# Component Availability for An Age Replacement Preventive Maintenance Policy

by

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# COMPONENT AVAILABILITY FOR AN AGE REPLACEMENT PREVENTIVE MAINTENANCE POLICY

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(ABSTRACT)

This research develops the availability function for a continuously demanded component which is maintained by an age replacement preventive maintenance policy. The availability function, $A(t)$, is a function of time and is defined as the probability that the component functions at time $t$. The component is considered to have two states: operating and failed. In this policy, the component is repaired or replaced at time of failure. Otherwise, if the component survives T time units, a preventive maintenance service is performed. T is known as the age replacement period or preventive maintenance policy. The component is considered to be as good as new after either service action is completed.

A renewal theory approach is used to develop $A(t)$. Past research has concerned infinite time horizons letting analysis proceed with limiting values. This research considers component economic life that is finite. The lifetime, failure service time and preventive maintenance service time probability distributions are unique and independent. Laplace transforms are used to simplify model development. The age replacement period, T, is treated as a

parameter during model development. The partial Laplace transform is developed to deal with truncated random time periods. A general model is developed in which the resulting availability function is dependent on both continuous time and T. An exact expression for the Laplace transform of $A(t, \mathrm{T})$ is developed.

Two specific cases are considered. In the first case, the lifetime, repair and preventive maintenance times are unique exponential distributions. This case is used to validate model performance. Tests are performed for $t \to 0$, $t \to \infty$ and for times in between these extremes. Results validate model performance. The second case models the lifetime as a Weibull distribution with exponential failure repair and preventive maintenance times. Results validate model performance in this case also. Exact infinite series for the partial and normal Laplace transform of the Weibull distribution and survivor function are presented.

Research results show that the optimum infinite time horizon age replacement period does not maximize average availability for all finite values of component economic life. This result is critical in lifecycle maintenance planning.

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# CHAPTER 1

# INTRODUCTION

## 1.1 Background

Maintenance costs represent a major portion of the total production costs in industrial environments. Surveys have shown the average percentage of cost attributed to factory maintenance activities ranges from about fifteen to forty per cent with an average around twenty-five per cent [1, 2]. Obviously, an effective maintenance planning program serves to reduce these costs thus increasing bottom-line profits. The key to effective maintenance planning is minimizing unplanned downtime. For example, an article appearing in *Service News* [3] and Jacobson and Arora [4] report that computer downtime has a huge impact on the bottom-line profits of major businesses. A survey detailed in [3] shows that the cost to American corporations due to computer downtime is about four billion dollars annually. The net impact of computer downtime is estimated at $78,191 per hour and $330,000 per outage. An extreme case is Wall Street, where one minute of computer downtime means losses measured in millions of dollars [3, 4].

The implication is clear: minimize unplanned downtime through effective planned maintenance. Planned or scheduled maintenance is referred to as preventive maintenance. Most preventive maintenance actions require a system or component to be non-operational. However, the advantage is that these actions may be scheduled when the impact on production and operational requirements is minimized. For instance, preventive maintenance actions might be scheduled during a night shift or on a week-end when production requirements are not high. The objective is to minimize unplanned downtime during peak times of production and operations. As stated in the examples above, unplanned non-operational periods during these peak times can be very costly.

## 1.2 Preventive Maintenance

Preventive maintenance policies may be thought of as the replacement, servicing, overhaul, etc. of one or more functioning components in a device at selected (scheduled) points in time. Thus, the preventive maintenance (PM) action is defined as required for the specific situation. Assumptions concerning the preventive maintenance action are discussed later. Effective maintenance planning must also deal with the unplanned downtime that will occur. As noted, these emergency or failure repairs, replacements, etc. may be very costly. The objective of a preventive maintenance policy is to avoid the high costs of failure service (unplanned downtime) by performing preventive maintenance on functioning (non-failed) components and thus accrue the usually lower costs of preventive maintenance service (planned downtime).

There are three common types of preventive maintenance policies. These are age replacement, block replacement and opportunistic preventive maintenance policies. An age replacement policy specifies a component age, T, at which a component undergoes a preventive maintenance action (planned downtime). If the component fails before time T (unplanned downtime), the component receives a failure maintenance. For the purposes of the research described here, the component is considered as good as new after either a failure repair or a preventive maintenance action. Thus, the clock tracking component age is reset after either a failure repair or preventive maintenance action.

A block replacement policy specifies a defined time interval, $T_{block}$. The component is preventively maintained at integer multiples of $T_{block}$ regardless of the age of the component. If the component fails between the block replacement times, the component receives a failure maintenance. Thus this policy does not consider when the last component failure service occurred. A common example of the block replacement policy is the periodic replacement of stadium lights. In this case the set-up costs tend to be high and thus it is more cost effective to replace lights at given intervals regardless of their age. Berg and Epstein [6] provide a concise description and comparison of age, block, and failure replacement policies. In this case, a failure replacement policy only performs failure repair/replacement. No preventive maintenance is performed.

The opportunistic replacement policy may follow any one of an infinitely large number of preventive maintenance policies. In this sense, it covers all other policies that do not fit the above definitions of age replacement and block replacement. An example is the servicing of an automobile. During a scheduled oil change, an inspection reveals that the brakes are nearing the end of their service life. The brakes are replaced even though they may have lasted an additional amount of time. The brake replacement was "opportunistic".

## 1.3 Preventive Maintenance Implementation Concerns

Administratively, the age replacement policy is more difficult to implement than the block replacement policy. Each component of each subsystem or system must be tracked and scheduled for preventive maintenance actions. Block replacement requires less administrative overhead since the components may be "grouped" for block preventive maintenance. Depending on the complexity of an opportunistic policy, the administrative overhead may range from small to large. For large complex systems, block replacement may be easier to implement. However, most companies have computer-based maintenance tracking systems that allow more complex preventive maintenance policies to be effectively executed. The preventive maintenance scheduling policies may be implemented and automatically tracked by a computer-based maintenance action system.

In reality, opportunistic maintenance policies will intertwine with age replacement or block replacement policies. Preventive maintenance performed according to an age replacement or block replacement policy will at times reveal other problems that require maintenance. These problems may be corrected at these "opportunistic" times.

The research reported in this dissertation concerns the age replacement preventive maintenance policy. The policy is examined for single component systems or systems that may be modeled as a single component. Unscheduled component failure repairs and scheduled preventive maintenance actions are considered to be "perfect". These service actions renew the component; thus the component is as good as new at the conclusion of a service action.

## 1.4 Component Availability

The effectiveness of repairable components and systems is appropriately measured by a parameter known as availability. Availability, $A(t)$, (a function of time) is defined as the probability that a system or component is operational at time $t$. The limiting availability, A $= \lim_{t\to\infty} A(t)$, may be thought of as the proportion of time the system is operational in the long term. Thus in the long term, the limiting availability, A, is [1, 7]:

$$\text{A} = \frac{E[uptime]}{E[uptime]+E[downtime]} \ .$$

Other measures involving $A(t)$ include [7]:

(1) Average availability, $\text{A}_{avg}(\tau)$:

$$\text{A}_{avg}(\tau) = \frac{1}{(\tau)} \int_0^{\tau} \text{A}(t)\, dt \ . \tag{1.1}$$

(2) Limiting average availability:

$$A_\infty = \lim_{\tau\to\infty} \text{A}_{avg}(\tau) = \lim_{\tau\to\infty} \frac{1}{(\tau)} \int_0^{\tau} \text{A}(t)\, dt \ .$$

Interval average availability may be found for an interval $[\tau_1, \tau_2]$ using

$$A_{avg}(\tau_2 - \tau_1) \ = \ \frac{1}{(\tau_2-\tau_1)} \int_{\tau_1}^{\tau_2} A(t)\, dt \ .$$

In the computer manufacturing area, many manufacturers are responding to customer requirements by guaranteeing availability levels for their systems [5]. Often the purchase and/or service contracts designate specific time intervals where availability levels are evaluated. If the contract availability levels are not met at these designated times, the manufacturer provides compensation (rebate, extended warranty, etc.) to the customer. In these cases, using limiting availability as a planning tool may not provide the expected results from either the manufacturer's or customer's point of view. A more appropriate measure may be the average availability for the time period under consideration. Furthermore, evaluation of an optimal age replacement preventive maintenance policy, T*, using the limiting case where time approaches infinity may not provide the expected results. The key to the proper evaluation is obtaining the function $A(t)$, where $A(t)$ provides the probability the system or component operates as a function of time.

## 1.5 Research

The focus of this research effort is the age replacement preventive maintenance policy. Previous work [7,12] in this area has concentrated on obtaining the optimal replacement period, T*, to minimize a cost rate function. Most models assume that failure maintenance cost is $c_1$ and a preventive maintenance action cost is $c_2$. Furthermore, research [7,12] has shown that for $c_1 > c_2$, an optimal T* exists while for $c_1 \leq c_2$, T* goes to infinity (i.e. perform no preventive maintenance). These results are obtained by considering the limit as time, t, approaches infinity. Additionally, if $c_1$ and $c_2$ are taken to be the expected failure repair time and the expected preventive maintenance time, then the results shown above also maximize the limiting availability.

It is emphasized that in both cases, a limiting result is developed by taking the limit as $t$ approaches infinity. When applied to actual systems and components, this implies an infinite time horizon for system or component operational use. Most components and systems have finite time horizons over which they are operated and repaired. At the end of the system or component life, the system is often replaced by newer technology. Many examples of this exist in the computer and telecommunications industries. Other studies have evaluated the case where failure repair times and preventive maintenance times are stochastic but are assumed to be independent and identically distributed random variables.

The objective of this research is to evaluate age replacement times, T*, that maximize average availability for finite time intervals. This maximization of the average availability proceeds unconstrained. The results obtained are compared to the optimal age replacement period obtained from the limiting case.

The first task of this research is the development of a stochastic model to allow the formulation of the function $A(t)$ for an age replacement preventive maintenance policy. The stochastic model is based on the assumption that the component lifetime, failure repair times, and preventive maintenance times are unique random variables having known and unique distribution functions. Even though past research has recognized that failure repair and preventive maintenance times may be distinct random variables with distinct distribution functions, the limiting case analysis allows the mean value of these distributions to be used in the model development. Thus, the model is developed to provide an optimal replacement period, $T^*_\infty$, for the infinite time horizon. However, the model development methodology in the limiting case does not allow the development of the availability function, $A(t)$. The availability function, $A(t)$, allows both the time interval case as well as the limiting cases to be evaluated. The availability function, $A(t)$, is developed analytically and is numerically estimated for specific lifetime, failure repair time and preventive maintenance time distributions.

The second task includes the evaluation of these models for specific component lifetime, failure repair time and preventive maintenance time distributions. First, the exponential case is evaluated. Unique exponential distributions are designated for the failure repair time and the preventive maintenance time.

The second case to be treated considers aging, i.e., the component's lifetime distribution has an increasing failure rate (IFR). This means that given the component survives until time $t$, the probability that a component fails in the next instant of time increases as $t$, the conditional survival time,increases. One widely accepted lifetime distribution exhibiting the IFR characteristic is the Weibull distribution with shape parameter greater than one ($\beta > 1$). Therefore, the Weibull failure case is studied in combination with exponential failure repair and preventive maintenance actions. These random service periods represent the component or system downtime (non-operational) while the component or system lifetime distribution function and the age replacement period govern the system uptime (operational).

Specific expressions are developed for $A(t)$ in each of the cases discussed above. For each case, T, the age replacement policy that maximizes the average availability over the interval [0, t] is found using a numeric search. Time interval values considered include the component's mean lifetime and characteristic life in the case of Weibull lifetime distribution. Furthermore, analysis is accomplished at small values of time as well as large values of time to verify availability model performance near $t = 0$ and the limiting case, $t \to \infty$. Whenever possible analytical results have been obtained before resorting to specific numerical techniques and examples.

## 1.6 Summary

This research develops the component availability function, $A(t)$, for the age replacement preventive maintenance policy where component lifetime, failure repair and preventive maintenance times are stochastic and have known but unique distribution functions. The model development assumes that service actions, whether a failure repair or preventive maintenance, provide complete component or system renewal. Thus the component is as good as new at the conclusion of a service action. T, the age replacement period, is treated as a parameter in the development of $A(t)$. We find the age replacement period, T, that maximizes average availability over an interval [0,t] (unconstrained) through numerical integration. Intervals examined include the component's mean lifetime as well as the characteristic life in the case of the Weibull lifetime distribution. The characteristic life for the Weibull lifetime distribution is defined as the point in time, $t$, where there is a cumulative probability of component failure occurring at or before time $t$, equal to 0.632 [95].

Both constant failure rate (CFR) (no aging effects) and increasing failure rate (IFR) (aging) component lifetime distributions are considered. Exponentially distributed failure repair times and preventive maintenance times are examined for an exponentially distributed lifetime as well as for Weibull lifetimes. Model validation is accomplished at values of $t$ near zero and for large values of $t$, to approximate the limiting case. Final results are compared to the limiting case.

# CHAPTER 2

# LITERATURE REVIEW

## 2.1 Introduction

Over the past thirty-five years there has been a continuing and growing interest in the use of preventive maintenance to increase system productivity, product quality, and cost effectiveness. This is not surprising considering the vast industrial complex engaged in producing goods and delivering services. The vast majority of these systems and components require periodic maintenance of some type to maintain production efficiency as well as product and service quality. With the exception of electronic components and systems, these systems and components age with use and time. Their performance may be termed deteriorating with respect to the probability of failure over time. When this characteristic is coupled with the large cost of unscheduled downtime, it is logical that research pertaining to preventive maintenance policies has produced many models for systems and components exhibiting deterioration.

The purpose of this review is to give an overview of the type of preventive maintenance models produced by past research. Specific attention is given to models dealing with age replacement policies. These preventive maintenance policies repair or replace a component or system at failure or perform a preventive maintenance action at a pre-specified time interval, T. This time interval is usually considered the age of the component since these models consider complete component or system renewal at the conclusion of a failure maintenance or preventive maintenance action. Hence, T is known as an age replacement policy or period. Discussions concerning model development and analysis techniques such as renewal theory and Laplace transforms are included. The review includes a short discussion concerning probability modeling for component lifetimes as well as for the time required for repair/replacement and preventive maintenance action. Finally, a summary of the presentation concludes this chapter.

## 2.2 Preventive Maintenance Models

Four major articles review the research performed concerning preventive maintenance models. These include the 1965 paper by McCall [8], the 1976 paper by Pierskalla and Voelker [9], the 1977 paper by Lie, et al [10], and the 1989 paper by Valdez-Flores and Feldman [11]. These papers provide a chronological review of the research performed concerning preventive maintenance. The content of each of these papers is discussed below in chronological order.

In addition to the articles named above, bibliographic references may be found in Osaki and Nakagawa [52], Sherif and Smith [53] and Sherif [54].

### 2.2.1 Preventive Maintenance Research Prior to 1965

In "Maintenance Policies For Stochastically Failing Equipment: A Survey" [8], McCall divides maintenance models into two distinct categories. These are preventive maintenance and preparedness models. In preventive maintenance models, the component, system or equipment is subject to stochastic failure and the state of the component is always known with certainty. In what McCall terms preparedness models, the component is subject to stochastic failure, however, the state of the component is not known with certainty. McCall's discussion is divided by whether the component's lifetime failure distribution is known or not known with certainty. The research performed in this dissertation pertains to age replacement preventive maintenance policies for which the component's lifetime probability density function is known. Thus discussion of past research in this time period is limited to this area.

The periodic preventive maintenance policy repairs or replaces components at failure or preventively maintains them at age T. This is the age replacement preventive maintenance policy defined above. The periodic preparedness policy inspects and/or repairs/replaces equipment at age T. In the simplest situation, both models consider the component to be in one of two states - either an operational state or a failed state. The preventive maintenance model detects component failure immediately while the preparedness model detects component failure only at the time of an inspection or repair/replacement.

The simplest and one of the best known preventive maintenance policies is the strictly periodic or age replacement policy. Recall that in this policy the component is repaired/replaced upon failure or when the component reaches an age replacement policy of T. The model considers complete component renewal at the completion of either service action. Thus, the component age T is the same as the time since the last service action was completed. Barlow and Hunter [12] as well as Barlow and Proschan [13] have shown this type of policy is only worthwhile to implement in the situation where components exhibit the effects of aging (strictly increasing failure rate, see [7]) and the cost of repair/replacement due to component failure (unscheduled downtime) is greater than the cost of a preventive maintenance action. These costs may also be represented as the time required to perform the component service actions described.

In the case of cost interpretation, the objective is to minimize cost while in the case of time interpretation, the objective is to maximize the proportion of time the component is operational. This is also known as limiting availability. Both analyses are equivalent and an infinite time span is considered. Even though the costs (times) for failure maintenance and preventive maintenance are considered to be random variables with known distributions, the infinite time span assumption allows the expected values of these quantities to be used. Thus, one may derive an optimal age replacement period that minimizes the cost rate or maximizes the limiting component availability for an infinite time horizon.

However, the analysis does not allow for the derivation of the cost or availability as a function of time. Therefore, the calculation of the cost or average availability over a specified time interval may not be found since the cost rate or availability function is not available for integration. Other research work referenced for the age replacement policy includes Barlow and Proschan [13], Campbell [14], Dean [15], Drenick [16], Kamins and McCall [17], Klein and Rosenberg [18], Lotka [19], Morse [20] and Welker [21 and 22]. Furthermore, both Barlow and Proschan [13] and Derman [23] have shown that for infinite time horizons, a strictly periodic replacement policy, i.e., non-random, is optimal over random replacement policies.

Additionally, the McCall article discusses two preventive maintenance policies for systems or equipment made up of several parts. An opportunistic model developed by Radner [24] considers a two component system consisting of component 1 and component 2. The components are considered to fail stochastically and independently. Component 1 has a strictly increasing failure rate (deterioration) while the component 2 has a constant failure rate. Component 2 does not exhibit deterioration and thus is modeled using the exponential probability density function for its lifetime. The model does recognize economies of scale in maintenance. This means that it is less costly to replace both components at one time than it is to replace each component separately. Radner [24] termed these model conditions as stochastic independence and economic dependence. He shows that the optimal policy is characterized by two decision parameters, n and N. If the age of the system, x, is less than n, $(x < n)$, then component 1 is replaced/repaired only if it fails. If $n \leq x < N$, then replace component 1 if either part fails. If $x \geq N$, then replace/repair component 1 at once.

A block replacement policy is defined by McCall [8]. In this policy, system components are preventively repaired/replaced at regular non-random intervals regardless of their age. The model assumes each component is repaired upon failure as well. Comparisons of the operating characteristics between block and age replacement policies may be found in Barlow and Proschan [7 and 13], Boivard [25], Cox [26], Flehinger [27, 28 and 29], Hunter and Proschan [30], and Arrow, et al [31].

An age replacement model for finite time horizon, $(0, t)$, has been researched and analyzed. This policy is known as a sequential preventive maintenance policy. This policy repairs or replaces the component upon failure or at the age replacement interval, T. However, it differs from the previous infinite time horizon case in that at each event (whether failure or preventive maintenance), the optimal replacement period, T, is recalculated for the remaining time period. Thus, this replacement period minimizes the expected cost over the remaining period (or maximizes the component availability depending on the interpretation). Barlow and Proschan [13] prove that the expected cost resulting from following an optimal sequential policy during the interval $(0, t)$ is always less than or equal to the expected cost of the corresponding optimal periodic policy. They also prove that the optimal sequence is non-random and illustrate a method for calculating the optimal intervals.

The periodic preparedness model, as discussed in McCall [8], and under the conditions where the component's lifetime distribution is known may concern either an optimal replacement period or an optimal inspection period. For the case where an optimal replacement period is found, the component's condition is only known at the time of replacement and the prescribed service action is always carried out regardless of the component's condition. References for research in this area include Jorgenson and McCall [32], Jorgenson [33], Savage [34], Solomon and Derman [35] and Veinott and Wagner [36]. In the other case, an optimal inspection period is found. The difference in this case is that the component is inspected for proper operation and repaired/replaced only if found to be in a failed state. References for work in this area include [32], [33], Barlow, et al [37], Coleman, et al [38], and Kamins [39]. Kamins [39] and Coleman, et al [38] have also extended the model to include imperfect inspection.

### 2.2.2 Preventive Maintenance Research Between 1965 and 1975

Two papers provide excellent summaries of the research accomplished concerning preventive maintenance during this time period. The first, "A

Survey of Maintenance Models: The Control and Surveillance of Deteriorating Systems" by Pierskalla and John A. Voelker [9] surveys the research accomplished since the 1965 paper by McCall [8]. Contrasting this survey is the paper by Lie, et al [10], titled "Availability of Maintained Systems: A State-of-the-Art Survey". This paper provides a systematic classification of the literature relevant to availability. One specific area is the effect of preventive maintenance on availability. An excellent discussion of the various definitions of availability is also given in this paper.

Lie, et al [10] classify availability depending on the time interval involved. These are: (1) instantaneous availability, $A(t)$, (2) average uptime or average availability for a time interval [0, $\tau$], designated $A_{avg}(\tau)$, and (3) steady state availability, $A$. These quantities are defined as follows:

(1) $A(t) \equiv Probability\, the\, component\, is\, operational\, at\, time,\ t$

(2) $A_{avg}(\tau) = \frac{1}{\tau}\int_0^{\tau} A(t)\,dt$

(3) $A = \lim\limits_{t\to\infty} A(t)$ .

These definitions may be found in Barlow and Proschan [7] and Sandler [40]. Barlow and Proschan [7] also define the limiting average availability, $A_{\infty}$. This measure, $A_{\infty}$ , is mathematically defined as

$$A_{\infty} = \lim_{\tau\to\infty} A_{avg}(\tau) = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^{\tau} A(t)\,dt \ .$$

They show that $A_\infty$ is equivalent to $A$ when the limit exists. Previously, limiting availability was often referred to the limiting efficiency (see Barlow and Hunter, 1960 [12]).

Lie, et al [10] claim that the most appropriate measure of availability depends on a component's or system's mission and its condition of use. They provide the following examples of application:

> The steady-state availability may be a satisfactory measure for systems which are to be operated continuously, for example a detection radar system. The average uptime [average availability] may the most satisfactory measure for systems whose usage is defined by a duty cycle, for example, a tracking radar system which is called upon only after an object has been detected, and is expected to track continuously during a given time period. For systems which are required to perform a function at any random time, the instantaneous availability may be the most satisfactory measure. An example of such a system is a data processing system used in air traffic control which is called upon to process flight paths, and then remain idle for a length of time.

Lie, et al [10] also note that in general, several types of preventive maintenance policies are possible. These are age replacement, block replacement, random periodic replacement and sequential determined replacement. Numerous references are cited for these types of preventive maintenance. They note that the earliest research on planned replacement was by Campbell [14].

Maintenance models for this period are extensively surveyed by Pierskalla and Voelker [9]. This paper includes only those models which involve an optimal decision to procure, inspect, repair and/or replace a single component exhibiting deterioration. Their presentation surveys discrete-time maintenance models and continuous-time maintenance models. In discrete-time maintenance models, a component is monitored and a decision to repair, replace and/or restock the component is made at discrete intervals in time. In continuous-time maintenance models, actions and events are not restricted, *a priori*, to occur only within a discrete subset of the time axis. The age replacement preventive maintenance policy falls into this latter category. Hence, discussion will be limited to this area.

Pierskalla and Voelker [9] note that research has extended the results of the Barlow and Proschan [41] optimal age replacement model for the infinite time horizon to include the age replacement policy under a total discounted cost criterion [42] as well as including an age-dependent cost [43]. This cost is defined as the increasing burden of routine maintenance as the component ages resulting from its diminishing productivity or reduced salvage value. It is also noted that Glasser [44] has extended the Barlow and Proschan's optimal age replacement period, $T^*$, for the truncated normal, gamma and Weibull density functions. Other types of models considered include age dependent costs that are incurred at discrete times [45], and a generalization to include, $c^*$, a maximum cost rate that when exceeded causes the component to be replaced [46]. This maximum cost rate, $c^*$, is set by the decision maker.

Multiple component systems are considered by Nakagawa and Osaki [47]. Their research considers a two-component redundant system with identical components. One component operates while the other is in standby status. The component in standby status does not age. Failure of the system occurs when both units are undergoing maintenance due to failure or scheduled preventive action. Under the conditions where mean failure repair time is greater than mean preventive maintenance time, they find the optimal age, T, that a component should be serviced. Nakagawa and Osaki find the optimum T that maximizes the limiting availability. Thus an infinite time horizon is considered.

Makabe and Morimura [48, 49 and 50] and Morimura [51] consider an age replacement model where the component is replaced at the $k^{th}$ failure. The $k-1$ previous failures are corrected by minimal repair. Under minimal repair a component is repaired to operating condition. However, the repair is assumed to not effect the aging of the component. Hence, after the minimal repair, the component restarts deterioration, according to its lifetime density, at the same point in time that it began minimal repair. For components with strictly increasing failure rates, the optimal policy consists of $t^*$ and $k$. If the component's $k^{th}$ failure occurs before $t^*$ then a minimal repair is completed and replacement is completed at the next component failure. If the $k^{th}$ failure occurs after $t^*$ then a replacement is carried out. Optimality is shown for the infinite time horizon case in which limiting component efficiency (availability) is maximized.

### 2.2.3 Preventive Maintenance Research From 1976 to Present

In "A Survey of Preventive Maintenance Models for Stochastically Deteriorating Single-Unit Systems," Valdez-Flores and Feldman [11] (1989) present a state-of-the-art survey of the research accomplished in preventive maintenance mathematical modeling since the survey accomplished in 1975 by Pierskalla and Voelker [9] as well as the bibliographic references provided by Osaki and Nakagawa [52], Sherif and Smith [53] and Sherif [54]. The maintenance models considered in this survey include those for systems that may be modeled as a single unit or component.

Valdez-Flores and Feldman classify age replacement models within what they term "inspection models". This group of models considers the case where the status of the component or system may not be continuously observed. However, it may be possible to perform an inspection to ascertain the status of the system. These models usually assume that the state of the system is unknown unless an inspection is performed. Initial models assumed that the inspection information was perfect. Later, research allowed for the case where inspections were not perfect. Thus, age replacement preventive maintenance research has progressed now to include inspection policies. Optimal policies are still developed by minimizing the cost rate function or the expected total cost per inspection cycle. Barlow, et al [37] presented this research first.

Beichelt [55] extends this basic model by finding optimal inspection times for the cases when system replacement and no system replacement of a failed system are permitted. He also obtains optimal inspection schedules when the lifetime distribution of the system is partially known and when it is completely unknown (system mean lifetime is considered known). Thus this research extends the inspection model to cases where incomplete or unknown initial conditions prevail. Luss [56] extends this further to the case where system or component operational state is multi-state and is only observable through inspection. The model is generalized for $L$ states of component operational condition. Optimal control limit policy with control state $\alpha$ and the optimal inspection intervals for states 0, 1, ..., $\alpha - 1$ are found through an iterative procedure by assuming exponentially distributed state transition times (Markovian assumptions).

Interestingly, Rosenfield [57 and 58] examines a model where component deterioration follows a discrete-time Markov chain. The operating and replacement costs increase with state number and inspections are considered perfect. Kander [59] also considers inspection for a component that deteriorates at discrete levels. He models the situation using semi-Markov processes to determine the optimal inspection schedule that minimizes long-run expected cost per unit time.

Nakagawa and Yasui [60] present an algorithm to compute sub-optimal inspection times for components that do not have exponential failure times. Their research demonstrates the validity of a numerical example for the case where the component's lifetime distribution is Weibull. A recursive approach computes successive inspection times backwards . For lifetime hazard functions characterized by an increasing hazard rate, Munford [61] shows that an inspection policy with decreasing intervals between successive inspections as a function of component age are better than strictly periodic inspection policies.

Minimal repair models have also been considered. Park [62] presents an entirely new concept of age replacement under minimal repair as that introduced by Barlow and Proschan [13]. Park [62] finds the number of failures and minimal repairs before the system is replaced instead of a fixed age to replacement. He develops a closed-form solution of his model for the case where the system's lifetime distribution is Weibull. Numerically, Park [62] shows his policy provides a lower long-run expected cost per unit time than a fixed-age replacement policy. However, a mathematical proof is not offered.

Recent research includes Nachlas, 1989 [63] and Lam and Yeh, 1994 [64]. Nachlas develops a constrained optimization model for selecting replacement policies for both age and block replacement. This model's solution finds the replacement policy that minimizes the cost rate function constrained by a confidence interval on required availability. Nachlas's model assumes failure repair and preventive maintenance times are random but identically and independently distributed. Lam and Yeh present algorithms for deriving optimal maintenance policies to minimize the mean long-run cost-rate for continuous-time Markov deteriorating systems. The model assumes the deterioration state is only known through inspection. They consider five maintenance strategies. These are failure replacement, age replacement, sequential inspection, periodic inspection and continuous inspection.

### 2.2.4 Summary of Preventive Maintenance Research

The models described above consider the infinite time horizon case in which cost rate functions are developed and minimized. The alternative approach presented uses Markov chain analysis. In the case of the infinite time horizon case, cost and/or availability functions with respect to time are not developed. Under Markovian assumptions, the lifetime, failure repair and preventive maintenance probability density functions are exponential. Under these restrictive assumptions, cost and availability functions with respect to time may be developed. Applicable density functions will be discussed later in this review. However, it is evident that the instantaneous availability function, $A(t)$, is a prime research area for the cases where lifetime, repair and preventive maintenance distributions are not all exponentially distributed.

## 2.3 Analytical Techniques

The summary of the preventive maintenance models presented above shows that with the exception of Markov models, the optimal preventive maintenance policy is derived by developing an expected total cost rate function. This function is minimized to find the optimal policy. The development of this function requires that an infinite time horizon be assumed. Often, this is not realistic as system lifecycles consist of a finite operational period. This section summarizes analytical techniques and methods pertinent to developing the component instantaneous availability, $A(t)$, for an age replacement policy.

## 2.3.1 Renewal Theory

Renewal theory deals with stochastic processes that have renewal points. At these renewal points, the stochastic process probabilistically starts again or renews itself. A renewal process is therefore made up of a finite number of random periods, $n$, each with density function, $f_i(t)\, \forall\, i = 1, \ldots, n$, which reoccur in a specific order (cycle) as the process progresses. Thus the renewal process consists of these well defined cycles that are probabilistically identical. Interestingly, renewal theory began as the study of probability problems connected with the failure and replacement of components, such as light bulbs [26]. Barlow and Proschan [13] and Cox [26] offer a mathematical development of renewal theory.

One important property stems from the definition of a renewal process given above. Since a time-based renewal process deals with some finite number of random time periods that occur in a specific order during a cycle, it is often necessary to find the density function for the time between renewals.

This may be accomplished by taking the convolution of the probability density functions represented in a renewal cycle, $f_i(t)\,\forall\, i = 1, ..., n$. For the time-based renewal process, this is known as the density function for time between renewals and is designated $h(t)$. Mathematically, $h(t)$ is stated as

$$h(t) = f_1(t) \otimes f_2(t) \otimes ... \otimes f_{n-1}(t) \otimes f_n(t) \,. \qquad (2.1)$$

where the symbol, $\otimes$, represents the convolution operator. The convolution of two functions, $f_1(t) \otimes f_2(t)$, is mathematically defined as

$$f_1(t) \otimes f_2(t) = \int_0^t f_1(t)\, f_2(t-u)\, du \, = \int_0^t f_2(t)\, f_1(t-u)\, du \,.$$

### 2.3.1.1 Laplace Transforms

The probability density function, $h(t)$, may be found mathematically by taking the convolution as defined in equation 2.1.

The Laplace transform convolution theorem provides a method for finding $h(t)$ without explicitly evaluating the multiple integrals required for equation 2.1. If we designate the Laplace transform of a function, $£[f(t)]$, as $f^*(s)$ then the following statements hold [97]:

(1) $£[f(t)] = f^*(s)$

(2) $£^{-1}[f^*(s)] = f(t)$

(3) $f_1(t) \otimes f_2(t) = £^{-1}[f_1^*(s)\, f_2^*(s)]$ .

For many cases, the inversion of the Laplace transform into a time domain function may be found through tables, partial fraction decomposition, contour integral evaluation as well as other methods. However, the case may arise where the inverse of $h^*(s)$ may not be found exactly. In this case, approximation methods must be used. These methods include numerical inversion techniques evaluated by Davies and Martin [65] as well as those evaluated and implemented in Mathematica (Wolfram Research Inc.) by Cheng, et al [66]. Several numerical inversion methods for Laplace transforms are evaluated. These include the Stehfest method [67], the Papoulis legendre polynomial method [68], the Durbin method [69], the Crump method [70], the Weeks method [71], and the Piessens gaussian quadrature method [72 and 73]. These methods provide an estimate of the time domain function at a specific value of time, $f(t = c)$. Functions using the Laplace transform function, $f^*(s)$, are developed and implemented [66].

## 2.3.2 Lifetime Probability Density Functions

A component's lifetime probability density function, $f_L(t)$, defines the probability density function for time to component failure. This function defines the probability of component failure at time $t + \epsilon$, where $\epsilon$ is an infinitesimally small increment in time. Much research has been accomplished concerning the proper probabilistic modeling of the component lifetime density function. The most frequently used density function is the exponential density function. This density function seems to model electronics well. The most advantageous aspect of this probability density function is its mathematical tractability and it is implemented often for this reason alone [88]. Justification for the use of the exponential lifetime density may be found in [74, 75, 76, 77, 78, 79, 80, 81 and 82].

Other references such as Bell, et al [83] maintain that density functions such as the truncated normal, log-normal and Weibull density functions are applicable in many cases. Bell, et al [83] claim that a significant portion of aircraft parts follow a normal lifetime density function. Lie, et al [10] make the following statements concerning lifetime density functions:

(1) In many practical situations, lifetime density functions are markedly skewed and thus are not normally distributed.

(2) The gamma and Weibull density functions provide the skewed characteristic discussed above.

(3) The Weibull density function is known to be suitable in describing fatigue failure such as that occurring in vacuum tubes and ball-bearings.

(4) The log-normal density function seems to fit repair times than failure times.

The Weibull density function is justified for use as a lifetime density function in [83, 84, 85 and 86] and the truncated normal in [83]. Lie, et al [10] provide additional references on other lifetime density functions such as the Erlang, gamma, Rayleigh, uniform, extreme value and arbitrary functions.

Proper modeling of a component's lifetime density must consider the component's family (mechanical, electronic, etc.) characteristics and the failure mechanisms at work on and/or within the component.

### 2.3.3 Repair Time Probability Density Functions

Repair time density functions have usually been approximated by an exponential density function for analytical purposes [87]. Rohn [88] considers the repair times of complex electronic equipment. He maintains, in the case of electronic repair, that the repair times consist of a high frequency of short times and a few long repair times. This characteristic, according to Rohn, suggests an exponential lifetime density function.

Other studies concerning airborne radar equipment and ground equipment for surface-to-air missile systems have shown that the log-normal density function fits the observed repair times best [89 and 90]. Additional references justifying the use of the log-normal probability density function to model repair times include [91, 92, 93 and 94].

# CHAPTER 3

# PROBLEM STATEMENT

## 3.1 Stochastic Process Description

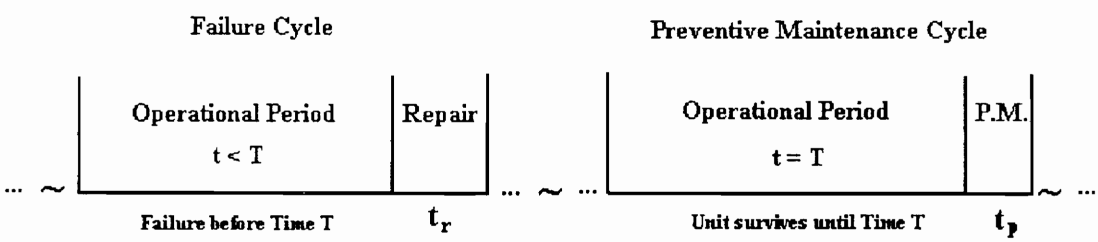


Figure 3.1 Age Replacement Preventive Maintenance Stochastic Process

The problem analyzed in this research is depicted in Figure 3.1. This figure represents a stochastic process based upon an age replacement preventive maintenance policy for a continuously demanded component. In this situation, the component is repaired or replaced if it fails before T time units have passed since the last maintenance action. This event is called a failure cycle. The component is preventively maintained whenever T time units have passed since the last maintenance action. This event is called a preventive maintenance cycle. The stochastic process consists of randomly occurring failure cycles and preventive maintenance cycles. Figure 3.1 does not imply an ordering in the occurrence of failure and preventive maintenance cycles. The occurrence of these cycles is completely random.

Failure repairs and preventive maintenance actions are perfect; thus at the completion of a failure repair or preventive maintenance action, the component is considered to be as good as new. At these points in time we have component renewal and thus have a stochastic process exhibiting renewal points. At these points the stochastic process renews itself and probabilistically repeats.

The notation used in this research is shown below. The symbol, $\equiv$, is read as "is defined as".

Notation:

$f_L(t) \equiv$ known component lifetime density

$F_L(t) \equiv$ known component lifetime distribution, $\int_0^t f_L(t)\,dt$

$R_L(t) \equiv$ known component survivor function, $1 - F_L(t)$

$g_r(t) \equiv$ known repair time density after component failure

$g_p(t) \equiv$ known preventive maintenance time density

T $\equiv$ age replacement preventive maintenance period.

The stochastic processes governing the component's lifetime, the time required to effect repair after a component failure and the time required to carry out a preventive maintenance action are considered to be governed by known probability density functions, which are all unique and independent functions. Furthermore, it is assumed these probability density functions are integrable in closed form to obtain their respective distribution functions.

## 3.2 Problem Discussion

As stated earlier, past research has concentrated on finding an optimal replacement period, $T^*_\infty$, for component use over an infinite time horizon. This optimal replacement period maximizes the limiting availability of the component. In this research, the availability function, $A(t)$, is defined to be the probability the component operates at time $t$. The limiting availability, $A$, is defined to be $\lim_{t\to\infty} A(t)$.

An infinite time horizon for the component's expected life cycle is not realistic. Realistically, a component is used over a prescribed time period. For a finite component life cycle, the average availability over the time period may be a more appropriate measure to be maximized. The average availability, $A_{avg}(\tau)$, is defined as

$$A_{avg}(\tau) = (\tfrac{1}{\tau}) \int_0^\tau A(t)\, dt \,. \tag{3.1}$$

Finding a value for the age replacement period, T, that maximizes $A_{avg}(\tau)$ requires an expression for the availability function, $A(t)$.

## 3.3 Development of the Availability Function

A notable feature of the stochastic process under consideration is the unique probability density functions representing the time required for repair (after component failure) and the time required for preventive maintenance after T time units have passed since the last maintenance action. Previous results only consider the expected values of these times since the analysis is for an infinite time horizon ($t \to \infty$). For finite time horizons, the nature of the probability density functions governing these respective service times must be taken into account.

To accommodate this, a renewal theory approach is taken to find the availability function for the specified stochastic process. This approach is based upon Laplace transform theory as well as the application of conditional probability theory. The result of the initial analysis is the Laplace transform of the availability function, $A^*(s)$, which is defined as

$$A^*(s) = \int_0^\infty (e^{-st})\, A(t)\, dt \,. \tag{3.2}$$

The availability function, $A(t)$, may be found by taking the inverse Laplace transform of $A^*(s)$. It is important to realize that throughout this development, T, the age replacement preventive maintenance period, is treated as a parameter. After inverting $A^*(s)$, we obtain the availability as a function of both time, $t$, and the age replacement preventive maintenance period, T.

As shown in Figure 3.1, two types of sample paths occur. The first type of cycle shown is the "failure cycle". In this event the component fails before T time units have elapsed since the last repair or preventive maintenance action. At failure, a failure service action immediately begins. The component's operational or uptime is always less than T in this cycle. Thus, the component's lifetime probability density function is truncated at time T.

The second type of cycle shown is the "preventive maintenance cycle". In this event the component survives exactly T time units since the last repair or preventive maintenance action. After the component reaches the "age" of T, the component undergoes a random repair period (preventive maintenance action). Thus the component's operational period or uptime in this case is a constant T time units.

The service time periods are not truncated and may range on $[0,\infty]$. In the failure cycle, we are dealing with an operating period of $t < T$ and in the preventive maintenance cycle we have a constant operating period of $t = T$. These limitations introduce special cases of the Laplace transform that complicate our expression for the Laplace transform of the availability function, $A^*(s)$. The following sections review pertinent properties of the Laplace transform applicable to analyzing the age replacement preventive maintenance policy.

## 3.4 Laplace Transforms

Equation (3.2) gives the classical definition of the Laplace transform. In some cases we may experience a truncated time period. Specifically, the time domain variable may have a limited range. This is true for the stochastic process under investigation and discussed above. For example consider the "failure cycle" and the component's lifetime probability density. Let $f_L(t)$ be the non-truncated probability density function for the component's lifetime in which $t$ has range [0,∞]. If operating period, $t$, is now limited to the range, [0,T], then the truncated lifetime probability density function is

$$f(t) = \frac{f_L(t)}{F_L(T)} \; for\, t \leq T, \; and$$
$$0 \qquad otherwise \; .$$

The partial or truncated Laplace transform for $f(t)$ is

$$f^*(s, T) \; = \; \int_0^T e^{-st} \, f(t) \, dt \;\; .$$

Basic Laplace transform properties include the useful convolution theorem. The convolution of two functions, $f(t)$ and $g(t)$ is

$$f(t) \otimes g(t) = \int_0^t f(t) \, g(t-u) \, du \; = \; \int_0^t f(t-u) \, g(t) \, du \, ,$$

where the symbol, $\otimes$, represents the convolution operator. The Laplace transform convolution theorem provides the following relationship:

$$f(t) \otimes g(t) = \pounds^{-1}[f^*(s)\ g^*(s)],$$

where $\pounds^{-1}[\bullet]$ is the inverse Laplace transform of the argument within the brackets [97].

Further discussion of the partial Laplace transform is appropriate. Due to the limited operating time range of the component, notably, $0 < t <$ T, we are forced to reconsider the limits on the Laplace transform for the truncated lifetime density. This range limitation produces the result shown for $f^*(s, T)$ above. Section 4.3 produces a significant result for the partial Laplace transform.

### 3.4.1 Approximate Laplace Transforms

Many of the most useful lifetime probability distributions as well as service time distributions do not have exact Laplace transforms. An approach similar to the one used to approximate Laplace transform inversion is often used. The time domain function, $f(t)$, is expanded into a series of functions with known, exact Laplace transforms. Thus an approximate Laplace transform, $f^*(s)$, may be found.

If the series of the approximate Laplace transform contains many terms, then the complexity of the $A^*(s)$ increases. Thus, there is a trade-off between the producing an accurate estimated Laplace transform and obtaining a manageable expression for $A^*(s)$. Issues pertaining to the inversion of $A^*(s)$ to obtain the time domain function, $A(t)$ are discussed next.

### 3.4.2 Approximate Laplace Transform Inversion

Due to the complicated nature of the Laplace transform of the availability function, $A^*(s)$, an exact inversion may not exist or may not be easily obtainable. In these cases, it is necessary to construct approximations of $A^*(s)$, to find an approximate expression for $A(t)$. Numerical approximation methods have been developed. These methods are applied through the assistance of Mathematica [96] to obtain the approximate Laplace transform inversion of $A^*(s)$.

Conceptually these methods attempt to expand the non-invertible $f^*(s)$ into functions of $s$ that have exact transforms. Various methods exist for this expansion and these methods are evaluated for use in this analysis.

## 3.5 Maximizing Average Availability Over A Finite Time Horizon

After obtaining an approximate expression for $A(t)$, the next step attempts to obtain a value for T that maximizes the $A_{avg}(t)$ through equation (3.1). Two options are investigated. The first step involves inverting, $A^*(s)$, the Laplace transform of the availability function. Recall that equation (3.1) defines $A_{avg}(t)$ and involves $\int_0^t A(u)\,du$. A useful property of the Laplace transform allows the direct development of $A_{avg}(t)$ through a simple transformation of $A^*(s)$ before inversion. If $f(t)$ is a function with $\int_0^t f(t)\,dt = F(t)$ and Laplace transform, $f^*(s)$, then the following holds:

$$£^{-1}\left[\ \frac{1}{s}\ f^*(s)\right] = F(t)\ . \qquad \text{i.e., } F^*(s) = \frac{1}{s} f^*(s)$$

Once an expression is obtained for $A^*(s)$, we simply multiply by $(\frac{1}{s})$ and invert to obtain $\int_0^t A(t)\,dt$ [97]. Finally, to obtain $A_{avg}(t)$ we multiply the resulting function, representing $\int_0^t A(t)\,dt$ by $(\frac{1}{t})$. This result gives a functional representation of $A_{avg}(t)$. Once this function is obtained, optimization operations may be performed to find values of T that maximize $A_{avg}(t)$ for specific values of $t$.

However, the resulting function obtained for $A_{avg}(t)$ may not be differentiable or of a form that is convenient to manipulate. An alternative method is to perform a numerical search. In this case, we would select a specific time horizon, $t$, and perform the integration for various values of T. (It is assumed specific values have been given to all probability density function parameters.) Ranges for the possible values of T are first limited by the time horizon, $t$. It is evident that T $< t$, because, if T $\geq t$, then no preventive maintenance would be performed during the time horizon period. Also the value for T obtained for the infinite time horizon case should provide some limit on T for the finite time horizon case. However, it is not clear whether this may be an upper or lower bound.

Clearly, the most interesting cases include the condition where aging is exhibited by the component. In reliability theory, this is called increasing hazard. The hazard function, $z(t) = \frac{f_L(t)}{R_L(t)}$ is strictly increasing if $\frac{dz(t)}{dt} > 0$. If the component exhibits constant hazard or decreasing hazard, preventive maintenance will adversely effect the component's availability. Under constant hazard ($\frac{dz(t)}{dt}$=0), no matter how long the component has survived, it is always as good as new (exponential lifetime density function). Thus there would never be an age at which the component is purposely brought out of operation for a preventive maintenance action. Under strictly decreasing hazard ($\frac{dz(t)}{dt} < 0$), the longer the component survives the more reliable the component becomes. Thus, it is easily seen that if the component actually becomes more reliable with time, then preventive maintenance would adversely affect component availability. If we consider how the probability of failure changes with time, we will find it actually decreases the longer the component survives. Some references term this probability of failure the "force of mortality" or "age-specific failure rate" [26]. For components with strictly increasing hazard, the force of mortality increases as the component ages; for components with constant hazard the force of mortality remains constant as the component ages; for components with strictly decreasing hazard, the force of mortality decreases as the component ages.

Additionally, the stochastic nature of the failure repair time and the preventive maintenance action time must be considered. In this analysis, we consider the case in which the failure repair time distribution is stochastically greater than the preventive maintenance time distribution. A random variable $X$ is stochastically greater than another random variable $Y$ if the following is true:

$$Prob[X > z] > Prob[Y > z],\ for\, all\, z,$$

or equivalently,

$$F_X(z) < G_Y(z),\ for\, all\, z,$$

where $F_X(u)$ and $G_Y(u)$ are the cumulative density functions for the random variables, $X$ and $Y$, respectively (see p.78 of [95]). Logically, it is easily argued that if this condition does not exist, then preventive maintenance will adversely affect the availability of the component. If the preventive maintenance action time distribution were stochastically greater than or equal to the failure repair time distribution, the option would be to let $T \to \infty$, and thus perform no preventive maintenance. In this case, the component is only repaired/replaced upon failure. Thus the important conditions to be analyzed include component lifetime distributions with strictly increasing hazard and repair time distributions that are stochastically greater than preventive maintenance action time distributions.

Another point needs to be addressed concerning repair and preventive maintenance time distributions. Only distributions from the same families are considered here. For example, if the failure repair distribution is modeled as exponential, then the preventive maintenance time distribution is also modeled as exponential. Analysis considering distributions with shape parameters (i.e., Weibull) use the same shape parameter for both distributions. In these cases, only distributions with strictly increasing hazard are considered. Repair time distributions exhibiting these characteristics are interpreted as follows: given that a repair task has taken $t$ time units, the probability of finishing the task in the next instant of time increases as $t$ increases.

## 3.6 Summary of Analytical Approach

The stochastic process shown in Figure 3.1 is analyzed as follows. First, a renewal theory based model is developed. The objective of this model is to evaluate the availability of the system or component. Expressions for the availability of the component or system are developed. These expressions require the evaluation of the process renewal density, $m(t)$, and/or the probability density function for the time between renewals, $h(t)$. The Laplace transform is used to find the convolutions of the probability density functions.

Once the Laplace transform of the availability function, $A^*(s)$, has been developed, it must be inverted. Numerical inversion packages are investigated for application to this analysis. Also methods to invert functions not having an exact closed-form Laplace transform are investigated.

These Laplace transform and inversion methodologies are applied to specific cases. These cases include an all exponential case for validation and a Weibull lifetime density with exponentially distributed repair times and preventive maintenance times. In all cases, the distributions are unique and independent.

The following ratios are established for this analysis:

$$\rho \equiv \frac{Mean\,Lifetime}{Mean\,Failure\,Repair\,Time}$$

$$\delta \equiv \frac{Mean\,Failure\,Repair\,Time}{Mean\,P.M.\,Action\,Time} \; .$$

Various values of $\rho$ and $\delta$ are placed in the model and evaluated. Typical values of $\rho$ range from two up to ten thousand while values of $\delta$ range from two up to ten. Lower values of $\rho$ are expected to provide more noticeable changes in the average availability as T is changed. This is easily understood by noting that the lower the value of $\rho$, the longer the repair and preventive maintenance periods are in relation to the operative lifetime. Thus we would expect the average availability to be less for lower values of $\rho$ given all other parameters are held constant. The values obtained for T for finite time horizons are compared to the values obtained for T for infinite time horizons.

The infinite time horizon values also allow the model to be evaluated and partially validated for large values of $t$. Since the $\lim_{t \to \infty} A_{avg}(t) = \lim_{t \to \infty} A(t)$, we may take a large time horizon and evaluate the average availability at the value of T found from the infinite time horizon and compare it to the limiting availability obtained from the infinite time horizon model. The model is also checked at very small values of time where the availability of the model should be close to unity.

# CHAPTER 4

# MODEL DEVELOPMENT

## 4.1 Component Availability

Barlow and Proschan [7] define the availability function, $A(t)$, as the probability that a component is functioning at time $t$. They develop the integral equation for the availability function of an entity modeled by an alternating renewal process. This process is the repair upon component failure case in which no preventive maintenance is performed (T→∞). The availability equation for the repair upon failure (alternating renewal) process is

$$A(t) = R_L(t) + \int_0^t R_L(t-u)\, m(u)\, du \ , \tag{4.1}$$

where $R_L(t)$ is the component's survivor function (1-$F_L(t)$) and $m(t)$ is the renewal density function [7]. This expression for the availability as a function of time, $t$, may be interpreted as follows: the probability the component is functioning at time t is equal to the probability that the unit has not failed since the time the component was put into service (represented by the $R_L(t)$ term) plus the probability that a renewal occurred at time $u$, (represented by the $m(u)$ term) and no component failure occurred in the remaining time period (represented by the $R_L(t-u)$ term). The integral sums the probabilities of these possible events over the time interval [0,$t$].

### 4.1.1 Laplace Transform of the Availability Integral Equation for the Repair Upon Failure Case

Equation (4.1) gives an integral expression for finding the availability of a component that is repaired upon failure. This equation may be solved by taking the Laplace transform of both sides. Before proceeding, recognize that the second term on the right-hand side of equation (4.1) is a convolution of the component survivor function and the renewal density function of the process. This convolution may be found by taking the inverse Laplace transform of the product of the Laplace transforms of the survivor function and the renewal density (Tierney, p.251, [97]). This relationship is

$$R_L(t) \otimes m(t) = \int_0^t R(t-u)\, m(u)\, du = £^{-1}[\, R_L^*(s)\; m^*(s)\,],$$

where $f^*(s) = £[\, f(t)\,] = \int_0^\infty e^{-s\,t} f(t)\, dt$. Using this relationship and taking the Laplace transform of both sides of equation (4.1) we obtain

$$A^*(s) = R_L^*(s) + (R_L^*(s)\; m^*(s)) \; .$$

Factoring the $R_L^*(s)$, we obtain

$$A^*(s) \;=\; R_L^*(s)\,(1 + m^*(s))\;. \tag{4.2}$$

The renewal density, $m(t)$ is defined by Cox [26] and Barlow and Proschan [7] as the sum of the n-fold convolutions of the probability density function for the time between renewals, $h(t)$. Proofs may be found in the references above as well as in Ross, p. 297 [101]. This relationship is

$$m(t) = \sum_{n=1}^{\infty} \{h(t)\}^n \;, \tag{4.3}$$

where $\{h(t)\}^n$ is the n-fold convolution of $h(t)$. By taking the Laplace transform of both sides of equation (4.3) , Cox [26] (p.54) has shown a simplification of this infinite sum as

$$m^*(s) \;=\; \sum_{n=1}^{\infty} (h^*(s))^n \quad, \tag{4.4}$$

where in this case $(h^*(s))^n$ is $h^*(s)$ raised to the $n^{\underline{th}}$ power. Barlow and Proschan [7] (p.166) reduce equation (4.4) to

$$m^*(s) = \frac{h^*(s)}{1-h^*(s)} \quad . \tag{4.5}$$

Substituting this result into equation (4.2), the following is obtained.

$$A^*(s) = R_L^*(s)\,(1 + \tfrac{h^*(s)}{1-h^*(s)})\ .$$

Simplifying, the desired result is

$$A^*(s) = \frac{R_L^*(s)}{1-h^*(s)} \quad . \tag{4.6}$$

Recall that $R_L^*(s)$ is the Laplace transform of the component's survivor function and that $h^*(s)$ is the Laplace transform of the probability density function for the time between renewals in the process. The main significance and advantage of equation (4.6) is that the Laplace transform expression for the component availability function does not require the Laplace transform of the renewal density function, $m^*(s)$.

## 4.2 The Availability Function for a Component Age Replacement Preventive Maintenance Policy

This development begins by breaking the time scale into two intervals. The first includes the interval where the availability function of the component is desired for $t <$ T, where $t$ is the time variable and T is the age replacement period. Recall that the component is preventively maintained whenever T time units pass since the last maintenance action, whether failure repair or preventive maintenance was last performed. The second interval considers $t \geq \mathrm{T}$.

For $t < \mathrm{T}$, a component repair/replacement due to preventive maintenance will not have occurred. All repair/replacements during this time interval are due to component failure. Thus, for $t < \mathrm{T}$, a repair upon failure process (alternating renewal) is experienced and the availability function as in equation (4.1) is expressed as

$$A(t) = R_L(t) + \int_0^t R_L(t-u)\, m(u)\, du\,, \qquad for\ t < \mathrm{T}.$$

For $t \geq \mathrm{T}$, component renewal may occur due to component failure or by preventive maintenance. In this case, the farthest point back in time a renewal may occur from time $t$, is $t - \mathrm{T}$ time units. This condition holds since an age replacement policy of T time units is in place. Thus, our time interval is limited to $[t - \mathrm{T}, t]$. The expression for the availability function for $t \geq \mathrm{T}$ is

$$A(t) = \int_{t-\mathrm{T}}^{t} R_L(t-u)\, m(u)\, du, \qquad for\ t \geq \mathrm{T}. \tag{4.7}$$

Equation (4.7) may be interpreted as follows: for $t \geq \mathrm{T}$, the probability that the component functions at time $t$, $A(t)$, is equal to the probability that a renewal occurs at time $u$ (represented by $m(u)$) **and** the component survives the remaining period of time (represented by $R_L(t-u)$). The renewals, which occur due to component failure or scheduled preventive maintenance, may occur continuously in time during the interval $[t - \mathrm{T}, t]$. Only this time interval is of interest in the case $t \geq \mathrm{T}$, since the longest time that may elapse after a renewal is T time units. This particular event has a renewal (preventive maintenance in this case) occurring at time $t - \mathrm{T}$ and no failures in the remaining period of time.

Summarizing, the availability function for a component age replacement preventive maintenance policy is

$$A(t) = \begin{cases} R_L(t) + \int_0^t R_L(t-u)\, m(u)\, du, & for\ t < \mathrm{T}, \\ \\ \int_{t-\mathrm{T}}^t R_L(t-u)\, m(u)\, du, & for\ t \geq \mathrm{T}. \end{cases} \tag{4.8}$$

## 4.3 Laplace Transform for the Availability Function for a Component Age Replacement Preventive Maintenance Policy

The following development shows the construction of the Laplace transform of the availability function, $A(t)$, stated in equation (4.8). The proof is started by noting the definition of the Laplace transform for the availability function is

$$A^*(s) = \int_0^\infty e^{-s\,t}\, A(t)\, dt.$$

Applying (4.8) to this relationship, the following is obtained:

$$A^*(s) = \left(\int_0^{\mathsf{T}} e^{-st}\left(R_L(t) + \int_0^t R_L(t-u)\,m(u)\,du\right)dt\right)$$
$$+ \left(\int_{\mathsf{T}}^{\infty} e^{-st}\left(\int_{t-\mathsf{T}}^t R_L(t-u)\,m(u)\,du\right)dt\right)$$

$$= \int_0^{\mathsf{T}} e^{-st} R_L(t)\,dt + \int_0^{\mathsf{T}} e^{-st}\left(\int_0^t R_L(t-u)\,m(u)\,du\right)dt$$
$$+ \int_{\mathsf{T}}^{\infty} e^{-st}\left(\int_{t-\mathsf{T}}^t R_L(t-u)\,m(u)\,du\right)dt$$
$$= \int_0^{\mathsf{T}} e^{-st} R_L(t)\,dt + \int_0^{\mathsf{T}}\int_0^t e^{-st} R_L(t-u)\,m(u)\,du\;dt$$
$$+ \int_{\mathsf{T}}^{\infty}\int_{t-\mathsf{T}}^t e^{-st} R_L(t-u)\,m(u)\,du\,dt \;.$$

Changing the order of integration on the last two terms involving the double integrals, the following is obtained:

$$A^*(s) = \int_0^{\mathsf{T}} e^{-st} R_L(t)\,dt + \int_0^{\mathsf{T}}\int_u^{\mathsf{T}} e^{-st} R_L(t-u)\,m(u)\,dt\;du$$
$$+ \int_0^{\mathsf{T}}\int_{\mathsf{T}}^{\mathsf{T}+u} e^{-st} R_L(t-u)\,m(u)\,dt\,du + \int_{\mathsf{T}}^{\infty}\int_u^{\mathsf{T}+u} e^{-st} R_L(t-u)\,m(u)\,dt\,du$$

$$= \int_0^{\mathsf{T}} e^{-st} R_L(t)\,dt + \int_0^{\mathsf{T}} m(u)\int_u^{\mathsf{T}} e^{-st} R_L(t-u)\,dt\,du$$
$$+ \int_0^{\mathsf{T}} m(u)\int_{\mathsf{T}}^{\mathsf{T}+u} e^{-st} R_L(t-u)\,dt\,du + \int_{\mathsf{T}}^{\infty} m(u)\int_u^{\mathsf{T}+u} e^{-st} R_L(t-u)\,dt\,du.$$

Performing a change of variable by substituting $y = t - u$, the following is obtained:

$$
\begin{aligned}
A^*(s) &= \int_0^{\mathsf{T}} e^{-st} R_L(t)\, dt + \int_0^{\mathsf{T}} m(u) \int_0^{\mathsf{T}-u} e^{-s(y+u)} R_L(y)\, dy\, du \\
&\quad + \int_0^{\mathsf{T}} m(u) \int_{\mathsf{T}-u}^{\mathsf{T}} e^{-s(y+u)} R_L(y)\, dy\, du + \int_{\mathsf{T}}^{\infty} m(u) \int_0^{\mathsf{T}} e^{-s(y+u)} R_L(y)\, dy\, du
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{\mathsf{T}} e^{-st} R_L(t)\, dt + \int_0^{\mathsf{T}} e^{-su} m(u) \int_0^{\mathsf{T}-u} e^{-sy} R_L(y)\, dy\, du \\
&\quad + \int_0^{\mathsf{T}} e^{-su} m(u) \int_{\mathsf{T}-u}^{\mathsf{T}} e^{-sy} R_L(y)\, dy\, du + \int_{\mathsf{T}}^{\infty} e^{-su} m(u) \int_0^{\mathsf{T}} e^{-sy} R_L(y)\, dy\, du
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{\mathsf{T}} e^{-st} R_L(t)\, dt + \int_0^{\mathsf{T}} e^{-su} m(u) \int_0^{\mathsf{T}} e^{-sy} R_L(y)\, dy\, du \\
&\qquad + \int_{\mathsf{T}}^{\infty} e^{-su} m(u) \int_0^{\mathsf{T}} e^{-sy} R_L(y)\, dy\, du
\end{aligned}
$$

$$
A^*(s) = \int_0^{\mathsf{T}} e^{-st} R_L(t)\, dt + \left( \left( \int_0^{\infty} e^{-su} m(u)\, du \right) \left( \int_0^{\mathsf{T}} e^{-sy} R_L(y)\, dy \right) \right) . \qquad (4.9)
$$

By definition the following holds:

$$
m^*(s) = \int_0^{\infty} e^{-su} m(u)\, du \quad .
$$

Define the partial or truncated Laplace transform of a function, $f(t)$, as

$$
f^*(s, T) = \int_0^{T} e^{-st} f(t)\, dt \, . \qquad (4.10)
$$

Thus, the following holds:

$$
R_L^*(s, \mathsf{T}) = \int_0^{\mathsf{T}} e^{-st} R_L(t)\, dt \quad .
$$

Using this notation, equation (4.9) becomes the following:

$$A^*(s) = R_L^*(s,\mathrm{T}) + \Big(m^*(s) \circledast R_L^*(s,\mathrm{T})\Big)$$

$$A^*(s) = R_L^*(s,\mathrm{T})\Big(1 + m^*(s)\Big) \ . \tag{4.11}$$

Substituting the results of equation (4.5) into (4.11) the following is obtained:

$$A^*(s) = R_L^*(s,\mathrm{T})\left(1 + \frac{h^*(s)}{1-h^*(s)}\right) .$$

Simplifying, the desired result is

$$A^*(s) = \frac{R_L^*(s,\mathrm{T})}{1-h^*(s)} \ . \tag{4.12}$$

Equation (4.12) gives the Laplace transform of the availability function for a component age replacement preventive maintenance policy. In this form, it does not require the Laplace transform of the renewal density. It depends on the partial Laplace transform of the component's survivor function and the Laplace transform of the probability density function for the time between renewals for the referenced stochastic process.

It is assumed that the distribution function for the component's lifetime is known and therefore the survivor function, $R_L(t)$, equivalent to (1-$F_L(t)$) is also known. Also, the probability density functions for the time to complete a repair due to component failure as well as the time to complete a preventive maintenance action are known. However, the probability density function for the time between renewals, referred to as $h(t)$, is not known. A renewal theory approach is applied to develop the Laplace transform of the probability density function for the time between renewals.

## 4.4 Laplace Transform for the Time Between Renewals

The notation used in this development is first presented in Section 3.1 and for convenience is reiterated below. The stochastic process under consideration is depicted by Figure 3.1 in Section 3.1.

Notation:

$f_L(t) \equiv$ known component lifetime density

$F_L(t) \equiv$ known component lifetime distribution, $\int_0^t f_L(t)\,dt$

$R_L(t) \equiv$ known component survivor function, $1 - F_L(t)$

$g_r(t) \equiv$ known repair time density after component failure

$g_p(t) \equiv$ known preventive maintenance time density

T $\equiv$ age replacement preventive maintenance period, i.e. if a component survives T time units then it is preventively maintained.

The following assumptions are made concerning this stochastic process:

(1) All probability densities (lifetime, failure repair, and preventive maintenance action) are known or may be estimated from available data.

(2) A single component is modeled and is considered to be continuously demanded when available. If the component is not continuously demanded, then it continues to age according to the lifetime density during periods of availability whether demanded or not.

(3) Repairs as a result of component failure and preventive maintenance actions are perfect. At their completion, the component is "as good as new" and thus these points are renewal points in the process. At the beginning of each cycle, the component exhibits identical probabilistic behavior as a "brand new" component.

(4) Upon completion of a repair due to component failure or a preventive maintenance action, the component is immediately placed back into operation.

(5) Upon completion of a repair due to component failure or a preventive maintenance action, the time counter for age replacement preventive maintenance policy is reset to zero.

Two types of events or cycles occur in this stochastic process. The first is termed a "failure cycle". In this event, the component fails before time T and undergoes a stochastic repair period subject to the probability density, $g_r(t)$. The second event is termed a "preventive maintenance cycle". In this event, the component survives until time T and then undergoes a stochastic preventive maintenance period subject to the probability density, $g_p(t)$. To develop the probability density function for the time between renewals in this process, an analysis of each cycle is first performed. The Laplace transform for the probability density function of the time between renewals within each cycle is developed. Then, by conditioning on the time of component failure, an expression for the Laplace transform for the time between renewals, $h(t)$, is developed.

### 4.4.1 Failure Cycle

In this event, the component fails before time T. Thus, if $t$ is the continuous time variable, then $t < \mathrm{T}$. Recall that we assume that the component lifetime density function is known. However, in this cycle the lifetime density function is truncated at time T. This requires the component lifetime density function, $f_L(t)$, to be normalized by the factor, $F_L(\mathrm{T})$. In fact, $F_L(\mathrm{T})$ is the probability that the component fails before time T. The truncated lifetime density function is $\frac{f_L(t)}{F_L(\mathrm{T})}$ $for\, t \leq \mathrm{T}$ and $0\, otherwise.$ Define the probability density function for the time between renewals in the failure cycle as $h_f(t)$. This density function, $h_f(t)$, is the convolution of the truncated lifetime density function, $\frac{f_L(t)}{F_L(\mathrm{T})}$, and the failure repair time probability density function, $g_r(t)$. This is expressed mathematically as

$$h_f(t) = \left(\frac{f_L(t)}{F_L(\mathrm{T})}\right) \otimes g_r(t) \quad , \tag{4.13}$$

where the convolution operator is designated by the symbol, $\otimes$. Taking the Laplace transform of both sides of equation (4.13) and applying the convolution theorem for truncated Laplace transforms yields the following:

$$h_f^*(s) = \left(\int_0^T e^{-st}\,\frac{f_L(t)}{F_L(\mathrm{T})}\,dt\right)\left(\int_0^\infty e^{-st}\,g_r(t)\,dt\right)$$

$$h_f^*(s) = \left(\int_0^T e^{-st}\,\frac{f_L(t)}{F_L(\mathrm{T})}\,dt\right)\left(g_r^*(s)\right)$$

$$h_f^*(s) = \frac{1}{F_L(\mathrm{T})}\left(\int_0^T e^{-st}\,f_L(t)\,dt\right)\left(g_r^*(s)\right) \ .$$

Note that the Laplace transform of the truncated lifetime density is also truncated at T. This is reflected in the upper limit of the integral since the interval for $t$ is [ 0,T ] in this cycle. The component fails before T time units have elapsed.

### 4.4.2 Preventive Maintenance Cycle

In this event, the component survives until time T. Thus, if $t$ is the continuous time variable, then $t =$ T. In this cycle the operation time of the component is a constant time period equal to T time units. Probabilistically, this constant time period may be treated as a Dirac Delta function (or Unit Impulse function) for the probability density function. This function is designated, $\delta(t-a)$ and has a singularity of infinite value at $t = a$ and is equal to zero at all other values of $t$. The Dirac Delta function for this constant time period, $\delta(t-\mathrm{T})$, has the following properties ([102], pp. 8-9):

(1) $\int_0^\infty \delta(t-\mathrm{T})\,dt = 1$

(2) $\int_0^\infty \delta(t-\mathrm{T})\,G(t)\,dt = G(\mathrm{T})$ for any continuous function.

Property 1 gives the Dirac Delta function the primary characteristic of a probability density function. Property 2 assists in taking the Laplace transform of the Dirac Delta function. The Laplace transform for the Dirac Delta function for this constant time period is

$$£\ [\delta(t-\mathrm{T})] = \int_0^\infty \delta(t-\mathrm{T})\,e^{-st}\,dt$$
$$£\ [\delta(t-\mathrm{T})] = \mathrm{e}^{-s\mathrm{T}}\ .$$

Define the probability density function for the time between renewals in the preventive maintenance cycle as $h_p(t)$. This density function is the convolution of the Dirac Delta function, $\delta(t-\mathrm{T})$, and the preventive maintenance time probability density function, $g_p(t)$. This is expressed mathematically in the following:

$$h_p(t) = \Big(\delta(t-\mathrm{T})\Big) \otimes g_p(t), \qquad (4.14)$$

where the convolution operator is designated by the symbol, $\otimes$. Taking the Laplace transform of both sides of equation (4.14) and applying the convolution theorem for Laplace transforms yields

$$h_p^*(s) = \left(\int_0^\infty e^{-st}\,\delta(t-\mathrm{T})\,dt\right)\left(\int_0^\infty e^{-st}\,g_p(t)\,dt\right)$$

$$h_p^*(s) = e^{-s\mathrm{T}}\left(g_p^*(s)\right) \quad . \tag{4.15}$$

### 4.4.3 Conditioning

To obtain the complete expression for the Laplace transform of the probability density function for the time between renewals, $h^*(s)$, for the stochastic process comprised of randomly occurring failure cycles and preventive maintenance cycles, we condition on the length of time the component has been in operation at the time of component failure (refer to Figure 3.1). Specifically, two conditions shall be used. These are:

(1) Component failure before the component has been in operation for T time units since the last service action ($t_{\mathrm{oper}} < \mathrm{T}$, where $t_{\mathrm{oper}}$ is elapsed time of component operation since the last service action). A service action is either a repair due to component failure or a preventive maintenance action.

(2) Component survival until the component has been in operation T time units since the last service action ($t_{\text{oper}} \geq \mathrm{T}$, where $t_{\text{oper}}$ is elapsed time of component operation since the last service action).

The $h^*(s)$ may be found by using the following conditional probability statement:

$$h^*(s) = h^*(s)\Big|_{t_{\text{oper}}<\mathrm{T}} \; Prob\{t_{\text{oper}} < \mathrm{T}\} \; + \; h^*(s)\Big|_{t_{\text{oper}}\geq\mathrm{T}} \; Prob\{t_{\text{oper}} \geq T\} \; .$$

Under the condition, $t_{\text{oper}} < \mathrm{T}$, the stochastic process experiences a failure cycle event. Thus, $h^*(s)\Big|_{t_{\text{oper}}<\mathrm{T}}$, is as shown in equation (4.13). Under the condition, $t_{\text{oper}} \geq \mathrm{T}$, the stochastic process experiences a preventive maintenance cycle event. Thus, $h^*(s)\Big|_{t_{\text{oper}}\geq\mathrm{T}}$, is as shown in equation (4.14).

The probability that the component fails before it operates T time units, $Prob\{t_{\text{oper}} < \mathrm{T}\}$, is $F_L(\mathrm{T})$. The probability that the component survives at least T time units, $Prob\{t_{\text{oper}} \geq T\}$, is $R_L(T) = 1 - F_L(\mathrm{T})$. Substituting these into equation (4.15), the following expression for the Laplace transform of the time between renewals for the referenced stochastic process is

$$h^*(s) = \left(\frac{1}{F_L(\mathrm{T})}\left(\int_0^T e^{-st} f_L(t)\,dt\right)\left(g_r^*(s)\right)F_L(\mathrm{T})\right) + \left(e^{-s\mathrm{T}}\left(g_p^*(s)\right)R_L(\mathrm{T})\right)$$

$$h^*(s) = f_L^*(s,\,\mathrm{T})\,g_r^*(s) + e^{-s\mathrm{T}}R_L(\mathrm{T})\,g_p^*(s) \quad , \qquad (4.16)$$

where $f_L^*(s, \mathrm{T})$ is the partial Laplace transform of $f_L(t)$ as defined by equation (4.10).

## 4.5 Availability Model Conclusion

The availability model for the age replacement preventive maintenance is complete. An expression is given for the Laplace transform of the availability function in terms of the Laplace transforms of the component's survivor function and the probability density function for the time between renewals. Restated, $A^*(s)$ is

$$A^*(s) = \frac{R_L^*(s, \mathrm{T})}{1-h^*(s)} \quad , \tag{4.17}$$

where $h^*(s)$ is shown in equation (4.16).

### 4.5.1 Model Issues

Three main issues affect the use of this model to obtain an expression for $A(t)$. These issues include finding partial (truncated) and normal Laplace transforms, finding an estimated inverse Laplace transform for $A^*(s)$ and the numerical integration of $A(t)$ to find the value of T that maximizes the average availability of a time interval $[0, t]$.

### 4.5.1.1 Partial and Normal Laplace Transforms

The first issue involves finding the partial Laplace transforms of the lifetime density, $f_L^*(s, \mathrm{T})$ and the partial Laplace transform of the component's survivor function, $R_L^*(s, \mathrm{T})$. For the most interesting types of lifetime densities (i.e., those exhibiting the ability to model increasing hazard or "component aging"), special power expansion techniques are employed. The lifetime density to be used in this case is the Weibull density function defined [95] as

$$f_L(t) = \alpha\,\beta\,t^{\beta-1}e^{-\alpha\,t^{\beta}}\ ,\ t \geq 0,$$

and
$$F_L(t) = 1 - e^{-\alpha\,t^{\beta}},$$

where $\alpha$ is the scale parameter and $\beta$ is the shape parameter. For values of $\beta > 1$, the Weibull lifetime density exhibits strictly increasing hazard.

Consider the Rayleigh density function which is a special case of the Weibull with $\beta = 2$. In this case, the hazard function, $z(t)$, is defined as follows:

$$z(t) = \frac{f_L(t)}{R_L(t)} = 2\,\alpha\,t\,.$$

Therefore, for the Rayleigh lifetime density function, the hazard function is strictly increasing in a linear fashion. Practically, this family of distributions models aging components but does not model a "wear out" period very well since the hazard function increases at same rate over all time. Wear out periods may be modeled using Weibull lifetime densities with values of $\beta > 2$. In these cases, the force of component mortality increases exponentially with time. Analysis here proceeds with the Rayleigh distribution and the application of power series expansions to find $f_L^*(s, \mathrm{T})$ and $R_L^*(s, \mathrm{T})$. These expansions are truncated to provide estimates for the preceding terms.

In the event that the Weibull distribution is used to model the repair time and preventive maintenance time distributions, a similar approach is followed. The difference in this case is that these service times are not truncated. Thus we must find an estimate for the full Laplace transform. Power series expansions may be applied to find $g_r(t)$ and $g_p(t)$. If successful, these expansion are truncated to provide estimates for the preceding terms.

#### 4.5.1.2 Laplace Transform Inversion Estimation

The complexity of the expression obtained for $A^*(s)$ will increase as more realistic distributions are used for the component's lifetime density as well as the repair and preventive maintenance action times. Due to this complexity, it is unlikely that an exact inversion of $A^*(s)$ may be accomplished. Therefore methods to estimate the time domain function, $A(t)$, are investigated. One family of methods provides an estimate of the value of the time domain function, $A(t)$, at specific points in time from the $A^*(s)$. One approach would be to iteratively run a numerical inversion technique over many values of time and then fit a function to the values to obtain an estimate of $A(t)$. Unfortunately, this methodology would become very time consuming and cumbersome. However, Cheng et al [66] have implemented several of the well known Laplace transform numerical inversion techniques in Mathematica. Their focus is on the numerical solution of linear partial differential equations relating to groundwater and reservoir engineering applications. Mathematica is used to compile numerical results for functions with known Laplace inverses for comparison purposes. Due to the analytical capabilities of Mathematica, this analysis uses these inversion estimation techniques in a somewhat different manner. In the analysis presented here, the time parameter, $t$, is left as a parameter in the inversion techniques. The result from Mathematica is an estimate of the time domain function, $A(t)$, which is an extension from a single estimated value of the time domain function at a specific time, $t$.

#### 4.5.1.3 Numerical Integration Search to Find Values of T to Maximize Average Availability

After obtaining an estimate of $A(t)$, a numerical integration search is attempted for a specific value of $t$, to find the value of T that maximizes average availability over the time interval $[0, t]$. This numerical search is performed in Mathematica. The results are compared to the values of T for the infinite time horizon case.

Additionally, methods for obtaining a direct functional form of the average availability with respect to time are investigated.

# CHAPTER 5

# THE EXPONENTIAL CASE

## 5.1 Exponential Case Description

In this case, the component's lifetime as well as the time required to perform repair upon component failure and the time required to perform preventive maintenance on the component are all considered to have exponential probability density functions (pdf). These exponential density functions are assumed to be unique and independent of each other. These functions are defined as

$f_L(t) = \lambda\, e^{-\lambda t}$ (the component lifetime pdf),

$g_r(t) = \mu_r\, e^{-\mu_r t}$ (the repair time pdf), and

$g_p(t) = \mu_p\, e^{-\mu_p t}$ (the preventive maintenance time pdf).

This particular case is selected to provide validation examples for the proposed model. The hazard function, $z(t)$, is defined as follows [7]:

$$z(t) = \frac{f_L(t)}{R_L(t)} \ .$$

For the exponential lifetime density, $z(t)$ is equal to $\lambda$, a constant. Thus, as the component ages, the hazard remains constant. Therefore the "memoryless" property of the exponential pdf applies. Given that the component survives $t$ time units, the probability that the component fails in the next instant of time remains constant for all $t$. In essence, the component does not exhibit aging since whether it has survived one time unit or one-hundred time units, the probability of it failing in the next instant of time is the same. Thus, given the component has survived $t$ time units, it is always as good as new. Barlow and Proschan [7] term this constant hazard.

Logically, it is easy to argue that if the component's lifetime is modeled in this manner then preventive maintenance should never be performed. There would not exist a component age at which one would desire to purposely remove the component from operation to perform preventive maintenance. Recall that given the component has survived $t$ time units, it is as good as new. Thus for the exponential case, the replacement period maximizing average availability should be infinity or in other words, no preventive maintenance should be performed. In this case, the process simplifies into an alternating renewal process where the component is repaired upon failure.

The model developed in Chapter 4 is used to analyze the exponential case described above. This case offers a method to assist in validating the model since the correct outcome is known.

## 5.2 Development of the Availability Function for the Exponential Case

Recalling equations (4.16) and (4.17) from Sections 4.4.3 and 4.5 respectively, the Laplace transform of the availability function, $A^*(s)$, is

$$A^*(s) = \frac{R_L^*(s, \mathsf{T})}{1-h^*(s)} \quad ,$$

where $$h^*(s) = f_L^*(s, \mathsf{T})\, g_r^*(s) + e^{-s\mathsf{T}} R_L(\mathsf{T})\, g_p^*(s)$$

and $$f^*(s, \mathsf{T}) = \int_0^{\mathsf{T}} e^{-st} f(t)\, dt \; .$$

Applying these equations to the referenced exponential case, the following is obtained:

$$A^*(s) = \frac{\int_0^{\mathsf{T}} e^{-st}\, e^{-\lambda t}\, dt}{1-\left(\left(\int_0^{\mathsf{T}} e^{-st}\, e^{-\lambda t}\, dt\right)\frac{\mu_r}{s+\mu_r} + \left(e^{-\mathsf{T}(s+\lambda)}\right)\frac{\mu_p}{s+\mu_p}\right)}$$

$$A^*(s) = \frac{\frac{1}{(s+\lambda)}\left(1-e^{-\mathsf{T}(s+\lambda)}\right)}{1-\left(\frac{\mu_r}{s+\mu_r}\;\frac{\lambda}{s+\lambda}\left(1-e^{-\mathsf{T}(s+\lambda)}\right) + \frac{\mu_p}{s+\mu_p}\left(e^{-\mathsf{T}(s+\lambda)}\right)\right)}$$

$$A^*(s) = \frac{(\mu_p+s)\,(\mu_r+s) - (\mu_p+s)\,(\mu_r+s)\, e^{-\mathsf{T}(s+\lambda)}}{(\lambda+s)\,(\mu_r+s)\,(\mu_p+s) - \lambda\mu_r(\mu_p+s) + \left(\lambda\mu_r(\mu_p+s) - \mu_p(\lambda+s)\,(\mu_r+s)\right) e^{-\mathsf{T}(s+\lambda)}} \; . \quad (5.1)$$

## 5.3 Laplace Transform Inversion to Obtain $A(t)$

Inversion of this function shown in equation (5.1) to the time domain is not simple. One simplifying condition is to note that the numerator has two terms that only differ by a factor of $e^{-T(s+\lambda)}$. This allows the inversion to be broken down into the inversion of two terms. This factor $e^{-T(s+\lambda)}$may be stated as $e^{-Ts}e^{-T\lambda}$, in which, $e^{-T\lambda}$, is a constant. A useful property of inverting Laplace transforms (see [98]) is

$$£^{-1}[e^{-Ts}e^{-T\lambda}f^{*}(s)] = e^{-T\lambda}\left(\text{Unitstep}[t-\text{T}]\right) F(t-\text{T}) , \tag{5.2}$$

where $$F(u) = £^{-1}[f(s)]$$

and $$\text{Unitstep}[t-\text{T}] = \begin{cases} 0 \text{ for } t-\text{T}<0 \\ 1 \text{ for } t-\text{T}\geq 0 \end{cases} .$$

This allows the inversion to proceed with only the first term and then use the rules shown above to obtain the inverse of the second term. We then sum the results to obtain the $A(t)$.

The work now lies with inverting the function shown below.

$$£^{-1}\left[\frac{(\mu_p+s)\,(\mu_r+s)}{(\lambda+s)\,(\mu_r+s)\,(\mu_p+s)-\lambda\mu_r(\mu_p+s)+\left(\lambda\mu_r(\mu_p+s)-\mu_p(\lambda+s)\,(\mu_r+s)\right)\,e^{-T(s+\lambda)}}\right] \tag{5.3}$$

Using Mathematica (the Apart function) this term may be broken down into the sum of the following fractions:

$$£^{-1}\left( \frac{(\mu_r+s)}{s(\mu_r+\lambda+s)} \right) +$$

$$£^{-1}\left( \frac{(\mu_r+s)(\mu_r\mu_p-\mu_r\lambda+\mu_p\lambda+\mu_p s)}{s(\mu_r+\lambda+s)(\mu_r\lambda-\mu_r\mu_p-\mu_p\lambda-\mu_p s+(e^{(\lambda+s)T}(\mu_r\mu_p+\mu_p\lambda+(\mu_r+\mu_p+\lambda)s+s^2)} \right) \quad (5.4)$$

The first term in this sum may be inverted exactly and is

$$£^{-1}\left( \frac{(\mu_r+s)}{s(\mu_{rr}+\lambda+s)} \right) = \frac{\mu_r}{\mu_r+\lambda} + \frac{\lambda}{\mu_r+\lambda}\, e^{-t(\mu_r+\lambda)} \quad .$$

The time domain function for the first term in equation (5.4) is identical to the time domain function for the alternating renewal process where repair upon failure is executed (i.e. preventive maintenance is not performed). The problem now is to invert the second term of the summation shown in equation (5.4). Note that this term in equation (5.4), which is designated as $g_2(s)$, may be represented as

$$g_2(s) = \frac{q(s)}{p(s)} \quad .$$

Since $q(s)$ is a second order polynomial with respect to $s$, and of the form $c_1 + c_2 s + c_3 s^2$, we may apply standard rules for inverting the Laplace transform. If $G_2(t)$ represents the $£^{-1}\{g_2(s)\}$ and $P(t)$ represents the $£^{-1}\{\frac{1}{p(s)}\}$ [97], we obtain

$$G_2(t) = c_1 P(t) + c_2 \frac{d}{dt}[P(t)] + c_3 \frac{d^2}{dt^2}[P(t)]\,. \qquad (5.5)$$

This decomposition process leaves the task of inverting $\frac{1}{p(s)}$. However, this term is successfully inverted exactly. An estimate of the inverted function was obtained using a numerical inversion package obtained from Wolfram Research, Inc., distributors of Mathematica.

### 5.3.1 Laplace Transform Inversion Estimation

A numerical inversion package for Mathematica written by Cheng, et al, [66] is implemented. The Stehfest method [67 and 98] is used to invert the ($\frac{1}{p(s)}$) term referenced in the previous section. The Stehfest method for approximating inverse Laplace transforms is designed to provide an estimate of the time domain function at a specific value of time from the Laplace transform function. For this reason, the technique is termed a "numerical" inversion package. The power of Mathematica allows the time specification to be left as a parameter. This allows Mathematica to analytically solve for the time domain function and hence give us an estimate of the time domain function, $A(t)$.

The Stehfest method [67 and 98] obtains an estimate of the time domain function by defining the time domain function, $F(t)$, as

$$F(t) = \frac{ln\,2}{t} \sum_{n=1}^{N} c_n f^*\left(\frac{n\,ln\,2}{t}\right),$$

$$where \qquad £[F(t)] = f^*(u) \qquad and$$

$$c_n = (\text{-}1)^{n+\frac{N}{2}} \sum_{k=\lfloor \frac{n+1}{2} \rfloor}^{min(n,\,\frac{N}{2})} \frac{k^{\frac{N}{2}}\,(2k)!}{(\frac{N}{2}-k)!\,k!\,(k-1)!\,(n-k)!\,(2k-n)!} \quad .$$

Tests were performed within Mathematica to verify the validity of the Stehfest method. The tests consisted of performing the numerical inversion for a specific value of time for a function, $f^*(s)$. The result of this test provided time domain values for $F(t)$ (note that $£^{-1}[f^*(s)] = F(t)$) . The next test consisted of performing the numerical inversion with the time specification left as the parameter, $t$. The result was an estimate of the function, $F(t)$ (as a function of time). The same values used in the previous test were evaluated by the estimated time domain function and were found to be the same as those obtained by direct numerical inversion. The test was successful and the "numerical" inversion package was implemented to provide an estimate of the time domain function. The results of these tests appear in Appendix 1. This appendix provides the annotated Mathematica input statements and output used in obtaining these results.

Using the relationship shown in equation (5.2), an estimate of the inversion of the first term in equation (5.1) shown in equation (5.2) is obtained. Finally an estimate of the availability function, $A(t)$, is obtained by applying equation (5.2). Appendix 2 shows the Mathematica statements used to find this estimate of $A(t)$ for the exponential case. This estimated function is long and complicated and is left as a function in Mathematica. If desired, the estimated function obtained from the Stehfest method may be observed within Mathematica. Note that in these results, T, the age replacement period is included as a parameter. Thus the estimated availability function obtained, $A(t, \mathrm{T})$, is actually a function of both continuous time, $t$ and the age replacement period, T.

## 5.4 Model Validation

Two methods are used to validate the estimated availability function, $A(t)$, obtained as detailed in Section 5.3.1. The first method involves evaluating the value of the availability function at very small values of time. Since the estimated function contains factors of $(\frac{1}{t})$, evaluating $t = 0$ is not possible. For the case where $\lambda$=1., $\rho = 1000.$ and $\delta = 5.$, the estimated availability function is evaluated for $t = 0.001$ for various values of T, the age replacement policy. In all cases the results are very close to unity. For example, with T $= 1.$, the function evaluated at $t = 0.001$ was 0.999363. Also for T=10., the function evaluated at $t = 0.00000001$ was 0.99999999. These values are sufficiently close to unity and provide availability model validation for small values of $t$.

The second technique used for validating the model is evaluation of the availability for large values of T at large values of continuous time, $t$. In this case, as T$\to\infty$, the stochastic process becomes a repair upon component failure, i.e. no preventive maintenance is accomplished. For evaluating this case, we set $\lambda$=1., $\rho = 1000.$, $\delta = 5.$ and T $= 1000.$ The estimated availability function is evaluated at $t = 500.$ The value obtained is $A(t = 500.) = 0.999001.$ The limiting availability for this case is given by

$$A = \lim_{t\to\infty} A(t) \quad ,$$

and for the repair upon failure case (alternating renewal process) is equivalent to

$$A = \frac{E[lifetime]}{E[lifetime]+E[repair\, time]}$$

$$A = \frac{1.}{1.+0.001} = 0.999001 \quad .$$

Thus for the values shown above the model estimates the limiting availability for the repair upon failure only process exactly.

These validation methods basically show the validity of the model at the end points. The first technique evaluated the model as $t \to 0$ while the second technique considered the case where both $t$ and T grew large. A major implication of this result is the development of an expression for the availability function for the component in this process. This availability function, $A(t,$ T), is a function of both time, $t$ and the replacement period, T. The literature search did not show this result in past research. Thus a method to evaluate the model's validity between the extremes discussed above has not been devised. However, the validation results presented are encouraging and analysis of the availability model will proceed.

## 5.5 Maximizing Average Availability Over Finite Time Intervals

The average availability, $A_{avg}(\tau)$, is defined as [7]

$$A_{avg}(\tau) = \frac{1}{\tau}\int_0^{\tau} A(t)\,dt \ .$$

The desired result is to find a value for T, the age replacement policy, that maximizes the average availability over a finite time horizon, $\tau$. Two methods to find the value of T are detailed in Chapter 3 . These methods were an integral equation search method and using Laplace transform inversion rules to obtain an analytical estimate of $A_{avg}(\tau)$ directly. The latter method is implemented and compared to cases in which the first method is used. Function values correspond well and therefore the method of using Laplace transform rules to obtain an analytical estimate of $A_{avg}(\tau)$ is not used.

Mathematica is used to perform searches for T that maximizes the value of $A_{avg}(\tau)$ for specific, finite values of $\tau$. An iterative, "double Do loop" is implemented through the Table function within Mathematica. The algorithm iterates through values of finite time horizon, $\tau$. At each value of $\tau$, $A_{avg}(\tau)$ function values are calculated for several values of T. Meaningful $A_{avg}(\tau)$ function values only exist for the case where T $\leq \tau$. The finite time horizon, $\tau$, is the operational period of the component's lifecycle. Thus if T $\geq \tau$, then no preventive maintenance is performed. This analysis is constrained to evaluating values of $A_{avg}(\tau)$ for which T $\leq \tau$. If the maximum value of $A_{avg}(\tau)$ occurs at T $= \tau$, then the age replacement period is equivalent to the finite time horizon. Therefore, no preventive maintenance would be performed.

For the exponential case, as explained earlier, no preventive maintenance should be performed. This is true regardless of the time horizon. An additional concern is the relationship between the mean repair time due to failure and the mean preventive maintenance time. If the mean preventive maintenance time exceeds the mean repair time, Barlow and Hunter [12], have shown that no preventive maintenance should be performed. The objective in obtaining this result is to maximize the limiting availability. The ratio, $\delta$, defined to be the mean repair time divided by the mean preventive maintenance time is always be greater than one ($\delta > 1$). Furthermore, the ratio, $\rho$, defined to be the mean lifetime divided by the mean repair time is characteristically large. Values of 10 to $10^3$ may be typical depending on the class of the component (i.e. mechanical, electronic). For demonstration purposes, smaller values of $\rho$ are used to exaggerate values of availability for ease of analysis.

### 5.5.1 Exponential Model Results

The purpose of these results is to provide additional model verification. The correct result for all values of $\tau$, is to obtain a model result that shows the maximum $A_{avg}(\tau)$ to be attained when preventive maintenance is not performed. It is expected that this result should be evident by $A_{avg}(\tau)$ being maximized when T $= \tau$.

The following exponential model is evaluated. The mean lifetime, $\frac{1}{\lambda} = 1.$; $\rho = 2.$, thus the mean repair time, $\frac{1}{\mu_r} = 0.5$ . The ratio of mean repair time to mean preventive maintenance time is $\delta = 5$. Thus, the mean preventive maintenance time, $\frac{1}{\mu_p} = 0.1$ . The results are shown below for values of $\tau$ between $0.5$ and $5$. Values of T include the upper end point. The results are shown in Table 5.1. Observe that the maximum value of $A_{avg}(\tau)$ occurs at the point where T $= \tau$. In some instances, the number of digits shown does not reflect this maximization. However, using sixteen digit accuracy within Mathematica reflected $A_{avg}(\tau)$ being maximized at T $= \tau$. This result confirms that for the exponential case, preventive maintenance should not be performed to maximize $A_{avg}(\tau)$ for finite time horizons. This provides additional evidence supporting the validity of the model.

Table 5.1 $A_{avg}(\tau)$ Values for Increasing Values of T, $0.5 \leq \tau \leq 5$, $0.5 \leq T \leq 5$

| | Values of T, the Age Replacement Preventive Maintenance Policy | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | t/T | 0.1 | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 | 4.5 | 5.0 |
| τ | 0.5 | .523 | .862 | X | X | X | X | X | X | X | X | X |
| | 1.0 | .472 | .710 | .794 | X | X | X | X | X | X | X | X |
| | 1.5 | .451 | .679 | .716 | .755 | X | X | X | X | X | X | X |
| | 2.0 | .440 | .661 | .700 | .713 | .732 | X | X | X | X | X | X |
| | 2.5 | .433 | .649 | .688 | .700 | .709 | .717 | X | X | X | X | X |
| | 3.0 | .429 | .641 | .680 | .692 | .698 | .703 | .707 | X | X | X | X |
| | 3.5 | .425 | .635 | .675 | .689 | .692 | .696 | .699 | .701 | X | X | X |
| | 4.0 | .423 | .631 | .670 | .683 | .688 | .691 | .693 | .695 | .696 | X | X |
| | 4.5 | .421 | .628 | .667 | .680 | .685 | .688 | .690 | .691 | .692 | .692 | X |
| | 5.0 | .420 | .626 | .664 | .677 | .682 | .685 | .687 | .688 | .689 | .689 | .689 |

# CHAPTER 6

# THE WEIBULL - EXPONENTIAL CASE

## 6.1 Introduction

In this case, I examine the age replacement preventive maintenance stochastic process with a Weibull component lifetime probability density function and exponential repair and preventive maintenance (p. m.) time probability density functions. These density functions are defined as

$f_L(t) = \alpha\,\beta\,t^{\beta-1}\,e^{-\alpha\,t^{\beta}}$ (component lifetime pdf),

$g_r(t) = \mu_r\,e^{-\mu_r t}$ (repair time pdf),

$g_p(t) = \mu_p\,e^{-\mu_p t}$ (preventive maintenance time pdf), and

T (age at which component receives preventive maintenance).

The stochastic process consists of randomly occurring failure cycles and preventive maintenance cycles. A failure cycle occurs when the component fails before it ages T time units. The Weibull distribution governs the random failure of the component. Given that the component fails before time T (before T time units have elapsed since the last service action), the component immediately begins failure repair. This random repair period is exponentially distributed. If the component survives T time units (since the last repair or preventive maintenance action) then the component is immediately removed from service and begins preventive maintenance service. The time required for preventive maintenance is random and is exponentially distributed. The exponential distributions modeling the preventive maintenance action and repair times are unique and independent of each other.

## 6.2 Application of Model

Recalling equations (4.16) and (4.17) from Sections 4.4.3 and 4.5, the Laplace transform of the availability function, $A^*(s)$, is

$$A^*(s) = \frac{R_L^*(s, \mathsf{T})}{1-h^*(s)} \;,$$

where $$h^*(s) = f_L^*(s, \mathsf{T})\, g_r^*(s) + e^{-s\mathsf{T}} R_L(\mathsf{T})\, g_p^*(s)$$

and $$f_L^*(s, \mathsf{T}) = \int_0^{\mathsf{T}} e^{-s\mathsf{T}} f_L(t)\, dt \,.$$

Applying these equations to the referenced Weibull-exponential case requires the development of expressions for the partial Laplace transform for the Weibull probability density function, $f_L^*(s, \mathrm{T})$, as well as for the Weibull survivor function, $R_L^*(s, \mathrm{T})$. The Laplace transforms for the exponentially distributed repair and preventive maintenance times are not truncated since the range on time variable in these cases is $0 < t < \infty$. These transforms are easily evaluated.

### 6.2.1 Laplace Transform of the Weibull Probability Density Function

In the form given in Section 6.1, the Laplace transform of the Weibull probability density function may not be found. However, the Weibull pdf may be expanded into a power series. The Laplace transform of the power series representation of the Weibull pdf may be found. Thus, an exact Laplace transform of the Weibull pdf is generated in the form of an infinite series.

The methodology used to obtain the Laplace transform for the Weibull pdf is also used to find the partial Laplace transforms of the Weibull pdf and survivor function.

### 6.2.1.1 Power Series Expansion of the Weibull Probability Density Function

Recall that the Weibull probability density function is defined as [95]

$$f_L(t) = \alpha\,\beta\,t^{\beta-1}\,e^{-\alpha\,t^{\beta}}\,. \tag{6.1}$$

Using a similar methodology as that shown by Lomnicki [99], this function may be expanded into a power series. First, let $x = t^{\beta}$ and expand the exponential term of the function as

$$e^{-\alpha\,x} = \sum_{n=0}^{\infty} (\text{-}1)^n \left(\frac{\alpha^n x^n}{n!}\right).$$

Substituting, $t^{\beta}$ for $x$, equation (6.2) is obtained.

$$e^{-\alpha\,t^{\beta}} = \sum_{n=0}^{\infty} (\text{-}1)^n \left(\frac{\alpha^n t^{(\beta n)}}{n!}\right) \quad . \tag{6.2}$$

Substituting this MacLaurin series expansion into equation (6.1), the power series expansion for the Weibull probability density function is

$$f_L(t) = \sum_{\mathrm{n=1}}^{\infty} (\text{-}1)^{n-1}\,\frac{(\alpha^n\,\beta\,t^{(n\beta-1)})}{(n-1)!}\,. \tag{6.3}$$

### 6.2.1.2 Laplace Transform of the Weibull Power Series Expansion

By definition, the Laplace transform of a function, $\pounds[f(t)]$ is

$$\pounds[f(t)] = f^{*}(s) = \int_0^{\infty} e^{-st} f(t)\, dt\,.$$

Applying this to equation (6.3), the following is derived:

$$f_L^{*}(s) = \int_0^{\infty} e^{-st} f_L(t)\, dt$$

$$f_L^{*}(s) = \int_0^{\infty} e^{-st} \left( \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(\alpha^n \beta\, t^{(n\beta-1)})}{(n-1)!} \right) dt\,. \tag{6.4}$$

Mathematica is used to solve the integral in equation (6.4). The resulting infinite series representation of the Weibull density function Laplace transform is

$$f_L^{*}(s) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\alpha^n \beta\, \Gamma[n\beta]}{(n-1)!\, s^{n\beta}}\,, \tag{6.5}$$

where $\Gamma[z]$ is the Gamma function (Abramowitz & Stegun [100], p. 255 and Mathematica Reference manual, [96], p. 365) and is defined as

$$\Gamma[z] = \int_0^{\infty} t^{(z-1)} e^{-z}\, dz\,. \tag{6.6}$$

The result shown in equation (6.5) is also useful for analyzing the age replacement preventive maintenance stochastic process with Weibull repair and preventive maintenance time distributions. Note that these time distributions are not truncated and may range, $0 < t < \infty$. However, the immediate requirements for this analysis are the partial Laplace transforms of the Weibull lifetime probability density function and the Weibull survivor function. Sections 6.2.2 and 6.2.3 develop these partial transforms.

### 6.2.2 Partial Laplace Transform of the Weibull Lifetime Density Function

In this case, we are concerned with the time interval for component failure within the failure cycle. This time interval is random and is distributed according to a Weibull probability density function. This known probability function, as shown in equation (6.1), has range $0 < t < \infty$. However, this time period is always less than T time units according to our age-replacement preventive maintenance policy. This fact is taken into consideration when the expression for $A^*(s)$ is developed for the general case in Chapter 4. This time truncation causes the Laplace transform to be truncated and is referred to as a partial Laplace transform. Recall the partial Laplace transform we are concerned with in general is

$$f_L^*(s, \mathrm{T}) = \int_0^{\mathrm{T}} e^{-st} f_L(t)\, dt \,. \tag{6.7}$$

Substituting the power series representation of the Weibull probability density function shown in equation (6.3) into equation (6.7) gives

$$f_L^*(s,\,\mathsf{T}) = \int_0^{\mathsf{T}} e^{-s\,t} \left( \sum_{n=1}^{\infty} (\text{-}1)^{n-1} \, \frac{(\alpha^n \, \beta \, t^{(n\beta-1)})}{(n-1)!} \right) dt \,. \tag{6.8}$$

Mathematica is used to solve the integral in equation (6.8). The resulting infinite series representation of the Weibull density function partial Laplace transform is

$$f_L^*(s,\,\mathsf{T}) = \sum_{n=1}^{\infty} (\text{-}1)^{n-1} \, \frac{\alpha^n \, \beta \, \gamma[\,n\beta,\, s\mathsf{T}\,]}{(n-1)!\, s^{n\,\beta}} \quad , \tag{6.9}$$

where $\gamma[\,a,\,z\,]$ is the alternative incomplete gamma function (Abramowitz and Stegun [100], p. 260 and by Wolfram [96], p. 365) and is defined as

$$\gamma[\,a,\,z\,] = \int_0^z t^{(a-1)} \, e^{-t} \, dt \tag{6.10}$$

Equation (6.10) gives an exact ( as $n \to \infty$) series representation of the partial Laplace transform of the Weibull lifetime probability density function. Practical use of this infinite series requires truncating the infinite series to obtain a finite representation of the Weibull pdf partial Laplace transform. This development provides the basis for obtaining the partial Laplace transform for the Weibull survivor function, $R_L^*(s,\,\mathsf{T})$. This development is presented in Section 6.2.3.

## 6.2.3 Partial Laplace Transform of the Weibull Survivor Function

The survivor function for a component's lifetime, $R_L(u)$, is defined as the probability that the component survives at least a period of time $u$. This is stated using probability notation, where $u$ represents the length of the component's life and $Y$ is some specified period of time, as

$$R_L(Y) = Prob\{t > Y\} = 1 - F(Y) \; .$$

In this case, the Weibull survivor function is defined by [7]:

$$R_L(t) = e^{-\alpha\, t^{\beta}} \; .$$

This survivor function may be expanded into an equivalent power series by the same method employed in Section 6.2.1.1 . The power series representation of the Weibull survivor function is

$$R_L(t) = \sum_{n=0}^{\infty} (\text{-}1)^n \, \frac{\alpha^n \, t^{n\beta}}{n!} \, . \qquad (6.11)$$

The partial Laplace transform of the survivor function, $R_L^*(t)$, is

$$R_L^*(s, \mathsf{T}) = \int_0^{\mathsf{T}} e^{-s\,t} \, R_L(t) \, dt \;\; . \qquad (6.12)$$

Substituting the power series representation of the Weibull survivor function shown in equation (6.11) into equation (6.12) gives

$$R_L^*(s,\ \mathsf{T}) = \int_0^{\mathsf{T}} e^{-st} \left( \sum_{n=0}^{\infty} (\text{-}1)^n \frac{\alpha^n t^{n\beta}}{n!} \right) dt\ . \tag{6.13}$$

Mathematica is used to solve the integral in (6.13). The resulting infinite series representation of the Weibull survivor function partial Laplace transform is

$$R_L^*(s,\ \mathsf{T}) = \frac{1}{s} - \frac{1}{se^{s\mathsf{T}}} + \sum_{n=1}^{\infty} (\text{-}1)^n \frac{\alpha^n\ \gamma[(1+n\beta),\, s\mathsf{T}]}{(n)!\, s^{(n\beta+1)}} \tag{6.14}$$

Equation (6.14) gives an exact ( as $n \to \infty$) series representation of the partial Laplace transform of the Weibull lifetime survivor function. Practical use of this infinite series requires truncating the infinite series to obtain a finite representation of the Weibull survivor function partial Laplace transform.

## 6.3 Numerical Results for the Weibull - Exponential Case

The availability model is constructed in Mathematica. The infinite series for the partial Laplace transforms of the Weibull lifetime density, $f_L^*(s,\ \mathsf{T})$, and the Weibull survivor function are truncated at 32 terms each. Then the availability expression developed in Chapter 4 is implemented in Mathematica to obtain an estimate of the availability function Laplace transform, $A^*(s)$. This transform is inverted in Mathematica using the numerical inversion package developed by Cheng, et al [66]. The Stehfest method was chosen and implemented for $n = 6$ and as described earlier is used to find an analytical estimate of $A(t, \mathrm{T})$.

A specific numerical case is used in this analysis. The lifetime, repair and preventive maintenance densities' parameters are shown below. The parameters for the repair and preventive maintenance densities reflect values of $\rho = 2.$ and $\delta = 5.$

Weibull lifetime density: $\alpha = \frac{1}{200}$, $\beta = 2$, $mean\, life = 12.5$,
$characteristic\ life = 14.14$

Exponential repair time density: $\mu_r = 0.16$, $mean = 6.25$

Exponential preventive maintenance density: $\mu_p = 0.8$, $mean = 1.25$

### 6.3.1 Model Validation for the Weibull - Exponential Case

Three methods are employed to validate the model for $A(t)$. First, the estimated value of $A(t)$ near $t = 0$ is evaluated for several different values of T, the age-replacement time interval. Since the model assumes the component is working at $t = 0$, the value of $A(t)$ for small values of $t$ should be approximately equal to one regardless of the value of the age replacement policy, T (T>0).

The second method is the computation of an estimate for the limiting availability value for the case in which T is set to the optimal age replacement period for an infinite time horizon, $T^*_\infty$. Barlow and Hunter [12] develop the analytical equations to solve for $T^*_\infty$ and the limiting availability, $A$. This analysis is carried out by first setting $T=T^*_\infty$ and then evaluating the average availability, $A_{avg}(t)$, for large values of time. These values should converge to the value of $A$ obtained analytically.

In the third method, the estimated availability function is used to compute values of average availability for several values of T, near and including $T^*_\infty$ for large values of continuous time, $t$. For large values of $t$, we expect that the estimated availability function should produce an average availability local maxima for $T= T^*_\infty$.

#### 6.3.1.1 Model Validation Results for $t \approx 0$

These values are evaluated in the estimated availability function, $A(t$, T), using Mathematica, for $t = 0.000001$. This evaluation is performed for values of T between 0.1 and 750. The expected results are values of $A(t = 0.000001$, T) close to unity for all values of T. Table 6.1 shows the results of this analysis. In each case, Mathematica returned a value of 1. According to the Mathematica documentation, the default accuracy is at least sixteen

Table 6.1 $A(t=0.000001)$ for $0.1 \leq T \leq 750$

| T (age replacement period) | $A(t=0.000001)$ |
|---|---|
| 0.10 | 1. |
| 0.25 | 1. |
| 0.50 | 1. |
| 0.75 | 1. |
| 1.00 | 1. |
| 2.00 | 1. |
| 5.00 | 1. |
| 10.00 | 1. |
| 20.00 | 1. |
| 50.00 | 1. |
| 100.00 | 1. |
| 200.00 | 1. |
| 500.00 | 1. |
| 750.00 | 1. |

decimal places. These results support the validity of the model for small values of time.

### 6.3.1.2 Validation Results for the Infinite Time Horizon

The results of this validation test are compared to the limiting availability, $A$, obtained from the infinite time horizon case. This value of $A$ depends on the optimal age replacement period, $T^*_\infty$. Barlow and Hunter [12] develop these analytical equations. $T^*_\infty$ is found by solving the following integral equation for $T^*_\infty$ (see [12]):

$$\left(z(T^*_\infty)\int_0^{T^*_\infty} R_L(t)\,dt\right) - F_L(T^*_\infty) = \frac{E[P.M.\,time]}{E[Repair\,time]-E[P.M.\,time]}\ , \tag{6.15}$$

where $z(t)$ is the hazard function and $E[\bullet]$ is the expected value operator. The limiting availability, $A$, is found by solving the following equation (see [12]):

$$A = \frac{1}{1+\left(E[Repair\,time]-E[P.M.\,time]\right)z(T^*_\infty)}\ . \tag{6.16}$$

These equations apply when the lifetime density function, $f_L(t)$, has a strictly increasing failure rate ($\frac{d}{dt}z(t) > 0$) and the mean time required to repair the component is greater than the mean time required to perform preventive maintenance ($E[Repair\,time] > E[P.M.\,time]$)[12].

The values defined above for the component lifetime density as well as the repair time and preventive maintenance time densities are evaluated in equation (6.15) to find $T^*_\infty$. Mathematica solves this equation quickly and produces $T^*_\infty = 7.22176$. This value of $T^*_\infty$ is used in equation (6.16) to obtain $A = 0.734706$.

The value of $T^*_\infty = 7.22176$ is used in the estimated availability function to find values of the average availability , $A_{avg}(t)$, for large values of $t$. Barlow and Proschan [7] show that the limiting average availability is equivalent to the limiting availability. Mathematically, this is

$$\lim_{T\to\infty} A_{avg}(t) = \lim_{T\to\infty} \frac{1}{T}\int_0^T A(t)\,dt = A = \lim_{t\to\infty} A(t).$$

Recall that $T^*_\infty = 7.22176$, for the parameter values given in Section 6.3. This value is entered for T in the availability function in Mathematica and $A_{avg}(t)$, is calculated for $t = 500.,\ 750.,\ and\,1000.$ These results are shown in Table 6.2 with the difference from the limiting availability, $A = 0.734706$, noted in the "error" column.

Table 6.2 Limiting Values of $A_{avg}(t)$

| $t$ | $A_{avg}(t)$ | $Error$ |
|---|---|---|
| 500. | 0.737231 | 0.002525 |
| 750. | 0.736391 | 0.001685 |
| 1000. | 0.735970 | 0.001264 |

As expected, the error decreases as $t$ increases. At $t = 1000.$, the error is approximately $10^{-3}$. Given the approximations employed to develop the estimated availability function, these values validate the performance of the estimated function as time increases to large values.

### 6.3.1.3 Model Recognition of Local Maxima

In this validation test, the parameter values for lifetime, repair and preventive maintenance densities are changed to the values shown below:

Weibull lifetime density: $\alpha = \frac{1}{200}$, $\beta = 2$, $mean\,life = 12.5$,

$characteristic\ life = 14.14$

Exponential repair time density: $\mu_r = 800.$, $mean = 0.00125$

Exponential preventive maintenance density: $\mu_p = 4000.$, $mean = 0.00025$

These parameter values for the lifetime, repair time and preventive maintenance time densities reflect values of $\rho = 10000.$ and $\delta = 5.$ Recall that $\rho$ is the ratio of the mean lifetime to mean repair time and $\delta$ is the ratio of the mean repair time to mean preventive maintenance time.

These parameter values are entered in Mathematica and the average availability at $t = 750.$ is evaluated for $T = 6.722,\ 7.222,\ 7.722,\ and\,8.222.$

Mathematica evaluates these average availabilities through numerical integration. Table 6.3 shows the results obtained.

Table 6.3 Local Maxima for $T^*_\infty$

| T | $A_{avg}(t = 750.)$ |
|---|---|
| 6.722 | 0.9999 2778 |
| $T^*_\infty = 7.222$ | 0.9999 2801 |
| 7.722 | 0.9999 2793 |
| 8.222 | 0.9999 2762 |

Due to the large value of $\rho$ used in this validation, availability values are very high. However, even with such a large ratio of mean lifetime to mean repair time, the estimated availability function still produced a local maximum at $T^*_\infty = 7.222$.

## 6.3.2 Availability Model Results

For the following plots, the Weibull lifetime density function is used. Parameter values are $\alpha=\frac{1}{200.}$ (scale parameter) and $\beta$=2 (shape parameter). The parameter values provide a mean component lifetime of approximately 12.5 and a characteristic life of 14.14. The repair time and preventive maintenance time are modeled as exponential distributions. The mean lifetime to mean repair time ratio, $\rho$, is set to a value of $\rho = 2.$ to accentuate the differences in the availability values. The value of the mean repair time to mean preventive maintenance time, $\delta$, is set to a value of $\delta = 5.$ Thus the mean repair time is 6.25 and the mean preventive maintenance time is 1.25

In the following series of plots, the first plot represents the estimated availability function for $0.1 < t < 15.$ with an age replacement period of T=7.22176. This T value is the optimal replacement period for an infinite time horizon. Note that the function converges rather quickly to the limiting availability value for the optimal value, $T^*_\infty$. This value of $A$ is approximately, $A$=0.7347.

The subsequent plots show the estimated availability function for T= 1, 3, 5, 6.75, 12.5, 14.14, 28.28, 35, 42.4, 50., and 55. as a function of time. For reference purposes, the $T^*_\infty$=7.22176 plot is included in each plot.

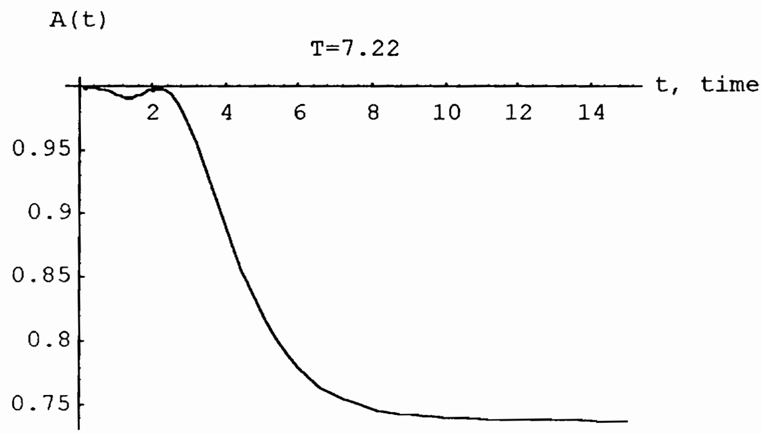


Figure 6.1 Availability Plot for $T^*_\infty$=7.22

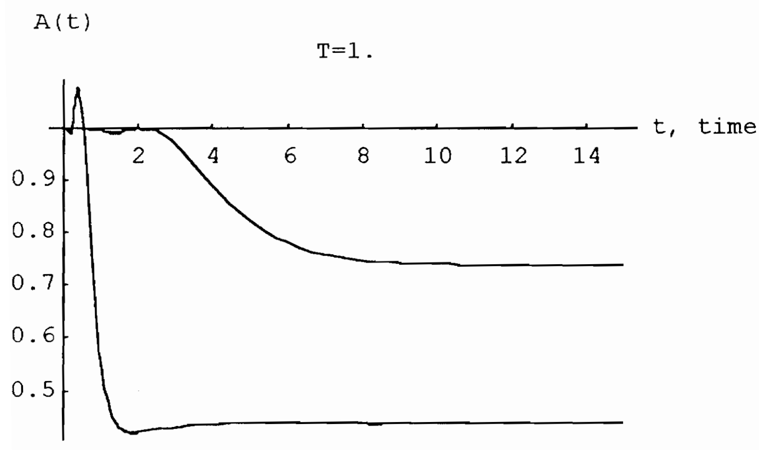


Figure 6.2 Availability Plots for T=1. and T=7.22

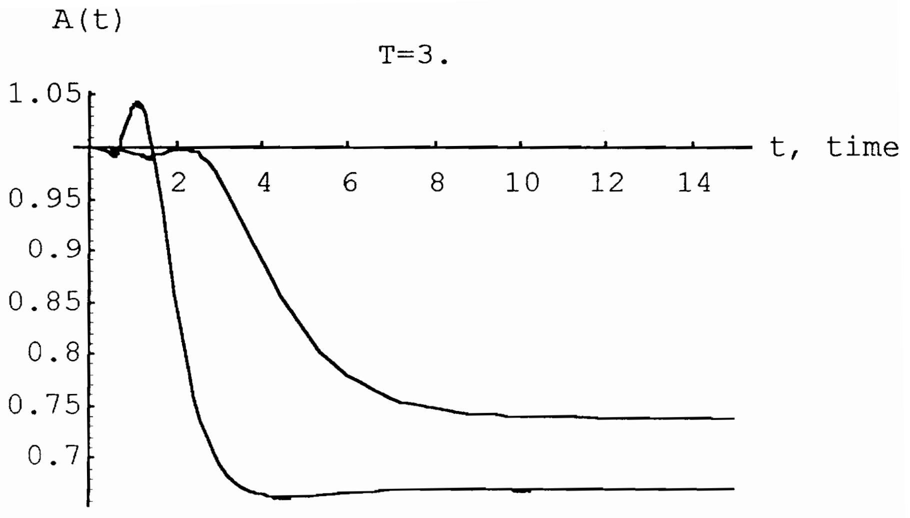


Figure 6.3 Availability Plots for T=3. and T=7.22

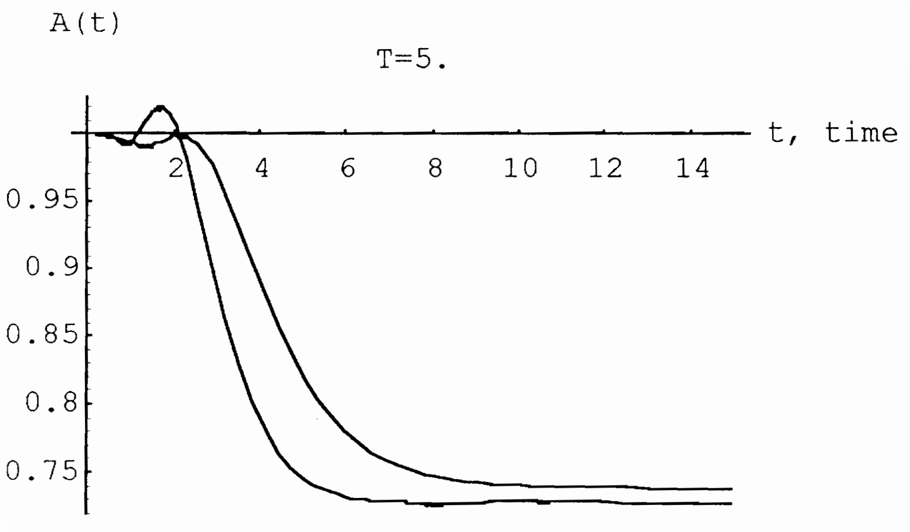


Figure 6.4 Availability Plots for T=5. and T=7.22

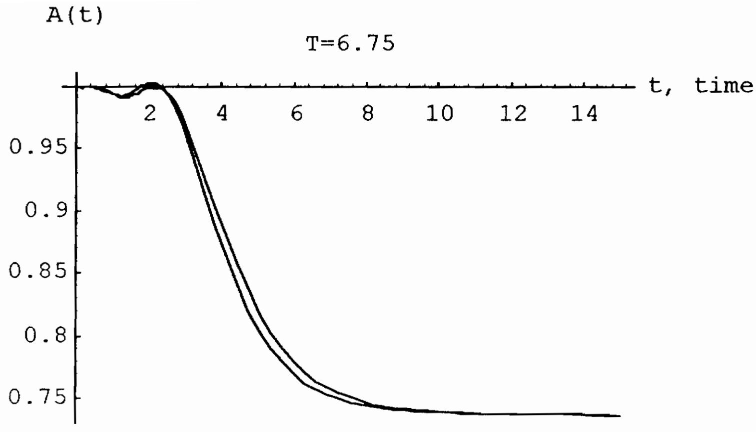


Figure 6.5 Availability Plots for T=6.75 and T=7.22

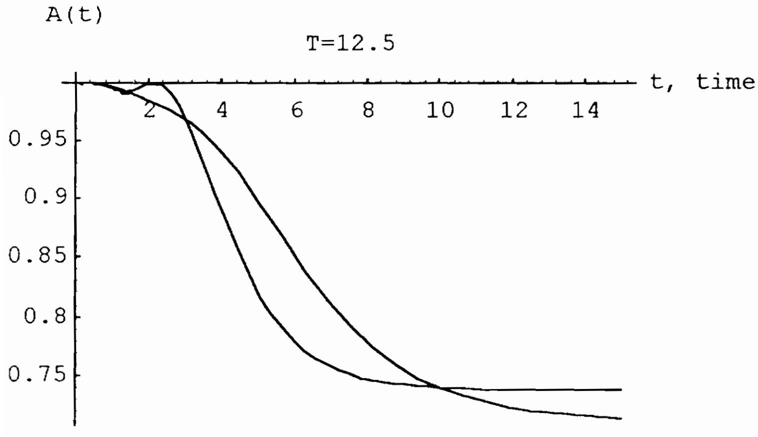


Figure 6.6 Availability Plots for T=12.5 and T=7.22

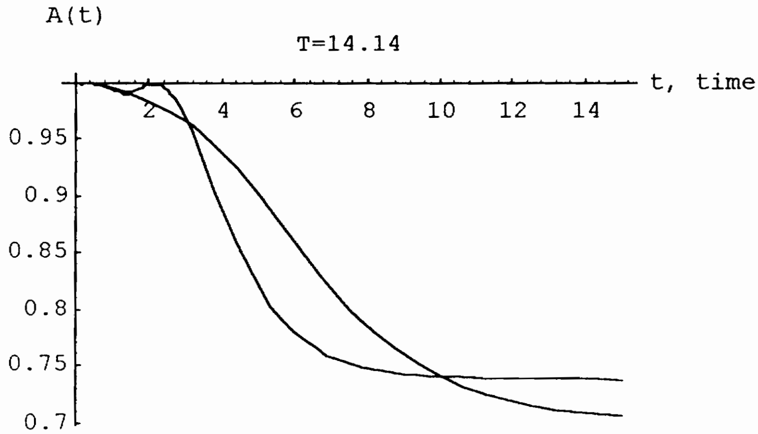


Figure 6.7 Availability Plots for T=14.14 and T=7.22

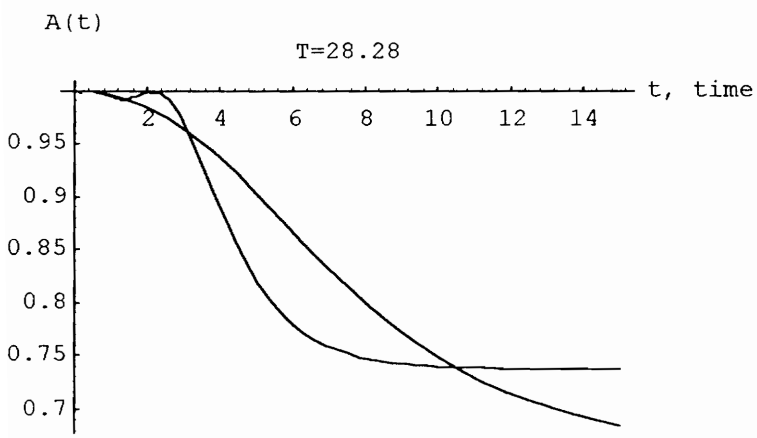


Figure 6.8 Availability Plots for T=28.28 and T=7.22

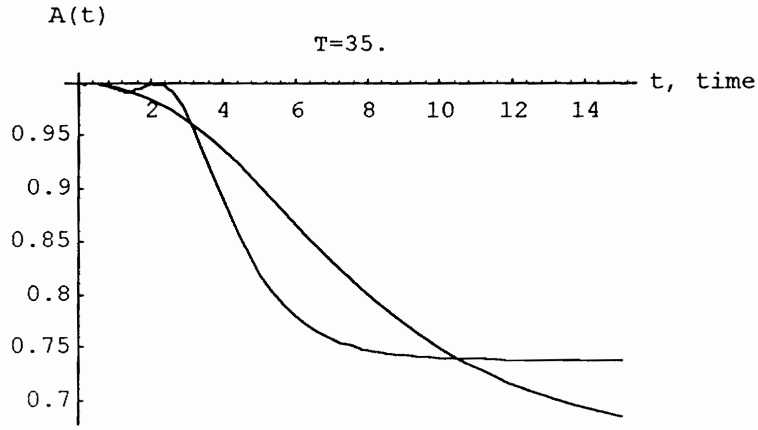


Figure 6.9 Availability Plots for T=35. and T=7.22

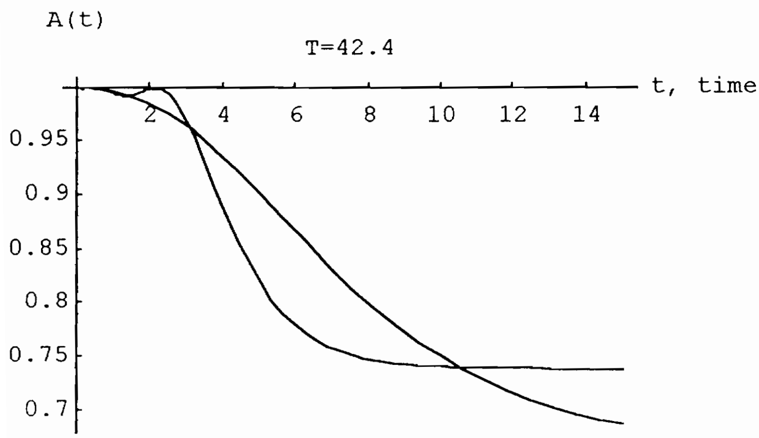


Figure 6.10 Availability Plots for T=42.4 and T=7.22

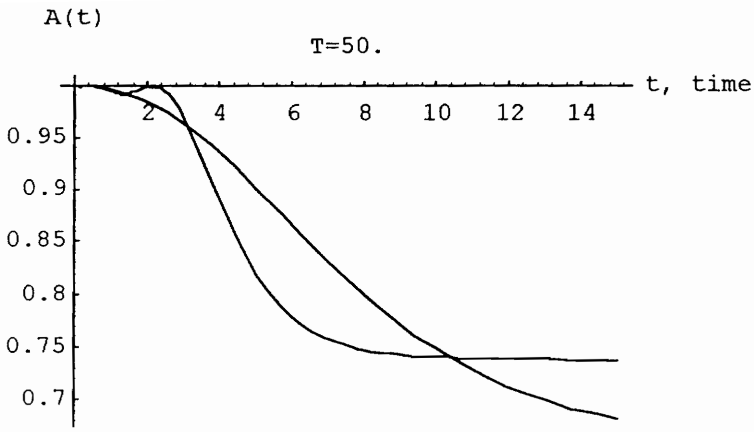


Figure 6.11 Availability Plots for T=50. and T=7.22

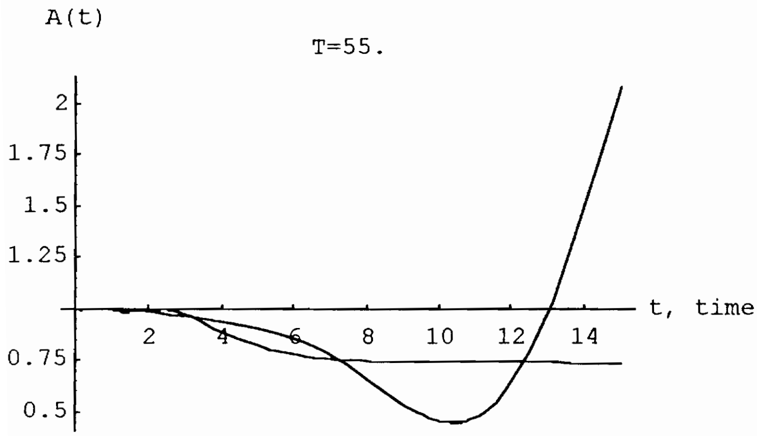


Figure 6.12 Availability Plots for T=55. and T=7.22

### 6.3.3 Availability Model Results Discussion

Figures 6.2, 6.3 and 6.4 show an anomaly in the estimated availability function, $A(t)$. Note the portion of the plot that goes above a value of one for $A(t)$. Since the availability function, $A(t)$, is the probability that the component is functioning at time $t$, the maximum $A(t)$ value is one. This error is small and ranges from about 0.5 for an age replacement value of T=1. to about 0.025 for an age replacement value of T=5. The error decreases as the age replacement value, T, increases. Additionally, the error occurs later in the time domain, $t$, as the age replacement value, T, increases. This small anomaly occurs due to the estimated Laplace transform inversion technique employed to estimate $A(t)$. Recall that the methodology employed to find the availability function, $A(t)$, provided an exact representation of the Laplace transform of $A(t)$. The only estimation technique employed is the use of the Stehfest numerical Laplace transform inversion technique. Thus, this estimated inversion technique caused a small anomaly to occur for small values of T.

Figure 6.12 demonstrates that the estimated availability function diverges for large values of T and large values of continuous time, $t$. The limitation of the model is T $<$ 55. This limitation occurs due to computational limitations such as computer round-off error and floating point limitations.

The average availability, $A_{avg}(\tau)$, for a finite time period $\tau$, may be evaluated graphically from Figures 6.1 through 6.11. Recall that $A_{avg}(\tau)$ is

$$A_{avg}(\tau) = \frac{1}{\tau}\int_0^{\tau} A_{avg}(u)\,du \ .$$

For a finite time period, $\tau$, $A_{avg}(\tau)$ may be graphically evaluated by considering the area under the availability function, $A(t)$, curve between $t = 0$ and $t = \tau$, where $t$ is the continuous time variable. Furthermore, if a comparison of the $A_{avg}(\tau)$ for two different functions shown on the same plot is desired, one may simply consider only the intersection of the areas bounded by both curves. For example consider a finite time period of 10 time units and refer to Figure 6.11. Note that for the time period from $t = 0.$ to $t = 2.$ that the curve corresponding to $T^*_\infty = 7.22$ bounds the area under the curve for T=50. Since the $T^*_\infty = 7.22$ curve bounds a larger area over this time period, $T^*_\infty = 7.22$ has a larger average availability than T=50. Now consider the time period from $t = 2.$ to $t = 10.$ Note that in this case, the T=50. curve bounds the $T^*_\infty = 7.22$ curve. Thus for this time interval, T=50. provides a larger average availability value than $T^*_\infty = 7.22$ since the area under its curve is greater. In fact if we consider the time period $t = 0.$ to $t = 10.$, it is graphically apparent from Figure 6.11, that the T=50. curve bounds more area than the $T^*_\infty = 7.22$ curve. Thus the age replacement value of T=50. provides a larger value of $A_{avg}(\tau = 10.)$ than the optimal infinite time horizon value, $T^*_\infty = 7.22$. This result is extended further in Chapter 7.

# Chapter 7

# Conclusion and Accomplishments

## 7.1 Availability Model Performance

The research methodology presented produces an exact representation of the Laplace transform of the availability function, $A(t, \mathrm{T})$. This availability function represents the probability that a component functions at time $t$ with a preventive maintenance period of T time units. T represents the age at which the component is preventively maintained. If the component operates T time units, it is preventively maintained. If the component fails before T time units have passed, it receives a failure maintenance action. This type of maintenance policy is commonly known as an age replacement preventive maintenance policy. However, this research does not define the preventive maintenance as a component replacement. This research requires that the preventive maintenance restore the component to as good as new condition. The actual action taken is defined by the specific situation.

The derived model treats the age replacement period, T, as a parameter. Thus, the component availability at a given time, $t$, is dependent on the age replacement period, T. The methodology presented produced the Laplace transform for $A(t, \mathrm{T})$. This transform could not be inverted exactly. Thus an inversion estimation technique is applied to obtain an estimate of $A(t, \mathrm{T})$. This estimation process causes some anomalies for small values of T. These are noted in Section 6.3.2. Validation tests of the model in both the exponential case as well as the Weibull-exponential case are successful and reviewed in Sections 7.1.1 and 7.1.2.

### 7.1.1 Exponential Validation Case

In this case, the component lifetime probability distribution as well as the failure repair time and preventive maintenance service time are modeled as exponential probability distributions. Each distribution is considered to be unique and independent of each other.

This case is considered a validation case. Recall that if a component's lifetime is considered to be exponential, then no benefit will result from performing preventive maintenance. Under exponential assumptions, the component does not exhibit the effects of aging. No matter how long the component has been in operation, the probability of failure in the next instant of time remains constant (constant failure rate, CFR). Since the component does not age under these assumptions, no benefit results from purposely removing an operating component for preventive maintenance.

Three cases are implemented to validate the availability model in the exponential case. All three cases support the validity of the availability model developed and implemented in this research. First, the model is tested for performance near time equal to zero $(t = 0)$. The model assumes that the component is working at $t = 0$ and the availability at this time is unity, $A(t = 0) = 1$. Results presented in Section 5.4 support model performance in this time region.

Second, the model is tested at very large values of time. Values of the estimated availability function at these large values of time should converge to the limiting availability value. Previous research results provide the values of the limiting availability, $A$. The model is evaluated at large values of continuous time, $t$, for several values of the age replacement period, T. Results presented in Section 5.4 support the availability model performance as the continuous time variable, $t$, grow large. In the tests, the model produces results matching the limiting availability value obtained from previous research results.

The first two methods of validation support the availability model validity as $t \to 0$ and as $t \to \infty$. However, model validity between these extremes is not supported. The third test provides model validation for times between these extremes. This test takes advantage of the exponential assumptions discussed above. If the component's lifetime is assumed to be exponential then the availability model should produce increasing values of availability as the age replacement period, T, is increased. In this case, maximum average availability as well as limiting availability is attained by performing no preventive maintenance. Thus, the age replacement period, T, should approach infinity (T→∞). The estimated availability function produces this result. Table 5.1 shows that as T is increased, the average availability, $A_{avg}(t,\mathrm{T})$, increases. This is demonstrated at several values of continuous time.

Overall, the exponential case validated the availability model produced by this research. The model exhibited the expected behavior at extreme values of continuous time, $t$, as well as at times between these extremes.

### 7.1.2 Weibull - Exponential Case

The Weibull-exponential case assumes that the component lifetime probability distribution is Weibull. Furthermore, the shape parameter for the Weibull is considered to be greater than zero. This causes the component lifetime distribution to have an increasing failure rate (IFR). Under this assumption, the component exhibits the effects of aging since the probability that the component fails in the next instant of time is now dependent on the component's age or in effect how long the component has been operated. Both failure repair time and preventive maintenance time probability distributions are modeled as unique exponential probability distributions. All probability distributions are considered to be independent of each other. Three validation tests are performed in this case. The model is tested at the extreme values of continuous time ($t \to 0$, $t \to \infty$). The model is also tested for recognition of average availability local maxima.

In the Weibull-exponential case, results validated the model performance at very small values of continuous time, $t \to 0$. Table 6.1 details the results. Values of the estimated availability function, $A(t, \mathrm{T})$ are taken at $t = 0.000001$ for values of T between 0.1 and 750. In all cases, a value of unity is obtained. This matches the expected theoretical value. The model assumes that the component is functioning at time, $t = 0$. By definition, the availability at $t = 0.$ is the probability the component is functioning at $t = 0$. Thus, the expected availability value at $t = 0.$ is one. Note that the estimated availability function, $A(t, \mathrm{T})$ for the Weibull-exponential case experiences discontinuities at $t = 0.$ due to the Laplace inversion estimation and the Weibull distribution transform estimation techniques employed. Therefore, a value of $t = 0.000001$ is used.

Results also validate the model performance at large values of continuous time ($t \to \infty$). For this case the optimal replacement period for the infinite time horizon, $\mathrm{T}^*_\infty$, is calculated per previous research results. The corresponding limiting value of availability, $A$ is also calculated. As discussed earlier, the limiting availability, $A = \lim_{t \to \infty} A(t)$ and the $\lim_{t \to \infty} A_{avg}(t)$ are the same when the limits exist. Thus, the values of the estimated $A_{avg}(t, \mathrm{T}^*_\infty)$ should converge to $A$ for large values of $t$. The value of $A_{avg}(t, \mathrm{T}^*_\infty)$ is calculated for large values of time from the estimated $A(t, \mathrm{T}^*_\infty)$. Mathematica is used to numerically integrate $A(t, \mathrm{T}^*_\infty)$ to obtain $A_{avg}(t, \mathrm{T}^*_\infty)$. Table 6.2 details the results and shows an error of $10^{-3}$ at $t = 1000$.

The third validation test performed for the Weibull-exponential case tested whether the model could recognize an average availability local maxima. The test calculated values of $A_{avg}(t, \mathrm{T})$ for values of T, close to and including the value of $\mathrm{T}^*_\infty$ discussed above. In this validation test, limiting values are required, thus, a large value of time, $t = 750.$, is used. Table 6.3 details the results. The model recognizes $\mathrm{T}^*_\infty$ as the replacement period that maximizes average availability for very large values of time ($t \rightarrow \infty$).

The validation tests for the Weibull-exponential case demonstrate model performance. Results validate model performance at small values of time ($t \rightarrow 0$) and at large values of time ($t \rightarrow \infty$). Also, the model is shown to recognize $\mathrm{T}^*_\infty$ as the age replacement period that maximizes average availability for very large values of time ($t \rightarrow \infty$).

## 7.2 Maximizing Average Availability for Finite Component Economic Lifespan

The results in Section 6.3.2 show that for finite time intervals, the optimal infinite time horizon replacement interval, $\mathrm{T}^*_\infty$, does not maximize average availability. This result is shown for example in Figure 6.11 and is discussed in Section 6.3.2. The conclusion is that the optimum infinite time horizon preventive maintenance policy, $\mathrm{T}^*_\infty$, does not maximize average availability for some finite time periods.

In actual operation, repairable components and systems are designed to be used over a finite time interval. These lifecycle design constraints are very important in determining the component and/or system maintenance requirements. This research concludes that using the age replacement period based upon an infinitely long operational interval does not maximize component and/or system availability for a finite operating design lifespan. This result is important to system and/or component lifecycle design considerations.

## 7.3 Summary of Significant Research Accomplishments

This research presents a unique and original approach to developing the component availability function for the age replacement preventive maintenance policy. The approach explicitly considered the effects of unique failure repair time and preventive maintenance action time probability distributions. The availability model included derivation of the partial Laplace transform for cases where the time variable is truncated (i.e. does not approach infinity). Normal Laplace transforms require the time variable to have a range of $[0,\infty]$. The end result is an estimate of component availability as a function of time, $t$, as well as the age replacement period, T. Thus, the component availability function is defined as $A(t, \mathrm{T})$.

The availability model derived in this research produces an exact representation of the Laplace transform the component availability function.

This is defined as $A^*(s, \mathrm{T})$. Even in the exponential case, $A^*(s, \mathrm{T})$ could not be inverted exactly to find $A(t, \mathrm{T})$. A numerical Laplace transform inversion estimation technique, the Stehfest method, is employed. The Stehfest method and other inversion estimation techniques have been used to find estimates of the time domain function at specific values of time. Thus, past research applied these techniques for numerical results. However, this research extended the use of the Stehfest method to estimating the function $A(t,\ \mathrm{T})$ from $A^*(s,\ \mathrm{T})$. The Wolfram Research, Inc. software package, Mathematica, allowed the Stehfest series estimation technique to be applied in general and thus produce an estimate of the availability function, $A(t,\ \mathrm{T})$.

In the Weibull-exponential case, the partial Laplace transform must be evaluated for the Weibull distribution and the Weibull survivor function. Recall that in the partial Laplace transform, the time variable is truncated and thus ranges over [0, $\tau$], where $\tau$ is finite. This research developed an exact infinite series representation for the partial Laplace transform of the Weibull distribution and survivor function. This development also yielded the normal Laplace transform for the Weibull distribution and survivor function. In the normal Laplace transform the time variable ranges over [0, $\infty$]. The latter result is particularly useful in many applied probability applications such as reliability studies.

Finally, this research demonstrated that average availability for a finite time periods, $\tau$, is not maximized by the optimal infinite time horizon replacement period, $\mathrm{T}^*_\infty$, at all values of $\tau$. This result is important to lifecycle design considerations for maintaining and servicing components and systems.

# Chapter 8

# Further Research

## 8.1 Finite Economic Life Model

In the availability model developed and presented in this research the component's economic life is not bounded. This model is useful for estimating the effects of the infinite time horizon age replacement period (preventive maintenance policy), $T^*_\infty$, on a finite economic component life. Definitive values of the age replacement period value, T, require further research and development of the availability model.

Let $\tau$ be defined as the economic life of the component. Equation (4.8) is modified to

$$A(t) = \begin{cases} R_L(t)+\int_0^t R_L(t-u)\, m(u)\, du, & for\ t < \mathrm{T}, \\ \\ \int_{t-\mathrm{T}}^t R_L(t-u)\, m(u)\, du, & for\ \mathrm{T} \leq t \leq \tau. \end{cases} \qquad (8.1)$$

Note that the continuous time variable, $t$, is now bounded by the economic component life, $\tau$. As in Chapter 4, the development proceeds with the construction of the Laplace transform of the availability function, $A(t)$, stated in equation (8.1). Recall that the definition of the Laplace transform for the availability function is

$$A^*(s) = \int_0^{\infty} e^{-st}\, A(t)\, dt.$$

Applying (8.1) to this relationship, the following is obtained:

$$\begin{aligned} A^*(s) &= \left( \int_0^{\mathsf{T}} e^{-st} \left( R_L(t) + \int_0^t R_L(t-u)\, m(u)\, du \right) dt \right) \\ &\qquad + \left( \int_{\mathsf{T}}^{T} e^{-st} \left( \int_{t-\mathsf{T}}^{t} R_L(t-u)\, m(u)\, du \right) dt \right) \\ &= \int_0^{\mathsf{T}} e^{-st}\, R_L(t)\, dt + \int_0^{\mathsf{T}} e^{-st} \left( \int_0^t R_L(t-u)\, m(u)\, du \right) dt \\ &\qquad + \int_{\mathsf{T}}^{T} e^{-st} \left( \int_{t-\mathsf{T}}^{t} R_L(t-u)\, m(u)\, du \right) dt \\ &= \int_0^{\mathsf{T}} e^{-st}\, (1\text{-}F_L(t))\, dt + \int_0^{\mathsf{T}} \int_0^t e^{-st}\, R_L(t-u)\, m(u)\, du\ dt \\ &\qquad + \int_{\mathsf{T}}^{T} \int_{t-\mathsf{T}}^{t} e^{-st}\, R_L(t-u)\, m(u)\, du\, dt \ . \end{aligned}$$

$$= \int_0^{\mathsf{T}} e^{-s\,t}\,dt - \int_0^{\mathsf{T}} e^{-s\,t}\,F_L(t)\,dt + \int_0^{\mathsf{T}}\int_u^{\mathsf{T}} e^{-s\,t}\,R_L(t-u)\,m(u)\,dt\,du$$

$$+ \int_0^{\mathsf{T}}\int_{\mathsf{T}}^{\mathsf{T}+u} e^{-s\,t}\,R_L(t-u)\,m(u)\,dt\,du \;+ \int_{\mathsf{T}}^{T}\int_u^{T} e^{-s\,t}\,R_L(t-u)\,m(u)\,dt\,du$$

$$= \frac{1-e^{-s\mathsf{T}}}{s} - F_L^*(s,\ \mathsf{T}) + \int_0^{\mathsf{T}} m(u)\ \int_u^{\mathsf{T}} e^{-s\,t}\,R_L(t-u)\,dt\,du$$

$$+ \int_0^{\mathsf{T}} m(u)\int_{\mathsf{T}}^{\mathsf{T}+u} e^{-s\,t}\,R_L(t-u)\,dt\,du \;+ \int_{\mathsf{T}}^{T} m(u)\int_u^{T} e^{-s\,t}\,R_L(t-u)\,dt\,du$$

$$= \frac{1-e^{-s\mathsf{T}}}{s} - F_L^*(s,\ \mathsf{T}) + \int_0^{\mathsf{T}} m(u)\ \int_0^{\mathsf{T}-u} e^{-s(y+u)}\,R_L(y)\,dy\,du$$

$$+ \int_0^{\mathsf{T}} m(u)\int_{\mathsf{T}-u}^{\mathsf{T}} e^{-s(y+u)}\,R_L(y)\,dy\,du \;+ \int_{\mathsf{T}}^{T} m(u)\int_0^{T-u} e^{-s(y+u)}\,R_L(y)\,dy\,du$$

$$= \frac{1-e^{-s\mathsf{T}}}{s} - F_L^*(s,\ \mathsf{T}) + \int_0^{\mathsf{T}} e^{-s\,u}m(u)\ \int_u^{\mathsf{T}-u} e^{-s\,y}\ R_L(y)\,dy\,du$$

$$+ \int_0^{\mathsf{T}} e^{-s\,u}m(u)\int_{\mathsf{T}-u}^{\mathsf{T}} e^{-s\,y}\,R_L(y)\,dy\,du \;+ \int_{\mathsf{T}}^{T} e^{-s\,u}m(u)\int_0^{T-u} e^{-s\,y}\ R_L(y)\,dy\,du$$

$$= \frac{1-e^{-s\mathsf{T}}}{s} - F_L^*(s,\ \mathsf{T}) + \int_0^{\mathsf{T}} e^{-s\,u}m(u)\ \int_0^{\mathsf{T}} e^{-s\,y}\ R_L(y)\,dy\,du$$

$$+ \int_{\mathsf{T}}^{T} e^{-s\,u}m(u)\int_0^{T-u} e^{-s\,y}\,R_L(y)\,dy\,du$$

$$= \left( \tfrac{1-e^{-s\mathsf{T}}}{s} - F_L^*(s,\ \mathsf{T}) \right) \left( 1 + \int_0^{\mathsf{T}} e^{-s\,u} m(u)\, du \right)$$

$$+ \int_{\mathsf{T}}^{\tau} e^{-s\,u} m(u) \int_0^{\tau-u} e^{-s\,y}\, R_L(y)\, dy\, du$$

$$= \left( \tfrac{1-e^{-s\mathsf{T}}}{s} - F_L^*(s,\ \mathsf{T}) \right) \left( 1 + m^*(s,\, \mathsf{T}) \right)$$

$$+ \int_{\mathsf{T}}^{\tau} e^{-s\,u} m(u) \int_0^{\tau-u} e^{-s\,y}\, (1 - F_L(y))\, dy\, du$$

$$= \left( \tfrac{1-e^{-s\mathsf{T}}}{s} - F_L^*(s,\ \mathsf{T}) \right) \left( 1 + m^*(s,\, \mathsf{T}) \right)$$

$$+ \int_{\mathsf{T}}^{\tau} e^{-s\,u} m(u) \left( \tfrac{1-e^{-s(\tau-u)}}{s} - F_L^*(s,\ \tau - u) \right) du$$

$$A^*(s) = \left( \tfrac{1-e^{-s\mathsf{T}}}{s} - F_L^*(s,\ \mathsf{T}) \right) \left( 1 + m^*(s,\, \mathsf{T}) \right)$$

$$+ \int_{\mathsf{T}}^{\tau} m(u) \left( \tfrac{e^{-su} - e^{-s\tau}}{s} \right) du \ - \int_{\mathsf{T}}^{\tau} e^{-s\,u} m(u)\ F_L^*(s,\ \tau - u)\, du \tag{8.2}$$

The second term of the sum shown in equation (8.2) may be restated as follows:

$$\int_{\mathrm{T}}^{\tau} m(u) \left(\frac{e^{-su}-e^{-s\tau}}{s}\right) du = \int_{\mathrm{T}}^{\tau} \left(\frac{e^{-su}}{s}\right) m(u)\, du - \frac{e^{-s\tau}}{s} \int_{\mathrm{T}}^{\tau} m(u)\, du$$

$$= \frac{1}{s}\Big(m^*(s,\tau) - m^*(s,\mathrm{T})\Big) - \frac{e^{-s\tau}}{s}\left(\int_0^{\tau} m(u)\, du - \int_0^{\mathrm{T}} m(u)\, du\right)$$

$$= \frac{1}{s}\Big(m^*(s,\tau) - m^*(s,\mathrm{T})\Big) - \frac{e^{-s\tau}}{s}\Big(M(\tau) - M(\mathrm{T})\Big)$$

Substituting this result into equation (8.2), we obtain

$$A^*(s) = \left(\frac{1-e^{-s\mathrm{T}}}{s} - F_L^*(s,\ \mathrm{T})\right)\Big(1 + m^*(s,\mathrm{T})\Big) + \frac{1}{s}\Big(m^*(s,\tau) - m^*(s,\mathrm{T})\Big)$$

$$- \frac{e^{-s\tau}}{s}\Big(M(\tau) - M(\mathrm{T})\Big) - \int_{\mathrm{T}}^{\tau} e^{-su} m(u)\ F_L^*(s,\ \tau - u)\, du \qquad (8.3)$$

Equation (8.3) provides the Laplace transform of the availability function for a finite economic life span, $\tau$. The renewal density, $m(u)$, is defined and discussed in Section 4.1.1. This density function may be found through the density function for the time between renewal, $h(t)$. $M(u)$ is the renewal function and is defined as

$$M(u) = \int_0^u m(x)\, dx.$$

$F_L^*(s,\ \mathrm{T})$ is the partial Laplace transform of the component's lifetime probability distribution.

Additional research of this model is required. Methods to evaluate this model need to be determined. This includes numerical approaches to evaluate the availability function as well as optimization techniques to find an optimal value of T that maximizes average availability over the component's finite economic life span. One specific complication is the requirement for the renewal density and renewal function. Numerical methods to accurately find or estimate these functions must be found.

## 8.2 Service Time Probability Distributions

The specific cases examined in this research assume that the repair time and the preventive maintenance service time probability distributions are exponential. Section 2.3.3 provides a reference for this assumption [88]. However, other distributions may more accurately model the randomness of service times since the exponential distribution is "memoryless". Under exponential assumption, if a component is serviced for $z$ time units, then the probability that the unit is repaired in the next instant of time remains constant for all values of $z$. Essentially, this means that if you have worked on repairing (or preventively maintaining) a component for 1 second, 1 minute, 1 hour, 1 day, 1 month or 1 year then the probability that you fix the component in the next instant of time remains constant no matter how long you have worked. In most practical applications, the probability the component is fixed increases as the repair time increases. Thus a more practical probability distribution will be IFR (increasing failure rate).

Two well known distributions may be IFR. The Weibull distribution is IFR for values of the shape parameter greater than 1 ($\beta > 1$). The log-normal distribution may be both IFR and DFR (decreasing failure rate) [95]. For example, for service times less than some threshold value, the log-normal distribution may be DFR and for service times above this value it may be IFR. However, the use of either distribution presents complications. The model developed in this research requires the Laplace transform of the repair time and preventive maintenance service time distributions. This research presents the Laplace transform for the Weibull distribution. However, the Laplace transform for the log-normal distribution needs to be developed. The Weibull Laplace transform is an infinite series and it is highly probable that the log-normal may also be an infinite series.

In either case, the infinite series must be truncated to estimate the required Laplace transform. Considering equation (4.16) this may require the multiplication of two expressions containing many terms. For example, consider a case where we have Weibull component lifetimes with unique Weibull repair and preventive maintenance times. In the Weibull-exponential case, 32 terms were used for the partial Weibull Laplace transform. If we use 32 terms in this new case, we are faced with multiplying a 32 term expression by another 32 term expression. This results from multiplying $f_L^*(s, \mathrm{T})$ and $g_r^*(s)$. This represents the first term in equation (4.16). Obviously, the complexity has increased. Methods to reduce the complexity need to be found.

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# APPENDIX 1

## Numerical Laplace Transform Inversion Test

The purpose of this appendix is to demonstrate that Mathematica allows the estimate of a time domain function through the use of the numerical Laplace transform inversion package, nlapinv.m . This numerical Laplace transform inversion package is available from the Wolfram Research, Inc. world wide web (WWW) site, http://www.wri.com.

As explained in the main text, the numerical inversion techniques were developed to obtain estimates of the time domain function at specific numerical values of time. These numerical Laplace transform inversion techniques were developed to be applied to Laplace transforms that cannot be inverted exactly through conventional mathematical techniques. The Stehfest method estimates this value through the application of a mathematical formula containing two levels of summations. By leaving the time specification as a parameter, t, in this case, Mathematica produces an estimate of the time domain function (as a function of t) by evaluating the series contained in the Stehfest formula explicitly. This methodology results in an estimate for the time domain function containing over 100 terms for N=6. For this reason, the time domain function is not displayed.

The first Mathematica statement defines the test function A(s). In this case it is equation (5.3). Mathematica input statements are represented by **boldface** type. Mathematica output resulting from the input statements is represented by *italicized* type.

```
A[s_]= ((P+s) (R+s))/(((P+s) (R+s) (L+s))-(L R (P+s))+
((L R (P+s))-(P (L+s) (R+s))) Exp[-T (L+s)])
```

```
((P + s) (R + s)) /
 (-(L R (P + s)) + (L + s) (P + s) (R + s) +
   L R (P + s) - P (L + s) (R + s)
   -------------------------------)
            (L + s) T
           E
```

The following statement loads the numerical Laplace transform inversion package available from Wolfram Research, Inc. In this case the Mathematica file containing the package is 'nlapinv.m' and is located in the 'wnmath22' subdirectory on drive c.

```
<<c:\wnmath22\nlapinv.m
```

The following statement shows the Mathematica syntax for using the Stehfest method for numerical Laplace transform inversion.

**?NLInvSteh**

*NLInvSteh[expr, s, t, n]*
*where expr = the Laplace transform expression to be inverted*
*s = the Laplace transform parameter*
*t = time*
*n = number of terms in series (!!! must be an even number !!!),*
*typically 6 < n < 20.*

The function, InvA1(t), is defined as the Stehfest numerical Laplace transform inversion of the Laplace transform function, A(s). Recall that InvA1(t) represents an estimate of the time domain function obtained by numerically inverting A(s).

**InvA1[t_]=NLInvSteh[A[s],s,t,6];**

The variables L, R and P define the exponential probability density parameters for the lifetime, repair time and preventive maintenance time densities respectively. These parameters were not numerically defined when A(s) was entered. The parameters are numerically defined below. The mean lifetime to mean failure repair time ratio is 1000 to 1 and the mean failure repair time to mean preventive repair time ratio is 2.5 to 1.

**L=1.**
**R=1000.**
**P=2500.**

*1.*
*1000.*
*2500.*

The age replacement parameter, T, is set to 0.6 for numerical evaluation. Recall that T=0.6 means that a component is preventively maintained when it attains an age of 0.6 time units. Complete renewal is assumed for either failure or preventive maintenance actions thus the component's age is reset to zero after either action is completed.

**T=0.6**

*0.6*

Evaluating InvA1(t) at t=5., we obtain a time domain function value of 2.19671.

**InvA1[5.]**

*2.19671*

The following statement takes the original Laplace transform function, A(s), and directly inverts the function numerically using the Stehfest method at t=5. The value obtained is exactly the same.

**NLInvSteh[A[s],s,5.,6]**

*2.19671*

The age replacement parameter, T, is reset to a variable through the Mathematica "Clear" statement.

**Clear[T]**

The following two plots demonstrate that InvA1(t) is equivalent to the direct Stehfest numerical inversion of A(s) at discrete points in time. In the first plot, the estimate time domain function, InvA1(t), is plotted for values of t between 0.1 and 15. and T between 0.05 and 10. In the second plot, the ranges of the variables t and T are exactly the same. However, note that the function evaluated is the direct Stehfest numerical inversion of the original Laplace transform function, A(s).

**Plot3D[InvA1[t],{t,0.1,15.},{T,0.05,10.}]**

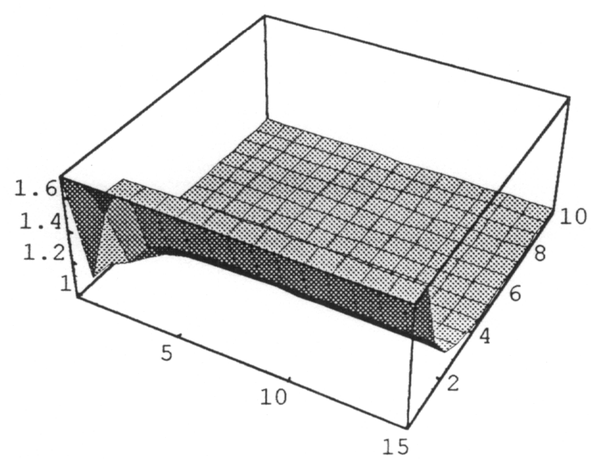


Fig. A1.1 Plot of InvA1(t), $0.1 \leq t \leq 15.$, $0.05 \leq T \leq 10.$

**Plot3D[NLInvSteh[A[s],s,t,6],{t,0.1,15.},{T,0.05,10.}]**

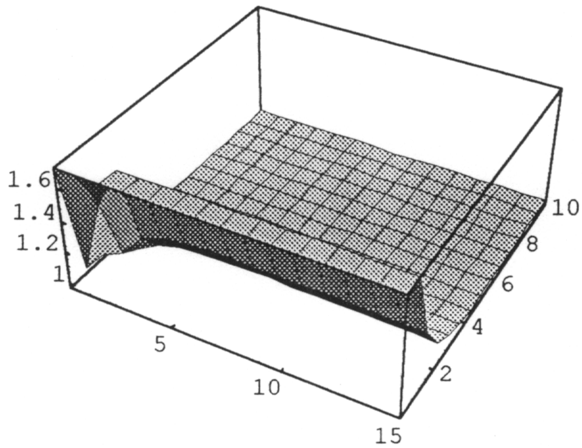


Fig. A1.2 Plot of Direct Stehfest Inversion, $0.1 \leq t \leq 15.$, $0.05 \leq T \leq 10.$

The following output further demonstrates that InvA1(t) is equivalent to the direct Stehfest numerical inversion of the original Laplace transform equation, A(s). The "Table" function provides specific function values for both InvA1(t) and NLInvSteh[A[s], s,t,6]. For the ranges of t and T evaluated, the results match exactly.

**Table[InvA1[t],{t,0.1,5.1,0.5},{T,0.1,5.1,0.5}]**

*{{1.42798, 1.0046, 0.999161, 0.999004, 0.999001, 0.999001, 0.999001, 0.999001, 0.999001, 0.999001, 0.999001},*
*{5.03848, 1.28497, 0.989437, 0.981246, 0.99531, 0.999234, 0.999514, 0.999277, 0.999116, 0.999045, 0.999017},*
*{7.19112, 1.66612, 1.18129, 1.03008, 0.993293, 0.99171, 0.995391, 0.997742, 0.998701, 0.998994, 0.99905},*
*{8.45334, 1.87848, 1.30932, 1.11227, 1.03173, 1.00353, 0.99663, 0.996403, 0.997421, 0.998233, 0.998681},*

*{9.20271, 2.00367, 1.38155, 1.16618, 1.06857, 1.02401, 1.00579,
0.999572, 0.998077 0.998084, 0.998398},
{9.65807, 2.07965, 1.42417, 1.19837, 1.09374, 1.04148, 1.01621,
1.00496, 1.00049, 0.998994, 0.998649},
{9.94162, 2.12695, 1.45032, 1.21786, 1.1098, 1.05396, 1.02486,
1.0103, 1.00346, 1.00049, 0.99933},
{10.1222, 2.15708, 1.46684, 1.22999, 1.11999, 1.06242, 1.03129,
1.01472, 1.00623, 1.00209, 1.00019},
{10.2394, 2.17664, 1.47752, 1.23773, 1.12653, 1.06807, 1.03588,
1.01812, 1.00855, 1.00356, 1.00106},
{10.3168, 2.18954, 1.48454, 1.24277, 1.13079, 1.07185, 1.03909,
1.02065, 1.01038, 1.0048, 1.00185},
{10.3685, 2.19817, 1.48922, 1.24609, 1.13361, 1.07441, 1.04134,
1.0225, 1.01179, 1.0058, 1.00252}}*

**Table[NLInvSteh[A[s],s,t,6],{t,0.1,5.1,0.5},{T,0.1,5.1,0.5}]**

*{{1.42798, 1.0046, 0.999161, 0.999004, 0.999001, 0.999001, 0.999001,
0.999001, 0.999001, 0.999001, 0.999001},
{5.03848, 1.28497, 0.989437, 0.981246, 0.99531, 0.999234, 0.999514,
0.999277, 0.999116, 0.999045, 0.999017},
{7.19112, 1.66612, 1.18129, 1.03008, 0.993293, 0.99171, 0.995391,
0.997742, 0.998701, 0.998994, 0.99905},
{8.45334, 1.87848, 1.30932, 1.11227, 1.03173, 1.00353, 0.99663,
0.996403, 0.997421, 0.998233, 0.998681},
{9.20271, 2.00367, 1.38155, 1.16618, 1.06857, 1.02401, 1.00579,
0.999572, 0.998077, 0.998084, 0.998398},
{9.65807, 2.07965, 1.42417, 1.19837, 1.09374, 1.04148, 1.01621,
1.00496, 1.00049, 0.998994, 0.998649},
{9.94162, 2.12695, 1.45032, 1.21786, 1.1098, 1.05396, 1.02486,
1.0103, 1.00346, 1.00049, 0.99933},
{10.1222, 2.15708, 1.46684, 1.22999, 1.11999, 1.06242, 1.03129,
1.01472, 1.00623, 1.00209, 1.00019},
{10.2394, 2.17664, 1.47752, 1.23773, 1.12653, 1.06807, 1.03588,
1.01812, 1.00855, 1.00356, 1.00106},
{10.3168, 2.18954, 1.48454, 1.24277, 1.13079, 1.07185, 1.03909,
1.02065, 1.01038, 1.0048, 1.00185},
{10.3685, 2.19817, 1.48922, 1.24609, 1.13361, 1.07441, 1.04134,
1.0225, 1.01179, 1.0058, 1.00252}}*

# APPENDIX 2

## Mathematica Statements for the Exponential Case

The purpose of Appendix 2 is to provide an example of the Mathematica input statements used for analysis of the exponential case referenced in Section 5.3.1. Note that Mathematica input statements appear in **boldface** type while Mathematica output, except graphic output, appears in *italicized* type.

The following statement loads the "DiracDelta" package in Mathematica. This package defines the diracdelta function also known as the Unitstep function for use in Mathematica.

**<<Calculus`DiracDelta`**

The following statement defines the equation (5.3) which is also the first part of equation (5.1) referenced in Section 5.3.

**A[s_]= ((P+s) (R+s))/(((P+s) (R+s) (L+s))-(L R (P+s))+((L R (P+s))-(P (L+s) (R+s))) Exp[-T (L+s)])**

```
((P + s) (R + s)) /
 (-(L R (P + s)) + (L + s) (P + s) (R + s) +
   L R (P + s) - P (L + s) (R + s)
   -------------------------------)
             (L + s) T
            E
```

The following statement loads the numerical Laplace transform inversion package, "nlapinv.m". Note that the package, "nlapinv.m", is located in the "wnmath22" subdirectory on drive c.

```
<<c:\wnmath22\nlapinv.m
```

This statement provides the correct syntax and use of the Stehfest numerical inversion package.

**?NLInvSteh**

```
NLInvSteh[expr, s, t, n]
 where expr = the Laplace transform expression to
  be inverted
      s = the Laplace transform parameter
      t = time
      n = number of terms in series (!!! must be
  an even number !!!),
         typically 6 < n < 20.
```

The following statement defines the function, InvA1(t). This function is a numerical inversion of the function A(s) defined above. Note that the Stehfest method is used for the numerical inversion. The resulting function, InvA1(t), is a function of time, t and the age replacement parameter, T.

```
InvA1[t_]=NLInvSteh[A[s],s,t,6];
```

To obtain a functional estimate of the availability function, equation (5.2) is applied as noted in Section 5.3.1. Note in this statement as well as the previous statement, the actual output of the functions are suppressed by the semicolon appearing at the end of the statement. These functions are created by applying the Stehfest series expansion noted in Section 5.3.1. and become very long due to nested summations within the Stehfest method.

```
TAvail[t_]=InvA1[t]-(Exp[-T L] UnitStep[t-T] InvA1[t-T]);
```

The following statements set the exponentially distributed lifetime, failure repair time, and the preventive maintenance time parameters to 1., 10. and 25. The ratio of mean lifetime to mean repair time is 10 and the ratio of mean repair time to mean preventive maintenance time is 2.5.

**L=1.**
**R=10.**
**P=25.**

*1.*
*10.*
*25.*

The following statement produces a three dimensional plot of the availability function for the exponential case for time, t, range of [0.1 20.] and age replacement, T, range of [0.05, 15.]

**Plot3D[TAvail[t],{t,0.1,20.},{T,0.05,15.}]**

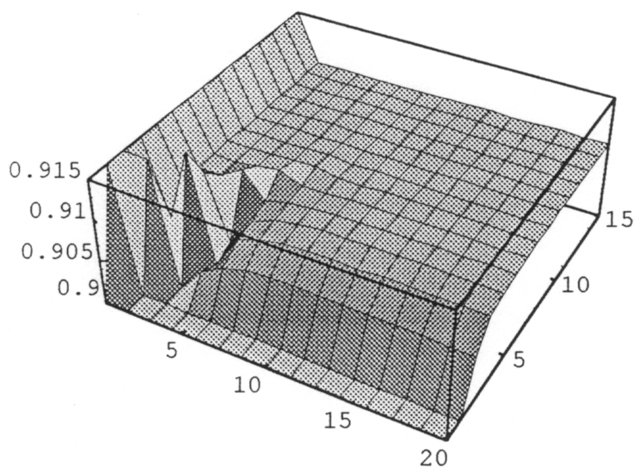


Fig. A2.1 A(t,T), $0.1 \leq t \leq 20.$, $0.05 \leq T \leq 15.$

The following two dimensional plots represent "slices" of the three dimension plot shown above. The variable representing continuous time, t, is held constant while the age replacement period is allowed to vary. The response is the availability at a given point in time and is plotted against the age replacement variable. Thus the graphs shown below are "slices" of Fig. A2.1 along the y-axis (representing the age replacement period, T) for constant points in time.

**Plot[TAvail[0.75],{T,0.001,0.749}]**

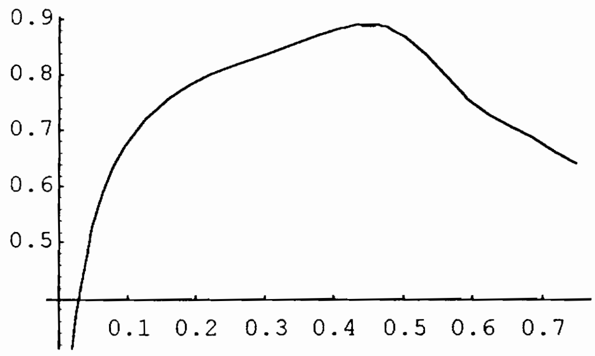


Fig. A2.2 A(t,T), t=0.75, 0.001 $\leq$ T $\leq$ 0.75

**Plot[TAvail[1.],{T,0.01,2.5}]**

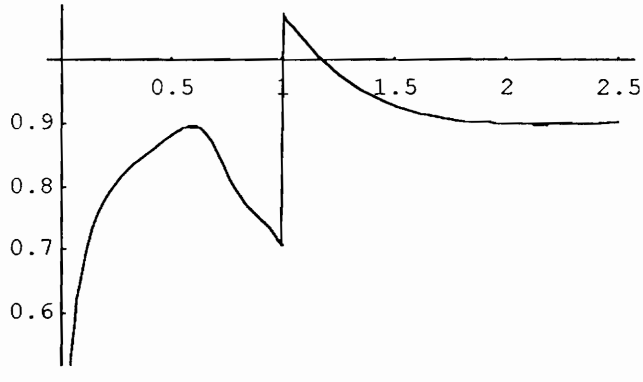


Fig. A2.3 A(t,T), t=1., 0.01 $\leq$ T $\leq$ 2.5

**Plot[TAvail[2.],{T,0.1,3.}]**

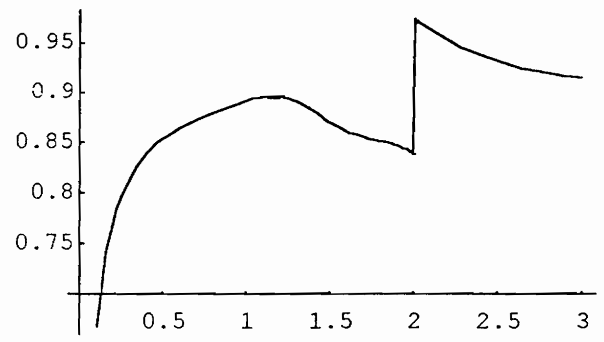


Fig. A2.4 A(t,T), t=2., 0.1 $\leq$ T $\leq$ 3.

**Plot[TAvail[5.],{T,0.1,6.}]**

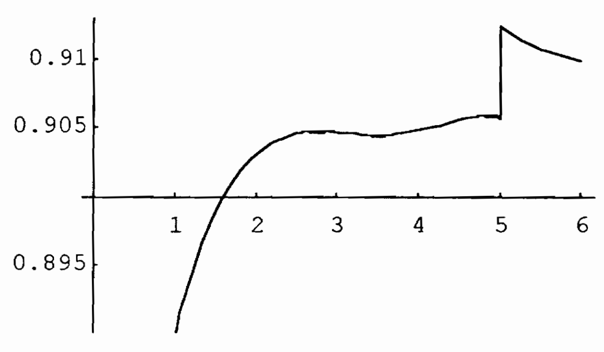


Fig. A2.5 A(t,T), t=5., 0.1 $\leq$ T $\leq$ 6.

```
Plot[TAvail[10.],{T,0.1,7.}]
```

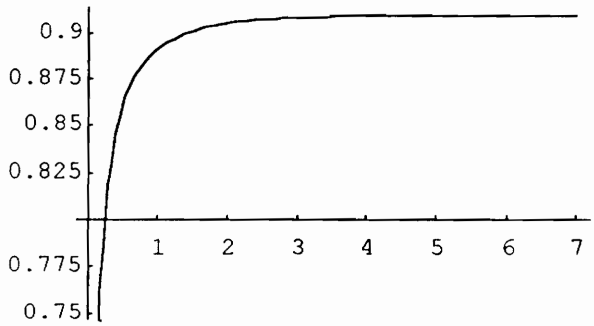


Fig. A2.6 A(t,T), t=10., $0.1 \leq T \leq 7.$

```
Plot[TAvail[20.],{T,0.1,10.}]
```

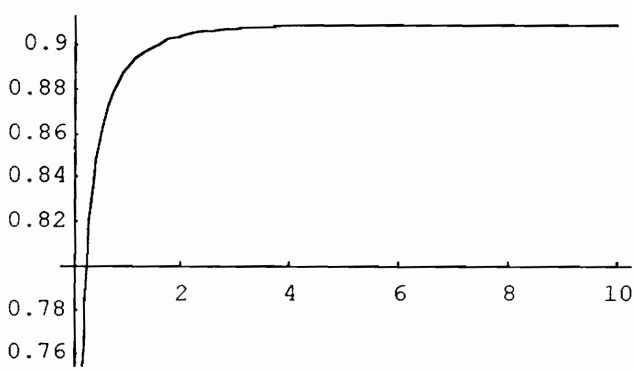


Fig. A2.7 A(t,T), t=20., $0.1 \leq T \leq 10.$

**Plot[TAvail[50.],{T,0.1,25.}]**

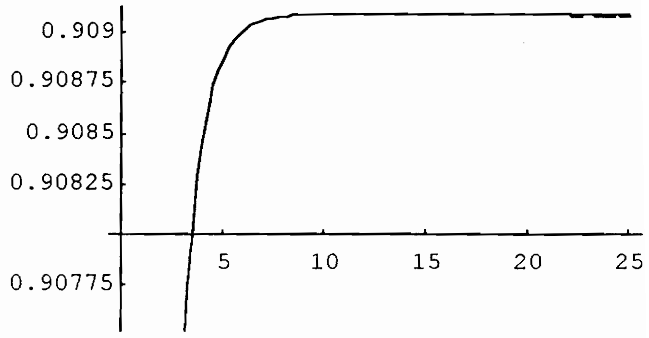


Fig. A2.8 A(t,T), t=50., $0.1 \leq T \leq 25.$

# VITA

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II. Educational Background

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William P. Murdock, Jr.