A Substructure Synthesis Formulation for Vibration Isolation

by

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(ABSTRACT)

The new modeling method presented here is classified as a substructure synthesis (SS) technique. The distinction between the new SS method and the component mode synthesis formulation is that no transformation between local coordinates and generalized coordinates occurs in the new SS method. The advantage of this is a retention of physical insight into the model and the ability to form equations of motions directly with generalized coordinates. The new formulation differs from other substrucutre synsthsis formulation because it satisfies geometric, natural, displacement and force constraints between substructures into one mathematical process, instead of using both kinematic chains and boundary condition approximation methods. This has the advantage of reducing the complexity of the integrals that are required in the computation. The new formulation also results in global eigenfunction approximations and global generalized coordinates, which eventually satisfies the inclusion principle which means eigenvalue estimates converge from above their actual values. The analysis method also facilitates the examination of boundary conditions in a unique manner. The method is unique

because constraints are explicitly examined and selectively satisfied. This allows the identification of extraneous constraints and provides guidance in the selection of admissible functions. The new SS formulation may be divided into two steps. The first step is to satisfy geometric boundary conditions of substructures with appropriate admissible functions. The second step is the modification of these admissible functions to minimally satisfy geometric constraints imposed by the interaction of substructures. Natural constraints can also be satisfied to improve convergence to the exact eigenvalues.

The MAF-SS formulation results in explicit knowledge of the constraints coupling substructures. Changing these constraints with active feedback results in a modified structure. The effect of active feedback of terms proportional to the coupling constraints is to lower the stiffness of the structure. This increases the isolation between substructures. The ability to improve isolation using this unique type of feedback is demonstrated. The concept of structural modification through substructure constraint alteration is applied to systems using a multivariable feedback method. This is accomplished by combing the MAF-SS method with a standard eigenstructure assignment technique. This method uses the MAF-SS formulation to define a system with substructure constraint eigenstructure properties, the active feedback gain that realizes these systems is calculated with an eigenstructure assignment method. The MAF-SS has application to active control formulation, the result of this control can be an improvement in substructure isolation.

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1. Overview

The following work is divided into two sections: the development of a new substructure synthesis (SS) formation called MAF-SS or "modified admissible function substructure synthesis," and the application of this new formulation to the active multiple-degree-of-freedom vibration isolation problem. The new substructure synthesis formulation has the advantages of global eigenfunction approximations and global generalized coordinates. These are advantages in a modeling formulation because, the global eigenfunction approximations leads to the satisfaction of the Rayleigh-Ritz convergence criteria and the global generalized coordinates allow a continuity of the physical interpretation of the model. The MAF-SS explicitly details the connection constraints between substructures. This attribute is used to form an active vibration isolation scheme. The control goal of the isolation feedback system is to eliminate or alter the coupling constraints between substructures. This technique is demonstrated on multiple-degree of freedom systems.

Chapter 2 presents a literature survey and a summary of the new substructures synthesis formulation. Chapters 3-8 primarily deal with background information on the theoretical considerations of modeling dynamic systems with distributed and discrete methods. Chapter 9 present the details of MAF-SS. Chapters 10-11 concern the application of MAF-SS to the vibration isolation problem. Concluding remarks are made in chapter 12.

2. Introduction and Overview of Substructure Synthesis Methods

2.1 Introduction

Substructure synthesis (SS) is a method used to model dynamic systems. A complex structure is broken down into a set of simple substructures. These substructures are modeled with eigenfunction approximations and time dependent coordinates. The substructures are recombined using constraint information related to the substructure connection points. This simplifies the classical Rayleigh-Ritz modeling process of modeling complex structures by requiring only eigenfunction approximations of substructures versus the entire structure. In comparison with the finite element modeling method (FEM) another Rayleigh-Ritz based modeling technique that defines approximation functions over smaller sections of a structure, fewer degrees of freedom can be used to achieve the same model accuracy. Improving the ability specify substructures and their constraint requirements leads to the enhancement of the substructures ability to model complex geometry. This ability leads to the modeling of larger areas of a structure with fewer degrees of freedom and the same accuracy. A model with fewer degrees of freedom reduces the mathematical computation required to model the system. While the substructure synthesis method has proven useful in modeling a variety of systems, deficiencies exist. Specifically, one of these deficiencies is the inability to methodically formulate approximations of the eigenfunctions. Approximations are formulated in a number of ways but all current methods have the same two faults: eigenfunction estimates are not guaranteed to satisfy substructure connection properties, and integration becomes complex because of the use of kinematics chains which are a necessity of relative coordinate motion definition.

A development of a new substructure synthesis method is presented in the following chapters. The development covers the aspects of forming the Lagrange equations for holonomic and nonholonomic systems, a discretization of the Lagrange equations, the assumed modes method, and finally the new substructure synthesis method. This new method has several unique and important features that are discussed. Specifically, these features are a guarantee that eigenfunction estimates will satisfy the substructure connection constraints though the modification of eigenfunction estimates using the constraint equations between substructures, a reduction in the complexity of potential and kinetic energy term integrals, and the definition of a precise and methodical modeling procedure.

2.2 Literature Review

Hurty is often credited with being the first researcher to formalize the ideas of substructure synthesis. Substructure synthesis is the concept that the ability to model segments of a structure is less complex than trying to model the entire structure. A paper written in 1960 illustrates the use of admissible functions to model a structure divided into substructures (Hurty, 1960). These substructure models are reformed in global model using a coordinate transformation matrix derived from substructure displacement and force boundary conditions. The principle is based on a variation of the Rayleigh-Ritz technique that eigenvalues estimates can be approximated by functions that approximate the eigenfunctions of a system. Hurty proposed that substructures should be modeled with three types of component modes, namely, rigid-body, constraint, normal modes

(Hurty, 1965). This work led to the foundation of the analysis method known as component modes synthesis (CMS).

Other researchers at the time were working on similar ideas, to model complex structures by breaking them into smaller, simpler structures. Przemieniecki developed a method to model substructures individually with fixed boundary conditions and then alter these equations of motion using constraint equations (Przemieniecki, 1961). (This method did not take advantage of the Rayleigh Ritz properties of convergence or completeness.) Craig further developed Hurty's work. Craig and Bampton presented a method that eliminated treating rigid body modes separately from other constraint modes (Craig, 1968). Gladwell and Kane developed methods that were shown to be mathematically similar to Hurty and Craig by Hintz (Hintz, 1975). Work in SS area through 1977 was summarized by Craig in a review-of-the field (Craig, 1977).

Meirovitch made an important contribution to the CMS field by defining component mode synthesis as a subclass of substructure synthesis (eg., Meirovitch, 1980a) Substructure synthesis is a more general problem formulation than component mode synthesis. Meirovitch formalized the theory of using admissible functions in a Rayliegh-Ritz technique modeling interacting bodies (Meirovitch, 1977). Convergence of the eigenvalue estimates improvements through the development of intermediate substructures was developed by Meirovitch and Hale (Meirovitch, 1980). Weighting factors were used to approximate the geometric connection conditions between substructures. The result of approximating geometric connections results in the loss of the eigenvalue convergence criteria from above their actual values.

Another review paper on component synthesis methods was presented by Craig in 1987 (Craig, 1987). This paper demonstrated that the solution formulation of CMS had not radically altered from Craig's 1977 paper. Advances occured were the ability to model damping in structures and incorporate experimental information. The use of experimental data is presented along with a technique to incorporate damping effects in the models. Luk and Mitchell presented work on incorporating experimental data into substructure synthesis model (Luk, 1989).

Baruh and Tadikonda produced a method that uses local coordinates to model elastic motions using approximations, the Lagrangian formed from writing the structures kinetic and potential energy terms is modified to include constraint conditions (Baruh and Tadikonda, 1989). A method to model a system without a coordinate transformation was presented by Zhang (Zhang, 1989). The method uses approximation functions that satisfy the geometric boundary conditions. The method does not produce "global" coordinates. Recent work was completed by Tadikonda that reemphasized the difficulty of forming appropriate admissible functions (Tadikonda, 1995). The new method differs from these works by using global generalized coordinates through out the modeling procedure. The new method also provides a methodical method to define admissible functions.

There is a mathematical similarity in all the CMS and SS methods when they are distilled to their basic properties, but the SS method has important advantages. These advantages include a simpler problem formulation (in most cases), a result of the fact that substructures can be approximated with eigenfunction approximations that minimally satisfy the admissible function criteria (i.e. geometric boundary conditions and are p

differential of a system whose kinetic and potential energy terms are a maximum of order p). Meirovitch has worked to improve convergence using approximations that attempt to satisfy both geometric and natural boundary conditions (Meirovitch, 1990). The use of kinematic chains that relate local coordinates and substructures was introduced by Meirovitch to approximate displacement and slope requirements between substructures. Quasicomparison functions are used to satisfy geometric and approximate natural boundary conditions (Meirovitch, 1990). The SS developed by Meirovitch was further enhanced by showing that eigenvalues converge from above the natural frequencies. This ability was unique at the time (Meirovitch, 1992). The inclusion principle states that eigenvalue approximations are greater than the smallest actual eigenvalue and smaller than the largest. It yields the conclusion that eigenvalue estimates converge from above their actual values. The inclusion principle was proven for the substructure method using quasicomparison functions and chains. Nonlinearity induced by rigid body motions are examined, and dealt with using quasi-coordinates in a recent paper (Meirovitch, 1995).

Another review of CMS methods has been completed by Craig (Craig, 1995). This review indicates SS methods are being incorporated into control formulations. CMS and SS methods are particularly applicable to decentralized control schemes.

The MAF-SS differs in some important ways from other formulations. In the new formation the admissible functions are modified to meet geometric and kinematic boundary conditions, eliminating the use of intermediate structures and approximate boundary conditions. Global generalized coordinates are maintained throughout the modeling process. In addition, global eigenfunctions are formed. The new formation is developed using one set of global coordinates, simplifying the coupling of substructures.

2.3 Illustration of a Substructure Technique

An early example of a substructure synthesis technique was worked by Hurty on a cantilever beam. This is a component mode synthesis method which is a subclass of the substructures synthesis method. Although the method has evolved since the publication of this work (Hurty, 1960) a review of this problem is useful since it contains the basic elements, including some of the faults of many of the current techniques. This procedure and example may be found not only in Hurty's paper but in Thomson's 1993 text as well (Hurty, 1960, Thomson, 1993).

The system modeled is shown in Figure 1.1. The system is a bent (in the middle) (90°) Euler-Bernoulli beam vibrating in the plane of the paper. The beam is divide into two substructures defined by the dashed lines. The motion of each substructure is approximated by polynomial series that are not required to be orthogonal or to satisfy the connection conditions for displacement or force. Transverse motion of substructure 1 (w_1) is motion in the y direction of the coordinates in Fig. 1.1. The transverse motion of substructure 2 (w_2) is in the direction of x. Axial motion of substructure 1 (u_1) is in the x direction and the axial motion of substructure 2 (u_2) is in the y direction. When the substructures are reformed in a the original global structure these constraints must be and are, satisfied.

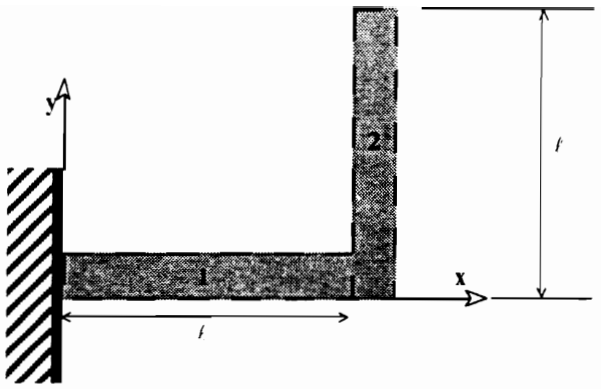


Figure 1.1. Beam system modeled using substructure method.

The assumed motion of the beam is defined as,

$$w_1(x,t) = \phi_1(x)p_1(t) + \phi_2(x)p_2(t)$$
$$or \qquad (1a, b)$$
$$w_1(x,t) = \left(\frac{x}{\ell}\right)^2 p_1(t) + \left(\frac{x}{\ell}\right)^3 p_2(t)$$

for the transverse motion of substructure 1. Were p_i are time dependent coordinates. The terms before the time dependent coordinates p_i are the admissble functions ϕ_i. The transverse motion of substructure 2 relative to an inertial coordinate system is approximated by,

$$w_2(x,t) = 1p_3(t) + \left(\frac{x}{\ell}\right)p_4(t) + \left(\frac{x}{\ell}\right)^4 p_5(t) \qquad (2)$$

and the longitudinal motion as,

$$u_2(x,t) = 1p_6(t) \qquad (3)$$

The approximations of the motion for substructure 1 must satisfy a clamped-boundary condition constraint.

The approximations for both substructures are now used to calculate the uncoupled mass and stiffness matrices. The form of each elements are,

$$m_{ij} = \int_o^{\ell} m(x)\phi_i(x)\phi_j(x)dx \tag{4}$$

for the mass matrix M, where m represents the mass per unit length. For the stiffness matrix K,

$$k_{ij} = \int_0^{\ell} EI \frac{d^2\phi_i(x)}{dx^2}\frac{d^2\phi_j(x)}{dx^2}dx \tag{5}$$

where ϕ_i represent the i^{th} eigenfunction approximation, E is the modulus of elasticity and I the mass moment of inertia. The uncoupled matrices are,

$$M = m\ell\begin{bmatrix} .2000 & .1666 & 0 & 0 & 0 & 0 \\ .1666 & .1428 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0000 & .5000 & .2000 & 0 \\ 0 & 0 & .5000 & .3333 & .2000 & 0 \\ 0 & 0 & .2000 & .1666 & .1666 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.000 \end{bmatrix} \tag{6}$$

and

$$K = \frac{EI}{\ell^3}\begin{bmatrix} 4 & 6 & 0 & 0 & 0 & 0 \\ 6 & 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 28.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7}$$

These matrices which model the uncoupled substructures must coupled. This is accomplished by forming a coordinate transformation matrix. Unfortunately, the result of a coordinate transformation is the further loss of physical meaning of the coordinates. This transformation matrix is based on the substructure displacement and force conditions that exist between the substructures. The constraints resulting from the analysis of the cut between substrucutre 1 and 2 arranged in matrix form are,

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & -1 & -4 & 0 \\ 2 & 6 & 0 & 0 & 12 & 0 \end{bmatrix}\begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ p_4(t) \\ p_5(t) \\ p_6(t) \end{bmatrix} = 0 \tag{8}$$

an analysis of the constraints for the same system is carried out in chapter 9. This system of equations must be reformed into a transformation matrix. To determine system models final size (M and K are nxn) the number of time dependent coordinates *(N)* and holonomic constraints *(m)* are used in the equation below where n is the number of generalized coordinates,

$$n = N - m \tag{9}$$

In this case, $n=2$, so the transformation matrix converts the *6* time dependent coordinates of the uncoupled matrices into *2* generalized coordinates. If the systems had nonholonomic constraints equation (9) is not valid which limits its utility. The time dependent coordinates coordinates p_1 and p_6 can be arbitrarily chosen as the final generalized coordinates. Rewriting equation (8) using the size information as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 3 & 0 & -1 & -4 \\ 6 & 0 & 0 & 12 \end{bmatrix} \begin{bmatrix} p_2(t) \\ p_3(t) \\ p_4(t) \\ p_5(t) \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 0 \\ -2 & 0 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_6(t) \end{bmatrix} \tag{10}$$

is the next step in the analysis. Thomson's notation is used to write the above equation in abbreviated notation as,

$$[s]\{p_{2-5}\} = [Q]\{q_{1,6}\} \tag{11}$$

This is next written as,

$$\{p_{2-5}\} = [s]^{-1}[Q]\{q_{1,6}\} \tag{12}$$

which is used to write the transformation matrix after incorporating the identity for $p_1=q_1$ and $p_6=q_6$ into the matrix $s^{-1}Q$. For this example the transformation matrix is calculated using equation (12) as,

$$\begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ p_4(t) \\ p_5(t) \\ p_6(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 2 & 4.50 \\ -2.333 & -5.0 \\ .333 & .50 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_6(t) \end{bmatrix} \equiv [C] \begin{bmatrix} q_1(t) \\ q_6(t) \end{bmatrix} \tag{13}$$

The uncoupled system is written as,

$$M\ddot{p} + Kp = 0 \tag{14}$$

So that using equation (13) results in he transformed equations of motion written as,

$$C^T MC\left\{\frac{d^2q}{dt^2}\right\} + C^T MC\{q\} = 0 \tag{15}$$

These equations yield the eigenvalues,

$$\begin{aligned} \omega &= 1.172\sqrt{\frac{EI}{m\ell^4}} \\ \omega &= 3.198\sqrt{\frac{EI}{m\ell^4}} \end{aligned} \tag{16}$$

The mode shapes are approximated by equations (1-3). These functions do not satisfy the connection conditions between substructures hence, they are not global eigenfunctions. A weakness of the method. The eigenvalue calculation results are fairly accurate with the actual ones, differing only by 0.5% and 1.0% from exact solutions worked by Bishop (1956).

2.4 Proposed Substructure Formulation Summary

The new modeling method presented here is classified as a substructure synthesis technique. The distinction between the new SS method and the component mode synthesis formulation is that no transformation between local coordinates and generalized coordinates occurs in the new SS method. The advantage of this is a retention of physical insight into the model and the ability to form equations of motions directly with generalized coordinates. The new formulation differs from other substrucutre synsthsis formulation because it satisfies geometric, natural, displacement and force constraints between substructures into one mathematical process, instead of using both kinematic chains and boundary condition approximation methods. This has the advantage of reducing the complexity of the integrals that are required in the computation. The new formulation also results in global eigenfunction approximations and global generalized coordinates, which eventually satisfies the inclusion principle which means eigenvalue estimates converge from above their actual values. The analysis method also facilitates the examination of boundary conditions in a unique manner. The method is unique because constraints are explicitly examined and selectively satisfied. This allows the identification of extraneous constraints and provides guidance in the selection of admissible functions. The new SS formulation may be divided into two steps. The first step is to satisfy geometric boundary conditions of substructures with appropriate admissible functions. The second step is the modification of these admissible functions as detailed in chapter 9 to minimally satisfy geometric constraints imposed by the interaction of substructures. Natural constraints can also be satisfied to improve convergence to the exact eigenvalues.

3. A Review of Holonomic and Nonholonomic System Definitions

3.1 Introduction

The MAF-SS formulation consists of understanding boundary conditions and constraint connections, it is therefore appropriate to review constraint types. Systems can be divided into two types depending on their constraints: holonomic and nonholonomic. A constraint on a system is the result of a physical restriction on the motion of the system. A definition is presented of holonomic and nonholonomic systems. The term *generalized coordinates* in the forthcoming definitions refers to the minimum number of independent coordinates *(n)* required to indicate a system's state uniquely. This is equal to the dimension of the configuration space or space required to model the system. A dot over a variable indicates a time derivative.

3.2 Holonomic and Nonholonomic System Definition

Kinematic constraints can be functions of the *(n)* generalized coordinates, time derivatives, and time. A general form of *(m)* kinematic constraints can be written as,

$$f_i\left(\dot{q}_1,\dot{q}_2,\dot{q}_3,\ldots,\dot{q}_n,q_1,q_2,q_3,\ldots,q_n,t\right) \quad i=1,2,3,\ldots,m \tag{1}$$

Geometric constraints are functions of the *(n)* generalized coordinates and time. A general format for *(m)* geometric constraints is,

$$f_i\left(q_1,q_2,q_3,\ldots,q_n,t\right) \quad i=1,2,3,\ldots,m \tag{2}$$

Nonholonomic systems can be defined as systems with a kinematic constraint that can not be integrated with respect to any of their components or reduced to a geometric constraint (Brown, 1972). It is necessary but not sufficient that a system have a nonintegrable constraint to be nonholonomic (Ne ĭmark, 1972). Constraints that are inequalities result in nonholonomic systems (Meirovitch, 1967). Systems without geometric or kinematic constraints are holonomic. Systems with geometric constraints are holonomic (Ne ĭmark, 1972). Constraint definitions relate to the MAF-SS formulation because the constraint type determines how the Lagrange equations should be applied. The MAF-SS method is applicable to either class of system with the only change being the alteration of the Lagrange equations. The CMS is incapable of handing nonholonomic constraints without an alteration of the procedure, since the transformation matrix correct size can not be determined by the standard methods as demonstrated in the beam example in chapter 2.

Constraints that contain the time parameter *(t)* explicitly are termed rheonomic and those that do not scleronomic. Example 3.1 details a possible constraint of each type.

Example 3.1

A rheonomic constraint (holonomic),

$$x\cos(\omega t) + y\sin(\omega t) = 0$$

which might relate the constrain of a slider mechanisms.

A scleronomic constraint (holonomic),

$$x^2 + y^2 = L^2$$

which might correspond to a beam of length L pinned at one end under going rotation.

Relationship of Geometric and Kinematic Constraints

A geometric constraint implies a kinematic constraint. All geometric constraints can be written as kinematic constraints since some form of,

$$\mathrm{d}f(q_1, q_2, q_3, \ldots, q_n, t) \tag{3}$$

exists. This means kinematic constraints that are integrable are actually geometric constraints. This can be seen since,

$$f_i(q_1, q_2, q_3, \ldots, q_n, t) = \text{constant} \tag{4}$$

implies

$$\frac{\mathrm{d}f_i(q_1, q_2, q_3, \ldots, q_n, t)}{\mathrm{d}t} = 0 \tag{5}$$

The converse is not true, because a kinematic constraint is not a geometric constraint because the possibility exists that the constraint cannot be integrated.

Integrability

This is a test to remove velocity terms from the constraints through integration. Integration is performed with respect to time.

Reducibility

Reducibility refers to the process of transforming or combining constraints so that they may become integrable.

Holonomic and Nonholonomic Systems Properties

The dimensions of the configuration space *(n)* relative to the number of constraints can yield insight into the previous definition of nonholonomic and holonomic systems. Nonholonomic system require more generalized coordinates than the number of degrees of freedom or primitive coordinates to describe the system. In other words, the number of degrees of freedom is less than the number of generalized coordinates, by the number of nonholonomic constraints. Systems that are holonomic have the same number of generalized coordinates as degrees of freedom. Further more, the variations of holonomic systems coordinates are independent in contrast to variations of nonholonomic coordinates which are not (Brown, 1972 1976, and 1981).

Several coordinate types exist. The definition of a few of the types will be useful for the discussion of holonomic and nonholonomic systems. Primitive coordinates can be defined as the Cartesian true coordinates of the system as represented by *(N)* in dimension. True coordinates are any generalized geometric coordinate system algebraically related to an inertial reference. An ignorable coordinate occurs when the potential energy of a generalized coordinate is zero for all time.

The original discussion of coordinate properties can be written in a more mathematical manner. For a nonholonomic system,

$$n = N - m + l \tag{6}$$

where *(n)* is the number of generalized coordinates, *(l)* represents the number of nonholonomic constraints and *(m)* is the number of holonomic constraints. A holonomic system has a configuration space dimension or minimal number of degrees of freedom defined by,

$$n = N - m \tag{7}$$

as stated by for example by Brown (Brown, 1981).

3.3 Illustrative Definitions

Several examples of holonomic and nonholonomic systems have been harvested from various sources (Kane, 1968, Meirovitch, 1967, Neĭmark, 1972). These are presented and expanded upon to clarify the previous definitions. The first example purpose is to present cases of holonomic and nonholonomic systems so that the two can be visualized. The second example examines how constraints can be written in different forms, details a general constraints form (i.e. Pfaffian) and discusses methods to examine if a constraint can be integrated for a specific constraint type. The last two examples examine rare and subtle cases of holonomic systems.

Conceptual Discussion

The example of a ball or sphere with a fixed radius *(r)* moving in contact with a rough or smooth plane is usually conceptually used to convey the difference between nonholonomic and holonomic systems. The plane that the ball moves on is defined by the *x, y* Cartesian coordinates. The configuration of the ball is completely specified by

five (5) coordinates, the x, y coordinates of the ball center and three (3) Eulerian angles (ϕ, φ, ψ) describing the orientation of the ball about its center.

In the case of a ball or sphere that initially is at rest on a rough plane or surface that has high friction, a force (perhaps the wind acting on the ball) will cause it to roll. The ball cannot move forward without rotating, this constraint is nonholonomic. The friction creates a constraint requiring the point of contact between the ball and plane to have zero displacement. This in turn can be written as two constraints of the form

$$\begin{aligned} dx &= r\cos\varphi d\psi \\ dy &= r\sin\varphi d\psi \end{aligned} \tag{8}$$

were ϕ, ψ are angular rotations. Applying equation (6) yields,

$$n = 5 - 0 + 2 = 7$$

If the same ball is imagined to be floating in a perfectly calm ocean and the wind blows again disturbing only the ball. The ball can move freely in any one direction independent of the others. This is a holonomic system. This is further illustrated by the statement that since there are no constraints the system is holonomic. General arbitrary independent infinitesimal displacements are possible. Therefore equation (7) yields,

$$n = 5 - 0 - 0 = 5 \tag{10}$$

which is also the number of degrees of freedom.

Constraint Forms

A pendulum system is a useful example to illustrate the fact that systems can have holonomic constraints and the different forms constraints can be written. A mass *(M)* suspended with a wire of length *(L)* from a fixed boundary point has the geometric constraint

$$x^2 + y^2 = L^2 \tag{11}$$

This constraint can be written as a kinematic constraint

$$x\dot{x} + y\dot{y} = 0 \tag{12}$$

which can be integrated to yield the geometric constraint equation (11). A further variation using the transformation to polor coordinates where (θ) is the angle between the attachment plane and the wire can be written as

$$\begin{aligned} x - L\cos(\theta) &= 0 \\ y - L\sin(\theta) &= 0 \end{aligned} \tag{13 a, b}$$

This equation can also be written as a differential form

$$\begin{aligned} \mathrm{d}x - L\cos(\theta)\mathrm{d}\theta &= 0 \\ \mathrm{d}y - L\sin(\theta)\mathrm{d}\theta &= 0 \end{aligned} \tag{14a, b}$$

which also can be written as a kinematic boundary condition after dividing by dt

$$\begin{aligned} \dot{x} - L\dot{\theta}\cos(\theta) &= 0 \\ \dot{y} - L\dot{\theta}\sin(\theta) &= 0 \end{aligned} \tag{15a, b}$$

With so many forms available for the constraints to take, the task of determining if they are directly integrable or reducible and ultimately holonomic or nonholonomic can become difficult. The existence of a generalized constraint form and methods to determine if a function can be transformed (usually integrated) to form a geometric constraint are very useful. This is the role of the Pfaffian constraint form.

Pfaffian form

Many constraints may be written in a Pfaffian form. A general Pfaffian form is,

$$a_x dx + a_y dy + a_z dz + a_o dt = 0 \tag{16}$$

where the terms a_x, a_y, a_z, and a_o are functions depending on *x, y, z, t.* or the generalized coordinates (q_1, q_2, q_3, t) (Lanczos, 1977, Pars, 1965, Whittaker, 1988). The partial derivatives with respect to each variable exists through order 1 and are continuous or can be said to be C_1 functions. Special definitions are applied to function where a_O equals zero. If a_O equals 0 $(a_O=0)$ the function is defined as a catastatic constraint. The alternative when a_O is not equal to zero ($a_o \neq 0$) the function is termed acatastatic. An example of this difference is below.

Example 3.2

Three catastatic constraints,

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dx + \frac{\partial f}{\partial z} dz = 0$$

$$t dx + t dy + t dz = 0 \tag{17a, b, c}$$

$$2x dx = 0$$

Two acatastatic constraints,

$$\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz+\frac{\partial f}{\partial t}dt=0 \qquad (18a, b)$$

$$2tdt=0$$

The Pfaffian form can be used to determine the integrablness of a catastatic constraint. The condition that needs to be met if equation (16) is written as (Lanczos, 1977),

$$B_n dq_n + B_{n-1}dq_{n-1} + B_{n-2}dq_{n-2}+\ldots+B_1 dq_1 + B_0 t = 0 \qquad (19)$$

is,

$$\frac{\partial B_j}{\partial q_i}=\frac{\partial B_i}{\partial q_j} \quad i,j=1,2,3,\ldots n \quad i\neq j \qquad (20)$$

which indicates that the terms can or can not be integrated.

Example 3.3

For one (1) constraint between three (3) variables of the form,

$$dq_3 = B_1(q_1,q_2)dq_1 + B_2(q_1,q_2)dq_2 \qquad (21)$$

The equation that must be satisfied so that integration is possible and therefore the constraint holonomic is,

$$\frac{\partial B_1}{\partial q_2}=\frac{\partial B_2}{\partial q_1} \qquad (22)$$

The constraint,

$$\left(y^2 - x^2 - z\right)\mathrm{d}x + \left(z - y^2 - xy\right)\mathrm{d}y + x\mathrm{d}z = 0 \tag{23}$$

is of the type in equation (21). It can be rewritten as,

$$\mathrm{d}z = -1\left\{\left(y^2 - x^2 - z\right)\mathrm{d}x + \left(z - y^2 - xy\right)\mathrm{d}y\right\} \tag{24}$$

The equations to be examined are *(q_1=x, q_2=y, q_3=z),*

$$\frac{\partial B_1}{\partial q_2} = \frac{\partial B_1}{\partial q_1}\frac{\partial q_1}{\partial q_2} + \frac{\partial B_1}{\partial q_2} + \frac{\partial B_1}{\partial q_3}\frac{\partial q_3}{\partial q_1} \tag{25}$$

and

$$\frac{\partial B_2}{\partial q_1} = \frac{\partial B_2}{\partial q_1} + \frac{\partial B_2}{\partial q_2}\frac{\partial q_2}{\partial q_1} + \frac{\partial B_2}{\partial q_3}\frac{\partial q_3}{\partial q_1} \tag{26}$$

which after some manipiulation using equation (22) yields,

$$\frac{\partial B_1}{\partial q_2} + \frac{\partial B_1}{\partial q_3}B_2 = \frac{\partial B_2}{\partial q_1} + \frac{\partial B_2}{\partial q_3}B_1 \tag{27}$$

indicating that the constraint is holonomic.

Nonintegrable But Reducible And Holonomic

A system can possess a nonintegratable constraint but still be holonomic. Other constraints may combine with the nonintegrable constraint forming an integratable constraints (Ne ĭmark, 1972).

Example 3.4

$$\begin{aligned}\left(x^2+y^2\right)\mathrm{d}x+xz\mathrm{d}z&=0\\ \left(x^2+y^2\right)\mathrm{d}y+yz\mathrm{d}z&=0\end{aligned} \qquad (29a,b)$$

These can be transformed to holonomic constraints. Solving equation (29a) for ($\mathrm{d}z$) and substituting into equation (29b) yields,

$$\frac{1}{y}\mathrm{d}y-\frac{1}{x}\mathrm{d}x=0 \qquad (30)$$

which can be written as the holonomic constraint,

$$\mathrm{d}\ln\left(\frac{x}{y}\right)=0 \qquad (31)$$

The next transformation occurs by adding equations (29a) and (28b) and manipulating this equation. The result

$$\left(x^2+y^2\right)d\left(x^2+y^2+z^2\right)=0 \qquad (32)$$

is a holonomic constraint.

Nonintegrable and Holonomic

It is possible for a system to be nonintegrable and non reducible but still be holonomic. These systems are rare but the possibility needs to be pointed out and remembered (Brown, 1981, Neĭmark, 1972). In fact examples of these system seem to be extinct.

The theoretical statements concerning this situation are defined by Brown (1981). This case is included here for completeness of the discussion.

3.4 Summary

Dynamic systems are divided into two classes: holonomic and nonholonomic. Identification of these classes can be accomplished by casting the constraints into a standard form (Pfaffian). Examples of all the possible constraint cases are presented as a tool to class particular systems. The type of constraint effects the method or way the system should be modeled. This is particularly evident in the application of Lagrange equations. This issue is examined in chapter 5.

4. Lagranges Equations

4.1 Introduction

Lagrange equations are an essential element of the MAF-SS formulation. The basic assumptions made in the derivation of Lagrange equations also applies to the MAF-SS method. The derivation of the Lagrange equations is traced back to Newton's law of motion. The contents of Fig. 1 detail the evolution of the Lagrange equations. A development of the Lagrange equations following this evolution is contained in the following section (Meirovitch, 1967, Shames, 1985, Junkins, 1993). Understanding the limitations or assumptions related the MAF-SS formulation depends on understanding the evolution and underlying assumptions of Lagrange equations this is the reason for this review.

Newton' s law → D' Alembert' s principle → Hamilton' s principle → Euler – Lagrange equations → Lagrange equations

Figure 4.1. Development of Lagrange Equations.

4.2 Lagrange Equations Development Details

Newton prescribed that a particle must (relative to an inertial reference) obey

$$F = m\frac{d^2u}{dt^2} \tag{1}$$

where F represent all external forces acting on the particle and m , t , u, are the mass, time and position respectively. Later it was recognized that this law holds if the following two conditions are satisfied:

1. The speed of sound is not approached.
2. The particle size is above the atomic scale.

D' Alembert rewrote equation (1) as

$$F - m\frac{d^2u}{dt^2} = 0 \tag{2}$$

to form the D'Alembert principle stated as; the resultant and inertia forces are in equilibrium (Shames, 1985). Extending D'Alemberts principle to a system of n particles and applying the idea of a virtual displacement yields

$$\sum_{i=1}^{n}\left(F_i - \frac{d^2u_i}{dt^2}\right)\cdot\delta u_i = 0 \tag{3}$$

where δ is a small hypothetical displacement or equivalently

$$\sum_{i=1}^{n}\left(m_i\frac{d^2u_i}{dt^2} - F_i\right)\cdot\delta u_i = \sum_{i=1}^{n}m_i\frac{d^2u_i}{dt^2}\cdot\delta u_i - \sum_{i=1}^{n}F\,\delta u_i = 0 \tag{4}$$

The equation

$$\frac{d}{dt}\left(\begin{bmatrix}\dot{u}_1 & \cdots & \dot{u}_n\end{bmatrix}\begin{bmatrix}\delta u_1\\ \vdots\\ \delta u_n\end{bmatrix}\right) = \begin{bmatrix}\ddot{u}_1 & \cdots & \ddot{u}_n\end{bmatrix}\begin{bmatrix}\delta u_1\\ \vdots\\ \delta u_n\end{bmatrix} + \begin{bmatrix}\dot{u}_1 & \cdots & \dot{u}_n\end{bmatrix}\begin{bmatrix}\delta \dot{u}_1\\ \vdots\\ \delta \dot{u}_n\end{bmatrix} \tag{5}$$

written using a preferable vector notion for the dot product ($a \cdot b = a^T b$) is useful for developing Hamilton's principle. The alternative form

$$\begin{bmatrix} \dot{u}_1 & \cdots & \dot{u}_n \end{bmatrix} \begin{bmatrix} \delta\dot{u}_1 \\ \vdots \\ \delta\dot{u}_n \end{bmatrix} = \delta \left(\begin{bmatrix} \frac{1}{2}\dot{u}_1 & \cdots & \frac{1}{2}\dot{u}_n \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_n \end{bmatrix} \right) \tag{6}$$

by remembering δ follows the rules of elementary calculus and assuming δ and $\frac{d}{dt}$ are interchangeable is also advantageous. In this particular case remembering

$$\frac{d}{d\dot{u}} \left(\frac{1}{2} \dot{u}^2 \right) = \dot{u} \tag{7}$$

and

$$\dot{u} \left(\frac{d}{d\dot{u}} (\dot{u}) \right) = \dot{u} \tag{8}$$

yields an insight into forming equation (6). Equation (6) is arranged into

$$\ddot{u}_i \cdot \delta u_i = \frac{d}{dt} (\dot{u}_i \cdot \delta u_i) - \delta \left(\frac{1}{2} \dot{u}_i \cdot \dot{u}_i \right) \tag{9}$$

and multiplied by m_i and a summation performed leading to

$$\sum_{i=1}^{n} m_i \ddot{u}_i \cdot \delta u_i = \sum_{i=1}^{n} m_i \frac{d}{dt} (\dot{u}_i \cdot \delta u_i) - \delta \sum_{i=1}^{n} \left(\frac{1}{2} m_i \dot{u}_i \cdot \dot{u}_i \right) \tag{10}$$

which is used to rewrite equation (4). Equation (4) is now written

$$\delta T + \delta W = \sum_{i=1}^{n} m_i \frac{d}{dt}\left(\dot{u}_i \cdot \delta u_i\right) \tag{11}$$

after substituting equation (10) and defining

$$\delta W = \sum_{i=1}^{n} F_i \cdot \delta u_i \tag{12}$$

$$\delta T = \delta \sum_{i=1}^{n} \frac{1}{2} m_i \left(\dot{u}_i \cdot \dot{u}_i\right) \tag{13}$$

The final steps in forming Hamilton's principle are to consider the variation and an integration between two points in time of equation (11). The knowledge of n generalized coordinates or n independent coordinates that specify the inertial position of every mass element in an n dimensional space at an instant in time defines the instantaneous configuration. As the configuration moves through time it traces the true, Newtonian or dynamic path in the n dimensional space. If a different path resulting from δu_i is followed synchronously with the true path, the path is called the varied path. The rule that must be regarded for these varied paths is that at two instants of time the varied and true paths must intersect. So that

$$\delta u_i(t_1) = \delta u_i(t_2) = 0 \tag{14}$$

is satisfied. Applying the above concepts to equation (11) requires multiplication by dt and an integration between two time points such that

$$\int_{t_1}^{t_2}(\delta T+\delta W)dt=\sum_{i=1}^{n}\int_{t_1}^{t_2}m_i\frac{d}{dt}(\dot{u}_i\cdot\delta u_i)dt=\sum_{i=1}^{n}m_i\dot{u}_i\cdot\delta u_i\Big|_{t_1}^{t_2}=0 \tag{15}$$

results in the extended Hamilton's principle. The generalized Hamilton's principle

$$\int_{t_1}^{t_2}\delta(T-V)dt+\int_{t_1}^{t_2}\delta W_{nc}dt=0 \tag{16}$$

is written when the conservative forces are written as a potential energy term δV and the non conservative work as δW_{nc}. A restriction on this equation is that the kinetic energy is expressed only in generalized coordinates and the first derivatives of the generalized coordinates and the potential energy are expressed only in the generalized coordinates. The non conservative virtual work is a linear function of virtual generalized coordinate displacements (Junkins, 1993). These restrictions are stated where Q is a generalized force term

$$T=T(u_1,u_2,u_3,\ldots,u_n,\dot{u}_1,\dot{u}_2,\dot{u}_3,\ldots,\dot{u}_n,t) \tag{18}$$

$$V=V(u_1,u_2,u_3,\ldots u_n,t) \tag{19}$$

$$\delta W_{nc}=Q_1\delta u_1+Q_2\delta u_2+Q_3\delta u_3+\ldots+Q_n\delta u_n \tag{20}$$

Sometimes , L called the Lagrangian, is used in place of T-V.

If the actual path corresponds to the varied path equation (16) is stationary or the rate of change any direction vanishes at that point with respect to all other possible imagined or varied paths. The stationary value is a minimum (Meirovitch, 1967).

When the applicable forms of equation (18, 19, 20) are substituted into equation (16) and variations are taken using the chain rule and an integration by parts is performed the result is the Lagrange equations.

4.3 Summary

Lagrange equations are a fundamental part of the MAF-SS procedure. The equation are used to form the system model after kinetic and potential energy terms.

5. Generalized Holonomic and Nonholonomic Lagrange Equations

5.1 Introduction

Lagrange equation formulation is dependent on the type of constraints that exist in the system. Two classes of systems defined by constraint type have been examined in chapter 3, holonomic and nonholonomic. Lagrange equations for treating both types of systems are presented in this chapter.

5.2 Generalized Holonomic Lagrange Equations

The origins of Lagrange equations are traced to Newton's law and therefore must obey the same stipulations for validity. The advantage of applying Lagrange equations to a system is that the spatial (geometric and natural) boundary conditions are automatically formed during the analysis. Application of Lagrange equations to distributed parameter systems results in an infinite model dimension. Lagrange equations applied to lumped parameter models form a finite number of equations. This section concentrates on applying Hamilton's principle to holonomic elastic and rigid systems forming the generalized Lagrange equations of the coupled system. The hybrid equations contain partial and ordinary differential equations as a result of modeling both the elastic *(w)* and rigid *(q)* body motions.

Generalizing Lagrange equations requires at the outset some assumptions or simplifications. Notably, connections are assumed to occur at a point between joined members.

The Lagrangian of a system composed of both rigid and elastic members is divide into three components by some researchers (Junkins, 1993). This form

$$L = L_D + \int_\Omega \hat{L} d\Omega + L_B \tag{1}$$

contains a terms L_D related to the discrete or lumped elements of the system, $\hat{L}$ indicating a distributed term integrated over the domain Ω and L_B dependent on boundary motions. The nonconservative work variation term for a similar system

$$\delta W_{nc} = Q^T \delta q + \int_\Omega \hat{f}^T \delta w d\Omega + f_1 \delta w(l) + f_2{}^T \delta w'(l) \tag{2}$$

contains the term Q representing a nonconservative generalized force vector, f a nonconservative generalized force density vector, and f_1 , f_2 which are associated with boundary conditions. The term $\delta w(l)$ indicates the variation evaluated at l or the end of the elastic member

Although some boundary forces may do noconservative work there exits several forces that do not do work. The kino-static relations in Lagrange equations are nonexistent. This is a result of the following force types not considered to do work (Whittaker, 1988). These 4 types are extracted from Whittaker directly,

> *1. The molecular forces which act between the particles of rigid bodies of the rigid bodies in the system.*
>
> *2. The pressures of connecting-rods of invariable length, the reactions at fixed pivots, and the tensions of taut in extensible strings.*

3. The reactions of any fixed smooth surfaces or curves which bodies of the system are constrained to remain in contact; or of perfectly rough surfaces, so far as these can enter into holonomic systems

4. The reactions of any smooth surfaces or curves with which bodies of the system are constrained to remain in contact, when these surfaces or curves are forced to move in some prescribed manner; for the displacement considered above is made on the supposition that t , so far as it is required for the specification of the system, is not varied, i.e. that such surfaces or curves are not moved during the displacement; so that this case reduces to the preceding.

(Whittaker, 1988)

Lagrange Equations: Rigid Bodies and a Single Elastic Body

A further system simplification is made to present the development of the Lagrange equations in a manageable format. This simplification, only one distributed parameter body is contained in the system and its deformation state can be described with one coordinate still yields a useful set of equations.

The kinetic energy term

$$T = T_D(q,\dot{q}) + \int_{l_o}^{l} \hat{T}(q,\dot{q},w,w',w'')dx + T_B\left(w(l),\dot{w}(l),w'(l),\dot{w}(l),q,\dot{q}\right) \tag{3}$$

is in a similar format to equation (1) consisting of discrete, distributed and boundary terms and describes the system considered. The potential energy function written

$$V = V_D(q,\dot{q}) + \int_{l_o}^{l} \hat{V}(q,\dot{q},w,w',w'')dx + V_B\left(w(l),\dot{w}(l),w'(l),\dot{w}'(l),q,\dot{q}\right) \tag{4}$$

also matches this format. If

$$L_D = T_D - V_D \tag{5}$$

$$\hat{L} = \hat{T} - \hat{V} \tag{6}$$

$$L_B = T_B - V_B \tag{7}$$

are applied after substituting equations (3) and (4) to equation (1) the extended Hamilton's principle may be formed for the system. Integration by parts and application of the rules for treating variations yields the Lagrange equations (Junkins, 1993).

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q^T \tag{8}$$

$$\frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial \dot{w}}\right) - \frac{\partial \hat{L}}{\partial w} + \frac{\partial}{\partial x}\left(\frac{\partial \hat{L}}{\partial w'}\right) - \frac{\partial^2}{\partial x^2}\left(\frac{\partial \hat{L}^2}{\partial w''}\right) = \hat{f}^T \tag{9}$$

and boundary conditions

$$\left\{\frac{\partial \hat{L}}{\partial w'} - \frac{\partial}{\partial x}\left(\frac{\partial \hat{L}}{\partial w''}\right)\delta w\Big|_{l_o}^{l} + \left\{\frac{\partial L_B}{\partial w(l)} - \frac{d}{dt}\left(\frac{\partial L_B}{\partial \dot{w}(l)}\right)\right\}\delta w(l) + f_1{}^T \delta w(l) = 0\right\} \tag{10}$$

$$\frac{\partial \hat{L}}{\partial w''}\delta w'\Big|_{l_o}^{l} + \left\{\frac{\partial L_B}{\partial w'(l)} - \frac{d}{dt}\left(\frac{\partial L_B}{\partial \dot{w}'(l)}\right)\right\}\delta w'(l) + f_2{}^T \delta w'(l) = 0 \tag{11}$$

The variations associated with the rigid body coordinates forms one set of equations while the variations associated with the elastic terms forms the equations of the flexible system.

Lagrange Equations Rigid Bodies and Multiple Elastic Bodies

A more general result occurs when more than one elastic member is considered. The development is similar to the above except a summation is placed around the distributed parameter terms. The results are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=Q^{T} \tag{12}$$

$$\frac{d}{dt}\left(\frac{\partial \hat{L}^{i}}{\partial \dot{w}_{i}}\right)-\frac{\partial \hat{L}^{i}}{\partial w_{i}}+\frac{\partial}{\partial x_{i}}\left(\frac{\partial \hat{L}^{i}}{\partial w_{i}'}\right)-\frac{\partial^{2}}{\partial x_{i}^{2}}\left(\frac{\partial^{2} L^{i}}{\partial w_{i}''}\right)=\hat{f}^{i\,T} \tag{13}$$

with the boundary conditions

$$\left\{\frac{\partial \hat{L}^{i}}{\partial w_{i}'}-\frac{\partial}{\partial x_{i}}\left(\frac{\partial \hat{L}^{i}}{\partial w_{i}''}\right)\delta w_{i}\Big|_{l_{0_i}}^{l_i}+\left\{\frac{\partial L_{B}}{\partial w_{i}(l_{i})}-\frac{d}{dt}\left(\frac{\partial L_{B}}{\partial \dot{w}_{i}(l_{i})}\right)\right\}\delta w_{i}(l_{i})+f^{i}{}_{1}{}^{T}\delta w_{i}(l_{i})=0\right\} \tag{14}$$

$$\frac{\partial \hat{L}^{i}}{\partial w_{i}''}\delta w_{i}'\Big|_{l_{0_i}}^{l_i}+\left\{\frac{\partial L^{i}{}_{B}}{\partial w_{i}'(l_{i})}-\frac{d}{dt}\left(\frac{\partial L^{i}{}_{B}}{\partial \dot{w}_{i}'(i)}\right)\right\}\delta w_{i}'(l_{i})+f^{i}{}_{2}{}^{T}\delta w_{i}'(l_{i})=0 \tag{15}$$

where *(i=1,2,3,...,n)* for the system containing rigid bodies and *n* coupled elastic members (Junkins).

Review of Integration of Parts

Certain integrals may be evaluated using the relation

$$\int u dv = uv - \int v du \tag{16}$$

The terms u and v are determined by trial and error.

5.3 Nonholonomic Lagrange Equations

A brief development of nonholonomic equations is presented to illustrate the effect of constraints type on the system model. Lagrange equations for nonholonomic systems are developed for a specific constraint form (Junkins, 1993). Namely, m Pfaffian constraints of the form

$$A\dot{q} + a_o = 0 \tag{17}$$

or

$$\begin{bmatrix} a_{11}(q_1,\ldots,q_n,t) & \cdots & a_{1n}(q_1,\ldots,q_n,t) \\ \vdots & \ddots & \vdots \\ a_{m1}(q_1,\ldots,q_n,t) & & a_{mn}(q_1,\ldots,q_n,t) \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} + \begin{bmatrix} a_{o_1}(q_1,\ldots,q_n,t) \\ \vdots \\ a_{o_m}(q_1,\ldots,q_n,t) \end{bmatrix} = 0 \tag{18}$$

where A is m x n matrix, and a_o is a m x 1 vector function. The modeling technique is to modify the Lagrange equations

$$\begin{bmatrix} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) \\ \vdots \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) \end{bmatrix} - \begin{bmatrix} \frac{\partial L}{\partial q_1} \\ \vdots \\ \frac{\partial L}{\partial q_n} \end{bmatrix} = \begin{bmatrix} Q_1 \\ \vdots \\ Q_n \end{bmatrix} \tag{19}$$

by adding generalized constraint forces to the right-side. The result of this is

$$\begin{bmatrix} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) \\ \vdots \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) \end{bmatrix} - \begin{bmatrix} \frac{\partial L}{\partial q_1} \\ \vdots \\ \frac{\partial L}{\partial q_n} \end{bmatrix} = \begin{bmatrix} Q_1 \\ \vdots \\ Q_n \end{bmatrix} + A \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix} \tag{20}$$

One method for developing the additional generalized force is differentiation of equation (17) forming

$$A\ddot{q} + \dot{A}\dot{q} + \dot{a}_o = 0 \tag{21}$$

and solving it simultaneously with equation (19) for $\ddot{q}$ and λ, the generalized coordinate acceleration and Lagrange multipliers. The results are then applied to equation (20) forming the nonholonomic system equations.

5.4 Summary

This chapter is important to the MAF-SS procedure since it outlines the methods to apply Lagrange equations to holonomic and nonholonomic constraints. The MAF-SS

procedure can model both types of constraints using the methods described in this chapter. The CMS method fails to successfully model nonholonomic constraints.

6. Quasi-Coordinates and Ignorable Coordinates

6.1 Introduction

Coordinates may be defined and interpreted in different ways. Quasi-coordinates and ignorable coordinates are encountered when modeling dynamic systems. Since the MAF-SS method distinguishes itself from other modeling methods because of its use of generalized coordinates it is important to understand the use and implications of other coordinate types.

6.2 Quasi-Coordinates

Coordinates can be transformed into different forms but still represent the same information. The transformation of rectangular coordinates into polor coordinates is an example of a coordinate transformation. In this section a transformation will be developed for Lagrange's equations (Whittaker, 1988). The concept is presented using an analysis that applies to rigid bodies that do not have potential and dissipative energy terms. A more comprehensive development is attributed to Meirovitch (Meirovitch, 1995).

A new coordinate system called quasi-coordinates (ω) is written as linear independent combinations of the original or true coordinate velocities ($\dot{u}$). The quasi-coordinates are written for n true-coordinates as,

$$\omega = \alpha\dot{u} \tag{1}$$

where ω is $n \times 1$, the coefficient matrix α is $n \times n$, and $\dot{u}$ is $n \times 1$. The coefficient matrix α is composed of arbitrary functions of the true coordinates u. The true coordinates are written in terms of the quasi-coordinates as,

$$\dot{u} = \alpha^{-1}\omega \tag{2}$$

or

$$\dot{u} = \beta\omega \tag{3}$$

where β represents α^{-1}.

The goal is to write Lagrange's equations,

$$\begin{bmatrix} \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{u}_1}\right) - \dfrac{\partial T}{\partial u_1} \\ \vdots \\ \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{u}_n}\right) - \dfrac{\partial T}{\partial u_n} \end{bmatrix} = \begin{bmatrix} Q_1 \\ \vdots \\ Q_n \end{bmatrix} \tag{4}$$

where T is the kinetic energy and Q the generalized forces in quasi-coordinate form. The first step is the elimination of $\dot{u}$ from T. This is accomplished with equation (2). The kinetic energy written in quasi-coordinates is represented by $\overline{T}$. The assumption has been made that t is not explicitly contained in T. The derivatives of T in equation (4) also must be converted. The relationship

$$\begin{bmatrix} \frac{\partial T}{\partial \dot{u}_1} \\ \vdots \\ \frac{\partial T}{\partial \dot{u}_n} \end{bmatrix} = \alpha \begin{bmatrix} \frac{\partial \bar{T}}{\partial \omega_1} \\ \vdots \\ \frac{\partial \bar{T}}{\partial \omega_n} \end{bmatrix} \tag{5}$$

and the chain rule is useful to begin this process. After multiplying equation (4) by b and substituting equation (5) the Lagrange equations are written as

$$\beta\alpha \begin{bmatrix} \frac{d}{dt}\left(\frac{\partial \bar{T}}{\partial \omega_1}\right) \\ \vdots \\ \frac{d}{dt}\left(\frac{\partial \bar{T}}{\partial \omega_n}\right) \end{bmatrix} + \beta\dot{\alpha} \begin{bmatrix} \frac{\partial \bar{T}}{\partial \omega_1} \\ \vdots \\ \frac{\partial \bar{T}}{\partial \omega_n} \end{bmatrix} - \beta \begin{bmatrix} \frac{\partial T}{\partial u_1} \\ \vdots \\ \frac{\partial T}{\partial u_n} \end{bmatrix} = \beta \begin{bmatrix} Q_1 \\ \vdots \\ Q_n \end{bmatrix} \tag{6}$$

The above equation can be simplified by realizing

$$\beta\alpha = \alpha^{-1}\alpha = \underset{(n \times n)}{\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}} \tag{7}$$

The simplified version is

$$\begin{bmatrix} \frac{d}{dt}\left(\frac{\partial \bar{T}}{\partial \omega_1}\right) \\ \vdots \\ \frac{d}{dt}\left(\frac{\partial \bar{T}}{\partial \omega_n}\right) \end{bmatrix} + \beta\dot{\alpha} \begin{bmatrix} \frac{\partial \bar{T}}{\partial \omega_1} \\ \vdots \\ \frac{\partial \bar{T}}{\partial \omega_n} \end{bmatrix} - \beta \begin{bmatrix} \frac{\partial T}{\partial u_1} \\ \vdots \\ \frac{\partial T}{\partial u_n} \end{bmatrix} = \beta \begin{bmatrix} Q_1 \\ \vdots \\ Q_n \end{bmatrix} \tag{8}$$

which still contains a term in the original coordinates. This final vector must be transformed into quasi-coordinates. Examining the vector it is apparent that

$$\begin{bmatrix} \frac{\partial T}{\partial u_1} \\ \vdots \\ \frac{\partial T}{\partial u_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial \bar{T}}{\partial u_1} \\ \vdots \\ \frac{\partial \bar{T}}{\partial u_n} \end{bmatrix} + \begin{bmatrix} \frac{\partial \omega_1}{\partial u_1} & \cdots & \frac{\partial \omega_n}{\partial u_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \omega_1}{\partial u_n} & \cdots & \frac{\partial \omega_n}{\partial u_n} \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{T}}{\partial \omega_1} \\ \vdots \\ \frac{\partial \bar{T}}{\partial \omega_n} \end{bmatrix} \tag{9}$$

Remembering

$$[\omega] = \alpha \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} \tag{10}$$

equation (9) is written as

$$\begin{bmatrix} \frac{\partial \omega}{\partial u_1} & \cdots & \frac{\partial \omega}{\partial u_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \omega}{\partial u_1} & \cdots & \frac{\partial \omega}{\partial u_1} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \frac{\partial \alpha_{11}}{\partial u_1} & \cdots & \frac{\partial \alpha_{1n}}{\partial u_1} \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_n \end{bmatrix} & \cdots & \begin{bmatrix} \frac{\partial \alpha_{11}}{\partial u_n} & \cdots & \frac{\partial \alpha_{1n}}{\partial u_n} \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_n \end{bmatrix} \\ \vdots & \ddots & \vdots \\ \begin{bmatrix} \frac{\partial \alpha_{n1}}{\partial u_1} & \cdots & \frac{\partial \alpha_{nn}}{\partial u_1} \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_n \end{bmatrix} & \cdots & \begin{bmatrix} \frac{\partial \alpha_{n1}}{\partial u_n} & \cdots & \frac{\partial \alpha_{nn}}{\partial u_n} \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_n \end{bmatrix} \end{bmatrix} \tag{11}$$

Before substituting the above result a useful relationship can be formed by recognizing

$$\dot{\alpha} = \frac{\partial \alpha}{\partial t} = \begin{bmatrix} \frac{\partial \alpha_{11}}{\partial t} & \cdots & \frac{\partial \alpha_{1n}}{\partial t} \\ \vdots & \ddots & \vdots \\ \frac{\partial \alpha_{n1}}{\partial t} & \cdots & \frac{\partial \alpha_{nn}}{\partial t} \end{bmatrix} \tag{12}$$

and then applying the chain rule. The relationship is

$$\begin{bmatrix} \frac{\partial \alpha_{11}}{\partial t} & \cdots & \frac{\partial \alpha_{1n}}{\partial t} \\ \vdots & \ddots & \vdots \\ \frac{\partial \alpha_{n1}}{\partial t} & \cdots & \frac{\partial \alpha_{nn}}{\partial t} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \frac{\partial \alpha_{11}}{\partial u_1} & \cdots & \frac{\partial \alpha_{11}}{\partial u_n} \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_n \end{bmatrix} & \begin{bmatrix} \frac{\partial \alpha_{1n}}{\partial u_1} & \cdots & \frac{\partial \alpha_{1n}}{\partial u_n} \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_n \end{bmatrix} \\ \begin{bmatrix} \frac{\partial \alpha_{n1}}{\partial u_1} & \cdots & \frac{\partial \alpha_{n1}}{\partial u_n} \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_n \end{bmatrix} & \begin{bmatrix} \frac{\partial \alpha_{nn}}{\partial u_1} & \cdots & \frac{\partial \alpha_{nn}}{\partial u_n} \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_n \end{bmatrix} \end{bmatrix} \tag{13}$$

Substituting equations (11) and (13), equation (8) yields the final form

$$\begin{bmatrix} \frac{d}{dt}\left(\frac{\partial \bar{T}}{\partial \omega_1}\right) \\ \vdots \\ \frac{d}{dt}\left(\frac{\partial \bar{T}}{\partial \omega_n}\right) \end{bmatrix} + \beta \begin{bmatrix} \left\{\begin{bmatrix} \frac{\partial \alpha_{11}}{\partial u_1} & \cdots & \frac{\partial \alpha_{11}}{\partial u_n} \end{bmatrix} - \begin{bmatrix} \frac{\partial \alpha_{11}}{\partial u_1} & \cdots & \frac{\partial \alpha_{1n}}{\partial u_1} \end{bmatrix}\right\} \begin{bmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_n \end{bmatrix} & \left\{\begin{bmatrix} \frac{\partial \alpha_{1n}}{\partial u_1} & \cdots & \frac{\partial \alpha_{1n}}{\partial u_n} \end{bmatrix} - \begin{bmatrix} \frac{\partial \alpha_{n1}}{\partial u_1} & \cdots & \frac{\partial \alpha_{nn}}{\partial u_1} \end{bmatrix}\right\} \begin{bmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_n \end{bmatrix} \\ \left\{\begin{bmatrix} \frac{\partial \alpha_{n1}}{\partial u_1} & \cdots & \frac{\partial \alpha_{n1}}{\partial u_n} \end{bmatrix} - \begin{bmatrix} \frac{\partial \alpha_{11}}{\partial u_n} & \cdots & \frac{\partial \alpha_{1n}}{\partial u_n} \end{bmatrix}\right\} \begin{bmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_n \end{bmatrix} & \left\{\begin{bmatrix} \frac{\partial \alpha_{nn}}{\partial u_1} & \cdots & \frac{\partial \alpha_{nn}}{\partial u_n} \end{bmatrix} - \begin{bmatrix} \frac{\partial \alpha_{n1}}{\partial u_n} & \cdots & \frac{\partial \alpha_{nn}}{\partial u_n} \end{bmatrix}\right\} \begin{bmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_n \end{bmatrix} \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{T}}{\partial \omega_1} \\ \vdots \\ \frac{\partial \bar{T}}{\partial \omega_n} \end{bmatrix} - \beta \begin{bmatrix} \frac{\partial \bar{T}}{\partial u_1} \\ \vdots \\ \frac{\partial \bar{T}}{\partial u_n} \end{bmatrix} = \beta Q \tag{14}$$

which is the Lagrange equations written in terms of quasi-coordinates. Alternatively,

$$\frac{d}{dt}\left(\frac{\partial \bar{T}}{\partial \omega_r}\right) + \sum_k \sum_s \sum_m \beta_{kr} \frac{\partial \bar{T}}{\partial \omega_s} \dot{u}_m \left(\frac{\partial \alpha_{ks}}{\partial u_m} - \frac{\partial \alpha_{ms}}{\partial u_k}\right) - \sum_k \beta_{kr} \frac{\partial \bar{T}}{\partial u_k} = \sum_k \beta_{kr} Q_k \tag{15}$$

$r, s, m = 1, 2, 3, \ldots, n$

represents the same equations.

When Are Quasi-Coordinates Equal To True Coordinates

Quasi-coordinates are true coordinates if

$$\gamma = \begin{bmatrix} \left\{ \begin{bmatrix} \frac{\partial \alpha_{11}}{\partial u_1} & \cdots & \frac{\partial \alpha_{11}}{\partial u_n} \end{bmatrix} - \begin{bmatrix} \frac{\partial \alpha_{11}}{\partial u_1} & \cdots & \frac{\partial \alpha_{1n}}{\partial u_1} \end{bmatrix} \right\} \begin{bmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_n \end{bmatrix} & \left\{ \begin{bmatrix} \frac{\partial \alpha_{1n}}{\partial u_1} & \cdots & \frac{\partial \alpha_{1n}}{\partial u_n} \end{bmatrix} - \begin{bmatrix} \frac{\partial \alpha_{n1}}{\partial u_1} & \cdots & \frac{\partial \alpha_{nn}}{\partial u_1} \end{bmatrix} \right\} \begin{bmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_n \end{bmatrix} \\ \left\{ \begin{bmatrix} \frac{\partial \alpha_{n1}}{\partial u_1} & \cdots & \frac{\partial \alpha_{n1}}{\partial u_n} \end{bmatrix} - \begin{bmatrix} \frac{\partial \alpha_{11}}{\partial u_n} & \cdots & \frac{\partial \alpha_{1n}}{\partial u_n} \end{bmatrix} \right\} \begin{bmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_n \end{bmatrix} & \left\{ \begin{bmatrix} \frac{\partial \alpha_{nn}}{\partial u_1} & \cdots & \frac{\partial \alpha_{nn}}{\partial u_n} \end{bmatrix} - \begin{bmatrix} \frac{\partial \alpha_{n1}}{\partial u_n} & \cdots & \frac{\partial \alpha_{nn}}{\partial u_n} \end{bmatrix} \right\} \begin{bmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_n \end{bmatrix} \end{bmatrix} \tag{16}$$

is equal to zero. This result implies the differential of equation (1) is integrable. The equations of motion reduce to the familiar form of

$$\begin{bmatrix} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\pi}_1}\right) - \frac{\partial T}{\partial \pi_1} \\ \vdots \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\pi}_n}\right) - \frac{\partial T}{\partial \pi_n} \end{bmatrix} = \begin{bmatrix} \Pi_1 \\ \vdots \\ \Pi_n \end{bmatrix} \tag{17}$$

when γ is equal to zero. The term π and Π are the coordinate values and generalized forces respectively.

The Practical Use Of Quasi-Coordinates

Quasi-coordinates are used to model rigid body and flexible systems. Their main purpose is to attempt to simply the modeling process of complex interacting bodies. Quasi-coordinates have the ability to uniquely express a particular motion of interest in a kinematic system.

6.3 Ignorable Coordinates

True and quasi coordinates (ex. $u_1, u_2, u_3, \ldots, u_n$) that are not explicitly contained in L the Lagrangian, but appear as the corresponding velocities (ex. $\dot{u}_1, \dot{u}_2, \dot{u}_3, \ldots \dot{u}_n$) are called ignorable coordinates or cyclic. The n Lagrange equations containing ignorable coordinates can be reduced by the number of ignorable coordinates. This process will be developed in the next section (Whittaker, 1988).

Another interpretation is to define ignorable coordinates as those coordinates that correspond to a potential energy term (V) equal to zero (Brown, 1981, Brown, 1976, Brown, 1972).

The Lagrange equations of motion for k ignorable coordinates is written,

$$\begin{bmatrix} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) \\ \vdots \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \end{bmatrix} = 0 \tag{19}$$

It is apparent after examining equation (19) that integrating yields

$$\begin{bmatrix} \frac{\partial L}{\partial q_1} \\ \vdots \\ \frac{\partial L}{\partial q_k} \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_k \end{bmatrix} \tag{20}$$

This relationship will be used later in the coordinate reduction process.

Begin the reduction process by defining a function

$$R = L - \left\{ \begin{bmatrix} \frac{\partial L}{\partial \dot{q}_1} & \cdots & \frac{\partial L}{\partial \dot{q}_n} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_k \end{bmatrix} \right\} \tag{21}$$

and a vector as

$$\begin{bmatrix} \frac{\partial L}{\partial \dot{q}_1} \\ \vdots \\ \frac{\partial L}{\partial \dot{q}_k} \end{bmatrix}^T = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}^T \tag{22}$$

An arbitrary infinitesimal change in R is written

$$\delta R = \delta \left(L - \begin{bmatrix} \frac{\partial L}{\partial \dot{q}_1} & \cdots & \frac{\partial L}{\partial \dot{q}_k} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_k \end{bmatrix} \right) = \delta L - \delta \left(\begin{bmatrix} \frac{\partial L}{\partial \dot{q}_1} & \cdots & \frac{\partial L}{\partial \dot{q}_k} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_k \end{bmatrix} \right) \tag{23}$$

The components of the above equation are rewritten using the chain rule as

$$\delta L = \begin{bmatrix} \delta q_{k+1} & \cdots & \delta q_n \end{bmatrix} \begin{bmatrix} \frac{\partial L}{\partial q_{k+1}} \\ \vdots \\ \frac{\partial L}{\partial q_n} \end{bmatrix} + \begin{bmatrix} \delta \dot{q}_1 & \cdots & \delta \dot{q}_k \end{bmatrix} \begin{bmatrix} \frac{\partial L}{\partial \dot{q}_1} \\ \vdots \\ \frac{\partial L}{\partial \dot{q}_k} \end{bmatrix} + \begin{bmatrix} \delta \dot{q}_{k+1} & \cdots & \delta \dot{q}_n \end{bmatrix} \begin{bmatrix} \frac{\partial L}{\partial \dot{q}_{k+1}} \\ \vdots \\ \frac{\partial L}{\partial \dot{q}_n} \end{bmatrix} \tag{24}$$

and

$$\delta\left(\begin{bmatrix}\frac{\partial L}{\partial \dot{q}_1} & \cdots & \frac{\partial L}{\partial \dot{q}_k}\end{bmatrix}\begin{bmatrix}\dot{q}_1 \\ \vdots \\ \dot{q}_k\end{bmatrix}\right)=\begin{bmatrix}\delta\dot{q}_1 & \cdots & \delta\dot{q}_k\end{bmatrix}\begin{bmatrix}\frac{\partial L}{\partial \dot{q}_1} \\ \vdots \\ \frac{\partial L}{\partial \dot{q}_k}\end{bmatrix}+\begin{bmatrix}\delta\beta_1 & \cdots & \delta\beta_k\end{bmatrix}\begin{bmatrix}\dot{q}_1 \\ \vdots \\ \dot{q}_k\end{bmatrix} \tag{25}$$

when equation (20) is remembered. Equations (24) and (25) substituted into equation (23) result in

$$\delta R=\begin{bmatrix}\delta q_{k+1} & \cdots & \delta q_n\end{bmatrix}\begin{bmatrix}\frac{\partial L}{\partial q_{k+1}} \\ \vdots \\ \frac{\partial L}{\partial q_n}\end{bmatrix}+\begin{bmatrix}\delta\dot{q}_{k+1} & \cdots & \delta\dot{q}_n\end{bmatrix}\begin{bmatrix}\frac{\partial L}{\partial \dot{q}_{k+1}} \\ \vdots \\ \frac{\partial L}{\partial \dot{q}_n}\end{bmatrix}-\begin{bmatrix}\delta\beta_1 & \cdots & \delta\beta_k\end{bmatrix}\begin{bmatrix}\dot{q}_1 \\ \vdots \\ \dot{q}_k\end{bmatrix} \tag{26}$$

Now, since the infinitesimal quantities are arbitrary and independent the above equation is written as a system of equations. These equations are

$$\begin{bmatrix}\frac{\partial L}{\partial \dot{q}_{k+1}} \\ \vdots \\ \frac{\partial L}{\partial q_n}\end{bmatrix}=\begin{bmatrix}\frac{\partial R}{\partial \dot{q}_{k+1}} \\ \vdots \\ \frac{\partial R}{\partial \dot{q}_n}\end{bmatrix} \tag{27}$$

$$\begin{bmatrix}\frac{\partial L}{\partial q_{k+1}} \\ \vdots \\ \frac{\partial L}{\partial q_n}\end{bmatrix}=\begin{bmatrix}\frac{\partial R}{\partial q_{k+1}} \\ \vdots \\ \frac{\partial R}{\partial q_n}\end{bmatrix} \tag{28}$$

$$\begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_k \end{bmatrix} = -\begin{bmatrix} \dfrac{\partial R}{\partial \beta_1} \\ \vdots \\ \dfrac{\partial R}{\partial \beta_k} \end{bmatrix} \tag{29}$$

Lagranges equations and the above equations can be used after substitution to write the n equations of motion using a reduced number of equations. The Lagrange equations for k ignorable coordinates for a system of n coordinates is

$$\begin{bmatrix} \dfrac{d}{dt}\left(\dfrac{\partial R}{\partial \dot{q}_{k+1}}\right) \\ \vdots \\ \dfrac{d}{dt}\left(\dfrac{\partial R}{\partial \dot{q}_n}\right) \end{bmatrix} - \begin{bmatrix} \dfrac{\partial R}{\partial q_{k+1}} \\ \vdots \\ \dfrac{\partial R}{\partial q_n} \end{bmatrix} = 0 \tag{30}$$

The equation

$$\begin{bmatrix} q_1 \\ \vdots \\ q_k \end{bmatrix} = -\begin{bmatrix} \int \dfrac{\partial R}{\partial \beta_1} dt \\ \vdots \\ \int \dfrac{\partial R}{\partial \beta_k} dt \end{bmatrix} \tag{31}$$

is used to obtained the ignored coordinate equations.

6.4 Summary

Two type of coordinates were examined, quasi-coordinates and ignorable coordinates. Dynamic system models potentially may incorporate both types of coordinates. Usually

quasi-coordinates are associated with rigid body motions and ignorable coordinates relate to potential energy terms equal to zero. The use of both types of coordinates may effect the model size. Quasi-coordinates usually expand the model size and ignorable coordinates reduce the size of the model. Both coordinates types are used to simplify the modeling process.

7. Discretization of the Lagrange Equations

7.1 Introduction

The MAF-SS formulation is based on distributed parameter modeling methods. The distributed parameter model is discretized using approximations and generalized coordinates. Discretization is a well established principle in the literature. A review of the methodology is conducted here.

7.2 Discretization

Lagrange equations may in some instances result in a nonlinear-hybrid system of equations written in quasi and generalized coordinates having an infinite solution form. By any standards this is a complicated system and difficult to solve. Discretization is a method to simply the solution process. Background is presented that yields an approximate solution theory to complicated system types.

The elastic motions of the distributed parameter elements are modeled using a finite series of functions ϕ_i^j that approximate the natural mode shapes of the elements. The functions must at least satisfy the definition of admissible functions. The functions consist of a spatial and time dependent terms of the form

$$\begin{bmatrix} u_1(u_2,t) \\ \vdots \\ u_n(u_2,t) \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \phi_1^1(u_2) & \cdots & \phi_m^1(u_2) \end{bmatrix} \begin{bmatrix} q_1^1(t) \\ \vdots \\ q_m^1(t) \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \phi_1^n(u_2) & \cdots & \phi_m^n(u_2) \end{bmatrix} \begin{bmatrix} q_1^n(t) \\ \vdots \\ q_m^n(t) \end{bmatrix} \end{bmatrix} \tag{1}$$

for the case of elastic motions *(u)* dependent on one spatial variable u_2 of n structures, approximated by m assumed mode shapes and m generalized time dependent coordinates q. The elastic motions are replaced by the summations or assumed modes in the energy terms of the Lagrange equations. The result is a system with a finite solution size and equations with only unknown generalized coordinates dependent on time, a much more solvable system.

The assumed modes method is based on the Rayleigh quotient. Rayleigh's quotient enables the approximation of the solution to a set of differential equations. It forms the theoretical foundation of approximating the solution through finding the stationary values of a function. The next sections will review the Rayleigh's quotient and then proceed to a more detailed discussion of the assumed mode method.

Rayleigh's Quotient

The Rayleigh quotient R defined

$$\lambda_{min} \leq R \leq \lambda_{max} \tag{2}$$

is the cornerstone of approximation method presented. It can be used to approximate the eigenvalues of a system. The eigenvalues are located by the fact that Rayleigh's quotient has a stationary value near an eigenfunction. It is sometimes used to estimate the eigenvalues of a system (Meirovitch, 1991, Meirovitch, 1967).

Discrete Rayleigh Quotient

The Rayleigh's quotient directly relates to the familiar discrete equation form

$$M\ddot{r} + Kr = 0 \tag{3}$$

of coordinates r where M is a mass matrix and K a stiffness matrix that are both assumed positive definite. Equation (3) can be reformulated by substituting $r=qf(t)$ and the solution form $f(t)=me^{\mu t}$ obtaining (Inman, 1989)

$$\left(M\mu^2 + K\right)qe^{\mu t} = 0 \tag{4}$$

Now, eliminating $e^{\mu t}$ yields,

$$\left(M\mu^2 + K\right)q = 0 \tag{5}$$

The previous equation is rearranged into

$$Kq = -\mu^2 Mq \tag{6}$$

The result of premultipling both sides of the equation by q^T, dividing by q^TMq and substituting $R(q) = -\mu^2$ is

$$R(q) = \frac{q^T Kq}{q^T Mq} \tag{7}$$

the Rayleigh quotient (Meirovitch, 1991). Posing the question, what happens to the Rayleigh quotient as q becomes an eigenvector of the system leads to establishing the benefits and characteristics of the Rayleigh quotient. It is shown that Rayleigh's quotient has a stationary value when the vector q is close to an eigenvector q^v. An arbitrary motion is written using the superposition rule and the eigenvectors as

$$q = q^v C \tag{8}$$

where C is a matrix of coefficients forming the eigenvectors into a specific displacement. The quantities

$$q^{vT} Mq^v = I \tag{9}$$

and

$$q^{vT} Kq^v = \lambda \tag{10}$$

are defined where λ are the eigenvalues. Substituting into equation (7) leaves

$$R(u) = \frac{C^T q^{vT} Kq^v C}{C^T q^{vT} Mq^v C} = \frac{C^T \lambda C}{C^T IC} = \lambda \tag{11}$$

The above equation indicates that if the q approximates q^v then the Rayleigh quotient approximates the eigenvalues. In addition equation (11) indicates that the eigenvalues bound the maximum and minimum values the Rayleigh quotient may take (Meirovitch, 1991).

If an eigenvector is approximated by the estimate q, an eigenvalue and some small error terms will equal the Rayleigh quotient.

The fact that Rayleigh's quotient also falls between the highest and lowest eigenvalue can be seen using the logic that errors in the eigenvector estimate can only move the quotient towards the next eigenvalue which is either above or below the maximum and minimum eigenvalues.

A stationary value of the quotient yields an eigenvalue because as the error is decreased in any direction the $R(u)$ gets closer to λ.

Distributed Rayleigh Quotient

For distributed parameter systems the development is similar equation (3) for the distributed system is written generally as

$$LW(q) = \lambda m(q) W(q) \tag{12}$$

where L is a linear self-adjoint differential operator of order $2p$ (Meirovitch, 1967). This means L is a linear operator mapping the elements which satisfy the conditions of a inner product space $\Re$ into itself and (Ortega, 1987)

$$(LW_1, q_2) = (W_1, LW_2) \tag{13}$$

for all

$$W_1, W_2 \in \Re \tag{14}$$

The other terms are defined as, W displacement of a specific point q along the length ℓ of the structure, $m(q)$ is the mass density, and λ is termed the eigenvalues. The term p refers to the highest order derivative in equation (12). The i boundary conditions are represented by

$$B_i W = 0 \tag{15}$$

where B_i are of maximum order $2p\text{-}1$ and are linear homogenous differential operators. The solution of the set (W, λ) formed by equation (5) and (8) is infinite. The eigenfunctions are orthonormal ($q_1 * q_2 = 0$ & $q_1 * q_1 = q_2 * q_2 = 1$ where $(*)$ indicates a conjugate transpose) by definition and can be normalized to satisfy

$$\int_0^{\ell} m W_r W_s dx = \delta_{rs} \tag{16}$$

and

$$\int_0^l W_r L W_s dx = \lambda_r \delta_{rs} \tag{17}$$

where

$$\delta = \begin{cases} 0 & r \neq s \\ 1 & r = s \end{cases} \tag{18}$$

is the Kronecker delta. Any arbitrary motion can be represented by a infinite linear combination of the eigenfuctions W^V. This is written

$$w = \sum_{i=1}^{\infty} c_i W_i^V \tag{19}$$

where it is seen after multiplying equation (19) by

$$\int_o^l m W_i^V dq \tag{20}$$

and recalling equations (16 and 17) as

$$c_r = \int_0^l m w W_r^V dq \tag{21}$$

The Raleigh's quotient is now written after multiplying by $W^V(x)$ and integrating equation (12) as

$$R(W^V) = \lambda = \frac{\int_0^l W^V L W dq}{\int_0^l m W^{V^2} dq} \tag{22}$$

A proof of for distributed system closely resembles the discrete development (Meirovitch, 1967).

7.3 Summary

Discretization of distributed parameter systems is discussed in this chapter. The MAF-SS relies on this theory to discretize distributed substructures. Discretization is achieved using eigenfunction approximations and generalized coordinates. The method of finding appropriate eigenfunction approximations set the MAF-SS method apart from other methods. The MAF-SS follows a specific methodical procedure leading to eigenfunction approximations that satisfy geometric and natural constraints.

8. Assumed-Mode Method

8.1 Introduction

The MAF-SS is very closely associated with the assumed modes method. They share all of the same convergence properties. The most important difference is that the assumed modes method is applicable to only a single substructure or structure.

The assumed-modes method approximates the solution of a set of differential equations. This is accomplished by choosing a finite series of functions and time dependent generalized coordinates to represent the motion of the elastic members. These series are substituted into the energy terms which are in turn used to form the Lagrange equations. The result is a set of equations with unknown time dependent generalized coordinates.

The assumed modes method is closely related to the Rayleigh-Ritz method. This will be briefly explored along with presentation of the issue of completeness of energy in the next sections.

8.2 The Assumed-Modes Method and the Rayleigh-Ritz Method

The assumed-modes method and the Rayleigh-Ritz method differ mostly in that the coefficients of the Rayleigh-Ritz method are of the form

$$\begin{bmatrix} u_1(u_2,t) \\ \vdots \\ u_n(u_2,t) \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \phi_1^1(u_2) & \cdots & \phi_m^1(u_2) \end{bmatrix} \begin{bmatrix} a_1^1 \\ \vdots \\ a_m^1 \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \phi_1^n(u_2) & \cdots & \phi_m^n(u_2) \end{bmatrix} \begin{bmatrix} a_1^n \\ \vdots \\ a_m^n \end{bmatrix} \end{bmatrix} \tag{1}$$

for the case of elastic motions *(u)* dependent on one spatial variable u_2 of n structures, approximated by m functions and m coefficients. The elastic motions are approximated by a set of functions ϕ and constant coefficients a. These functions are sometimes found by solving a simplified version of the actual system studied. The functions are then substituted into the Rayleigh quotient forming the algebraic Galerkin equations or Ritz system. The eigenvalues and coefficients are then solved. This last step is repeated until increasing the size of the series does not appreciably change the result (Meirovitch, 1980).

8.3 Basis Functions and Energy Issues

There exist three types of functions in a domain D commonly used to estimate elastic motions of a particular system of equations. The equation system considered here is

$$L_{2p}[w] = \lambda L_{2q}[w] \tag{2}$$

satisfying the boundary conditions

$$L_{2p-1}[w] = 0 \tag{3}$$

where L_{2p}, L_{2q} and L_{2p-1} are linear homogenous operators of order *2p*, *2q* and *2p-1* such that $p>q$. These functions are listed below (Meirovitch, 1967).

Admissible

Admissible functions satisfy all geometric boundary conditions and are p differentiable over D.

Comparison

Comparison functions satisfy all geometric and kinematic boundary conditions and are *2p* differentiable over D.

Eigenfunctions

Eigenfunctions satisfy all geometric and kinematic boundary conditions and equation (3).

8.4 Completeness of Energy

The above functions are all potentially complete in energy. Completeness of energy is defined using an energy norm (Meirovitch, 1980). The energy norm is defined

$$|u| = [u,u]^{\frac{1}{2}} = \left(\int_0^l \sum_{k=0}^{p} a_k \left(\frac{d^k u}{dq_2{}^k} \right)^2 dx + \sum_{j=0}^{p-1} b_j \left(\frac{d^j u}{dq_2{}^j} \right)^2 \Bigg|_0^l \right)^{\frac{1}{2}} \tag{4}$$

The terms a_k and b_j are coefficient terms, the term ℓ indicates the elastic member length, remaining terms are defined in the first section. A function is said to be complete in energy if

$$\left| u - \begin{bmatrix} \phi_1 & \cdots & \phi_m \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_m \end{bmatrix} \right|^2 < \varepsilon \tag{5}$$

where ε is a small number and converges in energy if

$$\lim_{m \to \infty} \left| u - \begin{bmatrix} \phi_1 & \cdots & \phi_m \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_m \end{bmatrix} \right|^2 = 0 \tag{6}$$

Functions that are complete in energy satisfy the equations under consideration and satisfy the inclusion principle that the eigenvalue estimates are bounded by the actual minimal and maximum eigenvalues. This concept associated with the Rayleigh-Ritz has been explored in chapter 7. This was demonstrated by Meirovitch for admissible functions (Meirovitch,1977).

8.5 Assumed-Modes Model Example

An example using a Euler-Bernoulli beam is presented to demonstrate the assumed-modes method.

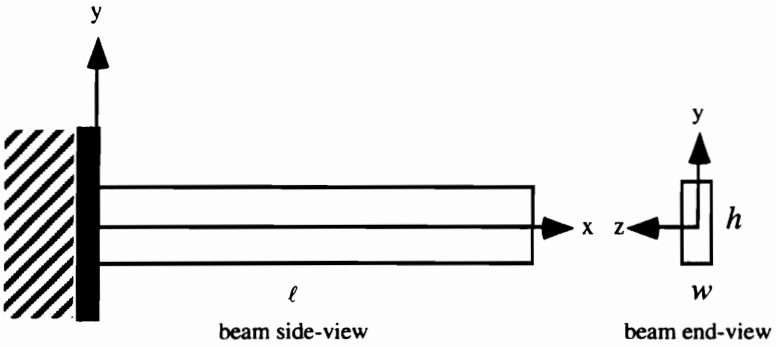


Figure 8.1. Beam System Model

The kinetic and potential energy terms for the system are

$$T = \frac{1}{2}\int_0^{\ell} \rho(x)\left(\frac{\partial y(x,t)}{\partial t}\right)^2 dx \tag{7}$$

and

$$V = \frac{1}{2}\int_0^{\ell} EI(x)\left(\frac{\partial^2 y(x,t)}{\partial x^2}\right)^2 dx \tag{8}$$

There are no non-conservative work terms considered or dissapative energy terms modeled. The next step is to replace the elastic motions *y(x,t)* with assumed functions composed of an admissible function and generalized time coordinates.

$$u_1 = \begin{bmatrix} x^2 & \cdots & x^{j+1} \end{bmatrix} \begin{bmatrix} q_1(t) \\ \vdots \\ q_j(t) \end{bmatrix} \tag{9}$$

The admissible functions satisfy the geometric boundary condition

$$y(0,t) = 0 \tag{10}$$

and

$$\left.\frac{\partial y(x,t)}{\partial x(x,t)}\right|_{x=0} = 0 \tag{11}$$

but don't satisfy the natural boundary conditions of

$$\left.\frac{\partial^2 y(x,t)}{\partial x^2(x,t)}\right|_{x=l} = \left.\frac{\partial^3 y(x,t)}{\partial x^3(x,t)}\right|_{x=l} = 0 \tag{12}$$

which is as an admissible functions are expected to behave. The boundary conditions and generalized Lagrange equations for the type of system considered have been previously defined by Junkins (1992). The Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial q_j}\right) - \frac{\partial \hat{L}}{\partial q_1} + \frac{\partial}{\partial q_2}\left(\frac{\partial \hat{L}}{\partial q_1'}\right) - \frac{\partial^2}{\partial q_2}\left(\frac{\partial \hat{L}}{\partial q_1''}\right) = 0 \tag{13}$$

The Lagrangian is

$$\hat{L} = \hat{T} - \hat{V} = \frac{1}{2}\rho\int_0^{\ell}\left(\begin{bmatrix} \phi_1 & \cdots & \phi_n \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} \begin{bmatrix} \phi_1 & \cdots & \phi_n \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} \right) dx - \frac{1}{2} EI\int_0^{\ell}\left(\begin{bmatrix} \phi_1'' & \cdots & \phi_n'' \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \begin{bmatrix} \phi_1'' & \cdots & \phi_n'' \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right) dx \tag{14}$$

where ρ and EI are assumed constant. The development will designate $j=2$ so that the method will not be lost in mathematical manipulations. Therefore, the Lagragian is written

$$\hat{L} = \frac{1}{2}\left\{\rho\int_0^{\ell}\left(\phi_1^2\dot{q}_1^2 + 2\phi_1\phi_2\dot{q}_1\dot{q}_2 + \phi_2^2\dot{q}_2^2\right)dx - EI\int_0^{\ell}\left(\phi_1''^2 q_1^2 + 2\phi_1''\phi_2'' q_1 q_2 + \phi_2''^2 q_2^2\right)dx\right\} \tag{15}$$

Substituting the admissible functions

$$\begin{aligned} \phi_1^2 &= \left(x^2\right)^2 = x^4 \\ \phi_2^2 &= \left(x^3\right)^2 = x^6 \\ \phi_1''^2 &= (2)^2 = 4 \\ \phi_2''^2 &= (3x)^2 = 9x^2 \\ \phi_1\phi_2 &= \left(x^2\right)\left(x^3\right) = x^6 \\ \phi_1''\phi_2'' &= (2)(3x) = 6x \end{aligned} \tag{16}$$

results in the Lagrangian

$$\hat{L} = \frac{1}{2}\left\{\rho\left(\frac{x^5}{5}\Big|_0^{\ell}\dot{q}_1^2 + \frac{x^6}{3}\Big|_0^{\ell}\dot{q}_1\dot{q}_2 + \frac{x^7}{7}\Big|_0^{\ell}\dot{q}_2^2\right) - EI\left(4x\Big|_0^{\ell}q_1^2 + 3x^3\Big|_0^{\ell}q_1 q_2 + 3x^3\Big|_0^{\ell}q_2^2\right)\right\} \tag{17}$$

which simplifies to

$$\hat{L} = \frac{1}{2}\left\{\rho\left(\frac{\ell^5}{5}\dot{q}_1^2 + \frac{\ell^6}{3}\dot{q}_1\dot{q}_2 + \frac{\ell^7}{7}\dot{q}_2^2\right) - EI\left(4\ell q_1^2 + 3\ell^2 q_1 q_2 + 3\ell^3 q_2^2\right)\right\} \tag{18}$$

The Lagrange equations of the clamped-free beam system are

$$\begin{bmatrix} \frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial q_1}\right) \\ \frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial q_2}\right) \end{bmatrix} - \begin{bmatrix} \frac{\partial \hat{L}}{\partial q_1} \\ \frac{\partial \hat{L}}{\partial q_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{19}$$

or after substitution of equation (18)

$$\rho\begin{bmatrix} \frac{2\ell^5}{5} & \frac{\ell^6}{3} \\ \frac{\ell^6}{3} & \frac{2\ell^7}{7} \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + EI\begin{bmatrix} 8\ell & 3\ell^2 \\ 3\ell^2 & 6\ell^3 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{20}$$

The eigenvalues of this system are (in hertz),

$$freq = \begin{bmatrix} 1.5936 \\ 15.7016 \end{bmatrix} \tag{21}$$

when ρ and EI are set eqaul to 1. These results compare fairly well to the solutions provided by Blevins of 1.5861 and 9.9399 hertz (Blevins, 1995). The accuracy of this model is limited by the fact that the series was truncated at $j=2$. Fortunately, increasing the number of admissible improves the systems model accuracy. An increase in the

series size only effect the amount of computation required the procedure remains the same.

8.6 Summary

The assumed modes method models distributed parameter systems using admissible functions and generalized coordinates. This modeling methods satisfies the Rayleigh-Ritz criteria so that eigenvalues converge from above their exact values. The MAF-SS method will be shown in the next chapter to share this property.

9. Details of the Modified Admissible Function Substructure Synthesis Formulation

9.1 Introduction

The following chapter illustrates and defines a new substructure synthesis (SS) method that modifies admissible functions (MAF) approximating substructure dynamics to form a structural model. The new method possesses some unique characteristics. One of these characteristics is the use of a global coordinate system to define all motions through out the analysis, other methods require a transformation of local coordinates either through a matrix transformation (CMS) or kinamatic analysis (SS). Another attribute is the modification of admissible functions to satisfy constraint conditions between substructures. The result of these two attributes is a modeling method that retains global generalized coordinates and eigenvector approximations throughout the analysis. The method satisfies the conditions of the inclusion principle and therefore eigenvalues converge from above the actual values. The mathematical procedure is straightforward. This reduces effort during system modeling.

A clamped-free beam is the first system used to present the MAF-SS method. This example highlights the effect of using a global coordinate system on the integration limits during model formation. The example also illustrates the improvement of convergence to actual eigenvalues values by improving the number of natural boundary conditions satisfied.

The next section devotes space to incorporating lumped elements such as a spring or mass into the MAF-SS formulation and presents the new concept of extraneous boundary conditions.

Finally a bent cantilever beam MAF-SS model is presented to illustrate the ability of the method to model complex systems.

9.2 Overview of MAF-SS

Substructure synthesis is a process of dividing a flexible structure into discrete portions, modeling these sections and then recombining them to form a model of the original system. There are many interpretations on how to accomplish this and with each interpretation comes subtle details that are both advantages and detracting to practical implementation. A new substructure synthesis scheme is presented here.

The new substructuring procedure is composed of 4 steps. The first step is the definition of admissible functions which satisfy the geometric boundary conditions of the substructures. This is followed by the identification of constraints (2) at the connection points between substructures. These constraints are then used to modify the original admissible functions (3). The modified functions approximating the eigenfunctions are used to form the energy terms of the system (4) which are used to form the Lagrange equations leading to the eigenvalues of the system.

The new method has the limitation that only point connections between substructures can efficiently be modeled. This limitation stems from the complexity of modeling line or surface connections. The advantages include the use of global eigenfunction, simplified

kinetic and potential energy integrals, general coordinate usage, and the delineation of connection constraints.

9.3 Clamped-Free Beam Example

A beam clamped at the root and free at the other end, is useful to illustrate the new SS formulation. The example illustrates how to apply the method to a series of substructures, the effect of not satisfying natural boundary constraints, and the integration limit procedure. The inclusion principle will also be shown to apply to MAF-SS formulation. In this example a beam will be divided into three substructures, this forms a foundation to apply MAF-SS to larger systems.

Figure 9.1 contains a schematic of the beam considered. The Euler-Bernoulli beam is clamped at the base and divided into three substructures as indicated by the dotted lines. The beam properties are ℓ=.3048, EI=.3077Nm^2, and $\overline{m}$=.0548 kg/m.

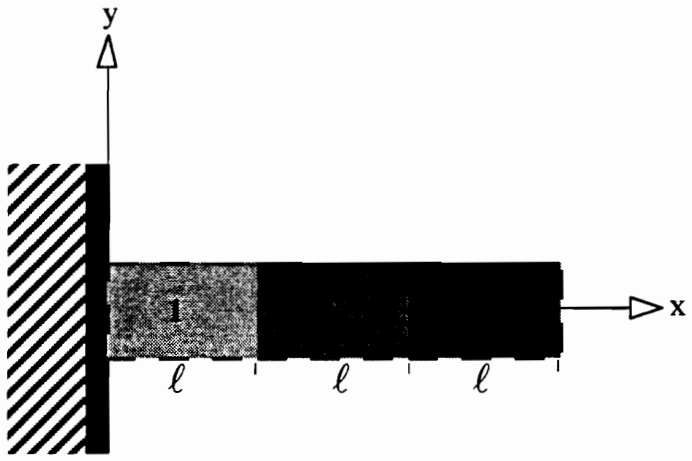


Figure 9.1. Euler-Bernoulli Beam System.

Each substructure needs to have an admissible function assigned. These functions must satisfy the geometric boundary conditions. Substructure 1 has the geometric boundary conditions of

$$w(0,t) = \frac{dw}{dx}(0,t) = 0 \tag{1}$$

the remaining substructures do not have geometric boundary conditions (Blevin, 1995). Appropriate admissible functions (any functions twice differentiable) using an n=2 approximation for substructure 1 are,

$$w_1(x,t) = x^2 q_1(t) + x^3 q_2(t) \tag{2}$$

and for substructures 2 and 3,

$$w_2(x,t) = w_3(x,t) = x q_1(t) + x^2 q_2(t) \tag{3}$$

The admissible functions now must be modified to incorporate connection constraints between substructures. These connection constraints consist of geometric and natural boundary constraints that arise from displacement and force elements. Figure 9.2 indicates the constraints for a cut through the beam between substructures 1 and 2. Moment terns are represented by M_i and shear and normal forces represented by V_i and N_i respectively.

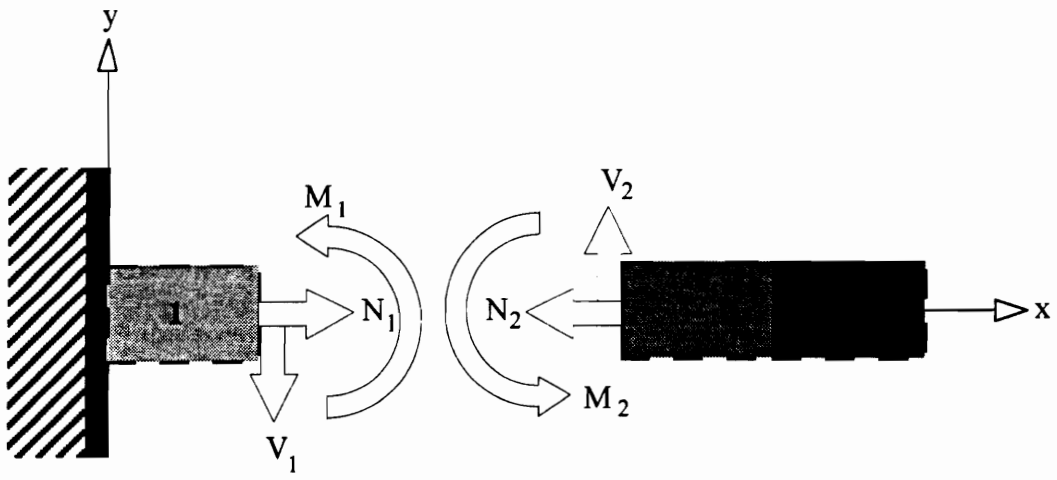


Figure 9.2. Cut Beam Showing Constraints.

Geometric and slope constraints dictate the displacement and slope must be continuous across SS connection points. These must be minimally satisfied during the modification phase. These are considered geometric boundary conditions and in order to retain the benefits of a Rayleigh-Ritz method they are required to be satisfied. Natural constraints are not required to be satisfied but if they are a better approximation of the eigenfunctions and better convergence to the exact eigenvalues occurs. The constraints (geometric and natural) imposed by the coupling of substructures are,

$$w_1(\ell,t) = w_2(\ell,t) \tag{4}$$

$$w_2(2\ell,t) = w_3(2\ell,t) \tag{5}$$

$$\frac{dw_1}{dx}(\ell,t) = \frac{dw_2}{dx}(\ell,t) \tag{6}$$

$$\frac{dw_2}{dx}(2\ell,t) = \frac{dw_2}{dx}(2\ell,t) \tag{7}$$

$$M_1(\ell,t) = -M_2(\ell,t) \tag{8}$$

$$M_2(2\ell,t) = -M_3(2\ell,t) \tag{9}$$

$$N_1(\ell,t) = N_2(\ell,t) \tag{10}$$

$$N_2(2\ell,t) = N_3(2\ell,t) \tag{11}$$

$$V_1(\ell,t) = V_2(\ell,t) \tag{12}$$

$$V_2(2\ell,t) = V_3(2\ell,t) \tag{13}$$

Some of these constraints can be neglected and transformed. According to Shames (1975) compatibility demands that,

$$\left[\frac{d^2w_1}{dx^2}(\ell,t)\right] = \left[\frac{d^2w_2}{dx^2}(\ell,t)\right] \tag{14}$$

this is related to the moment terms using the relation,

$$R = \frac{EI}{M} \tag{15}$$

where R, the radius of curvature which is approximated by,

$$\frac{1}{R} = \frac{d^2w}{dx^2} \tag{16}$$

The terms associated with normal forces (N) are neglected since the axial motion is considered rigid or negligible. The constraints associated with shear forces are neglected in the Euler-Bernoulli beam. The contribution could be quantified for a general unrestricted system using,

$$\frac{dw}{dx} = \frac{\alpha V}{GA} \tag{17}$$

where α is the shear coefficient a dimensionless quantity calculated from the cross section of the beam that accounts for the non-uniformity of the distribution of shear and stress over the cross section, G is the shear modulus and A the cross sectional area. Equation (17) would be used to replace V in equation (6). If the beam length is at least twice the height (long and slender) the contribution to transverse motion from shear effects is negligible (Shames, 1975). There for the imposed constraints that must be satisfied during the modification process are equations (4-9).

Examination of Moment Constraints

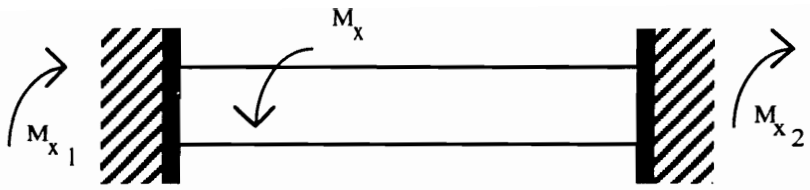
Constraint compatibility and their correct representation between substructures is an important concept in developing the modified eigenfunction approximations. The notation of moments in Fig. 2 is perhaps somewhat nonstandard although it does yield the correct equations of compatibility when the moment and force equilibrium conditions are written correctly. This section is intended to clarify the notation and explore moment constraints between substructures.

The goal of modifying constraints is to change eigenfunction approximations of substructures into global continuous (i.e. satisfying the geometric constraints between substructures) eigenfunction approximations. The result of modifying the constraints is twofold. Satisfying geometric constraints fulfills the requirement for satisfaction of the Rayleigh-Ritz principle (Meirovitch, 1977). The satisfaction of natural boundary conditions improves convergence of the eigenvalues to the exact values (Meirovitch, 1992).

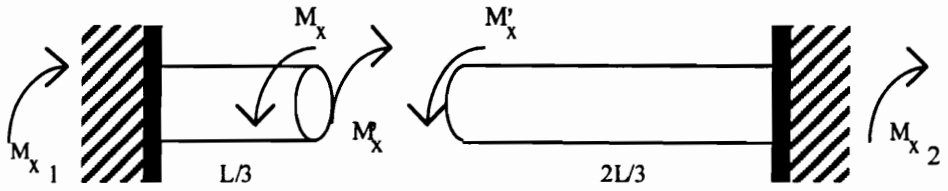
Substructuring or the process of dividing an original structure in to smaller portions leads to free body diagrams that contain geometric constraints that need to be satisfied in order to achieve compatibility between the substructures. Compatibility is the keyword to remember when thinking about the modification process. The reason compatibility is the keyword is that the goal is to achieve global continuous eigenfunction approximations. Shames illustrates the concept of compatibility of moments in a rod in torsion. The example is presented here along with some additional remarks.

Shames, Irving H., Introduction to Solid Mechanics, Prentice-Hall, Inc. Englewood Cliffs, New Jersey, pp. 278-279.

A circular shaft clamped at both ends and loaded by a torque M_x is illustrated below. The goal is to determine the reactive torque M_{x1} and M_{x2} developed by the applied torque M_x.



Fixed circular shaft with torque load.



Segmented circular shaft with transmitted torques and torque load.

Figure 9.1'. Circular shaft with torsion load.

The problem is statically indeterminate so the theory of strength of materials is required to solve the problem.

Equilibrium gives the relation

$$M_{x_1} + M_{x_2} = M_x \tag{1'}$$

Now the beam is cut and two free-body diagrams constructed. The angle of twist is written for the left end of the beam as

$$\phi_1 = \frac{(M_x - M'_x)\left(\frac{L}{3}\right)}{GJ} \tag{2'}$$

and the right end of the beam as

$$\phi_2 = \frac{M_x'\left(\frac{2L}{3}\right)}{GJ} \tag{3'}$$

Shames states, "clearly, compatibility demands, that the angles ϕ_1 and ϕ_2 must be equal" and so we write the angles of twist (ϕ_i) as,

$$\phi_1 = \phi_2 \tag{4'}$$

or

$$\frac{(M_x - M_x')}{GJ}\frac{L}{3} = \frac{M_x'\left(\frac{2L}{3}\right)}{GJ} \tag{5'}$$

from this equation it is determined that,

$$M_x' = \frac{1}{3}M_x \tag{6'}$$

and using the equilibrium equations for the left end of the cut section,

$$M_{x_1} = M_x - M_x' \tag{7'}$$

and using equations (6') yields,

$$M_{x_1} = \frac{2}{3} M_x \tag{8'}$$

Examining the right end section of the cut shaft, the equilibrium conditions yield,

$$M_{x_2} = M_x' = \frac{1}{3} M_x \tag{9'}$$

The relevant parts of the examples are, that correct definition of moment directions is detailed and that the compatibility equation is the key equation in establishing the correct constraints between segmented structures. This correlates to equation (14) of the main text. This is the compatibility equation that must be satisfied and is, when modifying the eigenfunction approximations. The angle of twist correlates to the strain in the beam example.

The example illustrates that the notation used in the beam example (although somewhat nonstandard) and the moment constraints are correct in the context they are presented. The essential equation is the compatibility equation between substructures. The concept of constraint satisfaction is important since satisfying these constraints is critical to the overall convergence of the model and therefore attention to it is warranted. Not satisfying natural or moment constraints reduces the accuracy of the model. This reduction in accuracy resulting from not satisfying the natural boundary conditions (ex. moment or strain in the beam) is demonstrated on the beam in table 1 of the main text.

Modification Procedure

The admissible functions are modified so that their left end satisfies the connection constraints. Moving left to right across the structure, a continuous global eigenfunction is

formed. Starting with substructure 1 no modification needs to occur, because the left end boundary conditions correspond with the structures geometric boundary conditions that are satisfied by the admissible functions. Substructure 2 does need modification to satisfy geometric and moment constraints. The admissible functions needed to be modified such that,

$$\begin{bmatrix} w(\ell,t) \\ \frac{dw}{dx}(\ell,t) \\ \frac{dw}{dx}(\ell,t) \end{bmatrix} = \begin{bmatrix} \ell^2 & \ell^3 \\ 2\ell & 3\ell^2 \\ 2 & 6\ell \end{bmatrix} \begin{bmatrix} q_1(x) \\ q_2(x) \end{bmatrix} = \begin{bmatrix} \ell + \Delta_{11} & \ell^2 + \Delta_{21} \\ 1 + \Delta_{21} & 2\ell + \Delta_{22} \\ 0 + \Delta_{31} & 6\ell + \Delta_{23} \end{bmatrix} \begin{bmatrix} q_1(x) \\ q_2(x) \end{bmatrix} \tag{18}$$

are satisfied. These equations must be solved remembering the fundamental theorem of calculus that

$$w(x,t) = \int \frac{dw}{dx}(x,t)dx + c \tag{19}$$

Examining the first admissible function the equations

$$\begin{aligned} \ell^2 &= \ell + \Delta_{11}(\ell) \Rightarrow \Delta_{11}(\ell) = \ell^2 - \ell \\ 2\ell &= 1 + \Delta_{21}(\ell) \Rightarrow \Delta_{21}(\ell) = 2\ell - 1 \\ 2 &= -\Delta_{31}(\ell) \Rightarrow \Delta_{31}(\ell) = 2 \end{aligned} \tag{20}$$

are written to find the modifier (Δ) at the connection point. Equation (19) is now recalled so that,

$$\begin{aligned} \Delta_{21}(x) &= \int \Delta_{31} dx + c_{21} = 2x + c_{21} \\ \Delta_{21}(\ell) &= 2\ell + c_{21} = -2\ell - 1 \Rightarrow c_{21} = -1 \\ \Delta_{21}(x) &= 2x - 1 \end{aligned} \tag{21}$$

can be written and the function modifiers found. This procedure is repeated to find the final element of the modifier,

$$\begin{aligned}
&\Delta_{11}(x) = \int \Delta_{21}(x)dx + c_{11} = -x^2 - x + c_{11} \\
&\Delta_{11}(\ell) = -\ell^2 - \ell + c_{11} = \ell^2 - \ell \Rightarrow c_{11} = 0 \\
&\Delta_{11}(x) = x^2 - x
\end{aligned} \tag{22}$$

which when combined with the original function,

$$\phi_1(x) = x \tag{23}$$

yields ,

$$\phi_1^{\Delta}(x) = x^2 \tag{24}$$

The remaining admissible functions are modified in a similar manner. These modified admissible functions are then used to write the kinetic and potential energy terms. These terms have the form,

$$
\begin{aligned}
T = {} & \frac{\overline{m}}{2}\int_0^{\ell}\begin{bmatrix}\phi_1 & \phi_2\end{bmatrix}\begin{bmatrix}\dfrac{dq_1}{dt}\\ \dfrac{dq_2}{dt}\end{bmatrix}\begin{bmatrix}\phi_1 & \phi_2\end{bmatrix}\begin{bmatrix}\dfrac{dq_1}{dt}\\ \dfrac{dq_2}{dt}\end{bmatrix}dx + \\
& \frac{\overline{m}}{2}\int_{\ell}^{2\ell}\begin{bmatrix}\phi_1^{\Delta} & \phi_2^{\Delta}\end{bmatrix}\begin{bmatrix}\dfrac{dq_1}{dt}\\ \dfrac{dq_2}{dt}\end{bmatrix}\begin{bmatrix}\phi_1^{\Delta} & \phi_2^{\Delta}\end{bmatrix}\begin{bmatrix}\dfrac{dq_1}{dt}\\ \dfrac{dq_2}{dt}\end{bmatrix}dx + \\
& \frac{\overline{m}}{2}\int_{2\ell}^{3\ell}\begin{bmatrix}\phi_1^{\Delta} & \phi_2^{\Delta}\end{bmatrix}\begin{bmatrix}\dfrac{dq_1}{dt}\\ \dfrac{dq_2}{dt}\end{bmatrix}\begin{bmatrix}\phi_1^{\Delta} & \phi_2^{\Delta}\end{bmatrix}\begin{bmatrix}\dfrac{dq_1}{dt}\\ \dfrac{dq_2}{dt}\end{bmatrix}dx
\end{aligned}
\tag{25}
$$

and

$$
\begin{aligned}
V = {} & \frac{1}{2}\int_0^{\ell}\begin{bmatrix}\dfrac{d^2\phi_1}{dx^2} & \dfrac{d^2\phi_2}{dx^2}\end{bmatrix}\begin{bmatrix}q_1\\ q_2\end{bmatrix}\begin{bmatrix}\dfrac{d^2\phi_1}{dx^2} & \dfrac{d^2\phi_2}{dx^2}\end{bmatrix}\begin{bmatrix}q_1\\ q_2\end{bmatrix}dx + \\
& \frac{1}{2}\int_{\ell}^{2\ell}\begin{bmatrix}\dfrac{d^2\phi_1^{\Delta}}{dx^2} & \dfrac{d^2\phi_2^{\Delta}}{dx^2}\end{bmatrix}\begin{bmatrix}q_1\\ q_2\end{bmatrix}\begin{bmatrix}\dfrac{d^2\phi_1^{\Delta}}{dx^2} & \dfrac{d^2\phi_2^{\Delta}}{dx^2}\end{bmatrix}\begin{bmatrix}q_1\\ q_2\end{bmatrix}dx + \\
& \frac{1}{2}\int_{2\ell}^{3\ell}\begin{bmatrix}\dfrac{d^2\phi_1^{\Delta}}{dx^2} & \dfrac{d^2\phi_2^{\Delta}}{dx^2}\end{bmatrix}\begin{bmatrix}q_1\\ q_2\end{bmatrix}\begin{bmatrix}\dfrac{d^2\phi_1^{\Delta}}{dx^2} & \dfrac{d^2\phi_2^{\Delta}}{dx^2}\end{bmatrix}\begin{bmatrix}q_1\\ q_2\end{bmatrix}dx
\end{aligned}
\tag{26}
$$

The energy terms are used to write the Lagrangian defined as,

$$L = T - V, \tag{27}$$

which in turn is used to form the Lagrange equations of motion for the system.

$$
\begin{bmatrix}\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_1}\right)\\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_2}\right)\end{bmatrix} - \begin{bmatrix}\dfrac{\partial L}{\partial q_1}\\ \dfrac{\partial L}{\partial q_2}\end{bmatrix} = 0 \tag{28}
$$

The mass and stiffness matrices resulting from the application of the Lagrange equations are,

$$M = \begin{bmatrix} .007006346976 & .004131922493 \\ .004131922493 & .002460910176 \end{bmatrix} \tag{29}$$

$$K = \begin{bmatrix} 1.125736128 & .8578109297 \\ .8578109297 & .7320901600 \end{bmatrix} \tag{30}$$

yielding natural frequency estimates of

$$\lambda = \begin{bmatrix} 1.6806 \\ 10.8882 \end{bmatrix} \tag{31}$$

Reduction in Boundary Condition Satisfaction

The natural boundary conditions are not required to be satisfied as shown by Meirovitch (1980). The effect of reducing them is now examined. Satisfying just the geometric imposed constraint yields admissible functions of the form,

$$\phi_1^{\Delta}(x) = 2\ell x - \ell^2 \tag{32}$$

and

$$\phi_2^{\Delta}(x) = x^2 + 3\ell^2 x - 2\ell x - 2\ell^3 + \ell^2 \tag{33}$$

which are used to form the kinetic and potential energy terms and ultimately the Lagrange equations of motion. This results in the mass and stiffness matrix of,

$$M = \begin{bmatrix} .003008212078 & .002775416827 \\ .002775416827 & .002722439722 \end{bmatrix} \tag{34}$$

$$K = \begin{bmatrix} .3752453760 & .1715621859 \\ .1715621859 & .8550750606 \end{bmatrix} \tag{35}$$

yielding eigenvalues approximations of,

$$\lambda = \begin{bmatrix} 1.6893 \\ 11.6023 \end{bmatrix} \tag{36}$$

A summary of these results is presented in table 9.1.

Table 9.1. Comparison of Modeling Methods.

frequency (cycles/second)	Assumed modes	Modified Admissible (Geometric)	Modified Admissible (Geometric & Natural)	Exact solution (Blevins, 1995)
1	1.5936	1.6893	1.6806	1.5861
2	15.7016	11.6023	10.8882	9.9399

The table indicates that the eigenvalues converge from above the exact natural frequencies and that satisfying more constraint conditions increases model accuracy. The approximation function used for each of these models are,

assumed modes	$w(x,t) = x^2 q_1(t) + x^3 q_2(t)$	$w(x,t) = x^2 q_1(t) + x^3 q_2(t)$
modified admissible†	$w(x,t) = x^2 q_1(t) + x^3 q_2(t)$	$w(x,t) = (2\ell x - \ell^2) q_1(t) + (x^2 + 3\ell^2 x - 2\ell x - 2\ell^3 + \ell) q_2(t)$
modified admissible‡	$\underbrace{w(x,t) = x^2 q_1(t) + x^3 q_2(t)}_{0 \to \ell}$	$\underbrace{w(x,t) = x^2 q_1(t) + (3\ell x^2 - 3\ell^2 x + \ell^3) q_2(t)}_{\ell \to 3\ell}$

(37)

where the † indicated that only geometric constraints between substructures are satisfied and the ‡ indicates both geometric and natural boundary conditions are satisfied.

A 4 mode Beam Model

The number of modes modeled is increased by expanding the number of terms and generalized coordinates approximating the system. The beam model is expanded to 4 modes through the use of the following modified admissible functions,

$$
\begin{aligned}
&w(x,t) = x^2 q_1(t) + x^3 q_2(t) + x^4 q_3(t) + x^5 q_4(t) \quad \text{for } 0 \le x \le \ell \\
&w(x,t) = x^2 q_1(t) + \left\{3\ell x^2 - 3\ell^2 x + \ell^3\right\} q_2(t) + \\
&\left\{x^3 + 6\ell^2 x^2 - 3\ell x^2 - 8\ell^3 x + 3\ell^2 x + 3\ell^4 - \ell^3\right\} q_3(t) + \\
&\left\{x^4 + 10\ell^3 x^2 - 6\ell^2 x^2 - 15\ell^4 x + 8\ell^3 x + 6\ell^5 - 3\ell^4\right\} q_4(t) \quad \text{for } \ell \le x \le 3\ell
\end{aligned}
\tag{38}
$$

The natural frequencies that result from these modified admissible function series are listed in Table 2.

Table 9.2. Model Comparison of 4 Modes.

frequency (Hertz)	MAF-SS (4 gen. cords.)	Blevin (exact solution)
1	1.5871	1.5861
2	10.0020	9.9399
3	28.2667	27.8319
4	100.3410	54.5395

The result again closely correspond to the exact formulation results compiled by Blevins. A convergence from above is demonstrated as expected.

Integral Limit Examination

The correspondence between the MAF-SS and the assumed modes methods is now examined. The correspondence occurs because the MAF-SS model is essentially an assumed modes model where eigenfunction approximations are developed using the modification procedure. A assumed modes model has been developed in chapter 8, this is used to compare the results of adjusting integral limits or integrands with the MAF-SS method.

The Lagrangian (L) is equal to the difference of the kinetic (T) and potential (V) energy terms of a system. In a distributed parameter system model using the assumed modes technique the Lagrangian is written,

$$L = \int_{x_1}^{x_2} T(x)dx - \int_{x_1}^{x_2} V(x)dx \tag{39}$$

were the integral limits span the distributed system. The Lagrangian for the MAF-SS system can be written,

$$L = \int_{x_1}^{x_2} T(x)dx + \int_{x_1}^{x_2} V(x)dx \tag{40}$$

since the modified eigenfunction approximations are continuous across the integral limits. The modification process insures that the eigenfunction approximations and general

coordinates are global. This allows the MAF-SS Lagrangian to be mathematically similar to the assumed modes Lagrangian.

A further illustrated is presented using the results from the chapter on the assumed modes method. The beam considered in the chapter 8 is separated into three substructures designated A, B, and C.

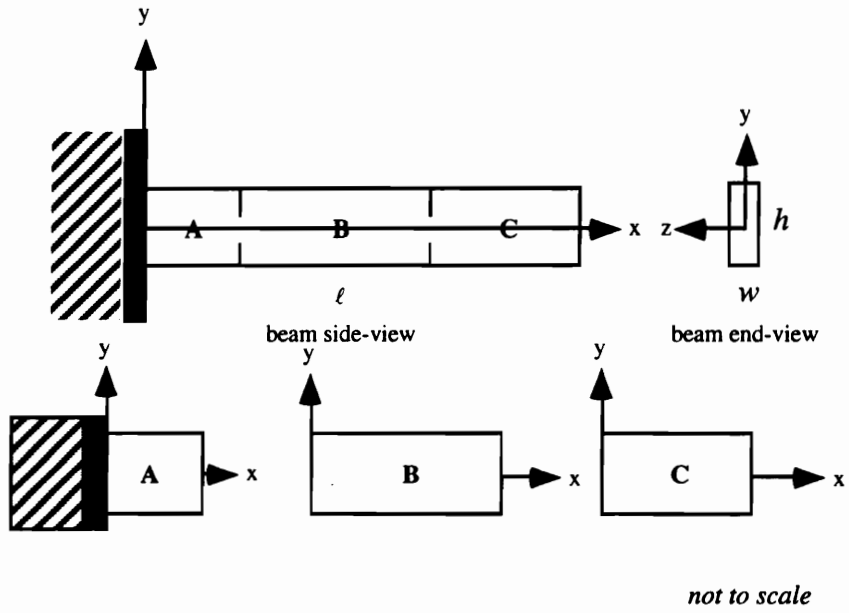


Figure 9.3. Beam System Substructured

The distributed terms must be integrated to develop the energy terms. An issue that arises, is which coordinate system to use (global or local) and how the motions should be defined (relative or absolute).

The admissible functions used to model the entire beam in the assumed modes model fulfill the role for each substructure. To see this, the kinetic and potential energies are

written for each substructure using the assumed modes. The integral limits must be in the original or global coordinate system. The energy functions for each substructure are

$$
\begin{aligned}
{}^{A}\hat{T} &= \frac{1}{2}\rho\int_{0}^{\frac{\ell}{3}}\left(\begin{bmatrix}{}^{A}\phi_1 & \ldots {}^{A}\phi_j\end{bmatrix}\begin{bmatrix}{}^{A}\dot{q}_1\\ {}^{A}\dot{q}_j\end{bmatrix}\begin{bmatrix}{}^{A}\phi_1 & \ldots {}^{A}\phi_j\end{bmatrix}\begin{bmatrix}{}^{A}\dot{q}_1\\ {}^{A}\dot{q}_j\end{bmatrix}\right)d({}^{A}x)\\
{}^{A}\hat{V} &= \frac{1}{2}EI\int_{0}^{\frac{\ell}{3}}\left(\begin{bmatrix}{}^{A}\phi_1'' & \ldots {}^{A}\phi_j''\end{bmatrix}\begin{bmatrix}{}^{A}q_1\\ {}^{A}q_j\end{bmatrix}\begin{bmatrix}{}^{A}\phi_1'' & \ldots {}^{A}\phi_j''\end{bmatrix}\begin{bmatrix}{}^{A}q_1\\ {}^{A}q_j\end{bmatrix}\right)d({}^{A}x)
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
{}^{B}\hat{T} &= \frac{1}{2}\rho\int_{\frac{\ell}{3}}^{\frac{2\ell}{3}}\left(\begin{bmatrix}{}^{B}\phi_1 & {}^{B}\phi_j\end{bmatrix}\begin{bmatrix}{}^{B}\dot{q}_1\\ {}^{B}\dot{q}_j\end{bmatrix}\begin{bmatrix}{}^{B}\phi_1 & {}^{B}\phi_j\end{bmatrix}\begin{bmatrix}{}^{B}\dot{q}_1\\ {}^{B}\dot{q}_j\end{bmatrix}\right)d({}^{B}x)\\
{}^{B}\hat{V} &= \frac{1}{2}EI\int_{\frac{\ell}{3}}^{\frac{2\ell}{3}}\left(\begin{bmatrix}{}^{B}\phi_1'' & {}^{B}\phi_j''\end{bmatrix}\begin{bmatrix}{}^{B}q_1\\ {}^{B}q_j\end{bmatrix}\begin{bmatrix}{}^{B}\phi_1'' & {}^{B}\phi_j''\end{bmatrix}\begin{bmatrix}{}^{B}q_1\\ {}^{B}q_j\end{bmatrix}\right)d({}^{B}x)
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
{}^{C}\hat{T} &= \frac{1}{2}\rho\int_{\frac{2\ell}{3}}^{\ell}\left(\begin{bmatrix}{}^{C}\phi_1 & {}^{C}\phi_j\end{bmatrix}\begin{bmatrix}{}^{C}\dot{q}_1\\ {}^{C}\dot{q}_j\end{bmatrix}\begin{bmatrix}{}^{C}\phi_1 & {}^{C}\phi_j\end{bmatrix}\begin{bmatrix}{}^{C}\dot{q}_1\\ {}^{C}\dot{q}_j\end{bmatrix}\right)d({}^{C}x)\\
{}^{C}\hat{V} &= \frac{1}{2}EI\int_{\frac{2\ell}{3}}^{\ell}\left(\begin{bmatrix}{}^{C}\phi_1'' & {}^{C}\phi_j''\end{bmatrix}\begin{bmatrix}{}^{C}q_1\\ {}^{C}q_j\end{bmatrix}\begin{bmatrix}{}^{C}\phi_1'' & {}^{C}\phi_j''\end{bmatrix}\begin{bmatrix}{}^{C}q_1\\ {}^{C}q_j\end{bmatrix}\right)d({}^{C}x)
\end{aligned}
\tag{43}
$$

Examining the above equations it is clear the Lagrangian of this system is equal to the assumed modes Lagrangian.

A question that arises is, can the local coordinates be used to form the correct Lagrangian. The three independent coordinated systems (A, B, and C) define the integral limits across each substructure as

$$
\begin{aligned}
0 &\rightarrow {}^{A}\frac{\ell}{3} \\
0 &\rightarrow {}^{B}\frac{\ell}{3} \\
0 &\rightarrow {}^{C}\frac{\ell}{3}
\end{aligned}
\tag{44}
$$

The redefinition of the limits has changed the Lagrangian of the system so that it is no longer required to be equal to equation (14) in chapter 8. This is corrected by writing relationship for the elastic motions as

$$
\begin{aligned}
y({}^{A}x) &= {}^{A}y({}^{A}x) \\
y({}^{B}x) &= {}^{A}y\left({}^{A}\frac{\ell}{3}\right) + {}^{B}y({}^{B}x) \\
y({}^{C}x) &= {}^{A}y\left({}^{A}\frac{\ell}{3}\right) + {}^{B}y\left({}^{B}\frac{2\ell}{3}\right) + {}^{C}y({}^{C}x)
\end{aligned}
\tag{45}
$$

It can be shown that the Lagrangian of this system is the same as equation (14) in chapter 8 if generalized coordinates and compatibility between substructures is maintained. The integration becomes more tedious with a more complex integrand using this method. Most other SS methods use local integration limits and correct the intagrand for kinematic compatibility.

The result of this example is that it appears mathematically simpler to correct integral limits verses kinematics relationship in the integrand. Writing elastic motions locally and limits globally appears more efficient. When the system model is developed using a global coordinate system, generalized coordinates, and global integral limits, the Lagrangian is (if the eigenfunction approximation satisfy constraint connections)

equivalent to an assumed modes Lagrangian formation. The MAF-SS fulfills these requirements. It then can be stated that they share the same convergence properties. The convergence properties are identified in chapter 8 for the assumed modes. The assumed modes method satisfies the inclusion principle, therefore the MAF-SS also has this beneficial property.

9.4 Discrete Elements and Extraneous Constraints

Introduction

This section contains two examples. Each example illustrates a different concept of the MAF-SS formulation using a beam and discrete element. The concepts demonstrated are the incorporation of discrete elements into the SS formulation and identification of extraneous connection constraints.

The beams used in both examples have the same physical parameters which along with the discrete element properties identified in Table 9.3. The terms q represent generalized coordinates and w transverse vibrations in the plane of the paper.

Table 9.3. Beam Parameters for Figs. 4 and 5.

Parameter	**Value**
mass per unit length	.0548 kg/m
modulus of elasticity, mass moment of inertia, *(EI)*	.30778 Nm^2
beam length *(ℓ)*	.3048 m
mass, *(m)*	.3048* ℓ kg
spring constant *(k)*	.30788/ ℓ^3 N/m

Discrete and Discretized System Models

Models can be formed using MAF-SS which are composed of both distributed parameter substructures that have been discretized and discrete elements. A mass attached at the end of a beam illustrates this concept.

Beam-Mass Example

A beam with a mass attached demonstrates the ability to incorporate discrete elements into the system model. The Euler-Bernoulli beam is clamped at the base and has a mass attached to the other end.

Figure 9.4. Beam-Mass System

The system is substructured into two parts, identified in the beam-mass figure. Each of these substructures is modeled using admissible functions. The admissible functions are defined for substructure 1 (the beam) as,

$$w_1(x,t) = x^2 q_1(t) + x^3 q_2(t) + x^4 q_3(t) + x^5 q_3(t) \tag{44}$$

The motion of the mass is approximated by,

$$w_2(x,t) = w_2(\ell,t) \tag{45}$$

Since the mass is a rigid element no mode approximations are necessary. The admissible functions are now modified to incorporate the constraints between substructures. Proceeding with substructures 1 it is apparent that no modifications needs to take place. The left of the substructures is clamped and modeled correctly by the admissible function series. The second substructure needs to be modified. Its left end is attached to the beam. The constraint equations are (Laura, 1974),

$$w_2(\ell,t) = w_1(\ell,t) \tag{46}$$

and

$$-\left[-EI\frac{\partial^3 w_1}{\partial x^3}(\ell,t)\right] = m\frac{\partial^2 w_2}{\partial t^2}(\ell,t) \tag{47}$$

The modified admissible functions satisfying these constraints are,

$$w_2^{\Delta}(x,t) = \ell^2 q_1(t) + \ell^3 q_2(t) + \ell^4 q_3(t) + \ell^5 q_4(t) \tag{48}$$

Using equations (44) and (45) to write the potential and kinetic energy for the system and then applying Lagrange equations results in equations of the form,

$$M\frac{d^2 q}{dt^2} + Kq = 0 \tag{49}$$

where

$$M = \begin{bmatrix} .0001729962216 & .00005126454700 & .00001530654752 & .4592538253e-5 \\ .00005126454700 & .00001530654752 & .4592538251e-5 & .1382524108e-5 \\ .00001530654752 & .4592538251e-5 & .1382524108e-5 & .4171794145e-6 \\ .4592538253e-5 & .1382524108e-5 & .41717941445e-6 & .1261054072e-6 \end{bmatrix} \tag{50}$$

and

$$K = \begin{bmatrix} .3752453760 & .1715621859 & .06972287235 & .02656441437 \\ .1715621859 & .1045843085 & .04781594586 & .01943240040 \\ .06972287235 & .04781594586 & .02331888048 & .009871659400 \\ .02656441437 & .01943240040 & .009871659400 & .004298402550 \end{bmatrix} \tag{51}$$

The corresponding eigenvalues are,

$$\lambda = \begin{bmatrix} 6.3225 \\ 66.0589 \\ 209.7043 \\ 750.3022 \end{bmatrix} \tag{52}$$

in cycles per second. These compare well with ones calculated by Laura (1974). The results are,

$$\lambda = \begin{bmatrix} 6.3225 \\ 65.9744 \\ 206.6343 \\ 427.0992 \end{bmatrix} \tag{53}$$

in cycles per second.

Extraneous Connection Constraints

Connections constraints can sometimes be ignored, these extraneous constraints are a result of the fact an imposed connection constraint is transformed into a natural boundary condition of a substructure that does not need or has already been modified. This new concept is demonstrated using the example that follows.

Beam-Spring Example

An Euler-Bernoulli beam with a spring attached at the end illustrates the fact that some constraints arising from substructure connections are extraneous.

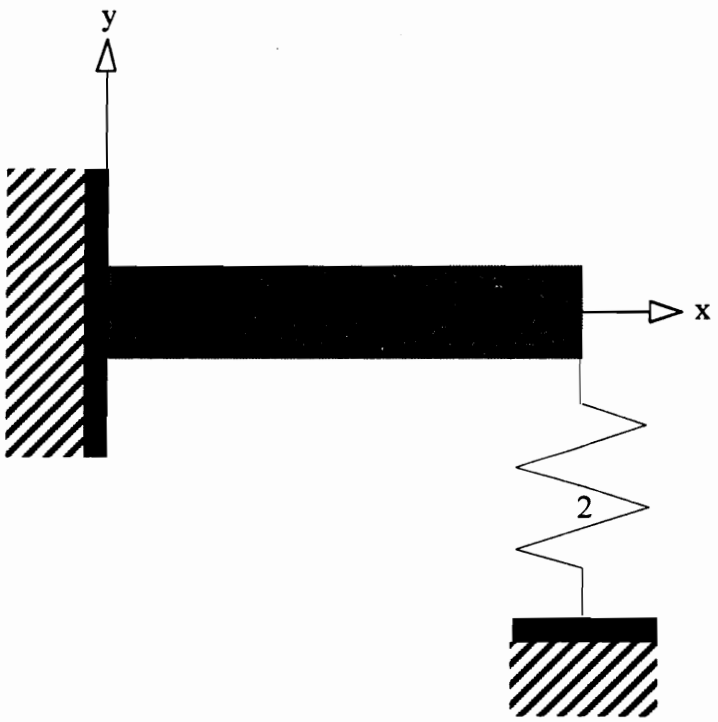


Figure 9.5. Beam-Spring System.

The solution development is similar to the beam mass example. First the structure is divided into two substructures, the beam and the spring. Admissible functions are then written for each substructure. In this case admissible functions for the beam are defined as,

$$w_1(x,t) = x^2 q_1(t) + x^3 q_2(t) + x^4 q_3(t) + x^5 q_4(t) \tag{54}$$

and the springs motion for now is identified as,

$$w_2(x,t) = w_2(\ell,t) \tag{55}$$

Since the spring is not a distributed parameter structure, modal approximations are not necessary. The displacement and force constraints according to Maurizi (1976) are,

$$w_1(\ell,t) = w_2(\ell,t) \tag{56}$$

and

$$EI\frac{\partial^3 w_1(\ell,t)}{\partial x^3} = kw_2(\ell,t) \tag{57}$$

These must be satisfied by modifying the admissible functions. Once again starting with substructure 1, it is determined since the left end boundary condition is clamped and this geometric boundary condition is satisfied by the admissible functions no modification needs to take place. Moving on to substructure 2 the spring, it appears that modification

must take place to satisfy equations (56) and (57). Upon reexamining these imposed constraints, if equation (56) is satisfied equation (57) can be rewritten as,

$$EI\frac{\partial^3 w_1}{\partial x^3}(\ell,t) = kw_1(\ell,t) \tag{58}$$

which can be viewed as a natural boundary condition of substructure 1. It therefore does not need to be satisfied. This previously unidentified concept of extraneous constraints touches the reason that sometimes inconsistent convergence occurs for boundary approximate methods. The other methods may exhibit better convergence than expected when an extraneous constraint is encountered. The formulation presented here methodically unveils such conditions resulting in a more understandable formulation. The modified function for substructure 2 after satisfying equation (56) only is,

$$w_2^{\Delta}(x,t) = \ell^2 q_1(t) + \ell^3 q_2(t) + \ell^4 q_3(t) + \ell^5 q_4(t) \tag{59}$$

Equations (54) and (59) are used to write the potential and kinetic energy terms which the Lagrange equations can be applied to resulting in the equations of motion of the form,

$$M\frac{d^2 q}{dt^2} + Kq = 0 \tag{60}$$

where ,

$$M = \begin{bmatrix} .00002883270362 & .7323506715e-5 & .1913318441e-5 & .5102820280e-6 \\ .7323506715e-5 & .1913318440e-5 & .5102820280e-6 & .1382524108e-6 \\ .1913318441e-5 & .5102820280e-6 & .1382524108e-6 & .3792540134e-7 \\ .5102820280e-6 & .1382524108e-6 & .3792540134e-7 & .1050878393e-7 \end{bmatrix} \tag{61}$$

and

$$K = \begin{bmatrix} .4690567200 & .2001558836 & .07843823140 & .02922085581 \\ .2001558836 & .1132996676 & .05047238730 & .02024208375 \\ .07843823140 & .05047238730 & .02412856383 & .01011845089 \\ .02922085581 & .02024208375 & .01011845089 & .004379536245 \end{bmatrix} \tag{62}$$

the eigenvalues (cycles per second) are,

$$\lambda = \begin{bmatrix} 16.4027 \\ 90.3369 \\ 257.3350 \\ 1143.4054 \end{bmatrix} \tag{63}$$

which compares well with the exact first eigenvalue (cycles per second) of,

$$\lambda_1 = 16.4026 \tag{64}$$

calculated by Maurizi (1976).

9.5 Complex System Model

A single bending elastic beam joined with another beam at a right angle is a system considered to demonstrate the substructure synthesis method (see Fig. 6). This system was investigated by Hurty (1960) during his development of substructure synthesis. The beam under consideration is assumed clamped-free and satisfies the Bernoulli-Euler beam requirements. Motions in the plane of the paper are only considered. Connections

are assumed to be internal single-point boundaries. The system is illustrated in Fig. 9.6 and is found in chapter 2 to illustrate the substructure formulation. Substructures 1 and 2 are identified by dotted lines.

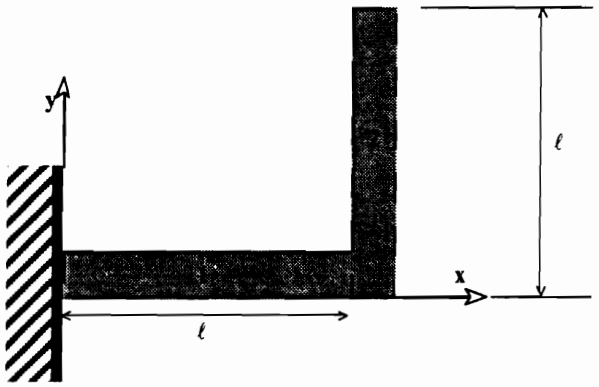


Figure 9.6. Bent Cantilever Beam.

The beam properties are defined as having a length of .3048 m, a modulus of elasticity multiplied by moment of inertia of .30778 Nm^2 and a mass per unit length of .0548 kg/m.

The substructuring procedure is composed of several steps. The first step is the definition of admissible functions which satisfy the geometric boundary conditions of the substructures. This is followed by the identification of displacement and force constraints at the connection points between substructures. These constraints are then used to modify the original admissible functions. The modified functions approximating the eigenfunctions are used to form the energy terms of the system which are used to form the Lagrange equations leading to the eigenvalues of the system.

The system is divided into two substructures. The minimal functions needed to satisfy the solutions are admissible (see for instance Meirovitch, 1980). Admissible functions

must satisfy geometric boundary conditions only (Meirovitch, 1980). The boundary conditions, both geometric and natural for each substructure are,

$$w_1(0,0,t) = \frac{\partial w_1(0,0,t)}{\partial x} = 0 \tag{65}$$

$$\frac{\partial^2 w_1(\ell,0,t)}{\partial x^2} = \frac{\partial^3 w_1(\ell,0,t)}{\partial x^3} = 0 \tag{66}$$

and

$$\frac{\partial^2 w_2(\ell,0,t)}{\partial y^2} = \frac{\partial^3 w_2(\ell,0,t)}{\partial y^3} = 0 \tag{67}$$

$$\frac{\partial^2 w_2(\ell,\ell,t)}{\partial y^2} = \frac{\partial^3 w_2(\ell,\ell,t)}{\partial y^3} = 0 \tag{68}$$

for the transverse vibrations. The functions,

$$w_1(x,0,t) = \left(\frac{x}{\ell}\right)^2 q_1(t) + \left(\frac{x}{\ell}\right)^3 q_2(t) + \left(\frac{x}{\ell}\right)^4 q_3(t) \tag{69}$$

$$w_2(\ell,y,t) = 1q_1(t) + \left(\frac{y}{\ell}\right) q_2(t) + \left(\frac{y}{\ell}\right)^4 q_3(t) \tag{70}$$

satisfy the geometric boundary conditions and are used to model the system. For the axial or longitudinal vibrations the geometric boundary conditions are,

$$u_1(0,0,t) = 0 \tag{71}$$

$$\frac{\partial u_1(\ell,0,t)}{\partial x} = 0 \tag{72}$$

and

$$\frac{\partial u_2(\ell,0,t)}{\partial y} = 0 \tag{73}$$

$$\frac{\partial u_2(\ell,\ell,t)}{\partial y} = 0 \tag{74}$$

The equations

$$u_1(x,t) = 0p_1(t) \tag{75}$$

$$u_2(y,t) = 1p_1(t) \tag{76}$$

satisfy the geometric boundary conditions for each substructure.

The imposed constraints arising from the connection of substructures are identified as,

$$u_1(\ell,0,t) + w_2(\ell,0,t) = 0 \tag{77}$$

$$w_1(\ell,0,t) - u_2(\ell,0,t) = 0 \tag{78}$$

$$\frac{\partial w_1(\ell,0,t)}{\partial x} - \frac{\partial w_2(\ell,0,t)}{\partial y} = 0 \tag{79}$$

$$\frac{\partial^2 w_1(\ell,0,t)}{\partial x^2} + \frac{\partial^2 w_2(\ell,0,t)}{\partial y^2} = 0 \tag{80}$$

In addition Hurty used the assumption that the there is infinite axial rigidity in order to compare with an exact solution worked by Bishop. This assumption is made here as well for the same reason.

This next step diverges from the other methods classed as substructure methods. The admissible functions are modified to satisfy the constraints arising from connections. These connection constraints are imposed into the admissible functions. The imposed constraints are usually included in most methods using a coordinate transformation after the Lagrange equations have been formed (Hurty, 1960, Craig, 1987).

The eigenfunction approximation method must insure geometric compatibility and not disrupt convergence to the correct eigenvalues and mode shapes. The ability to do this for the method presented stems from the formulation of correcting integral limits not integrands relative to the inertial axis. Integration occurs using integral limits that are relative to the inertial reference system. This differs from other methods which use local coordinate reference frames. This also means that the generalized coordinates are the same for each substructure. The size of the coupled matrices is equal to the number of admissible functions used to model the substructures. The assumption is made that the same number is used for each substructure. The admissible functions initially formed for

the model satisfy only the geometric boundary conditions of the substructures. Imposed compatibility conditions are satisfied using a straight forward modification process.

The transverse motions has three imposed connecting condition equations (77-78) that must be satisfied. This is easily done by modifying equation (70) using equations (77-78) and the form,

$$w_2^{\Delta}(\ell,0,t) = w_2(\ell,0,t) + \alpha_1 + \int \alpha_2 dy + \iint \alpha_3 dy dy = 0 \tag{81}$$

$$\frac{\partial w_2^{\Delta}(\ell,0,t)}{\partial y} = \frac{\partial w_2(\ell,0,t)}{\partial y} + \alpha_2 + \int \alpha_3 dy = \frac{\partial w_1(\ell,0,t)}{\partial x} \tag{82}$$

$$\frac{\partial^2 w_2^{\Delta}(\ell,0,t)}{\partial y^2} = \frac{\partial^2 w_2(\ell,0,t)}{\partial y^2} + \alpha_3 = \frac{\partial^2 w_1(\ell,0,t)}{\partial x^2} \tag{83}$$

where the D superscripts indicates the admissible functions series has been changed or modified. The modified admissible function approximating the transverse vibrations are,

$$w_2^{\Delta}(\ell,y,t) = \left(\frac{y}{\ell}\right)^4 q_3(t) + \left(\frac{y}{\ell}\right)^5 q_4(t) - \left\{\left(\frac{2}{\ell^2}\right)q_1(t) + \left(\frac{3}{\ell}\right)q_2(t) + \left(\frac{4}{\ell}\right)q_3(t) + \left(\frac{5}{\ell}\right)q_4(t)\right\}y - \left\{\frac{2}{\ell^2}q_1(t) + \frac{6}{\ell^2}q_2(t) + \frac{12}{\ell^2}q_3(t) + \frac{20}{\ell^2}q_4(t)\right\}\frac{y^2}{2} \tag{84}$$

and the modified longitudinal approximation is,

$$u_2(\ell,y,t) = \left(-\dot{q}_1(t) - \dot{q}_2(t) - \dot{q}_3(t) - \dot{q}_4(t)\right) \tag{85}$$

Lagrange equations now follow after writing the kinetic and potential energy terms for the system using equations (70, 84, 85). The Lagrangian is defined as,

$$L = T - V \tag{86}$$

were T and V refer to the kinetic and potential energy terms formed using the approximation series. The Lagrange equations are written,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{87}$$

The mass (M) and stiffness (K) matrices resulting from the application of the Lagrange equations are,

$$M = \begin{bmatrix} .06235801602 & .1004966240 & .1425326081 & .2054076229 \\ .1004966240 & .1744274606 & .2554968583 & .3767794878 \\ .1425326081 & .2554968583 & .3792915718 & .5642843680 \\ .2054076229 & .3767794878 & .5642843680 & .8449496714 \end{bmatrix} \tag{88}$$

and

$$K = \begin{bmatrix} 86.50495318 & 195.6446664 & 260.8595551 & 434.7659253 \\ 195.6446664 & 521.7191102 & 717.3637764 & 1239.082887 \\ 260.8595551 & 717.3637764 & 1147.782043 & 1956.446662 \\ 434.7659253 & 1239.082887 & 1956.446662 & 3416.017982 \end{bmatrix} \tag{89}$$

for the 4 generalized coordinate approximation model. The results of the analysis compared to Bishop are listed in Table 9.4.

Table 9.4. Results of the L-shaped Cantilever Beam Model.

	Modeling Method				
	Modified Admissible (number of gen. coords.)				Exact Solution*
	1	2	3	4	
frequency 1	37.3419	30.4247	30.06667	29.9097	29.7541
frequency 2		99.2495	87.3854	81.6174	80.8239
frequency 3			1181.3157	418.2432	400.0280
frequency 4				1223.6550	587.7362

* Blevins, 1995

The result compare extremely well with the exact solution developed by Bishop. This is a complex structure model using the MAF-SS method and yielding excellent results.

9.6 Summary

The MAF-SS method is detailed using several examples. The benefits include convergence from above the exact eigenvalues, global eigenfunction estimations, and a methodical means to model system using admissible functions. A unique feature is the delineation of connection constraints.

10. Isolation Introduction

10.1 Introduction

Traditionally most vibration isolation problems have been examined with simplified models and solved through a passive approach. In the majority of engineering applications an adequate solution can be found with this method. This unfortunately is changing with the evolution of more complex and precise engineering systems. The development of multiple-degree-of-freedom isolation systems is a current problem of interest. This chapter will present the concept of transmissibility, basic isolation principles and an overview of research in the isolation field.

10.2 Transmissibility

Transmissibility, is a common way to judge the effectiveness of a vibration isolation system. A more mathematical definition is described as the magnitude of the transmitted action divided by the magnitude of the impressed action or

$$\text{Transmissibility} = \left| \frac{\text{Transmitted Action}}{\text{Impressed Action}} \right| \tag{1}$$

There are more specific forms of transmissibility, the most important are absolute transmissibility and force transmissibility. Absolute transmissibility is the absolute measurement of a motion such as displacement, velocity, or acceleration from a fixed reference. Force transmissibility quantifies force as the transmitted and impressed action.

Remark On Transmissibility Equality

Transmissibility plots for displacement, force, acceleration, and velocity are equal for linear systems being excited by a harmonic disturbance. The term "displacement" transmissibility applies to measuring both the disturbance and response in terms of displacement. This meaning also applies to force, acceleration and velocity. This deserves some further examination. Referring to the original definition of transmissibility equation (1) a more generalized representation of the transmissibility for harmonic disturbances can be written as,

$$\text{Transmissibility} = \left| \frac{\text{Transmitted Action}}{\text{Impressed Action}} \right| = \left\{ \frac{\text{Amplitude of Responce}}{\text{Amplitude of Excitation}} \right\} \qquad (2)$$

A representation of a harmonic disturbance and response is written,

$$\begin{aligned} u(t) &= U\sin(\omega_{dr}(t)) \leftrightarrow \text{harmonic excitation} \\ x(t) &= X\sin(\omega_{dr}(t)+\theta) \leftrightarrow \text{harmonic response} \end{aligned} \qquad (3a,b)$$

where velocity and acceleration information is the first and second derivatives of these forms. When the amplitudes of these forms are placed into equation (2) the result reduces to equal the displacement transmissibility. Therefore it can be stated for harmonic motion,

Force Transmissibility = Displacement Transmissibility = Velocity Transmissibility = Acceleration Transmissibility .

The concept of transmissibility is used to examine the performance of the isolation control technique developed in chapter 11.

10.3 Basic Isolation Principles

The simplest isolator is a spring. An example of this type of isolator is a steel coil spring. Figure 10.1 is a schematic of a single-degree-of-freedom system. The diagram contains both base and force disturbances for illustrative purposes. These two types of disturbances will be considered separately.

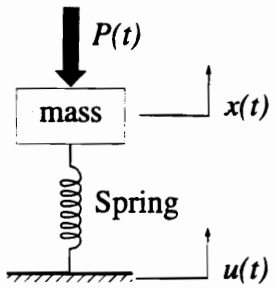


Figure 10.1. Single-Degree of Freedom Isolation System.

Force Excitation

A force disturbance scenario is developed first. The case of the rigid connection is easily solved. By observation the force transmissibility is 1 for all frequencies. The spring isolator is a little more complex. The equations of motion are detailed below. The spring constant is represented by k and the mass by m, $P(t)$ is the disturbance force.

$$m\ddot{x}(t) = -kx(t) - P(t) \rightarrow \ddot{x}(t) = -\omega^2 x(t) - \frac{P(t)}{m} \qquad (4a,b)$$
$$-F_T(t) = kx(t)$$

$F_T(t)$ represents the force transmitted to the base, and ω is equal to $\sqrt{\frac{k}{m}}$. These equations can be converted to the frequency domain.

$$s^2X(s) = -\omega^2X(s) - \frac{P(s)}{m}$$
$$-F_T(s) = kX(s) \qquad (5a, b)$$

The above equations can be combined and rewritten to form the transfer function between the transmitted and impressed force.

$$\frac{F_T(s)}{P(s)} = \frac{\omega^2}{s^2 + \omega^2} \qquad (6)$$

Base Excitation

Base excitation ($u(t)$ is a harmonic motion, $P(t)$ =0) can be examined in a similar fashion. Once again the case of a rigid connection will be examined first and then the simple spring isolator.

The result is the same for the rigid connection, force transmissibility equals 1. The base excitation transmissibility is developed below for a spring isolator.

$$m\ddot{x}(t) = -kx(t) + ku(t) \rightarrow \ddot{x}(t) = -\omega^2x(t) + \omega^2u(t) \qquad (7)$$

Converting the above equation into the frequency domain yields,

$$s^2X(s) = -\omega^2X(s) + \omega^2U(s) \qquad (8)$$

The transfer function between the transmitted and impressed displacement can now be written.

$$\frac{X(s)}{U(s)} = \frac{\omega^2}{s^2 + \omega^2} \tag{9}$$

Examining equation (6) and (9) it is evident that they are the same. When the driving or disturbance frequency divided by the natural frequency of the system is less than $\sqrt{2}$ no isolation occurs. Amplification actually occurs where the frequency ratio approaches 1. This situation is one of the key principles in vibration isolation design. Namely the natural frequency or corner frequency of the isolator must be set through design methods to be below the disturbance frequencies. The area above $\sqrt{2}$ or the cross-over point is the region where it is desired to have disturbance frequencies occur.

10.4 Research Review

The subject of vibration isolation is not considered by many to be a recondite field. It is easy to find references and gain an understanding of the basic problems and solutions methods used. This section overviews literature where more exacting isolation methods are required and established texts on isolation. Two applications have received substantial recent attention, vibration isolation of flexible structures in space and earthquake engineering problems.

Traditional Isolation Methods

Most references in this category present the problem and a passive solution methodology. The solution emphasis has been on the use of spring and damper systems. Several authors have help define and develop the vibration isolation field (Den Hartog, 1934, Crede, 195, Ruzicka and Derby, 1971, Snowden, 1973, Inman, 1994). The focus of much of the early work centered on the isolation of heavy machinery or engines, although the problem presents itself in many different applications. Snowden details many standard solutions to these types of problems. Mainly solutions using elastomeric isolators are examined. This trend of elastomeric passive solutions has continued. The current trend is to enhance these passive methods with active components.

Literature Survey

Researches are trying to more precisely examine isolation methods and find solutions that can deliver reliable, precise and practical solutions. Researchers are currently seeking semi-active or active solutions to many isolation problems.

A performance trade-off may occur between passive and active isolator methods. Karnopp (1973) considers low-order passive, semi-active (a combination of active and passive methods) and active isolator systems. He compares the advantages of each showing an active system may achieve better performance. Although Miller (1988) found by comparing passive, semi-active and fully active isolators based on absolute and relative velocity feedback that in some instance depending on the performance criteria the less sophisticated method proved better. The primary benefit of active isolation has been seen at low frequencies for a least-mean-square (LMS) filter algorithm (Jenkins, 1990, Chen, 1990). Active/passive isolation techniques may offer more precise and robust isolation. The issue of low and high frequency isolation may require different design

techniques (Tomlinson, 1982). This problem is evident in engine suspension systems. A successful approach has been through active control using a combination of actuators, phase angle control and with an active fluid mount (Ishihama, 1992, Vahadati, 1993). Broadband isolation was investigated by Waters (1988). The isolation of two flexible structures increases the complexity of the problem. The use of passive isolators in conjunction with tuned mass absorbers or the addition of base masses has proved effective (Snowden, 1973). Narrow band compensation when knowledge of the disturbance spectrum was available was investigated by Scribner (1990). Active control has been applied to developing microgravity experiment isolation systems. Tzou (1989) uses a PVDF active isolator. Stampleman (1990) also examines the use of PVDF film to develop a low frequency microgravity isolation system. The methods and control techniques for active solution are varied.

11. Substructure Synthesis Formulation and Active Isolation

11.1 Introduction

The active isolation method presented is linked directly to the substructure synthesis (SS) formulation that uses modified admissible functions (MAF) to form a structural model. The MAF-SS formulation is unique in that all coupling constraints are delineated during the model formulation in global generalized coordinates. Other methods approximate (substructure synthesis) boundary conditions or require further analysis (finite element analysis, component mode synthesis) to formulate the constraints. The most advanced substructure synthesis formulation only approximates geometric and kinematic constraints with separate entities which are kinematic chains and quasicoordinates (Meirovitch, 1981). The finite element analysis (for example presented by Paz, 1991) does not have the ability to adapt to different coupling constraints between finite elements and constraints are lumped together as forces. The component mode synthesis formulation requires a transformation using the coupling constraints but this results in a loss of physical meaning of the final model, constraints are written in local coordinate terms (see Thomson for instance, 1994). It is demonstrated in the following work that decoupling substructures results in the isolation of the substructures. The investigation also presents an analysis showing positive feedback of terms proportional to the constraints formed during the MAF-SS modeling reduce the coupling of the substructures resulting in increased isolation. Since this method centers on the reduction of constraints, the formulation is independent of disturbance information. This isolation method is applicable to multivariable feedback and multiple-degree-of-freedom (MDOF) systems. A standard eigenstructure assignment technique is combined with the MAF-SS technique

to produce a multivariable feedback control mechanism through which substructure constraints can be modified with precision. This may lead to improved isolation between MDOF substructures.

The isolation control formulation investigated here differs from others because feedback terms are directly related to the removal of constraints between substructures. The idea of decoupling structures is usually associated with decentralized control methods. Decoupling is a means to form local controllers that act independently. The decoupling usually occurs mathematically, for example a localized control is developed for an uncoupled system that is then coupled using component mode synthesis methods (Su and Craig, 1990, Babuska and Craig, 1993). Theoretical work also has been published where physical decoupling is a prerequisite to local control formulation (Su and Juang, 1994). Active isolation methods usually focus on a type or combination of signals such as force, position, or acceleration to form a control system that improves isolation performance through the minimization of an output measurement related to isolation performance (Leo and Anderson, 1995, Hyde and Anderson, 1994, Lurie, Fanson, and Laskin, 1993, Spanos, Rahman, 1993, Hyland and Phillips, 1995). The focus of this work is to use constraint information gathered in the MAF-SS formulation to develop feedback terms that reduce coupling, leading to modified structural dynamic characteristics or improved isolation performance.

11.2 Active Isolation

Control Goal

The goal of applying active feedback to a system is to isolate substructures. The proposed control achieves isolation through the removal or reduction of constraints

between substructures. The question of whether decoupling or reducing the coupling between substructures is a valid isolation technique is examined now. A two-mode decoupled system is written as,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{q} \\ \ddot{p} \end{bmatrix}+\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}\begin{bmatrix} q(t) \\ p(t) \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \end{bmatrix}u_d$$
$$y_o=\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} q \\ p \end{bmatrix} \qquad \text{(1a, b)}$$

where the input disturbance is on m_1 and output is measured by the motion of m_2. The system is transformed into,

$$\begin{bmatrix} \ddot{q} \\ \ddot{p} \end{bmatrix}+M^{-1}K\begin{bmatrix} q \\ p \end{bmatrix}=M^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}u_d \qquad (2)$$

and if a Laplace transformation is applied (assuming initial conditions of zero) it is written as,

$$\begin{bmatrix} Q(s) \\ P(s) \end{bmatrix}\left[s^2+M^{-1}K\right]=M^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}U_d$$
$$Y_o(s)=\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} Q(s) \\ P(s) \end{bmatrix} \qquad \text{(3a, b)}$$

further manipulations result in,

$$\left[s^2 + M^{-1}K\right]\begin{bmatrix} Q(s) \\ Y_o(s) \end{bmatrix} = M^{-1}\begin{bmatrix} U_d(s) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} Q(s) \\ Y_o(s) \end{bmatrix} = \left[s^2 + M^{-1}K\right]^{-1} M^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} U_d(s) \qquad \text{(4a, b)}$$

and recalling that the formula for a 2x2 matrix inverse is

$$M^{-1} = \frac{\begin{bmatrix} m_2 & 0 \\ 0 & m_1 \end{bmatrix}}{m_1 m_2} \qquad (5)$$

the system is cast as a transfer function as,

$$\begin{bmatrix} Q(s) \\ Y_o(s) \end{bmatrix} = \frac{\left[s^2 + M^{-1}K\right]^{-1}\begin{bmatrix} m_2 & 0 \\ 0 & m_1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{m_1 m_2} U_d(s)$$

$$\begin{bmatrix} Q(s) \\ Y_o(s) \end{bmatrix} = \frac{\left[s^2 + M^{-1}K\right]^{-1}\begin{bmatrix} m_2 \\ 0 \end{bmatrix}}{m_1 m_2} U_d(s) \qquad \text{(6a, b, c)}$$

$$Y_o(s) = \frac{\left[s^2 + M^{-1}K\right]^{-1} 0}{m_1 m_2} U_d(s) = 0 U_d(s)$$

The transfer function between the input $U_d(s)$ and $Y_o(s)$ is

$$\left|\frac{Y_o}{U_d}\right| = 0 \qquad (7)$$

which is zero for the uncoupled system. This is an intuitive result but demonstrates the validity the control goal, a decoupled structure is isolated.

Feedback System Model

The closed loop transfer function is useful for examining a positive feedback isolation control. The function indicates how the poles and zeros change. A state space system is written as,

$$\begin{aligned} \dot{z}(t) &= Az(t) + B_d u_d + B_f u_f \\ y_o(t) &= C_o z(t) \\ y_f(t) &= C_f z(t) \end{aligned} \qquad \text{(8a, b, c)}$$

where $z(t)$ are the system states, $u_d(t)$ and $u_f(t)$ the disturbance and feedback input terms and $y_f(t)$ and $y_o(t)$ the feedback output functions The matrix A is related to the system model. The matrices B_d, B_f, correspond to the disturbance input and feedback input. The matrices C_o and C_f are related to the output and feedback terms. Another relation,

$$u_f = G_f y_f(t) \qquad (9)$$

represents a feedback signal. This system is written compactly as,

$$\begin{aligned} \dot{z}(t) &= \left(A + B_f G_f C_f\right) z(t) + B_d u_d \\ y_o(t) &= C_o z(t) \end{aligned} \qquad \text{(10a, b)}$$

and taking the Laplace transform of this (assuming zero initial conditions) system results in,

$$\begin{aligned} Z(s)s &= \left(A + B_f G_f C_f\right)Z(s) + B_d U_d(s) \\ Y_o(s) &= C_o Z(s) \end{aligned} \qquad \text{(11a, b)}$$

Combining the above equations into,

$$Y_o(s) = C_o\left[sI - \left(A + B_f G_f C_f\right)\right]^{-1} B_d U_d(s) \qquad (12)$$

or

$$Y_o(s) = \frac{C_o adj\left(sI - \left(A + B_f G_f C_f\right)\right)B_d}{\left|sI - \left(A + B_f G_f C_f\right)\right|} U_d(s) \qquad (13)$$

results in the closed loop transfer function for a single feedback term.

Scalar Gain Feedback

A diagram of the active feedback system is presented in Fig. 11.1. This system contains a feedback signal operated on by a scalar gain.

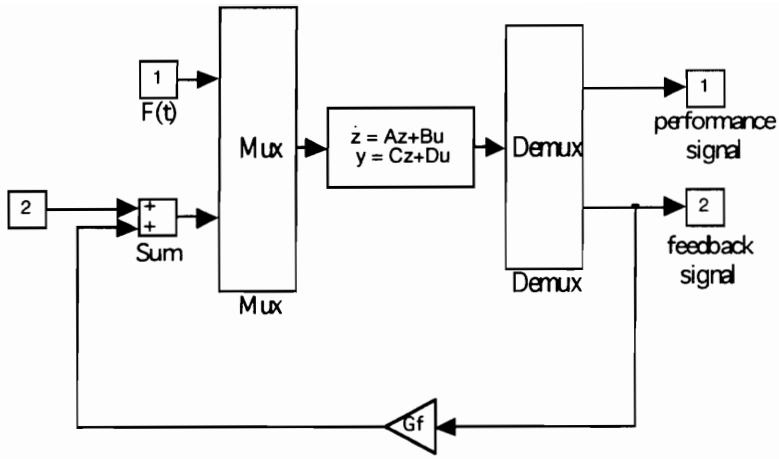


Figure 11.1. Feedback Diagram.

This is the block diagram that corresponds to the positive constraint feedback method. The method involves first choosing a constraint to remove from the constraints delineated by the MAF-SS method. The next step is to feed this term back using a positive gain. The results of the control is then examined using a transmissibility plot.

Decreasing Versus Increasing Coupling

The issue of whether to use negative feedback or positive feedback is subtle. It appears when examining a negative feedback magnitude response between force and displacement that increasing the negative feedback improves isolation by lowering the magnitude plot. This is because the magnitude plot between a force disturbance and a displacement in the limit is,

$$\frac{1}{stiffness} \tag{14}$$

and increasing the stiffness decreases the magnitude. The reality is that increasing stiffness increases coupling. The result of increasing the coupling is a system that in the limit will be a rigid connection. This has transmissibility in the best case of 1. The goal of isolation is to reduce or attenuate the coupling between substructures. This is accomplished by using positive feedback.

11.3 Isolation Demonstration

Equations of motion

A cantilever beam with a spring and mass attached at the end is used to demonstrate the control methods. The system is modeled with five generalized coordinates and approximates five modes of vibration. The model is modified to include nonconservative work terms associated with control and disturbance energy. The isolation goal is to isolate or decouple substructure 1 from substructures 2. The disturbance is towards the base of the beam on substructure 1. A point located at 2/3 the total length is used as the point of interest to perform the decoupling control.

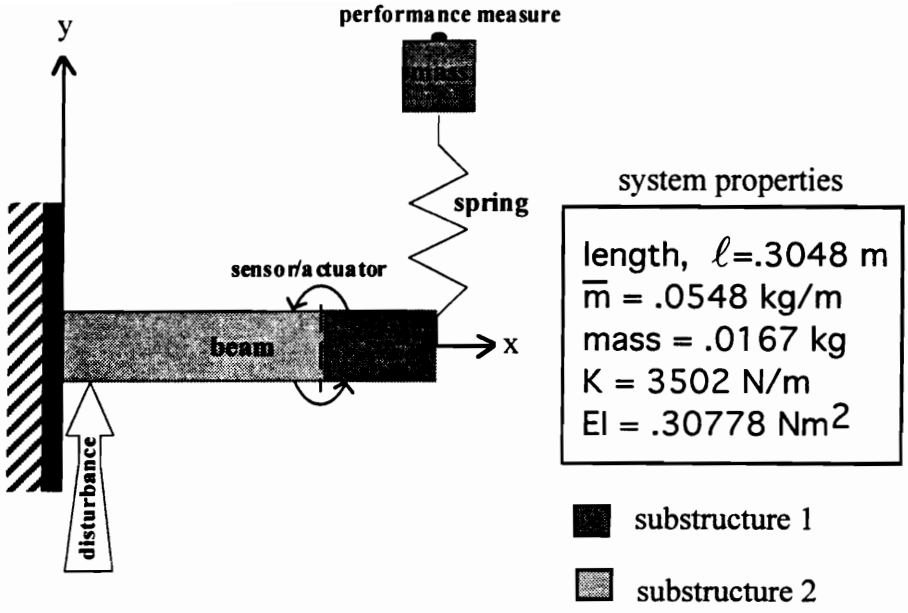


Figure 11.2. Cantilever beam-spring-mass system.

Incorporating the driving point force into the system model is briefly reviewed here. The variation of the nonconservative work (δW_{nc}) resulting from a positive force disturbance located at,

$$x = \frac{\ell}{10} \tag{15}$$

is written,

$$\delta W_{nc} = F\left(\frac{\ell}{10}, t\right)\left\{\sum_{k=1}^{m} \phi_k\left(\frac{\ell}{10}\right)\delta q_k\right\}_2 \tag{16}$$

when the system is modeled by a mode approximation of the form,

$$y_i(x,t) = \sum_{k=1}^{m} \phi_k(x) q_k(t) \tag{17}$$

where m is the number of generalized coordinates. The nonconservative work then becomes,

$$\delta W_{nc} = \sum_{k=1}^{m} Q_k \delta q_k \tag{18}$$

this form fits the generalized forces (Q) form of the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} - Q_k = 0 \tag{19}$$

where

$$Q_k = F\left(\frac{\ell}{10}\right)\phi_k \tag{20}$$

in this case.

Illustration of Constraint Feedback

The goal is to isolate substructures 2 from the disturbance on substructure 1 pictured in Fig. 11.2. The term isolate could be replaced with decouple. The idea is not to control the structure and mitigate vibrations the goal is to reduce interactions between substructures. This step stems from the MAF-SS model formulation where uncoupled or perfectly isolated substructures were coupled using constraint relations. Remembering

connection constraints between the substructure 1 and 2 for beam were determined to be in chapter 9,

$$w_1\left(\frac{2\ell}{3},t\right)=w_2\left(\frac{2\ell}{3},t\right) \tag{21}$$

$$\frac{dw_1\left(\frac{2\ell}{3},t\right)}{dx}=\frac{dw_2\left(\frac{2\ell}{3},t\right)}{dx} \tag{22}$$

$$\frac{d^2w_1\left(\frac{2\ell}{3},t\right)}{dx^2}=\frac{d^2w_2\left(\frac{2\ell}{3},t\right)}{dx^2} \tag{23}$$

during the MAF-SS formulation. Eliminating these connections would tend to decouple the substructures or create isolated structures. This is the control goal, reduce coupling to improve isolation. The moment constraint or natural boundary condition is one constraint to be eliminated or reduced. The goal is to eliminate the moment acting from substructure 1 on substructure 2 resulting from,

$$M=EI\frac{d^2w_2\left(\frac{2\ell}{3}\right)}{dx^2} \tag{24}$$

through active feedback. A term proportional to

$$\frac{d^2w_2\left(\frac{2\ell}{3},t\right)}{dx^2} \tag{25}$$

is a logical feedback choice. The feedback term is positive since the desire is to eliminate the reaction forces between the substructures. The feedback term is proportional to the second derivative at the point of connection. The work needed to reduce coupling between structures is for the $m=5$ model of the form,

$$\delta W_{nc} = G_f \left\{ \sum_{k=1}^{5} \frac{\partial^2 \phi_k \left(\frac{2\ell}{3} \right)}{\partial x^2} \delta q_k \right\} \tag{26}$$

where Gf is the feedback gain. The operator (") is used to replace,

$$\frac{\partial^2}{\partial x^2} \tag{27}$$

in some of the following equations. The feedback is positive, this has the effect of reducing the stiffness at the coupling point. The block diagram form in Fig. 11.1 is used to simulate the feedback method. The related state space form is,

$$\begin{aligned} \dot{z} &= Az + Bu \\ y &= Cz + Du \end{aligned} \tag{28}$$

where

$$A = \begin{bmatrix} zeros(5) & eye(5) \\ -M^{-1}K & zeros(5) \end{bmatrix} \tag{29}$$

and where *zeros(5)* is a 5x5 matrix of zeros and *eye(5)* is a diagonal 5x5 matrix of 1's. The output matrix has the form,

$$B = \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ M^{-1}\begin{bmatrix} b1 & b2 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \tag{30}$$

where

$$\begin{aligned} b1 &= \left[\phi_1\left(\frac{\ell}{10}\right) \quad \cdots \quad \phi_{m-1}\left(\frac{\ell}{10}\right) \right]^T \\ b2 &= \left[\phi_1''\left(\frac{2\ell}{3}\right) \quad \cdots \quad \phi_{m-1}''\left(\frac{2\ell}{3}\right) \right]^T \end{aligned} \tag{31}$$

where superscript *(T)* indicates a transpose operation.

$$C = \begin{bmatrix} c1 & zero \\ c2 & zero \end{bmatrix} \tag{32}$$

where

$$\begin{aligned} c1 &= \left[\phi_1\left(\frac{\ell}{10}\right) \quad \cdots \quad \phi_{m-1}\left(\frac{\ell}{10}\right) \right] \\ c2 &= \left[\phi_1\left(\frac{2\ell}{3}\right) \quad \cdots \quad \phi_{m-1}\left(\frac{2\ell}{3}\right) \right] \end{aligned} \tag{33}$$

and *zer* is a row vector of 6 zeros. The matrix *D* is a 3x2 matrix of zeros. The states and input terms are written,

$$z = \begin{bmatrix} q_{5x1} \\ \dot{q}_{5x1} \end{bmatrix} \tag{34}$$

and

$$u = \left[F(t) \quad G_f y_2\left(\frac{2\ell}{3}\right) \right] \tag{35}$$

receptively. A proportional damping matrix was added to the system to better approximate the structure. This alters the form of A to,

$$A = \begin{bmatrix} zeros(5) & eye(5) \\ -M^{-1}K & -\frac{1}{10^3} M^{-1}K \end{bmatrix} \tag{36}$$

This approximately sets the damping to a maximum of .3% in the system. These terms are incorporated into a Simulink model in the Matlab programming environment. Feedback terms proportional to the displacement and slope at $x = 2\ell/3$ are formed in the same manner. The system model for these cases requires the change of equation (31) and (32) to reflect strain or displacement.

Determining G_f

The maximum gain yielding a stable system is sought as the final feedback gain. This gain yields the greatest reduction in stiffness which in turn results in the greatest improvement in isolation between substructures. The gains were established through root locus examination and trial and error. Stability was insured by verifying the real part of closedloop eigenvalues were all negative.

Feedback Results

The results of applying the positive constraint feedback method are presented in Figs. 11.3 and 11.4. The information in Fig. 11.3 details the open and closedloop system in a

Bode format. The transmissibility of the openloop and closed loop systems is displayed in Fig. 11.4. This figure indicates that each of the feedback terms which are related to the constraint displacement, slope and strain improves the isolation performance. The displacement constraint feedback radically alters the low frequencies almost eliminating the first modes transmissibility. The other frequencies are lowered as well. The attenuation effect reduces as the frequency increase since more energy is required to effect these higher modes. The feedback term related to slope also show an improvement in isolation. Although at frequencies below the first mode the transmissibility is increased. This is a result of the structure having larger static bending due to the stiffness decrease. Strain feedback also demonstrated improvement in the isolation performance. It demonstrates a significant effect on the second mode of vibration. Overall an improvement in isolation performance was demonstrated by the concept of positive constraint feedback where the constraints are calculated using the MAF-SS modeling method.

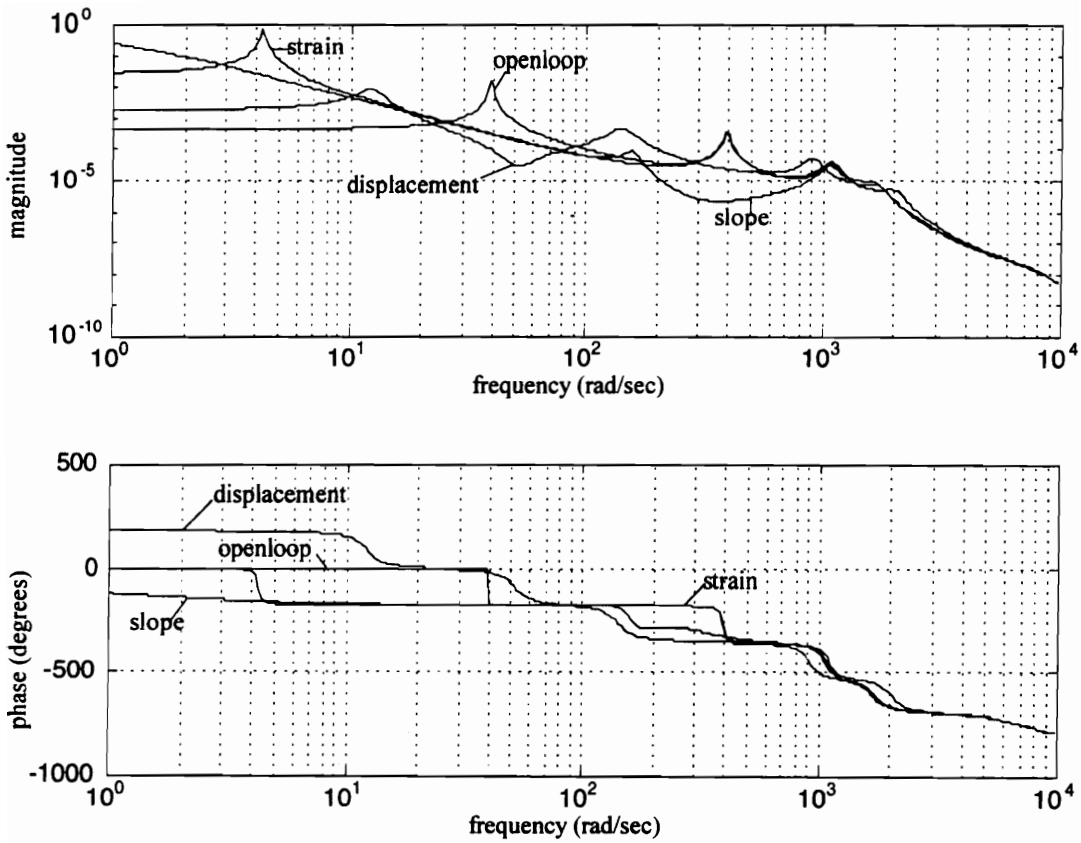


Figure 11.3. Bode plot of constraint feedback.

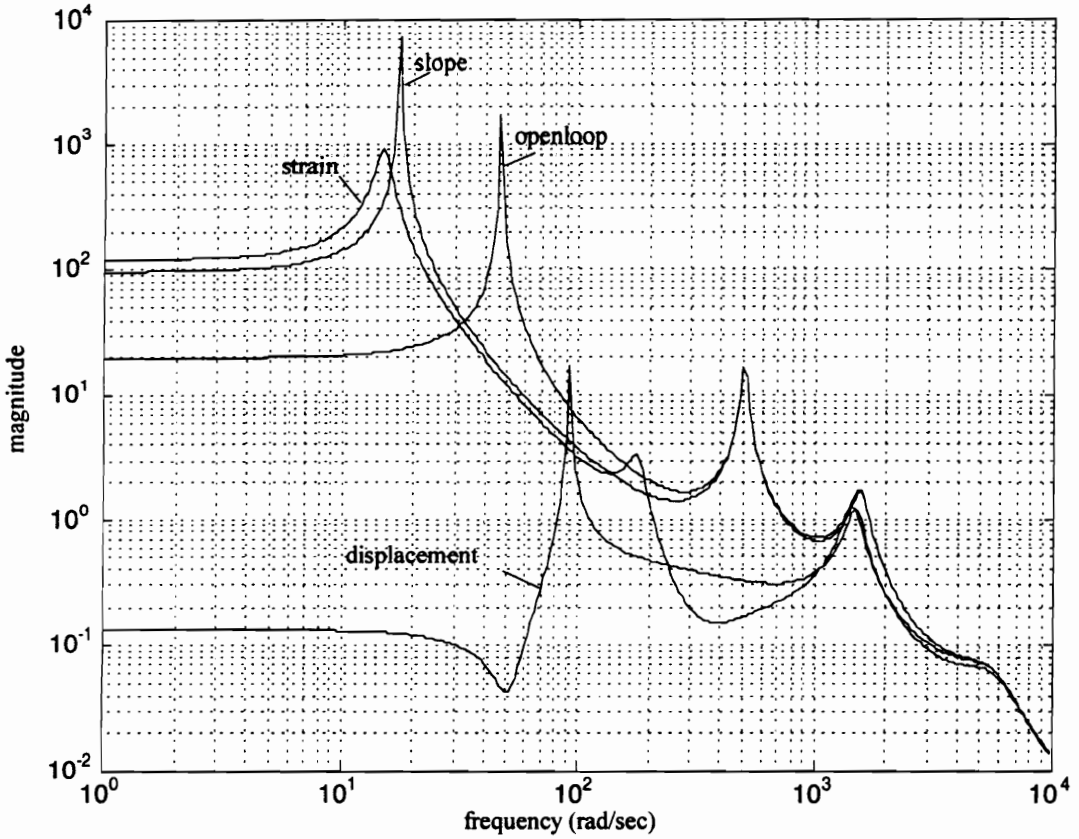


Figure 11.4. Transmissibilty of constraint feedback.

The MAF-SS method has the unique ability to examine the constraints between substructures. The information resulting from the MAF-SS modeling technique includes, what are the substructure constraints and to what extend do they influence the system dynamics. The positive constraint feedback method presented here takes advantage of this property to determine feedback signals that will reduce the coupling between substructures. This reduction in coupling yields substructures with improved isolation characteristics.

Eigenstructure Assignment Method

The new substructure synthesis (MAF-SS) technique allows modification of the constraints modeled between substructures. This ability may offer a unique tool to alter structures through the manipulation of substructure constraints using active feedback. The modification of the substructure constraints has application to the vibration isolation control problem. The approach of using altered substructure constraint models to form active controller parameters is summarized here. The beam-spring-mass system used to explore constraint feedback is used again to demonstrate the principle of using the MAF-SS technique to define a goal matrix that models the modified substructure constraints. A standard eigenstructure assignment method is then used to define a gain matrix that achieves this system with active feedback. The system is pictured in Fig. 11.5. A disturbance force is located near the clamped end at a driving point ($x = \ell/10$) and a performance measurement taken on the mass (w_{mass}). The goal is to isolate the mass from the disturbance by altering the substructures constraints at the point where the sensor/actuator pair is located ($x = 2\ell/3$).

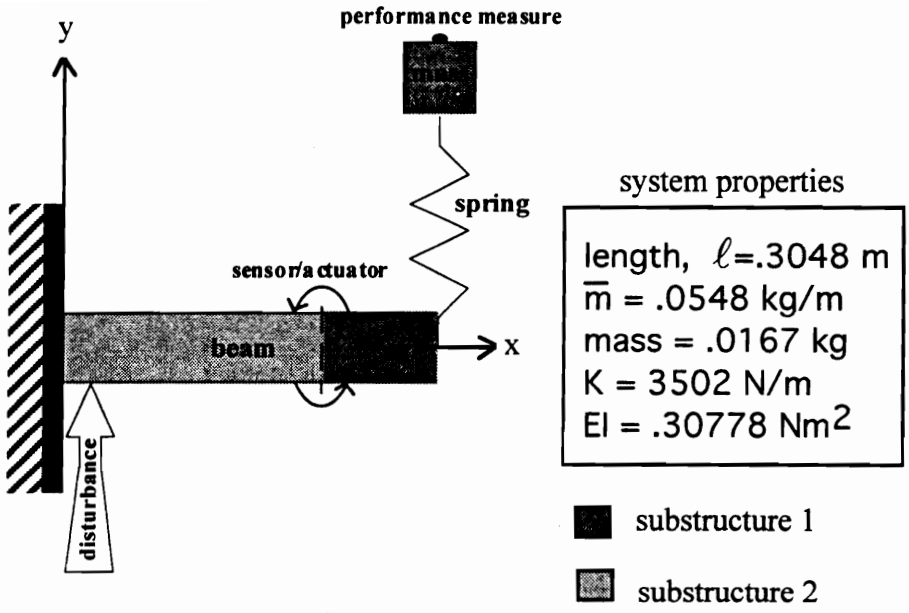


Figure 11.5. Beam-spring-mass system.

The substructure synthesis method presented in the dissertation has the unique property of allowing the methodical manipulation of the constraints between substructures during the modeling process. The ability to identify and modify substructure coupling constraints suggests it is useful to define a desired or goal system that could be obtained using active feedback leading to a modified structure or improved vibration isolation characteristics. The eigenstructure assignment method is useful to define a feedback gain matrix once a desired system is known. This is the procedure that is demonstrated. The system detailed in Fig. 11.5 is modified by active feedback. The feedback matrices are defined by a standard eigenstructure assignment technique. The goal or desired system is developed using the substructure synthesis procedure (MAF-SS).

The system model of Fig. 11.5 is comprised of mass, stiffness and damping matrices. These matrices are cast into a state space representation as,

$$\dot{z} = Az + B_d u_d + B_f u_f$$
$$y_o = C_o z + C_f z \tag{37}$$

The state space model may be transformed into a transfer function format of

$$Y_o(s) = C_o\left[sI - \left(A + B_f G_f C_f\right)\right]^{-1} B_d U(s) \tag{38}$$

the active feedback matrix is defined using this representation. The closedloop system matrix is,

$$A + B_f G_f C_f = A_g \tag{39}$$

where and Ag is the goal or desired system defined using MAF-SS. The matrix gain needed to realize Ag using equation (38) is,

$$G_f = \left(B_f^T B_f\right)^{-1} B_f^T \left(A_g - A\right) C_f^T \left(C_f C_f^T\right)^{-1} \tag{40}$$

The development of equation (40) is general enough to be applied to any system modeled in the state space form of equation (37).

The defined standard eigenstructure assignment technique and the MAF-SS method are used to examine the effect of changing constraints between substructures. Two examples sets are presented here. The first uses two generalized coordinates to represent the motion of Fig. 11.5. The goal of this example is to demonstrate the effect and ability to modify substructure boundary constraints using the control concept. This example

contains two situations where isolation performance is actually reduced. This should not be viewed as a weakness or problem with the method since the examples are intended not only to show the isolation application but a more general new active structural modification technique. The second case illustrates the concept of effecting the strain boundary, and uses a larger number of generalized coordinates in the model.

Four different constraint boundaries are examined. These conditions are listed below where subscripts relate the motion *(y)* to substructure 1 or 2.

<u>case 1 openloop system</u>

$$\begin{aligned} y_1\left(\frac{2}{3}\ell\right) &= y_2\left(\frac{2}{3}\ell\right) \\ y_1'\left(\frac{2}{3}\ell\right) &= y_2'\left(\frac{2}{3}\ell\right) \\ y_1''\left(\frac{2}{3}\ell\right) &= y_2''\left(\frac{2}{3}\ell\right) \end{aligned} \tag{41}$$

<u>case 2 pinned-constraint</u>

$$\begin{aligned} y_1\left(\frac{2}{3}\ell\right) &= y_2\left(\frac{2}{3}\ell\right) = 0 \\ y_1'\left(\frac{2}{3}\ell\right) &= y_2'\left(\frac{2}{3}\ell\right) \\ y_1''\left(\frac{2}{3}\ell\right) &= y_2''\left(\frac{2}{3}\ell\right) = 0 \end{aligned} \tag{42}$$

<u>case 3 sliding-constraint</u>

$$\begin{aligned} y_1\left(\frac{2}{3}\ell\right) &= y_2\left(\frac{2}{3}\ell\right) \\ y_1'\left(\frac{2}{3}\ell\right) &= y_2'\left(\frac{2}{3}\ell\right) = 0 \\ y_1''\left(\frac{2}{3}\ell\right) &= y_2''\left(\frac{2}{3}\ell\right) \\ y_1'''\left(\frac{2}{3}\ell\right) &= y_2'''\left(\frac{2}{3}\ell\right) = 0 \end{aligned} \tag{43}$$

case 4 clamped-constraint

$$y_1\left(\frac{2}{3}\ell\right) = y_2\left(\frac{2}{3}\ell\right) = 0$$
$$y_1'\left(\frac{2}{3}\ell\right) = y_2'\left(\frac{2}{3}\ell\right) = 0 \qquad (44)$$
$$y_1''\left(\frac{2}{3}\ell\right) = y_2''\left(\frac{2}{3}\ell\right)$$

These constraints are chosen because they represent the familiar physical constraint conditions possible for a beam (Blevins, 1995). The original model is represented by the openloop case. The MAF-SS method is applied to this original system, modifying it so that the goal models *(Ag)* represent cases 2-4. Equation (40) is then used to calculate the gain matrices necessary to realize these systems composed of modified substructure constraints. The eigenfunction approximations satisfying the MAF-SS connection conditions are for the distributed parameter elements of the four above cases,

case 1 openloop system

$$\phi_1 = \cosh\left(\frac{1.87510407x}{\ell}\right) - \cos\left(\frac{1.87510407x}{\ell}\right) -$$
$$.734095514\left(\sinh\left(\frac{1.87510407x}{\ell}\right) - \sin\left(\frac{1.87510407x}{\ell}\right)\right) \quad \text{for } 0 \le x \le 2\ell/3$$

$$\phi_2 = \cosh\left(\frac{1.87510407x}{\ell}\right) - \cos\left(\frac{1.87510407x}{\ell}\right) - \qquad (45)$$
$$.734095514\left(\sinh\left(\frac{1.87510407x}{\ell}\right) - \sin\left(\frac{1.87510407x}{\ell}\right)\right) \quad \text{for } 2\ell/3 \le x \le \ell$$

case 2 clamped-constraint

$$\phi_1^{\Delta} = \cosh\left(\frac{3(3.92660231)x}{2\ell}\right) - \cos\left(\frac{3(3.92660231)x}{2\ell}\right) -$$

$$1.000777304\left(\sinh\left(\frac{3(3.92660231)x}{2\ell}\right) - \sin\left(\frac{3(3.92660231)x}{2\ell}\right)\right) \quad \text{for } 0 \le x \le 2\ell/3$$

$$\phi_2^{\Delta} = \cosh\left(\frac{3(3.92660231)x}{\ell}\right) - \cos\left(\frac{3(3.92660231)x}{\ell}\right) - \tag{46}$$

$$1.000777304\left(\sinh\left(\frac{3(3.92660231)x}{\ell}\right) - \sin\left(\frac{3(3.92660231)x}{\ell}\right)\right) \quad \text{for } 2\ell/3 \le x \le \ell$$

case 3 sliding-constraint

$$\phi_1^{\Delta} = \cosh\left(\frac{3(2.36502037)x}{2\ell}\right) - \cos\left(\frac{3(2.36502037)x}{2\ell}\right) -$$

$$.982502207\left(\sinh\left(\frac{3(2.36502037)x}{2\ell}\right) - \sin\left(\frac{3(2.36502037)x}{2\ell}\right)\right) \quad \text{for } 0 \le x \le 2\ell/3$$

$$\phi_2^{\Delta} = \cosh\left(\frac{3(2.36502037)x}{\ell}\right) - \cos\left(\frac{3(2.36502037)x}{\ell}\right) -$$

$$.982502207\left(\sinh\left(\frac{3(2.36502037)x}{\ell}\right) - \sin\left(\frac{3(2.36502037)x}{\ell}\right)\right) \quad \text{for } 2\ell/3 \le x \le \ell$$

(47)

case 4 clamped-constraint

$$\phi_1^{\Delta} = .982502215\left(\sinh\left(\frac{3(4.73004074)x}{2\ell}\right) - \sin\left(\frac{3(4.73004074)x}{2\ell}\right)\right) - \cosh\left(\frac{3(4.73004074)x}{2\ell}\right) - \cos\left(\frac{3(4.73004074)x}{2\ell}\right) \quad \text{for } 0 \le x \le 2\ell/3$$

$$\phi_2^{\Delta} = .982502215\left(\sinh\left(\frac{3(4.73004074)x}{\ell}\right) - \sin\left(\frac{3(4.73004074)x}{\ell}\right)\right) - \cosh\left(\frac{3(4.73004074)x}{\ell}\right) - \cos\left(\frac{3(4.73004074)x}{\ell}\right) \quad \text{for } 2\ell/3 \le x \le \ell$$

(48)

These eigenfunction approximations correspond to an equation form of,

$$y_i = \sum_{k=1}^{m} \phi_k q_k \tag{49}$$

which represents the motion of the system, where *i* indicates a particular substructure and *m* the number of generalized coordinates of the system. The effect of applying equation (40) and the MAF-SS technique are demonstrated using two plots. These plots are displacement transmissibility, openloop and closedloop Bode. The transmissibility plot ordinate displays,

$$\left| \frac{\dfrac{y_{mass}}{F_d\left(\frac{\ell}{10}\right)}}{\dfrac{y\left(\frac{\ell}{10}\right)}{F_d\left(\frac{\ell}{10}\right)}} \right| \tag{50}$$

information and Fig. 11.7, the openloop and closedloop Bode plots contains information of the form,

$$\left| \frac{y_{mass}}{F_d\left(\frac{\ell}{10}\right)} \right| \tag{51}$$

on the ordinate. The information along the abscissa is frequency in rad/sec for both Figs. 6 and 7.

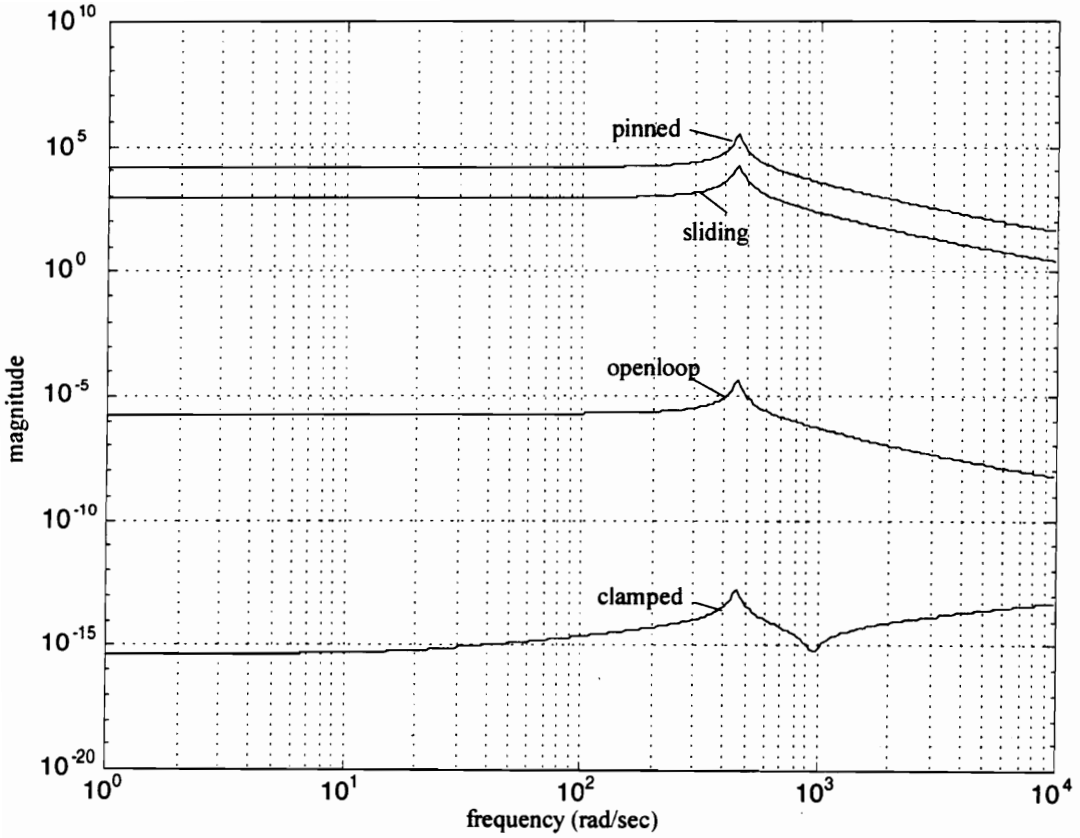


Figure 11.6. Transmissibility for substructure constraint modification.

The transmissibility plot in Fig. 11.6 indicates that the active feedback control has modified the system. The clamped boundary results in the lowering of the transmissibility magnitude. The system is essentially decoupled. This is the ideal isolation result if the goal is to prevent all energy from passing between substructures. The pinned and sliding constraints raised the overall transmissibility levels, this is a result that in order to obtain the specified substructure constraints the coupling stiffness in the off-diagonal terms are increased, this has the effect of increasing stiffness and increasing the coupling between substructures. The clamped-boundary conditions in contrast eliminated the off-diagonal stiffness terms, resulting in decoupling. They closedloop stiffness matrices are listed below to illustrate the effect of the active control. These matrices in conjunction with Fig. 11.6 support the concept that increasing stiffness, increases transmissibility.

case 1 openloop

$$K = \begin{bmatrix} 134.3683587 & .0002102 \\ .0002102 & 3502.512680 \end{bmatrix} \tag{52}$$

case 2 pined-constraint

$$K = \begin{bmatrix} .6679737989e9 & .1529555521e7 \\ .1529555521e7 & 3502.512680 \end{bmatrix} \tag{53}$$

case 3 sliding-constraint

$$K = \begin{bmatrix} .2531320269e7 & 94129.00813 \\ 94129.00813 & 3502.512680 \end{bmatrix} \tag{54}$$

case 4 clamped-constraint

$$K = \begin{bmatrix} 18362.37299 & 0 \\ 0 & 3502.512680 \end{bmatrix} \tag{55}$$

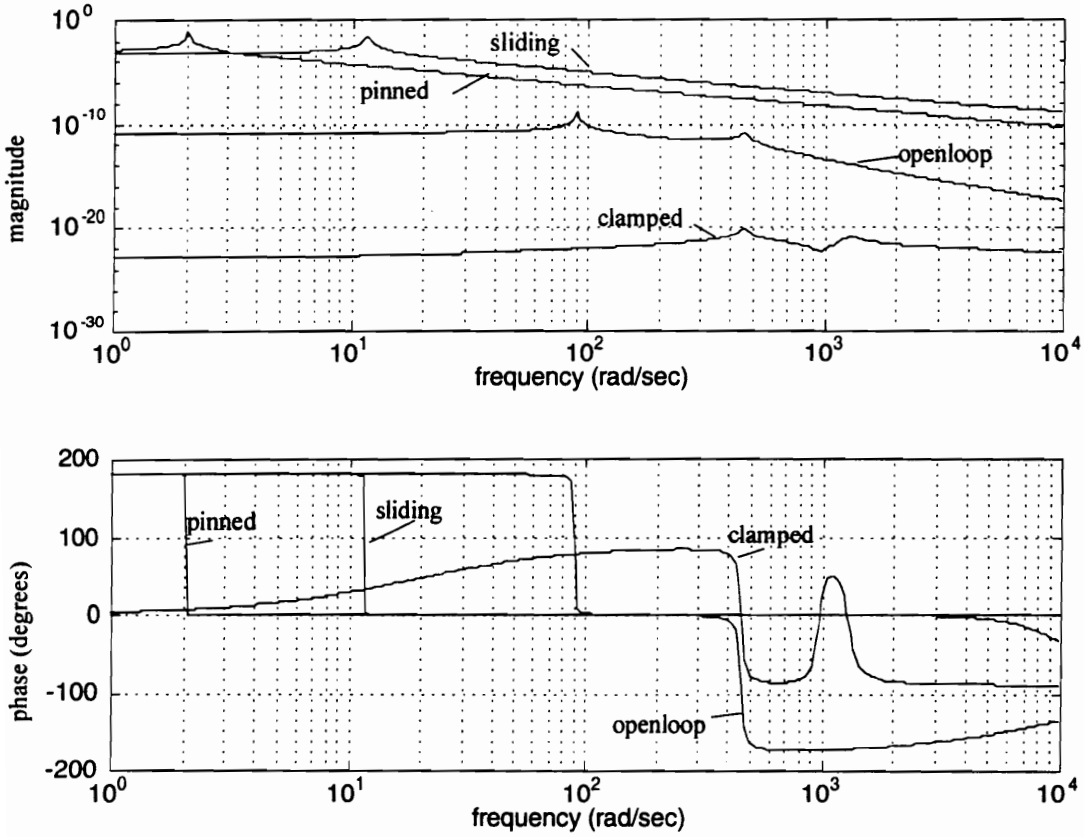


Figure 11.7. Beam-spring-mass magnitude and phase, cases 1-4.

An eigenvalue of case 4 indicted in Fig. 11.7 should correspond with Blevins (1995) solution for a clamped-clamped beam since this should be one of the resulting decoupled substructures. The eigenvalues do correspond when a comparison is made. The eigenvalues are listed in the table below.

Table 11.1. Substructure eigenvalue comparison for case 4.

modified structure, case 4	204.3772 hertz
Blevins, (1995)	201.2823 hertz

The MAF-SS method can be combined with a standard eigenstructure assignment method to form active feedback gains that alter substructure constraints in a predictable manner. This active modification of the constraints may lead to improved isolation characteristics. The ideal isolation case is achieved by creating a clamped-constraint boundary as demonstrated in Fig. 11.6. The new concept of using MAF-SS's unique ability to examine and manipulate substructure constraints and form a desired system model and a standard eigenstructure assignment technique has application to forming active vibration isolation design solutions. This is further demonstrated by examining the effect of modifying the constraint related to strain.

The system in Fig. 11.5 is again used to examine the effect of altering the boundary constraints between substructures using the MAF-SS modeling technique and a standard eigenstructure assignment technique. The difference between this example and the previous one is that different eigenfunction approximations are used and three generalized coordinates are related to the model in addition to different substructure constraints. The constraints for the open and closed loop system are,

case 5 openloop or original system

$$\left.\begin{aligned} y_1\left(\frac{2\ell}{3}\right) &= y_1\left(\frac{2\ell}{3}\right) \\ y_1'\left(\frac{2\ell}{3}\right) &= y_1'\left(\frac{2\ell}{3}\right) \\ y_1''\left(\frac{2\ell}{3}\right) &= y_2''\left(\frac{2\ell}{3}\right) \end{aligned}\right\} \tag{56}$$

case 6 modified strain-constraint

$$\left.\begin{aligned} y_1\left(\frac{2\ell}{3}\right) &= y_1\left(\frac{2\ell}{3}\right) \\ y_1'\left(\frac{2\ell}{3}\right) &= y_1'\left(\frac{2\ell}{3}\right) \\ y_1''\left(\frac{2\ell}{3}\right) &= y_2''\left(\frac{2\ell}{3}\right) = 0 \end{aligned}\right\} \tag{57}$$

The related eigenfunction approximation functions are,

case 5 openloop or original system

$$\left.\begin{aligned} \phi_1 &= x^2 \\ \phi_2 &= x^3 \end{aligned}\right\} \text{ for } 0 \le x \le \ell \tag{58}$$

case 6 modified strain-constraint

$$\left.\begin{aligned} \phi_1 &= x^2 \\ \phi_2 &= x^3 \end{aligned}\right\} \text{ for } 0 \le x \le 2\ell/3 \tag{59}$$

$$\left.\begin{aligned} \phi_1^{\Delta} &= \frac{4}{3}\ell x - \frac{4}{9}\ell^2 \\ \phi_2^{\Delta} &= x^3 - 2\ell x^2 + \frac{24}{9}\ell^2 x - \frac{24}{29}\ell^3 \end{aligned}\right\} \text{ for } 0 \le x \le \ell \tag{60}$$

which also correspond to an equation form of motion represented by equation (49). After the original model (formed using case 5 substructure constraints) is modified to satisfy the constraints of case 6 the eigenstructure assignment method is used to calculate the gain matrix which will achieve the desired system. The openloop and closedloop Bode and displacement transmissibility plots are pictured below.

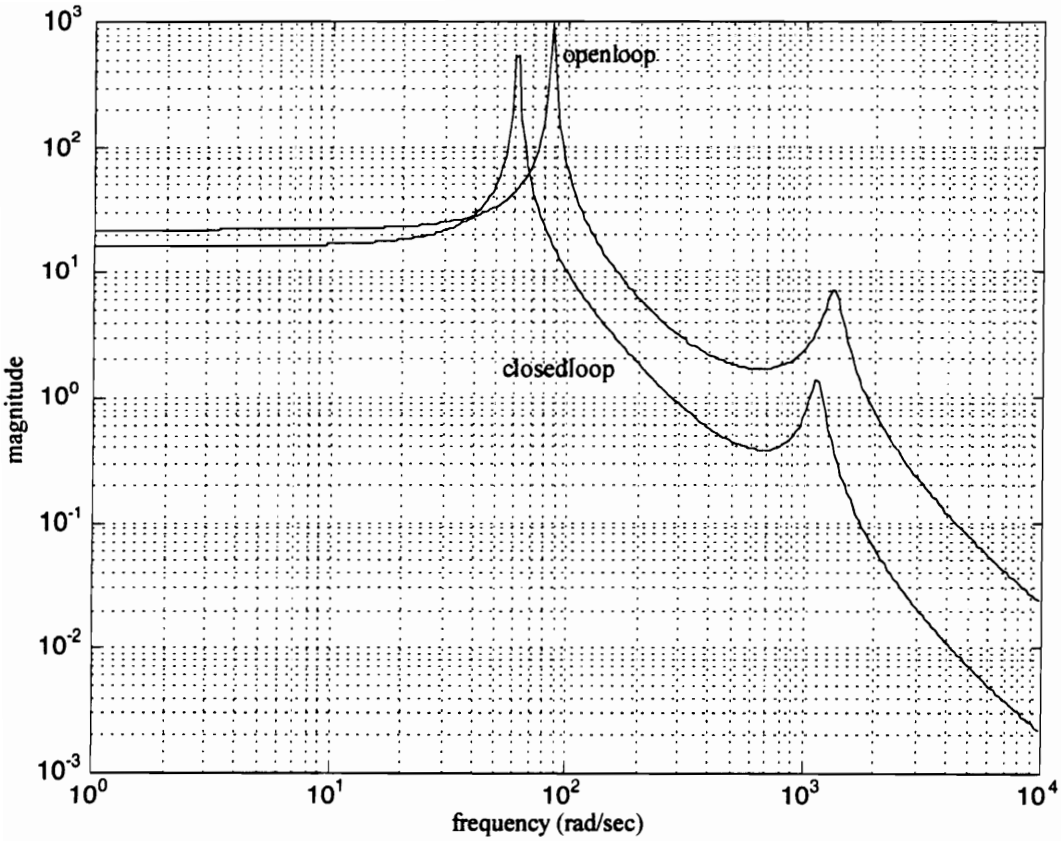


Figure 11.8. Displacement tranmissibilty, cases 5 &6.

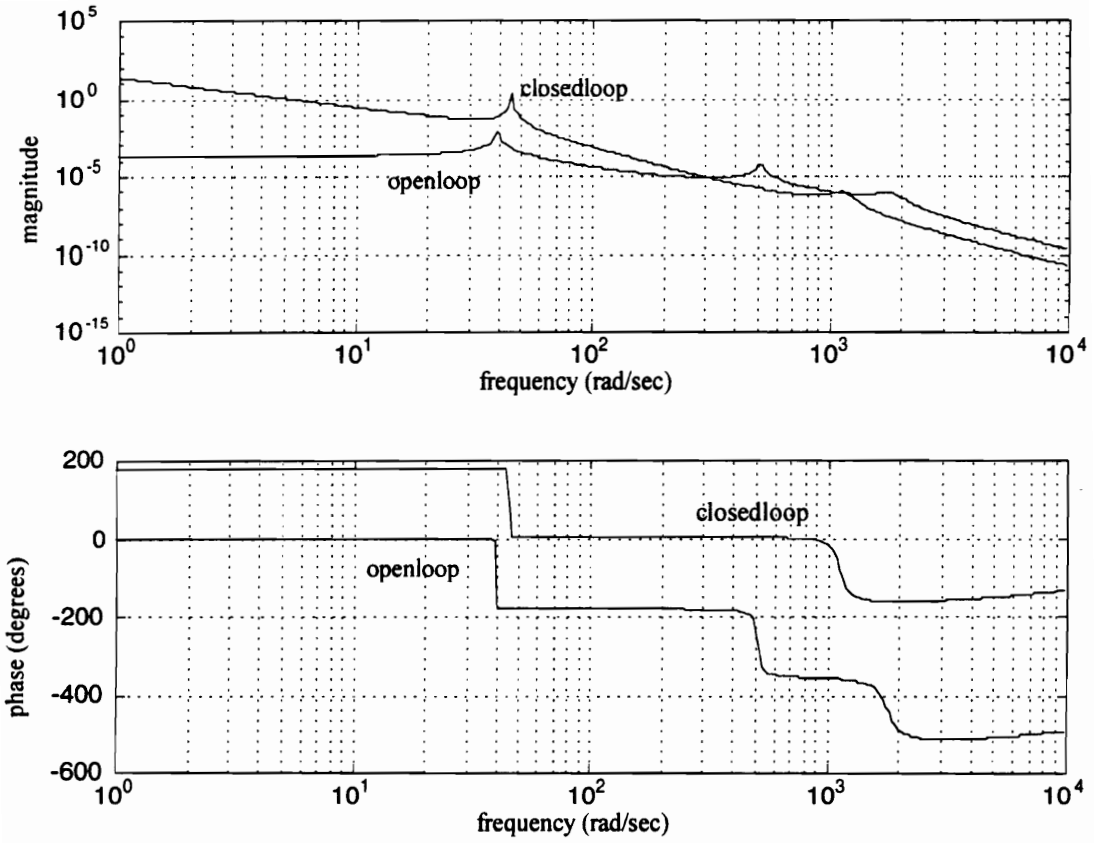


Figure 11.9. Magnitude and phase, cases 5 &6.

The results indicate that the transmissibility is lowered in the active feedback case. Improvement occurs in two ways first the comer frequency is lowered increasing the bandwidth of attenuated frequencies. The second improvement occurs through the attenuation of the magnitude of the transmissibility.

The MAF-SS modeling method combined with a standard eigenstructure assignment method results in a new and unique ability to methodically alter constraints between substructures. This technique changes system parameters and specifically may result in an improvement in vibration isolation performance between substructures.

11.4 Summary of Active Isolation

The MAF-SS formulation results in explicit knowledge of the constraints coupling substructures. Removing these constraints with active feedback results in a modified structure. The effect of active feedback of terms proportional to the coupling constraints is to lower the stiffness of the structure. This increases the isolation between substructures. The ability to improve isolation using this unique type of feedback is demonstrated. The concept of structural modification through substructure constraint alteration is applied to systems using a multivariable feedback method. This is accomplished by combing the MAF-SS method with a standard eigenstructure assignment technique. This method uses the MAF-SS formulation to define a system with desired eigenstructure properties, the active feedback gain that realizes these systems is calculated with an eigenstructure assignment method. The MAF-SS has application to

active control formulation, the result of this control can be an improvement in substructure isolation.

12 Summary

The MAF-SS differs in some important ways from other formulations. The new formation modifies admissible functions to meet geometric and kinematic boundary conditions. This eliminates the use of intermediate structures and approximate boundary conditions. Another difference is the use of a global coordinate system to define all motions. A unique attribute is the modification of admissible functions to satisfy constraint conditions between substructures. The result of these two attributes is a modeling method that retains global generalized coordinates and eigenvector approximations throughout the analysis. The method satisfies the conditions of the inclusion principle and therefore eigenvalues converge from above the actual values. The mathematical procedure is straight forward. This reduces effort during system modeling.

The active isolation method presented is linked directly to the substructure synthesis (SS) formulation that uses modified admissible functions (MAF) to form a structural model. The MAF-SS formulation is unique in that all coupling constraints are delineated during the model formulation in global generalized coordinates. It is demonstrated that decoupling substructures results in the isolation of the substructures. The investigation also presented an analysis showing positive feedback of terms proportional to the constraints formed during the MAF-SS modeling lessen the coupling of the substructures resulting in increased isolation. Since this method centers on the reduction of constraints, the formulation is independent of disturbance information. This isolation method is applicable to multivariable feedback and multiple-degree-of-freedom systems. This is accomplished by combining the MAF-SS method and a standard eigenstructures assignment technique. The isolation control formulation investigated here differs from

others because feedback terms are directly related to the removal of constraints between substructures. The focus of this work is to use constraint information gathered in the MAF-SS formulation to develop feedback terms that alter substructure coupling, leading to modified structural properties.

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Appendix

Review of the Chain Rule of Differentiation

Functions commonly occur that implicitly or explicitly depend on different variables. Such as

$$f(q_1, q_2, q_3, \ldots q_n, t) \tag{1}$$

which depends on n generalized coordinates q and time t.. This is just one example of an infinite number of possibilities. The differentiation of these functions is accomplished using the chain rule [1]. As applied to equation (1) with respect to t it is written

$$\frac{df}{dt} = \frac{\partial f}{\partial q_1}\frac{\partial q_1}{dt} + \frac{\partial f}{\partial q_2}\frac{\partial q_2}{dt} + \frac{\partial f}{\partial q_3}\frac{\partial q_3}{dt} + \ldots + \frac{\partial f}{\partial q_n}\frac{\partial q_n}{dt} + \frac{\partial f}{\partial t} \tag{2}$$

In the situation of a vector containing a set of m functions

$$f = \begin{bmatrix} f_1 & \cdots & f_m \end{bmatrix} \tag{3}$$

the derivatives are written in matrix form as

$$\frac{df}{dt} = \begin{bmatrix} \frac{df_1}{dt} \\ \vdots \\ \frac{df_m}{dt} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \cdots & \frac{\partial f_1}{\partial q_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial q_1} & \cdots & \frac{\partial f_m}{\partial q_n} \end{bmatrix} \begin{bmatrix} \frac{dq_1}{dt} \\ \vdots \\ \frac{dq_n}{dt} \end{bmatrix} + \begin{bmatrix} \frac{df_1}{dt} \\ \vdots \\ \frac{df_m}{dt} \end{bmatrix} \tag{4}$$

with respect to t.

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Vita

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