

Mathematical Analysis of a Large-Gap Electromagnetic Suspension System

by

Addison B. Jump

Dissertation submitted to the Faculty of the

Virginia Polytechnic Institute and State University

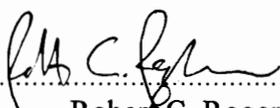
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

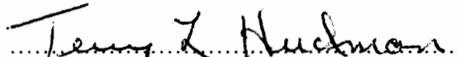
APPROVED:



Robert C. Rogers, chair



John A. Burns



Terry L. Herdman



David L. Russell



Christopher A. Beattie

April, 1996

Blacksburg, Virginia

C. 2

LD
5655
V856
1996
J367
c.2

MATHEMATICAL ANALYSIS OF A LARGE-GAP ELECTROMAGNETIC SUSPENSION SYSTEM

by

Addison B. Jump

Dr. Robert C. Rogers, Chairperson

Mathematics Department

In a form of controlled electromagnetic suspension, a permanent magnetic is levitated by a magnetic field; the field is produced by electrical currents passing through coils. These currents are the control input. In a Large-Gap system the coils are at some distance from the suspended body; in general, there is no closed form expression relating the currents to the flux at the point of the suspended body. Thus, in the general case, it is not possible to establish control-theoretic results for this kind of Large-Gap suspension system. It is shown, however, that if the coil placement configuration exhibits a particular cylindrically symmetric structure, expressions can be found relating the coil positions to the flux. These expressions are used to show the existence of a unique equilibrium point and controllability, in five dimensions of control, for a generic form of Large-Gap system. The results are shown to remain true if the suspended body is rotated about a particular axis. Closed form expressions are found for the currents required to suspend the body at these variable orientations. An inequality between difference classes of experimental inputs is shown to be a necessary condition for suspension of the body. It is demonstrated that the addition of coils to the system cannot lead to six dimensions of controllability.

Let the system be given by the standard control equation

$$\dot{x} = Ax + Bu$$

Closed form expressions are found for the eigenvalues of A . In the course of proving that some coil placement restrictions may be relaxed, B is shown to be related to the Vandermonde matrix.

Acknowledgments

I must first thank my principle advisor, Dr. Robert Rogers, not only for his encouragement and advice during the writing of this dissertation, but also while I was his student at Virginia Tech. I thank all the members of my committee; particular thanks go to Dr. Russell, who helped conceive the initial project, and Dr. Beattie, who also assisted me when I was his student at Tech. I am indebted to Dr. Rossi, and to Dr. Bowan of the physics department, for encouraging me to pursue a doctorate in mathematics; to the Defense Dept., and the taxpayers, who made it financially possible; and to all my supervisors at Naval Surface Warfare Center who signed the papers allowing me to continue in the program. I especially thank Dr. Richard Hartman, who was my immediate supervisor while most of this dissertation was being written. The dissertation work has not permitted me to spend as much time with my two children, Aaron and Kimberly, as I would have wished, so I must thank them for their patience.

Contents

1	Introduction.....	1
1.1	Electromagnetic Suspension and Its Applications.....	1
1.2	Background and Nature of the Dissertation; Experimental and General Systems...3	
2	The Large-Gap Magnetic Suspension System.....	5
2.1	General Definition of the Large-Gap, Light Load System.....	5
2.2	Description and Definitions of the Experimental and General Systems and Their Variations.....	5
2.3	Notation and Coordinate System.....	10
2.4	Electromagnetic Equations.....	11
2.5	Equations of Controlling Forces and Torques.....	12
2.6	Transformation of Force-Torque Equations Into the Body Coordinate System..	15
2.7	Magnetic Field as a Function of Current; Linearity Condition.....	17
2.8	Controlling Force-Torque as a Function of Current.....	19
2.9	Equations of Motion.....	20
2.10	Approximate Equations of Motion -- The Basic System.....	24
2.11	Linearized Equations of Motion -- Expressions for the State Vector.....	28
2.12	Linearized Equations of Motion -- Equilibrium Point.....	29
2.13	Linearized Equations of Motion.....	32
2.14	Suspension After Rotation Through a Finite Angle -- The Rotated System.....	37
2.15	Methods of Observation and Control.....	44
2.16	Additional Results of the Experiment; Notes on the Additional Coils System.....	46
2.17	Recent Experiments at NASA Langley.....	48

Contents (continued)

3	Goals and Difficulties in the Mathematical Treatment of Large-Gap Systems; A Brief Discussion of Three Other Systems.....	50
4	Invertibility of the Equilibrium Matrix and Related Results.....	52
4.1	Analytic Expression for the Inverse of the Equilibrium Matrix for the Basic System.....	52
4.2	Expression for the Equilibrium Currents; Methods of Verification.....	63
4.3	Analytic Expression for the Inverse of the Equilibrium Matrix for the Rotated System.....	64
4.4	Alternative Notation and Numerical Verification of Equations; Equilibrium Currents for the Rotated System.....	69
5	Proof of Controllability.....	70
6	Invariance of the Dynamics Matrix.....	75
7	Factorization and Eigenvalues of the Dynamics Matrix, Basic and Rotated Systems; Suspension Constraint.....	80
7.1	Expression for Equilibrium B-Values.....	80
7.2	Analysis Using a Dynamics Matrix of Reduced Dimension.....	84
7.3	Factorization of the Dynamics Matrix.....	85
7.4	Eigenvalues of the Dynamics Matrix.....	89
7.5	Suspension Constraint.....	92

Contents (continued)

8 The Additional Coils System.....	95
8.1 First Method, Five Degrees of Freedom of Control.....	96
8.2 Second Method, Six Degrees of Freedom of Control.....	100
8.3 Eigenvalues of the Dynamics Matrix of the Additional Coils System.....	103
9 Proof of Controllability for Arbitrary Angles System; Relation to the Vandermonde Matrix.....	104
10 Generalization of Results and Prospects for Future Work.....	116
10.1 Expressions for Flux Resulting from an Arbitrary Coil Configuration.....	116
10.2 More General Body Systems.....	119
10.3 Use of Less Than Five Coils.....	122
10.4 Other Prospects for Future Work.....	122
10.5 Summary.....	124
Appendix 1: Mathematical Symbols Used.....	125
Appendix 2: Assumptions Made.....	128
References.....	129
Figures.....	133
Vita.....	151

1 Introduction

1.1 Electromagnetic Suspension and Its Applications

ElectroMagnetic Suspension (EMS) is the science of levitating a body with electromagnetically generated forces. As a control problem, the output is a function of the position and orientation of the suspended body. The body must then somehow be observed. The control input is the electromagnetic field around the body; this field in turn is usually generated by a controlled set of currents or voltages. The control parameters are therefore the currents or voltages.

EMS has a surprisingly wide range of applications. Reference [1] contains citations to such pure science applications as the Search for Free Quark Production (pg 7) and Absolute Determination of the Magnetic Flux Quantum (pg 8). It includes references to fascinating, innovative uses of interest to environmentalists, such as Flywheel Energy-Storage-and-Conversion System for Photovoltaic Applications, an attempt to store solar energy in the form of a rotating, levitated body (pg 9). But in addition to its scientific and speculative uses, EMS has been shown to be viable in more practical settings. Also included in [1] are industrial applications, such as high-precision accelerometers (pgs 1 and 5), magnetic bearings (pgs 11 and 12), and shock and vibration isolation (pg 12). EMS is widely used for containerless heating and solidification of melts, and in investigations of the physical and chemical properties of materials [2]. EMS (as well as electrostatic suspension, which is similar) is used in gyroscopes, and as of June, 1991, the world's most accurate gyro was an electrostatically levitated system [3]. Probably the most publicized use of EMS is in Magnetically Levitated Trains (MAGLEV). Both [1] and [4] have numerous references to MAGLEV. EMS is valuable in the low gravity environment of space vehicles, and [4] has a number of papers on this, as well as articles on suspension of model aircraft in wind tunnel tests. The dissertation will examine an experiment relevant to both these applications.

The above list is by no means exhaustive. In fact, the use of EMS is indicated whenever we wish to control a body without having material contact with that body. The reasons for avoiding material contact are various. EMS is desirable in studies of the properties of materials because contact with a supporting structure may contaminate the body under study. It is used in wind tunnel tests because conventional supports distort the airflow around the model and because, with conventional supports, it is difficult to move the model to different orientations. But probably the commonest reason for using EMS is that the absence of material contact greatly reduces friction and vibration. Friction and vibration not only increase wear and dissipate energy; in delicate instruments, their effects must be accounted for in the dynamics and control equations of the system, and this is often impossible to do perfectly. As expressed in [5]:

Conventional mechanical structures suffer from friction and backlash so that advanced control is a mere theory or an impractical proposition. On the other hand, the contactless structure permits magnetic levitation to function as a precision force/position controllable actuator of multi-degrees of freedom possessing the capability of multi-axis force sensing without any additional mechanisms.

It might be surmised that EMS would be used even more widely than it is. For example, in almost all mechanical devices, friction and vibration are unwanted side effects. Why not use EMS to eliminate them? The answer is, of course, that EMS is a relatively high-tech engineering accomplishment. It is expensive and energy consuming. The forces generated are usually weak compared to more conventionally produced forces. Levitated trains will apparently become a reality in the United States but we should not expect ball bearings in automobiles to be eliminated in the near future.

But as the cost, performance, and operational life of mechanical devices increase, it becomes easier to justify the use of EMS components in them. This use becomes still

easier to justify as the materials and devices which go to make up EMS systems continue to become cheaper and more efficient. Also, some EMS applications have proved marginally successful in the past because completely adequate control methods could not be found for them [6, pgs 7-8]. With the advent of improved methods and hardware for observation and computation, and a higher state of control theory, these applications may become practical.

1.2 Background and Nature of the Dissertation; Experimental and General Systems

The initial goal of this dissertation was to examine in the abstract the control of a permanent magnet by electromagnetic forces alone. The shape of the magnet, the nature of the electromagnetism, the means of observation, and so on, were conceived as arbitrary. We hoped that general theorems could be established and that these could then be specialized to particular cases. However, we obtained very little in the way of significant results. It appears that, unlike some questions in physics, the study of EMS, at least in this stage of its development, can hardly be examined apart from particular physical devices. It is a little like studying, say, the general question of the control of heavier than air flying devices without specifying that they are assumed to have wings, tail, rudder, and ailerons, and the nature and position of these. Without this specificity, the control equations for such devices are so general as to be almost meaningless.

We then began a study of the extensive literature dealing with electromagnetic suspension. We found a particular experiment which was similar in some respects to our abstract problem. The dissertation is essentially the mathematical analysis of this single experiment. We establish control theoretic results which are more general and formal than those established in the paper describing this experiment; yet there is still a great deal of specificity in our results. We will try to make clear the greater generality of our results and, at the same time, their limitations. We will often refer to the specific experiment which was the starting point for our study. We will call it the "Experimental System". By

this we mean the specific devices and configuration of the experiment *together with* the specific numbers associated with these devices. By contrast, we will refer to our abstraction from the Experimental System as the "General System". There are variations within both these systems, which we will describe in Section 2.2. Since we feel we have abstracted the all the essential characteristics of the Experimental System into the General System, all results that are true of the General System are also true of the Experimental System. For those variations in the Experimental System which were successful, the constraints derived for the General System were satisfied by the Experimental System.

The results we derive will be seen to be of quite limited generality. We believe that this dissertation establishes the negative result that the nature of the system described below limits the control-theoretic formal development that can be obtained for it. (The specific difficulty is noted in Section 2.1 below and discussed more fully in Section 3.) We think, however, that these limits have not been reached. We hope our results indicate the fundamental requirements for future development (see Section 10).

Control-theoretic results for heavier than air devices are of course today quite extensive, powerful, and general. Probably the initial investigations in this area were for rather specific systems, but drew on the established general theories of physics as a framework. From the study of a great many devices, a general pool of control theory emerged. In this study we begin with a specific device. We then develop a formalism which allows us to draw on general control theory. The theory is applied to a system somewhat more general than the original one. We hope our results are a start toward a systematic study of this type of suspension system; and, also, that they may help to delimit and define the possibilities of using control theory for this class of system.

2 The Large-Gap Magnetic Suspension System

2.1 General Definition of the Large-Gap, Light Load System

Our Experimental System is an example of a Large-Gap Magnetic Suspension System (LGMSS). "Large-Gap" refers to the fact that there is a relatively large open space around the suspended object. Therefore, the devices which produce the electromagnetic field are at some distance from the suspended body. This often makes it difficult or impossible to find an analytic expression for the electromagnetic field at the body, and the fields must be calculated numerically. This appears to essentially limit the possibilities of control theory derivable for such systems. Also, whatever devices are used to observe the body must be some distance from it. In the general LGMSS, large external forces (or "loads") other than the control forces may act on the body; but in our Experimental System, the force of gravity will be the only external force. It is thus called a "light-load" LGMSS. Of course, it is only one type of light-load LGMSS.

2.2 Description and Definitions of the Experimental and General Systems and Their Variations

The Experimental System is but one in a series of experiments that have been carried out, and are ongoing, at NASA Langley Research Center, Hampton, Virginia. Most of the theory in this section has largely been taken from [7], with some changes in notation and emphasis. Figure 1 shows the physical configuration of the basic Experimental System, and all the systems we will consider in this dissertation have the same general features show here. A single small cylindrical core of permanently magnetized material is embedded in the suspended body. The body that the core is embedded in is shown schematically since it will play no essential role in the analysis. The magnetization of the core is of constant magnitude in a single direction. An electromagnetic field will act on

this core to produce forces and torques. The field is produced by coils through which slowly varying direct current passes. The coils are mounted in a planar array. In the case shown in Figure 1 there are five coils, but the number of coils is variable and will give rise to variations within the Experimental System. The LGMSS technology we will describe has a large range of applications, including microgravity and vibration isolation systems, magnetically suspended pointing mounts, large-angle magnetic suspension systems for advanced actuators, wind tunnel magnetic suspension systems, and remote manipulation, control, and positioning of objects in space [8].

Of course, the general LGMSS can vary greatly from this. The actual materials and physical principles used may be different. For example, iron magnets may be used rather than permanent magnets. Alternating current may be used rather than direct current. Reference [6] is a good starting point for information about the great variety of systems that may be used in wind tunnel applications alone. But it is important to note that even given the same basic physical situation, a great variety of configurations of both coils and magnets is possible. There may be more than one magnetized core embedded in the body; and the coils need not be in a planar array, as shown in Figure 2. In the final section, we touch on the application of our theory to such systems. However, all the systems we will study in a systematic way have a great deal in common with Figure 1.

Before describing and defining these systems and their variations, we will first define some variables and terms which will be used throughout the dissertation. In all systems the coils will lie in a plane with the center of each coil on a circle. See Figure 3. The circle is denoted C and its radius r . The suspended body will be on a line which passes through the center of C and is perpendicular to the plane of C . It will be at height h above the plane. The coils in any one system will be identical. Any individual coil has defining characteristics, such as height, inner radius, outer radius, and so on. While these characteristics are important design considerations, we will not be concerned with the specific details of coil parameters in this dissertation. We merely note that the type of coil used is an important variable for any system.

The variables mentioned above - the type of coil used, the radius r of the circle the coils are placed on, and the height h of the body above the coils - all deal with the coils themselves or how they are placed in relation to the body. It will be useful to group all such variables together, and we will call them Coil Values. A more accurate term would be "coil and/or r , h values" but we will use Coil Values in the interest of brevity. If any variable is a function of *only* Coil Values, we will call it also a Coil Value. We will see in Section 4 the *magnitude* of the flux produced by any coil at the body is a Coil Value. However, the *direction* of the horizontal component of this flux vector is determined solely by the angle λ in Figure 3. We will refer to the collection of all such angles in a coil configuration as the Angular Values for that system. Again, if any variable is determined solely by Angular Values, it too will be called an Angular Value.

One final type of parameter is needed to specify the physical elements of a system. These are the values that characterize the suspended body, such as its mass, volume, moments of inertia, magnetization, and so on. We will include the gravitation constant g among these parameters since it is needed to find the weight of the body. All these values, together with any variable which is determined solely by them, are called Body Values.

In any Experimental System, the Coil, Angular, and Body Values are considered to be fixed, specific numerical data. In any General System, these are considered to be variable, within certain limitations. One limitation we will make is the position of the suspended body in relation to the pattern of flux produced by the coils, which we now specify. In Figure 3, plane P passes through a typical coil, and this plane is reproduced schematically in Figure 4. One loop of flux produced by the coil is shown. As a matter of sign convention, we will consider that a positive current through the coil produces the orientation of the flux shown; a negative current would cause the arrows on the flux line to point in the opposite direction. B_H will denote the horizontal component of flux and B_z the vertical. In the drawing the suspended body is at point P_1 . Note that if the body were at point P_2 , the vertical flux would be zero. Since this would be undesirable for suspension, we will assume it is not the case for the General System. If the body were at point P_3 , the horizontal component of flux would be zero, which is also undesirable.

However, since we assume h is positive, this point is excluded. For later reference, we note that B_H will also be pointing in the positive x_p direction, toward the coil, so long as h is positive.

Now note that when the body is at a point like P_1 , B_z is pointing in the negative z direction. If the body were on the other side of P_2 , at P_1' , B_z would be positive. In the Experimental System, B_z is negative, as in Figure 4. There are additional considerations which lead us to believe that an actual suspension system could not suspend the body at a point like P_1' . We therefore assume the following:

The suspended body is in the upper half plane, that is, h is positive; the vertical component of flux due to each coil is negative. (Assumption 1)

Other restrictions must be placed on all the General System variations in order for them to be operational, or for the theory to apply. They can best be understood at the point in the exposition where the need for them arises and will be listed there. However, for convenience we have listed all the Assumptions in Appendix 2. All Assumptions apply to every General System. We now define each of our systems and subsystems. The listing is for reference and to provide the reader a very cursory notion of the variations:

Definition 2-2-1: The Basic Experimental System is as shown in Figure 1: five coils with equal angles between coils. The suspended body is fixed at the position shown in Figure 1. The Basic General System is the same as the corresponding Experimental System except that Body and Coils Values are variable, within the restrictions mentioned above. Thus, the Angular Values are the same as those of the Experimental System. The angles between the five coils are equal; each is $360/5 = 72$ degrees.

Definition 2-2-2: The Rotated Experimental System is the same as the Basic System except the body is rotated through a fixed (but arbitrary) angle θ_z about the z axis, as shown in Figure 5. The Rotated General System is again the same as the Experimental System, except with variable Body and Coil values.

Definition 2-2-3: The Additional Coils Experimental System is like the Basic System except that six, seven, or eight coils are used instead of five. The angles between coils are equal. See Figures 6, 7, and 8. This System has two subvariations: one in which five degrees of freedom of control was achieved, and another in which six degrees of freedom of control was attempted. The Additional Coils General System is like the corresponding Experimental system except that the number of coils is may be any number greater than five. As in the Experimental case, angles between coils are equal; Coil and Body Values are variable.

Definition 2-2-4: The Arbitrary Angles General System is the only system which has no experimental counterpart. It has five coils, with arbitrary Coil and Body Values, as in the Basic system; however, the Angular Values are now arbitrary; the angles between coils can be anything. See Figure 9.

We now develop the fundamental physics of our system and the mathematical apparatus. The reader may think of these as essentially the same for all systems. However, Sections 2.3 through 2.9 apply specifically to the both the Basic and Rotated Systems; 2.10 through 2.13 apply only to the Basic System and 2.14 is the corresponding exposition for the Rotated System. Results original in this dissertation begin in Section 4, and from this point on section headings indicate the systems to which the results apply.

2.3 Notation and Coordinate System

We first develop the notation and coordinate system for our example. A complete list of conventions is given in the Appendix 1; here we will describe only those conventions which may be unfamiliar to the reader. Referring to Figure 1, the \bar{x} , \bar{y} , \bar{z} system is fixed to the levitated body. (We will sometimes call this suspended element the "core" or simply the "body"). In general, a bar over a variable means that the variable is in body coordinates. Again referring to Figure 1, the x_b, y_b, z_b system is fixed to the coil configuration; it is motionless or "inertial". The x, y, z system is parallel to the x_b, y_b, z_b system and at a height h above it. Of course it is also inertial. For purposes of analysis, both the inertial systems are equivalent so when we speak an inertial system, we mean either the x, y, z or the x_b, y_b, z_b system. The only non-inertial system is the body system and variables in this system will have a bar over them. This greatly simplifies notation. We will call the $\bar{x}, \bar{y}, \bar{z}$ system the "body" system, x_b, y_b, z_b the "coil" system, and x, y, z the "lab" system. However, note that in Figure 1 the body and lab systems coincide. When this is the case, and the body is motionless, we will say the body is at the "equilibrium point" or "equilibrium". Note that the equilibrium point is not a point in three dimensional space. When the body is in the equilibrium point, its center is at the origin of the lab system, which is a point in three dimension space; but, also, its coordinate axes are parallel to the lab axes and it is motionless. We wish to distinguish the equilibrium point from the point in three dimension space just mentioned; this latter point, the origin of the lab system, will be called the "datum point". To give the reader a rough idea of the dimensions of the Experimental System, the height h that the body is suspended above the coils is about one meter and the entire suspended body weighs about 23 kilograms [7, pg 15].

The core is uniformly magnetized in the \bar{x} direction. In wind tunnel applications, this would be the direction that the nose of the aircraft would point. The \bar{y} direction would be the direction of the left wing and \bar{z} would be "up" for the pilot. We will later show that with uniform magnetization in the \bar{x} direction, it is not possible to induce any torque about the \bar{x} axis ("roll" in aerodynamics jargon). However, we wish to be able to slightly rotate the body about the \bar{z} axis ("yaw") and about the \bar{y} axis ("pitch"). We also want to be able to produce small translation of the body in all three directions. Thus, in some sense, the

body must have five degrees of freedom ("5DOF") of control. Intuitively, it would seem that a minimum of five coils would be required to make the system controllable in this sense. We discuss controllability in Section 5.

To indicate a variable is a column vector, we enclose it in braces: $\{ \}$; brackets indicate a matrix: $[]$. $\{B\}$ refers to the magnetic flux vector. Its components in the x, y, z direction will be given by B_x, B_y, B_z . We will sometimes use i as a subscript to refer to x, y, or z; that is, B_i is a B-component, where "i" is one of x, y, or z. When convenient, i may take integer values 1, 2, 3, where $1 = x, 2 = y, 3 = z$. Derivatives will also be indicated by subscripts: $B_{i,j}$ means $\frac{\partial B_i}{\partial j}$ where i and j are taken from the set containing x, y, and z. Similarly, $B_{i,j,k}$ means the partial derivative of the i,j partial derivative in the k direction, $\frac{\partial^2 B_i}{\partial j \partial k}$, and so on to higher space derivatives. We will use the term "B-values" when referring to a subset of the set of all $B_i, B_{i,j}, B_{i,j,k}$ values. The symbol Δ will refer to any of the subscripts i, i,j, i,j,k, i,j,k,l , etc. Thus B_Δ means any B-value.

2.4 Electromagnetic Equations

We now derive the electromagnetic equations that apply in our system. Let $\{B\}$ equal the magnetic field in the region of the core. Since the control and torque forces will be produced by magnetic rather than electric fields, only two of Maxwell's equations will be of interest. Using the usual notation ($\{H\}$ is the magnetic intensity, $\{D\}$ the electric displacement, and $\{J\}$ the current density), the equations are

$$(1) \quad \nabla \cdot \{B\} = 0$$

$$(2) \quad \nabla \times \{H\} = \{J\} + \frac{\partial \{D\}}{\partial t}$$

Since in the region of the core no current is flowing, we have $\{J\} = \{0\}$ in (2); also because the current in the coils is changing slowly, $\frac{\partial \{D\}}{\partial t} = \{0\}$ in (2). Finally, since we have $\{B\} = \mu_0 \{H\}$, where μ_0 is the permeability of free space, (2) reduces to

$$(3) \quad \nabla \times \{B\} = \{0\}$$

Equations (1) and (3) amount to four equations of constraint on the derivatives of the components of B. Equation (1) results in a single scalar equation:

$$(4) \quad \sum_{i=1}^3 B_{i,i} = B_{x,x} + B_{y,y} + B_{z,z} = 0$$

and (3) gives three equations which may be written

$$(5) \quad B_{i,j} = B_{j,i} \quad \text{where } i < j$$

2.5 Equations of Controlling Forces and Torques

We now find the torques and forces produced on the core by the magnetic field. We take as fundamental that the torque {T} exerted on a magnetic point dipole by a magnetic field {B} is given by

$$(6) \quad \{T\} = \{m\} \times \{B\}$$

where {m} is the magnetic moment of the dipole [9, pg 165]. It may be shown [10, pg 35]. that a relationship of this same form holds for permanent magnetic material, with {m} replaced by the magnetization of the permanent magnet core, {M}. This results in a relationship of the form $\{T\} = \text{volume} \{M\} \times \{B\}$. Applying this relation differentially gives

$$(7) \quad \{\delta T_c\} = (\{M\} \times \{B\})\delta v$$

where δT_c is a differential torque produced by a differential volume δv within the core and {B} is the flux density within the element of volume.

The force $\{F\}$ produced by magnetic field $\{B\}$ and dipole $\{m\}$ as above is given by [9, pg. 263]:

$$(8) \quad \{F\} = (\{m\} \bullet \nabla)\{B\}$$

Relations like (8) are commonly written by mathematicians as

$$(9) \quad \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} B_{xx} & B_{xy} & B_{xz} \\ B_{yx} & B_{yy} & B_{yz} \\ B_{zx} & B_{zy} & B_{zz} \end{bmatrix} \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix}$$

If we let $[\partial B]$ symbolize the 3x3 matrix of gradients of B above, (9) may be written

$$(10) \quad \{F\} = [\partial B]\{m\}$$

We note that, due to (5), $[\partial B]$ is symmetric. Moreover, due to (4), $[\partial B]$ has only five independent elements.

Reference [10], pg 35, again shows, as in (7), that a similar relationship of the form $\{F\} = volume[\partial B]\{M\}$ holds with permanent magnetic materials. Again applying the relationship differentially, the differential force produced on the core, $\{\delta F_c\}$, by a differential element of its volume is given by

$$(11) \quad \{\delta F_c\} = [\partial B]\{M\}\delta v$$

Where $\{M\}$ and δv are as in (7) and $[\partial B]$ as in (9).

The total torque on the core is found by integrating over its volume. Using (7),

$$(12) \quad \{T_c\} = \int_{vol} (\{M\} \times \{B\})dv + \int_{vol} \{r\} \times \{\delta F_c\}dv$$

where $\{r\}$ is a vector from the center of mass of the core to the element of volume. Now suppose in (12) that the B field varied over the space within the body. Then in all

likelihood it would be impossible to carry out the integration analytically. The same would be true if the magnetization M were not constant within the body. Similarly, we could not integrate (11) unless the gradient were constant within the space that the body occupies. We therefore make the following assumptions:

The body is so small that $\{B\}$, and its derivatives, may be regarded as constant throughout the body; $\{M\}$ is also constant. (Assumption 2)

In actual computations, the values of B and its derivations are found at the origin of the x, y, z system; that is, at the center of the core.

The second integral in (12) is produced by unbalanced forces about the center of mass. But using the assumptions above, and (11), the force $\{\delta F_c\}$ acting on every point of the body is constant. Also, the cylindrical body is symmetric in the following sense: for every $\{r\}$ drawn from the origin to a point in the body there is a $-\{r\}$ which also falls within the body. Since forces are constant throughout the body, the net torque produced by $\{r\}$ and $-\{r\}$ is

$$\{r\} \times \{\delta F_c\} - \{-r\} \times \{\delta F_c\} = \{0\}$$

Since all torques can be paired off in this manner, the total torque produced by forces is zero and the second term in (12) vanishes. (We remark parenthetically that in asymmetric cores, torques of this type, caused by unbalanced forces about the center of mass, can be used to produce a desired roll torque. See [6, pg 18] .) The remaining integral in (12) can be evaluated. Since M and B can be brought out of the integral, and we have

$$(13) \quad \{T_c\} = v(\{M\} \times \{B\})$$

where v is the volume of the core. Exactly the same process may be applied to the forces on the body given by (11). Since we also assume that the gradients of $\{B\}$ are constant over the core we can similarly integrate (11) to obtain

$$(14) \quad \{F_c\} = v[\partial B]\{M\}$$

2.6 Transformation of Force-Torque Equations Into the Body Coordinate System

Equations (13) and (14) express the control forces as functions of the electromagnetic field at the center of the core. However, they are expressed in the inertial coordinates of the fixed coil configuration. There is a great advantage to expressing them in body coordinates, since in this frame the magnetization of the core is given very simply by

$$(15) \quad \{\bar{M}\}^T = [M_{\bar{x}}, 0, 0]$$

where $\{\bar{M}\}^T$ refers to the transpose of the column vector $\{\bar{M}\}$ and $M_{\bar{x}}$ is the magnetization of the core. (Recall that bars over quantities mean they are expressed in body coordinates.) However, we wish to retain $\{B\}$ and $[\partial B]$ in coil coordinates. This is because the values of these variables are tabulated as functions of points in coil coordinates. Let $[T_M]$ be the Euler angle transformation matrix that takes inertial coordinates into core coordinates. The Euler rotation convention will be a 3,2,1 rotation sequence; θ_z will symbolize the rotation about the z axis, etc. Analysis of the 3,2,1 sequence (also called the x,y,z convention) is given in [12], on pgs. 608-610; see also [13], pg. 175 for a drawing. Since it is a standard Euler rotation matrix, and the expression for it is complex, $[T_M]$ will not be written out explicitly. $[T_M]^{-1}$ (which is $[T_M]^T$) is given explicitly on page 4 of [14]. If $\{U\}$ is any vector in inertial coordinates and $\{\bar{U}\}$ the same vector in body coordinates, then

$$(16) \quad [T_M]\{U\} = \{\bar{U}\} \text{ and } [T_M]^{-1}\{\bar{U}\} = \{U\}$$

We want to write (13) so that the left side is in body coordinates but $\{B\}$ remains in inertial (coil) coordinates. Now (13) is an example of a cross product relationship. It may be shown that cross product relationships hold true if every vector variable in them is

written in another coordinate system. Thus, as a first step, we write every vector in (13) in body coordinates:

$$(17) \quad \{\bar{T}_c\} = v(\{\bar{M}\} \times \{\bar{B}\})$$

At this point, it is convenient to break off our development a moment to verify the assertion made in Section 2.3 that, in this experiment, it is impossible to generate a torque about the \bar{x} axis. By (17), and the nature of the vector cross product, $\{\bar{T}_c\}$ is always perpendicular to $\{\bar{M}\}$. Since $\{\bar{M}\}$ is always in the \bar{x} direction, $\{\bar{T}_c\}$ is always perpendicular to \bar{x} . Thus the cross product nature of the torque equation means that no torque, and therefore no rotation, can be generated about the \bar{x} , or "roll", body axis.

Resuming our development, we now we use (16) to replace $\{\bar{B}\}$ by $[T_m]\{B\}$ in (17) to obtain

$$(18) \quad \{\bar{T}_c\} = v(\{\bar{M}\} \times [T_m]\{B\})$$

which is the required form of (13). We next write the force equation (14) in body coordinates. However, the force equation is a matrix expression, so does not transform like a cross product equation; it is not valid to simply write every variable in (14) in body coordinates. Instead, we first multiply both sides of (14) on the left by $[T_m]$; in the resulting expression, we use (16) to replace $[T_m]\{F_c\}$ by $\{\bar{F}_c\}$ and $\{M\}$ by $[T_m]^{-1}\{\bar{M}\}$ which gives us

$$(19) \quad \{\bar{F}_c\} = v[T_m][\partial B][T_m]^{-1}\{\bar{M}\}$$

as desired. Equations (18) and (19) express the control forces and torques in body coordinates. At this point, the control input is the B field and its gradients, expressed in the inertial coordinates of the coil configuration. But the actual control input will be currents through the five coils. Thus we must express $[B]$ and $[\partial B]$ as functions of these currents.

2.7 Magnetic Field as a Function of Current; Linearity Condition

We make the important assumption that the B field produced by a coil varies linearly as the current through that coil, provided that the current remains within a given range.

More precisely,

Given that when maximal positive current, I_{MAX} , flows through a coil, it produces a maximal flux $\{K\}$. Suppose a current, I , with $|I| \leq I_{MAX}$, flows through the coil. Then the flux $\{B\}$ produced by I is given by

$$(20) \quad \{B\} = \frac{1}{I_{MAX}} \{K\} I \quad (\text{Assumption 3})$$

Given (20), it is easy (but a little cumbersome) to take care of the fact that there are in fact five coils and five currents. Let the current flowing through the p^{th} coil be given by I_p , $p = 1, 2, \dots, 5$, and the maximal flux produced by the p^{th} coil be given by $\{K^p\}$. Then by the principle of superposition of flux, the total flux, $\{B\}$, produced by all coils is given by

$$(21) \quad \{B\} = \frac{1}{I_{MAX}} \sum_{p=1}^5 \{K^p\} I_p$$

We see that each component of $\{B\}$ is, so to speak, a "dot product" of a 5-vector of the corresponding component of K's with the 5-vector of currents. To reflect this in our notation, let

$$(22) \quad [K_i] = [K_i^1, K_i^2, \dots, K_i^5]$$

be a ROW vector of the i^{th} component of the K's of the five coils; thus, i is either x, y, or z. Let

$$(23) \quad \{I\} = [I_1, I_2, \dots, I_5]^T$$

be a COLUMN vector of the five currents. Then, by (21), the i^{th} component of B is given by

$$(24) \quad B_i = \frac{1}{I_{MAX}} [K_i] \{I\}$$

If we take the derivative of (24) with respect to space direction j, we have $B_{i,j} = \frac{1}{I_{MAX}} [K_{i,j}] \{I\}$, and so on for higher space derivatives, $B_{i,jk}$, $B_{i,jkl}$, etc. Recalling our definition of Δ as any subscript of this form, we may write

$$(25) \quad B_{\Delta} = \frac{1}{I_{MAX}} [K_{\Delta}] \{I\}$$

We note that, for the Experimental System, the maximal flux vector for the p^{th} coil, $\{K^p\}$, and all its derivatives, cannot be found by an analytic expression. Instead, they are computed by a numerical process [15]. Their values are tabulated on page 16 of [7]. We will regard them as given, constant data. We also assume that our General System is such that analytic expressions cannot be found for the flux. It is true that in certain limiting cases, or at certain points, quite accurate analytic expressions can be found by the flux produced by a single current loop or a coil. See Chapter 8 of [9]. The problem with the system under discussion is that the datum point is not located along the axis of symmetry of any coil; neither is it extremely distant from the coils. If we think of each coil as a series of concentric current loops, the flux contribution of each loop must be numerically calculated; and the contribution from each loop on a single coil will not be the same, since each loop is a different distance from the datum point. The total flux due to each coil can only be found by numerically summing the contributions from each loop of the coil.

2.8 Controlling Force-Torques as a Function of Current

These results give apparently cumbersome but actually quite simple expressions for torques and forces as functions of current. To obtain the torque expression, use (24) to write $\{B\}$ as

$$(26) \quad \{B\} = \frac{1}{I_{MAX}} \begin{bmatrix} [K_x] \\ [K_y] \\ [K_z] \end{bmatrix} \{I\} = \frac{1}{I_{MAX}} [K] \{I\}$$

Where the 3x5 constant matrix $[K]$ is defined by the above expression. If (26) is substituted into (18), the resulting expression gives torque as a function of current.

The force equation is found by using (25) to write ∂B as

$$(27) \quad [\partial B] = \frac{1}{I_{MAX}} \begin{bmatrix} [K_{x,x}] & [K_{x,y}] & [K_{x,z}] \\ [K_{y,x}] & [K_{y,y}] & [K_{y,z}] \\ [K_{z,x}] & [K_{z,y}] & [K_{z,z}] \end{bmatrix} \begin{bmatrix} \{I\} & \{0\} & \{0\} \\ \{0\} & \{I\} & \{0\} \\ \{0\} & \{0\} & \{I\} \end{bmatrix} = \frac{1}{I_{MAX}} [\partial K] [I]$$

where $[\partial K]$ is a 3x15 constant matrix defined by (27) and $[I]$ is a 15x3 matrix similarly defined.

If (27) is put into (19), force is found as a function of current. Again, the difficulties are purely notational. There should be a better way to move $\{I\}$ outside the matrix in (27) and "dot" it with the remaining matrix of K elements, which are tabulated constants.

However, when we linearize the force and torque equations, as we will shortly do, we will not need the full expressions for B and ∂B . Define the operator $\left[\frac{\partial}{\partial I} \right]$ as the following ROW vector:

$$(28) \quad \left[\frac{\partial}{\partial I} \right] = \left[\frac{\partial}{\partial I_1}, \frac{\partial}{\partial I_2}, \dots, \frac{\partial}{\partial I_5} \right]$$

Then by (24),

$$(29) \quad \left[\frac{\partial B_i}{\partial I} \right] = \frac{1}{I_{MAX}} [K_i]$$

and by (25),

$$(30) \quad \left[\frac{\partial B_\Delta}{\partial I} \right] = \frac{1}{I_{MAX}} [K_\Delta]$$

2.9 Equations of Motion

Equations (18) and (19), together with (26) and (27), give the torques and forces as functions of the current. We now describe the motions of the body they produce. These must be expressed in the body coordinate system, since the forces and torques are given in that system.

We assume the core is a rigid body and we first find the moment of inertia tensor for it. This tensor will be written in body coordinates; we will symbolize it by $[\bar{J}]$, where the bar indicates the values are taken in body coordinates. Since the body is cylindrical, the products of inertia will be zero. The reason for this parallels that dealing with the second term in (12). In each product of inertia integral the function to be integrated is of the form $\bar{x}_i \bar{x}_j, i \neq j$. However, for each \bar{x}_i in the body, $-\bar{x}_i$ is also in the body. Thus every $\bar{x}_i \bar{x}_j$ in the integral is matched by $\bar{x}_i (-\bar{x}_j) = -\bar{x}_i \bar{x}_j$ so the integral over the whole body is zero. Also due to symmetry, the moments of inertia about the \bar{y} and \bar{z} axes will be equal, and we will let I_c be this common value. If $I_{\bar{x}}$ is the moment of inertia about the \bar{x} , then the moment of inertia tensor will be given by

$$(31) \quad [\bar{J}] = \begin{bmatrix} I_{\bar{x}} & 0 & 0 \\ 0 & I_c & 0 \\ 0 & 0 & I_c \end{bmatrix}$$

and the angular momentum, which we will call H, is given by

$$(32) \quad \{\bar{H}\} = [\bar{J}]\{\bar{\Omega}\}$$

Here, $\{\Omega\}$ is the instantaneous rotation vector and $\{\bar{\Omega}\}$ is this vector written in body coordinates:

$$(33) \quad \{\bar{\Omega}\} = [\Omega_{\bar{x}}, \Omega_{\bar{y}}, \Omega_{\bar{z}}]^T$$

We take as fundamental that the rate of change of angular momentum, when observed from an inertial system, is equal to the total torque, T. [12, pg. 204]. This relation may be expressed in any frame, in particular the body frame, so we may write

$$(34) \quad \{\bar{T}\} = \frac{d\{\bar{H}\}}{dt}_{inertial}$$

The convenient simplicity of our notation has caused a slight difficulty in writing (34). A bar over a vector means merely that the vector has been resolved into the coordinate system of the body. We have no notation to indicate that a variable takes a certain value when observed from a particular coordinate system. When we later discuss the method of observation of the body, we will see that it is observed from an inertial system. But a vector so observed may be resolved into the coordinates of the body system. Thus the right-most expression in (34) means that the rate of change of angular momentum is measured by an inertial observer, and this vector is then resolved into the coordinate system of the body.

We also take as axiomatic [12, pg. 176] that the relation between the time derivative with respect to an inertial observer and an observer moving with the body is given by

$$(35) \quad \left(\frac{d}{dt}\right)_{inertial} = \left(\frac{d}{dt}\right)_{body} + \{\bar{\Omega}\} \times$$

We apply this relation to the right side of (34). Since $[\bar{J}]$ is constant with respect to time, we have

$$(36) \quad \{\bar{T}\} = [\bar{J}]\{\dot{\bar{\Omega}}\} + \{\bar{\Omega}\} \times ([\bar{J}]\{\bar{\Omega}\})$$

We saw in the note following (17) that no torque may be generated by electromagnetic forces about the \bar{x} axis. Thus $\{\bar{T}_c\}$ has a zero component in the \bar{x} direction. But this is the only torque acting on the body; the only external force is gravitational, and this produces no torque. Thus the total torque $\{\bar{T}\}$ equals $\{\bar{T}_c\}$ and this has a zero component in the \bar{x} direction, so $T_{\bar{x}}$ is zero. Thus there can be no angular acceleration or angular rate about the \bar{x} axis and $\Omega_{\bar{x}} = \dot{\Omega}_{\bar{x}} = 0$. When this is substituted into (36) the result is

$$(37) \quad \begin{bmatrix} 0 \\ T_{\bar{y}} \\ T_{\bar{z}} \end{bmatrix} = \begin{bmatrix} 0 \\ I_c \dot{\Omega}_{\bar{y}} \\ I_c \dot{\Omega}_{\bar{z}} \end{bmatrix} + \begin{bmatrix} 0 \\ \Omega_{\bar{y}} \\ \Omega_{\bar{z}} \end{bmatrix} \times \begin{bmatrix} 0 \\ I_c \Omega_{\bar{y}} \\ I_c \Omega_{\bar{z}} \end{bmatrix}$$

If the common I_c is placed in front of the second term on the right above, the remainder of the term is seen to be the cross product of a vector with itself, which is the zero vector. Thus the cross product term drops out. When we divide both sides of the remaining equation by I_c , we obtain the rotational equations of motion:

$$(38) \quad \{\dot{\bar{\Omega}}\} = \frac{1}{I_c} \{\bar{T}_c\}$$

We next find the translational equations of motion. The total force, $\{\bar{F}\}$, will be given by

$$(39) \quad \{\bar{F}\} = \{\bar{F}_c\} + \{\bar{F}_g\}$$

where $\{\bar{F}_c\}$ is given by (19) and $\{\bar{F}_g\}$ is gravitational force. In inertial coordinates,

$$(40) \quad \{F_g\}^T = [0, 0, -m_c g]$$

where m_c is the mass of the core and g is the acceleration of gravity. By (16), the gravitational force expressed in core coordinates is given by

$$(41) \quad \{\overline{F}_g\} = [T_M]\{F_g\}$$

Newton's Second Law applied to a rigid body is

$$(42) \quad \{\overline{F}\} = m_c \frac{d\overline{V}}{dt}_{inertial}$$

where $\{V\}$ is the velocity of the center of mass of the body. The discussion in regard to (34) applies here. $\frac{dV}{dt}_{inertial}$ is the acceleration observed from an inertial frame; $\frac{d\overline{V}}{dt}_{inertial}$ is this vector resolved into body coordinates. Solving (42) for the acceleration and using (39) gives the translational equations of motion:

$$(43) \quad \frac{d\overline{V}}{dt}_{inertial} = \frac{1}{m_c}\{\overline{F}\} = \frac{1}{m_c}\{\overline{F}_c\} + \frac{1}{m_c}\{\overline{F}_g\}$$

If we apply (35) to the left side of (43), we obtain

$$(44) \quad \{\dot{\overline{V}}\} + \{\overline{\Omega}\} \times \{\overline{V}\} = \frac{1}{m_c}\{\overline{F}_c\} + \frac{1}{m_c}\{\overline{F}_g\}$$

We now summarize all our results. We first write the equations of motion as functions of $[B]$ and $[\partial B]$. Substituting (18) into (38) results in

$$(45) \quad \{\dot{\overline{\Omega}}\} = \frac{v}{I_c}(\{\overline{M}\} \times [T_M]\{B\})$$

And by substituting (19) and (41) into the right side of (44), we obtain

$$(46) \quad \{\dot{\overline{V}}\} + \{\overline{\Omega}\} \times \{\overline{V}\} = \frac{v}{m_c}[T_M][\partial B][T_M]^{-1}\{\overline{M}\} + \frac{1}{m_c}[T_M]\{F_g\}$$

In the next section we apply approximations to the above equations of motion.

2.10 Approximate Equations of Motion -- The Basic System

Because of the notation complexities it would entail, we will not write (45) and (46) as explicit functions of current $\{I\}$. However, we remind the reader that $[B]$ in (45) and $[\partial B]$ in (46) are merely linear functions of $\{I\}$ and the numerically tabulated K-values. (See (26) and (27) .) Thus, other than $\{I\}$, the only variables on the right sides of (45) and (46) are the Euler angles contained in $[T_M]$ and $[T_M]^{-1}$. But these are matrices of trigonometric functions, so are non-linear. As a first step toward obtaining linear equations, it is therefore necessary to restrict the Euler angles to small variations near values that are fixed in any one experiment. In our initial formulation, the fixed values of the Euler angles will all be zero. That is, the body will remain very near the equilibrium orientation, with the body axes nearly parallel to the inertial axes. To denote that a variable angle is small, we will prefix it with a delta (δ). The rotation matrix $[T_M]$ above will now be an infinitesimal rotation matrix, which we will denote $[\delta T_M]$. Also, in the application, it will be assumed that the body will not rotate rapidly so that Euler rates will be small. The same restrictions apply to translations; the body will not translate far from the equilibrium position at the origin of the inertial axes and its linear velocity will be small. The product of any of these small quantities may therefore be dropped. We summarize these approximations, which are assumed to hold in all systems:

The Euler angle rotations from the equilibrium orientation will be small, so that, for example, we may use the approximations $\sin\delta\theta_i = \tan\delta\theta_i = \delta\theta_i$ and $\cos\delta\theta_i = 1$, where $i \in \{x,y,z\}$. Also angular rates, linear translations and velocities are small. We may therefore approximate the PRODUCT of any two of these quantities with zero. (Assumption 4)

We first apply these approximations to the left side of (46). Each term in the cross product expression will be the product of an angular rate with a velocity, and so may be neglected. Thus we may assume the following approximation:

$$(47) \quad \{\dot{\bar{V}}\} = \frac{v}{m_c} [T_M][\partial B][T_M]^{-1} \{\bar{M}\} + \frac{1}{m_c} [T_M] \{F_g\}$$

We now apply the approximations to the right sides of (45) and (47). In (45), the Euler angle accelerations are expressed in body coordinates. However, the system of observation will record the Euler angles themselves, measured from an inertial frame. Therefore, we wish to show that, under Assumption 4, the body coordinate system may be approximated by the Euler angle system. The first step is to find the transformation matrix from body rates to the Euler rates. (This is tricky because each of the Euler rates is expressed in a different coordinate system. The method for obtaining this transformation is shown on page 176 of [12], though for a different Euler angle convention than the one used here. Appendix B of [12], pages 608-610, gives the Euler angle relationship for our convention.) It works out that if $[T_E]$ is defined by

$$(48) \quad \{\Omega\}_{Euler} = [T_E] \{\bar{\Omega}\}$$

Then

$$(49) \quad [T_E] = \begin{bmatrix} 1 & \tan \delta\theta_y \sin \delta\theta_x & \tan \delta\theta_y \cos \delta\theta_x \\ 0 & \cos \delta\theta_x & -\sin \delta\theta_x \\ 0 & \sec \delta\theta_y \sin \delta\theta_x & \sec \delta\theta_y \cos \delta\theta_x \end{bmatrix}$$

We note two facts here which will be important in Section 2.14, where, in the Rotated System, θ_z is allowed to take any finite value. First, in (49) we have assumed infinitesimal delta-angles, but the relation (48) still holds if finite angles are substituted in (49). Second, by (49), $[T_E]$ is independent of θ_z .

In the discussion that followed (34), we showed that the torque about the \bar{x} axis is zero. Therefore, we assume that $\delta\theta_x$ is zero; if this is substituted in (49) and Assumption 4 is applied to the remaining entries in the matrix, the result is

$$(50) \quad [\delta T_E] = \begin{bmatrix} 1 & 0 & \delta\theta_y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Due to Assumption 4, we can ignore the product of $\delta\theta_y$ with the angular rate and (48), when $[T_E]$ is replaced by $[\delta T_E]$, becomes

$$(51) \quad \{\Omega\}_{Euler} \cong \{\bar{\Omega}\}$$

(We note that if we had used some other Euler angle conventions, as, for example, that on page 146 of [12], $[T_E]$ would not approach the identity matrix for small angles, and the approach used here would have to be modified.)

We use a similar method to show that linear motions expressed in the body frame may be taken as being expressed in an inertial frame. For this purpose, $[T_M]$ will play the part that $[T_E]$ played for angular rates. We refer the reader to page 4 of [14] for the standard but complicated expression for the full rotation matrix $[T_M]$. Using this complicated expression and the small angle part of Assumption 4, and the fact that $\delta\theta_x$ is zero, the infinitesimal rotation matrix $[\delta T_M]$ may be written as

$$(52) \quad [\delta T_M] = \begin{bmatrix} 1 & \delta\theta_z & -\delta\theta_y \\ -\delta\theta_z & 1 & 0 \\ \delta\theta_y & 0 & 1 \end{bmatrix}$$

With an inverse given by

$$(53) \quad [\delta T_M]^{-1} = \begin{bmatrix} 1 & -\delta\theta_z & \delta\theta_y \\ \delta\theta_z & 1 & 0 \\ -\delta\theta_y & 0 & 1 \end{bmatrix}$$

These results may also be derived from the section on infinitesimal rotations, pages 166 to 174 of [12]. Now, since, by (16),

$$(54) \quad \{V\} = [\delta T_M]^{-1} \{\bar{V}\}$$

We may use that part of Assumption 4 that states the product of rates and angles are zero to obtain

$$(55) \quad \{V\} \cong \{\bar{V}\}$$

We next apply Assumption 4 to find manageable equations of motion. On the right side of (45) we substitute our approximation (52) $[\delta T_M]$ for $[T_M]$. Recalling our simple expression (15) for $\{\bar{M}\}$, we have that

$$(56) \quad \{\dot{\bar{\Omega}}\} = \frac{vM_{\bar{x}}}{I_c} \begin{bmatrix} 0 \\ -\delta\theta_y B_x - B_z \\ -\delta\theta_z B_x + B_y \end{bmatrix}$$

On the right side of (47) we also use (15) and our infinitesimal approximation for $[T_M]$. In addition, we substitute in our approximation (53) for $[T_M]^{-1}$. When worked out, the result is

$$(57) \quad \{\dot{\bar{V}}\} = \frac{vM_{\bar{x}}}{m_c} \begin{bmatrix} B_{x,x} + \delta\theta_z B_{x,y} - \delta\theta_y B_{x,z} + \delta\theta_z B_{y,x} - \delta\theta_y B_{z,x} + \frac{\delta\theta_y m_c g}{vM_{\bar{x}}} \\ -\delta\theta_z B_{x,x} + B_{y,x} + \delta\theta_z B_{y,y} - \delta\theta_y B_{y,z} \\ \delta\theta_y B_{x,x} + B_{z,x} + \delta\theta_z B_{z,y} - \delta\theta_y B_{z,z} - \frac{m_c g}{vM_{\bar{x}}} \end{bmatrix}$$

Equation (57) may be simplified by using that, by (5), $B_{ij} = B_{ji}$. This give us

$$(58) \quad \{\dot{\bar{V}}\} = \frac{vM_{\bar{x}}}{m_c} \begin{bmatrix} B_{x,x} + 2\delta\theta_z B_{x,y} - 2\delta\theta_y B_{x,z} + \frac{\delta\theta_y m_c g}{vM_{\bar{x}}} \\ -\delta\theta_z B_{x,x} + B_{x,y} + \delta\theta_z B_{y,y} - \delta\theta_y B_{y,z} \\ \delta\theta_y B_{x,x} + B_{x,z} + \delta\theta_z B_{y,z} - \delta\theta_y B_{z,z} - \frac{m_c g}{vM_{\bar{x}}} \end{bmatrix}$$

2.11 Linearized Equations of Motion -- Expressions for the State Vector

Equations (56) and (58) are the approximate equations of motion when the body is near the equilibrium point. Our next step is to linearize these equations about the equilibrium point. To formalize this process, we first form the state vector $\{X\}$. Because, from (56), $\delta\theta_x$ and $\Omega_{\bar{x}}$ are zero, we will ignore these dimensions in our formulation. Thus $\{X\}$ will be a 10-vector given by

$$(59) \quad \{X\}^T = [\Omega_{\bar{y}}, \Omega_{\bar{z}}, \theta_y, \theta_z, V_{\bar{x}}, V_{\bar{y}}, V_{\bar{z}}, x, y, z] = [X_1, X_2, X_3, \dots, X_{10}]$$

(In the above we have dropped the δ in front of the angles, it being understood that all elements of the state vector take near-zero values.) Note that the elements of $\{X\}$ are expressed in different coordinate systems. The angular rates and linear velocities are in body coordinates; the positions are in inertial coordinates; and the orientation angles are Euler angles. However, by (51), we can set, for example, $\dot{X}_3 = \dot{\theta}_y = \Omega_{\bar{y}}$. By (55), we may set, for instance, $\dot{X}_8 = \dot{x} = V_{\bar{x}}$. We will define the vector function $\{g\}$ by setting it equal to the derivative of $\{X\}$:

$$(60) \quad \{\dot{X}\} = \{g(\{X\}, \{B\}, [\partial B])\}$$

Explicit expressions for $\{g\}$ are found as follows.

$[\dot{X}_1, \dot{X}_2] = [\dot{\Omega}_{\bar{y}}, \dot{\Omega}_{\bar{z}}] = [g_1, g_2]$ is given by the right side of (56). The right side of

(58) provides $[g_5, g_6, g_7]$. The remaining expressions, $g_3, g_4,$ and $g_8, g_9, g_{10},$ are trivial. This is because, as noted above, for example, $g_3 = \dot{\theta}_y = \Omega_{\bar{y}}$ and $g_8 = \dot{x} = V_x$, etc.

Thus $\{g\}$ is a function of $\{X\}, \{B\}, [\partial B]$. However, from (26) and (27), $\{B\}$ and $[\partial B]$ depend only on the constant matrices $[K]$ and $[\partial K]$ respectively, and on the current, so are really only functions of current. In this sense, $\{\dot{X}\}$ is a function of $\{X\}$ and current only. We formalize this as follows. Substitute (26) and (27) into (60) and the right side will be a function of variables $\{X\}$ and $\{I\}$ only. We will call this function $\{f\}$ so that

$$(61) \quad \{\dot{X}\} = \{g(\{X\}, \{B(\{I\})\}, [\partial B(\{I\})])\} = \{f(\{X\}, \{I\})\}$$

2.12 Linearized Equations of Motion -- Equilibrium Point

To linearize we must first find an equilibrium point ($\{X_0\}, \{I_0\}$) so that

$$(62) \quad \{f(\{X_0\}, \{I_0\})\} = \{0\}$$

But once we have found $\{I_0\}$, (26) and (27) mean we have also found the corresponding $\{B_0\}$ and $[\partial B_0]$ so that

$$(63) \quad \{g(\{X_0\}, \{B_0\}, [\partial B_0])\} = \{0\}$$

In doing our analysis, we will actually reverse the above line of reasoning. That is, we will solve (63) for the needed B-values, then, in a sense, invert (24) and (25) to find $\{I_0\}$. To solve (63), it is clear that we must force the right sides of (56) and (58) to be zero. Now in this formulation we are linearizing about the datum position point, and about the zero point of velocities and angular rates, so that $\{X_0\} = \{0\}$. Thus, in (56) and (58), $\delta\theta_y = \delta\theta_z = 0$. Hence for (56) and (58) to be zero we must have

$$(64) \quad B_y = B_z = B_{x,x} = B_{x,y} = 0 \quad \text{and} \quad B_{x,z} = \frac{mcg}{vM\bar{x}} = c$$

Where we have defined $\frac{mcg}{vM_X} = c$ for convenience

We seek an equilibrium current vector $\{I_0\}$ which will produce the above B-values. If we put the five B-values above into a five-vector, by using (24) and (25) we may write

$$(65) \quad \begin{bmatrix} B_y \\ B_z \\ B_{x,x} \\ B_{x,y} \\ B_{x,z} \end{bmatrix} = \begin{bmatrix} [K_y] \\ [K_z] \\ [K_{x,x}] \\ [K_{x,y}] \\ [K_{x,z}] \end{bmatrix} \frac{\{I\}}{I_{MAX}}$$

For the purpose of discussion, we will write the above equation as

$$(66) \quad \{B_{EQ}\} = [K_{EQ}] \frac{\{I\}}{I_{MAX}}$$

Where the 5-vector $\{B_{EQ}\}$ and the 5 times 5 matrix $[K_{EQ}]$ are defined by (65). We will call $[K_{EQ}]$ the "equilibrium matrix"; it is of paramount importance in the succeeding development. When $\{B_{EQ}\}$ takes the values required for equilibrium, we will call it $\{B_{EQ}\}_0$ so that

$$(67) \quad \{B_{EQ}\}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ c \end{bmatrix} = [K_{EQ}] \frac{\{I_0\}}{I_{MAX}}$$

A solution to the above equation is an equilibrium current ratio $\{I_0\}/I_{max}$. Whether (67) has no solution, one solution, or infinitely many solutions depends on $[K_{EQ}]$. We may think of $[K_{EQ}]$ as a mapping from a five dimensional set of current ratios to a five dimensional set of B-values. But the domain of $[K_{EQ}]$ will not be the entire vector space

R^5 ; for each individual current, I_p , the linearity condition (20) requires that we have $|I_p|/I_{MAX} \leq 1$. Therefore, the domain of $[K_{EQ}]$ will be a cube in R^5 of side 2. In the general case, (67) has a solution if and only if the image of this cube contains $\{B_{EQ}\}_0$. (We will later see that if (67) is satisfied for any vector of current ratios, the ratios will satisfy (20) provided the suspension constraint, (231) of Section 7.5, is satisfied.)

However, in Section 4.1 we will derive an analytic expression for $[K_{EQ}]^{-1}$. Thus a solution to (67) is unique if it exists and is given by

$$(68) \quad \{I_o\}/I_{MAX} = [K_{EQ}]^{-1} \{B_{EQ}\}_0$$

In the Experimental System [7], the equilibrium current ratio values, $\{I_o\}/I_{MAX}$, were found from (68) by numerically inverting $[K_{EQ}]$. They were valid solutions since they satisfied $|I_p| \leq I_{MAX}$. Their numerical values are presented on page 12 of [7]. We note that if more than five coils are used, there will normally be infinitely many solutions $\{I_o\}$. In this case, $\{I_o\}$ may be chosen according to energy minimization criteria. See [7], page 9.

Thus $\{I_o\}/I_{MAX}$ is determined by the design of the experiment; by (24), (25), and the note following, the equilibrium B-values, B_i , B_{ij} , B_{ijk} , are also determined at this time. We will indicate these values with an "0" subscript. (Note that using this notation scheme means that every B-value in (64) should have been "0" subscripted.) By (24) and (68),

$$(69) \quad B_{i_0} = \frac{1}{I_{MAX}} [K_i] \{I_o\} = [K_i] [K_{EQ}]^{-1} \{B_{EQ}\}_0$$

And by (25) and (68)

$$(70) \quad B_{\Delta_0} = \frac{1}{I_{MAX}} [K_{\Delta}] \{I_o\} = [K_{\Delta}] [K_{EQ}]^{-1} \{B_{EQ}\}_0$$

2.13 Linearized Equations of Motion

We next write the linearized expression for (61) about the equilibrium point we have found, using standard linearization methods. That is, we set

$$(71) \quad \{\dot{X}\} = [\Lambda(X_0, I_0)]\{X\} + [\Gamma(X_0, I_0)]\{I\}$$

We call the 10 by 10 matrix Λ the "dynamics" matrix and the 10 by 5 matrix Γ the "control" matrix. They are defined by

$$(72) \quad [\Lambda(X, I)] = [\alpha_{ij}] = \left[\frac{\partial f_i(X, I)}{\partial X_j} \right]$$

$$(73) \quad [\Gamma(X, I)] = [\gamma_{ij}] = \left[\frac{\partial f_i(X, I)}{\partial I_j} \right]$$

The linearization procedure above is standard, but some explanation is in order with respect to the notation. In the standard notation of control theory, $[A]$ usually designates the dynamics matrix rather than $[\Lambda]$ and $[B]$ the control matrix instead of $[\Gamma]$. We have used Greek letters to avoid confusion with the flux, B . Also, in the above we have suppressed the brackets $\{ \}$ around X , X_0 , I , and I_0 and will continue to do so when there was no danger of confusion. Finally, note that in (71), Λ and Γ are constant matrices whereas in (72) and (73) they are variable matrices. The constant matrices in (71) are found by evaluating (72) and (73) at the unique equilibrium point X_0, I_0 .

In (72) and (73) the dynamics and control matrices are written as functions of X and I . However, we will find it much more convenient to find them as functions of B and ∂B by using the equations of motions, (56) and (57). We can do this by writing then as derivatives of the function g defined by (61), $\{g(\{X\}, \{B(I)\}, [\partial B(I)])\} = \{f(X, I)\}$.

Thus,

$$(74) \quad \alpha_{ij} = \frac{\partial f_i}{\partial X_j} = \frac{\partial g_i}{\partial X_j}$$

$$(75) \quad \gamma_{ij} = \frac{\partial f_i}{\partial I_j} = \frac{\partial g_i}{\partial I_j}$$

(75) is computed by the chain rule:

$$(76) \quad \frac{\partial g_i}{\partial I_j} = \sum_{k=1}^3 \frac{\partial g_i}{\partial B_k} \frac{\partial B_k}{\partial I_j} + \sum_{k,l=1}^3 \frac{\partial g_i}{\partial B_{k,l}} \frac{\partial B_{k,l}}{\partial I_j}$$

These computations are not as complex as (76) might indicate. We will illustrate by computing typical rows of Λ and Γ . We will arbitrarily choose to compute the fifth row of each. By (59), the fifth row is $\dot{X}_5 = \dot{V}_x = g_5$ and g_5 is given by the top entry of the vector on the right side of (58). Thus,

$$(77) \quad g_5 = \frac{\nu M_{\bar{x}}}{m_c} \left(B_{x,x} + 2\delta\theta_z B_{x,y} - 2\delta\theta_y B_{x,z} + \frac{\delta\theta_y m_c g}{\nu M_{\bar{x}}} \right)$$

The fifth row of Λ , which we will call $[\Lambda_5]$, is defined as

$$(78) \quad [\Lambda_5] = \left[\frac{\partial g_5}{\partial X_1}, \frac{\partial g_5}{\partial X_2}, \dots, \frac{\partial g_5}{\partial X_{10}} \right]$$

This must be evaluated at the equilibrium point. Thus the derivatives are taken and then the equilibrium values $\{X_0\} = \{0\}$ and $\{I_0\}$ are substituted in. The substitution of the zero values for $\{X\}$ results in dropping terms which, after differentiation, contain $X_3 = \delta\theta_y$ or $X_4 = \delta\theta_z$. The substitution of $\{I_0\}$ forces the introduction of equilibrium B-values rather than variable B-values. Thus,

$$(79) \quad [\Lambda_5] = \frac{\nu M_{\bar{x}}}{m_c} \left[\begin{array}{cccccccc} 0 & 0 & -2B_{x,z_0} + \frac{m_c g}{\nu M_{\bar{x}}} & 2B_{x,y_0} & 0 & 0 & 0 & B_{x,x_0} & B_{x,y_0} & B_{x,z_0} \end{array} \right]$$

where, by (70) and the note following it, all the B-values in (79) are predetermined constants. Note that by (64), $B_{x,y_0} = 0$ and $B_{x,z_0} = \frac{m_c g}{\nu M_{\bar{x}}}$. Thus we have been slightly

inconsistent in (79) since we have substituted in the actual constant values for $\{X_0\}$, which were all zero, but we have not substituted in the actual values for the constant B-values. This is because in most cases they are constants which can only be computed using the constant K-values "dotted" with the equilibrium current ratio, $\{I_0\}/I_{MAX}$, as shown by (24) and (25).

Continuing our example, we now compute the fifth row of Γ , $[\Gamma_5]$. We first derive the general expression for the n^{th} row, $[\Gamma_n]$. By (75) and (76),

$$(80) \quad [\Gamma_n] = \left[\frac{\partial g_n}{\partial I_1}, \frac{\partial g_n}{\partial I_2}, \dots, \frac{\partial g_n}{\partial I_5} \right] = \sum_{i=1}^3 \frac{\partial g_n}{\partial B_i} \left[\frac{\partial B_i}{\partial I} \right] + \sum_{i,j=1}^3 \frac{\partial g_n}{\partial B_{ij}} \left[\frac{\partial B_{ij}}{\partial I} \right] =$$

$$= \frac{1}{I_{MAX}} \left(\sum_{i=1}^3 \frac{\partial g_n}{\partial B_i} [K_i] + \sum_{i,j=1}^3 \frac{\partial g_n}{\partial B_{ij}} [K_{ij}] \right)$$

where the operator $\left[\frac{\partial}{\partial I} \right]$ is defined by (28), and the value of this operator when it acts on B_i and B_{ij} is given by (29) and (30). When g_s , given by (77), is substituted into (80), a very simple expression is obtained, since g_s has no B_i terms. Thus the first summation term vanishes. In regard to the second summation, only the derivative with respect to $B_{x,x}$ survives. This is because the other two B_{ij} terms in (77) have delta-angle factors in them and these are zero at the equilibrium point. Thus we have

$$(81) \quad [\Gamma_5] = \frac{\upsilon M_{\bar{x}}}{m_c I_{MAX}} [K_{x,x}]$$

The other rows of $[\Gamma]$ are also very simple expressions.

We now give the Λ , Γ matrices which result from linearization. To avoid carrying along constants, we introduce the 10 by 10 constant matrix $[W]$:

$$(82) \quad [W] = \begin{bmatrix} \frac{\omega M_{\bar{x}}}{I_c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\omega M_{\bar{x}}}{I_c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\omega M_{\bar{x}}}{m_c} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\omega M_{\bar{x}}}{m_c} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\omega M_{\bar{x}}}{m_c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then $[\Lambda]$ is given by

$$(83) \quad [\Lambda] = [W] \begin{bmatrix} 0 & 0 & -B_x & 0 & 0 & 0 & 0 & -B_{x,z} & -B_{y,z} & -B_{z,z} \\ 0 & 0 & 0 & -B_x & 0 & 0 & 0 & B_{x,y} & B_{y,y} & B_{y,z} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{m_c g}{\omega M_{\bar{x}}} - 2B_{x,z} & 2B_{x,y} & 0 & 0 & 0 & B_{x,xx} & B_{x,xy} & B_{x,xz} \\ 0 & 0 & -B_{y,z} & B_{y,y} - B_{x,x} & 0 & 0 & 0 & B_{x,yx} & B_{x,yy} & B_{x,yz} \\ 0 & 0 & B_{x,x} - B_{z,z} & B_{y,z} & 0 & 0 & 0 & B_{x,zx} & B_{x,zy} & B_{x,zz} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= [W][\beta]$$

where we define $[\beta]$ to be the 10 by 10 matrix above consisting of only equilibrium B-values.

In (83), each B-value should have a "0" subscript to indicate that it is an equilibrium value. This had to be omitted in order to put rows of the matrix on one line. Also, as

indicated in the note following (79), many of these equilibrium B-values are zero. The control matrix Γ is given by

$$(84) \quad [\Gamma] = \frac{1}{I_{MAX}} [W] \begin{bmatrix} -[K_z] \\ [K_y] \\ [0] \\ [0] \\ [K_{x,x}] \\ [K_{x,y}] \\ [K_{x,z}] \\ [0] \\ [0] \\ [0] \end{bmatrix}$$

As shown in [7], the Basic Experimental System was tested for controllability and observability by a numerical process [16]. (We do not derive the observation matrix here. We give a brief note on the observation process in Section 2.15.) That is, using (69), (70), and the note that follows (70), numerical values were found for the B-values in (83) and (84). The numerically valued matrices were then submitted to one of the standard numerical routines in Matlab which tests for controllability and observability and passed both tests. In Section 4 we will prove controllability for the Basic General System. We next discuss the Rotated System.

2.14 Suspension After Rotation Through a Finite Angle -- The Rotated System

The development to this point has assumed the body was at the single equilibrium point shown in Figure 1. It is often advantageous in EMS applications to be able to suspend a body at more than one orientation. Thus, another goal of the experiment was to show that the body could be suspended and controlled after being rotated through an arbitrary fixed, finite angle, θ_z , about the z axis. See Figure 5. In a sense, the Rotated System is just the Basic System with a different equilibrium point. The new equilibrium point is identical to the old, except that the body is oriented differently with respect to the coil configuration; the only orientation difference is the rotation θ_z . The equations describing this situation will be virtually identical to the ones we have already developed, if we slightly expand the notation we have developed. We will form a new inertial coordinate system, which we will call the x', y', z' (primed) system which is rotated through the finite angle θ_z from the x, y, z system. Figure 10 shows the coordinate systems. The body system, $\bar{x}, \bar{y}, \bar{z}$, we consider to be perturbed through infinitesimal angles $\delta\theta_y, \delta\theta_z$ from the x', y', z' system. Thus the approximations (51) and (55) still hold between the primed and the body system. That is,

$$(85) \quad \{\Omega'\}_{Euler} \cong \{\bar{\Omega}\}$$

and

$$(86) \quad \{V'\} \cong \{\bar{V}\}$$

Since the equation of motion (45), (46) were derived for an arbitrary rotation, given by $[T_M]$, not necessarily an infinitesimal one, they still apply to the present situation. Now, however, the rotation from the x, y, z system to the body system is accomplished in two steps: first, a rotation through a finite angle θ_z to the primed system, and then an infinitesimal rotation from the primed system to the body system. Thus, $[T_M]$ will be given now by

$$(87) \quad [T_M] = [\delta T_M][T_{\theta_z}]$$

where $[T_{\theta_z}]$ is the standard rotation matrix

$$(88) \quad [T_{\theta_z}] = \begin{bmatrix} \cos \theta_z & \sin \theta_z & 0 \\ -\sin \theta_z & \cos \theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $[\delta T_M]$ is still given by (52) and its inverse by (53). As is generally true of rotation matrices,

$$(89) \quad [T_{\theta_z}]^{-1} = [T_{\theta_z}]^T$$

If (87) is substituted into (45) and (46) we have

$$(90) \quad \{\bar{\dot{\Omega}}\}_{inertial} = \frac{v}{I_c} \left(\{\bar{M}\} \times [\delta T_M][T_{\theta_z}]\{B\} \right)$$

and

$$(91) \quad \{\bar{V}\}_{inertial} = \frac{v}{m_c} \left([\delta T_M][T_{\theta_z}][\partial B][T_{\theta_z}]^{-1}[\delta T_M]^{-1}\{\bar{M}\} \right) + \frac{1}{m_c}[\delta T_M][T_{\theta_z}]\{F_g\}$$

Using (88) and our expression for $\{F_g\}$, (40), we find that the gravitation term is unaffected by the finite rotation:

$$(92) \quad [T_{\theta_z}]\{F_g\} = \{F_g\}$$

Let us define primed B-values by

$$(93) \quad \{B'\} = [T_{\theta_z}]\{B\} \quad \text{and}$$

$$(94) \quad [\partial B'] = [T_{\theta_z}][\partial B][T_{\theta_z}]^{-1}$$

It may be shown that (94) is the expression for the gradient matrix in the primed coordinate system, but this is not required for our analysis. Substituting (93) into (90) gives

$$(95) \quad \{\bar{\dot{\Omega}}\}_{inertial} = \frac{v}{I_c} \left(\{\bar{M}\} \times [\delta T_M]\{B'\} \right)$$

and (92) into (94) results in

$$(96) \quad \{\bar{V}\}_{inertial} = \frac{v}{m_c} [\delta T_M][\partial B'] [\delta T_M]^{-1} \{\bar{M}\} + \frac{1}{m_c} [\delta T_M]\{F_g\}$$

Now (95) and (96) are exactly the same as (45) and (46), with primed B-values substituted for B-values and $[\delta T_M]$ substituted for $[T_M]$. This latter substitution led to the approximate equations of motion, (56) and (58). Thus, (56) and (58) must hold for our current system, except that now we have primed B-values. But (56) and (58) led to the expression for equilibrium B-values, (64). Exactly the same process must lead to the same expression for equilibrium primed B-values, namely

$$(97) \quad B'_{y_0} = B'_{z_0} = B'_{x,x_0} = B'_{x,y_0} = 0; \quad B'_{x,z_0} = c$$

We will therefore adopt the notation for equilibrium primed B-values that we used for B-values. That is, the 5-vector $\{B_{EQ}'\}$ will be as defined by (65) and (66), except primed B-values will replace B-values. We would like to use an equation similar to (66) to find the equilibrium currents needed to hold the body in suspension after being rotated through

the angle θ_z . (We will call this equilibrium current vector $\{I'_0\}$.) However, (65) and (66) are not true for primed B-values because the relations (24) and (25) between B-values and K-values does not hold for primed B-values. Instead, we must substitute (26) into (93) to obtain

$$(98) \quad \{B'\} = \frac{1}{I_{MAX}} [T_{\theta_z}] [K] \{I'\}$$

Where, of course, $\{I'\}$ is simply a notational device to indicate those currents required to produce $\{B'\}$. To find the gradient B'-values, we put (27) into (94) to find

$$(99) \quad [\partial B'] = \frac{1}{I_{MAX}} [T_{\theta_z}] [\partial K] [I'] [T_{\theta_z}]^{-1}$$

Form (98) and (99), the elements of the $\{B'_{EQ}\}$ vector are found to be

$$(100) \quad \{B'_{EQ}\} = \begin{bmatrix} B'_y \\ B'_z \\ B'_{x,x} \\ B'_{x,y} \\ B'_{x,z} \end{bmatrix} =$$

$$\begin{bmatrix} -\sin \theta_z [K_x] + \cos \theta_z [K_y] \\ [K_z] \\ \cos^2 \theta_z [K_{x,x}] + 2 \cos \theta_z \sin \theta_z [K_{x,y}] + \sin^2 \theta_z [K_{y,y}] \\ -\cos \theta_z \sin \theta_z [K_{x,x}] + (\cos^2 \theta_z - \sin^2 \theta_z) [K_{x,y}] + \cos \theta_z \sin \theta_z [K_{y,y}] \\ \cos \theta_z [K_{x,z}] + \sin \theta_z [K_{y,z}] \end{bmatrix} \frac{\{I'\}}{I_{MAX}}$$

Thus the generalization of (66) is

$$(101) \quad \{B'_{EQ}\} = \left[K'_{EQ} \right] \frac{\{I'\}}{I_{MAX}}$$

Where the 5 by 5 matrix $[K_{EQ}']$ is defined by (100) and (101). The extension of (67) to an arbitrary yaw angle θ_z is

$$(102) \quad \{B'_{EQ}\}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{mcg}{vM_{\bar{x}}} \end{bmatrix} = [K'_{EQ}] \frac{\{I'_0\}}{I_{MAX}}$$

Just as in the discussion of (67), a unique equilibrium current $\{I'_0\}$ will exist if and only if $[K_{EQ}']^{-1}$ exists. Again, in Section 4.3, we will find an analytic expression for $[K_{EQ}']^{-1}$. Thus if a solution exists it is unique and is given by

$$(103) \quad \frac{\{I'_0\}}{I_{MAX}} = [K'_{EQ}]^{-1} \{B'_{EQ}\}_0$$

The equilibrium current ratios in the Experimental System [7] were numerically found from (103) and were acceptable since they satisfied $|I_p| \leq I_{MAX}$. They are shown graphically in Figure 15. With these values, we can find $\{B'\}_0$ by substituting $\{I'_0\}$ into (98) to obtain

$$(104) \quad \{B'\}_0 = \frac{1}{I_{MAX}} [T_{\theta_z}] [K] \{I'_0\}$$

The remaining primed B-values may be found by substituting $\{I'_0\}$ into (99) which gives

$$(105) \quad \{\partial B'\}_0 = \frac{1}{I_{MAX}} [T_{\theta_z}] [\partial K] [\partial I'_0] [T_{\theta_z}]^{-1}$$

However, we will show in Section 6 that it is not necessary to carry out the calculation given by (105). Due to certain properties of the system which we will discuss in Section 6, the equilibrium B-values will not vary with θ_z .

We next find the linearized equations, which correspond to (71) of the unprimed system. The state vector will now be

$$(106) \quad \{X'\} = [\Omega_{\bar{y}}, \Omega_{\bar{z}}, \theta'_y, \theta'_z, V_{\bar{x}}, V_{\bar{y}}, V_{\bar{z}}, x', y', z']$$

The analog to (71) is

$$(107) \quad \{\dot{X}'\} = [\Lambda']\{X'\} + [\Gamma']\{I'\}$$

We have noted that (56) and (58) hold for the primed system, except that primed B-values must be substituted for B-values. But (56) and (58) lead to the dynamics matrix $[\Lambda]$ in (71). Therefore, the dynamics matrix for the primed system, $[\Lambda']$, will be exactly the same as $[\Lambda]$, as given by (83), with primed B-values replacing B-values. That is

$$(108) \quad [\Lambda'] = [\Lambda(B', \partial B')]$$

But in fact the situation is even simpler than indicated by the above. As mentioned in the note just below (105), the equilibrium B-values are invariant. Thus we must have $[\Lambda'] = [\Lambda]$. This will be shown rigorously in Section 6.

To find the matrix corresponding to the control matrix $[\Gamma]$ in (71), we must take $\frac{\partial g'_i}{\partial I'_j}$ using the chain rule, as indicated by (76). Thus we must take derivatives of g' with respect to B'-values, times the derivatives of B'-values with respect to I' values. The first computation is just the same as the unprimed case, with, as usual, primed B-values substituting for unprimed. However, the second computation will not be same because the relationship between B'-values and I'-values is no longer given by (24) and (25). It appears that we must use the complicated equations (98) and (99) to find the derivatives of the B'-values with respect to the I' values. However, an examination of (56) and (57) shows that after the terms containing delta-angle factors are dropped, the only B-values

that remain are those in $[K_{EQ}]$. Thus (100) gives us the expression for all the primed B-values that we will need. To avoid writing this complicated matrix again, let us write

$$(109) \quad [K'_{EQ}] = \begin{bmatrix} [K'_y] \\ [K'_z] \\ [K'_{x,x}] \\ [K'_{x,y}] \\ [K'_{x,z}] \end{bmatrix}$$

Then

$$(110) \quad [\Gamma'] = [W] \begin{bmatrix} \left[\frac{\partial(-B'_z)}{\partial I'} \right] \\ \left[\frac{\partial B'_y}{\partial I'} \right] \\ [0] \\ [0] \\ \left[\frac{\partial B'_{x,x}}{\partial I'} \right] \\ \left[\frac{\partial B'_{x,y}}{\partial I'} \right] \\ \left[\frac{\partial B'_{x,z}}{\partial I'} \right] \\ [0] \\ [0] \\ [0] \end{bmatrix} = \frac{1}{I_{MAX}} [W] \begin{bmatrix} -[K'_z] \\ [K'_y] \\ [0] \\ [0] \\ [K'_{x,x}] \\ [K'_{x,y}] \\ [K'_{x,z}] \\ [0] \\ [0] \\ [0] \end{bmatrix}$$

which is the exact analog of (84). Thus the equations have the same form as in the unprimed case, except that primed values of $[K]$ and $\{I_0\}$ must be found for each θ_z . In the Experimental System [7], the primed system was also found to be controllable and observable, using the same methods that were used in the unprimed case. To recover the unprimed position coordinates, x, y, z , we must use the inverse of the finite rotation matrix given by (89). Thus,

$$(111) \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [T_{\theta_z}]^T \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

To recover the unprimed Euler angles, we use the analog of (48), which is

$$(112) \quad \{\Omega\}_{Euler} = [T_E] \{\Omega'\}_{Euler}$$

Where now the only non-zero rotation angle in the expression (49) for $[T_E]$ is the finite angle θ_z . But an examination of (49) reveals the rather surprising fact the $[T_E]$ is independent of θ_z . Thus $[T_E]$ is the identity matrix for the finite rotation, and we have that

$$(113) \quad \{\Omega\}_{Euler} = \{\Omega'\}_{Euler}$$

This equation, together with (92), (which shows that the gravitation term is not affected by the finite rotation) indicate why this particular rotation was an appealing one to use.

We emphasize that the Experimental System is only capable of suspending the body at a fixed θ_z ; it cannot suspend the body at one θ_z and then generate torque about the z axis to bring the body to a new θ_z which is finitely different from the old. This is because the original equations of motion, (45) and (46), depend non-linearly on θ_z .

This concludes our derivation of the equations of the system. We next sketch the method of observation and the controller design approaches employed in the NASA experiments.

2.15 Methods of Observation and Control

Adequate observation of the suspended body has proved to be a difficult problem for Large-Gap systems [11, pg. 7]. The observing system must provide quite accurate estimates of position and orientation. To meet the requirements of the control system, it

must also be capable of rapidly updating these estimates. Another complicating factor is that the body must be observed in a wide range of orientations. Finally, the observing system must not interfere with the form or function of the levitated body or suspension system. The system used in our experiment is quite complex and the whole of [13] is devoted to it. Here we will note only its external characteristics.

The system uses optical sensing. Light sensing units (cameras) are stationed in an inertial frame surrounding the suspended body. Small light emitting devices, called targets, are embedded in the body. When the body is in one fixed position and orientation, light emitted from any target will stimulate only some of the light-sensitive devices. Thus, the devices stimulated may be associated with the position and orientation of the body. Through an exceedingly complex process involving the coordinates of stimulated devices and the known previous position and orientation of the body, a current estimate of the Euler angles and x,y,z values of the center of the core may be found. Statistical estimation and linearization techniques are employed in this process. In the Experiment System, with sixteen sensors and eight targets, a total of 128 simultaneous nonlinear equations must be solved to produce a single position update. To satisfy the requirements of control, 20 updates per second must be provided. In addition to this computational requirement, consideration must be given to such matters as the signal-to-noise ratio of the light stimulus, motion of the body between light emissions, and camera calibrations. Since we are not dealing with real-time observation issues in this dissertation, or in fact any observation issues, we assume the system is observable and that the real-time difficulties of observation have been solved.

We indicate the papers which have dealt with real-time control calculations. When the body is perturbed away from the equilibrium position, a correcting B-field must be applied. To do this, a current vector must be computed. If, for example, the restoring force-torques which result from the new current vector are too strong, the body may return to equilibrium, but then overshoot the position. Unstable modes must be damped (see Section 2.16). We have not touched at all on these complex questions. Two papers

are devoted to them. Reference [17] discusses two types of controller, the Dual Phase Advance and Proportional-Integral-Derivative. The effect of white noise on both these systems is analyzed. Reference [18] considers two type of Linear Quadratic Regulator systems: a nonzero set point regulator with constant disturbance and an integral feedback regulator. Simulation results for positioning accuracy over large variations in θ_z , with fixed gains, are obtained for both regulators. While these studies have a bearing on the questions discussed in this paper, we will not summarize them here.

We hope that this brief description gives an idea of the difficulties involved in observation and control, and the constraints on the control design that they impose. We next describe some further results that are presented in [7].

2.16 Additional Results of the Experiment; Notes on the Additional Coils System

In addition to developing the equations of our system, [7] addresses some practical control problems. We omit the details of this investigation and give the results only because they are relevant to our discussion in Section 3. Suppose the body is suspended at the equilibrium point. If the body is perturbed slightly away from equilibrium, it will undergo some characteristic motions, called modes. The modes for this system are shown in Figure 11. Some modes tend to return the body to the equilibrium point, while others move the body far away from it. These are called stable and unstable mode, respectively. Associated with each mode is an eigenvalue of the dynamics matrix. The units of the eigenvalues are radians/second; this value is called a frequency. It is the angular rate that a mode associated with an eigenvalue occurs. The frequency of the highest frequency mode must be kept as low as possible because of time constraints in the system. For example, as discussed in 2.15, the body's minimum rate of observation may be 20 samples/second.

The qualitative nature of the modes of our system were determined by an analysis of the eigenvectors of the Λ matrix given by (83). We will not present this analysis here. (See [7], pgs. 10-13. We do show an analytic method for finding the eigenvalues in Section 7.4.) From this analysis and from heuristic examination of the equations of motion, it was

determined that the highest frequency modes were caused by the x component of the B-field, B_x . The attempts to control this component by adding coils resulted in the Additional Coils System. We treat this system in Section 8. For the present, we discuss only the case when one coil is added, so that there are six coils. The six coils are symmetrically positioned, as seen in Figure 6. K_x and B_x are added to (65). so that (65), (66) takes the form

$$(114) \quad \{B_{EQ}\} = \begin{bmatrix} B_x \\ B_y \\ B_z \\ B_{x,x} \\ B_{x,y} \\ B_{x,z} \end{bmatrix} = \begin{bmatrix} [K_x] \\ [K_y] \\ [K_z] \\ [K_{x,x}] \\ [K_{x,y}] \\ [K_{x,z}] \end{bmatrix} \frac{\{I\}}{I_{MAX}} = [K_{EQ}] \frac{\{I\}}{I_{MAX}}$$

Since there are now six coils, $[K_{EQ}]$ is now a 6 by 6 matrix. In the Experimental System, it proved invertible and equilibrium currents were found as before. (B_{x_0} was set to zero.) However, the solution vector of currents obtained by this process was obviously unacceptable because the currents were all very large. In [7], a physical explanation is given for this phenomenon: that the flux field at the datum point resulting from six coils consists of opposing and nearly canceling contributions from all coils. (In Section 8.2 we show that the occurrence can also be understood as a consequence of the structure of the equilibrium matrix.) These results indicated that control of B_x with this configuration of coils was not a viable approach. Seven and eight coils were tried in the same manner, with the same negative outcome. In addition, increasing the number of coils produced an insignificant effect on mode shapes and frequencies. The conclusion was that the general form of this experiment -- identical coils arranged in a single plane and the suspended body with a single horizontally magnetized core -- was not a viable approach to the problem of controlling the B_x component. We next note some of the recent research on LGMSS that has been done at NASA Langley.

2.17 Recent Experiments at NASA Langley

Since [7] was written, NASA has built a physical demonstration model and shown that it works. The model is much smaller than that of the Experimental System. For example, in the Experimental System, the body was suspended about one meter above the coils. In the demonstration model, the body is suspended one-tenth of a meter above the coils, with other parts of the model proportionately smaller. This laboratory system is called the Large-Angle Magnetic Suspension Test Fixture (LAMSTF). "Large-Angle" refers to a major goal of the experiment. This was to show that a configuration such as we have described is capable of rotating the body through a large angle about the vertical axis. In fact, as described in [19], a 360 degree rotation has been accomplished.

The reader is referred to [19], [20], [21], [22], and [23] for detailed descriptions of various aspects of the LAMSTF experiment. We will not attempt to summarize these papers, but to note some of the material in them this is relevant to this report. Some of them reveal a number of facets of the problem which we have not considered. For example, [21] discusses the discrepancies between various measurements and their predicted values. These differences are thought to be partly due to physical phenomena which we have neglected, such as eddy currents. Others, such as [23], deal with the areas of control briefly alluded to in sections 2.15 and 2.16 ; for example, the selection of a control approach to deal with unique features of the experiment, such as the unstable nodes, which we do not treat. In [23], in the process of developing a control approach, a result was stated which provides insight into the nature of the control equations of our system. Equation (17) of [23] essentially writes the control of angular and translational accelerations of the body as a function of a matrix times the current vector, $\{I\}$. That is, in essence the space state equation (71) is written with only the control terms included so that the dynamics matrix $[\Lambda]$ is omitted and we have

$$(115) \quad \{y\} = [\Gamma^*]\{I\}$$

where $\{y\}$ is the five vector of angular and translational accelerations. It does not include the five integrals of these accelerations. The five by five control matrix $[\Gamma^*]$ is much like the control matrix $[\Gamma]$ of the Basic System, if the zero rows of $[\Gamma]$ are omitted. Also $[\Gamma^*]$ turns out to be equivalent, from a linear algebraic point of view, to the equilibrium matrix $[K_{EQ}]$. Thus, for our system, the equilibrium matrix is very similar to the control matrix. This lead us to the proof of controllability in Section 5.

In the Rotated System, when the body is rotated through θ_z , the vector of equilibrium currents, $\{I'\}$, required to suspended the body at this orientation must be computed for each θ_z . In principle these are given by (103), but page 1916 of [19] indicates that in practice a slightly different but equivalent formulation was used. There, the methods used to demonstrate of the 360 degree rotation capability are discussed. To find the current $\{I\}$ corresponding to a desired $\{y\}$ in (115), the matrix is inverted, resulting in

$$(116) \quad \{I\} = [\Gamma^*]^{-1} \{y\}$$

But when the body is rotated through the finite angle θ_z , the matrix in (116) changes; so to demonstrate the 360 degrees rotation capability, it was necessary to compute 60 different matrices at 6 degree intervals. The controller interpolates between these matrices in real time. It is also stated on page 1916 of [19] that the algorithm is not capable of torquing the body through θ_z or of determining the value of this angle. Thus, in any one experiment with the Rotated System, we must think of θ_z as a fixed angle; the system is not capable of moving to other values of this angle. Finally, it is there noted that the currents required for various angles, when plotted as a function of the angle, are almost perfectly sinusoidal. We will see in Section 4.4 that in fact a simple analytic expression for these currents may be derived.

3 Goals and Chief Difficulty in the Mathematical Treatment of Large-Gap Systems; A Brief Discussion of Three Other Systems

We would like to develop a mathematical formalism in large-gap systems that would allow practitioners to establish general theorems in regard to such matters as controllability, observability, and stability. In our search through the literature, it appeared that much more in the way of formal mathematical results of this sort had been achieved in short-gap systems. We discuss a possible reason for this.

By the definition of a large-gap system, the body to be controlled must be at a relatively large distance from the controlling coils. This large gap creates what seems to be the great difficulty in developing a mathematical formalism for large-gap systems. At large distances from a coil, the B-field produced by it normally cannot be computed by other than a numerical process. This limits the results that can be formally deduced from the configuration of coils and magnets on the body. In short-gap systems much more can be done and has been done. To give an idea of the kind of results that we would like to obtain for large-gap systems, we will cite two papers from short-gap theory. Reference [24] deals with the suspension of a thin, rectangular, flat plate. The magnetic elements which provide the suspension are called actuators.; the elements which observe the suspended body are called sensors. The positions of these elements are specified as coordinates in a two dimensional plane. The physics of the actuators and sensors is not described. However, the control force of each actuator is given as a scalar function of time, times a two dimensional delta function of the position coordinates. The sensors are divided into three non-overlapping groups according to whether they measure the position, velocity, or acceleration of the plate. Using this simple formulation, and the physics of the plate, impressive results are obtained. For example, [24] states necessary and sufficient conditions for proper selection of the actuator and sensor positions to ensure controllability and observability and to reduce controller complexity.

Reference [5] also achieves valuable results in short-gap suspension using formal mathematics. In this case the suspended object is an arbitrary rigid three dimensional body. Control force actuators are located at arbitrary points on the surface of this body. With each actuator two vectors are associated: one from the origin to the actuator and a second in the direction of the force of the actuator. The latter vector is of unit magnitude; a positive scalar is also associated with each actuator, which gives the magnitude of its control force. Using this framework, the minimum number of actuators needed to ensure n degrees of freedom is determined, and conditions on the placement of these actuators are developed.

Much more was done in both these papers, but the essential characteristics of each system which made the development possible seem to be the following: (i) the location of each actuator was known and (ii) the exact force associated with each actuator was known. In large-gap systems using coils, we have (i) but usually not (ii). Reference [25] describes a large-gap system in which (ii) is satisfied; that is, an analytic expression for the B-field in the region of the body is known. In addition, the system is similar to ours in that the actuators are all located in a plane which is normal to the gravity vector; and the levitated object is a magnetic dipole. However, the actuators are not coils, but a parallel pair of infinitely long permanent bar magnets. Thus these magnets serve to levitate the body but not control it. Using the analytic expression for the field, [25] finds precise conditions for stability of the suspended object. This was done without computer simulation. The experiment shows that valuable theoretical results can be obtained for large-gap systems if an analytic expression for the B-field in the region of the body is known. However, this system was much simpler than ours and did not provide means to control the body. In the next section we will show that one can obtain analytic results using only assumptions about the cylindrical symmetry of the field produced by the coils.

4 Invertibility of the Equilibrium Matrix and Related Results

4.1 Analytic Expression For The Inverse Of the Equilibrium Matrix for the Basic System

As previously noted, the inverse of the equilibrium matrix for the Basic Experimental System was numerically computed. We will now show $[K_{EQ}]^{-1}$ exists for the Basic General System by deriving an analytic expression for it. The invertibility of the equilibrium matrix will then be seen to be independent of the type of coil used, the height of the body above the coils, and the radial distance of the coils from the origin of coil coordinate system. We will first find relationships that exist between the components of each row of $[K_{EQ}]$. For example, $\text{row}[K_y]$ of $[K_{EQ}]$ will be written as a scalar times another row vector. The scalar will contain all the information about the coils of this particular experiment, and the radial placement of the coil; that is, the Coil Value information. The row vector will have the information relative to the angular placement of the coil, or Angular Value information. If, for example, another experiment were done using different sized coils, or coils placed at a different radial distance from the origin, but the same angular coil placement, the scalar would change but the row vector would not. Similar expressions will be found for each row of $[K_{EQ}]$. These expressions may be arranged so that a factorization of the equilibrium matrix results:

Lemma 4.1-1: The equilibrium matrix for the Basic System may be factored

$$[K_{EQ}] = [C][L]$$

where the five by five matrix $[C]$ depends only on Coil Values and the five by five matrix $[L]$ depends only on Angular Values.

The proof of Lemma 4.1-1 requires equations (117) through (151), which develop the relationships within $[K_{EQ}]$. These relationships depend not only on the symmetric

placement of the coils but also on the symmetry of the flux field produced by each coil, which is a special kind of cylindrical symmetry. Rather than writing out these conditions of symmetry and deriving the relations rigorously (which then are rather hard to visualize), we will use an informal method of visualization. The results have been verified numerically for the Experimental System using the data given in [7].

Figure 12 shows a single coil in the plane. The coil has planar coordinates (r, λ) . Figure 13 shows these relations projected onto the lab x, y plane, and in more detail. The lab coordinates of the center of the coil are x_c, y_c . A line from the origin to x_c, y_c makes an arbitrary angle λ with the x axis. Let $\{K_H\}$ be the maximal horizontal flux at the datum point produced by the coil. This is the flux in the x, y plane produced when the maximal current, I_{MAX} , flows through the coil. As shown in Figures 12 and 13, the direction of $\{K_H\}$, as a function of the position of the coil, depends **entirely** upon λ , whereas the magnitude of $\{K_H\}$, which we will call K_H , has no λ - dependence at all. K_H will depend only on r and the vertical distance, h , that the datum point is above the origin of the coil system. That is, as shown in Figure 12,

$$(117) \quad K_H = f(r, h), \text{ for some function } f, \text{ where } f \text{ is determined by the particular coil}$$

Let K_x, K_y be the x, y components of $\{K_H\}$, as shown in Figure 13. The symmetry of the flux field means that $\{K_H\}$ is along the line from the origin to the center of the coil, so that

$$(118) \quad K_x = K_H \cos \lambda = f(r, h) \cos \lambda$$

$$(119) \quad K_y = K_H \sin \lambda = f(r, h) \sin \lambda$$

We will shortly show that (119) gives us the first row, $[K_y]$, of the equilibrium matrix $[K_{EQ}]$. We now use (118) to find the third and fourth rows. $K_{x,x}$ is the derivative of K_x with respect to x , and from (118), and since h does not vary with x ,

$$(120) \quad K_{x,x} = \frac{\partial f}{\partial r} \frac{\partial r}{\partial x} \cos \lambda - f \sin \lambda \frac{\partial \lambda}{\partial x}$$

Now the change in r due to a change in x means we are finding how r would vary if we moved a small distance away from the origin of the lab frame along the x axis. Clearly, this change would be the same as would occur if we moved the coil the same distance in the opposite direction; that is,

$$(121) \quad \frac{\partial r}{\partial x} = -\frac{\partial r}{\partial x_c}$$

and similarly,

$$(122) \quad \frac{\partial \lambda}{\partial x} = -\frac{\partial \lambda}{\partial x_c}$$

Since

$$(123) \quad r = \sqrt{x_c^2 + y_c^2}$$

and

$$(124) \quad \lambda = \arctan \frac{y_c}{x_c}$$

we have, using (120) through (124),

$$(125) \quad K_{x,x} = -\frac{\partial f(r, h)}{\partial r} \cos^2 \lambda - \frac{f(r, h)}{r} \sin^2 \lambda$$

By a similar process, starting with (118), one may show that

$$(126) \quad K_{x,y} = \left(\frac{f(r, h)}{r} - \frac{\partial f(r, h)}{\partial r} \right) \cos \lambda \sin \lambda$$

To find $K_{x,z}$ we must find the variation in flux when we move a small distance vertically above the datum point. But this is the same as increasing the height, h , of the datum point.

Thus

$$(127) \quad \frac{\partial K_H}{\partial z} = \frac{\partial K_H}{\partial h}$$

Then, by (118)

$$(128) \quad K_{x,z} = \frac{\partial f(r, h)}{\partial h} \cos \lambda - f \sin \lambda \frac{\partial \lambda}{\partial z}$$

But by Figure 12,

$$(129) \quad \frac{\partial \lambda}{\partial z} = 0 ; \text{ thus}$$

$$(130) \quad K_{x,z} = \frac{\partial f(r, h)}{\partial h} \cos \lambda$$

Finally we consider K_z , the flux in the z direction. If, in Figure 12, we imagine the position of the coil changing to different λ 's but r and h remaining constant, it is clear from the symmetry of the flux field that the flux in the z direction at the datum point will not change. Thus, as indicated in Figure 12, K_z is some function of r and h only, say

$$(131) \quad K_z = g(r, h)$$

Equations (119), (125), (126), (130), and (131) will enable us to find the desired expression for $[K_{EQ}]$ so we will write these equations in the order of the rows of $[K_{EQ}]$. The expressions for $K_{x,x}$ and $K_{x,y}$, given by (125) and (126) respectively, have been slightly altered for convenience to obtain (134) and (135) below. The alterations involve only trigonometric substitutions and rearrangement. Also, from this point on, the values r and h are understood, so we omit them.

$$(132) \quad K_y = f \sin \lambda$$

$$(133) \quad K_z = g$$

$$(134) \quad K_{xx} = 1/2 \left(\frac{f}{r} - \frac{\partial f}{\partial r} \right) \cos 2\lambda - 1/2 \left(\frac{f}{r} + \frac{\partial f}{\partial r} \right)$$

$$(135) \quad K_{xy} = 1/2 \left(\frac{f}{r} - \frac{\partial f}{\partial r} \right) \sin 2\lambda$$

$$(136) \quad K_{xz} = \frac{\partial f}{\partial h} \cos \lambda$$

We now specialize (132) through (136) to the case of the actual experiment, which has extreme symmetry. First, the coils are identical, so the functions f and g will each be the same for all coils. Then, since each coil is located at the same distance r from the origin of the coil system, and each coil is located distance h below the datum point, $f, \frac{f}{r}, g, \frac{\partial f}{\partial r}$, etc., will be the same for all coils. To reflect this, let us write

$$(137) \quad c_0 = f; \quad c_1 = g; \quad c_2 = 1/2 \left(\frac{f}{r} - \frac{\partial f}{\partial r} \right); \quad c_{2\#} = -1/2 \left(\frac{f}{r} + \frac{\partial f}{\partial r} \right);$$

$$c_3 = c_2; \quad c_4 = \frac{\partial f}{\partial h}$$

Finally, the coils are located at values of λ which are multiples of $\frac{2\pi}{5}$ radians. If we let $\phi = \frac{2\pi}{5}$, then for the j^{th} coil, the value of λ will be $(j-1)\phi$. Let us then define the following constants, for $j = 0, 1, 2, 3, 4$:

$$(138) \quad L_0^j = \sin lj\phi$$

$$(139) \quad L_1^j = 1$$

$$(140) \quad L_2^j = \cos 2j\phi$$

$$(141) \quad L_3^j = \sin 2j\phi$$

$$(142) \quad L_4^j = \cos 1j\phi$$

Note that j varies from 0 to 4 rather than 1 to 5 to avoid carrying along the factor $j-1$. (The reason that 1 is placed before $j\phi$ in (138) and (142) will be obvious in a moment.)

Then for $i = 0, 1, \dots, 4$, we define the five component row vector

$$(143) \quad [L_i] = [L_i^0, L_i^1, \dots, L_i^4]$$

Then by (132) through (143), we have

$$(144) \quad [K_y] = c_0[L_0]$$

$$(145) \quad [K_z] = c_1[L_1]$$

$$(146) \quad [K_{x,x}] = c_2[L_2] + c_2^\# [L_1]$$

$$(147) \quad [K_{x,y}] = c_3[L_3]$$

$$(148) \quad [K_{x,z}] = c_4[L_4]$$

Using the definition of the equilibrium matrix given by (8) and (9), we may write (144) through (148) in matrix form:

$$(149) \quad [K_{EQ}] = [C][L] \quad \text{where}$$

$$(150) \quad [C] = \begin{bmatrix} c_0 & 0 & 0 & 0 & 0 \\ 0 & c_1 & 0 & 0 & 0 \\ 0 & c_2^\# & c_2 & 0 & 0 \\ 0 & 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & 0 & c_4 \end{bmatrix} \quad \text{and}$$

$$(151) \quad [L] = \begin{bmatrix} [L_0] \\ [L_1] \\ [L_2] \\ [L_3] \\ [L_4] \end{bmatrix}$$

This prove Lemma 4.1-1. Now by (149), if $[K_{EQ}]^{-1}$ exists, it must be given by

$$(152) \quad [K_{EQ}]^{-1} = [L]^{-1}[C]^{-1}$$

provided that the inverses on the right exist. If each c_i on the main diagonal of the C matrix is non-zero, $[C]^{-1}$ obviously exists and is easily computed. Hence, we make the following assumption:

Each of the elements of the main diagonal of $[C]$ is non-zero. (Assumption 5)

The data in [7] shows that Assumption 5 holds for the Experimental System.

Assumption 5 is entirely reasonable for the General System but does impose a slight restriction in addition to Assumption 1. We refer again to Figure 4. The B_H of Figure 4 is the K_H of the present discussion; recall that K_H is the maximum value that B_H may take on. We have set $K_H = f = c_0$ in this discussion. As noted in the discussion of Assumption 1, since h is positive, $K_H = f = c_0$ is non-zero. The B_z of Figure 4 is now $K_z = g = c_1$; thus by Assumption 1, c_1 is non-zero. Now $c_4 = \partial f / \partial h$ and from Figure 4 it is not obvious that this could not be zero in a reasonable experiment. However, we will show in Section 7.5 that in order for the body to be suspended, this value must be bounded away from zero. Thus,

if we assume that the experiment has been designed so that the body is suspended, then $c_4 \neq 0$.

Finally, by (137), $c_2 = c_3$ is non-zero provided that $f/r \neq \partial f/\partial r$. This can be shown to hold provided $f = c_0 \neq 0$ (which is true from Assumption 1) and the datum point is in a region of the flux field in which the magnitude of horizontal flux decreases as a point moves radially away from the coil. That is, suppose at the datum point

$$(153) \quad \frac{\partial |f|}{\partial r} < 0$$

If $f > 0$ then $|f| = f$ and so $\frac{\partial f}{\partial r} = \frac{\partial |f|}{\partial r}$; thus, by (153), $\frac{1}{r} \frac{\partial f}{\partial r} < 0$ and hence cannot equal f .

If we assume $f < 0$, a similar chain of reasoning leads to $\frac{1}{r} \frac{\partial f}{\partial r} > 0$ so that again we have

$\frac{1}{r} \frac{\partial f}{\partial r} \neq f$. Thus, Assumption 1 plus the reasonable assumption that the horizontal flux decreases as we move radially away from the coil leads to the condition that $c_2 = c_3 \neq 0$.

However, we will not analyze the nature of the flux pattern to determine regions, if any, where these conditions might not hold. From a theoretical point of view, the C matrix will be singular only if one of the diagonal elements is exactly equal to zero; and for this to happen a flux component or derivative would have to be exactly zero when the maximum current is flowing through the coils, an extremely unlikely occurrence. Thus, Assumption 5 poses little restriction.

We now show that $[L]$ is invertible; in fact,

Lemma 4.1-2: The rows of $[L]$ are orthogonal; $[L][L]^T = [D]$, where $[D]$ is a non-singular diagonal matrix.

To demonstrate Lemma 4.1-2 we will prove some identities. These identities are well known, but we will use a method of proof using complex numbers; this form of proof suggests that the L matrix, though real, may be related through complex numbers to a more fundamental form, as is shown in Section 9. Let w be the fifth root of unity, that is

$$(154) \quad w = e^{i\phi}, \text{ where } i = \sqrt{-1} \text{ and } \phi = \frac{2\pi}{5}$$

Let n be an integer. Now

$$(155) \quad w^n = e^{in\phi}$$

Consider the sum s where

$$(156) \quad s = \sum_{j=0}^4 (w^n)^j = \sum_{j=0}^4 \cos jn\phi + i \sum_{j=0}^4 \sin jn\phi$$

s is the sum of a geometric series so that if $w^n \neq 1$,

$$(157) \quad s = \frac{1 - w^5}{1 - w} = 0$$

If, however, $w^n = 1$,

$$(158) \quad s = 5$$

w^n will equal 1 if and only if n is a multiple of 5. Thus by equating the real and imaginary parts of (156), we have

$$(159) \quad \sum_{j=0}^4 \cos nj\phi = 0, \quad \text{if } n \text{ is not a multiple of } 5$$

$$= 5, \quad \text{if } n \text{ is a multiple of } 5$$

$$(160) \quad \sum_{j=0}^4 \sin nj\phi = 0, \quad \text{for all integral values of } n$$

Now consider "dot products" of the rows of $[L]$, that is, expressions of the form

$$(161) \quad [L_r \cdot L_t]^T$$

We first show that $[L_1]$ is orthogonal to all other rows. By (139), $[L_1]$ consists of all 1's; the dot product of it with itself is 5. The dot product of $[L_1]$ with any other row will just be the sum of the elements in that row. Thus it will be a sum of the form (159) or (160) where the value of n is different for different rows of $[L]$. In (138) and (142), n equals 1; in (140) and (141) it equals 2. Thus in no case is n a multiple of 5. We have then for $t = 0, 1, \dots, 4$

$$(162) \quad [L_1][L_t]^T = 5\delta_{1t}, \text{ where } \delta_{rt} \text{ is the Kronecker delta symbol}$$

It remains to compute (161) for values of r and t not equal to 1. To do this, we need the following trigonometric identities:

$$(163) \quad \sin nj\phi \sin mj\phi = 1/2[-\cos(n+m)j\phi + \cos(n-m)j\phi]$$

$$(164) \quad \cos nj\phi \cos mj\phi = 1/2[\cos(n+m)j\phi + \cos(n-m)j\phi]$$

$$(165) \quad \sin nj\phi \cos mj\phi = 1/2[\sin(n+m)j\phi + \sin(n-m)j\phi]$$

The dot product (161) for both r and t not equal to 1 will consist of sums from $j = 0$ to $j = 4$ of terms like those on the left sides (163), (164), or (165). Consider sums of products of sines and cosines. Such a product is given by (165). By the right side of (165), the sum of such products will involve two sums of sine terms. By (160), such sums will always be zero no matter what the values of $(n+m)$ and $(n-m)$. That is

$$(166) \quad \sum_{j=0}^4 \sin nj\phi \cos mj\phi = \frac{1}{2} \sum_{j=0}^4 \sin(m+n)j\phi + \frac{1}{2} \sum_{j=0}^4 \sin(m-n)j\phi = \frac{1}{2}(0+0) = 0$$

Thus sums of sine times cosine products will always be zero, and we need only consider sine times sine and cosine times cosine products. Consider the sine times sine product given by (163). We show that the sum of such products is given by

$$(167) \quad \sum_{j=0}^4 \sin nj\phi \sin mj\phi = \frac{1}{2} \sum_{j=0}^4 -\cos(m+n)j\phi + \frac{1}{2} \sum_{j=0}^4 \cos(m-n)j\phi$$

$$= \frac{1}{2}(0 + 5\delta_{mn}) = \frac{5}{2}\delta_{mn}$$

As (138) through (142) show, n and m can only take on the values 1 and 2. Thus $(n+m)$ can't be a multiple of 5. Therefore, on the right side of (163) the sum of terms $-\cos(n+m)j\phi$ is zero by (159). For the same reason, $(n-m)$ can only be a multiple of 5 if $(n-m) = 0$; that is, if $n = m$. Then the sum of terms $\cos(n-m)j\phi$ will be zero if $n \neq m$; it will be 5 if $n = m$. Thus a sine row dotted with itself is 5/2 and dotted with any other sine row is zero. By considering (164), we see that exactly the same considerations hold for cosine times cosine products. Thus we have

$$(168) \quad [L_r \perp L_t]^T = 5/2\delta_{rt}, \text{ for } r, t \neq 1$$

This, together with (162), show that the rows of $[L]$ are orthogonal and Lemma 4.1-2 is established. We now prove

Theorem 4.1-3: Given Assumption 5, the inverse of the equilibrium matrix for the Basic System exists.

Proof: By Lemma 4.1-1, $[K_{EQ}] = [C][L]$. Assumption 5 together with (150) mean $[C]^{-1}$ exists. By Lemma 4.1-2, the rows of $[L]$ are orthogonal and (162), (168) show $[L][L]^T = [D]$, where $[D]$ is a non-singular diagonal matrix; $[D] = \text{diag}(5/2, 5, 5/2, 5/2, 5/2)$. Now

$$[L]([L]^T[D]^{-1}) = ([L][L]^T)[D]^{-1} = [D][D]^{-1} = [E]$$

where $[E]$ is the identity matrix. Thus, $[L]^{-1}$ exists and is given by $[L]^{-1} = [L]^T[D]^{-1}$. The inverse of $[C]$ will also be a diagonal matrix, save for one off-diagonal term. Using these expressions in (152), we compute the analytic expression for the equilibrium matrix:

$$(169) \quad [K_{EQ}]^{-1} = [L]^T[D]^{-1}[C]^{-1} =$$

$$2/5 \left[c_0^{-1} \{L_0\}, \frac{1}{2c_1} \{L_1\} - \frac{c_2^\#}{c_1 c_2} \{L_2\}, c_2^{-1} \{L_2\}, c_3^{-1} \{L_3\}, c_4^{-1} \{L_4\} \right]$$

This proves Theorem 4.1-3.

4.2 Expression For The Equilibrium Currents; Methods Of Verification

Equations (68) and (169) lead to a simple expression for the equilibrium current ratios, $\{I\}_0/I_{MAX}$. We note from (67) that the first four components of $\{B_{EQ}\}_0$ are zero, and the fifth component is c . Thus, by (68), the equilibrium current ratio vector is just c times the fifth column of $[K_{EQ}]^{-1}$, which can be read off from (169). We state this as

Corollary 4.2-1: Given Theorem 4.1-3, the equilibrium current ratios for the Basic System exist and are uniquely given by

$$(170) \quad \frac{\{I\}_0}{I_{MAX}} = [K_{EQ}]^{-1} \{B_{EQ}\}_0 =$$

$$[K_{EQ}]^{-1} \begin{bmatrix} 0 & 0 & 0 & 0 & c \end{bmatrix}^T = \frac{2c}{5c_4} \{L_4\} = \frac{2c}{5} \left(\frac{\partial f}{\partial h} \right)^{-1} \{L_4\}$$

provided Assumption 2 holds for the current ratios that result from the above computation.

Note that $c_4 = \partial f / \partial h$ in the above expression is determined by Coil Values, while $c = m_c g / \nu M_{\bar{x}}$ is given by Body Values. In Section 7.5 we will use (170) to show that Assumption 2 holds if and only if an inequality between Coil and Body Values is satisfied.

All of the expressions in this dissertation have been verified for the Experimental System using the data given in [7]. To show how this may be done, we indicate how (170) was verified. The value $c = 0.096$ is given directly in [7]. The value of c_4 is found as follows. Recalling the definition of c_4 given by (137), we have from (130)

$$(171) \quad K_{xz} = c_4 \cos \lambda$$

By Figure 1, $\lambda = 0$ for coil number 1, so $c_4 = K_{xz}^1$, and $K_{xz}^1 = -0.0497$ is given in [7]. The five equilibrium currents $\{I\}_0/I_{MAX}$ are also given in [7], and (170) was thus numerically verified. We will not reproduce these tabular results here. Instead, in Section 4.4, we will use the numbers given here to verify a more general case. Equation (170) is a special instance of this more general case.

4.3 Analytic Expression for the Inverse of the Equilibrium Matrix for the Rotated System

We now establish the above results for the Rotated System. The analogy to Lemma 4.1-1 is

Lemma 4.3-1: The equilibrium matrix for the Rotated System may be factored

$$[K'_{EQ}] = [C][L']$$

where [C] depends only on Coil Values and [L'] depends only on Angular Values, and the angle the body has been rotated through, θ . (From this point on we will denote the rotation angle θ rather than θ_z for convenience.)

The proof of Lemma 4.3-1 can be effected by simple re-definitions. To this end, the reader is requested to look at Figure 14 and then to re-examine equations (132) through (136). These are expressions for K-values of a coil which is at angle λ from the x axis. The coil in Figure 14 is at angle λ' from the x' axis, where

$$(172) \quad \lambda' = \lambda - \theta$$

The coil stands in exact relation to the primed coordinate system as it did to the unprimed, except that λ is replaced by λ' . An examination of (132) through (136) shows that the expressions for f , g and r are independent of the angle λ . Therefore, these equations are valid when λ is replaced by λ' and the K -values on the left sides of the equations are replaced by K' values. This gives us an alternate method of computing $[K'_{EQ}]$. For example, by (132), recalling (118) and (119), we have that

$$(173) \quad K'_y = f \sin(\lambda - \theta) = \cos \theta (f \sin \lambda) - \sin \theta (f \cos \lambda) = \cos \theta K_y - \sin \theta K_x$$

by which the top row of the matrix in (100) may be verified. However, if we look at (137), we see that the c_i values will not change in the primed case. These constants carry all the information about the distance r that the coils are from the origin of the coil system, and the particular coils that are used in the experiment. That is, they are Coil Values and obviously could not be changed by rotating the body. The angular information is carried in the matrix $[L]$. We show that a primed version of this matrix may be written so that the inverse to $[K'_{EQ}]$ is easily computed.

In the unprimed system, the value of λ for the j^{th} coil is $(j - 1)\phi$. In the primed system this value is given by $(j - 1)\phi - \theta$. We are led to the following definitions, which are analogous to (138) through (142):

For $j = 0, 1, 2, 3, 4$,

$$(174) \quad L''_0 = \sin 1(j\phi - \theta)$$

$$(175) \quad L''_1 = 1$$

$$(176) \quad L''_2 = \cos 2(j\phi - \theta)$$

$$(177) \quad L''_3 = \sin 2(j\phi - \theta)$$

$$(178) \quad L''_4 = \cos 1(j\phi - \theta)$$

Obviously, L^j_i is a special case of L''_i with $\theta = 0$.

Having explicitly written the above primed definitions, we will not write out every such definition when it is obvious what the primed definition would be. Thus the meaning of the five component row vector $[L'_i]$ may be gathered from (143) and the five times five matrix $[L']$ may then be defined in exact analogy from (151). When this is done, the equation corresponding to (149) is seen to be

$$(179) \quad [K'_{EQ}] = [C][L']$$

Thus Lemma 4.3-1 is established.

Now, since the c_i values do not change, $[C]$ is still given by (150). We again make Assumption 5, so $[C]$ is non-singular. Then, as in (152),

$$(180) \quad [K'_{EQ}]^{-1} = [L']^{-1}[C]^{-1}$$

provided $[L']^{-1}$ exists. We show that the rows of the primed L-matrix are also orthogonal:

Lemma 4.3-2: The rows of $[L']$ are orthogonal; in fact,

$$[L'] [L']^T = [L][L]^T = [D] = \text{diag}(5/2, 5, 5/2, 5/2, 5/2)$$

To prove this, we use the same definitions given in (154), (155), and (156) and multiply both side of (156) by $e^{-in\theta}$ to obtain

$$(181) \quad e^{-in\theta} s = 0, \text{ if } n \text{ is not a multiple of } 5$$

$$= 5e^{-in\theta}, \text{ if } n \text{ is a multiple of } 5$$

We equate the real and imaginary parts of (181) to obtain the equations corresponding to (159) and (160):

For arbitrary θ , if n is not a multiple of 5,

$$(182) \quad \sum_{j=0}^4 \cos n(j\phi - \theta) = 0 \quad \text{and}$$

$$(183) \quad \sum_{j=0}^4 \sin n(j\phi - \theta) = 0$$

If n is a multiple of 5,

$$(184) \quad \sum_{j=0}^4 \cos n(j\phi - \theta) = 5 \cos n\theta \quad \text{and}$$

$$(185) \quad \sum_{j=0}^4 \sin n(j\phi - \theta) = -5 \sin n\theta$$

We merely indicate the proof of the orthogonality of the rows of $[L']$, since it is essentially as in the unprimed case. When $[L'_1]$ is dotted with any other row, we obtain the sum of elements in that row; in other words, one of the sums given by (182) through (185). But by (174), (176), (177), and (178), n is always 1 or 2, and so is never a multiple of 5. Thus the sums are given by (182) or (183), which are both zero. Hence, $[L'_1]$ is orthogonal to every other row, and the dot product of it with itself is 5. (Note this is the same as $[L_1]$ dotted with itself.) We next consider the dot product of any of the other rows. For brevity, we consider the product of a sine row with a sine row, the other cases being entirely similar. For this, we need the analogy to (163) given by

$$(186) \quad \sin n(j\phi - \theta) \sin m(j\phi - \theta) = (1/2)[- \cos (n + m)(j\phi - \theta) + \cos (n - m)(j\phi - \theta)]$$

The dot product of a sine row with a sine row will be the sum from $j = 0$ to 4 of the left side of (186), with both n and m either 1 or 2. Since n and m are both either 1 or 2, $n + m$

can't be a multiple of 5, and $n - m$ will be a multiple of 5 if and only if $n - m = 0$, or $n = m$, in which case we must have the dot product of a row with itself. Applying these results to the right side of (186) we see that the dot product of a sine row with a distinct sine row will result a pair of cosine sums, each like (182) and so zero. The dot product of a sine row with itself will also result in a pair of cosine sums. The one corresponding to $n + m$ will still be zero since $n + m$ can't be a multiple of 5. But the one corresponding to $n - m$ will not be zero. This is because $n - m = 0$ and so is a multiple of 5. Thus this sum is like (184), and so the sum equals $5 \cos 0\theta = 5$. Therefore, when a sine function is dotted with itself, the right side of (186) is given by $(1/2)(-0 + 5)$ or $5/2$. (Note that this is the same as an unprimed sine row dotted with itself.) By a similar process it may be shown that the dot product of a cosine row with a distinct cosine row is zero, and the dot product of a cosine row with itself is $5/2$. (Again, the same value as an unprimed cosine row dotted with itself.) Thus the rows of $[L']$ are orthogonal and when dotted with themselves give the same values as in the unprimed case. Lemma 4.3-2 is established.

Thus we have

Theorem 4.3-3: Given Assumption 5, the inverse of the equilibrium matrix for the Rotated System exists.

Proof: The proof is exactly the same as the proof of Theorem 4.1-3, with $[K_{EQ}]$, $[L]$ replaced by their primed counterparts and the other matrices unchanged. We have, then, in exact analogy to (169),

$$(187) \quad [K'_{EQ}]^{-1} = 2/5 \left[c_0^{-1} \{L'_0\}, \frac{1}{2c_1} \{L'_1\} - \frac{c_2\#}{c_1c_2} \{L'_2\}, c_2^{-1} \{L'_2\}, c_3^{-1} \{L'_3\}, c_4^{-1} \{L'_4\} \right]$$

4.4 Alternative Notation And Numerical Verification Of Equations; Equilibrium Currents for the Rotated System

We see that to obtain (187) from (169), we merely replace $\{L_i\}$ with $\{L'_i\}$ throughout. In general, we may think of the unprimed equations as a special case of the primed ones in which $\theta = 0$. The Basic System may be thought of as a special case of the Rotated. Instead of writing primes we could carry along the variable θ and there would be only one formulation. For example, in our new notation $\{L'_i\}$ would be replaced by $\{L_i(\theta)\}$ and $\{L_i\}$ by $\{L_i(0)\}$. We do this with (170) to obtain the primed restatement of Corollary 4.2-1:

Corollary 4.4-1: Given Theorem 4.3-3, the equilibrium current ratios for the Rotated System exist and are uniquely given by

$$(188) \quad \frac{\{I'\}}{I_{MAX}} = \frac{2c}{5c_4} \{L'_4\}; \text{ alternative notation: } \frac{\{I(\theta)\}}{I_{MAX}} = \frac{2c}{5c_4} \{L_4(\theta)\}$$

provided Assumption 2 holds for the current ratios that result from the above computation.

Proof: Since by (95), $\{B_{EQ}\}_0 = \{B'_{EQ}\}_0$, the proof is exactly like the proof of Corollary 4.2-1 with the expression for the primed equilibrium matrix given by (187) replacing the unprimed.

Thus the equilibrium currents ratios may be considered to be functions of θ . Since by (178) each L'_4 is a cosine function of $-\theta$ with period 2π and phase $j\phi$, if the current ratios were plotted as functions of θ they should have this form. We must also note that, in the Experimental System, the constant before the vector on the right side of (188) is negative; the values for the Experimental System given just below (171) result in $\frac{2c}{5c_4} = -0.7726$. The current ratios were graphed as functions of θ in [7] and we have reproduced these graphs in Figure 15. The graphs were just as predicted by (188). For example, the graph for coil 1 (for which $j = 0$) approximates $-0.7726 \cos(-\theta)$. The graph for coil 2 (for which $j = 1$) is shifted to the right the value of ϕ , which is 72 degrees; thus it is the graph of $-0.7726 \cos(\phi - \theta)$, as expected. Thus (188) checked numerically with

the Rotated Experimental System. (We must parenthetically note here there are numerical discrepancies between a few of our figures and those in [7], but they are very small. Their order of magnitude may be indicated as follows: if a number is given to four significant figures in [7], it always agrees to three significant figures with ours. We believe these differences are due to round-off.)

The above analysis was more clearly expressed by using θ instead of primes, but often it is more convenient to write primes. In the following, we must ask the reader's indulgence in regard to the use of these two notations. We will use them interchangeably, according to which is more convenient.

5 Proof Of Controllability

We now show that the invertibility of the equilibrium matrix means the Basic System is controllable. Our proof depends on a peculiarity of the system, but it is a peculiarity which is probably shared by many control systems. It is actually easier to discuss this question in the abstract before considering the details of our control system. Suppose we have a control system of dimension $2n$ whose general control equation may be written in the following form:

$$(189) \quad \frac{d}{dt} \begin{bmatrix} z_1 \\ z_2 \\ \dots \\ z_n \\ \dot{z}_1 \\ \dot{z}_2 \\ \dots \\ \dot{z}_n \end{bmatrix} = [A] \begin{bmatrix} z_1 \\ z_2 \\ \dots \\ z_n \\ \dot{z}_1 \\ \dot{z}_2 \\ \dots \\ \dot{z}_n \end{bmatrix} + [B] \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix}$$

Now suppose:

- (i) each z_i is *only* the integral of \dot{z}_i and

(ii) each \ddot{z}_i is not a function of *any* \dot{z}_k .

Then if we let $\{z\}^T = [z_1, z_2, \dots, z_n]$ then (i) implies

$\frac{d\{z\}}{dt} = [E]\{\dot{z}\}$, where $[E]$ is the n by n identity matrix and (ii) implies that

$\frac{d\{\dot{z}\}}{dt} = [A_1]\{z\} + [B_1]\{u\}$, where $[A_1]$ and $[B_1]$ are some n by n matrices.

Thus (189) may be written in the following form:

$$(190) \quad \frac{d}{dt} \begin{bmatrix} \{z\} \\ \{\dot{z}\} \end{bmatrix} = \begin{bmatrix} 0 & E \\ A_1 & 0 \end{bmatrix} \begin{bmatrix} \{z\} \\ \{\dot{z}\} \end{bmatrix} + \begin{bmatrix} 0 \\ B_1 \end{bmatrix} \{u\}$$

It may be shown that the above system is controllable provided B_1 is invertible. (The proof will be obvious from the proof for our system.) Our system is of this type, with K_{EQ} playing the role of B_1 . We state this as

Lemma 5-1: The control equation for the Basic System may be written in the form (190), with $n = 5$.

Proof: Consider rows 3, 4, 7, 8, 9 of the matrices on the right sides of (83) and (84). In $[\Lambda]$ these rows are zero except for a single 1; in $[\Gamma]$ the rows are zero. A careful examination of the details shows that the 1's are in exactly the correct columns so that statement (i) of the abstract system is true for our system. Now consider columns 1, 2, 5, 6, 7 of $[\Lambda]$. These columns contain only zeros except for a single 1. Examination shows this 1 is in the row which makes statement (ii) true. This proves Lemma 5-1.

We now re-arrange our control system equations so they are in the form given by (190). First note that (190) implies

$$(191) \quad \{\ddot{z}\} = [A_1]\{z\} + [B_1]\{u\}$$

We eliminate the rows and columns from our system which convey no essential information to get it in the form (191). We then rearrange the rows so that $[B_1] = [K_{EQ}]$. The resulting expression is still in the form (191), which can then be written as in (190). The first step is to delete the rows of the matrix on the right side of (83) which are zero except for a single 1. These are rows 3,4,8,9,10; the same rows are all zero in the matrix on the right side of (84), and we also delete these. When this is done the resulting matrix would be $[K_{EQ}]$ is the top row were multiplied by -1 and interchanged with the second row. We do this with our rearranged system and also delete columns 1,2,5,6,7, which are zero except for a single 1, from the matrix on the right side of (83). Since these operations correspond to multiplication by a non-singular matrix, they will not affect controllability. Angles and positions form the upper five elements of the state vector of the rearranged system, angular rates and velocities the lower. Because it would involve too much detail, we will not reproduce the actual equations. Instead, we will merely specify the elements in the rearranged state vector. We start with the upper half of it. Let the five component vector $\{Y\} = [y_1, \dots, y_5]^T$ of the system be given in this order: angles about the \bar{z} , $-\bar{y}$ axes and positions in the \bar{x} , \bar{y} , \bar{z} directions. Then, up to an unimportant non-singular matrix of constants, the acceleration of $\{Y\}$ is given by

$$(192) \quad \{\ddot{Y}\} = [A_1]\{Y\} + [K_{EQ}] \frac{\{I\}}{I_{MAX}}$$

(Of course the units of the above system are not correct due to the omissions of constants.)

The rearrangement and multiplication by -1 was done so the equilibrium matrix K_{EQ} could appear in the equation above. The five by five constant matrix $[A_1]$ is found by

writing five of the ten linearized equations of motion (71) in the order of $\{Y\}$ given above. The five elements are those which correspond to angular and translational accelerations. We then define the rearranged ten component state vector $\{S\}$ by

$$(193) \quad s_i = y_i, \quad s_{i+5} = \dot{y}_i, \quad i = 1, 2, 3, 4, 5$$

The state space equation of the system can be written

$$(194) \quad \dot{\{S\}} = [A]\{S\} + [B]\{u\} \quad \text{where}$$

$$(195) \quad [A] = \begin{bmatrix} 0 & E \\ A_1 & 0 \end{bmatrix}, \quad [B] = \begin{bmatrix} 0 \\ K_{EQ} \end{bmatrix}, \quad \{u\}^T = \begin{bmatrix} \{0\}^T & \{I\}^T / I_{MAX} \end{bmatrix}$$

Where $[E]$ is the five by five identity matrix, $[0]$ is the five by five zero matrix, and $\{0\}$ is the five component zero vector. $[A]$ is the dynamics matrix of our equivalent system. We are now ready to prove

Theorem 5-2: The Basic System is controllable.

Proof: By [26] the system is given by (195) is controllable if the 10 by 50 controllability matrix $[Q]$ has rank 10 where

$$(196) \quad [Q] = \begin{bmatrix} B & AB & A^2B & \dots & A^9B \end{bmatrix}$$

By the definition of the rank of a matrix, a matrix has rank at least as great of any of its submatrices. Thus $[Q]$ has rank greater than or equal to the 10 by 10 submatrix $\begin{bmatrix} B & AB \end{bmatrix}$ But by (195)

$$(197) \quad \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & K_{EQ} \\ K_{EQ} & 0 \end{bmatrix}$$

Which is clearly non-singular since, by $[K_{EQ}]$ is non-singular. The system must therefore be controllable no matter what the entries of $[A_1]$ may be. The system given by (195) and (196) may be obtained from the control equations of the Basic System, given by (71), (83), and (84) by multiplication by non-singular matrices. Therefore the Basic System is controllable and the proof of Theorem 5-2 is complete. This result is easily shown to be true of the rotated system:

Theorem 5-3: The Rotated System is controllable.

Proof: The proof of Theorem 5-3 shows that controllability is independent of the dynamics matrix, provided that the equations can be put in the form (191). The control equations for the Rotated System, (107), (109), (110), have the same form as those for the Basic System, hence they can be put in the form (191). Every step of the proof of Theorem 5-2 may be applied to the Rotated System, and since by Theorem 4.3-3, $[K'_{EQ}]$ is non-singular, the result follows.

The following are some intuitive ideas related to the controllability of our systems. Since $[K_{EQ}]$ was defined to deal with the question of equilibrium, it may seem surprising that it should appear in a discussion of controllability. We will try to give an intuitive rationale for the fact that the existence of a unique solution to the equilibrium equation means the system is controllable. First, recall that for our systems, controllability does not mean that the body may be moved to any finite x, y, z position, or be oriented to any angle about the y or z axes. The goal of the experiment was to be able to suspend the body at the equilibrium point and to counteract deviations from this position by generating torques and forces. Intuitively, the system is controllable if this is possible.

To **suspend** the body at the equilibrium state, we must be able to solve the equilibrium equation (67). But to **control** the body, we must be able to solve the same equation for small deviations from the equilibrium state. That is, we must be able to solve (66). For example, to **hold** the body at the equilibrium state, we must be able to overcome the force of gravity by exerting a force in the z direction. But to **control** the z -dimension of

position, we must be able to exert forces in the z direction which are slightly greater or less than the force of gravity. This may be stated informally as

(198) External "forces" needed to suspend = $[K_{EQ}]\{I\}_0/I_{MAX}$ (Where "forces" also includes torques.)

(199) External "forces" needed to control = $[K_{EQ}](\{I\}_0 + \{\delta I\})/I_{MAX}$

Where δI is small deviations away from equilibrium current. Thus the invertibility of the equilibrium matrix means not only that (198) is solvable and we have an equilibrium point. It also means that we can solve (199) for $\{\delta I\}$ to generate small external "forces" greater than those needed for suspension. Now suppose we have a small deviation from the equilibrium point. For the sake of simplicity and definiteness, let us suppose this is a deviation in the z direction only. The solvability of (199) means we can find a δI which will exert a force in only the z direction. This will result in a non-zero \ddot{z} . Thus, of course, \dot{z} becomes non-zero. However, (ii) of Lemma 5-1 means that \dot{z} does not cause any additional forces or torques. Condition (i) of Lemma 5-1 means that \dot{z} causes only a change in z, not in any of the other coordinates. Hence the original deviation in the z direction can be corrected without affecting any of the other coordinates.

This is the intuitive meaning of controllability. Using these ideas, we could probably show that the abstract system given by (189) is controllable, provided the statements (i) and (ii) of Lemma 5-1 are true, and if an equilibrium point exists.

6 Invariance Of The Dynamics Matrix

In Section 4.3, the computation of the dot products of rows of the primed L matrix show they were the same as the unprimed. This is the same as saying that the dot product of rows of the L matrix do not vary with θ . We restate the equation of Lemma 4.3-1 in vector form

(200) $[L'_i]\{L'_j\} = [L_i]\{L_j\}$; alternative notation: $[L_i(\theta)]\{L_j(\theta)\} = [L_i(0)]\{L_j(0)\}$

Next, we state the linearized space state equation (107) as a function of θ :

$$(201) \quad \{\dot{X}(\theta)\} = [\Lambda(\theta)]\{X(\theta)\} + [\Gamma(\theta)]\{I(\theta)\}$$

By (195), the control matrix Γ is essentially the equilibrium matrix. Since, as we have shown, both the equilibrium matrix and the equilibrium current vector change with θ , the control matrix changes with θ . We would expect this to also be true of the dynamics matrix Λ . By (108), the entries in the dynamics matrix can be written as a functions of equilibrium B-values, which are assumed to vary with θ . We will not write out the 10 by 10 dynamics matrix as a function of B-values, but formally state the result as follows:

$$(202) \quad [\Lambda(\theta)] = [\Lambda(B_i(\theta)_0, B_{i,j}(\theta)_0, B_{i,jk}(\theta)_0)]$$

We will show, however, that due to (200), the equilibrium B-values are in fact invariant with respect to θ . That is

Theorem 6-1: When expressed in the primed coordinate system the equilibrium B-values are invariant with respect to θ ; that is

$$B'_{\Delta_0} = B_{\Delta_0}$$

Before proceeding to the proof, perhaps a few words of clarification are in order since the notation has become somewhat involved. The expression B_x means the flux in the x direction; B'_x means the flux in the x' direction. $B_{x,y}$ means the derivative, in the y direction, of the x component of flux; $B'_{x,y}$ means the derivative, in the y' direction, of the x' component of flux. Recall that Δ means any of such subscripts so B'_Δ means components and derivatives in the primed directions. Finally the "0" subscript attached to any such expression means the flux that is required to maintain the body in equilibrium. This equilibrium state may be rotated, in which case $\theta \neq 0$ and we have a B'_{Δ_0} ; if $\theta = 0$ we have B_{Δ_0} .

Continuing with the proof, we first wish to show that the primed analogue of (70) holds. We repeat (70) for convenience:

$$(70) \quad B_{\Delta_0} = \frac{1}{I_{MAX}} [K_{\Delta}] \{I\}_0$$

The basis for (70) was linearity condition (20). This equation is obviously true in any coordinate system so long as the variables on both sides are expressed in the same system; so it is true in the primed system.

$$(203) \quad B'_{\Delta} = \frac{1}{I_{MAX}} [K'_{\Delta}] \{I'\}$$

We now recall that the notation $\{I'\}_0$ merely means the currents required to hold the suspended body in the primed position. We may thus write

$$(204) \quad B'_{\Delta_0} = \frac{1}{I_{MAX}} [K'_{\Delta}] \{I'\}_0$$

Equation (204) is the primed analogue of (70). The proof rests on the fact that all rows $[K'_{\Delta}]$ may be written as linear combinations of rows of the L matrix, $[L'_i]$. Up to this point we have worked out such expressions for only the five given by (144) through (148). Also, we note that in (202) the expression $B'_{i,jk}$ occurs; that is, second derivatives of components of $\{B\}$ are taken in the linearization process. We have not worked out any such expressions for objects such as $K_{i,jk}'$ thus far, but we will derive these expressions in Section 7.1. In any case, since the rows of L matrix are orthogonal, they are linearly independent. They thus form a basis for R^5 . Hence ANY five-vector may be written as a linear combination of these rows. For example, $[K'_{x,yz}]$ is some linear combination of $[L'_i]$, $i = 0,1,2,3,4$. The notational difficulty is finding some way to symbolize the scalar coefficients in this expansion. The subscripts $i = 0,1,2,3,4$ were convenient for forming sums. However, they have nothing to do with the subscripts x,y,z . Hence we will write the coefficient of $[L'_i]$ in the expansion of $[K'_{x,yz}]$ as $c_{x,yz,i}$. Then we may write

$$(205) \quad [K'_{x,yz}] = \sum_{i=0}^4 c_{x,yz,i} [L'_i]$$

The coefficients, c_i , $i = 0, 1, 2, 3, 4$, which we found earlier, may be written in this more complex notation. For example, since by (146), $[K_{xx}] = c_2\#[L_1] + c_2[L_2]$, we have that $c_2\# = c_{xx_1}$ and $c_2 = c_{xx_2}$. The alternate notation for the other c_i 's may be read off from equations (144) through (148). However, the reader need much concern himself with these relations. All that matters is that now we may formally write c_{Δ_i} as the coefficient of $[L'_i]$ in the expansion of $[K'_\Delta]$, where Δ is any subscript. (Note that the c 's do not change with rotation through θ ; thus we may state the same relation for primed and unprimed expressions. The unprimed are just special cases of the primed.) Hence

$$(206) \quad [K'_\Delta] = \sum_{i=0}^4 c_{\Delta_i} [L'_i]; \quad \text{alternative notation: } [K_\Delta(\theta)] = \sum_{i=0}^4 c_{\Delta_i} [L_i(\theta)]$$

Where the c_{Δ_i} are worked out exactly as those in (137) were, and will be found in Section 7.1. Now by (188), the equilibrium current $\{I'\}_0$ is a constant times the column vector $\{L'_4\}$. Substituting both (188) and (206) into (204), we have that

$$(207) \quad B'_{\Delta_0} = \frac{2c}{5c_4} \sum_{i=0}^4 c_{\Delta_i} [L'_i] \{L'_4\} = \frac{2c}{5c_4} \sum_{i=0}^4 c_{\Delta_i} [L_i] \{L_4\} = B_{\Delta_0}$$

Where the primes may be removed to give the second equality because of (200). This proves Theorem 6-1. The following corollary is immediate:

Corollary 6-2: The entries in the dynamics matrix do not change when the body is rotated.

Proof: By Theorem 6-1 and (202),

$$(208) \quad [\Lambda'] = [\Lambda(\theta)] = [\Lambda(B_i(\theta)_0, B_{ij}(\theta)_0, B_{ijk}(\theta)_0)] =$$

$$[\Lambda(B_i(0)_0, B_{ij}(0)_0, B_{ijk}(0)_0)] = [\Lambda]$$

The invariance of the dynamics matrix may also be proved from physical principles. To do this, refer to Figure 16. The force of gravity vector, $\{G\}$, is seen to be at the origin, pointing in the $-z = -z'$ direction. The first step in this proof is to note from Figure 16 that for this particular rotation of the body, the force of gravity vector has the same entries whether it is expressed in the primed or unprimed system. The same will therefore be true of the five equilibrium B-values in $\{B'_{EQ}\}_0 = \{B_{EQ}\}_0$ required to exactly counteract the force of gravity; see equations (67) and (97). We next show that $\{B_{EQ}\}_0$ determines all B-values. By the invertibility of the equilibrium matrix, the equilibrium currents $\{I_0\}$ are uniquely determined by $\{B_{EQ}\}_0$; see (68). Then, of course, any one set of currents can produce only one B-field; and from this B-field all B-values are uniquely determined.. Thus $\{B_{EQ}\}_0$ uniquely determines all equilibrium B-values. Since $\{B_{EQ}\}_0$ is the same whether expressed in the primed or unprimed system, the same must be true of all equilibrium B-values. It is as if the unique B-field required to suspend the body were rigidly rotated through the angle θ . Note that the first step in the proof would not be true for ANY other class of rotations of the body.

Also note that in the above proof there was only one point in which our particular coil configuration was used. This was in the invertibility of the equilibrium matrix. It therefore appears that any five coils, of arbitrary size, and placed in arbitrary positions, would have sufficed for the same result, provided only that they were capable of suspending the body and that the equilibrium matrix corresponding to them was non-singular.

The computation of equilibrium B-values by (207) exhibits a slightly surprising feature. Since the rows of the L matrix, $[L_i]$, are orthogonal, $[L_i]\{L_4\} = 0$ unless $i = 4$. Thus when computing equilibrium B-values, in the expansion of K-rows into L-rows, all terms may be thrown out except $[L_4]$; by (142) these rows are formed by using the function $\cos \lambda$. If there are no such terms of course the B-value is zero. For example, there are no $\cos \lambda$ terms in the expansion of $[K_{x,x}]$ so by (207) $B'_{x,x_0} = B_{x,x_0} = 0$. According to (83), 23 of the 25 entries in the 5 by 5 dynamics submatrix A_1 are non-zero functions of equilibrium B-values. But due to the facts just noted, only 7 entries are actually non-zero.

7 Factorization and Eigenvalues of the Dynamics Matrix, Basic and Rotated Systems; Suspension Constraint

In the previous section we showed that the dynamics matrices, $[\Lambda]$, $[\Lambda']$ for the Basic and Rotated Systems are identical. Thus the following applies to both of these systems. The dynamics matrix appears to be more important than the equilibrium matrix with respect to practical engineering concerns. (See, for example, pgs. 10 to 13 of [7] .) We would like to obtain a characterization of it similar to the one we have found for the equilibrium matrix. However, since it is independent of θ , all angular dependence drops out of $[\Lambda]$. Thus, we can't obtain a factorization like that of the equilibrium matrix, (149), in which $[K_{EQ}]$ is written as $[C][L]$, where $[L]$ depends solely on the angular placement of the coils. Instead, we obtain a factorization of the dynamics matrix of the form $[\Lambda] = [W_2][C_2]$, where $[W_2]$ depends only on Body Values and $[C_2]$ on Coil Values.

We will indicate how to find expressions for the eigenvalues of the dynamics matrix as functions of these two classes of input. Unfortunately not all these eigenvalue expressions can be exactly factored as products of Coil Values and Body Values, although they can be approximately written in this form. We will make a suggestion as to how the eigenvalues may be shaped by the choice of Body Values. Finally, we will show that the choice of the Body Values imposes some bounds on the Coil Values in any experiment.

7.1 Expressions for Equilibrium B-values

By (83) the dynamics matrix $[\Lambda]$ is given by $[\Lambda] = [W][\beta]$ where $[\beta]$ consists entirely of equilibrium B-values. An examination of $[W]$ shows it is a function of Body Values. Our goal is to write the dynamics matrix as the product of a Body Value matrix and a Coil Value matrix. Thus we wish to relate the entries in $[\beta]$ to these two categories of input. We will first show that each entry can be written as product of Coil Values and Body Values:

Lemma 7.1-1: Every equilibrium B-value can be written as a product of Body Values and Coil Values.

Proof:

By (207) and (168),

$$(209) \quad B_{\Delta_0} = B'_{\Delta_0} = \frac{2c}{5c_4} \sum_{i=0}^4 c_{\Delta_i} [L_i] \{L_4\} = \frac{2c}{5c_4} \sum_{i=0}^4 c_{\Delta_i} \frac{5}{2} \delta_{i,4} = \frac{c}{c_4} c_{\Delta_4}$$

We recall that c_{Δ_4} is the coefficient of $[L_4]$ in the expansion of the equilibrium B-value with subscript symbolized by Δ , and hence is a Coil Value. (We will compute a number of these c_{Δ_4} values in a moment.) Also recall that by (137) $c_4 = \frac{\partial f(r, h)}{\partial h}$, so that it is also a Coil Value. Finally, $c = m_c g / \nu M_{\bar{x}}$, so is a Body Value. This proves the lemma.

All equilibrium B-values can be computed by this formula. As mentioned in the last paragraph of Section 6, most of them are zero. This is so because in most of the corresponding K_{Δ} expressions, the coefficient c_{Δ_4} is zero. Recall that a c_{Δ_4} coefficient corresponds to a $\cos \lambda$ term in the K_{Δ} expansion since $\cos \lambda$ is the trig function which is used to form $[L_4]$ of (142). Thus, for example, in all the K_{Δ} expressions computed in (131) through (136), only K_{xz} has a $\cos \lambda$ term, so only B_{x,z_0} is non-zero.

We will illustrate the above by showing that B_{y,y_0} is zero. By (131),

$$\begin{aligned} (210) \quad K_{y,y} &= \frac{\partial f}{\partial y} \sin \lambda + f \frac{\partial \sin \lambda}{\partial y} \\ &= \frac{\partial f}{\partial r} \frac{\partial r}{\partial y} \sin \lambda + f \cos \lambda \frac{\partial \lambda}{\partial y} \\ &= -\frac{\partial f}{\partial r} \sin^2 \lambda - \frac{f}{r} \cos^2 \lambda \\ &= -\frac{\partial f}{2\partial r} (1 - \cos 2\lambda) - \frac{f}{2r} (1 + \cos 2\lambda) \end{aligned}$$

Since this does not contain a $\cos \lambda$, $c_{y,y_4} = 0$, so $B_{y,y_0} = 0$. We have used these methods to compute numerical values for the non-zero entries in the dynamics matrix and verified them against the numerical values given in [7] for the Experimental System. We illustrate this by computing B_{x,y_0} . By (135),

$$(211) \quad K_{x,yy} = \frac{\partial}{2\partial y} \left(\frac{f}{r} - \frac{\partial f}{\partial r} \right) \sin 2\lambda + \frac{1}{2} \left(\frac{f}{r} - \frac{\partial f}{\partial r} \right) \frac{\partial \sin 2\lambda}{\partial y}$$

$$= -\frac{1}{4} \cos 3\lambda \left(\frac{3f}{r^2} - \frac{3\partial f}{r\partial r} + \frac{\partial^2 f}{\partial r^2} \right) - \frac{1}{4} \cos \lambda \left(\frac{f}{r^2} - \frac{1\partial f}{r\partial r} - \frac{\partial^2 f}{\partial r^2} \right)$$

By (209) only the term containing $\cos \lambda$ survives; $c_{x,y_4} = -\frac{1}{4} \left(\frac{f}{r^2} - \frac{1\partial f}{r\partial r} - \frac{\partial^2 f}{\partial r^2} \right)$; we have

$$(212) \quad B_{x,y_0} = -\frac{c}{4c_4} \left(\frac{f}{r^2} - \frac{1\partial f}{r\partial r} - \frac{\partial^2 f}{\partial r^2} \right)$$

Using the same methods, it works out that

$$(213) \quad B_{x,xx_0} = 3B_{x,y_0} \quad \text{and}$$

$$(214) \quad B_{x,zz_0} = \frac{c}{c_4} \frac{\partial^2 f}{\partial h^2}$$

By (118) and (137), $K_x = f \cos \lambda = c_0 \cos \lambda$; hence by (142), $[K_x] = c_0 [L_4]$ and so $c_{x_4} = c_0$; then by (209),

$$(215) \quad B_{x_0} = \frac{cc_0}{c_4}$$

By (136) and (137), $K_{x,y} = (\partial f / \partial h) \cos \lambda = c_4 \cos \lambda$; again by (142), $[K_{x,y}] = c_4 [L_4]$ which gives $c_{x,y_4} = c_4$ and again by (209)

$$(216) \quad B_{x,z_0} = \frac{cc_4}{c_4} = c$$

Equations (212) - (216) are explicit expressions for all the non-zero equilibrium B-values. Let us restate Lemma 7.1-1: By (209), every equilibrium B-value is a product of c times Coil Values, since c_{Δ_4}/c_4 is a Coil Values expression. (In the case of $B_{x,z_0} = c$ the Coil Value is one.) At this point we will introduce a change in notation. The symbol c_{Δ_4} is rather unfortunate. The "4" has no significance, other than to indicate the bottom row of the equilibrium matrix. Also, we wish to write the Coil Values as a single term. Now Coil Values may be thought of as arising largely from the function "small f", and its derivatives with respect to h and r, and division by powers of r. So in the remainder of this section we will write the Coil Values corresponding to a B-value as a "big F" with the B-value subscript attached. That is

$$(217) \quad F_{\Delta} = c_{\Delta_4}/c_4$$

So instead of (209) we write

$$(218) \quad B_{\Delta_0} = cF_{\Delta}, \text{ where } c \text{ is a Body Value, } F_{\Delta} \text{ a Coil Value}$$

Thus by (212) through (218), using the fact that c_4 is defined by (137) to be $\frac{\partial f}{\partial h}$,

$$(219) \quad F_x = f \left(\frac{\partial f}{\partial h} \right)^{-1}; \quad F_{xz} = 1; \quad F_{xxx} = \frac{3}{4} \left(-\frac{f}{r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2} \right) \left(\frac{\partial f}{\partial h} \right)^{-1};$$

$$F_{x,yy} = F_{xxx}/3; \quad F_{x,zz} = \frac{\partial^2 f}{\partial h^2} \left(\frac{\partial f}{\partial h} \right)^{-1}$$

The numerical values for the Experimental System for each factor in (218) can be by examination of the data given in [7]. Thus the numerical values of B_{Δ_0} , c , and F_{Δ} were each computed separately and in each case (218) was verified.

Reference [7] also shows that in the Experimental System, if we minimize the complex absolute values of the eigenvalues we will minimize the frequency of the open-loop modes

of the system. This analysis will not be reproduced here. However, none of the analysis was dependent on the particular numbers in the Experimental System, so would also apply to the General System. Thus, we would like to minimize the magnitude of the eigenvalues of the dynamics matrix. This was our motivation for finding expressions for the eigenvalues in terms of Coil and Body Values, which will be presented in Section 7.4. However, we first simplify our work by showing that we may work with a dynamics matrix of reduced dimension.

7.2 Analysis Using A Dynamics Matrix of Reduced Dimension

Some of the rows and columns of the dynamics matrix given by (83) consist of only zeros and ones. We will show that these convey no essential information with respect to the eigenvalues.

Lemma 7.2-1: Let (83) be rearranged so that the dynamics matrix is given by an expression which has the same form as the expression for the A-matrix of (195); that is

$$(220) \quad [\Lambda] = \begin{bmatrix} 0 & E \\ \Lambda_1 & 0 \end{bmatrix}$$

Where the E-matrix is again the 5 by 5 identity matrix. (This rearrangement can be formally accomplished by multiplying (83) by matrices which contain only zeros and ones, so will not affect the eigenvalues.) Then if μ is an eigenvalue of $[\Lambda_1]$ then $+\sqrt{\mu}$, $-\sqrt{\mu}$ are eigenvalues of $[\Lambda]$.

Proof:

Let $\mu, \{v_1\}$ be an eigenvalue, eigenvector pair of Λ_1 , i.e.,

$$(221) \quad [\Lambda_1]\{v_1\} = \mu\{v_1\}$$

Let η be either $+\sqrt{\mu}$ or $-\sqrt{\mu}$; that is, a value such that $\eta^2 = \mu$; define

$$(222) \quad \{v_2\} = \eta\{v_1\}$$

Let $\{v\}$ be a 10-vector given by

$$(223) \quad \{v\} = \begin{bmatrix} \{v_1\} \\ \{v_2\} \end{bmatrix}$$

Then by (215) through (209),

$$(224) \quad [\Lambda]\{v\} = \begin{bmatrix} \{v_2\} \\ [\Lambda_1]\{v_1\} \end{bmatrix} = \begin{bmatrix} \eta\{v_1\} \\ \eta^2\{v_1\} \end{bmatrix} = \eta \begin{bmatrix} \{v_1\} \\ \{v_2\} \end{bmatrix} = \eta\{v\}$$

Thus $\eta, \{v\}$ is an eigenvalue, eigenvector pair for $[\Lambda]$. Recalling the definition of η , we see that if we find an eigenvalue μ for the five by five matrix $[\Lambda_1]$, then $+\sqrt{\mu}, -\sqrt{\mu}$ will be eigenvalues for the ten by ten matrix $[\Lambda]$. Thus Lemma 7.2-1 is established.

7.3 Factorization of the Dynamics Matrix

We will require all of Section 7.3 to prove the following:

Theorem 7.3-1: The dynamics matrix $[\Lambda]$ for the Basic and Rotated Systems may be written as $[\Lambda] = [W_2][C_2]$, where the entries in $[W_2]$ are all Coil Values and those in $[C_2]$ all Body Values.

We first prove the corresponding assertion for the reduced dimensions matrix. When the rows containing only 1's and 0's are eliminated from (83) and the zero equilibrium B-values are replaced by 0's, we may write the resulting matrix Λ_1 as

$$(225) \quad [\Lambda_1] = [W'_1] \begin{bmatrix} -B_{x_0} & 0 & -B_{x,z_0} & 0 & 0 \\ 0 & -B_{x_0} & 0 & 0 & 0 \\ -B_{x,z_0} & 0 & B_{x,xx_0} & 0 & 0 \\ 0 & 0 & 0 & B_{x,yy_0} & 0 \\ 0 & 0 & 0 & 0 & B_{x,zz_0} \end{bmatrix}$$

$$= [W'_1][\beta_1]$$

Where the five by five matrix $[\beta_1]$ is defined by (225) and the five by five diagonal matrix W'_1 is found from the W matrix of (83) by eliminating the rows and columns of W that contain only 1's and 0's. An examination of (83) shows that we may write W'_1 as

$$(226) \quad [W'_1] = \upsilon M_{\bar{x}} \begin{bmatrix} 1/I_c & 0 & 0 & 0 & 0 \\ 0 & 1/I_c & 0 & 0 & 0 \\ 0 & 0 & 1/m_c & 0 & 0 \\ 0 & 0 & 0 & 1/m_c & 0 \\ 0 & 0 & 0 & 0 & 1/m_c \end{bmatrix}$$

By (218) and (219) we may write

$$(227) \quad [\beta_1] = \frac{m_c g}{\upsilon M_{\bar{x}}} \begin{bmatrix} -F_x & 0 & -1 & 0 & 0 \\ 0 & -F_x & 0 & 0 & 0 \\ -1 & 0 & F_{x,xx} & 0 & 0 \\ 0 & 0 & 0 & F_{x,yy} & 0 \\ 0 & 0 & 0 & 0 & F_{x,zz} \end{bmatrix} = \frac{m_c g}{\upsilon M_{\bar{x}}} [C_1]$$

Where the five by five matrix $[C_1]$, which is a function of only Coil Values, is defined by (227). Note that when we take the product of the matrices given by (226) and (227), the factors $\upsilon M_{\bar{x}}$ cancel.

By (225), (226), and (227) we may write

$$(228) \quad [\Lambda_1] = g[W_1][C_1]$$

where

$$(229) \quad [W_1] = \begin{bmatrix} m_c/I_c & 0 & 0 & 0 & 0 \\ 0 & m_c/I_c & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[C_1] = \begin{bmatrix} -F_x & 0 & -1 & 0 & 0 \\ 0 & -F_x & 0 & 0 & 0 \\ -1 & 0 & F_{x,xx} & 0 & 0 \\ 0 & 0 & 0 & F_{x,yy} & 0 \\ 0 & 0 & 0 & 0 & F_{x,zz} \end{bmatrix}$$

Equation (228) is thus a factorization of the reduced dimension dynamics matrix into the product of the gravitational constant g , the Body Values matrix W_1 , and the Coil Values matrix C_1 . Note that C_1 is a symmetric matrix which is almost diagonal except for the -1's. More importantly, note that W_1 depends only on the ratio of the mass of the body to its moment of inertia. Suppose that designers of the experiment wished to change the properties of the dynamics matrix. Further suppose they wished to do this by changing only Body Values and not Coil Values. It appears that changing the mass to moment of inertia ratio is the only possible way to do so. The values of v and $M_{\bar{x}}$ drop out. Though it may not seem reasonable physically, changing these values would not change the dynamics matrix at all. (We have analyzed the equations of motion and determined why these values do indeed cancel out; this analysis will not be presented here.) Thus shaping the eigenvalues solely by altering Body Values requires either changing the gravitation

constant g or the ratio of mass, m_c to moment of inertia, I_c . Since the mass density of the body is uniform, this ratio depends only on the geometry of the body.

The factorization of the original dynamics matrix is found by inserting g into the W_1 matrix and putting back the rows and columns which contains only zeros and ones. When this is done, we have

(230) $[\Lambda] = [W_2][C_2]$ where

$$(231) \quad [W_2] = \begin{bmatrix} gm_c/I_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & gm_c/I_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(232) \quad [C_2] = \begin{bmatrix} 0 & 0 & -F_x & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -F_x & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & F_{x,xx} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{x,yy} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{x,zz} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Thus Theorem 7.3-1 is established. Since the dynamics matrix is independent of two input design parameters, the following is immediate:

Corollary 7.3-2: The eigenvalues of the dynamics matrix are independent of v (the volume of the magnetized core) and $M_{\bar{x}}$ (the magnitude of magnetization of the core).

The reader will note that the actual dynamics matrix is the product of rather sparse matrices. In the following we will not deal with the actual 10 times 10 dynamics matrix. Instead, we will return to the smaller matrix given by (225) to find the eigenvalues of the dynamic matrix in terms entries in this smaller matrix.

7.4 Eigenvalues of the Dynamics Matrix

Merely to simplify, let us write (225) as

$$(233) \quad [\Lambda_1] = \begin{bmatrix} a & 0 & b_1 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ b_2 & 0 & k & 0 & 0 \\ 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

Using the fact that by (228), $[\Lambda_1] = g[W_1][C_1]$, where $[C_1]$ is defined by (227) and $[W_1]$ by (229), by substitution one has

$$(234) \quad a = \frac{-gm_c F_x}{I_c}, \quad b_1 = \frac{-gm_c}{I_c}, \quad b_2 = -g, \quad k = gF_{x,xx},$$

$$d = gF_{x,yy}, \quad e = gF_{x,zz}$$

The indicial equation corresponding to (233) is given by

$$(235) \quad \det([\Lambda_1 - \mu[E]]) = (a - \mu)(d - \mu)(e - \mu)[(a - \mu)(k - \mu) - b_1 b_2] = 0$$

The solutions to (235) are the eigenvalues of reduced dimension dynamics matrix and are given by

$$(236) \quad \mu_1 = a; \quad \mu_2 = d; \quad \mu_3 = e;$$

$$\mu_4 = \left(a + c + \sqrt{(a-k)^2 + 4b_1b_2} \right) / 2; \quad \mu_5 = \left(a + c - \sqrt{(a-k)^2 + 4b_1b_2} \right) / 2$$

Thus we have

Theorem 7.4-1: The ten eigenvalues of the dynamics matrix for the Basic and Rotated Systems are given by

$$(237) \quad +\sqrt{\mu_i}, -\sqrt{\mu_i}, \quad i = 1, 2, \dots, 5$$

where the μ_i are given in terms of a, b_1, b_2, k, d, e by (236); the a, b_1, b_2, k, d, e are given in terms of Body and Coil Values by (234).

Proof: By Lemma 7.2-1, the eigenvalues of the dynamics matrix are the positive and negative square roots of the reduced dimensions matrix $[\Lambda_1]$; but these are given by (236). Thus 7.4-1 is established.

Because they are not of simple form, we will not explicitly display here all the expressions for the eigenvalues that result from the above steps. They have been found numerically for the Experimental System and their values confirmed using the values in [7]. To find algebraic expressions for the eigenvalues of the dynamics matrix, first substitute the values for a, b_1, b_2, k, d, e into (236) to find the μ_i 's. For the Experimental System these turn out to be all real numbers, although some of them are negative. This gives us the eigenvalues of the reduced dimension dynamics matrix $[\Lambda_1]$, in terms of Body Values and the Coil Values, F_Δ . If desired, these F_Δ 's may be written out explicitly using (219). The positive and negative square roots of these μ_i 's, as shown by (237), give us the eigenvalues of the original dynamics matrix.

We illustrate this with a simple example. Using (236) and (234) we have

$$(238) \quad \mu_1 = -\frac{gm_c F_x}{I_c}$$

Substituting the value of F_x given by (219), we have

$$(239) \quad \mu_1 = -(gm_c/I_c)f\left(\frac{\partial f}{\partial h}\right)^{-1}$$

Finally, the positive and negative square roots of the above give two eigenvalues of the dynamics matrix. From (239), we see that in the case of μ_1 the expression for it can be written as the product of Body Values and Coil Values. For the Experimental System, every value on the right side of (239) can be found from [7]. The Body Values are g , m_c , and I_c , and these must be all positive for both the Experimental and General Systems. The Coil Values are f and its derivative with respect to h . For the Experimental System, the former is positive while the latter is negative. (At the end of this section we will conjecture that this must also be true of the General System.) Thus for the Experimental System, μ_1 is positive and its square roots are real numbers. In fact, the positive square root of μ_1 furnishes the second largest eigenvalue. As mentioned earlier, the size of the largest eigenvalue is critical to design considerations. We see that to reduce the size of the second largest eigenvalue by manipulating Body Values, the ratio of m_c to I_c is critical.

By the above process it can be shown that the square of the largest eigenvalue of the General System (Basic and Rotated) is given by

$$(240) \quad \mu_4 = \frac{g}{2} \left(F_{x,xx} - \frac{m_c}{I_c} F_x + \sqrt{\left(\frac{m_c}{I_c} F_x + F_{x,xx} \right)^2 + \frac{m_c}{I_c}} \right)$$

In this case, the eigenvalue can't be written as a product of Body and Coil Values. However, for the Experimental System, a consideration of the magnitude of the terms in

(240), which we will omit, shows that the expression for μ_4 approximately equals that for μ_1 , which can be written as such a product.

From an examination of (239) and (240), and the other eigenvalues (not shown), it appears that the simplest way to reduce the magnitude of the eigenvalues is to reduce the ratio of m_c to I_c . Of course, there may be other considerations in the experiment which make this change untenable.

We noted above that for the Experimental System, $f = c_0$ is positive while $\partial f / \partial h = c_4$ is negative. We think this must be true of the General System. To prove this rigorously would require very careful attention to sign conventions and coordinates systems. We apologize for not having developed all this apparatus sufficiently. However, at the risk of going into too much detail, we feel it worthwhile to make the following

Conjecture 7.4-2: In the Basic and Rotated Systems, $f > 0$ and $\partial f / \partial h < 0$

This appears plausible from examination of Figure 4. In Figure 4, f , the horizontal component of flux when a maximum current is applied, appears as B_H . From the drawing this is always towards the coil, which corresponds to positive. Thus it appears that $f > 0$. It is somewhat more difficult to make reasonable the assertion that $\partial f / \partial h = \partial B_H / \partial h$ is negative. Let B equal the total magnitude of the flux vector at P_i . That is $B = \sqrt{B_H^2 + B_z^2}$.

If the body were moved up, corresponding to larger h , the body would be further from the coil, so it is reasonable to suppose that B would grow smaller. However, as h increases and B grows smaller, B_H would appear to grow smaller relative to B , due to the flux vector turning down more steeply. That is, the angle α in Figure 4 grows in absolute value. Thus $\cos \alpha$ decreases and $B_H = B \cos \alpha$ decreases relative to B . On both counts, B_H decreases with increasing h . This makes us feel that in any system for which Assumption 1 holds, we have $\partial f / \partial h < 0$. The conjecture is made plausible. (We mention that a close examination of the data of in the Experimental System also tends to support this conclusion, but we will not present this analysis.)

7.5 Suspension Constraint

We have show that in choosing the parameters of this experiment, one may group the inputs into two categories. One category is the selection of coils and their placement, the

Coil Values. The other category is all other design parameters, such as the mass, volume, moment of inertia, and magnetization of the body, the Body Values. (We may think of the gravitational constant g as such an input, since it is possible the experiment may not be done on the surface of the earth.) In this section we derive a necessary relationship between these two classes of input.

By (137) and (170),

$$(241) \quad \frac{\{I_0\}}{I_{MAX}} = \frac{2}{5}c \left(\frac{\partial f}{\partial h} \right)^{-1} \{L_4\}$$

The linearity condition on current (Assumption 2) imply

$$(242) \quad \frac{|I_j|}{I_{MAX}} \leq 1, \text{ for } j = 1, 2, \dots, 5$$

Now the data of [7] show c is positive and $\partial f / \partial h$ negative for the Experimental System. The nature of the constants which make up c , such as mass, mean that c must also be positive in the General System. Let us assume Conjecture 7-4.2 is true, so that $\partial f / \partial h$ is negative for the General System. Then if we take the absolute value of both sides of (241) and apply (242) we have

$$(243) \quad \frac{-2}{5}c |L_4^j| \left(\frac{\partial f}{\partial h} \right)^{-1} \leq 1, \text{ } j = 0, 1, \dots, 4$$

By (142),

$$(244) \quad |L_4^j| = |\cos j\phi|, \text{ } \phi = \pi/5, \text{ } j = 0, 1, \dots, 4$$

The maximum of the values on the right side of (244) is 1 so that (243) implies

Theorem 7.5-1: If we assume Conjecture 7.4-2, then for suspension to occur we must have

$$(245) \quad \partial f / \partial h \leq -2c/5$$

The left side of (245) is a Coil Value and the right side is an Body Value. Thus, (245) shows that the choice of these two classes of values is not entirely independent. For example, once the Body Values are chosen, the Coil Values must be picked so that (245) holds. Also, of course, even if we do not assume Conjecture 7.4-2, the above steps would lead to $|\partial f / \partial h| \geq 2c/5$. However, we believe (245) is the more realistic and meaningful result.

Inequality (245) may be rationalized in the following way. In order to suspend the body, electromagnetic force in the h direction must be equal and opposite to gravitational force. An examination of the third component of (58) shows that this implies $B_{x,z_0} = c$. Equation (130) shows that the contribution of each coil to B_{x,z_0} is proportional to $\partial f / \partial h$, which implies some sort of lower bound on the magnitude of this derivative. But the contribution of each coil to B_{x,z_0} also depends on the current flowing through the coil, which is not arbitrary but determined by (170). When the dot product of $\{L_4\}$ with itself is computed and the linearity condition is taken into account, the result is (245). We therefore refer to (245) as a suspension constraint.

We think (245) might be of value to researchers, since once the Body Values are chosen, it must be satisfied for suspension to occur. By using (245) another inequality may be derived but we think its utility is questionable; we state it here for completeness. If we multiply both sides of (218) by $\partial f / \partial h (B_{\Delta_0})^{-1}$ we obtain $\partial f / \partial h = (\partial f / \partial h) c F_{\Delta} (B_{\Delta_0})^{-1}$, so by (245), $(\partial f / \partial h) c F_{\Delta} (B_{\Delta_0})^{-1} \leq -2c/5$; dividing both side by c, which is assumed positive, we have

$$(246) \quad (\partial f / \partial h) F_{\Delta} (B_{\Delta_0})^{-1} \leq -2/5$$

8 The Additional Coils System

The use of six, seven, and eight coils has been explored in the Experimental System [7]. The coil configurations are shown in Figures 6, 7, and 8. We discussed the rationale for adding coils in Section 2.16. Recall that it was hoped that the addition of coils would reduce the frequency of the highest frequency open-loop mode. The heuristic, physical, and numerical approach used in [7] indicated that reducing the equilibrium B-value, B_{x_0} , would accomplish this. An attempt to directly control this value is discussed in [7]. Due to experimental results, the approach was deemed untenable. We hope the following will give another, more theoretical prospective on these negative results.

For the General System, we will assume that the number of coils, N , is any integer greater than five; in the Experimental System, $N = 6, 7, \text{ or } 8$. In both systems, the angles between coils are equal, as in Figures 6, 7, and 8. Also for both Systems, for each N , the position of the suspended body and all parameters relating to it are exactly as in the five coil case. Thus the Body Values are the same. However, while the value of h was unchanged, the radius r of the coil configuration was different for each N , and in some cases different sized coils were used. Thus, Coil Values differed among the cases in the Experimental System.

Two radically different control approaches were used in the Experimental System, and we assume the same methods for the General System. In describing these approaches we will refer to the fundamental equilibrium equation (66), which we repeat for convenience:

$$(66) \quad \{B_{EQ}\} = [K_{EQ}]\{I\}/I_{MAX}$$

The control approach had no effect on the dynamics matrices, but did change the equilibrium matrix. In the next two sections, 8.1 and 8.2, we will describe the two control methods and the differing equilibrium matrices which resulted from each method. Since the control method did not affect the dynamics matrix, the discussion of this is deferred to

Section 8.3. The treatment of the dynamics matrix given there applies to both control methods.

8.1 First Method, Five Degree of Freedom of Control

In this methodology, no direct attempt is made to control the offending B-value, B_{x_0} . This value is therefore not included in the equilibrium vector, $\{B_{EQ}\}$ in (66). Thus, $\{B_{EQ}\}$ remains a 5-vector in (66) and the specific numbers that go into $\{B_{EQ}\}_0$ are the same as those given by (67). However, as there are now N coils there must of course be N currents, so $\{I\}$ must be an N-vector. Hence $[K_{EQ}]$ is no longer square but rather has dimensions 5 by N. Because the equilibrium vector is the same, $[K_{EQ}]$ will contain the same rows $[K_{\Delta}]$ as given by (65), but now each row will be of width N. Instead of (22) we will have

$$(247) \quad [K_{\Delta}] = [K_{\Delta}^1, K_{\Delta}^2, \dots, K_{\Delta}^N]$$

The angular spacing between rows is now $360/N$ degrees so the ϕ of (138) - (142) is $360/N$. Instead of (143) we now have

$$(248) \quad [L_i] = [L_i^1, L_i^2, \dots, L_i^N], \quad i = 0, 1, \dots, 4$$

but all other definitions regarding the L matrix remain the same. The equation for the matrix [C] will still be (150). However, the entries in [C] are functions of Coil Values. As mentioned, these will be different for each N in the Experimental System, so the entries in [C] will be different. However, [C] remains a 5 by 5 matrix of the same form and, in the Experimental System, its diagonal entries are still non-zero; we will make Assumption 4 for the General System. Hence [C] remains invertible for both systems.

To clarify the dimensions in regard to the factorization of the equilibrium matrix, we write (67) together with (149) as

$$(249) \quad \{B_{EQ}\}_0 \ 5 \times 1 = [K_{EQ}] \ 5 \times N \ \{I_0\} \ N \times 1 / I_{MAX}$$

$$= [C]_{5 \times 5} [L]_{5 \times N} \{I_0\}_{N \times 1} / I_{MAX}$$

We will suppress the number of coils subscript, N , in the following, for convenience. It will be understood that each vector and matrix which has an N subscript in the above should carry this subscript along in the following equations.

As this point in our prior development we inverted the equilibrium matrix to find the unique equilibrium currents $\{I_0\}$. However, since $[K_{EQ}]$ is no longer square its inverse is no longer defined and the current vector is under-determined; there is not a unique current vector satisfying (249). In the Experimental System, [7], to determine a unique current vector, the pseudoinverse [28] of the equilibrium matrix was used. As noted in [7] this leads to a current vector whose two-norm, $\|I_0\|$, is minimized. We assume the same methodology is used in the General System.

We find an analytic expression for the pseudoinverse of $[K_{EQ}]$ which is a precise generalization of the inverse of the five by five equilibrium matrix given by (169). Because the method used is so similar to the previous case, we omit validation of the details. The results have been verified for the Experimental System using the numbers in [7].

First, by using the methods of Section 4.1, it can be shown that the rows of $[L]$ are still orthogonal. The equations for the dot product of rows are generalizations of (162) and (168):

$$(250) \quad [L_1] \{L_t\} = N \delta_{1t} ,$$

$$(251) \quad [L_r] \{L_t\} = \frac{N}{2} \delta_{rt}$$

Where N is the number of coils, and $r, t \in \{0, 1, 2, 3, 4\}, r \neq 1$. Thus, for each N , we define a 5×5 diagonal matrix D_N , with the N suppressed, by

$$(252) \quad [L][L]^T = [D] = \frac{N}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Let $[L]^+$ denote the pseudoinverse of the L matrix. We claim

Lemma 8.1-1: The pseudoinverse of the L matrix is given by

$$(253) \quad [L]^+ = [L]^T[D]^{-1}$$

Proof: For the proof of this and the next lemma, equations (254) through (259), the brackets around matrices are omitted for convenience. The proof uses the characterization of the pseudoinverse given in [28]. There it is shown that the pseudoinverse, $[X]$, of a real matrix $[A]$, uniquely satisfies the following four equations:

$$(254) \quad (i) \quad AXA=A; \quad (ii) \quad XAX=X; \quad (iii) \quad (AX)^T=AX; \quad (iv) \quad (XA)^T=XA$$

We will verify only (i) and (iv) of the above, the proof of the rest being similar. Using (252) and (253),

$$(255) \quad LL^+L = L(L^TD^{-1})L = (LL^T)D^{-1}L = DD^{-1}L = L$$

which verifies (i). To show (iv), since D is diagonal we may use $D^{-T} = D^{-1}$, so that

$$(256) \quad (L^+L)^T = (L^TD^{-1}L)^T = L^TD^{-T}L = L^TD^{-1}L = L^+L$$

Thus Lemma 8.1-1 is established. We next find an expression for the pseudoinverse of the equilibrium matrix:

Lemma 8.1-2: The pseudoinverse of K_{EQ} is given by

$$(257) \quad K_{EQ}^+ = L^+ C^{-1}$$

Proof: We again verify this only for (i) and (iv) of (254). To prove (i),

$$(258) \quad K_{EQ} K_{EQ}^+ K_{EQ} = (CL)(L^+ C^{-1})(CL) = C(LL^+)(C^{-1}C)L = C(LL^+L) = CL = K_{EQ}$$

where the next to last equality follows from (255). Finally, to prove (iv),

$$(259) \quad (K_{EQ}^+ K_{EQ})^T = (L^+ C^{-1} CL)^T = (L^+ L)^T = L^+ L = L^+ C^{-1} CL = K_{EQ}^+ K_{EQ}$$

where the middle equality sign in the above follows from (256). This proves Lemma 8.1-2.

Since the expression for the pseudoinverse has exactly the same form as the expression for the inverse, the generalization of (169) holds:

Theorem 8.1-3: Given the same assumptions that were made for the Basic System, the pseudoinverse of the equilibrium matrix for the Addition Coils, five degree of freedom of control case, is given by

$$(260) \quad [K_{EQ}]^+ =$$

$$2/N \left[c_0^{-1} \{L_0\}, \frac{1}{2c_1} \{L_1\} - \frac{c_2^\#}{c_1 c_2} \{L_2\}, c_2^{-1} \{L_2\}, c_3^{-1} \{L_3\}, c_4^{-1} \{L_4\} \right]$$

Proof:

Recalling our derivation of the analytic expression for inverse of the equilibrium matrix in Section 4.1, by (169) $K_{EQ}^{-1} = L^T D^{-1} C^{-1}$, where D is the diagonal matrix given by (252) when $N = 5$. In the present discussion, if we insert (253) into (257), we have that $K_{EQ}^+ = L^T D^{-1} C^{-1}$. Now C has exactly the form same as in the Basic System case, although the actual numerical value of the entries are different. As noted before, with Assumption 4, C is invertible, so C^{-1} must have the same algebraic expression as in the Basic System case. D^{-1} is readily computed for arbitrary N, while L^T is simply the transpose of the rows of L, which are given by (248); the only difference between the Basic and Additional Coils expression for L is the length of each vector $\{L_i\}$, which is now N, and the Angular Values of L. These are now multiples of $360/N$ degrees rather than $360/5$ degrees. In short, the algebraic computation of K_{EQ}^{-1} is identical to that of K_{EQ}^+ , so the expression for the two matrices have identical forms. This proves Theorem 8.1-3.

The above shows that the methods developed for the $N = 5$ case readily generalize to other values of N, provided the control equation is unchanged. In the next section, we adduce mathematical reasons for the failure of a second control method.

8.2 Second Method, Six Degrees of Freedom of Control

Here the experimenters sought to directly control B_x so it was included in $\{B_{EQ}\}$. Equation (66) was thus set up as follows:

$$(261) \quad \begin{bmatrix} B_x \\ B_y \\ B_z \\ B_{x,x} \\ B_{x,y} \\ B_{x,z} \end{bmatrix} = \begin{bmatrix} [K_x] \\ [K_y] \\ [K_z] \\ [K_{x,x}] \\ [K_{x,y}] \\ [K_{x,z}] \end{bmatrix} \{I\} / I_{MAX}$$

We let $\{B_{EQ}\}$ and $[K_{EQ}]$ be defined for the present discussion by the above equation. Thus $\{B_{EQ}\}$ is now 6×1 and $[K_{EQ}]$ is $6 \times N$, where N is greater than 5. The equilibrium value B_{x_0} was set to zero and the other equilibrium values remained as before so that

$$(262) \quad \{B_{EQ}\}_0^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & c \end{bmatrix}$$

As in previous cases, the specific number in (262) we substituted into (261) and a solution current vector $\{I_0\}$ was sought for the resulting equation. However,

Theorem 8.2-1: The equation which results from substituting (262) into (261),

$$(263) \quad \{B_{EQ}\}_0 = [K_{EQ}]\{I_0\}/I_{MAX}$$

has no solution vector $\{I_0\}$

Proof:

Recalling (118) and (131),

$$(118) \quad K_x = f \cos \lambda = c_0 \cos \lambda$$

$$(131) \quad K_{x,z} = \frac{\partial f}{\partial h} \cos \lambda = c_4 \cos \lambda$$

Since, by (142), $[L_4]$ is the row vector corresponding to $\cos \lambda$ we have

$$(264) \quad [K_x] = c_0 [L_4] \quad \text{and}$$

$$(265) \quad [K_{x,z}] = c_4 [L_4]$$

The equation for the top element of the vector on the left side of (263) is

$$0 = B_{x_0} = [K_x]\{I_0\}/I_{MAX} = c_0[L_4]\{I_0\}/I_{MAX}$$

since c_0 is non-zero, by the above we must have

$$(266) \quad [L_4]\{I_0\} = 0$$

The equation for the bottom element of the vector on the left side of (263) is

$$c = B_{x_{z_0}} = [K_{xz}]\{I_0\}/I_{MAX} = c_4[L_4]\{I_0\}/I_{MAX}$$

since c_4 is non-zero, by the above we must have

$$(267) \quad [L_4]\{I_0\} = \frac{c}{c_4}I_{MAX}$$

Since (266) and (267) are incompatible, there is no equilibrium current which satisfies (263). This proves Theorem 8.2-1.

In the Experimental System, [7], the pseudoinverse (or inverse when $N = 6$) of $[K_{EQ}]$ was found numerically and the equilibrium current ratio vector was found according to

$$(268) \quad \{I_0\}/I_{MAX} = [K_{EQ}]^+ \{B_{EQ}\}_0 \quad ([K_{EQ}]^{-1} \text{ was used when } N = 6)$$

However, the current ratios found by this numerical process were in the hundreds. This was unacceptable because, according to the linearity assumption, each current ratio must have a maximum absolute value less than or equal to one. The method was deemed untenable for this reason. Physically, large currents of opposite sign and equal magnitude in adjacent coils tended to cancel one another.

Since (263) has no solution at all, of any magnitude, the question arises as to how the current ratios obtained from (268) could have resulted from the numerical process

described in [7]. We are not sure how the software used to compute the pseudoinverse would behave in cases of this kind. One would think that a warning would have been generated, noting an instability in the computation. However, as noted in Section 4.4, the numerical values we have obtained by our development do not always exactly match those given in [7]. It seems probable that these slight numerical differences lead to a pseudoinverse with very large values in its rightmost column, rather than a warning flag. This would have produced the large current ratios.

8.3 Eigenvalues of the Dynamics Matrix of the Additional Coils System

It was hoped that the use of more than five coils would reduce the frequency of the highest frequency mode. This is equivalent, in this particular experiment, to reducing the size of the largest eigenvalue. The expression for the square of the largest eigenvalue is given by (240); but since this is not a simple expression, for our analysis we will use the expression for the square of the second largest eigenvalue given by (239). It can be shown that both these expressions vary in the same manner and are close numerically. We restate (239) in a slightly different form:

$$(269) \quad \mu_1 = (gm_c/I_c) \frac{f}{-\frac{\partial f}{\partial h}}$$

The terms in parentheses are Body Values, which, as mentioned earlier, do not change with the number of coils. Thus the only hope of making μ_1 smaller lies in the Coil Values, which do change. In this case, this means making the ratio

$$(270) \quad F_x = \frac{f}{-\frac{\partial f}{\partial h}}$$

smaller. However, a computation using the figures given in [7] shows that in the Experimental System this ratio actually increased slightly with the number of coils. The magnitude of the corresponding eigenvalue (that is, $\sqrt{\mu_1}$) should have increased proportionately to the square root of F_x and the data in [7] show that this was the case. Hence changing the number of coils, and thereby changing the Coil Values, did not result in a reduction of the frequency of the highest frequency mode.

But is it possible that another selection of coils and/or value of r might have produced the desired reduction? We know that for the five coil case, any selection of Coil Values must satisfy the suspension constraint given by (245). Analysis of the derivation of (245) shows that it may be generalized to the N coil case by replacing 5 by N in (245). If we do this and also multiply both sides of the inequality by -1 , we obtain

$$(271) \quad -\frac{\partial f}{\partial h} \geq \frac{2}{N}c$$

Thus with increasing N the suspension constraint is relaxed. However, this merely permits us to make $-\frac{\partial f}{\partial h}$ smaller. To reduce the size of the eigenvalue, (269) shows we want to make this derivative bigger. Hence, our analysis does not suggest any method of Coil Value selection to obtain the desired result.

It seems likely that the ratio given by (270) depends on certain coil parameters, such as the height and width of the coil, which are not examined in the dissertation. Analysis of this dependence might suggest a coil selection process that would result in a smaller ratio and thus a smaller eigenvalue.

9 Proof of Controllability for Arbitrary Angles System; Relation to the Vandermonde Matrix

We now consider a variation of the General System which has no counterpart in the Experimental System. We still assume five coils in circular, planar array but we no longer

require the angles between coils to be equal. See Figure 17. The angle, θ_j , from the x axis to a line from the center of the j^{th} coil to the origin, is now arbitrary. We may still define the L matrix as we did in (138) and (143) provided the following change is made: in the definition of L_i^j the angle $j\phi$ is replaced by θ_j . Since the C matrix of (137) and (150) is independent of angular placement, it remains unchanged so is still non-singular. Thus $[K_{EQ}] = [C][L]$ is non-singular if $[L]$ is non-singular. Sections 4 and 5 showed that the invertibility of the equilibrium matrix implies that the system is controllable, with a unique equilibrium point; so these conditions will follow if $[L]$ is non-singular.

Since exchanging the rows of a matrix will not alter its invertibility, for convenience we will write $[L]$ as

$$(272) \quad [L] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \sin \theta_1 & \sin \theta_2 & \sin \theta_3 & \sin \theta_4 & \sin \theta_5 \\ \cos \theta_1 & \cos \theta_2 & \cos \theta_3 & \cos \theta_4 & \cos \theta_5 \\ \sin 2\theta_1 & \sin 2\theta_2 & \sin 2\theta_3 & \sin 2\theta_4 & \sin 2\theta_5 \\ \cos 2\theta_1 & \cos 2\theta_2 & \cos 2\theta_3 & \cos 2\theta_4 & \cos 2\theta_5 \end{bmatrix}$$

Theorem 9-1: Matrix $[L]$ given by (272) is non-singular if and only if all the θ_j 's are distinct.

Proof:

If two of the θ_j 's are equal this corresponds to two coils being in exactly the same position; these two columns of $[L]$ would also be identical so $[L]$ would be singular. This proves the "only if" part of the theorem. We next show that if all the θ_j 's are distinct then $[L]$ is non-singular. We will give two proofs. The first is not especially difficult but presents some notational difficulties. The second proof is straightforward and unambiguous.

We now give two proofs of the following:

If all the θ_j 's are distinct then $[L]$ is nonsingular.

First proof:

The proof will involve finding a relationship between the determinant of [L] and polynomials in $e^{i\theta}$ so we will prove two lemmas about the linear dependence of collections of polynomials.

Lemma 9-2: Given a collection of $n + 2$ complex polynomials in z , each of which is no greater than n^{th} degree ,

$$f_i(z) = \sum_{k=0}^n \alpha_{i,k} z^k, \quad i = 1, 2, \dots, n+2$$

It is always possible to find complex $\lambda_i, i = 1, 2, \dots, n+2$, not all zero, such that

$$\sum_{i=1}^{n+2} \lambda_i f_i(z) = 0, \quad \text{for all } z$$

Proof: This amounts to solving the matrix equation

$$(273) \quad \begin{bmatrix} \alpha_{1,0} & \alpha_{2,0} & \dots & \alpha_{n+2,0} \\ \dots & \dots & \dots & \dots \\ \alpha_{1,n} & \alpha_{2,n} & \dots & \alpha_{n+2,n} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \dots \\ \lambda_{n+2} \end{bmatrix} = \begin{bmatrix} 0 \\ \dots \\ 0 \end{bmatrix}$$

for the λ_i 's. Since the matrix on the left is $n+1$ by $n+2$, a non-trivial solution is always possible. This proves Lemma 9-2.

Lemma 9-3: Let $n \geq p \geq 2$. Given a collection of p complex polynomials in z , $f_i(z), i = 1, 2, \dots, p$, each of which is of no higher degree than n , which have $n - p + 2$ roots in common, it is always possible to find complex $\lambda_1, \dots, \lambda_p$, not all zero, such that

$$\sum_{i=1}^p \lambda_i f_i(z) = 0, \quad \text{for all } z.$$

Proof: Since the polynomials have $n - p + 2$ roots in common, it is possible to write each polynomial as

$$f_i = qg_i$$

where q is a polynomial of degree $n - p + 2$ and each polynomial g_i is of degree no greater than $p - 2$. Then

$$(274) \quad \sum_{i=1}^p \lambda_i f_i(z) = q \sum_{i=1}^p \lambda_i g_i(z) =$$

has a non-trivial solution since $\sum_{i=1}^p \lambda_i g_i(z) = 0$ has such a solution by Lemma 9-2. Lemma 9-3 is established. The following lemma will be used to find a relationship between the determinant of [L] and polynomials in $e^{i\theta}$.

Lemma 9-4: For each real five-vector $\{\alpha\}^T = [\alpha_0, \alpha_1, \dots, \alpha_4]$ there exists a unique complex five-vector $\{\beta\}^T = [\beta_0, \beta_1, \dots, \beta_4]$ defined by

$$(275) \quad \alpha_0 + \alpha_1 \cos \theta + \alpha_2 \sin \theta + \alpha_3 \cos 2\theta + \alpha_4 \sin 2\theta = e^{-2i\theta} \sum_{j=0}^4 \beta_j e^{ji\theta}, \text{ where}$$

$$i = \sqrt{-1}$$

Any collection of such real five-vectors, $\{\alpha^k\}$, $k = 1, 2, \dots$, is linearly dependent if and only if the corresponding collection $\{\beta^k\}$, $k = 1, 2, \dots$, is linearly dependent.

Proof: Write every trigonometric function on the left side of (275) as a complex sum of complex exponential functions and multiply the result by $1 = e^{-2i\theta} e^{2i\theta}$. The expression given by $e^{2i\theta}$ times the left side of (275) will contain exponential expressions $e^{ji\theta}$, $j = 0, 1, \dots, 4$ only. The coefficient of each of these will be β_j . When the calculation is carried out, the relationship between $\{\alpha\}$ and $\{\beta\}$ is given by

$$(276) \quad \{\beta\} = [\Xi]\{\alpha\}$$

where

$$[\Xi] = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 1 & i \\ 0 & 1 & i & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 & 0 \\ 0 & 0 & 0 & 1 & -i \end{bmatrix}$$

Since $[\Xi]$ is non-singular, the mapping is 1 to 1 and preserves linear dependence in both directions. The proof of Lemma 9-4 is complete.

We can now complete the first proof. Let

$$(277) \quad \{v(\theta)\}^T = \begin{bmatrix} 1 & \sin\theta & \cos\theta & \sin 2\theta & \cos 2\theta \end{bmatrix} \text{ and}$$

$$(278) \quad \{v_j\} = \{v(\theta_j)\}, \quad j = 1, 2, \dots, 5$$

where we suppose the θ_j 's are all distinct. For $p = 1, 2, \dots, 5$, we form the five by p matrices

$$(279) \quad [M_p] = [\{v_1\}, \{v_2\}, \dots, \{v_p\}]$$

The proof is by induction on p . By the form of $\{v(\theta)\}$, $[M_1]$ is of rank 1. We suppose that $[M_k]$ is of rank k , where $k < 5$, and show that $[M_{k+1}]$ is of rank $k+1$. Suppose $[M_{k+1}]$ is not of rank $k+1$. Thus $[M_k]$ contains at least one nonsingular k by k submatrix, $[N]$, while every $k+1$ by $k+1$ submatrix of $[M_{k+1}]$ is singular. Without loss of generality, suppose $[M_k]$ can be partitioned

$$(280) \quad [M_k] = \begin{bmatrix} R \\ N \end{bmatrix}$$

That is, we do not know that a non-singular submatrix really resides in the bottom rows of $[M_k]$, but since switching around the rows of a matrix does not change its rank, we may for convenience make this assumption. In the same vein, let us partition $\{v_k\}$ compatibly with (280), setting

$$(281) \quad \{v_k\} = \begin{bmatrix} \eta \\ \sigma \end{bmatrix} \text{ where we define}$$

$$\{\eta\}^T = [e_1, e_2, \dots, e_{5-k}], \quad \{\sigma\}^T = [e_{5-k+1}, e_{5-k+2}, \dots, e_5]$$

Note that this means that each e_j is one of the trig functions on the right side of (277), evaluated at $\theta = \theta_{k+1}$. The difficulties in notation arise due to our having no concise way of indicating these trig functions.

We now form $5 - k$ submatrices of $[M_{k+1}]$, each $k+1$ by $k+1$, given by

$$(281) \quad [P_j] = \begin{bmatrix} r_j & e_j \\ N & \sigma \end{bmatrix}, \quad j = 1, 2, \dots, 5 - k$$

where r_j is the j^{th} row of $[R]$. Note that it is possible to carry out this construction since we assume $k < 5$. Since the rank of $[M_{k+1}]$ is assumed to be k , the determinant of each $[P_j]$ is zero. Let

$$(282) \quad \alpha = (-1)^k \det[N]$$

We develop the determinant of each $[P_j]$ along its rightmost column. These developments may be written, for $j = 1, 2, \dots, 5 - k$, in the form

$$(283) \quad \det[P_j] = 0e_1 + \dots + 0e_{j-1} + \alpha e_j + 0e_{j+1} + \dots + 0e_{5-k} + \sum_{l=5-k+1}^5 \alpha_{j,l} e_l = 0$$

In the above, the $\alpha_{j,l}$'s are just the unknown appropriate cofactors and we have added in $0e_r$'s for $r < 5 - k + 1$ and $r \neq j$. Now if we define $\alpha_{j,l}$'s for all j, l by

$$(284) \quad \det[P_j] = \sum_{l=1}^5 \alpha_{j,l} e_l = 0$$

and define the $5 - k$ vectors $\{\alpha^j\}$ by

$$(285) \quad \{\alpha^j\}^T = [\alpha_{j,1}, \alpha_{j,2}, \dots, \alpha_{j,5}]$$

then we note that the collection of $5 - k$ vectors $\{\alpha^j\}$ *cannot be linearly dependent*. This is due to the following structure, shown by (283): for each j , when $l \leq 5 - k$, $\alpha_{j,l} = \alpha \delta_{j,l}$, where $\delta_{j,l}$ is the Kronecker delta symbol. We have that α cannot be zero due to (282) and our inductive assumption that $[N]$ is non-singular. We also note that each sum in (283) and (284) is just like the sum in (275) of Lemma 9-4. Hence, by that lemma, with each $\{\alpha^j\}$ we may associate a $\{\beta^j\}$ for which we have that

$$(286) \quad e^{-2i\theta_{k+1}} \sum_{l=0}^4 \beta_{j,l} e^{li\theta_{k+1}} = 0, \quad j = 1, 2, \dots, 5 - k$$

and since $e^{-2i\theta} \neq 0$ for all θ ,

$$(287) \quad \sum_{l=0}^4 \beta_{j,l} e^{li\theta_{k+1}} = 0, \quad j = 1, 2, \dots, 5 - k$$

Now, thinking for a moment of θ_{k+1} as a variable, suppose θ_{k+1} were to take on any of the values $\theta_1, \theta_2, \dots, \theta_k$. This would mean two columns of $[M_{k+1}]$ were equal, so that it would again have rank k . All the results we have obtained by assuming $[M_k]$ and $[M_{k+1}]$ were both of rank k would still be true; that is, (280) through (287) still hold. Thus by (287) we have

$$(288) \quad \sum_{l=0}^4 \beta_{j,l} e^{li\theta_r} = 0, \quad r = 1, 2, \dots, k, k+1, \quad j = 1, 2, \dots, 5 - k$$

For complex z , define

$$(289) \quad f_j(z) = \sum_{l=0}^4 \beta_{j,l} z^l, \quad j = 1, 2, \dots, 5 - k$$

and let

$$(289) \quad z_r = e^{i\theta r}, \quad r = 1, 2, \dots, k, k+1$$

By (287), $z_1, z_2, \dots, z_k, z_{k+1}$ are all roots of each f_j . Hence the f_j 's form a collection of $5 - k$ complex polynomials in z , each of which is of no higher degree than 4. Suppose $k = 4$. Then the collection consists of only f_1 , a polynomial of degree at most 4. But since $k + 1 = 5$, f_1 has five roots, an impossibility. Hence if $k = 4$, a contradiction is reached, so $[M_{k+1}]$ must be of rank $k + 1$ in this case.

Suppose $k < 4$. Using the notation of Lemma 9-3, $p = 5 - k$ so that $p > 1$. The n of Lemma 9-3 equals 4 so that $n - p + 1 = k + 1$. Since the collection has $k + 1$ roots in common, Lemma 9-3 applies. Thus there exist complex $\lambda_1, \dots, \lambda_{5-k}$, not all zero, such that

$$(290) \quad \sum_{j=1}^{5-k} \lambda_j f_j(z) = 0, \quad \text{for all } z$$

By (289) this means

$$(291) \quad \sum_{j=1}^{5-k} \lambda_j \sum_{l=0}^4 \beta_{j,l} z^l = 0, \quad \text{for all } z$$

But this is equivalent to

$$(292) \quad \sum_{j=1}^{5-k} \lambda_j \{\beta^j\} = 0$$

where the $\{\beta^j\}$'s correspond, as in Lemma 9-4, to the $\{\alpha^j\}$'s of (285). But by (292), the $\{\beta^j\}$'s are linearly dependent whereas by (283), (284) and (285), the $\{\alpha^j\}$'s are not linearly dependent. But this is in contradiction to Lemma 9-4. Thus, whether $k = 4$ or $k < 4$, by assuming that $[M_{k+1}]$ was not of rank $k + 1$ we have reached a contradiction. Hence

$[M_{k+1}]$ must be of rank $k + 1$ and so by induction $[M_5] = [L]$ is of rank 5. This completes the first method of proof.

The above proof shows that the determinant of $[L]$ is essentially a fourth order polynomial in $\exp i\theta_k$, for each k . The second method of proof exploits this fact.

Second proof: We regard all the θ_k 's as arbitrary; to assign a unique numerical value to each angle, we assume $0 \leq \theta_k < 2\pi$, for each k . Let $D = \det [L]$. To put the D into the desired form, refer to (272): multiply the second row of D by the imaginary unit i , and add the resulting row to the third row. Factor out $i/2$ from the second row and add the new third row to the new second row. The fourth and fifth rows are just like the second and third rows except they are functions of twice the angle. Hence, repeat the steps above for these rows, with the fourth row substituting for the second and the fifth for the third. When these steps are carried out, the k^{th} column of D will be the following:

$$(293) \quad \begin{bmatrix} 1 \\ \cos \theta_k - i \sin \theta_k \\ \cos \theta_k + i \sin \theta_k \\ \cos 2\theta_k - i \sin 2\theta_k \\ \cos 2\theta_k + i \sin 2\theta_k \end{bmatrix} = \begin{bmatrix} 1 \\ e^{-i\theta_k} \\ e^{i\theta_k} \\ e^{-2i\theta_k} \\ e^{2i\theta_k} \end{bmatrix}$$

Since we have factored out $i/2$ from the second and fourth rows, D will have $-1/4$ in front. Now we factor $e^{-2i\theta_k}$ from the k^{th} column, for each k . Each column of D will now be

$$(294) \quad \begin{bmatrix} e^{2i\theta_k} \\ e^{i\theta_k} \\ e^{3i\theta_k} \\ 1 \\ e^{4i\theta_k} \end{bmatrix}$$

We interchange the fourth row with the first then the new fourth row with the third.

Taking into account the exponentials we factored out, D will be given by

$$(295) \quad D = \frac{-1}{4} e^{-2i(\theta_1 + \theta_2 + \dots + \theta_5)} \det \begin{bmatrix} 1 & \dots & 1 \\ e^{i\theta_1} & \dots & e^{i\theta_5} \\ e^{2i\theta_1} & \dots & e^{2i\theta_5} \\ e^{3i\theta_1} & \dots & e^{3i\theta_5} \\ e^{4i\theta_1} & \dots & e^{4i\theta_5} \end{bmatrix}$$

The k,j element of the above determinant is z_k^{j-1} , where $z_k = e^{i\theta_k}$. This is the well-known Vandermonde determinant, V, in z_1, \dots, z_5 . (See reference [27].) The Vandermonde determinant is non-zero if and only if the z_k 's are distinct; in fact, in [27] it is shown that

$$(296) \quad V = \prod_{1 \leq r < s \leq 5} (z_s - z_r)$$

That is, V is equal to the product of all differences between the five z_k 's. In the present case, there are ten difference factors in the expression (296) for V. Since the z_k 's are distinct if and only if the θ_k 's are, and the factor preceding the Vandermonde determinant in (295) is always non-zero, D is non-zero. The second proof of Theorem 9-1 is complete. We now formalize the remarks made at the beginning of this section:

Theorem 9-2: Given Assumption 4, the equilibrium matrix for the Arbitrary Angles System is non-singular.

Proof: We still have that $[K_{EQ}] = [C][L]$ and [C] is identical to the [C] of the Basic System, which, given Assumption 4, is non-singular. By Theorem 9-1, [L] non-singular. Since the product of non-singular matrices is non-singular, Theorem 9-2 is proved.

Theorem 9-3: The Arbitrary Angles System is controllable.

Proof: The equations of motion for the Arbitrary Angles System are identical to those of the Basic. Thus the dynamics matrix is of identical form. In particular, it can still be written in the form which was used in the proof of controllability for the Basic System, (195)

$$[A] = \begin{bmatrix} 0 & E \\ A_1 & 0 \end{bmatrix}$$

Then, as in the proof for the Basic System, Theorem 5-2, controllability holds provided only that the equilibrium matrix is non-singular. Since this is true by Theorem 9-2, the proof of Theorem 9-3 is complete.

Two remarks are in order. First, since the Basic System is a special case of the Arbitrary Angles System, the above proofs could have used to prove the corresponding theorems for the Basic System. However, the L matrix in the Arbitrary Angles case is not orthogonal, so the explicit expression for the equilibrium matrix given by (169) for the Basic System would not hold for the Arbitrary Angles equilibrium matrix. Since much of the succeeding development for the Basic System depends on this explicit expression, they might not have been possible for the Arbitrary Angles System.

Second, in an actual physical system, the angles between coils cannot approach zero. The phrase "angles between coils" is a shortened expression for "the angles between lines to the center of coils". Since one coil can't physically occupy the same space as another coil, the angles between coils must be greater than some positive number; this number is determined by the coil radius and r . Thus the theorems in this section are primarily of theoretical rather than practical interest. The following corollary is of this type:

Corollary 9-4: $D = \det[L]$, where $[L]$ is given by (272), may be written

$$D = 2^8 \prod_{1 \leq r < s \leq 5} \sin(\theta_s - \theta_r)/2$$

Proof: Let $g_k = e^{i\theta_k/2}$; hence $z_k = g_k^2$ and $e^{-2i\theta_k} = g_k^{-4}$. If we put this into (296) and into (295), we have

$$(297) \quad D = \frac{-1}{4} \prod_{k=1}^5 g_k^{-4} \prod_{1 \leq r < s \leq 5} (g_s^2 - g_r^2)$$

While it might not be apparent, in the product

$$\prod_{1 \leq r < s \leq 5} (g_s^2 - g_r^2)$$

in (297) each g_k appears in exactly four factors. Since there are also exactly four factors of each g_k in the product

$$\prod_{k=1}^5 g_k^{-4}$$

of (297), $g_s^{-1}g_r^{-1}$ may be associated with each factor $(g_s^2 - g_r^2)$ and D becomes

$$(298) \quad D = \frac{-1}{4} \prod_{1 \leq r < s \leq 5} \frac{g_s^2 - g_r^2}{g_s g_r} = \frac{-1}{4} \prod_{1 \leq r < s \leq 5} (g_s g_r^{-1} - g_r g_s^{-1})$$

Now writing each g_k back into its exponential half-angle form and factoring $2i$ from each of the ten products, we have

$$(299) \quad D = \frac{-1}{2^2} (2i)^{10} \prod_{1 \leq r < s \leq 5} \left(\frac{e^{i(\theta_s - \theta_r)/2} - e^{-i(\theta_s - \theta_r)/2}}{2i} \right) = 2^8 \prod_{1 \leq r < s \leq 5} \sin(\theta_s - \theta_r)/2$$

This proves Corollary 9-4. Recalling our convention that each angle is greater than or equal to zero but strictly less than 360 degrees, we again see that D is zero if and only if at least two angles are equal. Theorem 9-1 is confirmed.

10 Generalization of Results and Prospects for Future Work

We have shown that mathematically simple expressions may be obtained for the flux field resulting from coils placed in a circularly symmetric configuration. We used these expressions to derive results for the particularly structured system under study - a single small magnet placed symmetrically with respect to the coils. Now we now consider generalizations of both these situations. We find expressions for the total flux generated by an arbitrary coil configuration. Then we consider more complex systems of magnetized bodies, such as multiple magnets or non-stationary magnets. Since all the results of this section are so tentative we will not dignify them by stating them as theorems.

10.1 Expressions for Flux Resulting from an Arbitrary Coil Configuration

We first must deal with a persistent difficulty that one encounters in trying to generalize results of the kind we have developed. Not knowing exactly what sort of system the results may be applied to, we are not sure what sort of flux expressions we are seeking. We assume they will be similar to the ones we have worked out previously, given by (144) through (148). For definiteness, we will find the expression corresponding to (144), which we now repeat for convenience:

$$(144) \quad [K_y] = c_0 [L_0]$$

We seek an expression like the above for n coils, not necessarily identical, placed arbitrarily in relation to the body, except for the following restrictions: we assume that the axes of all coils are parallel to the z axis of the coordinate system. We also assume, as previously, that the flux values are to be found at a single point, which is on z . We retain a cylindrical coordinate system to describe coil placement; it unlikely that any other system

would take advantage of the cylindrical symmetry of the flux field. The coordinate for the j^{th} coil will be λ_j, r_j, h_j . The coils are no longer in the same plane, so h_j is the vertical distance from the j^{th} coil to the plane of the datum point. Since the coils are no longer identical, instead a single function f describing all coils, the j^{th} coil will now have f_j to describe it. Then the analogy to (132) still holds:

$$(300) \quad K_y^j = f_j(r_j, h_j) \sin \lambda_j$$

Where K_y^j is the maximal flux in the y direction produced by the j^{th} coil. In analogy to (137), we write

$$(301) \quad c_0^j = f_j(r_j, h_j)$$

While the analogy to (138) is

$$(302) \quad L_0^j = \sin \lambda_j$$

By the above three equations, in place of (144) we have

$$(303) \quad [K_y] \equiv [K_y^1, K_y^2, \dots, K_y^n] = [c_0^1 L_0^1, c_0^2 L_0^2, \dots, c_0^n L_0^n]$$

If we now define

$$(304) \quad [c_0] = [c_0^1, c_0^2, \dots, c_0^n]$$

then (303) can be simply written by using the direct (or Hadamard) matrix product [29].

This product is defined for the general case as follows: let $[A]$ and $[B]$ be n by m matrices with entries a_{ij}, b_{ij} . Their direct matrix product is written

$$(305) \quad [C] = [A] \bullet [B]$$

where $[C]$ is an n by m matrix whose entries c_{ij} are given by

$$(306) \quad c_{ij} = a_{ij}b_{ij}$$

Since the vector of (303) is a one by n matrix, (303) becomes

$$(307) \quad [K_y] = [c_0] \bullet [L_0]$$

The remaining expressions (145) through (148) may be generalized in the same way that (144) is generalized to (307): the scalar c_i becomes the row vector $[c_i]$ and the scalar product becomes the direct matrix product. Obviously we could find $[K_\Delta]$, where Δ is any subscript, in a similar manner. If we improve on the notation of (307) by letting $[c_\Delta]$ be the Coil Value vector corresponding to any subscript Δ , and $[L_\Delta]$ the corresponding vector of the appropriate trig function evaluated at the n values of λ_j , we have, for any $\Delta \in \{i, i,j, i,jk, \dots\}$

$$(308) \quad [K_\Delta] = [c_\Delta] \bullet [L_\Delta]$$

Now let us assume we are still seeking five dimension of control. It is possible to put expressions like (308) together into a matrix equation, just as we put (144) through (148) together to form (149). Were it not for the troublesome $c_2^\#$ constant in (146), we could write, in analogy to (149),

$$(309) \quad [K_{EQ}] = [C] \bullet [L]$$

where now the K_{EQ} , C and L matrices are five by n . Because of the $c_2^\#$ term in (146), a slightly more complicated expression is required for $[K_{EQ}]$. We will not bother to reproduce this because even if (309) were valid, there is little we could do with it. It is extremely likely that in any application we would be interested in the properties of the equilibrium matrix with respect to the **regular** matrix product. If the equilibrium matrix

were found from (309), its properties with respect to the regular matrix product could not be deduced from the properties of the C and L matrices. We conclude that the factorization exemplified by (308) is not of practical value unless the coil placement exhibits more structure.

This is not unexpected. An almost arbitrary configuration of coils results in an almost arbitrary flux. We feel that the development in this section has value only as a point of departure; more useful expressions may be found by specializing equations like (307) for coil configurations that possess some form of structure. The structure must make possible the formation of some rule connecting, for instance, the set of c_0^j 's, L_0^j 's in expressions like $[c_0^1 L_0^1, c_0^2 L_0^2, \dots, c_0^n L_0^n]$. We will not present any such possible configurations here. Instead we will say a word about the structure such configuration must exhibit in order to produce structure on these constants.

It would seem likely that to be of value this structure would have to, in some way, reflect the structure in the flux field of an individual coil. The predominate structure of this field is cylindrical symmetry; however, the field may exhibit other structure which we have not explored in this paper. For example, as a test point is moved radially away from a coil, in most regions the magnitude of the flux at the point decreases monotonically. We merely note this as one possibility for obtaining structure on the Coil and Angular values through coil placement.

The reader should not suppose that our General System, in all its variations, exhausts all the possibilities of coil placement exhibiting circular symmetry. Figure 18 shows a coil configuration which has actually been adopted [30]. We have not applied our method to this system but it is likely that useful expressions could be found for it.

10.2 More General Body Systems

We now assume that the coil configuration allows us to write meaningful expressions for the flux. We need not specify what the coil placement structure is, but it must be such that equations like (144), reproduced in the previous section, can be written. We now consider

the class of body systems to which such expressions apply. We consider only large-gap electromagnetic suspension systems in which the flux is slowly varying. The suspension forces and torques are produced by the flux acting on a magnet. Often such systems employ more than one magnet, with the magnets some distance from one another. Our results probably would not directly apply to such systems due to Assumption 1 in Section 2.5, which essentially implies that the flux is computed at only one point in space.

However, one can imagine a locus of points in space within which point-dependent but simple expressions for the flux could be found. For example, consider the coil configuration our General System; here the desired locus would be all points along the z axis. Thus, two or more magnets could be positioned on this "locus of symmetry". The total flux at each magnet could be found by simple expressions, all of the same form. For n identical magnets located at points z_1, z_2, \dots, z_n along the z axis, instead of the single function $f(r, h)$ of (117), we would have $f(r, z_1), f(r, z_2), \dots, f(r, z_n)$.

In our General System, the body could make only infinitesimal translational motion. Now consider a single magnet moving through space. This is conceptually very similar to multiple magnets located at separate positions in space. If we are to find simple expressions for the flux, the body must move through a "locus of symmetry". To give a specific example, we may think of a magnet which moves along the z axis in the general experiment described in this report. Instead of the function f of (117), or the $f(r, z_1), f(r, z_2), \dots, f(r, z_n)$ of the previous example, we would have a function of time $f[r, z(t)]$. We see that it would be difficult, but not necessarily impossible, to apply our method to a magnet translating through space.

Similar conclusions apply with regard to finite rotations of the magnet. To find simple expression for the flux we must find an "axis of symmetry", analogous to the "locus of symmetry" of the above paragraphs. Again, for a specific example, we look at our General System. In this system the body could rotate, in some sense, about two axes: the body x axis, which was uncontrolled, and the z axis. The force-torque equations were independent of the body's orientation about the x axis, so rotation about this axis was

completely unrestricted, though probably of little practical utility since it could not be controlled. The force-torque equations were not independent of the body's orientation about the z axis, and it was evidently for this reason that body could not undergo controlled rotations through other than "infinitesimally" small values of θ . However, the dynamics matrix, when expressed in the rotated coordinate system, was shown in Section 6 to be invariant with respect of θ . It was thus possible to suspend the body at any fixed θ orientation. The function f of (117) is, of course, completely independent of θ . The form of the expression for the flux is $f(r,h)L(\theta)$, where L depends only on θ . From the point of view of finding the flux, controlled rotations would be possible, since we can find expressions of the form $f(r,h)L[\theta(t)]$, but the realities of the force-torque equations make this untenable.

All this points to the fact that in any particular case, a careful engineering analysis must be done using the equations of motion, control scheme, hardware constraints, and so on, before the applicability of our method can be determined. Any departure from the system detailed herein must be carefully scrutinized. For example, our magnet was cylindrical, so was symmetric about the datum point. There is apparently no reason why the body must be cylindrical, but if it is not symmetric about the datum point, a torque will be produced by the second integral in (12). However, there is no a priori reason why such a torque will cause the system to fail. Similarly, it is not a priori impossible to use induced magnetization rather than a permanent magnet, and the magnetization need not be in a single direction.

We hope the above has given some intuitive notion of the considerations involved in applying our method to other systems. It should be possible to write a compact set of general equations describing an entire general system. These equations would employ "generalized coordinates", that is, coordinates which may denote lengths, angles, or time; and "generalized forces", which could be either forces or torques. To be useful, the expressions for the flux must be independent of one or more of the generalized coordinates. If the body is to occupy more than one generalized coordinate position, the generalized force equations and the flux equations must "mesh" within the locus of generalized locus of points the body would occupy. This would identify our "locus of

symmetry" and our "axes of symmetry". Once the system formulation is written, the problem would be to express specifically what is meant by the vague term "mesh". We will not attempt such a formulation here but would appreciate any comments or suggestions on how to do it.

In a discussion of applications to other systems, we must note the following. The methods developed in Section 2 were, of course, developed for use in a magnetic suspension system. However, what they actually amount to are general methods to write the sum of the flux resulting from several symmetrically positioned coils. They are valid only at a point on the z axis, and would be of value only if an analytic expression for the flux field could not be found. However, there is nothing in their nature which restricts their use to a magnetic suspension system. They could be of use in any application in which it is necessary to find the flux at large distances from the coil configuration.

10.3 Use of Less Than Five Coils

We briefly discuss a natural generalization of our General System. We saw that five coils symmetrically placed give controllability in five dimensions. Suppose we have n coils symmetrically placed, where n is less than five; could we have controllability in n dimensions? The answer is no. The conditions for linearization imply that the three dimensions of space and the two dimension of angle are kept very small. If any one of these dimensions are uncontrolled, the experiment is untenable. The nature of our experiment ties it to at least five dimensions.

10.4 Other Prospects for Future Work

We saw in Section 9 that the equilibrium matrix for the General System is closely related to a very structured and well-known matrix, the Vandermonde matrix. From this we were able to prove that the Arbitrary Angles System is controllable, provided no coil is precisely

on top of another. This was interesting mathematically but of little value practically, since due to physical reality no coil could be on top of another or even come close to it. Using other properties of the Vandermonde matrix, we might be able to prove other interesting mathematical results. For example, it is possible to define mathematically a metric on the set of, say, five by five matrices. Using this metric, we may define how "near" these matrices are to one another. Given any equilibrium matrix of the Arbitrary Angles System, we could determine how "near" this matrix is to being singular, as a real-valued function of the angular placement of its coils. This would be mathematically interesting but of dubious practical value. However, the history of technology has shown that some developments undertaken purely in the spirit of intellectual inquiry do turn out later to have practical value.

We would like to pose a question which could have practical consequences. We must apologize to the reader because the question deals with the stability of the system, a subject we have not adequately dealt with here. The question is: what would be the effect on the stability of the General System if h , the height of body above the plane of the coils, were negative? Of course, this is the same as asking what would happen if the body were suspended beneath the coils, rather than above them. In the General System, the coils are between the suspended body and the object producing the gravitational force, that is, the earth. In this alternative system, the body is between the coils and earth. This alternative system might be significantly more stable. In [7], it is shown that if the body is perturbed by being rotated slightly about the z axis, in the absence of a correcting torque it will tend to rotate still further. This instability is referred to as the "compass needle effect" in [7], and is one of the major difficulties with the system. We hypothesize that in the alternative system, the force needed to suspend the body would be "the opposite" of that in the conventional system. This would result in the compass needle effect forcing the body back to the equilibrium position rather than away from it. However, we have found it difficult to deal with polarities of the systems and our hypothesis may be false. We mention it merely as a question to be explored.

10.5 Summary

We have shown that a mathematical formalism describing a Large Gap Magnetic Suspension System may be used to prove certain standard control-theoretic results about the system. The formalism, however, only applies to a restricted class of coil configurations. It appears that if designers of Large Gap systems seek a priori assurance of, for example, controllability, they must choose a coil placement scheme which exhibits some form of circular symmetry. Given any coil configuration it should not be difficult to determine whether or not useful expressions can be written for it using the methods developed here.

Appendix 1: Mathematical Symbols Used

Matrix and vector notations:

$[\]$	matrix
$[\]^{-1}$	inverse of a matrix
$[\]^{+}$	the pseudoinverse of a matrix
$[\]^T$	transpose of a matrix
$\{ \ }$	column vector
$\{ \ }^T$	transpose of a column vector
$[\]$	row vector

Special symbols:

i, j in subscripts, means the derivative in the j^{th} space direction of the i^{th} component of a vector; eg,

$$B_{i,j} = \frac{\partial B_i}{\partial x_j}; \text{ similarly, } B_{i,j,k} = \frac{\partial^2 B_i}{\partial x_j \partial x_k}, \text{ and so on for higher space derivatives}$$

δ when placed before a variable, means the variable is small; for example,

$$\sin \delta\theta \cong \delta\theta, \text{ since } \delta\theta \text{ is small}$$

Δ used as a subscript to mean any of the subscripts i ; i, j ; i, j, k ; i, j, k, l and so on. For example, the following means the equation is true for any value of the subscripts:

$$B_{\Delta} = \frac{1}{I_{MAX}} [K_{\Delta}] \{I\}$$

— bar, when placed over a variable, means the variable is expressed in the noninertial coordinate system of the body

$\left[\frac{\partial}{\partial t} \right]$ row vector of derivatives with respect to the five currents; for example,

$$\left[\frac{\partial B}{\partial t} \right] = \left[\frac{\partial B}{\partial I_1}, \frac{\partial B}{\partial I_2}, \dots, \frac{\partial B}{\partial I_5} \right]$$

$[\partial]$ matrix of gradients of a vector; for example $B_{i,j}$ is the i,j^{th} element of $[\partial B]$

$\{ \}_{inertial}$ the value of a dynamic vector when it is observed from an inertial frame. The components of the vector may be expressed with respect to an inertial or noninertial coordinate system.

$[] \bullet []$ the matrix direct (or Hadamard) product; if

$[A] = [a_{i,j}]$, $[B] = [b_{i,j}]$, $[C] = [c_{i,j}]$, and $[A]$, $[B]$ are both n by m , then

$$[C] = [A] \bullet [B] \Leftrightarrow c_{i,j} = a_{i,j} b_{i,j}$$

Common Mathematical Symbols:

$\dot{\{ \}}$ dot over vector (or scalar) variable means derivative with respect to time

$\{ \} \times \{ \}$ vector cross product

$\nabla \bullet \{ \}$ gradient of a vector function

$\nabla \times \{ \}$ curl of a vector function

$(\{\} \bullet \nabla)\{\}$ dot product of a vector with gradient operator acting on a vector; eg,

$$(\{m\} \bullet \nabla)\{B\} = [\partial B]\{m\}$$

Appendix 2: Assumptions Made

Note: Each of the following assumptions is made for every General System.

Assumption 1: The suspended body is in the upper half plane, that is, h is positive; the vertical component of flux due to each coil is negative. (See Figure 4.)

Assumption 2: The body is so small that $\{B\}$ and its derivatives may be regarded as constant throughout the body; $\{M\}$ is also constant.

Assumption 3: Given that when the maximal positive current I_{MAX} flows through a coil it produces a maximal flux $\{K\}$. Suppose a current I , with $|I| \leq I_{MAX}$, flows through the coil. Then the flux $\{B\}$ produced by I is given by

$$(20) \{B\} = \frac{1}{I_{MAX}} \{K\}$$

Assumption 4: The Euler angle rotations from the equilibrium orientation will be small, so that, for example, we may use the approximations $\sin \theta_i = \tan \theta_i = \theta_i$ and $\cos \theta_i = 1$, where $i \in \{x, y, z\}$. Also angular rates, linear translations and velocities are small. We may therefore approximate the PRODUCT of any two of these quantities with zero.

Assumptions 5: Each of the elements of the main diagonal of $[C]$ is non-zero.

References

1. Magnetic Levitation. Citations from the NTIS Database, Jan 1980 - Jan 1992. U.S. Dept of Commerce, Technology Administration, National Technical Information Service, Springfield, VA 22161.
2. Wouch, G.; Lord Jr., A.E.: Levitation, Electromagnetic. Encyclopedia of Physics, Second Edition, pgs. 627 - 630. VCH Publishers.
3. Yang, W. Q.; Zhang, Y.S.: Electrostatic Suspension System with Sliding Mode Control. IEEE Transactions of Aerospace and Electronic Systems, Vol 28, No. 2, April 1992.
4. Groom, Nelson J.; Britcher, Colin P. (editors): International Symposium on Magnetic Suspension Technology, Parts 1 and 2. NASA CP-3152, August, 1991.
5. Tsuda, Masahiro; Higuchi, Toshiro: Design and Control of Magnetic Servo Levitation. Report of the Institute of Industrial Science, the University of Tokyo, Vol. 37, No. 2, March 1992.
6. Britcher, Colin P.: Some Aspects of Wind Tunnel Magnetic Suspension Systems With Special Application at Large Physical Scales. NASA CR-172154, 1983.
7. Groom, Nelson J.; Britcher, Colin P.: Open-Loop Characteristics of Magnetic Suspension Systems Using Electromagnets Mounted in a Planar Array. NASA TP-3229, May 1992.

8. Britcher, Colin P.: Technical Backround for a Demonstration Magnetic Levitation System. Department of Mechanical Engineering and Mechanics, Old Dominion University, Norfolk, VA. Grant NAG1-716, May 1987.
9. Reitz, J.R.; Milford, F.J.; Christy, R.W.: Foundations of Electromagnetic Theory, Third Edition, Addison-Wesley Publishing, 1980.
10. Hutter, Kolumban; van de Ven, Alphons A.F.: Lecture Notes in Physics 88, Field Matter Interactions in Thermoelastic Solids, Springer-Verlag, 1970.
11. Parker, D.; Britcher, C. P.: Progress Toward Extreme Attitude Testing with Magnetic Suspension and Balance Systems. AIAA 15th Aerodynamic Testing Conference, San Diego, CA, May, 1988.
12. Goldstein, H.: Classical Mechanics, Second Edition, Addison-Wesley Publishing, 1981.
13. Welch, S.S.; Shelton, K.J.; Clemmons, J.I.: Optical Position Measurement for a Large Gap Magnetic Suspension System. Proceedings of the 37th International Instrumentation Symposium, paper #91-063, May 1991.
14. Groom, Nelson J.: Analytic Model of a Five Degree of Freedom Magnetic Suspension and Positioning System. NASA TM-100671, 1989.
15. The VF/GFUN Reference Manual. VF068894, Vector Fields Limited, June 20, 1988.
16. Groom, Nelson J., NASA Langley Research Center, Mail Code 161, Hampton, VA 23681-0001. Personal communication.

17. Haj, Anwar M.: Dynamics and Control of a Five Degree-Of -Freedom Magnetic Suspension System. Department of Mechanical Engineering and Mechanics, Old Dominion University, Norfolk, VA. Grant NAG-1-1056, June 1992.
18. Groom, Nelson J.; Schaffner, Philip R.: An LQR Controller Design Approach for a Large Gap Magnetic Suspension System (LGMSS). NASA TM-101606, 1990.
19. Britcher, Colin P.; Ghofrani, Mehran: A Magnetic Suspension System with a Large Angular Range. Rev. Sci. Instrum. 64 (7) , July, 1993.
20. Britcher, C.P.; Ghofrani, M.; Britton, T.C.; Groom, N.J.: The Large-Angle Magnetic Suspension Test Fixture. International Symposium of Magnetic Suspension Technology, Part 2, pg. 971. NASA CP-3152, August 1991.
21. Britcher, C.P.; Ghofrani, M.; Britton, T.C.; Haj,A.: Analysis, Modelling, and Simulation of the Large-Angle Magnetic Suspension Test Fixture. Proceedings of the Third International Symposium on Magnetic Bearings, July, 1992.
22. Groom, Nelson J.; Britcher, Colin P.: A description of a Laboratory Model Magnetic Suspension Test Fixture With Large Angular Capability. The First IEEE Conference on Control Applications, Volume 1, IEEE Catalog No: 92CH3000-7, Sep 1992.
23. Groom, Nelson J.: A Decoupled Control Approach for Magnetic Suspension Systems Using Electromagnets Mounted in a Planar Array. NASA Technical Memorandum 109011, August 1993.

24. Youcef-Toumi, K.; Yeh, T-J: Dynamics and Control of High Precision Magnetically Levitated Vibration Isolation Systems. International Symposium on Magnetic Suspension Technology, Part 1, pg. 149. NASA CP-3152, August 1991.
25. Chong, P.; Commandeur, C.; Davis, H.; Whitehead, L.: Large-Gap Magnetic Positioning System Having Advantageous Configuration. International Symposium on Magnetic Suspension Technology, Part 2, pg. 941. NASA CP-3152, August, 1991.
26. Friedland, Bernard: Control System Design, McGraw-Hill, 1986.
27. Strang, Gilbert: Introduction to Linear Algebra, Wellesley-Cambridge Press, 1993.
28. Penrose, R.: A Generalized Inverse for Matrices. Proc. Cambridge Philos. Soc., vol. 51, 1955, pp. 406-413.
29. Rao, C. R.; Mitra, S. K.: Generalized Inverse of Matrices and its Applications, John Wiley & Sons, 1971.
30. Groom, Nelson J.: Description of the Large-Gap Magnetic Suspension System (LGMSS) Ground Based Experiment, pg. 303. Aerospace Applications of Magnetic Suspension Technology, NASA CP 10066, Part 1, 1991.

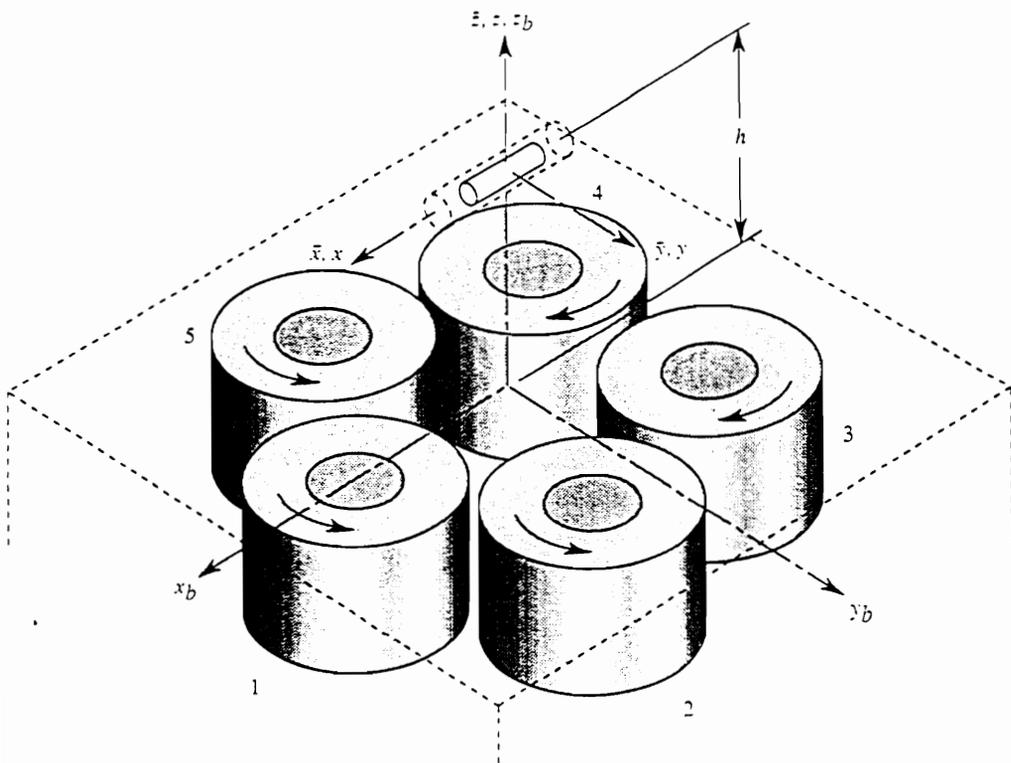


Figure 1. The Basic Experimental System. All systems which we analyze are very similar to this one. In all systems the coils lie in a plane, with the centers of the coils on a circle. The suspended body is on a line perpendicular to the plane of the coils, through the center of the circle. After [7].

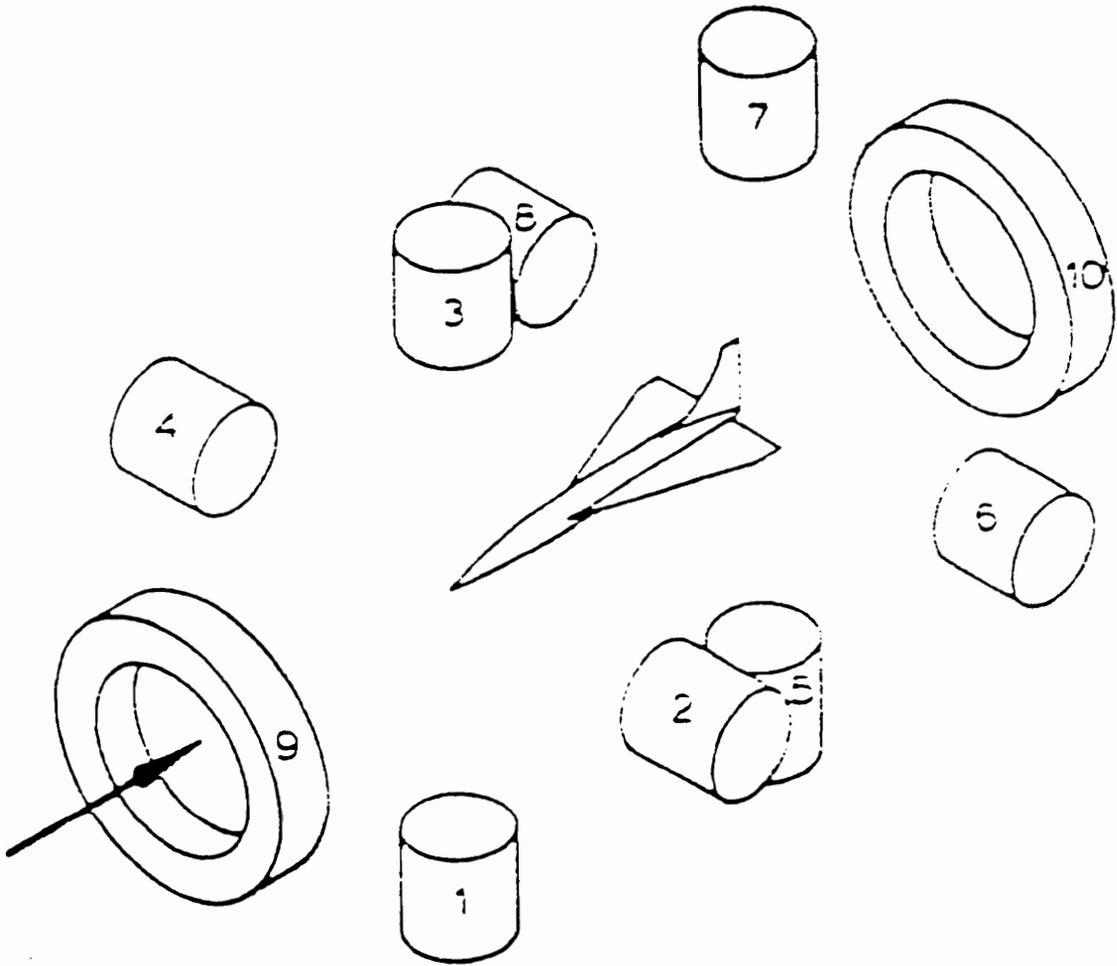


Figure 2. This is a coil and body configuration which has actually been used, but to which our analysis would not directly apply. Included to show the variety of configurations which are possible. After [11].

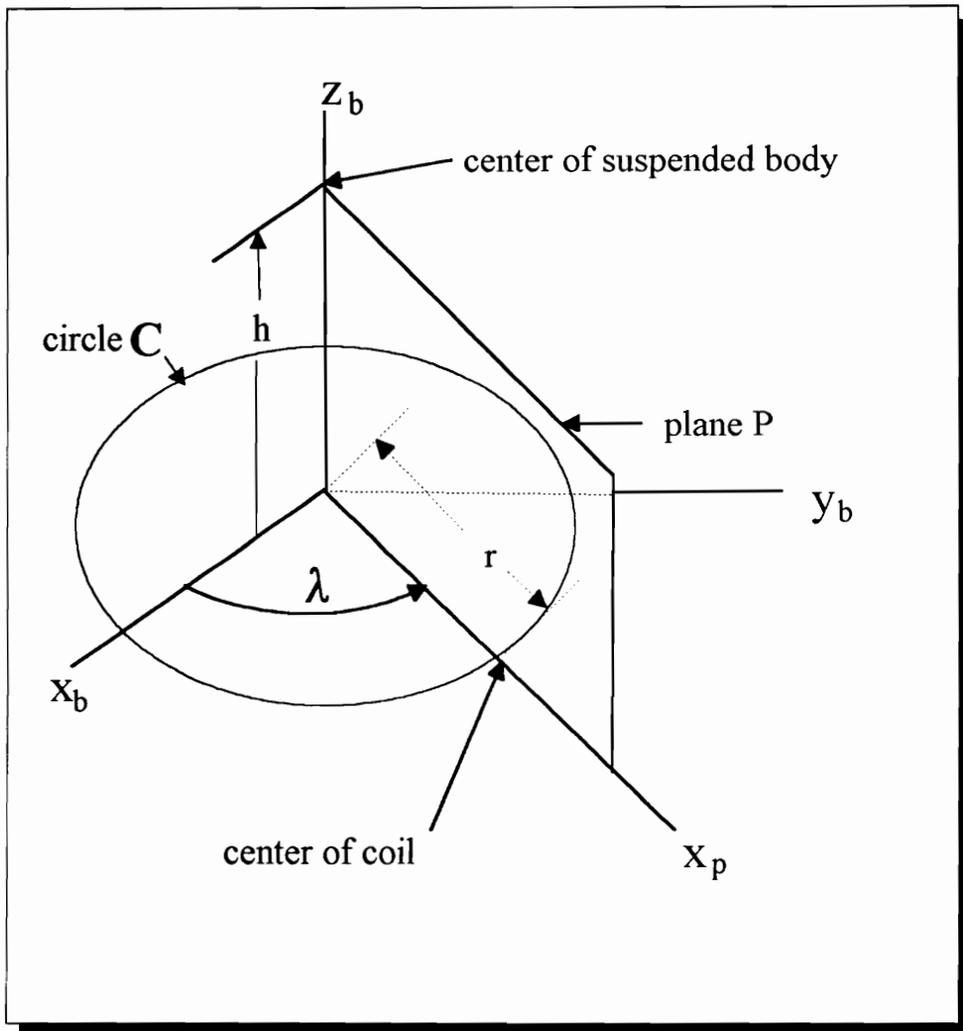


Figure 3. Features common to all systems. In general, there may be N coils where $N \geq 5$; for clarity, the planar coordinates, r, λ , of the center of only one coil are shown. The coils must be identical, and the center of each must lie on the circle C . C has radius r and lies in the x_b, y_b plane; the center of C is at the origin of the x_b, y_b, z_b system. The center of the suspended body must lie on the positive z_b axis, at $z_b = h$. Plane P passes through the center of the coil along the axis x_p ; it is used in Figure 4 to discuss the flux pattern of this coil.

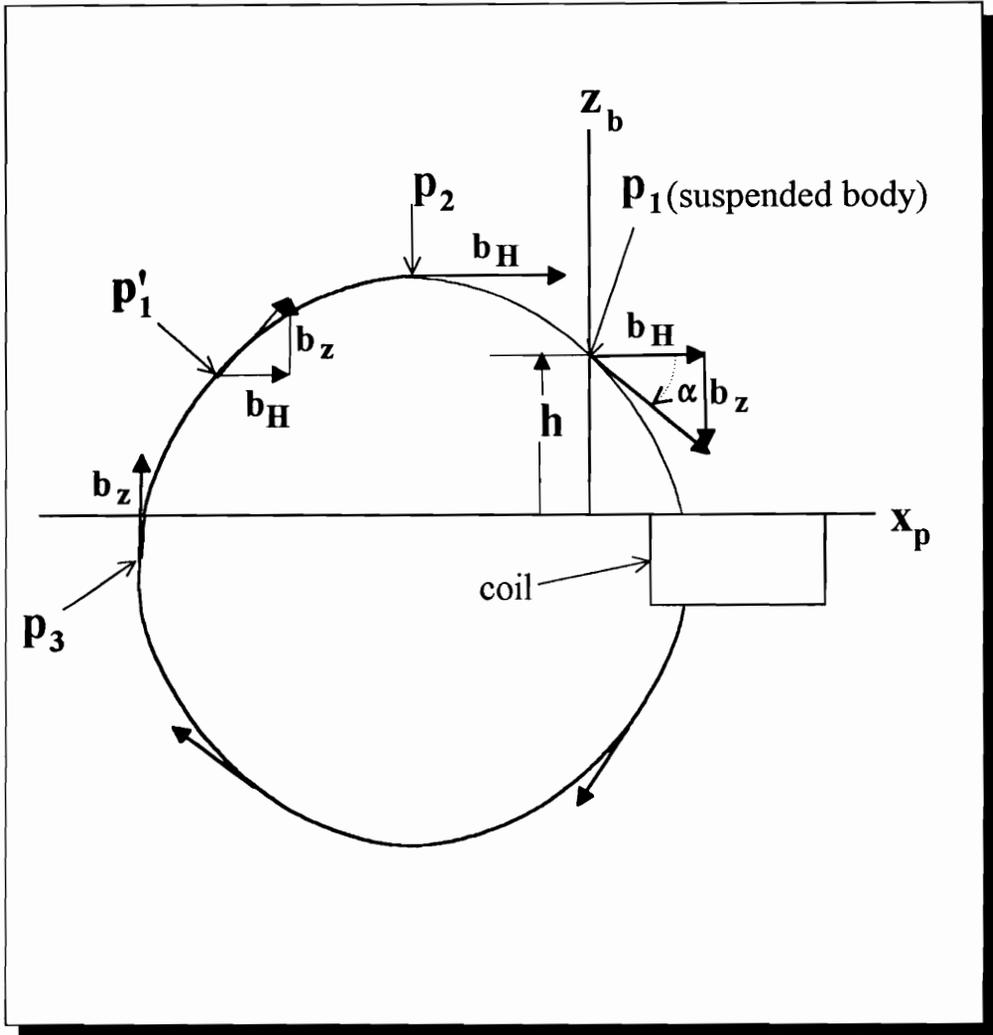


Figure 4. Schematic drawing of a single flux loop in Plane P of Figure 3. The center of the suspended body is at P_1 ; at this point both the horizontal flux B_H and the vertical flux B_z are nonzero; and B_z is negative. It is assumed the suspended body is not at P_2 because B_z is zero there; it may not be at P_3 because B_H is zero there. We assume further that the suspended body is not anywhere between P_2 and P_3 , such as at P_1' , where B_z is positive.

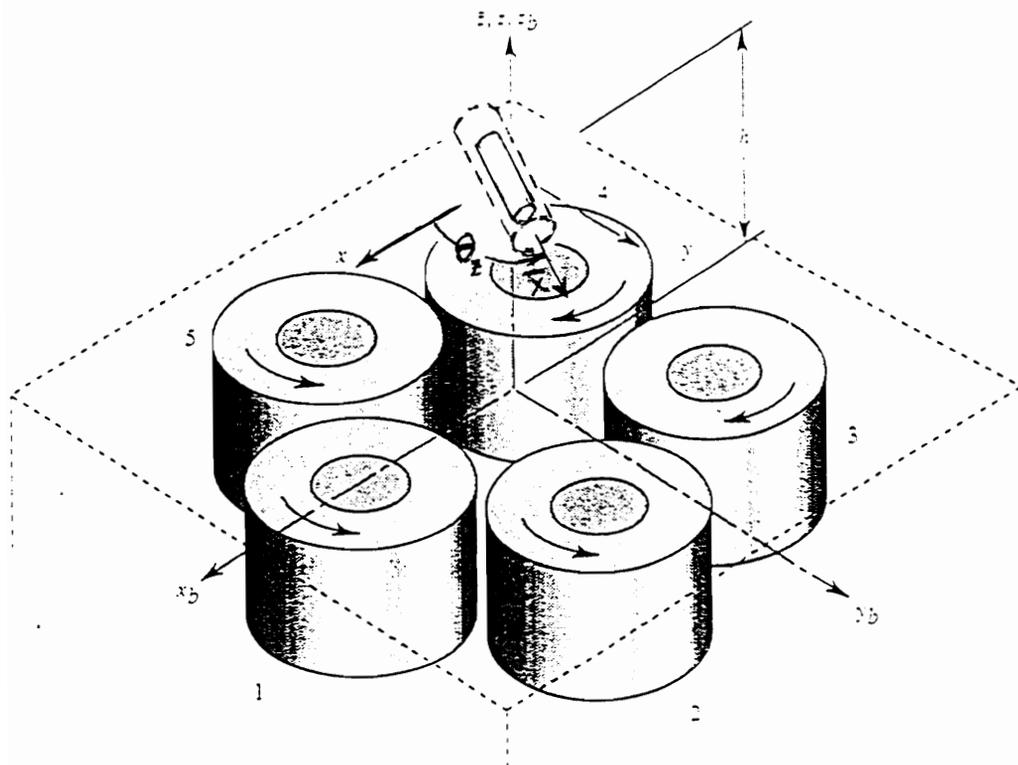


Figure 5. The Rotated Experimental System. The coil configuration is the same that of the Basic System but the body has been rotated about the z axis through the fixed, but arbitrary, angle θ_z . The equilibrium point is at this new orientation. After [7].

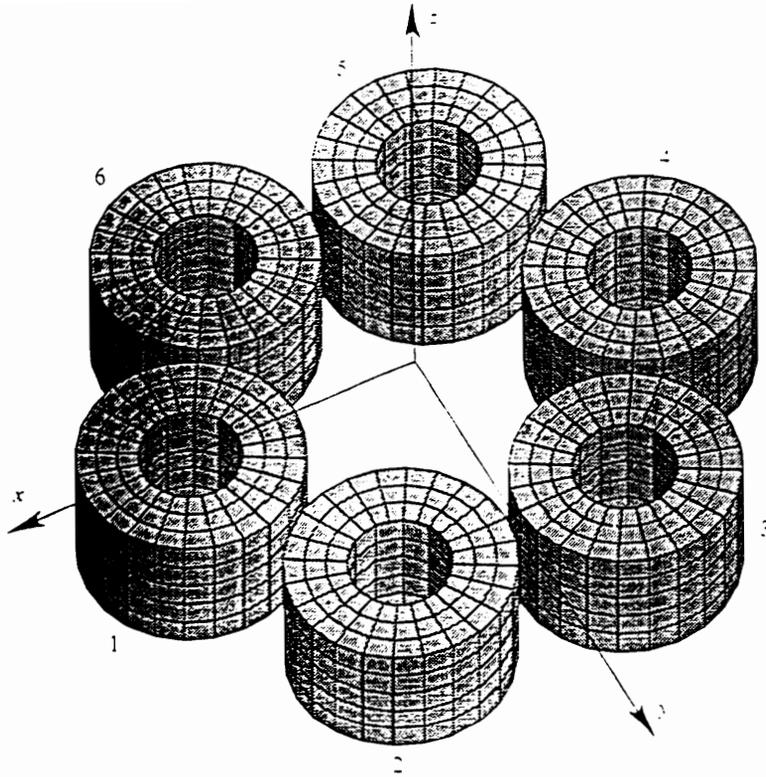


Figure 6. The six coil configuration. The coils are equally spaced and the center of coil 1 lies on the x_b axis. After [7].

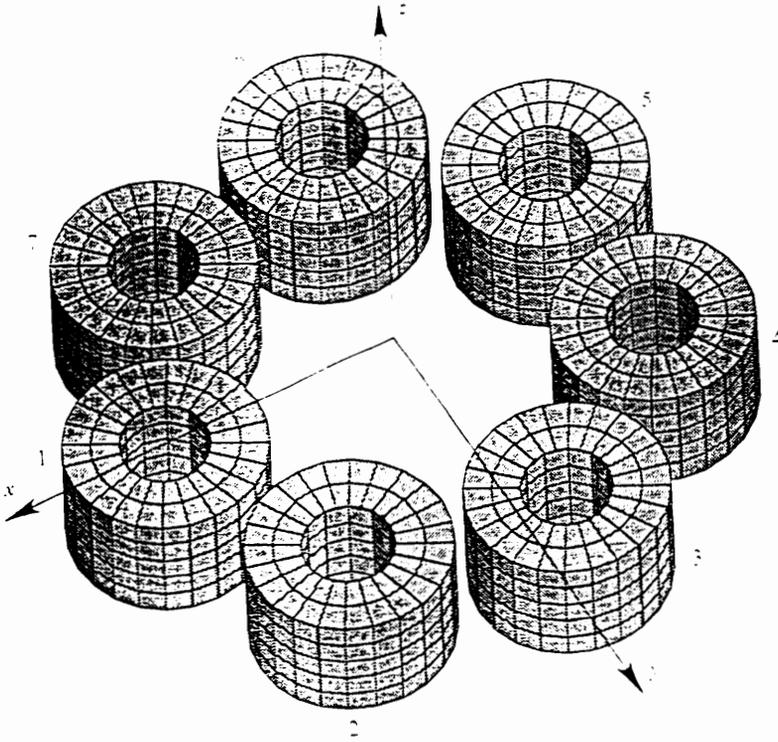


Figure 7. The seven coil configuration. After [7].

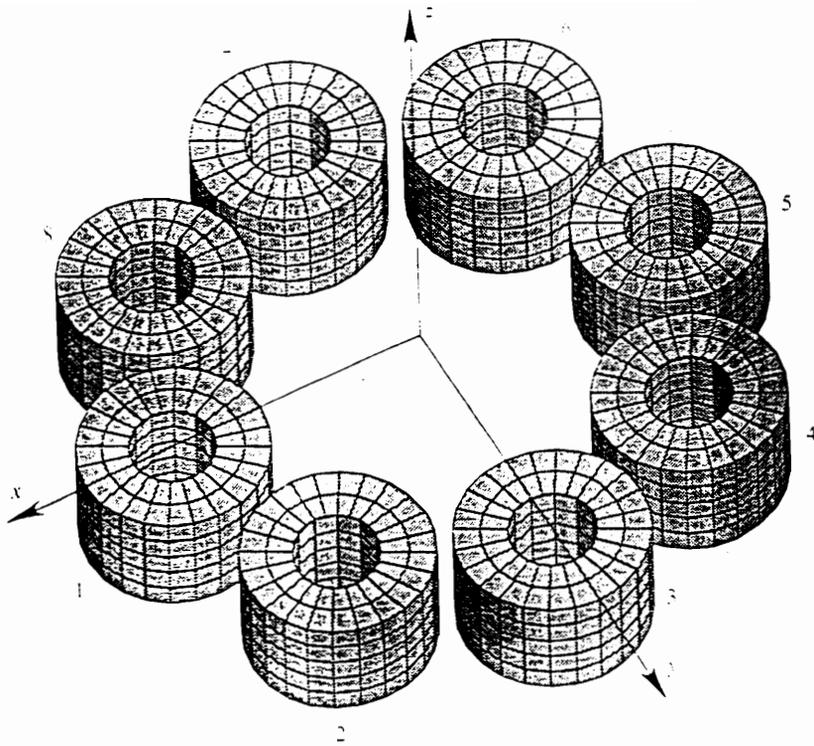


Figure 8. The eight coil configuration. After [7].

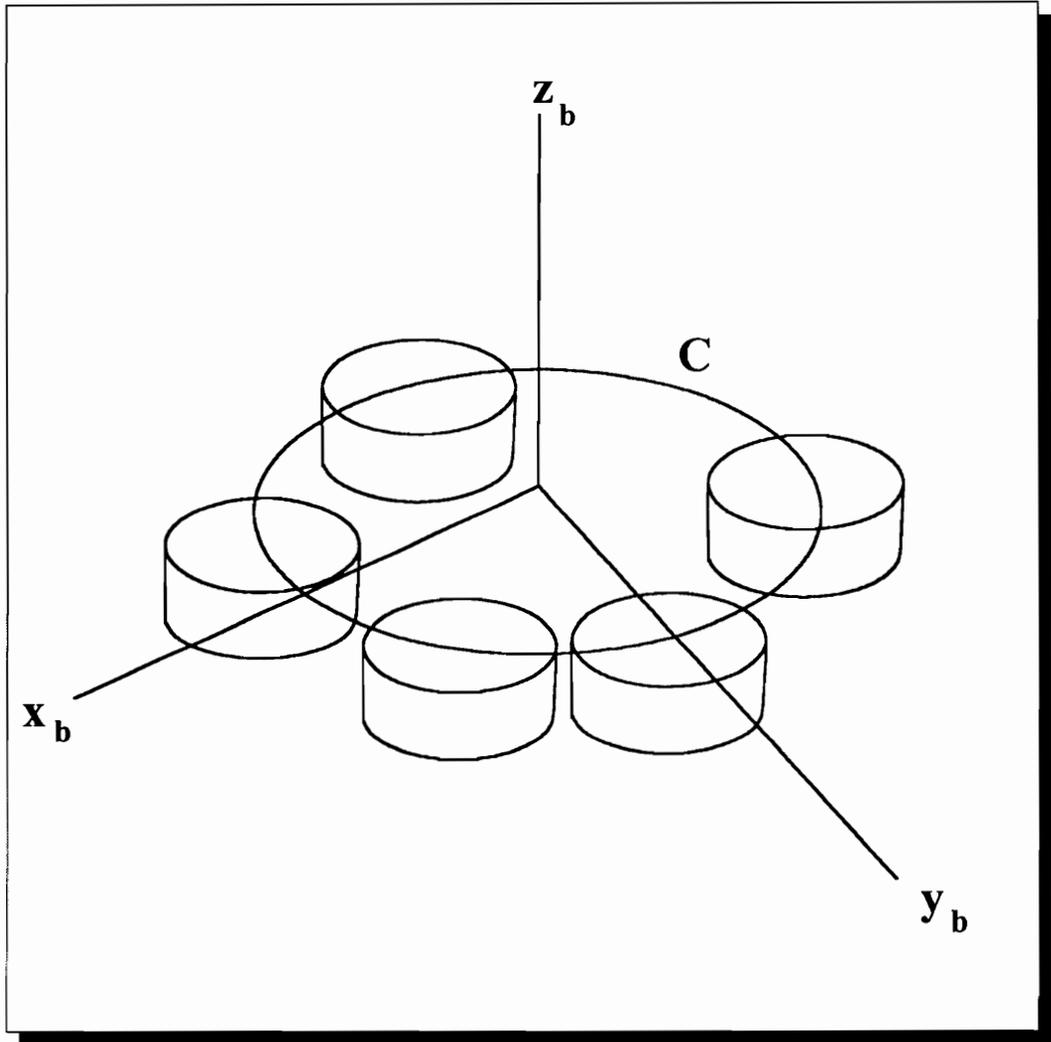


Figure 9. The Arbitrary Angles system. Like the Basic system, it consists of five identical coils in a plane with the center of each lying on circle C . However, the angular placement of the coils on C is now arbitrary.

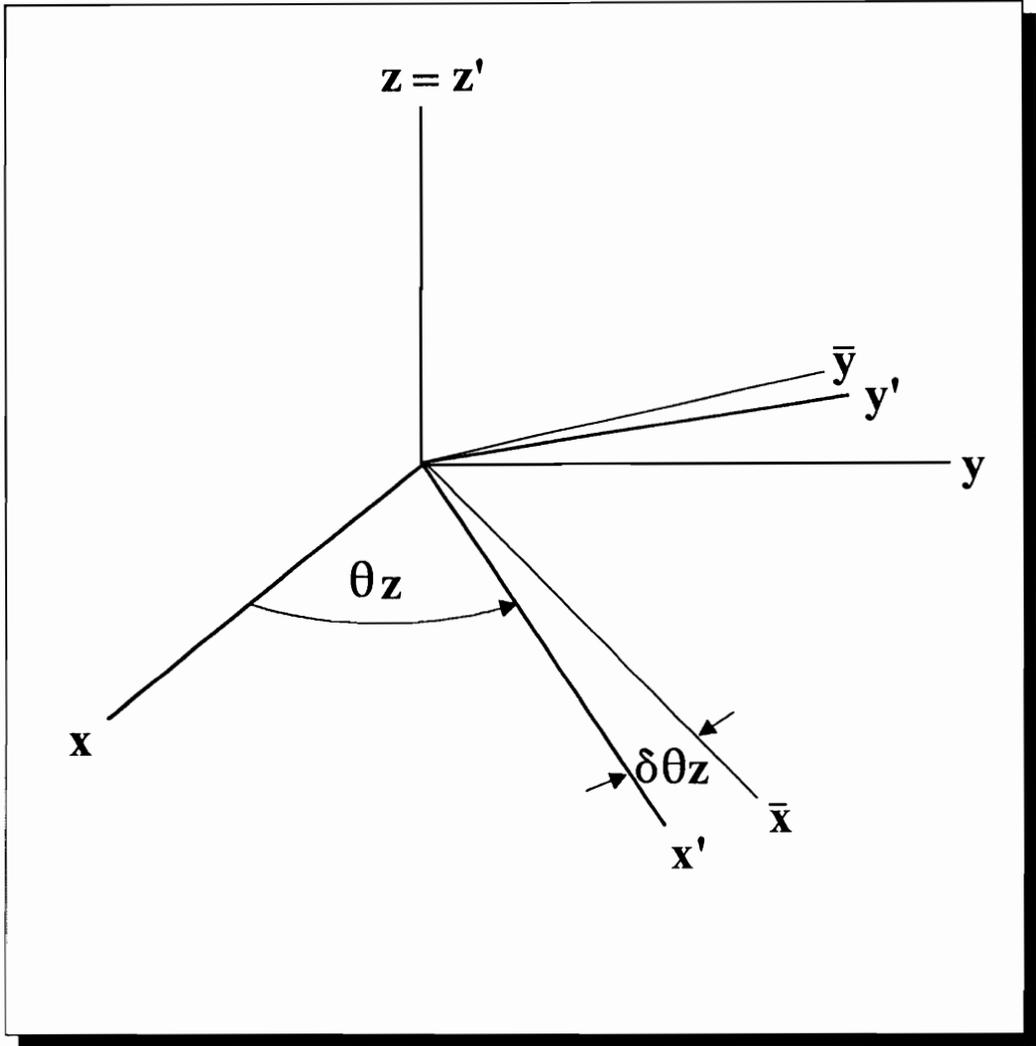


Figure 10. The coordinate system x', y', z' for the Rotated System is rotated about the z axis through the fixed angle θ_z from the x, y, z system. The coordinate system of the body, $\bar{x}, \bar{y}, \bar{z}$, is perturbed through infinitesimal angles $\delta\theta_y, \delta\theta_z$ away from the x', y', z' system. ($\delta\theta_z$ is shown in the figure but $\delta\theta_y$ is not.)

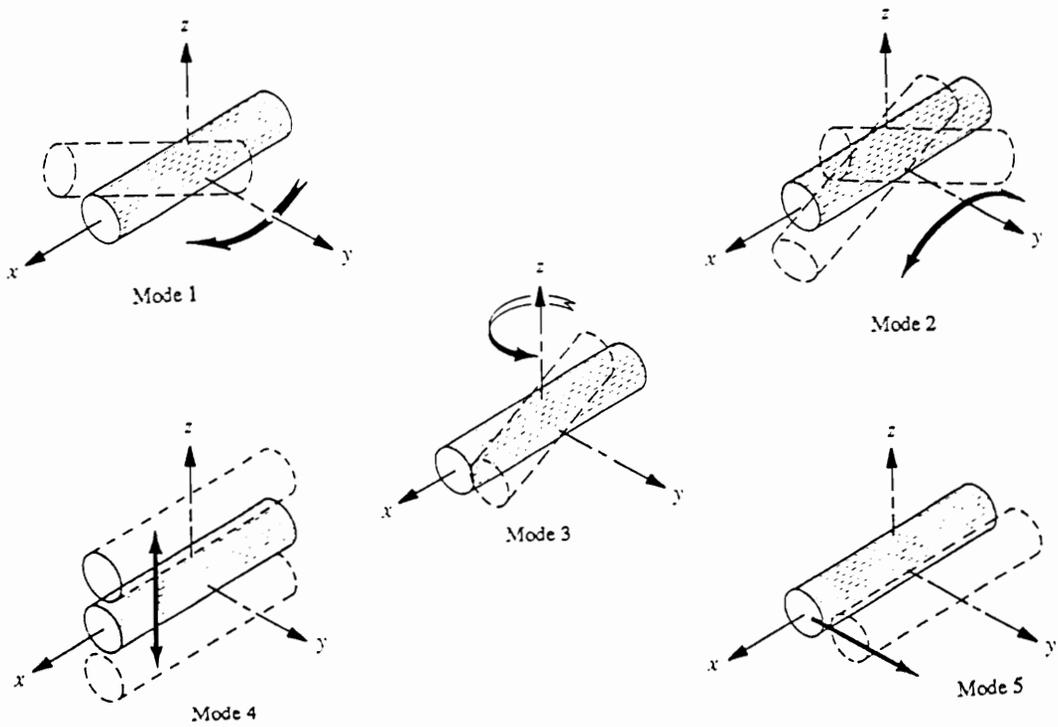


Figure 11. The characteristic perturbations of the suspended body from equilibrium in the absence of controlling forces and torques. These are sometimes called "modes". The shaded body in each figure is at equilibrium. After [7].

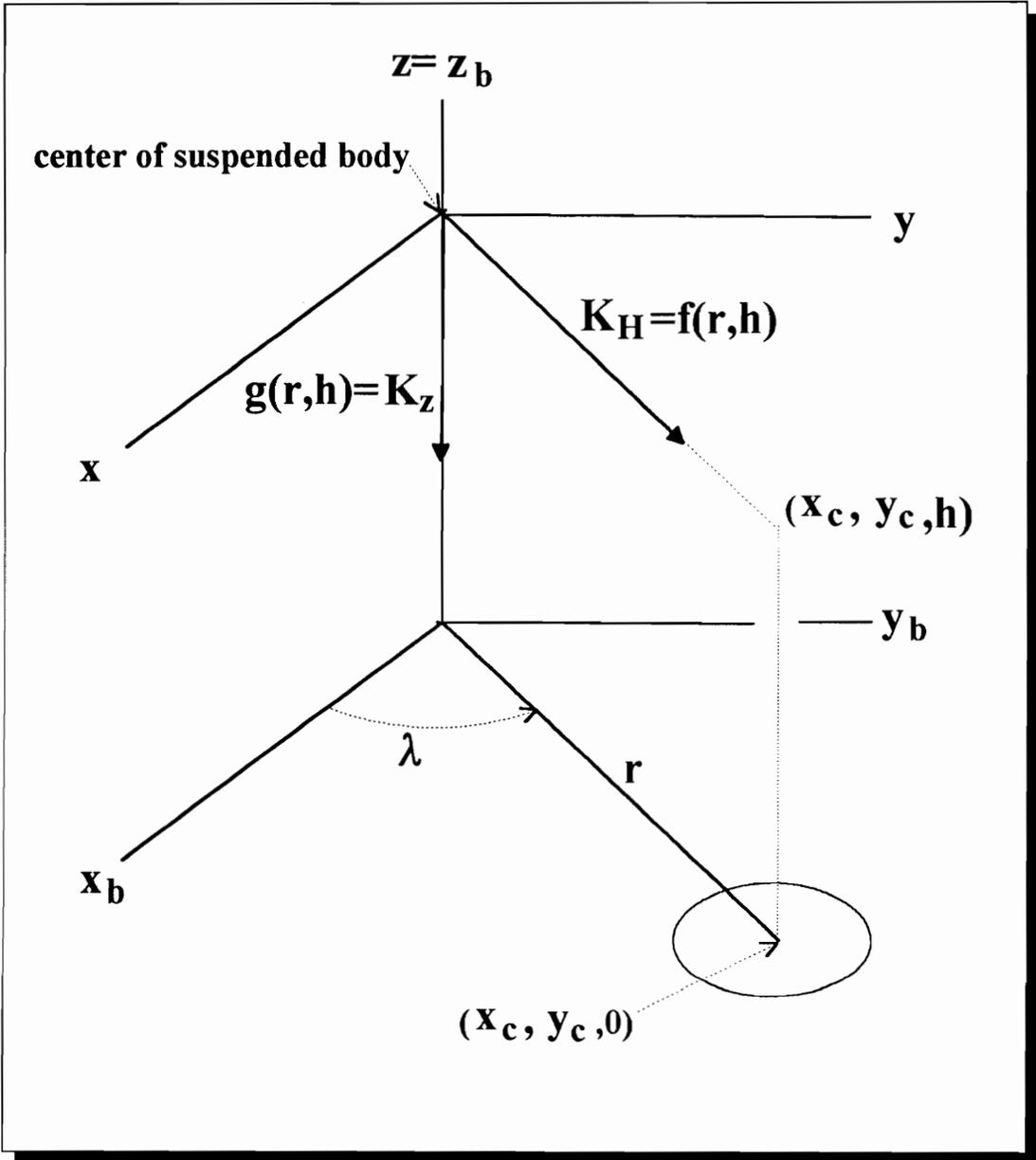


Figure 12. The center of a coil has rectangular coordinates $(x_c, y_c, 0)$ and planar polar coordinates r, λ . At the center of the suspended body ("datum point"), the coil produces the flux shown. The *magnitude* of the flux is independent of λ and depends only on r and h . The *magnitude* of the maximum horizontal flux is given by $K_H = f(r, h)$, the vertical by $K_z = g(r, h)$; the functions f and g depend on the coil used. Thus for identical coils in a planar array, as in all our systems, K_H and K_z are the same for all coils. Note that the *direction* of the horizontal flux depends only on λ , whereas, for all our systems, both the *magnitude* and *direction* of vertical flux is identical for all coils.

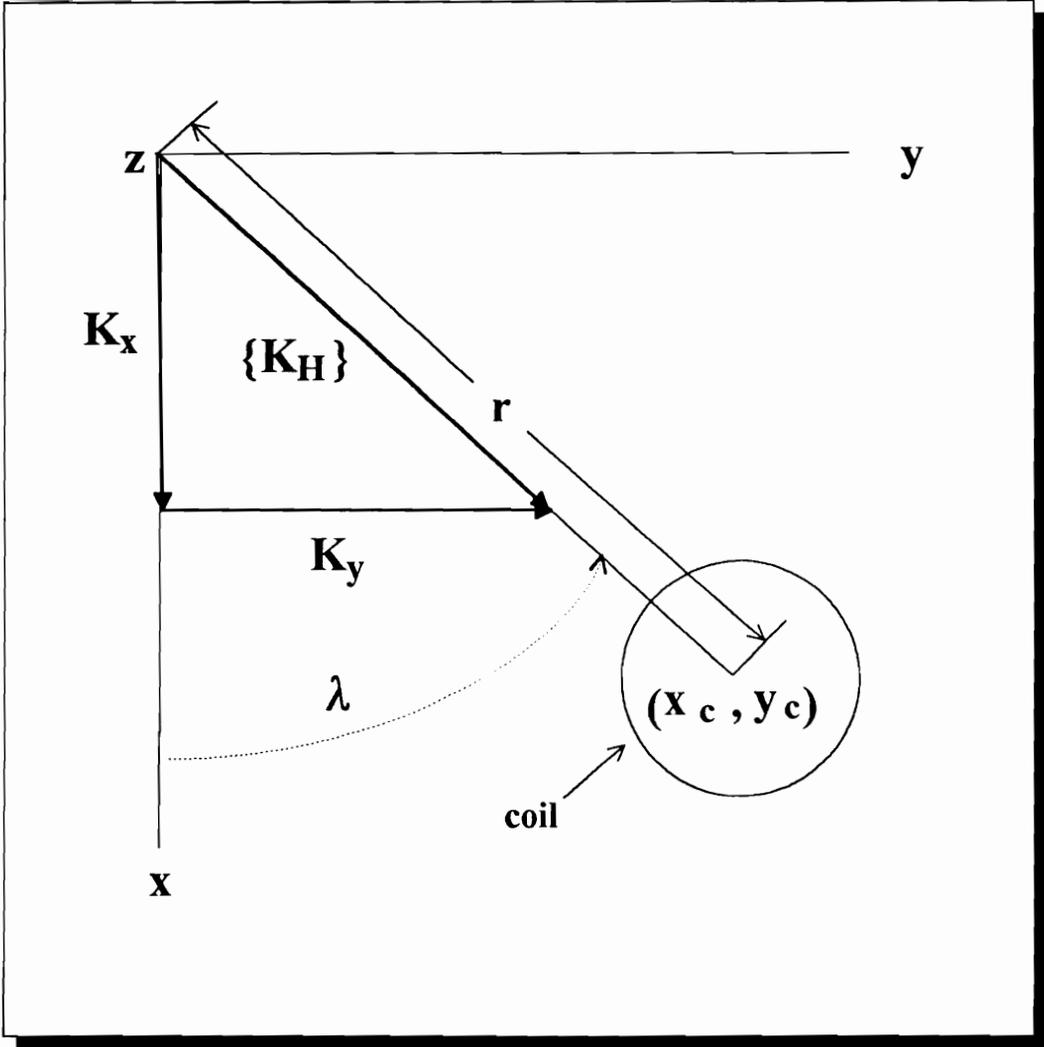


Figure 13. The flux relations of Figure 12 projected into the x,y plane of the lab coordinate system. The maximum flux in the x direction is given by $K_x = K_H \cos \lambda$, in the y direction by $K_y = K_H \sin \lambda$

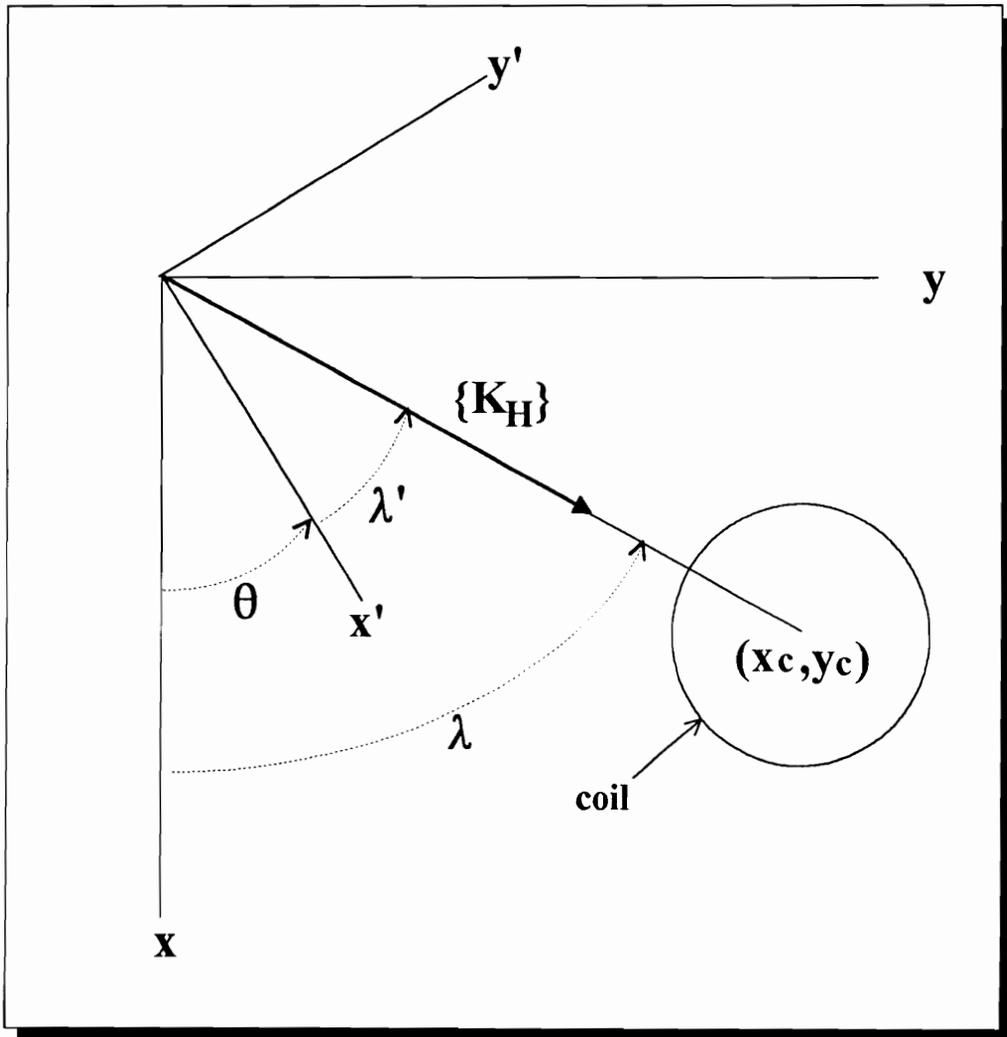


Figure 14. The vector of maximum horizontal flux $\{K_H\}$ resolved into the coordinates x' , y' of the Rotated coordinate system: $K'_x = K_H \cos \lambda'$, $K'_y = K_H \sin \lambda'$, where $\lambda' = \lambda - \theta$. Compare with the analogous expressions for the Basic system: $K_x = K_H \cos \lambda$, $K_y = K_H \sin \lambda$. Since K_H is the same for both Systems, the equations for the Basic are valid for the Rotated if λ is replace by λ' .

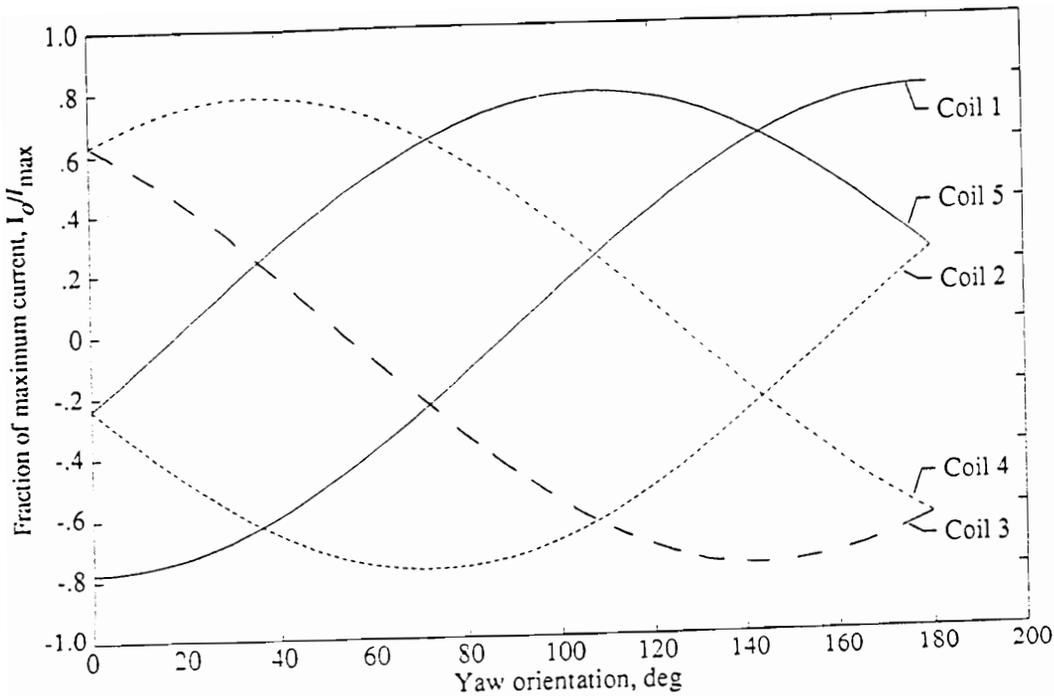


Figure 15. Equilibrium current ratios, I_0/I_{MAX} , for the five coils required to suspended the body when it is rotated about the z axis. (In aerospace jargon, a rotation about this axis is referred to as "yaw"; in Section 4.4 we called the rotation angle θ . In the figure above θ varies from 0 to 200 degrees.) The graph for Coil 1 is $-0.7726\cos(-\theta)$; for Coil 2, it is $-0.7726\cos(72 - \theta)$; and the general equation for the k^{th} coil is $-0.7726\cos[(k - 1)72 - \theta]$. Equation (178) may be written $L_4^j(\theta) = \cos[(k - 1)72 - \theta]$. Since in the Rotated Experimental System, $2c/5c_4 = -0.7726$, the graphs confirm equation (188):

$$\{I(\theta)\}_0/I_{MAX} = (2c/5c_4)\{L_4(\theta)\}$$

for this System. After [7].

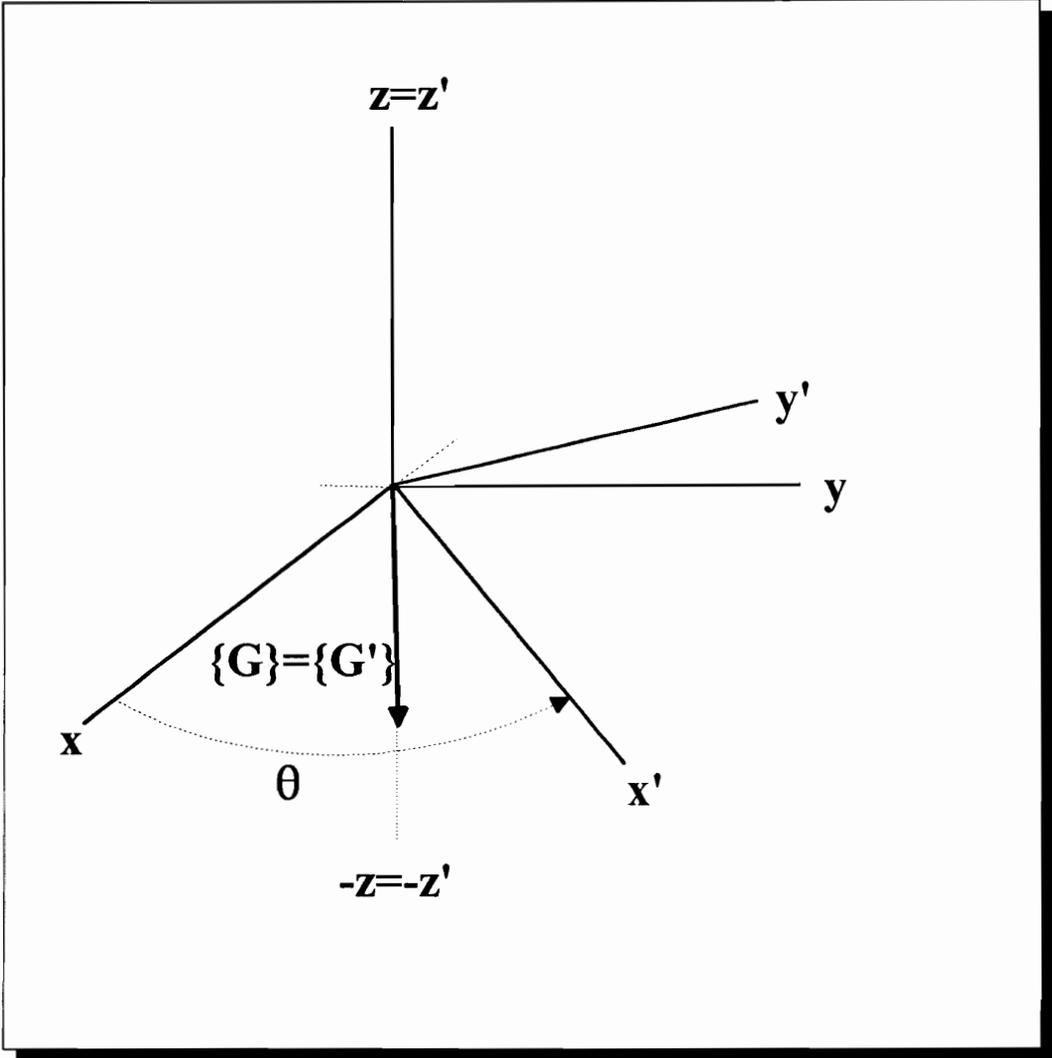


Figure 16. The force of gravity vector $\{G\}$ has the same entries whether expressed in the Basic (x,y,z) or Rotated (x',y',z') system; hence $\{G\} = \{G'\}$. Therefore the vector of forces and torques needed to suspend the body is the same in both Systems. This means that the vector of equilibrium flux is the same for both: $\{B_{EQ}\}_0 = \{B'_{EQ}\}_0$. Since this vector uniquely determines all equilibrium flux and flux derivative values, these are equal when expressed in their respective coordinate systems: $B_{\Delta_0} = B'_{\Delta_0}$.

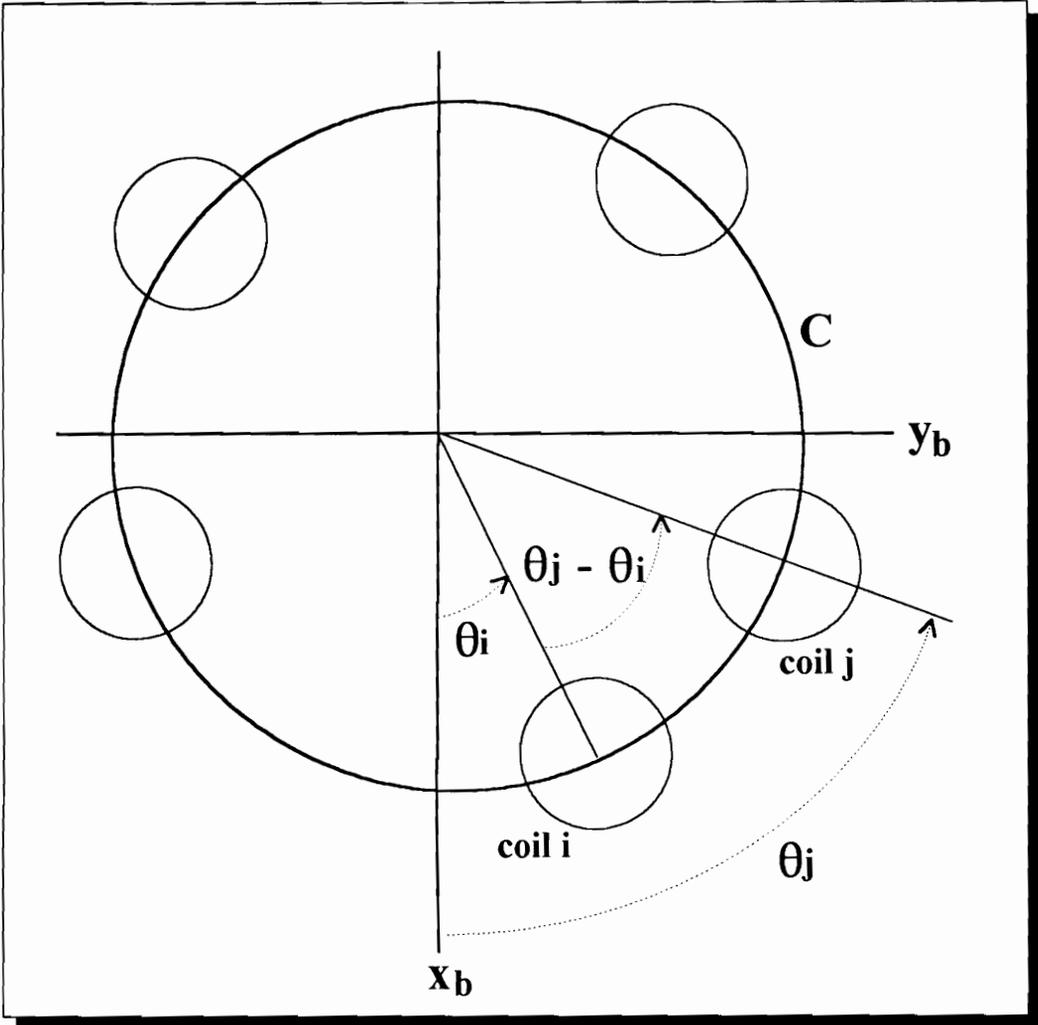


Figure 17. The Arbitrary Angles System. The center of each coil must lie on the circle C but angular placement is arbitrary. The angular placement of the j^{th} coil is given by θ_j . (In our previous notation, λ was used to designate angular placement of coils.)

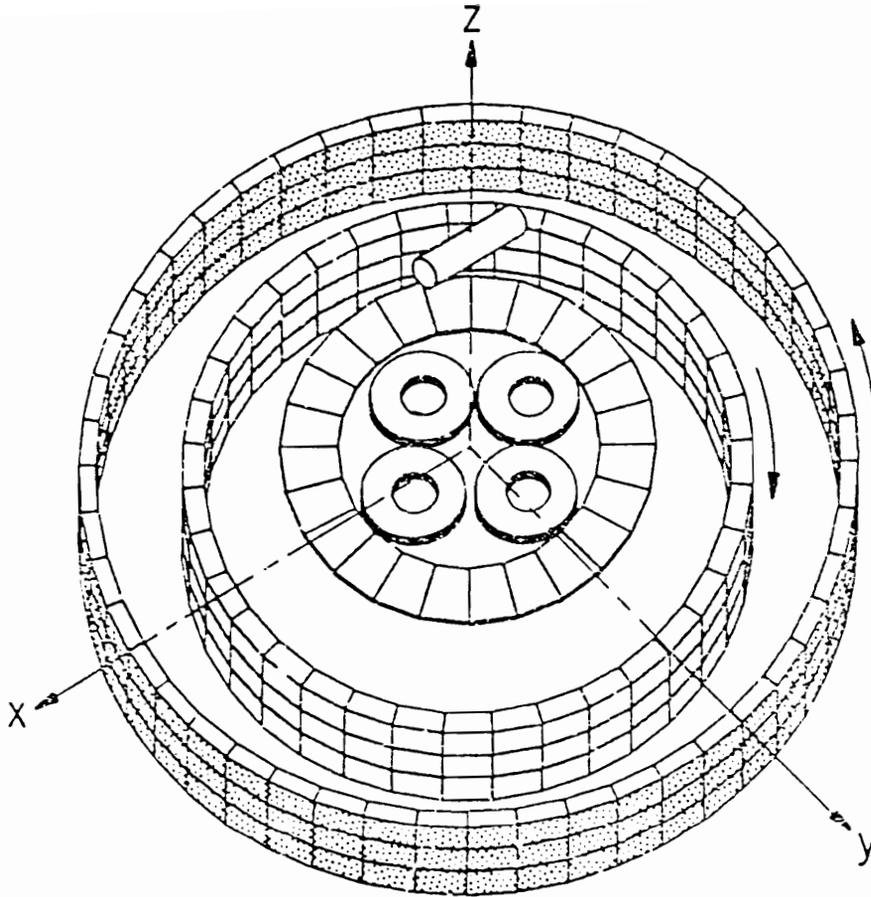
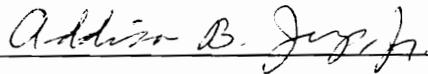


Figure 18. A coil and body configuration which has actually been implemented and to which our analysis would probably apply. After [30].

Vita

Addison B. Jump, Jr. 

Born July 24, 1941, Pawhuska, Oklahoma. Current residence: Shiloh, Virginia, near Fredericksburg. Two children: Aaron, 19, Kimberly, 17.

Education and employment:

Attended N.E. St. Coll., OK, 1959-60, and Kansas U., 1960 - 62, majoring in sociology and economics. Active duty, U.S. Army Reserves, 1963. Farm and construction laborer, 1963 - 66; engineering aide and lab technician, 1966 - 67.

Bureau of Indian Affairs Grant, 1967 - 69 to attend Tulsa U., 1967 - 68, and Oklahoma State U., 1968 - 69, majoring in math. **B. S., mathematics, Okla. St., 1969.** Graduate teaching assistant, math, Okla. St., 1969 - 71. **M. S., mathematics, Okla. St., 1971.**

Instructor of Mathematics, Haskell Indian Junior College, Lawrence, KS, 1971 - 82. Attended Kansas U., 1982 - 83, with major in computer science. Mathematician, Naval Surface Warfare Center, Dahlgren, VA, 1983 to present. Naval training programs to attend Virginia Tech off-campus physics classes, 1983 - 88. **M. S., physics, Virginia Tech, 1989.** Full time doctoral study, mathematics, Virginia Tech, 1990 - 92; dissertation, 1992 - 96. **Doctorate, mathematics, Virginia Tech, 1996.**

Areas of study and employment experience:

Employment: analysis, modeling, and simulation, of cruise missiles; Kalman filtering; geodesy. Study: control systems; electromagnetic suspension.

Address inquiries to: Commander, ATTN Addison Jump L41, Naval Surface Warfare Center, 17320 Dahlgren Rd, Dahlgren, VA 22448-5100. Phone: (540) 653-6276; FAX: (540) 653-8588; email: JUMP@RELAY.NSWC.NAVY.MIL