AVAILABILITY ANALYSIS OF OPPORTUNISTIC AGE REPLACEMENT POLICIES

by

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(ABSTRACT)

This research developes the availability function for a two component series system in which a component is replaced because of component failure or because it reaches a prescribed age. Also each component replacement provides an opportunity for the replacement of the other component. This last maintenance policy is called an opportunistic replacement strategy.

The system functions only if the both components of the system are functioning. The system fails if either of the components fails. Component i is replaced if it fails before attaining age T_i since it was last replaced or maintained. The component i is preventatively maintained if is has not failed by the age T_i. This type of replacement plan is called age replacement policy. When component i is being replaced or preventatively maintained, if the age of component $j \neq i$ exceeds τ_j then both components i and j are replaced at the same time. This type of replacement is called opportunistic replacement of component j and τ_j is called the opportunistic replacement time for component j. The time dependent and long run availability measures for the system are developed.

A nested renewal theory approach is used is used to develop the system availability function. The nesting is defined by considering the replacement of a specific one of the components as an elementary renewal event and the simultaneous replacement of both components as the macroscopic renewal event. More specifically, the renewal process for the system represents a starting point for the entire system and is in fact a renewal process. The intervals between system regeneration points are called "major intervals".

The age replacement time T_i and opportunistic replacement time τ_i are treated as decision parameters during the model developement. The probability distribution of the major interval is developed and the Laplace transform of the system availability is developed.

Four replacement models are obtained from the main opportunistic age replacement policy. These are a failure replacement policy, an opportunistic failure model, a partial opportunistic age replacement policy and an opportunistic age replacement policy. These models are obtained as specific cases of the general model.

The long run availability measure for the failure replacement model is proven to be the same measure as that developed by Barlow and Proschan. This proof validates the modeling approach.

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CHAPTER 1 INTRODUCTION

1.1 Problem Describtion

Failures occur in any system. Their place and time of occurrence cannot be predicted in advance. The effect of these failures on a system can be partial or complete loss of operation of the system. If it is possible to repair these failures in a reasonable amount of time, we call the system a maintained system.

With the steady increase in the complexity of systems, the increasingly stringent requirements for system effectiveness and the increase in cost of system maintenance, more emphasis is being placed on preventive maintenance. That is a maintenance policy to improve system effectiveness and reduce maintenance cost is sought.

Any activity undertaken to bring a unit into operation after a failure or to prevent failure is known as maintenance. This maintenance action may involve planned or unplanned actions carried out to retain a system in or restore it to an acceptable condition. Planned maintenance is done to retain a system in satisfactory order by providing inspection, detection, and correction of initial stage failures. In unplanned maintenance, repair is done when system fails . During these maintenance actions, units may be replaced or repaired.

Several maintenace policies that can be used are:

1. Age replacement policy
2. Block replacement policy
3. Opportunistic replacement policy.

In an age replacement policy, a unit is replaced or repaired upon failure or after it has accumulated a certain age since its last replacement. This is normally done to minimze the cost of system support or to maximize the average availability of the system.

Under a block replacement policy, the unit in operation is replaced or repaired at times T, 2T, 3T. . . . As in the age replacement policy, the aim is to minimze the expected cost of the system.

Both age and block replacement policies provide the neccessary system effectiveness or the reduction in the expected cost of maintenaning the system for cases in which the cost of replacing a functioning component is less than the cost of a component failure and the failure distribution of the components display an increasing failure rate.

Of the preventive maintenance policies, the opportunistic replacement policy has received very little attention. For a series system the repair or the replacement of one unit should sometimes be considered in conjunction with what happens to the other units. An opportunistic replacement policy exploits economies of scale in the repair or replacement activity. That is, two or more repair activities done concurrently may cost less than the cost of repairing each component seperately. Thus the necessity of performing at least one repair might provide the economic justification to perform a second one at the same time.

Quite often the main reason for constructing a model of preventive maintenance is to find a maintenance policy that optimizes the expected cost per unit time. In both age and block replacement policies we seek optimal replacement policies, T_i* for component i, that minimize the expected cost per unit time or maximize the average availability of the system. In opportunistic age replacement policy we seek an optimal age replacement policy, T_i*, and an opportunistic replacement policy, τ_i* for component i, that minimize the expected cost per unit time.

Many systems have components arranged in series, i.e when one of the components fails the system fails. When the components in the system are connected in series the frequency of system breakdowns increases and so will the cost of replacing the failed components. Under such circumstances, one needs a replacement policy that will minimize the frequency of system shutdown, reduce the length of time the system is shut down due to failure of a component and eventually reduce the cost of system maintenance. An opportunistic replacement policy provides the advantages of economies of scale. Two or more replacements are done concurrently according to a set criterion that will either minimize the cost or reduce the frequency of system breakdowns.

In most manufacturing systems the availability of the system is of very great importance since the breakdown of the system due to component failure may mean losing production capacity. Quite often the attempt is to optimize the average availability of the system. Optimizing average availability of the system may come at the expense of an increase in the cost of maintenance. An effective replacement policy is the one that yields a balance between the cost of maintenance and the systems effectiveness.

1.2 RESEARCH OBJECTIVES

The aim of this research is to develop a model for the operation of a system subject to an opportunistic age replacement policy. Expressions for the availabilty of a series system consisting of two components are developed. The availability model is a function of the age replacement times and the opportunistic replacement times of the components. Each component is replaced when it fails or after it has reached its age replacement time without failure. Also both components are replaced at the same time if either one of two conditions occur:

1. If at the time of replacing component 1, component 2 has accumulated an age equal to its opportunistic replacement age since it was last replaced then both are replaced at the same time.

2. If at the time of replacing component 2, component 1 has accumulated an age equal to its opportunistic replacement age since it was last replaced then both are replaced at the same time.

1.3 APPROACH

The system performance is modeled as a nested renewal process. The nesting is defined by considering the replacement of a specific one of the components as an elementary renewal event and the simultaneous replacement of both components as the macroscopic renewal event. More specifically, the renewal process for the system represents a starting point for the entire system and is in fact a renewal process. The intervals between system regeneration points are called "major intervals"

The unit of analysis is thus the minor interval. In a minor interval it is assumed that one of the components fails between the k^{th} and the $k+1^{st}$ failure time of the other component. These sequences of system failures are modeled as a renewal process. That is the sequence of component 1 replacements before a component 2 replacement is modeled as a renewal process. A minor interval may end at any point in time and may end with or without system renewal. When the end of the minor interval coincides with system renewal, it corresponds to the end of a major interval. In any case, the analytical approach used here is to define the probability that a minor interval ends by a particular time without system renewal. The corresponding probability of system renewal over time may then be determined indirectly. In the definition of a probability model for the lengths of minor intervals, the input quantities are the component life distributions, repair time distributions and residual life distributions. The derived probability measures are the distributions on operating time and repair time during a minor interval. Using these measures and their implied renewal density, expressions for the point and limiting availability are constructed. Since major intervals are comprised of minor intervals, the probability measures for the lengths of minor intervals and for the availability during the minor intervals are used to define the corresponding quantities for the major interval.

1.4 OUTLINE OF RESEARCH

Chapter 2 provides a review of some of the relevant literature on preventive maintenance, availability functions, cost models for age replacement and for opportunistic replacement. There is no article which treats availability of opportunistic replacement.

General developement of the availability model for a series system is presented in Chapter 3. In this chapter various definitions and notations used in the research are given. Theorems that help in the development of the model are stated and proven. The distribution for the length of the major interval is described. The time dependent availability function and the corresponding Laplace transform are given.

In chapter 4 two replacement models are considered, the failure replacement model and opportunistic failure replacement model. The relationship between the two models is dicussed. With the failure model there is no opportunistic replacement and there is no age replacement for either component. In opportunistic failure replacement model, we have opportunistic replacement but there are no age replacements.

Chapter 5 also treats two replacement models, the partial opportunistic age replacement policy and the full opportunistic age replacement policy. For a partial opportunistic age replacement policy both components have opportunistic replacement times but only one component has an age replacement time. In a full opportunistic age replacement policy, both components have age replacement times and opportunistic replacement times. In both cases the distributions on interval duration and the corresponding density functions are given. The relationship between the two models is disscussed.

Chapter 6 presents some examples of numerical result for the various models. The Laplace transforms of the density functions and distribution functions of the lengths of minor intervals are given. The mean of the operating period for the the initial and general minor intervals are given. Also the mean for the total length of the general and initial minor intervals are given. Numerical result are presented for the time dependent availability under an opportunistc age replacement policy, a failure replacement policy and the limiting availability measure for the opportunistic age replacement policy is computed.

Chapter 7 attempts to summarize all that has been acomplished in the research and suggest any further work that could be done in the future.

1.5 Summary

This research develops the system availability function, $\mathfrak{A}(t)$, for the failure replacement policy, the opportunistic failure replacement policy, partial opportunistic age replacement policy and the opportunistic age replacement model. A comparism of various replacement policies showed that the failure replacement policy provides a higher availability compared to the other replacement policies.

The limiting availability of the system was also developed. Using $T_i=2\ \mu_i$ and $\tau_i=\beta\ T_i$ for β between 0.4 and 0.8 provides a higher value for the limiting availability. Where μ_i is the mean operating time of component i .

We were also able to show that the limiting availability for failure replacement policy is the same as the result obtained by Barlow and Proschan[1].

CHAPTER 2 LITERATURE REVIEW

2.1 Introduction

This chapter provides an overview of the existing literature on replacement policies involving both cost and availability models. Little research has been done on the modeling of point availability of a series system or on cost models for series systems. Many of the existing availability and cost models for series system treat only limiting behaviour. An interest in the time dependent availability of a series system for which an age replacement and opportunistic replacement policies are used has motivated the current research.

2.2 Cost Analysis

Most of the research done on cost analysis of preventive maintenance treats the areas of age replacement and block replacement. Little attention has been given to opportunistic replacement models. In both age and block replacement, the aim is to find a policy, T, such that the expected cost of replacement is minimized.

Nachlas[23] compares the cost models of age and block replacement models. The total cost for an age replacement policy is:

$$C(T)=C_2\bar{F}(T)+C_1F(T) \tag{2.1.1}$$

so the total cost per unit time is:

$$\frac{C(T)}{E(r)+\int_0^T \bar{F}(t)dt} \tag{2.1.2}$$

where C_2 is the cost of replacing a functioning component and C_1 is the cost of replacing a failed component. The probability that the system is functioning at time T is $\bar{F}(T)$, and $F(T)$ is the probability that the system fails by time T. The expected repair time is $E(r)$ and the expected operating time of a component is $\int_0^T \bar{F}(t)dt$.

The total cost under block replacement is:

$$C(T)=C_2+M_H(T) \tag{2.1.3}$$

so the model for the total cost per unit time is:

$$\frac{C(T)}{E(r)+T} \tag{2.1.4}$$

where expected number of failures by time t is $M_H(t)$.

Scheafer[34] investigates the age replacement model for the case in which the cost of keeping the unit operating in a system increases with the age of the unit. In this model,

C_1 is the cost of a failed unit,

C_2 is the cost of replacing a unit which has not failed,

C_3 is the cost of proportionality,

$N_1(t)$ is the number of failed units until the time t,

$N_2(t)$ is the number of units replaced before failure by time t,

A cost factor which increases with age of the unit is added by introducing a factor proportional to Z^α, $\alpha>0$.

The expected total cost at time t is:

$$C(t)=C_2N_2(t)+C_1N_1(t)+C_3\left(\sum_{i=1}^{N(t)} Z_i^\alpha+(t-S_{N(t)})^\alpha\right) \tag{2.1.5}$$

and the expected total cost per unit time:

$$\frac{C(t)}{t}. \tag{2.1.6}$$

Where $N(t)=N_2(t)+N_1(t)$

Scheafer uses the expected cost per unit time over an infinite interval as a criterion for evaluating replacement policies. He assumes that the optimum value of T is the one which minimizes:

$$\lim_{t\to\infty}\frac{E[C(t)]}{t} \tag{2.1.7}$$

If no finite solution exists, then the optimum policy must be $T=\infty$.

For $\alpha=1$ the optimum policy is the same as that for the case in which Z^α is omitted from the cost model.

Almost all of the research on opportunistic replacement policies has been analyzed using using the expected cost of replacement per unit time with the aim of finding optimal values for age replacement time and opportunistic replacement time. All cost models for preventive maintenance assume stable conditions.

Dekker and Dijkstra[10] consider a component replacement model in which preventive replacements are only possible at maintenance opportunities. These opportunities arise according to a poisson process, independent of failures of components. Conditions for the existence of a unique average optimal control limit policy are established and the equation charaterizing the optimal policy and the average cost is derived. An important result is that the optimal policy can be described as a so called one-opportunity-look-ahead policy. The authors show that there is a correspondence with the well-known age replacement model. The component has a stochastic lifetime x with c.d.f $F_X(t)$ and the random variable Y denote the time between successive opportunities. In the opportunistic replacement model preventive replacements are allowed at opportunities only, which occurr only according to a poisson process with rate 1/E(Y). The time after a failure and after a preventive replacement the (residual) time to the next opportunity is again exponentially distributed with mean E(Y) and both events therefore can be considered as the end of a renewal cycle. The long term average cost $g_{op}(t)$ under a policy with control limit t is:

$$g_{op}(t)=\frac{C_{op}(t)}{L_{op}(t)}=\frac{C_P+(C_f-C_P)P(X<t+Y)}{L_{op}(t)} \tag{2.1.8}$$

where $C_{op}(t)$ and $L_{op}(t)$ denote the expected cycle cost and the length respectively . The latter is given by the formula:

$$L_{op}(t)=E(\min(X, t+Y))=\int_0^{\infty}\int_0^{t+y}(1-F_X(x))dx\, dF_Y(y). \qquad (2.1.9)$$

L'Ecuyer and Haurie[18] proposes a dynamic programming model (DP) for optimal preventive replacement in a multicomponent system. There is opportunistic replacement if economies of scale are possible in the replacement activity. The model assume the following:

1. System consistz of m components with non identical independent life-time distributions charaterized by discrete non decreasing failure rates.
2. State of the system is perfectly observed at discrete times. A strategy tells, for each possible state, which operative component should be replaced (preventive replacement) in addition to the mandatory replacement of failed components.
3. Replacements, if any, are instanteneous and by new components only.

At observation times, the system is in state $x \in X$ and the set R of components to be replaced are chosen. The set contains at least the set of failed components. The components in the set W(R) must be dismantled and a cost:

$$C(R)=\sum_{i\in R} C_i+\sum_{i\in W(R)} c_i \qquad (2.1.9)$$

is incurred. The transition probabilities are computed from failure probabilities.

Duncan and Scholnick[12] propose a model in which there are factory production epochs that allow components to be replaced at reduced cost. Over a long production run the unit cost of replacing these stochastically deteriorating components can be controlled by decisions which govern when production is to be interrupted for component replacement and when components are to be replaced at the reduced cost replacement opportunities. They develop and analyze models for optimizing "interrupt and opportunistic" replacement strategies in simple systems. The model is treated within the framework of a discrete Markov decision model and a dynamic progamming recurrence relation for the system is formed. Numerical results are given that illustrate the advantage of combining the interrupt replacement with opportunistic replacement. The interrupt opportunity policies studied are within the framework of the discrete Markov decision model. The process of production span between replacement opportunities is assumed to form a renewal process.

Marathe and Nair[23] compare two types of multistage planned replacement strategies, namely multistage planned replacement by age and a multistage block replacement strategy. The strategy brings economies by transfering failures from groups where replacement costs are higher to those in which it is smaller. This is facilitated by grouping the units according to replacement cost and ordering the groups or stages with increasing or decreasing cost according as the failure rate is increasing or decreasing with the age of the item respectively. Each is compared with the corresponding single stage replacement strategy in literature. The authors make the following conclusions.

1. The multistage planned replacement strategies in this paper are applicable to failing types of items with monotonic increasing failure rates.
2. The ordering of stages has to be done with decreasing replacement cost per unit. The optimal number of stages for a system can be determined by evaluating the possible groupings which would be rather few from a practical point of view.

3. For a given system, the strategy can be evaluated completely, optimzing the relevant cost of the strategy.

4. The possibilities of improving the economy of the strategies are indicated.

2.3 Opportunistic Replacement models

Berg[4] considers a two unit series system in series in two ways. First the opportunistic repair occurs only at failure epochs which he calls an opportunistic failure replacement policy and the other he calls opportunistic age replacement policy where opportunistic replacement apart from occuring at failure epochs may also occur when a unit reaches a predetermined critical age. That is, at planned as well as failure replacement epochs, we allow a replacement of a second unit if its age exceeds a control limit. He obtains appropraite integral equations for the stationary pdf of the ages of the units. These equations are then solved for units whose life is ditributed according to a general identical Erlang distribution. This is similar to the current research except for the following;

1. Instanteneous replacement of failed units,
2. Application is restricted to general erlang distribution, and
3. use of cost model.

The expected cost per unit time in the long run with control limits L_i and critical ages S_i of the model is:

$$C(L_1,L_2, S_1, S_2)=\sum_{i=1}^{\infty}(\bar{N}_i a_i+\tilde{N}_i \tilde{a}_i)+ \tilde{N}_{12}\tilde{b} +\bar{N}_{12}b+r. \qquad (2.1.10)$$

The expected number (per unit time in the long run) of: single failure replacement of unit 1, single failure replacement of unit 2 and common failure replacement of units, denoted by $\bar{N}_1, \bar{N}_2$ and $\bar{N}_{12}$, respectively. The expected number of single planned replacements of unit i (i = 1,2) is denoted by $\tilde{N}_i$ and common planned replacement is denoted by $\tilde{N}_{12}$.

a_i is the cost of a single (failure)

$\tilde{a}_i$ is the cost of a single planned replacement.

b is the cost of common failure replacement.

$\tilde{b}$ is the cost of common planned replacement.

r is the total expected running cost per unit time in the long run.

Zheng and Fard[39], present a cost analysis of an opportunistic replacement model for a system with several units. A unit is replaced at failure or when its hazard reaches L. All operating units with hazard rate falling in the interval (L-u,L) are replaced. This policy allows joint replacement of failed units. The long run cost is derived and optimal values for L and u are obtained in order to minimized the average total replacement cost rate. The authors use the following assumptions:

1. Hazard rates of the units are increasing in the cycle time t.
2. Replacing more than one unit at the same time is cheaper than replacing them separately.
3. The replacement times are negligble.
4. The planning horizon is infinite.
5. All failure events are statistically independent.
6. System is stable when system time t is large enough.

7. Given type i unit is renewed at t_0, the time from the cycle time $(T_i\text{-}w_i)$ to an active replacement of any unit other than the type i follows an exponential distribution.

In another paper Zheng and Fard[40] present another cost analysis of an opportunistic replacement policy with several units. A unit is repaired at failure when the hazard rate falls in (0, L-u). A unit is replaced at failure when its failure rate falls in (L-u, L). An operating unit is replaced because its hazard reaches L, all operating units with their hazard rate in (L-u, L) are replaced. The long run cost rate as a function of L and u is derived. Optimal values for L and u are obtained to minimize total maintenance cost rate. The authors make the following assumptions:

1. The hazard rates of the units are increasing in cycle time.
2. The replacement times are negligble and repair times are exponentially distrbuted with mean $1/\nu$.
3. The planning horizon is infinite.
4. All failure events are independent.
5. The system approaches steady state as $t \rightarrow \infty$.
6. There is no stanby unit in the system.
7. After repair, the hazard rate of the unit is not changed.
8. pdf of time failure of each unit is known.

McCall[24] investigates the operating characteristics of opportunistic replacement and inspection policies. An opportunistic replacement policy makes the replacement of a single uninspected part conditioned on the state of one or more continoustly inspected parts. He investigates charateristics such as the expected rate of opportunistic replacement of an uninspected part and one of the monitored parts and the expected rate of planned replacement of the uninspected part.

Epstein and Williamowsky[13] consider opportunistic replacement in a deterministic enviroment. Repair and failure times are non random. An opportunistic replacement problem dealing with life-limited parts is analyzed. Because of the high cost of unscheduled breakdowns, life limits on system component may be set low in order to insure performance during an allotted life span. If two components with different life-limits are present, each individually scheduled replacement point offers potential opportunity for monetary savings.

Pullen and Thomas[32], examine the operating characteristics of an opportunistic replacement policy with an approach similar to the model by Berg[2]. In their model, one of the parts is non failing. It is always replaced at a certain age or life limit. Replacement of this part therefore occurs at deterministic intervals if it is not replaced opportunistically. Its replacement could also represent a regularly occuring maintenance event or scheduled repair opportunity. The second part has a random life with increasing failure rate. An opportunistic replacement policy that is based on control-limit age or screen for each part is analyzed. The policy is to replace a part at a replacement opportunities if its age exceeds its screen. The scope of the paper is limited to determining the long run rate of the three events:

1. Single replacement of the non failing part, which occurs when its age equals its life limit but the ageing part has not aged past its screen.
2. Single replacement of the failing part, which occur when it fails but the non failing part has not age past its screen.

3. Joint replacement, when both parts are replaced at the same opportunity, either the random-life part failed and the age of the non-failing part exceeds its screen, or the non failing part is due for replacement and the age of the failing part exceeds its screen.

2.4 Other models

Khalil[17] considers the availability of a system with various shut-off rules. He considers a series system where failure of any component leads to system failure. However depending on the shut-off rule, some of the components continue to operate when the system is down. Limiting system availability under the various shut-off rules are calculated. In particular availability for 2- and 3- unit systems with constant failure and repair rates are calculated.

Tillman, Kuo, Nasser and Hwang[36] calculate instanteneous availability via renewal theory by representing the system as a two state stochastic process. The defined states are on and off. The on and off times combine to form the total cycle time. The paper presents a numerical approach for any distribution or any empirical data antecedent on a general renewal equation.

Ibe and Wein[16] model the availability of a system which when operational can fail in two modes. However the system operator does not always diagnose the system correctly. Given that one failure mode has occured he correctly diagnoses the the failure with a probability η and mis-diagnoses with with probability 1-η, where η may vary with failure mode. The problem is modeled using partially observable markov process.

Barlow and Proschan[1] model the availability of a system consisting of k components connected in series. In this they model the steady state availability, the average availability and the asymptotic distribution of N(t), the number of renewals.

Zhao[40] models the availabilty of a series system with imperfect repair. He generalizes the availability model of a repairable components and series system. The life time of the repaired component has a general distribution which can be different from that of a new component. Availability and some asymptotic quantities in these models are derived.

Osaki and Nakagawa[30] gives a brief bibliography for stochastic models use to analyze system-reliability and availability. They list selected references on system reliability using stochastic process such as Markov chains, renewal process and semi-Markov(general Markov) processes.

There are other articles that review the research concerning preventive maintenance models. These include the 1965 paper by McCall [25], the 1976 paper by Pierskalla and Voelker[31], the 1977 paper by Lie et al[20] and the 1989 paper by Valdez-Flores and Feldman[37]. These papers provide chronological review of research performed concerning preventive maintenace.

CHAPTER 3 Development of Availability model

3.1 Introduction

System with two components in series

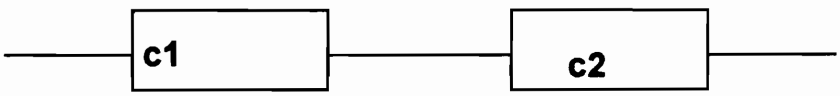

Fig 3.1.1

In this research, we wish to construct a model for the availability of a two component series system in which a component is replaced because of component failure or because it reaches a prescribed age. Also each component replacement provides an opportunity for the replacement of the other component. This last maintenance policy is called an opportunistic replacement strategy. The model is constructed in stages.

In the first stage we model only the failures of the system. It is assumed that there is no preventive maintenance and that there is no simultaneous replacement of both components. That is, we only have component renewal but no system renewal. We call this model a failure model.

In the second stage, we include opportunistic replacement. It is assumed that a component is replaced not only when it fails but also when the other component fails and the age of the unfailed component exceeds a certain threshold value. In the second stage, both single component replacement and the simultaneous replacement of both components can occur. Therefore both component renewals and system renewals are possible.

In the third stage it is assumed that one of the components is subject to age replacement as well as opportunistic and failure replacement. The other component is replaced only opportunistically or when it fails. Therefore in the third stage one component has an age replacement policy and the other a failure policy. Both component renewals and system renewals can occur.

In the final stage, it is assumed that both components have an age replacement policy and are also subject to opportunistic replacements. The model defined in the final stage is a generalization of those developed in the other three stages. The models of Stages 1 through 3 can be obtained by fixing the appropriate decision variables of the fourth stage. Stage one is used for validating the final model. In the next chapter we show that the long run availability under this failure model yields the same result as that obtained by Barlow and Proschan[1].

We proceed in this chapter as if we are dealing with the final model. This permits the full definition of the terms and notation. For the two component series system, each of the components have two decision variables, τ_i and T_i $i=1,2$. T_i is the age replacement time of component i, and τ_i is the opportunistic replacement time of component i. Necessarily $\tau_i<T_i$, $i=i,2$. The progressive development of the models are characterized by:

Stage one $T_i=\tau_i=\infty$. $i=1,2$

Stage two $T_1=T_2=\infty$.

Stage three $T_2=\infty$.

The system functions only if the both components of the system are functioning. The system fails if either of the components fails. Component i is replaced if it fails before attaining age T_i since it was last replaced or maintained. The component i is preventatively maintained if it has not failed by the age T_i. This type of replacement plan is called age replacement policy. When component i is being replaced or preventatively maintained, if the age of component $j \neq i$ exceeds τ_j then both components i and j are replaced at the same time. This type of replacement is called opportunistic replacement of component j, and τ_j is called the opportunistic replacement time for component j. Therefore a component is replaced in one of the following ways:

1. If a component fails before its age replacement time
2. The component has not failed by its age replacement time.
3. The component has attained its opportunistic replacement age before system operation is interrupted in order to replace the other component.

A nested renewal process is used here to model the system behavior. The nesting is defined by considering the replacement of a specific one of the components as an elementary renewal event and the simultaneous replacement of both components as the macroscopic renewal event. More specifically, the renewal process for the system represents a starting point for the entire system and is in fact a renewal process. The intervals between system regeneration points are called "major intervals" here.

In contrast, each major interval may be viewed as being comprised of sequence of operating periods that end with the replacement of a single one of the two components. The end points of these periods are not system renewal points. In order to construct a model for system renewal, one of the components (#2) is taken arbitrarily and the restart times of that component are treated as a delayed renewal process that is nested within the system level renewal process. The periods between component 2 restart events are called "minor intervals".

In this chapter we provide a general modeling structure using the final stage model defined above. We also provide a general set of notation, theorems, and formulas which apply to all four stages of the model development. Where there are differences in terms of the use of the theorems and the formulas it is noted and an attempt is made to explain the reason for the differences. The construction of the model begins with the development of probability expressions for the minor intervals. As noted earlier, a minor interval is the period between the component 2 restart events. There are two types of minor intervals, an initial minor interval and a general minor interval. The two types of minor interval are defined and explained below.

3.1.1 Initial Minor interval

An initial minor interval is a period of system operation in which the system starts operation with both components new. It may end with the replacement of one of the components or may end with both components being replaced at the same time. The interval ends with the replacement of one and not both of the components when either the following conditions are met:

a) one component fails at a time, say t, and the most recent replacement of the other component occurred at time u such that t-u is less than the opportunistic replacement policy time of the component that did not fail.

b) one component reaches its age replacement policy age, say T, and the most recent replacement of the other component occurred at u such that T-u is less than the opportunistic replacement policy time of the component that did not fail.

The complementary condition in either of the above events yields opportunistic replacement in which both components are replaced at the same time.

3.1.2 General Minor interval

A general minor interval is a period of system operation in which the system starts operation with one of the components new and the other already used. The life distribution for the previously used component is its residual life distribution. The interval may end with the replacement of one of the components or may end with both components being replaced at the same time.

A general minor interval is similar to an initial minor interval. The difference is that in an initial minor interval the system starts with both components new while in a general minor interval, the system starts with one of the components new and the other already used. The conditions for ending the interval are comparable to those of the initial minor interval except that the life of the used component must be modeled using a residual life distribution and in some cases the age of the used component must be considered.

3.1.3 Major interval

A major interval is a period of system operation comprised of an initial minor interval and zero or more of general minor intervals. It always starts with both components new and ends with the replacement of both components at the same time. An opportunistic replacement may occur either at the end of an initial minor interval or at the end of a general minor interval. This can be generalized by saying that in a major interval we have an initial minor interval followed by n=0,1,2... general minor intervals.

3.2 Assumptions and Notation

3.2.1 Assumptions

1. Failure repairs, preventive maintenance and opportunistic replacement actions are perfect. Each component is considered to be as good as new when any replacement or maintenance action is taken. Where there is opportunistic replacement, both components are considered to be as good as new and the entire system is considered to be as good as new.

2. Component replacements result in component renewal and opportunistic replacement results in system renewal.

3. The life length of each of the components has a known distribution function. Quite often when the failure time distributions on the components of a system have an increasing failure rate (IFR), then there might be the need to use a preventive maintenance policy. Therefore in this research only IFR distributions are considered.

4. The repair time, preventive maintenance times and opportunistic replacement time of each of the components have known probability distribution functions. There are several distributions that can be used to model repair or preventive maintenance times. For simplification purposes we use a distribution with a constant completion rate. We also assume that the repair time distribution and the age replacement time distribution are the same.

5. System aging is suspended when a component is undergoing repair or preventive maintenance.

3.2.2 Notation

The following is a list of the notation used. The various terms are explained as they are included in the models.

$X_{1,i}$, $X_{2,i}$ time between successive failures of component i

$S_{k,i}$ is the time of n^{th} component i failure.

$S_{0,i}=0$, $S_{k+1,i}=S_{k,i}+X_{k+1,i}$.

$H_I(t,k)$ cumulative distribution function on the replacement time of component 2 in the cases in which there is no system renewal and the replacement of component 2 occurs after the k^{th} component 1 replacement during an initial minor interval.

$H_G(t,k)$ cumulative distribution function on the replacement time of component 2 in the cases in which there is no system renewal and the replacement of component 2 occurs after the k^{th} component 1 replacement during a general minor interval.

$V_{I1}(t,k)$ cumulative distribution function on the replacement time of component 1 in the cases in which there is system renewal and the replacement occurs on the k^{th} component 1 replacement during an initial minor interval.

$V_{I2}(t,k)$ cumulative distribution function on the replacement time of component 2 in the cases in which there is system renewal and the replacement of component 2 occurs after the k^{th} component 1 replacement during an initial minor interval.

$V_{G1}(t,k)$ cumulative distribution function on the replacement time of component 1 in the cases in which there is system renewal and the replacement occurs on the k^{th} component 1 replacement during a general minor interval.

$V_{G2}(t,k)$ cumulative distribution function on the replacement time of component 2 in the cases in which there is system renewal and the replacement of component 2 occurs after the k^{th} component 1 replacement during a general minor interval.

$h_I(t,k)=\frac{d}{dt}H_I(t,k)$.

$h_G(t,k)=\frac{d}{dt}H_G(t,k)$.

$Q_I(t)$ is the corresponding distribution function on the lengths of the initial minor intervals that end without system renewal.

$q_I(t)$ is the density function on the lengths of the initial minor intervals where these interval that end without system renewal.

$U_I(t)$ is the corresponding distribution function on the lengths of the initial minor intervals that end with system renewal.

$u_I(t)$ is the density function on the lengths of the initial minor intervals where these interval that end with system renewal.

$Q_G(t)$ is the corresponding distribution function on the lengths of the general minor intervals that end without system renewal.

$q_G(t)$ is the density function on the lengths of the general minor intervals where these intervals ends without system renewal.

$U_G(t)$ is the corresponding distribution function on the lengths of the initial minor intervals that ends with system renewal.

$u_G(t)$ is the density function on the lengths of the initial minor intervals where these interval that end with system renewal.

$f_i(t)$ is the failure time density of component i.

$F_i(t),\bar{F}_i(t)$ C.d.f , reliability of component i.

$\tilde{f}_i(t)$ is the residual life density of component i.

$f_i^{(k)}(t)$ k th convolution of the failure time density component i.

$Z_I(t)$ distribution on the length of operating periods in initial minor intervals with or without system renewal.

$Z_G(t)$ distribution on the length of operating periods in general minor intervals with or without system renewal.

$g_i(t)$ repair time density of component i.

$g_i^{(k)}(t)$ k th convolution of the component i repair time density.

$g_{opp}(t)$ repair density during opportunistic replacement.

μ_i mean failure time of component i. (i =1,2)

ν_i mean repair time of component i. (i =1,2)

μ_I mean of initial minor interval that ends without system renewal.

μ_G mean of general minor interval that ends without system renewal.

μ_{IR} mean of initial minor interval that ends with system renewal.

μ_{GR} mean of general minor interval that ends with system

μ_L mean of the major interval.

ν_I mean operating time in an initial minor interval with or without system renewal .

ν_G mean operating time in an general minor interval with or without system renewal.

$M_Q(t)$ renewal function for the minor intervals.

$m_Q(t)$ renewal density function for minor intervals.

A(t) system availability during a sequence of minor intervals.

$\mathfrak{F}(t)$ C.d.f on the duration of major intervals (or the probability of an opportunistic replacement before time t).

$\phi(t)$ is the convolution of the major interval duration density and the opportunistic replacement time density

$M_\phi(t)$ renewal function for the system.

$\mathfrak{A}(t)$ overall system availability.

3.3 Development of System Availability function.

3.3.1 Approach

As indicated above, system performance is modeled as a nested renewal process. The unit of analysis is thus the minor interval. Observe that a minor interval may end at any point in time and may end with or without system renewal. When the end of the minor interval coincides with system renewal, it corresponds to the end of a major interval. In any case, the analytical approach used here is to define the probability that a minor interval ends by a particular time without system renewal. The corresponding probability of system renewal over time may then be determined indirectly. In the definition of a probability model for the lengths of minor intervals, the input quantities are the component life distributions, repair time distributions and residual life distributions. The derived probability measures are the distributions on operating time and repair time during a minor interval. Using these measures and their implied renewal density, expressions for the point and limiting availability are constructed. Since major intervals are comprised of minor intervals, the probability measures for the lengths of minor intervals and for the availability during the minor intervals are used to define the corresponding quantities for the major interval.

3.3.2 Density on the lengths of minor intervals.

The length of a minor interval is comprised of periods of operating time and periods of repair time. As stated earlier, within each minor interval we assume there are k component 1 replacements and one component 2 replacement. Consider first the periods of operation.

Let $H_I(t,k)$ represent the probability that an initial minor interval includes an accumulated operating time that is less than or equal to t, ends without a system renewal and ends after k component 1 replacements. Denote the associated density function(overtime) by $h_I(t,k)$ and let $H_G(t,k)$ and $h_G(t,k)$ represent the corresponding probability measures for a general minor interval.

If there are k component 1 replacements and 1 component 2 replacement then there will be k component 1 repair times and one component 2 repair time. The density on the total repair time during a minor interval is:

$$g(t,k)=\int_0^t g_1^{(k)}(x)g_2(t-x)dx. \tag{3.3.1}$$

For both types of operating intervals, the probability density on the length of the interval is the convolution of the density on the repair time and that on the operating time. One may either not have a component 1 replacement before the component 2 replacement (k=0) or have at least one component 1 replacement before the component 2 replacement ($k \geq 1$). Therefore summing over all possible values of k yields:

$$q_I(t)=\sum_{k=0}^{\infty}\int_0^t h_I(x,k)g(t-x,k)dx. \qquad (3.3.2)$$

for the density function on the duration of initial minor intervals that end without system renewal. Using similar arguments for the general minor interval, the convolution of the repair time and the operating time of the system is:

$$q_G(t)=\sum_{k=0}^{\infty}\int_0^t h_G(x,k)g(t-x,k)dx. \qquad (3.3.3)$$

Thus $q_G(t)$ is the density function on the lengths of the general minor intervals where these intervals ends without system renewal. The corresponding distribution function on the length of the initial minor interval is:

$$Q_I(t)=\int_0^t q_I(x)dx. \qquad (3.3.4)$$

The corresponding distribution on the length of the general minor interval is:

$$Q_G(t)=\int_0^t q_G(x)dx. \qquad (3.3.5)$$

Now the cumulative distribution on the length of the operating time during a minor interval with or without system renewal is the cumulative probability of interval completion with or without system renewal taken over all possible values of k. Therefore the distribution function on the combined length of the operating periods in an initial minor interval with or without system renewal is:

$$Z_I(t) = \sum_{k=0}^{\infty} H_I(t,k) + \sum_{k=0}^{\infty} V_{I1}(t,k) + \sum_{k=0}^{\infty} V_{I2}(t,k). \tag{3.3.6}$$

Similarly the distribution on the combined length of the operating periods in a general minor interval with or without system renewal is:

$$Z_G(t) = \sum_{k=0}^{\infty} H_G(t,k) + \sum_{k=0}^{\infty} V_{G1}(t,k) + \sum_{k=0}^{\infty} V_{G2}(t,k). \tag{3.3.7}$$

Combining the probability measures on periods of operation and on interval duration leads to availability measures.

3.4 System Renewal

System renewal occurs when there is opportunistic replacement. Opportunistic replacement may occur at the end of an initial minor interval or at the end of a general minor interval. We must construct the distribution on the lengths of the minor intervals in order to be able to construct the distribution on the length of the major interval.

Under the defined modeling approach, system behavior during a sequence of minor intervals looks like a delayed renewal process $Y=\{Y_n : n \in N\}$, where q_I is the distribution for Y_1-Y_0, and q_G is the common distribution of Y_2-Y_1, Y_3-Y_2,. . ..The Y_n are the renewal points of this process. They correspond to component 2 restart times. That is Y_n is the end of the n^{th} component 2 replacement(repair) time. The interval Y_n-Y_{n-1}, is comprised of k component 1 failure times, k component 1 repair times and one component 2 operating time and repair time.

3.4.1 Renewal density of minor intervals

A major interval consists of an initial minor interval and n general minor intervals (n=0,1,2. . .). This means that a major interval always starts with both components new. Both components go through some number of failures and repairs. The major interval ends when both components are replaced at the same time. Cox[9] defines a modified renewal process as a renewal process in which the duration of all intervals other than the first are i.i.d. and the duration of the first interval has a differrent distribution. Cox further shows that for a modified renewal process the renewal density is the convolution of the density on the length of the initial interval and that on the lengths of n general intervals for all possible values of n. Therefore the renewal density for the restart of a minor interval without also having system renewal is given by:

$$m_Q(t)=\sum_{n=0}^{\infty}\int_0^t q_I(x)q_G^{(n)}(t-x)dx. \tag{3.4.1}$$

Observe that this is the convolution of an initial minor interval and some number of general minor intervals.

Since the sample paths are modeled such that there is no opportunistic replacement, $Q_I(\infty)$ and $Q_G(\infty)$ are the probabilities that an opportunistic replacement never occurs in the initial minor and the general minor intervals respectively.

If $Q_I(\infty)<1$ and $Q_G(\infty)<1$, then there is a positive probability that a system renewal (opportunistic replacement) occurs in an initial minor interval and a general minor interval respectively. In such a situation the minor intervals are said to be transient. Keeping in mind the fact that the models defined here are to be used to analyze several specific cases, the values of $Q_I(\infty)$ and $Q_G(\infty)$ can be used to characterize the cases studied.

If $Q_I(\infty)=1$ and $Q_G(\infty)<1$, then the major interval cannot end at the time the initial interval ends but may end with a general interval. Therefore the initial minor interval is recurrent and the general minor interval is transient.

If $Q_I(\infty)=Q_G(\infty)=1$, the probability of opportunistic replacement in either an initial or a general minor interval is zero. That is, there will never be an opportunistic replacement. The minor intervals are recurrent. The model is a failure model with no opportunistic replacement.

If $Q_G(\infty)<1$, then as time becomes very large the total number of renewals N is finite with probability 1 and its expected value is:

$$E[N]=1+\frac{Q_I(\infty)}{1-Q_G(\infty)}. \tag{3.4.2}$$

$$P[N=k]=\begin{cases} 1-Q_I(\infty) & k=1 \\ Q_I(\infty)Q_G(\infty)^{k-2}(1-Q_I(\infty)) & k\geq 2 \end{cases}$$

$$E[N]=\sum_{k=1}^{\infty} kP[N=k]=(1-Q_I(\infty))1+\sum_{k=2}^{\infty} k\, Q_I(\infty)Q_G(\infty)^{k-2}(1-Q_G(\infty))$$

simplifying the above expression yields:

$$E[N]=\sum_{k=1}^{\infty} kP[N=k]=(1-Q_I(\infty))1+Q_I(\infty)\sum_{k=1}^{\infty}(k+1)\, Q_G(\infty)^{k-1}(1-Q_G(\infty))$$

$$\sum_{k=1}^{\infty}(k+1)\, Q_G(\infty)^{k-1}(1-Q_G(\infty))=\frac{1}{1-Q_G(\infty)}+1$$

Therefore

$$E[N]=(1-Q_I(\infty))+\frac{Q_I(\infty)}{1-Q_G(\infty)}+Q_I(\infty)=1+\frac{Q_I(\infty)}{1-Q_G(\infty)}$$

3.4.2 Renewal density for major intervals

Let L represent the length of a major interval. Then L is the time between opportunistic replacements and is the system renewal time. Let $\mathfrak{F}(t)= P[L \leq t]$ be the distribution function on L.

Theorem: 3.4.1

If $Q_G(t)<1$, then:

$$\mathfrak{F}(t)=P\{L \leq t\}=U_I(t)+\int_0^t m_Q(x)U_G(t-x)dx. \qquad (3.4.3)$$

where

$$\mathfrak{f}(t)=\frac{d}{dt}\mathfrak{F}(t)$$

and $M_Q(t)$ is the renewal function associated with the renewal density given in (3.4.1).

$\mathfrak{F}(t)=P[L \leq t]$ is the probability that there is opportunistic replacement before time t. The probability that there is opportunistic is given as the probability that there is opportunistic replacement in an initial minor interval plus the probability that there are several minor intervals(may include an initial minor interval and some number of general minor intervals) and opportunistic replacement in a general interval.

There are several ways of finding the expected length of a major interval. The simplest is to evaluate the derivative of the Laplace transform $-\frac{d}{ds}f^*(s)$ at s=0. This yields the expression below which is the mean length of a major interval. When $Q_I(\infty)=Q_G(\infty)=1$, then $\mathfrak{F}(t)$ is not defined.

$$f^*(s)=s\,F^*(s)=s\,U_I^*(s)+m_Q^*(s)U_G^*(s)\,s$$

where

$$u_I^*(s)=s\,U_I^*(s)$$

and

$$u_G^*(s)=s\,U_G^*(s).$$

Therefore:

$$f^*(s)=u_I^*(s)+m_Q^*(s)u_G^*(s).$$

$$\frac{d}{ds}f^*(s)=\frac{d}{ds}u_I^*(s)+m_Q^*(s)\frac{d}{ds}u_G^*(s)+u_G^*(s)\,\frac{d}{ds}m_Q^*(s)$$

from (3.4.1)

$$m_Q^*(s)=\frac{q_I^*(s)}{1-q_G^*(s)}$$

and

$$\frac{d}{ds}m_Q^*(s)=\frac{\frac{d}{ds}q_I^*(s)}{1-q_G^*(s)}+\frac{\frac{d}{ds}q_G^*(s)q_I^*(s)}{(1-q_G^*(s))^2}$$

By definition:

$$\mu_I=-\frac{d}{ds}q_I^*(s)\Big|_{s=0}$$

and

$$\mu_G=-\frac{d}{ds}q_G^*(s)\Big|_{s=0}$$

and

$$u_G^*(s)\Big|_{s=0}=U_{GR}(\infty)=1-Q_G(\infty)$$

and

$$M_Q(\infty)=m_Q^*(s)\Big|_{s=0}=\frac{q_I^*(s)}{1-q_G^*(s)}\Big|_{s=0}=\frac{Q_I(\infty)}{1-Q_G(\infty)}$$

Therefore we have:

$$u_{GR}^*(s)\,\frac{d}{ds}m_Q^*(s)\Big|_{s=0}=-\left(\mu_I+\frac{Q_I(\infty)}{1-Q_G(\infty)}\mu_G\right)$$

Now

$$E[L]=\mu_L=\int_0^{\infty} t\, f(t)dt = -\frac{d}{ds}f^*(s)\Big|_{s=0} \tag{3.4.10}$$

$$E[L]=\mu_L=\mu_{IR}+\frac{Q_I(\infty)}{1-Q_G(\infty)}\mu_{GR}+\mu_I+\frac{Q_I(\infty)}{1-Q_G(\infty)}\mu_G$$

The above expression simplifies to:

$$E[L]=\mu_L=(\mu_{IR}+\mu_I)+\frac{Q_I(\infty)}{1-Q_G(\infty)}(\mu_{GR}+\mu_G)$$

From (3.4.2) we know that $\frac{Q_I(\infty)}{1-Q_G(\infty)}$ is the expected number of minor interval renewals in a major interval. Therefore the expected length of a major interval is the expected length of the initial minor interval (includes the mean of initial minor interval without renewal and the mean of initial minor interval with renewal) plus the expected total length of the general minor intervals in the major interval.(includes the mean of general minor interval without renewal and the mean of general minor interval with renewal).

3.5 Availability

The system availability consists of two types availability. There is one type during the sequence of minor intervals in a major interval. This is the availability within the major interval. The other type of availability is at the system level and depends on system renewal. We now construct the availability function for the model. We do this by first constructing the availability of within a major interval. Then system availability is constructed by using the distribution on the length of a major interval, the distribution function on the length of an opportunistic repair and the availability within the major interval. First we construct the availability within the major interval.

Now $Z_I(t)$ and $Z_G(t)$ are respectively the distribution functions on the total length of the operating time in an initial minor interval and general minor interval which ends with or without system renewal. From (3.3.6) and (3.3.7) we have:

$$Z_I(\infty)=\sum_{k=0}^{\infty}H_I(\infty,k)+\sum_{k=0}^{\infty}V_{I1}(\infty,k)+\sum_{k=0}^{\infty}V_{I2}(\infty,k)=1. \tag{3.5.1}$$

$$Z_G(\infty)=\sum_{k=0}^{\infty}H_G(\infty,k)+\sum_{k=0}^{\infty}V_{G1}(\infty,k)+\sum_{k=0}^{\infty}V_{G2}(\infty,k)=1. \tag{3.5.2}$$

Also (3.3.2) implies:

$$q_I^*(s)=\sum_{k=0}^{\infty}h_I^*(s,k)g^*(s,k). \tag{3.5.3}$$

and we note that:

$$g^*(s,k)\Big|_{s=0}=\left(g_1^*(s)\right)^k g_2(s)\Big|_{s=0}=1. \tag{3.5.4}$$

so:

$$Q_I(\infty)=q_I^*(s)\Big|_{s=0}=\sum_{k=0}^{\infty}h_I^*(0,k)g^*(0,k)=\sum_{k=0}^{\infty}H_I(\infty,k). \tag{3.5.5}$$

Similarly from (3.3.3)

$$Q_G(\infty)=\sum_{k=0}^{\infty}H_G(\infty,k). \tag{3.5.6}$$

Let $\bar{Z}_I(t)= 1- Z_I(t)$ represent the probability that the length of the operating period in an initial minor interval that ends with or without system renewal is longer than t. Similarly $\bar{Z}_G(t)=1-Z_G(t)$ is the probability that the length of the operating period in a general minor interval that ends with or without system renewal is longer than t. Then the availability during a sequence of minor intervals is:

$$A(t)=\bar{Z}_I(t)+\int_0^t \bar{Z}_G(t-x)m_Q(x)dx. \tag{3.5.7}$$

A(t) is the probability that the system is functioning at any time t during a sequence of minor intervals. This availability measure for a sequence of minor intervals can now be used to define the overall system availability. To construct the availability function for the system, let L_i be the duration of the i^{th} major interval and D_i the down time for the i^{th} opportunistic replacement. D_i is the extra time it takes perform the second maintenance activity on the i^{th} system renewal(opportunistic replacement).

Let $W_n=\sum_{i=1}^{n}(L_i+D_i)$, then $W=\{W_n: n\epsilon N\}$ is a random process and W_n is the time of the nth renewal. Let $X_i=W_{i+1}-W_i=L_i+D_i$. Let Φ denote the common distribution of the X_i.

$$M_\Phi(t)=\sum_{n=1}^{\infty}\Phi^n(t). \tag{3.5.8}$$

is the renewal function corresponding to the underlying distribution Φ.

Theorem: 3.5.1(System Availability)

If $Q_I(\infty)<1$ and $Q_G(\infty)<1$, then the system availability function is:

$$\mathfrak{A}(t)=A(t)+\int_0^t A(x)m_\Phi(t-x)dx. \tag{3.5.9}$$

Proof:

The system is up at time t either because the length of the first major interval exceeds time t ($W_1>t$) or $W_n \leq t<W_{n+1}$. That is, there are n system renewals and there is no system renewal before t in the $(n+1)^{st}$ th major interval taken over all possible values of n. From (3.5.7) the probability that the system is functioning and the length of the major interval is longer than t is A(t).

$$\text{Prob[system is up and } t<W_1] = \text{Prob[system is up and } t<X_1] = A(t) \tag{3.5.10}$$

Now

$$\begin{aligned}\mathfrak{A}(t) &= \text{Prob[system is up at time t]}\\ &= \text{Prob[system is up and } t<W_1]+\\ &\quad \sum_{n=1}^{\infty}\text{Prob[system is up at t } \& \ W_n \leq t<W_{n+1}].\end{aligned} \tag{3.5.11}$$

Consider Prob[system is up at t & $W_n \leq t<W_{n+1}$]. Condition on the time of the nth renewal. That is $W_n=u$ yields:

$$\begin{aligned}&\text{Prob[system is up at t } \& \ W_n \leq t<W_{n+1}]\\ &=\int_0^{\infty}\text{Prob[system is up at t } \& \ W_n \leq t<W_{n+1}|W_n=u]d\Phi^{(n)}(u)\\ &=\int_0^{\infty}\text{Prob[system is up at t -u } \& \text{ t-u}<X_{n+1}]d\Phi^{(n)}(u)\end{aligned}$$

From (3.5.10):

$$\text{Prob[system is up at and } t\text{-}u<X_{n+1} \ \& \ t\text{-}u \geq 0]=A(t\text{-}u).$$

Therefore:

$$\text{Prob[system is up at t } \& \ W_n \leq t<W_{n+1}]=\int_0^{\infty} A(t\text{-}u)d\Phi^{(n)}(u) \qquad (3.5.12)$$

Now substituting (3.5.10) and (3.5.12) into (3.5.11) yields:

$$\mathfrak{A}(t)=A(t)+\sum_{n=1}^{\infty}\int_0^{\infty} A(t\text{-}u)d\Phi^{(n)}(u) \qquad (3.5.13)$$

$$=A(t)+\int_0^{\infty} A(t\text{-}u)d\sum_{n=1}^{\infty}\Phi^{(n)}(u)du$$

$$=A(t)+\int_0^{t} A(t\text{-}u)m_{\Phi}(u)dx.$$

When $Q_I(\infty)=1$ and $Q_G(\infty)=1$, then there is no opportunistic replacement in either the initial minor or the general minor intervals. Hence there is no major interval and the availability of the system is the same as the availability during a major interval. That is, the probability of a system renewal at any time t is not defined. The system availability $\mathfrak{A}(t)$ in (3.5.9) reduces to A(t).

Quite often it is easier to analyze the availability function by the use of Laplace transforms. This is especially true when the inverse Laplace transform is available. Where the inverse of the Laplace transform cannot be easily manipulated, numerical approximations can sometimes be used.

Laplace Transform of $\mathfrak{A}(t)$

Taking the Laplace transform of $\mathfrak{A}(t)$ in (3.5.9) yields:

$$\mathfrak{A}^*(s)=A^*(s)+A^*(s)m_\Phi{}^*(s). \qquad (3.5.14)$$

which simplifies to

$$\mathfrak{A}^*(s)=A^*(s)(1+m_\Phi{}^*(s)). \qquad (3.5.15)$$

Differentiating $M_\Phi(t)$ in (3.5.8) yields:

$m_\Phi(t)=\sum_{k=1}^{\infty}\phi^{(k)}(t)$, and taking the Laplace transforms yields:

$$m_\Phi(s)=\frac{\phi^*(s)}{1-\phi^*(s)}. \qquad (3.5.16)$$

Substituting (3.5.15) into (3.5.16) yields:

$$\mathfrak{A}^*(s)=\frac{A^*(s)}{1-\phi^*(s)} \qquad (3.5.17)$$

From (3.5.7) the Laplace transform of $A(t)$ is :

$$A^*(s)=\overline{Z}_I^*(s)+\overline{Z}_G^*(s)m_Q{}^*(s). \qquad (3.5.18)$$

Substituting $m_Q^*(s)$ in (3.4.14) into (3.5.18) yields:

$$A^*(s)=\overline{Z}_I^*(s)+\overline{Z}_G^*(s)\frac{q_I^*(s)}{1-q_G^*(s)}, \tag{3.5.19}$$

and substituting the $A^*(s)$ in (3.5.17) into (3.5.18) yields:

$$\mathfrak{A}^*(s)=\frac{\overline{Z}_I^*(s)+\overline{Z}_G^*(s)\frac{q_I^*(s)}{1-q_G^*(s)}}{1-\phi^*(s)},$$

Simplifying the above expression yields:

$$\mathfrak{A}^*(s)=\frac{\overline{Z}_I^*(s)\left(1-q_G^*(s)\right)+\overline{Z}_G^*(s)q_I^*(s)}{\left(1-\phi^*(s)\right)\left(1-q_G^*(s)\right)}. \tag{3.5.20}$$

The inverse Laplace transform of the above expression gives the time dependent system availability for the specific model used. Often evaluating time dependent availabiltiy is cumbersome so instead the limiting availability is used. Therefore the limiting is also constructed.

Theorem:

The long run system availability $\mathfrak{A}$ is:

$$\mathfrak{A}=\lim_{t\to\infty}\mathfrak{A}(t)=\frac{\int_0^\infty A(t)dt}{\mu_L+\mu_{opp}}, \tag{3.5.22}$$

where

μ_L=E[L]=mean duration of a major interval,

μ_{opp}=E[D]=mean opportunistic replacement repair time.

Proof:

From (3.5.9)

$$\mathfrak{A}(t)=A(t)+\int_0^t A(x)m_\Phi(t-x)dx.$$

To prove the above theorem we need to show that as time becomes very large the availability in a minor interval is zero. That is:

$$\lim_{t\to\infty} A(t)=Prob[X(t)=1, \text{there is no system renewal}]=0.$$

Now the probability that the system is functioning and the length of the major interval is longer than t is:

$$Prob[\text{system is up, there is no system renewal}]=Prob[\text{system is up}, L>t]. \quad (3.5.23)$$

Now:

$$Prob[\text{system is up}, L>t] \leq P[L>t], \quad (3.5.24)$$

from (3.5.23) and (3.5.24) we have:

$$\lim_{t\to\infty} A(t)< \lim_{t\to\infty} P[L>t]=0.$$

Therefore using the key renewal theorem we obtain the required result.

Now :

$$\int_0^\infty A(t)dt=\lim_{s\to 0} A^*(s)=\lim_{s\to 0}\left(\overline{Z}_I^*(s)+\overline{Z}_G^*(s)m_Q^*(s)\right),$$

$$=\left(\nu_I+\nu_G{}^*M_Q(\infty)\right)=\left(\nu_I+\frac{\nu_G Q_I(\infty)}{1-Q_G(\infty)}\right). \quad (3.5.25)$$

where:

$$\nu_I = \int_0^\infty t\, Z_I(t)dt = \int_0^\infty \overline{Z}_I(t)dt = \overline{Z}_I^*(s)\Big|_{s=0}$$

and

$$\nu_G = \int_0^\infty t\, Z_G(t)dt = \int_0^\infty \overline{Z}_G(t)dt = \overline{Z}_G^*(s)\Big|_{s=0}$$

This is the expected operating time in a major interval. Substituting the average length of the major interval in (3.4.16) and the expression in (3.5.25) into (3.5.22) yields:

$$\mathfrak{A} = \lim_{t\to\infty} \mathfrak{A}(t) = \frac{\int_0^\infty A(t)dt}{\mu_L + \mu_{opp}} = \frac{\left(\nu_I + \frac{\nu_G Q_I(\infty)}{1-Q_G(\infty)}\right)}{\mu_L + \mu_{opp}}. \tag{3.5.26}$$

$$= \frac{\left(\nu_I + \frac{\nu_G Q_I(\infty)}{1-Q_G(\infty)}\right)}{(\mu_{IR} + \mu_I) + \frac{Q_I(\infty)}{1-Q_G(\infty)}(\mu_{GR} + \mu_G) + \mu_{opp}}$$

Simplification of (3.5.26) yields:

$$\mathfrak{A} = \frac{(\nu_I(1-Q_G(\infty)) + \nu_G Q_I(\infty))}{(\mu_{IR} + \mu_I + \mu_{opp})(1-Q_G(\infty)) + (\mu_{GR} + \mu_G)Q_I(\infty)}. \tag{3.5.27}$$

This is the availability for the general opportunistic replacement model if the minor intervals are transient. If the minor intervals are recurrent, that is, if $Q_I(\infty) = Q_G(\infty) = 1$, then there are no opportunistic replacements. In that case $\mathfrak{A}(t) = A(t)$, and from (3.5.27),

$$\mathfrak{A}=\mathrm{A}=\lim_{t\to\infty}\mathrm{A}(t)=\frac{\nu_G}{\mu_G} \tag{3.5.28}$$

3.6 Summary

The system availability constructed in this chapter applies to all realizations of the model stated earlier in the chapter. For each of the stages, the construction of H follows a specific set of charcteristics. Therefore the key to the construction of availability measure for the system for the various stages is the construction of H. The limiting availability expression in (3.5.27) is applicable when the minor intervals are either transient or recurrent. The same can be said about the time dependent case.

Chapter 4 contains the contruction of H for a failure model in which replacement of a component is made only at failure and also the construction of H for an opportunistic failure model in which there is single replacement of a component at failure and simultaneous replacement of both components at the failure of a single component.

Chapter 5 contains the construction of H for a partial opportunistic age replacement policy in which where there is single replacement of a component at failure for both components and also for one of the components when it has attained a time equal to its age replacement time without failure. There is also the opportunistic replacement of both components.

CHAPTER 4 Failure Type Models

4.1 Introduction

In this chapter we construct a failure replacement model and an opportunistic failure replacement model for a two component series system. The failure model is used as a basis for the expanded models and to validate the modeling approach. A failure replacement model represents the behavior of the system when the only interruptions to the system operation are those that result from the failure of a component. For this case, there is no point in time at which both components are replaced simultaneously. Thus there are no system renewals.

4.2 Failure Model

4.2.1 System Renewal distribution:

To define a model for the behavior of the two component series system, note first that numbering of the components is arbitrary. Therefore, as stated in section 3.1, adopt the convention that component 2 renewals are to be portrayed. Clearly, each component 2 renewal occurs between two successive component 1 replacements, say the k^{th} and $k+1^{st}$. System operation starts with both components new. Subsequently, system restarts following a component 2 replacement occur with component 2 being new and component

1 being used. We model the first operating period until the first component 2 repair completion as an "initial" interval. We call the subsequent intervals "general " intervals. The probability models for the two types of intervals are defined in the following paragraphs. The definitions are based on the general model structure presented in chapter 3. The model is made specific to a failure replacement policy or an opportunistic failure replacement policy by the definition of the distribution functions H_I and H_G.

4.2.2 Initial Interval

In the initial operating interval, both components start new. The probability that component 2 fails between the k^{th} and the $k+1^{st}$ failure time of component 1 before time t is:

$$H_I(t,k)=P[S_{k,1} \le X_{1,2} < S_{k+1,1}], \tag{4.2.1}$$

Conditioning on $S_{k,1}=u$ and $X_{12}=w$ yields:

$$H_I(t,k)=\int_w \int_u P[S_{k,1} \le X_{1,2} < S_{k,1}+X_{k+1,1} | S_{k,1}=u; X_{1,2}=w] f_i^{(k)}(u) f_2(w) du dw$$

$$=\int_{w-u\ge 0} \int_u P[w-u < X_{k+1,1}] f_1^{(k)}(u) f_2(w) du dw .$$

Since w-u ≥ 0 and u ≥ 0, the $0 \le u \le w$, the expression (4.2.1) reduces to:

$$H_I(t,k) = \begin{cases} \int_0^t \int_0^w \bar{F}_1(w-u) f_1^{(k)}(u) f_2(w) du dw, & k \ge 1 \\ \int_0^t \bar{F}_1(w) f_2(w) du dw, & k=0 \end{cases} . \tag{4.2.2}$$

Now differentiating the above expression yields:

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=\begin{cases} \int_0^t \bar{F}_1(t\text{-}u)f_1^{(k)}(u)f_2(t)du, & k\geq 1 \\ \bar{F}_1(t)f_2(t)du, & k=0 \end{cases} \quad . \tag{4.2.3}$$

Observe that $H_I(t,k)$ is a cumulative probability measure on t conditioned on k, the number of component 1 replacements.

4.2.3 General Interval

For the particular case of the failure replacement policy, there are no points in time at which both components are replaced simultaneously. Consequently, there are no true system renewal points and there are no major intervals. As we model the system in terms of component 2 renewals, each period of system operation following the first component 2 replacement starts with component 2 new and component 1 used. Therefore, the time to the next component 1 failure has the corresponding residual life distribution. Cox[9] defines this distribution function to be :

$$\tilde{F}(t)=(1/\mu)\int_0^t \bar{F}(x)dx. \tag{4.2.4}$$

in general. For the model developed here, the application of the form given by Cox is:

$$\tilde{F}_1(t)=(1/\mu_1)\int_0^t \bar{F}_1(x)dx. \tag{4.2.5}$$

with the corresponding density of $\tilde{f}_1(t)$. For component 1, then, the probability density on the time of the k^{th} failure following a restart is the convolution of $\tilde{f}_1(t)$ and $f_1^{(k-1)}(t)$. This density is represented here as $\tilde{f}_1^{(k)}(t)$.

Arguments for the general interval are similar to those for the initial interval. For the general interval, the probability that component 2 failure occurs between the k^{th} and $k+1^{st}$ component 1 failure is :

$$H_G(t,k)=P[\tilde{S}_{k,1} \le X_{1,2} < \tilde{S}_{k+1,1}]\,.$$

Conditioning on $\tilde{S}_{k,1}=u$ and $X_{12}=w$ yields

$$H_G(t,k)=\int_w\int_u P[\tilde{S}_{k,1} \le X_{1,2} < \tilde{S}_{k,1}+X_{k+1,1}\,|\,\tilde{S}_{k,1}=u;X_{1,2}=w]\tilde{f}_1^{(k)}(u)f_2(w)dudw$$

$$=\int_{w-u\ge 0}\int_u P[w-u < X_{k+1,1}]\tilde{f}_1^{(k)}(u)f_2(w)dudw. \qquad (4.2.6)$$

Since $w-u \ge 0$ and $u \ge 0$, then $0 \le u \le w$, and:

$$H_G(t,k)=\begin{cases} \int_0^t f_2(w)\int_0^w \overline{F}_1(w-u)\tilde{f}_1^{(k)}(u)dudw, & k \ge 1 \\ \int_0^t f_2(w)\,\overline{\overline{F}}_1(w)dw, & k=0. \end{cases} \qquad (4.2.7)$$

Differentiating the above equation yields:

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=\begin{cases} f_2(t)\int_0^w \overline{F}_1(t-u)\tilde{f}_1^{(k)}(u)dt, & k \ge 1 \\ f_2(t)\,\overline{\overline{F}}_1(t)dw, & k=0. \end{cases} \qquad (4.2.8)$$

Again observe that $H_G(t,k)$ is a cumulative probability measure on t conditioned on k, the number of component 1 replacements.

4.2.4 Convolution of repair time and failure time of the system

If during either an initial or a general minor interval, there are k component 1 failures and 1 component 2 failure then the duration of the interval will include k component 1 repair times and 1 component 2 repair time. The total amount of time devoted to repair during the interval must therefore have the density function defined as:

$$g(t,k)=\int_0^t g_1^{(k)}(x)g_2(t-x)dx \tag{4.2.9}$$

For both types of operating interval, the probability density on the length of the interval is the convolution of the density on the repair time and that on operating time. This may be constructed by taking the convolution and summing over k. From (3.3.2) the density function on the length of an initial minor interval is:

$$q_I(t)=\sum_{k=0}^{\infty}\int_0^t h_I(x,k)g(t-x,k)dx$$

and substituting (4.2.3) and (4.2.9) into the above expression yields:

$$q_I(t)=\int_0^t f_2(w)\bar{F}_1(w)g_2(t-w)dw+\sum_{k=1}^{\infty}\int_0^t\int_0^w \bar{F}_1(w-u)f_1^{(k)}(u)f_2(w)g(t-w,k)dudw \tag{4.2.10}$$

The corresponding distribution function on the length of the initial minor interval as defined is :

$$Q_I(t)=\int_0^t q_I(x)dx. \tag{4.2.11}$$

Similarly from (3.3.3) for the general interval, the convolution of the repair time and the operating time of the system is:

$$q_G(t)=\sum_{k=0}^{\infty}\int_0^t h_G(x,k)g(t-x,k)dx.$$

Substituting (4.2.4) and (4.2.9) into the above expression yields:

$$q_G(t)=\int_0^t f_2(w)\bar{\bar{F}}_1(w)g_2(t-w)dw+\sum_{k=1}^{\infty}\int_0^t\int_0^w \bar{F}_1(w-u)\tilde{f}_1^{(k)}(u)f_2(w)g(t-w,k)dudw \quad (4.2.12)$$

for which the corresponding distribution function is:

$$Q_G(t)=\int_0^t q_G(x)dx. \quad (4.2.13)$$

Observe finally that the average length of a general minor interval is computed as:

$$\mu_G=\int_0^{\infty} tq_G(t)dt. \quad (4.2.14)$$

4.2.5 Renewal density:

The model defined above for the period of system operation corresponds to a "modified renewal process". Cox[9] defines a modified renewal process as a renewal process in which the duration of all intervals other than the first are i.i.d. and the duration of the first interval is independent of the other intervals and has a different distribution. Cox further shows that for a modified renewal process, the renewal density is the convolution of the density on the length of the initial minor interval and the length of n general minor intervals for all possible values of n. Thus as in (3.4.1) the renewal density is:

$$m_Q(t)=\sum_{n=0}^{\infty}\int_0^t q_I(x)q_G^n(t-x)dx. \quad (4.2.15)$$

In the model defined above, $q_G(t)$ is the density on the duration of the general minor intervals without system renewal. It includes both operating and repair times. In order to obtain some of the interesting system availability measures, it is useful to construct the density on the total duration of the system operation during an interval.

From (3.3.6) and returning to expression (4.2.2), the cumulative distribution on the total operating time during an initial minor interval without system renewal is:

$$Z_I(t)=\sum_{k=0}^{\infty}H_I(t,k)=\sum_{k=0}^{\infty}\int_0^t\int_0^w \bar{F}_1(w-u)f_1^{(k)}(u)f_2(w)dudw$$

$$=\int_0^t f_2(w)\sum_{k=0}^{\infty}\int_0^w (1-F_1(w-u))f_1^{(k)}(u)dudw$$

$$=\int_0^t f_2(w)\sum_{k=0}^{\infty}\left\{\int_0^w f_1^k(u)du-\int_0^w F_1(w-u)f_1^{(k)}(u)du\right\}dw$$

$$=\int_0^t f_2(w)\sum_{k=0}^{\infty}\left\{F_1^{(k)}(w)du-F_1^{(k+1)}(w)du\right\}dw=\int_0^t f_2(w)\sum_{k=0}^{\infty}\left\{P[N(w)=k]\right\}dw$$

$$Z_I(t)=\int_0^t f_2(w)dw=F_2(t). \tag{4.2.16}$$

Similarly from (3.3.7) and (4.2.3), for the general minor interval:

$$Z_G(t)=\sum_{k=0}^{\infty}H_G(t,k)=\sum_{k=0}^{\infty}\int_0^t\int_0^w \bar{F}_1(w-u)\tilde{f}_1^{(k)}(u)f_2(w)dudw=\int_0^t f_2(w)dw=F_2(t). \tag{4.2.17}$$

These are important results that are intuitively appealing but were not apparent in the model construction. An important implication of the results in (4.2.16) and (4.2.17) is that for all points in time, the average probability that the duration of system operation during an interval is greater than or equal to t units of time is $\bar{F}_2(t)$. It should be noted that the result of (4.2.16) and (4.2.17) apply specifically to the failure replacement models. In effect, these two results provide partial validation of the model of chapter 3 because they show that simplification of the overall model to the simplest case yields an appropraite model form.

Given the result of (4.2.17) and the identies of (3.4.23) and (3.4.24), observe that $Z_I(\infty)=Z_G(\infty)=1$ and $Q_I(\infty)=Q_G(\infty)=1$. Hence the system is recurrent and there are no opportunistic replacements at any point.

4.2.6 Availability

System availability is the appropriate measure of effectiveness for a repairable system. As stated above the system is recurrent. Therefore the availability function $\mathfrak{A}(t)$ defined in (3.5.3) reduces to A(t), the first term in the availability function. From (4.2.16), (4.2.17) and (3.5.3), the availability measure for the two component system defined above is:

$$\mathfrak{A}(t)=A(t)=\bar{F}_2(t)+\int_0^t \bar{F}_2(x)m_Q(t-x)dx. \qquad (4.2.18)$$

Observe that this is a new result that has not been developed previously. To verify its accuracy, we evaluate the limit of the expression as the limiting availability which has been determined and given in Barlow and Proschan[1] . The result validates expression (4.2.18).

Theorem 4.2.1: For the above defined two component series system,

$$\lim_{t\to\infty} \mathfrak{A}(t) = \left(1+\frac{\nu_1}{\mu_1}+\frac{\nu_2}{\mu_2}\right)^{-1}$$

where ν_i is the mean of the distribution $G_i(t)$ and μ_i is the mean of the distribution $F_i(t)$.

Proof:

Application of the key renewal theorem to (4.2.18) yields:

$$\lim_{t\to\infty} \mathfrak{A}(t) = \lim_{t\to\infty} \int_0^t \bar{F}_2(t-x) m_Q(x)\, dx$$

$$= \frac{1}{\mu_G} \int_0^\infty \bar{F}_2(t) dt = \frac{\mu_2}{\mu_G} \tag{4.2.20}$$

Thus to establish the result, it is necessary to evaluate μ_G which is defined in (4.2.14). Of course, μ_G is also equal to the first derivative of the Laplace transform of $q_G(t)$ multiplied by -1 and evaluated at s=0. That is:

$$\mu_G = -\frac{d}{ds} q_G^*(s)\Big|_{s=0} \tag{4.2.21}$$

where

$$q_G^*(s) = \int_0^\infty e^{-st} q_G(t) dt \tag{4.2.22}$$

Now expanding (4.2.12) yields:

$$\frac{d}{ds}q_G^*(s)= \sum_{k=0}^{\infty}h_G^*(s,k)\frac{d}{ds}g^*(s,k) +\sum_{k=0}^{\infty}g^*(s,k)\frac{d}{ds}h_G^*(s,k) \tag{4.2.23}$$

To analyze this derivative, observe that:

$$\int_0^t \overline{F}_1(t-u)\widetilde{f}_1^{(k)}(u)\, du =\int_0^t (1-F_1(t-u))\, \widetilde{f}_1^{(k)}(u)\, du=\widetilde{F}_1^{(k)}(t) - \widetilde{F}_1^{(k+1)}(t)$$

$$=P[\widetilde{N}(t)=k]. \tag{4.2.24}$$

and:

$$h_G^*(s,k)=\int_0^{\infty} e^{-st}h_G(t,k)dt=\int_0^{\infty} e^{-st}f_2(t) \int_0^t \overline{F}_1(t-u)\widetilde{f}_1^{(k)}(u)\, du\, dt$$

$$=\int_0^{\infty} e^{-st}f_2(t)\, P[\widetilde{N}_1(t)=k]dt. \tag{4.2.25}$$

Then:

$$\frac{d}{ds}h_G^*(s,k)=\int_0^{\infty} -t\, e^{-st}f_2(t)\, P[\widetilde{N}_1(t)=k]dt. \tag{4.2.26}$$

Similarly:

$$g^*(s,k)=\int_0^{\infty} e^{-st}g(t,k)dt$$

$$=\int_0^{\infty} e^{-st} \int_0^{\infty} g_1^{(k)}(x)g_2(t-x)dxdt$$

$$= [g_1^*(s)]^k g_2(s). \tag{4.2.27}$$

so:

$$\frac{d}{ds}g^{*}(s,k)=\frac{d}{ds}[g_1^{*}(s)]^{k}g_2(s)=g_2^{*}(s)\frac{d}{ds}\left(g_1^{*}(s)\right)^{k}+\left(g_1^{*}(s)\right)^{k}\frac{d}{ds}g_2^{*}(s)$$

$$=k\ g_2(s)\left(g_1^{*}(s)\right)^{k-1}\frac{d}{ds}g_1^{*}(s)+\left(g_1^{*}(s)\right)^{k}\frac{d}{ds}g_2^{*}(s).$$

$$=k\ g_2^{*}(s)\left(g_1^{*}(s)\right)^{k-1}\int_0^{\infty}-t\ e^{-st}g_1(t)dt+\left(g_1^{*}(s)\right)^{k}\int_0^{\infty}-t\ e^{-st}g_2(t)dt. \qquad (4.2.28)$$

Now, returning to (4.2.23) and substituting (4.2.24) to (4.2.28) into(4.2.23) yields:

$$\frac{d}{ds}q_G^{*}(s)=\sum_{k=0}^{\infty}h_G^{*}(s,k)\frac{d}{ds}g^{*}(s,k)+\sum_{k=0}^{\infty}g^{*}(s,k)\frac{d}{ds}h_G^{*}(s,k)$$

$$=\sum_{k=0}^{\infty}\left(\int_0^{\infty}e^{-st}f_2(t)\ P[\tilde{N}_1(t)=k]dt\right)*$$

$$\left(kg_2^{*}(s)[g_1^{*}(s)]^{k-1}\int_0^{\infty}-t\ e^{-st}g_1(t)dt+[g_1^{*}(s)]^{k}\int_0^{\infty}-t\ e^{-st}g_2(t)dt\right)+$$

$$\sum_{k=0}^{\infty}\left([g_1^{*}(s)]^{k}g_2^{*}(s)\right)\left(\int_0^{\infty}-t\ e^{-st}f_2(t)\ P[\tilde{N}_1(t)=k]dt\right) \qquad (4.2.29)$$

Therefore

$$\frac{d}{ds}q_G^{*}(s=0)=\sum_{k=0}^{\infty}\left(\int_0^{\infty}f_2(t)\ P[\tilde{N}_1(t)=k]dt\right)*\left(k\int_0^{\infty}-t\ g_1(t)dt+\int_0^{\infty}-t\ g_2(t)dt\right)+$$

$$\sum_{k=0}^{\infty}\int_0^{\infty}-t\ f_2(t)\ P[\tilde{N}_1(t)=k]dt$$

$$=\int_0^{\infty} f_2(t)\left(\sum_{k=0}^{\infty} P[\tilde{N}_1(t)=k]\,(-k\,\nu_1-\nu_2)\right)dt+\int_0^{\infty} -t\, f_2(t)dt$$

$$=-\nu_1\int_0^{\infty} f_2(t)\sum_{k=0}^{\infty} k\, P[\tilde{N}_1(t)=k]dt-\nu_2\int_0^{\infty} f_2(t)\sum_{k=0}^{\infty} P[\tilde{N}_1(t)=k]dt-\int_0^{\infty} t\, f_2(t)dt$$

$$=-\nu_1\int_0^{\infty} f_2(t)\frac{t}{\mu_1}dt-\nu_2-\mu_2=-\frac{\nu_1}{\mu_1}\mu_2-\nu_2-\mu_2 \qquad (4.2.30)$$

This implies that:

$$\mu_G=\frac{\nu_1}{\mu_1}\mu_2+\nu_2+\mu_2$$

and the corresponding expression for the limiting availability is:

$$\mathfrak{A}=\lim_{t\to\infty}\mathfrak{A}(t)=\frac{\mu_2}{\mu_G}=\frac{\mu_2}{\frac{\nu_1}{\mu_1}\mu_2+\nu_2+\mu_2}=\left(1+\frac{\nu_1}{\mu_1}+\frac{\nu_2}{\mu_2}\right)^{-1}. \qquad (4.2.31)$$

This completes the proof. As indicated above, the fact that the limiting value of the availability function for the defined model matches the general result given by Barlow and Proschan is considered to validate the model developed here. It is therefore considered that the new model summarized in the expression (4.2.18) represents the time dependent availability of a two component series system. It is also observed that the model is general in that no specific probability distributions are assumed during the development of the model. As a final point, it can be noted that if the positions of the components are interchanged the long run availability is not affected.

4.3 OPPORTUNISTIC FAILURE REPLACEMENT POLICY

The first extension of the failure model that is examined here is the opportunistic failure replacement policy. Under an opportunistic failure replacement strategy, each of the components is replaced at failure and both components are replaced at the same time when at the failure of one component, the other component has an age greater than a certain threshold value (the opportunistic replacement time of the component). As explained in the general case in Chapter 3, the behavior of the system operated under an opportunistic failure replacement strategy is modeled as a nested renewal process. It is assumed that the system is renewed whenever both components are replaced. The renewal intervals over which the system is renewed are called major intervals and they are comprised of a number of intervals which end with component 2 replacement only. These are called minor intervals, and as stated in chapter 3 these intervals are analyzed for the case in which there are no opportunistic replacements.

A component 2 replacement may occur as a result of a failure of component 2 or an implementation of the opportunistic replacement strategy. The probability of one of these is the complement of the other. Therefore the following model is developed for the case in which the minor intervals end with a component 2 failure. As in the case of the failure replacement model, the first minor interval, the "initial minor interval", starts with both components new. Subsequent general minor intervals starts with component 1 used. The following is the construction of the renewal model of system operation for the case in which system renewal does not occur.

4.3.1 Initial minor interval(No Renewal)

As in the failure replacement model in section 4.2, assume that the time of the component 2 failure lies between the times of the k^{th} and $k+1^{st}$ component 1 failure. In an initial minor interval that ends without system renewal the time between the k^{th} component 1 failure time and the component 2 failure time is less than the opportunistic replacement time of component 1. Otherwise both components are replaced at the same time and system renewal occurs. Also the time of the k^{th} component 1 failure is less than the opportunistic replacement time of component 2.

If both components start new, then the probability that component 2 fails between the k^{th} and the $k+1^{st}$ failure time of component 1 is:

$$H_I(t,k)=P[S_{k,1} \leq X_{1,2} < S_{k+1,1}]$$

$$=\int_{A_1}\int P[S_{k,1} \leq X_{1,2} < S_{k,1}+X_{k+1,1} \mid S_{k,1}=u; X_{1,2}=w] f_1^{(k)}(u) f_2(w)\,du\,dw$$

$$=\int_{A_1}\int P[w-u < X_{k+1,1}] f_1^{(k)}(u) f_2(w)\,du\,dw$$

$$=\int_{A_1}\int \bar{F}_1(w-u) f_1^{(k)}(u) f_2(w)\,du\,dw \tag{4.3.1}$$

For the case in which k=0, there is no component 1 failure before the time of the component 2 failure. In such a situation, we have:

$$H_I(t,0)=P[S_{0,1} \leq X_{1,2} < S_{1,1}]=\int_0^t \bar{F}_1(w) f_2(w)\,dw. \tag{4.3.2}$$

To represent the cases in which system renewal does not occur, the time of the component 2 failure should not exceed τ_1. Therefore $0 \leq t < \tau_1$ and the corresponding density function is:

$$h_I(t,0)=\bar{F}_1(t)f_2(t) \qquad 0 \leq t<\tau_1 \tag{4.3.3}$$

For the case in which $k \geq 1$, system renewal occurs if any of the component 1 failures occur after τ_2 time units. Alternatively, system renewal occurs if the interval between the k^{th} component 1 failure and the time of the component 2 failure exceeds τ_1. Therefore, to represent the cases in which system renewal does not occur, expression (4.3.1) applies over the time intervals in A_1. Where in general:

$$A_1=\{(w,u)|\{0 \leq w-u < \tau_1,\ 0 \leq u< \tau_2\}\}. \tag{4.3.4}$$

However, this general statement must be specified more carefully and in terms of relationship between τ_1 and τ_2. As indicated above, when k=0, the failure times of component 2 that do not imply system renewal are those in $0 \leq t <\tau_1$. This is true regardless of the relative magnitudes of τ_1 and τ_2 so expression (4.3.3) applies in both the $\tau_1<\tau_2$ and the $\tau_1 \geq \tau_2$ cases. For the values $k \geq 1$, when $\tau_1<\tau_2$, expression (4.3.1) applies over the ranges defined by :

$$A_1=\{(w,u)|(0 \leq w < \tau_1,\ 0 \leq u < w) \cup (\tau_1 \leq w < \tau_2,\ w-\tau_1 \leq u < w)$$
$$\cup\ (\tau_2 \leq w < \tau_1+\tau_2,\ w-\tau_1 \leq u < \tau_2)\} \tag{4.3.5}$$

The reason for this definition is the following:

i) If component 2 fails at time w $(0 \leq w < \tau_1)$, any number of component 1 failures could occur before time w without opportunistic replacement. That is, if the k^{th} component 1 failure time is at u, then $0 \leq u < w$.

ii) If component 2 fails at time w $(\tau_1 \leq w < \tau_2)$ and the k^{th} component 1 failure is at time u, then opportunistic replacement does not occur if the time between the component 2 failure and the time of the last component 1 failure does not exceed τ_1. That is, there is no system renewal if $w-u < \tau_1$ or $u > w-\tau_1$. By construction, $0 \leq u < w$.

iii) Finally, if component 2 fails at time w $(w \geq \tau_2)$ and the k^{th} component 1 failure occurs at time u, then there is no system renewal if component 1 fails at time u where $u < \tau_2$ and $w-u < \tau_1$. These two conditions imply that $w-\tau_1 < u < \tau_2$. These conditions hold if $w \leq \tau_1+\tau_2$.

The combined applications of the conditions enumerated yield (4.3.5) and the corresponding full statement of (4.3.1) is:

$$H_I(t,k)=\int_0^t \int_0^w \overline{F}_1(w-u)f_1^{(k)}(u)f_2(w)dudw \qquad 0 \leq t<\tau_1$$

$$H_I(t,k)=\int_0^{\tau_1} \int_0^w \overline{F}_1(w-u)f_1^{(k)}(u)f_2(w)dudw + \int_{\tau_1}^t \int_{w-\tau_1}^w \overline{F}_1(w-u)f_1^{(k)}(u)f_2(w)dudw \qquad \tau_1 \leq t<\tau_2$$

$$H_I(t,k)=\int_0^{\tau_1} \int_0^w \overline{F}_1(w-u)f_1^{(k)}(u)f_2(w)dudw + \int_{\tau_1}^{\tau_2} \int_{w-\tau_1}^w \overline{F}_1(w-u)f_1^{(k)}(u)f_2(w)dudw + \int_{\tau_2}^t \int_{w-\tau_1}^{\tau_2} \overline{F}_1(w-u)f_1^{(k)}(u)f_2(w)dudw \qquad \tau_2 \leq t<\tau_1+\tau_2 \qquad (4.3.6).$$

Hence:

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=\int_0^t \bar{F}_1(t-u)f_1^{(k)}(u)f_2(t)du \qquad 0 \le t<\tau_1$$

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=\int_{t-\tau_1}^t \bar{F}_1(t-u)f_1^{(k)}(u)f_2(t)du \qquad \tau_1 \le t<\tau_2$$

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=\int_{t-\tau_1}^{\tau_2} \bar{F}_1(t-u)f_1^{(k)}(u)f_2(t)du \qquad \tau_2 \le t<\tau_1+\tau_2. \qquad (4.3.7)$$

A similar but simpler construction applies when $\tau_1 \ge \tau_2$. Again under the cases in which $k \ge 1$. The reasoning is as follows:

i) If component 2 fails at time w $(0 \le w<\tau_2)$ and u is the time of the last component 1 failure before component 2 fails, then no system renewal occurs when $0 \le u<w$.

ii) If component 2 fails at time w $(\tau_2 \le w<\tau_1)$ and u is the time of the last component 1 failure before component 2 fails, system renewal does not occur if $0 \le u<\tau_2$.

iii) Finally if component 2 fails at time w $(w \ge \tau_1)$ and u is the time of the last component 1 failure before component 2 failure, no system renewal occurs if the time of the component 1 failure does not exceed τ_2 $(u<\tau_2)$. However it must also be the case that $w-u<\tau_1$. The combined application of these conditions yields $w-\tau_1<u<\tau_2$ which in turn imply that $w \le \tau_1+\tau_2$.

Therefore for $\tau_1>\tau_2$, then the set A_1 is qualified as:

$$A_1=\{(w,u)|(0 \le w<\tau_2, 0 \le u<w) \cup (\tau_2 \le w<\tau_1, 0 \le u<\tau_2) \cup$$

$$(\tau_1 \le w<\tau_1+\tau_2,\ w-\tau_1 \le u<\tau_2)\}. \tag{4.3.8}$$

Implementing this definition in expression (4.3.1) yields:

$$H_I(t,k)=\int_0^t\int_0^w \overline{F}_1(w-u)f_1^{(k)}(u)f_2(w)dudw \qquad 0 \le t<\tau_2$$

$$H_I(t,k)=\int_0^{\tau_2}\int_0^w \overline{F}_1(w-u)f_1^{(k)}(u)f_2(w)dudw + \int_{\tau_2}^t\int_0^{\tau_2} \overline{F}_1(w-u)f_1^{(k)}(u)f_2(w)dudw$$

$$\tau_2 \le t<\tau_1$$

$$H_I(t,k)=\int_0^{\tau_2}\int_0^w \overline{F}_1(w-u)f_1^{(k)}(u)f_2(w)dudw + \int_{\tau_2}^{\tau_1}\int_0^{\tau_2} \overline{F}_1(w-u)f_1^{(k)}(u)f_2(w)dudw +$$

$$\int_{\tau_1}^t\int_{w-\tau_1}^{\tau_2} \overline{F}_1(w-u)f_1^{(k)}(u)f_2(w)dudw \qquad \tau_1 \le t<\tau_1+\tau_2 \tag{4.3.9}$$

Hence:

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=\int_0^t \overline{F}_1(t-u)f_1^{(k)}(u)f_2(t)du \qquad 0 \le t<\tau_2$$

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=\int_0^{\tau_2} \overline{F}_1(t-u)f_1^{(k)}(u)f_2(t)du \qquad \tau_2 \le t<\tau_1$$

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=\int_{t-\tau_1}^{\tau_2} \overline{F}_1(t-u)f_1^{(k)}(u)f_2(t)du \qquad \tau_1 \le t<\tau_1+\tau_2 \tag{4.3.10}$$

4.3.2 General Minor Interval(No Renewal)

The same reasoning that is used for the initial minor interval can be applied to the general minor interval. At the start of a general minor interval, component 1 has a residual life distribution and component 2 starts new. As always the interval ends with a component 2 failure. The generic statement for the probability that a general minor interval ends without system renewal at a time not exceeding t and following the k^{th} component 1 failure is given in expression (4.3.11) below. This expression is the conceptual analog of (4.3.1).

$$H_G(t,k)=P[\tilde{S}_{k,1} \leq X_{1,2} < \tilde{S}_{k+1,1}]$$

$$=\int_{A_2}\int P[\tilde{S}_{k,1} \leq X_{1,2} < \tilde{S}_{k,1}+X_{k+1,1} \mid \tilde{S}_{k,1}=u; X_{1,2}=w]\tilde{f}_i^{(k)}(u)f_2(w)dudw$$

$$=\int_{A_2}\int P[w\text{-}u < X_{k+1,1}]\tilde{f}_1^{(k)}(u)f_2(w)dudw$$

$$=\int_{A_2}\int \bar{F}_1(w\text{-}u)\tilde{f}_1^{(k)}(u)f_2(w)dudw \qquad (4.3.11)$$

The applicable sample paths are defined by the set A_2 which can be constructed using the same conditions as applied in the definition of A_1. The single difference is that component 1 is considered to have age corresponding to the average backward recurrence time. Using the results of Cox[9]; the average age of component 1 is:

$$a_1=\frac{\mu_1^2+\sigma_1^2}{2\,\mu_1} \qquad (4.3.12)$$

where the construction of this form is shown in appendix 1.

Now, using this form, the expression corresponding to (4.3.2) and (4.3.3) for the general minor interval when k=0 are:

$$H_G(t,0)=P[\tilde{S}_{0,1} \leq X_{1,2} < \tilde{S}_{1,1}]=\int_0^t \bar{\bar{F}}_1(w)f_2(w)dw \quad 0 \leq t < \tau_1\text{-}a_1 \qquad (4.3.13)$$

$$h_G(t,0)=\frac{d}{dt}H_G(t,0)=\bar{\bar{F}}_1(t)f_2(t) \qquad 0 \le t<\tau_1-a_1 \qquad (4.3.14)$$

For the case in which $k \ge 1$, the reasoning that applied to A_1 also applies to A_2. In fact $A_2=A_1$. Therefore applying the sample path set conditions to expression (4.3.11) for the situation in which $\tau_1<\tau_2$ yields:

$$H_G(t,k)=\int_0^t\int_0^w \bar{F}_1(w-u)\tilde{f}_1^{(k)}(u)f_2(w)dudw \qquad 0 \le t<\tau_1$$

$$H_G(t,k)=\int_0^{\tau_1}\int_0^w \bar{F}_1(w-u)\tilde{f}_1^{(k)}(u)f_2(w)dudw+\int_{\tau_1}^t\int_{w-\tau_1}^w \bar{F}_1(w-u)\tilde{f}_1^{(k)}(u)f_2(w)dudw$$

$$\tau_1 \le t<\tau_2$$

$$H_G(t,k)=\int_0^{\tau_1}\int_0^w \bar{F}_1(w-u)\tilde{f}_1^{(k)}(u)f_2(w)dudw+\int_{\tau_1}^{\tau_2}\int_{w-\tau_1}^w \bar{F}_1(w-u)\tilde{f}_1^{(k)}(u)f_2(w)dudw+$$

$$\int_{\tau_2}^t\int_{w-\tau_1}^{\tau_2} \bar{F}_1(w-u)\tilde{f}_1^{(k)}(u)f_2(w)dudw \qquad \tau_2 \le t<\tau_1+\tau_2 \qquad (4.3.15)$$

Hence:

$$h_G(t,k)=\frac{d}{dt}H_I(t,k)=\int_0^t \bar{F}_1(t-u)\tilde{f}_1^{(k)}(u)f_2(t)du \qquad 0 \le t<\tau_1$$

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=\int_{t-\tau_1}^t \bar{F}_1(t-u)\tilde{f}_1^{(k)}(u)f_2(t)du \qquad \tau_1 \le t<\tau_2$$

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=\int_{t-\tau_1}^{\tau_2} \bar{F}_1(w-u)\tilde{f}_1^{(k)}(u)f_2(t)du \qquad \tau_2 \le t<\tau_1+\tau_2 \qquad (4.3.16)$$

For the cases in which $\tau_1 \geq \tau_2$ and $k \geq 1$, A_2 is again identically equal to A_1. Therefore, the probability model is:

$$H_G(t,k)=\int_0^{\tau_2}\int_0^{w}\overline{F}_1(w-u)\widetilde{f}_1^{(k)}(u)f_2(w)dudw \qquad 0 \leq t<\tau_2$$

$$H_G(t,k)=\int_0^{\tau_2}\int_0^{w}\overline{F}_1(w-u)\widetilde{f}_1^{(k)}(u)f_2(w)dudw+\int_{\tau_2}^{t}\int_0^{\tau_2}\overline{F}_1(w-u)\widetilde{f}_1^{(k)}(u)f_2(w)dudw \qquad \tau_2 \leq t<\tau_1$$

$$H_G(t,k)=\int_0^{\tau_2}\int_0^{w}\overline{F}_1(w-u)\widetilde{f}_1^{(k)}(u)f_2(w)dudw+\int_{\tau_2}^{\tau_1}\int_0^{\tau_2}\overline{F}_1(w-u)\widetilde{f}_1^{(k)}(u)f_2(w)dudw+$$
$$\int_{\tau_1}^{t}\int_{w-\tau_1}^{\tau_2}\overline{F}_1(w-u)\widetilde{f}_1^{(k)}(u)f_2(w)dudw \qquad \tau_1 \leq t<\tau_1+\tau_2 \qquad (4.3.17)$$

Hence:

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=\int_0^{t}\overline{F}_1(t-u)\widetilde{f}_1^{(k)}(u)f_2(t)du \qquad 0 \leq t<\tau_2$$

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=\int_0^{\tau_2}\overline{F}_1(t-u)\widetilde{f}_1^{(k)}(u)f_2(t)du \qquad \tau_2 \leq t<\tau_1$$

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=\int_{t-\tau_1}^{\tau_2}\overline{F}_1(w-u)\widetilde{f}_1^{(k)}(u)f_2(t)du \qquad \tau_1 \leq t<\tau_1+\tau_2 \qquad (4.3.18)$$

4.3.3 Initial minor interval (Renewal)

The sample paths for the renewal interval are the compliment of the sample paths for the non renewal cases. All the sample paths for the renewal and the non renewal sample paths are enumerated in appendix 2A.

We denote by $V_{I1}(t,k)$ the probability that a component 1 replacement produces a system renewal and by $V_{I2}(t,k)$ the probability that a system renewal is the result of a component 2 replacement. $U_I(t)$ is distribution on the length of initial minor interval that leads to a system renewal. For the case in which k=0, there is no component 1 failure before the time of the component 2 failure and there is system renewal. In such a situation we have:

$$V_{I2}(t,0)=\int_{\tau_1}^{t}\overline{F}_1(w)f_2(w)du \qquad \tau_1 \le t<\infty \qquad (4.3.19)$$

The corresponding density function is:

$$v_{I2}(t,0)=\begin{cases}\overline{F}_1(t)f_2(t) & \tau_1 \le t<\infty \\ 0 & \text{otherwise}\end{cases} \qquad (4.3.20)$$

For the case $\tau_1<\tau_2$ and $k \ge 1$, the sample path for the system with no renewal is stated in (4.3.5). The complimentary sample paths for system renewal are :

$$\begin{aligned}\overline{A}_1=\{(w,u)|&(\tau_1 \le w<\tau_2,\ 0 \le u \le w-\tau_1)\cup\\ &(\tau_2 \le w<\tau_1+\tau_2, 0 \le u \le w-\tau_1 \cup \tau_2 \le u<w)\cup\\ &(\tau_1+\tau_2 \le w<\infty,\ 0 \le u<\tau_2\ \cup \tau_2 \le u<w).\end{aligned} \qquad (4.3.21)$$

It is easier when constructing the distribution function for the case in which system renewal is due to component 2 replacement because the the minor intervals ends with a component 2 replacement. When the system renewal is due to a component 1 replacement the construction of the distribution function is not obvious. When $0 \leq t<\tau_1$, there is no system renewal. Therefore:

$$V_{I1}(t,k)=0 \quad \text{and} \quad V_{I2}(t,k)=0 \qquad\qquad 0 \leq t<\tau_1 \qquad\qquad (4.3.22)$$

When $\tau_1 \leq t<\tau_2$ we consider $(\tau_1 \leq w<\tau_2,\ 0 \leq u \leq w-\tau_1)$. That is a component 2 failure produces a system renewal. $V_{I1}(t,k)=0$. Hence the probability that component 2 fails between the k^{th} and the $k+1^{st}$ failure time is:

$$V_{I2}(t,k)=\int_{\tau_1}^{t}\int_{0}^{w-\tau_1} f_2(w)f_1^{(k)}(u)\bar{F}_1(w-u)dudw \qquad \tau_1 \leq t<\tau_2$$

$$v_{I2}(t,k)=\int_{0}^{t-\tau_1} f_2(t)f_1^{(k)}(u)\bar{F}_1(t-u)du \qquad \tau_1 \leq t<\tau_2 \qquad\qquad (4.3.23)$$

In the renewal sample path whenever $u \geq \tau_2$ then system renewal is the result of component 1 replacement. Otherwise the system renewal is due component 2 replacement. When $\tau_2 \leq t<\tau_1+\tau_2$ we consider $(\tau_2 \leq w<\tau_1+\tau_2, 0 \leq u \leq w-\tau_1 \cup \tau_2 \leq u<w)$. This can be separated into two cases $(\tau_2 \leq w<\tau_1+\tau_2, 0 \leq u \leq w-\tau_1)$ and $(\tau_2 \leq w<\tau_1+\tau_2, \tau_2 \leq u<w)$. The case $(\tau_2 \leq w<\tau_1+\tau_2, 0 \leq u \leq w-\tau_1)$ means a component 2 failure produces a system renewal and the case $(\tau_2 \leq w<\tau_1+\tau_2, \tau_2 \leq u<w)$ means a component 1 failure produces a system renewal. That is a failure(either component1 or component 2) produces a system renewal.

The probability that component 2 replacement produces a system renewal is:

$$V_{I2}(t,k)=V_{I2}(\tau_2,k)+\int_{\tau_2}^{t}\int_{0}^{w-\tau_1} f_2(w)f_1^{(k)}(u)\bar{F}_1(w-u)dudw$$

The above expression simplifies to:

$$V_{I2}(t,k)=\int_{\tau_1}^{t}\int_{0}^{w-\tau_1} f_2(w)f_1^{(k)}(u)\bar{F}_1(w-u)dudw \qquad \tau_2 \leq t<\tau_1+\tau_2 \qquad (4.3.24)$$

The corresponding density function is:

$$v_{I2}(t,k)=\int_{0}^{t-\tau_1} f_2(t)f_1^{(k)}(u)\bar{F}_1(t-u)du \quad \tau_2 \leq t<\tau_1+\tau_2 \qquad (4.3.25)$$

When component 1 failure produces system renewal we have k component 1 failures and no component 2 failure. Also the time of the k-1[st] component 1 failure must be less than τ_2 so that system renewal occurs only on the k[th] component 1 replacement. The probability that component 1 replacement produces a system renewal is:

$$V_{I1}(t,k)=\int_{\tau_2}^{t}\bar{F}_2(u)\int_{0}^{\tau_2} f_1^{(k-1)}(v)f_1(u-v)dvdu \qquad \tau_2 \leq t<\tau_1+\tau_2 \qquad (4.3.26)$$

The corressponding density function is:

$$v_{I1}(t,k)=\bar{F}_2(t)\int_{0}^{\tau_2} f_1^{(k-1)}(v)f_1(t-v)dv \qquad \tau_2 \leq t<\tau_1+\tau_2 \qquad (4.3.27)$$

When $\tau_1+\tau_2 \leq t<\infty$ any of the components that is replaced produces a system renewal.

$$V_{I2}(t,k)=V_{I2}(\tau_1+\tau_2,k)+\int_{\tau_1+\tau_2}^{t}\int_{0}^{\tau_2} f_2(w)f_1^{(k)}(u)\bar{F}_1(w-u)dudw$$

$$\tau_1+\tau_2 \leq t<\infty \qquad (4.3.28)$$

The corressponding density function is:

$$v_{I2}(t,k)=\int_0^{\tau_2} f_2(t)f_1^{(k)}(u)\overline{F}_1(t-u)dt \qquad \tau_1+\tau_2 \le t<\infty \tag{4.3.29}$$

$$V_{I1}(t,k)=V_{I1}(\tau_1+\tau_2,k)+\int_{\tau_1+\tau_2}^{t} \overline{F}_2(u)\int_0^{\tau_2} f_1^{(k-1)}(v)f_1(u-v)dvdu$$

$$\tau_1+\tau_2 \le t<\infty \tag{4.3.30}$$

The above expression simplifies to:

$$V_{I1}(t,k)=\int_{\tau_2}^{t} \overline{F}_2(u)\int_0^{\tau_2} f_1^{(k-1)}(v)f_1(u-v)dv \qquad \tau_1+\tau_2 \le t<\infty \tag{4.3.31}$$

The corresponding density function is:

$$v_{I1}(t,k)=\overline{F}_2(t)\int_0^{\tau_2} f_1^{(k-1)}(v)f_1(t-v)dv \qquad \tau_1+\tau_2 \le t<\infty \tag{4.3.32}$$

From expressions (4.3.22), (4.3.23),(4.3.25) and (4.3.29) the density function for $V_2(t,k)$ is:

$$v_{I2}(t,k)=\begin{cases} f_2(t)\int_0^{t-\tau_1} f_1^{(k)}(u)\overline{F}_1(t-u)du & \tau_1 \le t<\tau_1+\tau_2 \\ f_2(t)\int_0^{\tau_2} f_1^{(k)}(u)\overline{F}_1(t-u)du & \tau_1+\tau_2 \le t<\infty \\ 0 & \text{otherwise} \end{cases} \tag{4.3.33}$$

From expressions(4.3.22), (4.3.27) and (4.3.32) the density function for $V_1(t,k)$ is:

$$v_{I1}(t,k)=\begin{cases} \overline{F}_2(t)\int_0^{\tau_2} f_1^{(k-1)}(v)f_1(t-v)dv & \tau_2 \le t<\infty \\ 0 & \text{otherwise} \end{cases} \tag{4.3.34}$$

When component 1 replacement produces system renewal we assume that component 1 fails k times and there is system renewal at the k^{th} component 1 replacement. Therefore we have k component 1 repair times and 1 component 2 repair time. When component 2 produces a system renewal we assume component 2 replacement occurs after the k^{th} component 1 replacement. Therefore we assume k+1 component 1 repair times and 1 component 2 repair time.

When component 2 replacement produces system renewal, the density on the length of the repair time is:

$$g_{opp}(t,k+1)=\int_0^t g_1^{(k+1)}(x)g_2(t-x)dx \qquad (4.3.35)$$

On the other hand when a component 1 replacement produces a system renewal densityon the length of the repair time is:

$$g_{opp}(t,k)=\int_0^t g_1^{(k)}(x)g_2(t-x)dx \qquad (4.3.36)$$

The convolution on the length operating time and the repair time is:

$$u_I(t)=\sum_{k=0}^{\infty}\int_0^t v_{I1}(x,k)g_{opp}(t-x)dx+\sum_{k=0}^{\infty}\int_0^t v_{I2}(x,k)g_{opp}(t-x,k+1)dx \qquad (4.3.37)$$

For the case in which $\tau_1 \geq \tau_2$ and $k \geq 1$, the sample path with no system renewal is stated in (4.3.8) The complimentary sample paths for the system renewals are:

$\bar{A}_1=\{(w,u)|(\tau_2 \leq w<\tau_1,\ \tau_2 \leq u \leq w) \cup$

$(\tau_1 \leq w<\tau_1+\tau_2, 0 \leq u \leq w-\tau_1 \cup \tau_2 \leq u<w) \cup$

$(\tau_1+\tau_2 \le w<\infty,\ 0 \le u<\tau_2\ \cup \tau_2 \le u<w)$

Using the same argument as above for the case in which $\tau_1 \ge \tau_2$. We have:

$$V_{I2}(t,k)=0 \qquad 0 \le t<\tau_1$$

$$V_{I2}(t,k)=\int_{\tau_1}^{t}\int_{0}^{w-\tau_1} f_2(w)f_1^{(k)}(u)\bar{F}_1(w-u)dudw \qquad \tau_1 \le t<\tau_1+\tau_2$$

$$V_{I2}(t,k)=V_{I2}(\tau_1+\tau_2,k)+\int_{\tau_1+\tau_2}^{t}\int_{0}^{\tau_2} f_2(w)f_1^{(k)}(u)\bar{F}_1(w-u)dudw$$

$$\tau_1+\tau_2 \le t<\infty \qquad (4.3.38)$$

and the corressponding density function is:

$$v_{I2}(t,k)=\begin{cases} \int_0^{t-\tau_1} f_2(t)f_1^{(k)}(u)\bar{F}_1(t-u)du & \tau_1 \le t<\tau_1+\tau_2 \\ \int_0^{\tau_2} f_2(t)f_1^{(k)}(u)\bar{F}_1(t-u)du & \tau_1+\tau_2 \le t<\infty \\ 0 & \text{otherwise} \end{cases} \qquad (4.3.39)$$

This is the same as the case for $\tau_1<\tau_2$ in (4.3.33). Also when component 1 replacement produces system renewal we have:

$$V_{I1}(t,k)=0 \qquad 0 \le t<\tau_2$$

$$V_{I1}(t,k)=\int_{\tau_2}^{t}\bar{F}_2(u)\int_{0}^{\tau_2} f_1^{(k-1)}(v)f_1(u-v)dv \qquad \tau_2 \le t<\tau_1$$

$$V_{I1}(t,k)=V_{I1}(\tau_1,k)+\int_{\tau_1}^{t}\bar{F}_2(u)\int_{0}^{\tau_2} f_1^{(k-1)}(v)f_1(u-v)dv \quad \tau_1 \le t<\tau_1+\tau_2$$

The above expression simplifies to :

$$V_{I1}(t,k)=\int_{\tau_2}^{t}\bar{F}_2(u)\int_{0}^{\tau_2} f_1^{(k-1)}(v)f_1(u-v)dv \qquad \tau_1 \le t<\tau_1+\tau_2$$

$$V_{I1}(t,k)=V_{I1}(\tau_1+\tau_2,k)+\int_{\tau_1+\tau_2}^{t}\bar{F}_2(u)\int_0^{\tau_2}f_1^{(k-1)}(v)f_1(u-v)dvdu$$

$$\tau_1+\tau_2 \leq t<\infty \qquad (4.3.40)$$

Simplifying again the corressponding density function is:

$$v_{I1}(t,k)=\begin{cases}\bar{F}_2(t)\int_0^{\tau_2}f_1^{(k-1)}(v)f_1(t-v)dv & \tau_2 \leq t<\infty \\ 0 & \text{otherwise}\end{cases} \qquad (4.3.41)$$

This result is the same as the result above is also the same for the case $\tau_1<\tau_2$ in expression (4.3.34). Hence $u_{IR}(t)$ is the same as the expression in (4.3.37).

4.3.4 General Minor interval (Renewal)

The same reasoning that is used for the initial minor interval with renewal can be applied to the general minor interval with renewal.The sample paths are the same. We denote by $V_{G1}(t,k)$ the probability that a component 1 replacement produces a system renewal and $V_{G2}(t,k)$ the probability that system renewal is the result of component 2 replacement. $U_{GR}(t)$ is distribution on the length of general minor interval that leads to a system renewal.

For the case in which k=0, then there is no component 1 failure before the time of component 1 failure. In such a situation we have:

$$V_{G1}(t,k)=0 \qquad 0 \leq t<\infty$$

$$V_{G2}(t,0)=\int_{\tau_1-a_1}^{t}\bar{\bar{F}}_1(w)f_2(w)du \qquad \tau_1-a_1 \leq t<\infty \qquad (4.3.42)$$

The corresponding density function is:

$$v_{G2}(t,0)=\bar{\bar{F}}_1(w)f_2(w)du \qquad \tau_1\text{-}a_1 \leq t<\infty \tag{4.3.43}$$

For $k \geq 1$, using the same argument for the minor interval and the same sample paths the density fuction when a component 2 replacement produces a system renewal in a general minor interval is:

$$v_{G2}(t,k)=\begin{cases} \int_0^{t-\tau_1} f_2(t)\tilde{f}_1^{(k)}(u)\bar{F}_1(t-u)du & \tau_1 \leq t<\tau_1+\tau_2 \\ \int_0^{\tau_2} f_2(t)\tilde{f}_1^{(k)}(u)\bar{F}_1(t-u)du & \tau_1+\tau_2 \leq t<\infty \\ 0 & \text{otherwise} \end{cases} \tag{4.3.44}$$

Also the density function when a component 1 replacement produces a system renewal in a general minor interval is:

$$v_{G2}(t,k)=\begin{cases} \bar{F}_2(t)\int_0^{\tau_2}\tilde{f}_1^{(k-1)}(v)f_1(t-v)dv & \tau_2 \leq t<\infty \\ 0 & \text{otherwise} \end{cases} \tag{4.3.45}$$

The convolution on the length operating time and the repair time is:

$$u_G(t)=\sum_{k=0}^{\infty}\int_0^t v_{G1}(x,k)g_{opp}(t-x)dx+\sum_{k=0}^{\infty}\int_0^t v_{G2}(x,k)g_{opp}(t-x,k+1)dx \tag{4.3.46}$$

4.3.5 Convolution of repair time and failure time of the system

For the cases in which system renewal does not occur, the expressions stated earlier as (3.3.2) and (3.3.3) apply directly to the opportunistic failure replacement strategy. In addition these expressions apply equally well for both the $\tau_1<\tau_2$ and the $\tau_1 \geq \tau_2$ cases. For the initial minor interval, the convolution of the densities on duration of operation and the repair time along the set of sample paths defined by A_1 is:

$$q_I(t)=\sum_{k=0}^{\infty}\int_0^t h_I(x,k)\ g(t-x,k)dx \tag{4.3.48}$$

where $h_I(t,k)$ is the expression in (4.3.3) and (4.3.7) when $\tau_1<\tau_2$ or (4.3.3) and (4.3.10) when $\tau_1 \geq \tau_2$ and $g(t,k)$ is the expression in (4.2.9).

As indicated in chapter 3, $Z_I(t)$ is the distribution function on the combined length of all operating periods during an initial minor interval for the set of sample paths defined by A_1 and $\bar{A}_1$. Using $H_I(t,k)$ as defined in (4.3.2) and (4.3.6) when $\tau_1<\tau_2$ or (4.3.2) and (4.3.9) when $\tau_1 \geq \tau_2$, $Z_I(t)$ is:

$$Z_G(t)=\sum_{k=0}^{\infty}H_G(t,k)+\sum_{k=0}^{\infty}V_{G1}(t,k)+\sum_{k=0}^{\infty}V_{G2}(t,k) \tag{4.3.49}$$

For the general minor interval the convolution of the interval duration and the repair time for the set of sample paths defined by A_2 is:

$$q_G(t)=\sum_{k=0}^{\infty}\int_0^t h_G(x,k)\ g(t-x,k)dx \tag{4.3.50}$$

where $h_G(t,k)$ is the expression in (4.3.14) and (4.3.16) when $\tau_1<\tau_2$ or (4.3.14) and (4.3.18) when $\tau_1 \geq \tau_2$ and $g(t,k)$ is the expression in (4.2.9).

Using $H_G(t,k)$ as defined in (4.3.13) and (4.3.15) when $\tau_1<\tau_2$ or (4.3.13) and (4.3.17) when $\tau_1 \geq \tau_2$, the corresponding distribution on the combined length of the operating periods for the set of sample paths defined by A_2 and $\bar{A}_2$is:

$$Z_G(t)=\sum_{k=0}^{\infty} H_G(t,k) + \sum_{k=0}^{\infty} V_{G1}(t,k) + \sum_{k=0}^{\infty} V_{G2}(t,k) \tag{4.3.51}$$

4.3.5 Major Interval

A major interval consists of an initial minor interval and n general minor intervals (n=0,1,2. . .). This means that a major interval always starts with both components new. Both components go through several failures and repairs. The major interval ends when both components are replaced at the same time.

The renewal density function for the minor intervals that end without system renewal is the convolution of the density on the length of the initial minor interval and that on the lengths of n general minor intervals. Using (4.3.19) and (4.3.21) the renewal density is:

$$m_Q(t)=\sum_{n=0}^{\infty} \int_0^t q_I(x) q_G^{(n)}(t-x)dx \tag{4.3.52}$$

From (3.5.7) the availability is the probability that the system is functioning and the length of the major interval is longer than t. That is:

$$A(t)=\bar{Z}_I(t)+\int_0^t \bar{Z}_G(x) m_Q(t-x)dx \tag{4.3.53}$$

where $\overline{Z}_I(t)=1-Z_I(t)$ and $\overline{Z}_G(t)=1-Z_G(t)$ as defined in chapter 3. The minor interval renewal process is transient. Also from (3.5.9) the time dependent system availability is:

$$\mathfrak{A}(t)=A(t)+\int_0^t A(x)m_\Phi(t-x)dx.$$

where

$$\mathfrak{F}(t)=U_I(t)+\int_0^t m_Q(x)U_G(t-x)dx.$$

and

$$\Phi(t)=\int_0^t \mathfrak{F}(x)g_{opp}(t-x)dx.$$

A(t) is the expression in (4.3.24) and the long run availability is:

$$\mathfrak{A}=\lim_{t\to\infty}\mathfrak{A}(t)=\frac{(\nu_I(1-Q_G(\infty))+\nu_G Q_I(\infty))}{(\mu_{IR}+\mu_I+\mu_{opp})(1-Q_G(\infty))+(\mu_{GR}+\mu_G)Q_I(\infty)} \tag{4.3.54}$$

The expressions (4.3.24) and (4.3.25) constitute the time dependent and the limiting availability measures for the specific case of the opportunistic replacement strategy. As in the case of failure replacement, this is general in that no specific distributions are assumed.

Partial validation of the model is possible. Observe that if the opportunistic replacement policy ages are set at infinity ($\tau_1=\tau_2=\infty$), the model reduces to the failure replacement model. The minor intervals become recurrent and the limiting availability expression reduces to (4.3.25).

Chapter 5 Opportunistic Replacement Models

5.1 Introduction

In this chapter we discuss two types of opportunistic replacement policies, a partial opportunistic age replacement policy and a full opportunistic age replacement policy. Under a partial opportunistic age replacement policy, each component is subject to opportunistic replacement. When this occurs both components are replaced so the system is renewed. For those cases in which there is no system renewal one component is replaced only at failure and the replacement of the other component may be the result of failure or of age replacement. Thus only one component is subject to age replacement. In contrast, under a full opportunistic age replacement policy, both components are subject to age replacement and to opportunistic replacement.

5.2 Partial Opportunistic Age Replacement

In order to define a model for a partial opportunistic age replacement policy, assume it is component 2 that has an age replacement time T_2. Assume further that both components are replaced upon failure and at their opportunistic replacement ages when appropriate. For this policy system renewal occurs if:

a) component 1 fails and the time since component 2 was last replaced exceeds τ_2, the opportunistic replacement time of component 2, or

b) component 2 fails or attains an age T_2, and the time since component 1 was last replaced exceeds τ_1, the opportunistic replacement time of component 1.

5.2.1 Initial minor interval(No Renewal)

In this interval both components start new and the interval ends with a component 2 failure or with component 2 attaining age T_2. In an initial minor interval without system renewal the time between the k^{th} component 1 failure time and the component 2 replacement(failure or age replacement) time is less than the opportunistic replacement time of component 1. If we condition on whether the component 2 replacement is due to failure or is the result of age replacement, then the probability that component 2 replacement occurs between the k^{th} and the $k+1^{st}$ replacement time of component 1 is:

$$H_I(t,k)=P[S_{k,1} \leq X_{1,2} < S_{k+1,1}]$$

$$=P[S_{k,1} \leq X_{1,2} < S_{k+1,1} \cap X_{1,2} \leq T_2]+P[S_{k,1} \leq X_{1,2} < S_{k+1,1} \cap X_{1,2} > T_2] \quad (5.2.1)$$

We represent the cases in which system renewal does not occur. For the case in which k=0, there is no component 1 failure before the time of the component 2 failure(or age replacement). In such a situation:

$$H_I(t,0)=P[S_{0,1} \leq X_{1,2} < S_{1,1} \cap X_{1,2} < T_2]+P[S_{0,1} \leq X_{1,2} < S_{1,1} \cap X_{1,2} \geq T_2] \quad (5.2.2)$$

If component 2 failure occurs before time T_2, then :

$$P[S_{0,1} \leq X_{1,2} < S_{1,1} \cap X_{1,2} < T_2]=\int_0^t \overline{F}_1(w)f_2(w)dudw \quad (5.2.3)$$

If $T_2 > \tau_1$ then expression (5.2.3) applies in the interval $0 \leq t < \tau_1$. Otherwise, for $T_2 \leq \tau_1$, the expression applies when $0 \leq t < T_2$.

If component 2 is attains an age T_2 without failure and there is no system renewal, then for $T_2 < \tau_1$:

$$P[S_{0,1} \leq X_{1,2} < S_{1,1} \cap X_{1,2} \geq T_2] = \bar{F}_2(T_2)\bar{F}_1(T_2), t \geq T_2. \qquad (5.2.4)$$

In contrast for $T_2 \geq \tau_1$:

$$P[S_{0,1} \leq X_{1,2} < S_{1,1} \cap X_{1,2} \geq T_2] = 0, \quad t \geq T_2. \qquad (5.2.5)$$

Combining expressions (5.2.3),(5.2.4) and (5.2.5) yields:

When $T_2 > \tau_1$:

$$H_I(t,0) = \int_0^t \bar{F}_1(w) f_2(w) dw, \qquad 0 \leq t < \tau_1$$

$$H_I(t,0) = \int_0^{\tau_1} \bar{F}_1(w) f_2(w) dw, \qquad t \geq \tau_1 \qquad (5.2.6)$$

and the corresponding density function is:

$$h_I(t,0) = \frac{d}{dt} H_I(t,0) = \bar{F}_1(t) f_2(t), \qquad 0 \leq t < \tau_1$$

$$h_I(t,0) = \frac{d}{dt} H_I(t,0) = 0, \qquad t \geq \tau_1 \qquad (5.2.7)$$

Also for $T_2 \leq \tau_1$ the distribution is:

$$H_I(t,0)=\int_0^t \bar{F}_1(w)f_2(w)dw, \qquad 0 \le t<T_2$$

$$H_I(t,0)=\int_0^{T_2} \bar{F}_1(w)f_2(w)dw+\bar{F}_2(T_2)\bar{F}_1(T_2), \qquad t \ge T_2 \qquad (5.2.8)$$

and the corresponding density is:

$$h_I(t,0)=\frac{d}{dt}H_I(t,0)=\bar{F}_1(t)f_2(t), \qquad 0 \le t<T_2$$

$$h_I(t,0)=\frac{d}{dt}H_I(t,0)=\bar{F}_2(T_2)\bar{F}_1(T_2), \qquad t=T_2$$

$$h_I(t,0)=\frac{d}{dt}H_I(t,0)=\ 0, \qquad \text{otherwise.} \qquad (5.2.9)$$

For the case in which $k \ge 1$, assume component 2 fails before T_2. Then:

$$P[S_{k,1} \le X_{1,2} < S_{k+1,1} \cap X_{1,2} \le T_2]=$$

$$=\int_{w<T_2}\int_u P[X_1^2-u<X_{k+1}^1|X_1^2=w, 0 \le w<T_2]dF_1^{(k)}(u)dF_2(w)$$

$$=\int_{B_1}\int \bar{F}_1(w-u)dF_1^{(k)}(u)dF_2(w)\,. \qquad (5.2.10)$$

System renewal occurs if any of the component 1 failures occur after τ_2 time units. Alternatively, system renewal occurs if the interval between the k th component 1 failure and the time of the component 2 replacement exceeds τ_1. Therefore, to represent the cases in which system renewal does not occur, expression (5.2.10) applies over the time intervals in B_1 where in general:

$$B_1=\{(w,u)|\{w<T_2,\ 0 \leq w-u<\tau_1,\ 0 \leq u<\tau_2\}\}. \qquad (5.2.11)$$

However, this general statement must be specified more carefully and in terms of relationships between τ_1 and τ_2, τ_1 and T_2 and also the relationship between $\tau_1+\tau_2$ and T_2.

For the values $k \geq 1$:

When $\tau_1<\tau_2$ and $\tau_1+\tau_2<T_2$, expression (5.2.10) applies over the ranges defined by :

$$B_1=\{(w,u):(0 \leq w<\tau_1, 0 \leq u<w) \cup (\tau_1 \leq w<\tau_2,\ w-\tau_1 \leq u<w) \cup$$

$$(\tau_2 \leq w<\tau_1+\tau_2, w-\tau_1 \leq u<\tau_2)\}. \qquad (5.2.12)$$

When $\tau_1<\tau_2$ and $\tau_1+\tau_2 \geq T_2$, expression (5.2.10) applies over the ranges defined by :

$$B_1=\{(w,u):(0 \leq w<\tau_1, 0 \leq u<w) \cup (\tau_1 \leq w<\tau_2,\ w-\tau_1 \leq u<w) \cup$$

$$(\tau_2 \leq w<T_2, w-\tau_1 \leq u<\tau_2)\}. \qquad (5.2.13)$$

The reasons for the definitions in (5.2.12) and (5.2.13) are the following:

1. If component 2 fails at time w ($0 \leq w<\tau_1$), any number of component 1 failures could occur before time w without opportunistic replacement. That is, if the k th component 1 failure time is at u, then $0 \leq u<w$.

2. If component 2 fails at time w ($\tau_1 \leq w<\tau_2$) and the k th component 1 failure is at time u, then opportunistic replacement does not occur if the time between the component 2 failure and the time of the last component 1 failure does not exceed τ_1. That is, there is no system renewal if $w-u<\tau_1$ or $u>w-\tau_1$. By construction, $0 \leq u<w$.

3. Finally, if component 2 fails at time $w \geq \tau_2$ and $\tau_1+\tau_2<T_2$, the k th component 1 failure occurs at time u, then there is no system renewal if component 1 fails at time u, where $u<\tau_2$ and $w-u<\tau_1$ .These two conditions imply that $w-\tau_1<u<\tau_2$. These conditions hold if $w \leq \tau_1+\tau_2$. In contrast if component 2 fails at time $w \geq \tau_2$ and $\tau_1+\tau_2 \geq T_2$, the k th component 1 failure occurs at time u, then there is no system renewal if component 1 fails at time u, where $u<\tau_2$ and $w-u<\tau_1$ .These two conditions imply that $w-\tau_1<u<\tau_2$. These conditions hold if $w \leq T_2$.

When $\tau_1 \geq \tau_2$ and $\tau_1+\tau_2<T_2$, expression (5.2.10) applies over the ranges defined by :

$$B_1=\{(w,u)|(0 \leq w<\tau_2,\ 0 \leq u<w) \cup (\tau_2 \leq w<\tau_1,\ w-\tau_1 \leq u<\tau_2) \cup$$
$$(\tau_1 \leq w<\tau_1+\tau_2, w-\tau_1 \leq u<\tau_2)\}. \tag{5.2.14}$$

When $\tau_1 \geq \tau_2$, $\tau_1 \leq T_2$ and $\tau_1+\tau_2 \geq T_2$, expression (5.2.10) applies over the ranges defined by :

$$B_1=\{(w,u):|(0 \leq w<\tau_2,\ 0 \leq u<w) \cup (\tau_2 \leq w<\tau_1,\ 0 \leq u<\tau_2) \cup$$
$$(\tau_1 \leq w<T_2,\ w-\tau_1 \leq u<\tau_2)\}. \tag{5.2.15}$$

When $\tau_1 \geq \tau_2$, and $\tau_1>T_2$ expression (5.2.10) applies over the ranges defined by :

$$B_1=\{(w,u):|\ (0 \leq w<\tau_2,\ 0 \leq u<w) \cup (\tau_2 \leq w<T_2,\ 0 \leq u<\tau_2)\} \tag{5.2.16}$$

The reasons for the definitions in (5.2.14) and (5.2.15) are the following:

1. If component 2 fails at time w $(0 \leq w<\tau_2)$ and u is the time of the last component 1 failure before component 2 fails, then no system renewal occurs when $0 \leq u<w$.

2. If component 2 fails at time w ($\tau_2 \leq w < \tau_1$) and u is the time of the last component 1 failure before component 2 fails, system renewal does not occur if $0 \leq u < \tau_2$.

3. If component 2 fails at time w ($w \geq \tau_1$) and u is the time of the last component 1 replacement before component 2 failure, no system renewal occurs if the time of the component 1 replacement does not exceed τ_2 ($u < \tau_2$). However it must also be the case that $w-u < \tau_1$. The combined application of these conditions yields $w-\tau_1 < u < \tau_2$ which in turn imply that $w \leq \tau_1+\tau_2$ and $\tau_1+\tau_2 < w \leq T_2$ makes the interval when $w-\tau_1 < u < \tau_2$ infeasible when $\tau_1+\tau_2 < T_2$. In contrast when $\tau_1+\tau_2 \geq T_2$ then $w \leq T_2$.

The reasoning behind (5.2.16) when $\tau_1 \geq \tau_2$ and $\tau_1 > T_2$ is:.

4. If component 2 fails at time w ($0 \leq w < \tau_2$) and u is the time of the last component 1 failure before component 2 fails, then no system renewal occurs when $0 \leq u < w$.

5. If component 2 fails at time w ($\tau_2 \leq w < T_2$) and u is the time of the last component 1 failure before component 2 fails, system renewal does not occur if $0 \leq u < \tau_2$.

Next consider the case in which $k \geq 1$ and component 2 attains age T_2 without failure. Then, the probability that component 2 age replacement occurs between the k^{th} and the $k+1^{st}$ component 1 failure is:

$$P[S_{k,1} \leq X_{1,2} < S_{k+1,1} \cap X_{1,2} > T_2] =$$

$$\bar{F}_2(T_2) P[T_2 - u < X^1_{k+1}] dF_1^{(k)}(u) =$$

$$\bar{F}_2(T_2)\int_{B_2}\bar{F}_1(T_2-u)dF_1^{(k)}(u). \tag{5.2.17}$$

When $T_2 \geq \tau_2$ and $\tau_1+\tau_2 \geq T_2$ and $T_2>\tau_1$, expression (5.2.17) applies over the ranges defined by :

$$B_2=\{u|T_2-\tau_1 \leq u<\tau_2\}. \tag{5.2.18}$$

This is true for both $\tau_1<\tau_2$ and $\tau_1 \geq \tau_2$. When $T_2<\tau_1$, expression (5.2.17) applies over the ranges defined by:

$$B_2=\{u|0 \leq u<\tau_2\}. \tag{5.2.19}$$

When $\tau_1+\tau_2 \leq T_2$, expression (5.2.17) applies over the ranges defined by:

$$B_2=\{u|\ \emptyset\}. \tag{5.2.20}$$

Combining (5.2.10) and (5.2.17) we have the following:

For the case in which $\tau_1 \leq \tau_2$, $\tau_1+\tau_2<T_2(\Rightarrow \tau_1<T_2)$, the combined application of the conditions enumerated yield (5.2.12) and (5.2.20) and the corresponding joint full statement of (5.2.10) and (5.2.17) is:

$$H_I(t,k)=\int_0^t\int_0^w \bar{F}_1(w-u)dF_1^{(k)}(u)dF_2(w), \qquad 0 \leq t<\tau_1$$

$$H_I(t,k)=\int_0^{\tau_1}\int_0^w \bar{F}_1(w-u)dF_1^{(k)}(u)dF_2(w)+\int_{\tau_1}^t\int_{w-\tau_1}^w \bar{F}_1(w-u)dF_1^{(k)}(u)dF_2(w),$$

$$\tau_1 \leq t<\tau_2$$

$$H_I(t,k)=\int_0^{\tau_1}\int_0^{w}\overline{F}_1(w-u)dF_1^{(k)}(u)dF_2(w)+\int_{\tau_1}^{\tau_2}\int_{w-\tau_1}^{w}\overline{F}_1(w-u)dF_1^{(k)}(u)dF_2(w)+$$

$$\int_{\tau_2}^{t}\int_{w-\tau_1}^{\tau_2}\overline{F}_1(w-u)dF_1^{(k)}(u)dF_2(w), \qquad \tau_2 \le t<\tau_1+\tau_2$$

$$H_I(t,k)=\int_0^{\tau_1}\int_0^{w}\overline{F}_1(w-u)dF_1^{(k)}(u)dF_2(w)+\int_{\tau_1}^{\tau_2}\int_{w-\tau_1}^{w}\overline{F}_1(w-u)dF_1^{(k)}(u)dF_2(w)+$$

$$\int_{\tau_2}^{\tau_1+\tau_2}\int_{w-\tau_1}^{\tau_2}\overline{F}_1(w-u)dF_1^{k}(u)dF_2(w)\,, \qquad \tau_1+\tau_2 \le t. \qquad (5.2.21)$$

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=\int_0^{t}\overline{F}_1(t-u)dF_1^{(k)}(u)dF_2(t), \qquad 0 \le t<\tau_1$$

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=\int_{t-\tau_1}^{t}\overline{F}_1(t-u)dF_1^{(k)}(u)dF_2(t), \qquad \tau_1 \le t<\tau_2$$

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=\int_{t-\tau_1}^{\tau_2}\overline{F}_1(t-u)dF_1^{(k)}(u)dF_2(t), \qquad \tau_2 \le t<\tau_1+\tau_2$$

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=0, \qquad \text{otherwise.} \quad (5.2.22)$$

For the case in which $\tau_1 \le \tau_2$, $\tau_1+\tau_2 \ge T_2$ ($\Rightarrow \tau_1<T_2$), the combined application of the conditions enumerated yield (5.2.13) and (5.2.18) and the corresponding joint full statement of (5.2.10) and (5.2.17) is:

$$H_I(t,k)=\int_0^{t}\int_0^{w}\overline{F}_1(w-u)dF_1^{(k)}(u)dF_2(w), \qquad 0 \le t<\tau_1$$

$$H_I(t,k)=\int_0^{\tau_1}\int_0^{w}\bar{F}_1(w-u)dF_1^{(k)}(u)dF_2(w)+\int_{\tau_1}^{t}\int_{w-\tau_1}^{w}\bar{F}_1(w-u)dF_1^{(k)}(u)dF_2(w),$$

$$\tau_1 \le t<\tau_2$$

$$H_I(t,k)=\int_0^{\tau_1}\int_0^{w}\bar{F}_1(w-u)dF_1^{(k)}(u)dF_2(w)+\int_{\tau_1}^{\tau_2}\int_{w-\tau_1}^{w}\bar{F}_1(w-u)dF_1^{(k)}(u)dF_2(w)+$$

$$\int_{\tau_2}^{t}\int_{w-\tau_1}^{\tau_2}\bar{F}_1(w-u)dF_1^{(k)}(u)dF_2(w), \qquad \tau_2 \le t<T_2$$

$$H_I(t,k)=\int_0^{\tau_1}\int_0^{w}\bar{F}_1(w-u)dF_1^{(k)}(u)dF_2(w)+\int_{\tau_1}^{\tau_2}\int_{w-\tau_1}^{w}\bar{F}_1(w-u)dF_1^{(k)}(u)dF_2(w)+$$

$$\int_{\tau_2}^{T_2}\int_{w-\tau_1}^{\tau_2}\bar{F}_1(w-u)dF_1^{(k)}(u)dF_2(w)+\bar{F}_2(T_2)\int_{T_2-\tau_1}^{\tau_2}\bar{F}_1(T_2-u)dF_1^{(k)}(u), \tag{5.2.23}$$

$$T_2 \le t.$$

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=\int_0^{t}\bar{F}_1(t-u)dF_1^{(k)}(u)dF_2(t), \qquad 0 \le t<\tau_1$$

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=\int_{t-\tau_1}^{t}\bar{F}_1(t-u)dF_1^{(k)}(u)dF_2(t), \qquad \tau_1 \le t<\tau_2$$

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=\int_{t-\tau_1}^{\tau_2}\bar{F}_1(t-u)dF_1^{(k)}(u)dF_2(t), \qquad \tau_2 \le t<T_2$$

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=\bar{F}_2(T_2)\int_{T_2-\tau_1}^{\tau_2}\bar{F}_1(T_2-u)dF_1^{(k)}(u), \qquad t=T_2$$

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=0, \qquad \text{otherwise.} \tag{5.2.24}$$

For the case in which $\tau_1 \geq \tau_2$, $\tau_1+\tau_2<T_2$ and $\tau_1<T_2$, the combined application of the conditions enumerated yield (5.2.14) and (5.2.20) and the corresponding joint full statement of (5.2.10) and (5.2.17) is:

$$H_I(t,k)=\int_0^t\int_0^w \overline{F}_1(w-u)dF_1^{(k)}(u)dF_2(w), \qquad 0 \leq t<\tau_2$$

$$H_I(t,k)=\int_0^{\tau_2}\int_0^w \overline{F}_1(w-u)dF_1^{(k)}(u)dF_2(w)+\int_{\tau_2}^t\int_0^{\tau_2} \overline{F}_1(w-u)dF_1^{(k)}(u)dF_2(w),$$

$$\tau_2 \leq t<\tau_1$$

$$H_I(t,k)=\int_0^{\tau_2}\int_0^w \overline{F}_1(w-u)dF_1^{(k)}(u)dF_2(w)+\int_{\tau_2}^{\tau_1}\int_0^{\tau_2} \overline{F}_1(w-u)dF_1^{(k)}(u)dF_2(w)+$$

$$\int_{\tau_1}^t\int_{w-\tau_1}^{\tau_2} \overline{F}_1(w-u)dF_1^{(k)}(u)dF_2(w), \qquad \tau_1 \leq t<\tau_1+\tau_2$$

$$H_I(t,k)=\int_0^{\tau_2}\int_0^w \overline{F}_1(w-u)dF_1^{(k)}(u)dF_2(w)+\int_{\tau_2}^{\tau_1}\int_0^{\tau_2} \overline{F}_1(w-u)dF_1^{(k)}(u)dF_2(w)+$$

$$\int_{\tau_1}^{\tau_1+\tau_2}\int_{w-\tau_1}^{\tau_2} \overline{F}_1(w-u)dF_1^{(k)}(u)dF_2(w), \qquad T_2 \leq t. \qquad (5.2.25)$$

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=\int_0^t \overline{F}_1(t-u)dF_1^{(k)}(u)dF_2(t), \qquad 0 \leq t<\tau_2$$

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=\int_0^t \overline{F}_1(t-u)dF_1^{(k)}(u)dF_2(t), \qquad \tau_2 \leq t<\tau_1$$

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=\int_{t-\tau_1}^{\tau_2}\bar{F}_1(t-u)dF_1^{(k)}(u)dF_2(t), \qquad \tau_1 \leq t<\tau_1+\tau_2$$

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=0, \qquad \text{otherwise.} \qquad (5.2.26)$$

For the case in which $\tau_1 \geq \tau_2$ and $\tau_1+\tau_2 \geq T_2$ implies $\tau_1<T_2$, the combined applications of the conditions enumerated yield (5.2.14) and (5.2.18) and the corresponding joint full statement of (5.2.10) and (5.2.17) is:

$$H_I(t,k)=\int_0^t\int_0^w\bar{F}_1(w-u)dF_1^{(k)}(u)dF_2(w), \qquad 0 \leq t<\tau_2$$

$$H_I(t,k)=\int_0^{\tau_2}\int_0^w\bar{F}_1(w-u)dF_1^{(k)}(u)dF_2(w)+\int_{\tau_2}^t\int_0^{\tau_2}\bar{F}_1(w-u)dF_1^{(k)}(u)dF_2(w),$$

$$\tau_2 \leq t<\tau_1$$

$$H_I(t,k)=\int_0^{\tau_2}\int_0^w\bar{F}_1(w-u)dF_1^{(k)}(u)dF_2(w)+\int_{\tau_2}^{\tau_1}\int_0^{\tau_2}\bar{F}_1(w-u)dF_1^{(k)}(u)dF_2(w)+$$

$$\int_{\tau_1}^t\int_{w-\tau_1}^{\tau_2}\bar{F}_1(w-u)dF_1^k(u)dF_2(w), \qquad \tau_1 \leq t<T_2$$

$$H_I(t,k)=\int_0^{\tau_2}\int_0^w\bar{F}_1(w-u)dF_1^{(k)}(u)dF_2(w)+\int_{\tau_2}^{\tau_1}\int_0^{\tau_2}\bar{F}_1(w-u)dF_1^{(k)}(u)dF_2(w)+$$

$$\int_{\tau_1}^{T_2}\int_{w-\tau_1}^{\tau_2}\bar{F}_1(w-u)dF_1^{(k)}(u)dF_2(w)+\bar{F}_2(T_2)\int_{T_2-\tau_1}^{\tau_2}\bar{F}_1(T_2-u)dF_1^{(k)}(u), \quad T_2 \leq t.$$

$$(5.2.27)$$

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=\int_0^t\bar{F}_1(t-u)dF_1^{(k)}(u)dF_2(t), \qquad 0 \leq t<\tau_2$$

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=\int_0^{\tau_2}\bar{F}_1(t-u)dF_1^{(k)}(u)dF_2(t), \qquad \tau_2 \le t<\tau_1$$

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=\int_{t-\tau_1}^{\tau_2}\bar{F}_1(t-u)dF_1^{(k)}(u)dF_2(t), \qquad \tau_1 \le t<T_2$$

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=\bar{F}_2(T_2)\int_{T_2-\tau_1}^{\tau_2}\bar{F}_1(T_2-u)dF_1^{(k)}(u), \qquad t=T_2$$

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=0, \qquad \text{otherwise.} \qquad (5.2.28)$$

For the case in which $\tau_1 \ge \tau_2$ and $\tau_1 \ge T_2$, the combined application of the conditions enumerated yield (5.2.14) and (5.2.19) and the corresponding joint full statement of (5.2.10) and (5.2.17) is:

$$H_I(t,k)=\int_0^t\int_0^w\bar{F}_1(w-u)dF_1^{(k)}(u)dF_2(w), \qquad 0 \le t<\tau_2$$

$$H_I(t,k)=\int_0^{\tau_2}\int_0^w\bar{F}_1(w-u)dF_1^{(k)}(u)dF_2(w)+\int_{\tau_2}^t\int_0^{\tau_2}\bar{F}_1(w-u)dF_1^{(k)}(u)dF_2(w),$$

$$\tau_2 \le t<T_2$$

$$H_I(t,k)=\int_0^{\tau_2}\int_0^w\bar{F}_1(w-u)dF_1^{(k)}(u)dF_2(w)+\int_{\tau_2}^{T_2}\int_0^{\tau_2}\bar{F}_1(w-u)dF_1^{(k)}(u)dF_2(w)+$$

$$\bar{F}_2(T_2)\int_0^{\tau_2}\bar{F}_1(T_2-u)dF_1^{k}(u), \qquad t \ge T_2. \qquad (5.2.29)$$

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=\int_0^t\bar{F}_1(t-u)dF_1^{(k)}(u)dF_2(t), \qquad 0 \le t<\tau_2$$

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=\int_0^{\tau_2}\overline{F}_1(t-u)dF_1^{(k)}(u)dF_2(t), \qquad \tau_2 \leq t<T_2$$

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=\overline{F}_2(T_2)\int_0^{\tau_2}\overline{F}_1(T_2-u)dF_1^{(k)}(u), \qquad t=T_2$$

$$h_I(t,k)=\frac{d}{dt}H_I(t,k)=0, \qquad \text{otherwise.} \quad (5.2.30)$$

5.2.2 General interval(No Renewal)

The same reasoning that is used for the initial minor interval can be applied to the general minor interval. At the start of a general minor interval, component 1 has a residual life distribution and component 2 start new. As above the interval ends with a component 2 failure or age replacement.

The generic statement for the probability that a general minor interval ends without system renewal at a time not exceeding t and following the kth component 1 failure is given in expression (5.2.31) below. This expression is the conceptual analog of (5.2.1).

$$H_G(t,k)=P[\widetilde{S}_{k,1} \leq X_{1,2} <\widetilde{S}_{k+1,1}]$$

$$=P[\widetilde{S}_{k,1} \leq X_{1,2} <\widetilde{S}_{k+1,1} \cap X_{1,2} \leq T_2]+P[\widetilde{S}_{k,1} \leq X_{1,2} <\widetilde{S}_{k+1,1} \cap X_{1,2}>T_2].$$

$$(5.2.31)$$

The explanation given for the use of a_1 in (4.3.12) is the same explanation that is given the use of a_1 in this chapter. They are in fact the same. Now, using this form, the expressions corresponding to (5.2.6) and (5.2.8) for the general minor interval when k=0 are:

for $T_2>\tau_1-a_1$:

$$H_G(t,0)=\int_0^t \bar{\bar{F}}_1(w)f_2(w)dw, \qquad 0\leq t<\tau_1-a_1$$

$$H_G(t,0)=\int_0^{\tau_1-a_1} \bar{\bar{F}}_1(w)f_2(w)dw, \qquad \tau_1-a_1\leq t \qquad (5.2.32)$$

and the corresponding density function is:

$$h_G(t,0)=\frac{d}{dt}H_G(t,0)=\bar{\bar{F}}_1(t)f_2(t), \qquad 0\leq t<\tau_1-a_1$$

$$h_G(t,0)=\frac{d}{dt}H_G(t,0)=0, \qquad \tau_1-a_1\leq t. \qquad (5.2.33)$$

Also for the case $T_2\leq\tau_1-a_1$ the distribution is:

$$H_G(t,0)=\int_0^t \bar{\bar{F}}_1(w)f_2(w)dw, \qquad 0\leq t<T_2$$

$$H_G(t,0)=\int_0^{T_2} \bar{\bar{F}}(w)f_2(w)dw+\bar{F}_2(T_2)\bar{\bar{F}}_1(T_2), \qquad t\geq T_2 \qquad (5.2.34)$$

and the corresponding density is:

$$h_G(t,0)=\frac{d}{dt}H_G(t,0)=\bar{\bar{F}}(t)f_2(t), \qquad 0\leq t<T_2$$

$$h_G(t,0)=\frac{d}{dt}H_G(t,0)=\bar{\bar{F}}_2(T_2)\bar{F}_1(T_2), \qquad t=T_2$$

$$h_G(t,0)=\frac{d}{dt}H_G(t,0)= 0, \qquad \text{otherwise.} \qquad (5.2.35)$$

Consider the cases in which $k \geq 1$ and assume component 2 failure occurrs before T_2. The probability that component 2 failure occurs between the k^{th} and the $k+1^{st}$ component 1 replacement is:

$$P[\widetilde{S}_{k,1} \leq X_{1,2} < \widetilde{S}_{k+1,1} \cap X_{1,2} \leq T_2]=$$

$$=\int_{w<T_2}\int_u P[X_1^2\text{-}u<X_{k+1}^1|X_1^2=w,0 \leq w<T_2]d\widetilde{F}_1^{(k)}(u)dF_2(w)$$

$$\int_{D_1}\int \bar{F}_1(w\text{-}u)d\widetilde{F}_1^{(k)}(u)dF_2(w) \qquad (5.2.36)$$

The reasoning that applied to B_1 also applies to D_1. In fact $D_1=B_1$.

The probability that component 2 age replacement occurs between the k^{th} and the $k+1^{st}$ component 1 failure is:

$$P[\widetilde{S}_{k,1} \leq X_{1,2} < \widetilde{S}_{k+1,1} \cap X_{1,2}>T_2]=$$

$$\bar{F}_2(T_2)P[T_2\text{-}u<X_{k+1}^1]d\widetilde{F}_1^{(k)}(u)=$$

$$\bar{F}_2(T_2)\int_{D_2} \bar{F}_1(T_2\text{-}u)d\widetilde{F}_1^{(k)}(u) \qquad (5.2.37)$$

The reasoning that applied to B_2 also applies to D_2. In fact $D_2=B_2$.

For the case in which $\tau_1 \leq \tau_2$, $\tau_1+\tau_2<T_2$ ($\Rightarrow \tau_1<T_2$), the combined application of the conditions enumerated yield (5.2.12) and (5.2.20) and the corresponding full statement of (5.2.36) and (5.2.37) is:

$$H_G(t,k)=\int_0^t \int_0^w \bar{F}_1(w-u)d\tilde{F}_1^{(k)}(u)dF_2(w), \qquad 0 \leq t<\tau_1$$

$$H_G(t,k)=\int_0^{\tau_1} \int_0^w \bar{F}_1(w-u)d\tilde{F}_1^{(k)}(u)dF_2(w)+\int_{\tau_1}^t \int_{w-\tau_1}^w \bar{F}_1(w-u)d\tilde{F}_1^{(k)}(u)dF_2(w),$$

$$\tau_1 \leq t<\tau_2$$

$$H_G(t,k)=\int_0^{\tau_1} \int_0^w \bar{F}_1(w-u)d\tilde{F}_1^{(k)}(u)dF_2(w)+\int_{\tau_1}^{\tau_2} \int_{w-\tau_1}^w \bar{F}_1(w-u)d\tilde{F}_1^{(k)}(u)dF_2(w)+$$

$$\int_{\tau_2}^t \int_{w-\tau_1}^{\tau_2} \bar{F}_1(w-u)dF_1^k(u)dF_2(w), \qquad \tau_2 \leq t<\tau_1+\tau_2$$

$$H_G(t,k)=\int_0^{\tau_1} \int_0^w \bar{F}_1(w-u)d\tilde{F}_1^{(k)}(u)dF_2(w)+\int_{\tau_1}^{\tau_2} \int_{w-\tau_1}^w \bar{F}_1(w-u)d\tilde{F}_1^{(k)}(u)dF_2(w)+$$

$$\int_{\tau_2}^{\tau_1+\tau_2} \int_{w-\tau_1}^{\tau_2} \bar{F}_1(w-u)d\tilde{F}_1^k(u)dF_2(w), \qquad t \geq \tau_1+\tau_2 \qquad (5.2.38)$$

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=\int_0^t \bar{F}_1(t-u)d\tilde{F}_1^{(k)}(u)dF_2(t), \qquad 0 \leq t<\tau_1$$

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=\int_{t-\tau_1}^t \bar{F}_1(t-u)d\tilde{F}_1^{(k)}(u)dF_2(t), \qquad \tau_1 \leq t<\tau_2$$

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=\int_{t-\tau_1}^{\tau_2} \bar{F}_1(t-u)d\tilde{F}_1^{(k)}(u)dF_2(t), \qquad \tau_2 \leq t<\tau_1+\tau_2$$

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=0, \qquad \text{otherwise.} \qquad (5.2.39)$$

For the case in which $\tau_1 \le \tau_2$, $\tau_1+\tau_2 \ge T_2$ ($\Rightarrow \tau_1<T_2$), the combined application of the conditions enumerated yield (5.2.13) and (5.2.18) and the corresponding full statement of (5.2.36) and (5.2.37) is:

$$H_G(t,k)=\int_0^t\int_0^w \overline{F}_1(w-u)d\widetilde{F}_1^{(k)}(u)dF_2(w), \qquad 0 \le t<\tau_1$$

$$H_G(t,k)=\int_0^{\tau_1}\int_0^w \overline{F}_1(w-u)d\widetilde{F}_1^{(k)}(u)dF_2(w)+\int_{\tau_1}^t\int_{w-\tau_1}^w \overline{F}_1(w-u)d\widetilde{F}_1^{(k)}(u)dF_2(w),$$

$$\tau_1 \le t<\tau_2$$

$$H_G(t,k)=\int_0^{\tau_1}\int_0^w \overline{F}_1(w-u)d\widetilde{F}_1^{(k)}(u)dF_2(w)+\int_{\tau_1}^{\tau_2}\int_{w-\tau_1}^w \overline{F}_1(w-u)d\widetilde{F}_1^{(k)}(u)dF_2(w)+$$

$$\int_{\tau_2}^t\int_{w-\tau_1}^{\tau_2} \overline{F}_1(w-u)d\widetilde{F}_1^{(k)}(u)dF_2(w), \qquad \tau_2 \le t<T_2$$

$$H_G(t,k)=\int_0^{\tau_1}\int_0^w \overline{F}_1(w-u)d\widetilde{F}_1^{(k)}(u)dF_2(w)+\int_{\tau_1}^{\tau_2}\int_{w-\tau_1}^w \overline{F}_1(w-u)d\widetilde{F}_1^{(k)}(u)dF_2(w)+$$

$$\int_{\tau_2}^{T_2}\int_{w-\tau_1}^{\tau_2} \overline{F}_1(w-u)d\widetilde{F}_1^{(k)}(u)dF_2(w)+\overline{F}_2(T_2)\int_{T_2-\tau_1}^{\tau_2} \overline{F}_1(T_2-u)d\widetilde{F}_1^{(k)}(u), \qquad (5.2.40)$$

$$t \ge T_2.$$

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=\int_0^t \overline{F}_1(t-u)d\widetilde{F}_1^{(k)}(u)dF_2(t), \qquad 0 \le t<\tau_1$$

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=\int_{t-\tau_1}^t \overline{F}_1(t-u)d\widetilde{F}_1^{(k)}(u)dF_2(t), \qquad \tau_1 \le t<\tau_2$$

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=\int_{t-\tau_1}^{\tau_2}\bar{F}_1(t-u)d\tilde{F}_1^{(k)}(u)dF_2(t), \qquad \tau_2 \leq t<T_2$$

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=\bar{F}_2(T_2)\int_{T_2-\tau_1}^{\tau_2}\bar{F}_1(T_2-u)d\tilde{F}_1^{(k)}(u), \qquad t=T_2$$

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=0, \qquad \text{otherwise.} \qquad (5.2.41)$$

For the case in which $\tau_1 \geq \tau_2$, $\tau_1+\tau_2<T_2(\Rightarrow \tau_1<T_2)$, the combined application of the conditions enumerated yield (5.2.14) and (5.2.20) and the corresponding full statement of (5.2.36) and (5.2.37) is:

$$H_G(t,k)=\int_0^t\int_0^w\bar{F}_1(w-u)d\tilde{F}_1^{(k)}(u)dF_2(w), \qquad 0 \leq t<\tau_2$$

$$H_G(t,k)=\int_0^{\tau_2}\int_0^w\bar{F}_1(w-u)d\tilde{F}_1^{(k)}(u)dF_2(w)+\int_{\tau_2}^t\int_0^{\tau_2}\bar{F}_1(w-u)d\tilde{F}_1^{(k)}(u)dF_2(w),$$

$$\tau_2 \leq t<\tau_1$$

$$H_G(t,k)=\int_0^{\tau_2}\int_0^w\bar{F}_1(w-u)d\tilde{F}_1^{(k)}(u)dF_2(w)+\int_{\tau_2}^{\tau_1}\int_0^{\tau_2}\bar{F}_1(w-u)d\tilde{F}_1^{(k)}(u)dF_2(w)+$$

$$\int_{\tau_1}^t\int_{w-\tau_1}^{\tau_2}\bar{F}_1(w-u)d\tilde{F}_1^{k}(u)dF_2(w), \qquad \tau_1 \leq t<\tau_1+\tau_2$$

$$H_G(t,k)=\int_0^{\tau_2}\int_0^w\bar{F}_1(w-u)d\tilde{F}_1^{(k)}(u)dF_2(w)+\int_{\tau_2}^{\tau_1}\int_0^{\tau_2}\bar{F}_1(w-u)d\tilde{F}_1^{(k)}(u)dF_2(w)+$$

$$\int_{\tau_1}^{\tau_1+\tau_2}\int_{w-\tau_1}^{\tau_2}\overline{F}_1(w-u)d\widetilde{F}_1^{(k)}(u)dF_2(w), \qquad t \geq \tau_1+\tau_2 \qquad (5.2.42)$$

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=\int_0^t\overline{F}_1(t-u)d\widetilde{F}_1^{(k)}(u)dF_2(t), \qquad 0 \leq t<\tau_2$$

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=\int_0^t\overline{F}_1(t-u)d\widetilde{F}_1^{(k)}(u)dF_2(t), \qquad \tau_2 \leq t<\tau_1$$

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=\int_{t-\tau_1}^{\tau_2}\overline{F}_1(t-u)d\widetilde{F}_1^{(k)}(u)dF_2(t), \qquad \tau_1 \leq t<\tau_1+\tau_2$$

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=0, \qquad \text{otherwise.} \qquad (5.2.43)$$

For the case in which $\tau_1 \geq \tau_2$, $\tau_1+\tau_2 \geq$ T2 and $\tau_1<T_2$, the combined applications of the conditions enumerated yield (5.2.14) and (5.2.18) and the corresponding full statement of (5.2.36) and (5.2.37) is:

$$H_G(t,k)=\int_0^t\int_0^w\overline{F}_1(w-u)d\widetilde{F}_1^{(k)}(u)dF_2(w), \qquad 0 \leq t<\tau_2$$

$$H_G(t,k)=\int_0^{\tau_2}\int_0^w\overline{F}_1(w-u)d\widetilde{F}_1^{(k)}(u)dF_2(w)+\int_{\tau_2}^t\int_0^{\tau_2}\overline{F}_1(w-u)d\widetilde{F}_1^{(k)}(u)dF_2(w),$$

$$\tau_2 \leq t<\tau_1$$

$$H_G(t,k)=\int_0^{\tau_2}\int_0^w\overline{F}_1(w-u)d\widetilde{F}_1^{(k)}(u)dF_2(w)+\int_{\tau_2}^{\tau_1}\int_0^{\tau_2}\overline{F}_1(w-u)d\widetilde{F}_1^{(k)}(u)dF_2(w)+$$

$$\int_{\tau_1}^t\int_{w-\tau_1}^{\tau_2}\overline{F}_1(w-u)d\widetilde{F}_1^{(k)}(u)dF_2(w), \qquad \tau_1 \leq t<T_2$$

$$H_G(t,k)=\int_0^{\tau_2}\int_0^{w}\overline{F}_1(w-u)d\widetilde{F}_1^{(k)}(u)dF_2(w)+\int_{\tau_2}^{\tau_1}\int_0^{\tau_2}\overline{F}_1(w-u)d\widetilde{F}_1^{(k)}(u)dF_2(w)+$$

$$\int_{\tau_1}^{T_2}\int_{w-\tau_1}^{\tau_2}\overline{F}_1(w-u)d\widetilde{F}_1^{(k)}(u)dF_2(w)+\overline{F}_2(T_2)\int_{T_2-\tau_1}^{\tau_2}\overline{F}_1(T_2-u)d\widetilde{F}_1^{(k)}(u),\quad t\geq T_2.$$

(5.2.44)

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=\int_0^{t}\overline{F}_1(t-u)d\widetilde{F}_1^{(k)}(u)dF_2(t),\qquad 0\leq t<\tau_2$$

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=\int_0^{\tau_2}\overline{F}_1(t-u)d\widetilde{F}_1^{(k)}(u)dF_2(t),\qquad \tau_2\leq t<\tau_1$$

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=\int_{t-\tau_1}^{\tau_2}\overline{F}_1(t-u)d\widetilde{F}_1^{(k)}(u)dF_2(t),\qquad \tau_1\leq t<T_2$$

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=\overline{F}_2(T_2)\int_{T_2-\tau_1}^{\tau_2}\overline{F}_1(T_2-u)d\widetilde{F}_1^{(k)}(u),\qquad t=T_2$$

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=0,\qquad \text{otherwise.}\qquad (5.2.45)$$

For the case in which $\tau_1\geq\tau_2$ and $\tau_1\geq T_2$, the combined application of the conditions enumerated yield (5.2.14) and (5.2.19) and the corresponding full statement of (5.2.36) and (5.2.37) is:

$$H_G(t,k)=\int_0^{t}\int_0^{w}\overline{F}_1(w-u)d\widetilde{F}_1^{(k)}(u)dF_2(w),\qquad 0\leq t<\tau_2$$

$$H_G(t,k)=\int_0^{\tau_2}\int_0^{w}\overline{F}_1(w-u)d\widetilde{F}_1^{(k)}(u)dF_2(w)+\int_{\tau_2}^{t}\int_0^{\tau_2}\overline{F}_1(w-u)d\widetilde{F}_1^{(k)}(u)dF_2(w),$$

$$\tau_2\leq t<T_2$$

$$H_G(t,k)=\int_0^{\tau_2}\int_0^{w}\overline{F}_1(w-u)d\widetilde{F}_1^{(k)}(u)dF_2(w)+\int_{\tau_2}^{T_2}\int_0^{\tau_2}\overline{F}_1(w-u)d\widetilde{F}_1^{(k)}(u)dF_2(w)+$$

$$\overline{F}_2(T_2)\int_0^{\tau_2}\overline{F}_1(T_2-u)d\widetilde{F}_1^{(k)}(u), \qquad t \geq T_2. \qquad (5.2.46)$$

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=\int_0^{t}\overline{F}_1(t-u)d\widetilde{F}_1^{(k)}(u)dF_2(t), \qquad 0 \leq t<\tau_2$$

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=\int_0^{\tau_2}\overline{F}_1(t-u)d\widetilde{F}_1^{(k)}(u)dF_2(t), \qquad \tau_2 \leq t<T_2$$

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=\overline{F}_2(T_2)\int_0^{\tau_2}\overline{F}_1(T_2-u)d\widetilde{F}_1^{(k)}(u), \qquad t=T_2$$

$$h_G(t,k)=\frac{d}{dt}H_G(t,k)=0, \qquad \text{otherwise.} \qquad (5.2.47)$$

5.2.3 Convolution of repair time and failure time of the system

For the cases in which system renewal does not occur, the expressions stated earlier as (3.3.2) and (3.3.3) apply directly to the partial opportunistic age replacement strategy. In addition these expressions apply equally well for all cases enumerated. For an initial minor interval the convolution of the operating interval duration and the repair time along the set of sample paths defined by B_1 and B_2 is:

$$q_I(t)=\sum_{k=0}^{\infty}\int_0^{t}h_I(x,k)\, g(t-x,k)dx \qquad (5.2.48)$$

where:

1. $h_I(t,k)$ is a combination of expression in (5.2.22) and (5.2.7) when $\tau_1<\tau_2$ and $\tau_1+\tau_2<T_2$.

2. $h_I(t,k)$ is a combination of expression in (5.2.24) and (5.2.7) when $\tau_1<\tau_2$ and $\tau_1+\tau_2 \geq T_2$.

3. $h_I(t,k)$ is a combination of expression in (5.2.26) and (5.2.7) when $\tau_1 \geq \tau_2$ and $\tau_1+\tau_2<T_2$ and $\tau_1<T_2$.

4. $h_I(t,k)$ is a combination of expression in (5.2.28) and (5.2.7) when $\tau_1 \geq \tau_2$ and $\tau_1+\tau_2 \geq T_2$ and $\tau_1<T_2$.

5. $h_I(t,k)$ is a combination of expression in (5.2.30) and (5.2.9) when $\tau_1 \geq \tau_2$ and $\tau_1 \geq T_2$.

6. g(t,k) is the same as the expression in (4.2.9).

As indicated in chapter 3, $Z_I(t)$ is the distribution function on the combined length of all operating periods during an initial minor interval for the set of sample paths defined by B_1 and B_2. $H_I(t,k)$ is defined as follows:

1. $H_I(t,k)$ is a combination of expression in (5.2.21) and (5.2.6) when $\tau_1<\tau_2$ and $\tau_1+\tau_2<T_2$.

2. $H_I(t,k)$ is a combination of expression in (5.2.23) and (5.2.6) when $\tau_1<\tau_2$ and $\tau_1+\tau_2 \geq T_2$.

3. $H_I(t,k)$ is a combination of expression in (5.2.25) and (5.2.6) when $\tau_1 \geq \tau_2$ and $\tau_1+\tau_2<T_2$ and $\tau_1<T_2$.

4. $H_I(t,k)$ is a combination of expression in (5.2.27) and (5.2.6) when $\tau_1 \geq \tau_2$ and $\tau_1+\tau_2 \geq T_2$ and $\tau_1<T_2$.

5. $H_I(t,k)$ is a combination of expression in (5.2.29) and (5.2.8) when $\tau_1 \geq \tau_2$ and $\tau_1 \geq T_2$.

The distribution on the combined length of the operating periods for the set of all possible sample paths is:

$$Z_I(t)=\sum_{k=0}^{\infty}H_I(t,k)+\sum_{k=0}^{\infty}V_{I1}(t,k)+\sum_{k=0}^{\infty}V_{I2}(t,k) \tag{5.2.49}$$

For a general minor interval, the convolution of the operating interval duration and the repair time for the set of sample paths defined by B_1 and B_2 is:

$$q_G(t)=\sum_{k=0}^{\infty}\int_0^t h_G(x,k)\, g(t-x,k)dx \tag{5.2.50}$$

where :

1. $h_G(t,k)$ is a combination of expression in (5.2.39) and (5.2.33) when $\tau_1<\tau_2$ and $\tau_1+\tau_2<T_2$.
2. $h_G(t,k)$ is a combination of expression in (5.2.41) and (5.2.33) when $\tau_1<\tau_2$ and $\tau_1+\tau_2 \geq T_2$.
3. $h_G(t,k)$ is a combination of expression in (5.2.43) and (5.2.33) when $\tau_1 \geq \tau_2$ and $\tau_1+\tau_2<T_2$ and $\tau_1<T_2$.
4. $h_G(t,k)$ is a combination of expression in (5.2.45) and (5.2.33) when $\tau_1 \geq \tau_2$ and $\tau_1+\tau_2 \geq T_2$ and $\tau_1<T_2$.
5. $h_G(t,k)$ is a combination of expression in (5.2.47) and (5.2.35) when $\tau_1 \geq \tau_2$ and $\tau_1 \geq T_2$.
6. $g(t,k)$ is the same as the expression in (4.2.9).

$H_G(t,k)$ is defined as follows:

1. $H_G(t,k)$ is a combination of expression in (5.2.38) and (5.2.32) when $\tau_1<\tau_2$ and $\tau_1+\tau_2<T_2$.

2. $H_G(t,k)$ is a combination of expression in (5.2.40) and (5.2.32) when $\tau_1<\tau_2$ and $\tau_1+\tau_2 \geq T_2$.

3. $H_G(t,k)$ is a combination of expression in (5.2.42) and (5.2.32) when $\tau_1 \geq \tau_2$ and $\tau_1+\tau_2<T_2$ and $\tau_1<T_2$.

4. $H_G(t,k)$ is a combination of expression in (5.2.44) and (5.2.32) when $\tau_1 \geq \tau_2$ and $\tau_1+\tau_2 \geq T_2$ and $\tau_1<T_2$.

5. $H_G(t,k)$ is a combination of expression in (5.2.46) and (5.2.34) when $\tau_1 \geq \tau_2$ and $\tau_1 \geq T_2$.

The corresponding distribution on the combined length of the operating periods for all possible set of sample is:

$$Z_G(t)=\sum_{k=0}^{\infty} H_G(t,k)+\sum_{k=0}^{\infty} V_{G1}(t,k)+\sum_{k=0}^{\infty} V_{G2}(t,k) \qquad (5.2.51)$$

5.2.3 Initial minor interval (Renewal)

The sample paths for the renewal interval are the compliment of the sample paths for the non renewal paths. All the sample paths for the renewal and the non renewal sample paths are enumerated in appendix 2B.

For the case in which k=0, there is no component 1 failure before the time of the component 2 failure. In such a situation we have:

$$V_{I1}(t,0)=0$$

and

$$V_{I2}(t,0)=\int_{\tau_1}^{t}\bar{F}_1(w)f_2(w)du \qquad \tau_1 \le t<\infty \qquad (5.2.52)$$

The corresponding density function is:

$$v_{I2}(t,0)=\begin{cases}\bar{F}_1(t)f_2(t) & \tau_1 \le t<\infty \\ 0 & \text{otherwise}\end{cases} \qquad (5.2.53)$$

For the broad case in which $\tau_1+\tau_2 \le T_2$ and $k \ge 1$ the probability distributions and the density functions are similar to case in the opportunistic failure replacement policy.

$$v_{I2}(t,k)=\begin{cases}\int_0^{t-\tau_1} f_2(t)f_1^{(k)}(u)\bar{F}_1(t\text{-}u)du & \tau_1 \le t<\tau_1+\tau_2 \\ \int_0^{\tau_2} f_2(t)f_1^{(k)}(u)\bar{F}_1(t\text{-}u)du & \tau_1+\tau_2 \le t<T_2 \\ 0 & \text{otherwise}\end{cases} \qquad (5.2.54)$$

and

$$v_{I1}(t,k)=\begin{cases}\bar{F}_2(t)\int_0^{\tau_2} f_1^{(k-1)}(v)f_1(t\text{-}v)dv & \tau_2 \le t<T_2 \\ 0 & \text{otherwise}\end{cases} \qquad (5.2.55)$$

On the other hand when $k \ge 1$, $\tau_1+\tau_2 \ge T_2$ and $\tau_1<T_2$ then probability distributions and the density functions are:

$$v_{I2}(t,k)=\begin{cases}\int_0^{t-\tau_1} f_2(t)f_1^{(k)}(u)\bar{F}_1(t\text{-}u)du & \tau_1 \le t<T_2 \\ 0 & \text{otherwise}\end{cases} \qquad (5.2.56)$$

and

$$v_{I1}(t,k)=\begin{cases} \bar{F}_2(t)\int_0^{\tau_2} f_1^{(k-1)}(v)f_1(t-v)dv & \tau_2 \le t<T_2 \\ 0 & \text{otherwise} \end{cases} \tag{5.2.57}$$

When $k \ge 1$, $\tau_1+\tau_2 \ge T_2$ and $\tau_1>T_2$ then probability distributions and the density functions are:

$$v_{I2}(t,k)=0 \qquad 0 \le t \le T_2$$

and

$$v_{I1}(t,k)=\begin{cases} \bar{F}_2(t)\int_0^{\tau_2} f_1^{(k-1)}(v)f_1(t-v)dv & \tau_2 \le t<T_2 \\ 0 & \text{otherwise} \end{cases} \tag{5.2.58}$$

Also

$$V_{I2}(t,k)=\int_0^t v_2(x,k)dx$$

and

$$V_{I1}(t,k)=\int_0^t v_1(x,k)dx$$

When the initial minor interval ends with component 2 age replacement then when we consider the case $\tau_1+\tau_2<T_2$ and the case k=0 we have:

$$v_{I2}(t,0)=\bar{F}_2(T_2)\bar{F}_1(T_2) \qquad t=T_2 \tag{5.2.59}$$

For $k \ge 1$ we have:

$$v_{I2}(t,k)=\overline{F}_2(T_2)\int_0^{\tau_2} f_1^{(k)}(u)\overline{F}_1(T_2-u)du \qquad t=T_2 \tag{5.2.60}$$

When we have age replacement of component 2. when we consider the case $\tau_1+\tau_2 \geq T_2$ and the case k=0 we have:

$$v_{I2}(t,k)=\delta(T_2-\tau_1)\overline{F}_2(T_2)\overline{F}_1(T_2) \quad t=T_2 \tag{5.2.61}$$

and for $k \geq 1$ we have:

$$v_{I2}(t,k)=\overline{F}_2(T_2)\int_0^{T_2-\tau_1} f_1^{(k)}(u)\overline{F}_1(T_2-u)du \tag{5.2.62}$$

When component 1 replacement produces system renewal we assume that component 1 fails k times and there is system renewal at the k^{th} component 1 replacement. Therefore we have k component 1 repair times and 1 component 2 repair time. When component 2 produces a system renewal we assume component 2 replacement occurs after the k^{th} component 1 replacement. Therefore we k+1 component 1 repair time and 1 component 2 repair time.

The density on the length operating time and the repair time is:

$$u_I(t)=\sum_{k=0}^{\infty}\int_0^t v_{I1}(x,k)g_{opp}(t-x)dx+\sum_{k=0}^{\infty}\int_0^t v_{I2}(x,k)g_{opp}(t-x,k+1)dx \tag{5.2.63}$$

5.2.4 General Minor interval (Renewal)

The same reasoning that is used for the initial minor interval with renewal can be applied to the general minor interval with renewal.The sample paths are the same. Hence we have the followind results:

For the case in which k=0, then there is no component 1 failure before the time of component 1 failure. In such a situation we have:

$$V_{G1}(t,0)=0$$

$$V_{G2}(t,0)=\int_{\tau_1-a_1}^{t} \bar{\bar{F}}_1(w)f_2(w)du \qquad \tau_1-a_1 \leq t<\infty \qquad (5.2.64)$$

The corresponding density function is:

$$v_{G2}(t,0)=\bar{\bar{F}}_1(w)f_2(w)du \qquad \tau_1-a_1 \leq t<\infty \qquad (5.2.65)$$

For the broad case in which $k \geq 1$ and $\tau_1+\tau_2 \leq T_2$ the probability distributions and the density functions are similar to case in the opportunistic failure replacement policy.

$$v_{G2}(t,k)=\begin{cases} \int_0^{t-\tau_1} f_2(t)\tilde{f}^{(k)}{}_1(u)\bar{F}_1(t-u)du & \tau_1 \leq t<\tau_1+\tau_2 \\ \int_0^{\tau_2} f_2(t)\tilde{f}_1^{(k)}(u)\bar{F}_1(t-u)du & \tau_1+\tau_2 \leq t<T_2 \\ 0 & \text{otherwise} \end{cases} \qquad (5.2.66)$$

and

$$v_{G1}(t,k)=\begin{cases} \bar{F}_2(t)\int_0^{\tau_2}\tilde{f}_1^{(k-1)}(v)f_1(t-v)dv & \tau_2 \leq t<T_2 \\ 0 & \text{otherwise} \end{cases} \tag{5.2.67}$$

On the other hand when $k \geq 1$ and $\tau_1+\tau_2 \geq T_2$ and $\tau_1<T_2$ the probability distributions and the density functions are:

$$v_{G2}(t,k)=\begin{cases} \int_0^{t-\tau_1} f_2(t)\tilde{f}_1^{(k)}(u)\bar{F}_1(t-u)du & \tau_1 \leq t<T_2 \\ 0 & \text{otherwise} \end{cases} \tag{5.2.68}$$

and

$$v_{G1}(t,k)=\begin{cases} \bar{F}_2(t)\int_0^{\tau_2}\tilde{f}_1^{(k-1)}(v)f_1(t-v)dv & \tau_2 \leq t<T_2 \\ 0 & \text{otherwise} \end{cases} \tag{5.2.69}$$

When $k \geq 1$ and $\tau_1+\tau_2 \geq T_2$ and $\tau_1>T_2$ the probability distributions and the density functions are:

$$v_{G2}(t,k)=0 \qquad 0 \leq t \leq T_2$$

and

$$v_{G1}(t,k)=\begin{cases} \bar{F}_2(t)\int_0^{\tau_2}f_1^{(k-1)}(v)f_1(t-v)dv & \tau_2 \leq t<T_2 \\ 0 & \text{otherwise} \end{cases} \tag{5.2.70}$$

Also

$$V_{G2}(t,k)=\int_0^t v_2(x,k)dx$$

and

$$V_{G1}(t,k)=\int_0^t v_1(x,k)dx$$

When the general minor interval ends with component 2 age replacement then when we consider the case $\tau_1+\tau_2<T_2$ and the case k=0 we have:

$$v_{G2}(t,k)=\bar{F}_2(T_2)\bar{\bar{F}}_1(T_2)\delta(T_2-\tau_1+a_1) \quad t=T_2 \tag{5.2.71}$$

For $k \geq 1$ we have:

$$v_{G2}(t,k)=\bar{F}_2(T_2)\int_0^{\tau_2}\tilde{f}_1^{(k)}(u)\bar{F}_1(T_2-u)du \quad t=T_2 \tag{5.2.72}$$

When we have age replacement of component 2. when we consider the case $\tau_1+\tau_2 \geq T_2$ and the case k=0 we have:

$$v_{G2}(t,k)=\delta(T_2-\tau_1+a_1)\bar{F}_2(T_2)\bar{\bar{F}}_1(T_2) \quad t=T_2 \tag{5.2.73}$$

and for $k \geq 1$ we have:

$$v_{G2}(t,k)=\bar{F}_2(T_2)\int_0^{T_2-\tau_1}\tilde{f}_1^{(k)}(u)\bar{F}_1(T_2-u)du \tag{5.2.74}$$

$$u_I(t)=\sum_{k=0}^{\infty}\int_0^t v_{G1}(x,k)g_{opp}(t-x)dx+\sum_{k=0}^{\infty}\int_0^t v_{G2}(x,k)g_{opp}(t-x,k+1)dx \tag{5.2.75}$$

5.2.5 Major Interval

A major interval consists of an initial minor interval and n general minor intervals (n=0,1,2. . .). This means that a major interval always starts with both components new.

Both components go through several failures and repairs. The major interval ends when both components are replaced at the same time.

The renewal density function for the minor intervals that end without system renewal is the convolution of the density on the length initial interval and the lengths of n general intervals. The minor intervals are constructed such that there are no system renewals. Using (5.2.48) and (5.2.50) the renewal density is:

$$m_Q(t)=\sum_{n=0}^{\infty}\int_0^t q_I(x)q_G^{(n)}(t-x)dx \tag{5.2.76}$$

From (3.5.1) the availability is the probability that the system is functioning and the length of the major interval is longer than t. This is:

$$A(t)=\bar{Z}_I(t)+\int_0^t \bar{Z}_G(x)m_Q(t-x)dx \tag{5.2.77}$$

where $\bar{Z}_I(t)=1-Z_I(t)$ and $\bar{Z}_G(t)=1-Z_G(t)$ are as defined in chapter 3. Also from (3.5.3) the time dependent system availability is:

$$\mathfrak{A}(t)=A(t)+\int_0^t A(x)m_\Phi(t-x)dx.$$

and the long run availability is:

$$\mathfrak{A}=\lim_{t\to\infty}\mathfrak{A}=\frac{(\nu_I(1-Q_G(\infty))+\nu_G Q_I(\infty))}{(\mu_{IR}+\mu_I+\mu_{opp})(1-Q_G(\infty))+(\mu_{GR}+\mu_G)Q_I(\infty)} \tag{5.2.78}$$

Note that when $T_2=\infty$, there is no age replacement for either component and the model is the same as the opportunistic failure replacement model.

5.3 Full Opportunistic Age Replacement Policy

In a full opportunistic age replacement policy, both components have independent age replacement policy times. Any such replacement of a unit necessitates the stopping of the machine and thus creates a planned replacement opportunity for the second unit. Failure times provide the same opportunity. The replacement of the unfailed unit is actually carried out if its age exceeds the opportunistic replacement policy time.

In the three previous models we assume that component 2 replacement occurs between the k^{th} and the $k+1^{st}$ component 1 replacement. It is also shown that system behaviour can be effectively modeled in terms of the probabilities associated with sample paths that yield no system renewal. Using that modeling approach, the models developed previously subsume a majority of those needed to represent a full opportunistic replacement policy. In particular the models constructed in section 5.2 subsume and apply directly to all feasible full opportunistic age replacement policies in which $T_1>T_2$. More specially, each of the models developed is defined in terms of the relative magnitude of τ_1,τ_2 and T_2. For each such model, the corresponding full opportunistic age replacement policy having $T_1>T_2$ has the exact same model.

For example, for the partial opportunistic age replacement strategy when $\tau_1<\tau_2<T_2$ and $\tau_1+\tau_2 \geq T_2$, $h_I(t,k)$ is defined by (5.2.24) and (5.2.7). $H_I(t,k)$ is defined by (5.2.23) and

(5.2.6), $h_G(t,k)$ is defined by (5.2.41) and (5.2.33), and $H_G(t,k)$ is defined by (5.2.40) and (5.2.32). For the full opportunistic age replacement strategy, when $\tau_1<\tau_2<T_2<T_1$ and $\tau_1+\tau_2>T_2$, the exact same equations apply. The reasoning for this is that the component 2 age replacement policy time, T_2, bounds the lengths of the minor intervals. Consequently, age replacement of component 1 cannot occur and the model reduces to that for the partial case.

The implications of the above statements concerning model equivalence are essential. First it is noted that a complete enumeration of all possible model cases in which minor intervals end with the restart of only one component indicates that there are only three cases beyond those already treated. These cases are:

1. $0 \leq \tau_2 \leq \tau_1 \leq T_1 \leq T_2$
2. $0 \leq \tau_1 \leq \tau_2 \leq T_1 \leq T_2$
3. $0 \leq \tau_1 \leq T_1 \leq \tau_2 \leq T_2$.

A further implication of the model equivalence is that prior knowledge permits the application of models of section 5.3 by simply intercahanging the indices on the components. This comment raises questions of the use of the models. If the models are to be used to evaluate system availability under a proposed (or specified) policy (τ_1,τ_2, T_1,T_2), then the relative magnitudes of the decision variables are known and the models of section 5.3 apply directly given the appropraite assignment of indices. If, on the other hand, the models are analyzed in order to select the decision variables (τ_1,τ_2, T_1,T_2), then the models can be applied under both possible assignment of indices and the results can be compared. Thus the set of models of section 5.3 are sufficient for the analysis and design of all opportunistic replacement policies.

The single deficiency in simply exploiting the model equivalence is that the set of models is not exhaustive of all concievable cases despite the fact that it is exhaustive of all practically useful cases. The cases excluded are the ones in which the age replacement policy time for the component whose replacement ends a minor interval exceeds the age replacement time for the other component. Given the absence of intrinsic practical value for the three excluded cases, their developement is not considered worthwhile and is not persued here.

5.4 Conclusions:

When $T_1>T_2$ assume that component 2 replacement time occurs between the k^{th} and the $k+1^{st}$ component 1 replacement time and when $T_1 \leq T_2$ we will reverse the component indices.

For all the possible cases enumerated the expressions (5.3.6) and (5.3.7) are respectively the time dependent and long run availability measures for full opportunistic age replacement replacement strategy.

When $T_1=T_2 \Rightarrow \infty$, the full opportunistic replacement model becomes the failure opportunistic replacement model. When $T_1=\tau_1$ and $T_2=\tau_2$, then there is no opportunistic replacement for either component, and the full opportunistic replacement becomes model an age replacement model. In the age replacement model the renewal process of the minor interval are recurrent.

We also make the following conclusions about the model:

1. We have a consistent set of models

2. We have a "practically" exhaustive set of models

3. We have both time dependent and limiting availability measures.

4. We have a general model that applys to any definition of $F_i(t)$ and $G_i(t)$.

In the next chapter, numerical examples are presented for the Weibull and the Gamma failure distributions.

CHAPTER 6 Analysis and Results.

6.1 Introduction

In this chapter we find a numerical approximation for the convolution of the failure times of the components. Numerical approximations to the distribution of the initial and the general intervals are computed. Numerical examples of the models are given. Comparisions are made between the failure model and the opportunistic replacement model. Suggestions for possible improvement in the opportunistic replacement model are given. An explanation of how to interpret the numerical results from the model is provided. The Weibull and Gamma distribution functions are used for the failure times of the components. The Laplace transforms for the system availability are given for both cases. An attempt is made to estimate the time dependent availability and the long run availability measures for the system.

6.2 Convolutions for Failure and Repair distributions.

We convert both $q_I(t)$ and $q_G(t)$ to their respective Laplace Transforms, since it will be easier working through the Laplace transforms.

From (3.3.2)

$$q_I(t)=\sum_{k=0}^{\infty}\int_0^t h_I(x,k)g(t-x,k)dx.$$

and

$$q_G(t)=\sum_{k=0}^{\infty}\int_0^t h_G(x,k)g(t-x,k)dx.$$

The Laplace transforms of $q_I(t)$ and $q_G(t)$ are respectively:

$$q_I^*(s)=\sum_{k=0}^{\infty}h_I^*(s,k)g^*(s,k). \tag{6.2.1}$$

and

$$q_G^*(s)=\sum_{k=0}^{\infty}h_G^*(s,k)g^*(s,k). \tag{6.2.2}$$

where

$$g^*(s,k)=(g^*{}_1(s))^k g_2{}^*(s). \tag{6.2.3}$$

and the Laplace transform for $h_I(t,k)$ is defined as:

$$h_I^*(s,k)=\int_0^{\infty} e^{-st}\, h_I(t,k)dt \tag{6.2.4}$$

The set A will denote the sample paths in which component 2 fails before T_2 and there is no system renewal and the set B will denote the sample paths in which there is an age replacement of component 2 but there is no system renewal. The set $\bar{A}_1$ will denote the sample paths that produce a system renewal because of component 1 replacement. The set $\bar{A}_2$ will denote the sample paths that produce a system renewal because of component 2 replacement. Also the set $\bar{B}_2$ will denote the sample paths that cause a system renewal as a result of component 2 age replacement.

If we condition on the age replacement time of component 2 the Laplace transform for $h_I(t,k)$ is defined as:

$$h_I^*(s,k)=\int_A\int \bar{F}_1(w-u)f_2(w)f_1^{(k)}(u)e^{-ws}dudw+e^{-T_2s}\bar{F}_2[T_2]\int_B \bar{F}_1(T_2-u)f_1^{(k)}(u)du. \qquad (6.2.5)$$

Also the Laplace transform for $h_G(t,k)$ is defined as:

$$h_G^*(s,k)=\int_0^\infty e^{-st}\, h_G(t,k)dt \qquad (6.2.6)$$

$$h_G^*(s,k)=\int_A\int \bar{F}_1(w-u)f_2(w)\tilde{f}_1^{(k)}(u)e^{-ws}dudw+e^{-T_2s}\bar{F}_2[T_2]\int_B \bar{F}_1(T_2-u)\tilde{f}_1^{(k)}(u)du. \qquad (6.2.7)$$

where A and B are the sample paths and they depend on the type of model. In chapters 4 and 5 these sample paths are clearly defined for both k=0 and for $k \geq 1$ with respect to the type of model.

For the initial interval:

$$q^*_I(s)= g_2^*(s)\int_{A|k=0} \bar{F}_1(w)f_2(w)e^{-ws}dw+ e^{-T_2s}\bar{F}_2[T_2]\bar{F}_1[T_2]\delta(\tau_1-T_2)+$$

$$\sum_{k=1}^{\infty}(g^*{}_1(s))^k g_2{}^*(s)\int_A\int \overline{F}_1(w-u)f_2(w)f_1^{(k)}(u)e^{-ws}dw+$$

$$e^{-T_2s}\overline{F}_2[T_2]\sum_{k=1}^{\infty}(g^*{}_1(s))^k g_2{}^*(s)\int_B \overline{F}_1(T_2-u)f_1^{(k)}(u)du. \qquad (6.2.8)$$

and for the general interval :

$$q^*{}_G(s)= g_2{}^*(s)\int_{A|k=0}\overline{\overline{F}}_1(w)f_2(w)e^{-ws}dw+e^{-T_2s}\overline{F}_2[T_2]\overline{\overline{F}}_1[T_2]\delta(\tau_1-a_1-T_2)+$$

$$\sum_{k=1}^{\infty}(g^*{}_1(s))^k g_2{}^*(s)\int_A\int \overline{F}_1(w-u)f_2(w)\tilde{f}_1^{(k)}(u)e^{-ws}dw+$$

$$e^{-T_2s}\overline{F}_2[T_2]\sum_{k=1}^{\infty}(g^*{}_1(s))^k g_2{}^*(s)\int_B \overline{F}_1(T_2-u)\tilde{f}_1^{(k)}(u)du. \qquad (6.2.9)$$

where

$$\delta(x)\begin{cases}1 & x\geq 0\\ 0 & \text{otherwise.}\end{cases}$$

Now using distribution functions on the combined lengths of the operating periods in the minor intervals without renewal in chapter 4 and 5, we find the Laplace transforms, the probabilities and the expected values for both initial minor and general minor intervals.

The probability distribution on the operating time during an of the initial interval is:

$$Z_I(t)=\sum_{k=0}^{\infty}(H_I(t,k)+V_{I1}(t,k)+V_{I2}(t,k)) \tag{6.2.10}$$

The Laplace transform of $z_I(t)=\frac{d}{dt}Z_I(t)$ is:

$$z_I^*(s)=\int_0^{\infty} z_I(t)\, e^{-st}dt$$

$$z_I^*(s)=\sum_{k=0}^{\infty}(h_I^*(s,k)+v_{I1}^*(s,k)+v_{I2}^*(s,k)) \tag{6.2.11}$$

where $h_I^*(s,k)$ is the same expression in (6.2.5) and

$$v_{I1}^*(s,k)=\int_0^{\infty} v_{I1}(t,k)\, e^{-st}dt$$

$$v_{I1}^*(s,k)=\int_{\bar{A}1}\int \bar{F}_2(u)f_1^{(k-1)}(v)f_1(u-v)dv\, e^{-su}du$$

and

$$v_{I2}^*(s,k)=\int_0^{\infty} v_{I2}(t,k)\, e^{-sw}dw$$

$$v_{I2}^*(s,k)=\int_{\bar{A}2} f_2(w)\int f_1^{(k)}(u)\bar{F}_1(w-u)\, e^{-sw}dw+$$

$$\bar{F}_2(T_2)\int_{\bar{B}2} f_1^{(k)}(u)\bar{F}_1(T_2-u)\, e^{-sT_2} \tag{6.2.12}$$

The Laplace transform of $z_I(t)$ is:

$$z_I^*(s)=\sum_{k=0}^{\infty}\int_A\int \overline{F}_1(w-u)f_2(w)f_1^{(k)}(u)e^{-ws}dw+$$

$$\int_{\bar{A}2} f_2(w)\int f_1^{(k)}(u)\overline{F}_1(w-u)\ e^{-sw}dw+$$

$$\overline{F}_2(T_2)\sum_{k=0}^{\infty}\int_{\bar{B}2} f_1^{(k)}(u)\overline{F}_1(T_2-u)\ e^{-sT_2}+$$

$$e^{-T_2 s}\overline{F}_2[T_2]\sum_{k=0}^{\infty}\int_B \overline{F}_1(T_2-u)f_1^{(k)}(u)du+$$

$$\int_{\bar{A}1}\int \overline{F}_2(u)f_1^{(k-1)}(v)f_1(u-v)dv\ e^{-su}du \qquad (6.2.13)$$

Differentiating $z_I^*(s)$ with respect to s :

$$\frac{d}{ds}z_I^*(s)=\sum_{k=0}^{\infty}\int_A\int -w\ \overline{F}_1(w-u)f_2(w)f_1^{(k)}(u)e^{-ws}dw+$$

$$\int_{\bar{A}2} -w\ f_2(w)\int f_1^{(k)}(u)\overline{F}_1(w-u)\ e^{-sw}dw+$$

$$-T_2\ \overline{F}_2(T_2)\sum_{k=0}^{\infty}\int_{\bar{B}2} f_1^{(k)}(u)\overline{F}_1(T_2-u)\ e^{-sT_2}$$

$$-T_2 e^{-T_2 s}\overline{F}_2[T_2]\sum_{k=0}^{\infty}\int_B \overline{F}_1(T_2-u)f_1^{(k)}(u)du+$$

$$\int_{\bar{A}1}\int -u\ \overline{F}_2(u)f_1^{(k-1)}(v)f_1(u-v)dv\ e^{-su}du \qquad (6.2.14)$$

The expected operating time in an initial interval:

$$\nu_I = -\left.\frac{d\ z_I^*(s)}{ds}\right|_{s=0}$$

$$\nu_1=\sum_{k=0}^{\infty}\int_A\int w\ \overline{F}_1(w-u)f_2(w)f_1^{(k)}(u)dw+$$

$$\sum_{k=0}^{\infty}\int_{\bar{A}2} w\ f_2(w)\int f_1^{(k)}(u)\overline{F}_1(w-u)\ dw+$$

$$T_2\ \overline{F}_2(T_2)\sum_{k=0}^{\infty}\int_{\bar{B}2} f_1^{(k)}(u)\overline{F}_1(T_2-u)+$$

$$\overline{F}_2[T_2]\sum_{k=0}^{\infty}\int_B \overline{F}_1(T_2-u)f_1^{(k)}(u)du$$

$$\int_{\bar{A}1}\int -u\overline{F}_2(u)f_1^{(k-1)}(v)f_1(u-v)dvdu \qquad (6.2.15)$$

Similarly for the general minor interval. The probability distribution on the length of the opreating time is:

$$Z_G(t)=\sum_{k=0}^{\infty}(H_G(t,k)+V_{G1}(t,k)+V_{G2}(t,k)) \qquad (6.2.16)$$

The Laplace transform of $z_G(t)=\frac{d}{dt}Z_G(t)$ is:

$$z_G^*(s)=\int_0^{\infty} z_G(t)\ e^{-st}dt$$

$$z_G^*(s)=\sum_{k=0}^{\infty}\left(h_G^*(s,k)+v_{G1}^*(s,k)+v_{G2}^*(s,k)\right) \qquad (6.2.17)$$

where $h_G^*(s,k)$ is the same expression in (6.2.7) and

$$v^*_{G1}(s,k)=\int_0^{\infty} v_{G1}(t,k)\, e^{-st}dt$$

$$v^*_{G1}(s,k)=\int_{\bar{A}1}\int_0^{T_2} \bar{F}_2(u)\tilde{f}_1^{(k-1)}(v)f_1(u-v)dv\, e^{-su}du$$

and

$$v^*_{G2}(s,k)=\int_0^{\infty} v_{G2}(t,k)\, e^{-sw}dw$$

$$v^*_{G2}(s,k)=\int_{\bar{A}2} f_2(w)\int f_1^{(k)}(u)\bar{F}_1(w-u)\, e^{-sw}dw+$$

$$\bar{F}_2(T_2)\int_{\bar{B}2} f_1^{(k)}(u)\bar{F}_1(T_2-u)\, e^{-sT_2} \qquad (6.2.18)$$

$$z^*_G(s)=\int_{A|k=0} e^{-sw}\,\bar{\bar{F}}_1(w)f_2(w)dw+\sum_{k=1}^{\infty}\int_A\int \bar{F}_1(w-u)f_2(w)\tilde{f}_1^{(k)}(u)e^{-ws}dw+$$

$$\int_{\bar{A}_2|k=0} e^{-sw}\,\bar{\bar{F}}_1(w)f_2(w)dw+\sum_{k=1}^{\infty}\int_{\bar{A}2} f_2(w)\int \tilde{f}_1^{(k)}(u)\bar{F}_1(w-u)\, e^{-sw}dw+$$

$$e^{-sT_2}\bar{F}_2(T_2)\bar{\bar{F}}_1(T_2)\delta(T_2-\tau_1+a_1)+\bar{F}_2(T_2)\sum_{k=1}^{\infty}\int_{\bar{B}2} \tilde{f}_1^{(k)}(u)\bar{F}_1(T_2-u)\, e^{-sT_2}+$$

$$e^{-T_2s}\bar{F}_2[T_2]\bar{F}_1[T_2]+e^{-T_2s}\bar{F}_2[T_2]\sum_{k=1}^{\infty}\int_B \bar{F}_1(T_2-u)\tilde{f}_1^{(k)}(u)du+$$

$$\int_{\bar{A}1|k=0} \bar{F}_2(u)\tilde{f}(u)\, e^{-su}du+\int_{\bar{A}1}\int_0^{T_2} \bar{F}_2(u)\tilde{f}_1^{(k-1)}(v)f_1(u-v)dv\, e^{-su}du \qquad (6.2.19)$$

Differentiating $z^*_G(s)$ with respect to s :

$$\frac{d}{ds}z_G^*(s)=\sum_{k=0}^{\infty}\int_A\int -w\,\overline{F}_1(w-u)f_2(w)\tilde{f}_1^{(k)}(u)e^{-ws}dw+$$

$$\sum_{k=0}^{\infty}\int_{\bar{A}2} -w\,f_2(w)\int \tilde{f}_1^{(k)}(u)\overline{F}_1(w-u)\,e^{-sw}dw+$$

$$-T_2\,\overline{F}_2(T_2)\sum_{k=0}^{\infty}\int_{\bar{B}2}\tilde{f}_1^{(k)}(u)\overline{F}_1(T_2-u)\,e^{-sT_2}-$$

$$T_2e^{-T_2s}\overline{F}_2[T_2]\sum_{k=0}^{\infty}\int_B \overline{F}_1(T_2-u)\tilde{f}_1^{(k)}(u)du+$$

$$\int_{\bar{A}1}\int -u\,\overline{F}_2(u)\tilde{f}_1^{(k-1)}(v)f_1(u-v)dv\,e^{-su}du \qquad (6.2.20)$$

The expected operating time in general interval is:

$$\nu_G=-\left.\frac{d\,z_G^*(s)}{ds}\right|_{s=0}$$

$$\nu_G=\int_{A|k=0} w\,\overline{\overline{F}}_1(w)f_2(w)dw+\sum_{k=1}^{\infty}\int_A\int w\,\overline{F}_1(w-u)f_2(w)\tilde{f}_1^{(k)}(u)dw+$$

$$\int_{\bar{A}|k=0} w\,\overline{\overline{F}}_1(w)f_2(w)dw+\sum_{k=1}^{\infty}\int_{\bar{A}} w\,f_2(w)\int \tilde{f}_1^{(k)}(u)\overline{F}_1(w-u)\,dw+$$

$$T_2\,\overline{F}_2(T_2)\overline{\overline{F}}_1(T_2)\delta(T_2-\tau_1+a_1)+T_2\,\overline{F}_2(T_2)\sum_{k=1}^{\infty}\int_{\bar{B}}\tilde{f}_1^{(k)}(u)\overline{F}_1(T_2-u)+$$

$$T_2\,\overline{F}_2(T_2)\,\overline{\overline{F}}_1(T_2)\delta(\tau_1-a_1-T_2)+\overline{F}_2[T_2]\sum_{k=1}^{\infty}\int_B \overline{F}_1(T_2-u)\tilde{f}_1^{(k)}(u)du+$$

$$\int_{\bar{A}_1|k=0} u\,\overline{F}_2(u)\tilde{f}_1(u)du+\sum_{k=1}^{\infty}\int_{\bar{A}1}\int u\,\overline{F}_2(u)\tilde{f}_1^{(k)}(v)f_1(u-v)dv\,e^{-su}du \qquad (6.2.21)$$

$\sum_{k=1}^{\infty}\tilde{f}_1^{(k)}(t)$ is the renewal density of a stationary renewal process and

$$\sum_{k=1}^{\infty}\tilde{f}_1^{(k)}(t)=\frac{d}{dt}\sum_{k=1}^{\infty}\tilde{F}_1^{(k)}(t)=\frac{d}{dt}\tilde{M}(t)=\frac{1}{\mu} \tag{6.2.22}$$

Now simplifying (6.2.19) and subtituting $\frac{1}{\mu}$ for $\sum_{k=1}^{\infty}\tilde{f}_1^{(k)}(t)$ yields:

$$z_G^*(s)=\int_{A|k=0} e^{-sw}\,\overline{\overline{F}}_1(w)f_2(w)dw+\int_A\int \overline{F}_1(w-u)f_2(w)f\frac{1}{\mu_1}e^{-ws}dw+$$

$$\int_{\bar{A}_2|k=0} e^{-sw}\,\overline{\overline{F}}_1(w)f_2(w)dw+\int_{\bar{A}2} f_2(w)\int \tilde{f}_1^{(k)}(u)\overline{F}_1(w-u)\,e^{-sw}dw+$$

$$e^{-sT_2}\overline{F}_2(T_2)\overline{\overline{F}}_1(T_2)\delta(T_2-\tau_1+a_1)+\overline{F}_2(T_2)\int_{\bar{B}2}\frac{1}{\mu_1}\overline{F}_1(T_2-u)\,e^{-sT_2}+$$

$$e^{-T_2s}\overline{F}_2[T_2]\overline{F}_1[T_2]+e^{-T_2s}\overline{F}_2[T_2]\int_B \overline{F}_1(T_2-u)\frac{1}{\mu_1}du+$$

$$\int_{\bar{A}1|k=0}\overline{F}_2(u)\tilde{f}(u)\,e^{-su}du+\int_{\bar{A}1}\int_0^{T_2}\overline{F}_2(u)\frac{1}{\mu_1}f_1(u-v)dv\,e^{-su}du \tag{6.2.23}$$

Also from (6.2.21) we have:

$$\nu_G=\int_{A|k=0} w\,\overline{\overline{F}}_1(w)f_2(w)dw+\int_A\int w\,\overline{F}_1(w-u)f_2(w)\frac{1}{\mu_1}dw+$$

$$\int_{\bar{A}|k=0} w\,\overline{\overline{F}}_1(w)f_2(w)dw+\sum_{k=1}^{\infty}\int_{\bar{A}} w\,f_2(w)\int\frac{1}{\mu_1}\overline{F}_1(w-u)\,dw+$$

$$T_2\ \bar{F}_2(T_2)\bar{\bar{F}}_1(T_2)\delta(T_2-\tau_1+a_1)+T_2\ \bar{F}_2(T_2)\sum_{k=1}^{\infty}\int_{\bar{B}}\frac{1}{\mu_1}\bar{F}_1(T_2-u)\ +$$

$$T_2\ \bar{F}_2(T_2)\ \bar{\bar{F}}_1(T_2)\delta(\tau_1-a_1-T_2)+\bar{F}_2[T_2]\sum_{k=1}^{\infty}\int_{B}\bar{F}_1(T_2-u)\frac{1}{\mu_1}du+$$

$$\int_{\bar{A}_1|k=0}u\ \bar{F}_2(u)\tilde{f}_1(u)du+\sum_{k=1}^{\infty}\int_{\bar{A}1}\int u\ \bar{F}_2(u)\frac{1}{\mu_1}f_1(u-v)dv\ e^{-su}du \qquad (6.2.24)$$

Now using the distribution functions on the combined lengths of the operating periods and repair times in the general minor intervals with system renewal in chapters 4 and 5, we find the Laplace transform for the distribution on the length both the initial minor and the general minor intervals. For the initial interval we have:

$$u_I(t)=\sum_{k=0}^{\infty}\int_0^t v_{I1}(x,k)g_{opp}(t-x,k)dx+\sum_{k=0}^{\infty}\int_0^t v_{I2}(x,k)g_{opp}(t-x,k+1)dx \qquad (6.2.25)$$

and the Laplace transform is:

$$u_I^*(s)=\sum_{k=0}^{\infty}v_{I1}^*(s,k)g_1(s)^k g_2(s)+\sum_{k=0}^{\infty}v_{I_2}^*(s,k)g_1(s)^{k+1}g_2(s) \qquad (6.2.26)$$

From expression (6.2.12) the above expression is:

$$u_I^*(s)=\sum_{k=0}^{\infty}g_1(s)^k g_2(s)\int_{\bar{A}1}\int\bar{F}_2(u)f_1^{(k-1)}(v)f_1(u-v)dv\ e^{-su}du+$$
$$\sum_{k=0}^{\infty}g_1(s)^{k+1}g_2(s)\int_{\bar{A}2}f_2(w)\int f_1^{(k)}(u)\bar{F}_1(w-u)\ e^{-sw}dw+$$
$$\bar{F}_2(T_2)\ \sum_{k=0}^{\infty}g_1(s)^{k+1}g_2(s)\int_{\bar{B}2}f_1^{(k)}(u)\bar{F}_1(T_2-u)\ e^{-sT_2} \qquad (6.2.27)$$

and for the general interval we have:

$$u_G(t)=\sum_{k=0}^{\infty}\int_0^t v_{G1}(x,k)g_{opp}(t-x,k)dx+\sum_{k=0}^{\infty}\int_0^t v_{G2}(x,k)g_{opp}(t-x,k+1)dx \qquad (6.2.28)$$

and the Laplace transform is:

$$u_G^*(s)=\sum_{k=0}^{\infty}v_{G1}^*(s,k)g_1(s)^k g_2(s)+\sum_{k=0}^{\infty}v_{G_2}^*(s,k)g_1(s)^{k+1}g_2(s)$$

$$u_G^*(s)=\sum_{k=0}^{\infty}g_1(s)^k g_2(s)\int_{\bar{A}1}\int \bar{F}_2(u)\tilde{f}_1^{(k-1)}(v)f_1(u-v)dv\; e^{-su}du+$$
$$\sum_{k=0}^{\infty}g_1(s)^{k+1}g_2(s)\int_{\bar{A}2}f_2(w)\int \tilde{f}_1^{(k)}(u)\bar{F}_1(w-u)\; e^{-sw}dw+$$
$$\bar{F}_2(T_2)\sum_{k=0}^{\infty}g_1(s)^{k+1}g_2(s)\int_{\bar{B}2}\tilde{f}_1^{(k)}(u)\bar{F}_1(T_2-u)\; e^{-sT_2} \qquad (6.2.29)$$

6.3 THE WEIBULL DISTRIBUTION FOR FAILURE TIME.

The failure distributions for both components are considered to be Weibull distributions. The k fold convolution of a Weibull distribution has no closed algebraic form. We use numerical approximations to estimate $f^{(k)}(t)$. The idea is to express $f^{(k)}(t)$ as an infinite series. For application purposes only a finite number of terms are used. The Weibull distribution is represented as:

$$F(t)=1-e^{-\frac{\alpha}{\beta}t^{\beta}} \qquad (6.3.1)$$

The k fold convolution of the above density distribution is:

$$f^{(k)}(t)=\beta\, \mathrm{Exp}[-\frac{\alpha}{\beta}t^{\beta}]\sum_{v=k}^{\infty}\frac{t^{\beta v-1}}{(v-1)!}\gamma_k(v) \qquad (6.3.2)$$

where the algorithm for finding the $\gamma_k(s)$ can be found the paper by Lominicki [21].

From (6.2.8) we can treat the expression below as one unit and simplify it for computation purposes.

$$\sum_{k=1}^{\infty}(g^*{}_1(s))^k g_2{}^*(s) f_1^{(k)}(u) \tag{6.3.3}$$

Substituting $f_1^k(u)$ in (6.3.2) into (6.3.3) yields:

$$\left(\sum_{k=1}^{\infty}(g^*{}_1(s))^k g_2{}^*(s)\beta \operatorname{Exp}[-\frac{\alpha}{\beta}u^{\beta}] \sum_{v=k}^{\infty}\frac{u^{\beta v-1}}{(v-1)!}\gamma_k(v)\right). \tag{6.3.4}$$

Regrouping the terms in equation (6.3.4) :

$$g_2{}^*(s)\beta \operatorname{Exp}[-\frac{\alpha}{\beta}u^{\beta}] \sum_{k=1}^{\infty}(g^*{}_1(s))^k \sum_{v=k}^{\infty}\frac{u^{\beta v-1}}{(v-1)!}\gamma_k(v) \tag{6.3.5}$$

and this simplifies to:

$$g_2{}^*(s)\beta \operatorname{Exp}[-\frac{\alpha}{\beta}u^{\beta}] \sum_{v=1}^{\infty}\frac{u^{\beta v-1}}{(v-1)!}\left(\sum_{k=1}^{v}\gamma_k(v)(g^*{}_1(s))^k\right). \tag{6.3.6}$$

Substituting (6.3.6) for (6.3.3) into (6.2.8) we express the Laplace transform for the density on the length of the initial interval with no system renewal as:

$$q^*{}_1(s)=g_2{}^*(s)\int_{A|k=0}\bar{F}_1(w)f_2(w)e^{-ws}dw+$$

$$\int_A\int \overline{F}_1(w-u)f_2(w)\left(g_2{*}(s)\beta\,\mathrm{Exp}[-\frac{\alpha}{\beta}u^{\beta}]\sum_{v=1}^{\infty}\frac{u^{\beta v-1}}{(v-1)!}\left(\sum_{k=1}^{v}\gamma_k(v)(g{*}_1(s))^k\right)\right)e^{-ws}dw+$$

$$e^{-T_2 s}\overline{F}_2[T_2]\int_B \overline{F}_1(T_2-u)\left(g_2{*}(s)\beta\,\mathrm{Exp}[-\frac{\alpha}{\beta}u^{\beta}]\sum_{v=1}^{\infty}\frac{u^{\beta v-1}}{(v-1)!}\left(\sum_{k=1}^{v}\gamma_k(v)(g{*}_1(s))^k\right)\right)du.$$

$$+e^{-T_2 s}\,g_2^{*}(s)\,\overline{F}_2[T_2]\overline{F}_1[T_2]\delta(\tau_1-T_2) \qquad (6.3.7)$$

The expected value $\mu_I$of $q_I(t)$ is $\mu_I = -\left.\frac{dq_I(s)}{ds}\right|_{s=0}$. This reduces to the expression

$$\mu_I = \int_{A|k=0} \overline{F}_1(w)f_2(w)(\eta_2+w)\,dw +$$

$$\int_A\int \overline{F}_1(w-u)f_2(w)w\,dw\left(\beta\,\mathrm{Exp}[-\frac{\alpha}{\beta}u^{\beta}]\sum_{v=1}^{\infty}\frac{u^{\beta v-1}}{(v-1)!}\left(\sum_{k=1}^{v}\gamma_k(v){*}(k\eta_1+\eta_2+w)\right)\right).$$

$$\overline{F}_2[T_2]\int_{B|w\geq T_2} \overline{F}_1(T_2-u)\left(\beta\,\mathrm{Exp}[-\frac{\alpha}{\beta}u^{\beta}]\sum_{v=1}^{\infty}\frac{u^{\beta v-1}}{(v-1)!}\left(\sum_{k=1}^{v}\gamma_k(v)(k\eta_1+\eta_2+T_2)\right)\right)du+$$

$$(T_2+\eta_2)\,\overline{F}_2[T_2]\overline{F}_1[T_2]\delta(\tau_1-T_2) \qquad (6.3.8)$$

From (6.2.9), with regard to the general minor interval,

$$\tilde{f}_1^{(k)}(t) = \begin{cases} \tilde{f}_1(t) & k=1 \\ \int_0^t \tilde{f}_1(x)f_1^{(k-1)}(t-x)dx & k>1 \end{cases} \qquad (6.3.9)$$

and:

$$\sum_{k=1}^{\infty}(g^*{}_1(s))^k g_2{}^*(s)\tilde{f}_1^{(k)}(u)$$

which can be expressed as:

$$g^*{}_1(s)g_2{}^*(s)\tilde{f}_1(u)+\sum_{k=2}^{\infty}(g^*{}_1(s))^k g_2{}^*(s)\int_0^t \tilde{f}_1(x)f_1^{(k-1)}(t-x)dx. \qquad (6.3.10)$$

This can be rewritten as:

$$g^*{}_1(s)g_2{}^*(s)\tilde{f}_1(u)+\sum_{k=1}^{\infty}(g^*{}_1(s))^{k+1} g_2{}^*(s)\int_0^t \tilde{f}_1(t-x)f_1^{(k)}(x)dx \qquad (6.3.11)$$

Simplifying (6.3.11) yields:

$$g^*{}_1(s)g_2{}^*(s)\tilde{f}_1(u)+\int_0^t \tilde{f}_1(t-x)\sum_{k=1}^{\infty}\left((g^*{}_1(s))^{k+1} g_2{}^*(s)f_1^{(k)}(x)\right)dx \qquad (6.3.12)$$

The expression:

$$\sum_{k=1}^{\infty}\left((g^*{}_1(s))^{k+1} g_2{}^*(s)f_1^{(k)}(x)\right)$$

is similar to (6.3.3). Hence can be expressed as:

$$g_2{}^*(s)\beta\,\mathrm{Exp}[-\frac{\alpha}{\beta}x^{\beta}\,]\;\sum_{v=1}^{\infty}\frac{x^{\beta v-1}}{(v-1)!}\left(\sum_{k=1}^{v}\gamma_k(v)(g^*{}_1(s))^{k+1}\right) \qquad (6.3.13)$$

The expression (6.3.12) can be simplified to yield:

$$g^*{}_1(s)g_2{}^*(s)\tilde{f}_1(u)+$$

$$\int_0^t \tilde{f}_1(t-x)g_2{}^*(s)\beta\,\mathrm{Exp}[-\frac{\alpha}{\beta}x^{\beta}\,]\;\sum_{v=1}^{\infty}\frac{x^{\beta v-1}}{(v-1)!}\left(\sum_{k=1}^{v}\gamma_k(v)(g^*{}_1(s))^{k+1}\right)dx \qquad (6.3.14)$$

and (6.2.9) becomes:

$$q_G^*(s)=g_2{}^*(s)\int_{A|k=0}\overline{\overline{F}}_1(w)f_2(w)e^{-ws}dw+$$

$$\int_A\int\overline{F}_1(w\text{-}u)f_2(w)\left(g^*{}_1(s)g_2{}^*(s)\widetilde{f}_1(u)\right)e^{-ws}dw+$$

$$\int_A\int\overline{F}_1(w\text{-}u)f_2(w)\int_0^u\widetilde{f}_1(u\text{-}x)g_2{}^*(s)\beta\text{Exp}[-\frac{\alpha}{\beta}x^\beta]\sum_{v=1}^{\infty}\frac{x^{\beta v-1}}{(v-1)!}\left(\sum_{k=1}^{v}\gamma_k(v)(g^*{}_1(s))^{k+1}\right)dxe^{-ws}dw$$

$$+e^{-T_2s}\overline{F}_2[T_2]\int_B\overline{F}_1(T_2\text{-}u)\left(g^*{}_1(s)g_2{}^*(s)\widetilde{f}_1(u)\right)du$$

$$e^{-T_2s}\overline{F}_2[T_2]\int\overline{F}_1(T_2\text{-}u)\int_0^u\widetilde{f}_1(u\text{-}x)g_2{}^*(s)\beta\text{Exp}[-\frac{\alpha}{\beta}x^\beta]\sum_{v=1}^{\infty}\frac{x^{\beta v-1}}{(v-1)!}\left(\sum_{k=1}^{v}\gamma_k(v)(g^*{}_1(s))^{k+1}\right)dxdu$$

$$+e^{-T_2s}g_2^*(s)\ \overline{F}_2[T_2]\overline{\overline{F}}_1[T_2]\delta(\tau_1\text{-}a_1\text{-}T_2) \qquad (6.3.15)$$

The mean repair time for component i is:

$$\eta_i=-\left.\frac{dg_i(s)}{ds}\right|_{s=0} i=1,2.$$

The expected value μ_G of $q_G(t)$ will be $\mu_G=-\left.\frac{dq_G(s)}{ds}\right|_{s=0}$. This reduces to the expression

$$\mu_G=\int_{A|k=0}\overline{\overline{F}}_1(w)f_2(w)(w+\eta_2)dw+$$

$$\int_A\int \overline{F}_1(w-u)f_2(w)\tilde{f}_1(u)(w+\eta_1+\eta_2)dw +$$

$$\int_A\int \overline{F}_1(w-u)f_2(w)\int_0^u \tilde{f}_1(u-x)\beta\, Exp[-\frac{\alpha}{\beta}x^\beta]\sum_{v=1}^{\infty}\frac{x^{\beta v-1}}{(v-1)!}\left(\sum_{k=1}^{v}\gamma_k(v)((k+1)\eta_1+\eta_2+w)\right)dx+$$

$$(T_2+\eta_1+\eta_2)\overline{F}_2[T_2]\int_B \overline{F}_1(T_2-u)\tilde{f}_1(u)du +$$

$$\overline{F}_2[T_2]\int \overline{F}_1(T_2-u)\int_0^u \tilde{f}_1(u-x)\beta Exp[-\frac{\alpha}{\beta}x^\beta]\sum_{v=1}^{\infty}\frac{x^{\beta v-1}}{(v-1)!}\left(\sum_{k=1}^{v}\gamma_k(v)(T_2+(k+1)\eta_1+\eta_2)\right)dxdu+$$

$$(T_2+\eta_2)\,\overline{F}_2[T_2]\overline{\overline{F}}_1[T_2]\delta(\tau_1-a_1-T_2) \qquad (6.3.16)$$

Now for the combined length of the operating period in the minor interval with or without renewal, from (6.3.2) we can write

$$\sum_{k=1}^{\infty} f_1^{(k)}(u) \qquad \text{as:}$$

$$\sum_{k=1}^{\infty}\beta\, Exp[-\frac{\alpha}{\beta}u^\beta]\sum_{v=k}^{\infty}\frac{u^{\beta v-1}}{(v-1)!}\gamma_k(v) \qquad (6.3.17)$$

Rearranging (6.3.17) yields:

$$\beta\, Exp[-\frac{\alpha}{\beta}u^\beta]\sum_{k=1}^{\infty}\ \sum_{v=k}^{\infty}\frac{u^{\beta v-1}}{(v-1)!}\gamma_k(v) \qquad (6.3.18)$$

The expression (6.3.18) simplifies to:

$$\beta \,\mathrm{Exp}[-\frac{\alpha}{\beta}u^{\beta}]\sum_{v=1}^{\infty}\frac{u^{\beta v-1}}{(v-1)!}\left(\sum_{k=1}^{v}\gamma_k(v)\right) \tag{6.3.19}$$

Substituting (6.3.19) in place of $\sum_{k=1}^{\infty} f_1^{(k)}(u)$ into (6.2.13). The expected initial failure interval is:

$$\nu_1=\int_{A|k=0}\bar{F}_1(w)f_2(w)wdw$$

$$+\int_A\int\bar{F}(w-u)\left(\beta\,\mathrm{Exp}[-\frac{\alpha}{\beta}u^{\beta}]\sum_{s=1}^{\infty}\frac{u^{\beta s-1}}{(s-1)!}\left(\sum_{k=1}^{s}\gamma_k(s)\right)\right)f_2(w)\,wdudw+$$

$$T_2\bar{F}_2[T_2]\bar{F}_1[T_2]\delta(\tau_1-T_2)+$$

$$T_2\bar{F}_2[T_2]\int_B\bar{F}_1(T_2-u)\left(\beta\,\mathrm{Exp}[-\frac{\alpha}{\beta}u^{\beta}]\sum_{s=1}^{\infty}\frac{u^{\beta s-1}}{(s-1)!}\left(\sum_{k=1}^{s}\gamma_k(s)\right)\right)+$$

$$\int_{\bar{A}2|k=0}\bar{F}_1(w)f_2(w)wdw$$

$$+\int_{\bar{A}2}\int\bar{F}(w-u)\left(\beta\,\mathrm{Exp}[-\frac{\alpha}{\beta}u^{\beta}]\sum_{s=1}^{\infty}\frac{u^{\beta s-1}}{(s-1)!}\left(\sum_{k=1}^{s}\gamma_k(s)\right)\right)f_2(w)\,wdudw+$$

$$T_2\bar{F}_2[T_2]\bar{F}_1[T_2]\delta(T_2-\tau_1)+$$

$$T_2\bar{F}_2[T_2]\int_{\bar{B}2}\bar{F}_1(T_2-u)\left(\beta\,\mathrm{Exp}[-\frac{\alpha}{\beta}u^{\beta}]\sum_{s=1}^{\infty}\frac{u^{\beta s-1}}{(s-1)!}\left(\sum_{k=1}^{s}\gamma_k(s)\right)\right)+$$

$$\int_{\bar{A}1}\int\bar{F}_2(u)\left(\beta\,\mathrm{Exp}[-\frac{\alpha}{\beta}v^{\beta}]\sum_{s=1}^{\infty}\frac{v^{\beta s-1}}{(s-1)!}\left(\sum_{k=1}^{s}\gamma_k(s)\right)\right)f_1(w)\,udvdu+$$

$$\int_{\bar{A}1|k=0} u\, \bar{F}_2(u) f_1(u) du.$$

Once again substituting (6.3.19) in place of $\sum_{k=1}^{\infty} f_1^{(k)}(u)$ into (6.2.10). Substituting (6.3.19) into (6.2.11) the Laplace transform for the density on the combined length of the operating period in initial minor interval is:

$$z_I^*(s) = \int_{A|k=0} \bar{F}_1(w) f_2(w) e^{-ws} dw +$$

$$\int_A \int \bar{F}_1(w-u) f_2(w) \left(\beta\, \mathrm{Exp}[-\frac{\alpha}{\beta} u^\beta] \sum_{s=1}^{\infty} \frac{u^{\beta s-1}}{(s-1)!} \left(\sum_{k=1}^{s} \gamma_k(s) \right) \right) e^{-ws} dw +$$

$$e^{-T_2 s} \bar{F}_2[T_2] \bar{F}_1[T_2] \delta(\tau_1 - T_2) +$$

$$e^{-T_2 s} \bar{F}_2[T_2] \int_B \bar{F}_1(T_2 - u) \left(\beta\, \mathrm{Exp}[-\frac{\alpha}{\beta} u^\beta] \sum_{s=1}^{\infty} \frac{u^{\beta s-1}}{(s-1)!} \left(\sum_{k=1}^{s} \gamma_k(s) \right) \right) du +$$

$$\int_{\bar{A}2|k=0} \bar{F}_1(w) f_2(w) e^{-ws} dw +$$

$$\int_{\bar{A}2} \int \bar{F}_1(w-u) f_2(w) \left(\beta\, \mathrm{Exp}[-\frac{\alpha}{\beta} u^\beta] \sum_{s=1}^{\infty} \frac{u^{\beta s-1}}{(s-1)!} \left(\sum_{k=1}^{s} \gamma_k(s) \right) \right) e^{-ws} dw +$$

$$e^{-T_2 s} \bar{F}_2[T_2] \bar{F}_1[T_2] \delta(T_2 - \tau_1) + \int_{\bar{A}1|k=0} e^{-us}\, \bar{F}_2(u) f_1(u) du.$$

$$e^{-T_2 s}\bar{F}_2[T_2]\int_{\bar{B}2}\bar{F}_1(T_2-u)\left(\beta\,\mathrm{Exp}[-\frac{\alpha}{\beta}u^{\beta}]\sum_{s=1}^{\infty}\frac{u^{\beta s-1}}{(s-1)!}\left(\sum_{k=1}^{s}\gamma_k(s)\right)\right)du+$$

$$\int_{\bar{A}1}\int\bar{F}_2(u)f_1(u-v)\left(\beta\,\mathrm{Exp}[-\frac{\alpha}{\beta}v^{\beta}]\sum_{s=1}^{\infty}\frac{v^{\beta s-1}}{(s-1)!}\left(\sum_{k=1}^{s}\gamma_k(s)\right)\right)e^{-ws}dvdu \quad (6.3.22)$$

The Laplace transform for the density function on the combined length of the operating period in a general minor interval is already simplified in (6.2.23). Now we attempt to use the Laplace transform to find $Q_I(\infty)$ and $Q_G(\infty)$. If $q_I^*(s)$ and $q_G^*(s)$ are the Laplace transforms of the distributions of the initial minor and the general minor intervals without system renewal respectively then

$$Q_I(\infty)=q_I^*(s)\Big|_{s=0} \text{ and } Q_G(\infty)=q_G^*(s)\Big|_{s=0}.$$

6.4 THE GAMMA DISTRIBUTION FOR FAILURE TIME.

Suppose the failure time distributions of the two components are Gamma distribution functions. Unlike the Weibull distribution the k fold convolution of the Gamma distribution has a closed form. The Gamma density function is represented as:

$$f(t)=\frac{\lambda^{\alpha}t^{\alpha-1}}{\Gamma(\alpha)}e^{-\lambda t} \quad (6.4.1)$$

and the k fold convolution $f^{(k)}(t)$ on the density function $f(t)$ is:

$$f^{(k)}(t)=\frac{\lambda^{\alpha k}\,t^{\alpha k-1}}{\Gamma(\alpha k)}e^{-\lambda t} \quad (6.4.2)$$

From (6.2.8) we can treat the expression below as one unit and simplify it for computation purposes.

$$\sum_{k=1}^{\infty}(g^*_1(s))^k g_2^*(s) f_1^{(k)}(u) \tag{6.4.3}$$

Substituting $f_1^k(u)$ in (6.4.2) into (6.4.3) yields:

$$\sum_{k=1}^{\infty}(g^*_1(s))^k g_2^*(s) \frac{\lambda^{\alpha k}\, u^{\alpha k-1}}{\Gamma(\alpha k)} e^{-\lambda u} \tag{6.4.4}$$

Simplifying the above expression yields:

$$g_2^*(s) e^{-\lambda u} \sum_{k=1}^{\infty}(g^*_1(s))^k \frac{\lambda^{\alpha k}\, u^{\alpha k-1}}{\Gamma(\alpha k)} \tag{6.4.5}$$

Substituting (6.4.5) for (6.4.3) into (6.2.8). Now we express the Laplace transform for the density on the length of the initial interval as:

$$q^*_{I(s)} = g_2^*(s) \int_{A|k=0} \overline{F}_1(w) f_2(w) e^{-ws} dw +$$

$$\int_A\int \overline{F}_1(w-u) f_2(w) \left(g_2^*(s) e^{-\lambda u} \sum_{k=1}^{\infty}(g^*_1(s))^k \frac{\lambda^{\alpha k}\, u^{\alpha k-1}}{\Gamma(\alpha k)} \right) e^{-ws} dw +$$

$$e^{-T_2 s} \overline{F}_2[T_2] \int_B \overline{F}_1(T_2-u) \left(g_2^*(s) e^{-\lambda u} \sum_{k=1}^{\infty}(g^*_1(s))^k \frac{\lambda^{\alpha k}\, u^{\alpha k-1}}{\Gamma(\alpha k)} \right) du.$$

$$+ e^{-T_2 s}\, g_2^*(s)\, \overline{F}_2[T_2] \overline{F}_1[T_2] \delta(\tau_1 - T_2) \tag{6.4.6}$$

The expected value μ_I of $q_I(t)$ is $\mu_I = \left.\frac{dq_I(s)}{ds}\right|_{s=0}$. This reduces to the expression

$$\mu_I = \int_{A|k=0} \bar{F}_1(w) f_2(w)(\eta_2 + w)\, dw +$$

$$\int_A \int \bar{F}_1(w-u) f_2(w) dw \left(e^{-\lambda u} \sum_{k=1}^{\infty} (k\eta_1 + \eta_2 + w) \frac{\lambda^{\alpha k} u^{\alpha k - 1}}{\Gamma(\alpha k)} \right).$$

$$\bar{F}_2[T_2] \int_B \bar{F}_1(T_2 - u) \left(e^{-\lambda u} \sum_{k=1}^{\infty} (k\eta_1 + \eta_2 + T_2) \frac{\lambda^{\alpha k} u^{\alpha k - 1}}{\Gamma(\alpha k)} \right) du$$

$$(T_2 + \eta_2)\, \bar{F}_2[T_2] \bar{F}_1[T_2] \delta(\tau_1 - T_2) \tag{6.4.7}$$

Using the simplification in expressions (6.3.9) through (6.3.12) and the k fold distribution for the Gamma function in (6.4.2) the expression

$$\sum_{k=1}^{\infty} (g^*_1(s))^k g_2^*(s) \tilde{f}_1^{(k)}(u) \tag{6.4.8}$$

simplifies to the expression:

$$g^*_1(s) g_2^*(s) \tilde{f}_1(u) + \int_0^t \tilde{f}_1(t-x) \left(g_2^*(s) e^{-\lambda u} \sum_{k=1}^{\infty} (g^*_1(s))^{k+1} \frac{\lambda^{\alpha k} u^{\alpha k - 1}}{\Gamma(\alpha k)} \right) dx \tag{6.4.9}$$

The expression (6.2.9) becomes:

$$q^*_G(s) = g_2^*(s) \int_{A|k=0} \bar{\bar{F}}_1(w) f_2(w) e^{-ws} dw +$$

$$\int_A \int \bar{F}_1(w-u) f_2(w) \left(g^*_1(s) g_2^*(s) \tilde{f}_1(u) \right) e^{-ws} dw +$$

$$\int_A\int \overline{F}_1(w\text{-}u)f_2(w)\int_0^u \tilde{f}_1(u\text{-}x)\, g_2{*}(s)e^{-\lambda x}\sum_{k=1}^{\infty}(g{*}_1(s))^{k+1}\frac{\lambda^{\alpha k}\, x^{\alpha k-1}}{\Gamma(\alpha k)}dx e^{-ws}dw$$

$$+e^{-T_2 s}\overline{F}_2[T_2]\int_B \overline{F}_1(T_2\text{-}u)\big(g{*}_1(s)g_2{*}(s)\tilde{f}_1(u)\big)du+$$

$$e^{-T_2 s}\overline{F}_2[T_2]\int \overline{F}_1(T_2\text{-}u)\int_0^u \tilde{f}_1(u\text{-}x)g_2{*}(s)e^{-\lambda x}\sum_{k=1}^{\infty}(g{*}_1(s))^{k+1}\frac{\lambda^{\alpha k}\, x^{\alpha k-1}}{\Gamma(\alpha k)}dxdu+$$

$$e^{-T_2 s}g_2^{*}(s)\,\overline{F}_2[T_2]\overline{\overline{F}}_1[T_2]\delta(\tau_1\text{-}a_1\text{-}T_2) \qquad (6.4.10)$$

The expected value μ_G of $q_G(t)$ is

$$\mu_G = -\left.\frac{dq_G(s)}{ds}\right|_{s=0}.$$

This reduces to the expression

$$\mu_G = \int_{A|k=0} \overline{\overline{F}}_1(w)f_2(w)(w+\eta_2)dw + \int_A\int \overline{F}_1(w\text{-}u)f_2(w)\tilde{f}_1(u)(w+\eta_1+\eta_2)dw +$$

$$\int_A\int \overline{F}_1(w\text{-}u)f_2(w)\int_0^u \tilde{f}_1(u\text{-}x)e^{-\lambda x}\sum_{k=1}^{\infty}(w+(k+1)\eta_1+\eta_2)\frac{\lambda^{\alpha k}\, x^{\alpha k-1}}{\Gamma(\alpha k)}dx +$$

$$(T_2+\eta_1+\eta_2)\overline{F}_2[T_2]\int_B \overline{F}_1(T_2\text{-}u)\tilde{f}_1(u)du + (T_2+\eta_2)\,\overline{F}_2[T_2]\overline{\overline{F}}_1[T_2]\delta(\tau_1\text{-}a_1\text{-}T_2)+$$

$$\overline{F}_2[T_2]\int \overline{F}_1(T_2\text{-}u)\int_0^u \tilde{f}_1(u\text{-}x)e^{-\lambda x}\sum_{k=1}^{\infty}(T_2+(k+1)\eta_1+\eta_2)\frac{\lambda^{\alpha k}\, x^{\alpha k-1}}{\Gamma(\alpha k)}dxdu \qquad (6.4.11)$$

Now for the combined length of the operating period with or without renewal in a minor interval, from (6.4.2) we can write $\sum_{k=1}^{\infty} f_1^{(k)}(u)$ as

$$e^{-\lambda u}\sum_{k=1}^{\infty}\frac{\lambda^{\alpha k}\, u^{\alpha k-1}}{\Gamma(\alpha k)} \tag{6.4.12}$$

Substituting (6.4.12) in place of $\sum_{k=1}^{\infty} f_1^{(k)}(u)$ into (6.2.13), the expected initial failure interval is:

$$\nu_I = \int_{A|k=0} \bar{F}_1(w) f_2(w) w\,du\,dw + \int_A\int \bar{F}(w-u)\left(e^{-\lambda u}\sum_{k=1}^{\infty}\frac{\lambda^{\alpha k}\, u^{\alpha k-1}}{\Gamma(\alpha k)}\right) f_2(w)\, w\,du\,dw \, +$$

$$T_2 e^{-T_2 s}\bar{F}_2[T_2]\bar{F}_1[T_2]\delta(\tau_1 - T_2) \, +$$

$$T_2 e^{-T_2 s} F_2[T_2] \int_B \bar{F}_1(T_2-u)\left(e^{-\lambda u}\sum_{k=1}^{\infty}\frac{\lambda^{\alpha k}\, u^{\alpha k-1}}{\Gamma(\alpha k)}\right) +$$

$$\int_{\bar{A}2|k=0} \bar{F}_1(w) f_2(w) w\,du\,dw + \int_{\bar{A}2}\int \bar{F}(w-u)\left(e^{-\lambda u}\sum_{k=1}^{\infty}\frac{\lambda^{\alpha k}\, u^{\alpha k-1}}{\Gamma(\alpha k)}\right) f_2(w)\, w\,du\,dw \, +$$

$$T_2 e^{-T_2 s}\bar{F}_2[T_2]\bar{F}_1[T_2]\delta(T_2 - \tau_1) \, +$$

$$T_2 e^{-T_2 s} F_2[T_2] \int_{\bar{B}2} \bar{F}_1(T_2-u)\left(e^{-\lambda u}\sum_{k=1}^{\infty}\frac{\lambda^{\alpha k}\, u^{\alpha k-1}}{\Gamma(\alpha k)}\right) +$$

$$\int_{\bar{A}1|k=0} \bar{F}_2(u) f_2(u) u\,du + \int_{\bar{A}1}\int \bar{F}_2(u)\left(e^{-\lambda v}\sum_{k=1}^{\infty}\frac{\lambda^{\alpha k}\, v^{\alpha k-1}}{\Gamma(\alpha k)}\right) f_1(u-v)\, u\,dv\,du \tag{6.4.13}$$

Substituting (6.4.12) into (6.2.11) the Laplace transform for the density on the combined length of the operating period in initial minor interval is:

$$z_I^*(s)=\int_{A|k=0}\overline{F}_1(w)f_2(w)e^{-ws}dw+$$

$$\sum_{k=1}^{\infty}\int_{A|w<T_2}\int\overline{F}_1(w-u)f_2(w)\left(e^{-\lambda u}\sum_{k=1}^{\infty}\frac{\lambda^{\alpha k}u^{\alpha k-1}}{\Gamma(\alpha k)}\right)e^{-ws}dw+$$

$$e^{-T_2s}\overline{F}_2[T_2]\overline{F}_1[T_2]\delta(\tau_1-T_2)+$$

$$e^{-T_2s}\overline{F}_2[T_2]\sum_{k=1}^{\infty}\int_B\overline{F}_1(T_2-u)\left(e^{-\lambda u}\sum_{k=1}^{\infty}\frac{\lambda^{\alpha k}u^{\alpha k-1}}{\Gamma(\alpha k)}\right)du+$$

$$\int_{\bar{A}2|k=0}\overline{F}_1(w)f_2(w)e^{-ws}dw+$$

$$\sum_{k=1}^{\infty}\int_{\bar{A}2|w<T_2}\int\overline{F}_1(w-u)f_2(w)\left(e^{-\lambda u}\sum_{k=1}^{\infty}\frac{\lambda^{\alpha k}u^{\alpha k-1}}{\Gamma(\alpha k)}\right)e^{-ws}dw+$$

$$e^{-T_2s}\overline{F}_2[T_2]\overline{F}_1[T_2]\delta(T_2-\tau_1)+$$

$$e^{-T_2s}\overline{F}_2[T_2]\sum_{k=1}^{\infty}\int_{\bar{B}2}\overline{F}_1(T_2-u)\left(e^{-\lambda u}\sum_{k=1}^{\infty}\frac{\lambda^{\alpha k}u^{\alpha k-1}}{\Gamma(\alpha k)}\right)du+$$

$$\int_{\bar{A}1|k=0}\overline{F}_2(u)f_1(u)e^{-us}du+\int_{\bar{A}1}\int\overline{F}_2(u)\left(e^{-\lambda v}\sum_{k=1}^{\infty}\frac{\lambda^{\alpha k}v^{\alpha k-1}}{\Gamma(\alpha k)}\right)f_1(u-v)\,e^{-ws}dvdu \quad (6.4.15)$$

6.5 Availability

From (3.5.20) the Laplace transform of the system availability function is:

$$\mathfrak{A}^*(s)=\frac{\overline{Z}_I^*(s)\left(1-q_G^*(s)\right)+\overline{Z}_G^*(s)q_I^*(s)}{\left(1-\phi^*(s)\right)\left(1-q_G^*(s)\right)}. \quad (6.5.1)$$

Now

$$\bar{Z}_I(t)=1-Z_I(t) \tag{6.5.2}$$

The Laplace transform of the above expression is:

$$\bar{Z}_I^*(s)=\frac{1-z_I^*(s)}{s} \tag{6.5.3}$$

Similarly the Laplace transform for $Z_G(t)$ is:

$$\bar{Z}_G^*(s)=\frac{1-z_G^*(s)}{s} \tag{6.5.4}$$

From (3.4.3) we have:

$$\mathfrak{F}^*(s)=U_I^*(s) + U_G^*(s)m_Q^*(s). \tag{6.5.5}$$

$$\mathfrak{F}^*(s)=U_I^*(s) + U_G^*(s)\frac{q_I^*(s)}{1-q_G^*(s)}.$$

The Laplace transform of the corresponding density distribution is:

$$f^*(s)=s\,\mathfrak{F}^*(s)$$

$$=u_I^*(s) + u_G^*(s)\frac{q_I^*(s)}{1-q_G^*(s)}. \tag{6.5.6}$$

From (3.5.20) the laplace tranform of the system availability is:

$$\mathfrak{A}^*(s)=\frac{\bar{Z}_I^*(s)\left(1-q_G^*(s)\right)+\bar{Z}_G^*(s)q_I^*(s)}{(1-\phi^*(s))(1-q_G^*(s))}$$

and

$$\phi^*(s)=f^*(s)g^*_{opp}(s)$$

Since the construction of $\mathfrak{F}(t)$ includes the opportunistic replacement times. Therefore

$$f^*(s)=\phi^*(s) \tag{6.5.7}$$

therefore

$$\mathfrak{A}^*(s)=\frac{\overline{Z}^*_I(s)\left(1-q^*_G(s)\right)+\overline{Z}^*_G(s)q^*_I(s)}{(1-f^*(s))(1-q^*_G(s))}. \tag{6.5.8}$$

Substituting (6.5.6) into (6.5.8) yields:

$$\mathfrak{A}^*(s)=\frac{\overline{Z}^*_I(s)\left(1-q^*_G(s)\right)+\overline{Z}^*_G(s)q^*_I(s)}{(1-q^*_G(s)-u_I^*(s)-u^*_G(s)q^*_I(s))} \tag{6.5.9}$$

From (3.5.27) the long run system availability measure is:

$$\text{Availability}=\frac{(\nu_I(1-Q_G(\infty))+\nu_G Q_I(\infty))}{(\mu_I+\mu_{IR})(1-Q_G(\infty))+(\mu_G+\mu_{GR})Q_I(\infty)} \tag{6.5.10}$$

6.6 Observations

Some numerical results are presented here for the Opportunistic Replacement policy model. Use is made of model 4 to demonstrate its relation to the other 3 models. The limiting availability is computed. To compute the time dependent availability, the Laplace transform of the availability in expession (6.5.9) has to be inverted. There is no closed inversion for the expression. Therefore we use a numerical method developed by Stehfest[35]

Examples are given for the case in which both component have a Weibull failure distribution and also for the case in which both components have a Gamma failure distribution.

For the case in which failure density functions for both components are Weibull. The density function for failure time of component i is:

$$f_i(t)=\alpha_i t^{\beta_i-1}\ \mathrm{Exp}[-\frac{\alpha_i}{\beta_i}t^{\beta_i}] \qquad i=1,2. \tag{6.6.1}$$

For the case in which failure density functions for both components are Gamma. The density function for failure time of component i is:

$$f_i(t)=\frac{\lambda_i^{\alpha_i}t^{\alpha_i-1}}{\Gamma(\alpha_i)}e^{-\lambda_i t} \tag{6.6.2}$$

The repair time age replacement and opportunistic replacement density functions are exponential and assumed to be the same. The density function for repair time of component i is:

$$g_i(t)=\lambda_i t\ \mathrm{Exp}[-\lambda_i t] \qquad i=1,2. \tag{6.6.3}$$

6.6.1 Failure model

When we set $T_1=T_2=\tau_1=\tau_2=\infty$ in model 4, then $Q_I(\infty)=Q_G(\infty)=1$. The probability of no opportunistic replacement is 1.0 for both the initial minor and the general minor interval. There is no opportunistic replacement in either the initial or the general interval. Therefore there is only replacement of individual components and no simultaneous replacement of both components. This is a failure model. The system has no renewal points.

As noted earlier in the failure model replacements are made only at component failures and no two components are replaced at the same opportunity. Therefore this can be compared to a general series model where components are replaced only at failure.

The Table (6.1) below shows three numerical examples for the long run availability for the failure model with a Weibull failure distribution. Also in the same table is the availability using the formula by Barlow and Proschan. The numerical answers from the model are the same as the numerical results from Barlow and Proschan.

Table 6.1 Limiting Availability of a Weibull Failure Replacement Model

Parameters	Model	Barlow	Q_I	Q_G
$\alpha_1=0.5;\eta_1=0.67;\beta_1=2,\alpha_2=1.0;\ \eta_2=0.4,\beta_2=2$	0.589873	0.589873	1.00	1.00
$\alpha_1=0.5;\eta_1=0.12;\beta_1=2,\alpha_2=0.8;\ \eta_2=0.11,\beta_2=1.5$	0.875077	0.874845	1.00	1.00
$\alpha_1=0.5;\eta_1=0.095;\beta_1=2,\alpha_2=0.8;\ \eta_2=0.069,\beta_2=1.6$	0.906221	0.906118	1.00	1.00

6.6.2 Opportunistic replacement model.

In opportunistic replacement $Q_I(\infty)<1$ and $Q_G(\infty)<1$. Therefore there is always a positive probability that either the initial interval or the general interval are of infinite length or we could say that there is a positive probability of having an opportunistic replacement in either the initial interval or the general interval. Appendix 3 gives numerical evaluation of opportunistic replacement for some selected values of the decision variables. The set of decision variables are selected so that they are related to the mean failure time of the components. This is to see if there is some kind of relationship between the age replacement times and the corresponding opportunistic replacement times. The following relationship is used for the decision variables:

$T_1=n\ \mu_1$, $T_2=(m-n)\mu_2$, $\tau_1=\beta T_1$ and $\tau_2=\beta T_2$. m takes the values 2,3 and 4 while n<m. These are not optimal values. The following values are used for the parameters of the failure and the repair time distributions. For the Weibull failure distributions:

Component 1:

Failure parameters; $\alpha_1=0.5;\beta_1=2$.

Repair parameters; $\eta_1=\frac{1}{1.5}$

Component 2:

Failure parameters; $\alpha_2=1.0;\beta_2=2$.

Repair parameters;$\eta_2=\frac{1}{2.5}$.

For the Gamma failure distribution:

Component 1:

Failure parameters; $\alpha_1=0.5;\gamma_1=2$.

Repair parameters; $\eta_1=\frac{1}{1.5}$

Component 2:

Failure parameters; α_2=1.0;γ_2=2.

Repair parameters; $\eta_2=\frac{1}{2.5}$

In Figures 6.1 and 6.2, M1 to M6 represent sets of age decision varriables. These variables are summarized as:

$$M1=\{T_1=\mu_1\ ,\ \ T_2=\mu_2\}$$

$$M2=\{T_1=\mu_1\ ,\ T_2=2\mu_2\}$$

$$M3=\{T_1=2\mu_1\ ,\ T_2=\mu_2\}$$

$$M4=\{T_1=\mu_1\ ,\ T_2=3\mu_2\}$$

$$M5=\{T_1=2\mu_1\ ,\ T_2=2\mu_2\}$$

$$M6=\{T_1=3\mu_1\ ,\ T_2=\mu_2\}$$

From Figure 6.1, M5 has a higher availability than the other set of decision variables for Beta values between 0.2 and 0.6. For Beta values beyond 0.8 M1 has a higher availability. M1 has a lower availability compared to the other decision variables when Beta is between 0.2 and 0.6. All the decision variables seems to converging towards to a point as the value of beta approaches 1. All the decision varaible have their highest availability for values of Beta between 0.4 and 0.8.

From Figure 6.2, M5 has a higher availability compared to the other decision variables for Beta greater than 0.4. All the decision variables seems to be converging towards a common availability as beta approaches 1. Similar to the case of the Weibull the decision variables have their highest availability between 0.4 and 0.8.

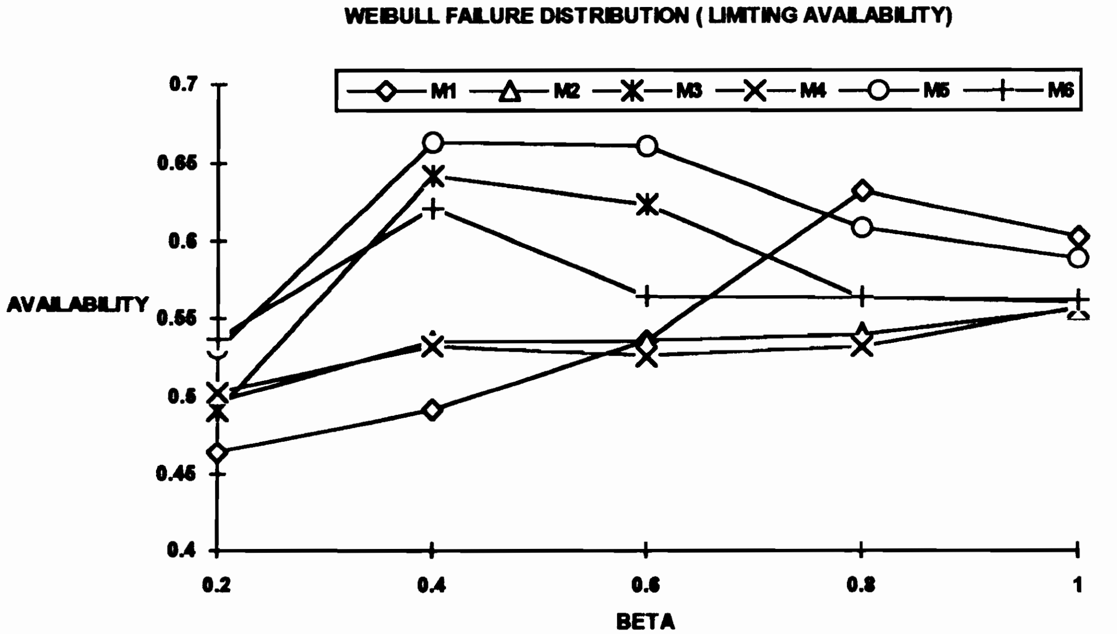

Fig. 6.1 Limiting Availability For Weibull Failure Time

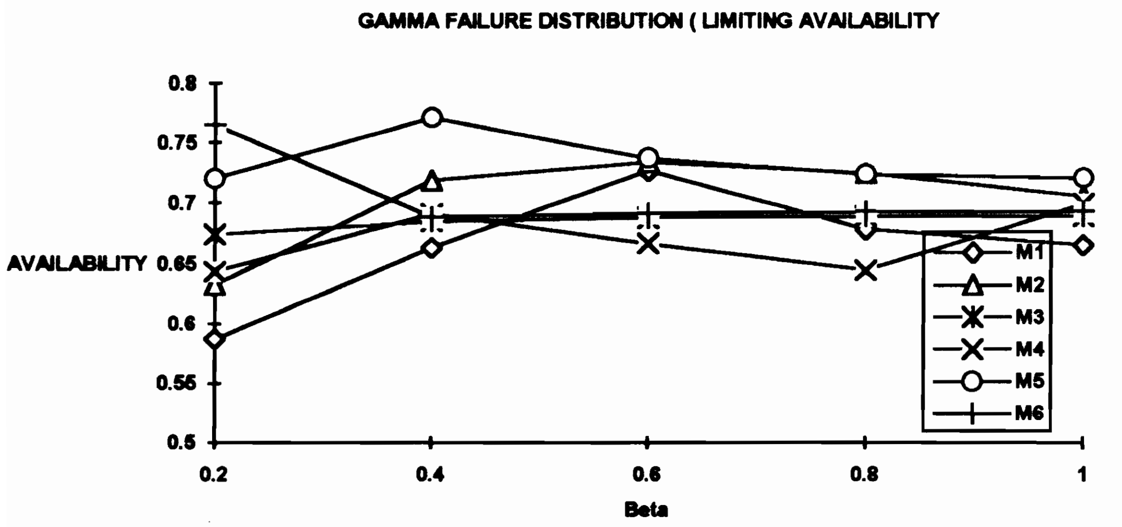

Fig 6.2 Limiting Availability For Gamma Failure Time

When Beta=1, the opportunistic replacement model becomes an age replacement model. From both Figures 6.1 and 6.2 we notice that opportunistic replacement model has a better availability when compared to the age replacement model. In both cases the decision variables that have the highest availability values have Beta values less than 0.6.

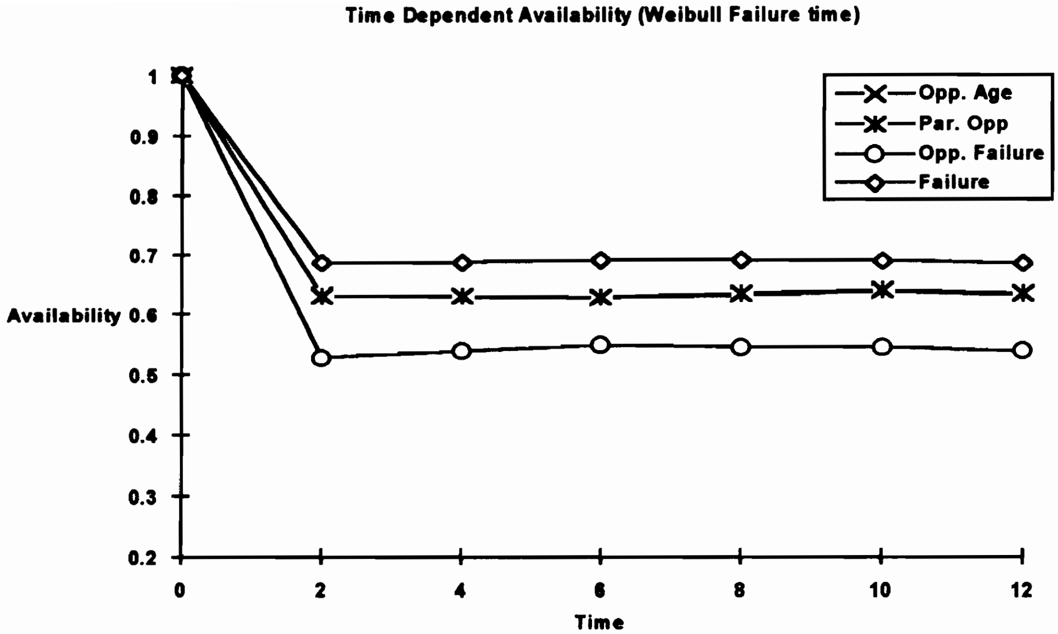

Figure 6.3 Time Dependent Availability For Weibull Failure Time

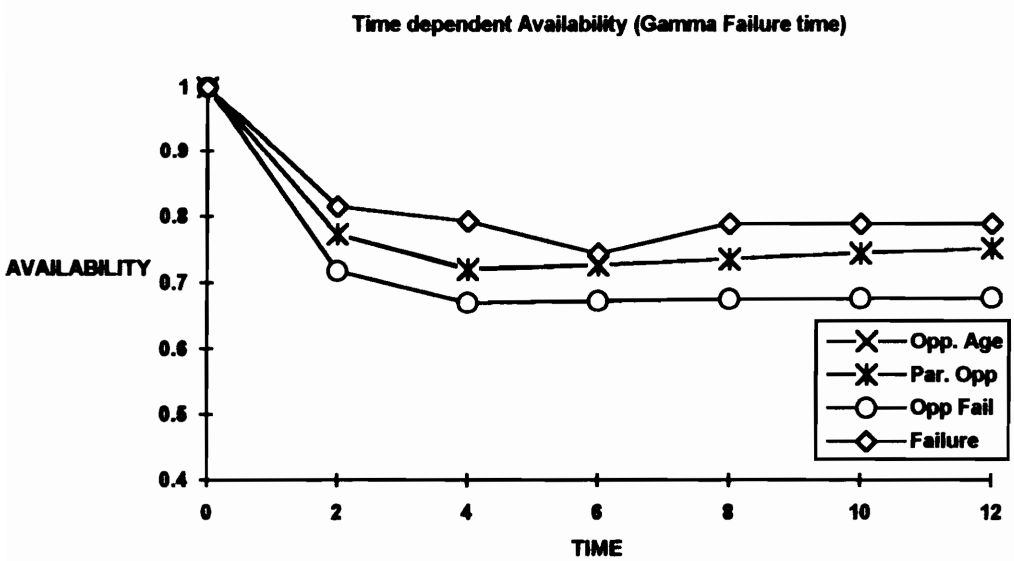

Figure 6.4 Time Dependent Availability For Gamma Failure Time

Fig 6.3 shows the graphs of the time dependent availability for an oppotunistic age replacement policy, partial opportunistic age replacement policy, opportunistic failure replacement policy and failure replacement policy for the Weibull distributions. The set of decision variables used are:

For the case of Opportunistic Age Replacement Policy we have:

$\tau_1=2.13$ $\tau_2=1.50$ $T_1=3.55$ and $T_2=2.51$.

For the case of Partial Opportunistic Age Replacement Policy we have:

$\tau_1=2.13$ $\tau_2=1.50$ $T_1=\infty$ and $T_2=2.51$.

For the case of Opportunistic failure replacement policy we have:

$\tau_1=2.13$ $\tau_2=1.50$ $T_1=\infty$ and $T_2=\infty$.

For the case of the failure replacement policy we have:

$\tau_1=\infty$ $\tau_2=\infty$ $T_1=\infty$ and $T_2=\infty$.

We notice that the opportunistic age replacement model is almost the same as the partial opportunistic age replacement model. The failure replacement policy has better availability than all the other replacement policies followed by the partial opportunistic age replacement and the opportunistic age replacement. The opportunistic failure replacement policy has the lowest avaibility.

Fig 6.4 shows the graphs of the time dependent availability for an oppotunistic age replacement policy, partial opportunistic age replacement policy, opportunistic failure replacement policy and failure replacement policy for the Gamma distributions. The following set of decision variables used are:

For the case of Opportunistic Age Replacement Policy we have:

τ_1=4.8 τ_2=2.4 T_1=8.0 and T_2=4.0.

For the case of Partial Opportunistic Age Replacement Policy we have:

τ_1=4.8 τ_2=2.4 $T_1=\infty$ and T_2=4.0.

For the case of Opportunistic failure replacement policy we have:

τ_1=4.8 τ_2=2.4 $T_1=\infty$ and $T_2=\infty$.

For the case of the failure replacement policy we have:

$\tau_1=\infty$ $\tau_2=\infty$ $T_1=\infty$ and $T_2=\infty$.

As in the case with weibull failure distribution we notice that the opportunistic age replacement model is almost the same as the partial opportunistic age replacement model. The failure replacement policy has better availability than all the other replacement policies followed by the partial opportunistic age replacement and the opportunistic age replacement. The opportunistic failure replacement policy has the lowest avaibility.

6.7 Interpretation of results from the model:

With Weibull failure distributions when the decision variables M_2 and Beta=0.8. That is the decision variables are $T_1=\mu_1$=1.77, $T_2=2\mu_2$=2.51, τ_1=0.8*1.772=1.42 and τ_2=2*0.8*2.253= 2.01. The availability was found to be 0.5401. This means components 1 is either replaced at failure or after it has been operation for a time of 1.77 units. Also component 2 is replaced either at failure or after it has been in operation for a time of 2.51 units. If at the time of replacing component 1 component 2 has aged 2.01 units then both are replaced at the same time. Also if at the time of replacing component 2 component 1 has aged 1.42 unit then both components are replace at the same time. The result of such a decision is a system availability of 0.5401.

When the decision variables all set at such that $T_1=T_2=\tau_1=\tau_2=\infty$ then we have a failure replacement policy. Components are only replaced at failure and there is no simultaneous replacement of both components.

6.8 Summary

In this chapter we succeeded in developing numerical approximations to the distribution for operating period periods in both the initial and the general minor intervals and also numerical approximations to the convolutions on the length of the operating period and the repair time distributions. Numerical examples are given for the long run availability and also graphs for the time dependent availability of the different policies are included.

CHAPTER 7 CONCLUSIONS AND FUTURE WORK

7.1 Introduction

The developement and analysis of the models carried out in the previous chapters gives several insights into these models. The findings point to some interesting observations in the models.

7.2 Conclusions

7.2.1 Model developement.

The research presented an exact representation of the Laplace transform of the system avaialbility function, $\mathfrak{A}(t)$ and the limiting availability $\mathfrak{A}=\lim_{t\to\infty}\mathfrak{A}(t)$. The availability function is the probability that the system functions at time t. T_i represents the age replacement time of component i and τ_i is the opportunistic replacement time of component i. The availability of opportunistic age replacement has not been previously investigated.

The derived model treats the age replacement period T_i and the opportunistic replacement period τ_i as parameters. The methodology presented produces the Laplace tranform for

$\mathfrak{A}(t)$ and the limiting availability $\mathfrak{A}$. The Laplace transform cannot be inverted exactly. Therefore a numerical inversion technique is applied to obtain an estimate for $\mathfrak{A}(t)$. The numerical inversion technique is less accurate when the function that is being inverted has discontinuties near t. In the developement of the minor interval, the intervals had to separated into renewal and non renewal minor interval.s The distribution on the length of the initial and general minor intervals $q_I(t)$ and $q_G(t)$ were made up of discontinuous sets of intervals.

There is no exact analytical expression for the k-fold convolution for a Weibull distribution. Numerical approximations have to be made for k-fold convolution and the renewal function of the Weibull distribution. Using the first 10 terms of the infinite series representation gives very accurate answers for the failure model, though it seems to increase the amount of computing time required. Most of the integrations involved could not be solved in closed algebraic forms. Therefore numerical integrtaion is used. The minor intervals also have an infinite number of terms. Using 10 to 12 terms works very well. The accuracy of these approximations can be seen in the computation of the limiting availability of the failure replacement policy. The numerical results in Table 6.1 show that when these approximations are use to compute the limiting availability of the failure replacement policy the results are close to at least four decimal places.

Though there is a closed form expression for the k-fold convolution for the Gamma distribution. The minor intervals have an infinite number of terms. The terms are truncated to 12 and this also provides accurate results.

Using either the Weibull distribution or the Gamma distribution for component failure time we had to use numerical integration to calculate the system availability. This takes a lot of computing time especially for the case of Weibull distribution for component failure because the k-fold convolution is an infinite series. The computing time becomes excessive when we apply the numerical inversion technique to invert $\mathfrak{A}(t)$.

In the general model, that is model 4, by making the neccessary changes to the appropraite decision variables, we can have any of the other three models. Setting $T_1=T_2=\tau_1=\tau_2=\infty$ in the opportunistic age replacement model, we will have the failure model in model 1. For $T_1=T_2=\infty$ in the opportunistic age replacement model, we have model 2. To have model 3 we need to set $T_1=\infty$. There are other versions of the model. For example any time we $T_i=\tau_i$, then we have an age replacement model.

In the failure model, the long run availability gives us a solution similar to the formula given by Barlow and Proschan[1]. This means the different approach used in the model seems to work very well.

7.3 Extension and Future Work

The model needs to be generalized so that one can analyze more than two components at the same time. Most systems have more than two components in series. Since the more components we have in the system the more decision variables the model will be more complicated. For example each component has two decision variables, an age replacement time and an opportunistic replacement time, and therefore if we have n components in the system, then there will be 2*n decision variables. The large number of decision variables will makes the model more complicated.

Another consideration that has to be made in any future work is to use the availability function as a constraint to a cost model, or use a cost model as a constraint to the availability model. The availability model attempts to measure the proportion of time that system will be functioning with no consideration to the cost of keeping the system in an optimal operational state. Therefore there may be the situation in which we may have high availability but at a very high cost of keeping it operational. On the other hand, there is the case in which the optimal cost is usually found without the consideration of the proportion of time that the system will be functioning. Hence for the optimal solution in an operating system neither situation should be done in isolation.

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Backward Recurrence-Time

Let Z(t) be the time from t since the last renewal. That is:

$$Z(t)=t-S_N(t)$$

where $S_N(t)$ is the time of the last renewal. The from Cox[]

$$P\{Z(t) \leq x\}=\begin{cases} F(t)-\int_0^{t-x}[1-F(t-y)]m_F(y)dy & x<t \\ 1 & x \geq t \end{cases}$$

then:

$$U(x)=\lim_{t\to\infty} P\{Z(t) \leq x\}=\int_0^x \frac{[1-F(y)]}{\mu}dy$$

$$u(x)=\frac{d}{dx}U(x)=\frac{1-F(x)}{\mu}$$

We will use the Laplace transform to find the mean of the backward recurrence time. The Laplace transform of u(x) is:

$$u^*(s)=\frac{1/s-F^*(s)}{\mu}=\frac{1-s\,F^*(s)}{s\,\mu}=\frac{1-f^*(s)}{s\mu}$$

The mean $a_1= -\lim_{s\to 0}\frac{d}{ds}u^*(s)$. Differentiating $u^*(s)$ yields:

$$\frac{d}{ds}u^*(s)=\frac{-s\,\frac{d}{ds}f^*(s)-\left(1-f^*(s)\right)}{s^2\mu}$$

Applying L'Hospital rule yields:

$$a_1 = -\lim_{s\to 0}\frac{d}{ds}u^*(s) = -\lim_{s\to 0}\frac{-\frac{d}{ds}f^*(s) - s\frac{d^2}{ds^2}f^*(s) + \frac{d}{ds}f^*(s)}{s^2\mu} = \lim_{s\to 0}\frac{\frac{d^2}{ds^2}f^*(s)}{2\mu}$$

$$a_1 = \frac{\mu^2+\sigma^2}{2\,\mu}$$

where μ and σ are the mean and standard deviation of the distribution F(t)

APPENDIX 2A

Sample Paths For Failure Opportunistic Replacement

Case 1: k=0

A	$\bar{A}$	$A \cup \bar{A}$
$0 \leq w<\tau_1$	$\tau_1 \leq w<\infty$	$0 \leq w<\infty$

Case 2: $k \geq 1$, $\tau_1<\tau_2$

A	$\bar{A}$	$A \cup \bar{A}$
$0 \leq w<\tau_1$, $0 \leq u<w$	$\emptyset$	$0 \leq u<w$
$\tau_1 \leq w<\tau_2$, $w-\tau_1 \leq u<w$	$0 \leq u<w-\tau_1$	$0 \leq u<w$
$\tau_2 \leq w<\tau_1+\tau_2$, $w-\tau_1 \leq u<\tau_2$	$0 \leq u<w-\tau_1 \cup \tau_2 \leq u<w$	$0 \leq u<w$
$\tau_1+\tau_2 \leq w<\infty$, $\emptyset$	$0 \leq u<\tau_2 \cup \tau_2 \leq u<w$	$0 \leq u<w$

Case3: $k \geq 1$, $\tau_1 \geq \tau_2$

A	$\bar{A}$	$A \cup \bar{A}$
$0 \leq w<\tau_2$, $0 \leq u<w$	$\emptyset$	$0 \leq u<w$
$\tau_2 \leq w<\tau_1$, $0 \leq u<\tau_2$	$\tau_2 \leq u<w$	$0 \leq u<w$
$\tau_1 \leq w<\tau_1+\tau_2$, $w-\tau_1 \leq u<\tau_2$	$0 \leq u<w-\tau_1 \cup \tau_2 \leq u<w$	$0 \leq u<w$
$\tau_1+\tau_2 \leq w<\infty$, $\emptyset$	$0 \leq u<\tau_2 \cup \tau_2 \leq u<w$	$0 \leq u<w$

APPENDIX 2B

Partial Opportunistic Age replacement

Case 1:k=0

B	$\bar{B}$	$B \cup \bar{B}$
$0 \leq w<\tau_1$	$\tau_1 \leq w<T_2$	$0 \leq w<T_2$

Case 2: $T_2<\tau_1$, k=0

B	$\bar{B}$	$B \cup \bar{B}$
$0 \leq w<\tau_1$	$\tau_1 \leq w<T_2$	$0 \leq w<T_2$

Case 3: $\tau_1<\tau_2<T_2$ and $\tau_1+\tau_2<T_2$, $k \geq 1$

B	$\bar{B}$	$B \cup \bar{B}$
$0 \leq w<\tau_1$, $0 \leq u<w$	$\emptyset$	$0 \leq u<w$
$\tau_1 \leq w<\tau_2$, $w-\tau_1 \leq u<w$	$0 \leq u<w-\tau_1$	$0 \leq u<w$
$\tau_2 \leq w<\tau_1+\tau_2$, $w-\tau_1 \leq u<\tau_2$	$0 \leq u<w-\tau_1 \cup \tau_2 \leq u<w$	$0 \leq u<w$
$\tau_1+\tau_2 \leq w<T_2$, $\emptyset$	$0 \leq u<\tau_2 \cup \tau_2 \leq u<w$	$0 \leq u<w$

Case4: $k \geq 1$, $T_2>\tau_1 \geq \tau_2$ and $\tau_1+\tau_2<T_2$, $k \geq 1$

B	$\bar{B}$	$B \cup \bar{B}$
$0 \leq w<\tau_2$, $0 \leq u<w$	$\emptyset$	$0 \leq u<w$
$\tau_2 \leq w<\tau_1$, $0 \leq u<\tau_2$	$\tau_2 \leq u<w$	$0 \leq u<w$
$\tau_1 \leq w<\tau_1+\tau_2$, $w-\tau_1 \leq u<\tau_2$	$0 \leq u<w-\tau_1 \cup \tau_2 \leq u<w$	$0 \leq u<w$
$\tau_1+\tau_2 \leq w<T_2$, $\emptyset$	$0 \leq u<\tau_2 \cup \tau_2 \leq u<w$	$0 \leq u<w$

Case 5: $\tau_1<\tau_2<T_2$ and $\tau_1+\tau_2 \geq T_2$, $k \geq 1$

B	$\bar{B}$	$B \cup \bar{B}$
$0 \leq w<\tau_1$, $0 \leq u<w$	$\emptyset$	$0 \leq u<w$
$\tau_1 \leq w<\tau_2$, $w-\tau_1 \leq u<w$	$0 \leq u<w-\tau_1$	$0 \leq u<w$
$\tau_2 \leq w<T_2$, $w-\tau_1 \leq u<\tau_2$	$0 \leq u<w-\tau_1 \cup \tau_2 \leq u<w$	$0 \leq u<w$

Case6: $T_2>\tau_1 \geq \tau_2$ and $\tau_1+\tau_2 \geq T_2$, $k \geq 1$

B	$\bar{B}$	$B \cup \bar{B}$
$0 \leq w<\tau_2$, $0 \leq u<w$	$\emptyset$	$0 \leq u<w$
$\tau_2 \leq w<\tau_1$, $0 \leq u<\tau_2$	$\tau_2 \leq u<w$	$0 \leq u<w$
$\tau_1 \leq w<T_2$, $w-\tau_1 \leq u<\tau_2$	$0 \leq u<w-\tau_1 \cup \tau_2 \leq u<w$	$0 \leq u<w$

Case7: $\tau_1>T_2 \geq \tau_2$ and $\tau_1+\tau_2 \geq T_2$, $k \geq 1$

B	$\bar{B}$	$B \cup \bar{B}$
$0 \leq w<\tau_2$, $0 \leq u<w$	$\emptyset$	$0 \leq u<w$
$\tau_2 \leq w<T_2$, $0 \leq u<\tau_2$	$\tau_2 \leq u<w$	$0 \leq u<w$

Appendix 3A

Weibull Failure Time. μ_1=1.772 μ_2=1.253 $\tau_i=\beta T_i$

Beta	M1	M2	M3	M4	M5	M6
0.2	0.4642	0.4969	0.4911	0.5029	0.5304	0.5374
0.4	0.4915	0.5355	0.6416	0.5322	0.6628	0.6214
0.6	0.5358	0.5361	0.6233	0.5259	0.6606	0.5643
0.8	0.6321	0.5401	0.5633	0.5322	0.6084	0.5636
1.0	0.6024	0.5558	0.5603	0.557	0.5885	0.5617

Appendix 3B

Gamma Failure Time μ_1=4.0 μ_2=2.0 $\tau_i=\beta T_i$

Beta	M1	M2	M3	M4	M5	M6
0.2	0.5872	0.6326	0.6742	0.6437	0.7202	0.7645
0.4	0.6634	0.7185	0.685	0.6911	0.7705	0.6885
0.6	0.7265	0.7334	0.688	0.6668	0.7371	0.6928
0.8	0.6785	0.7245	0.6891	0.6448	0.7235	0.6928
1.0	0.6654	0.7056	0.6894	0.6991	0.7205	06932

Limiting Availability

Gamma Failure & Renewals

```
Off[NIntegrate::slwcon];
Zed[t_,b_,a_]:=(a^b) t^(b-1) Exp[-a t]/Gamma[b];
REL[t_,b_,a_]:=Sum[((a t)^i) Exp[-a t]/i!,{i,0,b-1}];
CDF[t_,b_,a_]:=CDF[t,b,a]=1-REL[t,b,a];
mean[b_,a_,T_]:=Sum[Gamma[1+i,0,a T]/i!, {i, 0, -1 + b}]/a;
f[y_,t_,b_,a_,Mu1_,Mu2_]:=Exp[-a t] Sum[
(a^(b n)) t^(b n-1) (n Mu1+Mu2+y)/Gamma[b n],{n,1,12}];

Y[t_,b_,a_,T_]:=If[t<=T,REL[t,b,a]/mean[b,a,T],0.0];
WKK[y_,t_,b_,a_,Mu1_,Mu2_]:=Exp[-a t] Sum[
(a^(b n)) t^(b n-1) ((n+1) Mu1+Mu2+y)/Gamma[b n],{n,1,12}];
VMM[y_,t_,b_,a_,Mu1_,Mu2_]:=Exp[-a t] Sum[
(a^(b n)) t^(b n-1) ((n+2) Mu1+Mu2+y)/Gamma[b n],{n,1,12}];
W[y_,t_,b_,a_,Mu1_,Mu2_,T_]:=Y[t,b,a,T]*(Mu1+Mu2+y)+
NIntegrate[Y[x,b,a,T]*WKK[y,t-x,b,a,Mu1,Mu2],{x,0.0,t},
PrecisionGoal->3,AccuracyGoal->6];
WZ[y_,t_,b_,a_,Mu1_,Mu2_,T_]:=Y[t,b,a,T]*(2 Mu1+Mu2+y)+
NIntegrate[Y[x,b,a,T]*VMM[y,t-x,b,a,Mu1,Mu2],{x,0.0,t},
PrecisionGoal->3,AccuracyGoal->6];
RESDL[t_,b_,a_,T_]:=If[t<=T,1-mean[b,a,t]/mean[b,a,T],0.0];
```

Weibull Failures & Renewals

```
(*Off[NIntegrate::slwcon];
Zed[t_,b_,a_]:=a t^(b-1) Exp[-a/b t^b];
REL[t_,b_,a_]:=Exp[-(a/b) t^b];
CDF[q_,b_,T_]:=CDF[q,b,T]=1-Exp[-(q/b) T^b]
mean[b_,a_,T_]:=mean[b,a,T]=NIntegrate[REL[t,b,a],{t,0,T}];
Z[m_,b_]:=Gamma[b m + 1]/Gamma[m+1];
B[k_,s_,b_]:=If[
                    k==0,Z[s,b],
                          Sum[B[k-1,r,b] Z[s-r,b],{r,k-1,s-1}]];
A[k_,s_,b_]:=A[k,s,b]=Sum[(-1)^(p+k) Binomial[s,p] B[k,p,b]/Z[p,b],
{p,k,s}];
BBpha[k_,s_,b_]:=If[s==k,A[k,s,b],
             Sum[A[r,s,b],{r,k,s}]-Sum[A[r,s-1,b],{r,k,s-1}]];
Alpha[y_,m_,b_,Mu1_,Mu2_]:=Sum[BBpha[k,m,b]*(k*Mu1+Mu2+y),
{k,1,m}]
f[y_,t_,b_,a_,Mu1_,Mu2_]:=Exp[-(a/b) t^b]*
Sum[((a/b)^(m)) Alpha[y,m,b,Mu1,Mu2] b t^(b m-1)/((m-1)!),
{m,1,10}];
LPpha[y_,m_,b_,Mu1_,Mu2_]:=Sum[BBpha[k,m,b]*((k+1)*Mu1+Mu2+y),
{k,1,m}];
UKM[y_,m_,b_,Mu1_,Mu2_]:=Sum[BBpha[k,m,b]*((k+2)*Mu1+Mu2+y),
{k,1,m}];
Y[t_,b_,a_,T_]:=REL[t,b,a]/mean[b,a,T];
WKK[y_,t_,b_,a_,Mu1_,Mu2_]:=Exp[-(a/b) t^b]*
Sum[((a/b)^(m)) LPpha[y,m,b,Mu1,Mu2] b t^(b m-1)/((m-1)!),
{m,1,10}];
VMM[y_,t_,b_,a_,Mu1_,Mu2_]:=Exp[-(a/b) t^b]*
Sum[((a/b)^(m)) UKM[y,m,b,Mu1,Mu2] b t^(b m-1)/((m-1)!),
{m,1,10}];
W[y_,t_,b_,a_,Mu1_,Mu2_,T_]:=Y[t,b,a,T]*(Mu1+Mu2+y)+
NIntegrate[Y[x,b,a,T]*WKK[y,t-x,b,a,Mu1,Mu2],{x,0.0,t},
PrecisionGoal->3,AccuracyGoal->6];
WZ[y_,t_,b_,a_,Mu1_,Mu2_,T_]:=Y[t,b,a,T]*(2 Mu1+Mu2+y)+
NIntegrate[Y[x,b,a,T]*VMM[y,t-x,b,a,Mu1,Mu2],{x,0.0,t},
PrecisionGoal->3,AccuracyGoal->6];
RESDL[t_,b_,a_,T_]:=1-NIntegrate[REL[x,b,a],{x,0,t}]/mean[b,a,T];
ST[x_]:=1/;x>0
ST[x_]:=0/;x<=0;*)
```

Parameters

```
ld1=1.5;ld2=2.5;b1=2.0;b2=2.0;q1=0.5;q2=1.0;Mu1=1/ld1;
Mu2=1/ld2;ldp=2.0; MuP=1/(ldp);Opp:=1/(ld1+ld2);
var[a_,b_]=Gamma[1 + 2/b]/(a/b)^(2/b)-(Gamma[b^(-1)]/((a/b)^b^(-1)*b
vt[a_,b_]=Gamma[b^(-1)]/((a/b)^b^(-1)*b);
PU[a_,b_]:=PU[a,b]=(vt[a,b]^2 + var[a,b]^2)/(2 vt[a,b]);

(*ld2=1.5;ld1=2.5;b1=2.0;b2=2.0;q2=0.5;q1=1.0;Mu1=1/ld1;
Mu2=1/ld2;ldp=2.0; MuP=1/(ldp);Opp:=1/(ld1+ld2);*)
```

Initial Interval

```
A3:=Function[{Tau1,Tau2,T1,T2},
If[T2<=T1,
     If[T2>=Tau1,
          NIntegrate[
          Zed[w,b2,q2]*REL[w,b1,q1]*(Mu2+w),{w,0,Tau1},
          PrecisionGoal->6,AccuracyGoal->6],
          NIntegrate[
          Zed[w,b2,q2]*REL[w,b1,q1]*(Mu2+w),{w,0,T2},
          PrecisionGoal->6,AccuracyGoal->6]
          ],
     If[T1>=Tau2,
          NIntegrate[
          Zed[w,b1,q1]*REL[w,b2,q2]*(Mu1+w),{w,0,Tau2},
          PrecisionGoal->6,AccuracyGoal->6],
          NIntegrate[
          Zed[w,b1,q1]*REL[w,b2,q2]*(Mu1+w),{w,0,T1},
          PrecisionGoal->6,AccuracyGoal->6]
          ]
     ]
];
A1:=Function[{Tau1,Tau2,T1,T2},
If[T2<=T1,
     If[T2<Tau1,
          REL[T2,b1,q1]*REL[T2,b2,q2]*(Mu2+T2),0],
     If[T1<Tau2,
          REL[T1,b2,q2]*REL[T1,b1,q1]*(Mu1+T1),0]
     ]
];
AM1:=Function[{Tau1,Tau2,T1,T2},
If[T2<=T1,
     If[T2<Tau1,
          REL[T2,b1,q1]*REL[T2,b2,q2]*T2,0],
     If[T1<Tau2,
          REL[T1,b2,q2]*REL[T1,b1,q1]*T1,0]
     ]
];
AP1:=Function[{Tau1,Tau2,T1,T2},
If[T2<=T1,
     If[T2<Tau1,
          REL[T2,b1,q1]*REL[T2,b2,q2],0],
     If[T1<Tau2,
          REL[T1,b2,q2]*REL[T1,b1,q1],0]
     ]
];
AM3:=Function[{Tau1,Tau2,T1,T2},
If[T2<=T1,
     If[T2>=Tau1,
          NIntegrate[
          Zed[w,b2,q2]*REL[w,b1,q1]*(w),{w,0,Tau1},
          PrecisionGoal->6,AccuracyGoal->6],
          NIntegrate[
          Zed[w,b2,q2]*REL[w,b1,q1]*(w),{w,0,T2},
```

```
          PrecisionGoal->6,AccuracyGoal->6]
          ],
     If[T1>=Tau2,
          NIntegrate[
          Zed[w,b1,q1]*REL[w,b2,q2]*(w),{w,0,Tau2},
          PrecisionGoal->6,AccuracyGoal->6],
          NIntegrate[
          Zed[w,b1,q1]*REL[w,b2,q2]*w,{w,0,T1},
          PrecisionGoal->6,AccuracyGoal->6]
          ]
     ]
];
AP3:=Function[{Tau1,Tau2,T1,T2},
If[T2<=T1,
     If[T2>=Tau1,
          NIntegrate[
          Zed[w,b2,q2]*REL[w,b1,q1],{w,0,Tau1},
          PrecisionGoal->6,AccuracyGoal->6],
          NIntegrate[
          Zed[w,b2,q2]*REL[w,b1,q1],{w,0,T2},
          PrecisionGoal->6,AccuracyGoal->6]
          ],
     If[T1>=Tau2,
          NIntegrate[
          Zed[w,b1,q1]*REL[w,b2,q2],{w,0,Tau2},
          PrecisionGoal->6,AccuracyGoal->6],
          NIntegrate[
          Zed[w,b1,q1]*REL[w,b2,q2],{w,0,T1},
          PrecisionGoal->6,AccuracyGoal->6]
          ]
     ]
];
DK:=Function[{Tau1,Tau2,T1,T2},
If[T2<=T1,
  If[Tau1<Tau2,
          If[T2<=Tau1+Tau2,
               NIntegrate[Zed[w,b2,q2]*
               f[w,u,b1,q1,Mu1,Mu2]*
               REL[w-u,b1,q1],{w,0,Tau1},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*
               f[w,u,b1,q1,Mu1,Mu2]*REL[w-u,b1,q1],
               {w,Tau1,Tau2},{u,w-Tau1,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*
               f[w,u,b1,q1,Mu1,Mu2]*REL[w-u,b1,q1],
               {w,Tau2,T2},{u,w-Tau1,Tau2},
               PrecisionGoal->3,AccuracyGoal->6],
                    NIntegrate[Zed[w,b2,q2]*f[w,u,b1,q1,Mu1,Mu2]*
                    REL[w-u,b1,q1],{w,0,Tau1},{u,0,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
                    NIntegrate[Zed[w,b2,q2]*
                    f[w,u,b1,q1,Mu1,Mu2]*REL[w-u,b1,q1],
                    {w,Tau1,Tau2},{u,w-Tau1,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
                    NIntegrate[Zed[w,b2,q2]*
```

```
                 f[w,u,b1,q1,Mu1,Mu2]*REL[w-u,b1,q1],
                 {w,Tau2,Tau1+Tau2},{u,w-Tau1,Tau2},
                 PrecisionGoal->3,AccuracyGoal->6]
      ],
   If[T2<=Tau1+Tau2,
      If[Tau1<T2,
           NIntegrate[Zed[w,b2,q2]*
           f[w,u,b1,q1,Mu1,Mu2]*
           REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
           PrecisionGoal->3,AccuracyGoal->6]+
           NIntegrate[Zed[w,b2,q2]*
           f[w,u,b1,q1,Mu1,Mu2]*REL[w-u,b1,q1],
           {w,Tau2,Tau1},{u,0.0,Tau2},
           PrecisionGoal->3,AccuracyGoal->6]+
           NIntegrate[Zed[w,b2,q2]*
           f[w,u,b1,q1,Mu1,Mu2]*REL[w-u,b1,q1],
           {w,Tau1,T2},{u,w-Tau1,Tau2},
           PrecisionGoal->3,AccuracyGoal->6],
            NIntegrate[Zed[w,b2,q2]*
                 f[w,u,b1,q1,Mu1,Mu2]*
                 REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
                 PrecisionGoal->3,AccuracyGoal->6]+
           NIntegrate[Zed[w,b2,q2]*
                 f[w,u,b1,q1,Mu1,Mu2]*
                 REL[w-u,b1,q1],{w,Tau2,T2},{u,0,Tau2},
                 PrecisionGoal->3,AccuracyGoal->6]
                 ],
                 NIntegrate[Zed[w,b2,q2]*
                 f[w,u,b1,q1,Mu1,Mu2]*
                 REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
                 PrecisionGoal->3,AccuracyGoal->6]+
                 NIntegrate[Zed[w,b2,q2]*
                 f[w,u,b1,q1,Mu1,Mu2]*REL[w-u,b1,q1],
                 {w,Tau2,Tau1},{u,0.0,Tau2},
                 PrecisionGoal->3,AccuracyGoal->6]+
                 NIntegrate[Zed[w,b2,q2]*
                 f[w,u,b1,q1,Mu1,Mu2]*REL[w-u,b1,q1],
                 {w,Tau1,Tau1+Tau2},{u,w-Tau1,Tau2},
                 PrecisionGoal->3,AccuracyGoal->6]
       ]
],
If[Tau2<Tau1,
     If[T1<=Tau1+Tau2,
          NIntegrate[Zed[w,b1,q1]*
          f[w,u,b2,q2,Mu2,Mu1]*
          REL[w-u,b2,q2],{w,0,Tau2},{u,0,w},
          PrecisionGoal->3,AccuracyGoal->6]+
          NIntegrate[Zed[w,b1,q1]*
          f[w,u,b2,q2,Mu2,Mu1]*REL[w-u,b2,q2],
          {w,Tau2,Tau1},{u,w-Tau2,w},
           PrecisionGoal->3,AccuracyGoal->6]+
           NIntegrate[Zed[w,b1,q1]*
          f[w,u,b2,q2,Mu2,Mu1]*REL[w-u,b2,q2],
          {w,Tau1,T1},{u,w-Tau2,Tau1},
          PrecisionGoal->3,AccuracyGoal->6],
               NIntegrate[Zed[w,b1,q1]*
```

```
                    f[w,u,b2,q2,Mu2,Mu1]*
                    REL[w-u,b2,q2],{w,0,Tau2},{u,0,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
                    NIntegrate[Zed[w,b1,q1]*
                    f[w,u,b2,q2,Mu2,Mu1]*REL[w-u,b2,q2],
                     {w,Tau2,Tau1},{u,w-Tau2,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
                    NIntegrate[Zed[w,b1,q1]*
                    f[w,u,b2,q2,Mu2,Mu1]*REL[w-u,b2,q2],
                    {w,Tau1,Tau1+Tau2},{u,w-Tau2,Tau1},
                    PrecisionGoal->3,AccuracyGoal->6]
                ],
        If[T1<=Tau1+Tau2,
           If[Tau2<T1,
           NIntegrate[Zed[w,b1,q1]*
           f[w,u,b2,q2,Mu2,Mu1]*
           REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
           PrecisionGoal->3,AccuracyGoal->6]+
           NIntegrate[Zed[w,b1,q1]*
           f[w,u,b2,q2,Mu2,Mu1]*REL[w-u,b2,q2],
           {w,Tau1,Tau2},{u,0.0,Tau1},
           PrecisionGoal->3,AccuracyGoal->6]+
           NIntegrate[Zed[w,b1,q1]*
           f[w,u,b2,q2,Mu2,Mu1]*REL[w-u,b2,q2],
           {w,Tau2,T1},{u,w-Tau2,Tau1},
           PrecisionGoal->3,AccuracyGoal->6],
             NIntegrate[Zed[w,b1,q1]*
                    f[w,u,b2,q2,Mu2,Mu1]*
                    REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b1,q1]*
                    f[w,u,b2,q2,Mu2,Mu1]*
                    REL[w-u,b2,q2],{w,Tau1,T1},{u,0,Tau1},
                    PrecisionGoal->3,AccuracyGoal->6]
                    ],
               NIntegrate[Zed[w,b1,q1]*
               f[w,u,b2,q2,Mu2,Mu1]*
               REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b1,q1]*
               f[w,u,b2,q2,Mu2,Mu1]*REL[w-u,b2,q2],
               {w,Tau1,Tau2},{u,0.0,Tau1},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b1,q1]*
               f[w,u,b2,q2,Mu2,Mu1]*REL[w-u,b2,q2],
               {w,Tau2,Tau1+Tau2},{u,w-Tau2,Tau1},
               PrecisionGoal->3,AccuracyGoal->6]
           ]
      ]
   ]
];
A4:=Function[{Tau1,Tau2,T1,T2},
           (DK[Tau1,Tau2,T1,T2])
      ];
A2:=Function[{Tau1,Tau2,T1,T2},
      If[T2<=T1,
```

```
        If[T2>=Tau1,
             If[T2<=Tau1+Tau2,
                  NIntegrate[REL[T2,b2,q2]*
                  f[T2,u,b1,q1,Mu1,Mu2]*REL[T2-u,b1,q1],
                  {u,T2-Tau1,Tau2},
                  PrecisionGoal->3,AccuracyGoal->6],0],
                  NIntegrate[REL[T2,b2,q2]*
                  f[T2,u,b1,q1,Mu1,Mu2]*
                  REL[T2-u,b1,q1],{u,0,Tau2},
                  PrecisionGoal->3,AccuracyGoal->6]
             ],
        If[T1>=Tau2,
             If[T1<=Tau1+Tau2,
                  NIntegrate[REL[T1,b1,q1]*
                  f[T1,u,b2,q2,Mu2,Mu1]*REL[T1-u,b2,q2],
                  {u,T1-Tau2,Tau1},
                  PrecisionGoal->3,AccuracyGoal->6],0],
        NIntegrate[REL[T1,b1,q1]*f[T1,u,b2,q2,Mu2,Mu1]*
                  REL[T1-u,b2,q2],{u,0,Tau1},
                  PrecisionGoal->3,AccuracyGoal->6]
        ]
    ]
];
AM2:=Function[{Tau1,Tau2,T1,T2},
        If[T2>=T1,
             If[T2>=Tau1,
                  If[T2<=Tau1+Tau2,
                  NIntegrate[REL[T2,b2,q2]*
                  f[T2,u,b1,q1,0,0]*REL[T2-u,b1,q1],
                  {u,T2-Tau1,Tau2},
                  PrecisionGoal->3,AccuracyGoal->6],0],
                  NIntegrate[REL[T2,b2,q2]*
                  f[T2,u,b1,q1,0,0]*REL[T2-u,b1,q1],
                  {u,0,Tau2},
                  PrecisionGoal->3,AccuracyGoal->6]
             ],
        If[T1>=Tau2,
             If[T1<=Tau1+Tau2,
                  NIntegrate[REL[T1,b1,q1]*
                  f[T1,u,b2,q2,0,0]*REL[T1-u,b2,q2],
                  {u,T1-Tau2,Tau1},
                  PrecisionGoal->3,AccuracyGoal->6],0],
             NIntegrate[REL[T1,b1,q1]*
                  f[T1,u,b2,q2,0,0]*REL[T1-u,b2,q2],
                  {u,0,Tau1},
                  PrecisionGoal->3,AccuracyGoal->6]
        ]
    ]
];

AP2:=Function[{Tau1,Tau2,T1,T2},
    If[T2<=T1,
        If[T2>=Tau1,
             If[T2<=Tau1+Tau2,
                  NIntegrate[REL[T2,b2,q2]*
                  f[1,u,b1,q1,0,0]*REL[T2-u,b1,q1],
```

```
                    {u,T2-Tau1,Tau2},
                    PrecisionGoal->3,AccuracyGoal->6],0],
                    NIntegrate[REL[T2,b2,q2]*
                    f[1,u,b1,q1,0,0]*REL[T2-u,b1,q1],{u,0,Tau2},
                    PrecisionGoal->3,AccuracyGoal->6]
               ],
          If[T1>=Tau2,
               If[T1<=Tau1+Tau2,
                    NIntegrate[REL[T1,b1,q1]*
                    f[1,u,b2,q2,0,0]*REL[T1-u,b2,q2],
                    {u,T1-Tau2,Tau1},
                    PrecisionGoal->3,AccuracyGoal->6],0],
               NIntegrate[REL[T1,b1,q1]*
                    f[1,u,b2,q2,0,0]*REL[T1-u,b2,q2],
                    {u,0,Tau1},
                    PrecisionGoal->3,AccuracyGoal->6]
          ]
     ]
];

DKFF:=Function[{Tau1,Tau2,T1,T2},
If[T2<=T1,
  If[Tau1<Tau2,
          If[T2<=Tau1+Tau2,
               NIntegrate[Zed[w,b2,q2]*
               f[w,u,b1,q1,0,0]*
               REL[w-u,b1,q1],{w,0,Tau1},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*
               f[w,u,b1,q1,0,0]*REL[w-u,b1,q1],
               {w,Tau1,Tau2},{u,w-Tau1,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*
               f[w,u,b1,q1,0,0]*REL[w-u,b1,q1],
               {w,Tau2,T2},{u,w-Tau1,Tau2},
               PrecisionGoal->3,AccuracyGoal->6],
                    NIntegrate[Zed[w,b2,q2]*f[w,u,b1,q1,0,0]*
                    REL[w-u,b1,q1],{w,0,Tau1},{u,0,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
                    NIntegrate[Zed[w,b2,q2]*
                    f[w,u,b1,q1,0,0]*REL[w-u,b1,q1],
                    {w,Tau1,Tau2},{u,w-Tau1,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
                    NIntegrate[Zed[w,b2,q2]*
                    f[w,u,b1,q1,0,0]*REL[w-u,b1,q1],
                    {w,Tau2,Tau1+Tau2},{u,w-Tau1,Tau2},
                    PrecisionGoal->3,AccuracyGoal->6]
       ],
       If[T2<=Tau1+Tau2,
          If[Tau1<T2,
               NIntegrate[Zed[w,b2,q2]*
               f[w,u,b1,q1,0,0]*
               REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*
               f[w,u,b1,q1,0,0]*REL[w-u,b1,q1],
```

```
          {w,Tau2,Tau1},{u,0.0,Tau2},
          PrecisionGoal->3,AccuracyGoal->6]+
          NIntegrate[Zed[w,b2,q2]*
          f[w,u,b1,q1,0,0]*REL[w-u,b1,q1],
          {w,Tau1,T2},{u,w-Tau1,Tau2},
          PrecisionGoal->3,AccuracyGoal->6],
           NIntegrate[Zed[w,b2,q2]*
               f[w,u,b1,q1,0.0,0.0]*
               REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
          NIntegrate[Zed[w,b2,q2]*
               f[w,u,b1,q1,0.0,0.0]*
               REL[w-u,b1,q1],{w,Tau2,T2},{u,0,Tau2},
               PrecisionGoal->3,AccuracyGoal->6]
               ],
               NIntegrate[Zed[w,b2,q2]*
               f[w,u,b1,q1,0,0]*
               REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*
               f[w,u,b1,q1,0,0]*REL[w-u,b1,q1],
               {w,Tau2,Tau1},{u,0.0,Tau2},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*
               f[w,u,b1,q1,0,0]*REL[w-u,b1,q1],
               {w,Tau1,Tau1+Tau2},{u,w-Tau1,Tau2},
               PrecisionGoal->3,AccuracyGoal->6]
  ]
],
If[Tau2<Tau1,
     If[T1<=Tau1+Tau2,
          NIntegrate[Zed[w,b1,q1]*
          f[w,u,b2,q2,0,0]*
          REL[w-u,b2,q2],{w,0,Tau2},{u,0,w},
          PrecisionGoal->3,AccuracyGoal->6]+
          NIntegrate[Zed[w,b1,q1]*
          f[w,u,b2,q2,0,0]*REL[w-u,b2,q2],
          {w,Tau2,Tau1},{u,w-Tau2,w},
          PrecisionGoal->3,AccuracyGoal->6]+
          NIntegrate[Zed[w,b1,q1]*
          f[w,u,b2,q2,0,0]*REL[w-u,b2,q2],
          {w,Tau1,T1},{u,w-Tau2,Tau1},
          PrecisionGoal->3,AccuracyGoal->6],
               NIntegrate[Zed[w,b1,q1]*
               f[w,u,b2,q2,0,0]*
               REL[w-u,b2,q2],{w,0,Tau2},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b1,q1]*
               f[w,u,b2,q2,0,0]*REL[w-u,b2,q2],
                {w,Tau2,Tau1},{u,w-Tau2,w},
                PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b1,q1]*
               f[w,u,b2,q2,0,0]*REL[w-u,b2,q2],
               {w,Tau1,Tau1+Tau2},{u,w-Tau2,Tau1},
               PrecisionGoal->3,AccuracyGoal->6]
  ],
```

```
        If[T1<=Tau1+Tau2,
           If[Tau2<T1,
               NIntegrate[Zed[w,b1,q1]*
               f[w,u,b2,q2,0,0]*
               REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b1,q1]*
               f[w,u,b2,q2,0,0]*REL[w-u,b2,q2],
               {w,Tau1,Tau2},{u,0.0,Tau1},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b1,q1]*
               f[w,u,b2,q2,0,0]*REL[w-u,b2,q2],
               {w,Tau2,T1},{u,w-Tau2,Tau1},
               PrecisionGoal->3,AccuracyGoal->6],
             NIntegrate[Zed[w,b1,q1]*
                   f[w,u,b2,q2,0.0,0.0]*
                   REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
                   PrecisionGoal->3,AccuracyGoal->6]+
              NIntegrate[Zed[w,b1,q1]*
                   f[w,u,b2,q2,0.0,0.0]*
                   REL[w-u,b2,q2],{w,Tau1,T1},{u,0,Tau1},
                   PrecisionGoal->3,AccuracyGoal->6]
                   ],
                   NIntegrate[Zed[w,b1,q1]*
                   f[w,u,b2,q2,0,0]*
                   REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
                   PrecisionGoal->3,AccuracyGoal->6]+
                   NIntegrate[Zed[w,b1,q1]*
                   f[w,u,b2,q2,0,0]*REL[w-u,b2,q2],
                   {w,Tau1,Tau2},{u,0.0,Tau1},
                   PrecisionGoal->3,AccuracyGoal->6]+
                   NIntegrate[Zed[w,b1,q1]*
                   f[w,u,b2,q2,0,0]*REL[w-u,b2,q2],
                   {w,Tau2,Tau1+Tau2},{u,w-Tau2,Tau1},
                   PrecisionGoal->3,AccuracyGoal->6]
          ]
       ]
   ]
];
AM4:=Function[{Tau1,Tau2,T1,T2},
         (DKFF[Tau1,Tau2,T1,T2])
    ];
DKP:=Function[{Tau1,Tau2,T1,T2},
If[T2<=T1,
  If[Tau1<Tau2,
         If[T2<=Tau1+Tau2,
              NIntegrate[Zed[w,b2,q2]*
              f[1,u,b1,q1,0,0]*
              REL[w-u,b1,q1],{w,0,Tau1},{u,0,w},
              PrecisionGoal->3,AccuracyGoal->6]+
              NIntegrate[Zed[w,b2,q2]*
              f[1,u,b1,q1,0,0]*REL[w-u,b1,q1],
              {w,Tau1,Tau2},{u,w-Tau1,w},
              PrecisionGoal->3,AccuracyGoal->6]+
              NIntegrate[Zed[w,b2,q2]*
              f[1,u,b1,q1,0,0]*REL[w-u,b1,q1],
```

```
            {w,Tau2,T2},{u,w-Tau1,Tau2},
            PrecisionGoal->3,AccuracyGoal->6],
                NIntegrate[Zed[w,b2,q2]*f[1,u,b1,q1,0,0]*
                REL[w-u,b1,q1],{w,0,Tau1},{u,0,w},
                PrecisionGoal->3,AccuracyGoal->6]+
                NIntegrate[Zed[w,b2,q2]*
                f[1,u,b1,q1,0,0]*REL[w-u,b1,q1],
                {w,Tau1,Tau2},{u,w-Tau1,w},
                PrecisionGoal->3,AccuracyGoal->6]+
                NIntegrate[Zed[w,b2,q2]*
                f[1,u,b1,q1,0,0]*REL[w-u,b1,q1],
                {w,Tau2,Tau1+Tau2},{u,w-Tau1,Tau2},
                PrecisionGoal->3,AccuracyGoal->6]
      ],
      If[T2<=Tau1+Tau2,
        If[Tau1<T2,
            NIntegrate[Zed[w,b2,q2]*
            f[1,u,b1,q1,0,0]*
            REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
            PrecisionGoal->3,AccuracyGoal->6]+
            NIntegrate[Zed[w,b2,q2]*
            f[1,u,b1,q1,0,0]*REL[w-u,b1,q1],
            {w,Tau2,Tau1},{u,0.0,Tau2},
            PrecisionGoal->3,AccuracyGoal->6]+
            NIntegrate[Zed[w,b2,q2]*
            f[1,u,b1,q1,0,0]*REL[w-u,b1,q1],
            {w,Tau1,T2},{u,w-Tau1,Tau2},
            PrecisionGoal->3,AccuracyGoal->6],
              NIntegrate[Zed[w,b2,q2]*
                f[1,u,b1,q1,0,0]*
                REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
                PrecisionGoal->3,AccuracyGoal->6]+
            NIntegrate[Zed[w,b2,q2]*
                f[1,u,b1,q1,0,0]*
                REL[w-u,b1,q1],{w,Tau2,T2},{u,0,Tau2},
                PrecisionGoal->3,AccuracyGoal->6]
                ],
                NIntegrate[Zed[w,b2,q2]*
            f[1,u,b1,q1,0,0]*REL[w-u,b1,q1],{w,0,Tau2},
            {u,0,w},PrecisionGoal->3,AccuracyGoal->6]+
                NIntegrate[Zed[w,b2,q2]*
                f[1,u,b1,q1,0,0]*REL[w-u,b1,q1],
                {w,Tau2,Tau1},{u,0.0,Tau2},
                PrecisionGoal->3,AccuracyGoal->6]+
                NIntegrate[Zed[w,b2,q2]*
                f[1,u,b1,q1,0,0]*REL[w-u,b1,q1],
                {w,Tau1,Tau1+Tau2},{u,w-Tau1,Tau2},
                PrecisionGoal->3,AccuracyGoal->6]
      ]
    ],
If[Tau2<Tau1,
    If[T1<=Tau1+Tau2,
            NIntegrate[Zed[w,b1,q1]*
            f[1,u,b2,q2,0,0]*
            REL[w-u,b2,q2],{w,0,Tau2},{u,0,w},
            PrecisionGoal->3,AccuracyGoal->6]+
```

```
        NIntegrate[Zed[w,b1,q1]*
        f[1,u,b2,q2,0,0]*REL[w-u,b2,q2],
        {w,Tau2,Tau1},{u,w-Tau2,w},
        PrecisionGoal->3,AccuracyGoal->6]+
        NIntegrate[Zed[w,b1,q1]*
        f[1,u,b2,q2,0,0]*REL[w-u,b2,q2],
        {w,Tau1,T1},{u,w-Tau2,Tau1},
        PrecisionGoal->3,AccuracyGoal->6],
              NIntegrate[Zed[w,b1,q1]*
             f[1,u,b2,q2,0,0]*
             REL[w-u,b2,q2],{w,0,Tau2},{u,0,w},
              PrecisionGoal->3,AccuracyGoal->6]+
             NIntegrate[Zed[w,b1,q1]*
             f[1,u,b2,q2,0,0]*REL[w-u,b2,q2],
             {w,Tau2,Tau1},{u,w-Tau2,w},
             PrecisionGoal->3,AccuracyGoal->6]+
             NIntegrate[Zed[w,b1,q1]*
             f[1,u,b2,q2,0,0]*REL[w-u,b2,q2],
             {w,Tau1,Tau1+Tau2},{u,w-Tau2,Tau1},
         PrecisionGoal->3,AccuracyGoal->6]
],
If[T1<=Tau1+Tau2,
    If[Tau2<T1,
        NIntegrate[Zed[w,b1,q1]*
        f[1,u,b2,q2,0,0]*
        REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
        PrecisionGoal->3,AccuracyGoal->6]+
        NIntegrate[Zed[w,b1,q1]*
        f[1,u,b2,q2,0,0]*REL[w-u,b2,q2],
        {w,Tau1,Tau2},{u,0.0,Tau1},
        PrecisionGoal->3,AccuracyGoal->6]+
        NIntegrate[Zed[w,b1,q1]*
        f[1,u,b2,q2,0,0]*REL[w-u,b2,q2],
        {w,Tau2,T1},{u,w-Tau2,Tau1},
        PrecisionGoal->3,AccuracyGoal->6],
          NIntegrate[Zed[w,b1,q1]*
             f[1,u,b2,q2,0.0,0.0]*
             REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
             PrecisionGoal->3,AccuracyGoal->6]+
        NIntegrate[Zed[w,b1,q1]*
             f[1,u,b2,q2,0.0,0.0]*
             REL[w-u,b2,q2],{w,Tau1,T1},{u,0,Tau1},
             PrecisionGoal->3,AccuracyGoal->6]
             ],
             NIntegrate[Zed[w,b1,q1]*
             f[1,u,b2,q2,0,0]*
             REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
             PrecisionGoal->3,AccuracyGoal->6]+
             NIntegrate[Zed[w,b1,q1]*
             f[1,u,b2,q2,0,0]*REL[w-u,b2,q2],
             {w,Tau1,Tau2},{u,0.0,Tau1},
             PrecisionGoal->3,AccuracyGoal->6]+
             NIntegrate[Zed[w,b1,q1]*
             f[1,u,b2,q2,0,0]*REL[w-u,b2,q2],
             {w,Tau2,Tau1+Tau2},{u,w-Tau2,Tau1},
             PrecisionGoal->3,AccuracyGoal->6]
```

```
          ]
      ]
   ]
];
AP4:=Function[{Tau1,Tau2,T1,T2},
          (DKP[Tau1,Tau2,T1,T2])
      ];
```

GENERAL INTERVAL

```
B1:=Function[{Tau1,Tau2,T1,T2},
If[T2<=T1,
     If[T2<Max[(Tau1-PU[q1,b1]),0],
          RESDL[T2,b1,q1,T1]*REL[T2,b2,q2]*(Mu2+T2),0],
     If[T1<Max[(Tau2-PU[q2,b2]),0],
          RESDL[T1,b2,q2,T2]*REL[T1,b1,q1]*(Mu1+T1),0]
     ]
];

BJ1:=Function[{Tau1,Tau2,T1,T2},
If[T2<=T1,
     If[T2<Max[(Tau1-PU[q1,b1]),0],
          RESDL[T2,b1,q1,T1]*REL[T2,b2,q2],0],
     If[T1<Max[(Tau2-PU[q2,b2]),0],
          RESDL[T1,b2,q2,T2]*REL[T1,b1,q1],0]
     ]
];

BI1:=Function[{Tau1,Tau2,T1,T2},
If[T2<=T1,
     If[T2<Max[(Tau1-PU[q1,b1]),0],
          RESDL[T2,b1,q1,T1]*REL[T2,b2,q2]*(T2),0],
     If[T1<Max[(Tau2-PU[q2,b2]),0],
          RESDL[T1,b2,q2,T2]*REL[T1,b1,q1]*(T1),0]
     ]
];
B3:=Function[{Tau1,Tau2,T1,T2},
     If[T2<=T1,
     If[T2>=Max[(Tau1-PU[q1,b1]),0],
          NIntegrate[
          Zed[w,b2,q2]*RESDL[w,b1,q1,T1]*(Mu2+w),
          {w,0,Max[(Tau1-PU[q1,b1]),0]},
          PrecisionGoal->6,AccuracyGoal->6],
          NIntegrate[
          Zed[w,b2,q2]*RESDL[w,b1,q1,T1]*(Mu2+w),{w,0,T2},
          PrecisionGoal->6,AccuracyGoal->6]
          ],
     If[T1>=Max[(Tau2-PU[q2,b2]),0],
          NIntegrate[
          Zed[w,b1,q1]*RESDL[w,b2,q2,T2]*(Mu1+w),
          {w,0,Max[(Tau2-PU[q2,b2]),0]},
          PrecisionGoal->6,AccuracyGoal->6],
          NIntegrate[
          Zed[w,b1,q1]*RESDL[w,b2,q2,T2]*(Mu1+w),{w,0,T1},
          PrecisionGoal->6,AccuracyGoal->6]
          ]
     ]
];
BJ3:=Function[{Tau1,Tau2,T1,T2},
If[T2<=T1,
     If[T2>=Max[(Tau1-PU[q1,b1]),0],
          NIntegrate[
          Zed[w,b2,q2]*RESDL[w,b1,q1,T1],{w,0,Tau1},
          PrecisionGoal->6,AccuracyGoal->6],
```

```
          NIntegrate[
          Zed[w,b2,q2]*RESDL[w,b1,q1,T1],{w,0,T2},
          PrecisionGoal->6,AccuracyGoal->6]
          ],
     If[T1>=Max[(Tau2-PU[q2,b2]),0],
          NIntegrate[
          Zed[w,b1,q1]*RESDL[w,b2,q2,T2],{w,0,Tau2},
          PrecisionGoal->6,AccuracyGoal->6],
          NIntegrate[
          Zed[w,b1,q1]*RESDL[w,b2,q2,T2],{w,0,T1},
          PrecisionGoal->6,AccuracyGoal->6]
          ]
     ]
];
BI3:=Function[{Tau1,Tau2,T1,T2},
If[T2<=T1,
     If[T2>=Max[(Tau1-PU[q1,b1]),0],
          NIntegrate[
          Zed[w,b2,q2]*RESDL[w,b1,q1,T1]*(w),{w,0,Tau1},
          PrecisionGoal->6,AccuracyGoal->6],
          NIntegrate[
          Zed[w,b2,q2]*RESDL[w,b1,q1,T1]*(w),{w,0,T2},
          PrecisionGoal->6,AccuracyGoal->6]
          ],
     If[T1>=Max[(Tau2-PU[q2,b2]),0],
          NIntegrate[
          Zed[w,b1,q1]*RESDL[w,b2,q2,T2]*(w),{w,0,Tau2},
          PrecisionGoal->6,AccuracyGoal->6],
          NIntegrate[
          Zed[w,b1,q1]*RESDL[w,b2,q2,T2]*(w),{w,0,T1},
          PrecisionGoal->6,AccuracyGoal->6]
          ]
     ]
];
Jacb:=Function[{Tau1,Tau2,T1,T2},
If[T2<=T1,
  If[Tau1<Tau2,
          If[T2<=Tau1+Tau2,
               NIntegrate[Zed[w,b2,q2]*
               W[w,u,b1,q1,Mu1,Mu2,T1]*
               REL[w-u,b1,q1],{w,0,Tau1},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*
               W[w,u,b1,q1,Mu1,Mu2,T1]*REL[w-u,b1,q1],
               {w,Tau1,Tau2},{u,w-Tau1,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*
               W[w,u,b1,q1,Mu1,Mu2,T1]*REL[w-u,b1,q1],
               {w,Tau2,T2},{u,w-Tau1,Tau2},
               PrecisionGoal->3,AccuracyGoal->6],
                    NIntegrate[Zed[w,b2,q2]*
                    W[w,u,b1,q1,Mu1,Mu2,T1]*
                    REL[w-u,b1,q1],{w,0,Tau1},{u,0,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
                    NIntegrate[Zed[w,b2,q2]*
                    W[w,u,b1,q1,Mu1,Mu2,T1]*REL[w-u,b1,q1],
```

```
                {w,Tau1,Tau2},{u,w-Tau1,w},
                PrecisionGoal->3,AccuracyGoal->6]+
                NIntegrate[Zed[w,b2,q2]*
                W[w,u,b1,q1,Mu1,Mu2,T1]*REL[w-u,b1,q1],
                {w,Tau2,Tau1+Tau2},{u,w-Tau1,Tau2},
                PrecisionGoal->3,AccuracyGoal->6]
     ],
   If[T2<=Tau1+Tau2,
     If[Tau1<T2,
          NIntegrate[Zed[w,b2,q2]*
          W[w,u,b1,q1,Mu1,Mu2,T1]*
          REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
          PrecisionGoal->3,AccuracyGoal->6]+
          NIntegrate[Zed[w,b2,q2]*
          W[w,u,b1,q1,Mu1,Mu2,T1]*REL[w-u,b1,q1],
          {w,Tau2,Tau1},{u,0.0,Tau2},
          PrecisionGoal->3,AccuracyGoal->6]+
          NIntegrate[Zed[w,b2,q2]*
          W[w,u,b1,q1,Mu1,Mu2,T1]*REL[w-u,b1,q1],
          {w,Tau1,T2},{u,w-Tau1,Tau2},
          PrecisionGoal->3,AccuracyGoal->6],
           NIntegrate[Zed[w,b2,q2]*
               W[w,u,b1,q1,Mu1,Mu2,T1]*
               REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
          NIntegrate[Zed[w,b2,q2]*
               W[w,u,b1,q1,Mu1,Mu2,T1]*
               REL[w-u,b1,q1],{w,Tau2,T2},{u,0,Tau2},
               PrecisionGoal->3,AccuracyGoal->6]
               ],
               NIntegrate[Zed[w,b2,q2]*
               W[w,u,b1,q1,Mu1,Mu2,T1]*
               REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*
               W[w,u,b1,q1,Mu1,Mu2,T1]*REL[w-u,b1,q1],
               {w,Tau2,Tau1},{u,0.0,Tau2},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*
               W[w,u,b1,q1,Mu1,Mu2,T1]*REL[w-u,b1,q1],
               {w,Tau1,Tau1+Tau2},{u,w-Tau1,Tau2},
               PrecisionGoal->3,AccuracyGoal->6]
      ]
],
If[Tau2<Tau1,
     If[T1<=Tau1+Tau2,
          NIntegrate[Zed[w,b1,q1]*
          W[w,u,b2,q2,Mu2,Mu1,T2]*
          REL[w-u,b2,q2],{w,0,Tau2},{u,0,w},
          PrecisionGoal->3,AccuracyGoal->6]+
          NIntegrate[Zed[w,b1,q1]*
          W[w,u,b2,q2,Mu2,Mu1,T2]*REL[w-u,b2,q2],
          {w,Tau2,Tau1},{u,w-Tau2,w},
           PrecisionGoal->3,AccuracyGoal->6]+
           NIntegrate[Zed[w,b1,q1]*
          W[w,u,b2,q2,Mu2,Mu1,T2]*REL[w-u,b2,q2],
```

```
            {w,Tau1,T1},{u,w-Tau2,Tau1},
            PrecisionGoal->3,AccuracyGoal->6],
                 NIntegrate[Zed[w,b1,q1]*
                 W[w,u,b2,q2,Mu2,Mu1,T2]*
                 REL[w-u,b2,q2],{w,0,Tau2},{u,0,w},
                 PrecisionGoal->3,AccuracyGoal->6]+
                 NIntegrate[Zed[w,b1,q1]*
                 W[w,u,b2,q2,Mu2,Mu1,T2]*REL[w-u,b2,q2],
                  {w,Tau2,Tau1},{u,w-Tau2,w},
                 PrecisionGoal->3,AccuracyGoal->6]+
                 NIntegrate[Zed[w,b1,q1]*
                 W[w,u,b2,q2,Mu2,Mu1,T2]*REL[w-u,b2,q2],
                 {w,Tau1,Tau1+Tau2},{u,w-Tau2,Tau1},
                 PrecisionGoal->3,AccuracyGoal->6]
            ],
    If[T1<=Tau1+Tau2,
     If[Tau2<T1,
       NIntegrate[Zed[w,b1,q1]*
       W[w,u,b2,q2,Mu2,Mu1,T2]*
       REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
       PrecisionGoal->3,AccuracyGoal->6]+
       NIntegrate[Zed[w,b1,q1]*
       W[w,u,b2,q2,Mu2,Mu1,T2]*REL[w-u,b2,q2],
       {w,Tau1,Tau2},{u,0.0,Tau1},
       PrecisionGoal->3,AccuracyGoal->6]+
       NIntegrate[Zed[w,b1,q1]*
       W[w,u,b2,q2,Mu2,Mu1,T2]*REL[w-u,b2,q2],
       {w,Tau2,T1},{u,w-Tau2,Tau1},
       PrecisionGoal->3,AccuracyGoal->6],
        NIntegrate[Zed[w,b1,q1]*
                 W[w,u,b2,q2,Mu2,Mu1,T2]*
                 REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
                 PrecisionGoal->3,AccuracyGoal->6]+
            NIntegrate[Zed[w,b1,q1]*
                 W[w,u,b2,q2,Mu2,Mu1,T2]*
                 REL[w-u,b1,q1],{w,Tau1,T1},{u,0,Tau1},
                 PrecisionGoal->3,AccuracyGoal->6]
                 ],
            NIntegrate[Zed[w,b1,q1]*
            W[w,u,b2,q2,Mu2,Mu1,T2]*
            REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
            PrecisionGoal->3,AccuracyGoal->6]+
            NIntegrate[Zed[w,b1,q1]*
            W[w,u,b2,q2,Mu2,Mu1,T2]*REL[w-u,b2,q2],
            {w,Tau1,Tau2},{u,0.0,Tau1},
            PrecisionGoal->3,AccuracyGoal->6]+
            NIntegrate[Zed[w,b1,q1]*
            W[w,u,b2,q2,Mu2,Mu1,T2]*REL[w-u,b2,q2],
            {w,Tau2,Tau1+Tau2},{u,w-Tau2,Tau1},
            PrecisionGoal->3,AccuracyGoal->6]
       ]
    ]
  ]
];
B4:=Function[{Tau1,Tau2,T1,T2},
         (Jacb[Tau1,Tau2,T1,T2])
```

```
        ];

B2:=Function[{Tau1,Tau2,T1,T2},
    If[T2<=T1,
        If[T2>=Tau1,
            If[T2<=Tau1+Tau2,
                NIntegrate[REL[T2,b2,q2]*
            W[T2,u,b1,q1,Mu1,Mu2,T1]*REL[T2-u,b1,q1],
                {u,T2-Tau1,Tau2},
            PrecisionGoal->3,AccuracyGoal->6],0],
            NIntegrate[REL[T2,b2,q2]*
        W[T2,u,b1,q1,Mu1,Mu2,T1]*REL[T2-u,b1,q1],{u,0,Tau2},
        PrecisionGoal->3,AccuracyGoal->6]
            ],
        If[T1>=Tau2,
            If[T1<=Tau1+Tau2,
                NIntegrate[REL[T1,b1,q1]*
                W[T1,u,b2,q2,Mu2,Mu1,T2]*REL[T1-u,b2,q2],
                {u,T1-Tau2,Tau1},
                PrecisionGoal->3,AccuracyGoal->6],0],
            NIntegrate[REL[T1,b1,q1]*
        W[T1,u,b2,q2,Mu2,Mu1,T2]*REL[T1-u,b2,q2],{u,0,Tau1},
        PrecisionGoal->3,AccuracyGoal->6]
        ]
    ]
];
BM2:=Function[{Tau1,Tau2,T1,T2},
    If[T2<=T1,
        If[T2>=Tau1,
            If[T2<=Tau1+Tau2,
        NIntegrate[REL[T2,b2,q2]*T2*REL[T2-u,b1,q1],
        {u,T2-Tau1,Tau2},
    PrecisionGoal->3,AccuracyGoal->6]/mean[b1,q1,T1],0],
    NIntegrate[REL[T2,b2,q2]*T2*REL[T2-u,b1,q1],
    {u,0,Tau2},
    PrecisionGoal->3,AccuracyGoal->6]/mean[b1,q1,T1]
            ],
        If[T1>=Tau2,
        If[T1<=Tau1+Tau2,
            NIntegrate[REL[T1,b1,q1]*T1*REL[T1-u,b2,q2],
        {u,T1-Tau2,Tau1},
    PrecisionGoal->3,AccuracyGoal->6]/mean[b2,q2,T2],0],
NIntegrate[REL[T1,b1,q1]*T1*REL[T1-u,b2,q2],{u,0,Tau1},
    PrecisionGoal->3,AccuracyGoal->6]/mean[b2,q2,T2]
        ]
    ]
];

BF2:=Function[{Tau1,Tau2,T1,T2},
    If[T2<=T1,
        If[T2>=Tau1,
            If[T2<=Tau1+Tau2,
                NIntegrate[REL[T2,b2,q2]*REL[T2-u,b1,q1],
                {u,T2-Tau1,Tau2},
        PrecisionGoal->3,AccuracyGoal->6]/mean[b1,q1,T1],0],
```

```
          NIntegrate[REL[T2,b2,q2]*REL[T2-u,b1,q1],
          {u,0,Tau2},
          PrecisionGoal->3,AccuracyGoal->6]/mean[b1,q1,T1]
               ],
          If[T1>=Tau2,
               If[T1<=Tau1+Tau2,
                    NIntegrate[REL[T1,b1,q1]*REL[T1-u,b2,q2],
               {u,T1-Tau2,Tau1},
          PrecisionGoal->3,AccuracyGoal->6]/mean[b2,q2,T2],0],
          NIntegrate[REL[T1,b1,q1]*REL[T1-u,b2,q2],{u,0,Tau1},
          PrecisionGoal->3,AccuracyGoal->6]/mean[b2,q2,T2]
          ]
     ]
];
Jac:=Function[{Tau1,Tau2,T1,T2},
If[T2<=T1,
  If[Tau1<Tau2,
          If[T2<=Tau1+Tau2,
               NIntegrate[Zed[w,b2,q2]*(w/mean[b1,q1,T1])*
               REL[w-u,b1,q1],{w,0,Tau1},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*
               (w/mean[b1,q1,T1])*REL[w-u,b1,q1],
               {w,Tau1,Tau2},{u,w-Tau1,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*
               (w/mean[b1,q1,T1])*REL[w-u,b1,q1],
               {w,Tau2,T2},{u,w-Tau1,Tau2},
               PrecisionGoal->3,AccuracyGoal->6],
               NIntegrate[Zed[w,b2,q2]*(w/mean[b1,q1,T1])*
               REL[w-u,b1,q1],{w,0,Tau1},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
                    NIntegrate[Zed[w,b2,q2]*
                    (w/mean[b1,q1,T1])*REL[w-u,b1,q1],
                    {w,Tau1,Tau2},{u,w-Tau1,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
                    NIntegrate[Zed[w,b2,q2]*
                    (w/mean[b1,q1,T1])*REL[w-u,b1,q1],
                    {w,Tau2,Tau1+Tau2},{u,w-Tau1,Tau2},
                    PrecisionGoal->3,AccuracyGoal->6]
       ],
       If[T2<=Tau1+Tau2,
          If[Tau1<T2,
               NIntegrate[Zed[w,b2,q2]*
               (w/mean[b1,q1,T1])*
               REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*
               (w/mean[b1,q1,T1])*REL[w-u,b1,q1],
               {w,Tau2,Tau1},{u,0.0,Tau2},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*
               (w/mean[b1,q1,T1])*REL[w-u,b1,q1],
               {w,Tau1,T2},{u,w-Tau1,Tau2},
               PrecisionGoal->3,AccuracyGoal->6],
                 NIntegrate[Zed[w,b2,q2]*
```

```
                        (w/mean[b1,q1,T1])*
                        REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
                        PrecisionGoal->3,AccuracyGoal->6]+
                    NIntegrate[Zed[w,b2,q2]*
                        (w/mean[b1,q1,T1])*
                        REL[w-u,b1,q1],{w,Tau2,T2},{u,0,Tau2},
                        PrecisionGoal->3,AccuracyGoal->6]
                        ],
                        NIntegrate[Zed[w,b2,q2]*
                        (w/mean[b1,q1,T1])*
                        REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
                        PrecisionGoal->3,AccuracyGoal->6]+
                        NIntegrate[Zed[w,b2,q2]*
                        (w/mean[b1,q1,T1])*REL[w-u,b1,q1],
                        {w,Tau2,Tau1},{u,0.0,Tau2},
                        PrecisionGoal->3,AccuracyGoal->6]+
                        NIntegrate[Zed[w,b2,q2]*
                        (w/mean[b1,q1,T1])*REL[w-u,b1,q1],
                        {w,Tau1,Tau1+Tau2},{u,w-Tau1,Tau2},
                        PrecisionGoal->3,AccuracyGoal->6]
      ]
    ],
If[Tau2<Tau1,
    If[T1<=Tau1+Tau2,
                NIntegrate[Zed[w,b1,q1]*
                (w/mean[b2,q2,T2])*
                REL[w-u,b2,q2],{w,0,Tau2},{u,0,w},
                PrecisionGoal->3,AccuracyGoal->6]+
                NIntegrate[Zed[w,b1,q1]*
                (w/mean[b2,q2,T2])*REL[w-u,b2,q2],
                {w,Tau2,Tau1},{u,w-Tau2,w},
                PrecisionGoal->3,AccuracyGoal->6]+
                NIntegrate[Zed[w,b1,q1]*
                (w/mean[b2,q2,T2])*REL[w-u,b2,q2],
                {w,Tau1,T1},{u,w-Tau2,Tau1},
                PrecisionGoal->3,AccuracyGoal->6],
                    NIntegrate[Zed[w,b1,q1]*
                    (w/mean[b2,q2,T2])*
                    REL[w-u,b2,q2],{w,0,Tau2},{u,0,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
                    NIntegrate[Zed[w,b1,q1]*
                    (w/mean[b2,q2,T2])*REL[w-u,b2,q2],
                    {w,Tau2,Tau1},{u,w-Tau2,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
                    NIntegrate[Zed[w,b1,q1]*
                    (w/mean[b2,q2,T2])*REL[w-u,b2,q2],
                    {w,Tau1,Tau1+Tau2},{u,w-Tau2,Tau1},
                    PrecisionGoal->3,AccuracyGoal->6]
      ],
      If[T1<=Tau1+Tau2,
         If[Tau2<T1,
              NIntegrate[Zed[w,b1,q1]*
              (w/mean[b2,q2,T2])*
              REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
              PrecisionGoal->3,AccuracyGoal->6]+
              NIntegrate[Zed[w,b1,q1]*
```

```
              (w/mean[b2,q2,T2])*REL[w-u,b2,q2],
              {w,Tau1,Tau2},{u,0.0,Tau1},
              PrecisionGoal->3,AccuracyGoal->6]+
              NIntegrate[Zed[w,b1,q1]*
              (w/mean[b2,q2,T2])*REL[w-u,b2,q2],
              {w,Tau2,T1},{u,w-Tau2,Tau1},
              PrecisionGoal->3,AccuracyGoal->6],
                NIntegrate[Zed[w,b1,q1]*
                   (w/mean[b2,q2,T2])*
                   REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
                   PrecisionGoal->3,AccuracyGoal->6]+
                NIntegrate[Zed[w,b1,q1]*
                   (w/mean[b2,q2,T2])*
                   REL[w-u,b2,q2],{w,Tau1,T1},{u,0,Tau1},
                   PrecisionGoal->3,AccuracyGoal->6]
                   ],
                   NIntegrate[Zed[w,b1,q1]*
                   (w/mean[b2,q2,T2])*
                   REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
                   PrecisionGoal->3,AccuracyGoal->6]+
                   NIntegrate[Zed[w,b1,q1]*
                   (w/mean[b2,q2,T2])*REL[w-u,b2,q2],
                   {w,Tau1,Tau2},{u,0.0,Tau1},
                   PrecisionGoal->3,AccuracyGoal->6]+
                   NIntegrate[Zed[w,b1,q1]*
                   (w/mean[b2,q2,T2])*REL[w-u,b2,q2],
                   {w,Tau2,Tau1+Tau2},{u,w-Tau2,Tau1},
                   PrecisionGoal->3,AccuracyGoal->6]
      ]
    ]
  ]
];

BM4:=Function[{Tau1,Tau2,T1,T2},
          Jac[Tau1,Tau2,T1,T2]
          ];

LD:=Function[{Tau1,Tau2,T1,T2},
  If[T2<=T1,
     If[Tau1<Tau2,
          If[T2<=Tau1+Tau2,
               NIntegrate[Zed[w,b2,q2]*
     REL[w-u,b1,q1]/mean[b1,q1,T1],{w,0,Tau1},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*
               REL[w-u,b1,q1]/mean[b1,q1,T1],
               {w,Tau1,Tau2},{u,w-Tau1,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*
               REL[w-u,b1,q1]/mean[b1,q1,T1],
               {w,Tau2,T2},{u,w-Tau1,Tau2},
               PrecisionGoal->3,AccuracyGoal->6],
                    NIntegrate[Zed[w,b2,q2]*
                    REL[w-u,b1,q1]/mean[b1,q1,T1],
                    {w,0,Tau1},{u,0,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
```

```
                    NIntegrate[Zed[w,b2,q2]*
                    REL[w-u,b1,q1]/mean[b1,q1,T1],
                    {w,Tau1,Tau2},{u,w-Tau1,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
                    NIntegrate[Zed[w,b2,q2]*
                    REL[w-u,b1,q1]/mean[b1,q1,T1],
                    {w,Tau2,Tau1+Tau2},{u,w-Tau1,Tau2},
                    PrecisionGoal->3,AccuracyGoal->6]
        ],
        If[T2<=Tau1+Tau2,
           If[Tau1<T2,
                NIntegrate[Zed[w,b2,q2]*
                REL[w-u,b1,q1]/mean[b1,q1,T1],
                {w,0,Tau2},{u,0,w},
                PrecisionGoal->3,AccuracyGoal->6]+
                NIntegrate[Zed[w,b2,q2]*
                REL[w-u,b1,q1]/mean[b1,q1,T1],
                {w,Tau2,Tau1},{u,0.0,Tau2},
                PrecisionGoal->3,AccuracyGoal->6]+
                NIntegrate[Zed[w,b2,q2]*
                REL[w-u,b1,q1]/mean[b1,q1,T1],
                {w,Tau1,T2},{u,w-Tau1,Tau2},
                PrecisionGoal->3,AccuracyGoal->6],
                 NIntegrate[Zed[w,b2,q2]*
                     (1/mean[b1,q1,T1])*
                     REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
                     PrecisionGoal->3,AccuracyGoal->6]+
                  NIntegrate[Zed[w,b2,q2]*
                     (1/mean[b1,q1,T1])*
                     REL[w-u,b1,q1],{w,Tau2,T2},{u,0,Tau2},
                     PrecisionGoal->3,AccuracyGoal->6]
                     ],
                     NIntegrate[Zed[w,b2,q2]*
                     REL[w-u,b1,q1]/mean[b1,q1,T1],
                     {w,0,Tau2},{u,0,w},
                     PrecisionGoal->3,AccuracyGoal->6]+
                     NIntegrate[Zed[w,b2,q2]*
                     REL[w-u,b1,q1]/mean[b1,q1,T1],
                     {w,Tau2,Tau1},{u,0.0,Tau2},
                     PrecisionGoal->3,AccuracyGoal->6]+
                     NIntegrate[Zed[w,b2,q2]*
                     REL[w-u,b1,q1]/mean[b1,q1,T1],
                     {w,Tau1,Tau1+Tau2},{u,w-Tau1,Tau2},
                     PrecisionGoal->3,AccuracyGoal->6]
        ]
      ],
  If[Tau2<Tau1,
          If[T1<=Tau1+Tau2,
               NIntegrate[Zed[w,b1,q1]*
               REL[w-u,b2,q2]/mean[b2,q2,T2],
               {w,0,Tau2},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b1,q1]*
               REL[w-u,b2,q2]/mean[b2,q2,T2],
               {w,Tau2,Tau1},{u,w-Tau2,w},
               PrecisionGoal->3,AccuracyGoal->6]+
```

```
            NIntegrate[Zed[w,b1,q1]*
            REL[w-u,b2,q2]/mean[b2,q2,T2],
            {w,Tau1,T1},{u,w-Tau2,Tau1},
            PrecisionGoal->3,AccuracyGoal->6],
                 NIntegrate[Zed[w,b1,q1]*
                 REL[w-u,b2,q2]/mean[b2,q2,T2],
                 {w,0,Tau2},{u,0,w},
                 PrecisionGoal->3,AccuracyGoal->6]+
                 NIntegrate[Zed[w,b1,q1]*
                 REL[w-u,b2,q2]/mean[b2,q2,T2],
                 {w,Tau2,Tau1},{u,w-Tau2,w},
                 PrecisionGoal->3,AccuracyGoal->6]+
                 NIntegrate[Zed[w,b1,q1]*
                 REL[w-u,b2,q2]/mean[b2,q2,T2],
                 {w,Tau1,Tau1+Tau2},{u,w-Tau2,Tau1},
                 PrecisionGoal->3,AccuracyGoal->6]
    ],
    If[T1<=Tau1+Tau2,
       If[Tau2<T1,
            NIntegrate[Zed[w,b1,q1]*
  REL[w-u,b2,q2]/mean[b2,q2,T2],{w,0,Tau1},{u,0,w},
            PrecisionGoal->3,AccuracyGoal->6]+
            NIntegrate[Zed[w,b1,q1]*
  REL[w-u,b2,q2]/mean[b2,q2,T2],
            {w,Tau1,Tau2},{u,0.0,Tau1},
            PrecisionGoal->3,AccuracyGoal->6]+
            NIntegrate[Zed[w,b1,q1]*
  REL[w-u,b2,q2]/mean[b2,q2,T2],
            {w,Tau2,T1},{u,w-Tau2,Tau1},
            PrecisionGoal->3,AccuracyGoal->6],
              NIntegrate[Zed[w,b1,q1]*
                 (1/mean[b2,q2,T2])*
                 REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
                 PrecisionGoal->3,AccuracyGoal->6]+
              NIntegrate[Zed[w,b1,q1]*
                 (1/mean[b2,q2,T2])*
                 REL[w-u,b1,q1],{w,Tau1,T1},{u,0,Tau1},
                 PrecisionGoal->3,AccuracyGoal->6]
                 ],
                 NIntegrate[Zed[w,b1,q1]*
                 REL[w-u,b2,q2]/mean[b2,q2,T2],
                 {w,0,Tau1},{u,0,w},
                 PrecisionGoal->3,AccuracyGoal->6]+
                 NIntegrate[Zed[w,b1,q1]*
                 REL[w-u,b2,q2]/mean[b2,q2,T2],
                 {w,Tau1,Tau2},{u,0.0,Tau1},
                 PrecisionGoal->3,AccuracyGoal->6]+
                 NIntegrate[Zed[w,b1,q1]*
                 REL[w-u,b2,q2]/mean[b2,q2,T2],
                 {w,Tau2,Tau1+Tau2},{u,w-Tau2,Tau1},
                 PrecisionGoal->3,AccuracyGoal->6]
    ]
  ]
 ]
];
```

```
BF4:=Function[{Tau1,Tau2,T1,T2},
          LD[Tau1,Tau2,T1,T2]
          ];
```

Inintial Renewal Interval

```
BG1:=Function[{Tau1,Tau2,T1,T2},
If[T1>=T2,
     If[T2>=Tau1,
NIntegrate[Zed[w,b2,q2] REL[w,b1,q1]*(Mu1+Mu2+w),
{w,Tau1,T2},PrecisionGoal->3,AccuracyGoal->6]+
REL[T2,b2,q2] REL[T2,b1,q1]*(Mu1+Mu2+T2),
0],
     If[T1>=Tau2,
NIntegrate[Zed[w,b1,q1] REL[w,b2,q2]*(Mu1+Mu2+w),
{w,Tau2,T1},PrecisionGoal->3,AccuracyGoal->6]+
REL[T1,b1,q1] REL[T1,b2,q2]*(Mu1+Mu2+T1),0]
     ]
];

BG2:=Function[{Tau1,Tau2,T1,T2},
If[T1>=T2,
     If[T2>=Tau1+Tau2,
          NIntegrate[Zed[w,b2,q2]*WKK[w,u,b1,q1,Mu1,Mu2]*
          REL[w-u,b1,q1],{w,Tau1,Tau1+Tau2},{u,0,w-Tau1},
          PrecisionGoal->3,AccuracyGoal->6]+
          NIntegrate[Zed[w,b2,q2]*WKK[w,u,b1,q1,Mu1,Mu2]*
          REL[w-u,b1,q1],{w,Tau1+Tau2,T2},{u,0,Tau2},
          PrecisionGoal->3,AccuracyGoal->6],
          If[T2>=Tau1,
          NIntegrate[Zed[w,b2,q2]*WKK[w,u,b1,q1,Mu1,Mu2]*
          REL[w-u,b1,q1],{w,Tau1,T2},{u,0,Tau2},
          PrecisionGoal->3,AccuracyGoal->6],0]
     ],
     If[T1>=Tau1+Tau2,
          NIntegrate[Zed[w,b1,q1]*WKK[w,u,b2,q2,Mu2,Mu1]*
          REL[w-u,b2,q2],{w,Tau2,Tau1+Tau2},{u,0,w-Tau2},
          PrecisionGoal->3,AccuracyGoal->6]+
          NIntegrate[Zed[w,b1,q1]*WKK[w,u,b2,q2,Mu2,Mu1]*
          REL[w-u,b2,q2],{w,Tau2+Tau1,T1},{u,0,Tau1},
          PrecisionGoal->3,AccuracyGoal->6],
          If[T1>=Tau2,
          NIntegrate[Zed[w,b1,q1]*WKK[w,u,b2,q2,Mu2,Mu1]*
          REL[w-u,b2,q2],{w,Tau2,T1},{u,0,Tau1},
          PrecisionGoal->3,AccuracyGoal->6],0]
          ]
     ]
];

BG3:=Function[{Tau1,Tau2,T1,T2},
     If[T1>=T2,
          If[T2>=Tau1+Tau2,
          NIntegrate[REL[T2,b2,q2]*WKK[T2,u,b1,q1,Mu1,Mu2]*
          REL[T2-u,b1,q1],{u,0,Tau2},
          PrecisionGoal->3,AccuracyGoal->6],0]+
          If[T2<Tau1+Tau2 && T2>Tau1,
          NIntegrate[REL[T2,b2,q2]*WKK[T2,u,b1,q1,Mu1,Mu2]*
          REL[T2-u,b1,q1],{u,0,T2-Tau1},
          PrecisionGoal->3,AccuracyGoal->6],0],
```

```
If[T1>=Tau1+Tau2,
          NIntegrate[REL[T1,b1,q1]*WKK[T1,u,b2,q2,Mu2,Mu1]*
          REL[T1-u,b2,q2],{u,0,Tau1},
          PrecisionGoal->3,AccuracyGoal->6],0]+
     If[T1<Tau1+Tau2 && T1>Tau2,
          NIntegrate[REL[T1,b1,q1]*WKK[T1,u,b2,q2,Mu2,Mu1]*
          REL[T1-u,b2,q2],{u,0,T1-Tau2},
          PrecisionGoal->3,AccuracyGoal->6],0]
     ]
];


BG4:=Function[{Tau1,Tau2,T1,T2},
If[T1>=T2,
     NIntegrate[REL[u,b2,q2] Zed[u,b1,q1] (Mu1+Mu2+u),
     {u,Tau2,T2}]+ NIntegrate[REL[u,b2,q2]*
     f[u,v,b1,q1,Mu1,Mu2]*
     Zed[u-v,b1,q1],{u,Tau2,T2},{v,0,Tau2}],
     NIntegrate[REL[u,b1,q1] Zed[u,b2,q2] (Mu1+Mu2+u),
     {u,Tau1,T1}]+ NIntegrate[REL[u,b1,q1]*
     f[u,v,b2,q2,Mu2,Mu1]*
     Zed[u-v,b2,q2],{u,Tau1,T1},{v,0,Tau1}]
     ]
];
MRENINIT[Tau1_,Tau2_,T1_,T2_]:=MRENINIT[Tau1,Tau2,T1,T2]=
BG1[Tau1,Tau2,T1,T2]+BG2[Tau1,Tau2,T1,T2]
+BG3[Tau1,Tau2,T1,T2]+BG4[Tau1,Tau2,T1,T2];
```

General Renewal Interval

```
MG1:=Function[{Tau1,Tau2,T1,T2},
If[T1>=T2,
If[T2>=Max[(Tau1-PU[q1,b1]),0],
NIntegrate[Zed[w,b2,q2] RESDL[w,b1,q1,T1]*(Mu1+Mu2+w),
{w,Tau1,T2},PrecisionGoal->3,AccuracyGoal->6]+
REL[T2,b2,q2] RESDL[T2,b1,q1,T1]*(Mu1+Mu2+T2),0],
If[T1>=Max[(Tau2-PU[q2,b2]),0],
NIntegrate[Zed[w,b1,q1] RESDL[w,b2,q2,T2]*(Mu1+Mu2+w),
{w,Tau2,T1},PrecisionGoal->3,AccuracyGoal->6]+
REL[T1,b1,q1] RESDL[T1,b2,q2,T2]*(Mu1+Mu2+T1),0]
      ]
];


MG2:=Function[{Tau1,Tau2,T1,T2},
If[T1>=T2,
     If[T2>=Tau1+Tau2,
          NIntegrate[Zed[w,b2,q2]*WZ[w,u,b1,q1,Mu1,Mu2,T1]*
          REL[w-u,b1,q1],{w,Tau1,Tau1+Tau2},{u,0,w-Tau1},
          PrecisionGoal->3,AccuracyGoal->6]+
          NIntegrate[Zed[w,b2,q2]*WZ[w,u,b1,q1,Mu1,Mu2,T1]*
          REL[w-u,b1,q1],{w,Tau1+Tau2,T2},{u,0,Tau2},
          PrecisionGoal->3,AccuracyGoal->6],
          If[T2>=Tau1,
          NIntegrate[Zed[w,b2,q2]*WZ[w,u,b1,q1,Mu1,Mu2,T1]*
          REL[w-u,b1,q1],{w,Tau1,T2},{u,0,Tau2},
          PrecisionGoal->3,AccuracyGoal->6],0]
     ],
     If[T1>=Tau1+Tau2,
          NIntegrate[Zed[w,b1,q1]*WZ[w,u,b2,q2,Mu2,Mu1,T2]*
          REL[w-u,b2,q2],{w,Tau2,Tau1+Tau2},{u,0,w-Tau2},
          PrecisionGoal->3,AccuracyGoal->6]+
          NIntegrate[Zed[w,b1,q1]*WZ[w,u,b2,q2,Mu2,Mu1,T2]*
          REL[w-u,b2,q2],{w,Tau2+Tau1,T1},{u,0,Tau1},
          PrecisionGoal->3,AccuracyGoal->6],
          If[T1>=Tau2,
          NIntegrate[Zed[w,b1,q1]*WZ[w,u,b2,q2,Mu2,Mu1,T2]*
          REL[w-u,b2,q2],{w,Tau2,T1},{u,0,Tau1},
          PrecisionGoal->3,AccuracyGoal->6],0]
          ]
     ]
];

MG3:=Function[{Tau1,Tau2,T1,T2},
     If[T1>=T2,
          If[T2>=Tau1+Tau2,
          NIntegrate[REL[T2,b2,q2]*WZ[T2,u,b1,q1,Mu1,Mu2,T1]*
          REL[T2-u,b1,q1],{u,0,Tau2},
          PrecisionGoal->3,AccuracyGoal->6],0]+
          If[T2<Tau1+Tau2 && T2>Tau1,
          NIntegrate[REL[T2,b2,q2]*WZ[T2,u,b1,q1,Mu1,Mu2,T1]*
          REL[T2-u,b1,q1],{u,0,T2-Tau1},
```

```
          PrecisionGoal->3,AccuracyGoal->6],0],
If[T1>=Tau1+Tau2,
          NIntegrate[REL[T1,b1,q1]*WZ[T1,u,b2,q2,Mu2,Mu1,T2]*
          REL[T1-u,b2,q2],{u,0,Tau1},
          PrecisionGoal->3,AccuracyGoal->6],0]+
     If[T1<Tau1+Tau2 && T1>Tau2,
          NIntegrate[REL[T1,b1,q1]*WZ[T1,u,b2,q2,Mu2,Mu1,T2]*
          REL[T1-u,b2,q2],{u,0,T1-Tau2},
          PrecisionGoal->3,AccuracyGoal->6],0]
     ]
];
MG4:=Function[{Tau1,Tau2,T1,T2},
If[T1>=T2,
     NIntegrate[REL[u,b2,q2] Y[u,b1,q1,T1] (Mu1+Mu2+u),
     {u,Tau2,T2}]+ NIntegrate[REL[u,b2,q2]*
     W[u,v,b1,q1,Mu1,Mu2,T1]*
     Zed[u-v,b1,q1],{u,Tau2,T2},{v,0,Tau2}],
     NIntegrate[REL[u,b1,q1] Y[u,b2,q2,T2] (Mu1+Mu2+u),
     {u,Tau1,T1}]+ NIntegrate[REL[u,b1,q1]*
     W[u,v,b2,q2,Mu2,Mu1,T2]*
     Zed[u-v,b2,q2],{u,Tau1,T1},{v,0,Tau1}]
     ]
];
MRENGEN[Tau1_,Tau2_,T1_,T2_]:=MRENGEN[Tau1,Tau2,T1,T2]=
MG1[Tau1,Tau2,T1,T2]+MG2[Tau1,Tau2,T1,T2]+
MG3[Tau1,Tau2,T1,T2]+MG4[Tau1,Tau2,T1,T2];
```

EXPECTED TIME IN AN INITIAL INTERVAL

```
(*PHI(General)                                          *)

PHIInitial[Tau1_,Tau2_,T1_,T2_]:=
PHIInitial[Tau1,Tau2,T1,T2]=A4[Tau1,Tau2,T1,T2]+
A3[Tau1,Tau2,T1,T2]+A1[Tau1,Tau2,T1,T2]+
A2[Tau1,Tau2,T1,T2];
```

EXPECTED FAILURE TIME IN AN INITIAL INTERVAL

```
(*PHI(General)                                          *)

ReInitial[Tau1_,Tau2_,T1_,T2_]:=
ReInitial[Tau1,Tau2,T1,T2]=AM4[Tau1,Tau2,T1,T2]+
AM3[Tau1,Tau2,T1,T2]+AM1[Tau1,Tau2,T1,T2]+
AM2[Tau1,Tau2,T1,T2];
```

EXPECTED FAILURE TIME IN A GENERAL INTERVAL

```
(*PHI(General)                                          *)

ReGeneral[Tau1_,Tau2_,T1_,T2_]:=
ReGeneral[Tau1,Tau2,T1,T2]=BM4[Tau1,Tau2,T1,T2]+
BI3[Tau1,Tau2,T1,T2]+BM2[Tau1,Tau2,T1,T2]+
BI1[Tau1,Tau2,T1,T2];
```

PROBABILITY OF AN INITIAL INTERVAL

```
(*PHI(General)                                        *)

PHIIPROB[Tau1_,Tau2_,T1_,T2_]:=
PHIIPROB[Tau1,Tau2,T1,T2]=AP4[Tau1,Tau2,T1,T2]+
AP3[Tau1,Tau2,T1,T2]+AP2[Tau1,Tau2,T1,T2]+
AP1[Tau1,Tau2,T1,T2];
```

EXPECTED TIME IN A GENERAL INTERVAL

```
(*PHI general                                              *)
PHIGENERAL[Tau1_,Tau2_,T1_,T2_]:=
PHIGENERAL[Tau1,Tau2,T1,T2]=B4[Tau1,Tau2,T1,T2]+
B3[Tau1,Tau2,T1,T2]+B1[Tau1,Tau2,T1,T2]+
B2[Tau1,Tau2,T1,T2];
```

PROBABILITY OF A GENERAL INTERVAL

```
PHIJJPROB[Tau1_,Tau2_,T1_,T2_]:=
PHIJJPROB[Tau1,Tau2,T1,T2]=BF4[Tau1,Tau2,T1,T2]+
BJ3[Tau1,Tau2,T1,T2]+BF2[Tau1,Tau2,T1,T2]+
BJ1[Tau1,Tau2,T1,T2];
```

AVAILABILITY

```
Avail[Tau1_,Tau2_,T1_,T2_]:=
(((1-PHIJJPROB[Tau1,Tau2,T1,T2])*
(ReInitial[Tau1,Tau2,T1,T2]))+
(PHIIPROB[Tau1,Tau2,T1,T2]*ReGeneral[Tau1,Tau2,T1,T2]))/
(((1-PHIJJPROB[Tau1,Tau2,T1,T2])*
(MRENINIT[Tau1,Tau2,T1,T2]+PHIInitial[Tau1,Tau2,T1,T2])+
(PHIGENERAL[Tau1,Tau2,T1,T2]+MRENGEN[Tau1,Tau2,T1,T2])*
PHIIPROB[Tau1,Tau2,T1,T2]));
```

Time Dependent Availability

Gamma Failure & Renewals

```
(*Off[NIntegrate::slwcon];
g[s_,ld_]:=ld/(ld+s);
Zed[t_,b_,a_]:=(a^b) t^(b-1) Exp[-a t]/Gamma[b];
f[t_,b_,a_,s_]:=g[s,ld2] Exp[-a t] Sum[
(a^(b n)) t^(b n-1) (g[s,ld1]^n)/Gamma[b n],{n,1,12}];
f1[t_,b_,a_,s_]:=g[s,ld1] Exp[-a t] Sum[
(a^(b n)) t^(b n-1) (g[s,ld2]^n)/Gamma[b n],{n,1,12}];
f2[t_,b_,a_,s_]:=f[t,b,a,s] g[s,ld1];
f3[t_,b_,a_,s_]:=f1[t,b,a,s] g[s,ld2];
REL[t_,b_,a_]:=Sum[((a t)^i) Exp[-a t]/i!,{i,0,b-1}];
CDF[t_,b_,a_]:=CDF[t,b,a]=1-REL[t,b,a];
mean[T_,b_,a_]:=Sum[Gamma[1+i,0,a T]/i!, {i, 0, -1 + b}]/a;
L[t_,T_,b_,a_]:=If[t<=T,REL[t,b,a]/mean[T,b,a],0.0];
LM[t_,T_,b_,a_,s_]:=NIntegrate[L[x,T,b,a]*f[t-x,b,a,s],{x,0,t}];
LN[t_,T_,b_,a_,s_]:=NIntegrate[L[x,T,b,a]*f1[t-x,b,a,s],{x,0,t}];
W[t_,b_,a_,s_,T_]:=g[s,ld2]*L[t,T,b,a]+LM[t,T,b,a,s]*g[s,ld1];
W1[t_,b_,a_,s_,T_]:=g[s,ld1]*L[t,T,b,a]+LN[t,T,b,a,s]*g[s,ld2];
W2[t_,b_,a_,s_,T_]:=W[t,b,a,s,T] g[s,ld1];
W3[t_,b_,a_,s_,T_]:=W1[t,b,a,s,T] g[s,ld2];
RESDL[t_,T_,b_,a_]:=If[t<=T,1-mean[t,b,a]/mean[T,b,a],0.0];
gopp[s_]:=(ld1+ld2)/((ld1+ld2)+s);*)
```

Weibull Failure & Renewals

```
Off[NIntegrate::slwcon];
g[s_,ld_]:=ld/(ld+s);
Zed[t_,b_,a_]:=a t^(b-1) Exp[-a/b t^b];

Z[m_,b_]:=Gamma[b m + 1]/Gamma[m+1];
B[k_,s_,b_]:=If[
                    k==0,Z[s,b],
                Sum[B[k-1,r,b] Z[s-r,b],{r,k-1,s-1}]];
A[k_,s_,b_]:=
A[k,s,b]=Sum[(-1)^(p+k) Binomial[s,p] B[k,p,b]/Z[p,b],
{p,k,s}];
BBpha[k_,s_,b_]:=If[s==k,A[k,s,b],
          Sum[A[r,s,b],{r,k,s}]-Sum[A[r,s-1,b],{r,k,s-1}]];
Alpha[s_,m_,b_]:=Sum[BBpha[k,m,b]*(g[s,ld1]^k),
{k,1,m}];
GHAlpha[s_,m_,b_]:=Sum[BBpha[k,m,b]*(g[s,ld2]^k),
{k,1,m}]
f[t_,b_,a_,s_]:=g[s,ld2] Exp[-(a/b) t^b]*
Sum[((a/b)^(m)) Alpha[s,m,b] b t^(b m-1)/((m-1)!),
{m,1,10}];
f1[t_,b_,a_,s_]:=g[s,ld1] Exp[-(a/b) t^b]*
Sum[((a/b)^(m)) GHAlpha[s,m,b] b t^(b m-1)/((m-1)!),
{m,1,10}];
f2[t_,b_,a_,s_]:=f[t,b,a,s] g[s,ld1];
f3[t_,b_,a_,s_]:=f1[t,b,a,s] g[s,ld2];
REL[t_,b_,a_]:=Exp[-(a/b) t^b];
CDF[t_,b_,a_]:=CDF[t,b,a]=1-REL[t,b,a];
mean[T_,b_,a_]:=mean[T,b,a]=
NIntegrate[REL[t,b,a],{t,0,T}];
L[t_,T_,b_,a_]:=If[t<=T,REL[t,b,a]/mean[T,b,a],0.0];
LM[t_,T_,b_,a_,s_]:=
NIntegrate[L[x,T,b,a]*f[t-x,b,a,s],{x,0,t}];
LN[t_,T_,b_,a_,s_]:=
NIntegrate[L[x,T,b,a]*f1[t-x,b,a,s],{x,0,t}];
W[t_,b_,a_,s_,T_]:=
g[s,ld2]*L[t,T,b,a]+LM[t,T,b,a,s]*g[s,ld1];
W1[t_,b_,a_,s_,T_]:=
g[s,ld1]*L[t,T,b,a]+LN[t,T,b,a,s]*g[s,ld2];
W2[t_,b_,a_,s_,T_]:=W[t,b,a,s,T] g[s,ld1];
W3[t_,b_,a_,s_,T_]:=W1[t,b,a,s,T] g[s,ld2];
RESDL[t_,T_,b_,a_]:=
If[t<=T,1-mean[t,b,a]/mean[T,b,a],0.0];
gopp[s_]:=(ld1+ld2)/((ld1+ld2)+s);
```

Parameters

```
ld1=1.5;ld2=2.5;b1=2.0;b2=2.0;q1=0.5;q2=1.0;Mu1=1/ld1;
Mu2=1/ld2;ldp=2.0; MuP=1/(ldp);Opp:=1/(ld1+ld2);
var[a_,b_]:=b/(a^2);
vt[a_,b_]:=b/a;
PU[a_,b_]:=PU[a,b]=(vt[a,b]^2 + var[a,b]^2)/(2 vt[a,b]);

(*ld2=1.5;ld1=2.5;b1=2.0;b2=2.0;q2=0.5;q1=1.0;Mu1=1/ld1;
Mu2=1/ld2;ldp=2.0; MuP=1/(ldp);Opp:=1/(ld1+ld2);*)
```

Initial Interval

```
A3:=Function[{Tau1,Tau2,T1,T2,s},
If[T2<=T1,
     If[T2>=Tau1,
     NIntegrate[
Zed[w,b2,q2]*REL[w,b1,q1]*g[s,ld2]*Exp[-w s],{w,0,Tau1},
     PrecisionGoal->6,AccuracyGoal->6],
NIntegrate[
Zed[w,b2,q2]*REL[w,b1,q1]*g[s,ld2]*Exp[-w s],{w,0,T2},
          PrecisionGoal->6,AccuracyGoal->6]
          ],
     If[T1>=Tau2,
     NIntegrate[
Zed[w,b1,q1]*REL[w,b2,q2]*g[s,ld1]*Exp[-w s],{w,0,Tau2},
PrecisionGoal->6,AccuracyGoal->6],
     NIntegrate[
Zed[w,b1,q1]*REL[w,b2,q2]*g[s,ld1]*Exp[-w s],{w,0,T1},
     PrecisionGoal->6,AccuracyGoal->6]
          ]
     ]
];
A1:=Function[{Tau1,Tau2,T1,T2,s},
If[T2<=T1,
     If[T2<Tau1,
     REL[T2,b1,q1]*REL[T2,b2,q2]*(g[s,ld2]*Exp[-T2 s]),0],
     If[T1<Tau2,
     REL[T1,b2,q2]*REL[T1,b1,q1]*(g[s,ld1]*Exp[-T1 s]),0]
     ]
];
AM1:=Function[{Tau1,Tau2,T1,T2,s},
If[T2<=T1,
     If[T2<Tau1,
          REL[T2,b1,q1]*REL[T2,b2,q2]*Exp[-T2 s],0],
     If[T1<Tau2,
          REL[T1,b2,q2]*REL[T1,b1,q1]*Exp[-T1 s],0]
     ]
];
AP1:=Function[{Tau1,Tau2,T1,T2},
If[T2<=T1,
     If[T2<Tau1,
          REL[T2,b1,q1]*REL[T2,b2,q2],0],
     If[T1<Tau2,
          REL[T1,b2,q2]*REL[T1,b1,q1],0]
     ]
];
AM3:=Function[{Tau1,Tau2,T1,T2,s},
If[T2<=T1,
     If[T2>=Tau1,
          NIntegrate[
          Zed[w,b2,q2]*REL[w,b1,q1]*(Exp[-s w]),{w,0,Tau1},
          PrecisionGoal->6,AccuracyGoal->6],
          NIntegrate[
          Zed[w,b2,q2]*REL[w,b1,q1]*(Exp[-s w]),{w,0,T2},
```

```
          PrecisionGoal->6,AccuracyGoal->6]
          ],
     If[T1>=Tau2,
          NIntegrate[
          Zed[w,b1,q1]*REL[w,b2,q2]*(Exp[-s w]),{w,0,Tau2},
          PrecisionGoal->6,AccuracyGoal->6],
          NIntegrate[
          Zed[w,b1,q1]*REL[w,b2,q2]*Exp[-s w],{w,0,T1},
          PrecisionGoal->6,AccuracyGoal->6]
          ]
     ]
];
AP3:=Function[{Tau1,Tau2,T1,T2},
If[T2<=T1,
     If[T2>=Tau1,
          NIntegrate[
          Zed[w,b2,q2]*REL[w,b1,q1],{w,0,Tau1},
          PrecisionGoal->6,AccuracyGoal->6],
          NIntegrate[
          Zed[w,b2,q2]*REL[w,b1,q1],{w,0,T2},
          PrecisionGoal->6,AccuracyGoal->6]
          ],
     If[T1>=Tau2,
          NIntegrate[
          Zed[w,b1,q1]*REL[w,b2,q2],{w,0,Tau2},
          PrecisionGoal->6,AccuracyGoal->6],
          NIntegrate[
          Zed[w,b1,q1]*REL[w,b2,q2],{w,0,T1},
          PrecisionGoal->6,AccuracyGoal->6]
          ]
     ]
];
DK:=Function[{Tau1,Tau2,T1,T2,s},
If[T2<=T1,
  If[Tau1<Tau2,
          If[T2<=Tau1+Tau2,
               NIntegrate[Zed[w,b2,q2]*
               f[u,b1,q1,s]*Exp[-s w]*
               REL[w-u,b1,q1],{w,0,Tau1},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*Exp[-s w]*
               f[u,b1,q1,s]*REL[w-u,b1,q1],
               {w,Tau1,Tau2},{u,w-Tau1,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*Exp[-s w]*
               f[u,b1,q1,s]*REL[w-u,b1,q1],
               {w,Tau2,T2},{u,w-Tau1,Tau2},
               PrecisionGoal->3,AccuracyGoal->6],
            NIntegrate[Zed[w,b2,q2]*f[u,b1,q1,s]*Exp[-s w]*
                    REL[w-u,b1,q1],{w,0,Tau1},{u,0,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
                    NIntegrate[Zed[w,b2,q2]*Exp[-s w]*
                    f[u,b1,q1,s]*REL[w-u,b1,q1],
                    {w,Tau1,Tau2},{u,w-Tau1,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
                    NIntegrate[Zed[w,b2,q2]*Exp[-s w]*
```

```
                f[u,b1,q1,s]*REL[w-u,b1,q1],
                {w,Tau2,Tau1+Tau2},{u,w-Tau1,Tau2},
                PrecisionGoal->3,AccuracyGoal->6]
     ],
   If[T2<=Tau1+Tau2,
      If[Tau1<T2,
           NIntegrate[Zed[w,b2,q2]*
           f[u,b1,q1,s]*Exp[-s w]*
           REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
           PrecisionGoal->3,AccuracyGoal->6]+
           NIntegrate[Zed[w,b2,q2]*Exp[-s w]*
           f[u,b1,q1,s]*REL[w-u,b1,q1],
           {w,Tau2,Tau1},{u,0.0,Tau2},
           PrecisionGoal->3,AccuracyGoal->6]+
           NIntegrate[Zed[w,b2,q2]*Exp[-s w]*
           f[u,b1,q1,s]*REL[w-u,b1,q1],
           {w,Tau1,T2},{u,w-Tau1,Tau2},
           PrecisionGoal->3,AccuracyGoal->6],
            NIntegrate[Zed[w,b2,q2]*
                f[u,b1,q1,s]*Exp[-s w]*
                REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
                PrecisionGoal->3,AccuracyGoal->6]+
           NIntegrate[Zed[w,b2,q2]*
                f[u,b1,q1,s]*Exp[-s w]*
                REL[w-u,b1,q1],{w,Tau2,T2},{u,0,Tau2},
                PrecisionGoal->3,AccuracyGoal->6]
                ],
                NIntegrate[Zed[w,b2,q2]*
                f[u,b1,q1,s]*Exp[-s w]*
                REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
                PrecisionGoal->3,AccuracyGoal->6]+
                NIntegrate[Zed[w,b2,q2]*Exp[-s w]*
                f[u,b1,q1,s]*REL[w-u,b1,q1],
                {w,Tau2,Tau1},{u,0.0,Tau2},
                PrecisionGoal->3,AccuracyGoal->6]+
                NIntegrate[Zed[w,b2,q2]*Exp[-s w]*
                f[u,b1,q1,s]*REL[w-u,b1,q1],
                {w,Tau1,Tau1+Tau2},{u,w-Tau1,Tau2},
                PrecisionGoal->3,AccuracyGoal->6]
      ]
],
If[Tau2<Tau1,
     If[T1<=Tau1+Tau2,
          NIntegrate[Zed[w,b1,q1]*
          f1[u,b2,q2,s]*Exp[-s w]*
          REL[w-u,b2,q2],{w,0,Tau2},{u,0,w},
          PrecisionGoal->3,AccuracyGoal->6]+
          NIntegrate[Zed[w,b1,q1]*Exp[-s w]*
          f1[u,b2,q2,s]*REL[w-u,b2,q2],
          {w,Tau2,Tau1},{u,w-Tau2,w},
           PrecisionGoal->3,AccuracyGoal->6]+
           NIntegrate[Zed[w,b1,q1]*Exp[-s w]*
          f1[u,b2,q2,s]*REL[w-u,b2,q2],
          {w,Tau1,T1},{u,w-Tau2,Tau1},
          PrecisionGoal->3,AccuracyGoal->6],
               NIntegrate[Zed[w,b1,q1]*
```

```
                    f1[u,b2,q2,s]*Exp[-s w]*
                    REL[w-u,b2,q2],{w,0,Tau2},{u,0,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
                    NIntegrate[Zed[w,b1,q1]*Exp[-s w]*
                    f1[u,b2,q2,s]*REL[w-u,b2,q2],
                     {w,Tau2,Tau1},{u,w-Tau2,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
                    NIntegrate[Zed[w,b1,q1]*Exp[-s w]*
                    f1[u,b2,q2,s]*REL[w-u,b2,q2],
                    {w,Tau1,Tau1+Tau2},{u,w-Tau2,Tau1},
                    PrecisionGoal->3,AccuracyGoal->6]
                ],
         If[T1<=Tau1+Tau2,
            If[Tau2<T1,
            NIntegrate[Zed[w,b1,q1]*
            f1[u,b2,q2,s]*Exp[-s w]*
            REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
            PrecisionGoal->3,AccuracyGoal->6]+
            NIntegrate[Zed[w,b1,q1]*Exp[-s w]*
            f1[u,b2,q2,s]*REL[w-u,b2,q2],
            {w,Tau1,Tau2},{u,0.0,Tau1},
            PrecisionGoal->3,AccuracyGoal->6]+
            NIntegrate[Zed[w,b1,q1]*Exp[-s w]*
            f1[u,b2,q2,s]*REL[w-u,b2,q2],
            {w,Tau2,T1},{u,w-Tau2,Tau1},
            PrecisionGoal->3,AccuracyGoal->6],
              NIntegrate[Zed[w,b1,q1]*
                    f1[u,b2,q2,s]*Exp[-s w]*
                    REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
                NIntegrate[Zed[w,b1,q1]*
                    f1[u,b2,q2,s]*Exp[-s w]*
                    REL[w-u,b2,q2],{w,Tau1,T1},{u,0,Tau1},
                    PrecisionGoal->3,AccuracyGoal->6]
                    ],
                NIntegrate[Zed[w,b1,q1]*
                f1[u,b2,q2,s]*Exp[-s w]*
                REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
                PrecisionGoal->3,AccuracyGoal->6]+
                NIntegrate[Zed[w,b1,q1]*Exp[-s w]*
                f1[u,b2,q2,s]*REL[w-u,b2,q2],
                {w,Tau1,Tau2},{u,0.0,Tau1},
                PrecisionGoal->3,AccuracyGoal->6]+
                NIntegrate[Zed[w,b1,q1]*Exp[-s w]*
                f1[u,b2,q2,s]*REL[w-u,b2,q2],
                {w,Tau2,Tau1+Tau2},{u,w-Tau2,Tau1},
                PrecisionGoal->3,AccuracyGoal->6]
            ]
        ]
    ]
 ];
A4:=Function[{Tau1,Tau2,T1,T2,s},
           (DK[Tau1,Tau2,T1,T2,s])
       ];
A2:=Function[{Tau1,Tau2,T1,T2,s},
      If[T2<=T1,
```

```
          If[T2>=Tau1,
               If[T2<=Tau1+Tau2,
                    NIntegrate[REL[T2,b2,q2]*Exp[-s T2]*
                    f[u,b1,q1,s]*REL[T2-u,b1,q1],
               {u,T2-Tau1,Tau2},
                    PrecisionGoal->3,AccuracyGoal->6],0],
                    NIntegrate[REL[T2,b2,q2]*Exp[-s T2]*
                    f[u,b1,q1,s]*REL[T2-u,b1,q1],{u,0,Tau2},
                    PrecisionGoal->3,AccuracyGoal->6]
               ],
          If[T1>=Tau2,
               If[T1<=Tau1+Tau2,
                    NIntegrate[REL[T1,b1,q1]*Exp[-s T1]*
          f1[u,b2,q2,s]*REL[T1-u,b2,q2],{u,T1-Tau2,Tau1},
                    PrecisionGoal->3,AccuracyGoal->6],0],
               NIntegrate[REL[T1,b1,q1]*Exp[-s T1]*
                    f1[u,b2,q2,s]*REL[T1-u,b2,q2],{u,0,Tau1},
                    PrecisionGoal->3,AccuracyGoal->6]
          ]
     ]
];
AM2:=Function[{Tau1,Tau2,T1,T2,s},
          If[T2>=T1,
               If[T2>=Tau1,
                    If[T2<=Tau1+Tau2,
                    NIntegrate[REL[T2,b2,q2]*Exp[-s T2]*
          f[u,b1,q1,0]*REL[T2-u,b1,q1],{u,T2-Tau1,Tau2},
                    PrecisionGoal->3,AccuracyGoal->6],0],
                    NIntegrate[REL[T2,b2,q2]*Exp[-s T2]*
                    f[u,b1,q1,0]*REL[T2-u,b1,q1],{u,0,Tau2},
                    PrecisionGoal->3,AccuracyGoal->6]
               ],
          If[T1>=Tau2,
               If[T1<=Tau1+Tau2,
                    NIntegrate[REL[T1,b1,q1]*Exp[-s T1]*
          f1[u,b2,q2,0]*REL[T1-u,b2,q2],{u,T1-Tau2,Tau1},
                    PrecisionGoal->3,AccuracyGoal->6],0],
               NIntegrate[REL[T1,b1,q1]*Exp[-s T1]*
                    f1[u,b2,q2,0]*REL[T1-u,b2,q2],{u,0,Tau1},
                    PrecisionGoal->3,AccuracyGoal->6]
          ]
     ]
];

AP2:=Function[{Tau1,Tau2,T1,T2},
     If[T2<=T1,
          If[T2>=Tau1,
               If[T2<=Tau1+Tau2,
                    NIntegrate[REL[T2,b2,q2]*
          f[u,b1,q1,0]*REL[T2-u,b1,q1],{u,T2-Tau1,Tau2},
                    PrecisionGoal->3,AccuracyGoal->6],0],
                    NIntegrate[REL[T2,b2,q2]*
                    f[u,b1,q1,0]*REL[T2-u,b1,q1],{u,0,Tau2},
                    PrecisionGoal->3,AccuracyGoal->6]
               ],
          If[T1>=Tau2,
```

```
            If[T1<=Tau1+Tau2,
                 NIntegrate[REL[T1,b1,q1]*
            f[u,b1,q1,0]*REL[T1-u,b2,q2],{u,T1-Tau2,Tau1},
                 PrecisionGoal->3,AccuracyGoal->6],0],
            NIntegrate[REL[T1,b1,q1]*
                 f[u,b1,q1,0]*REL[T1-u,b2,q2],{u,0,Tau1},
                 PrecisionGoal->3,AccuracyGoal->6]
        ]
    ]
];

DKFF:=Function[{Tau1,Tau2,T1,T2,s},
If[T2<=T1,
  If[Tau1<Tau2,
         If[T2<=Tau1+Tau2,
              NIntegrate[Zed[w,b2,q2]*
              f[u,b1,q1,0]*Exp[-s w]*
              REL[w-u,b1,q1],{w,0,Tau1},{u,0,w},
              PrecisionGoal->3,AccuracyGoal->6]+
              NIntegrate[Zed[w,b2,q2]*Exp[-s w]*
              f[u,b1,q1,0]*REL[w-u,b1,q1],
              {w,Tau1,Tau2},{u,w-Tau1,w},
              PrecisionGoal->3,AccuracyGoal->6]+
              NIntegrate[Zed[w,b2,q2]*Exp[-s w]*
              f[u,b1,q1,0]*REL[w-u,b1,q1],
              {w,Tau2,T2},{u,w-Tau1,Tau2},
              PrecisionGoal->3,AccuracyGoal->6],
           NIntegrate[Zed[w,b2,q2]*f[u,b1,q1,0]*Exp[-s w]*
                 REL[w-u,b1,q1],{w,0,Tau1},{u,0,w},
                 PrecisionGoal->3,AccuracyGoal->6]+
                 NIntegrate[Zed[w,b2,q2]*Exp[-s w]*
                 f[u,b1,q1,0]*REL[w-u,b1,q1],
                 {w,Tau1,Tau2},{u,w-Tau1,w},
                 PrecisionGoal->3,AccuracyGoal->6]+
                 NIntegrate[Zed[w,b2,q2]*Exp[-s w]*
                 f[u,b1,q1,0]*REL[w-u,b1,q1],
                 {w,Tau2,Tau1+Tau2},{u,w-Tau1,Tau2},
                 PrecisionGoal->3,AccuracyGoal->6]
      ],
      If[T2<=Tau1+Tau2,
         If[Tau1<T2,
              NIntegrate[Zed[w,b2,q2]*Exp[-s w]*
         f[u,b1,q1,0]*REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
              PrecisionGoal->3,AccuracyGoal->6]+
              NIntegrate[Zed[w,b2,q2]*Exp[-s w]*
              f[u,b1,q1,0]*REL[w-u,b1,q1],
              {w,Tau2,Tau1},{u,0.0,Tau2},
              PrecisionGoal->3,AccuracyGoal->6]+
              NIntegrate[Zed[w,b2,q2]*Exp[-s w]*
              f[u,b1,q1,0]*REL[w-u,b1,q1],
              {w,Tau1,T2},{u,w-Tau1,Tau2},
              PrecisionGoal->3,AccuracyGoal->6],
               NIntegrate[Zed[w,b2,q2]*Exp[-s w]*
         f[u,b1,q1,0.0]*REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
                 PrecisionGoal->3,AccuracyGoal->6]+
              NIntegrate[Zed[w,b2,q2]*
```

```
                f[u,b1,q1,0.0]*Exp[-s w]*
                REL[w-u,b1,q1],{w,Tau2,T2},{u,0,Tau2},
                PrecisionGoal->3,AccuracyGoal->6]
                ],
                NIntegrate[Zed[w,b2,q2]*
                f[u,b1,q1,0]*Exp[-s w]*
                REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
                PrecisionGoal->3,AccuracyGoal->6]+
                NIntegrate[Zed[w,b2,q2]*Exp[-s w]*
                f[u,b1,q1,0]*REL[w-u,b1,q1],
                {w,Tau2,Tau1},{u,0.0,Tau2},
                PrecisionGoal->3,AccuracyGoal->6]+
                NIntegrate[Zed[w,b2,q2]*Exp[-s w]*
                f[u,b1,q1,0]*REL[w-u,b1,q1],
                {w,Tau1,Tau1+Tau2},{u,w-Tau1,Tau2},
                PrecisionGoal->3,AccuracyGoal->6]
  ]
],
If[Tau2<Tau1,
    If[T1<=Tau1+Tau2,
        NIntegrate[Zed[w,b1,q1]*
        f1[u,b2,q2,0]*Exp[-s w]*
        REL[w-u,b2,q2],{w,0,Tau2},{u,0,w},
        PrecisionGoal->3,AccuracyGoal->6]+
        NIntegrate[Zed[w,b1,q1]*
        f1[u,b2,q2,0]*REL[w-u,b2,q2],
        {w,Tau2,Tau1},{u,w-Tau2,w},
        PrecisionGoal->3,AccuracyGoal->6]+
        NIntegrate[Zed[w,b1,q1]*Exp[-s w]*
        f1[u,b2,q2,0]*REL[w-u,b2,q2],
        {w,Tau1,T1},{u,w-Tau2,Tau1},
        PrecisionGoal->3,AccuracyGoal->6],
            NIntegrate[Zed[w,b1,q1]*Exp[-s w]*
            f1[u,b2,q2,0]*REL[w-u,b2,q2],
            {w,0,Tau2},{u,0,w},
            PrecisionGoal->3,AccuracyGoal->6]+
            NIntegrate[Zed[w,b1,q1]*Exp[-s w]*
            f1[u,b2,q2,0]*REL[w-u,b2,q2],
             {w,Tau2,Tau1},{u,w-Tau2,w},
             PrecisionGoal->3,AccuracyGoal->6]+
            NIntegrate[Zed[w,b1,q1]*Exp[-s w]*
            f1[u,b2,q2,0]*REL[w-u,b2,q2],
            {w,Tau1,Tau1+Tau2},{u,w-Tau2,Tau1},
            PrecisionGoal->3,AccuracyGoal->6]
 ],
 If[T1<=Tau1+Tau2,
    If[Tau2<T1,
        NIntegrate[Zed[w,b1,q1]*
        f1[u,b2,q2,0]*Exp[-s w]*
        REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
        PrecisionGoal->3,AccuracyGoal->6]+
        NIntegrate[Zed[w,b1,q1]*Exp[-s w]*
        f1[u,b2,q2,0]*REL[w-u,b2,q2],
        {w,Tau1,Tau2},{u,0.0,Tau1},
        PrecisionGoal->3,AccuracyGoal->6]+
        NIntegrate[Zed[w,b1,q1]*Exp[-s w]*
```

```
                 f1[u,b2,q2,0]*REL[w-u,b2,q2],
                 {w,Tau2,T1},{u,w-Tau2,Tau1},
                 PrecisionGoal->3,AccuracyGoal->6],
               NIntegrate[Zed[w,b1,q1]*
                     f1[u,b2,q2,0.0]*Exp[-s w]*
                     REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
                     PrecisionGoal->3,AccuracyGoal->6]+
                NIntegrate[Zed[w,b1,q1]*
                     f1[u,b2,q2,0.0]*Exp[-s w]*
                     REL[w-u,b2,q2],{w,Tau1,T1},{u,0,Tau1},
                     PrecisionGoal->3,AccuracyGoal->6]
                     ],
                     NIntegrate[Zed[w,b1,q1]*
                     f1[u,b2,q2,0.0]*Exp[-s w]*
                     REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
                     PrecisionGoal->3,AccuracyGoal->6]+
                     NIntegrate[Zed[w,b1,q1]*Exp[-s w]*
                     f1[u,b2,q2,0]*REL[w-u,b2,q2],
                     {w,Tau1,Tau2},{u,0.0,Tau1},
                     PrecisionGoal->3,AccuracyGoal->6]+
                     NIntegrate[Zed[w,b1,q1]*Exp[-s w]*
                     f1[u,b2,q2,0]*REL[w-u,b2,q2],
                     {w,Tau2,Tau1+Tau2},{u,w-Tau2,Tau1},
                     PrecisionGoal->3,AccuracyGoal->6]
            ]
        ]
    ]
];
AM4:=Function[{Tau1,Tau2,T1,T2,s},
          (DKFF[Tau1,Tau2,T1,T2,s])
     ];
DKP:=Function[{Tau1,Tau2,T1,T2},
If[T2<=T1,
  If[Tau1<Tau2,
          If[T2<=Tau1+Tau2,
                NIntegrate[Zed[w,b2,q2]*
                f[u,b1,q1,0]*REL[w-u,b1,q1],
                {w,0,Tau1},{u,0,w},
                PrecisionGoal->3,AccuracyGoal->6]+
                NIntegrate[Zed[w,b2,q2]*
                f[u,b1,q1,0]*REL[w-u,b1,q1],
                {w,Tau1,Tau2},{u,w-Tau1,w},
                PrecisionGoal->3,AccuracyGoal->6]+
                NIntegrate[Zed[w,b2,q2]*
                f[u,b1,q1,0]*REL[w-u,b1,q1],
                {w,Tau2,T2},{u,w-Tau1,Tau2},
                PrecisionGoal->3,AccuracyGoal->6],
                     NIntegrate[Zed[w,b2,q2]*f[u,b1,q1,0]*
                     REL[w-u,b1,q1],{w,0,Tau1},{u,0,w},
                     PrecisionGoal->3,AccuracyGoal->6]+
                     NIntegrate[Zed[w,b2,q2]*
                     f[u,b1,q1,0]*REL[w-u,b1,q1],
                     {w,Tau1,Tau2},{u,w-Tau1,w},
                     PrecisionGoal->3,AccuracyGoal->6]+
                     NIntegrate[Zed[w,b2,q2]*
                     f[u,b1,q1,0]*REL[w-u,b1,q1],
```

```
                  {w,Tau2,Tau1+Tau2},{u,w-Tau1,Tau2},
                  PrecisionGoal->3,AccuracyGoal->6]
       ],
       If[T2<=Tau1+Tau2,
          If[Tau1<T2,
               NIntegrate[Zed[w,b2,q2]*
               f[u,b1,q1,0]*
               REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*
               f[u,b1,q1,0]*REL[w-u,b1,q1],
               {w,Tau2,Tau1},{u,0.0,Tau2},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*
               f[u,b1,q1,0]*REL[w-u,b1,q1],
               {w,Tau1,T2},{u,w-Tau1,Tau2},
               PrecisionGoal->3,AccuracyGoal->6],
                 NIntegrate[Zed[w,b2,q2]*
          f[u,b1,q1,0]*REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
                  PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*
      f[u,b1,q1,0]*REL[w-u,b1,q1],{w,Tau2,T2},{u,0,Tau2},
                  PrecisionGoal->3,AccuracyGoal->6]
                  ],
                  NIntegrate[Zed[w,b2,q2]*
          f[u,b1,q1,0]*REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
                  PrecisionGoal->3,AccuracyGoal->6]+
                  NIntegrate[Zed[w,b2,q2]*
                  f[u,b1,q1,0]*REL[w-u,b1,q1],
                  {w,Tau2,Tau1},{u,0.0,Tau2},
                  PrecisionGoal->3,AccuracyGoal->6]+
                  NIntegrate[Zed[w,b2,q2]*
                  f[u,b1,q1,0]*REL[w-u,b1,q1],
                  {w,Tau1,Tau1+Tau2},{u,w-Tau1,Tau2},
                  PrecisionGoal->3,AccuracyGoal->6]
       ]
     ],
If[Tau2<Tau1,
     If[T1<=Tau1+Tau2,
               NIntegrate[Zed[w,b1,q1]*
               f1[u,b2,q2,0]*
               REL[w-u,b2,q2],{w,0,Tau2},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b1,q1]*
               f1[u,b2,q2,0]*REL[w-u,b2,q2],
               {w,Tau2,Tau1},{u,w-Tau2,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b1,q1]*
               f1[u,b2,q2,0]*REL[w-u,b2,q2],
               {w,Tau1,T1},{u,w-Tau2,Tau1},
               PrecisionGoal->3,AccuracyGoal->6],
                   NIntegrate[Zed[w,b1,q1]*
            f1[u,b2,q2,0]*REL[w-u,b2,q2],{w,0,Tau2},{u,0,w},
                   PrecisionGoal->3,AccuracyGoal->6]+
                  NIntegrate[Zed[w,b1,q1]*
                  f1[u,b2,q2,0]*REL[w-u,b2,q2],
```

```
                    {w,Tau2,Tau1},{u,w-Tau2,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
                    NIntegrate[Zed[w,b1,q1]*
                    f1[u,b2,q2,0]*REL[w-u,b2,q2],
                    {w,Tau1,Tau1+Tau2},{u,w-Tau2,Tau1},
                PrecisionGoal->3,AccuracyGoal->6]
        ],
        If[T1<=Tau1+Tau2,
            If[Tau2<T1,
                NIntegrate[Zed[w,b1,q1]*
                f1[u,b2,q2,0]*
                REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
                PrecisionGoal->3,AccuracyGoal->6]+
                NIntegrate[Zed[w,b1,q1]*
                f1[u,b2,q2,0]*REL[w-u,b2,q2],
                {w,Tau1,Tau2},{u,0.0,Tau1},
                PrecisionGoal->3,AccuracyGoal->6]+
                NIntegrate[Zed[w,b1,q1]*
                f1[u,b2,q2,0]*REL[w-u,b2,q2],
                {w,Tau2,T1},{u,w-Tau2,Tau1},
                PrecisionGoal->3,AccuracyGoal->6],
                  NIntegrate[Zed[w,b1,q1]*
                     f1[u,b2,q2,0.0]*
                     REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
                     PrecisionGoal->3,AccuracyGoal->6]+
                NIntegrate[Zed[w,b1,q1]*
      f1[u,b2,q2,0.0]*REL[w-u,b2,q2],{w,Tau1,T1},{u,0,Tau1},
                     PrecisionGoal->3,AccuracyGoal->6]
                     ],
                     NIntegrate[Zed[w,b1,q1]*
           f1[u,b2,q2,0]*REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
                     PrecisionGoal->3,AccuracyGoal->6]+
                     NIntegrate[Zed[w,b1,q1]*
                     f1[u,b2,q2,0]*REL[w-u,b2,q2],
                     {w,Tau1,Tau2},{u,0.0,Tau1},
                     PrecisionGoal->3,AccuracyGoal->6]+
                     NIntegrate[Zed[w,b1,q1]*
                     f1[u,b2,q2,0]*REL[w-u,b2,q2],
                     {w,Tau2,Tau1+Tau2},{u,w-Tau2,Tau1},
                     PrecisionGoal->3,AccuracyGoal->6]
           ]
      ]
  ]
];
AP4:=Function[{Tau1,Tau2,T1,T2},
           (DKP[Tau1,Tau2,T1,T2])
      ];
```

Inintial Renewal Interval

```
YG1:=Function[{Tau1,Tau2,T1,T2,s},
If[T1>=T2,
     If[T2>=Tau1,
NIntegrate[Zed[w,b2,q2] REL[w,b1,q1]* Exp[-w s],
{w,Tau1,T2},PrecisionGoal->3,AccuracyGoal->6]+
REL[T2,b2,q2] REL[T2,b1,q1]* Exp[-T2 s],
0],
     If[T1>=Tau2,
NIntegrate[Zed[w,b1,q1] REL[w,b2,q2]* Exp[-w s],
{w,Tau2,T1},PrecisionGoal->3,AccuracyGoal->6]+
REL[T1,b1,q1] REL[T1,b2,q2]*(g[s,ld1] g[s,ld2]) E
xp[-T1 s],0]
     ]
];

YG2:=Function[{Tau1,Tau2,T1,T2,s},
If[T1>=T2,
     If[T2>=Tau1+Tau2,
          NIntegrate[Zed[w,b2,q2]*f2[u,b1,q1,0]*Exp[-w s]*
          REL[w-u,b1,q1],{w,Tau1,Tau2+Tau1},{u,0,w-Tau1},
          PrecisionGoal->3,AccuracyGoal->6]+
          NIntegrate[Zed[w,b2,q2]*f2[u,b1,q1,0]*Exp[-w s]*
          REL[w-u,b1,q1],{w,Tau1+Tau2,T2},{u,0,Tau2},
          PrecisionGoal->3,AccuracyGoal->6],
     If[T2>=Tau1,
          NIntegrate[Zed[w,b2,q2]*f2[u,b1,q1,0]*Exp[-w s]*
          REL[w-u,b1,q1],{w,Tau1,T2},{u,0,w-Tau1},
          PrecisionGoal->3,AccuracyGoal->6],0]
     ],
     If[T1>=Tau2+Tau1,
          NIntegrate[Zed[w,b1,q1]*f3[u,b2,q2,0]*Exp[-w s]*
          REL[w-u,b2,q2],{w,Tau2,Tau1+Tau2},{u,0,w-Tau2},
          PrecisionGoal->3,AccuracyGoal->6]+
          NIntegrate[Zed[w,b1,q1]*f3[u,b2,q2,0]*Exp[-w s]*
          REL[w-u,b2,q2],{w,Tau2+Tau1,T1},{u,0,w-Tau2},
          PrecisionGoal->3,AccuracyGoal->6],
          If[T1>=Tau2,
          NIntegrate[Zed[w,b1,q1]*f3[u,b2,q2,0]*Exp[-w s]*
          REL[w-u,b2,q2],{w,Tau2,T1},{u,0,w-Tau2},
          PrecisionGoal->3,AccuracyGoal->6],0]
          ]
     ]
];

YG3:=Function[{Tau1,Tau2,T1,T2,s},
     If[T1>=T2,
          If[T2>=Tau1+Tau2,
          NIntegrate[REL[T2,b2,q2]*f2[u,b1,q1,0]*Exp[-T2 s]*
          REL[T2-u,b1,q1],{u,0,Tau2},
          PrecisionGoal->3,AccuracyGoal->6],0]+
          If[T2<Tau1+Tau2 && T2>Tau1,
          NIntegrate[REL[T2,b2,q2]*f2[u,b1,q1,0]*Exp[-T2 s]*
          REL[T2-u,b1,q1],{u,0,T2-Tau1},
```

```
          PrecisionGoal->3,AccuracyGoal->6],0],
If[T1>=Tau1+Tau2,
          NIntegrate[REL[T1,b1,q1]*f3[u,b2,q2,0]*Exp[-T1 s]*
          REL[T1-u,b2,q2],{u,0,Tau1},
          PrecisionGoal->3,AccuracyGoal->6],0]+
     If[T1<Tau1+Tau2 && T1>Tau2,
          NIntegrate[REL[T1,b1,q1]*f3[u,b2,q2,0]*Exp[-T1 s]*
          REL[T1-u,b2,q2],{u,0,T1-Tau2},
          PrecisionGoal->3,AccuracyGoal->6],0]
     ]
];
YG4:=Function[{Tau1,Tau2,T1,T2,s},
If[T1>=T2,
     NIntegrate[REL[u,b2,q2] Zed[u,b1,q1]*Exp[-s u],
     {u,Tau2,T2},PrecisionGoal->3,AccuracyGoal->6]+
     NIntegrate[REL[u,b2,q2]*f[v,b1,q1,0]*
     Zed[u-v,b1,q1]*Exp[-s u],{u,Tau2,T2},{v,0,Tau2},
     PrecisionGoal->3,AccuracyGoal->6],
     NIntegrate[REL[u,b1,q1] Zed[u,b2,q2]*Exp[-s u],
     {u,Tau1,T1},PrecisionGoal->3,AccuracyGoal->6]+
     NIntegrate[REL[u,b1,q1]*f[v,b2,q2,0]*
     Zed[u-v,b2,q2]*Exp[-s u],{u,Tau1,T1},{v,0,Tau1},
     PrecisionGoal->3,AccuracyGoal->6]
     ]
];
MYTTINIT[Tau1_,Tau2_,T1_,T2_,s_]:=
MYTTINIT[Tau1,Tau2,T1,T2,s]=
YG1[Tau1,Tau2,T1,T2,s]+YG2[Tau1,Tau2,T1,T2,s]+
YG3[Tau1,Tau2,T1,T2,s]+
YG4[Tau1,Tau2,T1,T2,s];
```

```
BG1:=Function[{Tau1,Tau2,T1,T2,s},
If[T1>=T2,
     If[T2>=Tau1,
NIntegrate[Zed[w,b2,q2] REL[w,b1,q1]*
(g[s,ld1] g[s,ld2]) Exp[-w s],
{w,Tau1,T2},PrecisionGoal->3,AccuracyGoal->6]+
REL[T2,b2,q2] REL[T2,b1,q1]*
(g[s,ld1] g[s,ld2]) Exp[-T2 s],
0],
     If[T1>=Tau2,
NIntegrate[Zed[w,b1,q1] REL[w,b2,q2]*
(g[s,ld1] g[s,ld2]) Exp[-w s],
{w,Tau2,T1},PrecisionGoal->3,AccuracyGoal->6]+
REL[T1,b1,q1] REL[T1,b2,q2]*
(g[s,ld1] g[s,ld2]) Exp[-T1 s],0]
     ]
];

BG2:=Function[{Tau1,Tau2,T1,T2,s},
If[T1>=T2,
     If[T2>=Tau1+Tau2,
          NIntegrate[Zed[w,b2,q2]*f2[u,b1,q1,s]*Exp[-w s]*
          REL[w-u,b1,q1],{w,Tau1,Tau2+Tau1},{u,0,w-Tau1},
          PrecisionGoal->3,AccuracyGoal->6]+
          NIntegrate[Zed[w,b2,q2]*f2[u,b1,q1,s]*Exp[-w s]*
          REL[w-u,b1,q1],{w,Tau1+Tau2,T2},{u,0,Tau2},
          PrecisionGoal->3,AccuracyGoal->6],
     If[T2>=Tau1,
          NIntegrate[Zed[w,b2,q2]*f2[u,b1,q1,s]*Exp[-w s]*
          REL[w-u,b1,q1],{w,Tau1,T2},{u,0,w-Tau1},
          PrecisionGoal->3,AccuracyGoal->6],0]
     ],
     If[T1>=Tau2+Tau1,
          NIntegrate[Zed[w,b1,q1]*f3[u,b2,q2,s]*Exp[-w s]*
          REL[w-u,b2,q2],{w,Tau2,Tau1+Tau2},{u,0,w-Tau2},
          PrecisionGoal->3,AccuracyGoal->6]+
          NIntegrate[Zed[w,b1,q1]*f3[u,b2,q2,s]*Exp[-w s]*
          REL[w-u,b2,q2],{w,Tau2+Tau1,T1},{u,0,w-Tau2},
          PrecisionGoal->3,AccuracyGoal->6],
          If[T1>=Tau2,
          NIntegrate[Zed[w,b1,q1]*f3[u,b2,q2,s]*Exp[-w s]*
          REL[w-u,b2,q2],{w,Tau2,T1},{u,0,w-Tau2},
          PrecisionGoal->3,AccuracyGoal->6],0]
          ]
     ]
];

BG3:=Function[{Tau1,Tau2,T1,T2,s},
     If[T1>=T2,
          If[T2>=Tau1+Tau2,
          NIntegrate[REL[T2,b2,q2]*f2[u,b1,q1,s]*Exp[-T2 s]*
          REL[T2-u,b1,q1],{u,0,Tau2},
          PrecisionGoal->3,AccuracyGoal->6],0]+
          If[T2<Tau1+Tau2 && T2>Tau1,
          NIntegrate[REL[T2,b2,q2]*f2[u,b1,q1,s]*Exp[-T2 s]*
```

```
          REL[T2-u,b1,q1],{u,0,T2-Tau1},
          PrecisionGoal->3,AccuracyGoal->6],0],
If[T1>=Tau1+Tau2,
          NIntegrate[REL[T1,b1,q1]*f3[u,b2,q2,s]*Exp[-T1 s]*
          REL[T1-u,b2,q2],{u,0,Tau1},
          PrecisionGoal->3,AccuracyGoal->6],0]+
    If[T1<Tau1+Tau2 && T1>Tau2,
          NIntegrate[REL[T1,b1,q1]*f3[u,b2,q2,s]*Exp[-T1 s]*
          REL[T1-u,b2,q2],{u,0,T1-Tau2},
          PrecisionGoal->3,AccuracyGoal->6],0]
    ]
];
BG4:=Function[{Tau1,Tau2,T1,T2,s},
If[T1>=T2,
    NIntegrate[REL[u,b2,q2] Zed[u,b1,q1]*
(g[s,ld1] g[s,ld2])*Exp[-s u],
    {u,Tau2,T2},PrecisionGoal->3,AccuracyGoal->6]+
    NIntegrate[REL[u,b2,q2]*f[v,b1,q1,s]*
    Zed[u-v,b1,q1]*Exp[-s u],{u,Tau2,T2},{v,0,Tau2},
    PrecisionGoal->3,AccuracyGoal->6],
    NIntegrate[REL[u,b1,q1] Zed[u,b2,q2]*
(g[s,ld1] g[s,ld2])*Exp[-s u],
    {u,Tau1,T1},PrecisionGoal->3,AccuracyGoal->6]+
    NIntegrate[REL[u,b1,q1]*f[v,b2,q2,s]*
    Zed[u-v,b2,q2]*Exp[-s u],{u,Tau1,T1},{v,0,Tau1},
    PrecisionGoal->3,AccuracyGoal->6]
    ]
];
MRENINIT[Tau1_,Tau2_,T1_,T2_,s_]:=
MRENINIT[Tau1,Tau2,T1,T2,s]=
BG1[Tau1,Tau2,T1,T2,s]+BG2[Tau1,Tau2,T1,T2,s]+
BG3[Tau1,Tau2,T1,T2,s]+
BG4[Tau1,Tau2,T1,T2,s];
```

GENERAL INTERVAL

```
B1:=Function[{Tau1,Tau2,T1,T2,s},
If[T2<=T1,
     If[T2<Max[(Tau1-PU[q1,b1]),0],
          RESDL[T2,T1,b1,q1]*REL[T2,b2,q2]*
     (Exp[-T2 s] g[s,ld2]),0],
     If[T1<Max[(Tau2-PU[q2,b2]),0],
          RESDL[T1,T2,b2,q2]*REL[T1,b1,q1]*
     (Exp[-T1 s] g[s,ld1]),0]
     ]
];

BJ1:=Function[{Tau1,Tau2,T1,T2},
If[T2<=T1,
     If[T2<Max[(Tau1-PU[q1,b1]),0],
          RESDL[T2,T1,b1,q1]*REL[T2,b2,q2],0],
     If[T1<Max[(Tau2-PU[q2,b2]),0],
          RESDL[T1,T2,b2,q2]*REL[T1,b1,q1],0]
     ]
];

BI1:=Function[{Tau1,Tau2,T1,T2,s},
If[T2<=T1,
     If[T2<Max[(Tau1-PU[q1,b1]),0],
          RESDL[T2,T1,b1,q1]*REL[T2,b2,q2]*(Exp[-s T2]),0],
     If[T1<Max[(Tau2-PU[q2,b2]),0],
          RESDL[T1,T2,b2,q2]*REL[T1,b1,q1]*(Exp[-s T1]),0]
     ]
];
B3:=Function[{Tau1,Tau2,T1,T2,s},
     If[T2<=T1,
     If[T2>=Max[(Tau1-PU[q1,b1]),0],
          NIntegrate[
          Zed[w,b2,q2]*RESDL[w,T1,b1,q1]*(Exp[-w s] g[s,ld2]),
     {w,0,Max[(Tau1-PU[q1,b1]),0]},
          PrecisionGoal->6,AccuracyGoal->6],
          NIntegrate[
          Zed[w,b2,q2]*RESDL[w,T1,b1,q1]*
     (Exp[-w s] g[s,ld2]),{w,0,T2},
          PrecisionGoal->6,AccuracyGoal->6]
          ],
     If[T1>=Max[(Tau2-PU[q2,b2]),0],
          NIntegrate[
          Zed[w,b1,q1]*RESDL[w,T2,b2,q2]*(Exp[-w s] g[s,ld1]),
     {w,0,Max[(Tau2-PU[q2,b2]),0]},
          PrecisionGoal->6,AccuracyGoal->6],
          NIntegrate[
          Zed[w,b1,q1]*RESDL[w,T2,b2,q2]*
     (Exp[-w s] g[s,ld1]),{w,0,T1},
          PrecisionGoal->6,AccuracyGoal->6]
          ]
     ]
];
BJ3:=Function[{Tau1,Tau2,T1,T2},
If[T2<=T1,
```

```
     If[T2>=Max[(Tau1-PU[q1,b1]),0],
          NIntegrate[
          Zed[w,b2,q2]*RESDL[w,T1,b1,q1],{w,0,Tau1},
          PrecisionGoal->6,AccuracyGoal->6],
          NIntegrate[
          Zed[w,b2,q2]*RESDL[w,T1,b1,q1],{w,0,T2},
          PrecisionGoal->6,AccuracyGoal->6]
          ],
     If[T1>=Max[(Tau2-PU[q2,b2]),0],
          NIntegrate[
          Zed[w,b1,q1]*RESDL[w,T2,b2,q2],{w,0,Tau2},
          PrecisionGoal->6,AccuracyGoal->6],
          NIntegrate[
          Zed[w,b1,q1]*RESDL[w,T2,b2,q2],{w,0,T1},
          PrecisionGoal->6,AccuracyGoal->6]
          ]
     ]
];
BI3:=Function[{Tau1,Tau2,T1,T2,s},
If[T2<=T1,
     If[T2>=Max[(Tau1-PU[q1,b1]),0],
          NIntegrate[
          Zed[w,b2,q2]*RESDL[w,T1,b1,q1]*
     (Exp[-s w]),{w,0,Tau1},
          PrecisionGoal->6,AccuracyGoal->6],
          NIntegrate[
          Zed[w,b2,q2]*RESDL[w,T1,b1,q1]*
     (Exp[-s w]),{w,0,T2},
          PrecisionGoal->6,AccuracyGoal->6]
          ],
     If[T1>=Max[(Tau2-PU[q2,b2]),0],
          NIntegrate[
          Zed[w,b1,q1]*RESDL[w,T2,b2,q2]*
     (Exp[-s w]),{w,0,Tau2},
          PrecisionGoal->6,AccuracyGoal->6],
          NIntegrate[
          Zed[w,b1,q1]*RESDL[w,T2,b2,q2]*
     (Exp[-s w]),{w,0,T1},
          PrecisionGoal->6,AccuracyGoal->6]
          ]
     ]
];
Jacb:=Function[{Tau1,Tau2,T1,T2,s},
If[T2<=T1,
  If[Tau1<Tau2,
          If[T2<=Tau1+Tau2,
               NIntegrate[Zed[w,b2,q2]*
               W[u,b1,q1,s,T1]*Exp[-s w]*
               REL[w-u,b1,q1],{w,0,Tau1},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*Exp[-s w]*
               W[u,b1,q1,s,T1]*REL[w-u,b1,q1],
               {w,Tau1,Tau2},{u,w-Tau1,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*Exp[-s w]*
               W[u,b1,q1,s,T1]*REL[w-u,b1,q1],
```

```
          {w,Tau2,T2},{u,w-Tau1,Tau2},
          PrecisionGoal->3,AccuracyGoal->6],
               NIntegrate[Zed[w,b2,q2]*
          W[u,b1,q1,s,T1]*Exp[-s w]*
               REL[w-u,b1,q1],{w,0,Tau1},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*Exp[-s w]*
               W[u,b1,q1,s,T1]*REL[w-u,b1,q1],
               {w,Tau1,Tau2},{u,w-Tau1,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*Exp[-s w]*
               W[u,b1,q1,s,T1]*REL[w-u,b1,q1],
               {w,Tau2,Tau1+Tau2},{u,w-Tau1,Tau2},
               PrecisionGoal->3,AccuracyGoal->6]
     ],
  If[T2<=Tau1+Tau2,
     If[Tau1<T2,
          NIntegrate[Zed[w,b2,q2]*
          W[u,b1,q1,s,T1]*Exp[-s w]*
          REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
          PrecisionGoal->3,AccuracyGoal->6]+
          NIntegrate[Zed[w,b2,q2]*Exp[-s w]*
          W[u,b1,q1,s,T1]*REL[w-u,b1,q1],
          {w,Tau2,Tau1},{u,0.0,Tau2},
          PrecisionGoal->3,AccuracyGoal->6]+
          NIntegrate[Zed[w,b2,q2]*Exp[-s w]*
          W[u,b1,q1,s,T1]*REL[w-u,b1,q1],
          {w,Tau1,T2},{u,w-Tau1,Tau2},
          PrecisionGoal->3,AccuracyGoal->6],
           NIntegrate[Zed[w,b2,q2]*
               W[u,b1,q1,s,T1]*Exp[-s w]*
               REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
          NIntegrate[Zed[w,b2,q2]*
               W[u,b1,q1,s,T1]*Exp[-s w]*
               REL[w-u,b1,q1],{w,Tau2,T2},{u,0,Tau2},
               PrecisionGoal->3,AccuracyGoal->6]
               ],
               NIntegrate[Zed[w,b2,q2]*
               W[u,b1,q1,s,T1]*Exp[-s w]*
               REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*Exp[-s w]*
               W[u,b1,q1,s,T1]*REL[w-u,b1,q1],
               {w,Tau2,Tau1},{u,0.0,Tau2},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*Exp[-s w]*
               W[u,b1,q1,s,T1]*REL[w-u,b1,q1],
               {w,Tau1,Tau1+Tau2},{u,w-Tau1,Tau2},
               PrecisionGoal->3,AccuracyGoal->6]
      ]
],
If[Tau2<Tau1,
     If[T1<=Tau1+Tau2,
          NIntegrate[Zed[w,b1,q1]*
          W1[u,b2,q2,s,T2]*Exp[-s w]*
```

```
        REL[w-u,b2,q2],{w,0,Tau2},{u,0,w},
        PrecisionGoal->3,AccuracyGoal->6]+
        NIntegrate[Zed[w,b1,q1]*Exp[-s w]*
        W1[u,b2,q2,s,T2]*REL[w-u,b2,q2],
        {w,Tau2,Tau1},{u,w-Tau2,w},
         PrecisionGoal->3,AccuracyGoal->6]+
         NIntegrate[Zed[w,b1,q1]*Exp[-s w]*
        W1[u,b2,q2,s,T2]*REL[w-u,b2,q2],
        {w,Tau1,T1},{u,w-Tau2,Tau1},
        PrecisionGoal->3,AccuracyGoal->6],
             NIntegrate[Zed[w,b1,q1]*
             W1[u,b2,q2,s,T2]*Exp[-s w]*
             REL[w-u,b2,q2],{w,0,Tau2},{u,0,w},
             PrecisionGoal->3,AccuracyGoal->6]+
             NIntegrate[Zed[w,b1,q1]*Exp[-s w]*
             W1[u,b2,q2,s,T2]*REL[w-u,b2,q2],
              {w,Tau2,Tau1},{u,w-Tau2,w},
             PrecisionGoal->3,AccuracyGoal->6]+
             NIntegrate[Zed[w,b1,q1]*Exp[-s w]*
             W1[u,b2,q2,s,T2]*REL[w-u,b2,q2],
             {w,Tau1,Tau1+Tau2},{u,w-Tau2,Tau1},
             PrecisionGoal->3,AccuracyGoal->6]
        ],
If[T1<=Tau1+Tau2,
 If[Tau2<T1,
   NIntegrate[Zed[w,b1,q1]*
   W1[u,b2,q2,s,T2]*Exp[-s w]*
   REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
   PrecisionGoal->3,AccuracyGoal->6]+
   NIntegrate[Zed[w,b1,q1]*Exp[-s w]*
   W1[u,b2,q2,s,T2]*REL[w-u,b2,q2],
   {w,Tau1,Tau2},{u,0.0,Tau1},
   PrecisionGoal->3,AccuracyGoal->6]+
   NIntegrate[Zed[w,b1,q1]*Exp[-s w]*
   W1[u,b2,q2,s,T2]*REL[w-u,b2,q2],
   {w,Tau2,T1},{u,w-Tau2,Tau1},
   PrecisionGoal->3,AccuracyGoal->6],
    NIntegrate[Zed[w,b1,q1]*
             W1[u,b2,q2,s,T2]*Exp[-s w]*
             REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
             PrecisionGoal->3,AccuracyGoal->6]+
        NIntegrate[Zed[w,b1,q1]*
             W1[u,b2,q2,s,T2]*Exp[-s w]*
             REL[w-u,b1,q1],{w,Tau1,T1},{u,0,Tau1},
             PrecisionGoal->3,AccuracyGoal->6]
             ],
        NIntegrate[Zed[w,b1,q1]*
        W1[u,b2,q2,s,T2]*Exp[-s w]*
        REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
        PrecisionGoal->3,AccuracyGoal->6]+
        NIntegrate[Zed[w,b1,q1]*Exp[-s w]*
        W1[u,b2,q2,s,T2]*REL[w-u,b2,q2],
        {w,Tau1,Tau2},{u,0.0,Tau1},
        PrecisionGoal->3,AccuracyGoal->6]+
        NIntegrate[Zed[w,b1,q1]*Exp[-s w]*
        W1[u,b2,q2,s,T2]*REL[w-u,b2,q2],
```

```
                {w,Tau2,Tau1+Tau2},{u,w-Tau2,Tau1},
                PrecisionGoal->3,AccuracyGoal->6]
           ]
      ]
   ]
];
B4:=Function[{Tau1,Tau2,T1,T2,s},
          (Jacb[Tau1,Tau2,T1,T2,s])
          ];

B2:=Function[{Tau1,Tau2,T1,T2,s},
     If[T2<=T1,
          If[T2>=Tau1,
               If[T2<=Tau1+Tau2,
                    NIntegrate[REL[T2,b2,q2]*Exp[-s T2]*
                    W[u,b1,q1,s,T1]*REL[T2-u,b1,q1],
               {u,T2-Tau1,Tau2},
                    PrecisionGoal->3,AccuracyGoal->6],0],
                    NIntegrate[REL[T2,b2,q2]*Exp[-s T2]*
                    W[u,b1,q1,s,T1]*REL[T2-u,b1,q1],{u,0,Tau2},
                    PrecisionGoal->3,AccuracyGoal->6]
               ],
          If[T1>=Tau2,
               If[T1<=Tau1+Tau2,
                    NIntegrate[REL[T1,b1,q1]*Exp[-s T1]*
                    W1[u,b2,q2,s,T2]*REL[T1-u,b2,q2],
               {u,T1-Tau2,Tau1},
                    PrecisionGoal->3,AccuracyGoal->6],0],
               NIntegrate[REL[T1,b1,q1]*Exp[-s T1]*
                    W1[u,b2,q2,s,T2]*REL[T1-u,b2,q2],{u,0,Tau1},
                    PrecisionGoal->3,AccuracyGoal->6]
          ]
     ]
];
BM2:=Function[{Tau1,Tau2,T1,T2,s},
     If[T2<=T1,
          If[T2>=Tau1,
               If[T2<=Tau1+Tau2,
                    NIntegrate[REL[T2,b2,q2]*Exp[-s T2]*
                    REL[T2-u,b1,q1]/mean[T1,b1,q1],
                    {u,T2-Tau1,Tau2},
                    PrecisionGoal->3,AccuracyGoal->6],0],
                    NIntegrate[REL[T2,b2,q2]*Exp[-s T2]*
                    REL[T2-u,b1,q1]/mean[T1,b1,q1],
                    {u,0,Tau2},
                    PrecisionGoal->3,AccuracyGoal->6]
               ],
          If[T1>=Tau2,
               If[T1<=Tau1+Tau2,
                    NIntegrate[REL[T1,b1,q1]*Exp[-s T1]*
                    REL[T1-u,b2,q2]/mean[T2,b2,q2],
                    {u,T1-Tau2,Tau1},
                    PrecisionGoal->3,AccuracyGoal->6],0],
               NIntegrate[REL[T1,b1,q1]*Exp[-s T1]*
                    REL[T1-u,b2,q2]/mean[T2,b2,q2],{u,0,Tau1},
                    PrecisionGoal->3,AccuracyGoal->6]
```

```
          ]
     ]
];

BF2:=Function[{Tau1,Tau2,T1,T2},
     If[T2<=T1,
          If[T2>=Tau1,
               If[T2<=Tau1+Tau2,
                    NIntegrate[REL[T2,b2,q2]*
                    REL[T2-u,b1,q1]/mean[T1,b1,q1],
                    {u,T2-Tau1,Tau2},
                    PrecisionGoal->3,AccuracyGoal->6],0],
                    NIntegrate[REL[T2,b2,q2]*
                    REL[T2-u,b1,q1]/mean[T1,b1,q1],{u,0,Tau2},
                    PrecisionGoal->3,AccuracyGoal->6]
               ],
          If[T1>=Tau2,
               If[T1<=Tau1+Tau2,
                    NIntegrate[REL[T1,b1,q1]*
                    REL[T1-u,b2,q2]/mean[T2,b2,q2],
               {u,T1-Tau2,Tau1},
                    PrecisionGoal->3,AccuracyGoal->6],0],
               NIntegrate[REL[T1,b1,q1]*
                    REL[T1-u,b2,q2]/mean[T2,b2,q2],{u,0,Tau1},
                    PrecisionGoal->3,AccuracyGoal->6]
          ]
     ]
];
Jac:=Function[{Tau1,Tau2,T1,T2,s},
If[T2<=T1,
  If[Tau1<Tau2,
          If[T2<=Tau1+Tau2,
               NIntegrate[Zed[w,b2,q2]*(
               Exp[-s w]/mean[T1,b1,q1])*
               REL[w-u,b1,q1],{w,0,Tau1},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*
               (Exp[-s w]/mean[T1,b1,q1])*REL[w-u,b1,q1],
               {w,Tau1,Tau2},{u,w-Tau1,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*(
                    Exp[-s w]/mean[T1,b1,q1])*REL[w-u,b1,q1],
               {w,Tau2,T2},{u,w-Tau1,Tau2},
               PrecisionGoal->3,AccuracyGoal->6],
                    NIntegrate[Zed[w,b2,q2]*(
                    Exp[-s w]/mean[T1,b1,q1])*
                    REL[w-u,b1,q1],{w,0,Tau1},{u,0,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
                    NIntegrate[Zed[w,b2,q2]*
                    (Exp[-s w]/mean[T1,b1,q1])*REL[w-u,b1,q1],
                    {w,Tau1,Tau2},{u,w-Tau1,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
                    NIntegrate[Zed[w,b2,q2]*
                    (Exp[-s w]/mean[T1,b1,q1])*REL[w-u,b1,q1],
                    {w,Tau2,Tau1+Tau2},{u,w-Tau1,Tau2},
                    PrecisionGoal->3,AccuracyGoal->6]
```

```
        ],
        If[T2<=Tau1+Tau2,
           If[Tau1<T2,
               NIntegrate[Zed[w,b2,q2]*
               (Exp[-s w]/mean[T1,b1,q1])*
               REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*
               (Exp[-s w]/mean[T1,b1,q1])*REL[w-u,b1,q1],
               {w,Tau2,Tau1},{u,0.0,Tau2},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b2,q2]*
               (Exp[-s w]/mean[T1,b1,q1])*REL[w-u,b1,q1],
               {w,Tau1,T2},{u,w-Tau1,Tau2},
               PrecisionGoal->3,AccuracyGoal->6],
                 NIntegrate[Zed[w,b2,q2]*
                    (Exp[-s w]/mean[T1,b1,q1])*
                    REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
                 NIntegrate[Zed[w,b2,q2]*
                    (Exp[-s w]/mean[T1,b1,q1])*
                    REL[w-u,b1,q1],{w,Tau2,T2},{u,0,Tau2},
                    PrecisionGoal->3,AccuracyGoal->6]
                    ],
                    NIntegrate[Zed[w,b2,q2]*
                    (Exp[-s w]/mean[T1,b1,q1])*
                    REL[w-u,b1,q1],{w,0,Tau2},{u,0,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
                    NIntegrate[Zed[w,b2,q2]*
                    (Exp[-s w]/mean[T1,b1,q1])*REL[w-u,b1,q1],
                    {w,Tau2,Tau1},{u,0.0,Tau2},
                    PrecisionGoal->3,AccuracyGoal->6]+
                    NIntegrate[Zed[w,b2,q2]*
                    (Exp[-s w]/mean[T1,b1,q1])*REL[w-u,b1,q1],
                    {w,Tau1,Tau1+Tau2},{u,w-Tau1,Tau2},
                    PrecisionGoal->3,AccuracyGoal->6]
        ]
      ],
If[Tau2<Tau1,
     If[T1<=Tau1+Tau2,
               NIntegrate[Zed[w,b1,q1]*
              (Exp[-s w]/mean[T2,b2,q2])*
              REL[w-u,b2,q2],{w,0,Tau2},{u,0,w},
              PrecisionGoal->3,AccuracyGoal->6]+
              NIntegrate[Zed[w,b1,q1]*
              (Exp[-s w]/mean[T2,b2,q2])*REL[w-u,b2,q2],
              {w,Tau2,Tau1},{u,w-Tau2,w},
              PrecisionGoal->3,AccuracyGoal->6]+
              NIntegrate[Zed[w,b1,q1]*
              (Exp[-s w]/mean[T2,b2,q2])*REL[w-u,b2,q2],
              {w,Tau1,T1},{u,w-Tau2,Tau1},
              PrecisionGoal->3,AccuracyGoal->6],
                   NIntegrate[Zed[w,b1,q1]*
                   (Exp[-s w]/mean[T2,b2,q2])*
                   REL[w-u,b2,q2],{w,0,Tau2},{u,0,w},
                   PrecisionGoal->3,AccuracyGoal->6]+
```

```
                    NIntegrate[Zed[w,b1,q1]*
                    (Exp[-s w]/mean[T2,b2,q2])*REL[w-u,b2,q2],
                    {w,Tau2,Tau1},{u,w-Tau2,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
                    NIntegrate[Zed[w,b1,q1]*
                    (Exp[-s w]/mean[T2,b2,q2])*REL[w-u,b2,q2],
                    {w,Tau1,Tau1+Tau2},{u,w-Tau2,Tau1},
                    PrecisionGoal->3,AccuracyGoal->6]
       ],
       If[T1<=Tau1+Tau2,
          If[Tau2<T1,
               NIntegrate[Zed[w,b1,q1]*
               (Exp[-s w]/mean[T2,b2,q2])*
               REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b1,q1]*
               (Exp[-s w]/mean[T2,b2,q2])*REL[w-u,b2,q2],
               {w,Tau1,Tau2},{u,0.0,Tau1},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b1,q1]*
               (Exp[-s w]/mean[T2,b2,q2])*REL[w-u,b2,q2],
               {w,Tau2,T1},{u,w-Tau2,Tau1},
               PrecisionGoal->3,AccuracyGoal->6],
                 NIntegrate[Zed[w,b1,q1]*
                    (Exp[-s w]/mean[T2,b2,q2])*
                    REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
                 NIntegrate[Zed[w,b1,q1]*
                    (Exp[-s w]/mean[T2,b2,q2])*
                    REL[w-u,b2,q2],{w,Tau1,T1},{u,0,Tau1},
                    PrecisionGoal->3,AccuracyGoal->6]
                    ],
                    NIntegrate[Zed[w,b1,q1]*
                    (Exp[-s w]/mean[T2,b2,q2])*
                    REL[w-u,b2,q2],{w,0,Tau1},{u,0,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
                    NIntegrate[Zed[w,b1,q1]*
                    (Exp[-s w]/mean[T2,b2,q2])*REL[w-u,b2,q2],
                    {w,Tau1,Tau2},{u,0.0,Tau1},
                    PrecisionGoal->3,AccuracyGoal->6]+
                    NIntegrate[Zed[w,b1,q1]*
                    (Exp[-s w]/mean[T2,b2,q2])*REL[w-u,b2,q2],
                    {w,Tau2,Tau1+Tau2},{u,w-Tau2,Tau1},
                    PrecisionGoal->3,AccuracyGoal->6]
       ]
     ]
  ]
];

BM4:=Function[{Tau1,Tau2,T1,T2,s},
          Jac[Tau1,Tau2,T1,T2,s]
          ];

LD:=Function[{Tau1,Tau2,T1,T2},
  If[T2<=T1,
     If[Tau1<Tau2,
```

```
  If[T2<=Tau1+Tau2,
       NIntegrate[Zed[w,b2,q2]*
       REL[w-u,b1,q1]/mean[T1,b1,q1],
       {w,0,Tau1},{u,0,w},
       PrecisionGoal->3,AccuracyGoal->6]+
       NIntegrate[Zed[w,b2,q2]*
       REL[w-u,b1,q1]/mean[T1,b1,q1],
       {w,Tau1,Tau2},{u,w-Tau1,w},
       PrecisionGoal->3,AccuracyGoal->6]+
       NIntegrate[Zed[w,b2,q2]*
       REL[w-u,b1,q1]/mean[T1,b1,q1],
       {w,Tau2,T2},{u,w-Tau1,Tau2},
       PrecisionGoal->3,AccuracyGoal->6],
            NIntegrate[Zed[w,b2,q2]*
            REL[w-u,b1,q1]/mean[T1,b1,q1],
            {w,0,Tau1},{u,0,w},
            PrecisionGoal->3,AccuracyGoal->6]+
            NIntegrate[Zed[w,b2,q2]*
            REL[w-u,b1,q1]/mean[T1,b1,q1],
            {w,Tau1,Tau2},{u,w-Tau1,w},
            PrecisionGoal->3,AccuracyGoal->6]+
            NIntegrate[Zed[w,b2,q2]*
            REL[w-u,b1,q1]/mean[T1,b1,q1],
            {w,Tau2,Tau1+Tau2},{u,w-Tau1,Tau2},
            PrecisionGoal->3,AccuracyGoal->6]
],
If[T2<=Tau1+Tau2,
  If[Tau1<T2,
       NIntegrate[Zed[w,b2,q2]*
       REL[w-u,b1,q1]/mean[T1,b1,q1],
       {w,0,Tau2},{u,0,w},
       PrecisionGoal->3,AccuracyGoal->6]+
       NIntegrate[Zed[w,b2,q2]*
       REL[w-u,b1,q1]/mean[T1,b1,q1],
       {w,Tau2,Tau1},{u,0.0,Tau2},
       PrecisionGoal->3,AccuracyGoal->6]+
       NIntegrate[Zed[w,b2,q2]*
       REL[w-u,b1,q1]/mean[T1,b1,q1],
       {w,Tau1,T2},{u,w-Tau1,Tau2},
       PrecisionGoal->3,AccuracyGoal->6],
        NIntegrate[Zed[w,b2,q2]*
            REL[w-u,b1,q1]/mean[T1,b1,q1],
            {w,0,Tau2},{u,0,w},
            PrecisionGoal->3,AccuracyGoal->6]+
         NIntegrate[Zed[w,b2,q2]*
            REL[w-u,b1,q1]/mean[T1,b1,q1],
            {w,Tau2,T2},{u,0,Tau2},
            PrecisionGoal->3,AccuracyGoal->6]
            ],
            NIntegrate[Zed[w,b2,q2]*
            REL[w-u,b1,q1]/mean[T1,b1,q1],
            {w,0,Tau2},{u,0,w},
            PrecisionGoal->3,AccuracyGoal->6]+
            NIntegrate[Zed[w,b2,q2]*
            REL[w-u,b1,q1]/mean[T1,b1,q1],
            {w,Tau2,Tau1},{u,0.0,Tau2},
```

```
                    PrecisionGoal->3,AccuracyGoal->6]+
                    NIntegrate[Zed[w,b2,q2]*
                    REL[w-u,b1,q1]/mean[T1,b1,q1],
                    {w,Tau1,Tau1+Tau2},{u,w-Tau1,Tau2},
                    PrecisionGoal->3,AccuracyGoal->6]
        ]
      ],
If[Tau2<Tau1,
         If[T1<=Tau1+Tau2,
              NIntegrate[Zed[w,b1,q1]*
              REL[w-u,b2,q2]/mean[T1,b2,q2],
              {w,0,Tau2},{u,0,w},
              PrecisionGoal->3,AccuracyGoal->6]+
              NIntegrate[Zed[w,b1,q1]*
              REL[w-u,b2,q2]/mean[T1,b2,q2],
              {w,Tau2,Tau1},{u,w-Tau2,w},
              PrecisionGoal->3,AccuracyGoal->6]+
              NIntegrate[Zed[w,b1,q1]*
              REL[w-u,b2,q2]/mean[T1,b2,q2],
              {w,Tau1,T1},{u,w-Tau2,Tau1},
              PrecisionGoal->3,AccuracyGoal->6],
                   NIntegrate[Zed[w,b1,q1]*
                   REL[w-u,b2,q2]/mean[T1,b2,q2],
                   {w,0,Tau2},{u,0,w},
                   PrecisionGoal->3,AccuracyGoal->6]+
                   NIntegrate[Zed[w,b1,q1]*
                   REL[w-u,b2,q2]/mean[T1,b2,q2],
                   {w,Tau2,Tau1},{u,w-Tau2,w},
                   PrecisionGoal->3,AccuracyGoal->6]+
                   NIntegrate[Zed[w,b1,q1]*
                   REL[w-u,b2,q2]/mean[T1,b2,q2],
                   {w,Tau1,Tau1+Tau2},{u,w-Tau2,Tau1},
                   PrecisionGoal->3,AccuracyGoal->6]
       ],
       If[T1<=Tau1+Tau2,
          If[Tau2<T1,
               NIntegrate[Zed[w,b1,q1]*
          REL[w-u,b2,q2]/mean[T1,b2,q2],{w,0,Tau1},{u,0,w},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b1,q1]*
          REL[w-u,b2,q2]/mean[T1,b2,q2],
               {w,Tau1,Tau2},{u,0.0,Tau1},
               PrecisionGoal->3,AccuracyGoal->6]+
               NIntegrate[Zed[w,b1,q1]*
          REL[w-u,b2,q2]/mean[T1,b2,q2],
               {w,Tau2,T1},{u,w-Tau2,Tau1},
               PrecisionGoal->3,AccuracyGoal->6],
                 NIntegrate[Zed[w,b1,q1]
                    REL[w-u,b2,q2]/mean[T1,b2,q2],
          {w,0,Tau1},{u,0,w},
                    PrecisionGoal->3,AccuracyGoal->6]+
                 NIntegrate[Zed[w,b1,q1]*
                    REL[w-u,b2,q2]/mean[T1,b2,q2],
               {w,Tau1,T1},{u,0,Tau1},
                    PrecisionGoal->3,AccuracyGoal->6]
                    ],
```

```
                        NIntegrate[Zed[w,b1,q1]*
                        REL[w-u,b2,q2]/mean[T1,b2,q2],
                        {w,0,Tau1},{u,0,w},
                        PrecisionGoal->3,AccuracyGoal->6]+
                        NIntegrate[Zed[w,b1,q1]*
                        REL[w-u,b2,q2]/mean[T1,b2,q2],
                        {w,Tau1,Tau2},{u,0.0,Tau1},
                        PrecisionGoal->3,AccuracyGoal->6]+
                        NIntegrate[Zed[w,b1,q1]*
                        REL[w-u,b2,q2]/mean[T1,b2,q2],
                        {w,Tau2,Tau1+Tau2},{u,w-Tau2,Tau1},
                        PrecisionGoal->3,AccuracyGoal->6]
         ]
       ]
   ]
];

BF4:=Function[{Tau1,Tau2,T1,T2},
          LD[Tau1,Tau2,T1,T2]
          ];
```

General Renewal Interval

```
PG1:=Function[{Tau1,Tau2,T1,T2,s},
If[T1>=T2,
If[T2>=Max[(Tau1-PU[q1,b1]),0],
NIntegrate[Zed[w,b2,q2] RESDL[w,T1,b1,q1]*Exp[-w s],
{w,Tau1,T2},PrecisionGoal->3,AccuracyGoal->6]+
REL[T2,b2,q2] RESDL[T2,T1,b1,q1]*Exp[-T2 s],0],
If[T1>=Max[(Tau2-PU[q2,b2]),0],
NIntegrate[Zed[w,b1,q1] RESDL[w,T2,b2,q2] Exp[-w s],
{w,Tau2,T1},PrecisionGoal->3,AccuracyGoal->6]+
REL[T1,b1,q1] RESDL[T1,T2,b2,q2]*Exp[-T1 s],0]
     ]
];


PG2:=Function[{Tau1,Tau2,T1,T2,s},
If[T1>=T2,
     If[T2>=Tau1+Tau2,
     NIntegrate[Zed[w,b2,q2]*W2[u,b1,q1,0,T1]*Exp[-w s]*
     REL[w-u,b1,q1],{w,Tau1,Tau2+Tau1},{u,0,w-Tau1},
     PrecisionGoal->3,AccuracyGoal->6]+
     NIntegrate[Zed[w,b2,q2]*W2[u,b1,q1,0,T1]*Exp[-w s]*
     REL[w-u,b1,q1],{w,Tau1+Tau2,T2},{u,0,Tau2},
     PrecisionGoal->3,AccuracyGoal->6],
     If[T2>=Tau1,
     NIntegrate[Zed[w,b2,q2]*W2[u,b1,q1,0,T1]*Exp[-w s]*
     REL[w-u,b1,q1],{w,Tau1,T2},{u,0,w-Tau1},
     PrecisionGoal->3,AccuracyGoal->6],0]
     ],
     If[T1>=Tau2+Tau1,
     NIntegrate[Zed[w,b1,q1]*W3[u,b2,q2,0,T2]*Exp[-w s]*
     REL[w-u,b2,q2],{w,Tau2,Tau1+Tau2},{u,0,w-Tau2},
     PrecisionGoal->3,AccuracyGoal->6]+
     NIntegrate[Zed[w,b1,q1]*W3[u,b2,q2,0,T2]*Exp[-w s]*
     REL[w-u,b2,q2],{w,Tau2+Tau1,T1},{u,0,w-Tau2},
     PrecisionGoal->3,AccuracyGoal->6],
          If[T1>=Tau2,
     NIntegrate[Zed[w,b1,q1]*W3[u,b2,q2,0,T2]*Exp[-w s]*
     REL[w-u,b2,q2],{w,Tau2,T1},{u,0,w-Tau2},
          PrecisionGoal->3,AccuracyGoal->6],0]
          ]
     ]
];
PG3:=Function[{Tau1,Tau2,T1,T2,s},
     If[T1>=T2,
          If[T2>=Tau1+Tau2,
     NIntegrate[REL[T2,b2,q2]*W2[u,b1,q1,0,T1]*Exp[-T2 s]*
     REL[T2-u,b1,q1],{u,0,Tau2},
     PrecisionGoal->3,AccuracyGoal->6],0]+
     If[T2<Tau1+Tau2 && T2>Tau1,
     NIntegrate[REL[T2,b2,q2]*W2[u,b1,q1,0,T1]*Exp[-T2 s]*
     REL[T2-u,b1,q1],{u,0,T2-Tau1},
     PrecisionGoal->3,AccuracyGoal->6],0],
If[T1>=Tau1+Tau2,
```

```
     NIntegrate[REL[T1,b1,q1]*W3[u,b2,q2,0,T2]*Exp[-T1 s]*
     REL[T1-u,b2,q2],{u,0,Tau1},
     PrecisionGoal->3,AccuracyGoal->6],0]+
     If[T1<Tau1+Tau2 && T1>Tau2,
     NIntegrate[REL[T1,b1,q1]*W3[u,b2,q2,0,T2]*Exp[-T1 s]*
     REL[T1-u,b2,q2],{u,0,T1-Tau2},
     PrecisionGoal->3,AccuracyGoal->6],0]
     ]
];
PG4:=Function[{Tau1,Tau2,T1,T2,s},
If[T1>=T2,
     NIntegrate[REL[u,b2,q2] L[u,T1,b1,q1]*Exp[-s u],
     {u,Tau2,T2},PrecisionGoal->3,AccuracyGoal->6]+
     NIntegrate[REL[u,b2,q2]*W[v,b1,q1,0,T1]*
     Zed[u-v,b1,q1]*Exp[-s u],{u,Tau2,T2},{v,0,Tau2},
     PrecisionGoal->3,AccuracyGoal->6],
     NIntegrate[REL[u,b1,q1] L[u,T2,b2,q2]*Exp[-s u],
     {u,Tau1,T1},PrecisionGoal->3,AccuracyGoal->6]+
     NIntegrate[REL[u,b1,q1]*W[v,b2,q2,0,T1]*
     Zed[u-v,b2,q2]*Exp[-s u],{u,Tau1,T1},{v,0,Tau1},
     PrecisionGoal->3,AccuracyGoal->6]
     ]
];
MTTYGEN[Tau1_,Tau2_,T1_,T2_,s_]:=MTTYGEN[Tau1,Tau2,T1,T2,s]=
PG1[Tau1,Tau2,T1,T2,s]+PG2[Tau1,Tau2,T1,T2,s]+
PG3[Tau1,Tau2,T1,T2,s]+
PG4[Tau1,Tau2,T1,T2,s];

MG1:=Function[{Tau1,Tau2,T1,T2,s},
If[T1>=T2,
If[T2>=Max[(Tau1-PU[q1,b1]),0],
NIntegrate[Zed[w,b2,q2]
RESDL[w,T1,b1,q1]*(g[s,ld1] g[s,ld2]) Exp[-w s],
{w,Max[(Tau1-PU[q1,b1]),0],T2},
PrecisionGoal->3,AccuracyGoal->6]+
REL[T2,b2,q2]
RESDL[T2,T1,b1,q1]*(g[s,ld1] g[s,ld2]) Exp[-T2 s],0],
If[T1>=Max[(Tau2-PU[q2,b2]),0],
NIntegrate[Zed[w,b1,q1]
RESDL[w,T2,b2,q2]*(g[s,ld1] g[s,ld2]) Exp[-w s],
{w,Max[(Tau2-PU[q2,b2]),0],T1},
PrecisionGoal->3,AccuracyGoal->6]+
REL[T1,b1,q1] RESDL[T1,T2,b2,q2]*
(g[s,ld1] g[s,ld2]) Exp[-T1 s],0]
     ]
];

MG2:=Function[{Tau1,Tau2,T1,T2,s},
If[T1>=T2,
     If[T2>=Tau1+Tau2,
     NIntegrate[Zed[w,b2,q2]*W2[u,b1,q1,s,T1]*Exp[-w s]*
     REL[w-u,b1,q1],{w,Tau1,Tau2+Tau1},{u,0,w-Tau1},
     PrecisionGoal->3,AccuracyGoal->6]+
     NIntegrate[Zed[w,b2,q2]*W2[u,b1,q1,s,T1]*Exp[-w s]*
     REL[w-u,b1,q1],{w,Tau1+Tau2,T2},{u,0,Tau2},
```

```
      PrecisionGoal->3,AccuracyGoal->6],
      If[T2>=Tau1,
      NIntegrate[Zed[w,b2,q2]*W2[u,b1,q1,s,T1]*Exp[-w s]*
      REL[w-u,b1,q1],{w,Tau1,T2},{u,0,w-Tau1},
      PrecisionGoal->3,AccuracyGoal->6],0]
      ],
      If[T1>=Tau2+Tau1,
      NIntegrate[Zed[w,b1,q1]*W3[u,b2,q2,s,T2]*Exp[-w s]*
      REL[w-u,b2,q2],{w,Tau2,Tau1+Tau2},{u,0,w-Tau2},
      PrecisionGoal->3,AccuracyGoal->6]+
      NIntegrate[Zed[w,b1,q1]*W3[u,b2,q2,s,T2]*Exp[-w s]*
      REL[w-u,b2,q2],{w,Tau2+Tau1,T1},{u,0,w-Tau2},
      PrecisionGoal->3,AccuracyGoal->6],
      If[T1>=Tau2,
      NIntegrate[Zed[w,b1,q1]*W3[u,b2,q2,s,T2]*Exp[-w s]*
      REL[w-u,b2,q2],{w,Tau2,T1},{u,0,w-Tau2},
      PrecisionGoal->3,AccuracyGoal->6],0]
           ]
      ]
];
MG3:=Function[{Tau1,Tau2,T1,T2,s},
      If[T1>=T2,
           If[T2>=Tau1+Tau2,
NIntegrate[REL[T2,b2,q2]*W2[u,b1,q1,s,T1]*Exp[-T2 s]*
      REL[T2-u,b1,q1],{u,0,Tau2},
      PrecisionGoal->3,AccuracyGoal->6],0]+
      If[T2<Tau1+Tau2 && T2>Tau1,
      NIntegrate[REL[T2,b2,q2]*W2[u,b1,q1,s,T1]*Exp[-T2 s]*
      REL[T2-u,b1,q1],{u,0,T2-Tau1},
      PrecisionGoal->3,AccuracyGoal->6],0],
If[T1>=Tau1+Tau2,
      NIntegrate[REL[T1,b1,q1]*W3[u,b2,q2,s,T2]*Exp[-T1 s]*
      REL[T1-u,b2,q2],{u,0,Tau1},
      PrecisionGoal->3,AccuracyGoal->6],0]+
      If[T1<Tau1+Tau2 && T1>Tau2,
      NIntegrate[REL[T1,b1,q1]*W3[u,b2,q2,s,T2]*Exp[-T1 s]*
      REL[T1-u,b2,q2],{u,0,T1-Tau2},
      PrecisionGoal->3,AccuracyGoal->6],0]
      ]
];
MG4:=Function[{Tau1,Tau2,T1,T2,s},
If[T1>=T2,
      NIntegrate[REL[u,b2,q2]
L[u,T1,b1,q1]*(g[s,ld1] g[s,ld2])*Exp[-s u],
      {u,Tau2,T2},PrecisionGoal->3,AccuracyGoal->6]+
      NIntegrate[REL[u,b2,q2]*W[v,b1,q1,s,T1]*
      Zed[u-v,b1,q1]*Exp[-s u],{u,Tau2,T2},{v,0,Tau2},
      PrecisionGoal->3,AccuracyGoal->6],
      NIntegrate[REL[u,b1,q1]
L[u,T2,b2,q2]*g[s,ld1]*g[s,ld2]*Exp[-s u],
      {u,Tau1,T1},PrecisionGoal->3,AccuracyGoal->6]+
      NIntegrate[REL[u,b1,q1]*W[v,b2,q2,s,T2]*
      Zed[u-v,b2,q2]*Exp[-s u],{u,Tau1,T1},{v,0,Tau1},
      PrecisionGoal->3,AccuracyGoal->6]
      ]
];
```

```
MRENGEN[Tau1_,Tau2_,T1_,T2_,s_]:=
MRENGEN[Tau1,Tau2,T1,T2,s]=
MG1[Tau1,Tau2,T1,T2,s]+MG2[Tau1,Tau2,T1,T2,s]+
MG3[Tau1,Tau2,T1,T2,s]+
MG4[Tau1,Tau2,T1,T2,s];
```

TIME IN AN INITIAL INTERVAL

```
(*PHI(General)                                              *)

PHIInitial[Tau1_,Tau2_,T1_,T2_,s_]:=
PHIInitial[Tau1,Tau2,T1,T2,s]=
A4[Tau1,Tau2,T1,T2,s]+A3[Tau1,Tau2,T1,T2,s]+
A1[Tau1,Tau2,T1,T2,s]+A2[Tau1,Tau2,T1,T2,s];
```

FAILURE TIME IN AN INITIAL INTERVAL

```
(*PHI(General)                                              *)

ReInitial[Tau1_,Tau2_,T1_,T2_,s_]:=
ReInitial[Tau1,Tau2,T1,T2,s]=
AM4[Tau1,Tau2,T1,T2,s]+AM3[Tau1,Tau2,T1,T2,s]+
AM1[Tau1,Tau2,T1,T2,s]+AM2[Tau1,Tau2,T1,T2,s];
```

FAILURE TIME IN A GENERAL INTERVAL

```
(*PHI(General)                                              *)

ReGeneral[Tau1_,Tau2_,T1_,T2_,s_]:=
ReGeneral[Tau1,Tau2,T1,T2,s]=
BM4[Tau1,Tau2,T1,T2,s]+BI3[Tau1,Tau2,T1,T2,s]+
BM2[Tau1,Tau2,T1,T2,s]+BI1[Tau1,Tau2,T1,T2,s];
```

PROBABILITY OF AN INITIAL INTERVAL

```
(*PHI(General)                                              *)

PHIIPROB[Tau1_,Tau2_,T1_,T2_]:=
PHIIPROB[Tau1,Tau2,T1,T2]=AP4[Tau1,Tau2,T1,T2]+AP3[Tau1,Tau2,T1,T2]+
AP2[Tau1,Tau2,T1,T2]+AP1[Tau1,Tau2,T1,T2];

ZI[Tau1_,Tau2_,T1_,T2_,s_]:=1/s -
(ReInitial[Tau1,Tau2,T1,T2,s]+
MYTTINIT[Tau1,Tau2,T1,T2,s])/s;

QI[Tau1_,Tau2_,T1_,T2_,s_]:=PHIIPROB[Tau1,Tau2,T1,T2]/s-
PHIInitial[Tau1,Tau2,T1,T2,s]/s;
```

TIME IN A GENERAL INTERVAL

```
(*PHI general                                                     *)
PHIGENERAL[Tau1_,Tau2_,T1_,T2_,s_]:=
PHIGENERAL[Tau1,Tau2,T1,T2,s]=
B4[Tau1,Tau2,T1,T2,s]+B3[Tau1,Tau2,T1,T2,s]+
B1[Tau1,Tau2,T1,T2,s]+B2[Tau1,Tau2,T1,T2,s];
```

PROBABILITY OF A GENERAL INTERVAL

```
PHIJJPROB[Tau1_,Tau2_,T1_,T2_]:=
PHIJJPROB[Tau1,Tau2,T1,T2]=
BF4[Tau1,Tau2,T1,T2]+BJ3[Tau1,Tau2,T1,T2]+
BF2[Tau1,Tau2,T1,T2]+BJ1[Tau1,Tau2,T1,T2];

ZG[Tau1_,Tau2_,T1_,T2_,s_]:=1/s -
(ReGeneral[Tau1,Tau2,T1,T2,s]+
MTTYGEN[Tau1,Tau2,T1,T2,s])/s;

QG[Tau1_,Tau2_,T1_,T2_,s_]:=PHIJJPROB[Tau1,Tau2,T1,T2]/s -
PHIGENERAL[Tau1,Tau2,T1,T2,s]/s;
```

AVAILABILITY

```
Avail[Tau1_,Tau2_,T1_,T2_,s_]:=
(ZI[Tau1,Tau2,T1,T2,s]*
(1-PHIGENERAL[Tau1,Tau2,T1,T2,s])+
ZG[Tau1,Tau2,T1,T2,s]*PHIInitial[Tau1,Tau2,T1,T2,s])/
((1-PHIGENERAL[Tau1,Tau2,T1,T2,s])-(
(MRENGEN[Tau1,Tau2,T1,T2,s]*PHIInitial[Tau1,Tau2,T1,T2,s]+
(1-PHIGENERAL[Tau1,Tau2,T1,T2,s])*
MRENINIT[Tau1,Tau2,T1,T2,s])));


csteh[n_, i_] := (-1)^(i + n/2) Sum[ k^(n/2) (2 k)! /
     ( (n/2 - k)! k! (k - 1)! (i - k)! (2 k - i)! ),
     { k, Floor[ (i+1)/2 ], Min[ i, n/2] } ] //N;

Availability[Tau1_,Tau2_,T1_,T2_,t_]:=
(Log[2]/t) Sum[csteh[6,i] Avail[Tau1,Tau2,T1,T2,(Log[2] i/t)],
{i,1,6}]//N
```

VITA

Alfred Tsatsu Degbotse was born in Accra, Ghana. He received his B.s degree in Mathematics from the University of Science and Technology, Kumasi Ghana in 1987, and the M.S degree in Statistics from West Virginia University in 1992. Since 1993 he has been a graduate student in the Industrial and System Engineering Department at Virginia Tech where he recently completed the requirements for the Ph.D. degree.