpha_i(t)) &= f(t). & (4.13) \end{aligned}$$

From (4.13), $\alpha_0(t)$ can be written in terms of $\alpha_1(t), \ldots, \alpha_n(t)$:

$$\alpha_0(t) = \sum_{i=1}^{n-1} \frac{\delta_1}{g_1}[\frac{g_i}{\delta_i} - \frac{g_{i+1}}{\delta_{i+1}}]\alpha_i(t) + \frac{\delta_1 g_n}{g_1 \delta_n}\alpha_n(t) + \frac{\delta_1}{g_1} f(t). \tag{4.14}$$

Combining (4.12) and (4.14), we construct a first order ordinary differential equation:

$$\frac{d}{dt}X(t) = AX(t) + G(t), \tag{4.15}$$

where

$$X(t) = [\alpha_1(t) \ \ \alpha_2(t) \ \ \cdots \ \ \alpha_n(t)]^T,$$

$$A = \begin{bmatrix} -\frac{g_2}{g_1\delta_2} & \frac{1}{g_1}(\frac{g_2}{\delta_2} - \frac{g_3}{\delta_3}) & \cdots & \cdots & \frac{1}{g_1}(\frac{g_{n-1}}{\delta_{n-1}} - \frac{g_n}{\delta_n}) & \frac{g_n}{g_1\delta_n} \\ \frac{1}{\delta_2} & -\frac{1}{\delta_2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{\delta_3} & -\frac{1}{\delta_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\delta_n} & -\frac{1}{\delta_n} \end{bmatrix},$$

and

$$G(t) = \left[\frac{f(t)}{g_1} \ \ 0 \ \ \cdots \ \ 0\right]^T.$$

The initial condition for the system (4.15) comes from the initial condition (4.2). By combining (4.2), (4.6), (4.9) and fixing $t = 0$,

$$\begin{aligned} x(s) &= \sum_{i=0}^{n} \alpha_i(0) B_i(s) && (4.16) \\ &= \phi(s). && (4.17) \end{aligned}$$

From (4.16), (4.17) and the structure of $B_0(s)$, $B_1(s)$, ..., $B_n(s)$, we can find $\alpha_i(0)$, $i = 1, \ldots, n$. Therefore we use an ordinary differential equation (ode) solver to find $\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t)$.

In order to solve $x(\tilde{t})$, $0 < \tilde{t} \leq 1$, we fix $t = 1$ or $s = 0$ in (4.9). Since $B_j(s) = 1$ when $s = \tau_j$, we have either
Case 1: $t = 1$

$$\begin{aligned} x(1+\tau_i) &= \sum_{j=0}^{n} \alpha_j(1) B_j(\tau_i) \\ &= \alpha_i(1), && (4.18) \end{aligned}$$

or Case 2: $s = 0$

$$\begin{aligned} x(1+\tau_i) &= \sum_{j=0}^{n} \alpha_j(1+\tau_i) B_j(0) \\ &= \alpha_0(1+\tau_i). && (4.19) \end{aligned}$$

A similar idea can be applied to solve $x(\tilde{t})$, $1 < \tilde{t} < \infty$.

In Case 1, we use an ode solver to find $\alpha_i(1)$, $i = 1, \cdots, n$, which will be the corresponding solution $x(1 + \tau_i)$.

In Case 2, we solve for $\alpha_j(1 + \tau_i)$, $j = 1, \ldots, n$, then substitute $\alpha_j(1 + \tau_i)$, $j = 1, \ldots, n$, into equation (4.14) to obtain $\alpha_0(1 + \tau_i)$, which is the solution $x(1 + \tau_i)$.

To demonstrate the idea above, we solve Example 4.1 using the Matlab ode23 solver:

Example 4.1
Consider the equation

$$\frac{d}{dt} \int_{-1}^{0} |s|^{-0.5} x(t+s) ds = 1, \quad t > 0, \tag{4.20}$$

with

$$x(s) = 0, \quad -1 \leq s \leq 0.$$

By Lemma 3.3, $F(t) = t$, the solution is

$$x(t) = \frac{\Gamma(2)}{\Gamma(0.5)\Gamma(1.5)} t^{0.5} = \frac{2}{\pi} t^{0.5}, \quad 0 < t \leq 1.$$

The approximate solution by using uniform mesh points for $t = 1$ is given in Tables 4.1 and 4.2. For the errors, we compute the absolute value of the difference between the exact solution and the computed solution. For $s = 0$, see Tables 4.3 and 4.4. The results for using nonuniform mesh points are in Tables 4.5, 4.6 and 4.7. In the algorithms, nonuniform mesh points need to be computed first, therefore even for the same number of points, the nonuniform cases take more time than the uniform cases do.

Note that the solutions run by ode45 and ode23 are not that much different (Table 4.8).

4.4 Full discretization

The other way to solve (4.1)−(4.2) is to use full discretization, i.e., we discretize space and time. (In semi-discretization, we discretized space only.) For space, $-1 = \tau_n < \tau_{n-1} < \cdots < \tau_1 < \tau_0 = 0$ are defined as above and $\delta_i = \tau_{i-1} - \tau_i$. For time, we discretize $T^0, T^1, \ldots, T^m$ as $0 = T^0 < T^1 < \cdots < T^m = 1$ with $\triangle^k = T^{k+1} - T^k$, $k = 0, \ldots, m-1$. Then (4.12)−(4.13) can be written as

$$\begin{aligned} \frac{1}{\triangle^k}(\alpha_i^{k+1} - \alpha_i^k) &= \frac{1}{\delta_i}(\alpha_{i-1}^k - \alpha_i^k), \qquad (4.21) \\ \sum_{i=1}^{n} \frac{g_i}{\delta_i}(\alpha_{i-1}^{k+1} - \alpha_i^{k+1}) &= f(T^{k+1}), \quad i = 1, \ldots, n; \quad k = 0, \ldots, m-1. \qquad (4.22) \end{aligned}$$

We apply uniform and nonuniform mesh points defined in Section 4.2 to (4.21)−(4.22).

4.4.1 Uniform mesh points

In this section, we assume

$$\tau_i = -\frac{i}{n}, \quad i = 0, \ldots, n,$$

and

$$T^k = \frac{k}{m}, \quad k = 0, \ldots, m.$$

We assume $\triangle^k = T^{k+1} - T^k$ and $\delta_i = \tau_{i-1} - \tau_i$, therefore $\triangle^k = 1/m$ and $\delta_i = 1/n$. Moreover, if we set $m = n$, we obtain $\triangle^k = \delta_i = 1/n$, $k = 0, \ldots, n-1$; $i = 1, \ldots, n$.

Table 4.1: $p = 0.5$, uniform mesh for space, fix t = 1.

t+s	n=10	n=100	n=1000	exact sol.
0.10	0.2260	0.1860	0.1990	0.2013
0.20	0.2841	0.2773	0.2842	0.2847
0.30	0.3431	0.3463	0.3485	0.3487
0.40	0.4002	0.4019	0.4026	0.4026
0.50	0.4540	0.4501	0.4502	0.4502
0.60	0.5042	0.4934	0.4932	0.4931
0.70	0.5518	0.5332	0.5327	0.5326
0.80	0.5981	0.5705	0.5695	0.5694
0.90	0.6444	0.6078	0.6040	0.6040
1.00	0.6918	0.6507	0.6408	0.6366

Table 4.2: $p = 0.5$, uniform mesh for space, fix t = 1.

t+s	error(n=10)	error(n=100)	exact sol.
0.10	2.4708e-2	1.5309e-2	0.2013
0.20	6.1648e-4	7.3798e-3	0.2847
0.30	5.6315e-3	2.3846e-3	0.3487
0.40	2.4143e-3	7.3682e-4	0.4026
0.50	3.8411e-3	4.3308e-5	0.4502
0.60	1.1088e-2	2.9751e-4	0.4931
0.70	1.9126e-2	5.8768e-4	0.5326
0.80	2.8642e-2	1.0461e-3	0.5694
0.90	4.0480e-2	3.8016e-3	0.6040
1.00	5.5215e-2	1.4097e-2	0.6366

Table 4.3: $p = 0.5$, uniform mesh for space, fix s = 0.

t+s	n=10	n=100	n=1000	exact sol.
0.10	0.2408	0.2039	0.2016	0.2013
0.20	0.3085	0.2865	0.2849	0.2847
0.30	0.3663	0.3502	0.3488	0.3487
0.40	0.4172	0.4039	0.4028	0.4026
0.50	0.4639	0.4513	0.4503	0.4502
0.60	0.5083	0.4942	0.4932	0.4931
0.70	0.5524	0.5336	0.5327	0.5326
0.80	0.5973	0.5705	0.5695	0.5694
0.90	0.6437	0.6074	0.6040	0.6040
1.00	0.6918	0.6507	0.6408	0.6366

Table 4.4: $p = 0.5$, uniform mesh for space, fix s = 0.

t+s	error(n=10)	error(n=100)	exact sol.
0.10	3.9470e-2	2.6130e-3	0.2013
0.20	2.3789e-2	1.8119e-3	0.2847
0.30	1.7563e-2	1.4703e-3	0.3487
0.40	1.4615e-2	1.2691e-3	0.4026
0.50	1.3743e-2	1.1330e-3	0.4502
0.60	1.5211e-2	1.0330e-3	0.4931
0.70	1.9725e-2	9.5516e-4	0.5326
0.80	2.7856e-2	1.0883e-3	0.5694
0.90	3.9774e-2	3.4984e-3	0.6040
1.00	5.5215e-2	1.4097e-2	0.6366

Table 4.5: $p = 0.5, n = 10$, nonuniform mesh for space.

t+s	s=0	t=1	exact sol.
0.19	0.2890	0.2721	0.2775
0.36	0.3934	0.3714	0.3820
0.51	0.4674	0.4568	0.4546
0.64	0.5263	0.5237	0.5093
0.75	0.5756	0.5756	0.5513
0.84	0.6165	0.6169	0.5835
0.91	0.6490	0.6492	0.6073
0.96	0.6727	0.6728	0.6238
0.99	0.6870	0.6870	0.6334
1.00	0.6918	0.6918	0.6366

Table 4.6: $p = 0.5$, nonuniform mesh for space, fix s = 0.

t+s	error(n=10)	error(n=100)	exact sol.
0.19	1.1476e-2	1.0602e-3	0.2775
0.36	1.1411e-2	1.1206e-3	0.3820
0.51	1.2792e-2	1.0629e-3	0.4546
0.64	1.7049e-2	1.0879e-3	0.5093
0.75	2.4234e-2	1.1495e-3	0.5513
0.84	3.3029e-2	2.1309e-3	0.5835
0.91	4.1737e-2	5.1804e-3	0.6073
0.96	4.8935e-2	9.8513e-3	0.6238
0.99	5.3607e-2	1.3860e-2	0.6334
1.00	5.5174e-2	1.5399e-2	0.6366

Table 4.7: $p = 0.5$, nonuniform mesh for space, fix t = 1.

t+s	error(n=10)	error(n=100)	exact sol.
0.19	5.4170e-3	1.1241e-2	0.2775
0.36	1.0576e-2	1.7269e-3	0.3820
0.51	2.1698e-3	2.6454e-5	0.4546
0.64	1.4365e-2	6.1572e-4	0.5093
0.75	2.4284e-2	1.0200e-3	0.5513
0.84	3.3466e-2	2.2077e-3	0.5835
0.91	4.1921e-2	5.3281e-3	0.6073
0.96	4.9050e-2	9.8910e-3	0.6238
0.99	5.3546e-2	1.3841e-2	0.6334
1.00	5.5174e-2	1.5399e-2	0.6366

Table 4.8: $p = 0.5$, uniform mesh for space, fix t = 1.

t+s	ode23(n=10)	ode45(n=10)	ode23(n=100)	ode45(n=100)
0.10	0.2260	0.2261	0.1860	0.1861
0.20	0.2841	0.2842	0.2773	0.2773
0.30	0.3431	0.3432	0.3463	0.3463
0.40	0.4002	0.4003	0.4019	0.4019
0.50	0.4540	0.4540	0.4501	0.4501
0.60	0.5042	0.5042	0.4934	0.4935
0.70	0.5518	0.5517	0.5332	0.5332
0.80	0.5981	0.5980	0.5705	0.5705
0.90	0.6444	0.6444	0.6078	0.6078
1.00	0.6918	0.6918	0.6507	0.6507

Then equations (4.21)–(4.22) yield an extremely simple scheme:

$$\alpha_i^{k+1} = \alpha_{i-1}^k, \quad i = 1, \ldots, n; \quad k = 0, \ldots, n-1, \tag{4.23}$$

$$\alpha_0^{k+1} = \alpha_1^{k+1} - \sum_{i=2}^{n} \frac{g_i}{g_1}(\alpha_{i-1}^{k+1} - \alpha_i^{k+1}) + \frac{1}{ng_1} f(T^{k+1}). \tag{4.24}$$

Note that if we use a second order implicit trapezoidal scheme in time and space ([15]), (4.12) can be written as

$$\frac{1}{\triangle^k}\left(\frac{\alpha_{i-1}^{k+1} + \alpha_i^{k+1}}{2} - \frac{\alpha_{i-1}^k + \alpha_i^k}{2}\right) = \frac{1}{2\delta_i}(\alpha_{i-1}^{k+1} - \alpha_i^{k+1} + \alpha_{i-1}^k - \alpha_i^k), \tag{4.25}$$

by choosing $\triangle^k = \delta_i = 1/n$ for $i = 1, \ldots, n$; $k = 0, \ldots, n-1$, we obtain the same algebraic equations (4.23) and (4.24). In other words, by using equations (4.23) and (4.24), we should have first order or second order convergence. This observation coincides with the numerical results.

With algebraic equations (4.23) and (4.24) and initial condition (4.2), the procedure to find α_i^k is

1. From the initial condition $\phi(s)$, $-1 \le s \le 0$, we can find α_i^0, $i = 1, \ldots, n$, then we obtain α_0^0 by (4.24). Namely, we find α_i^k, $i = 0, \ldots, n$, at $k = 0$ first,
2. Use (4.23) to find α_i^{k+1}, $i = 1, \ldots, n$; $k = 0$,
3. Substitute α_i^{k+1}, $i = 1, \ldots, n$, into (4.24) to find α_0^{k+1},
4. Set $k = k+1$, if $k = n+1$ then stop,
5. Apply (4.23) to find α_i^{k+1}, $i = 1, \ldots, n$, go to 3.

To find $x(\tilde{t})$, $0 < \tilde{t} \le 1$, by (4.6) and (4.9), we have

$$\begin{aligned} x(\frac{j}{n}) &= x(1 - 1 + \frac{j}{n}) \\ &= \xi(1, -1 + \frac{j}{n}) \\ &= \sum_{i=0}^{n} \alpha_i(1) B_i(-1 + \frac{j}{n}) \\ &= \sum_{i=0}^{n} \alpha_i^n B_i(\frac{-n+j}{n}) \\ &= \sum_{i=0}^{n} \alpha_i^n B_i(\tau_{n-j}) \\ &= \alpha_{n-j}^n. \end{aligned}$$

Note that α_{n-j}^n actually is α_0^j, $j = 1, \ldots, n$, by (4.23).

4.4.2 Examples using uniform mesh points

In this section, we apply the idea of full discretization with uniform mesh points to solve (4.1)–(4.2). Especially, we concentrate on how close the approximation is. In other words, by increasing the number of mesh points, we should obtain a better approximation. The following is a definition of the rate of convergence for uniform mesh points:

Definition 4.1 *Suppose x is the exact solution and $\bar{x}$ is the computed solution at some time $t_0 > 0$. We assume the error between x and $\bar{x}$ has order $(h)^\alpha$, where h is the step size, i.e.,*

$$|x - \bar{x}| = Ch^\alpha.$$

Then α is the rate of convergence at t_0 for the numerical scheme.

Since we choose how many mesh points, say n, in $(0, 1]$ first, then h (step size) is fixed, we therefore can find α.

Example 4.2

We begin with the same example as we solved by an ode solver in Example 4.1:

$$\frac{d}{dt}\left(\int_{-1}^{0} |s|^{-p} x(t+s)ds\right) = 1, \quad p \in (0,1), \quad t > 0$$

$$x(s) = \phi(s) = 0, \quad -1 \leq s \leq 0.$$

By Lemma 3.3, the exact solution is

$$x(t) = \frac{\sin(p\pi)}{p\pi} t^p, \quad t \in (0, 1].$$

For $p = 0.5,\ 0.25,\ 0.75$, $t \in (0, 1]$, see Figures A.1, A.2, A.3 for the graphs of the exact solution and the approximate solution. Tables A.1, A.2, A.3 show the rates of convergence obtained by changing the number of mesh points. The rate of convergence is about 1.

Example 4.3

For the case $F(t) = t^a$, $0 < a < 1$, i.e., we are solving

$$\begin{aligned}
\frac{d}{dt}\left(\int_{-1}^{0} |s|^{-p} x(t+s)ds\right) &= at^{a-1}, \quad p \in (0,1), \quad t > 0, \\
x(s) &= \phi(s) = 0, \quad -1 \leq s \leq 0.
\end{aligned}$$

Note that with (4.5), the problem can be changed to

$$\begin{aligned} \int_{-1}^{0} |s|^{-p} x(t+s)ds &= D\phi + \int_0^t f(\tau)d\tau \\ &= t^a \\ &= F(t), \\ x(s) &= \phi(s) = 0, \quad -1 \le s \le 0. \end{aligned}$$

By Lemma 3.3, the exact solution is

$$x(t) = \frac{\Gamma(a+1)}{\Gamma(1-p)\Gamma(p+a)} t^{p+a-1}, \quad t \in (0,1].$$

Note that for $p + a = 1$,

$$\begin{aligned} x(t) &= \frac{\Gamma(a+1)}{\Gamma(1-p)\Gamma(1)} \\ &= \frac{a\Gamma(a)}{\Gamma(a)\Gamma(1)} \\ &= a. \end{aligned}$$

For $p = 0.5$, $a = 0.5, 0.25, 0.75$, $t \in (0,1]$, see Appendix B for the graph of the exact solution, the graph of the approximate solution and the rate of convergence. We observe that the rates of convergence basically depend on the power of the right-hand side, i.e., the constant a.

We have some other results:

$$\begin{aligned} p &= 0.25, \quad a = 0.75, \text{ rate of convergence is about } 0.75, \\ p &= 0.75, \quad a = 0.25, \text{ rate of convergence is about } 0.25, \\ p &= 0.75, \quad a = 0.75, \text{ rate of convergence is about } 0.75. \end{aligned}$$

Example 4.4

Let's consider a higher power of $F(t)$, i.e., $1 < a < 2$:

$$\begin{aligned} \frac{d}{dt}\left(\int_{-1}^{0} |s|^{-p} x(t+s)ds\right) &= at^{a-1}, \quad p \in (0,1), \quad t > 0, \\ x(s) &= \phi(s) = 0, \quad -1 \le s \le 0. \end{aligned}$$

By Lemma 3.3, the exact solution is

$$x(t) = \frac{\Gamma(a+1)}{\Gamma(1-p)\Gamma(p+a)} t^{p+a-1}, \quad t \in (0,1].$$

From the algebraic equations (4.23) and (4.24), we claim that for $p + a = 2$, the approximate solution is equal to the exact solution:

Proof Since $f(t) = at^{a-1}$, from equation (4.24), we have

$$\alpha_0^{k+1} = \sum_{i=1}^{n} \alpha_i^{k+1} \ \frac{g_i - g_{i+1}}{g_1} + \frac{1}{ng_1} \ a \ (\frac{k+1}{n})^{a-1},$$

where

$$\begin{aligned} g_{n+1} &= 0, \\ g_1 &= \int_{\tau_1}^{0} |s|^{-p} ds \\ &= \int_{\frac{1}{n}}^{0} (-s)^{-p} ds \\ &= \frac{-1}{1-p}(-s)^{1-p}|_{-\frac{1}{n}}^{0} \\ &= \frac{1}{(1-p)n^{1-p}}, \end{aligned}$$

and by a simple calculation,

$$\frac{g_i - g_{i+1}}{g_1} = 2i^{1-p} - (i-1)^{1-p} - (i+1)^{1-p}.$$

Therefore, the numerical solutions α_{n-i}^{n} ($= \alpha_0^i$), $i = 1, \ldots, n$, are

$$\begin{aligned} \alpha_0^0 &= 0, \ \ (\text{ since } \phi = 0) \\ \alpha_0^1 &= \frac{1}{ng_1} \ a \ (\frac{1}{n})^{a-1} \\ &= \frac{1}{n} \ n^{1-p} \ (1-p) \ a \ (\frac{1}{n})^{a-1} \\ &= \frac{1}{n^{1-1+p+a-1}} \ (1-p) \ a \\ &= a \ (1-p) \ \frac{1}{n}, \ \ (\text{ since } p + a = 2) \\ \alpha_0^2 &= \alpha_1^2(2 - 2^{1-p}) + \frac{1}{ng_1} \ a \ (\frac{2}{n})^{a-1} \\ &= \alpha_1^2(2 - 2^{1-p}) + \frac{1}{n} \ (1-p) \ n^{1-p} \ a \ 2^{a-1} \ \frac{1}{n^{a-1}} \\ &= \alpha_0^1(2 - 2^{1-p}) + \frac{1}{n^{1-1+p+a-1}} \ (1-p) \ a \ 2^{a-1} \\ &= a \ (1-p) \ \frac{1}{n} \ (2 - 2^{1-p}) + (1-p) \ a \ 2^{1-p} \ \frac{1}{n} \ \ (\text{ since } p + a = 2) \\ &= a \ (1-p) \ \frac{2}{n}, \end{aligned}$$

$$\begin{aligned}
\vdots &= \vdots \\
\alpha_0^n &= a\ (1-p).
\end{aligned}$$

The exact solution for $p + a = 2$ is

$$\begin{aligned}
x(t) &= \frac{\Gamma(a+1)}{\Gamma(1-p)\Gamma(p+a)} t^{p+a-1} \\
&= \frac{\Gamma(3-p)}{\Gamma(1-p)\Gamma(2)} t \\
&= \frac{(2-p)\Gamma(2-p)}{\Gamma(1-p)\Gamma(2)} t \\
&= (2-p)\ (1-p)\ t \\
&= a\ (1-p)\ t, \quad t \in (0,1].
\end{aligned}$$

□

For $p = 0.5$, $a = 1.25$, 1.75, by Theorem 3.2, we have the exact solution $x(t)$, $0 < t \leq 3$. Therefore, we find the rate of convergence for $t \in (0, 3]$. See Appendix C for the graphs and rates of convergence. The rate of convergence for $p = 0.5$, $a = 1.25$ is about 1.25. For $p = 0.5$, $a = 1.75$, the rate is approximately 1.5. We are interested in other values of p and a, the results are

$$\begin{aligned}
p &= 0.75,\ \ a = 1.75,\ \text{rate of convergence is about } 1.25, \\
p &= 0.25,\ \ a = 1.25,\ \text{rate of convergence is about } 1.25, \\
p &= 1/3,\ \ a = 4/3,\ \text{rate of convergence is about } 4/3, \\
p &= 2/3,\ \ a = 5/3,\ \text{rate of convergence is about } 4/3, \\
p &= 0.25,\ \ a = 1.5,\ \text{rate of convergence is about } 1.5, \\
p &= 0.75,\ \ a = 1.5,\ \text{rate of convergence is about } 1.25, \\
p &= 1/3,\ \ a = 1.5,\ \text{rate of convergence is about } 1.5, \\
p &= 0.5,\ \ a = 1.49,\ \text{rate of convergence is about } 1.49, \\
p &= 0.75,\ \ a = 1.55,\ \text{rate of convergence is about } 1.25, \\
p &= 0.5,\ \ a = 1.45,\ \text{rate of convergence is about } 1.45, \\
p &= 0.75,\ \ a = 1.6,\ \text{rate of convergence is about } 1.25, \\
p &= 0.25,\ \ a = 1.6,\ \text{rate of convergence is about } 1.6, \\
p &= 0.65,\ \ a = 1.25,\ \text{rate of convergence is about } 1.25, \\
p &= 0.25,\ \ a = 1.55,\ \text{rate of convergence is about } 1.55.
\end{aligned}$$

We observe that the rate of convergence seems to have the pattern $Min(2-p, a)$.

Example 4.5

We increase the value of a to $2 < a < 3$, and consider the same equation

$$\begin{aligned} \frac{d}{dt}\left(\int_{-1}^{0} |s|^{-p} x(t+s)ds\right) &= at^{a-1}, \quad p \in (0,1), \quad t > 0, \\ x(s) &= \phi(s) = 0, \quad -1 \le s \le 0. \end{aligned}$$

By Lemma 3.3, the exact solution is

$$x(t) = \frac{\Gamma(a+1)}{\Gamma(1-p)\Gamma(p+a)} t^{p+a-1}, \quad t \in (0,1].$$

For $p = 0.5$, $a = 2.5$, 2.75, 2.25, $t \in (0,3]$, see Appendix D. The rate of convergence is 1.5. We have other results

$$\begin{aligned} p &= 0.25, \quad a = 2.75, \text{ rate of convergence is about } 1.75, \\ p &= 0.75, \quad a = 2.25, \text{ rate of convergence is about } 1.25, \\ p &= 1/3, \quad a = 8/3, \text{ rate of convergence is about } 5/3, \\ p &= 2/3, \quad a = 7/3, \text{ rate of convergence is about } 4/3. \end{aligned}$$

Note that for $p = 0.5$, we use Theorem 3.2 to find the exact solution $x(t)$, $t \in (0,3]$. The rate of convergence for $2 < a < 3$ follows the pattern: $Min(2-p, a) = 2-p$.

How about the values of $a = 2$ and $a = 3$, or even $a = 4$? Here we use right-hand side $f(t) = t$, $t^2/2$ and $t^3/3!$, therefore $F(t) = D\phi + \int_0^t f(\tau)d\tau$ equals $t^2/2$, $t^3/3!$ and $t^4/4!$, respectively. Our conjecture for the rate of convergence for these cases is $Min(2-p, a) = 2-p$.

Example 4.6

$$\frac{d}{dt}\left(\int_{-1}^{0} |s|^{-p} x(t+s)ds\right) = f(t), \quad p \in (0,1), \quad t > 0,$$
$$x(s) = \phi(s) = 0, \quad -1 \le s \le 0.$$

For $a = 2$, by Lemma 3.3, the exact solution is

$$x(t) = \frac{\sin(p\pi)}{p(p+1)\pi} t^{p+1}, \quad t \in (0,1],$$

for $a = 3$, the solution is

$$x(t) = \frac{\sin(p\pi)}{p(p+1)(p+2)\pi} t^{p+2}, \quad t \in (0,1],$$

and for $a = 4$, the solution is

$$x(t) = \frac{\sin(p\pi)}{p(p+1)(p+2)(p+3)\pi} t^{p+3}, \quad t \in (0, 1].$$

For $p = 0.5$, $F(t) = t^2/2!$, $t \in (0, 3]$, see Appendix E for the graph of the exact solution, the graph of the approximate solution and the rate of convergence. A summary for rates of convergence is

$p = 0.5$, $F(t) = t^2/2$, rate of convergence are about $1.5, 1.5, 1.5$ for $t \in (0,1], (1,2], (2,3]$,

$p = 0.25$, $F(t) = t^2/2$, rate of convergence is about 1.75,

$p = 0.75$, $F(t) = t^2/2$, rate of convergence is about 1.25,

$p = 0.5$, $F(t) = t^3/3!$, rate of convergence is about 1.5,

$p = 0.25$, $F(t) = t^3/3!$, rate of convergence is about 1.75,

$p = 0.75$, $F(t) = t^3/3!$, rate of convergence is about 1.25,

$p = 0.5$, $F(t) = t^4/4!$, rate of convergence is about 1.5,

$p = 0.25$, $F(t) = t^4/4!$, rate of convergence is about 1.75,

$p = 0.75$, $F(t) = t^4/4!$, rate of convergence is about 1.25.

In all cases, the rate of convergence is $Min(2 - p, a) = 2 - p$.

After we solve the case of the right-hand side $F(t)$ equal t^a, with $a > 0$ and initial condition $\phi = 0$, we should find the solution where $F(t)$ is a combination of t^a, for different values of a. We consider the four examples in Chapter 3 with $p = 0.5$ (we use Theorem 3.2 to find the exact solutions for $t \in (0, 3]$).

Example 4.7

$$\frac{d}{dt}\left(\int_{-1}^{0} |s|^{-p} x(t+s)ds\right) = \frac{d}{dt}F(t), \quad p \in (0,1), \quad t > 0,$$
$$x(s) = \phi(s) = 0, \quad -1 \le s \le 0,$$

with

1. $F(t) = t + t^2/2!$,
2. $F(t) = \sin(t) = t - t^3/3! + t^5/5! - \cdots$,
3. $F(t) = \cos(t) - 1 = -t^2/2! + t^4/4! - \cdots$,

4. $F(t) = e^t - 1 = t + t^2/2! + t^3/3! + \cdots$.

For $F(t) = t + t^2/2!$, by Theorem 3.2, the exact solution is

$$x(t) = \begin{cases} \frac{2}{\pi}t^{0.5}(1+\frac{2t}{3}) & \text{for} \quad 0 < t \le 1, \\ \\ -\frac{4}{3\pi} + \frac{4t^{0.5}}{\pi} - \frac{2t}{\pi} + \frac{8t^{1.5}}{3\pi} & \text{for} \quad 1 < t \le 2, \\ \\ -\frac{4}{3\pi} + \frac{4t^{0.5}}{\pi} - \frac{2t}{\pi} + \frac{8t^{1.5}}{3\pi} & \\ +\frac{4}{\pi^2}[(1+t)\int_2^t s^{-0.5}\arctan\left(\frac{s-2}{s}\right)^{0.5} ds & \\ -\int_2^t s^{0.5}\arctan\left(\frac{s-2}{s}\right)^{0.5} ds] & \text{for} \quad 2 < t \le 3. \end{cases}$$

See Figure F.1 for graphs of the exact solution and the approximate solution when $t \in (0,3]$. As we can predict, for $F(t) = t + t^2/2!$, the lower power term t will be the main factor which dominates the rate of convergence, i.e., the rate of convergence is 1. We can see the results in Tables F.1 and F.2.

For $F(t) = \sin(t)$, the exact solution can be obtained as

$$x(t) = \begin{cases} \int_0^t \frac{1}{\pi}s^{-0.5}\cos(t-s)ds & \text{for} \quad 0 < t \le 1, \\ \\ \int_0^1 \frac{1}{\pi}s^{-0.5}\cos(t-s)ds + \int_1^t \frac{2}{\pi}s^{-0.5}\cos(t-s)ds & \text{for} \quad 1 < t \le 2, \\ \\ \int_0^1 \frac{1}{\pi}s^{-0.5}\cos(t-s)ds + \int_1^2 \frac{2}{\pi}s^{-0.5}\cos(t-s)ds & \\ +\int_2^t \frac{2}{\pi}s^{-0.5}\cos(t-s)[1+\frac{2}{\pi}\arctan\left(\frac{s-2}{s}\right)^{0.5}]ds & \text{for} \quad 2 < t \le 3, \end{cases}$$

for $F(t) = \cos(t) - 1$, the exact solution is

$$x(t) = \begin{cases} -\int_0^t \frac{1}{\pi}s^{-0.5}\sin(t-s)ds & \text{for} \quad 0 < t \le 1, \\ \\ -\int_0^1 \frac{1}{\pi}s^{-0.5}\sin(t-s)ds - \int_1^t \frac{2}{\pi}s^{-0.5}\sin(t-s)ds & \text{for} \quad 1 < t \le 2, \\ \\ -\int_0^1 \frac{1}{\pi}s^{-0.5}\sin(t-s)ds - \int_1^2 \frac{2}{\pi}s^{-0.5}\sin(t-s)ds & \\ -\int_2^t \frac{2}{\pi}s^{-0.5}\sin(t-s)[1+\frac{2}{\pi}\arctan\left(\frac{s-2}{s}\right)^{0.5}]ds & \text{for} \quad 2 < t \le 3, \end{cases}$$

and for $F(t) = e^t - 1$, the exact solution is

$$x(t) = \begin{cases} \int_0^t \frac{1}{\pi} s^{-0.5} e^{t-s} ds & \text{for} \quad 0 < t \le 1, \\ \\ \int_0^1 \frac{1}{\pi} s^{-0.5} e^{t-s} ds + \int_1^t \frac{2}{\pi} s^{-0.5} e^{t-s} ds & \text{for} \quad 1 < t \le 2, \\ \\ \int_0^1 \frac{1}{\pi} s^{-0.5} e^{t-s} ds + \int_1^2 \frac{2}{\pi} s^{-0.5} e^{t-s} ds & \\ + \int_2^t \frac{2}{\pi} s^{-0.5} e^{t-s} [1 + \frac{2}{\pi} \arctan \left(\frac{s-2}{s}\right)^{0.5}] ds & \text{for} \quad 2 < t \le 3. \end{cases}$$

The rates of convergence can also be found in Appendix F. They follow our conjecture, $Min(2-p, a)$. In other words, for

$$F(t) = \sin(t) = t - t^3/3! + \cdots$$

and

$$F(t) = e^t - 1 = t + t^2/2! + \cdots,$$

the lowest power term t dominates rate of convergence and the rate of convergence is $Min(2-p, 1) = 1$. For

$$F(t) = \cos(t) - 1 = -t^2/2! + t^4/4! - \cdots,$$

the lowest power is 2, therefore the rate of convergence is $Min(2-p, a) = Min(2 - 0.5, 2) = 1.5$.

4.4.3 Comments on the rate of convergence

In this section, we study the rate of convergence for the full discretization scheme with uniform mesh points. Our numerical examples for (4.1)–(4.2) with $\phi(s) = 0$ and $F(t) = t^a$ have rates of convergence $Min(2-p, a)$, i.e., we need to show that the error between the exact solution and the approximate solution has order $C_1(1/n)^{2-p} + C_2(1/n)^a$. We have not been able to establish the convergence rate, $Min(2-p, a)$, that we have observed in all of our numerical examples. However, we have made progress in this effort and summarize here our partial results. We believe that our results support the conjecture.

By section 4.4.1, we have

$$\bar{x}(\frac{i}{n}) = \alpha_{n-i}^n = \alpha_0^i, \quad i = 1, 2, \ldots, n,$$

where $\bar{x}$ is the approximate solution. By the initial condition $\phi(s) = 0$, $s \in [-1, 0]$ and $f(t) = at^{a-1}$, equation (4.24) implies

$$\begin{aligned}
\alpha_0^0 &= \frac{a}{ng_1}(\frac{1}{n})^{a-1}, \\
\alpha_0^{k+1} &= \sum_{j=1}^{n} \alpha_j^{k+1} \frac{g_j - g_{j+1}}{g_1} + \frac{a}{ng_1}(\frac{k+1}{n})^{a-1}, \qquad (4.26) \\
k &= 0, 1, 2, \ldots, n-1,
\end{aligned}$$

with $g_{n+1} = 0$ and

$$\begin{aligned}
g_j &= \int_{\tau_j}^{\tau_{j-1}} |s|^{-p} ds \\
&= \int_{-j/n}^{-(j-1)/n} (-s)^{-p} ds \\
&= \frac{-(-s)^{-p+1}}{-p+1} \Big|_{-j/n}^{-(j-1)/n} \\
&= \frac{1}{-p+1}[-(\frac{j-1}{n})^{-p+1} + (\frac{j}{n})^{-p+1}] \\
&= \frac{1}{-p+1}(\frac{1}{n})^{-p+1}[j^{-p+1} - (j-1)^{-p+1}].
\end{aligned}$$

Combining the initial condition $\phi(s) = 0$, $s \in [-1, 0]$, and equation (4.23), we have

$$\alpha_j^i = 0, \qquad \text{for } j > i,$$

then (4.26) becomes

$$\begin{aligned}
\alpha_0^i &= \sum_{j=1}^{n} \alpha_j^i \, \frac{\frac{1}{-p+1}(\frac{1}{n})^{-p+1}[j^{-p+1} - (j-1)^{-p+1} - (j+1)^{-p+1} + j^{-p+1}]}{\frac{1}{-p+1}(\frac{1}{n})^{-p+1}} \\
&\quad + \frac{a}{ng_1}(\frac{i}{n})^{a-1} \\
&= \sum_{j=1}^{i} \alpha_j^i [2j^{-p+1} - (j-1)^{-p+1} - (j+1)^{-p+1}] + \frac{a}{ng_1}(\frac{i}{n})^{a-1} \\
&= \sum_{j=1}^{i} \alpha_0^{i-j} [2j^{-p+1} - (j-1)^{-p+1} - (j+1)^{-p+1}] + \frac{a}{ng_1}(\frac{i}{n})^{a-1}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\alpha_0^1 &= \alpha_0^0[2 - 2^{-p+1}] + \frac{a}{ng_1}(\frac{1}{n})^{a-1}, \\
\alpha_0^2 &= \alpha_0^1[2 - 2^{-p+1}] + \alpha_0^0[2 \cdot 2^{-p+1} - 1 - 3^{-p+1}] + \frac{a}{ng_1}(\frac{2}{n})^{a-1}
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_0^0[2-2^{-p+1}]^2 + \alpha_0^0[2\cdot 2^{-p+1} - 1 - 3^{-p+1}] \\
&\quad + \frac{a}{ng_1}(\frac{1}{n})^{a-1}[2-2^{-p+1}] + \frac{a}{ng_1}(\frac{2}{n})^{a-1} \\
&= \alpha_0^0([2-2^{-p+1}]^2 + 2^{-p+2} - 1 - 3^{-p+1}) + \frac{a}{n^a g_1}[2^{a-1} + 2 - 2^{-p+1}], \\
\alpha_0^3 &= \alpha_0^2[2-2^{-p+1}] + \alpha_0^1[2\cdot 2^{-p+1} - 1 - 3^{-p+1}] \\
&\quad + \alpha_0^0[2\cdot 3^{-p+1} - 2^{-p+1} - 4^{-p+1}] + \frac{a}{ng_1}(\frac{3}{n})^{a-1} \\
&= \alpha_0^0([2-2^{-p+1}]^3 + [2-2^{-p+1}][2^{-p+2} - 1 - 3^{-p+1}] \\
&\quad + [2-2^{-p+1}][2^{-p+2} - 1 - 3^{-p+1}] + [2\cdot 3^{-p+1} - 2^{-p+1} - 4^{-p+1}]) \\
&\quad + \frac{a}{n^a g_1}[2^{a-1} + 2 - 2^{-p+1}][2-2^{-p+1}] + \frac{a}{n^a g_1}[2^{-p+2} - 1 - 3^{-p+1}] + \frac{a}{ng_1}(\frac{3}{n})^{a-1} \\
&= \alpha_0^0([2-2^{-p+1}]^3 \\
&\quad + 2\cdot[2-2^{-p+1}][2^{-p+2} - 1 - 3^{-p+1}] + [2\cdot 3^{-p+1} - 2^{-p+1} - 4^{-p+1}]) \\
&\quad + \frac{a}{n^a g_1}[2^a + 2^2 - 2^{-p+2} - 2^{a-p} - 2^{-p+2} + 2^{-2p+2} + 2^{-p+2} - 1 - 3^{-p+1} + 3^{a-1}] \\
&= \alpha_0^0([2-2^{-p+1}]^3 \\
&\quad + 2\cdot[2-2^{-p+1}][2^{-p+2} - 1 - 3^{-p+1}] + [2\cdot 3^{-p+1} - 2^{-p+1} - 4^{-p+1}]) \\
&\quad + \frac{a}{n^a g_1}[2^a - 2^{a-p} - 2^{-p+2} + 2^{-2p+2} + 3 - 3^{-p+1} + 3^{a-1}], \\
&\quad \vdots
\end{aligned}
$$

We need to show

$$
e(\frac{i}{n}) = O[(\frac{1}{n})^{2-p} + (\frac{1}{n})^a], \quad i = 1, 2, \ldots, n.
$$

Here we have

$$
\begin{aligned}
e(\frac{i}{n}) &= |x(\frac{i}{n}) - \alpha_0^i| \\
&= |\frac{\Gamma(a+1)}{\Gamma(1-p)\Gamma(p+a)} \cdot (\frac{i}{n})^{p+a-1} \\
&\quad - \sum_{j=1}^{i} \alpha_0^{i-j}[2j^{-p+1} - (j-1)^{-p+1} - (j+1)^{-p+1}] - \frac{a}{ng_1}(\frac{i}{n})^{a-1}| \\
&= |\frac{\Gamma(a+1)}{\Gamma(1-p)\Gamma(p+a)} \cdot (\frac{i}{n})^{p+a-1} \\
&\quad - \sum_{j=1}^{i} \alpha_0^{i-j}[2j^{-p+1} - (j-1)^{-p+1} - (j+1)^{-p+1}] - \frac{a}{g_1} \cdot i^{a-1} \cdot (\frac{1}{n})^a| \\
&= |\frac{\Gamma(a+1)}{\Gamma(1-p)\Gamma(p+a)} \cdot (\frac{i}{n})^{p+a-1}
\end{aligned}
$$

$$
\begin{aligned}
& -\sum_{j=1}^{i} \alpha_0^{i-j}[2j^{-p+1}-(j-1)^{-p+1}-(j+1)^{-p+1}] \\
& -\frac{a}{\frac{1}{-p+1}(\frac{1}{n})^{-p+1}} \cdot i^{a-1} \cdot (\frac{1}{n})^a| \\
= & \ |\frac{\Gamma(a+1)}{\Gamma(1-p)\Gamma(p+a)} \cdot (\frac{i}{n})^{p+a-1} \\
& -\sum_{j=1}^{i} \alpha_0^{i-j}[2j^{-p+1}-(j-1)^{-p+1}-(j+1)^{-p+1}] \\
& -a \cdot (-p+1) \cdot (\frac{1}{n})^{p-1} \cdot i^{a-1} \cdot (\frac{1}{n})^a| \\
= & \ |\frac{\Gamma(a+1)}{\Gamma(1-p)\Gamma(p+a)} \cdot (\frac{i}{n})^{p+a-1} \\
& -\sum_{j=1}^{i} \alpha_0^{i-j}[2j^{-p+1}-(j-1)^{-p+1}-(j+1)^{-p+1}] \\
& -a \cdot (-p+1) \cdot i^{a-1} \cdot (\frac{1}{n})^{p+a-1}|.
\end{aligned}
$$

Note that α_0^{i-j} depends on

$$
\begin{aligned}
\alpha_0^0 &= \frac{a}{ng_1} \cdot (\frac{1}{n})^{a-1} \\
&= \frac{a}{n \cdot (\frac{1}{1-p})(\frac{1}{n})^{-p+1}} \cdot (\frac{1}{n})^{a-1} \\
&= \frac{a(1-p)}{n^p} \cdot (\frac{1}{n})^{a-1} \\
&= a(1-p) \cdot (\frac{1}{n})^{p+a-1}.
\end{aligned}
$$

At this time, we have obtained $(1/n)^{p+a-1}$ as an estimate for the error, but we have not been able to obtain $(1/n)^a$ and $(1/n)^{2-p}$, appearing in our conjecture.

4.4.4 More examples

In this section, we continue applying the full discretization numerical scheme with uniform mesh points to singular integrodifferential equations with initial conditions $\phi(s) \neq 0$.

Example 4.8

We consider the problem

$$
\frac{d}{dt}\left(\int_{-1}^{0} |s|^{-p} x(t+s) ds\right) = \frac{d}{dt}(\frac{1}{1-p}), \quad p \in (0,1), \quad t > 0,
$$

$$x(s) = \phi(s) = 1, \quad -1 \le s \le 0.$$

Comparing to (4.1), (4.2) and (4.5), we have

$$\begin{aligned}
f(t) &= \frac{d}{dt}(\frac{1}{1-p}) = 0, \\
F(t) &= D\phi \\
&= \int_{-1}^{0} |s|^{-p} ds \\
&= \frac{-1}{1-p}(-s)^{-p+1}|_{-1}^{0} \\
&= \frac{1}{1-p}.
\end{aligned}$$

By a simple calculation, the exact solution is $x(t) = 1$, $t \in (0, \infty)$. The approximate solution is also 1. Namely, the numerical solution is the same as the approximate solution.

Proof From (4.24),

$$\alpha_0^{k+1} = \sum_{i=1}^{n} \alpha_i^{k+1} \frac{g_i - g_{i+1}}{g_1},$$

with $g_{n+1} = 0$. By applying initial condition, $\alpha_i^0 = 1$, $i = 1, \ldots, n$, then

$$\begin{aligned}
\alpha_0^0 &= \sum_{i=1}^{n} \alpha_i^0 \frac{g_i - g_{i+1}}{g_1} \\
&= \sum_{i=1}^{n} \frac{g_i - g_{i+1}}{g_1} \\
&= \frac{(g_1 - g_2) + (g_2 - g_3) + \cdots + (g_n - g_{n+1})}{g_1} \\
&= \frac{g_1}{g_1} \\
&= 1.
\end{aligned}$$

With the same argument, we find

$$\begin{aligned}
\alpha_0^1 &= 1 \\
\alpha_0^2 &= 1 \\
\vdots &= \vdots \\
\alpha_0^n &= 1.
\end{aligned}$$

□

Example 4.9

We consider the problem

$$\begin{aligned}\frac{d}{dt}\left(\int_{-1}^{0}|s|^{-p}x(t+s)ds\right) &= 0,\quad p\in(0,1),\quad t>0,\\ x(s) &= \phi(s)=|s|^{p},\quad -1\le s\le 0.\end{aligned}$$

Here $D\phi=\int_{-1}^{0}|s|^{-p}|s|^{p}ds=1$, $f(t)=0$ and $F(t)=1$, i.e., we are solving

$$\begin{aligned}\int_{-1}^{0}|s|^{-p}x(t+s)ds &= 1,\\ x(s) &= \phi(s)=|s|^{p}.\end{aligned}$$

The exact solution can be obtained by Lemma 3.2 for $t\in(0,1]$. The rates of convergence for $p=0.5$, 0.25, 0.75 are in Tables G.1, G.2, G.3.

Example 4.10

Consider the problem

$$\begin{aligned}\frac{d}{dt}\left(\int_{-1}^{0}|s|^{-p}x(t+s)ds\right) &= 0,\quad p\in(0,1),\quad t>0,\\ x(s) &= \phi(s)=(1-\frac{p}{2})|s|^{\frac{p}{2}},\quad -1\le s\le 0.\end{aligned}$$

We have

$$\begin{aligned}D\phi &= \int_{-1}^{0}|s|^{-p}(1-\frac{p}{2})|s|^{\frac{p}{2}}ds\\ &= (1-\frac{p}{2})\int_{-1}^{0}(-s)^{-\frac{p}{2}}ds\\ &= (1-\frac{p}{2})\frac{-(-s)^{1-\frac{p}{2}}}{1-\frac{p}{2}}|_{-1}^{0}\\ &= (1-\frac{p}{2})\frac{1}{1-\frac{p}{2}}\\ &= 1,\end{aligned}$$

with $f(t)=0$ and $F(t)=1$. The associated equation is

$$\begin{aligned}\int_{-1}^{0}|s|^{-p}x(t+s)ds &= 1,\\ x(s) &= \phi(s)=(1-\frac{p}{2})|s|^{\frac{p}{2}}.\end{aligned}$$

The exact solution is given by Lemma 3.2 for $t\in(0,1]$. The rates of convergence for $p=0.5$, 0.25, 0.75 can be found in Tables G.4, G.5, G.6.

Example 4.11
Consider

$$\begin{aligned} \frac{d}{dt}\left(\int_{-1}^{0} |s|^{-p} x(t+s)ds\right) &= 0, \quad p \in (0,1), \quad t > 0, \\ x(s) &= \phi(s) = (2-p)|s|, \quad -1 \le s \le 0. \end{aligned}$$

Then

$$\begin{aligned} D\phi &= \int_{-1}^{0} |s|^{-p}(2-p)|s|ds \\ &= \int_{-1}^{0} (-s)^{1-p}(2-p)ds \\ &= (2-p)\frac{-(-s)^{2-p}}{2-p}\Big|_{-1}^{0} \\ &= 1, \end{aligned}$$

with $f(t) = 0$ and $F(t) = 1$. We change the equation to

$$\begin{aligned} \int_{-1}^{0} |s|^{-p} x(t+s)ds &= 1, \\ x(s) &= \phi(s) = (2-p)|s|. \end{aligned}$$

The exact solution can be obtained by Lemma 3.2 for $t \in (0,1]$. Tables G.7, G.8, and G.9 contain the rates of convergence for $p = 0.5$, 0.25 and 0.75.

4.4.5 Nonuniform mesh points

Another way to solve (4.1)–(4.2) is by applying a full discretization with nonuniform mesh points. In particular, we compare the rate of convergence for nonuniform mesh points to the rate of convergence for uniform mesh points. For the examples of uniform mesh points, we observed that the rate of convergence is $2-p$, especially for the case $F(t) = t^a$, $a > 2$. Our goal is, by choosing nonuniform mesh points, to regain the second order convergence. The motivation of this approach is due to the conjecture in [4] (p.398) from solving the (linear) first-kind Volterra integral equation:

Conjecture 4.1
Assume that $g(t)$ and $K(t,s)$ in

$$g(t) + \int_{0}^{t} (t-s)^{-p} K(t,s) y(s) ds = 0, \quad t \in I = [0,T], \tag{4.27}$$

satisfy $g \in C^{m+1}(I)$, $K \in C^{m+1}(S)$, with $g^{(k)}(0) = 0$ $(k = 0, 1, \cdots, m)$, and $K(t,t) \neq 0$ for all $t \in I$. Let the collocation parameters $\{c_j\}$ be such that $0 < c_1 < \cdots < c_m = 1$.

If $u \in S^{(-1)}_{m-1}(Z_N)$ and $\hat{u} \in S^{(-1)}_{m-1}(Z_N)$ are the collocation approximations determined by certain collocation equations, and if the underlying mesh sequence is quasi-uniform, then

$$\|y - u\|_\infty = O(N^{-m})$$

and

$$\|y - \hat{u}\|_\infty = O(N^{-m}).$$

The relevance of this conjecture to our study is seen by changing (4.3) in the form of (4.27),

$$\begin{aligned}
& \int_{-1}^{0} |s|^{-p} x(t+s)ds = F(t) \\
\Rightarrow\ & \int_{t-1}^{t} |u-t|^{-p} x(u)du = F(t) \\
\Rightarrow\ & \int_{t-1}^{0} |u-t|^{-p} x(u)du + \int_{0}^{t} |u-t|^{-p} x(u)du = F(t) \\
\Rightarrow\ & (\int_{t-1}^{0} (t-u)^{-p} x(u)du - F(t)) + \int_{0}^{t} (t-u)^{-p} x(u)du = 0 \\
\Rightarrow\ & g(t) + \int_{0}^{t} (t-s)^{-p} y(s)ds = 0,
\end{aligned}$$

with

$$\begin{aligned}
g(t) &= \int_{t-1}^{0} (t-u)^{-p} x(u)du - F(t) \\
K(t,s) &= 1.
\end{aligned}$$

Note that $g(0) = 0$ implies

$$\begin{aligned}
F(0) &= \int_{-1}^{0} (-u)^{-p} x(u)du \\
&= \int_{-1}^{0} (-u)^{-p} \phi(u)du \\
&= D\phi,
\end{aligned}$$

this result coincides the property of $F(t)$ from equation (4.5).

We begin with choosing nonuniform mesh points. For space, $-1 = \tau_n < \tau_{n-1} < \cdots < \tau_1 < \tau_0 = 0$ are defined according to the equal area. i.e., τ_i is chosen so that

$$\int_{\tau_i}^{\tau_{i-1}} g(s)ds = \frac{1}{n} \int_{-1}^{0} g(s)ds \equiv g, \quad i = 1, \ldots, n,$$

and $\delta_i = \tau_{i-1} - \tau_i$, $i = 1, \ldots, n$. For time, we discretize $T^0, T^1, \cdots, T^m$ as $0 = T^0 < T^1 < \cdots < T^m = 1$ with $\triangle^k = T^{k+1} - T^k$, $k = 0, \ldots, m-1$. We assume $m = n$, and note that by (4.6), $d/dt[x(t+s)] = d/ds[x(t+s)]$, therefore the relation between $\triangle^k$ and δ_i is

$$\triangle^k = \delta_{n-k}, \quad k = 0, \ldots, n-1.$$

We use the second order implicit trapezoidal scheme in time and space. Then (4.12) and (4.13) can be written as

$$\begin{aligned}
\frac{1}{\triangle^k}\left(\frac{\alpha_{i-1}^{k+1}+\alpha_i^{k+1}}{2} - \frac{\alpha_{i-1}^k+\alpha_i^k}{2}\right) &= \frac{1}{2\delta_i}(\alpha_{i-1}^{k+1} - \alpha_i^{k+1} + \alpha_{i-1}^k - \alpha_i^k), \\
\sum_{i=1}^{n} \frac{g_i}{\delta_i}(\alpha_{i-1}^{k+1} - \alpha_i^{k+1}) &= f(T^{k+1}), \quad i = 1, \ldots, n; \quad k = 0, \ldots, n-1.
\end{aligned}$$

Therefore, we have

$$\alpha_i^{k+1} = \alpha_{i-1}^{k+1}\frac{\triangle^k - \delta_i}{\delta_i + \triangle^k} + \alpha_i^k \frac{\delta_i - \triangle^k}{\delta_i + \triangle^k} + \alpha_{i-1}^k \tag{4.28}$$

and

$$\begin{aligned}
& \alpha_0^{k+1}[\frac{2g}{\triangle^k+\delta_1} + \frac{2g(\triangle^k - \delta_1)}{(\triangle^k+\delta_1)(\triangle^k+\delta_2)} \\
& + \cdots + \frac{2g(\triangle^k-\delta_1)(\triangle^k-\delta_2)\cdots(\triangle^k-\delta_{n-1})}{(\triangle^k+\delta_1)(\triangle^k+\delta_2)\cdots(\triangle^k+\delta_n)}] \\
= \; & f(T^{k+1}) + \frac{g}{\delta_1}(\alpha_1^k \cdot \frac{\delta_1 - \triangle^k}{\delta_1+\triangle^k} + \alpha_0^k) \\
+ \; & \frac{g}{\delta_2}[(\alpha_1^k \cdot \frac{\delta_1-\triangle^k}{\delta+\triangle^k} + \alpha_0^k)\frac{\triangle^k-\delta_2}{\triangle^k+\delta_2} + \alpha_2^k \cdot \frac{\delta_2-\triangle^k}{\delta_2+\triangle^k} + \alpha_1^k \\
- \; & (\alpha_1^k \cdot \frac{\delta_1-\triangle^k}{\delta_1+\triangle^k} + \alpha_0^k)] \\
+ \; & \frac{g}{\delta_3}[((\alpha_1^k \cdot \frac{\delta_1-\triangle^k}{\delta_1+\triangle^k} + \alpha_0^k)\frac{\triangle^k-\delta_2}{\triangle^k+\delta_2} + \alpha_2^k \cdot \frac{\delta_2-\triangle^k}{\delta_2+\triangle^k} + \alpha_1^k)\frac{\triangle^k-\delta_3}{\triangle^k+\delta_3} \\
+ \; & \alpha_3^k \cdot \frac{\delta_3-\triangle^k}{\delta_3+\triangle^k} + \alpha_2^k - ((\alpha_1^k \cdot \frac{\delta_1-\triangle^k}{\delta_1+\triangle^k} + \alpha_0^k)\frac{\triangle^k-\delta_2}{\triangle^k+\delta_2} + \alpha_2^k \cdot \frac{\delta_2-\triangle^k}{\delta_2+\triangle^k} \\
+ \; & \alpha_1^k)] + \cdots \\
+ \; & \frac{g}{\delta_n}[(\cdots)\frac{\triangle^k-\delta_n}{\triangle^k+\delta_n} + \alpha_n^k \cdot \frac{\delta_n-\triangle^k}{\delta_n+\triangle^k} + \alpha_{n-1}^k \\
- \; & (\cdots + \alpha_{n-1}^k \cdot \frac{\delta_{n-1}-\triangle^k}{\delta_{n-1}+\triangle^k} + \alpha_{n-2}^k)].
\end{aligned} \tag{4.29}$$

From equation (4.28),

$$\alpha_0^0 \;=\; f(T^0) \cdot [\frac{2g}{\triangle^{-1}+\delta_1} + \frac{2g(\triangle^{-1}-\delta_1)}{(\triangle^{-1}+\delta_1)(\triangle^{-1}+\delta_2)}$$

$$+ \quad \cdots + \frac{2g(\Delta^{-1}-\delta_1)\cdots(\Delta^{-1}-\delta_{n-1})}{\delta_n(\Delta^{-1}+\delta_1)\cdots(\Delta^{-1}+\delta_{n-1})}]^{-1}. \tag{4.30}$$

The value of Δ^{-1} turns out to be crucial. We try three different values:

$$\Delta^{-1} = \begin{cases} \max(\delta_i), & 1 \le i \le n, \\ \min(\delta_i), & 1 \le i \le n, \\ 0. \end{cases}$$

With equations (4.28), (4.29), (4.30) and the initial condition, we can solve for α_i^k, $i = 0, \ldots, n$; $k = 0, \ldots, n$. The procedure is

1. From the initial condition $\phi(s)$, $-1 \le s \le 0$, we can find α_i^0, $i = 1, \ldots, n$, then we obtain α_0^0 from (4.30) by specifying Δ^{-1}. i.e., we find α_i^k, $i = 0, \ldots, n$, at $k = 0$ first,

2. Apply (4.29) to find α_0^k, $k = 1$,

3. Use (4.28) to find α_i^k, $i = 1, \ldots, n$,

4. Apply (4.29) to find α_0^{k+1},

5. Apply (4.28) to find α_i^{k+1}, $i = 1, \ldots, n$,

6. Set $k = k + 1$. If $k = n + 1$ then stop, otherwise go to 4.

To find $x(\tilde{t})$, $0 < \tilde{t} \le 1$, by (4.6) and (4.9), we have

$$\begin{aligned} x(\sum_{j=1}^{i} \delta_{n-j+1}) &= x(1 - 1 + \sum_{j=1}^{i} \delta_{n-j+1}) \\ &= \xi(1, -1 + \sum_{j=1}^{i} \delta_{n-j+1}) \\ &= \sum_{k=0}^{n} \alpha_k(1) B_k(-1 + \sum_{j=1}^{i} \delta_{n-j+1}) \\ &= \sum_{k=0}^{n} \alpha_k^n B_k(-\sum_{j=i+1}^{n} \delta_{n-j+1}) \\ &= \sum_{k=0}^{n} \alpha_k^n B_k(\tau_{n-i}) \\ &= \alpha_{n-i}^n. \end{aligned}$$

4.4.6 Examples using nonuniform mesh points

One goal for using nonuniform mesh points is to recover the rate of convergence. For the uniform case, we recall section 4.4.2 that the "h" is fixed after we decide the number of mesh points. For nonuniform mesh points, there is no such fixed "h". Although we do not have fixed "h", we still know the total number of mesh points. There are two ways to evaluate the rate of convergence for nonuniform cases. The following is the definition for the rate of convergence of nonuniform mesh points which we shall employ. Our decision to use this definition was made after consultation with Dr. Hermann Brunner.

Definition 4.2 *Let x be the exact solution and $\bar{x}$ be the computed solution at nonuniform mesh points, i.e., x and $\bar{x}$ are $1 \times n$ vectors. If*

$$\|x-\bar{x}\|_\infty = \max_{i=1,\ldots,n} |x(\tau_i) - \bar{x}(\tau_i)| = C(\frac{1}{n})^\alpha,$$

or

$$\frac{\|x-\bar{x}\|_1}{n} = \frac{\sum_{i=1}^{n} |x(\tau_i) - \bar{x}(\tau_i)|}{n} = C(\frac{1}{n})^\alpha.$$

Then the rate of convergence for the nonuniform mesh scheme is α.

For examples, we employ the nonuniform mesh points to (4.1)–(4.2) with $\phi = 0$, $f(t) = at^{a-1}$, therefore $F(t) = D\phi + \int_0^t a\tau^{a-1}d\tau = t^a$. Examples include $0 < a < 1$, $a = 1$, $F(t) = t^2/2!$, $F(t) = t^3/3!$ and $F(t) = t^4/4!$. Singularity p is 0.5 for all cases. By Theorem 3.2, we have the exact solution $x(t)$ for $t \in (0, 3]$.

Example 4.12

For the case $F(t) = t^a$, $0 < a < 1$,

$$\begin{aligned} \frac{d}{dt}\left(\int_{-1}^{0} |s|^{-0.5} x(t+s)ds\right) &= at^{a-1}, \quad t > 0, \\ x(s) &= \phi(s) = 0, \quad -1 \le s \le 0. \end{aligned}$$

By Lemma 3.3, the exact solution is

$$x(t) = \frac{\Gamma(a+1)}{\Gamma(1-p)\Gamma(p+a)} t^{p+a-1}, \quad t \in (0,1].$$

See the numerical results for $a = 1/2$ in Appendix H. For $\triangle^{-1} = \max(\delta_i)$, we do not have good results. As a matter of fact, they are worse than the uniform mesh case. For $\triangle^{-1} = \min(\delta_i)$ and 0, the rate of convergence is about a.

Example 4.13

We try $a = 1$, $\triangle^{-1} = \max(\delta_i)$, $\min(\delta_i)$ and 0.

$$\begin{aligned} \frac{d}{dt}\left(\int_{-1}^{0} |s|^{-0.5} x(t+s)ds\right) &= 1, \quad t > 0, \\ x(s) &= \phi(s) = 0, \quad -1 \le s \le 0. \end{aligned}$$

See the results in Appendix I. For $\triangle^{-1} = \min(\delta_i)$, we recover the rate of convergence for $t \in (1, 3]$. For $\triangle^{-1} = \max(\delta_i)$ and 0, we do not recover the rate of convergence.

Example 4.14

We try the following example with $\triangle^{-1} = \max(\delta_i)$, $\min(\delta_i)$ and 0.

$$\begin{aligned} \frac{d}{dt}\left(\int_{-1}^{0} |s|^{-0.5} x(t+s)ds\right) &= \frac{d}{dt}(\frac{t^2}{2}), \quad t > 0, \\ x(s) &= \phi(s) = 0, \quad -1 \le s \le 0. \end{aligned}$$

Since $f(t) = t$, then $f(T^0) = 0$, this implies $\alpha_0^0 = 0$. See the results in Appendix J. We recover the rate of convergence.

Example 4.15

We have $\alpha_0^0 = 0$ for the following equation

$$\begin{aligned} \frac{d}{dt}\left(\int_{-1}^{0} |s|^{-0.5} x(t+s)ds\right) &= \frac{d}{dt}(\frac{t^3}{3!}), \quad t > 0, \\ x(s) &= \phi(s) = 0, \quad -1 \le s \le 0. \end{aligned}$$

See the results in Appendix K. We recover the rate of convergence.

Example 4.16

We have $\alpha_0^0 = 0$ in the following problem

$$\begin{aligned} \frac{d}{dt}\left(\int_{-1}^{0} |s|^{-0.5} x(t+s)ds\right) &= \frac{d}{dt}(\frac{t^4}{4!}), \quad t > 0, \\ x(s) &= \phi(s) = 0, \quad -1 \le s \le 0. \end{aligned}$$

See the results in Appendix L. We recover the rate of convergence.

4.5 Summary

In this chapter, we solved (4.1)–(4.2) by four different numerical schemes: semi-discretization with uniform and nonuniform mesh points, full discretization with uniform and nonuniform mesh points. For the semi-discretization method, we changed the equations to a first order differential equation and used an ode solver (Matlab ode23). For full discretization with uniform mesh points, we observed the rate of convergence was not the same order as the numerical scheme. By using nonuniform mesh points, with $F(t)$ smooth enough, we regain the rate of convergence.

Chapter 5

Conclusions and Future Work

In this study, we discussed the singular integrodifferential equations

$$\frac{d}{dt}\int_{-b}^{0}|s|^{-p}x(t+s)ds = f(t), \quad t>0, \tag{5.1}$$

$$x(s) = \phi(s), \quad -b\leq s\leq 0, \tag{5.2}$$

where $f(t)$ is locally integrable on $t\geq 0$, initial function $\phi\in L_{2,g}$ and $b=1$. We introduced the semigroup setting on the state space $L_{2,g}$. For the associated abstract Cauchy problem, we characterized the properties of the infinitesimal generator and the generated semigroup. We showed that (5.1)–(5.2) is well-posed. Assuming $x_t(s)=x(t+s)$ and the difference operator D defined as

$$D\zeta = \int_{-1}^{0}|s|^{-p}\zeta(s)ds,$$

$$p\in(0,1),$$

equations (5.1)–(5.2) can be written as

$$\begin{aligned} Dx_t &= D\phi+\int_0^t f(\tau)d\tau & (5.3)\\ &\equiv F(t), \quad t>0, & (5.4)\\ x_0 &= \phi, & (5.5)\end{aligned}$$

provided Dx_t is absolutely continuous. For the equations (5.4)–(5.5), if $\phi=0$ and $F(t)=t^a$, $a>0$, we had the exact solution $x(t)$, $t\in(0,1]$. Since D is a linear operator, we extended the solution (with $\phi=0$) to the case where $F(t)$ is a linear combination of t^a, $a>0$. For the special case $p=0.5$, we found the exact solution $x(t)$ for $t\in(0,3]$. For $\phi\in C$, $F(t)\in C^1(0,1)$, the solution has a closed form. Once we solved the solution $x(t)$, $t \in (0,1]$, to (5.4)–(5.5), the method of steps could be applied to find $x(t)$, $t \in (1,\infty)$.

In Chapter 4, we introduced numerical methods. By choosing a semi-discretization method, we adopted an ode solver. Our full discretization code could approximate the solution $x(t)$ of (5.4)–(5.5) to $t \in (0,\infty)$, and by checking several examples using uniform mesh points, we determined the numerical rates of convergence using uniform mesh points. The rates depend on the singularity of the kernel, p, the smoothness of $F(t)$ and the initial condition, ϕ.

When $\phi = 0$ and $F(t)$ is a linear combination of t^a, $a > 0$, the rate of convergence is $Min(2-p, a)$, where a is the smallest power in the linear combination. Therefore, if $F(t)$ is smooth enough, the rate of convergence is $2-p$ and we have lost second order convergence. We were able to recover the optimal rate of convergence by using nonuniform mesh points. We emphasized this observation by noting the conjecture of [4] using the collocation method.

In the future, we would like to try more cases with $\phi \neq 0$ and $F(t)$ in more general form and complete the proof of the rate of convergence for both uniform and nonuniform meshes.

Appendix A

$\phi = 0,\ F(t) = t$

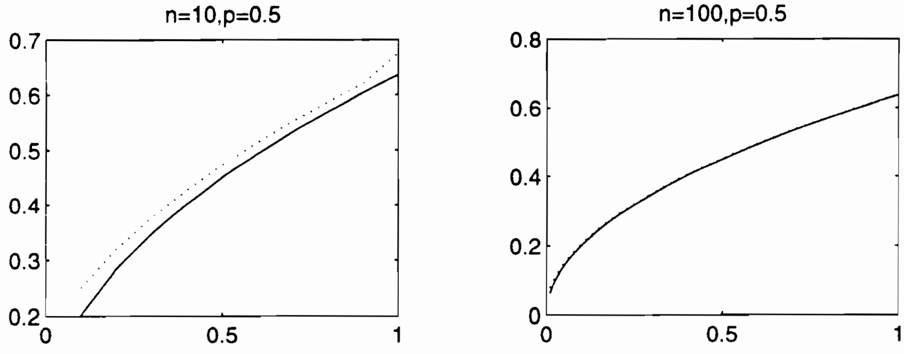

Figure A.1: $p = 0.5$, solid line: exact solution

Table A.1: rate of convergence for $p = 0.5$

t	n=10,100	n=100,1000
0.10	0.9920	1.0001
0.20	0.9992	1.0000
0.30	1.0001	1.0000
0.40	1.0002	1.0000
0.50	1.0002	1.0000
0.60	1.0001	1.0000
0.70	1.0001	1.0000
0.80	1.0001	1.0000
0.90	1.0001	1.0000
1.00	0.9943	0.9994

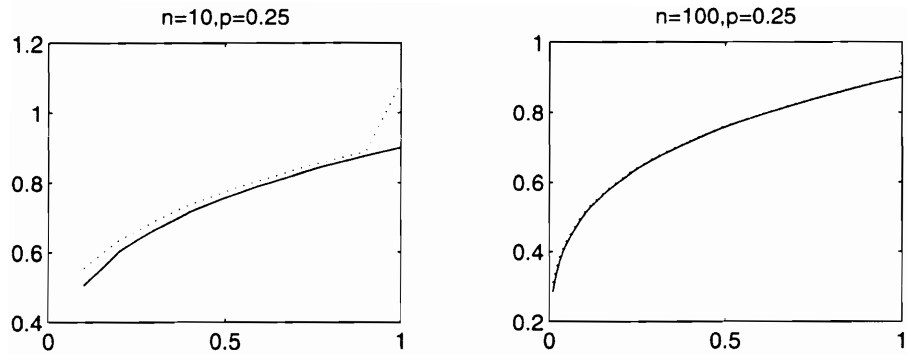

Figure A.2: $p = 0.25$, solid line: exact solution

Table A.2: rate of convergence for $p = 0.25$

t	n=10	n=100	exact sol.	r.o.c (n=10,100)
0.10	0.5560	0.5123	0.5063	0.9175
0.20	0.6341	0.6057	0.6021	0.9439
0.30	0.6909	0.6690	0.6663	0.9567
0.40	0.7362	0.7182	0.7160	0.9644
0.50	0.7744	0.7589	0.7571	0.9695
0.60	0.8077	0.7940	0.7924	0.9732
0.70	0.8373	0.8250	0.8235	0.9760
0.80	0.8640	0.8528	0.8515	0.9782
0.90	0.8884	0.8781	0.8769	0.9800
1.00	1.0867	0.9576	0.9003	0.5124
(sec)	CPU=0.0227	CPU=0.4171		

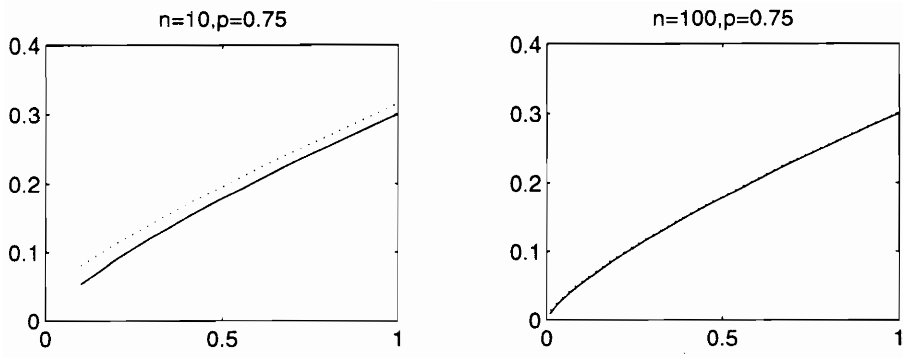

Figure A.3: $p = 0.75$, solid line: exact solution

Table A.3: rate of convergence for $p = 0.75$

t	n=10	n=100	exact sol.	r.o.c. (n=10,100)
0.10	0.0805	0.0559	0.0534	1.0292
0.20	0.1125	0.0918	0.0898	1.0415
0.30	0.1420	0.1235	0.1217	1.0440
0.40	0.1696	0.1523	0.1509	1.0443
0.50	0.1960	0.1800	0.1784	1.0439
0.60	0.2212	0.2061	0.2046	1.0433
0.70	0.2455	0.2311	0.2297	1.0427
0.80	0.2691	0.2552	0.2539	1.0420
0.90	0.2920	0.2786	0.2773	1.0414
1.00	0.3163	0.3015	0.3001	1.0749
(sec)	CPU=0.0227	CPU=0.4163		

Appendix B

$\phi = 0,\ 0 < a < 1$

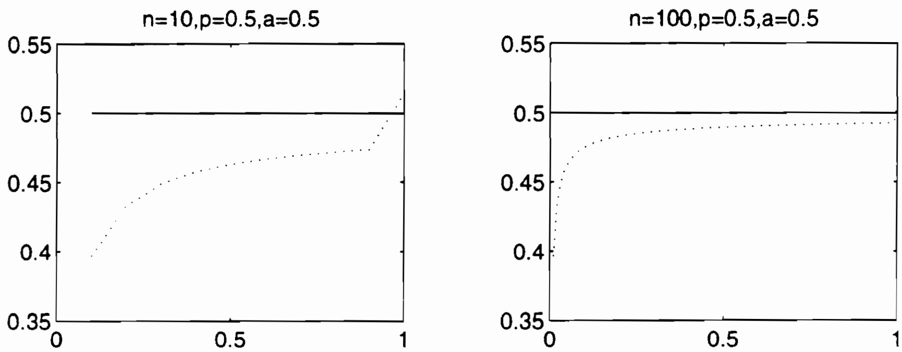

Figure B.1: $p = 0.5$, $a = 0.5$, solid line: exact solution

Table B.1: rate of convergence for $p = 0.5$, $a = 0.5$

t	n=10,100	n=100,1000
0.10	0.6218	0.5255
0.20	0.5965	0.5130
0.30	0.5739	0.5087
0.40	0.5588	0.5066
0.50	0.5485	0.5053
0.60	0.5412	0.5044
0.70	0.5357	0.5038
0.80	0.5316	0.5033
0.90	0.5282	0.5029
1.00	0.4345	0.4938

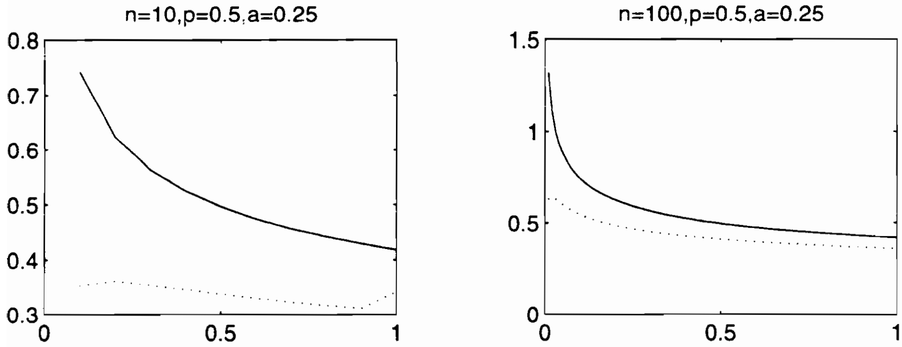

Figure B.2: $p = 0.5$, $a = 0.25$, solid line: exact solution

Table B.2: rate of convergence for $p = 0.5$, $a = 0.25$

t	n=10,100	n=100,1000
0.10	0.2960	0.2557
0.20	0.2808	0.2526
0.30	0.2713	0.2516
0.40	0.2659	0.2512
0.50	0.2625	0.2509
0.60	0.2602	0.2507
0.70	0.2586	0.2506
0.80	0.2574	0.2505
0.90	0.2564	0.2505
1.00	0.2626	0.2511

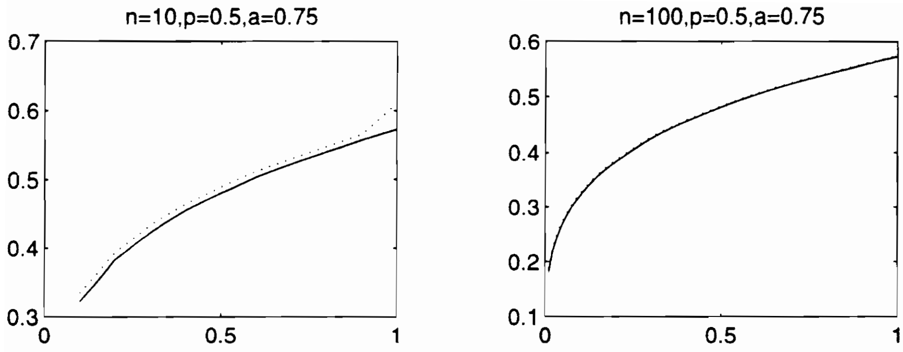

Figure B.3: $p = 0.5$, $a = 0.75$, solid line: exact solution

Table B.3: rate of convergence for $p = 0.5$, $a = 0.75$

t	n=10,100	n=100,1000
0.10	0.5125	0.7011
0.20	0.5743	0.7220
0.30	0.6186	0.7294
0.40	0.6455	0.7333
0.50	0.6631	0.7356
0.60	0.6753	0.7372
0.70	0.6843	0.7383
0.80	0.6912	0.7391
0.90	0.6967	0.7398
1.00	0.7334	0.7474

Appendix C

$\phi = 0,\ 1 < a < 2$

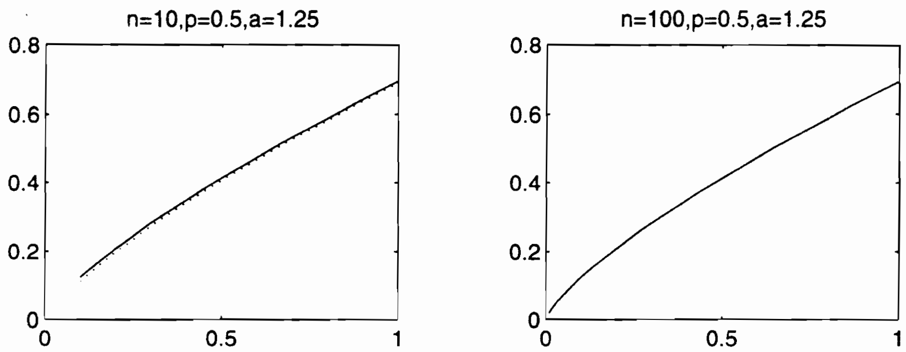

Figure C.1: $p = 0.5$, $a = 1.25$, solid line: exact solution

Table C.1: rate of convergence for $p = 0.5$, $a = 1.25$

t	n=10,100	t	n=10,100	t	n=10,100
0.10	1.0862	1.10	1.0677	2.10	1.1301
0.20	1.1513	1.20	1.0928	2.20	1.1161
0.30	1.1730	1.30	1.1114	2.30	1.1119
0.40	1.1837	1.40	1.1244	2.40	1.1115
0.50	1.1903	1.50	1.1340	2.50	1.1128
0.60	1.1948	1.60	1.1414	2.60	1.1149
0.70	1.1981	1.70	1.1474	2.70	1.1173
0.80	1.2007	1.80	1.1523	2.80	1.1199
0.90	1.2028	1.90	1.1565	2.90	1.1225
1.00	1.2046	2.00	1.1601	3.00	1.1251

Table C.2: rate of convergence for $p = 0.5$, $a = 1.25$

t	n=100,1000	t	n=100,1000	t	n=100,1000
0.10	1.2046	1.10	1.1626	2.10	1.1817
0.20	1.2141	1.20	1.1785	2.20	1.1802
0.30	1.2184	1.30	1.1865	2.30	1.1808
0.40	1.2211	1.40	1.1917	2.40	1.1819
0.50	1.2229	1.50	1.1955	2.50	1.1832
0.60	1.2243	1.60	1.1985	2.60	1.1845
0.70	1.2254	1.70	1.2009	2.70	1.1859
0.80	1.2263	1.80	1.2030	2.80	1.1872
0.90	1.2271	1.90	1.2047	2.90	1.1884
1.00	1.2277	2.00	1.2062	3.00	1.1896

Table C.3: rate of convergence for $p = 0.5$, $a = 1.25$

t	n=1000,10000	t	n=1000,10000	t	n=1000,10000
0.10	1.2277	1.10	1.2065	2.10	1.2141
0.20	1.2313	1.20	1.2138	2.20	1.2139
0.30	1.2328	1.30	1.2175	2.30	1.2143
0.40	1.2336	1.40	1.2200	2.40	1.2150
0.50	1.2340	1.50	1.2218	2.50	1.2156
0.60	1.2342	1.60	1.2233	2.60	1.2163
0.70	1.2343	1.70	1.2244	2.70	1.2170
0.80	1.2342	1.80	1.2254	2.80	1.2176
0.90	1.2340	1.90	1.2262	2.90	1.2182
1.00	1.2338	2.00	1.2270	3.00	1.2188

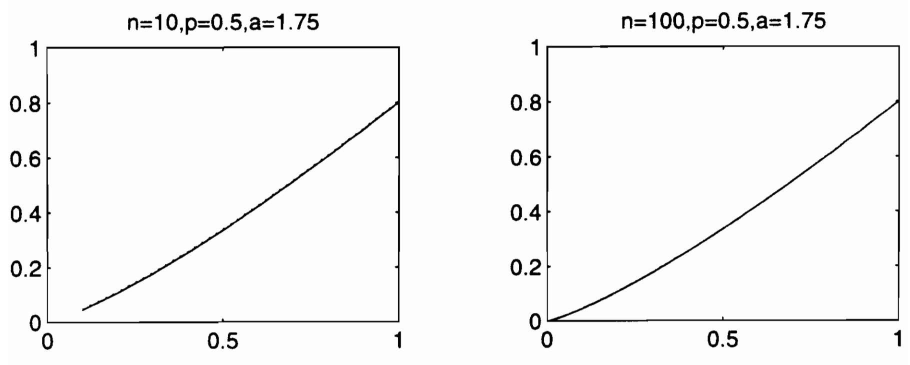

Figure C.2: $p = 0.5$, $a = 1.75$, solid line: exact solution

Table C.4: rate of convergence for $p = 0.5$, $a = 1.75$

t	n=10,100	t	n=10,100	t	n=10,100
0.10	1.2720	1.10	1.3697	2.10	1.4278
0.20	1.3475	1.20	1.3916	2.20	1.4282
0.30	1.3768	1.30	1.4047	2.30	1.4302
0.40	1.3928	1.40	1.4133	2.40	1.4325
0.50	1.4031	1.50	1.4196	2.50	1.4347
0.60	1.4105	1.60	1.4245	2.60	1.4368
0.70	1.4161	1.70	1.4283	2.70	1.4387
0.80	1.4205	1.80	1.4315	2.80	1.4404
0.90	1.4241	1.90	1.4342	2.90	1.4419
1.00	1.4272	2.00	1.4366	3.00	1.4434

Table C.5: rate of convergence for $p = 0.5$, $a = 1.75$

t	n=100,1000	t	n=100,1000	t	n=100,1000
0.10	1.4272	1.10	1.4519	2.10	1.4701
0.20	1.4436	1.20	1.4594	2.20	1.4710
0.30	1.4509	1.30	1.4633	2.30	1.4721
0.40	1.4553	1.40	1.4659	2.40	1.4730
0.50	1.4583	1.50	1.4678	2.50	1.4739
0.60	1.4605	1.60	1.4694	2.60	1.4746
0.70	1.4622	1.70	1.4706	2.70	1.4753
0.80	1.4636	1.80	1.4716	2.80	1.4759
0.90	1.4647	1.90	1.4725	2.90	1.4765
1.00	1.4657	2.00	1.4733	3.00	1.4770

Table C.6: rate of convergence for $p = 0.5$, $a = 1.75$

t	n=1000,10000	t	n=1000,10000	t	n=1000,10000
0.10	1.4657	1.10	1.4789	2.10	1.4863
0.20	1.4697	1.20	1.4815	2.20	1.4867
0.30	1.4697	1.30	1.4830	2.30	1.4871
0.40	1.4681	1.40	1.4839	2.40	1.4874
0.50	1.4656	1.50	1.4846	2.50	1.4877
0.60	1.4623	1.60	1.4852	2.60	1.4880
0.70	1.4586	1.70	1.4856	2.70	1.4882
0.80	1.4544	1.80	1.4859	2.80	1.4884
0.90	1.4499	1.90	1.4862	2.90	1.4885
1.00	1.4450	2.00	1.4864	3.00	1.4887

Appendix D

$\phi = 0,\ 2 < a < 3$

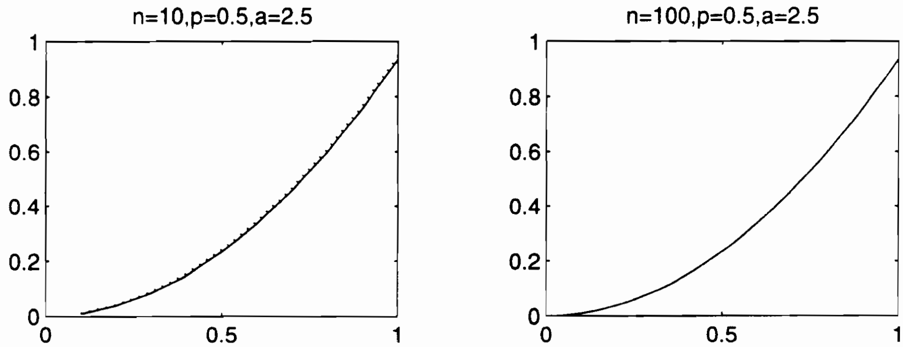

Figure D.1: $p = 0.5$, $a = 2.5$, solid line: exact solution

Table D.1: rate of convergence for $p = 0.5$, $a = 2.5$

t	n=10,100	t	n=10,100	t	n=10,100
0.10	1.3494	1.10	1.4582	2.10	1.4675
0.20	1.4019	1.20	1.4574	2.20	1.4675
0.30	1.4243	1.30	1.4583	2.30	1.4677
0.40	1.4369	1.40	1.4596	2.40	1.4679
0.50	1.4452	1.50	1.4610	2.50	1.4682
0.60	1.4510	1.60	1.4624	2.60	1.4686
0.70	1.4554	1.70	1.4637	2.70	1.4690
0.80	1.4589	1.80	1.4649	2.80	1.4694
0.90	1.4617	1.90	1.4660	2.90	1.4698
1.00	1.4640	2.00	1.4670	3.00	1.4702

Table D.2: rate of convergence for $p = 0.5$, $a = 2.5$

t	n=100,1000	t	n=100,1000	t	n=100,1000
0.10	1.4640	1.10	1.4876	2.10	1.4905
0.20	1.4759	1.20	1.4878	2.20	1.4905
0.30	1.4809	1.30	1.4882	2.30	1.4906
0.40	1.4837	1.40	1.4886	2.40	1.4906
0.50	1.4855	1.50	1.4890	2.50	1.4907
0.60	1.4869	1.60	1.4894	2.60	1.4908
0.70	1.4879	1.70	1.4897	2.70	1.4910
0.80	1.4888	1.80	1.4900	2.80	1.4911
0.90	1.4895	1.90	1.4903	2.90	1.4912
1.00	1.4900	2.00	1.4906	3.00	1.4913

Table D.3: rate of convergence for $p = 0.5$, $a = 2.5$

t	n=1000,10000	t	n=1000,10000	t	n=1000,10000
0.10	1.4900	1.10	1.4963	2.10	1.4971
0.20	1.4931	1.20	1.4963	2.20	1.4971
0.30	1.4944	1.30	1.4964	2.30	1.4971
0.40	1.4952	1.40	1.4965	2.40	1.4971
0.50	1.4957	1.50	1.4966	2.50	1.4972
0.60	1.4961	1.60	1.4967	2.60	1.4972
0.70	1.4964	1.70	1.4968	2.70	1.4972
0.80	1.4966	1.80	1.4968	2.80	1.4972
0.90	1.4968	1.90	1.4969	2.90	1.4972
1.00	1.4970	2.00	1.4969	3.00	1.4972

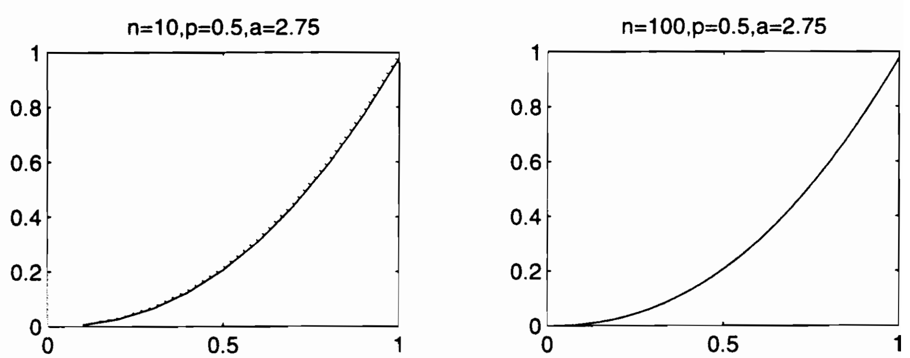

Figure D.2: $p = 0.5$, $a = 2.75$, solid line: exact solution

Table D.4: rate of convergence for $p = 0.5$, $a = 2.75$

t	n=10,100	t	n=10,100	t	n=10,100
0.10	1.3501	1.10	1.4586	2.10	1.4661
0.20	1.3974	1.20	1.4580	2.20	1.4665
0.30	1.4190	1.30	1.4585	2.30	1.4668
0.40	1.4316	1.40	1.4593	2.40	1.4671
0.50	1.4401	1.50	1.4603	2.50	1.4674
0.60	1.4462	1.60	1.4614	2.60	1.4678
0.70	1.4508	1.70	1.4625	2.70	1.4682
0.80	1.4545	1.80	1.4635	2.80	1.4685
0.90	1.4575	1.90	1.4645	2.90	1.4689
1.00	1.4600	2.00	1.4654	3.00	1.4692

Table D.5: rate of convergence for $p = 0.5$, $a = 2.75$

t	n=100,1000	t	n=100,1000	t	n=100,1000
0.10	1.4600	1.10	1.4877	2.10	1.4900
0.20	1.4729	1.20	1.4877	2.20	1.4901
0.30	1.4783	1.30	1.4879	2.30	1.4902
0.40	1.4814	1.40	1.4882	2.40	1.4903
0.50	1.4835	1.50	1.4885	2.50	1.4904
0.60	1.4850	1.60	1.4888	2.60	1.4905
0.70	1.4861	1.70	1.4891	2.70	1.4906
0.80	1.4870	1.80	1.4894	2.80	1.4907
0.90	1.4877	1.90	1.4897	2.90	1.4908
1.00	1.4883	2.00	1.4899	3.00	1.4909

Table D.6: rate of convergence for $p = 0.5$, $a = 2.75$

t	n=1000,10000	t	n=1000,10000	t	n=1000,10000
0.10	1.4883	1.10	1.4963	2.10	1.4970
0.20	1.4910	1.20	1.4963	2.20	1.4970
0.30	1.4915	1.30	1.4963	2.30	1.4970
0.40	1.4912	1.40	1.4964	2.40	1.4970
0.50	1.4906	1.50	1.4965	2.50	1.4971
0.60	1.4896	1.60	1.4966	2.60	1.4971
0.70	1.4884	1.70	1.4966	2.70	1.4971
0.80	1.4870	1.80	1.4967	2.80	1.4971
0.90	1.4855	1.90	1.4968	2.90	1.4971
1.00	1.4839	2.00	1.4968	3.00	1.4972

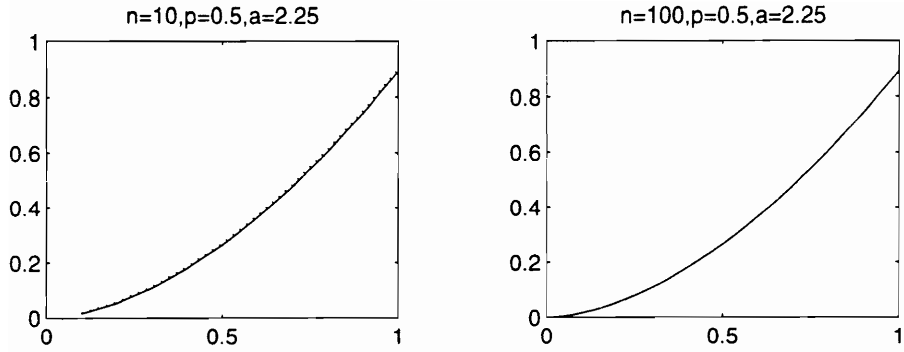

Figure D.3: $p = 0.5$, $a = 2.25$, solid line: exact solution

Table D.7: rate of convergence for $p = 0.5$, $a = 2.25$

t	n=10,100	t	n=10,100	t	n=10,100
0.10	1.3400	1.10	1.4501	2.10	1.4662
0.20	1.4001	1.20	1.4511	2.20	1.4658
0.30	1.4244	1.30	1.4536	2.30	1.4659
0.40	1.4377	1.40	1.4562	2.40	1.4662
0.50	1.4462	1.50	1.4585	2.50	1.4666
0.60	1.4522	1.60	1.4606	2.60	1.4672
0.70	1.4567	1.70	1.4624	2.70	1.4677
0.80	1.4602	1.80	1.4640	2.80	1.4683
0.90	1.4631	1.90	1.4654	2.90	1.4688
1.00	1.4654	2.00	1.4666	3.00	1.4694

Table D.8: rate of convergence for $p = 0.5$, $a = 2.25$

t	n=100,1000	t	n=100,1000	t	n=100,1000
0.10	1.4654	1.10	1.4858	2.10	1.4903
0.20	1.4772	1.20	1.4867	2.20	1.4903
0.30	1.4821	1.30	1.4875	2.30	1.4903
0.40	1.4849	1.40	1.4882	2.40	1.4905
0.50	1.4867	1.50	1.4888	2.50	1.4906
0.60	1.4881	1.60	1.4893	2.60	1.4907
0.70	1.4892	1.70	1.4898	2.70	1.4909
0.80	1.4901	1.80	1.4901	2.80	1.4910
0.90	1.4908	1.90	1.4905	2.90	1.4912
1.00	1.4915	2.00	1.4908	3.00	1.4913

Table D.9: rate of convergence for $p = 0.5$, $a = 2.25$

t	n=1000,10000	t	n=1000,10000	t	n=1000,10000
0.10	1.4915	1.10	1.4959	2.10	1.4971
0.20	1.4955	1.20	1.4961	2.20	1.4971
0.30	1.4983	1.30	1.4962	2.30	1.4971
0.40	1.5008	1.40	1.4963	2.40	1.4970
0.50	1.5033	1.50	1.4963	2.50	1.4970
0.60	1.5060	1.60	1.4963	2.60	1.4969
0.70	1.5087	1.70	1.4962	2.70	1.4969
0.80	1.5116	1.80	1.4962	2.80	1.4968
0.90	1.5147	1.90	1.4960	2.90	1.4967
1.00	1.5179	2.00	1.4959	3.00	1.4966

Appendix E

$\phi = 0,\ a = 2$

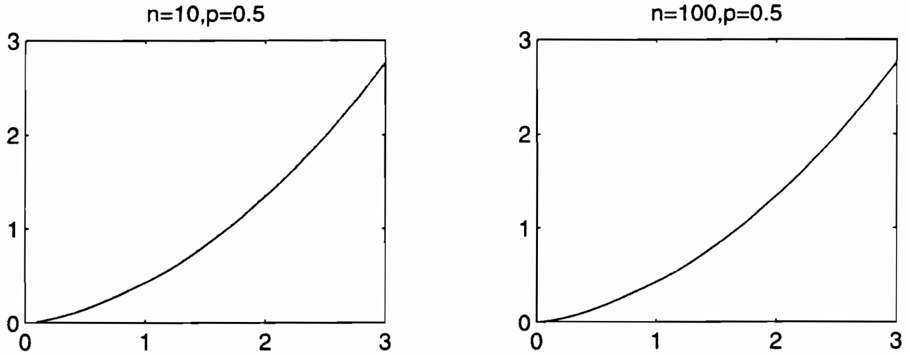

Figure E.1: $p = 0.5$, $F(t) = t^2/2$, solid line: exact solution

Table E.1: rate of convergence for $p = 0.5$, $F(t) = t^2/2$

t	n=10,100	t	n=10,100	t	n=10,100
0.10	1.3166	1.10	1.4249	2.10	1.4566
0.20	1.3854	1.20	1.4321	2.20	1.4560
0.30	1.4125	1.30	1.4386	2.30	1.4564
0.40	1.4272	1.40	1.4436	2.40	1.4572
0.50	1.4366	1.50	1.4476	2.50	1.4581
0.60	1.4433	1.60	1.4508	2.60	1.4592
0.70	1.4482	1.70	1.4535	2.70	1.4602
0.80	1.4521	1.80	1.4558	2.80	1.4611
0.90	1.4553	1.90	1.4576	2.90	1.4620
1.00	1.4579	2.00	1.4595	3.00	1.4629

Table E.2: rate of convergence for $p = 0.5$, $F(t) = t^2/2$

t	n=100,1000	t	n=100,1000	t	n=100,1000
0.10	1.4579	1.10	1.4777	2.10	1.4867
0.20	1.4714	1.20	1.4808	2.20	1.4868
0.30	1.4770	1.30	1.4827	2.30	1.4871
0.40	1.4802	1.40	1.4841	2.40	1.4874
0.50	1.4824	1.50	1.4851	2.50	1.4877
0.60	1.4840	1.60	1.4859	2.60	1.4880
0.70	1.4853	1.70	1.4866	2.70	1.4882
0.80	1.4862	1.80	1.4872	2.80	1.4885
0.90	1.4871	1.90	1.4878	2.90	1.4887
1.00	1.4877	2.00	1.4882	3.00	1.4889

Appendix F

$\phi = 0,\ F(t) = $ a combination of t^a

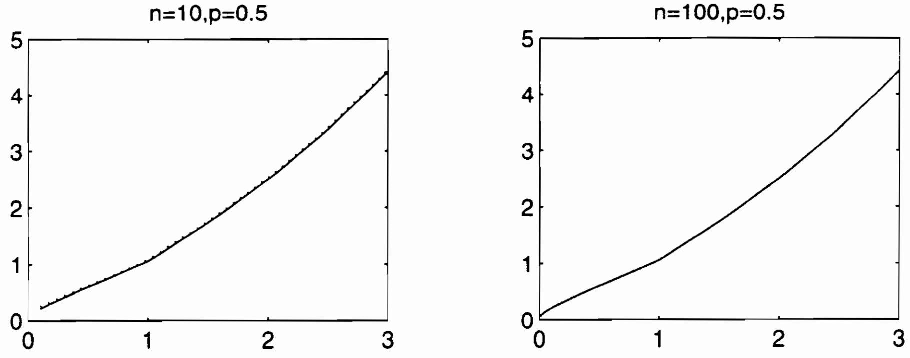

Figure F.1: $p = 0.5$, $F(t) = t + t^2/2!$, solid line: exact solution

Table F.1: rate of convergence for $p = 0.5$, $F(t) = t + t^2/2!$

t	n=10,100	t	n=10,100	t	n=10,100
0.10	1.0027	1.10	1.0712	2.10	1.1011
0.20	1.0191	1.20	1.0727	2.20	1.1011
0.30	1.0271	1.30	1.0736	2.30	1.1019
0.40	1.0330	1.40	1.0750	2.40	1.1030
0.50	1.0380	1.50	1.0767	2.50	1.1043
0.60	1.0424	1.60	1.0786	2.60	1.1058
0.70	1.0463	1.70	1.0806	2.70	1.1073
0.80	1.0499	1.80	1.0827	2.80	1.1089
0.90	1.0532	1.90	1.0849	2.90	1.1106
1.00	1.0186	2.00	1.1159	3.00	1.1181

Table F.2: rate of convergence for $p = 0.5$,$F(t) = t + t^2/2!$

t	n=100,1000	t	n=100,1000	t	n=100,1000
0.10	1.0064	1.10	1.0360	2.10	1.0440
0.20	1.0095	1.20	1.0323	2.20	1.0435
0.30	1.0119	1.30	1.0314	2.30	1.0436
0.40	1.0139	1.40	1.0314	2.40	1.0438
0.50	1.0156	1.50	1.0318	2.50	1.0443
0.60	1.0171	1.60	1.0324	2.60	1.0448
0.70	1.0185	1.70	1.0331	2.70	1.0455
0.80	1.0199	1.80	1.0339	2.80	1.0461
0.90	1.0211	1.90	1.0347	2.90	1.0468
1.00	1.0083	2.00	1.0490	3.00	1.0484

Table F.3: rate of convergence for $p = 0.5$, $F(t) = \sin(t)$

t	n=10,100	t	n=10,100	t	n=10,100
0.10	0.9908	1.10	1.0147	2.10	0.9743
0.20	0.9949	1.20	1.0036	2.20	0.9711
0.30	0.9913	1.30	0.9925	2.30	0.9673
0.40	0.9858	1.40	0.9822	2.40	0.9635
0.50	0.9792	1.50	0.9725	2.50	0.9600
0.60	0.9718	1.60	0.9633	2.60	0.9570
0.70	0.9636	1.70	0.9545	2.70	0.9544
0.80	0.9753	1.80	0.9461	2.80	0.9524
0.90	0.9458	1.90	0.9383	2.90	0.9510
1.00	0.9712	2.00	0.9891	3.00	0.9612

Table F.4: rate of convergence for $p = 0.5$, $F(t) = \sin(t)$

t	n=100,1000	t	n=100,1000	t	n=100,1000
0.10	0.9994	1.10	1.0113	2.10	0.9931
0.20	0.9981	1.20	1.0028	2.20	0.9916
0.30	0.9964	1.30	0.9977	2.30	0.9901
0.40	0.9944	1.40	0.9938	2.40	0.9888
0.50	0.9921	1.50	0.9905	2.50	0.9876
0.60	0.9897	1.60	0.9876	2.60	0.9867
0.70	0.9871	1.70	0.9849	2.70	0.9859
0.80	1.2577	1.80	0.9825	2.80	0.9854
0.90	0.9819	1.90	0.9803	2.90	0.9850
1.00	0.9916	2.00	0.9957	3.00	0.9859

Table F.5: rate of convergence for $p = 0.5$, $F(t) = \sin(t)$

t	n=1000,10000	t	n=1000,10000	t	n=1000,10000
0.10	0.9998	1.10	1.0037	2.10	0.9979
0.20	0.9993	1.20	1.0008	2.20	0.9973
0.30	0.9987	1.30	0.9991	2.30	0.9969
0.40	0.9981	1.40	0.9979	2.40	0.9964
0.50	0.9974	1.50	0.9969	2.50	0.9961
0.60	0.9866	1.60	0.9960	2.60	0.9958
0.70	0.9858	1.70	0.9951	2.70	0.9956
0.80	1.0958	1.80	0.9944	2.80	0.9954
0.90	0.9942	1.90	0.9938	2.90	0.9953
1.00	0.9974	2.00	0.9985	3.00	0.9953

Table F.6: rate of convergence for $p = 0.5$, $F(t) = \cos(t) - 1$

t	n=10,100	t	n=10,100	t	n=10,100
0.10	1.3167	1.10	1.4113	2.10	1.4242
0.20	1.3860	1.20	1.4208	2.20	1.3906
0.30	1.4137	1.30	1.4292	2.30	1.3368
0.40	1.4294	1.40	1.4358	2.40	1.1287
0.50	1.4399	1.50	1.4410	2.50	2.2682
0.60	1.4479	1.60	1.4454	2.60	1.6227
0.70	1.4544	1.70	1.4490	2.70	1.5537
0.80	1.4602	1.80	1.4523	2.80	1.5272
0.90	1.4657	1.90	1.4553	2.90	1.5135
1.00	1.4714	2.00	1.4584	3.00	1.5055

Table F.7: rate of convergence for $p = 0.5$, $F(t) = \cos(t) - 1$

t	n=100,1000	t	n=100,1000	t	n=100,1000
0.10	1.4580	1.10	1.4734	2.10	1.4736
0.20	1.4716	1.20	1.4778	2.20	1.4666
0.30	1.4774	1.30	1.4804	2.30	1.4556
0.40	1.4809	1.40	1.4822	2.40	1.4235
0.50	1.4834	1.50	1.4835	2.50	1.7169
0.60	1.4854	1.60	1.4847	2.60	1.5491
0.70	1.4870	1.70	1.4856	2.70	1.5191
0.80	1.4885	1.80	1.4864	2.80	1.5092
0.90	1.4899	1.90	1.4872	2.90	1.5045
1.00	1.4914	2.00	1.4879	3.00	1.5019

Table F.8: rate of convergence for $p = 0.5$, $F(t) = \cos(t) - 1$

t	n=1000,10000	t	n=1000,10000	t	n=1000,10000
0.10	1.4878	1.10	1.4921	2.10	1.4917
0.20	1.4915	1.20	1.4933	2.20	1.4898
0.30	1.4931	1.30	1.4940	2.30	1.4866
0.40	1.4942	1.40	1.4945	2.40	1.4779
0.50	1.4949	1.50	1.4949	2.50	1.2302
0.60	1.4955	1.60	1.4952	2.60	1.5174
0.70	1.4959	1.70	1.4955	2.70	1.5065
0.80	1.4964	1.80	1.4957	2.80	1.5032
0.90	1.4968	1.90	1.4959	2.90	1.5016
1.00	1.4972	2.00	1.4960	3.00	1.5007

Table F.9: rate of convergence for $p = 0.5$, $F(t) = e^t - 1$

t	n=10,100	t	n=10,100	t	n=10,100
0.10	1.0039	1.10	1.1108	2.10	1.2285
0.20	1.0232	1.20	1.1207	2.20	1.2374
0.30	1.0354	1.30	1.1308	2.30	1.2477
0.40	1.0468	1.40	1.1420	2.40	1.2587
0.50	1.0582	1.50	1.1540	2.50	1.2699
0.60	1.0700	1.60	1.1667	2.60	1.2812
0.70	1.0822	1.70	1.1800	2.70	1.2924
0.80	1.0949	1.80	1.1936	2.80	1.3035
0.90	1.1082	1.90	1.2074	2.90	1.3143
1.00	1.0529	2.00	1.2333	3.00	1.3262

Table F.10: rate of convergence for $p = 0.5$, $F(t) = e^t - 1$

t	n=100,1000	t	n=100,1000	t	n=100,1000
0.10	1.0071	1.10	1.0558	2.10	1.1266
0.20	1.0116	1.20	1.0569	2.20	1.1335
0.30	1.0159	1.30	1.0612	2.30	1.1420
0.40	1.0203	1.40	1.0671	2.40	1.1516
0.50	1.0250	1.50	1.0739	2.50	1.1620
0.60	1.0301	1.60	1.0816	2.60	1.1731
0.70	1.0356	1.70	1.0901	2.70	1.1848
0.80	1.0416	1.80	1.0993	2.80	1.1969
0.90	1.0482	1.90	1.1092	2.90	1.2093
1.00	1.0227	2.00	1.1284	3.00	1.2224

Table F.11: rate of convergence for $p = 0.5$, $F(t) = e^t - 1$

t	n=1000,10000	t	n=1000,10000	t	n=1000,10000
0.10	1.0026	1.10	1.0202	2.10	1.0513
0.20	1.0041	1.20	1.0205	2.20	1.0547
0.30	1.0055	1.30	1.0222	2.30	1.0590
0.40	1.0071	1.40	1.0246	2.40	1.0641
0.50	1.0087	1.50	1.0274	2.50	1.0698
0.60	1.0105	1.60	1.0306	2.60	1.0761
0.70	1.0125	1.70	1.0343	2.70	1.0831
0.80	1.0147	1.80	1.0383	2.80	1.0906
0.90	1.0172	1.90	1.0428	2.90	1.0987
1.00	1.0079	2.00	1.0520	3.00	1.1074

Appendix G

$\phi \neq 0$

Table G.1: rate of convergence for $p = 0.5$, $\phi(s) = |s|^p$, $F(t) = 1$

t	n=10,100	n=100,1000	n=1000,10000
0.10	0.8255	0.8877	0.9138
0.20	0.8368	0.8897	0.9142
0.30	0.8400	0.8905	0.9143
0.40	0.8412	0.8910	0.9156
0.50	0.8414	0.8910	0.9156
0.60	0.8408	0.8908	0.9146
0.70	0.8392	0.8906	0.9161
0.80	0.8359	0.8895	0.9151
0.90	0.8281	0.8870	0.9126
1.00	0.8383	0.8846	0.9089

Table G.2: rate of convergence for $p = 0.25$, $\phi(s) = |s|^p$, $F(t) = 1$

t	n=10,100	n=100,1000
0.10	0.7598	0.8652
0.20	0.7849	0.8712
0.30	0.7954	0.8733
0.40	0.8004	0.8742
0.50	0.8023	0.8746
0.60	0.8016	0.8744
0.70	0.7974	0.8737
0.80	0.7858	0.8716
0.90	0.7462	0.8638
1.00	0.3603	0.3926

Table G.3: rate of convergence for $p = 0.75$, $\phi(s) = |s|^p$, $F(t) = 1$

t	n=10,100	n=100,1000
0.10	0.8937	0.9322
0.20	0.9035	0.9293
0.30	0.9053	0.9277
0.40	0.9053	0.9265
0.50	0.9048	0.9257
0.60	0.9041	0.9249
0.70	0.9031	0.9242
0.80	0.9020	0.9235
0.90	0.9005	0.9226
1.00	0.9321	0.9329

Table G.4: rate of convergence for $p = 0.5$, $\phi(s) = (1 - p/2)|s|^{p/2}$, $F(t) = 1$

t	n=10,100	n=100,1000	n=1000,10000
0.10	0.6568	0.7134	0.7329
0.20	0.6690	0.7158	0.7334
0.30	0.6729	0.7168	0.7335
0.40	0.6746	0.7172	0.7336
0.50	0.6752	0.7174	0.7336
0.60	0.6748	0.7174	0.7336
0.70	0.6732	0.7171	0.7335
0.80	0.6695	0.7164	0.7334
0.90	0.6593	0.7143	0.7330
1.00	0.6771	0.7154	0.7319

Table G.5: rate of convergence for $p = 0.25$, $\phi(s) = (1 - p/2)|s|^{p/2}$, $F(t) = 1$

t	n=10,100	n=100,1000	n=1000,10000
0.10	0.6922	0.7925	0.8259
0.20	0.7165	0.7978	0.8253
0.30	0.7265	0.7995	0.8242
0.40	0.7312	0.8001	0.8231
0.50	0.7330	0.8003	0.8219
0.60	0.7321	0.8001	0.8208
0.70	0.7277	0.7993	0.8198
0.80	0.7151	0.7972	0.8188
0.90	0.6693	0.7894	0.8174
1.00	0.2927	0.3207	0.3371

Table G.6: rate of convergence for $p = 0.75$, $\phi(s) = (1 - p/2)|s|^{p/2}$, $F(t) = 1$

t	n=10,100	n=100,1000
0.10	0.5905	0.6177
0.20	0.5980	0.6171
0.30	0.5997	0.6167
0.40	0.6001	0.6165
0.50	0.6000	0.6163
0.60	0.5996	0.6160
0.70	0.5989	0.6158
0.80	0.5978	0.6155
0.90	0.5956	0.6149
1.00	0.6205	0.6214

Table G.7: rate of convergence for $p = 0.5$, $\phi(s) = (2 - p)|s|$, $F(t) = 1$

t	n=10,100	n=100,1000
0.10	0.9920	1.0003
0.20	0.9992	1.0002
0.30	1.0001	1.0002
0.40	1.0002	1.0006
0.50	1.0002	1.0006
0.60	1.0002	1.0006
0.70	1.0002	1.0006
0.80	1.0002	1.0006
0.90	1.0001	1.0006
1.00	0.9943	0.9996

Table G.8: rate of convergence for $p = 0.25$, $\phi(s) = (2 - p)|s|$, $F(t) = 1$

t	n=10,100	n=100,1000
0.10	0.9175	0.9816
0.20	0.9439	0.9891
0.30	0.9567	0.9920
0.40	0.9644	0.9936
0.50	0.9696	0.9946
0.60	0.9733	0.9954
0.70	0.9761	0.9959
0.80	0.9783	0.9963
0.90	0.9800	0.9966
1.00	0.5124	0.5053

Table G.9: rate of convergence for $p = 0.75$, $\phi(s) = (2-p)|s|$, $F(t) = 1$

t	n=10,100	n=100,1000
0.10	1.0292	1.0408
0.20	1.0415	1.0363
0.30	1.0440	1.0336
0.40	1.0443	1.0317
0.50	1.0440	1.0310
0.60	1.0434	1.0298
0.70	1.0427	1.0289
0.80	1.0421	1.0281
0.90	1.0414	1.0274
1.00	1.0749	1.0402

Appendix H

$0 < a < 1$, nonuniform case

Table H.1: exact solution for $p = 0.5$, $F(t) = t^{1/2}$

t		t		t	
0.19	5.0000e-1	1.19	6.3084e-1	2.19	7.9148e-1
0.36	5.0000e-1	1.36	6.7202e-1	2.36	8.2312e-1
0.51	5.0000e-1	1.51	6.9740e-1	2.51	8.4770e-1
0.64	5.0000e-1	1.64	7.1478e-1	2.64	8.6691e-1
0.75	5.0000e-1	1.75	7.2719e-1	2.75	8.8189e-1
0.84	5.0000e-1	1.84	7.3612e-1	2.84	8.9337e-1
0.91	5.0000e-1	1.91	7.4250e-1	2.91	9.0186e-1
0.96	5.0000e-1	1.96	7.4675e-1	2.96	9.0771e-1
0.99	5.0000e-1	1.99	7.4920e-1	2.99	9.1113e-1
1.00	5.0000e-1	2.00	7.5000e-1	3.00	9.1226e-1

Table H.2: rate of convergence for $\|\cdot\|_\infty$, $\max(\delta_i)$, $F(t) = t^{1/2}$

t	n=10	n=100(10 pts)	rate of convergence
$0 < t \leq 1$	2.6891e-1	9.6073e-2	4.4700e-1
$1 < t \leq 2$	5.2741e-2	1.7384e-1	-5.1801e-1
$2 < t \leq 3$	4.3084e-2	1.5191e-1	-5.4727e-1

t	n=10	n=100(100 pts)	rate of convergence
$0 < t \leq 1$	2.6891e-1	1.7715e+0	-8.1874e-1
$1 < t \leq 2$	5.2741e-2	1.8702e-1	-5.4974e-1
$2 < t \leq 3$	4.3084e-2	1.5191e-1	-5.4727e-1

Table H.3: rate of convergence for $\|\cdot\|_1/n$, $\max(\delta_i)$, $F(t) = t^{1/2}$

t	n=10	n=100(10 pts)	rate of convergence
$0 < t \leq 1$	6.0458e-2	9.4279e-2	-1.9296e-1
$1 < t \leq 2$	4.4510e-2	1.4611e-1	-5.1623e-1
$2 < t \leq 3$	4.2088e-2	1.4836e-1	-5.4716e-1
t	n=10	n=100(100 pts)	rate of convergence
$0 < t \leq 1$	6.0458e-2	1.5013e-1	-3.9503e-1
$1 < t \leq 2$	4.4510e-2	1.4844e-1	-5.2309e-1
$2 < t \leq 3$	4.2088e-2	1.4822e-1	-5.4675e-1

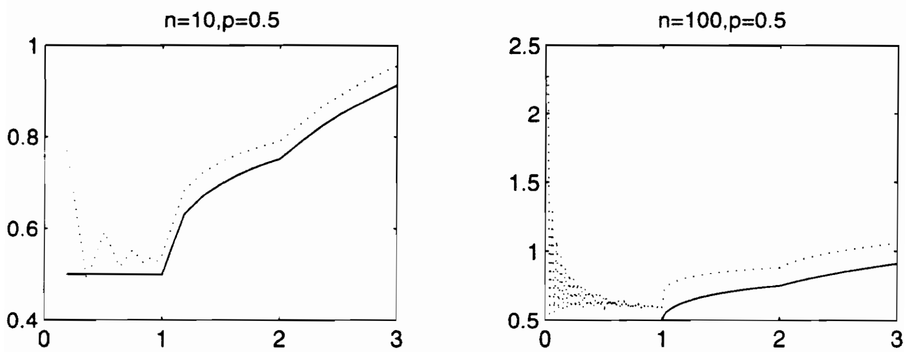

Figure H.1: $\max(\delta_i)$, $F(t) = t^{1/2}$, solid line: exact solution

Table H.4: rate of convergence for $\|\cdot\|_\infty$, $\min(\delta_i)$, $F(t) = t^{1/2}$

t	n=10	n=100(10 pts)	rate of convergence
$0 < t \leq 1$	1.3705e-1	6.1015e-2	3.5143e-1
$1 < t \leq 2$	1.2250e-1	3.9511e-2	4.9143e-1
$2 < t \leq 3$	1.0559e-1	3.4700e-2	4.8330e-1
t	n=10	n=100(100 pts)	rate of convergence
$0 < t \leq 1$	1.3705e-1	1.4438e-1	-2.2631e-2
$1 < t \leq 2$	1.2250e-1	4.3519e-2	4.4947e-1
$2 < t \leq 3$	1.0559e-1	3.4703e-2	4.8326e-1

Table H.5: rate of convergence for $\|\cdot\|_1/n$, $\min(\delta_i)$, $F(t) = t^{1/2}$

t	n=10	n=100(10 pts)	rate of convergence
$0 < t \leq 1$	8.4356e-2	3.0344e-2	4.4405e-1
$1 < t \leq 2$	1.0174e-1	3.3246e-2	4.8573e-1
$2 < t \leq 3$	1.0314e-1	3.3866e-2	4.8366e-1
t	n=10	n=100(100 pts)	rate of convergence
$0 < t \leq 1$	8.4356e-2	3.1542e-2	4.2723e-1
$1 < t \leq 2$	1.0174e-1	3.3796e-2	4.7862e-1
$2 < t \leq 3$	1.0314e-1	3.3853e-2	4.8383e-1

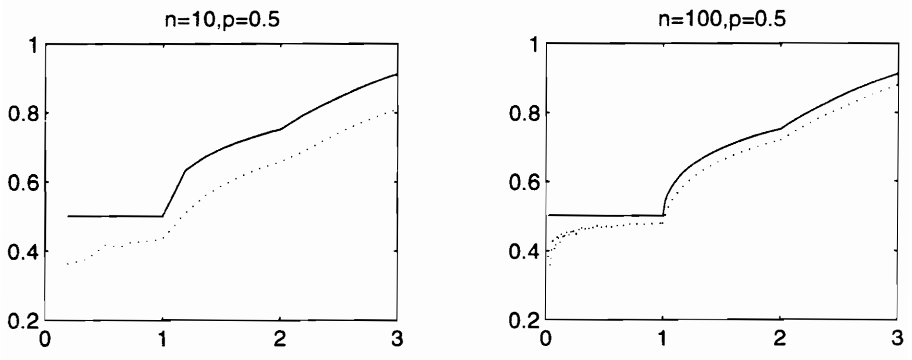

Figure H.2: $\min(\delta_i)$, $F(t) = t^{1/2}$, solid line: exact solution

Table H.6: rate of convergence for $\|\cdot\|_\infty$, $\Delta^{-1} = 0$, $F(t) = t^{1/2}$

t	n=10	n=100(10 pts)	rate of convergence
$0 < t \leq 1$	1.9213e-1	6.6133e-2	4.6319e-1
$1 < t \leq 2$	1.4629e-1	4.7018e-2	4.9294e-1
$2 < t \leq 3$	1.2575e-1	4.1266e-2	4.8392e-1
t	n=10	n=100(100 pts)	rate of convergence
$0 < t \leq 1$	1.9213e-1	1.8287e-1	2.1465e-1
$1 < t \leq 2$	1.4629e-1	5.1631e-2	4.5229e-1
$2 < t \leq 3$	1.2575e-1	4.1268e-2	4.8389e-1

Table H.7: rate of convergence for $\|\cdot\|_1/n$, $\triangle^{-1}=0$, $F(t)=t^{1/2}$

t	n=10	n=100(10 pts)	rate of convergence
$0<t\leq 1$	1.0372e-1	3.4729e-2	4.7519e-1
$1<t\leq 2$	1.2158e-1	3.9557e-2	4.8764e-1
$2<t\leq 3$	1.2285e-1	4.0278e-2	4.8430e-1

t	n=10	n=100(100 pts)	rate of convergence
$0<t\leq 1$	1.0372e-1	3.7934e-2	4.3684e-1
$1<t\leq 2$	1.2158e-1	4.0208e-2	4.8056e-1
$2<t\leq 3$	1.2285e-1	4.0260e-2	4.8450e-1

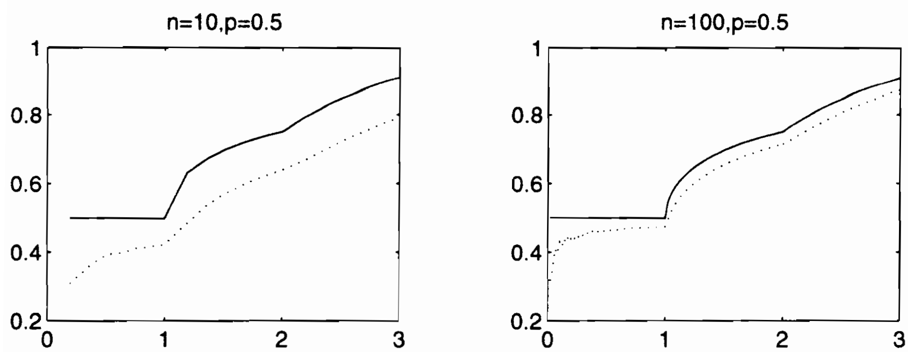

Figure H.3: $\triangle^{-1}=0$, $F(t)=t^{1/2}$, solid line: exact solution

Appendix I

$a = 1$, nonuniform case

Table I.1: exact solution for $p = 0.5$, $F(t) = t$

t		t		t	
0.19	0.2775	1.19	0.7523	2.19	1.2580
0.36	0.3820	1.36	0.8482	2.36	1.3450
0.51	0.4546	1.51	0.9280	2.51	1.4217
0.64	0.5093	1.64	0.9939	2.64	1.4878
0.75	0.5513	1.75	1.0477	2.75	1.5432
0.84	0.5835	1.84	1.0905	2.84	1.5882
0.91	0.6073	1.91	1.1230	2.91	1.6229
0.96	0.6238	1.96	1.1460	2.96	1.6475
0.99	0.6334	1.99	1.1595	2.99	1.6623
1.00	0.6366	2.00	1.1640	3.00	1.6672

Table I.2: rate of convergence for $\|\cdot\|_\infty$, $\max(\delta_i)$, $F(t) = t$

t	n=10	n=100(10 pts)	rate of convergence
$0 < t \leq 1$	3.9282e-1	3.7842e-2	1.0239e00
$1 < t \leq 2$	1.5697e-1	6.0239e-2	4.1594e-1
$2 < t \leq 3$	2.2649e-1	1.4890e-1	1.8217e-1

t	n=10	n=100(100 pts)	rate of convergence
$0 < t \leq 1$	3.9282e-1	5.5107e-1	-1.0239e-1
$1 < t \leq 2$	1.5697e-1	6.5089e-2	3.8231e-1
$2 < t \leq 3$	2.2649e-1	5.2683e-2	6.3338e-1

Table I.3: rate of convergence for $||\cdot||_1/n$, $\max(\delta_i)$, $F(t) = t$

t	n=10	n=100(10 pts)	rate of convergence
$0 < t \leq 1$	1.3056e-1	3.4492e-2	5.7810e-1
$1 < t \leq 2$	1.3097e-1	5.0639e-2	4.1269e-1
$2 < t \leq 3$	1.9561e-1	1.1716e-1	2.2260e-1
t	n=10	n=100(100 pts)	rate of convergence
$0 < t \leq 1$	1.3056e-1	5.1445e-2	4.0447e-1
$1 < t \leq 2$	1.3097e-1	5.1450e-2	4.0579e-1
$2 < t \leq 3$	1.9561e-1	5.1401e-2	5.8041e-1

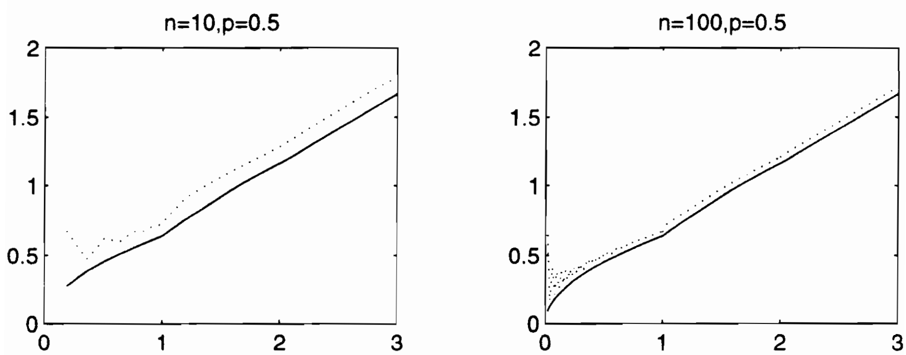

Figure I.1: $\max(\delta_i)$, $F(t) = t$, solid line: exact solution

Table I.4: rate of convergence for $||\cdot||_\infty$, $\min(\delta_i)$, $F(t) = t$

t	n=10	n=100(10 pts)	rate of convergence
$0 < t \leq 1$	3.8920e-2	3.1997e-3	1.0851e00
$1 < t \leq 2$	4.1957e-3	4.4453e-5	1.9749e00
$2 < t \leq 3$	3.4281e-3	3.3724e-5	2.0071e00
t	n=10	n=100(100 pts)	rate of convergence
$0 < t \leq 1$	3.8920e-2	1.8409e-2	3.2513e-1
$1 < t \leq 2$	4.1957e-3	4.7938e-5	1.9421e00
$2 < t \leq 3$	3.4281e-3	3.3724e-5	2.0071e00

Table I.5: rate of convergence for $\|\cdot\|_1/n$, $\min(\delta_i)$, $F(t) = t$

t	n=10	n=100(10 pts)	rate of convergence
$0 < t \leq 1$	8.2919e-3	6.8078e-4	1.0857e00
$1 < t \leq 2$	3.4760e-3	3.5346e-5	1.9927e00
$2 < t \leq 3$	3.2853e-3	3.3081e-5	1.9970e00

t	n=10	n=100(100 pts)	rate of convergence
$0 < t \leq 1$	8.2919e-3	1.2201e-3	8.3226e-1
$1 < t \leq 2$	3.4760e-3	3.5521e-5	1.9906e00
$2 < t \leq 3$	3.2853e-3	3.2612e-5	2.0032e00

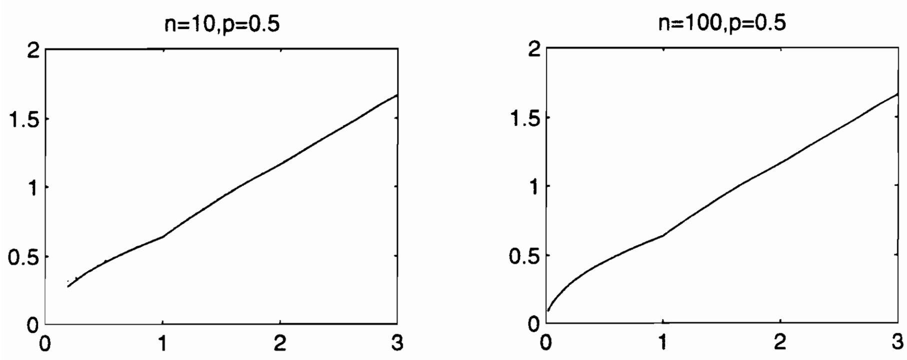

Figure I.2: $\min(\delta_i)$, $F(t) = t$, solid line: exact solution

Table I.6: rate of convergence for $\|\cdot\|_\infty$, $\Delta^{-1} = 0$, $F(t) = t$

t	n=10	n=100(10 pts)	rate of convergence
$0 < t \leq 1$	2.2275e-2	4.6438e-3	6.8093e-1
$1 < t \leq 2$	1.6536e-2	2.0735e-3	9.0172e-1
$2 < t \leq 3$	1.4183e-2	1.8191e-3	8.9193e-1

t	n=10	n=100(100 pts)	rate of convergence
$0 < t \leq 1$	2.2275e-2	1.1666e-2	2.8090e-1
$1 < t \leq 2$	1.6536e-2	2.2424e-3	8.6771e-1
$2 < t \leq 3$	1.4183e-2	1.8192e-3	8.9189e-1

Table I.7: rate of convergence for $\|\cdot\|_1/n$, $\triangle^{-1} = 0$, $F(t) = t$

t	n=10	n=100(10 pts)	rate of convergence
$0 < t \leq 1$	1.0738e-2	1.9055e-3	7.5094e-1
$1 < t \leq 2$	1.3825e-2	1.7452e-3	8.9883e-1
$2 < t \leq 3$	1.3895e-2	1.7759e-3	8.9346e-1

t	n=10	n=100(100 pts)	rate of convergence
$0 < t \leq 1$	1.0738e-2	1.6261e-3	8.1980e-1
$1 < t \leq 2$	1.3825e-2	1.7735e-3	8.9183e-1
$2 < t \leq 3$	1.3895e-2	1.7748e-3	8.9371e-1

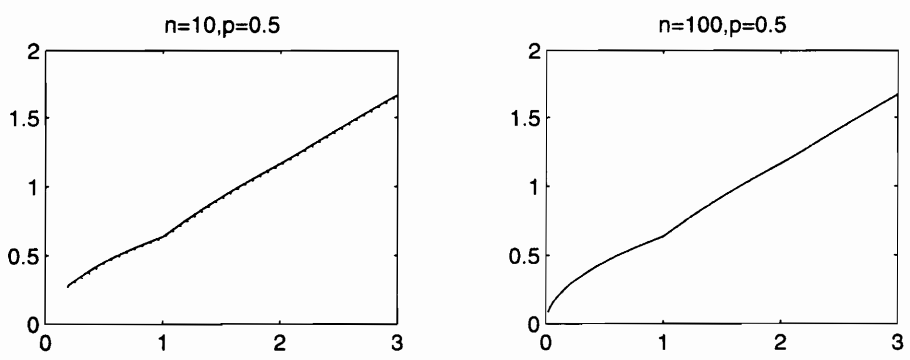

Figure I.3: $\triangle^{-1} = 0$, $F(t) = t$, solid line: exact solution

Appendix J

$F(t) = t^2/2$, nonuniform case

Table J.1: exact solution for $p = 0.5$, $F(t) = t^2/2$

t		t		t	
0.19	3.5150e-2	1.19	5.5652e-1	2.19	1.5698e+0
0.36	9.1673e-2	1.36	6.9266e-1	2.36	1.7910e+0
0.51	1.5458e-1	1.51	8.2592e-1	2.51	1.9985e+0
0.64	2.1730e-1	1.64	9.5088e-1	2.64	2.1877e+0
0.75	2.7566e-1	1.75	1.0632e+0	2.75	2.3544e+0
0.84	3.2674e-1	1.84	1.1594e+0	2.84	2.4953e+0
0.91	3.6843e-1	1.91	1.2369e+0	2.91	2.6077e+0
0.96	3.9920e-1	1.96	1.2936e+0	2.96	2.6894e+0
0.99	4.1806e-1	1.99	1.3282e+0	2.99	2.7391e+0
1.00	4.2441e-1	2.00	1.3398e+0	3.00	2.7557e+0

Table J.2: rate of convergence for $\|\cdot\|_\infty$, $f(T^0) = 0$, $F(t) = t^2/2$

t	n=10	n=100(10 pts)	rate of convergence
$0 < t \leq 1$	1.6721e-3	6.0072e-6	2.4446e+0
$1 < t \leq 2$	1.3474e-3	1.3452e-5	2.0007e+0
$2 < t \leq 3$	2.2086e-3	2.2067e-5	2.0004e+0

t	n=10	n=100(100 pts)	rate of convergence
$0 < t \leq 1$	1.6721e-3	7.0095e-5	1.3776e+0
$1 < t \leq 2$	1.3474e-3	1.3452e-5	2.0007e+0
$2 < t \leq 3$	2.2086e-3	2.2067e-5	2.0004e+0

Table J.3: rate of convergence for $\|\cdot\|_1/n$, $f(T^0)=0$, $F(t)=t^2/2$

t	n=10	n=100(10 pts)	rate of convergence
$0 < t \leq 1$	4.9462e-4	3.7535e-6	2.1198e+0
$1 < t \leq 2$	1.1033e-3	1.1009e-5	2.0009e+0
$2 < t \leq 3$	1.9832e-3	1.9796e-5	2.0008e+0
t	n=10	n=100(100 pts)	rate of convergence
$0 < t \leq 1$	4.9462e-4	5.0275e-6	1.9929e+0
$1 < t \leq 2$	1.1033e-3	1.0651e-5	2.0153e+0
$2 < t \leq 3$	1.9832e-3	1.9446e-5	2.0085e+0

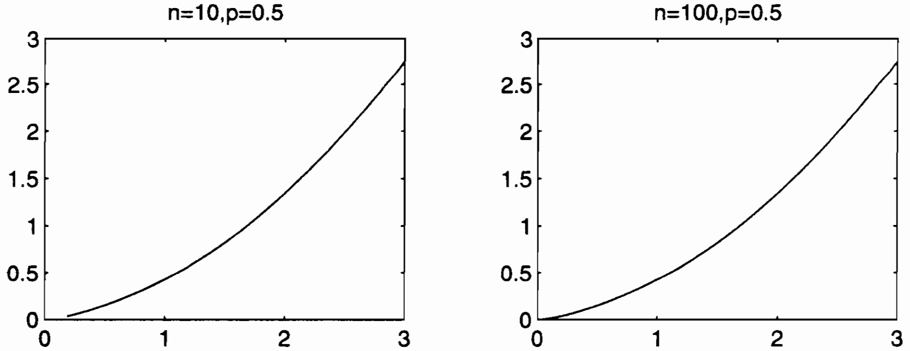

Figure J.1: $F(t)=t^2/2$, solid line: exact solution

Appendix K

$F(t) = t^3/3!$, nonuniform case

Table K.1: exact solution for $p = 0.5$, $F(t) = t^3/3!$

t		t		t	
0.19	2.6714e-3	1.19	2.6261e-1	2.19	1.2843e+0
0.36	1.3201e-2	1.36	3.6856e-1	2.36	1.5698e+0
0.51	3.1534e-2	1.51	4.8230e-1	2.51	1.8538e+0
0.64	5.5629e-2	1.64	5.9770e-1	2.64	2.1259e+0
0.75	8.2699e-2	1.75	7.0842e-1	2.75	2.3756e+0
0.84	1.0979e-1	1.84	8.0841e-1	2.84	2.5938e+0
0.91	1.3411e-1	1.91	8.9226e-1	2.91	2.7724e+0
0.96	1.5329e-1	1.96	9.5552e-1	2.96	2.9048e+0
0.99	1.6555e-1	1.99	9.9485e-1	2.99	2.9863e+0
1.00	1.6977e-1	2.00	1.0082e+0	3.00	3.0137e+0

Table K.2: rate of convergence for $\|\cdot\|_\infty$, $f(T^0) = 0$, $F(t) = t^3/3!$

t	n=10	n=100(10 pts)	rate of convergence
$0 < t \leq 1$	1.0593e-3	1.0610e-5	1.9993e+0
$1 < t \leq 2$	2.9057e-3	2.9100e-5	1.9994e+0
$2 < t \leq 3$	5.5506e-3	5.5572e-5	1.9995e+0

t	n=10	n=100(100 pts)	rate of convergence
$0 < t \leq 1$	1.0593e-3	1.0610e-5	1.9993e+0
$1 < t \leq 2$	2.9057e-3	2.9100e-5	1.9994e+0
$2 < t \leq 3$	5.5506e-3	5.5572e-5	1.9995e+0

Table K.3: rate of convergence for $\|\cdot\|_1/n$, $f(T^0)$, $F(t) = t^3/3!$

t	n=10	n=100(10 pts)	rate of convergence
$0 < t \leq 1$	8.7972e-4	8.7654e-6	2.0016e+0
$1 < t \leq 2$	2.4584e-3	2.4620e-5	1.9994e+0
$2 < t \leq 3$	4.8542e-3	4.8604e-5	1.9994e+0

t	n=10	n=100(100 pts)	rate of convergence
$0 < t \leq 1$	8.7972e-4	8.3852e-6	2.0208e+0
$1 < t \leq 2$	2.4584e-3	2.3832e-5	2.0135e+0
$2 < t \leq 3$	4.8542e-3	4.7461e-5	2.0098e+0

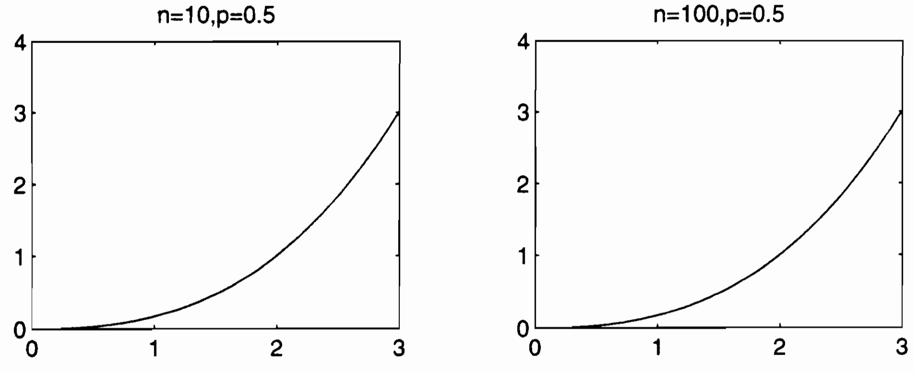

Figure K.1: $F(t) = t^3/3!$, solid line: exact solution

Appendix L

$F(t) = t^4/4!$, nonuniform case

Table L.1: exact solution for $p = 0.5$, $F(t) = t^4/4!$

t		t		t	
0.19	1.4502e-4	1.19	8.9182e-2	2.19	7.7804e-1
0.36	1.3578e-3	1.36	1.4250e-1	2.36	1.0201e+0
0.51	4.5949e-3	1.51	2.0607e-1	2.51	1.2765e+0
0.64	1.0172e-2	1.64	2.7609e-1	2.64	1.5349e+0
0.75	1.7721e-2	1.75	3.4781e-1	2.75	1.7823e+0
0.84	2.6349e-2	1.84	4.1601e-1	2.84	2.0058e+0
0.91	3.4868e-2	1.91	4.7550e-1	2.91	2.1936e+0
0.96	4.2047e-2	1.96	5.2168e-1	2.96	2.3355e+0
0.99	4.6828e-2	1.99	5.5093e-1	2.99	2.4239e+0
1.00	4.8504e-2	2.00	5.6095e-1	3.00	2.4539e+0

Table L.2: rate of convergence for $\|\cdot\|_\infty$, $f(T^0) = 0$, $F(t) = t^4/4!$

t	n=10	n=100(10 pts)	rate of convergence
$0 < t \leq 1$	4.2467e-4	4.2442e-6	2.0003e+0
$1 < t \leq 2$	2.2216e-3	2.2223e-5	1.9999e+0
$2 < t \leq 3$	6.2430e-3	6.2464e-5	1.9998e+0

t	n=10	n=100(100 pts)	rate of convergence
$0 < t \leq 1$	4.2467e-4	4.2442e-6	2.0003e+0
$1 < t \leq 2$	2.2216e-3	2.2223e-5	1.9999e+0
$2 < t \leq 3$	6.2430e-3	6.2464e-5	1.9998e+0

Table L.3: rate of convergence for $||\cdot||_1/n$, $f(T^0)=0$, $F(t)=t^4/4!$

t	n=10	n=100(10 pts)	rate of convergence
$0<t\le 1$	2.9961e-4	2.9884e-6	2.0011e+0
$1<t\le 2$	1.7589e-3	1.7579e-5	2.0002e+0
$2<t\le 3$	5.2052e-3	5.2064e-5	1.9999e+0
t	n=10	n=100(100 pts)	rate of convergence
$0<t\le 1$	2.9961e-4	2.7991e-6	2.0295e+0
$1<t\le 2$	1.7589e-3	1.6808e-5	2.0197e+0
$2<t\le 3$	5.2052e-3	5.0337e-5	2.0146e+0

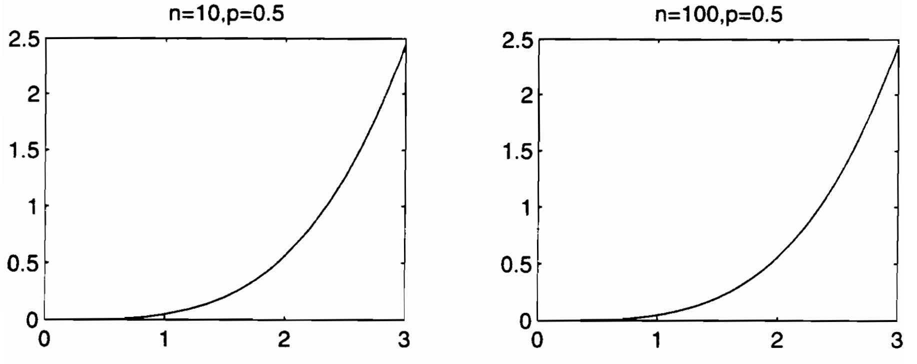

Figure L.1: $F(t)=t^4/4!$, solid line: exact solution

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