# *An Examination of Specification Error in Modern United States Growth Processes*

*By*

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## Abstract

This dissertation involves an empirical reexamination of US growth with the purpose of explaining growth usually attributed to advances in productivity. First, retaining the assumption of exogenous technological progress, I attempt to improve upon existing empirical models through new functional form assumptions. Next, I employ recent models of endogenous growth. Later chapters explore the issues of non-stationarity and international dependence. A significant generalization of the Gumbel Exponential distribution is developed and applied to the statistical modeling of economic growth. My chief objective is to characterize more accurately recent growth experience so that we may determine the most effective policy actions.

Current empirical studies of growth behavior have concentrated on a cross sectional approach. I believe, in addition, much can be learned about individual growth processes through a time series approach. This approach avoids many complicated issues in cross sectional analysis including changes in institutions within and between countries. Better understanding the nature of growth in a particular country and relating this process to other nations should yield valuable insight into the nature of growth, convergence and divergence and provide implications for public policy.

Many empirical studies have downplayed the crucial issue of examining the data in order to find the most appropriate econometric model specification. Through misspecification testing, we can identify and avoid faulty assumptions. Instead of viewing our data set as uncooperative, we should value the rich information our data contain. If

our usual specification assumptions are invalid, more information can be extracted from our series through the inclusion of additional variables or through a Maximum Likelihood approach based upon an alternative distribution.

This is the approach I follow in reexamining commonly utilized US input and output series. Utilizing the statistical and graphical abilities of the computer packages GAUSS and MATLAB, I am able to examine both graphically and analytically the validity of various assumptions about the underlying distributions of the data. With this approach, I can show that the Solow Residual contains a great deal of additional information about the dynamic pattern of growth of macroeconomic aggregates.

# *Acknowledgment*

This has been a long journey and a large number of persons played a vital role in helping me to complete it.

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# *I. An Introduction To Modeling Economic Growth*

With increasing technological innovations, it is no wonder the effects of accumulating resources and advancing technology have been a principal issue of economic theorizing throughout the latter twentieth century.

Early in this century, however, economic growth was viewed very differently. In 1931, Emil Lederer (1931) proposed that the persistent unemployment of the great depression in the late 1920's and early 1930's could be attributed to technological progress. In Lederer's view, inventions such as the assembly line and other labor saving devices lead to the replacement of laborers by machines. Since machines are doing the work of humans, such innovations will cause unemployment and may lower national income.

In his rebuttal of Lederer, Kaldor points out that innovations such as the adoption of new machinery will reduce costs.(See Kaldor, 1961 for summary). If they are not cost-reducing, they will not be implemented by profit seeking entrepreneurs and, therefore, technological improvements will not reduce national income. In 1932, Kaldor specifies the now common notion that unemployment will occur if wages exceed the marginal value product of labor regardless of the cause of this imbalance. So long as wages are free to adjust at some period, technical progress is not the evil that Lederer feared.

In retrospect, it is hard to imagine such a fear of innovation and technological advance. Today, productivity and growth in national output are compared as measures of the success and vitality of an economy. We study rapidly growing nations, not in fear for their declining welfare and falling national income, but to learn and imitate their means of rapid advancement.

Issues of replacement of workers by machinery do still receive attention, however not so much by economists. From a social stand point, we recognize the plight of displaced workers and certain productive techniques are blocked at the labor relations table. However, theoretically, we know that displacement of workers in an advancing industry is sometimes (but not always) necessary for improving productivity and, further, displaced workers will be absorbed into other industries so long as wages are not rigid. The reallocation of workers into industries in which they become relatively more valuable is a necessary adjustment for maximizing national output.

As we approach the twenty first century, we are in agreement that technological progress is an important field of study in economics. Such a recognition has led to tremendous advances and the application of new methodology and techniques throughout economics. In the study of rates of change of output, we now employ dynamic concepts of equilibrium and transition, thus opening a new and more sophisticated means of studying the nature of economies over time.

With the end of the cold war and the finding of our so called peace dividend, the question of how to stimulate growth through industrial and educational policies takes on a new significance. It is with the hope of improving our understanding of the processes of growth that I undertake this study. Only through an understanding of the factors crucial in our past and current growth may we best increase our productivity.

## 1.1 History Of United States Productivity Performance

Today, the United States leads the world in productivity. That is, the quantity of real output created by a unit of labor or capital in the United States exceeds the quantity created elsewhere. (see Table 1) Since the beginning of capitalism around 1700, only two other nations have earned this distinction. As reported by Maddison (1982), the Netherlands had the highest productivity rate in all the world from 1700 to 1780. Around 1780, Britain with its imperialism and vast industrial resources became the leader. The United States grew rapidly, however, and achieved the rank of number one around 1890. Approximately one hundred years later, the dominance of the United States is threatened by Japan whose rate of productivity increase has exceeded that of all industrialized nations since the mid-twentieth century.(Dennison, 1972)

| Leading Country | Date | GDP per Manhour | Gross Fixed Non-Residential Capital Stock per Manhour |
|---|---|---|---|
| Netherlands | 1700-1785 | -0.07 | (n.a.) |
| United Kingdom | 1785-1820 | 0.50 | 0.00 |
| United Kingdom | 1820-1890 | 1.40 | 0.90 |
| United States | 1890-1979 | 2.30 | 2.40 |
| Japan | Early 21st ? | ? | ? |

Table 1. Productivity Measures for Leading Nations : 1700-1979 (Source : Maddison, 1982)

Our desire to determine the cause of this pattern of leadership and decline leads us to study the causes of productivity growth. Maddison (1982) points out four common factors leading to rapid productivity advance. These are:

(1) Natural Resource Wealth

(2) Ample Labor Supply

(3) Stock of Capital & Technical Progress

(4) Efficiency through Specialization, Economies of Scale, Etc.

It is easy to see the influence of these factors on the rise and fall of productivity in the early United States. While being colonized by the powers of Europe, the young new world provided vast resources of raw materials as well as new goods. In the early days of United States independence, vast natural resources in the form of untouched fertile land were rapidly utilized with the opening of the rail system and the push westward. Also, rapid population growth through immigration provided a vast supply of labor resources, many with skills upon arrival in the United States. Explosions of skilled workers from European emigration combined with the technological knowledge settlers brought to the new nation and abundant natural resources to spur high rates of growth in output. Strong work ethics and minimal protection for workers and their rights also encouraged high output in the early part of this century. High levels of investment by entrepreneurial individuals and by large corporations with equally large research budgets led to product and process innovations as well as means of distribution. Managerial innovations increasing specialization such as the assembly line dramatically increased output per unit of input.

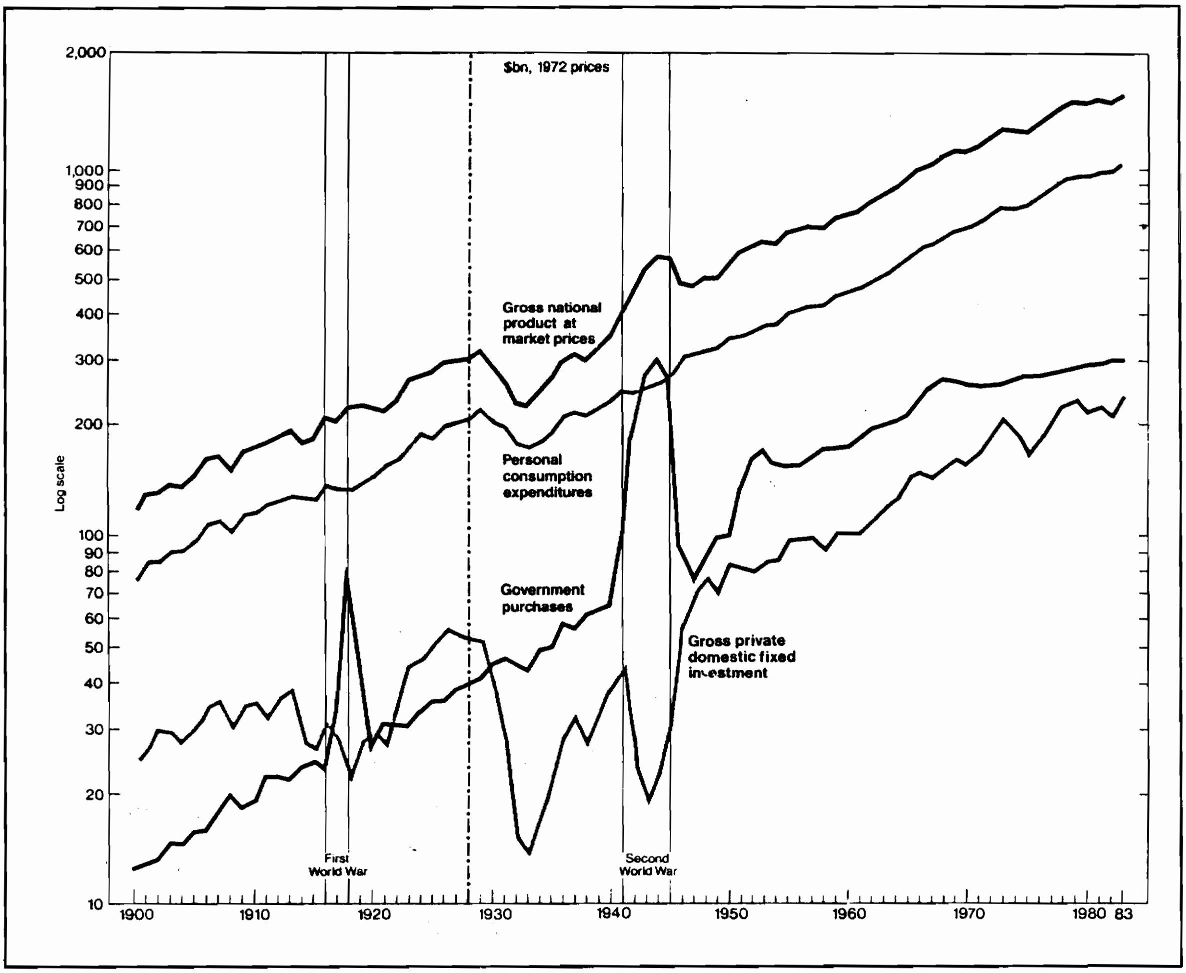


Figure 1 : US Gross National Product and its components. Source : *Economic Statistics*, 1985

Changes in institutional arrangements and the crowding of natural resources, however, were inevitable. As the US decreased its potential for discovery of untapped natural resources, so did its potential for economic growth decline. The isolationist policies of the mid twentieth century may also be partly to blame for the slowdown in US productivity growth. From the perspective of Klein (1977), we have entered a period of slow history, just as the Netherlands and England before us.

Since 1973, US productivity growth has slowed substantially. Output per worker has increased at an annual rate of only 0.8% compared to the previous rate of 2.5% (Farrell & Mandel, 1992) The countries of Europe and Japan have rapidly closed the gap in productivity. With the competitive pressures of emerging trade blocks, the issue of productivity increase is of vital importance.

Maddison's list of factors give us some idea as to the cause of lagging productivity growth in recent US history. Maintaining high productivity growth is a key objective of policy makers. Only by understanding the factors influencing growth can we direct resources toward the most beneficial activities. That is, we wish to discover where to invest our resources to best stimulate economic growth.

## 1.2 Growth Economics Defined

A primary goal of growth economics is to determine and study the characteristics of a dynamic equilibrium. Just as in static equilibria, we impose certain requirements under which an economy is said to be in a dynamic equilibrium. In this case, we find certain characteristics which lead us to a steady state pattern. Upon recognizing these equilibrium conditions, we then turn to questions of existence (as in the static study), stability, comparative dynamics, and so on.

Steady state growth roughly corresponds to our static definition of equilibrium. In a steady state, all variables are growing at a proportional rate or not growing at all. (Jones, 1975) Just as in static equilibrium, a dynamic equilibrium is an approximation to reality. Within a steady state, we find no incentive for change. As we model this, we see that the returns to factors of production have stabilized in such a matter to prevent readjustments in the proportion of various inputs utilized. How near an economy is to a true steady state is a question for econometrics.

A closely related dynamic equilibrium process is that of balanced growth. Here, all variables are growing at the same constant rate or not growing at all. This is a special case of a steady state growth pattern. In this case, all factors of production increase at exactly the same rate. The ratio of output to labor and output to capital in such a case is a constant rather than a constantly trending value. (Blanchard and Fisher, 1989)

## 1.3 The Aggregate Production Function

In growth theory we wish to determine the relationships between economic variables and the productivity of inputs, both now and in the future. If we think of simply one firm, such a relationship is described by the production function. That is, we specify the quantity of output produced given the quantities of inputs utilized and the current state of technology. We usually divide these inputs into the categories of land, labor and capital. For simplification, we can summarize the effects of the fixed input, land, through the diminishing marginal productivity of our other inputs. Any improvements in the quality of land may also be considered by labeling the growth in natural resources as a capital investment such as developing better mining so that we can extract more coal. This is one of many ways of accommodating the complications imposed by fixed land resources.

From the above, the output of a firm can be given by:

$$Y = F(K, L) \tag{1.1}$$

We should note that such a simple functional form incorporates numerous calculation difficulties. We have created a single measure capital to account for the many different types of machine resources which may vary in age, productivity, longevity, salvageable value and other attributes. Similarly, labor resources also vary in education, experience, work day length, and so on. A more complete form of the production function would, therefore, be :

$$Y = F(K_1, K_2, \ldots, K_n, L_1, L_2, \ldots, L_m) \tag{1.2}$$

where the firm employs *n* categories of capital and *m* categories of labor.

Such aggregation is difficult and cumbersome, even if all necessary data are available. Instead we assume we have an accurate summary measure for each of these inputs. This may be thought of as a weighted average of capital or labor inputs where weights are based upon the productivity of that asset.

In the study of a nation's productive potential, we wish to determine national output given the use of inputs in the production process. We, therefore, aggregate across all firms in all industries to determine our inputs utilized and output gained. Such a relationship is the aggregate production function.

Such an approach is attributed to Keynes and his *The General Theory of Employment, Interest and Money*. Keynes (1935) first utilized aggregate concepts in describing macroeconomic phenomena, but was very careful to note the difficulties involved in such a simplification. After Keynes, however, many economists utilized "heroic" aggregates in macroeconomic models without regard for the lack of applicability of output, labor and capital measures.(Kaldor, 1961) This is an area of enthusiastic research as differing measures of these macroeconomic quantities may significantly alter our views of the production relationship.

Due to data limitations, we will however follow suit and assume that we can specify an aggregate production function which expresses the relationship between national output or income and the inputs of capital and labor. Through this method, we will utilize commonly analyzed data and can display the improvements which can be made even with the faulty data available through respecification of the dynamic production relationship.

As in the case of a single firm, the aggregate production function takes the form : $Y=F(K,L)$ but now $K$ and $L$ represent the aggregate quantities of capital and labor in the economy. Since we are interested in the relative changes in these quantities over time, variables of interest are :

$$\frac{\partial Y/\partial t}{Y} = \frac{\dot{Y}}{Y}; \qquad \frac{\partial K/\partial t}{K} = \frac{\dot{K}}{K}; \qquad \frac{\partial L/\partial t}{L} = \frac{\dot{L}}{L}; \tag{1.3}$$

In a steady state, these proportional growth rates must remain constant. In addition, if we are to obtain balanced growth, these growth rates must also be equal. (Jones, 1975) These points will be discussed in detail in models of growth (See Chapter 2).

## 1.4 Sources Of Economic Growth

We can be more precise in defining two general categories of the causes of growth through reference to this production function. An increase in inputs will move us along a given production function resulting in higher output. Consider the case of $Y = F(K,L)$ shown below. An increase in the quantity of labor utilized in production will increase aggregate production of $Y$ with no change in the production function itself. Similar analysis may be constructed for an increase in quantity of capital employed.

The growth of output may be attributed to changes in either the number and quality of inputs employed or in the techniques of production. An increase in the quantity or quality of inputs employed will, most likely, increase the quantity of output produced. Also, improvements in the technology utilized in production should increase quantity of output gained for a certain input combination.

This may also be seen graphically through a look at the production function. In this case, we can visualize the three dimensions (*K-L-Y* combination) of the production relationship by fixing the quantity of one of our inputs (say capital) to a certain level and viewing combinations of the other factors (labor and output). An increase in labor will increase output through the existing production function and, therefore, move us to a new point on an existing production frontier. This is displayed in figure 2 below.

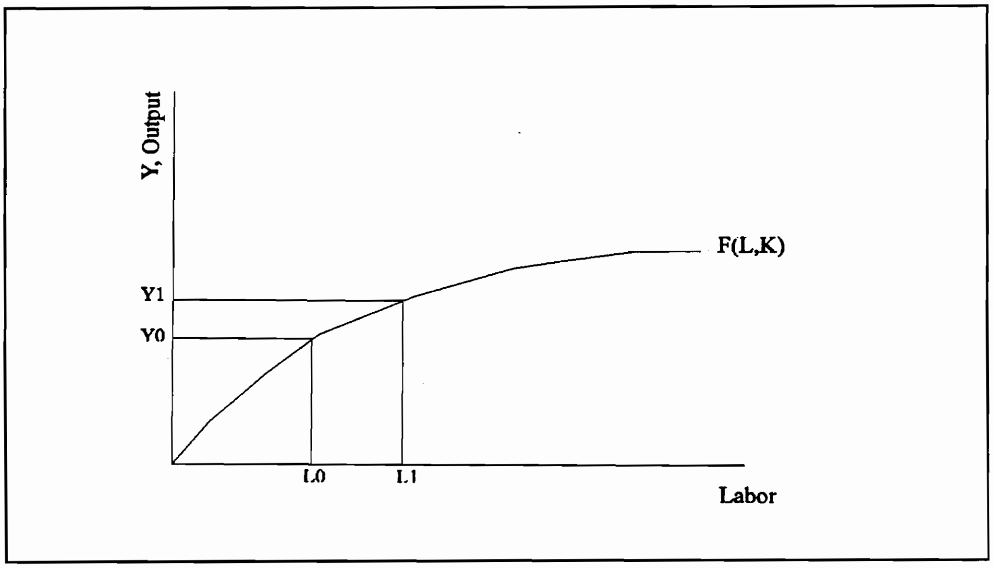



Figure 2 : Growth in Output due to an Increase in Input Employed

Technological improvements differ from simple increases in the quantity of inputs. Technological change will change the production function itself. Such change may be neutral in the sense that the improvement affects all input productivity in the same way. This would be represented as a parallel shift in the production function. This is termed disembodied technical progress as it is not attributed to any factor of production. However, technological change may also increase the productivity of one factor of production only or may lead to unequal growth in the productivity of several factors. In this case, the slope of the production function will also change. Such innovations are said to be labor saving if the productivity of labor increases (that is, less labor is required to produce a given level of output). Similarly, improvements in the productivity of capital are termed capital saving or capital augmenting innovations. Improvements in the quality of inputs may be viewed as changes in technology as such improvements will change the production function itself as opposed to moving us to the new point on an existing curve.

Some authors, however, have dealt with this type of improvement in productivity of inputs by a redefinition of input resources. Through the use of efficiency units of labor and capital which have fixed productivity, an improvement in the productivity of an input is noted by a larger number of capital or laborers when they are measured in efficiency units. This would then be interrupted as a movement along the same production function.

In either case, the factors which affect the production relationship are the same. Increases in the amount of resources available and in the productivity or quality of these resources will increase output. As noted earlier, we usually do not model the primarily fixed resource, land, in our aggregate production function. Increases in the quantity, quality or productivity of land are treated as a technological change as the coefficients or form of the production function will change.

Disembodied technological innovations will be represented by upward shifts of the production function. Embodied technical innovation, on the other hand, will cause a shift in the marginal efficiency of our inputs and will change the slope of the function.

Technical innovation changes the form or parameters of the production function. Such changes, therefore, may occur through changes in the production method or through changes in the quality of inputs (which will in turn cause a change in the marginal product of a factor). Technical progress as defined here includes both technological improvements in the production method and improvements in the productivity of factors of production. Such improvements include education and training of the labor force and improvement in the productivity of capital goods. An increase in output as a result of technological improvement is displayed in figure 3.

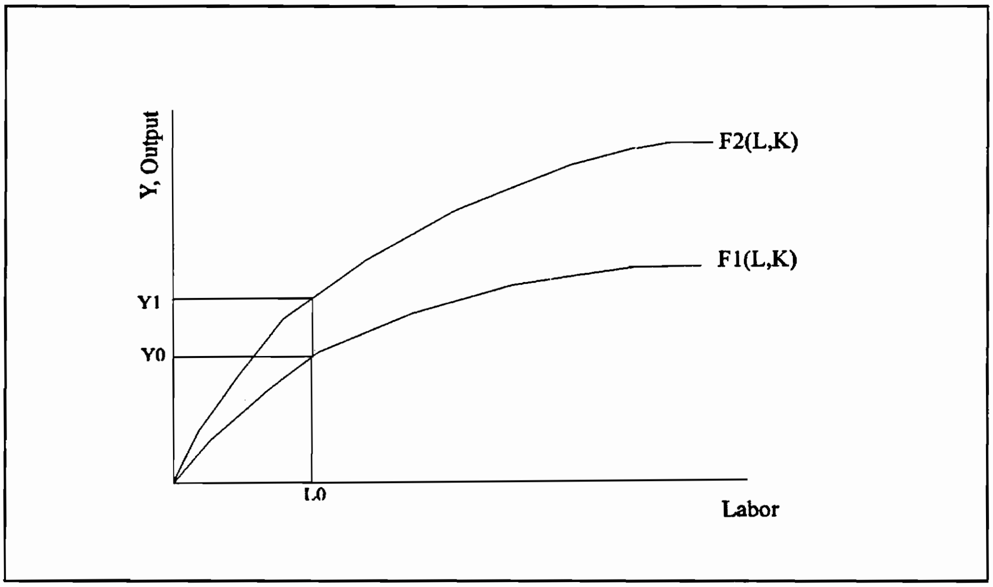



Figure 3: Growth in Output due to Technological Advance

## 1.5 Statement Of Objectives

It is my objective to analyze the nature of economic growth in the twentieth century United States with reference to NeoClassical Models of growth as well as more modern endogenous growth models. It is the intention of this research to yield policy directives which may increase the rate of growth of output. Given our government and society's limited resources, I would like to determine where invested resources may lead to the greatest increases in output.

In order to study US performance, a time series of output and inputs will be developed. These inputs include the traditional inputs of labor and capital as discussed above as well as some factors which affect the production function itself when specified in the usual capital-labor format. A more general production function will incorporate new factors which play a role in the creation of output.

This study will point out misspecification errors in the usual growth model representation. These early empirical growth models have spurred a set of "Stylized Facts" which originally were noted as generalizations of economic growth behavior.(Kaldor, 1961) Now, however, these generalizations are taken to be truthful representation of the growth behavior of all economies. Since these results were based upon 1960's examination of a cross section of studies, such reliance on these Stylized Facts without a careful examination of more recent and more specific behavior is misleading at best.

We will discuss alternatives to more properly define growth processes. Both exogenous and endogenous models of growth will be tested. Important specification issues to be discussed include the dynamics of economic growth processes as related to the notion of Evolutionary models of growth (Nelson and Winter, 1985) and the functional form underlying the production relationship. Also, the issues of stationarity and international growth relationships will be discussed as these also guide us to a more complete picture of production relationships over time.

Since specification also involves the selection of an appropriate conditional distribution assumption, carefully noting the dynamics of a production relationship and repairing the damage of heteroskedasticity and non-linearity may not give us the best understanding of the relationship between our variables of interest. It may be the case that the distribution assumption of conditional normality is not valid. This study concludes with an examination of specification under an alternative conditional distribution assumption. Through an appropriate distribution assumption, we may be able to draw from our data a better picture of the conditional mean and the conditional variance and therefore develop the best possible representation of the statistical properties relating these quantities. Such respecification also has important interpretations in terms of the theoretical production function.

## 1.6 Plan of Study

In Chapter 2, we will examine theoretical models of economic growth. These models encompass both growth when technological change does not occur (that is, growth through the accumulation of inputs) and also growth through technological change (when the production function itself changes). In this chapter, we will also review major empirical studies of economic growth.

In Chapter 3, the methodology utilized in this study is outlined. The empirical model is developed from the theory discussed in Chapter 2 and then assumptions of the linear regression model and the notion of misspecification testing are examined. This methodology is contrasted with many other empirical studies of economic growth. Also, Chapter 3 describes the data sets utilized through this dissertation and the sources of economic data.

In Chapter 4, we examine the validity of certain Stylized Facts which have been a measuring rod for the success of various theoretical models. These Stylized Facts, originally developed in the 1960's, are shown to fail to explain modern US growth performance. Misspecification testing shows the importance of dynamics in understanding the current path of growth and of non-linearity in the growth path.

In Chapter 5, we consider some alternative sources of economic growth. Growth driven by human capital development, public capital investment and innovation are all considered in turn. This is an empirical examination of much of the developing theory of endogenous economic growth.

In Chapter 6, the growth pattern found to examine US performance is applied to several other industrialized nations. Misspecification testing in the case of Sweden, the United Kingdom and Japan finds similar dynamic and functional form patterns. The importance of simultaneity in determining international growth patterns is discussed.

In Chapter 7, the stationarity of our estimates of output responsiveness to changes in inputs is explored. It is shown that our estimates do not remain stationary over the entire sample period. The literature on Unit Roots and Cointegration is discussed and the Unit Root Hypothesis is tested for various aggregate series. By splitting the sample, we find great improvements in the stationarity of our estimates. Finally, the Hamilton switching regime model is employed to determine when changes of models will take place.

In Chapter 8, we reexamine the standard growth model under a different distribution assumption. Guided by our misspecification results, we adopt a non-linear conditional mean and reexamine the growth path of output. Here we find that, while attractive in theory, the assumption of a conditional exponential distribution does not perform well in explaining output growth.

Concluding remarks follow in Chapter 9.

# *II. Studies of Economic Growth*

## 2.1 Growth Without Technological Progress

As discussed in Chapter 1, growth in output may occur without any improvement in the methods of production simply through increases in the amount of resources available. We can, therefore, create a model to describe growth without technological innovation. In this case, the production function does not change over time in either functional form or in coefficients.

We begin with models of the Harrod-Domar type. These represent the first models of economic growth. Then, the relaxation of the key assumption regarding the input-output relationship will bring us to NeoClassical models of the Solow type. We will discuss the main results of these models. However, it is important to note that some key questions such as existence and stability have been omitted. This is not a complete survey of even the most pivotal models of growth. The models presented here have been selected due to their clarity and their ability to highlight principle assumptions, results and questions for the following empirical evaluation. For an excellent survey of theoretical economic growth models, see Hahn and Matthews (1964). More modern growth models are well documented in Romer (1989).

### 2.1A The Harrod - Domar Model

The following description is a simplification of the Harrod Domar Model as discussed in Harrod (1939), Domar (1946), and Jones (1975). Consider an economy producing one good which is used for consumption in the current period or becomes a perfectly malleable capital good in the future. This good is produced by the inputs of labor and capital. We are interested in the growth path taken by this output good in relation to the growth paths of inputs.

In this model, therefore, we are interested in output, $Y$, Capital, $K$, and Labor, $L$. We assume that labor grows at a steady rate $n$ so that :

$$L_{t+1} - L_t = nL_t \tag{2.1}$$

In continuous time, the relationship above becomes

$$L_t = L_0 e^{nt} \tag{2.2}$$

We also must specify the means by which capital is accumulated. In the Harrod model, we assume all members of the population save some constant proportion of output given by the marginal propensity to save, $s$. That is :

$$S_t = sY_t \tag{2.3}$$

and

$$K_t = K_{t+1} + I_t \tag{2.4}$$

We assume there is no depreciation of the capital good. In equilibrium (and disregarding time lags to be discussed later), investment will equal savings so that :

$$I_t = S_t \tag{2.5}$$

therefore,

$$\Delta K(t) = I(t) = s \times Y(t) \tag{2.6}$$

The Harrod model also assumes a fixed coefficient technology so that

$$Y_t = vK_t \tag{2.7}$$

or

$$Y_t = uL_t \tag{2.8}$$

disregarding non-utilized inputs implied by the Leontief Production function

$$Y_t = Min[uL_t, vK_t] \tag{2.9}$$

Given that production occurs at full-capacity of capital (that is, the labor supply is not constraining), then

$$vY_t = K_t \tag{2.10}$$

So we have, in total, the following equilibrium conditions :

$$vY_t = K_t \tag{2.11i}$$

$$K_t - K_{t-1} = sY_t \tag{2.11ii}$$

$$L_t = L_0 \times e^{nt} = uY_t \tag{2.11iii}$$

so that

$$K_t - K_{t-1} = s(\frac{K_t}{v}) \tag{2.12}$$

and the growth rate of capital is given by

$$\Delta K/K = s/v \tag{2.13}$$

Output will also grow at the same rate as capital.

$$v\Delta Y = \Delta K = sY \tag{2.14}$$

therefore,

$$\Delta Y/Y = s/v = g \tag{2.15}$$

Given the full employment requirement above,

$$L = uY = L_0 e^{nt} \tag{2.16}$$

$$u(Y_0 e^{gt}) = L_0 e^{nt} \tag{2.17}$$

$$LnL = Lnu + LnY \tag{2.18}$$

$$n = g = s/v \tag{2.19}$$

In full employment, labor demand grows at the rate of output growth given by the full capacity constraint on the use of capital. For labor market equilibrium, we must have balanced growth where $g = n = s/v$. The growth paths in the steady state are given by :

$$L_t = L_0 e^{gt} \quad Y_t = Y_0 e^{gt} \quad K_t = K_0 e^{gt} \tag{2.20}$$

where the initial values are: $$Y_0 = vK_0 = uL_0 \tag{2.21}$$

These results are fairly intuitive given the fixed coefficients nature of the production function. If capital is always used to capacity, then the amount of extra output gained in a period depends upon the amount of extra capital accumulated. We assume here that we have a non-constraining supply of labor so that the labor market can be disregarded. Without this assumption, the labor market will be in equilibrium when labor growth is consistent with growth in capital. Otherwise, some labor will be unemployed.

We can generate the same equilibrium conditions without the use of the production function using a different interpretation of the full-capacity requirement on capital. Instead of reference to capital stock, we may think of investment as the factor linking output and capital goods.

Here, we assume investment occurs in proportion to the prior change in output so that :

$$\frac{\text{Investment}}{\text{Change in Output}} = v = \frac{I_t}{Y_t - Y_{t-1}} \tag{2.22}$$

Also, we assume savings are based upon our past income or

$$S_t = sY_{t-1} \tag{2.23}$$

This version of the Harrod story is termed the multiplier-accelerator version where the multiplier is derived from the savings function and equals $1/s$ while the accelerator comes from the output equation and equals $1/v$.

Equilibrium conditions in the case are:

$$I_t = v(Y_t - Y_{t-1}) \tag{2.24i}$$

$$I_t = S_t = sY_{t-1} \tag{2.24ii}$$

$$L_t = L_0(1+n)^t \tag{2.24iii}$$

By substitution, we find $$v \times (Y_t - Y_{t-1}) = s \times Y_{t-1} \tag{2.25}$$

or $$\frac{\Delta Y_t}{Y_{t-1}} = s/v = g \tag{2.26}$$

The labor condition again implies $g = n$ and $L_0 = uY_0$. The capital accumulation equation, however, is slightly different due to changes in the time dimension. Investment cannot be defined for time zero as it depends upon previous output. For the same reason; savings begin in the first period.

$$I_t = I_1(1+g)^{t-1} \tag{2.27}$$

and $$I_1 = sY_0 \tag{2.28}$$

Models of the Harrod-Domar type fix the relationship between changes in capital goods and changes in output by means of a fixed coefficient production function, $Y=vK$, or by a multiplier relationship, $\Delta Y/\Delta K = 1/v$. The ratio of growth in capital and growth in output is always fixed so, if we assume all capital is used in production or actual and planned investment are equivalent, then capital and output must necessarily grow at the same rate. In addition, if we assume labor remains fully employed, the labor supply must be growing at the same rate as labor demand. Labor demand grows with output, therefore, $L$ grows at the same rate as capital and output. This will yield a balanced steady state growth path.

It is interesting to consider what would happen if full employment were not obtained. That is, if $n$ does not equal $g$. If $n$ is greater than $s/v$ over time, increasing labor resources would be unemployed so that output growth would continue at $s/v$ with more and more labor remaining unemployed in successive periods. If n is less than $s/v$, then the labor market will become a binding constraint on output once all unemployed labor, if any exists, is used up. Output growth, in this case, would be reduced due to a relative shortfall of labor resources. This link to the labor market arises once again due to the fixed coefficients production requirement.

A drawback of the Harrod-Domar models arising from fixed proportion production technology is that we are forced to abandon any marginal product analysis and, therefore, the concepts of wages and returns to capital. We speak of equilibrium in input markets without reference to prices in these markets.

An important extension of the above model, therefore, is the relaxation of the fixed coefficient requirement. We do this in two stages to clarify the result. In the first, we specify a range of possible input combinations as a linear combination of two fixed coefficient production processes. In the second, we allow for smooth production processes as long as they satisfy certain regularity conditions. This is the NeoClassical Approach of Solow (1956, 1957).

## 2.1B Linear Combinations - The Generalized Leontief Form

Following Jones (1975), consider two fixed coefficient production functions which are capable of producing the same output, $Y$. One of the Leontief combinations can be considered labor intensive while the other is capital intensive. That is,

$$Y_t = Min[u_1 L_t,\ v_1 K_t] \tag{2.29}$$

or

$$Y_t = Min[u_2 L_t,\ v_2 K_t] \tag{2.30}$$

where

$$u_1 > u_2 \quad \text{and} \quad v_1 < v_2 \tag{2.31}$$

A firm may produce a total output $Y$ utilizing either method of production. They also may produce part of total output utilizing one method and the remainder by the other means. By this choice, we move away from fixed coefficients by allowing some mix of inputs. However, only linear combinations of two production techniques are possible, therefore, we still cannot employ marginal analysis.

The usefulness of this extension is its avoidance of the so called accidental equilibrium in which the rate of labor force growth just happens to coincide with the rate of output growth as in the Harrod Domar model. By selecting a certain mix of production techniques, we may select a production pattern so that steady state growth is achieved.

$$n = \frac{s}{v} = \frac{s}{\lambda v_1 + (1-\lambda)v_2} \; ; \; \lambda \in [0,1] \tag{2.32}$$

so

$$\lambda v_1 + (1-\lambda)v_2 = s/n \tag{2.33}$$

therefore

$$\lambda = \frac{(s/n - v_2)}{(v_1 - v_2)} \tag{2.34}$$

By selection of the correct combination of production techniques (that is, the correct $\lambda$), steady state growth may be achieved. Given the fixed range of possible capital-output ratios, however, it may not be possible to select a production mix to achieve *s/n*. It is preferable, therefore, to have an infinite range of *v*'s available.

Once again we haven't discussed prices of inputs or marginal analysis due to the linear nature of our production technology. However, prices of inputs play a more intuitive role in the choice between capital and labor intensive methods of production. As relative prices of these inputs change, we will observe a shift between our production technologies via a change in $\lambda$.

We can also generalize this model to any number of production processes. As more and more production functions are added, the chance that we can select a single production function which will equate the natural rate of growth of output and capital to the rate of growth of labor increases. That is the probability that $\lambda$ will equal 1 for a particular fixed coefficient technology will increase. This notion brings us to the NeoClassical case.

## 2.1C NeoClassical Models of Growth

The following description is of a very simple Solow Model. This is based on Solow (1970) which then extends this simplified model in numerous directions. The interested reader is referred to this excellent overview of the NeoClassical approach for more information.

In the NeoClassical framework, there exist a range of production functions leading to a continuous range of capital requirements, *v*. In this case, it is possible to select a single method which will equate *g* and *n*.

Assumptions other than the production function will remain the same as in the Harrod-Domar model. This is not necessary but it clarifies the role of the variable capital output ratio. Specifically, saving will be a fixed proportion of income and the labor force grows at an exogenous rate *n*.

The production function :

$$Y_t = F(K_t, L_t) \tag{2.35}$$

is linear homogenous or follows Constant Returns to Scale. This restriction implies that any increase in inputs will lead to a linear increase in output. It is possible, therefore, to speak in terms of per capita variables.

$$\frac{Y_t}{L_t} = F(\frac{K_t}{L_t}, \frac{L_t}{L_t}) \tag{2.36}$$

$$y_t = f(k_t, 1) = f(k_t) \tag{2.37}$$

This function is also assumed to be well behaved so that the function is everywhere upward-sloping and concave and corner solutions are not permitted.

We find the same equilibrium conditions requiring

$$Y_t / k_t = 1/v = n/s \tag{2.38}$$

so that the growth rate of labor and capital coincide. However, the generality of the production function allows a greater range of choice in the variable $v$. However, this same generality prohibits solution of the system of equations seen in the earlier models except in specific cases such as the Cobb Douglas Production Function.

In general, we see the equilibrium conditions as in the Harrod-Domar model. That is,

$$y_t = f(k_t) \tag{2.39i}$$

$$\Delta K_t = sY_t \tag{2.39ii}$$

$$L_t = L_0 e^{nt} \tag{2.39iii}$$

so that

$$\Delta K_t = sF(K_t, L_t) \tag{2.40}$$

For steady state growth,

$$F(K_0 e^{nt}, L_0 e^{nt}) = \frac{n}{sK_0 e^{nt}} \tag{2.41}$$

Since $F$ is assumed to follow constant returns to scale

$$F(K_0 e^{nt}, L_0 e^{nt}) = e^{nt} F(K_0, L_0) = Y_0 e^{nt} \tag{2.42}$$

Therefore, the capital output ratio remains $n/s$ or $v = s/n$.

The main advantage of the NeoClassical model lies in its generality of the production function as we may now employ marginal analysis and can speak of marginal product and prices of inputs. We can relate income shares of capital and labor owners at equilibrium to our previously determined steady state outcomes.

In perfect competition,

$$\rho = \delta Y / \delta K \tag{2.43}$$

and

$$w = \delta Y / \delta L \tag{2.44}$$

and $$Y = \rho K + wL \tag{2.45}$$

divides the total product between capital and labor owners in accordance to their marginal productivity. In per capita terms, and assuming CRS we have

$$\delta Y / \delta K = f'(k) \tag{2.46}$$

and $$\delta Y / \delta L = f(k) - kf'(k) \tag{2.47}$$

and $$y = pk + w \tag{2.48}$$

In this case, steady state growth implies that capital, labor and output grow at the constant rate $n$. The capital to labor, $k$, and output to labor, $y$, ratios remain constant.

## 2.1D Implications Of Models Without Technological Advance

We have discussed the "grass roots" models of growth in the absence of technological change. We have found that the only stable steady state is when the growth of the labor force and capital are equal. Since the growth rate of labor is exogenously given as a rate equal to $n$, steady state capital accumulation must occur at a rate $n$. Output growth, due to constant returns to scale, also occurs at a rate $n$. Capital accumulation depends upon the proportion of savings to income given by the constant marginal propensity to save $s$, and the capital requirement per unit of output, $v$.

We can note a determinant relationship between the growth of capital and the growth of output. The output produced from a certain quantity of capital depends upon the production requirement given by the capital per output ratio. Following production, investment is determined by the quantity of output and the constant marginal propensity to save, $s$. It is intuitively appealing, therefore, that capital and output grow at a rate given by these parameters $v$ and $s$. Due to the constant returns to scale technology and the fixed investment-output relationship, equal growth for output and capital is necessary. This is due in part also to our other simplifications (i.e. no depreciation,...).

I have, however, simplified these relationships in their time dimensions. Harrod was very cautious in this respect in noting that planned and actual investment may differ and, therefore, steady state growth would occur only if the expectations of investors were correct. Our growth paths depend upon assumptions made regarding lags in the output-investment relationship. As seen earlier in the multiplier-accelerator Domar model, investment may depend upon past income which implies that $K(0)$ does not exist. However, the main implications of growth and specifically the growth rates, will remain the same.

Since the growth rate of labor is exogenous to our models, little can be said about this quantity. Adjustment of production methods (that is, change in $v$) or in savings behavior (a change in $s$) may bring us to steady state growth, but labor growth is assumed to be fixed. The finding of Romer(1989) that population growth (and, therefore, labor supply growth under commonly accepted assumptions) is inversely related to income implies the exogeniety in population growth may be an inappropriate assumption.

In summary, the assumptions of constant returns to scale, the investment-output relationship, and a fixed labor growth rate yield a number of possibilities for our dynamic growth pattern. In the case of fixed proportions production, $v$ is fixed so that steady state growth without an exogenous change in some parameter is a coincidence. Only by changes in savings behavior or movement among a variety of production functions to yield some weighted average of capital requirements may we adjust the rate of growth of output and capital to reach steady state growth. With the generality of the production function of the NeoClassical model, a production technology may be selected to achieve steady state growth.

## 2.2A Growth Models with Technological Advance

As discussed in Chapter 1, output growth may also occur as a result of an improvement in production processes. The production function will shift so that more output may be produced from a given combination of inputs.

We may experience a neutral shift in technology which will affect the productivity of all inputs equally. This would result in a parallel shift of our production function. Solow has incorporated such a technological change in the form of a time dependent index of technical progress.

Alternatively, an improvement in production may cause a shift in the marginal productivity of one or more of the factors or the influence on the marginal product of inputs may not be proportional. Such a shift in the production function will affect the marginal conditions through changes in the slope of the function.

In general, if technical progress occurs via disembodied or neutral improvements, the steady state growth conditions remain basically unchanged. The growth of autonomous factors changes growth in output without affecting our capital requirements. That is, our steady state rule becomes $s/v = n + m$ where $m$ is the rate of technical improvement.

If, however, technological progress impacts the capital output ratio through embodied innovation, steady state growth rules become more complex. The change in our growth rules will depend upon the nature of the technical progress and its impact on the marginal productivity of each factor of production.

Specifically, we see a production function in the form

$$Y_t = F(A_t, K_t, L_t) \tag{2.49}$$

where $A$ measures the level of technology. We see that the marginal products of capital and labor both depend on the value of $A_t$. In the Cobb Douglas Case, the marginal product of labor and capital are given by

$$\frac{\partial Y}{\partial L} = (1-\alpha)A_t K_t^{\alpha} L_t^{-\alpha} \quad and \quad \frac{\partial Y}{\partial K} = \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} \tag{2.50}$$

Equation (2.50) shows that each factor displays diminishing marginal product with respect to increases in its own factor and increasing marginal product as a result of increases in the other input or due to increases in the production technology. We find a similar steady state solution, but now the rate of growth of output depends upon increases in $A$ as well.

The case of growth due to improvements in the quality of inputs is not easily classified. The measurement of our input becomes the key factor in determining if growth of this type results in a shift of the production function or simply movement along it. We may define a new measure of inputs which imposes a certain level of productivity (termed an efficiency unit). (Denison, 1962) Any improvement in the quality of existing inputs would lead to an increase in the number of efficiency units utilized even if the actual quantity of the input did not change. If the input is measured in efficiency units, we simply increase the quantity of input utilized so that growth occurs without a shift of the production relationship.

If, however, we do not consider the improvement in productivity of our input in measurement, then growth will be a result of technical advance with a shift of the production function. We can, therefore, deal with this complex improvement in factor productivity with two methods, with differing interpretation and application. Many empirical models attempt to measure inputs in efficiency units. However, this many result in a loss of information since many assumptions are imposed by such a practice.

## 2.2B Results Of Growth With Progress Of Technology

Modeling economic growth resulting from changes in the productivity of inputs as well as increases in the number of resources is reliant upon the nature of the change in productivity. If the shock which alters the production function affects the productivity of all inputs in the same manner, then the level of economic growth may change, while the growth paths determining this behavior will not. However, when the relationship between the marginal productivity of individual factors is altered, then we will see a change in the pattern determining the rate of growth of output.

The Solow model allows for exogenous progress in the production function through the inclusion of a multiplicative constant which appears in the marginal product functions of the inputs in the same way. An increase in the value of this technological constant will change the level of output produced, but will not alter the relationship between the marginal products of labor and capital and therefore will not alter equilibrium behavior. We will not see a change in the proportion of output attributed to each factor and steady state economic growth in output will now depend upon the rate of growth of inputs as well as the rate of change in technology as captured by this technological function.

An alternative to neutral technological growth involves an increase in the productivity of one input only or a disproportionate increase in the productivity of inputs. In this case, the ratio of the marginal product of the inputs will be altered which will influence producers to reallocate their resource decisions from the relatively less productive input to the relatively more productive factor. Such a change will alter both the level of output as well as the steady state growth patterns.

While the Solow model with a simple exogenous growth constant as shown in Equations (2.49) and (2.50) cannot directly encompass capital or labor saving innovations in technology, a new definition of inputs utilized in production may. By redefining labor and capital in efficiency units with constant marginal products, then increases in the productivity of one factor alone can be captured by an increase in the number of efficiency units available. Through this modeling practice, we may still find a steady state relationship between factors of production and output which remains unchanged, but the number of actual inputs hired will differ.

## 2.3 The Empirical Modeling Of Economic Growth

The majority of empirical growth studies concerning the estimation of growth paths took place either in the 1960's or in the early 1990's. There is continued interest in this rebirth of economic growth relationship estimation. However, we are not seeing a reestimation of the 1960's studies. Instead, we see significant differences in the methods of study in the current period as compared to early works.

In 1961, Nicholas Kaldor noted a series of 5 Stylized Facts regarding the nature of output growth as compared to increases in inputs and other factors. Kaldor noted that these general patterns appear to be robust across nations, although any individual nation may be an exception to the rule. Therefore, application of such general notions must be done only after substantial examination of the true behavior of the nation. While Kaldor's warnings about blindly applying Stylized Facts were lost, the Stylized Facts remain as a basis for the judgment of theoretical models of growth performance.(Solow, 1970) In fact, these warnings of the lack of applicability of usual growth behavior to individual countries leads this thesis off the standard route of examining growth performance through an increased emphasis on growth factors unique to a particular nation.

In the 1960's, major studies of United States economic growth concentrated on the engines which could drive such progress. Edward Denison was a key researcher in this area. In his 1962 text, *Sources of Economic Growth in the United States and the Alternatives Before Us*, Denison looked to a standard input-output growth framework with exogenous technological progress. He assumes a constant returns to scale production technology relating two inputs, labor and capital. In other words, Denison began with the Solow Model.

Results of this simplified analysis indicate that increases in the number of laborers and in the capital stock (as defined by Goldberg's Perpetual Inventory Method) fall short of explaining the true rate of growth of output in the US. Instead of abandoning the standard 2 input production model, Denison redefines inputs in terms of efficiency units. Denison creates an artificial unit of labor which has a constant marginal product throughout the time period. Improvements in the quality of labor, increases in the length of the work week and other changes which increase the productivity of labor are considered increases in the number of efficiency units of labor, even if the number of workers remains unchanged. Throughout his study, Denison discusses factors which may increase the productivity of labor and capital, adjusts his measures of inputs to capture this increase in efficiency units, and reexamines the relationship between inputs and output growth.

In this study, Denison identifies numerous sources for economic growth beyond increases in the number of inputs. However, his method of adjusting inputs to efficiency units is not necessarily a scientific one. For example, due to lower wages, Denison considers women in the work place to equate to the equivalent of one half of a man in efficiency units. Also, teenage workers are equated to one fourth of a man in efficiency units. If wages are a perfect measure of the true marginal product as would be suggested in a perfectly competitive labor market, then we can justify such an adjustment. However, this is quite an assumption especially given the level of discrimination which we would expect as women and youth first enter the workforce.

Denison's text is of great value even today, not necessarily for the numeric results yielded, but for the guidance into sources of growth. The methodology for modeling such complex relationships is much more advanced today. We should be reexamining the Denison work to determine the true impact of forces noted in his numerous "Methods to Increase Economic Growth"[1]. A few of these include:

1. Cut time lost from work due to sickness, accidents, labor-management disputes.
2. Increase the labor participation rate, the length of the work week, and the number of net immigrants.
3. Reduce cyclical and structural unemployment. That is, when worker's skills become obsolete in a given industry, increase the rate of absorption into other industries
4. Increase the time spent in school or increase the current education system's quality
5. Increase the amount of private investment and government owned productive assets.
6. Replace the current tax system with one which is neutral with respect to the allocation of resources among uses
7. Eliminate all misallocation of resources resulting from barriers to international trade.

8. Eliminate all misallocation of resources resulting from private monopolies in output and input markets.
9. Eliminate obstructions of the most productive process caused by labor organizations.
10. Eliminate crime, criminal rehabilitation and racial discrimination in hiring.
11. Reduce the lag of production practice behind the technology leaders
12. Increase the rate of the advancement of knowledge related to production advances

Denison quantifies each of these effects into an expected change in output. We must be careful in accepting these numbers as factual in a particular economy, but they do give an indication of the ability of each to increase output.

Denison's second text examines the differences in United States growth performance as compared to growth in other nations. In his 1969 text, *Why Do Growth Rates Differ?: Post War Experience In Nine Western Countries,* he outlines the movement of the US from a follower of United Kingdom performance to a leader in economic growth rates. This is a similar approach as taken by Angus Maddison in his 1982 text, *Phases of Capitalist Development*. Once again, the time series performance of the United States is examined to identify sources of technological progress capable of explaining the rate of growth of output.

Denison (1972, 1974 and 1985) reexamines economic growth performance in the United States, in Post World War Two Japan and in an international setting. He found that the overall determinants of economic growth have not changed dramatically, while individual performance does vary.

Other significant studies of economic performance in the early period of empirical growth studies can be attributed to Kendrick (1961) and Kuznets (1971) who emphasized regional studies of growth performance within a nation and within industries.

An alternative view of economic growth was stressed by Nelson and Winter in their 1985 text *An Evolutionary Theory of Economic Change*. In their evolutionary approach to economic growth, these authors stress that growth in the current period implies that growth will be more likely in the future due to increased accumulation of physical capital and increased investment in human capital and innovative activities. As more output is available, more may be reinvested in activities which promote growth through improvements in the technology of production (via research and development expenditures) and through improvements in the quality of inputs and through the accumulation of more inputs. Such a model strongly emphasizes the time dependence of growth rates.

However, modern empirical analysis has not taken this approach in general. Instead, we see numerous studies of economic growth utilizing a cross sectional rather than time series approach. The argument is that business cycle behavior interferes so much with the modeling of economic growth, that the only way to understand the true patterns of economic growth is to compare various countries level of economic growth during certain good or bad years.

The issues I examine in this literature differ significantly from Denison's time series questions and from my approach at examining growth. We cannot determine the nature of growth paths by comparing growth rates in a number of countries. Other interesting issues can be addressed including the long lived question of convergence of growth rates. We can compare the behavior of less developed nations to the industrialized grouping, but we can not say anything about the behavior of individual nations.

A key study in this group of papers was conducted by Paul Romer and published in *Modern Business Cycle Theory* (Barro, 1989). In his study of the cross sectional behavior of 7 nations originally studied in Maddison (1982), Romer finds that the Stylized Facts of Nicholas Kaldor are generally acceptable in twentieth century performance. Romer also notes seven new Stylized Facts including the presence of a Solow Residual in most countries. That is, similar to the findings of Denison (1962, 1969), increases in labor and capital are not capable of explaining the rate of growth of output.

The definitive source of economic data for cross sectional studies such as Romer (1986) is found in Summers and Heston (1988). In this paper, the authors develop consistent estimates of output and inputs for a wide variety of industrialized and developing nations. The authors also rank the relative quality and reliability of these data sources. Due to differences in data gathering mechanisms and definitions, such a measure of quality can greatly improve the consistency of the sample of international growth rates.

The issue of defining a sample of nations to include in an international study is not a trivial one. This problem goes far beyond quality and comparability of measures. Also, we must consider that nations with different underlying characteristics such as level and type of resource endowments, political and legal infrastructure and sociological characteristics will behave differently. This heterogeneity in the sample is interesting for the study of convergence of growth patterns, but it leads estimates to be inapplicable to any particular nation. These issues of heterogeneity have not been addressed by the vast literature of cross sectional empirical growth studies.

In addition, various studies of economic growth have included endogenous growth drivers. Many of these were noted by Denison (1962). Such papers include empirical analysis as well, including these new factors. Most papers take a cross-sectional approach to the analysis of economic growth rates including the influence of these endogenous factors.

The first key paper in the endogenous growth literature is Romer (1986) which looks at the effect of technological change in production to develop a model explaining increasing returns to scale. In this model, the accumulations of knowledge leads to improvements inn productivity and sustained growth. This builds upon the "learning by doing" work of Arrow (1962). Becker, Murphy and Tamura (1990) look to the influence of fertility rates on economic growth. Lucas (1988) and Mankiw, Romer and Weil (1990) develop a model of endogenous growth driven by the accumulation of human capital. Through the production of education as well as output goods, we find sustained and predictable growth driven by improvements in the quality of labor. Barro (1974), Barro

(1990) and Aschauer (1988) include government accumulated public capital stocks as an influence on the growth rate of output in a time series framework. Judd (1985), Stokey (1989) and Grossman and Helpman (1991) discuss the impact of innovation and new product development on economic growth rates.

A survey of these cross sectional studies of economic growth with endogenous factors along with numerous other growth examinations is found in Levine and Renault (1991) and (1992). These excellent surveys enable the reader to examine a wide variety of cross sectional studies of economic growth based upon a wide range of potential growth engines. The authors note the common usage of the Summers and Heston (1988) data which is quite popular due in part to a ranking of the reliability of data sources. They find that results of cross sectional analysis are mixed with inconsistencies when new variables are introduced or when the sample of countries surveyed is altered. This is not surprising.

In the following chapter, I will discuss in detail the benefits and limitations of this method of study. While we have developed more sophisticated models of economic performance, our concentration upon cross sectional studies and intercountry effects has prohibited us from truly evaluating the validity of numerous "Stylized Facts" and the applicability of theoretical formulations.

## 2.4 Notes

1 Denison continues to explore the issues surrounding economic growth performance in the United States, Japan and Europe in a series of texts. These include *"How Japan's Economy Grew so Fast : the Sources of Post War Expansion"* (1972), *"Accounting for US Economic Growth : 1929-1969"* (1974), and "*Trends in American Economic Growth : 1929-1982*" (1985).

# *III. Methodology and Data Collection*

## 3.1 Methodology

### 3.1A The Empirical Approach

We have observed in the previous chapter that most modern empirical studies of economic growth have concentrated upon cross country comparison of economic performance. See the discussion of Barro (1989), Romer (1989), and Levine and Renault (1990) for a survey of related studies. While such studies are necessary to test hypotheses such as the convergence of growth rates, I believe we must first understand the nature of economic growth within a country. This approach to empirical testing is very consistent with the theoretical framework which has grown through the Solow Model. The Solow framework involves the temporal dependence of output and inputs. It is this concentration on temporal patterns that easily implies a time series formulation of growth models.

Cross country studies are also subject to many difficulties that panel and time series analysis avoid. Exchange rate variations, inflationary periods which differ between countries, differences in accounting practices and so forth further complicate the already complex issues of growth measurement. Instead of questioning the implications of cross sectional studies, I instead question the assumptions under which these studies are conducted.

Differences in political regimes and their stability and other institutional factors may have a very significant impact on growth behavior in some countries in the sample and, therefore, will detract from the reliability of our estimates. Many potentially interesting countries must be excluded from our sample as they deviate widely from average country performance (the outliers) or have inadequate data. Often we exclude nations with poor quality data. If we do so, we may bias our results toward high income nations. Summers and Heston (1988) rank the quality of data from various nations and, therefore, encourage the exclusion of poor data sources. We should not believe that data collection quality is independent of the income of a nation and therefore excluding nations with low quality data collection forces us to eliminate a particular type of country.

In applying our results, we must be very cautious in taking our results seriously in relation to an individual nation. Due to differences in the underlying characteristics of the countries sampled, we must take great care in applying any general result to a specific country. Countries far from the average growth performance may behave very differently than our empirical model would lead us to believe.

In my approach, I attempt to minimize many of these problems. Time series analysis provides a relatively stable political and institutional regime within which we can answer more specific questions regarding the mechanisms of economic growth. The choice of country to study depends upon our objectives. We can limit our study to countries for which ample and reliable data are available. By holding many environmental factors fixed (or comparatively more so), we can increase the accuracy of our estimates of the effects of changes in select variables which may drive growth. We are simply eliminating some external variation in our model which narrows its focus and applicability.

However, narrowing our focus is not always a blessing. In some cases, we may require cross country comparisons to answer theoretical questions. In such a case, a panel study would be preferable to a simple cross section analysis. In this way, we can utilize information obtained over time on individual countries. This is the approach taken in Chapter 6 of my dissertation.

## 3.1B Developing the Empirical Model

The basic time series model of growth stems from NeoClassical Growth Theory. This is a benchmark model in any study of economic growth with exogenous productivity changes. I will utilize the Solow Growth Model (Solow, 1970) with technological advancement as developed in the previous chapter as the starting point for empirical study.

In this model, labor and technology increase at a constant rate of growth represented in the general form by

$$X_{t+1} = X_t e^{gt} \tag{3.1}$$

where $X$ represents the variable of interest and $g$ is its rate of growth. The non-linear form of this growth pattern would normally prevent us from utilizing the standard methods of optimization including ordinary least squares regression and closed forms of Maximum Likelihood. Due to the non-linearity of the conditional mean, we would not be able to capture the exponential path. However, using a logarithmic transformation of this theoretical model, we can estimate the growth path in a log linear form utilizing standard optimization techniques.

$$LogX_t = a_0 + a_1 t + \varepsilon_t \tag{3.2}$$

This is the starting point of our theory. We will test if this growth pattern accurately reflects the rate of growth of a variable, $X$.

But how do we know if the model accurately reflects the behavior of $X$? This is a very important issue and one of the centerpieces of this dissertation. The statistical properties of residuals, $\varepsilon$, in (3.2) will be examined to determine if the above is a justifiable representation of growth behavior. Such analysis is not only an important way to learn more from a given set of data, but also is essential to ensure that our model is truthfully estimated and contains information about the phenomena we wish to study. This issue will be fully discussed in sections 3.3 and 3.4.

NeoClassical growth theory continues to utilize a capital and labor production relationship even though previous empirical studies have noted significant growth in technology. This would imply that a log linear representation of the growth of output as a function of labor and capital only is not adequate.

The modern theoretical growth literature has identified various sources of this growth in productivity. In fact, Denison (1962), which developed the commonly used techniques of growth accounting, pointed out many of these same factors 20 years in advance of modern application of these ideas. Through the same log-linear production technology with the addition of explanatory variables believed to be related to the productivity of capital and labor inputs, I hope to specify a statistically meaningful growth relationship.

In addition, to capture growth in output due to advances in a coefficient of productivity of these terms, we can utilize a trend term. As was discussed in Chapter 2, this will capture exogenous technological progress through an exogenous term $A$.

$$A_t = A_0 e^{gt} \qquad or \qquad LnA_t = LnA_0 + gt \tag{3.3}$$

In this way, we can estimate also information related to changing productivity.

Therefore, we can include a multiplicative constant $A$ together with a basic constant returns to scale production technology in order to account for technological progress. We would find

$$Y_t = F(A_t, K_t, L_t) \tag{3.4}$$

or more specifically,

$$Y_t = A_t K_t^{\alpha} L_t^{\beta} \tag{3.5}$$

As $A$ increases, the marginal products of both capital and labor increase. Since the technological advance affects both factors in the same way, the advancement is considered to be Hicksian neutral or disembodied (Jones, 1975).

The addition of a term which increases with increases in technology in a log-linear production relationship would then result in an empirical model of the form

$$LnY_t = c + \alpha_1 t + \alpha_2 LnK_t + \alpha_3 LnL_t + \varepsilon_t \tag{3.6}$$

where $\varepsilon$ represents a stochastic variation in observations of output which cannot be accounted for in our model. In the special case that our production relationship between capital and labor is Cobb Douglas, $\alpha 3=1-\alpha 2$. This empirical model given in Equation (3.6) is fully explored in Chapter 4 of this study.

In the case that other factors may account for significant growth in output, we find the inclusion of the log of these additional variables in our log linear growth equation.

$$LnY_t = c + \alpha_1 t + \alpha_2 LnK_t + \alpha_3 LnL_t + \alpha_4 LnX_t + v_t \tag{3.7}$$

Once again, a Cobb Douglas production relationship, now between capital, labor and the unspecified additional factor, would imply that $\alpha2+\alpha3+\alpha4=1$. The empirical model in equation (3.7) is the focus of Chapter 5 of this dissertation.

Also included in this study will be reference to Nelson and Winter's (1982) evolutionary model of economic growth and the influence of dynamic effects on the rate of output growth. Strong autocorrelation in real gross national product has long been realized, but the application of temporal dependence to rates of economic progress is also important. A dynamic pattern of growth will also be explored through

$$LnY_t = c + \alpha_1 t + \alpha_2 LnK_t + \alpha_3 LnL_t + \sum_{i=1}^{m} \beta_i LnY_{t-m} + \lambda_i LnK_{t-m} + \gamma_i LnL_{t-m} \tag{3.8}$$

The selection of a precise empirical model will be directed by the characteristics of the data utilized based on misspecification testing. This includes selection of an appropriate ***m*** in the Dynamic Linear Regression Model given in equation (3.8) above.

## 3.1C The Specification Issue : Restrictions in the Haavelmo Framework

While we have developed an empirical model based on the Solow Model formulation, we are far from finished when it comes to determining the relationship between the rate of growth of inputs and output performance. We have utilized economic theory to determine our data of interest and give us a general method of modeling growth relationships. Now we turn to the data for more information which will tell us how to find a model that represents their various statistical properties.

For our approach to specification, we look primarily to the methodology provided by Aris Spanos in his 1986 text, his 1994 text and a strong contingent of papers (Spanos, 1987, 1989a, 1989b). Through these studies, Spanos has clearly outlined a versatile approach to take us from all possible data points contained in our data set, to a representative model with fewer parameters to estimate. Spanos (1989b) and Spanos

(1986) are recommended for descriptions of this method. Our method here also relies heavily upon the Haavelmo reduction framework and it forms the cornerstone of empirical model choices and misspecification testing alternatives.

So far we have specified a theoretical model. That is very different from building a statistical model which accurately captures the characteristics of our inputs and output. We have only just begun to address the issue of building a statistical, or empirical, model. Economic theory indicates which questions to examine and may indicate which variables are important in this analysis. The basic functional form may also be suggested by our theoretical beliefs. From here, however, we must not allow economic theory to blind us to the many choices necessary to build an empirical model which is statistically adequate.

What do we mean by statistically adequate? The model must satisfy the assumptions implied by the model we have selected. Since we will be utilizing the Multivariate Linear Regression Model and Multivariate Dynamic Linear Regression Model, we will concentrate upon the assumptions for Linear Regression.

The characteristics of the Linear Regression Model are best summarized by Spanos (1986). The specific assumptions for a linear regression model are given by *1-8* below. Without adherence to these assumptions, we cannot believe that the estimates obtained utilizing this model are adequately describing the process which generates our economic data. For a detailed description of each assumption see Spanos, 1986.

We begin with all the data available to us which includes

$$D(Z_1, Z_2, \ldots, Z_T : \theta_1, \theta_2, \ldots \theta_T) = D(Y_1, Y_2, \ldots, Y_T, X_1, X_2, \ldots X_T : \theta_1, \theta_2, \ldots \theta_T) \qquad (3.9)$$

where $Y$ and $X$ are observations of random variables ($X$ may be a matrix of explanatory variables in the multivariate case) and $\theta$ is a vector of the parameters characterizing the particular density function within the $D$ distribution family. $Z$ is a matrix containing all the observations of $Y$ and $X$ and is referred to as the Haavelmo distribution.

Our entire data set ($Z$) displays distinct statistical characteristics. These characteristics can be thought of in three categories : *Distributional, Heterogeneity and*

*Memory*. Through a particular combination of assumptions about these characteristics, we form a particular empirical model. Departures of these assumptions from the statistical behavior of *Z* indicate statistical inadequacy in our empirical model.

(I) **Statistical GM,** $y_t = \beta' x_t + u_t, \quad t \in \mathrm{T}$

*[1]* $\mu_t = E(y_t / X_t = x_t)$ *– is the systematic component;*

$u_t = y_t - E(y_t / X_t = x_t)$*- is the non-systematic component.*

*[2]* $\equiv (\beta, \sigma^2), \quad \beta = \Sigma_{22}^{-1}\sigma_{21}, \quad \sigma^2 = \sigma_{11} - \sigma_{12}\Sigma_{22}^{-1}\sigma_{21}$, *are the statistical parameters of interest when*

$\Sigma_{22} = Cov(X_t), \quad \sigma_{21} = Cov(X_t, y_t), \quad \sigma_{11} = Var(y_t)$.)

*[3]* $X_t$ *is weakly exogenous with respect to* $\theta$, $t = 1, 2, ..., T$.

*[4]* *No a priori information on* $\theta$.

*[5]* $Rank(X) = k$, $X = (x_1, x_2, ..., x_t)'$; *T x k data matrix,* $(T>k)$

(II) **Probability model**

$$\Phi = \left\{ D(y_t / X_t; \theta) = \frac{1}{\sigma\sqrt{(2\pi)}} \exp\left[ \frac{1}{2\sigma^2}(y_t - \beta' x_t)^2 \right], \theta \equiv (\beta, \sigma^2) \in \Re^2 \times \Re \right.$$

*[6](i)* $D(y_t / X_t; \theta)$ *is normal;*

*(ii)* $E(y_t / X_t = x_t) = \beta' x_t$ *- linear in* $x_t$

*(iii)* $Var(y_t / X_t = x_t) = \sigma^2$ *- homoskedastic (free of* $x_t$*);*

*[7]* $\theta$ *is time invariant.*

(III) **Sampling model**

*[8]* $y \equiv (y_1, \ldots, y_T)'$ *represents an independent sample sequentially drawn from* $D(y_t / X_t; \theta)$, $t = 1, 2, ..., T$.

*(Spanos, 1986)*

There are many combinations of assumptions which lead us to an empirical model. These assumptions are often closely related and misspecification of one assumption may adversely affect other assumptions. All three assumptions should be viewed as a whole with any false assumption grounds for invalidating an empirical model.

Distribution assumptions specify the $D$ in (3.9). We see that in the Linear Regression Model (Simple, Multivariate and Dynamic), the assumption of conditional normality is required. This enables us to limit our focus to the first two moments. For a Normal distribution, higher odd moments are zero and higher even moments are proportional to the variance. This distribution assumption also inherently implies a particular functional form for the conditional mean (or regression) equation and conditional variance (or skedastic) function. For Normal distributions, we observe a conditional mean which is linear in $X$ and a conditional variance which is free of $X$ or homoskedastic. These specific assumptions are noted explicitly by Spanos (1986) in assumption *6*. These requirements are also key in the definition of the Statistical Model (such as assumptions *1*, *2* and *3*).

In terms of our Haavelmo decomposition problem, the assumption of a Normal distribution allows us to say that our data set is :

$$N(Z_1, Z_2, \dots, Z_T : \theta_1, \theta_2, \dots \theta_T) \tag{3.10}$$

In addition, we recognize above that we have $T$ different observations of each variable given in the data set. While we generally like large volumes of data (long or frequently sampled time series) such temporal dimensions may cause us difficulties. For our estimates to have meaning, the fundamental relationships between our variables must not be changing over time. That is, we must enjoy some form of homogeneity. In general, we speak of the stationarity of our series. The principle of weak stationarity (the constancy of the first two moments) may be sufficient if we have already assumed a Normal distribution assumption. In order to learn from multiple observations of $Y$ and $X$ in our data set, the relationships governing $Y$ and $X$ must not change. In the case of our Linear Regression Models, homogeneity is assured by our requirement that $\theta$ is time invariant (assumption *7*). This changes our representation of our data set to

$$N(Z_1, Z_2, \dots, Z_T : \theta) \tag{3.11}$$

Our final category of assumptions relates to the Memory of our data series. Again, to estimate relationships from our data series, we must have enough information. This requires us to limit the number of relationships in (3.11) which we may possibly estimate with $T$ observations. To do so, we impose some sort of restriction upon the Memory or temporal dependence of our data. In the Linear Regression model, we impose that our variable $Y$ is independent of the past values of $Y$ and $X$. In the Dynamic Linear Regression Model, we relax this assumption somewhat by imposing independence past a certain lag length. This is usually in the form of an asymptotic independence assumption. Once again, such restrictions on the memory of the process are necessary to represent the characteristics of our data set by an estimated empirical model.

In the case of the Simple Linear Regression Model, we see that Assumption *8* will ensure that $Y_t$ is independent of its own past and the past values of $X$. Such an assumption allows us to decompose the distribution of data given in (3.11) as the product of $T$ independent density functions. We therefore see the total Haavelmo Decomposition in the case of the Linear Regression Model to be :

$$N(Z_1, Z_2, \ldots Z_T : \theta_1, \theta_2, \ldots \theta_T) = \prod_{t=1}^{T} N(Z_t : \theta_t) = \prod_{t=1}^{T} N(Z_t : \theta) = \prod_{t=1}^{T} N(Y_t | X_t : \varphi_1) * N(X_t : \varphi_2) \tag{3.12}$$

where the first expression of (3.12) notes the normal joint distribution of Y and X (the distribution assumption), the second term indicates independence (the memory assumption) and the third term shows the homogeneity through a constant $\theta$. The final expression results from the decomposition of any joint distribution into a conditional and marginal distribution. (See Spanos, 1986). A similar decomposition is possible to account for alternative distribution, memory and homogeneity assumptions.

## 3.1D Misspecification Testing - Omitted Variables to Specific Tests

We often speak in econometrics of "Omitted Variables". It is often recognized that the failure to include an important variable in analysis may lead to biased and inefficient estimates of the actual relationship between variables (See Judge et all, 1988). The Haavelmo framework is a methodology consistent with the theory of omitted variables. Through this method, we impose restrictions on the distribution, memory and homogeneity of a *Z* through the omission of particular variables. For example, in order to utilize the Simple Linear Regression Model, we omit the past values of *Y* and *X*. This is a conscious choice based on the characteristics of the Joint Distribution of our entire data set (the Haavelmo distribution as we have referred to it in the previous section). Our assumption of a normal distribution implies the omission of non-linear terms from the conditional mean or regression equation.

Misspecification testing of our empirical model will play a vital part is assessing how valid these assumptions may be. We will examine the impact of omitting particular pieces of data. In this way, our misspecification tests in general are significance tests when we decompose the Haavelmo distribution in alternative ways. We determine if a restriction imposed during our decomposition of the Haavelmo distribution was justified based on the observations of *Z* we have available. Through misspecification testing, our data will lead us to the most accurate representation of the relationship between our variables of interest.

If we obtain a statistical model which meets all of the above assumptions, only then can we look to our results for information relating to the actual relationship between variables. Once the statistical properties of our empirical model have been verified, we can proceed to interpret the model in a theoretically meaningful way.

Often we feel frustrated by the presence of complexities in our economic data. We desperately search for a way to "clean-up" our data. However, by disregarding misspecification errors, we also disregard information inherent in the Haavelmo distribution. Through misspecification tests, we learn more about the nature of the relationship underlying a variable's behavior. We can learn much through this careful examination of data. So not only is avoiding the statistical properties of data wrong in the sense that our estimates may not be accurate, but we also loose information which is able to improve our understanding of the underlying characteristics of our economy.

Specific misspecification tests which will be used throughout this dissertation can be thought of as significance tests when we include more information from $Z$ in estimation. These tests have been generated using the computer program SAM (Statistically Adequate Model) written by Dr. John Robertson, Dr. Aris Spanos, and Dr. Anya McGuirk. This extensive menu driven program for the statistical package GAUSS (Aptech Industries) allows the user to manipulate and graphically view data, estimate single and simultaneous equation models and then test restrictions which have been imposed in the selection of each particular model. These tests are also very well documented and related to the Haavelmo methodology in the Ph.D. dissertation of Dr. John Robertson completed at Virginia Polytechnic Institute and State University ("Misspecification Testing in Systems of Equations",1992) and in Spanos, 1986.

The Test for Conditional Mean Linearity

For tests of Linearity (a requirement of a conditional normal distribution), a RESET test was employed. Another commonly utilized test for the presence of non-linearity is the Kolmogorov-Gabor (KG) Polynomial Test. In both cases, the null hypothesis of a linear conditional mean is compared to an alternative hypothesis which allows non-linear terms to appear, such as the squares and higher orders of the regressors and interaction terms between variables. For example, lets consider a Linear Regression model expressing $Y_t$ as a function of 2 variables $X_{1t}$ and $X_{2t}$. A RESET test of order 2 in this case would utilize the auxiliary regression:

$$H_0: E(Y_t | X_{1t}, X_{2t}) = \beta_0 + \beta_1 X_{1t} + \beta_2 X_{2t} \;\; vs. \;\; H_1: \beta_0 + \beta_1 X_{1t} + \beta_2 X_{2t} + \beta_3 X_{1t}^2 + \beta_4 X_{2t}^2 + \beta_5 X_{1t} X_{2t} \tag{3.13}$$

and then test the joint significance of $\beta 3$, $\beta 4$ and $\beta 5$. The KG Polynomial test allows for greater interaction between terms through an alternative formulation of $H_1$.

The Test for Conditional Variance Homoskedasticity

Tests for conditional variance heteroskedasticity take a very similar form. In fact, we can look to a RESET Test for Heteroskedasticity in much the same way. Here, however, even linear dependence on $X$ will invalidate our assumption of homoskedasticity. In this case we test :

$$H_0 : \sigma^2 = \beta_0 \quad vs. \quad H_1 : \beta_0 + \beta_1 X_{1t} + \beta_2 X_{2t} \tag{3.14}$$

where we may also include higher order terms of $X$. We test the significance of the coefficients on $X$ terms in (3.14) to determine if the conditional variance is really homoskedastic. This is accomplished through a sample estimate of the conditional variance. A particularly important Heteroskedasticity test is the White Test which includes higher order terms in the alternative hypothesis. (See Spanos, 1986)

The Test for Independence

We test for independence using this same methodology. Through the inclusion of lagged values of both dependent and independent variables, we can determine if our assumption of independence (or asymptotic independence of some order) is adequate. The specific independence test employed is the LM error autocorrelation test. This examines the properties of the term

$$e_t = Y_t - \hat{\beta} X_t \tag{3.15}$$

and then tests the restriction that ρ=0 in the auxiliary regression

$$\hat{e}_t = \sum_{i=1}^{p} \rho_i ' \hat{e}_{t-i} + \upsilon_t \tag{3.16}$$

Many texts (Judge, et al, 1988, Johnston, 1984) view misspecification in the characteristics of the residuals. If the Linear Regression Model is appropriate for our data series, then the residual series will be Normal, Independent and Identically Distributed.

$$
\begin{aligned}
&e_t \sim Normal \\
&E(e_t) = 0 \\
&e_t e_s = 0 \ \ if \ \ t \neq s \\
&e_t e_s = \sigma^2 \ \ if \ \ t = s
\end{aligned}
\tag{3.17}
$$

The auxiliary regression given in (3.16) is a test of the restrictions implied by (3.17).

The Test for Normality of the Conditional Distribution

Our test of conditional normality is of a somewhat different form. We assume that the distribution is a normal one. In our test, we wish to see if this is a close approximation to our observations. To do so, we look at the higher moments of the conditional distribution as represented by the residuals of the empirical model. If these higher moments are close enough to those of a Normal distribution (Skewness = 0, Kurtosis = 3), then we cannot reject the distribution assumption. The specific form of the test is due to D'Agostino and Stephens (1986) and D'Agostino, Belanger and D'Agostino (1990). This test uses a normalizing transformation of the raw skewness and kurtosis coefficients :

$$
Skewness\colon \ \alpha_3 = \frac{\mu_3}{\sigma_3}; \quad Kurtosis\colon \ \alpha_4 = \frac{\mu_4}{\sigma_4} \tag{3.18}
$$

$$
\text{where } \ \mu_r = E[e^r] \tag{3.19}
$$

Normalizing $\alpha 3$ and $\alpha_4$ allows us to find a test statistic with a Standard Normal Distribution. Then, we compare the skewness and kurtosis statistics from the residual series to the skewness and kurtosis coefficients of a Normal distribution.

Test for Parameter Time Invariance

Finally, we must also discuss a test for the parameter stability requirement stated in Assumption 7. This is conducted utilizing both numerical tests and graphical examination. Through Recursive Least Squares, we see how the parameter estimates change as sample length increases from some arbitrary small level to full sample size. If the parameters are stationary, then we should see the parameter estimates leveling off and the variance of these estimates closing as sample length increases. Through Window Least Squares, we calculate parameter estimates for a series of "windows" of observations of equal size. This sample size remains constant and the window is slid over our entire data set. These parameter estimates and the variance of these estimates should remain constant if stationarity exists. Tests of stationarity will be discussed in Chapter 7.

In all cases, we are examining the difference between observed variable behavior and the assumed behavior. If there is a significant difference (which depends upon the confidence bounds specified), then the assumption is rejected and an alternative empirical model based upon a different decomposition of the Haavelmo distribution is required.

## 3. 2 Data Set Information

In this study, I wish to examine the relationships between output, labor and capital in a basically unadjusted form. The presence of stability in the linear growth patterns of these quantities will indicate if constant returns to scale is an valid assumption and if the productivity of our factors of production have changed over the sample

Upon determination of the growth pattern of output and inputs, we will be better able to make predictions regarding our rate of technological advance. In addition, we can evaluate the resemblance of our economy to a pattern of steady state growth that has been predicted by NeoClassical Growth Theory. In the following section, I will highlight the variables of interest in our study and their importance. The sources of data and issues regarding each series will also be pointed out.

### 3.2A. Labor Force Measures

*Population* - The residential population of the United States is necessary to calculate per capita welfare measures. It is assumed by growth theory that the labor force grows at a rate equal to the rate of increase in the population and, therefore, population can be disregarded in per capita analysis by substituting labor measures. If, however, there is a difference in the growth of labor and the growth of population, this simplification is no longer accurate. We would expect there to be a significant difference in these growth rates due to the entrance of an increasing number of women in the United States work force during our time frame. In addition, Romer (1989) has shown that population growth is negatively correlated with national output growth. In other words, as national output growth accelerates, the population growth rate may fall. Population growth has been assumed to be exogenous in growth models. A relationship between output and population will invalidate such an assumption.

Data: Actual population statistics including armed forces overseas are available for the period 1900-1970 in *Historical Statistics of the United States : Colonial Times to 1970* and, for 1971-1989, in *The Economic Report to the President, 1992*. Numbers referred to in this study will be in millions. Also of interest is the logarithm of Population, which will be constantly trending (have a linear trend) if population follows a steady exponential growth pattern.

*Labor Force* : Labor force includes all persons employed or unemployed and in search of employment. Armed forces are also counted in this measure. The inclusion of Armed Forces in the labor force does not present a problem of interpretation since we can assume if a person were not in the military, they would be employed or in search of employment in the private sector. In terms of input and output relationships, the inclusion of Armed Forces personnel is not as clear. The question relies upon the belief that military personnel contribute to national output in the same way as civilian laborers. In the early years of the sample, labor force statistics were based upon the number of individuals in the country of age 16 (or 14) or greater. It is assumed most of these people would be employed or in search of work. It is the typical case for these individuals to work rather than continue education as is the more recent years. As sophisticated surveys were not employed prior to 1929, such approximations are necessary.

Data: The quantity of persons in the labor force (in thousands) can be found in *Historical Statistics of the US* and the *Statistical Abstract of the US*. Information gathered by the Bureau of Labor Statistics has also been reported in *The Economic Report to the President*. In order to complete the series on labor force, some derivation was necessary for the years 1971-1974. Specifically, the *Economic Report to the President* provided both the percentage of persons unemployed and the number of persons unemployed, from which it was possible to find the number of persons in the labor force.

Why examine labor force? The labor force is of interest to us through our input output relationship. We wish to determine productivity trends of output per unit of labor. We wish to examine the rate of increase in the size of the labor force. Growth theory assumes a constant proportional rate of growth, $n$, in the labor force. This rate of growth is assumed to remain constant over long periods of time. We also should compare this rate of growth to the rate of growth of the total population in assessing the reliability of per capita variables in predicting the welfare of society.

Also of importance, however, is the rate of employment of labor resources. We are not interested in simply the number of persons who would have worked at full employment, we wish to determine the true number of labor inputs utilized to produced the actual quantity of output in GNP. This gives us a more accurate measure of productivity. This issue will be explored more in the following labor series.

*Unemployment and Employment Rates* - In order to determine the quantity of labor inputs actually utilized in production, it is necessary to calculate the number of members of the labor force who have obtained work. Average annual unemployment (and, therefore, employment) rates allow us to calculate on average the number of persons working in the economy for a given year. Note that this is a yearly average and, therefore, is subject to a loss of information compared to monthly or quarterly series. On the whole, however, such a figure of persons actually employed improves greatly upon our measure of the number of persons in the labor force.

Data: Yearly average unemployment rates are found in *Historical Statistics of the US* and in *The Economic Report to the President, 1992*.

*Average Weekly Hours Worked* - This series will give us an even more accurate account of the utilization of labor input. The number of hours worked per week has declined, on average, since the beginning of the century (*Historical Statistics of the US*). That is, on average current laborers spend less time per week in actual production of goods and services. Therefore, we must consider the length of the work week in assessing the quantity of labor supplied to the production of goods and services. Average hours per week is utilized to determine the quantity of labor input in total hours worked per week, rather than in individuals. This series is calculated as follows:

# Hours Labor per Week in 19XX =Aver Wkly Hours in 19XX * # Employed in 19XX

Data: Average weekly hours statistics are found in the *Historical Statistics of the US* and *Economic Statistics, 1900-1983.*

*Manhours Per Year* - This is our final measure of labor input in the US in a given year. While some economists also adjust for the quality of labor by including measures of education or experience, we will not employ these techniques. Such improvements actually shift the production function and, therefore, can be considered improvements in technology.

Our main focus will be upon the growth rate of labor inputs in production after adjusting for changing demand conditions (via the employment rate) and changes in the length of the work week. Growth theory assumes that the labor force will grow at a constant proportional rate of growth.

Data: Manhours per year is a derived quantity based upon the previous data. This quantity will show the thousands of hours worked by all employed in the US in a given year.

MANHOURS in 19XX = #Employed in 19XX * Hrs Per Week in 19XX * 52

## 3.2B. Output Measures

*National Output* - Gross national product will give us a measure of the quantity of output produced in the US in a given year. According to growth theory, the rate of growth of output is relatively constant and, in a steady state, equal to the rate of growth of labor input and/or technology.

Some studies have pointed out that government production differs significantly from private production, and therefore the most accurate measure of output is gross private output. Since the input measures are not adjusted for the inputs utilized in government production, such a measure will understate our productivity and, therefore, is not optimal for our purposes.

We will standardize our prices to 1982 dollars so that output will be comparable across the sample. Current values of GNP will be deflated (or inflated) to 1982 figures using the GNP deflator. Alternative price indexes are available. These indexes, however, are useful for more limited groups of products.

Various options are available for the series on national output. Specifically, we must decide how to handle depreciation or the capital consumption allowance. While the inclusion of depreciation will account partially for the difference in more modern goods, the questionable measurement of capital consumption and the lack of reliable data lead economists to utilize measures of gross output. Another theoretical issue lies in the relative improvement of goods which is not accounted for by nominal or real measures. For example, calculators today are not simply improved calculators of the past, they are also cheaper. Such development would not be displayed by our measure of aggregate output. Many studies of cross country growth utilize the more widely available measure of Gross Domestic Product. However, data for GDP have only recently gained preference over GNP and, therefore, are not available for time series analysis.

Data: Gross output data is readily accessible for the entire period, 1900-1990 from sources such as *The Economic Report to the President* and *Historical Statistics of the US*. *The Economic Report to the President*, in addition, provided GNP in both current and 1982 dollars. This was necessary for the derivation of price indexes for periods not covered by *Historical Statistics of the US*.

*Exports* - Increases in the volume of trade may increase the growth rate of output. This hypothesis is intuitively appealing as high demand outside of the country will stimulate production internally. Also, the presence of substantial imported substitute goods indicates that competitive pressures should spur innovation toward production methods which are more efficient. Such innovation may also stimulate growth in the domestic economy. To measure the volume of trade, we may concentrate on total trade, the trade balance or measures of exports and imports separately.

Volume of trade has been very volatile over the entire sample due to changing foreign policies in the United States and due to changes in the policies of our trade partners. The accuracy of conclusions regarding trade is questionable for the purposes of forecasting due to this high volatility. However, an established relationship between quantity traded and the growth in national output would be in support of previous findings.

Data: Data on volume of trade is available in *Historical Statistics of the US* and in *The Economic Report to the President, 1992*. Export and Import volume must be adjusted via the implicit price deflator to remove the effects of price changes. Recent productive methods which combine the intermediate products of various countries make the distinction between imports and exports much more difficult, but we will consistently use Department of Commerce methods and rely upon their consistency.

*Price*: The aggregate price level in the economy is related to a base year through the use of price indexes. Specifically, the implicit GNP deflator will compare the price of a bundle of specified goods over time in order to obtain a generalized indicator of inflation or deflation. It may be possible that a period of high or low inflation will stimulate competitive pressures toward more or less productive techniques.

The choice of a measure of price must include products most similar to our measure of output. For that reason, the implicit GNP deflator was selected. Other indexes are more useful for more specific studies.

Data: The Implicit Price Deflator data was available in *The Economic Report to the President, 1992*. Omitted observations were calculated based upon GNP in current and 1982 dollars. The base year for this index is assumed to be 1982.

## 3.2C. Capital Measures

*Capital Inputs*: Probably the most controversial variable we attempt to measure, capital goods have created difficulties throughout the study of productivity. Similar to labor, capital goods are heterogeneous. However, due to data limitability, we combine all capital goods to find some average productive unit.

Unlike labor, however, capital goods are not used up as soon as we purchase them. Machinery will gradually age and become obsolete. Capital consumption is a very important issue in the measurement of this input. Net measures of capital goods take into consideration the age of the good and its workable life. This is concurrent with the view of the vintage capital model (see Jones, 1975 for an excellent summary). However, the gross measure of capital assumes that firms will maintain machinery in its original state through regular upkeep. In this view, capital should be counted at its full value. Due to the poor sources of depreciation data, gross measures are often employed. Sometimes these are supplemented with calculations constructed by economists and other researchers. Differences in the depreciation practices of agencies yield many capital series incomparable. This problem is amplified as most do not specify their means of calculation. For this reason, I will employ a measure of capital as Gross Fixed Capital.

Also, we should try to concentrate our measure on capital actually used in production of other goods. This means we should attempt to exclude capital purchases not used in production. The largest category of this would be residential capital, or homes. We exclude home sales through the use of Gross Fixed Non-Residential Capital.

Potential idle capacity in capital also raises doubts as to the accuracy of our capital measure. While we may deduct unemployed labor from our measure of the labor force, such information is not available for capital, with the exception of capacity utilization rates now available from 1948. Also, the length of the work week for capital is usually not considered. Foss (1981) is an exception. Foss develops estimates of the hours of employment of machines for the same short sample period.

Data: The Fixed Non-Residential Capital series was found in *Historical Statistics of the US* and *The Statistical Abstract of the US.* A related series on Gross Investment was found in *Economic Statistics, 1900-1983.* Fixed non-residential capital measures the stock of structures and equipment, including inventories. Fixed capital is based upon Goldberg's perpetual inventory method where total investment is accumulated less a measure of obsolesce (see *Historical Statistics of the US* for an excellent discussion). This seems to be a standard method utilized by economists in dealing with capital ( Aschauer, 1987).

### 3.2D. Measures of Productivity Change

*Output per Manhour:* This series measures the productivity of our labor force. Given the assumptions and implications of NeoClassical growth theory, we would anticipate a relatively constant positive rate of growth of this quantity. This follows from the assertion that output grows at a constant rate greater than the exogenous rate of growth of labor when technological progress occurs.

*Output per Fixed Capital Unit:* This measure of the output capital relationship should confirm the theoretical prediction that output and capital grow at approximately the same rate, even in the presence of technological change. If this NeoClassical theory is shown to be true, we should find no systematic trend in this variable.

*Capital to Labor Ratio :* If indeed capital is growing at a near constant rate greater than the rate of growth of labor input, then we should see a positive trend in this ratio.

## 3.2E. Sources Of Technological Progress Measures

*Government Capital Provision :* Just as private capital is an input to the production of goods and services, we may consider government capital as an input. If we consider only the simple output capital labor relationship, then increases in government capital may appear as increases in the productivity of private capital and labor and, therefore, as sources of technological progress. Inclusion of publicly provided capital should reduce the effect of technological change if government capital is productive.

We wish to see if government capital accumulation is sufficient to model technological progress. Specifically, if the addition of government capital stock in the usual constant returns to scale production technology will result in a statistically well specified empirical model.

Data: Complete data for government capital stock have only recently been derived. Various agencies have utilized the Goldberg Perpetual Inventory system to derive measures of government capital which are comparable to those of private capital. The BEA have compiled data on government capital. These values along with data derived by the authors is available for the sample 1947-1987 in Boskin, Robinson and Roberts (1989).

*Patents as a Proxy for Innovation:* In an attempt to proxy for the level of innovative developments, we may choose the number of patents applied for or granted. The choice between these two measures is somewhat objective. Patents applied for is the more stable of the measures and may be less influenced by governmental regime or changes in standards. However, this measure may include duplicate or fictitious innovations.

As with government capital, we wish to include the number of patents in the standard growth model to test if patents can account for technological improvement. If the number of patents can explain technological change, we may find the extended empirical model is well specified.

Data: The US Patent Office supplies information for these two series from 1870 to present. These patents and applications include new products, designs, plants and pharmaceuticals. All innovations may have a significant impact upon productivity to some extent. Governmental innovations are included to some degree, however, I am uncertain of the role of defense developments. Publications of the Patent office listed as references provided these series and descriptions.

*Education:* The role of human capital formulation has been very important in theories of endogenous growth. The inclusion of some measure of the quality of labor has been utilized since the 1960's (See Denison, 1962). Which measure of education and experience to employ is much more difficult. Alternatives include the literacy rate, expenditures on public or private education and the percentage of the school age populations in school. The literacy rate in the US has varied so little over the entire sample period that this measure seems inappropriate for capturing improvements in labor quality. The unavailability of expenditure information, as well as lack of correlation between expenditure and quality, make this a questionable measure as well. Changing social institutions now allow education long after a typical school age. The return of adults to education and the provision of training and further education for adults by employers makes the school age population a very ambiguous figure in the US.

Instead, I have determined the number of post-secondary degrees granted per year and the proportion of the population earning such degrees to be a more suitable measure of labor quality in the United States. Variability of this measure is sufficient and educational differences in the population are commonly attributed to collegiate education since secondary education is nearly universal in the US. Such is not the case in many countries so these other measures may be appropriate for cross sectional studies.

We may use education as an input in the production of national output by inclusion of this term in the aggregate production relationship. If education is sufficient to account for technological progress, then the systematic portion of the Solow Residual should vanish or be significantly reduced once education is included as a dependent variable.

Data: The number of degrees (subdivided into First, Master and Doctorate) are available for the years 1925-1990 with several missing observations. The compact series is 1950-1987. This series is available in *Historical Statistics to the US* and the *Statistical Abstract of the US* and in *Economic Statistics, 1900-1983.*

## 3.2F Quarterly Data Set Sources

Standard Department of Commerce statistics relating to many of these data series are also available on a quarterly and monthly basis. Quarterly series on GNP and its components including Gross Fixed Non-Residential Capital, Exports and Prices are available as well as monthly series relating to Labor force, Employment rate and Average Earnings. While our output and capital series are defined in the same way for the quarterly sample as was utilized for the annual, the labor variable now becomes labor force as opposed to Manhours. The limited availability of Average Hours Worked prohibits us from using a Manhours series.

Data : In order to assess the stationarity of our previous estimates, these series were obtained through the University of Maryland's databank of educational resources. *ECONDATA* is available for anonymous FTP over the Internet. This is just one of many organizations offering this data on-line. Internet data sources will be discussed in more detail later in this chapter.

## 3.2G. International Data Set Sources

Data for the International comparisons conducted in Chapter 6 was located in *Economic Statistics, 1900-1983*. Also, the Japanese Data series was supplemented by information found in Denison (1976). Some international data is available on-line at the University of Maryland in *ECONDATA*. This is the commonly utilized data of Summers and Heston (1988). Input Output data for other nations may also be found on-line.

## 3.2H. Internet Sources of Economic Data

As you can see above, some data for this study were obtained from the Internet. This is a vital new source becoming available to most economists as governmental, research and educational institutions are eagerly entering the cyberspace era. As the

Internet increases in size and complexity, it is increasingly important to keep track of the resources available to us.

The definitive guide for economics data sources as well as information on economics related mailing lists and collections was composed by Bill Gaffe of the University of Southern Mississippi. This document is updated every 2 months and is a collection of submissions of other economists. Included in this document entitled *Resources for Economists on the Internet* (http: //gopher.econ.lsa.umich.edu/ ResourceHTML /FAQ_7_11.html) is a directory of economics related data available on line as well as a listing of government documents available for download, listings of economists which are maintained at various organizations and various Economics related Gopher Servers with an abundance of information. Some other very useful Internet sites for Economics related information include the Census Bureau HomePage (http://www/census.gov) and the Economics Gopher at Sam Houston State University (gopher://Niord.SHSU.edu:70/11gopher_root:[_data.economics]).

# *IV. Respecification of the Solow Model of Exogenous Growth*

In 1961, Nicholas Kaldor noted some commonly observed traits of input and output growth which should be considered in theoretical growth models. While any theory, by definition, abstracts from economic reality to create a solvable problem, a grounding in the behavior of the economy studied is necessary if we are to apply our model in matters of policy, forecasting, and even in understanding economic phenomena.

For this reason, Kaldor (1961) highlights certain important characteristics of the macroeconomic aggregates. However, the author very carefully cautions the reader as to the generality of such statements. Says Kaldor, "[the theorist should] construct a hypothesis that could account for these "stylized" facts, without necessarily committing himself on the historical accuracy or sufficiency of the facts."

What were these obvious facts that serve as a benchmark for economic theory?

(1) output and labor grow at a near constant and equal rate so that the productivity of labor, (Y/L), does not fall.
(2) the ratio of output to utilized capital remains constant implying that output (or income) and capital grow at the same rate. (Y/K) is a constant.
(3) there is a steady increase in the capital per worker, (K/L).
(4) capital earns a steady rate of profit.
(5) there is a high correlation between the share of profits to capital and the rate of investment.
(6) there is a large difference in the rate of growth of labor productivity and in output in various societies.

In recent history, Kaldor's "Stylized Facts" have withstood an informal test of accuracy and are now commonly accepted as truths (Romer, 1989). Even in this day of advancing statistical techniques and the recognition of the importance of accurate model specification, we tend not to criticize these overall facts. We instead build models with similar growth pattern assumptions and are critical of differences as compared to the "facts". Often we assume our theoretical constructs are incorrect as opposed to questioning the validity of these observations. However, it simply may be the fact that stylized facts are not displayed by our sample.

The present day theorist should also consider the change in the economic growth behavior of many industrialized countries. In the early days of economic growth theory, England surpassed the world in output per worker (Maddison, 1982). In the 1960's, the period in which Kaldor's Facts were first reported, the United States was in the midst of the cold war and was experiencing rapid growth. Now, we see a persistently low United States growth rate as compared to previous periods and the US may soon lose its status as a world leader (See Chapter 1).

Obviously, the economies of the world have undergone great changes since the early days of economic growth theory. The rebirth of economic growth theory in the 1980's should be accompanied by a rebirth of empirical studies of growth performance. New theoretical models and improving empirical techniques may very well lead to a better understanding of economic growth processes. A reexamination of these stylized "facts" is necessary for three important reasons :

(1) the models underlying these stylized facts may have been misspecified and we are now in a better position to recognize these statistical properties.

(2) both theoretical models of growth and empirical techniques have progressed and may be better able to capture the nature of progress.

(3) the world is a very different place than it was 30 years ago.

## 4.1 Methodology

Our objective is to examine the characteristics of early growth models and how the Solow model with and without technological progress compares to these stylized facts in the twentieth century United States. Utilizing time series data, we can estimate key parameters relating to theoretical growth models in an attempt to relate US growth performance to a predicted pattern. Assuming the theoretical model accurately describes the true growth pattern, such estimation will yield insight as to where we are along a predicted growth path. However, before we put stock in such predictions, it is also necessary to examine the empirical evidence (see Chapter 3). We must verify which, if any, of these facts are actually acceptable generalizations of the US economy's behavior.

In this chapter, I will utilize US output, capital and labor force data for periods ranging from 1900 to 1992 as collected by the US Department of Commerce to determine the accuracy of Kaldor's Stylized Facts in the specific case of the US. The utilization of time series data differs considerably from the usual approach in empirical growth literature which often examines cross sectional data. Through a more accurate characterization of the growth behavior of an individual country, we avoid the problems imposed by differences between countries and emphasize the behavior of consistent macroeconomic aggregates.

The following section describes the ability of standard log linear growth patterns to accurately reflect US growth. The Solow Model without technological progress is seen to be consistent with facts noted by Kaldor. This model also provides us with a simple framework in which to test the accuracy of these facts. These facts will be tested in section 3.

In sections 4 and 5, we build upon the basic Solow Model through the inclusion of exogenous technological change. The ability of this model with additional descriptive variables to describe performance is evaluated through a similar econometric procedures.

Section 6 suggests alternative models of growth which stem from our results in modeling technological progress. Conclusions and directions for future study follow.

## 4.2 Growth Without Technological Change

### 4.2A The Theoretical Building Blocks

The Harrod Domar Model

As early as 1939, R. F. Harrod and, working independently, E. D. Domar provided a straight forward model of economic growth performance. Using various simplifying assumptions including a fixed proportions production technology, these authors create a framework to analyze growth in output as a result of growth in inputs.

With the rise of the marginalist, the Harrod-Domar model was found to lack the ability to explain how a steady state evolves. Due to the lack of differentiability in the fixed proportions production technology, the model cannot be used to discuss the returns to inputs and, therefore, the incentives to increase or decrease the factors of production. The rate of growth of capital and output is solely determined by saving behavior and the fixed capital output ratio. There is no room for adjustment. If a coincidental equilibrium occurs at which this rate of growth is equal to the rate of growth of labor, then we have a balanced growth pattern with full employment. However, this model can not guarantee such an outcome nor explain circumstances which will lead to this result.

The Addition of Variable Coefficients Production Technologies : The Solow Model

Robert Solow (1956) substantially amended this basic approach to growth theory with the inclusion of a continuous production relationship between inputs. With differentiability, we can now discuss the evolution of the returns to capital and labor. Such a framework is more satisfying since, through economic incentives, we can motivate the movement to a steady state or a steady state growth path .

The Solow formulation of this variable coefficients approach provides us with the underpinnings of an estimable econometric model.[1] To begin, we have an economy with an aggregate production function in which national output is produced utilizing labor and a homogeneous capital good. This production relationship is subject to constant returns to scale overall and diminishing returns to individual inputs. For simplicity, we often consider a Cobb Douglas Production function.

$$Y_t = K^{\alpha} L^{1-\alpha} \text{ with } \alpha<1 \tag{4.1}$$

Note that overall the function is linear homogeneous. However, we still find a decline in marginal productivity if an input is increased individually. This a driving force toward steady state solutions. Given constant returns to scale technology, we can simplify notation so that the production relationship is in per worker terms while retaining the characteristic of constant returns to scale.

$$y = \frac{Y}{L} = f(k) = k^{\alpha} \text{ where } k = {K}/{L} \tag{4.2}$$

The assumption of homogenous, non-depreciated capital we utilize here is designed to simplify the model and has been extended in many directions.[2] We consider here a one good economy so capital is created by the saving of output. This negates the Harrod's complication that saving may not equal investment. We will follow the pattern of Solow (1956) and assume no depreciation of capital. It is very straightforward to add this term directly into the framework. Specifically, we are considering a simplified case in which $\delta$ equals 0 in

$$K_{t+1} = sY_t + (1-\delta)K_t \tag{4.3}$$

The pattern of growth of labor is predetermined and is assumed to progress at some constant exogenous rate, $n$. As a particular case, we can assume that the labor force is fixed so that the value of $n = 0$ in

$$L_t = L_0 e^{nt} \text{ for all t given initial value } L_0 \tag{4.4}$$

Combining these growth paths with the aggregate production function, we find the fundamental equation of growth for our economy. To find the evolutionary path of the capital labor ratio, $k$, and therefore the path of output growth, we look at the capital and labor adjustment paths,

$$K_{t+1} = sY_t = s(K_t^{\alpha}, L_t^{(1-\alpha)}); \quad L_{t+1} = L_0 e^{nt} \tag{4.5}$$

so that dk/dt = $(s/_{(1-\alpha)}) - nk$ and dl/dt = $n$. The capital labor ratio reaches a steady state when $k^* = (s/n)^{(1/1-\alpha)}$.

Given this steady state, output will achieve steady state value of Y* = f($k$*). In this model, the level of output is purely determined by the productivity of inputs reflected in the constant, α, and the determinants of the growth of inputs, $s$ and $n$.

This equilibrium does not imply that the stock of capital or output remain constant. Instead, *per capita* output and capital remain constant indicating that both capital and output are growing at the rate of growth of the labor force or a balanced growth path. In the special case of $n$=0, output and capital will also stop growing at some point leading to a steady state value of output.[3]

Also, it is interesting to consider the role of the returns to inputs. If we consider a competitive market economy, the returns to labor and capital should determine the adjustment to a steady state rate of growth. Specifically, with diminishing returns to each input individually, if capital grows at a rate greater than $n$, we will see a decline in marginal productivity and therefore diminishing rates of return. Once we reach the steady state rate of growth, there is no incentive to move away from a steady state capital labor ratio unless some underlying characteristic, such as the productivity of inputs, changes.

This Solow model without technological improvement results in the growth of output determined solely by the growth in $L$ and $K$ and the fixed productivity of these inputs through the coefficients of the production function. This is consistent with the Stylized Facts 1 and 2 given in the introduction. However, as we will see in the following section, this behavior is inconsistent with real world growth performance in the twentieth century United States.

## 4.2b The Empirical Model

The Solow model describes the underlying process of labor growth as

$$L_t = L_0 e^{nt} \tag{4.6}$$

where $n$ is the exogenous rate of growth of the labor force.[4] In order to obtain an estimate of this growth rate and to examine its properties utilizing the Multivariate Linear Regression Model, we must transform this equation to a linear form. Specifically, our dependent variable must exhibit a linear conditional mean and homoskedastic variance in order for the Linear Regression Model to be applicable.

Such a transformation is readily available. Taking the logarithm of each side we find,

$$LnL_t = LnL_0 + nt \tag{4.7}$$

This growth pattern indicates the rate of growth is a linear function of time with the slope determined by the rate of growth, $n$.

To prepare this theoretical formulation for econometric application, we recall that the rate of growth of labor as we observe it is subject to stochastic variation. So that the growth path above becomes

$$LnL_t = Const + nt + \varepsilon_t \tag{4.8}$$

If our pattern of growth as described by theory is accurate for this data series, we will find the series ε is free of non-linearity, non-normality, is homoskedastic and free of autocorrelation and time dependence.

These are conditions required by the assumption that our conditional distribution of data is normal and that these particular dependent variables are capable of capturing the systematic information contained in $L$ (see Chapter 3). Equivalently, we are imposing white noise conditions upon the distribution of the error. By examining the characteristics of the residual, we determine the characteristics of the conditional distribution giving rise to these observations. If the residuals are normal and if they are independent of both time dependence and autocorrelation, only then may we be confident of our model results.

We begin by estimating the empirical model given above. To examine the properties of the residuals we will employ the following tests. The standard errors of coefficients are reported in parenthesis in each equation. Test results are listed in the form of p values.[5]

The tests utilized all have the same basic premise. In each case, we compare a functional form or value consistent with the assumption made by the linear regression model with an alternative which allows violation of that assumption. If the difference between these two models is large, then the assumption is viewed as not valid. This "omitted variable" type test generalizes into a test for conditional mean linearity, conditional variance homoskedasticity and independence. The values of the skewness and kurtosis coefficients for each model are compared to the value of skewness and kurtosis for a normal distribution. In this way, we see if the conditional distribution can be called a conditional normal subject to a certain degree of error.

Skewness : The D'Agostino test for Skewness
Kurtosis: The D'Agostino test for Kurtosis
Homoskedasticity: Reset test of order 2 for heteroskedasticity
Linearity: Reset test of order 2 for conditional non-linearity
Independence: Modified LM residual autocorrelation test of order 2

These are not the only tests available to us. Occasionally, we may find alternative misspecification tests giving a different result. The form of the alternative hypothesis each test examines may differ making each test somewhat different. Some alternative forms of these tests were indicated in Chapter 3.

Specifically, I have examined the conditional variance for the presence of linear and quadratic terms and the conditional mean for quadratic terms. These alternative hypotheses are in violation of the assumption of conditional normality. If the null that linear and quadratic terms in the conditional variance are insignificant, then it can be said that the conditional variance is homoskedastic. If the null that the non-linear terms in the conditional mean are insignificant is not rejected, then we cannot reject the assumption of conditional mean linearity.

Misspecification testing was conducted on all relationships studied utilizing the statistical package GAUSS386 and GAUSSI by Aptech industries. This battery of misspecification tests was programmed through the diligent work of Dr. John Robertson (Australia National University), Dr. Aris Spanos (University of Cyprus) and Dr. Anya McGuirk (Virginia Tech).

It is crucial that we recognize the difference between a statistical model and the theoretical models we begin with. The theoretical empirical model provides us a starting point for the estimation of economic growth relationships. In order for us to discuss the validity of this model, however, we must verify that assumptions made regarding the functional form, memory and stationarity of the statistical model are indeed accurate. To do so, we abandon economic theory for the time being and concentrate of the validity of these assumptions in describing the statistical characteristics of the growth in output as captured through growth in inputs. Once we characterize the statistical model, then we will be able to analyze the validity of our theoretical model as well.

The objectives and methodology of this study are very similar to Aris Spanos' (1988) examination of the consumption function relationship. In this paper, the author shows that some commonly held beliefs regarding the empirical model explaining a consumption function are unfounded due to misspecification errors displayed.

## 4.3 Testing Kaldor's Stylized Facts

**Stylized Fact 1 : The labor force grows at a constant rate.**

We will employ two measures of the input labor. The first represents the average number of persons willing to work in a given year. This figure is based upon Commerce Department estimates of the labor force. The second, average man-hours per year, will measure the amount of labor resource actually employed on average in a year. This

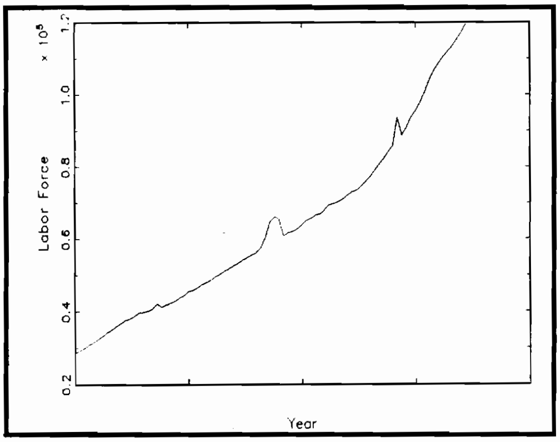



Figure 4: United States Labor Force, 1900-90

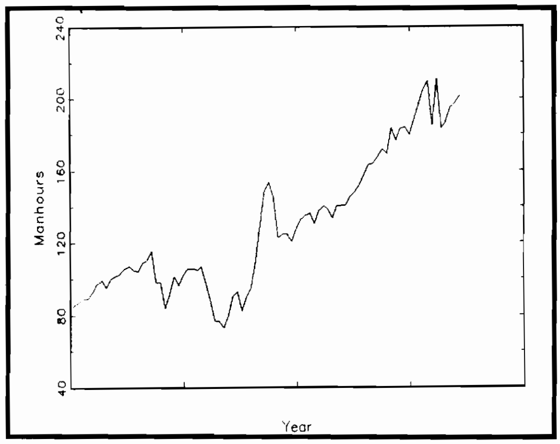



Figure 5 : US Average Weekly Manhours, 1900-90

number is calculated based upon the average size of the labor force, the average percentage employed that year, and the average number of hours per year worked. This measure is intended to capture differences in the employment pattern related to business cycles and changes in work behavior.

We can see both employment measures have shown an increase in labor inputs. The question of which has increased at the greater rate can be determined by the coefficient of *t* in (4.8).

To begin, we estimate the empirical model given above with each of our measures of the labor input. We begin with a measure of the labor force. We find

$$LnLab_t = \underset{(.0042)}{10.983} + \underset{(.0017)}{0.15}\, t \tag{4.9}$$

| T=87 | | Log Likelihood 160.34 | | Adjusted $R^2$.9897 |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.336 | 0.199 | 0.249 | 0.041 | 0.000 |

Our alternative measure of the labor input is yearly man-hours.[5] This is calculated as the average number of persons employed times the average hours per week worked.

This measure displays growth trends very similar to the labor force measure but is characterized by more variability (see figure 4 and 5).

A similar regression with the log of man-hours indicates

$$LnMan_t = \underset{(.015)}{4.82} + \underset{(.006)}{0.10}\,t \tag{4.10}$$

| T=87 | | Log Likelihood 49.148 | Adjusted $R^2$ .7720 | |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.000013 | 0.011105 | 0.000 | 0.1018 | 0.000 |

The p values above indicate that the rate of growth of labor, when measured by either of these quantities, is not well specified and therefore the above function of these variables is not capable of explaining labor growth.[6] Given the rejection the hypotheses of linearity and independence, we can propose an alternative model to capture observed performance in a statistically adequate manner.

Non linearity indicates that higher functions of time may be important in explaining growth. The inclusion of a higher degree trend polynomial should tell us if labor is growing at an increasing or decreasing rate as well as capture periodic variations through the cubic term.

Non-Independence indicates that the rate of growth in period *t* is dependent upon the rate of growth in period *t-1* or on more distant autoregressive terms. In a statistical sense, we see that dynamic terms are necessary to have a well specified empirical model.[7]

We can further verify the misspecification results shown in the tables through a graphical analysis of residuals. By looking at a time plot of the scaled residual from each regression, we can assess the appropriateness of our assumptions. In the below graphs, we find evidence of non-normality and temporal dependence. Positive dependence is evident due to the long swings in the residual series. These indicate that the past value of the residual in some way predicts roughly what the future residual will be. Non-normality is evident through the skewed nature of the empirical frequency distribution which can be visualized by piling the observations on the vertical axis. (Spanos, 1986)

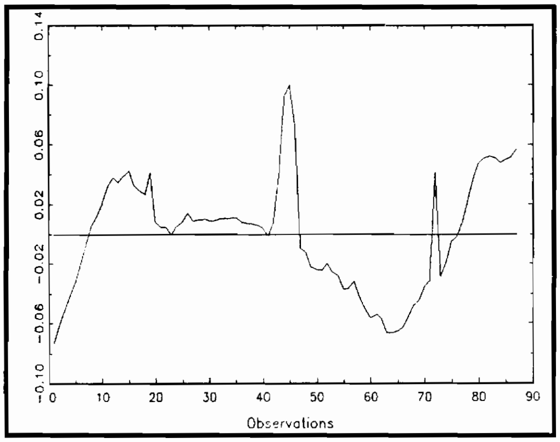


Figure 6: Residuals of labor force detrended

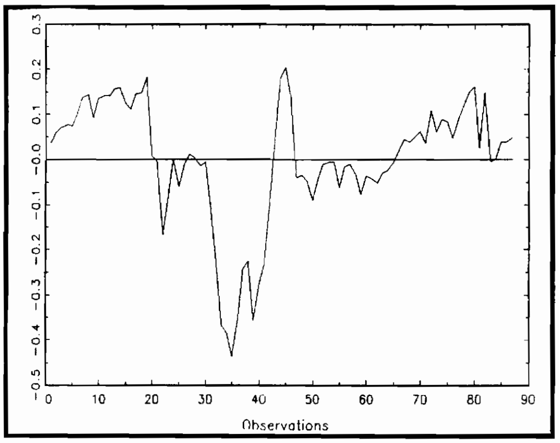


Figure 7 : Residuals from Manhours detrended

We improve our econometric model with the addition of these three regressors.

$$LnLab_t = \underset{(.557)}{1.48} + \underset{(.007)}{0.02}\,t - \underset{(.00004)}{0.0007}\,t^2 + \underset{(.112)}{0.909}\,LnLab_{t-1} - \underset{(.110)}{0.042}\,LnLab_{t-2} \quad (4.11)$$

| T=85 | | Log Likelihood 228.30 | Adjusted $R^2$ .9979 | |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.060 | 0.0000 | 0.490 | 0.208 | 0.529 |

$$LnMan_t = \underset{(.297)}{0.880} + \underset{(.006)}{0.02}\,t + \underset{(.0009)}{0.003}\,t^2 + \underset{(.110)}{0.980}\,LnMan_{t-1} - \underset{(.110)}{0.161}\,LnMan_{t-2} \quad (4.12)$$

| T=85 | | Log Likelihood 115.77 | Adjusted $R^2$ .9511 | |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.001 | 0.030 | 0.004 | 0.680 | 0.091 |

Residual analysis of these two series indicate that we have improved upon the memory characteristics of the residuals, but we still see some non-normality through the skewness of the empirical frequency distribution.

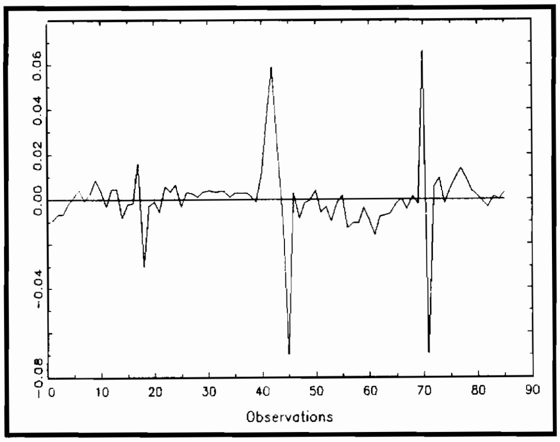



Figure 8 : Residuals of Labor Force- dememorized

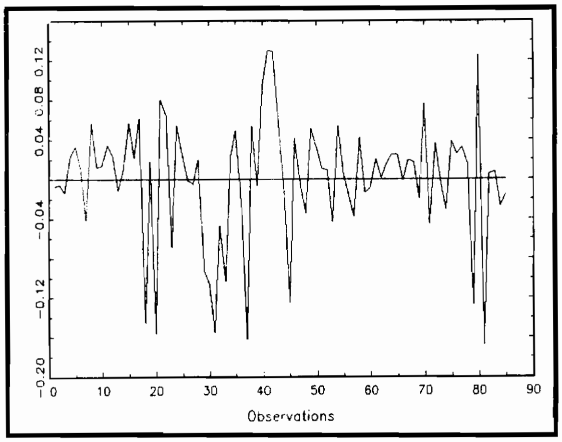



Figure 9 : Residuals Of Manhours - dememorized

Our results indicate, therefore, we have not found a statistically valid characterization of labor force growth.. Due to the remaining presence of non-linearity, it is not possible to determine accurately the coefficients of our model. The log linear model of growth of labor resources is commonly employed, but, as displayed above, is not statistically well specified and therefore may lead to false conclusions.

In conclusion, the rate of growth of labor cannot be assumed to be constant. This stylized fact of Kaldor does not hold up to empirical testing due to the presence of non-linearity in the conditional mean of our statistical model and also due to temporal dependence. The theoretical model of Solow simplifies the growth pattern of labor and population to a simple log linear formulation. Such a model, however, does not fit current US labor growth as measured through man-hours and labor force measures.

**Stylized Fact 1 (cont.): (1) Output and labor grow at a near constant and equal rate so that the productivity of labor, (Y/L), does not fall.**

If indeed output and labor grow at a constant and equal rate, we would find ourselves along a balanced growth path as opposed to a fixed steady state. This is consistent with the theoretical model developed previously.

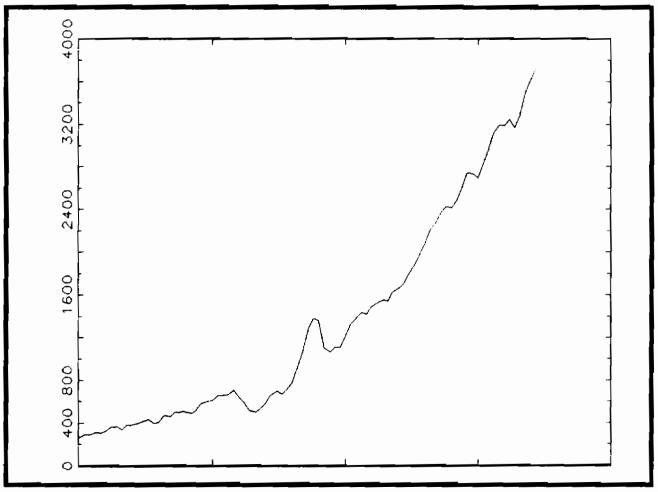



Figure 10 : U. S. Real Gross National Product 1900-90

A constant rate of output growth would be characterized by an empirical model of the same form as the labor model. Specifically, the log of GNP will follow a linear trend. If growth of output as measured by Real GNP grows exponentially at a steady rate, then we should be able to characterize output growth in the following statistical model.

$$LnGnp_t = Const + gt + e_t \tag{4.13}$$

$$LnGnp_t = \underset{(.012)}{6.913} + \underset{(.005)}{0.31t} \tag{4.14}$$

| T=87 | | Log Likelihood 65.352 | Adjusted $R^2$ .9785 | |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.00002 | 0.0006 | 0.036 | 0.214 | 0.0000 |

Once again, a simple linear trend is not sufficient to explain the growth path of this variable. Specifically, we find the same assumption violations as in the labor case. The conditional mean in this regression is non-linear and we find significant autocorrelation.

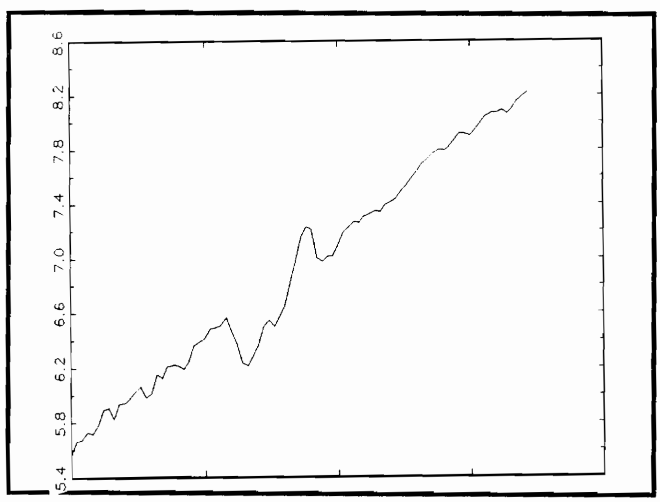



Figure 11: Log US Real Gross National Product, 1990-90

To attempt to satisfy the normality, linearity and independence assumptions of the linear regression model, we add a higher order trend and lagged terms of output.

$$LnGnp_t = \underset{(.380)}{1.47} + \underset{(.02)}{0.01}t + \underset{(.00001)}{0.0009}t^2 + \underset{(.100)}{1.214}LnGnp_{t-1} - \underset{(.100)}{0.425}LnGnp_{t-2} \tag{4.15}$$

| T=85 | | Log Likelihood 128.78 | Adjusted $R^2$ .9950 | |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.0153 | 0.030 | 0.578 | 0.007 | 0.269 |

The specification tests above indicate the presence of heteroskedasticity and non-normality. Once again, a simple representation of the growth pattern of output is appropriate in explaining the US. experience. This is verified in the residual series in Figure 12.

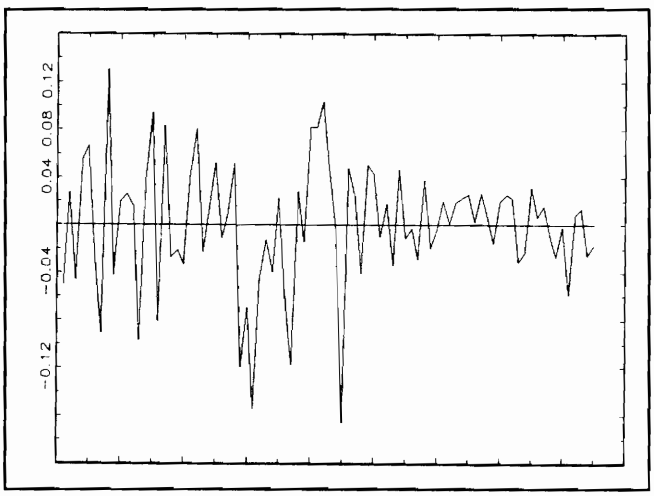



**Figure 12. Residuals from Log US Real GNP - dememorized**

We can continue to test Kaldor's stylized fact that the ratio of output to labor grows at a constant rate. Since we have not been able to characterize the pattern of growth of either of these terms, *Y/L* will remain constant only if the unknown growth paths of labor and output remain proportional to one another.

We estimate the same growth path for the output labor ratio and find that

$$Ln(Gnp / L)_t = \underset{(.007)}{2.10} + \underset{(.029)}{0.21}t \tag{4.16}$$

| T=87 | | Log Likelihood 111.60 | Adjusted $R^2$ .9835 | |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.053 | 0.060 | 0.000 | 0.066 | 0.000 |

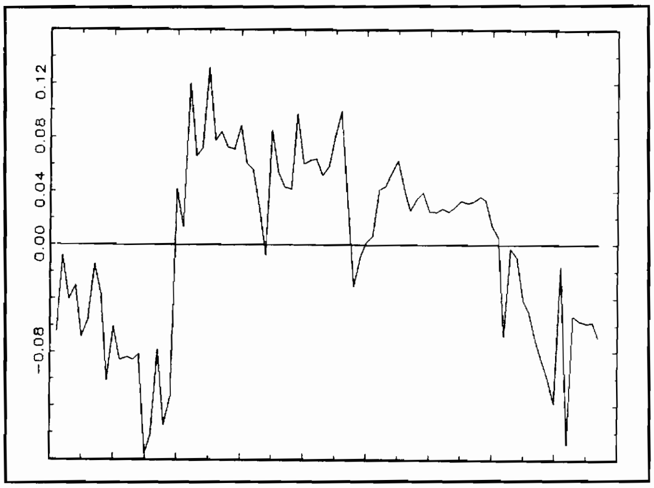


Figure 13: Residuals of Output per laborer - dememorized

Not surprisingly, this series suffers the same misspecification error. Specifically, we notice conditional mean non-linearity and non-independence. Due to the above errors, we know that our statistical model is not accurate and that the ratio of output to labor does not grow at a constant rate.[8]

**Stylized Fact 2: Capital resources grow at a constant rate approximately equal to the rate of growth of output**

According to Kaldor, capital resources will grow at approximately a constant rate equal to the rate of growth of output regardless of our method of measurement of capital. We define our measure of capital as Fixed Gross Non-Residential Capital Stock as calculated by Goldberg's perpetual inventory method.

The constant trending of capital is testable using the same framework developed for output and labor. Specifically, we estimate

$$LnK_t = Const + \kappa t + \varepsilon_t \tag{4.17}$$

where κ is the rate of growth of capital. Using the gross capital measure, this yields

$$LnK_t = \underset{(.015)}{7.78} + \underset{(.0009)}{0.030}t \tag{4.18}$$

| T=62 | | Log Likelihood 43.684 | Adjusted $R^2$ .9505 | |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.063 | 0.0002 | 0.000 | 0.123 | 0.000 |

We can once again reject our statistical model that gross capital increases at a constant exponential rate due to non-linearity and non-independence. Including a higher order trend and lagged capital terms, we find,

$$LnK_t = \underset{(.586)}{1.23} + \underset{(.02)}{0.05}\,t + \underset{(.0003)}{0.0005}\,t^2 + \underset{(.123)}{1.204}\,LnK_{t-1} - \underset{(.132)}{0.360}\,LnK_{t-2} \tag{4.19}$$

| T=60 | | Log Likelihood 143.63 | | Adjusted R2 .9982 |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.165 | 0.009 | 0.071 | 0.167 | 0.545 |

Even with this more sophisticated dynamic mechanism, we still find functional form problems. The rate of growth of capital is changing over time as indicated by the importance of non-linear terms in the Reset Test of Order 2.

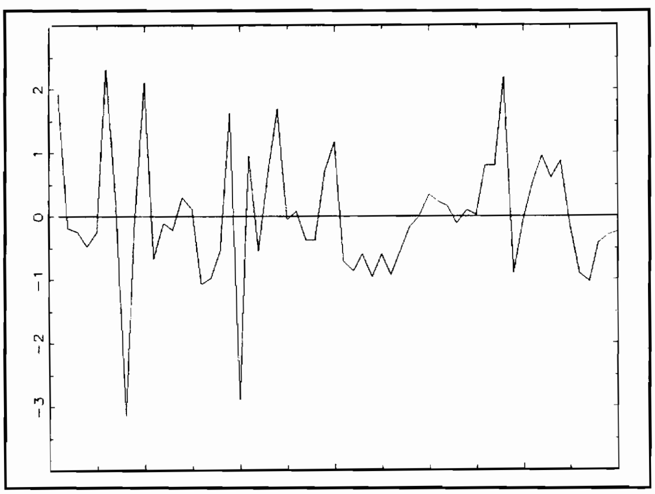



Figure 14: Residuals of Gross Capital Stock - dememorized

The congruence of the functional forms of the growth processes indicates the same factors are important. That is, both dynamics and non-linear trends are important in capturing the rate of growth of output and capital.

**Stylized Fact 3: The Capital to Labor Ratio grows at a constant rate.**

Assuming the earlier facts of a constant rate of growth of capital and labor, this would indicate that capital grows at a steady rate faster than the rate of growth of labor. However, we have found that neither capital nor labor grow at a steady rate. Both follow a non-linear pattern strongly dependent upon previous growth rates. The capital to labor ratio will be constant only if capital changes proportionally with labor whatever the true growth path.

We can test the constancy of the trend of K/L by the formulation

$$Ln(K / L)_t = Const + dt + \varepsilon_t \tag{4.20}$$

$$Ln(K / L)_t = \underset{(.024)}{2.76} + \underset{(.013)}{0.18}\,t \tag{4.21}$$

| T=62 | | Log Likelihood 16.266 | Adjusted $R^2$ .7582 | |
|---|---|---|---|---|
| Skewed | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.471 | 0.304 | 0.000 | 0.031 | 0.000 |

This regression indicates the same non-linearity and non-independence characteristics. We include a non-linear regressor and autoregressive terms to find

$$Ln(K / L)_t = \underset{(.187)}{.563} + \underset{(.013)}{0.038}\,t + \underset{(.0005)}{0.001}\,t^2 + \underset{(.129)}{1.095}\,Ln(K / L)_{t-1} - \underset{(.126)}{0.295}\,Ln(K / L)_{t-2} \tag{4.22}$$

| T=60 | | Log Likelihood 83.32 | Adjusted $R^2$ .9715 | |
|---|---|---|---|---|
| Skewed | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.277 | 0.036 | 0.002 | 0.481 | 0.091 |

Once again, we know that the growth rate of the capital to labor ratio is not a constant. We have determined that both dynamic and higher order trends are significant in explaining the pattern of growth. However, we still face non-linearity in the conditional mean which we have not been able to characterize. Therefore, we know that the growth path of the capital to labor ratio is not linear as predicted by the Stylized Facts and by the Solow Model without technological progress. However, as with our other series, we cannot say that even our improved empirical model represents the path of growth due to existing non-normality in the conditional distribution.

The rejection of the above Stylized Facts for twentieth century US performance implies that the rate of growth of output and capital exceed the growth rate of labor as shown through the trending of our output to capital and labor ratios. The Solow Model without technological progress is not supported by twentieth century United States growth experience. This finding is not surprising given our rising standard of living, however, the result does imply a need to speak very carefully of the Stylized Facts of growth.[8]

## 4.4 Growth With Technological Change

As noted in Romer (1989), the rate of growth of output exceeds the level explained by growth in inputs. It is this potential for change in productivity of inputs which can lead us to this sustained economic growth in excess of the growth rate of inputs. Through technological improvements, we may be able to experience permanent increases in output per capita. Modeling such sustained growth, however, is more complicated. The answer for Solow was neutral and exogenous technological progress.

To add the element of exogenous technological advance, we need to modify our simplified aggregate production function. Changes in technology which increase both the productivity of labor and capital can be captured through the inclusion of a multiplicative constant to our formulation.

$$Y = AK^{\alpha}L^{(1-\alpha)} \tag{4.23}$$

An increase in $A$ will increase the marginal productivity of both $K$ and $L$. In a competitive market, as assumed above, the returns to factors increase, thereby increasing the level of inputs as well as output. The maximum growth in $Y$ will again be determined by the underlying characteristics of growth in inputs as well as, due to the addition of technological change in the term $A$, the growth rate of technology.

If $A$ increases one time only, we would find the growth rates of $K$ and $Y$ exceed the rate of labor force growth for some period of time. Eventually, diminishing returns would return us to the balanced growth path with the rate of growth of all variables equal to $n$.

However, if $A$ continuously increases through some mechanism, say its own exogenous growth path, then we can expect sustained growth in both output per worker and capital per worker. We would find a steady state growth path in which the rate of growth of $K$, and therefore $K/L$ and $Y$, would exceed $n$. This new rate would be determined by the previous rate of growth, $n$, plus the rate of growth of the technological factor $A$.

## 4.5 Romer's Observation : The Solow Residual

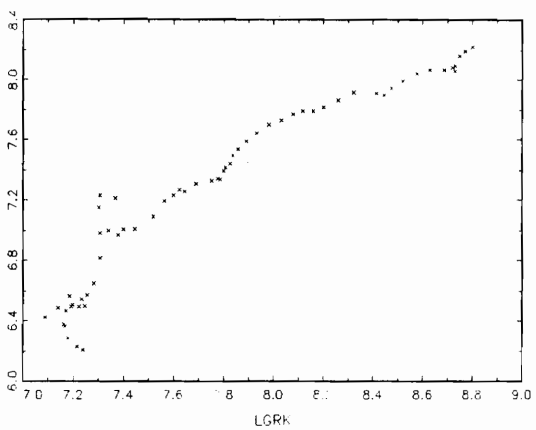
Paul Romer (1989) identified additional Stylized Facts which we should add to Kaldor's early list. Specifically, one of these facts deals with the time series growth path of output and its inputs as represented in the Solow Model.

Romer notes that the rate of growth of inputs combined is not sufficient to explain the rate of growth of output. That is, we do not observe constant returns to scale over time. Instead, the rate of growth of output exceeds the rate of growth of its inputs. Theoretically, we interpret such a result by noting that the marginal productivity of capital and labor have improved or, more commonly stated, technological progress has occurred.

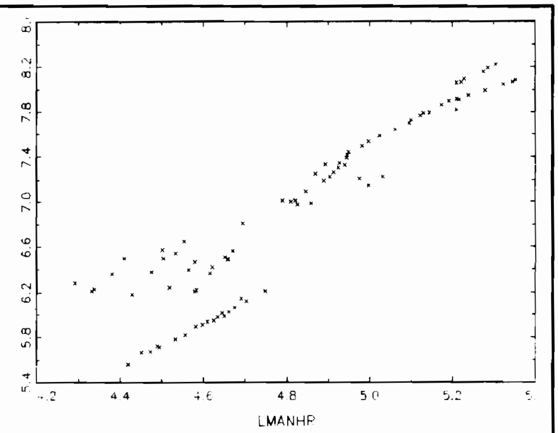
Empirical economists term this effect as the Solow Residual, defined as the growth of output not explained by the growth in inputs. The Solow residual is generally interpreted as the amount of technological progress as would be shown by the Solow Model with technological change. However, the Solow residual may not necessarily display only technological progress. This would be the case only if the constant returns to scale production technology was indeed accurate so that any residual is the effect of technological change. But, the Solow residual is essentially just that, *a residual*. *Any* influence from the data sample not captured by our empirical model appears in the residual, regardless of its cause. Through an improved specification of our model, we can actually better capture growth resulting from changes in the number of inputs.

If the Constant Returns to Scale relationship is applicable, then we should find a linear relationship between the logarithms of output and each input.[9] Examination of the cross plots of the logarithms of Real GNP, Gross Capital and Average Man-hours per year shows that a linear relationship should not be assumed. If these variables were linearly related, a best fit line through the cross plot would be a straight line at an angle of approximately 45 degrees. We see, however, a non-linear relationship between these pairs

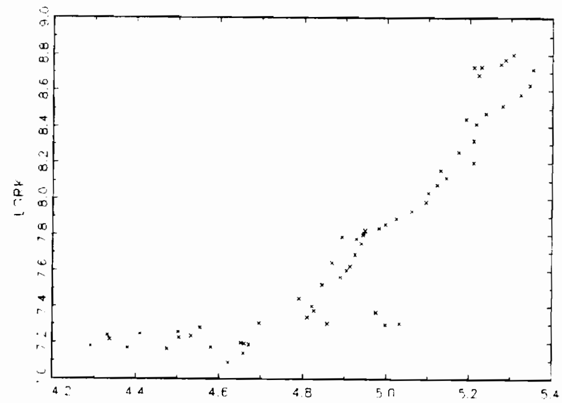
If the Constant Returns to Scale relationship is applicable, then we should find a linear relationship between the logarithms of output and each input.[9] Examination of the




a. Output and Gross Fixed Non-Residential Capital




b. Output and Average ManHours Employed



c. Gross Fixed Non-Residential Capital and Average Manhours Employed

Figure 15: Cross Plots of Log Output, Capital and Labor

cross plots of the logarithms of Real GNP, Gross Capital and Average Man-hours per year shows that a linear relationship should not be assumed. If these variables were linearly related, a best fit line through the cross plot would be a straight line at an angle of approximately 45 degrees. We see, however, a non-linear relationship between these pairs of variables.

Due to the faulty assumption that output growth is log linear in the growth in inputs, the Solow residual is troubling in a statistical sense. The residual does not display the hoped for characteristics of a remainder such as normality and independence. Our residual is systematically trending upward, is non-linear and non-normal as seen through testing the relationship :

$$LGnp_t = \alpha_0 + \alpha_1 LK_t + \alpha_2 LL_t + \varepsilon_t \tag{4.24}$$

where ε is actually the vector of Solow Residuals. .

Estimation yields

$$LnGnp_t = \underset{(.185)}{-2.552} + \underset{(.044)}{0.428}\, LnK_t + \underset{(.083)}{1.323}\, LnL_t \tag{4.25}$$

| T=62 | | Log Likelihood 66.488 | Adjusted $R^2$ .9805 | |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.0357 | 0.4413 | 0.0078 | 0.0175 | 0.0000 |

We find this empirical model is rejected due to non-linearity of the conditional mean, heteroskedasticity in the conditional variance and non-independence. The calculation of the Solow residual is inaccurate because the value of the coefficients of this production relationship are not statistically justified.

Looking at the Solow Residual as it is usually defined, we find numerous misspecification errors. Graphically, we see long swings in the residual series indicating positive memory which has not been accounted for. We also note the skewed empirical frequency distribution.

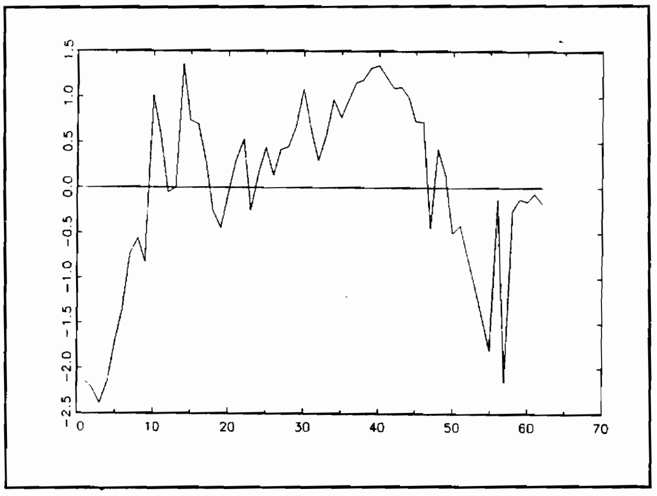



Figure 16: Solow Residuals, 1925-1990

The Solow formulation with the inclusion of technological progress leads us to a slightly different empirical model. The Solow model allows exogenous technological progress which can be represented by a series At which follows its own exponential growth path. In order to include technological progress, we can capture this growth indirectly through a trend term. An index of technological growth as defined by Solow is not available, so we rely on a trend to capture increases in output not accounted for by labor and capital.

This empirical model is of the form:

$$LnGnp_t = Const + \delta t + \alpha LnL_t + \beta LnK_t + v_t \tag{4.26}$$

and when estimated yields:

$$LnGnp_t = \underset{(.309)}{3.308} + \underset{(.011)}{0.22}\,t + \underset{(.034)}{1.021}\,LnL_t - \underset{(.033)}{0.134}\,LnK_t \tag{4.27}$$

| T=62 | | Log Likelihood 128.95 | Adjusted $R^2$ .9973 | |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.0006 | 0.0961 | 0.0295 | 0.3635 | 0.0365 |

We can once again attempt to improve our statistical model. To possibly account for conditional mean non-linearity, we add a quadratic trend to find :

$$LnGnp_t = \underset{(.695)}{3.425} + \underset{(.026)}{0.224}\,t + \underset{(.0004)}{0.0001}\,t^2 - \underset{(.088)}{0.149}\,LnK_t + \underset{(.035)}{1.022}\,LnL_t \tag{4.28}$$

| T=62 | | Log Likelihood 128.93 | Adjusted $R^2$ .9974 | |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.0006 | 0.0910 | 0.002 | 0.387 | 0.0375 |

Furthermore, the addition of dynamic terms in the growth equation may improve on model misspecification based on the results of independence tests. This yields

$$LnGnp_t = \underset{(.775)}{1.593} + \underset{(.038)}{0.108}\,t - \underset{(.0001)}{0.0003}\,t^2 - \underset{(.146)}{0.116}\,LnK_t + \underset{(.058)}{0.745}\,LnL_t + \underset{(.126)}{0.607}\,LnGnp_{t-1} - \underset{(.120)}{0.125}\,LnGnp_{t-2}$$
$$- \underset{(.219)}{0.03}\,LnK_{t-1} + \underset{(.152)}{0.104}\,LnK_{t-2} - \underset{(.113)}{0.114}\,LnL_{t-1} - \underset{(.113)}{0.171}\,LnL_{t-2} \tag{4.29}$$

| T=60 | | Log Likelihood 143.39 | Adjusted $R^2$ .9984 | |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.129 | 0.392 | 0.048 | 0.954 | 0.621 |

A look at our revised Solow Residuals shows that we have accounted for many of the misspecification errors of the original series. We have accounted for the positive memory of the process. However, we still have skewness in the empirical frequency distribution as a result of possible conditional mean non-linearity. This is shown in the below residual plots.

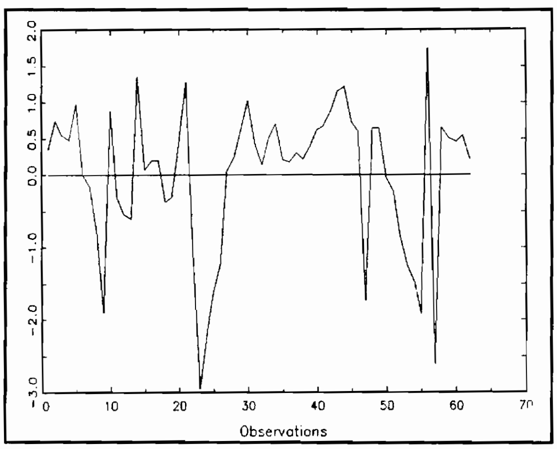


Figure 17 : Solow Residuals - detrended

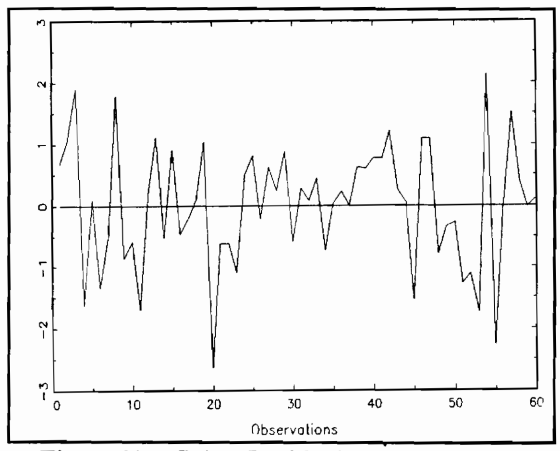


Figure 18 : Solow Residuals - dememorized

The results indicate that we cannot say that our model of the output growth solely as a simple function of the growth of inputs is accurate. This creates difficulty is assigning the output growth to the growth of any input. The simple statistical growth model without technology as well as the model with technological change are both misspecified. Therefore, we cannot determine how an increase in some input will affect output using this sort of functional form. Through technological change, the marginal productivity of each factor would change with changes in technology. If so, then the contribution of each input to output production changes.[10]

We may also consider the inclusion of non-linear regressors on the right hand side. While the test above indicates that a Reset Test of linearity nearly is not rejected, this the test value is so near our rejection criteria that further exploration is in order. Non-linearity as a result of non-linear variables on the right hand side does not create estimation problems for us. However, if the coefficients were to enter our regression function in a non-linear manner, then ordinary least squares regression would not be possible. Non-linear optimization methods would then be called for. These methods are explored in chapter 8.

Including non-linear forms of the capital and labor series, we find

$$LnGnp_t = \underset{(2.262)}{-5.501} + \underset{(.004)}{0.171}\,t - \underset{(.005)}{0.007}\,t^2 + \underset{(.545)}{1.096}\,LnK_t - \underset{0.036}{0.081}\,LnK_t^2 + \underset{(.595)}{2.076}\,LnL_t - \underset{.062}{0.138}\,LnL_t^2$$
$$+ \underset{(.129)}{0.507}\,LnGnp_{t-1} - \underset{(.118)}{0.253}\,LnGnp_{t-2} - \underset{(.202)}{0.035}\,LnK_{t-1} + \underset{(.141)}{0.144}\,LnK_{t-2} - \underset{(.110)}{0.035}\,LnL_{t-1} - \underset{(.111)}{0.048}\,LnL_{t-2} \quad (4.30)$$

| T=60 | | Log Likelihood 149.59 | Adjusted $R^2$ .9987 | |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.328 | 0.292 | 0.348 | 0.198 | 0.667 |

This leads to greatly improved characteristics in our residual series below figure.

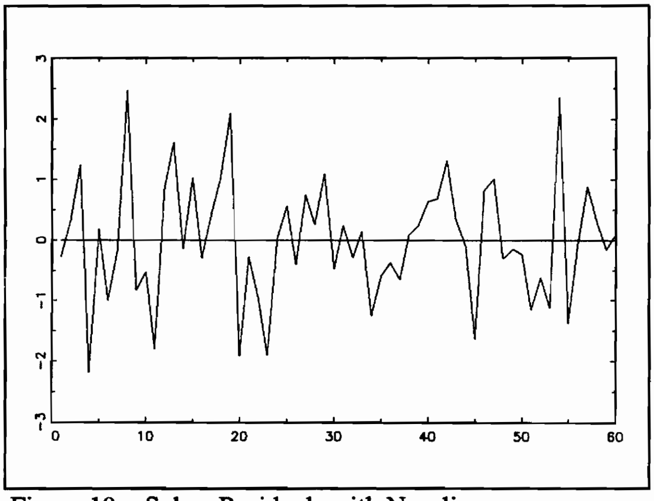


Figure 19 : Solow Residuals with Non-linear regressors

Output growth models have usually accepted a very simple form of the output growth pattern with poor fit and attributed the residual to exogenous technological progress. Throughout this paper, we have observed the importance of non-linear terms and dynamics in explaining patterns of growth. To attribute all of our residual to changes in the technological capabilities of the economy will miss vital information.

We have seen in this examination that a constant returns to scale production technology even with the inclusion of an exogenous technological growth term is not sufficient to model recent US growth performance. Specifically, we find a conditional variance which is heteroskedastic and non-linearity in the conditional mean. This is not characteristic of a normal distribution. If we examine a non-parametric representation of the Solow Residuals, we find skewness (in evidence of non-linearity) as well as kurtosis (which supports a heteroskedastic conditional variance). In the following non-parametric representation, the bold line represents a standard normal distribution. If our error terms conformed to the requirements of the Linear Regression model, then our residuals would fit this line. Instead, we find our distribution displays skewness and leptokurtosity.

We believe that the conditional mean may be dependent upon non-linear forms of the independent variables. This was indicated by the Reset test of Order 2 and was further supported by regression (4.30). We also believe this variance to be dependent upon our independent variables. Further examination of this variance using the sum of squared residuals as a measure finds that the variance may be captured by a linear formulation of the regressors from the conditional mean (regression) equation. This would indicate that the specific type of heteroskedasticity is linear. Such heteroskedasticity exists in a variety of distribution models, but not in the normal. For example, a generalized gamma distribution and various special cases of other distributions with a non-linear conditional mean exhibit a linear conditional variance. This model therefore, would improve upon our fit in the normal case through the added information provided in the conditional variance.

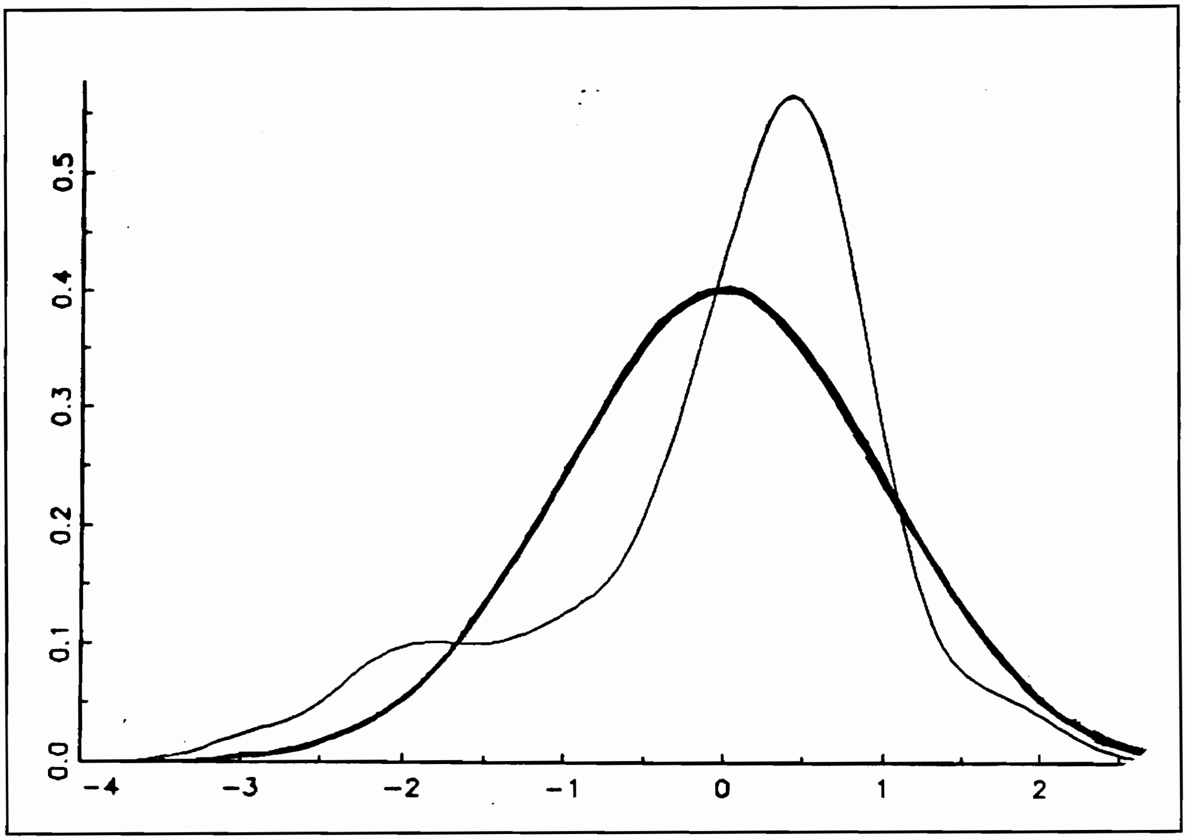


Figure 20: Normal Kernal Estimate for Solow Residual (detrended and dememorized)

Specifically, lets examine the most sophisticated of our models of growth. That is, lets consider the growth in output as modeled by exogenous linear and non-linear trends, growth in inputs and dynamic terms. Utilizing the sum of squared residuals, we have a sample estimate of our conditional variance. We can then test the hypothesis that the conditional variance is indeed homoskedastic by testing :

$$H_0\!:\sigma^2 = const \quad vs. \quad H_1\!:\sigma^2 = const + \beta X \tag{4.31}$$

We find that a homoskedastic conditional variance does not apply to this growth in output regression since the coefficient β is significant. Misspecification testing of this regression, indicates the presence of skewness, heteroskedasticity, and non-linearity. Not only is our conditional variance not a constant, it is not a linear function of the regressors of the conditional mean.

Specifically, we find the following.

$$uhat_t^2 = -4.202 + 2.600t - 0.0260t^2 - 7.00\,LK_t - 2.026\,LL_t$$
$$+ 5.413\,LGnp_{t-1} - 12.776\,LGnp_{t-2} + 17.410\,LK_{t-1} - 9.029\,LK_{t-2} \qquad (4.32)$$
$$+ 3.915\,LL_{t-1} + 15.282\,LL_{t-2}$$

| T=62 | | Log Likelihood 76.05 | Adjusted $R^2$ .1620 | |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.0036 | 0.1339 | 0.0067 | 0.0008 | 0.123 |

This leads to a rejection of the assumptions of normality, homoskedasticity and linearity. The conditional variance has yet to be determined, but it is not homoskedastic and therefore the standard linear regression model is inappropriate. By capturing the characteristics of this true conditional variance, we can select the distribution form most appropriate for the data at hand.

We also may explore the issue of non-linearity, just as in the case of the conditional mean, through the inclusion of squared levels of the capital and labor variables. This yields:

$$uhat_t^2 = \underset{(105.05)}{229.57} + \underset{(1.871)}{1.622}t - \underset{(.022)}{0.014}t^2 - \underset{(25.290)}{34.237}\,LK_t + \underset{(1.682)}{1.589}\,LnK_t^2 - \underset{(27.613)}{48.319}\,LL_t$$
$$+ \underset{(2.90)}{4.500}\,LL_t^2 + \underset{(5.617)}{4.604}\,LGnp_{t-1} - \underset{(5.617)}{12.101}\,LGnp_{t-2} + \underset{(9.317)}{13.147}\,LK_{t-1} - \underset{(6.544)}{6.343}\,LK_{t-2} \quad (4.33$$
$$+ \underset{(5.118)}{3.278}\,LL_{t-1} + \underset{(5.138)}{11.132}\,LL_{t-2}$$

| T=60 | | Log Likelihood 80.692 | Adjusted $R^2$ .1768 | |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.0009 | 0.0643 | 0.0191 | 0.0009 | 0.1937 |

Continuing misspecification in the form of distributional assumptions makes inference regarding the conditional variance function impossible. We have not developed a satisfactory empirical model to explain the characteristics of the conditional variance of the Solow Residual.

## 4.6 Some Alternatives In Model Selection

Our results still leave a gap in determination of growth behavior. Specifically, the exogeniety of key factors including technological progress limits the application of our model. From a perspective of policy, we wish to sustain growth through incentives working in the marketplace. Given Solow's model of technological progress, only non-modeled changes in efficiency will lead to increasing standards of living.

Recent theoretical models of growth have tried to determine the source of technological progress. By endogenizing the mechanism leading to increasing efficiency, economists have been able to not only explain the behavior of an economy without reliance on exogenous development but also have been able to explore questions which ask how to increase the rate of growth.

Building models of growth with a changing production relationship between inputs and output presented theoreticians with a challenging problem. The answer lies in the theory of public goods and related externalities. Through the inclusion of effects external to individual decision makers, we can find sustainable increases in growth.

One such source of growth is an increase in human capital as in Becker, Murphy and Tamura (1990). As an individual gains knowledge, all of society can produce better through the component of this increase which the individual shares with society. Other sources include increases in government capital which lead to improvements in the productivity of privately held inputs as in Barro (1990) and innovation by some part of society which increases the overall efficiency of the economy (Grossman and Helpman, 1989).

Such models can provide continuous predictable growth in output. The empirical testing of such models has just begun. The testing of these endogenous models of growth will lead to advances in our understanding of the power of policy variables to affect growth. Even the smallest increase in the rate of growth may lead to large changes in the level of production so a fuller understanding of the sources of technological advancement can be potentially beneficial for generations to come. Empirical examination of these new growth models follows in Chapter 5.

We may also examine the possibility of structural change in our output growth models. Maddison (1982) notes that economies tend to enter various stages of growth. Economies progress in some growth pattern until some critical point is reached at which time the economy would enter a new growth stage. Maddison points to the behavior of the Netherlands in the 1800's and the United Kingdom in the early twentieth century. Such behavior may also be exhibited at this time by the United States and Japan.

Azariadas and Drazen (1992) support this aspect of structural change. They propose a pattern of thresholds at which an economy jumps to a new growth path. Such an aspect of structural change can also be empirically tested. Specifically, we examine the growth pattern over a variety of sub samples to determine if the rate of growth of a variable changes.

Such analysis can be conducted by recursive least squares which begins with a small sample size and progressively adds observations calculating coefficients for each subsample. Another method will slide a window of a given sample size over the entire sample length once again calculating coefficients for each subsample. The value of these coefficients should be indifferent in statistical testing in threshold economies exist. With yearly observations, this is difficult due to the resulting small sample sizes. However, the availability of quarterly data for output and some inputs indicate this may be possible. This issue of structural change and non-stationarity is explored in Chapter 7.

## 4.7 Conclusion

Empirical analysis of US yearly observations of GNP, Capital and Labor indicate that the growth paths of these quantities, both separately and jointly, are not able to be captured through the use of a linear empirical model. By testing Kaldor's Stylized Facts, we have found that there has been a substantial increase in the productivity of these inputs so that growth in output is not simply a matter of growth in capital and labor.

The explanation most commonly given for this residual effect is the exogenous growth of technology. Given the nature of the production function we have specified, it is a fact that the productivity of labor and capital has increased. Any residual is assumed to result from an exogenous increase in technology which our model cannot represent.

However, through specification testing of the growth paths of inputs and of the Solow Residual, we can show a decrease in the magnitude of the Solow Residual when we include dynamic and non-linear terms indicating the standard representation of exogenous technological growth is inaccurate. This leads to an overestimation of the size of the Solow Residual, or the amount of growth due to technological progress. The inclusion of non-linear and autoregressive terms adds to our understanding of the pattern of growth of output, capital and labor and to the importance of technological change in output growth.

We have found that a constant returns to scale production function is not supported due to the importance of non-linear terms in the representation of the growth path of output. We also find support for the evolutionary theory of economic growth through the influence of dynamics in the growth path.

## Future Directions

An alternative approach to the modeling of technological progress and growth has emerged in the recent theoretical growth literature through the use of simple production relationships with additional variables including human capital accumulation, innovation and government capital provision These new factors may be sufficient to drive sustainable growth and explain the Solow Residual.

Also, the importance of structural change in growth patterns has been suggested as an alternative in time series modeling of growth. If threshold economics exist, then the growth rates of quantities are constant until a certain level of output is reached and a steady state can be defined for a certain range of output.

Finally, as we have shown, growth relationships do not display the standard log linear relationship as predicted by theory and necessary for empirical modeling with the Multivariate Linear Regression and Multivariate Dynamic Linear Regression models. An alternative would be to select a particular distribution whose attributes closely resemble the traits of the conditional moments examined. Then a maximum likelihood estimation of the new model may accurately capture the nature of growth relationships.

## 4.8 Notes

[1]See Solow (1970) for a more complete development of the Solow model with an without technological progress.

[2] For example, the vintage capital model considers a vector of capital goods which differ in characteristics important to their productivity.

[3]Alternative specifications of the Solow model will lead to different steady state values or possibly no steady state at all. For example, if we chose to model the deprecation of capital stock directly by specifying the rate of depreciation of capital, δ, to be greater than zero, then the steady state value of the capital to labor ratio would differ. Specifically, the growth rate of capital would have to increase enough to compensate for the loss of capital each period through depreciation and, therefore, the rate of capital growth would be greater than the growth rate of labor.

[4]This may or may not be equal to the rate of population growth. If these rates of growth differ substantially, then output per worker and output per capita also differ. Such a distinction is important to speak of changes in standard of living. The following test compares the rate of population growth to the rate of labor increase.

[5]Modeling the log of population now as opposed to the log of the labor force, we find

$$LnPop_t = \underset{(.003)}{11.86} + \underset{(.0011)}{.131}\,t \tag{4.34}$$

| T= 87 | Log Likelihood 200.19 | | Adjusted $R^2$ .9946 | |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.053 | 0.041 | 0.000 | 0.067 | 0.000 |

indicating similar problems to our preliminary regressions on labor force. This actually is a beneficial result indicating that the labor force and population follow similar paths of growth. To correct for non-linearity and non-independence, we include higher order and autoregressive terms. Residuals for this series are also available from the author.

$$LnPop_t = \underset{(.149)}{0.350} + \underset{(.002)}{0.0037}t - \underset{(.0001)}{0.001}t^2 + \underset{(.0734)}{1.714}LPop_{t-1} - \underset{(.0721)}{.744}LPop_{t-2} \qquad (4.35)$$

| T = 85 | | Log Likelihood 386.06 | | Adjusted $R^2$ .9999 |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.033 | 0.00001 | 0.031 | 0.065 | 0.0215 |

Here we notice non-normality exists so meaningful conclusions from the coefficients cannot be drawn. We have not corrected for the non-linearity and, therefore, the skewed nature of the conditional distribution. The true nature of the growth path has not been explained by this model and drawing further conclusions would be invalid.

[6]For my analysis, I will consider a 95% confidence level, however, the reader is free to impose individual standards. p values less than the desired a indicate a rejection of the null hypothesis and, therefore, a violation of our distribution assumption.

7 As a theoretical explanation for such behavior we can turn to evolutionary growth theory (Nelson and Winter, 1985) which proposes that increasing income in a period of rapid growth will contribute to increasing growth in the future and vice-versa.

8 Through misspecification testing, we have shown that the growth path of these variables are not log linear. Such a model does not accurately reflect the characteristics of the data which show non-linearity and autocorrelation. While our new models of the growth path consider these factors, our models are still not statistically adequate.

[9] We can choose a variety of measures for both of these variables. To exploit the comparably more stable series, I will use the labor force as my $L$ variable. I will continue to utilize gross private non-residential capital as my $K$ series.

[10]Growth accountants have developed a vast study of output growth performance based upon a simple constant returns to scale formulation. By these tests, we have seen that both inputs and outputs grow in complicated, non-linear patterns indicating constant returns to scale is not necessarily the case.

# *V. An Empirical Analysis of Endogenous Growth Models: The Role of Education, Public Infrastructure and Innovation in Output Growth*

We have seen in the previous chapter that the Solow Residual could not be completely explained by even a richer characterization of the growth processes of individual inputs. The rate of technological progress is not a constant as predicted by the Solow model, but instead is increasing at an increasing rate as shown by the apparent significance of the non-linear trend and input terms.

Modern theoretical growth models have sought to reduce our reliance upon exogenous technological progress. A variety of factors have been suggested which lead to sustained growth of the economy. Such so called growth engines include education, government capital provision and innovation.

In this chapter, I will examine the ability of these growth drivers to explain the Solow Residual or the portion of output growth in excess of the growth of inputs. In Section 1, I briefly outline the standard economic growth model of Robert Solow and highlight important results from the previous chapter. In Section 2, I discuss the theoretical basis of endogenous growth models. I amend the Solow model through the inclusion of additional factors and develop an empirical model of endogenous growth. In Section 3, I develop an empirical model capable of capturing the effect of these factors. In Section 4, each of our growth engines are considered in turn. In each case, we will discuss the data utilized and test the ability of this additional productive factor to explain the Solow Residual. Section 5 discusses the implications of our findings and explores alternative approaches to the modeling of economic growth. Conclusions then follow.

## 5.1 The Solow Model Revisited

According to the Solow model of economic growth, the increase in output is a function of growth in inputs (specifically capital and labor) and improvements in technology (See Solow, 1970 for a summary). A simple, Constant Returns to Scale production relationship is used to characterize the relationship between inputs and output in the absence of technological progress. Generally, we consider a Cobb Douglas production technology in order to simplify the model to a solvable form. A multiplicative constant is utilized to show the increase in productivity of factors as a result of improving technology.

In order to account for technological progress, we include a multiplicative constant $A$ together with a basic constant returns to scale production technology (in this case, Cobb Douglas).

$$Y = F(A, K, L) \text{ or more specifically } Y = AK^{\alpha}L^{(1-\alpha)} \tag{5.1}$$

As $A$ increases, the marginal products of both capital and labor increase. Since the technological advance affects both factors in the same way, the advancement is considered to be Hicksian neutral or disembodied. The productivity of both labor and capital are improved so that we find no substitution effects between inputs.[1]

If we choose an empirical model without a technological constant, we would observe increasing productivity of the labor and capital only through changes in the coefficients of the production relationship. Specifically, in the form,

$$LnY_t = \alpha LnK_t + \beta LnL_t \tag{5.2}$$

which we find in the Solow model without exogenous progress. The elasticity of output with respect to capital and labor shown by $\alpha$ and $\beta$ would increase as productive efficiency increases. This is due to a shift in the production function. In fact, an increase in an input to the production process other than capital or labor, would result in a change in the values of $\alpha$ and $\beta$ as increasing inputs change the productivity of these factors.

Even with the inclusion of a constant, changes in productivity would result from increases in inputs to the production function not specifically modeled. The amount of change in these coefficients depends upon the effect of the non-modeled input on the productivity of capital and labor respectively. The inclusion of the constant $A$ will capture disembodied technological progress.[2]

So, we have defined a model of growth including technological progress to be of the form

$$Y_t = F(A_t, K_t, L_t) = A_t K_t^{\alpha} L_t^{1-\alpha} \tag{5.3}$$

Such a model yields itself easily to empirical examination. By taking a logarithmic transformation of the Cobb Douglas Production relationship, we find the Log of real output is equal to the sum of the logs of inputs plus the log of technological progress.

$$LnY_t = LnA_t + \alpha LnK_t + (1-\alpha)LnL_t \tag{5.4}$$

Utilizing this model, any growth in the log of output not explained by the log of inputs is considered to be the result of changing productivity.

We can now consider how increases in technology as shown by increases in $A_t$ will impact the growth path of output. If $A$ increases at one point in time only, we will see an increase in the marginal product of both labor and capital. This will result in an increase in the use of labor and capital as well as an increase in the level of output. However, in this case we will not see a change in the steady state growth path predicted for output. If $A$ increases regularly based upon its own growth mechanism, then will see continuing improvements in productivity so that total growth in output will depend upon increases in the number of inputs and increases in their productivity as determined by the growth path of $A$. We may assume that $A$ is increasing at its own exponential rate given by

$$A_t = A_0 e^{gt} \qquad or \qquad LnA_t = LnA_0 + gt \tag{5.5}$$

This is attractive since the value of the technological constant $A$ is not an observable. The coefficient of a trend term in this model can indicate the growth of technology.

In the strictest sense of modeling, growth in output can only result due to increases in inputs and improvements in technology. Output increases either due to an increase in the amount of inputs employed utilizing a constant technological relationship or due to an increase in the productivity of these inputs. So, in a strict sense, if the rate of output growth exceeds the rate of growth of inputs, then the productivity of those inputs has improved.

However, this assumes that the production relationship is accurately characterized by a constant returns to scale technology. If increasing returns exist, then the rate of output growth will exceed the rate of growth of inputs even without technological improvement. A testing of the constant return to scale model does indicate misspecification with respect to recent US performance as shown in the previous chapter. Specifically, we find non-linearity in the relationship between the rate of growth of outputs and the rate of growth of inputs. The Solow residual as usually defined in the growth literature does not capture the implications of increases in inputs, but is poorly specified. (Please see the discussion of the Solow Residual of Chapter 4 for more information.)

Also, we find the history of economic growth has explanatory power. Dynamic patterns of growth in inputs and output are evident. Drawing upon the evolutionary theory of growth of Nelson and Winter (1982), past rates of growth of capital, labor and output influence the future rate of growth. If a country experiences greater than normal in some period, the country has a tendency to experience greater than normal growth in the future as it has more output resources to invest in capital accumulation and technology enhancing activities such as research and development. This implies a misspecification in the standard growth model of Solow when applied to US data.

## An Application Of The Solow Formulation - Growth Accounting

The log-linear formulation of the production function of the Solow model lends itself easily to analysis of the effect of various components on the growth of output. Specifically, by multiplying the growth in labor by labor's share in output, we can determine the contribution of labor growth to output growth. Such a calculation, however, relies upon the validity of the linear production function. If this function is misspecified or changes over time, then we cannot determine the true effect of the growth of an input on output growth.

Also, even if correctly specified, we must limit our analysis of the effects of growth of components to those utilized as inputs in the function. We cannot, for example, study the effects of education or changes in labor composition as our function does not relate these characteristics of the economy to output.

Contrary to these limitations, Edward Denison (1962) uses a log linear formulation to determine the effects of labor and capital growth on output growth. If the log linear relationship between output and input growth is correctly specified and the shares of labor and capital do not change over time, then this is justified and valuable information and policy implications may be derived. However, we have seen that specified input-output growth equations contain apparently significant non-linear terms. Also, the growth equation depends significantly upon growth in previous periods. This would lead to an understatement of the effects of input growth since as output increases today due to an increase in a factor, output will also increase in the future.

The influence of improving technology will also distort our growth accounting equations. Denison recognized that inputs may not exhibit constant productivity over the sample period, so his results utilizing measures of labor and capital would result in poor accounting relationships. In order to compensate for changes in the productivity of inputs, Denison redefined inputs in terms of efficiency units of constant productivity. The utilization of efficiency units of inputs in theory would capture increases in productivity simply by increases in the number of well defined inputs of constant productivity.

However the construction of efficiency units is difficult in the case of heterogeneous resources. We see some ad hoc methods of calculating the comparative productivity for inputs. By adjusting labor for the education and skills of the labor force, for the number of female and adolescent workers in the labor force (which he assumes equal a fraction of a male worker based upon relative earnings) and so on, Denison clouds the effect of changes in the economy's underlying characteristics. The same log-linear function cannot be assumed to hold regardless of the choice of input measures. As we adjust the data for factors which alter efficiency, a new production function emerges which may or may not be of the same form and certainly will imply differing measures of the effect of increases in labor and capital.

More information can be drawn from examining these characteristics as inputs in the production process on their own. Increases in education, for example, may increase the marginal product of labor as well as other inputs.[3] More specific knowledge of the effects of variables on output growth will be available and easily interpretable for policy.

To be able to accept the usual growth accounting equations, it is essential for us to assume a stationary log linear production function determining the impact of sources on output growth. Before we can analyze the impact of input growth on output, we will test the statistical adequacy of such a model. Theoretical intuition or models are not enough to suggest the functional form of the growth relationship. To be meaningful, the data must support the empirical process we have in mind.

The studies of Denison in the 1960's, however, clearly mark the path of testing the effects of various changes in the economy on the growth of output. His summary of guidelines for increasing the rate of growth of output clearly display policy actions which may be useful in fighting the productivity slowdown. However, we must be cautious in applying the numerical results of his analysis. By just how much a policy will have an effect is very difficult to ascertain. This depends upon both the empirical formulation of Denison and the particular format chosen for adjusting inputs to efficiency units.

However, the contribution of Denison in identifying sources of output growth and the changing nature of inputs provides valuable information.

## 5.2 Alternative Sources Of Growth In Theory

Modern economic growth theory has sought to augment the Solow model by endogenizing the rate of improvements in technology. Such a task is very challenging since a model with endogenous growth quickly becomes untraceable. The model is required to specify a framework in which sustained growth is possible based upon the rational decisions of agents in the economy. In the usual exogenous growth models, diminishing marginal productivity of inputs drives us to a steady state result with persistent growth in output as a result of exogenous growth in labor and technology only. With an increasing returns to scale production technology, we do not have this influence leading us to a dynamic equilibrium.

Such a challenge has been met through the use of externalities in the production of some input. Agents acting individually are able to overcome the hurdle of diminishing marginal productivity since each investment adds to the productivity of the overall economy. In such a way, the marginal product does not diminish to a point of steady state growth.

In Romer (1986), a straight forward theoretical model is derived which exhibits increasing returns to scale in the production of the consumption good. This is achieved through the use of externalities in the accumulation of knowledge. As firms invest individually in research and development, the overall level of knowledge in the entire economy increases. As this knowledge increases, the marginal product of physical capital, which usually displays diminishing returns, actually increases. This accumulation of knowledge, therefore, can lead to continual growth. However, since firms do not enjoy all the returns from innovation, the social gain from investment in knowledge exceeds the private return and research in new knowledge will take place at a less than socially optimal level. Romer, however, points out that taxes and other policy tools may be employed to achieve a second best solution.

Utilizing a secondary production technology which may or may not be CRS, Romer was able to show that non-linear or sustained patterns of growth can be represented by a simple output-labor-capital formulation. By endogenizing the rate of growth, we may actually derive policy implications which can lead to increasing in the standard of living (that is, output may grow more rapidly than population).

## 5.3 Modeling Technological Progress : Empirical Framework

To model such a system of externalities empirically would be a troubling task since we cannot observe this social gain from investment. Instead, however, we can build a simple model which attempts to capture technological progress. To obtain such a framework, we return to the simplified Solow model with exogenous technological change. To capture the increase in productivity due to some factor other than capital and labor, we can explicitly add these factors to our production relationship. In this way, we do not force the change in productivity resulting from an increase in a previously omitted factor into a change in the production relationship but instead we explicitly model the change in output growth due to the factor. The effect of this output enhancing increase will be summarized, not simply by an increase in the constant and productivity term but through a specific elasticity and related increases in the elasticity of other factors.

To be more specific in the form of the empirical model, return to the case of the Log linear growth formulation in the absence of other factors. Here

$$LnY_t = c + \alpha_1 t + \alpha_2 LnK_t + \alpha_3 LnL_t \tag{5.6}$$

Any advances in productivity will lead to increases in c, $\alpha_1$, $\alpha_2$, $\alpha_3$ or all of the above.

To be more explicit, let us include the source of this increase in production, say education.

$$Y_t = A_t K_t^{\alpha} L_t^{\beta} Edu_t^{\gamma} \tag{5.7}$$

or, in an empirical framework

$$LnY_t = c + \alpha_1 t + \alpha_2 LnK_t + \alpha_3 LnL_t + \alpha_4 LnEdu_t \tag{5.8}$$

Increases in education in the original empirical model (5.6) would have resulted in increases in exogenous technological change, $\alpha_1$, as well as increases in the productivity of capital and labor, $\alpha_2$ and $\alpha_3$ In our new empirical model assuming $\alpha > 0$, increases in educational attainment will lead to higher productivity in capital and labor, $\alpha_2$ and $\alpha_3$ as well as a positive $\alpha_4$.

For insight into the sources of growth, we can refer back to previous studies such as those carried out by Denison (1962, 1974, 1985). Here we see what factors have an effect on the growth of output but the specific impact is clouded by adjustments to equivalency units.

Specific sources of endogenous growth have also been advanced by the theoretical growth literature. Three such sources will be introduced below and the empirical impact of these sources will be examined using our new empirical model of endogenous growth.

## 5.4 The Impact of Alternative Factors

Alternative sources of growth, many initially identified in Denison (1962), have continued to play a vital role in the development of models of economic growth. Denison's listing of 20 prescriptions for increasing growth are a good starting point to test the implications of controllable variables upon the rate of growth of output (see a list of these factors in Chapter 1). With these new models, the evolution of output is not solely determined by the presence of labor force growth or exogenous increases in productivity. Theoretical economists have sought to increase the role of factors outside of the standard capital-labor-output model.

## 5.4A. Education

The accumulation of human capital has been the driving force behind early endogenous growth models. Beginning with Arrow (1962), learning by doing or the accumulation of knowledge which increases the productivity of labor has lead to models of continuous growth without exogenous factors.

In 1986, Romer shows also how sustained endogenous growth may be generated by the accumulation of human capital. Education is modeled as an only partially excludable good so that externalities exist in human capital accumulation and less than the social optimal is produced. Human capital accumulation is considered an input into innovative improvements in technology noted by Romer as new designs. With new designs available, the productivity of capital and labor does not diminish. Improving technology can be modeled with a dynamic relationship based on the amount of human capital invested in the search for new designs. This improving technology then results in a violation of diminishing returns to scale.

Romer employs a theoretical model of output growth which is very useful in an empirical sense. $H$ represents labor in the production of new designs, $K$ the amount of capital and $L$ the amount of labor in production of $Y$. Romer applies a model simple enough for us to estimate the parameters directly in the form of

$$Y = H^{\alpha} K^{\beta} L^{(1-\alpha-\beta)} \tag{5.9}$$

where $K$ actually is a vector of multiple capital, intermediate goods.[4] Such a model can be utilized in as a linear regression model as a logarithmic transform. We will retain the log-normal conditional distribution in regression and examine the ability of this simple model to explain the growth of output.

Lucas (1988) relies on a mechanism of human capital production to explain the persistence of growth in output when the production function of output as a function of capital and labor continues to show constant returns to scale. Human capital accumulation increases efficiency and can lead to a rate of growth of output exceeding the rate of population increase. Once again, the accumulation of human capital yields social benefits to society.

Becker, Murphy and Tamura (1990) endogenize the growth process by modeling the rate of fertility, and therefore labor force growth, based upon endogenous characteristics of the economy. Human capital is produced with returns which are not strictly diminishing. This complicated marginal product leads to the formulation of multiple steady states, in cases of low education or high education societies. Since the production of human capital today provides more resources for the production of human capital in the future (i.e., more teachers), and because the education sector utilizes relatively more educated workers, we again observe an externality in human development and sustained growth is possible.

The empirical model of Mankiw, Romer and Weil (1990) finds that models of output growth based upon growth in inputs are greatly improved by the inclusion of education. Using the proxy, percentage of the working age population in secondary school, the authors find an increase in the ability of their model to track the development of output over time.

Testing the Impact of Education on Output Growth

To test if human capital is capable of capturing the impact of technological advance on the standard production relation, we would like to see advances in education reduce the Solow Residual. We hope that education will accurately characterize the growth path of technological improvement observed in the previous chapter. Given the theoretical formulation given above, the growth of output is a log linear function of capital, labor and education.

Measures of human capital are difficult to find. A measure of the value of education is not readily available through the market since the cost of education is spread over the economy as is it paid by government sources as well as individuals.[5] Also, students incur the opportunity cost of education in the form of foregone wages. Less formal sources of human capital formulation including on the job training and experience often are unmeasured or immeasurable. Recent trends in industry concentrating on job training make this aspect increasingly important.

Accumulating the stock of human capital also causes measurement difficulty. Education is of value throughout the employees entire work life assuming that the use of education is not job specific. Measurement of educational expenditure or attainment should be a stock rather than flow concept. To the best of my knowledge, no such series has been developed. If we consider education to be a stock variable, we would also want to consider the depreciation of this quantity. Depreciation would occur either as educated employees leave the work force or as their skills become obsolete. However, we can avoid the issue of human capital stock measurement by considering the rate of change of human capital, just as we can look at increases in labor and physical capital.

In the past, the percentage of the population which is literate and the percentage of the working age population in secondary school have served as measures of human capital. These measures may be relevant in certain cases; specifically, developing countries. However, for the United States time series studied in this paper, these sources are not satisfactory. The literacy rate has not changed greatly during the sample period studied here. Also, the portion of the working age population in secondary school does not account for a major source of schooling, the increasingly common college education.

I will utilize the number of collegiate degrees (Associate through Ph.D.) issued per year in the United States as my measure of the quantity of human capital accumulated. It can be assumed that substitute forms of education have also increased in a proportionate manner. This, however, is an assumption which cannot, to the best of my knowledge, be tested due to the unavailability of data. In addition, the ratio of degrees obtained to total population will be used to scale the increase in education to population figures. Just as we look at literacy rates, we will look at total graduation rate for collegiate degrees.[6]

We would expect the rate of growth of output to depend upon the rate of growth of education as well as the growth in capital and labor. Dynamic effects may also play an important role due to the long service lives of human capital. We have choices with respect to the measure of capital and labor we choose as well. For consistency, I will continue to utilize gross private non-residential capital stock and yearly man-hours in analysis.

In order to test if education is able to explain technological change, we estimate :

$$LnY_t = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 LnK_t + \alpha_4 LnL_t + \alpha_5 LnEdu_t + \varepsilon_t \tag{5.10}$$

which yields the following estimates:

$$LnGnp_t = \underset{(.714)}{5.596} + \underset{(.029)}{0.281}\,t - \underset{(.0001)}{0.001}\,t^2 - \underset{(.084)}{0.122}\,LnK_t + \underset{(.106)}{0.635}\,LnL_t - \underset{(.021)}{0.023}\,LnEdu_t \tag{5.11}$$

| T = 40 | | Log Likelihood 97.234 | | Adjusted R .9958 |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.418 | 0.048 | 0.0001 | 0.316 | 0.047 |

By examining the statistical properties of the residuals, we find this model to exhibit non-linearity in the conditional mean and related skewness as well as autocorrelation. To attempt to better model this autocorrelation, it will be necessary to include dynamic forms of our regressors. This is not unexpected given the results of previous study of the pattern of growth of output (see Chapter 4). Including two lags of the dependent variable, LGNP, and the independent terms LK, LL, and LEdu, the new empirical model yields:

$$\begin{aligned} LnGnp_t = {} & \underset{(1.252)}{5.436} + \underset{(.059)}{0.262}\,t + \underset{(.0004)}{0.0002}\,t^2 + \underset{(.204)}{0.216}\,LnK_t + \underset{(.083)}{0.329}\,LL_t \\ & - \underset{(.038)}{0.085}\,LnEdu_t + \underset{(.199)}{0.390}\,LnGnp_{t-1} - \underset{(.157)}{0.066}\,LnGnp_{t-2} - \underset{(.306)}{0.630}\,LnK_{t-1} + \underset{(.215)}{0.082}\,LnK_{t-2} \\ & + \underset{(.110)}{.042}\,LnL_{t-1} + \underset{(.102)}{0.112}\,LnL_{t-2} + \underset{(.052)}{0.083}\,LnEdu_{t-1} + \underset{(.036)}{0.003}\,LnEdu_{t-2} \end{aligned}$$

(5.12)

| T = 38 | | Log Likelihood 116.06 | | Adjusted R .9983 |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.3753 | 0.2769 | 0.0783 | 0.6706 | 0.8564 |

The results imply that education is just barely capable of capturing the effect of technological change at a 95% confidence level. Examination of the scaled residuals of (5.11) and (5.12) indicate that inclusion of lagged variables has aided us in capturing the dependence structure, and overall the residual series seems to conform to our linear regression assumptions.

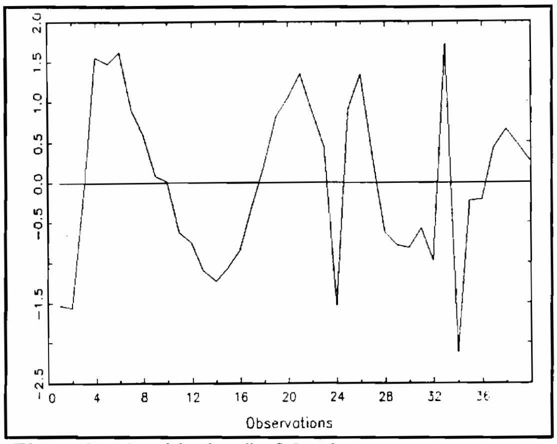


Figure 21 Residuals : # of Graduates

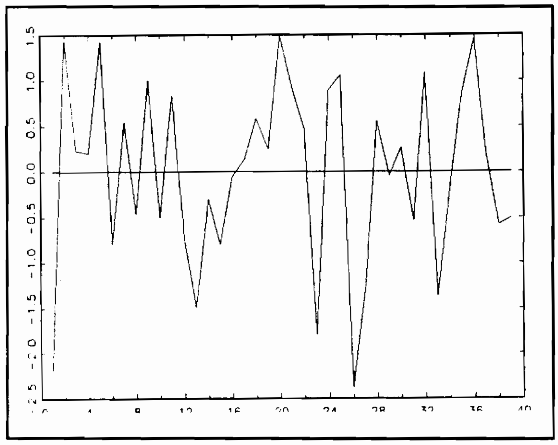


Figure 22 Residuals : # of Graduates - dememorized

While the misspecification test results indicate that the model is statistically adequate, we see an undesirable sign on education. This may be due to the fact that this variable will inherently pick up characteristics of population growth as well as educational obtainment. This is consistent with Romer's (1989) observation that population and output growth may be negatively correlated.

There is a significant danger in using a quantity of degrees granted. This increasing quantity of degrees may not signify an increase in level of education of the population is growing also. To avoid this, most measures of education are in the form of percentages or ratios (such as the literacy ratio). We can also follow this method by looking at the number of degrees granted relative to the population size.

By estimating our above regression with an education ratio, we find

$$LnGnp_t = \underset{(.845)}{5.318} + \underset{(.0297)}{0.2771}t - \underset{(.0004)}{0.001}t^2 - \underset{(.084)}{0.121}LnK_t + \underset{(.107)}{0.635}LnL_t - \underset{(.021)}{0.022}LnEdr_t \quad (5.13)$$

| T = 40 | | Log Likelihood 95.743 | | Adjusted R .9958 |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.4169 | 0.0423 | 0.0002 | 0.3101 | 0.04679 |

Again, we find non-linearity and memory misspecification. With dynamic terms, we have:

$$\begin{aligned} LnGnp_t = {} & \underset{(.893)}{5.068} + \underset{(.041)}{0.231}t - \underset{(.0003)}{0.0002}t^2 + \underset{(.208)}{0.303}LnK_t + \underset{(.088)}{0.327}LnL_t - \underset{(.031)}{0.118}LnEru_t \\ & + \underset{(.126)}{0.443}Lngnp_{t-1} - \underset{(.201)}{0.560}LnK_{t-1} - \underset{(.105)}{.071}LnL_{t-1} + \underset{(.032)}{0.118}LnEru_{t-1} \end{aligned} \quad (5.14)$$

| T = 39 | | Log Likelihood 107.926 | | Adjusted R .9982 |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.0678 | 0.4362 | 0.0068 | 0.6043 | 0.4244 |

The continuing presence of non-linearity indicates the coefficients of this model may not be an accurate presentation of the relationship between these variables and output. Somehow, education has made it easier to account for the original temporal dependence since only 1 lagged period was necessary.

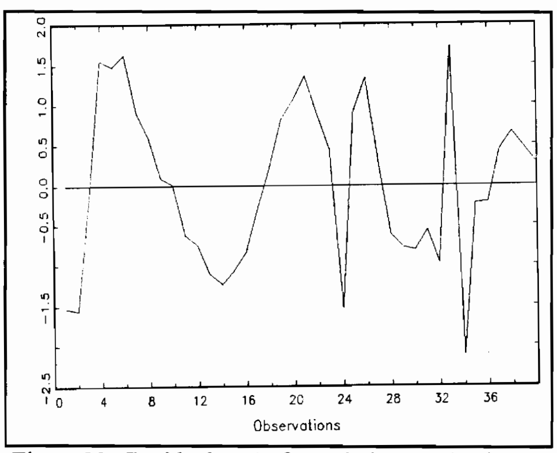


Figure 23 Residuals : % of Population Graduating

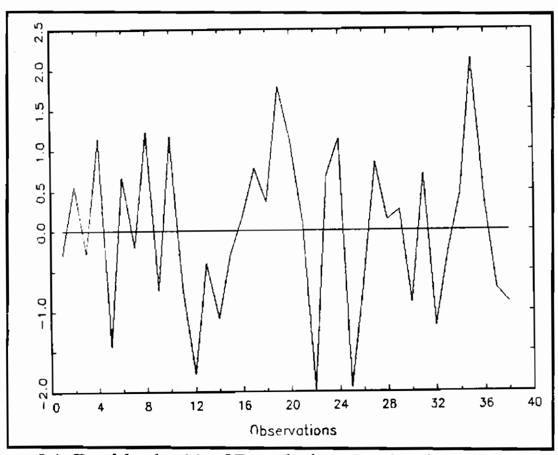


Figure 24 Residuals: % of Population Graduating-dememorized

## 5.4B. Government Investment

Another explanation of the advance of technological efficiency relies upon the presence of not only private productive capital, but also public capital. Due to the nature of some goods, products important in the creation of output cannot be efficiently supplied by private markets. For example, the provision of electricity and other utilities often follows the cost structure of a natural monopoly indicating that the lowest cost production technique involves only one firm. However, a monopolist will not provide goods at this efficient price. Government intervention in the form of regulation or actual operation can maintain this low cost. The government also supplies structures and equipment such as roadways, communication networks and so on. Such factors may have an influence on the productivity of private inputs such as capital and labor. Once a new road is constructed, the lowered transportation costs increase the amount of output a given amount of labor and capital can produce. Therefore, if we continue to model output by a standard capital and labor production function, an increase in government capital can lead to increasing returns to scale or technological progress.

This approach was taken by Barro (1990) which adds government capital stock to the log linear production function. Through increases in the stock of government provided capital, the marginal product of other factors is increased and the rate of growth of output exceeds the rate of growth of capital and labor. This, therefore, can explain the characteristics observed in many economies.

However, governmental capital stocks may be a substitute for private capital. That is, as government capital increases, private capital may be crowded out through increasing real interest rates. The total result of increasing government capital, assumed to be a substitute for private capital is somewhat unclear. First, if government capital borrowing competes for funds in the same market as private capital borrowing, then higher demand for loans may increase the cost of investment resulting in lower private capital formulation. However, increases in public capital which increase the productivity of private capital may counteract this effect by raising the marginal product of private capital investment. The total effect is ambiguous in theory. It depends upon the substitutability between these forms of capital, the financing of governmental investments and the impact of public capital on productivity.

Aschauer (1988) examines the empirical evidence linking increases in governmental capital stock with private capital stocks. He finds that increases in government capital lead to small decreases in private capital. This implies that both effects are at work. There is no evidence of one-for-one crowding out, but private capital stock does fall to some extent. If government capital is as productive as private capital, then the effect of increases in government capital on output should be positive. However, if public capital contributes less to the production of output, total output may fall as government capital increases.[7]

It is vital, however, that we make a distinction between government capital investment versus current consumption. Not all government spending increases the productivity of private market resources. However, making a distinction between public consumption and public investment is not a trivial task. Such information is difficult to obtain given the budget reporting system of the US federal government. State level data may be more accessible due to the large number of states with Balanced Budget Amendments. These amendments to state constitutions limit the debt a state may incur. Borrowing is permitted for long term capital projects only. However, such a measure would not be perfect by far. For example, expenditure to repair a roadway should be treated as a capital investment just as maintenance of factory equipment is treated as private investment. Despite data difficulties, recent statistics have been developed for the federal level, as well as state data.

Aschauer (1987) develops a national series for governmental capital stock in an attempt to treat government investment the same as private investment. That is, utilizing the Goldberg perpetual inventory method. Boskin, Robinson and Roberts (1989) also use the perpetual inventory method and have made their data widely available.

In formulating a model emphasizing the role of government capital in output, we utilize a similar model as in the case of human capital. Now, however, changes in the productivity of labor and capital will result from increases in the amount of government capital stock. Just as in the case of education, we will treat investment as a third factor in our log linear production relationship so that

$$LnY_t = a_0 + a_1 t + \alpha_2 LnK_t + a_3 LnL_t + a_4 LnGov_t \tag{5.15}$$

If government capital has a positive impact upon output, then the rate of growth of output will exceed the rate of growth of labor and capital.

Testing the Impact of Government Capital Accumulation on Output Growth

Following the same format as developed in the education model, we may also test if the formulation of government capital will reduce the Solow residual. If technological progress can be accounted for by the increase in public capital sources, then we should better specify the statistical characteristics of our growth equation as well as capture some or all of the exogenous change in technology as shown by the trend term.

Data for government capital has been recently derived by Boskin, Robinson and Roberts (1989). Such a series is derived using similar techniques as the perpetual inventory system for private capital of Goldberg. This assures us that comparability between our private and public capital series exists. Utilizing the growth in Gross Total Non-Military government capital stock, we will try to account for differences in productivity.[8]

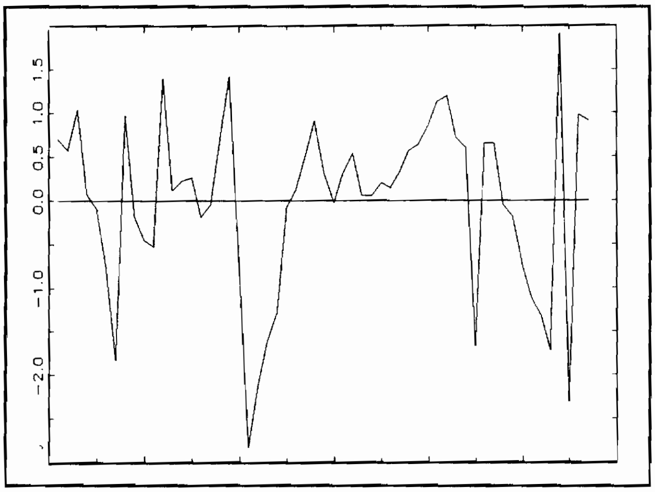
By estimating

$$LnY_t = a_0 + a_1 t + a_2 t^2 + a_3 LnK_t + a_4 LnL_t + a_5 LnGov_t + \eta_t \tag{5.16}$$

we find

$$LGnp_t = \underset{(1.083)}{3.186} + \underset{(.041)}{.214}\, t - \underset{(.0001)}{0.0004}\, t^2 - \underset{(.127)}{0.105}\, LnK_t + \underset{(.047)}{1.024}\, LnL_t + \underset{(.027)}{1.011}\, LnGov_t \tag{5.17}$$

| T = 57 | | Log Likelihood 123.48 | | Adjusted R .9968 |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.0050 | 0.2042 | 0.0091 | 0.3106 | 0.1074 |




Once again we observe conditional mean non-linearity of this series. It is interesting to note that in this case we do not experience autocorrelation as in all previous models so an alternative specification with dynamics will not remove our violations of the linear regression model.

**Figure 25 Residuals - Government non-military expenditure included**

## 5.4C. Research and Development and Innovation

The accumulation of new knowledge may also lead to increases in productivity. Romer (1986) includes new knowledge as an input in the production process. In this model, Romer assumes that new knowledge has its own production technology subject to decreasing returns to scale. As new knowledge is obtained, further advances in research are less likely. Knowledge cannot be perfectly patented and therefore the productivity of all firms can be enhanced by the discovery of new knowledge. It is this externality that can lead to an equilibrium with sustained growth. This form of externality was also utilized in Arrow (1962).

Grossman and Helpman (1993) also find that growth may be driven by innovation. Research and development is considered a partially public good. Again the presence of externalities leads to a non-pareto optimal result without subsidies. However, research and development may increase the productivity of inputs by creating new capital products and therefore increase output growth.

Testing the Implications of Innovation on Output Growth

A measure of innovation is very difficult to find. Research and development expenditures are not accurate since these expenditures may or may not lead to discoveries and discoveries certainly do not increase in a deterministic manner with respect expenditure. Only if the discovery process is a given function of the amount spent would expenditure provide information regarding the true amount of innovation. I believe these are correlated, but I do not think any expenditure can assure success in discovery. An alternative measure would take into consideration the outcomes of investment in R & D and therefore would capture the stochastic behavior of discovery. By looking at the outcome of the project, as opposed to input (funds expended), we avoid this lagging and stochastic effect.[9]

The number of patents issued by the US patent office in a particular year is a measure of innovation which increases as more discoveries are made. So long as patent standards remain unchanged, the reliance on this governmental body for a judgment of quality of innovations seems reasonable. This measure accounts for discoveries which are deemed significant by the supervising agency and rewards the creation of a new technology only once.

Using the same empirical model, but by including the log of the number of patents issued as a measure of growth in innovation as our growth driver, we can determine the significance of innovation and development in output growth.

By estimating

$$LnY_t = a_0 + a_1 t + a_2 t^2 + a_3 LnK_t + a_4 LnL_t + a_5 LnInno_t + \varepsilon_t \tag{5.18}$$

we find

$$LGnp_t = \underset{(.679)}{3.343} + \underset{(.003)}{0.234}\,t + \underset{(.0004)}{0.0001}\,t^2 - \underset{(.091)}{0.209}\,LnK_t + \underset{(.035)}{1.041}\,LnL_t + \underset{(.021)}{0.042}\,LnInno_t \tag{5.19}$$

| T = 62 | Log Likelihood 131.04 | | Adjusted R .9975 | |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.0202 | 0.2340 | 0.0001 | 0.2571 | 0.2220 |

We find misspecification similar to those in our previous models. These include non-normality, non-linearity and heteroskedasticity

Just as in the case of government expenditures, the inclusion of the endogenous growth factor somehow has accounted for the dynamic structure of the Solow Residual. Utilizing a dynamic model is not in order given the value of our independence test statistic.

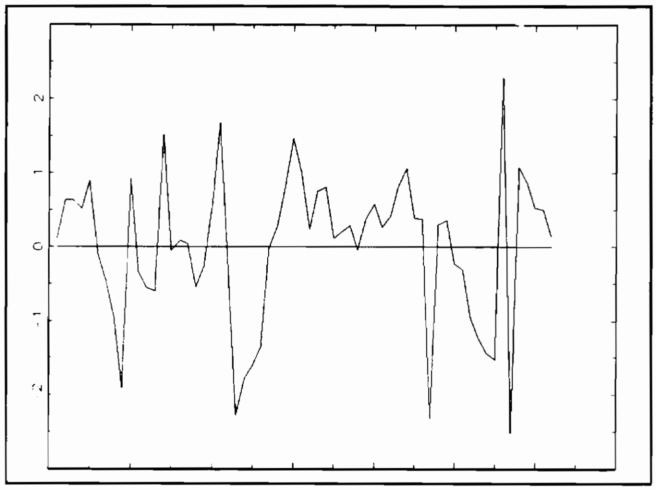


Figure 26 Residuals : # of Patents Granted Included

Our hypothesis is not supported by this model, but the continual presence of non-normality through both conditional mean non-linearity and conditional heteroskedasticity in the variance indicate misspecification error data in our sample and therefore drawing meaningful conclusions from our data are not possible.

## 5.5 Misspecification Alternatives

We have observed each of these growth engines can have an influence upon the growth of output. However, due to the presence of continuing misspecification in these models, it is not possible to draw inferences from the magnitude or direction of each effect. Specification testing has indicated in each case that the coefficients in the regression relationship cannot be trusted as the functional form of the distribution is somehow nonlinear and therefore our assumptions are not valid.

## 5.5A The Issue of Stationarity

Azariadis and Drazen (1990) employ technological externalities with a threshold property to explain the divergence of growth rates among various countries. In this model, as some input increases, in their case human capital, output will reach a range of critical mass. At that point, the economy can leave one growth path and move to another with an increased (or decreased in the case of lowered inputs) productivity. The notion of social returns of development as used by Romer (1986) is also important here. Increases in human capital benefit the society as a whole. It is the stock of human capital in the society as a whole which may lead to a higher growth rate through this jump to a new path.

The authors continue on to test this hypothesis using cross section data. By including a measure of human capital, the literacy ratio, they show that countries in low and medium income groups converge. The lack of variation in literacy rates in high income countries lead to insignificant results on convergence.

However, I propose an alternative test of this hypothesis which is more directly in line with the theoretical formulation. Using a time series approach, we can test if indeed a country has progressed from one balanced growth path to another. If a jump from one rate of growth to another has occurred, our coefficients will not be independent of time. By constructing window estimates, where the sample remains of the same size while the actual years utilized change, then we may find a change in the rate of growth.

Actually, the rate of growth is notoriously non-stationary. As we move across the sample, coefficients in our models vary widely, first increasing and then decreasing. The productivity is changing throughout the sample. This indicates a relationship more complex than the structural change suggested by Azariadis and Drazen (1990).[10]

## 5.5B. The Presence of Conditional Mean Non-Linearity

Throughout our study, we have found empirical evidence of conditional mean non-linearity. To employ the log linear model in regression, we must be attempting to capture a relationship with conditional mean linearity as implied by the log normal distribution. Due to the results of our analysis, we find that growth paths in the twentieth century United States do not follow a simple exponential path with a constant rate of growth. This has been evidenced by the ineffectiveness of the logarithmic transformation of this model to capture the rate of output growth.

We must find a way of estimating a more complex growth pattern which will be statistically valid. We can utilize the information given to us in this analysis by assuming that the growth rate of output (and inputs as well as shown in the previous paper) exhibits mean non-linearity.

Models with alternative distribution assumptions have been utilized in various circumstances are the reverse of growth. In modeling depreciation, economists have utilized a Weibull distribution which allows a rate of decay which varies with the age of the capital good. Also, medical statisticians have turned to non-linear distributions to model the effects of various diseases. In some cases, the rate of death is held constant and does not depend upon duration. In this case, the hazard function is a constant and an exponential distribution is employed. In other cases, a more complex hazard function is used which relates age or the duration of the disease to the rate of growth and a Weibull distribution is utilized. In all cases, a maximum likelihood function is used to determine the rate of decline.

Similar modeling should be possible with a model of growth. The growth rate can be a function of various attributes of the economy we already believe are significant. The growth path of the economy, however, can be assumed to follow a more complex pattern and modeled using a more appropriate distribution which allows for a non-linear conditional mean.

## 5.6 Conclusion

Modern growth theory has lead to the endogeniuty of the rate of technological progress. Rather than just acknowledging growth in technology and assigning any residual to its effect, modern theorists have attempted to model the change in productivity as well.

As shown in this paper, various possible sources of this growth are important. However, they are not sufficient for modeling the rate of growth of output. Non-linearity implies the rate of growth of output still exceeds the rate of growth of inputs, even when productivity enhancing inputs (in the Solow Framework) are included.

Rather than continue to search for some factor to account for the seemingly increasing returns to scale relationship between output and inputs, why not search instead for an alternative functional form? By utilizing techniques already applied to decaying processes, we may be better able to relate the growth of output to input increases. We relax the standard theoretical assumption of constant returns to scale and explore other possibilities. While this assumption is important in theoretical models to assure solvability through convergence to a steady state pattern, such a requirement is not required in an empirical sense.

Kaldor's Stylized Facts (1962) were a motivating factor in the formulation of growth models. These facts, however, do not conform to economic reality. It is essential we determine the process of output growth so that we may control it to some extent. With the endogenuity of various factors like education and government policy, we can now influence growth. However, we need to determine the nature of the relationship between inputs and outputs once this relationship is determined. We then can utilize our results to influence economic growth in a predictable manner.

## 5.7 Notes

**1** Labor saving or capital saving technological advances may also be modeled by modifying the basic constant returns to scale production function. In this case, however, the technological advance affects one factor by more than the other. For example,

$Y = F(AK, L)$ or $Y = (AK)^{\alpha} L^{(1-\alpha)}$ in the capital saving case

or $Y = F(K, AL)$ or $Y = K^{\alpha}(AL)^{(1-\alpha)}$ in the labor saving case.

**2** This specific form of the technological change measure will lead only to neutral changes in productivity. That is, as technology improves through increases in $A$, the productivity of capital and labor will change proportionately. Both labor and capital are made more productive so we do not see substitution between the factors of production. It is possible to incorporate labor-saving or capital-saving productivity improvements, but this would require an alternative mechanism of technological improvement.

**3** In our formulation, this too will be represented as a neutral improvement in technology which increases the productivity of both labor and capital. Obviously, we could amend this approach through our choice of representing the technological shift as labor-saving, which would be more realistic. This is a future extension of interest.

**4** Romer allows capital goods to be heterogeneous, an example of the what are referred to as vintage capital models. We can consider *m* varieties of capital each with its own productivity.

$$K_t = (K_{1t}, K_{2t}, \ldots, K_{mt})$$

However, the empirical use of vintage models is limited to the availability of accurate data on the stock of different types of capital goods. We instead discuss a homogenous capital good whose productivity is considered to be a weighted average of the productivity of the categories of capital.

**5** Educational funding is provided by Federal, State and Local levels of government in the United States. The amount of funding from each sector of government varies depending upon state and locality.

**6** We cannot assume that all learning occurs in colleges and universities. Learning gained at these institutions may or may not increase productivity on the job or have a positive externality effect on society. Ideally, I would prefer a cumulative measure of human capital. However, a perpetual inventory system would be much more difficult to derive. Human capital remains productive and actually may increase due to learning by doing effects. However, eventually these educated persons will leave the workforce or hold only obsolete knowledge. In this sense, human capital does follow a depreciation path. It is subject to the same debate as gross and net capital stock due to the characteristics described above.

**7** In a competitive market setting, the allocation decision between various forms of capital is determined by the firm based upon the marginal product of the capital input. This assures the most efficient allocation of resources among capital goods. However, in this setting, we are not in a competitive environment. The cost of pubic capital is paid by all members of society. Also, the decision of how much public capital to employ is not in the hands of the firm. Increases in public capital are determined in the political arena and firms simply react to potential increasing costs of private capital as well as the increasing marginal product.

**8** Government military expenditures are not included due to the spiky nature and also because the productivity of private resources may rise as a result of national defense, but by less than the increase due to non-military spending. Again, we could treat this as a heterogeneous capital input with differing marginal products and effects on firm productivity. If more and more military oriented projects aid households and firms in the United States (such as the activities of the Army Corp. of Engineers), then the use of military capital would be warranted.

**9** This decision is made to strengthen the relationship between innovation and growth. That is, we want to see the benefits of actual improvements in technology. The relationship between research expenditure and growth is the key question of interest in terms of policy. We may partially control the success of R&D through rewards for innovations and other company level incentives. However, there still exists some luck in creating new goods and technology. Mechanisms which increase the likelihood of successful discovery will lead to the effects displayed by this model. Increasing R&D expenditure may be one such mechanism.

**10** This model may be an indication of how we need to model such structural change. If some factor is driving occasional increases in the productivity of inputs as shown by altered coefficients on our regressors, then by inclusion of this variable, we should eliminate variability in the coefficients. We would in effect be modeling this structural change from one model to another. For example, lets consider expenditure on military. Specifically, what if increases in military expenditure of more than some percentage of GNP were sufficient to increase the productivity of inputs in an significant way permanently. Then we include percentage expenditure on military in our growth linear growth regressions. If this value is large enough (military expenditure is large), then the coefficients would change with it. The formulation of this discontinuity would be critical. In contrast, if military expenditures continuously effected the productivity of inputs, we would account for this in our model and would not find non stationarity of our parameters.

# *VI. International Comparisons of Output Growth Patterns*

Numerous studies of economic growth behavior have utilized cross sectional studies of output behavior in order to determine the pattern of development experienced by nations. For example, Romer (1989) looks at key productivity indexes in a variety of nations in particular years. Many other studies have used this same data, a very important data set developed by Summers and Heston (1988).

While this cross sectional approach can answer particular questions regarding the convergence of various nations to a uniform growth path, the estimates of growth relationships here cannot be applied to any nation in particular since the coefficients derived are some combination of the production characteristics of the numerous nations in the sample. This issue was discussed thoroughly in Chapter 3.

Through a time series examination, we can assess the differences and similarities in the functional form of the growth equation in each country and the coefficients relating input increases to output growth in each nation. Implications for the interpretation of cross sectional estimates will follow based upon these similarities and differences.

In Section 1 below, we will review the formulation of an empirical model based on the Solow model. We will then apply this empirical model to the United States in Section 2. This same empirical model is then examined utilized data from three other nations in Section 3. Section 4 discusses the possibility of simultaneity in international growth patterns. Conclusions then follow.

## 6.1 Theoretical Foundations : Solow Theory to an Empirical Model

As a starting point of our empirical analysis, we will look at the standard Solow model of economic growth with exogenous technological change. This standard model has been extended in a number of ways to include richer characteristics, but we will utilize a simple form of the model.

In the Solow model (see Solow, 1970 for an excellent development), a homogenous output product is created through a constant returns to scale production technology which utilizes capital and labor. We see constant marginal productivity overall, but diminishing productivity of capital and labor when either capital or labor is increased respectively. In addition, the production function is also indexed by a measure of the overall technology level given in an exogenous constant, $A$. This production function, therefore, takes the form

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} \tag{6.1}$$

We assume that capital is a homogenous good created by the saving of output. We assume that capital does not depreciate, but this assumption can be modified with small changes in results. Labor increases at an exogenous rate of growth $n$. Diminishing returns to scale in each input will lead us to a steady state growth path with output increasing at the exogenous rate n plus the rate of growth of technological progress shown by the series $A$ in the production function.

Using the Solow model, we see that output growth can be predicted under these assumptions. The rate of growth of output depends upon the rate of growth of labor, the rate of increase in technology, the parameters underlying the production function, $\alpha$, and the rate of saving or capital formulation.

In order for a theoretical model of economic growth to be consistent with the Linear Regression Model, we need to find a linear formulation of the Solow growth process. Since growth proceeds continuously, we have

$$Y_t = Y_0 e^{gt}; L_t = L_0 e^{nt}; K_t = K_0 e^{gt}; A_t = A_0 e^{at} \text{ where } g=n+a. \tag{6.2}$$

By a logarithmic transformation of the above growth paths and through the use of (6.1), we find that

$$LnY_t = \beta_0 + \beta_1 t + \beta_2 LnK_t + \beta_3 LnL_t \tag{6.3}$$

In the case that the production function exhibits a Cobb Douglas Constant Return to Scale format, then $\beta_2$=1-$\beta_3$ in (6.3). In this general representation of the output growth pattern, $\beta_1$ corresponds with a exogenous rate of technological progress and $\beta_2$ and $\beta_3$ are the elasticity of output with respect to labor and capital. This forms an estimable form of the Solow model since all are observable. Since output patterns may vary due to unobserved factors in the economy or since we may not be able to measure output and inputs without some error, this process also contains a stochastic error series, $\varepsilon_t$. I will not specify exactly what factor or factors cause this error, just that we cannot predict the pattern of output growth deterministically.

Methodology - the Difference between Statistical and Theoretical models

Before we proceed with the estimation of our model developed in the previous section, it is important to think about how we should interpret our estimated results. We will be utilizing the above empirical model as a means of capturing the behavior of output with respect to increasing productivity and input usage. However, to be certain that this model is adequate in capturing the behavior of output, it is necessary to view our model as a statistical approximation. We must be certain that the model satisfies the assumptions imposed by our estimation method and that all systematic information relating output and inputs is captured by this model.

To examine the performance of this model, we will utilize misspecification testing of the statistical model relative to the assumptions imposed by the linear regression model and our estimation method, ordinary least squares. OLS performs a linear approximation relating our dependent variable to the regressors. Linearity, is therefore a requirement in OLS. Also, ordinary least squares is capable of only finding a constant variance not dependent upon the regressors so heteroskedasticity is required. In this model, we have included no dynamic terms therefore we have assumed temporal independence in the model. We also have assumed that the underlying distribution assumptions do not change across the time period of our model so our random variables remain identically distributed or at least to second order stationarity.

Only after we have determined that these statistical qualities have been met by our model may we put some degree of confidence in the coefficients estimated and go beyond the statistical model to reexamine theory. If these assumptions are not met, then analysis beyond the statistical model is not meaningful.

## 6.2 Misspecification Testing with United States Data

In this section, we will briefly review the results of estimation given in Chapter 4. We begin by estimating the statistical model

$$LnY_t = \beta_0 + \beta_1 t + \beta_2 LnK_t + \beta_3 LnL_t + \varepsilon_t \tag{6.4}$$

for Real Gross National Product, Real Private Non-Residential Capital Stock and Average Manhours Per Year for the time period 1925 to the present (measured in constant 1982 dollars) in the United States and we find

$$LnGNP_t = \underset{(.309)}{3.308} + \underset{(.001)}{0.022}\, t + \underset{(.034)}{1.021}\, LnL_t - \underset{(.033)}{0.134}\, LnK_t \tag{6.5}$$

However, before we place any sort of theoretical interpretation upon these coefficients, we must determine the ability of this model to satisfy the assumptions imposed in estimation. That is, we must conduct misspecification tests of the model.

Our misspecification testing is guided by the assumptions imposed in the above statistical model. We must test to determine if the conditional distribution can be considered to be normal (through the assumptions of conditional mean linearity and conditional variance heteroskedasticity as well as the moments of the conditional distribution), if we see second order stationarity and if we see temporal independence. To test for heteroskedasticity, we determine if the regressors and higher order terms of these regressors are capable of systematically explaining the conditional variance, as measured by the sample equivalent. We examine the explanatory power of non-linear and dynamic terms in capturing the behavior of the logarithm of output to judge the assumptions of linearity and independence respectively.

Selection of the specific test to utilize in misspecification analysis may in some circumstances make a difference. The particular form of the alternative hypothesis may or may not matter depending upon the functional form of the actual conditional moments. The following tests were utilized.

Normality by Skewness : The D'Agostino test for Skewness
Normality by Kurtosis: The D'Agostino test for Kurtosis
Homoskedasticity: Reset test of order 2 for conditional heteroskedasticity
Linearity: Reset test of order 2 for conditional non-linearity
Independence: Modified LM residual autocorrelation test of order 2

Misspecification test results are reported in terms of p values. A p value is a very versatile tool that allows the reader to impose his or her own level of significance in test results. High p values indicate a high level of confidence that we cannot reject the null hypothesis. I will utilize a 95 percent confidence level for my analysis, but due to the continuous nature of the p value measure, the reader may reinterpret these results.

Utilizing misspecification tests and the statistical model estimated above, we find

$$LnGNP_t = \underset{(.309)}{3.308} + \underset{(.011)}{0.22}\,t + \underset{(.034)}{1.021}\,LnL_t - \underset{(.033)}{0.134}\,LnK_t \tag{6.6}$$

| T=62 | Log Likelihood 128.95 | | Adjusted $R^2$ .9973 | |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.0006 | 0.0961 | 0.0295 | 0.3635 | 0.0365 |

By the above table, we see the original Solow Model violates our assumption of conditional normality through skewness and conditional mean non-linearity. We also have violations of the assumption that past values of the regressors are independent from the current value of logarithm output. That is, we have violated the assumption of temporal independence. Just as in Chapter 4, these misspecification results can also be supported through a graphical examination of the residuals.

However, we should not view these misspecification results as a necessarily unwelcome outcome. Instead, we can learn from our errors. These misspecification tests have directed us toward a richer specification of our statistical model. This richer specification, using the same underlying random variables, may be better suited to the data we have on hand and may better imitate the behavior of output. We can attempt to improve our statistical model by adding a quadratic trend and dynamics to find :

$$LnGnp_t = \underset{(.775)}{1.593} + \underset{(.038)}{0.108}\,t - \underset{(.0001)}{0.0003}\,t^2 - \underset{(.146)}{0.116}\,LnK_t + \underset{(.058)}{0.745}\,LnL_t + \underset{(.126)}{0.607}\,LnGnp_{t-1} - \underset{(.120)}{0.125}\,LnGnp_{t-2}$$
$$- \underset{(.219)}{0.03}\,LnK_{t-1} + \underset{(.152)}{0.104}\,LnK_{t-2} - \underset{(.113)}{0.114}\,LnL_{t-1} - \underset{(.113)}{0.171}\,LnL_{t-2} \tag{6.7}$$

| T=60 | | Log Likelihood 143.39 | | Adjusted $R^2$ .9984 |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.129 | 0.392 | 0.048 | 0.954 | 0.621 |

From this table, we note that the new estimated model is nearly statistically adequate. We can see above that we still see the non-linearity in the conditional mean is still troubling since this assumption is only acceptable at the 5% confidence level through rounding. Utilizing alternative tests of linearity, we may further test this assumption and find that the KG Polynomial test of order 2 rejects the assumption of linearity with a p value of .02.

However, it is important to note the vast improvement in our model resulting from the inclusion of dynamic terms. This was not predicted by the Solow model, but is an important component to the growth pattern.

Interpreting the US model

Due to the possible presence of conditional mean non-linearity, we must view the coefficients and standard errors of our statistical model with suspicion and avoid using these estimates to draw theoretical interpretations. However, the presence of temporal dependence indicates an important characteristic not predicted by the Solow Model. There is an evolutionary element to the growth process. The non-linear trend terms may be significant indicating that the rate of output growth is increasing at a decreasing rate. However, due to continuing misspecification error, we cannot be certain of the importance or even the sign of these terms.

One conclusion that we can safely draw is that the Solow Model does not explain fully the pattern of output growth in the twentieth century United States. The formulation without dynamic terms is misspecified given the data we have utilized and therefore the Solow formulation which assumes a constant rate of growth is not capable of explaining all of the important characteristics of output growth.

## 6.3 The Growth Pattern of Other Nations - UK, Sweden, Japan

The United States provides us with just one sample observation of a nation's growth pattern in the twentieth century. For comparison purposes, we can also examine the same output growth relationship in other nations.

We would not expect the exact same pattern of growth in each nation. Differences in resource endowments, the nature of the production function and even the type of output produced would result in differing production function characteristics. Institutional characteristics which affect markets would also be expected to alter values in the production function as well as in our ability to determine output in any particular year (that is the error terms).[1]

It is exactly this list of characteristics which leads me away from the cross sectional approach of many empirical growth studies. A requirement of any sample is that the observations in each sample are identically distributed. When we define a sample, it is important for the random variable (i.e. output, labor,...) to retain the same probability structure across the sample period. In time series, this retaining of the same underlying structure is referred to as stationarity. In cross sectional studies, we also must retain this property of identical distributions in our observations, or application to any element in the sample is not meaningful. Defining elements of the same sample is important in cross sectional as well as time series studies. Due to differing institutional factors, it is not possible to assume that output given the value of inputs is distributed in the same way across all nations.

Despite differences in parameter values which are anticipated, it is still interesting to estimate the same statistical model for various nations. In this way, we can observe similarities and differences in the functional form and parameters of the growth function. Both the similarities and differences are of interest to us. I would also like to see if any nation conforms to the empirical model implied by the Solow framework.

### 6.3A The Data Series

The countries which I have selected for this study are those referred to by Klein (1977) and Maddison (1980) as the leaders in particular periods of economic development (see Table 1, Chapter 1). All are industrialized nations throughout the sample, and therefore some similarity in their institutional arrangements is expected. Comparing industrialized and less developed nations can be expected to have even more pronounced differences. We do not, however, expect to see exactly the same pattern of development in even these industrialized nations due to differences in resources and in positive and negative shocks to inputs and productivity.

The nations considered here are Sweden (pre-twentieth century), the United Kingdom (early twentieth century leader), the United States (mid-late twentieth century leader) and Japan (potentially, the leader to come). Data on inputs and output was gathered from various hard copy and on-line sources including an excellent summary by *Economic Statistics, 1900-1983* published by The Economist. The sample period varies by nation with the following available. The sample periods are:

| | |
|---|---|
| Sweden | 1890 - 1990 |
| United Kingdom | 1890 - 1990 |
| United States | 1925 - 1991 |
| Japan | 1948 - 1991 |

Pre-World War Two data is also available for Japan, however, a sizable gap from 1944-1947 makes this data less useful.[2]

## Modeling Output Growth in Different Nations

We estimate the same growth equation now with each nation in turn. That is,

$$LnY_t = \beta_0 + \beta_1 t + \beta_2 LnK_t + \beta_3 LnL_t + \varepsilon_t \tag{6.8}$$

The results relating to the international models are contained in Table 2. We can see through this table that we do find misspecification errors very similar to those we found in models of United States output growth. Specifically, we find consistent violations of our assumptions of normality in the conditional distribution, usually due to non-linearity in the conditional mean. We also find temporal dependence as in the United States case.

Table 2 International Growth Regressions

| | US | UK | Sweden | Japan |
|---|---|---|---|---|
| Constant | 3.425 (.696) | -10.44 (1.55) | -0.024 (.176) | 10.952 (2.16) |
| Trend | 0.022 (.003) | 0.007 (.001) | 0.022 (.002) | 0.058 (.008) |
| Trend - Sq | 0.0000 (.000) | 0.0001 (.000) | 0.0001 (.000) | - 0.001 (.000) |
| Ln Employed | 1.021 (.035) | 1.446 (.152) | -0.018 (.023) | - 0.599 (.538) |
| Ln Gross K | -0.149 (.088) | 0.095 (.019) | 0.264 (.034) | 0.274 (.033) |
| | | | | |
| Skewness | .001 | .00002 | .192 | .206 |
| Kurtosis | .091 | .00007 | .0003 | .422 |
| Linearity | .0001 | .00000 | .020 | .001 |
| Homoskedastic | .375 | .00010 | .141 | .182 |
| Dependence | .038 | .00000 | .00000 | .00000 |

This similarity in misspecification error is interesting in its own way. This shows us that not only is the log-linear growth relationship between current output and current inputs statistically inadequate in the United States, but it is statistically inadequate in these industrialized nations as well. Specifically, this model fails in the same respect in each nation, conditional mean non-linearity and temporal dependence.[3]

Just as with the United States model, we can utilize the results of our misspecification tests to respecify a statistical model which may be statistically well suited to the data on hand. Conditional mean non-linearity leads us to include a non-linear trend term in the regression (or conditional mean) equation. The importance of temporal dependence leads us to include dynamic terms in each regression.

The respecified model takes the form

$$\begin{aligned} LnGnp_t = {} & \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 LnK_t + \beta_4 LnL_t + \beta_5 LnGnp_{t-1} + \beta_6 LnGnp_{t-2} \\ & + \beta_7 LnK_{t-1} + \beta_8 LnK_{t-2} + \beta_9 LnL_{t-1} + \beta_{10} LnL_{t-2} \end{aligned} \tag{6.9}$$

These results are reported below. Note from Table 3 that we still see misspecification error often in the form of conditional mean non-linearity. This is not the case for Japan. However, the small degrees of freedom in this regression given the limited Japanese data employed should caution us in placing confidence in these p values.

<u>Interpreting International Growth Patterns</u>

While we do find continuing misspecification error in our models of output growth, we do find some interesting results. The inclusion of dynamic terms, especially the lagged value of output, is very important in explaining output growth in both the United States and in other nations. This fact is in support of an evolutionary model of economic growth proposed by Nelson and Winter (1985). Our continuing misspecification also seems to be of a similar form to the error in the United States growth model. Such similarity was not predicted a priori and does, to a certain point, indicate that these industrialized nations may follow similar growth patterns with differing parameter values dependent on resource endowments.

Numerical values of coefficients should be interpreted with caution. First, with continuing misspecification in the UK and Swedish models tells us that these coefficients are not trustworthy. Also, the numerical values of data utilized are misleading in comparing coefficient values. However, due to statistical adequacy in the case of United States and Japan, we can interpret these coefficients, standard errors and signs.

Table 3: Respecified International Regressions

| | US | UK | Sweden | Japan |
|---|---|---|---|---|
| Constant | 1.593 (.775) | -2.865 (1.36) | 0.004 (.156) | 3.866 (2.075) |
| Trend | 0.011 (.004) | 0.001 (.001) | 0.007 (.003) | 0.017 (.009) |
| Trend-Sq | -0.000 (.000) | 0.00002 (.000) | .00001 (.000) | -0.0002 (.000) |
| Ln Employed | 0.755 (.059) | 0.195 (.106) | 0.011 (.040) | -0.473 (.568) |
| Ln Gross K | -0.116 (.146) | 0.084 (.014) | 0.180 (.056) | 0.201 (.072) |
| Ln Gnp (-1) | 0.607 (.126) | 1.134 (.106) | 0.449 (.105) | 0.998 (.250) |
| Ln Gnp (-2) | -0.126 (.120) | -0.390 (.103) | 0.170 (.105) | -0.188 (.202) |
| Ln Empl (-1) | -0.114 (.113) | 0.004 (.112) | -0.005 (.055) | 0.029 (.494) |
| Ln Empl (-2) | -0.171 (.113) | 0.157 (.117) | -0.008 (.040) | -0.107 (.518) |
| Ln Gr K (-1) | -0.003 (.220) | -0.075 (.019) | -0.066 (.075) | -0.186 (.076) |
| Ln Gr K (-2) | 0.104 (.152) | 0.053 (.014) | 0.023 (.059) | 0.028 (.006) |
| | | | | |
| Skewness | .129 | .000000 | .00000 | .308 |
| Kurtosis | .329 | .000000 | .00000 | .422 |
| Linearity | .074 | .891 | .641 | .157 |
| Homoskedastic | .942 | .462 | .588 | .815 |
| Dependence | .542 | .325 | .426 | .449 |

## 6.4 Simultaneity in Economic Growth

It has been widely noted that output growth rates tend to be highly correlated across nations (Romer, 1989). Apparent similarities in the paths of output growth in these nations may indicate an important joint dependence in growth. This may exist due to a particular form of shocks to the level of inputs or productivity.

We would anticipate that the shocks affecting macroeconomic variables in each country could be linked. This may lead to similarities in the error series of nations. That is, a negative productivity shock which affects each of our nations could result in a particularly large error term for that particular time period and this error shock exists in each nation.

In addition, we may believe that growth performance in each nation may affect the others through trade ties. We have noticed, particularly in the last few years, a movement toward greater global dependence. Reduction in the costs of transporting both input and output goods as well as reductions in the transaction costs associated with trade barriers will lead to a greater reliance on other nations. This increasingly close relationship could explain our observed correlation of growth rates.

This idea has been explored extensively in the literature related to regional growth For a key reference, see Baumol (1967). While not directly applied to international trade, this model of trade within regions of a nation seems to be increasingly more appropriate for international applications. Due to decreasing barriers to trade on an international scale, we may be able to consider nations as simply 'regions of the world' with differing resource endowments.

The wealth of results developed in regional models of growth provides a rich starting point for international models. We would expect to see similar results with international, as opposed to interregional, trade. Baumol (1967) studies two regions with differing growth patterns. One region is stagnant while the other grows. This study finds that the non-progressive region will tend to decline in output while the growing region continues output growth. Simon (1982) shows that this will not always be the case, unless the goods produced in each region are close substitutes.

Empirical evidence of this increasing income inequality among regions of a nation has been supported by Amos (1988) in a study of the states of the US. Baumol (1986) examined the growth performance of 60 economies and concluded that these 60 countries could be divided into 3 growth clubs, each with a different rate of progress. We see convergence to a single growth rate within these growth clubs, however, we see increasing or constant deviations in the rates achieved by these nations.

Dowrick and Gemmel (1991) have explored possible reasons for the simultaneity in interregional and international growth. They sight four possible explanations, (1) technological spillover, (2) intersectoral disequilibrium in factor markets, (3) sectoral differences in technical progress and (4) capital deepening. An empirical evaluation of international disequilibrium growth models suggests that growth clubs do exists, that differing rates of technological progress are evident between growth clubs and that there is substantial spillover from the industrialized sectors and nations to non-industrialized sectors.

Regardless of the reason for high output correlation among nations, we may be able to take advantage of this increased information through a simultaneous equation approach. By allowing the growth rate in country A to dependent upon the rate of growth in country B, then we may measure and anticipate this codependency.

## 6.5 Conclusion

It has been shown in this paper that the Solow Model of economic growth even with the inclusion of exogenous technological change is not able to capture output growth as a function of increases in inputs. This model fails in explaining United States performance, as well as output performance in the United Kingdom, Sweden and Japan. Through the inclusion of evolutionary growth terms, we approach an adequate statistical model explaining economic growth. However, the continuing presence of conditional mean non-linearity leads to a rejection of even this improved model.

Similarity in misspecification of the Solow model across nations indicates that a similar growth path may exist in each country. In addition, we see high correlation among output growth rates in nations indicating that there may be some significant simultaneity in growth rates.

The simultaneity of growth rates can be related to simultaneity in interregional rates of growth. We may extend our model of growth rates to incorporate these international patterns, much as has been done in interregional models of growth.

Further study is required to understand the nature and importance of this international codependency. I would like to extend this to a simultaneous model of output growth in two or more closely linked nations.

## 6.6 Notes

**1** We would expect, however, for these error terms to be linked in some way. That is, we would expect that some shocks will affect all nations while other shocks are nation specific. This element of simultaneity will be discussed more later in this paper and hopefully in subsequent research.

**2** These missing observations are just one problem with Japanese data. Also, we would expect large structural changes to affect the Japanese economy after World War Two and therefore, should not expect pre-war data to reflect the current production relationship. While this limits our ability to estimate growth patterns in post war Japan, to make assumptions about the war years and estimate throughout all data available would cause errors in interpreting current production and growth relationships. However, a more careful study of structural change in this economy is in order.

**3** This fact does lend some support to the use of cross sectional modeling. However, we must be cautious here. Just because two conditional means exhibit non-linearity, does not necessarily mean that they have a conditional mean of the same functional form. In fact, one may have a quadratic conditional mean, another may have a conditional mean made up of trigonometric functions. However, violations of the same assumptions and apparent acceptance of others (although we cannot be certain of this due to the misspecification) do lead us to believe these growth patterns may be somewhat similar.

# *VII. The Stationarity of United States Growth Processes*

In addition to assumptions regarding the functional form of our growth pattern and the dynamics involved in the estimation of such a relationship, empirical relationships must also be examined for another potential faulty assumption. This is Assumption *7* in the Spanos (1986) framework discussed in Chapter 3.

The underlying characteristics of our stochastic process must remain constant over our sample period. In this chapter, we will see why stationarity is a vital assumption in economic theory and in the estimation of our output growth function. We will look at the literature which has sought to explain various forms of non-stationarity and to deal with the implications of these results. We will examine some of these "fixes" with relation to our data and also look to quarterly samples as a possible method of satisfying the assumption of stationarity.

In Section 1 below, we will discuss the theoretical need for stationarity . We see that this assumption is necessary for the predictable movement to a steady state. In Section 2, we examine the requirements on the stationarity of moments imposed by at estimation. We will develop an empirical model to explain the increases in output as a function of improving technology and increases in capital and labor. In order to find a valid description of the relationship between output and inputs, the characteristics of the variables of interest must remain relatively fixed over the time period examined. In Section 3, we discuss methods of assessing the stationarity of a series. We look at Recursive Least Squares and Ordinary Least Squares estimates of windowed observations. These tests are applied to our empirical model developed in the previous section. Next, in the Sections 4 and 5, we discuss particular types of non-stationarity which have been popular in the theory of economic growth. In the first of these sections, we look at unit root non-stationarity by discussing some methods of diagnosing unit roots and then methods to compensate for non-stationarity through unit roots and cointegrating vectors. Then in Section 5, we examine the theoretical model of structural change or threshold economies. In Section 6, we attempt to compensate for changing distribution parameters by decreasing the sample period through quarterly observations. This is consistent with the structural change literature noted in Section 5. In Section 7, the Hamilton Switching Regime model, a particular model of structural change which allows a probabilistic framework, is considered and applied to United States Growth Processes. A summary of results and conclusions will then follow.

## 7.1 Why Stationarity? The Theory Side

In statistics, we define stationarity as a condition where the distribution underlying our sample does not change throughout all periods of the sample. That is, in the notation introduced in Chapter 3,

$$D_1(Z;\theta) = D_2(Z;\theta) = ... = D_T(Z;\theta) \tag{7.1}$$

We will discuss this definition of stationarity in great detail in later sections. This statistical interpretation will be the primary focus of Chapter 7.

In addition to this statistical meaning of stationarity, stationarity is also often referred to in theoretical economics. This is a common assumption found in economic models. There is a difference between theoretical stationarity and stationarity defined in statisitics. However, both definitions have a similar intuition that is interesting to explore. A classic example of this deals with a shock to income (Hall, 1978).

Assume that income is a stationary series. Lets consider what would happen if we were to win a lottery. It is permissible for income to vary from the norm for a period of time, however the response to this shock must die away in order for us to return to the same distribution. We will see that this effect must be transitory for income to retain its original characteristics. The length of decay is not really important for the theoretical result we seek. However, it must be the case that the influence of a change in income becomes, on average, smaller in each period. Eventually the elevated average of income will die out so that the stationarity of income is retained.

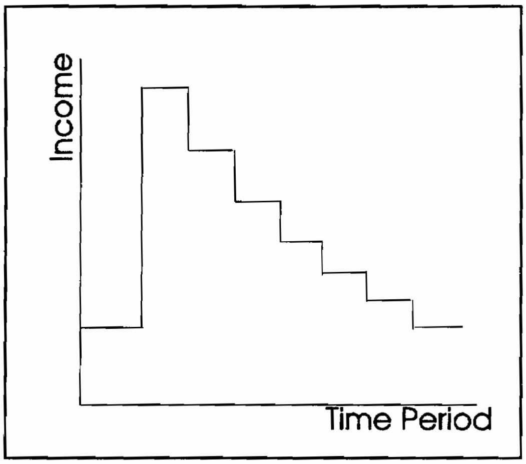



Figure 27 Behavior of Income when Stationary

We can see one such example on Figure 27. There are numerous forms of stationary response functions. In the first graph, we find a function that is typical in the sense that the response of income over time dies out slowly. After 7 periods, in this case, we have returned to the regular level. [1]

As an alternative, consider that income is non-stationary. In this case, we need not see transitory shocks. If we win the lottery in a given period, income will increase and can influence the average income as well. Without the assumption of stationarity, it is possible for the mean of income to change and therefore the effect of the shock need not dissipate over time. The characteristics of income

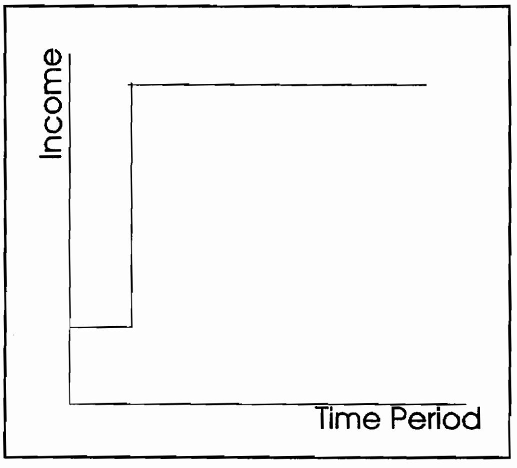



Figure 28 Behavior of Non-Stationary Income

are altered permanently by a single event in the history of income. This is displayed in Figure 28.[2]

The Real Business Cycle literature has dealt to a great extent with the behavior of income with respect to shocks as well as the implications on saving and consumption (See Kyland and Prescott, 1980). For the consistency of equilibrium, we require that our series be stationary. Otherwise, any shock to a variable will substantially change the underlying characteristics which in part determine the equilibrium value or path.

In addition to complicating theoretical models often to a point of insolvability, non-stationarity also causes severe complications in estimation of the empirical or statistical relationship underlying variables. However, this problem deals with the statistical interpretation of stationarity.

## 7.2 Stationarity Assumptions - the Empirical Side

Stationarity plays an important role in the estimation of economic relationships. This is true regardless of our method of estimation, be it Ordinary Least Squares (OLS), the Method of Moments or Maximum Likelihood. In order to get an accurate picture of the coefficients determining the relationship between two or more variables, the underlying characteristics of the data series must remain constant across the sample period.

We define $X(t)$ to be a *strictly stationary series if for any subset (t1, t2,..tn) and any* $\tau$*, then* $F(X(t_1),X(t_2),\ldots,X(T_n)) = F(X(t_1+\tau),X(t_2+\tau),\ldots X(t_n+\tau))$ (Spanos, 1986). This means that the distribution function does not vary regardless of the period of observation of $X$. The distribution itself remains unchanged. As such, stationarity is a restriction of the time heterogenuity of the stochastic series $X$ and therefore satisfies Assumption *7* in Spanos's framework (see Chapter 3).

Another very useful definition of a stationary series speaks of the moments of the distribution as opposed to the distribution itself. Since the moments of the distribution are much easier to observe, this is a more practical definition. This is referred to as *$l^{th}$-order* *stationarity*. Specifically, as *a series X(t) is said to be $l^{th}$-order stationary if, for any subset (t1, t2,..tn) and any $\tau$, $F(X(t_1),X(t_2),...,X(T_n))$ must have bounded moments up to the order l and have joint moments equal to the corresponding moments of $F(X(t_1+\tau),X(t_2+\tau),...X(t_n+\tau))$* (Spanos, 1986).

Very commonly in statistical analysis, we require a series to be second-order stationary. That is, the mean, variance and cotemporaneous covariance of the series must be constant. We require this order of stationarity often since this corresponds with strict stationarity in the case of the Normal stochastic process. A Normal distribution is fully defined by its first two moments. The higher odd moments of a Normal distribution are zero and the higher even moments are proportional to the variance (see Chapter 3 for more information on the importance of and testing of Normality assumptions). For alternative distribution assumptions, second-order stationarity may not be sufficient to ensure stationarity as these distributions are not characterized by the first two moments.

Lets take Ordinary Least Squares estimation of a Simple Linear Regresion Model as an example of the importance of stationarity in estimation. The simplicity of this method and the generality of its application will give us a picture of the importance of homogeneity assumptions.

When we relate two variables *X* and *Y* via Ordinary Least Squares we select the value of parameters β0 and β1 to minimize the expression

$$\underset{\beta_0,\beta_1}{Min} \quad \sum_{t=1}^{T}(y_t-(\beta_0+\beta_1 x_t))^2 \tag{7.2}$$

If a conditional normal distribution exists for *Y* and *X*, it can be shown that these coefficients are simply of combination of the means, variances of *X* and *Y* and the covariance of $X_t$ and $Y_t$ at any t (Spanos, 1994).

Specifically, we see that if

$$\begin{matrix} Y \\ X \end{matrix} \sim N\left(\begin{matrix} m_y \\ m_x \end{matrix}, \begin{matrix} \sigma_y^2 & \sigma_{12} \\ \sigma_{12} & \sigma_x^2 \end{matrix}\right) \tag{7.3}$$

then $$E(y|x) = \beta_0 + \beta_1 x \tag{7.4}$$

and $$V(y|x) = \sigma_2 \tag{7.5}$$

so that $$\beta_0 = m_y - \beta_1 m_x \text{ and } \quad \beta_1 = \frac{\sigma_{12}}{\sigma_x^2} \tag{7.6}$$

However, without second order stationarity, these numerical estimates mean nothing. The means and variances of $X$ and $Y$ must be constants if we can place any value on the results of these calculations. If we do not have second order stationarity, the accuracy of our coefficients for any pair of $X_t$ and $Y_t$, and therefore the overall coefficients, will be based entirely upon how close that particular observation is to the average mean and variance.

Similar reasoning applies to estimation by alternative methods. In the Method of Moments, we substitute the moments of the sample for the moments of the population as a whole (Judge et al, 1988). Such substitution has no meaning if we cannot determine accurately the moments of the sample alone. In maximum likelihood estimation, we search for values of the parameters of the conditional distribution which will lead to the most satisfactory characterization of the distribution function given the data we have at hand. Again, if the characteristics of our data change greatly throughout the sample, then we cannot trust our maximum likelihood estimates of these quantities since our estimates will simply be weighted averages of the parameters of a variety of different underlying distribution functions.

## 7.3 Testing for the Presence of Non-Stationarity

Stationarity implies that the value of moments relating to a distribution are constant. This also implies that mathematical combinations of these moments will also be constant. We can, therefore, test the assumption of stationarity either by directly observing the moments of the distribution across the sample or by examining the characteristics of estimated parameters. If our parameters estimators show no departure from consistency, then we can support the assumption of stationarity. These parameters need not be absolutely identical, since we recognize that the values observed may differ from the true mean or variance. However, as the size of the sample increases, we should find greater and greater similarity in our estimates (Spanos, 1986).

We will examine the stationarity of the coefficients involved in the standard economic growth model. In our empirical model, we allow exogenous technological growth as shown by Solow (1970) and as developed in Chapter 3.

## 7.3A Estimation of the Growth Path of Output

In deriving our empirical model, we look to Solow's model of economic growth resulting from increases in inputs and/or increases in technology. Increases in inputs result in higher output subject to the limitation of diminishing returns with respect to that factor. Since the Solow model proposes that these quantities follow constant exponential growth paths, we will use a logarithmic transformation to be consistent with the Multivariate Linear Regression Model discussed in Chapter 3.

Our first empirical model is the following:

$$LnY_t = \alpha_0 + \alpha_1 LnL_t + \alpha_2 LnK_t + \varepsilon_t \tag{7.7}$$

where $\alpha_1 = 1 - \alpha_2$ is a special case of a Cobb Douglas Production relationship and $A$ is a factor relating the level of technology. In this form, increases in $A$ will increase the productivity of both Labor and Capital. In order for us to believe that this theoretical model is appropriate for our data set we must examine all the underlying assumptions. We must see that the error term is free of non-normality through the examination of the conditional mean and variance. Also, the error term must not contain information relating to the behavior of past output or of the history of the inputs. We have examined our model's behavior relating to these assumptions in previous chapters, now we will look at the assumption of time stationarity.

Through a battery of misspecification tests as shown in Chapter 4, we find that this model is not statistically valid given twenieth century US data. We find non-normality through conditional mean non-linearity as well as strong autocorrelation of the errors indicating information remains in these residuals which can explain the time path of economic growth. The specific results of some of these misspecification tests are displayed below in the form of p values for the following tests.

Normality - Skew: The D'Agostino test for Skewness
Normality - Kurt: The D'Agostino test for Kurtosis
Homoskedasticity: Reset test of order 2 for conditional heteroskedasticity
Linearity: Reset test of order 2 for conditional non-linearity
Independence: Modified LM residual autocorrelation test of order 2

The specific results for this regression describing the relationship between the growth in output and growth in inputs follows.

$$LnGnp_t = \underset{(.185)}{-2.552} + \underset{(.044)}{0.428}\, LnK_t + \underset{(.083)}{1.323}\, LnL_t \tag{7.8}$$

| T=62 | | Log Likelihood 66.488 | | Adjusted $R^2$ .9805 |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.0357 | 0.4413 | 0.0078 | 0.0175 | 0.0000 |

The residual from this regression is usually termed the Solow Residual and is generally attributed to technological advance. However, we see our model does not capture all the information possible through changes in inputs. We can extract more information through the use of functional forms of variables.

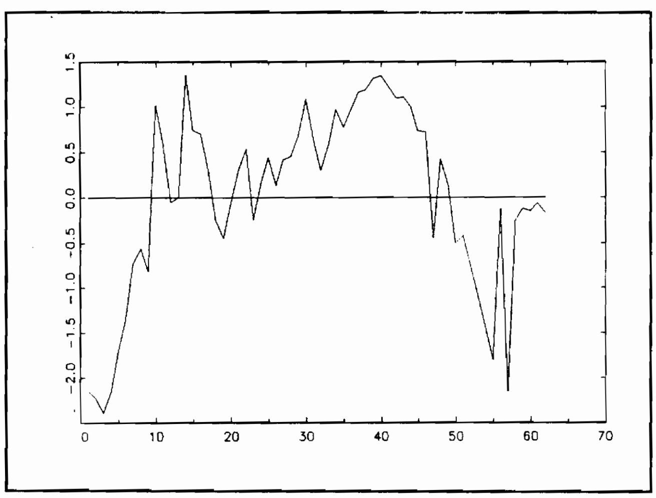



Figure 29 Solow Residuals for Annual Growth Data

Analysis of the residuals shown in Figure 30 also reinforce our statistical tests of misspecification. We see long swings in our residual series which indicate positive temporal dependence. We see non-normality indicated by the skewness of an empirical distribution function of these residuals.

If we have more information about the nature of technological growth, the Solow formulation with the inclusion of technological progress leads us to a slightly different empirical model. We cannot directly observe the true level of technological progress. However if a measure of technology, $A$, follows an exponential growth path with some exogenous rate of technological improvement, then we can extend the above to include the non-observed path of technical change. We can include the logarithm of the unobservable quantity $A$ indirectly through the inclusion of a constant indicating the natural logarithm of the technological constant and a deterministic trend term. The coefficient in relating the trend to output changes will determine the rate of productivity advancement, assuming this statistical model is accurately depicting the growth pattern.

This empirical model is of the form:

$$LnY_t = \alpha_0 + \alpha_1 t + \alpha_2 LnL_t + \alpha_3 LnK_t + v_t \tag{7.9}$$

and when estimated with our data, we find:

$$LnY_t = \underset{(.309)}{3.308} + \underset{(.01)}{0.22}\,t + \underset{(.034)}{1.021}\,LnL_t - \underset{(.033)}{0.134}\,LnK_t \tag{7.10}$$

| T=62 | | Log Likelihood 128.95 | | Adjusted $R^2$ .9973 |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.0006 | 0.0961 | 0.0295 | 0.3635 | 0.0365 |

This model is an improvement, however, it is still misspecified. Due to the low p values above, we see the presence of dependence, non-linearity in the conditional mean and non-normality in this conditional distribution.[3]

Utilizing our misspecification tests of Equation (7.10) as a guide, we can attempt to capture these characteristics by modifying our empirical model. In order to eliminate dependence, we can include the partial memory of output and inputs in the growth path. It was found that 2 periods of memory are necessary to capture the major influences of dynamics. To account for non-linearity, we include a higher order trend term. With these changes, the model now is

$$LnGnp_t = \underset{(.775)}{1.593} + \underset{(.038)}{0.108}\,t - \underset{(.0001)}{0.0003}\,t^2 - \underset{(.146)}{0.116}\,LnK_t + \underset{(.058)}{0.745}\,LnL_t + \underset{(.126)}{0.607}\,LnGnp_{t-1} - \underset{(.120)}{0.125}\,LnGnp_{t-2}$$
$$- \underset{(.219)}{0.03}\,LnK_{t-1} + \underset{(.152)}{0.104}\,LnK_{t-2} - \underset{(.113)}{0.114}\,LnL_{t-1} - \underset{(.113)}{0.171}\,LnL_{t-2} \qquad (7.11)$$

| T=60 | | Log Likelihood 143.39 | | Adjusted $R^2$ .9984 |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.129 | 0.392 | 0.048 | 0.954 | 0.621 |

As you can see above, have greatly improved upon the misspecification found originially. We may still have a remaining problem of conditional mean non-linearity depending upon the misspecification test and level of confidence chosen. We have not been fully able to capture this effect in our linear regression model. However, the inclusion of additional important factors has removed all other misspecification error.

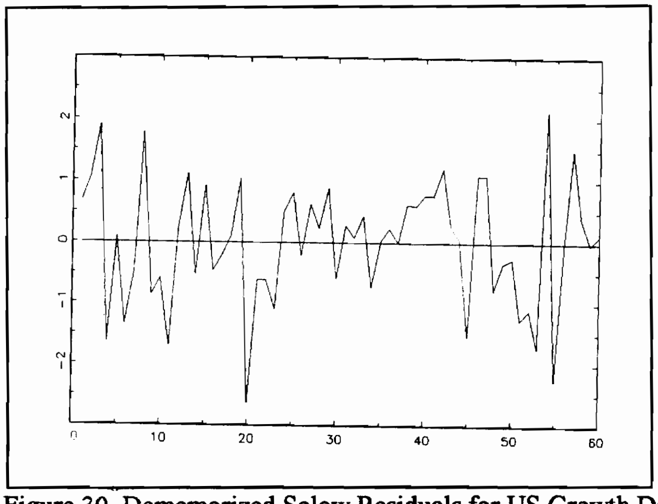


Figure 30 Dememorized Solow Residuals for US Growth Data

Our residual series in Figure 30 continues to display this feature of non-normality again through the skewed shape of the empirical density function, however compared to Figure 29, this residual series more closely approachs a residual consistent with the assumptions of a Multivariate Linear Regression Model.

## 7.3B Graphically Examining the Stability of β Estimates

As mentioned above, the coefficients in the statistical model are simply arithmetic combinations of the moments of the joint distribution of $Y$ and $X$. If these coefficients are consistent, then we have stationarity in our moments underlying the distribution. By varying the sample from which these coefficients were calculated, we can examine their consistency properties.[4]

We will consider two graphical methods to examine the stability of our estimates of the coefficients. One of these involves Recursive Ordinary Least Squares Regression (See Spanos 1986). We begin by selecting a subsample of our data (with $t >$ number of regressors) and calculating the growth regression as in the previous case. The variance and mean of each coefficient is calculated utilizing this small set of sample observations. This process is repeated by adding one more observation to the subsample of our data until the subsample increases to include all the observations. If stationarity exists, we should see convergence to a mean value of the coefficient and a decreasing variance indicating unbiasedness and that additional observations bring us closer and closer to the true value of the parameter.

Recursive Least Squares examined for Regression (7.11) shows non-stationarity in the moments of the joint distribution as indicated by inconsistency in the estimates. Our estimates of these coefficients do not settle to a single mean, nor does the variance remain stable. Example graphs are shown in Figures 31 and 32. Figure 31 is an example of a stationary RLS plot. Figure 32 is an example of one of our coefficients from Equation (7.11). In this case, we see the coefficient of Ln L. Other coefficients from (7.11) were similar in behavior.

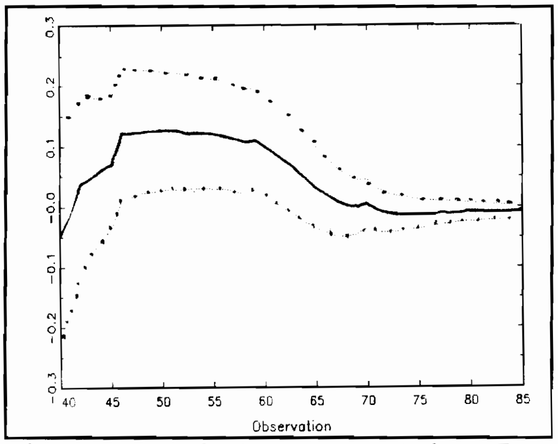



Figure 31 : Recursive Least Squares : Stationary Data

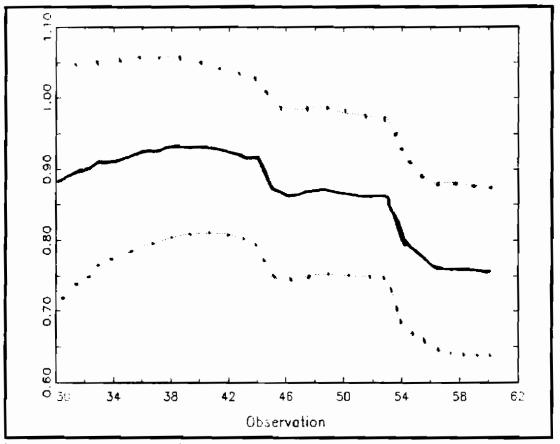



Figure 32 : Recursive Least Squares : US Growth Data

This lack of convergence in the coefficients estimated occurs for all the parameters of even our best growth model (Equation 7.11). That is, even when we consider the dynamic effects of former values of inputs and output and non-linear trending, we still do not see stationarity in the estimates of these relationships.

Similar evidence can be displayed through Ordinary Least Squares based upon a window of estimates. In this case also, we begin with a subsample of our total data set. This window is of a given size, greater than the number of independent variables. We calculate OLS estimates given this smaller sample size. In order to examine the constancy of these estimates, we slide this window of a given size across the entire sample period, adding one observation and taking one away. In this case, the number of observations utilized to calculate parameters remains constant, as opposed to Recursive Least Squares where the number of observation points increased in each successive iteration.

Due to this constant number of observations in the sample, we are not looking for convergence. That is, we do not expect to see a reduction in the variance. Instead, stationarity will be displayed when the estimates of the coefficients remain relatively fixed and the variance remains stable. An ideal WOLS (windows ordinary least squares) diagram would show a near constant value of the coefficient and a near constant variance. Graphical analysis of WOLS estimates can indicate if indeed our sample displays stationarity.

Through examination of a sample plot of the windows estimates of the parameters β in our dynamic growth path, we find evidence which rejects the assumption of stationarity. This is the coefficient of Ln L in (7.11). Other coefficients displayed similar behavior.

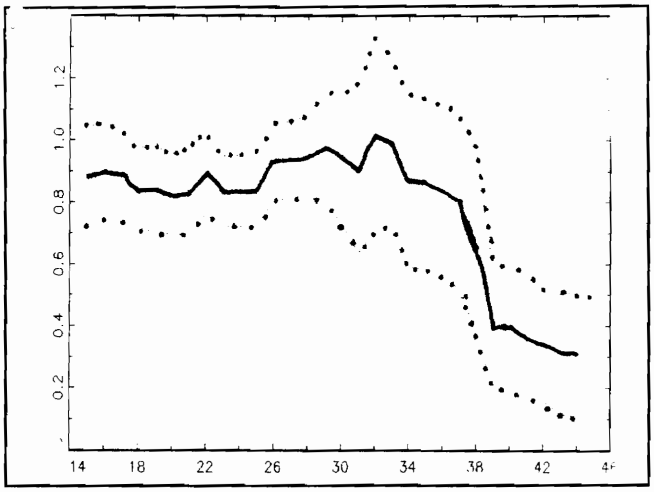



Figure 33 Windows Least Squares Estimate of (7.11)

Through examination graphical examination of RLS and WOLS estimates of coefficients in the United States growth model, we find evidence of non-stationarity. Diagnosising the form of non-stationarity and its source takes us to two branches of literature. Structural Change is explored in Section 4 and Section 5 looks at Unit Roots and Cointegration.

## 7.4 Unit Roots and Cointegration

In Chapter 3 and 4 we noted that non-linearity of the conditional mean or heteroskedasticity in the conditional variance allows for numerous possible representations. Eliminating one option does not give us grounds to accept the alternative utilized in testing. We now have a wide range of choices for the functional form, with only one specific form eliminated.

We see the same issue in time heterogenuity. We can reject the hypothesis that θ is time invariant (Assumption 7 discussed in Chapter 3, Spanos 1986). However, we do not as yet know how to characterize the non-stationarity of our time series.

There are many possible methods by which non-stationary behavior may be generated. One model of a non-stationary stochastic process is a particular AR(1) representation with a unit root. In this case, the value of current output depends entirely, on average, upon the value of output in the previous period. In this case, the estimate of the coefficient on lagged output in the AR representation would be one.

$$LnY_t = \beta_0 + \beta_1 LnY_{t-1} + \varepsilon_t \tag{7.12}$$

Whenever we encounter a unit root, AR(1) process, we find non-stationarity. To test for unit root non-stationarity, we can determine the estimate of β1 and utilize the estimate in a test to determine the presence of a unit root.[5]

Many economists have noted the presence of numerous unit roots in economic time series including output, capital, prices and trade behavior. A key summary of these results is discussed by Nelson and Plosner (1982). Hall (1978) also shows the unit root behavior of consumption and income. However, if we examine these unit root representation for the presence of misspecification errors, we find that the unit root hypothesis cannot be supported by most macroeconomic data. That is, the past value is not capable of capturing the behavior of the variable and therefore the empirical models are misspecified.

In order to test for non-stationarity, I examine United States Real Gross National Product, Real Gross Non-Residential Capital and Average Man-hours per Year for the presence of unit roots. All data was obtained or derived from Department of Commerce estimates for the period 1900-1990 for output and labor and 1925-1990 for capital.
On initial examination of Table 4, it does appear as though unit roots could be a problem based on the estimates of the coefficient on these lag terms. They do appear very close to one. However, before we fail to reject the null hypothesis that unit roots exist in these series, we must determine if the models are well specified. If not, any inference drawn from the models are invalid.

Table 4. AR(1) Results

| Variable (X) | Ln Real Gross National Product | Ln Gross Non-Residential Capital | Ln Average Man Hours |
|---|---|---|---|
| Constant | 0.061 (.060) | -0.088 (.048) | 0.1073 (.119) |
| Ln X(t-1) | 0.996 (.009) | 1.015 (.006) | 0.9798 (.025) |
| | | | |
| p value Test Results | | | |
| D'Agostino-Skewness | 0.001 | .1634 | .0046 |
| D'Agostino - Kurtosis | 0.0009 | .1044 | .0254 |
| Reset - Linearity | 0.7856 | .0517 | .7876 |
| Reset - Homosked | 0.051 | .1572 | .6586 |
| LM Test for AutoCor | .0061 | .0704 | .3446 |

Before we interpet coefficients, we must first view them simply as a statistical model which is attempting to summarize the characteristics of the growth relationship. In order to examine statistical adequacy, we must look at the assumptions we have made about this empirical model. In other words, we return to the assumptions that the conditional distribution is log-normal, implying a linear conditional mean and constant conditional variance. Also, we assume that we have captured the entire memory of the process through the inclusion of one dynamic term. We test this by looking at the autocorrelation of the variables beyond the lag length utilized.

Results of these misspecification tests follow in the tables. As you can see, all of these models are misspecified. The AR(1) representations of these series fail to capture the characteristics of the data and, therefore, we cannot have any confidence in the value of these parameters.

We can extend the simple unit root model to include additional lags. It is possible that our series are not AR(*1*) but instead are AR(*K*) where *K* represents the maximum lag value of the variable which has a significant impact on the current value. In this framework, we will discover a unit root if the sum of the coefficients on the lagged variables of the model total one. In other words, if a given number of past values of the series solely determine the current value, then we also have non-stationarity of the same form.

In this case, we will estimate

$$LnY_t = \beta_0 + \beta_1 LnY_{t-1} + \beta_2 LnY_{t-2} + ... + \beta_k LnY_{t-k} + \varpi_t \tag{7.13}$$

and we would test for a unit root under the by examining hypothesis that $\beta_1 + \beta_2 + ... \beta_k = 1$. Again, however, this result only has validity if $\varpi$ is white noise, i.e. there is no misspecification error in this representation.[6]

We see in the below table of coefficients that the the sum of β1 and β2 are very near to the unit root level. However, we also see significant misspecification in these relationships. Misspecification error implies that we cannot test the nearness of the sum of coefficients to 1. Due to the presence of misspecification error, we care not certain of the distribution of our test statistics.

Table 5. AR(2) Results

| Variable (X) | Ln Real Gross National Product | Ln Gross Non-Residential Capital | Ln Average Man Hours |
|---|---|---|---|
| Constant | 0.041 (.058) | -0.066 (.049) | 0.115 (.122) |
| Ln X(t-1) | 1.318 (.104) | 1.311 (.125) | 1.063 (.110) |
| Ln X(t-2) | -0.321 (.103) | -0.2998 (.127) | -0.085 (.111) |
| | | | |
| p Value Test Results: | | | |
| D'Agostino-Skewness | .0017 | .1015 | .0042 |
| D'Agostino - Kurtosis | .0035 | .0173 | .0352 |
| Reset - Linearity | .8764 | .0936 | .810 |
| Reset - Homosked. | .3070 | .3430 | .406 |
| LM Test for AutoCor. | .4091 | .6029 | .318 |

Also, it is possible that the high correlation between any variable $X$ and its past values is attributed to the trending nature of these terms. The impact of the increase in both values will be incorporated into the covariance of $X_t$ and $X_{t-1}$ which would overstate the impact of a change in $X$. We can examine this property by the estimation of

$$LnX_t = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 X_{t-1} + \upsilon_t \tag{7.14}$$

or

$$LnX_t = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 X_{t-1} + \beta_4 X_{t-2} + \varsigma_t \tag{7.15}$$

The results of these empirical models are displayed in the Table 6. Again, we find non-normality even in this richer model. However, it is interesting to note that our coefficients on the lags of the dependent variable move away from even the appearance of a unit root.

A closely related topic has become very popular in econometric analysis of macroeconomic variables. That is the use of cointegrating vectors to transform non-stationary data into a stationary form. This is possible under certain restrictions about the behavior of each variable on its own and the joint behavior of these variables.

Before discussing the theoretical implications of cointegration and applying this idea to our model, we need to more fully define this concept. To do so, we introduce first the notion of an integrated series. A variable is said to be *integrated of order l if l differences are required to produce a stationary series*. The series must be represented as a unit root with a lag length of *l*.

A cointegrating vector refers to the linear relationship between two or more integrated series. In the case of multiple variables, we say that *a series is cointegrated of order (A,B) if we can represent these non-stationary variables as a stationary series through a linear combination of the (A,B) differenced series*. Much of the method of testing for cointegration can be attributed to the work of Johanssen (1985).

Cointegrated series have a very meaningful theoretical interpretation. If we can represent two variables as cointegrated with a certain linear combination, then the coefficients in this linear combination yield us with a long-term relationship between these variables. In other words, we see that these variables display an equilibrium relationship which is exhibited by the coefficients in this linear equation. These coefficients are termed a cointegrating vector. Analysis of cointegration between income and consumption, for example, shows a certain linear relationship (Perman, 1991). The coefficient of this linear relationship, therefore, can be interpreted as a marginal propensity to consume.

Table 6 : AR(2) with Trend Results

| Variable (X) | Ln Real Gross National Product | Ln Gross Non-Residential Capital | Ln Average Man Hours |
|---|---|---|---|
| Constant | 1.465 (.380) | 1.227 (.586) | 0.8798 (.297) |
| Trend | 0.70 (.2) | 0.49 (.2) | 0.19 (.07) |
| Trend Squared | 0.001 (.001) | 0.005 (.003) | 0.003 (.002) |
| Ln X(t-1) | 1.214 (.100) | 1.2036 (.1234) | 0.9799 (.110) |
| Ln X(t-2) | -0.425 (.100) | -0.3596 (.1319) | -0.161 (.110) |
| | | | |
| p Value Test Results: | | | |
| D'Agostino-Skewness | .0152 | .1653 | .0011 |
| D'Agostino - Kurtosis | .0166 | .0089 | .0298 |
| Reset - Linearity | .5781 | .4520 | .0043 |
| Reset - Homosked. | .0664 | .3147 | .125 |
| LM Test for AutoCor. | .7646 | .5830 | .0912 |

Similar application of cointegration would be possible to our growth model. If a cointegrating vector between output and its inputs exists, then this vector would determine the coefficients of our production relationship in dynamic equilibrium. In addition, this gives us information about the state of technological change to influence output through the coefficient on the trend term.

However, just as we had to take care in interpreting our coefficients as unit roots, we must also pay attention to misspecification in cointegrating relationships. First, our data must be integrated. This hypothesis has been rejected since a unit root representation of our vector autoregressive series show continuing misspecification. In this case, we cannot determine a cointegrated relationship between our series.

Through misspecification testing, we have reason to be that unit roots do not capture the characteristics shown in macroeconomic series. The importance of trends in macroeconomic aggregates indicate that the simple unit root representation is not appropriate. Due to the presence of misspecification error, we cannot take the empirical results shown in the above tables as conclusive and the unit root representation is rejected. With a rejection of the unit root model, cointegration does not apply to this data either.

While a first look at the coefficients does tell the story of non-stationarity due to unit root style temporal dependence, we know that these relationships do not statistically capture the true nature of our data.

## 7.5 Structural Change

A particular type of non-stationarity can also be diagnosed from windows estimation of the coefficients. This type of non-stationarity, called structural change, is also consistent with certain theoretical models of economic growth and certain empirical cross-section results.

Azariadas and Drazen (1992) develop a model in which structural change is predicted to occur. In this model, growth in output follows a given path until a critical level of some factor is obtained. This factor may be output, output per capita, capital, education or any other specified variable. At this point, the economy can begin to produce along a new production function. When the production technology changes, output growth leaps to a new path and discrete technological change has occured. This level of the critical variable is termed a threshold.

This is consistent with models of less developed countries which predict that underdeveloped nations may suffer a "development trap" as they cannot increase a given input or total level output to a high enough level to jump to a new growth path. Economies with low physical or human capital, for example, may not be able to grow at the levels of more developed nations due to this structural aspect of the growth process.

Maddison (1982) and Klein (1977) have also alluded to this type of growth pattern. Maddison notes that the growth behavior of various nations changes as these countries become leaders or followers in terms of growth rates. Klein terms various stages of growth in terms of categories such as rapid advancement, slow history, and so on. The characteristics of the nation in terms of the quantity and quality of inputs, the level of output and the underlying determinants of productivity including institutional factors will determine the nature of the country's economic growth path.

Observation of this type of structural change could be evidenced in graphical analysis utilizing windows ordinary least squares regression as well as tests of structural change which compare the estimates of two or more subsamples for equality within a given error range.

In the case of the windows graphical tool, we would look to see if the parameter values of our model jumped drastically from one section of the sample to another. If, for example, slower growth after 1970 indicated a very different path relative to the quantity of inputs, we would see a sharp change in our estimates of $\beta$. The estimates of $\beta$ reflecting the productivity of factors before 1970 would regularly exceed the value of our estimates in the post 1970 sample.

In looking at our WOLS estimates given in Figure 33, we see a significant decline in the value of the coefficient on Ln L toward the end of the sample period. However, we do not observe a distinct jump in parameter value which would be consistent with structural change due to a threshold style economy.

In the next section, we examine United State growth performance in a subsample of the previous data set. If we observe substantial differences in parameter values and our subsample models are well specified, then we would find evidence of structural change after all.

## 7.6 Decreasing the Time Span : Utilizing Quarterly Economic Data

We have noted so far that persistent non-stationarity exists in our models of output growth. The characteristics of the underlying distribution change throughout the sample so it is not possible to get consistent estimates of the coefficients of the regression equation. We have noticed that this is not due to unit root representations, but is due to the characteristics of the joint distribution.

Another method of coping with this non-stationarity problem is to decrease the length of the sample in the hope that the underlying relationship is stationary over a shorter period of time. By restricting the period of time, we may limit the amount of structural or technological change and therefore find more stable estimates of the productivity of labor and capital.

However, we face a limitation in how small our sample can become if we want to gain accurate information from our sample observations. Since yearly time series for a short period of time would result in a small number of observations, we can instead look to quarterly series on output, capital stock, and employment.

Due to limited data availability, labor here is defined as the number of persons employed on average in a three month period. Specific information on average weekly hours by quarter is not available. Output and capital measures are composed quarterly by the Department of Commerce. The data set includes output and input measures by quarter from 1959 to the first quarter of 1994 (T=140).

Quarterly data, however, we would expect to suffer from greater variablity. Short term demand and supply effects may move us away from the growth path of output. It is for this reason, that the variance of our estimates would be expected to increase. However, the purpose of this study will be to examine if the shorter sample period with an equal number of observations avoids the non-stationarity problems in the larger series.

7.6A Estimation of the Quarterly Model

In our investigation of the empirical model for the quarterly data set, we will follow a methodology similiar to that used for the total sample. Plots of the scaled residuals of each regression are found in the Appendix.

We estimate our usual output growth pattern with a deterministic trend terms for the entire quarterly data series to find

$$LnY_t = \alpha_0 + \alpha_1 t + a_2 t^2 + \alpha_3 LnK_t + \alpha_4 LnL_t + \varepsilon_t \tag{7.16}$$

where $t$ still corresponds to the observation number. Estimation by OLS results in

$$LnY_t = \underset{(.950)}{1.34} + \underset{(.041)}{.257}\,t - \underset{(.001)}{0.001}\,t^2 + \underset{(.093)}{0.511}\,LL_t + \underset{(.030)}{0.172}\,LK_t \tag{7.17}$$

| T=140 | | Log Likelihood 370.41 | | Adjusted $R^2$ .9963 |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.000 | 0.413 | 0.000 | 0.693 | 0.000 |

As an alternative model, we may attempt to remove some variability associated with seasonal behavior through the use of quarterly dummy variables. The addition of quarterly dummies to the previous model yields

$$LnY_t = \underset{(.961)}{1.354} + \underset{(.041)}{0.2574}\,t - \underset{(.000)}{0.001}\,t^2 + \underset{(4.23)}{.322}\,Q_1 - \underset{(.423)}{0.029}\,Q_2 - \underset{(.422)}{0.118}\,Q_3 + \underset{(.095)}{0.510}\,LL_t + \underset{(.030)}{0.172}\,LK_t$$

(7.18)

| T=140 | | Log Likelihood 370.41 | | Adjusted $R^2$ .9962 |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.000 | 0.409 | 0.000 | 0.713 | 0.000 |

Continuing misspecification leads us to include dynamic and non-linear terms. Due to the small improvement in the statistical properties of the empirical model and in the interest of reducing requirements placed on our data from estimating so many coefficients, quarterly dummies are eliminated and we find a new statistical model.

$$\begin{aligned} LnY_t = {} & \underset{(.322)}{0.799} + \underset{(.016)}{0.058}\,t - \underset{(.002)}{0.002}\,t^2 + \underset{(.151)}{1.44}\,LnL_t + \underset{(.030)}{0.070}\,LnK_t \\ & + \underset{(.081)}{0.831}\,LnY_{t-1} + \underset{(.079)}{0.134}\,LnY_{t-2} - \underset{(.259)}{1.960}\,LnL_{t-1} + \underset{(.169)}{0.485}\,LnL_{t-2} \\ & - \underset{(.045)}{0.084}\,LnK_{t-1} + \underset{(.030)}{0.008}\,LnK_{t-2} \end{aligned} \tag{7.19}$$

| T=138 | | Log Likelihood 534.11 | | Adjusted $R^2$ .9997 |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.347 | 0.286 | 0.244 | 0.136 | 0.674 |

We see here that all assumptions are satisfied within this statistical model at a 95% confidence level. We now can move to the issue of stationarity. If our estimates of the parameters of this model are also stationary, then we have found a statistically accurate representation.

In order to examine the stationarity of our sample, we once again examine the Recursive Least Squares and Windowed Least Squares values of our parameters. Graphically, we see below that we have not satisfied this assumption.

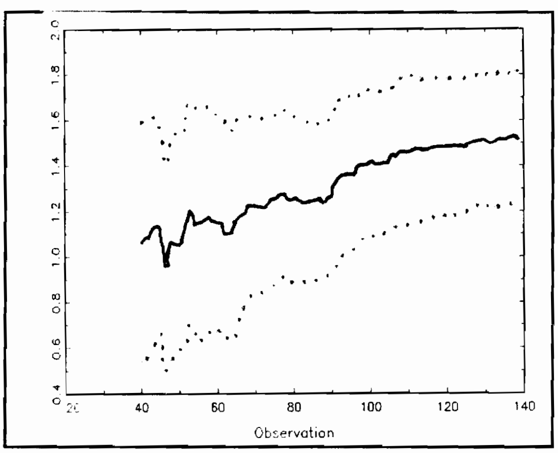


Figure 34 Recursive Least Squares: Quarterly Data

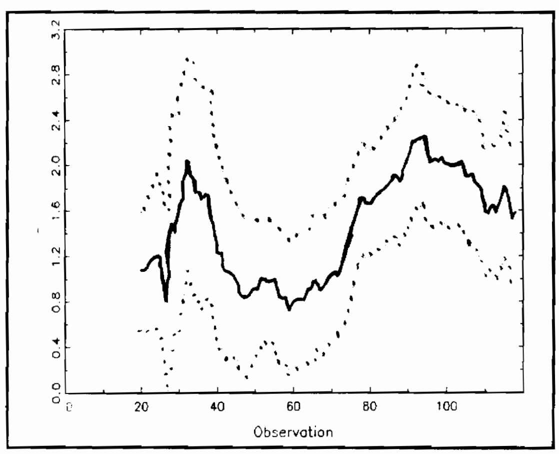


Figure 35 Windows Least Squares: Quarter Data

To examine the stationarity of our series, we will compute ordinary least squares estimates of various subsamples. If these estimates are within the sample and also among various subsamples, then we have evidence of the stationarity of our data process. If these coefficients vary greatly while displaying stationarity within the subsample, then we have shown a case for structural change and an improvement in the stationarity of our series can be accomplished by viewing various sub samples. If however, recursive least squares and windowed estimates display the same non-stationarity even in the smaller samples, then we have not adequately satisfied this requirement.

### 7.6B Breaking the Quarterly Data Set into Sub Samples

An alternative available to us is to divide the total sample into shorter time periods. In this way, we may hold some technological characteristics more constant and therefore, find more stable estimates of the productivity of the growth driving factors.

To do so, however, requires the choice of a breaking point. This issue will be discussed throughly in Section 7.7. We can utilize our WOLS estimates in order to select the Sub Samples most likely to have stationary parameters. We do not notice precise structural change, however, the values of our parameters tend to be fairly stable from the start of our sample until the mid-1970's and from the mid-1970's until the end of the sample.

The exact cut off year I have choosen is the mid-point of 1976. This is consistent with what our windows estimates tell us and is also attractive due to the equal sample periods. This, however, is not a requirement for selecting our sample cutoff points.

### 7.6C SubSample Estimation Results

In order to examine the stationarity of the parameters of our economic growth regression, we will estimate our empirical model with two Sub Samples of our quarterly data set. It is possible to estimate the growth model with more than two subsets, however sample size and the number of variables as well as graphical analysis will determine the appropriate number of subsamples.

We will then test the statistical properties of these models for statistical adequacy. This will include an examination of the stationarity of our estimates. Then, once satisfactory statistical models have been developed, we will look for any significant differences in parameter values. If we find stationarity within the subsamples and substantial differences in the mean of these estimates from one sample to another, then the hypothesis that these samples come from the same population will be rejected.

Estimation of the empirical model including dynamic and a non-linear trend term yields:

For Sub-Sample 1 (1959-1976.2):

$$LnY_t = \underset{(.966)}{3.92} + \underset{(0.054)}{0.260}\,t - \underset{(.0006)}{0.003}\,t^2 + \underset{(.097)}{0.260}\,LnL_t + \underset{(.031)}{0.208}\,LnK_t \tag{7.20}$$

| T=70 | | Log Likelihood 222.13 | | Adjusted $R^2$ .9967 |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.197 | 0.450 | 0.347 | 0.064 | 0.000 |

and with the inclusion of dynamic terms to remove the temporal dependence observed in the residuals, we have

$$\begin{aligned} LnY_t = {} & \underset{(.618)}{1.88} + \underset{(0.039)}{0.124}\,t - \underset{(.0004)}{0.002}\,t^2 + \underset{(.207)}{1.269}\,LnL_t + \underset{(.050)}{0.110}\,LnK_t + \underset{(.133)}{0.502}\,LnY_{t-1} \\ & + \underset{(.115)}{0.146}\,LnY_{t-1} - \underset{(.334)}{1.397}\,LnL_{t-1} + \underset{(.234)}{0.195}\,LnL_{t-2} - \underset{(.074)}{0.046}\,LnK_{t-1} - \underset{(.048)}{0.033}\,LnK_{t-2} \end{aligned} \tag{7.21}$$

| T=68 | | Log Likelihood 265.85 | | Adjusted $R^2$ .9991 |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.173 | 0.223 | 0.770 | 0.800 | 0.074 |

We see that we have indeed found a statistical adequate representation of the growth of output with the inclusion of trend and dynamic terms. If this model also passes the test of the parameter homogenuity assumption (7 in Spanos, 1986) then we can interpret coefficients in this model in a theoretically meaningful way.

We repeat this procedure for the Sub Sample 2 : 1976.2 to 1994.1. We are looking once again for a statistically well specified empirical model. We find

$$LnY_t = \underset{(.801)}{-1.938} + \underset{(.043)}{0.024}\,t + \underset{(.0001)}{0.002}\,t^2 + \underset{(.080)}{0.870}\,LnL_t + \underset{(.031)}{0.047}\,LnK_t \qquad (7.22)$$

| T=70 | | Log Likelihood 218.27 | | Adjusted $R^2$ .9917 |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.004 | 0.259 | 0.372 | 0.010 | 0.000 |

and once again with dynamic terms this becomes

$$\begin{aligned} LnY_t = {} & \underset{(.402)}{-0.555} + \underset{(.019)}{0.062}\,t + \underset{(.0003)}{0.001}\,t^2 + \underset{(.215)}{2.116}\,LnL_t + \underset{(.033)}{0.001}\,LnK_t + \underset{(.129)}{0.480}\,LnY_{t-1} \\ & - \underset{(.098)}{0.088}\,LnY_{t-2} - \underset{(.433)}{1.512}\,LnL_{t-1} - \underset{(.285)}{0.129}\,LnL_{t-2} - \underset{(.047)}{0.049}\,LnK_{t-1} + \underset{(.036)}{.072}\,LnK_{t-2} \end{aligned} \qquad (7.23)$$

| T=68 | | Log Likelihood 282.74 | | Adjusted $R^2$ .9988 |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.478 | 0.316 | 0.813 | 0.116 | 0.803 |

In this case, also, we have a well specified empirical model. The absence of specification error is further supported by the residuals found in the appendix.

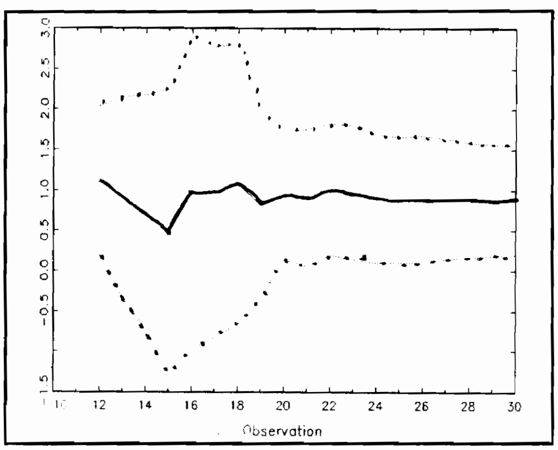


Figure 36 Sample 1 Recursive Least Squares

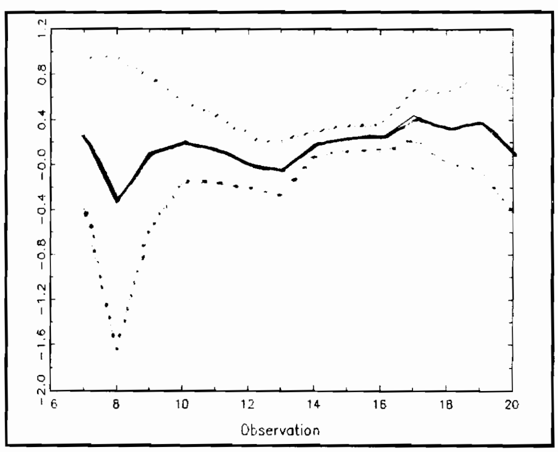


Figure 37 Sample 1 Windows Least Squares

Our next question, therefore, is to examine the stationarity of the parameter estimates to determine if they are stationary.

We again conduct RLS and WOLS with the dynamic growth model in each time period. The results as shown in Figures 36-39, the stationarity of the coefficients has been greatly improved through a reduction in time length.

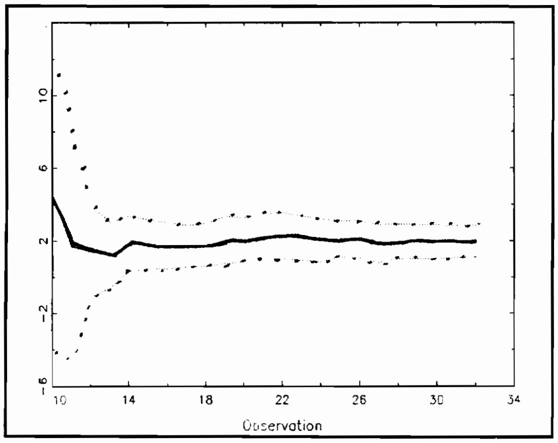


Figure 38 Sample 2 Recursive Least Squares

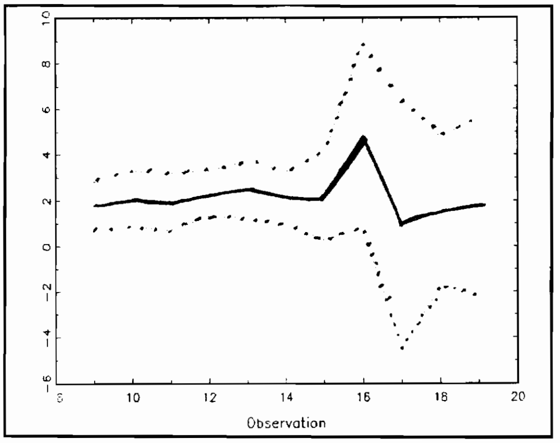


Figure 39 Sample 2 Windows Least Squares

Also, since our empirical models satisfy all the assumptions of the linear regression model and therefore are statistically adequate representations of the US growth process, we can make some theoretical observations. We find substantial differences in some parameters. The trend terms in particular result in very different behavior in each series. It is interesting to note that the differences in the parameters relating the dynamic terms do not change substantially from one sample to the next. Explanations for these differences in growth behavior rely upon our theoretical explanations of growth. In this case, we find data which support the notion of a structural change in the nature of our production technology.

## 7.7 A New Method : Hamilton's Switching Regime Model

In the previous section, we have observed at dividing the data set into two independent samples greatly improves the stationarity properities of our empirical model. We see that two empirical models describing United States growth behavior actually apply to our total data series, one to the observations from 1959 to 1976 and another from 1976 to 1994. This complicates our theoretical understanding of the growth process and we would prefer a model which could account for this structural change.

The decision of where to break the data set was based upon graphical analysis. We looked at the behavior of the coefficients in windows and recursive least squares and determined that the empirical model behaved differently in the pre-1976 period as compared to the post-1976 period. That is, by looking at the behavior of the coefficients in our linear regression model, we made a decision that the underlying model must have changed around the period 1976.

Utilizing this method, we allow for different coefficients and functional forms to exist in the two sub periods examined. By estimating the models separately, we place no requirements on the coefficients, or equivalently on the underlying means, variances and covariances seen in the joint distribution.

Another method of addressing non-stationary samples employed extensively in econometrics does allow for relationships to remain constant from one sample to the next. This is through the use of dummy variables. In this way, we may assign a dummy to account for intercept changes in our linear regression model (changing $\beta 0$) or changes in the elasticity of output with respect to changes in any particular explanatory variable. Including a dummy variable term for the intercept and for all the dummy variables will yeild the same result as estimating the regression for the two sub samples separately. (Judge et al, 1988). In the section above, we have allowed both the intercept and slope terms to change from one sample to the next.

Both methods, however, involve the choice of when and what terms to allow to vary. In the previous section, we were forced to make a decision regarding the cut off point. In the case of dummy variables, we must decide when and on what terms to include a dummy variable. We would prefer a method in which such arbritary decisions are not necessary. Specifically, this observation that 1976 is a cut off point separating two models tells us nothing about the behavior of output growth in the future. We would prefer instead a model in which the shift in the regime is actually endogenous. That is, the empirical model itself is able to determine in which regime we are most likely operating.

Hamilton's (1994) switching regime methodology is able to select among various models and determine the probability that we are in regime *n* given a certain observation. In this way, future observations may signal some unknown threshold and move us from one regime to another. Hopefully, if applied to my growth model, the Hamilton switching regime regression would determine that we moved from regime 1 to regime 2 in 1976.

## 7.7A Hamilton Model Formulation

We recognize from our previous examination that the parameters of our linear regression model are not time invariant as required by Assumption *7* discussed in Chapter 3. We have found substanial improvement in homogenuity through the use of an arbitrary breakpoint separating the sample into two regimes. Our objective in this section is to find a mechanism to endogenize the this breakpoint.

To do so, we need to think of our growth relationship depending upon, not only technological progress and the output elasticity with respect to our inputs, but also upon a state variable. This variable will determine in what regime we are operating. That is,

$$Y_t = f(K_t, L_t, A_t, S_t) \tag{7.24}$$

or in our dynamic framework,

$$Y_t = f(K_t, \ldots, K_{t-m}, L_t, \ldots, L_{t-m}, A_t, S_t, \ldots, S_{t-m}) \tag{7.25}$$

This methodology is well established thanks mainly to the work of James Hamilton (see Hamilton, 1994 for an excellent development of the basic model). Diebold, Lee and

Weinbach (1994) have extended this method to include variables capable of explaining the probability of being in a given regime. This is closely related to the notion of a threshold economy as previously discussed by Azardias and Drazen (1990).

We first will consider a very simple model of state space. Let's look at the case in which there are two regimes, $S_1$ and $S_2$. This methodology can easily be extended to cases in which there are more than two regimes. In fact, the choice of the optimal number of regimes is of great interest and should be evaluated based on sample information (Shelly, 1991).

We consider the case when our state random variable is a Markov process. That is, the value of $S_t$ depends solely upon the past only through the most recent state value, $S_{t-1}$. Using this Markov process in the case where we have two regimes only, we find a transistion matrix of the form :

$$P = \begin{Bmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{Bmatrix} \tag{7.26}$$

where $p_{ij}$ represents the probability that state *i* will be followed by state *j*. For example, *p11* tells us that state *1* will be followed by state *1* and *p12* the probability that state *1* will be followed by state *2*. Due to the axioms of probability, *p12* must equal *1-p11*. Due to the Markovness of our state space process, this matrix fully defines all the probabilities of transistion.

It may be the case, as examined in Diebold, Lee and Weinbach (1994) that these probabilities are not constant throughout the sample period. It may be the case that $p_{ij}$ depends upon some observable variable.

This method has been applied in several areas of macroeconomics. These areas include inflation expectations (Hamilton, 1986), yields on three month and ten year Treasury bonds (Hamilton, 1988), exchange rates (Engel and Hamilton, 1990) and the behavior of Quarterly United States Output (Hamilton, 1989, Lam, 1990, Goodwin, 1993 and Hamilton, 1994). In these output papers, the emphasis has been on Business Cycle behavior.

Hamilton (1989) points out that a Markov Switching model of Quarterly output data may determine periods of High growth and Low growth much as a threshold model would predict. This is one possible outcome of such a model. However, Hamilton found instead that the Markov Switching model instead found two regimes highly correlated with the business cycle. One regime was applicable in areas of economic recovery and boom while the other regime corresponded to periods of economic decline. The correlation of the Markov Switching Regime model to the National Bureau of Economic Research's somewhat arbritary divisions of business cycles indicates that this method performs very well in determining output performance within business cycles. The NBER cut off points are based on a variety of macroeconomic variables and the Hamilton model's naive correspondence to these dividing lines shows the power of this methodology.

## 7.7B Hamilton's Autoregressive Model of Output Behavior

In this section, we will consider one of two formulations of the Hamilton Model with Markov Switching. We will reexamine Hamilton's output growth analysis. Then, in the next section, we will consider the growth pattern of the Solow Residual itself. In this way, we may move away from cyclical behavior through the inclusion of the cotemporaneous regressors capital and labor.

Hamilton (1989) considers a model in which the growth rate of real output may exhibit different means depending upon regime. Output today is assumed to depend upon past values of output ($m$=4). This was noted throughout our study as well (Chapter 4). In order to satisfy the independence assumption of the Linear Regression Model, two lags of our variables were required in the empirical model.

Hamilton's formulation is therefore

$$Y_t - \mu_{s_t} = \phi_1(y_{t-1} - \mu_{s_{t-1}}) + \phi_2(y_{t-2} - \mu_{s_{t-2}}) + \phi_3(y_{t-3} - \mu_{s_{t-3}}) + \phi_4(y_{t-4} - \mu_{s_{t-4}}) + v_t \qquad (7.27)$$

where $s$ can equal 1 or 2 for regimes 1 and 2. The order of Autoregression necessary can be determined through exclusion testing or, as recommended by Hamilton, through the use of the autocorrelogram.

This estimation problem is much more complex compared to the estimation of models with a dummy variable or to the estimation of multiple regressions as shown in Section *6*. The reason is that the value of the variable $s$ is unknown in any period. We must instead base our choice of $s$ on the probability that our data exhibit behavior most consistent with regime one or two. This is a probabilistic statement and is subject to error itself.

Estimation of a Hamilton model is based upon iterative optimization of a distance function which now depends upon the parameters of the distribution function and also the transistion probabilities given in Equation (7.26). The unknowns are the means for each state, $\mu 1$ and $\mu 2$, the variance $\sigma 2$ and the unconditional and conditional probabilities associated with state *1* and state *2*. In order to accomodate all possible state patterns, we define a new state specific variable to consider both the current and past value of s. In our AR(1), 2 regime model, s would become

| | | |
|---|---|---|
| s=1 | $s_t=0, s_{t-1}=0$ | *p11* |
| s=2 | $s_t=0, s_{t-1}=1$ | *p12* |
| s=3 | $s_t=1, s_{t-1}=0$ | *p21* |
| s=4 | $s_t=1, s_{t-1}=1$ | *p22* |

An extension of this model also allows the variance to differ between states so that we would have 1 additional parameter in the 2 state example.

Estimation begins with starting values for all of these parameters. These starting values are often based on underlying characteristics of the data. We then select estimates of parameters, $\mu 1$, $\mu 2$, and $\sigma 2$ to minimize the squared deviation or maximize the likelihood function. Next, utilizing the estimates of $\mu$ and $\sigma$, we estimate new values of the transistion and unconditional probabilities. Utilizing these new probabilities, we search for new estimates of the means and variances. When the value of the objective function reaches a constant (hopefully a global optimal), then the search ends and parameter values of probabilities for each state can be recorded. Related search mechanisms look for combinations of parameters which instead will minimize the gradient function.

There are limitations to this form of optimization, particularly if the objective function is poorly shaped. In this case, selection of starting values or optimization routine may lead to different solutions. The possibility of finding a local rather than global optimal always exists as does the chance of a corner type solution. Examination of the gradient functions at the optimal combination of parameters should alert us to corner solutions. However, examining the gradient will not tell us if we have reached a local rather than global solution.

Another difficult issue which must be addressed is the requirement that all of our probabilities must be greater than zero and less than one. In addition, the unconditional probability of states 1 and 2 must sum to one to be consistent with the axioms of probability. In a similar way, the conditional probability of being in state 1 given state 1 in the past plus the conditional probability of being in state 2 given state 1 in the past must also sum to 1. Our optimization routine must be formed in a way to prevent violations of these assumptions. These characteristics and limitations of non-linear optimization are discussed in Hamilton (1994) and in Hendry (1995). Hamilton discusses the very specific application of the EM algorhythm to this problem while Hendry includes a chapter discussing the general principle as well as specific types of algorhytms employed in non-linear optimization problems.

James Hamilton has supplied his GAUSS2.0 code for optimization of an *n*-state Markov switching model. Through the manipulation of several global variables, it is possible to estimate this model to any degree of autoregression and with constant or state dependent variances. Reexamination of the Hamilton model showed that substantially longer Autoregressive series did not greatly alter the AR(4) representation chosen by the author. Also, estimation with state-varying variances did not converge to even a local optima but instead failed to update after very few iterations.

One factor to note with the Hamilton model with Markov switching is that it is extremely sensitive to starting values (especially the values of unconditional state probabilities). By changing these starting values, the predicted probability of being in any particular state changes drastically. This is a strong limitation and care should be utilized in intrepreting the output probabilities.

Hamilton found by estimating (7.27) that the mean of output growth does differ greatly between the two regimes with the estimates of $\mu 1$ and $\mu 2$ being 1.16 and -0.36, with very small standard errors. The probabilities of remaining in the current regime are very high with p11 = 0.75 and p22 = 0.59. However, changing the starting values of these probabilities (moving them away from extreme values) does change the estimate substantially.

If we consider the probability of being in a current state to be certain if the probability is greater than .70 (this is open to interpretation), then it is shown in Hamilton (1989, 1994) that the Markov switching model projections are very much in line with the NBER cut off points for recoveries and contractions.

In terms of our growth model, this is not a preferred result. Instead, we see that the Markov switching model is picking up cyclical occurences as opposed to growth regimes. Instead, we would have preferred a model in which long periods of regime 1 are followed by long periods of regime 2. In this way, we would say that the economy has leaped from one growth path to another through some threshold behavior.

This is not what was observed by Hamilton. Instead, we see the switching model as a naive method to determine the turning points of the business cycle. This is itself is very interesting, but not from a growth perspective.

## 7.7C Reformulation to a Model of Productivity Change

As an alternative, we may try to remove cyclical behavior through the use of regressors which vary also with the business cycle. That is, instead of a strictly autoregressive model, we can look to a series which may reflect productivity rather than total output. That is, we can look at the Solow Residual.

Since we are interested in capturing the temporal behavior of productivity, we will not include a trend term in our regression. We will find a residual series which shows the increase in output not explained by increases in labor and capital alone. Assuming a constant elasticity of output with respect to each input (given by the coefficients $\beta_1$ and $\beta_2$), we find a measure of change in the production function. Since the amount of labor and capital tend to change over time and throughout the business cycle, this may separate our regime shifts from purely cyclical behavior.

To find the Solow residual series, we estimate

$$LnY_t = \beta_0 + \beta_1 LnK_t + \beta_2 LnL_y + \varepsilon_t \tag{7.28}$$

We will then employ the residual series, $\varepsilon_t$, in Hamilton's Switching Regime model. We will again assume there are two possible states, high growth and low growth. The mean value of the residual (showing amount of productivity increase) may differ in each. We will also allow the variance (deviation of productivity in each model) to differ.

The selection of the number of regimes is still an unknown. Graphical analysis of the Solow residual may lead us in the correct direction. Also, examination of the log likelihood value achieved by models with differing numbers of regimes may answer this question.

An additional unknown is the number of lags to include in the Hamilton model. This determines the number of periods on which the transistion probabilities will depend. In our previous analysis of the Solow Residual for US quarterly data, we found that 2 lagged values of the independent variables were necessary to remove autocorrelation (Chapter 4). We will continue with a two state Markov switching process defining the transisiton probabilities.

We now estimate the Markov switching model to be

$$SR_t - \mu_{s_t} = \phi_1(SR_{t-1} - \mu_{s_{t-1}}) + \phi_2(SR_{t-2} - \mu_{s_{t-2}}) + v_t \tag{7.29}$$

and the conditional variance in period $t$ depends upon the value of $s_t$.

Estimation with US quarterly data leads us to :

$$\begin{aligned} \hat{\mu}_1 &= 0.5832 \\ \hat{\mu}_2 &= 0.9745 \\ \hat{\sigma}_1^2 &= 0.005 \\ \hat{\sigma}_2^2 &= 1.350 \end{aligned} \tag{7.30}$$

and autoregressive coefficients:

$$\begin{aligned} \hat{\phi}_1 &= 0.3566 \\ \hat{\phi}_2 &= 0.3344 \end{aligned} \tag{7.31}$$

Together, we find a log likelihood value for this switching regression to be 150.01. A variety of alternatives were examined. Estimating the same switching regression with constant variance caused the algorhythm to fail in the second iteration. Examining the gradient matrix indicated a very unrealistic value for the change in the log likelihood with respect to the variance parameter. Allowing the variance to change counteracted this difficulty. Additional autoregressive terms were included in the regression. While the log likelihood value did increase with the inclusion of these terms, the difference was very small. The model is still somewhat sensitive to starting values especially in the estimation of the probability any particular observation is in state 1 or state 2.

The matrix of transistion probabilities found for model (7.29) is

$$\begin{aligned} p_{11} &= 0.183 \quad & p_{21} &= 0.771 \\ p_{12} &= 0.817 \quad & p_{22} &= 0.229 \end{aligned} \tag{7.32}$$

This suggests that there is a high probability of switching from one regime to another ($p_{12}$ and $p_{21}$ are large). This differs from Hamilton's results which show a high persistence as well as from the theoretical threshold economy model. Such a theoretical model would also suggest that the likelihood of switching regimes, especially $p_{21}$ would be very small.

The overall probability of state 1 existing, unconditioned on any past values, is 0.4854 and the probability of observing state 2 is 0.5146. However, alternative starting values for these parameters did result in slightly different probabilities. When the $P(s_t=1)$ was above 0.95, then the final probability of observing state *1* was around 0.70.

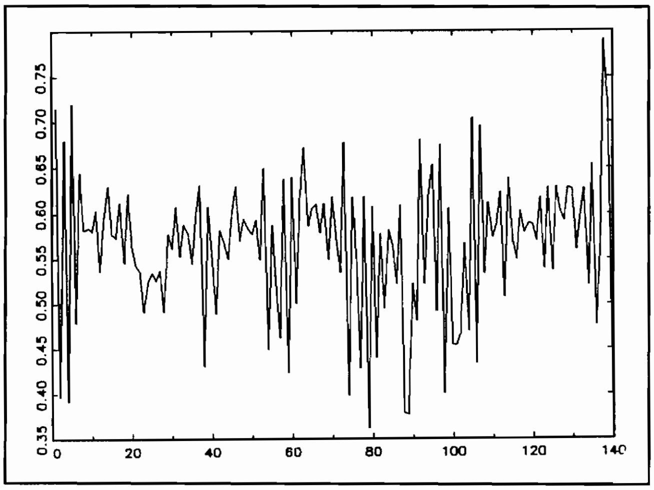


Figure 40 : Probability that economy is in State 1

We can also look at a graph which shows the probability of any particular observation falling in State 1. We could also display the probability that any observation belongs to state 2, but the same information would be given since $P(s_t=2)=1-P(s_t=1)$. Such a graph is shown in Figure *40*. As reported and recommended by Hamilton (1993, 1994), smoothed probabilities were utilized. We also see in Figure *41* a plot of quarterly residuals of (7.28).

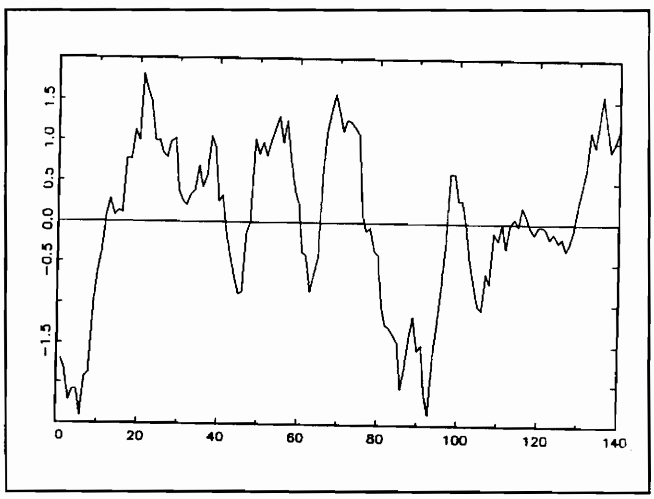


Figure 41 : Solow Residual for Quarterly Data without Trend

## 7.7D Further Directions : Inclusion of Varying Regime Shift Probability

An interesting extension of this model has been recently considered by Diebold, Lee and Weinbach (1994) which allows the transistion probabilities such as those in (7.32) to change. We can think of this in a threshold economy framework. For example, if the the value of capital stock reachs some amount $K^*$, then the probability of jumping from state 1 to state 2 increases. Also, it may be that the transistion probabilities themselves depend upon the level of capital stock or some other variable.

One variable which may be important in transisitions from high productivity to low productivity may be the amount of military expenditure. We notice that the level of productivity in the United States and in Europe increased greatly in World War Two. We saw countries devoting massive resources including previously unutilized labor sources (women leaving the home). Motivation was high as workers strived to support the country's military effort. It is proposed, therefore, that the transistion probability matrix may be a function of military or government expenditure.

Other possible determinents may include institutional factors. These factors are notoriously difficult to measure. However, major transistions in institutional and political infrastructure may also provide a clue to changing transistion probabilities.

This is a further direction of research in the Hamilton framework and extensions to Diebold et al's 1994 work are an important step in understanding and employing the Hamilton type model.

## 7.8 Conclusion

Stationarity of economic time series is a vital assumption in both theoretical and empirical terms. Without stationarity assumptions imposed on the first two moments of the joint distribution, we cannot justify the value of our OLS estimates as our ordinary least squares estimates will only represent an average of various different parameter values.

The approach of looking for unit roots in economic time series and then utilizing cointegrating vectors may improve our stationarity behavior in some cases. However, this relies completely upon the statistical properties of the vector autoregressive properties of the series studied. If a VAR model results in misspecification, then this is not the answer.

Dividing a sample into various sections may solve the problem of non-stationarity if the changes in the parameters result from structural change or threshold economies. If we see distinct changes in the parameter values, we may find we have abandoned one growth path. In this case, to sample from one population only, we need to reduce the length of the sample period. To do so, it may be necessary to utilize quarterly data, especially in a period in which structural change occurs quickly.

Estimation of breaking points by Hamilton's Markov switching regimes model found no support for the threshold economy framework. Instead, we see either a business cycle determinent of regime (as shown by the growth rate) or a model with frequent switching (as shown by the Solow Residual). An alternative method was suggested where the transistion probability is determined by some other observable variable.

There are no easy answers when we find persistent, continual non-stationarity unless that non-stationarity can be accounted for by differencing. If we have a particular type of non-stationarity that we can account for in terms of an error correcting mechanism, then a transformation of the data can be modeled and we can interpret our results in a meaningful way.

Examination of models with alternative conditional distributions through the method of maximum likelihood estimation may lead us to better specification of the relationship between two or more variables. By accurately portraying this relationship, we can hope to eliminate this time dependent change in our parameters.

## 7.9 NOTES

**1** More complex mechanisms of adjustments are permissible as long as the moments of the series return to normal is some finite time period. For example, the series could remain at the higher income without any change for some time, and then jump back to the original level. Such a pattern, however, as no meaningful theoretical derivation.

**2** We also may consider a case in which the response to income dies out for a few periods, but then remains at a higher level. In this case also, we have non-stationarity

**3** I have in general assumed a p value of .05. This corresponds with a standard test of significance at the 95% confidence level. However, the reader is free to determine any significance level. Stricter assumptions would be characterized by higher p value requirements for significance while more relaxed assumptions would allow p values lower than .05

**4** The accuracy of our tests for stationarity also rely upon an accurate statistical representation of the true process explaining output growth performance. Misspecification of conditional mean non-linearity is a significant concern in examining other properties of this distribution. For this reason, I concentrate my analysis on general graphical techniques as opposed to calculations.

**5** A reparameterization of this model can allow us to utilize a t-test for unit root. If we estimate,

$$LnY_t - LnY_{t-1} = \alpha + \varepsilon_t$$

we determine if $\alpha$ is significantly different from zero through a t-test.

**6** A similar representation in terms of differences could also be tested by a t-test. In this case the regression equation becomes

$$LnY_t - LnY_{t-1} - LnY_{t-2} - \ldots - LnY_{t-k} = \varpi_t$$

and we test $\varpi$ to determine if it is significantly different from zero.

## 7.10 Appendix : Residual Plots from Quarterly Analysis

Subset 1 Solow Residuals

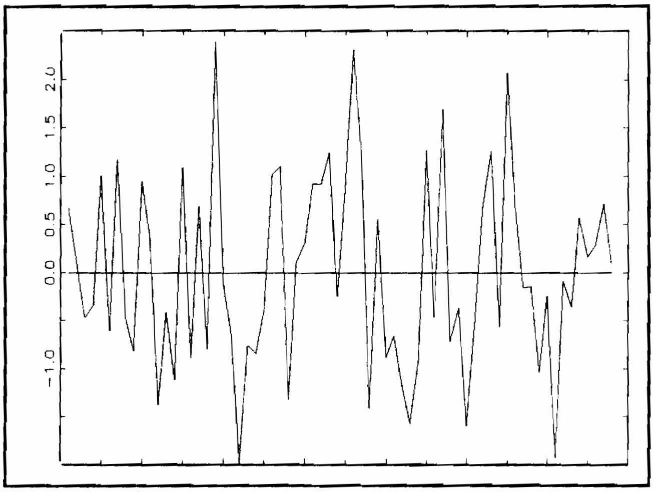



**Figure 42 : Solow Residuals for Sub Section 1**

Subset 2 Solow Residuals

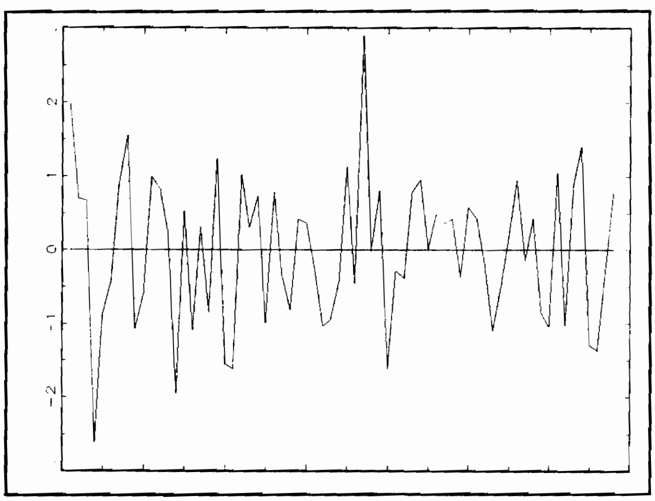



**Figure 43 : Solow Residuals for Sub Section 2**

# *VIII. Modeling Economic Phenomena with Alternative Distribution Assumptions*

Throughout this study of output performance, we have found that a dynamic linear regression model based upon the assumption of conditional normality is incapable of capturing the behavior of output growth. Even with the inclusion of factors believed to drive technological change and the inclusion of factors which allow for the evolution of output, we still have not found a model which is capable of capturing the characteristics of output growth in the United States and other nations.[1]

This failure to satisfy the assumptions of this distribution family leads us now to consider an alternative assumption regarding the distribution of output and other macroeconomic aggregates. If these variables exhibit distinct departures from a normal distribution (and therefore a conditional normal in regression), then we must seek a different distributional assumption.

In this paper, we begin with a brief review of misspecification results relating to Solow's model of output growth in Section 1. The specific forms of misspecification error will lead us to an alternative distribution function which may better represent our data series. In order to employ this distribution function, we will estimate through the method of Maximum Likelihood. In Section 2, the characteristics of the distribution selected will be discussed. Section 3 applies the Gumbel exponential distribution to our growth model. Then, due to limitations of Gumbel's distribution, I will derive an alternative distribution not yet utilized in the literature in Section 4. This new distribution function allows us to capture more variability in the marginal distributions and therefore adds flexibility to the conditional model. In Section 5, we look to a constrained optimization model based on the three parameter Gumbel distribution. Section 6 discusses model evaluation and selection. Conclusions and directions for further research follow.

## 8.1 The Need for Alternative Distribution Assumptions

Nicholas Kaldor (1961) pointed out that most every economy conforms to a list of stylized facts regarding the performance of output. These patterns have been evaluated in previous chapters and we have found persistent misspecification errors which lead us to believe that these facts are not an accurate representation of United States growth performance in the twentieth century. However, these facts have been a foundation for many of the assumptions imposed by Solow in his seminal model of economic growth (see Solow (1970) for a summary of major models and extensions).

In this chapter, we will consider a slightly altered form of the Solow model. Instead of the Multivariate empirical model considered in chapters 4-7, we will look in this chapter to a bivariate empirical model. We do this for simplicity in working with alternative distributions.

## 8.1A A Bivariate Formulation of the Empirical Model

In the Solow Model of economic growth, a homogenous output good is produced through the use of a homogenous (non-depreciating, in this example) capital product and labor. For ease of solution, we assume that this production occurs with constant returns to scale overall and diminishing returns to each factor individually. This constant returns to scale technology allows us to reduce the dimensionality of the production function to a function of the capital labor ratio only.

$$\begin{aligned} Y_t &= F(K_t, L_t) \\ y_t &= Y_t/L_t = F(K_t/L_t;1) \\ y_t &= f(k_t) = f(K_t/L_t) \end{aligned} \tag{8.1}$$

If we include parameters to represent the level of technology in this constant returns to scale framework, we find output per capita remains a function of the capital-labor ratio, and also a function of the technology parameter, $A$.

$$Y_t = F(A_t, k_t) \tag{8.2}$$

Kaldor's Stylized Facts tell us something of the expected behavior of output per capita, the capital to labor ratio and the ratio of output to capital and labor. These "facts" are consistent with the Solow model with a productivity term $\alpha$ that increases over time. These facts include

(1) output and labor grow at a constant and near equal rate so that (Y/L) does not fall.
(2) the ratio of output to utilized capital remains constant implying that output and capital grow at the same rate. (Y/K) is a constant.
(3) there is a steady increase in the capital per worker, (K/L).

These Stylized Facts are implied by a Solow model which exhibits an unbalanced steady state growth path. This unbalanced growth pattern is made possible through improvements in the productivity of capital inputs. Improvements in technology would lead to such a pattern if capital is dependent upon past values of output while labor is not.

In our statistical models of the output performance, we will look at output per capita in each year as related to the capital to labor ratio. The Solow model could also be estimated with a capital and a labor series separately, but that will require additional parameter estimates (2 per each variable added) and computer calculations and convergence are difficult with just 3 parameters.

The model which we will estimate will be

$$y_t = \beta_0 + \beta_1(\frac{K_t}{L_t}) + \varepsilon_t \tag{8.3}$$

If Kaldor's Stylized facts are an accurate representation and we are truly along a steady state growth path, then this statistical model will be able to specify the value of output per capita conditional upon the value of the capital to labor ratio with only white noise error terms remaining. That is, the assumptions of the simple linear regression model will be satisfied.(see Chapter 3).

## 8.1B Misspecification Testing of the Revised Empirical Model

Estimation of this statistical model utilizing United States Quarterly Data (T=140) yields the following. The standard errors are reported under each coefficient. Test statistics are reported in convenient $p$ values. If the $p$ value is less than your significance level, then the null hypothesis is rejected and the assumption fails.

$$y_t = \underset{(.545)}{3.819} + \underset{(.030)}{0.390}\left(K/L\right)_t \tag{8.4}$$

| T = 60 | Log Likelihood -114.61 | | | $R^2$ .7356 |
|---|---|---|---|---|
| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
| 0.180431 | 0.025078 | 0.005283 | 0.033678 | 0.000000 |

Residual Sum of Squares : 152.96

We see from the above that we have misspecification error in our conditional mean and variance and in the temporal independence of our residuals at a 5% significance level. We are not able to define a statistical model in this form which accounts for all but normal, identically distributed and independent error terms.

We may modify our above statistical model to account for these misspecification errors. Rather than view the misspecification errors as a ending point, we will utilize this information to improve upon our model.

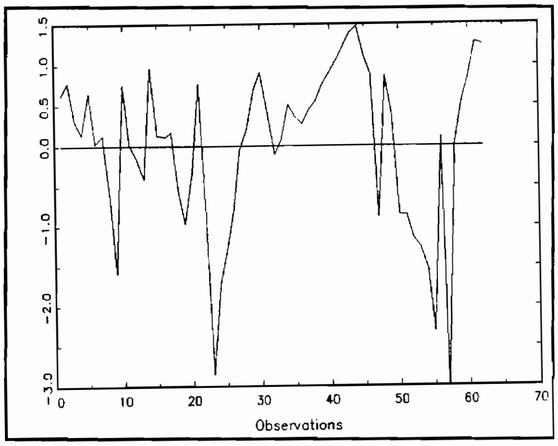



Figure 44: Residuals - Output per capita

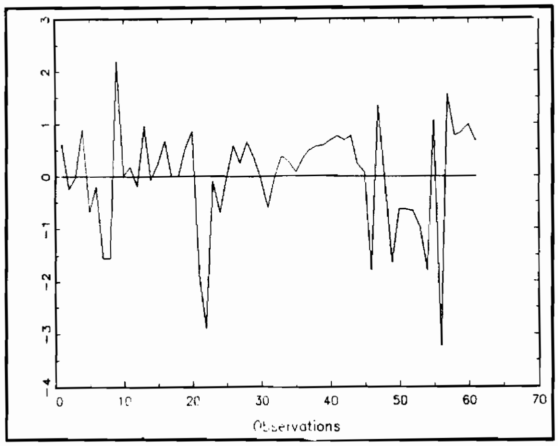


Figure 45 : Residuals-Output/capita dememorized

This is further supported by graphical analysis of the residual series. We see long swings in the residual series, an indication of positive memory. We also see a more pointed (leptokurtic) empirical density function which indicates that the assumption of normality is not appropriate. Analysis of residual plots is discussed in Dissertation Appendix I.

Through the inclusion of lagged values of output per capita and the capital to labor ratio, we may be able to capture the positive memory remaining in our series. In addition, by including a non-linear trend term, we may be able to gather the non-linearity with respect to our original regressors. This estimation yields

$$y_t = \beta_0 + \beta_1(\frac{K}{L})_t + \beta_2 y_{t-1} + \beta_3(\frac{K}{L})_{t-1} + \beta_4 t + \beta_5 t^2 + \upsilon_t \tag{8.5}$$

$$y_t = \underset{(1.39)}{2.506} + \underset{(.046)}{0.196}(K/L)_t + \underset{(.118)}{0.734}\, y_{t-1} - \underset{(.042)}{0.117}(K/L)_{t-1} + \underset{(.024)}{0.040}\, t - \underset{(.0004)}{0.0001}\, t^2 \tag{8.6}$$

| Skewness | Kurtosis | Linearity | Homoskedasticity | Independence |
|---|---|---|---|---|
| 0.000756 | 0.034474 | 0.010136 | 0.000388 | 0.055138 |

Once again we see difficulty in terms of misspecification error. This is consistent with the plot of the residual series. All of our assumptions regarding normality are in violation. In previous chapters, we have attempted to compensate for misspecification error in the output-input relationship through the inclusion of factors which may enhance productivity and through a study of the stationarity of these estimates through shorter time spans. However, we are often still faced with conditional mean non-linearity.

In this paper, we will attempt to improve upon our statistical model by abandoning the common assumption of a normal distribution of random variables and consider an alternative distribution assumption which may display these desired conditional moment characteristics.

## 8.2 Selecting An Alternative Distribution Function

Throughout this study, we have found that the conditional mean of macroeconomic aggregates does not conform to the conditional mean linearity assumption imposed by the linear regression model. We see persistent non-linearity even when we include technology enhancing variables in our analysis.

We have also found misspecification in the conditional variance. According to the normal distribution, we should find a conditional variance which is independent of the regressors, or homoskedastic. However, this violation is not as wide spread as our finding of non-linearity. In some cases, we have been able to satisfy the assumption of conditional variance homoskedasticity at a 95% confidence level. In other cases, we have found heteroskedasticity in the variance. Often we have discovered this heteroskedasticity in a non-linear form as shown in a Reset Test of Order 2 (see Chapter 4).

Through these misspecification test and a graphical examination of data and residual plots, we realize what characteristics the Haavelmo distribution of this data should have. We have rejected the assumption of linearity in the conditional mean so we should utilize a distribution with a non-linear conditional mean. We also have rejected the assumption of homoskedasticity so we expect a heteroskedastic conditional variance. This is our starting point for choosing a parametric form which is more suited to our data series. Through misspecification testing and graphical analysis of our data and residuals, we have discovered some areas in which the conditional normal distribution goes wrong. Using this information, we are directed toward a more appropriate distribution assumption.

However, defining a particular form of non-linearity is not a simple task. When we say, the conditional distribution has a linear conditional moment, we specify exactly what that functional form will be. That is,

$$E(Y|X) = \alpha + \beta X \tag{8.7}$$

However, when we can only state that the conditional mean is non-linear, we now only exclude one possible functional form and allow a multitude of possibilities. The conditional mean could now be a quadratic, a cubic, a ratio of any variety of polynomials, and possibly some combination of trigonometric functions.

It is not possible to specify through a misspecification testing which form of non-linearity we have encountered. We are only able to reject the null hypothesis that the conditional mean is linear. We cannot simply accept the alternative of some particular functional form.

Similarly, for a test for the presence of heteroskedasticity, we only reject the null that homoskedasticity exists as opposed to accepting the particular alternative. We know that the conditional variance is not a constant, but we cannot say with certainty that the conditional variance follows any particular functional form.

However, that does not mean that misspecification testing cannot aid us in identifying an alternative distribution. When we find we have a linear conditional mean and a heteroskedastic conditional variance, then the use of a Student's T or Pearson Type Two distribution may be most appropriate due to the linearity of the conditional mean. If we find a non-linear conditional mean and a homoskedastic conditional variance, then we should examine a Logistic distribution which is the only common parametric distribution with a non-linear conditional mean and constant conditional variance.(Evans, Hastings and Peacock, 1993).

## 8.2A Alternative Distributions Considered

To fully utilize results developed through previous studies of distribution characteristics, we should consider some of the more common distributions.[2] The conditional Student's T distribution has been utilized recently in the modeling of heteroskedastic data (Spanos, 1991). This has been a valuable contribution to the literature regarding models in which the conditional variance is of great interest. For example, in the study of exchange rates and stock returns, the variability of a series is as interesting as the rate itself (McGuirk , Robertson, and Spanos,.1992). Through the assumption that these variables follow a conditional Student's T distribution, we can actually predict the variance of a series with respect to values of the independent variables (Spanos, 1992).

We can derive the conditional moments of the Student's T distribution to be:

$$E(Y|X=x)=\beta_0+\beta_1 x \text{ — linear in } x \tag{8.8}$$

$$Var(Y|X=x)=\left(\frac{\nu}{\nu-1}\right)\sigma^2\left\{1+\frac{1}{\nu\sigma_{22}}[x-\mu_2]^2\right\} \text{ — heteroskedastic} \tag{8.9}$$

Note that this distribution does have non-linear conditional variance. However, we also find that this distribution yields a linear conditional mean. Since we have found persistent non-linearity in our economic data, we can see that the Student's T Distribution is not appropriate.

When we consider the class of distributions with a non-linear conditional mean, we see distributions with both homoskedastic and heteroskedastic conditional variances. The logistic distribution in particular has conditional moments of the form.

$$E(Y|X=x)=1-\log_e[1+\exp(-x)] \text{— non–linear in } x \tag{8.10}$$

$$Var(Y|X=x)=\frac{1}{3}\pi^2-1 \text{— homoskedastic} \tag{8.11}$$

Since this distribution function leads to a homoskedastic conditional variance, it is not appropriate. A listing of various distributions and graphical displays under alternative parameter values is contained in Dissertation Appendix 2.

One of the mostly commonly discussed conditional distributions with non-linearity is the Exponential Distribution. This distribution has the characteristic of a non-linear conditional mean in the form of a ratio of polynomials and a non-linear conditional variance. The conditional variance is also a ratio of polynomials. We will derive these conditional moments for multiple exponential distributions, but all have these characteristics. This distribution, therefore, may be able to capture the non-linearity present in our data series.

In fact, since the conditional mean of the exponential distribution is a ratio of polynomial expressions in the regressors, we are able to capture a wide range of possible functional forms (Marshall and Olkin, 1967). The flexibility of this approach as compared to a simple quadratic (for example) should allow us to utilize this distribution with diverse data series. We may find for some series that we do have a constant conditional variance or that we do have linearity in the conditional mean. However, instead of imposing these constraints as is done with the normal linear regression and Student's T model, instead we see these situations as special cases of a generalized distribution function.

In the case of the Normal and Student's T distribution functions, there is one standard functional form which bears these names. However, with the bivariate exponential distribution, we do not find this congruence (McCullaugh and Nelder, 1983). Instead, we see numerous bivariate exponential distributions. All of these functional forms for the exponential distribution are similar in one very important characteristic - they all must have marginal distributions which are exponential. Since the exponential distribution for a univariate series is well known, any bivariate exponential distribution must reduce through marginalization to the this particular distribution form.

The commonly accepted form for a univariate exponential distribution is

$$f(X;\sigma)=\frac{1}{\sigma}\exp(-\frac{x}{\sigma});\quad x\geq 0,\sigma>0 \tag{8.12}$$

with the moments $E(X) = \sigma$; $V(X) = \sigma^2$. The univariate distribution provides us with a starting point for distributions of more than one random variable. We see numerous bivariate and multivariate distributions which satisfy the requirement of having univariate marginals of this particular form.

We will consider three forms of the bivariate exponential distribution which have been most commonly employed (Fruend 1961, Block and Basu 1974, Marshall and Olkin 1967, Gumbel, 1960). We will see that two of these distributions are inappropriate for the empirical model discussed in Section 1. [3]

Due to similarity with the functional form characteristics displayed through misspecification testing and through graphical analysis of the residuals, we will concentrate on the bivariate exponential distribution throughout this chapter. We will return to the issue of model selection in the conclusion.

## 8.2B Forms of the Bivariate Exponential Distribution

Fruend's bivariate exponential distribution (1961) arises through the a life distribution setting. If two components of an instrument have lifetimes $X$ and $Y$ described by density functions, $f(X)$ and $f(Y)$, and the failure of one component changes the life distribution of the other, then the bivariate exponential distribution becomes

$$f(x,y)=\begin{matrix}\alpha\beta'\exp\left[-\beta'y-(\alpha+\beta-\beta')x\right] & (0\le x\le y)\\ \alpha'\beta\exp\left[-\alpha'y-(\alpha+\beta-\alpha')y\right] & (0\le y\le x)\end{matrix} \tag{8.13}$$

where $\beta'$ is the parameter denoting probability of *Y* failure after *X* fails.

This is a bivariate exponential distribution with exponential marginals only in the special case that the probability of *X* failing after *Y* fails is equal to the probability of *Y* failing after *X* fails ($\alpha'=\beta'$). While this particular exponential distribution would be of great interest in durability studies as well as in biostatistics, it is not directly applicable to the type of process we have in mind. [4]

Another very commonly utilized exponential model is that of Marshall and Olkin (1967). This distribution function has also been employed in studies of durability and life cycles. Like the Fruend distribution, this distribution is derived based on the failure of closely related components. In this model, shocks may affect either random variable independently or may affect both. The joint probability density function is

$$F_{x_1,x_2}(x_1,x_2)=\exp\left[+\lambda_1x_1+\lambda_{21}x_2+\lambda_{12}\min(x_1,x_2)\right]\quad(x_1,x_2<0) \tag{8.14}$$

Unlike the Fruend distribution, this does satisfy our requirement that the marginal distributions are exponential in all cases. However, this distribution is not a continuous function and therefore optimization through maximum likelihood methods is not possible. Ryu (1993) has extended this to allow the density function to be continuous under specific conditions. However, to utilize Ryu's method, we must be able to differentiate between the independent and joint shocks and can total each occurrence individually. This need to distinguish between shocks affecting one or both random variables yields this method unusable for our applications.

An alternative exponential distribution which suits our data generating process is the Gumbel Bivariate Exponential Distribution (Gumbel, 1960). This is a continuous distribution with standard exponential marginal distributions and a single parameter denoting the relationship between our random variables. While not directly derived from a life studies problem, this distribution function is applicable to life and survival studies as well as our growth analysis.

We find the Gumbel bivariate exponential distribution function is

$$f(x,y;\theta)=\left[(1+\theta x)(1+\theta y)-\theta\right]\exp[-x-y-\theta xy] \quad \left(x>0,y>0,\theta\in[0,1]\right) \qquad (8.15)$$

and therefore, the joint density function is

$$\frac{\partial^2}{\partial x\partial y}F(x,y;\theta)=e^{-x-y-\theta xy}\left\{(1+\theta x)(1+\theta y)-\theta\right\} \qquad (x,y\geq 0) \qquad (8.16)$$

This density is defined for $x\geq 0; y\geq 0; \theta\in[0,1]$. These requirements do limit our use of this model to experiments which yield positive values only. For our input-output pattern analysis, this condition is satisfied. However, we cannot model growth rates directly through this framework since the rate of growth of a series may often be a negative (for example, when output falls during a recession). Since output, capital and labor will take non-negative values only, this distribution function is accurate for our experiment and actually would be preferred to the normal which would allow negative observations, even though these are theoretically unappealing.

Continuing with the Gumbel bivariate density function, we can verify that this is truly an exponential representation by examination of the marginal density functions.

$$f(y)=\int_0^\infty f(x,y;\theta)dx=\int_0^\infty((1+\theta x)(1+\theta y)-\theta)\exp(-x-y-\theta xy)dx$$

$$f(y)=\exp(-y), \quad y\geq 0 \qquad (8.17)$$

and

$$f(x)=\int_0^\infty f(x,y;\theta)dy=\int_0^\infty((1+\theta x)(1+\theta y)-\theta)\exp(-x-y-\theta xy)dy$$

$$f(x) = \exp(-x), \quad x \geq 0 \tag{8.18}$$

Note that the parameter $\theta$ captures the non-linear dependence between the random variables $X$ and $Y$. If $\theta = 0$, then $f(X=x, Y=y) = f(X=x)\,f(Y=y)$ and the random variables are independent.

Note also that this joint density function implies marginal exponential distributions which are parameter free (see equations (8.17) and (8.18)). We will later extend this to allow a richer formulation of the marginal densities.

The Gumbel Exponential distribution will be our starting point for analysis, provided that the implied conditional distribution has the desired moments. Based upon our misspecification test results (the rejection of a linear conditional mean and a homoskedastic conditional variance) and the analysis of scaled residuals and crossplots, we have developed a possible alternative distribution assumption. We wish to find a distribution function with a non-linear conditional mean and heteroskedastic conditional variance. Before we employ the Gumbel distribution, we must next find its conditional distribution and conditional moments and verify that they meet our requirements.

For the one parameter Gumbel distribution, we find

$$f(Y|X) = e^{-y(1+\theta x)}\{(1+\theta x)(1+\theta y) - \theta\} \qquad (y \geq 0) \tag{8.19}$$

$$E[Y|X = x] = (1+\theta+\theta x)(1+\theta x)^{-2} \text{ and } V(Y|X = x) = \frac{r!(1+\theta x + r\theta)}{(1+\theta x)^{r+1}} \tag{8.20}$$

In fact, this distribution does contain all of our desired characteristics. We can model a continuous range of two random variables, $X$ and $Y$ which may or may not have a dependence structure shown by the value of $\theta$. $\theta$ will equal zero for independent series. We should take care in interpreting this parameter. $\theta$ differs from a correlation coefficient in the sense that it characterizes a complex, non-linear dependence between $X$ and $Y$. Correlation involves only linear dependence.

## 8.3 Estimation of the Empirical Model

We now turn our attention to the issue of estimation of our empirical model. Now that we have formed an alternative distribution assumption, we also must reconsider our method of estimation of the empirical model.

### 8.3A Method of Maximum Likelihood versus Ordinary Least Squares

In the case of the normal linear regression model, estimates of the parameters of the statistical model found by the method of maximum likelihood correspond directly to our least squares estimators and therefore OLS estimation is generally applied. A proof of the equivalence of OLS and Maximum Likelihood estimates is given in Section 8.9. Estimation by both OLS and Maximum Likelihood requires the use of differentiation. For this reason, our bivariate distributions must be continuous. Non-continuous distribution functions can be estimated through the use of alternative techniques such as the method of moments, but cannot utilize the above techniques.

In this paper, we note that not all economic data conform to our assumption of a normal distribution. We are interested in extending this notion beyond the normal requirement. In order to estimate alternative distribution forms, we must abandon the estimates derived through ordinary least squares, and turn instead to maximum likelihood methods. Where as these approaches yielded identical estimates in the normal case, they do not in the case of alternative distribution assumptions.

To derive the maximum likelihood estimates of our Gumbel bivariate exponential distribution, we find the log likelihood function corresponding with the density function

$$f(x,y;\theta)=\left[(1+\theta x)(1+\theta y)-\theta\right]\exp\left[-x-y-\theta xy\right]\quad \left(x>0,y>0,\theta\in[0,1]\right) \tag{8.21}$$

$$L=\prod_{i=1}^{T}\left[(1+\theta x_i)(1+\theta y_i)-\theta\right]\exp\left\{-x_i-y_i-\theta x_i y_i\right\} \tag{8.22}$$

$$\log L = \sum_{i=1}^{T} \log\left[(1+\theta x_i)(1+\theta y_i)-\theta\right] - x_i - y_i - \theta x_i y_i \tag{8.23}$$

$$\frac{\partial}{\partial \theta_i} = \frac{x_i + y_i + 2\theta x_i y_i - 1}{\left((1+\theta x_i)(1+\theta y_i)-\theta\right)} - x_i y_i = 0 \tag{8.24}$$

In this case, we are estimating the parameter $\theta$ which is our only parameter in the conditional mean and variance given in equation (8.20). Estimation of $\theta$ through a least squares formulation which implicitly conforms to estimation with a conditional normal distribution would not be satisfactory due to the additional information contained in the exponential distribution function. Through maximum likelihood estimation, we take advantage of this extra information.

## 8.3B Estimation of Gumbel's Bivariate Exponential Distribution

Recall that we have found output per capita dependent upon the value of the capital to labor ratio to display non-linearity in the conditional mean and some form of heteroskedasticity in the conditional variance. We realize that the Gumbel bivariate exponential distribution is continuous, is defined for values similar to those found in our data series, and is characterized by a non-linear conditional mean. Based upon our misspecification test results and our study of the conditional moments of our distribution of choice, we can proceed now to estimate our output-input relationship given this distribution assumption.

Using the maximum likelihood procedure in equations (8.21) - (8.24) above, we can determined the value of theta which most accurately captures the dependence between output and the K/L ratio, assuming each has a standard exponential distribution and they follow a bivariate exponential together. We will relax the very restrictive standard exponential marginal distribution assumption in the next section.

In terms of procedure, we are ultimately interested in the behavior of output with respect to the capital labor ratio so we will use maximum likelihood estimation to find our estimate of theta, and then define the conditional mean and the conditional variance based upon this estimate. This is analogous to the normal distribution case in which we find estimates for $\beta_0$ and $\beta_1$ and then substitute these estimates into our conditional mean (or regression) equation to determine a mechanism of predicting the expected value of *Y* given the value of the variable *X*.

## 8.3C The Estimation Algorithm : A Simulation

Estimation of the exponential model is somewhat different than in the normal linear regression model. This arises due to the non-linear form of the first order condition. Here we find it is not possible to obtain a closed form solution for the maximum likelihood estimator.

However, with the use of computers, solutions can still be found. A naive method of finding the maximum likelihood estimator would be to calculate the log likelihood value for many possible values of theta and then finding the theta corresponding with the maximum likelihood value. This takes advantage of the fact that for a given data set, the only unknown in the log likelihood function is the parameter.

Such a simple method was tried with the bivariate Gumbel distribution given a simulated data set. This data set was created utilizing the statistical package GAUSS by Aptech Systems. An algorithm for generating bivariate exponential data (Dagpunar, 1988) allowed me to generate exponentially distributed data with a particular relationship determined by theta.

By calculating the log likelihood value corresponding with a range of theta given by (.001, .005, ..., .995), I found that the theta corresponding with the highest log likelihood value was approximately 0.20. By plotting the log likelihood function versus the value of theta we see that this function has a clearly defined optimum around this value of theta. This is termed the Hildreth-Lu method (1960).

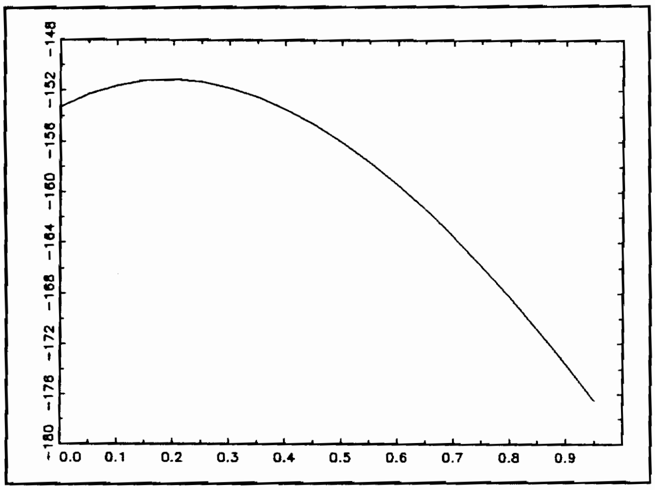



Figure 46 : Log Likelihood Value of Gumbel Distribution for given values of Theta

The data sets generated included sample sizes of 50, 100, 500, 1000, and 1500 observations. Through examination of computer performance as the sample size increased, I was able to determine how reliable my estimation routine performed in the case of small samples.

This method does have limitations. The accuracy of our estimate in this case depends upon the values of theta tried. For example, if we utilized a set of thetas (.25, .5, .75) in the above example, then we would expect to find a estimator of theta to be .25. Instead, if we utilized a set of thetas (.1, .2, .3, .4,...,.9), we would anticipate an estimator of .20. If we again used a "tighter grid" of thetas, we would expect a more accurate value of theta (maybe .17 or .22 in the next case). Our estimator will depend upon the thetas utilized to search. Even in the case of a limited range of values, the possible thetas are infinite and grid searches will always make some amount of compromise.

In the case of the Gumbel distribution, theta is defined to fall in the range (0,1). This is not true in the case of all distributions. A grid search such as the above would be very difficult since decisions on how large and how small of a range to cover are necessary. Also, if the log likelihood function was defined by two parameters, then a three dimensional graph would be required to visualize the search. Additionally, the search would now consider the value of two different parameters.

The grid search provides us with a useful visualization tool and a method of verifying the behavior of our estimating algorithm. Limitations in its ability to find the best theta value and in the large number of calculations required encourage us to find alternative methods. Algorithms for the maximization (or minimization) of a function have been developed long before computers simplified grid searches. In addition to limiting the number of calculations required, these methods also are applicable in cases where the range of theta is unbounded. These methods rely upon a slightly different objective - the seek to minimize the gradient function.

Such methods include the Newton-Raphson technique which utilizes the first two derivatives to search the function for the direction of increase in the function. This algorithm can be thought of as a blind person climbing a hill. If the individual is walking uphill, they continue in the current direction. Once the individual begins to walk down the hill, they know they have passed the peak and back up. Some degree of "flatness" must be set and some starting point on the hill selected. Once these two criteria are decided, the algorithm causes the decision maker to search the hill based on the value of the first and second derivatives at any point.

There are a variety of alternative algorithms, some with greater or lesser requirements on the differentiability of the function. The statistical package GAUSS386 provides several different algorithms with "on the fly" switching possible in the add-on package MAXMUM. In this way, general routines may be utilized when strict routines fail to obtain enough information to take the next step. Some other methods utilized include the Davidson-Fletcher-Powell (DFP) method which requires less of the Hessian matrix, the Berndt, Hall, Hall and Hausman (BHHH) method which utilizes only the first derivatives and the method of conjugate directions which requires no derivative functions.

For more information on these non-linear optimization methods, the reader is encouraged to read David Hendry's appendix on non-linear methods in his recent text, *Dynamic Econometrics* (1995).

Estimation with our simulated data showed that the BHHH algorithm converged regularly to the expected value of theta as found by the grid search method. Larger sample sizes allowed theta to converge in fewer iterations in general. However, overall time of estimation was not greatly improved due to the longer sample length.

## 8.3D Estimation of our Empirical Growth Model

When the statistical model was estimated utilizing GAUSS386 and the Standard bivariate exponential distribution, the estimate of theta was found to be insignificantly different from zero. This insignificance from zero implies that $Y$ is independent of $X$. This is not supported by a general observation of the data or theoretical beliefs. This insignificance I blame on the improper assumption that our marginal distributions are standard. In this next section, a 3 parameter model which allows non-standard distribution assumptions is introduced and then estimated.

## 8.4 A Modified Gumbel Distribution with Additional Parameters

It is not always the case that our random variables will follow a standard exponential distribution. It may be the case that they follow an exponential distribution for which $\sigma$ does not equal 1 in

$$f(X;\sigma)=\frac{1}{\sigma}\exp(-\frac{x}{\sigma});\quad x\geq 0,\sigma>0 \qquad (8.25)$$

Through the inclusion of this additional parameter in our univariate distributions, we can use estimation methods to learn the characteristics of the marginal distribution. These characteristics include the moments of the distribution

$$E(X) = \sigma \quad \text{and} \quad V(X) = \sigma^2 \tag{8.26}$$

However, our main focus in this paper is to explore the behavior of output per capita with respect to changes in the capital to labor ratio. For this reason, we would like to focus on the bivariate and conditional exponential densities corresponding with the above univariate densities.

## 8.4A Derivation of the Three Parameter Density Function

In order to derive the density function for a transformation of Gumbel's density function of standard exponential marginals, we employ a Change of Variable technique (Judge, Hill, Griffiths, Lutkepohl and Lee, 1988). This technique requires the use of the Jacobian. In this way, we redistribute the mass of the density over the new random variable ranges in such a way to conform to the axioms of probability.

The axioms of probability are necessary to assure the consistency of our mathematical formulation with the commonly derived notion of likelihood which we may reach by observing a random experiment a large number of times and noting the number of each outcome. In this case, the probability of each outcome occurring is the number of times it occurred divided by the total number of trials. Summing all these relative frequencies we find 1. Therefore, the mass of our density is distributed in such a way that the sum of all individual probabilities will be 1. Without employing the Jacobian transformation, we would not conform to this property.

We begin with our original random variables, $X$ and $Y$. We wish to substitute for each $X$, the scaled random variable $X/s_1$. In the same way, we will substitute a scaled measure of $Y$ as $Y/s_2$. The Jacobian therefore gives us the term $1/s_1s_2$ which we will also substitute into the original density function to find

$$f(X,Y;\theta,s_1,s_2) = \frac{1}{s_1s_2}((1+\frac{\theta}{s1}x)(1+\frac{\theta}{s2}y)-\theta)\exp(-\frac{x}{s1}-\frac{y}{s2}-\theta\frac{xy}{s1s2}) \tag{8.27}$$

Conditions required for any probability distribution can be verified. To be a proper density function, the sum of all probabilities must be one with no probability value outside the interval [0,∞]. We confirm that this is a true bivariate exponential distribution by examining the marginal distributions. To marginalize, we 'sum' the probability of obtaining some *X* over the entire region of *Y* to find

$$f(x;s1)=\int_0^\infty f(x,y)dy=\frac{1}{s1}\exp(-\tfrac{1}{s1}x) \tag{8.28}$$

$$f(y;s2)=\int_0^\infty f(x,y)dy=\frac{1}{s2}\exp(-\tfrac{1}{s2}y) \tag{8.29}$$

This does satisfy our requirement of exponential marginal distributions. Now we wish to consider the characteristics of the conditional distribution of *Y* given *X* to see if again we find non-linearity in the conditional mean and heteroskedasticity in the conditional variance.

The conditional distribution of Y given X is found through the bivariate and marginal distributions to be

$$f(y|x)=\left[\left(1+\theta\frac{x}{s_1}\right)\left(1+\theta\frac{y}{s_2}\right)-\theta\right]\exp\left\{-\frac{y}{s_2}-\theta\frac{xy}{s_1s_2}\right\} \tag{8.30}$$

This conditional distribution gives us the following conditional moments (See Section 8.10 for derivations).

$$E(y|x)=\int_0^\infty\left(y+\theta\frac{xy}{s_1}+\theta\frac{y}{s_2}+\theta^2\frac{xy^2}{s_1s_2}-\theta y^2\right)\exp\left\{-\frac{y}{s_2}\left(1+\theta\frac{x}{s_1}\right)\right\}\frac{1}{s_2}dy$$

$$=\frac{s_1s_2(s_1+\theta x+\theta s_1)}{(s_1+\theta x)^2} \tag{8.31}$$

$$V(y|x)=\int_0^{\infty}\left[y-\frac{(s_1+\theta x+\theta s_1)s_1s_2}{(s_1+\theta x)^2}\right]^2\times\left[\left(1+\theta\frac{x}{s_1}+\theta\frac{y}{s_2}+\theta^2\frac{xy}{s_1s_2}-\theta\right)\exp\left\{-\frac{y}{s_2}\left(1+\theta\frac{x}{s_1}\right)\right\}\right]dy$$

$$=E(y|x)\left[s_2\left(E(y|x)\right)+2s_2\right] \tag{8.32}$$

Indeed, we do find the same characteristics in our conditional distribution. These are characteristics which we observed in our data series based on misspecification tests.

## 8.4B Estimation with the Three Parameter Model

Estimating the moments of the bivariate distribution with these scaling factors is not as straight forward as in the case with one parameter only. Now we must satisfy multiple first order conditions to find the maximum of the log likelihood function. In addition, we also have constraints which we must conform to such as the value of each $s$ must be greater than zero. This invalidates many solution routines which set one parameter equal to zero and then solve.

In computer applications, the satisfaction of three first order conditions involves a repetitive search over a four dimensional surface for the optimal combination of parameters. Luckily, the statistical package GAUSS386 allows for searching in higher order spaces through a wide range of solution algorithms. Many of these methods are based on Newton's method and allow for manipulation of step size and tolerance.

Specifically, let us now examine the log likelihood function and first order conditions for our estimators of the bivariate distribution parameters.

$$f(x,y)=\left(1+\theta\frac{x}{s_1}+\theta\frac{y}{s_2}+\theta\frac{xy}{s_1s_2}-\theta\right)\exp\left\{-\frac{x}{s_1}-\frac{y}{s_2}-\theta\frac{xy}{s_1s_2}\right\} \tag{8.33}$$

$$L\approx\prod_{i=1}^{T}\left(1+\theta\frac{x_i}{s_1}+\theta\frac{y_i}{s_2}+\theta^2\frac{x_iy_i}{s_1s_2}-\theta\right)\exp\left\{-\frac{x_i}{s_1}-\frac{y_i}{s_2}-\theta\frac{x_iy_i}{s_1s_2}\right\} \tag{8.34}$$

$$\log L = \sum_{i=1}^{T} \ln\left(1+\theta\frac{x_i}{s_1}+\theta\frac{y_i}{s_2}+\theta^2\frac{x_i y_i}{s_1 s_2}-\theta\right)-\frac{x_i}{s_1}-\frac{y_i}{s_2}-\theta\frac{x_i y_i}{s_1 s_2} \tag{8.35}$$

$$\frac{\partial}{\partial\theta}=\frac{1}{\left(1+\theta\frac{x_i}{s_1}+\theta\frac{y_i}{s_2}+\theta^2\frac{x_i y_i}{s_1 s_2}-\theta\right)}\left(\frac{x_i}{s_1}+\frac{y_i}{s_2}+2\theta\frac{x_i y_i}{s_1 s_2}-1\right)-\frac{x_i y_i}{s_1 s_2}=0 \tag{8.36}$$

$$\frac{\partial}{\partial s_1}=\frac{1}{\left(1+\theta\frac{x_i}{s_1}+\theta\frac{y_i}{s_2}+\theta^2\frac{x_i y_i}{s_1 s_2}-\theta\right)}\left(\theta\frac{x_i}{s_1^2}+\frac{\theta^2 x_i y_i}{s_1 s_2}\right)-\frac{x_i}{s_1^2}=0 \tag{8.37}$$

$$\frac{\partial}{\partial s_2}=\frac{1}{\left(1+\theta\frac{x_i}{s_1}+\theta\frac{y_i}{s_2}+\theta^2\frac{x_i y_i}{s_1 s_2}-\theta\right)}\left(\theta\frac{y_i}{s_2^2}+\frac{\theta^2 x_i y_i}{s_1 s_2}\right)-\frac{y_i}{s_2^2}=0 \tag{8.38}$$

We can now apply this to our data series to examine the relationship between output per capita and the capital - labor ratio when these random variables do not necessarily follow a standard exponential distribution. Through this model, we will extract information about both the relationship between these quantities, still primarily through the variable theta, and information about the distribution of each variable separately, through the sigma terms.

## 8.4C Estimation of the Empirical Growth Model

Estimation our own model with the 3 parameter Gumbel distribution yields the following parameter estimates:

| Parameter | Theta | *s1* | *s2* |
|---|---|---|---|
| Estimated Value | 4.8397 | 1.1915 | 1.6459 |

and the conditional moments

$$E(Y|X) = \frac{22.409 + 15.595x}{(1.101 + 4.840x)^2} \tag{8.39}$$

$$V(Y|X) = \frac{22.409 + 15.595x}{(1.191 + 4.840x)^2}\left[\frac{36.863 + 25.654x}{(1.191 + 4.840x)^2} + 3.29\right] \tag{8.40}$$

In interpreting these conditional moments, we see that output per capita in each time period is related to the contemporaneous capital to labor ratio in a complex manner. Increases in the value of $X$ will increase both the numerator and the denominator. In addition, the variation of output per capita from its norm is also indicated as a function of the capital to labor ratio as shown by the conditional variance.

Analysis of our residuals shows that our model is not accurately characterizing the United States quarterly output performance. The sum of squared residuals from this model is considerably larger than the SSR obtained by the normal linear regression model utilized in Section 1. In the Exponential model, this SSR was 7745.90 as compared to a measure of 152.96 found when a Linear Regression model is utilized. While this is not the most accurate measure of fit for non-linear models (see Section 6), graphical analysis of the residuals in Figure 47 provides no confidence that this model is suitably capturing the behavior of output per capita. We consistently overestimate the size of output per capita as shown by the always positive residual series. We have not even achieved a residual series with zero mean, as we will have in a linear regression model.

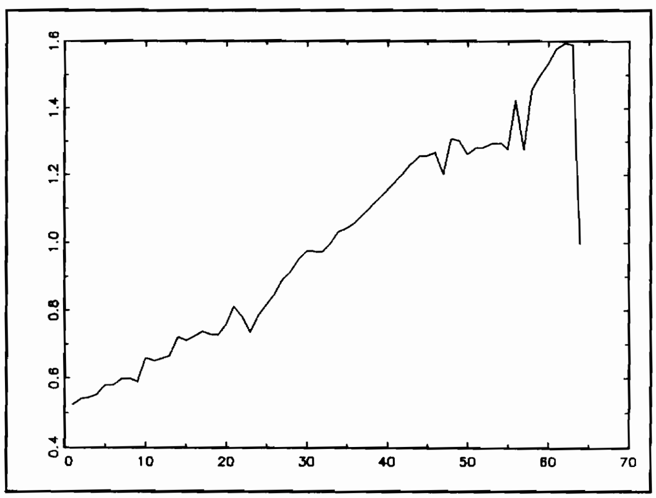



Figure 47 : Residuals of Three Parameter Gumbel Model

## 8.5 The Constrained Optimization Problem

The Gumbel distribution is subject to one serious limitation. We often find models which are not robust to starting values or models which break down at some point in optimization. The reason is that the Gumbel distribution implies certain constraints on the values and combinations of values of theta possible.

To address this problem, even in the case of multiple parameters and constraints, we now have computer algorithms which conquer the problem of constrained optimization. That is, given particular combinations of starting values, the conditions required to solve the log likelihood function were often violated in the three parameter Gumbel model. For example, negative sigma terms were selected by the computer. In the next step, the computer found that the log of sigma was undefined and exited the algorithm.

### 8.5A Formulation of the Empirical Model with Constraints

Non-negativity constraints are easily included in optimization through a reforming of the density function. This enables us to conduct constrained optimization without the use of constrained objective functions and specialized computer packages.

To constrain the terms *s1* and *s2* to be positive, we substitute *(δ1*δ1)* for each *s1* and then solve for the optimal value of *δ1*. Once *δ1* is found, we square it to find the estimate of *s1*. This is repeated for the parameter *s2* as well. In this way, we rewrite the log likelihood function in terms of the parameters θ, *δ1* and *δ2*. Once estimates of *δ1* and *δ2* are found, *s1* and *s2* estimates are also known.

However, we do have one constraint which cannot be accommodated by transformation of the parameter. That is,

$$\left(1+\theta\frac{x_i}{s_1}+\theta\frac{y_i}{s_2}+\theta^2\frac{x_i y_i}{s_1 s_2}-\theta\right)>0 \tag{8.41}$$

Formulation of the constrained optimization problem is based on the Lagrangian function. We reform our log likelihood function to be :

$$\log L = \sum_{i=1}^{T} \ln\left(1+\theta\frac{x_i}{s_1}+\theta\frac{y_i}{s_2}+\theta^2\frac{x_i y_i}{s_1 s_2}-\theta\right)-\frac{x_i}{s_1}-\frac{y_i}{s_2}-\theta\frac{x_i y_i}{s_1 s_2}+\lambda\left(1+\theta\frac{x_i}{s_1}+\theta\frac{y_i}{s_2}+\theta^2\frac{x_i y_i}{s_1 s_2}-\theta\right) \tag{8.42}$$

with an additional first order condition regarding the parameter $\lambda$.

To estimate by constrained optimization, the mainframe version of the statistical package Matlab (by Mathworks, Inc) was utilized. Matlab includes a Toolbox named OPTIM which includes procedures for constrained and unconstrained optimization of linear and non-linear functions. The log likelihood (objective) and constraining functions are fed into a procedure and Matlab searches utilizing a variety of algorithms just as Gauss has done previously.

## 8.5B Results of the Constrained Three Parameter Model

Results for this optimization differ greatly from the unconstrained case. We find a predicted value of $\theta$ to be zero which again indicates that *Y* and *X* are independent random variables. The estimates of *s1* and *s2* are not close at all to the value of 1. They are closer to the standard errors of the series.

These numerical estimates for constrained optimization are :

| Parameter | Theta | *s1* | *s2* |
|---|---|---|---|
| Estimated Value | 0.000 | 15.427 | 22.716 |

The lack of dependence between *Y* and *X* implies that our predicted values are simply a constant dependent upon the estimates of *s1* and *s2*. These constants consistently underestimate the true output per capita value and lead to a *SSR* of over 10000 as opposed to 7746 in the unconstrained model. Constrained optimization is difficult for computers to conduct, particularly with such small sample sizes (discussion with Ron Schoenberg of Aptech Industries, on GAUSS newsgroup, 1995). Often constrained optimization estimation may result in lowered objective function values since the search is now more limited.

We believe the capital-labor ratio is somewhat correlated to the amount of output per capita produced. This relationship appears in the Linear Regression Model through the parameter $\beta 1$. In the exponential model, we would anticipate the parameter $\theta$ to include this correlation along with a higher form of dependence. However, our constrained optimization estimation was not able to pick up any information relating $Y$ and $X$ and instead shows these variables to be independent. It is possible the requirements for estimation of these parameters was simply too much for our limited data.

The common finding that our estimate of $\theta$ equals zero is not that surprising when we consider the characteristics and shape of an exponential distribution. Exponential distributions tend to place a lot of weight around the origin (they are highly skewed to the right). This is displayed in Dissertation Appendix 2. The tendency therefore to find estimates of parameters to be very small is not unlikely.

An alternative distribution family with less weight around the origin while still remaining skewed may show a stronger relationship between $Y$ and $X$.

## 8.6 Model Selection with Competing Families of Distributions

Just as in the case of regression with a normal conditional mean, we must evaluate the properties of our estimators and the ability of our model to capture the behavior of the $Y$ variable. In most cases, we have conducted particular misspecification tests looking for deviations from the linear regression model and also analyzed the characteristics of the residual series.

## 8.6A Misspecification Testing Framework

Analysis of departures from the assumption of an exponential conditional distribution are not as straightforward. Test statistics for the misspecification tests employed in Section I assume that under the null hypothesis, we have a conditional normal so that the behavior of the test statistics is known. In the case of the exponential distribution, these misspecification tests are inappropriate since the distribution of the test statistic under the null would not be normal. There is room here for the development of the distribution of test statistics under the assumption that the distribution truly is exponential.

We can only employ the group of log likelihood ratio tests in cases in which the models to be examined have the same underlying log likelihood function which restrictions imposed in some cases. That is, we compare the value of the log likelihood function when the parameters are unrestricted to the log likelihood value when particular parameters are restricted. While this cannot help us to decide between a normal distribution family and an exponential family it does enable us to look at various exponential models and determine if the 1 or 3 parameter Gumbel distribution is best supported in this particular case.

The logic is as follows. We first estimate the 3 parameter Gumbel distribution to find estimates of $\theta$, $s1$ and $s2$. This results in a particular log likelihood value. We then estimate the 1 parameter Gumbel distribution. This form of the Gumbel corresponds with the three parameter model with $s1=s2=1$. This results in a log likelihood value. If the 3 parameter (or unrestricted) model corresponds perfectly with the one parameter model, then the log likelihood values will be equivalent and the restriction is not rejected. If the three parameter estimates of $s1$ and $s2$ differ significantly from one, then the log likelihood value in the restricted case will be significantly smaller and the restriction is rejected. (Judge, Hill, Griffiths, Luktepohl and Lee (1988)).

## 8.6B Selection Among Alternative Distributions

How then do we determine if the exponential model is appropriate? The answer requires a generalization of the our notions of goodness of fit. This has been explored to a great extent by Linhart and Zucchini's *Model Selection* (1986) and McCallagh and Nelder's *Generalized Linear Models (*1989).

### Generalizations of Goodness of Fit : Scaled Deviance and Pearson X2 Statistics

Both books utilize measurements of goodness of fit involving the general concept of a deviance function. The deviance function is closely related to likelihood estimation methods. In both the log likelihood and the deviance function, a particular objective function tries to link parameter estimates to data characteristics shown by a sample.

Empirical models attempt to approximate the behavior of $Y$ based on estimation by some number of parameters. We may only estimate one parameter such as the mean of $Y$ and assume that all other errors are random. Or we may estimate one parameter for each observation and therefore perfectly match the sample observations but not summarize information about what may happen in the future. Our employed methods usually fall somewhere in between, with the models providing better estimates than a constant but through the use of less than T estimators.

McCallagh and Nelder define a new function to be the *scaled deviance which is the maximum obtainable likelihood value (if the predicted values correspond perfectly with the data) minus the obtained likelihood value.* The difference will be smaller the better the empirical model approximates the observations. As more information is added, this measure should shrink (such as through the inclusion of new explanatory variables.) The discrepancy measure has a particular form regardless of the density.

Another strongly recommended measure is the Generalized Pearson $X^2$ statistic which includes information based on the estimated variance function as well as the squared deviation of $Y$ from predicted values. This measure employs information in both the conditional mean (or regression) equation and the conditional variance (skedastic) function. In the case of the Linear Regression Model, the skedastic function was simply a constant and the Pearson $X^2$ measure corresponded with the usual $R^2$. However, in the case of distributions with a heteroskedastic variance, the Pearson $X^2$ measure contains additional information. The Pearson measure has the advantage of using a very standard sum of squared deviations format, but accounts for the additional information we have found in the skedastic function.

Both the scaled deviance and the Pearson $X^2$ statistics have well defined distributions in the case that the normal model is appropriate. Asymptotic results are available for other distributions, but rely on large samples to be realistic.

## Comparing the Normal and Exponential Models : Discrepancy Measures

In Linhart and Zucchini's *Model Selection*, the authors discuss exactly the problem we need to address - is the exponential model appropriate, especially compared to the linear regression model we were using in the past? The intuition utilized to derive distribution free statistics is simply based on how accurate are our choices of distribution family and estimation method.

There may be a large number of distribution functions which we may choose to approximate the behavior of some unknown data generating process. Each of these distribution functions has its own log likelihood function which is used to derive estimates of some number of parameters. From here, we want to determine which of these models is most closely approximating the observed behavior. Comparisons of log likelihood values is like apples to oranges since these are such different models. Instead, we need a measure which will allow comparisons across different distributions.

Linhart and Zucchini begin by discussing the problem of fitting a model to observations and strategies to select an approximating family when the true family of distribution is unknown. In this section, the authors note the limitations of *"attempting to find the simplest approximating family which is not inconsistent with the data."* This approach may lead to us, not toward the best approximating model, but to one which satisfices under what may be ad hoc assumptions. This is very consistent with the approach utilized throughout this dissertation. We wish to find an empirical model which is consistent with assumptions implied. We wish the restrictions imposed through the Haavelmo reduction to be accurate. This includes the selection of distribution family.

The authors then continue to develop a measure of expected discrepancy which is minimized through the selection of an approximating family of distributions. However, not knowing the underlying data generation process means that even the most complete measures of discrepancies are based on assumptions and may be imprecise.

They consider the overall discrepancy to depend upon two components - the *Discrepancy Due To Approximation* and the *Discrepancy Due To Estimation*. The Approximation Discrepancy arises because we must choose an approximating distribution which may not be correct (this is not dependent on the data or the sample size). The Discrepancy Due To Estimation is the error because our estimates are imperfect and it does depend on the data and the sample size.

Discrepancies are larger the more the approximating model (based on parameter estimates) differs from the distribution generating the data. To find a discrepancy, two choices are necessary - one is the approximating family and the other is the method of estimation. The first choice determines the Discrepancy Due To Approximation. The second determines the Discrepancy Due To Estimation.

The limitation of this method is that the discrepancy is a random variable and its distribution depends upon the true data generating model (called an operating model by these authors). Expected values of discrepancies under certain assumptions about the operating models lead to criterion which can be used to select between models. If the same method of estimation is used, then one can select between approximating families through the use of criterion. If the same approximating family is used, then one can select the method of estimation to minimize discrepancy. Since the operating model is still unknown, however, the significance level corresponding to these tests is very low.

These measures of discrepancy due to approximation in the simple case of 1 parameter are based on the estimated covariance matrix. For additional parameters, the measures also include an interaction term . If it is believed that the operating model is contained in or is near to the approximating model, then simpler criteria are available. The authors state that these simple criteria tend to perform as well and sometimes better than the more detailed criteria, so long as the models are close.

Residual Analysis of Figure 47 shows that indeed the Exponential model does not out perform the conditional normal distribution assumption. This indicates that the exponential distribution assumption is no better than that of the normal for this particular data series.

To generate and understand the behavior of a discrepancy measure for the Gumbel Exponential distribution, it will be necessary to generate bootstrap estimates of the discrepancy statistic under the assumptions that the true operating model is a Gumbel Exponential distribution. This distribution of data can then be utilized in a Monte Carlo type study to determine the properties of a discrepancy statistic.

## 8.7 Conclusion

We began with a simple model of output growth which could be represented in a bivariate distribution. An extension to multivariate distributions would be ideal, but this strains our often limited data resources, especially for yearly observations. Using this statistical model, we examined the relationship between US output per capita and the capital to labor ratio under the usual normality assumption and found misspecification error. Specifically we found conditional mean non-linearity and variance heteroskedasticity which indicates that our series may not be normally distributed.

Using misspecification tests as a guide, I proposed one of many possible bivariate exponential distributions. The conditional distribution and its conditional moments were derived and shown to conform to our expectations about the characteristics of the conditional density. Next, we utilized Maximum Likelihood estimation to find an estimate of the parameter of the bivariate distribution.

A further extension examined the same relationship using a more generalized Gumbel distribution function allowing for non-standard exponential marginal distributions. This new density function has 3 parameters to be estimated. These parameter estimates were then substituted into the conditional mean and variance functions to determine the relationship between output per capita and the capital labor ratio. Such flexibility in accounting for the behavior of each variable separately would enhance our understanding of each variable individually and also in accurately portraying their relationship.

Constrained optimization revealed a methodology to avoid computer algorithm failure by stipulating conditions necessary for computation of the objective function and for restricting parameter values to those consistent with the probability model. However, our solution in this case corresponded closely with the one parameter Gumbel distribution and was not preferred to the Linear Regression Model.

It has been shown through this paper that alternative distribution assumptions can be easily employed to examine the behavior of variables not conforming to our normality assumption. However, the successfulness of this approach for this particular model cannot be advocated. As opposed to developing complex methods to "fix up" the conditional variance, we can instead utilize the characteristics of the data series itself to select a more appropriate distribution assumption. Both parametric misspecification testing and non-parametric graphical and numerical analysis can guide us in this choice.

As a future extension, I would like to see this method applied to additional distribution functions. Also, methods of testing the validity of non-normal conditional distribution assumptions has not been explored to the extent of testing with the normal. In general, we still look to asymptotic results. With increasing computing technology and exploration of individual distributions, research in the evaluation and comparison of models with alternative distributions would be very useful in a variety of applications.

## 8.8 Notes

1. Exogenous models of growth, including their relation to Kaldor's Stylized Facts, were explored in Chapter 4. Models with endogenous growth or growth engines were discussed in Chapter 5. In Chapter 6, international patterns were discussed.

2 There are a wide variety of distributions available to consider in this stage. Even if we consider only continuous distributions, we still have many difficult choices to make. As a guide, I recommend the excellent reference for Univariate distributions, *Statistical Distributions* by Merran Evans, Nicholas Hastings and Brian Peacock(1993). For Bivariate and Multivariate Distributions, I recommend Johnson and Kotz (1972) *Distributions in Statistics: Continuous Multivariate Distributions* although this listing is not quite as extensive. The distributions discussed by these resources include:

*Univariate Distributions* : Beta, Cauchy, Chi-Squared (central and non-central), Dirchlet, Erlang, Error, Exponential, Extreme Value, F (central and non-central), Gamma, Geometric, Hypergeometric, Laplace, Logrhytmic Series, Logistic, LogNormal, Normal, Pareto, Power Function, Uniform, Student's T, Weibull, Wishart.

*Bivariate/Multivariate Distributions* : Normal, Student's T, Wishart, Beta, Gamma, Extreme Value, Exponential, Pareto, Logistic

3 In the future, it would be interesting to generalize this framework even more by estimating parameters of a Gamma distribution. Many exponential distributions are special cases of a Gamma Distribution. Moments and properties of the Gamma distribution are well developed. However, the Gamma has a linear conditional variance by assumption, so this may not be appropriate for all data series.

4 Most common applications of exponential models deal with the life of a product or machine, or with the life of individuals. Many exponential distributions, in the form of hazard and survival functions, have been applied in the field of Biostatistics. The growth of an infectious organism can be predicted and a measure of likeliness of this growth to remain at a livable level is calculated. We are utilizing the exponential in a very similar way. We wish to obtain knowledge about the growth of output given values of our

growth engine (in this case, the capital to labor ratio). While biostatistics is very concerned with the cumulative probability distribution, we will primarily focus on the conditional density.

**5** I have shown derivations here for the standard normal bivariate distribution. An extension to the non-standard distribution is obvious, but more involved in notation.

**6** The linear regression model and the exponential model were also compared utilizing the log of output and the log of the capital to labor ratio. The lack of improvement in the Sum of Squared Residuals was displayed by the Log model as well. In this case, the Log linear regression model had a SSR of 1.840 while the exponential model had an SSR of 5.663.

## 8.9 First Appendix. An Examination of Maximum Likelihood and Ordinary Least Squares Estimate Equivalence

Given the linear regression model and the knowledge that $X$ and $Y$ are normal random variables with the distribution

$$\begin{matrix} Y \\ X \end{matrix} \sim N \begin{matrix} \bar{Y} \\ \bar{X} \end{matrix}, \begin{matrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{2} \end{matrix}$$

and our conditional mean is $Y = \beta_0 + \beta_1 X$.

Then

$$\beta_1 = \frac{\sigma_{12}}{\sigma_2} \quad \text{and} \quad \beta_0 = \bar{Y} - \beta_1 \bar{X}$$

through either the minimization of the ordinary least squares objective function

$$Min_{\beta_0,\beta_1}\sum_{t=1}^{T}(Y_t-\beta_0-\beta_1X_t)^2$$

$$\partial/_{\partial\beta_0}=\sum_t 2(Y_t-\beta_0-\beta_1X)=0$$

$$\bar{\beta}_0=1/_T\sum(Y_t-\beta_1X_t)$$

and

$$\partial/_{\partial\beta_1}=\sum_t 2X_t(Y_t-\beta_0-\beta_1X_t)=0$$

$$\bar{\beta}_1=\sum X_tY_t\,/\,X_t^2$$

or from a maximization of the log likelihood function of the bivariate density function.[5]

$$f(x,y;p)=\frac{\left(1-p^2\right)^{-\frac{1}{2}}}{2\pi}\exp\left\{-\frac{1}{2\left(1-p^2\right)}\left(x^2-2pxy+y^2\right)\right\},$$

$$\log L(p;x,y)=c-n\log 2\pi-\frac{n}{2}\log\left(1-p^2\right)$$

$$-\frac{1}{2\left(1-p^2\right)}\sum_{i-1}^{n}\left(x_i^2-2px_iy_i+y_i^2\right),$$

$$\frac{d\log L}{d\rho}=+\frac{n(-2)}{2\left(1-\rho^2\right)}\rho-\rho\frac{\sum_{i=1}^{n}\left(x_i^2+y_i^2\right)}{\left(1-\rho^2\right)^2}+\frac{\left(1-\rho^2\right)}{\left(1-\rho^2\right)^2}\sum_{i=1}^{n}x_iy_i=0$$

$$=n\bar{\rho}\left(1-\bar{\rho}^2\right)+\left(1+\bar{\rho}^2\right)\sum_{i=1}^{n}x_iy_i-\bar{\rho}\left(\sum_{i=1}^{n}x_i^2+\sum_{i=1}^{n}y_i^2\right)=0.$$

## 8.10 Second Appendix : Derivation of the Conditional Moments for the Three Parameter Gumbel Distribution:

$$E(y|x) = \int_0^{\infty}\left(y+\theta\frac{xy}{s_1}+\theta\frac{y}{s_2}+\theta^2\frac{xy^2}{s_1s_2}-\theta y^2\right)\exp\left\{-\frac{y}{s_2}\left(1+\theta\frac{x}{s_1}\right)\right\}-\frac{1}{s_2}dy$$

$$=\frac{\left(y+\theta\frac{xy}{s_1}+\theta\frac{y}{s_2}+\theta^2\frac{xy^2}{s_1s_2}-\theta y^2\right)}{\frac{-(s_1+\theta x)}{s_1s_2}}\exp\left\{-\frac{y}{s_2}\left(1+\theta\frac{x}{s_1}\right)\right\}-$$

$$\frac{\left(1+\theta\frac{x}{s_1}+2\theta\frac{y}{s_2}+2\theta^2\frac{xy}{s_1s_2}-\theta\right)}{\left(\frac{-(s_1+\theta x)}{s_1s_2}\right)^2}\exp\left\{-\frac{y}{s_2}\left(1+\theta\frac{x}{s_1}\right)\right\}-$$

$$\frac{2\theta}{\left(\frac{-(s_1+\theta x)}{s_1s_2}\right)^2}\exp\left\{-\frac{y}{s_2}-\theta\frac{xy}{s_1s_2}\right\}\Bigg|_{y=0}^{\infty}$$

$$=0-\left[\frac{-\left(1+\theta\frac{x}{s_1}-\theta\right)}{\left(\frac{-(s_1+\theta x)}{s_1s_2}\right)^2}\right]+\frac{2\theta}{\left(\frac{-(s_1+\theta x)}{s_1s_2}\right)^2}$$

$$=\frac{s_1s_2(s_1+\theta x+\theta s_1)}{(s_1+\theta x)^2}$$

and

$$V(y|x) = \int_0^{\infty} \left[ y - \frac{(s_1 + \theta x + \theta s_1)s_1 s_2}{(s_1 + \theta x)^2} \right]^2 \times$$

$$\left[\left(1 + \theta\frac{x}{s_1} + \theta\frac{y}{s_2} + \theta^2\frac{xy}{s_1 s_2} - \theta\right)\exp\left\{-\frac{y}{s_2}\left(1 + \theta\frac{x}{s_1}\right)\right\}\right]dy$$

$$= \left[ y - \frac{(s_1 + \theta x + \theta s_1)s_1 s_2}{(s_1 + \theta x)^2} \right]^2 \left\{ \frac{1 + \theta\frac{x}{s_1} + \theta\frac{y}{s_2} + \theta^2\frac{xy}{s_1 s_2}}{-\left(\frac{s_1 + \theta x}{s_1 s_2}\right)} \right\} \times$$

$$\exp\left\{-\frac{y}{s_2}\left(1 + \theta\frac{x}{s_1}\right)\right\} - 2\left[ y - \frac{(s_1 + \theta x + \theta s_1)s_1 s_2}{(s_1 + \theta x)^2} \right] \times$$

$$\left\{ \frac{\left(1 + \theta\frac{x}{s_1} + \theta\frac{y}{s_2} + \theta^2\frac{xy}{s_1 s_2}\right)}{-\left(\frac{s_1 + \theta x}{s_1 s_2}\right)} \exp\left\{-\frac{y}{s_2}\left(1 + \theta\frac{x}{s_1}\right)\right\} \right\}$$

$$= \left(\frac{(s_1 + \theta x + \theta s_1)s_1 s_2}{(s_1 + \theta x)^2}\right)\left[\frac{(s_1 + \theta x + \theta s_1)s_1 s_2^2}{(s_1 + \theta x)^2} + 2s_2\right]$$

$$= E(y|x)\left[s_2\left(E(y|x)\right) + 2s_2\right]$$

# *IX. Conclusion*

Nelson and Winter (1985) have likened our search for a Solow Residual to the physicist's search for a neutrino. A neutrino is a particle which has not been observed, but yet we believe it exists and utilize its existence to explain the failure of models. The search for a neutrino is a popular research topic in modern physics. Economists have similarly looked to an unobservable quantity, technological progress, to account for differences in input and output growth. It is generally believed that productivity changes do occur, but we must use caution when attributing all of our model's error to "the neutrino effect" or, in our case, to productivity enhancement.

## 9.1 An Overview Of This Study

The purpose of this study was to explore the pattern of United States output growth in the twentieth century. We wished to examine the ability of various inputs to increase the rate of growth and to learn about the Solow Residual. Particularly, we wished to see if our output behavior was consistent with the Stylized Facts mentioned by Nicholas Kaldor in the 1930's.

We noticed through an examination of theoretical growth models in Chapter 2 that the Solow model of economic growth with exogenous economic progress is consistent with these observations of Kaldor. However, empirical testing of this statistical model in Chapter 4 showed misspecification in terms of our distribution assumption (through the non-linearity of the conditional mean) and through temporal dependence. A respecified empirical model which included dynamic terms as well as non-linear terms did improve upon our earlier model. However, this statistical representation was still inadequate due to continuing non-linearity in the conditional mean.

In order to learn more about this process of technological change, Chapter 5 examined endogenous models of economic progress. The effects of education, public infrastructure development and innovation on output growth were evaluated. Through an inclusion of these terms in the output growth model, we account for some amount of productivity improvements. In each statistical model, we found the presence of temporal dependence indicating the need for lagged values. Also, we again saw misspecification in terms of the conditional mean. From a policy perspective, the impact of these variables on output growth is very informative. However, due to condition mean misspecification, we should not put too much reliance upon the values of these coefficients.

We recognize that the United States does not exist in a vacuum. Growth performance in other nations is important as well. Many growth models have concentrated on cross-sectional analysis. While this is useful for questions of convergence, changes in institutions and fundamental differences between nations caution its cross sectional interpretation. Instead, a panel study of several nations may help us to understand the similarities and differences in growth performance of different countries. This was evaluated in Chapter 6. Similarity in the functional form of the statistical model in four industrialized nations was not predicted and indicates that production characteristics may be similar across nations. However, there are apparently significant differences in parameter values. We notice similar misspecification remaining. In each case, we see evidence of conditional mean non-linearity, just as in the United States models.

In order for a statistical model to adequately represent the process generating our data, we must find stationarity in our parameter estimates. The need for this assumption in both economic theory and in estimation was explored in Chapter 7. Non-stationarity in our yearly estimates lead us to a shorten sample period. This greatly enhanced the stationarity of our parameters as shown by recursive and windowed least squares estimates. The Hamilton model which allows for endogenous switching was discussed and applied to the Solow Residual. Analysis was consistent with Hamilton's findings that output behavior tends to change regimes relative to the business cycle as opposed to threshold growth model predictions.

Finally, to account for this continuing result of non-linearity in the conditional mean, we looked toward a different method of estimation. We abandon the assumption of conditional normality and instead assume a more appropriate distribution - that is, a distribution with a non-linear conditional mean. Through the development of a bivariate Gumbel distribution with scaling parameters, we can estimate a non-linear conditional mean and variance of a particular form. Through the use of Maximum Likelihood Estimation, we find a new method for anticipating the form of the growth relationship. Both Constrained and Unconstrained nonlinear estimation are discussed. Also highlighted is the issue of model selection when we see competing distribution assumptions.

## 9.2 Major Contributions

This dissertation I believe makes contributions to both the understanding of United States and International growth processes and to the methodology of econometrics.

First, a systematic study of US growth models allowing for possible misspecification error has shown the important role of dynamics in prediction of current growth rates. This is a factor which must be included in our "growth accounting" else we underestimate the impact of an increase in a factor on total output. Also shown in this study was the persistent non-linearity of Solow residuals. We observed that international studies of economic growth showed the nations of the US, UK, Sweden and Japan all influenced by similar variables and subject to similar misspecification error.

In addition to this particular application, however, this research is an example of how important misspecification testing is in the understanding of a statistical relationship. We see how much the empirical models characterizing an important economic process have changed through a careful examination of information remaining in the residuals of an empirical model.

In terms of econometric theory, the major contributions of this dissertation are in the development and implementation of exponential regression shown in Chapter 8. This is just one possible direction of research in expanding our maximum likelihood tools. Through careful analysis of our data, an informed choice was made for the distribution assumption. Then the Exponential model was estimated and a new exponential model allowing for more generality in the parameters was derived and estimated. The methodology for constrained optimization was also discussed and implemented.

The movement away from Normal distribution assumptions follows logically from probabilistic theory and encompasses a variety of recent literature (such as the Generalized Least Squares and ARCH methodology). This dissertation shows how we can learn from our data series through a meticulous examination of information contained in the data itself and also how to implement alternative assumptions in the form of the dynamic linear regression model and exponential regression.

## 9.3 Future Directions

This study has identified some misconceptions in the empirical growth literature. Not only were problems diagnosed, but often alternative model assumptions were applied. My extensions of this research are taking off in two fields, theoretical and applied econometrics.

In terms of econometric methodology and theory, I am continuing to develop this maximum likelihood method displayed in Chapter 8. I would like to develop diagnostic tools to analyze this output further as well as develop even more generalized distribution assumptions. For example, a gamma distribution includes the exponential as a special case, and therefore should be able to estimate the relationships displayed here as well as be suitable for a wider variety of conditional relationships. Some possible alternative distributions were given in Appendix 2.

In terms of applications to growth theory, I am currently exploring the simultaneity of growth patterns between nations as displayed in Chapter 7. I am exploring this dependency between nations in the hopes of developing a simultaneous model of growth for 2 nations. I hope that this model will be applied to the United States-Canada and United States-Mexico relationships particularly due to interest in a North America Free Trade Alliance. Hopefully, this model can be extended to multiple nations in the future.

I hope to continue to strengthen the misspecification testing framework presented throughout this research. This work involves a further exploration of stationarity and distributional model assumptions. I would like to further endogenize the probability of regime shift in the Hamilton model as discussed in Chapter 7. I believe there still may be threshold style variables which will indicate switches from low growth to high growth periods. I addition, I hope to continue to apply the misspecification testing utilized throughout this dissertation in a variety of areas of economics to promote exposure to the benefits of reading the residuals.

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# *Appendix 1. Graphical Analysis of Model Misspecification*

In addition to analytical tests, much can be learned about the behavior of a data generating mechanism through an examination of plots. Through the use of residual plots in particular, knowledge regarding information retained in error terms may lead us to improved model specification.

Most misspecification tests are performed by adding information to the empirical model and then observing the improvement in some criteria such as the value of the log likelihood ratio or the sum of squared residuals. Through graphical analysis, we may be able to determine which variables are most likely to capture systematic information remaining in the error term of an empirical model.

In this appendix, we will discuss graphical tools as an aid in misspecification testing. The information presented in this chapter closely follows Spanos (1986) and Spanos (1994).

A word of caution is in order. Graphical tools enable us to anticipate the presence of misspecification error and to guess at variables which may remove influences from the error term. However, graphical analysis is inherently arbitrary. Systematic tests with known significance level are necessary to avoid this arbitrary decision making. In this way, formal misspecification testing and informal graphical analysis work together to lead us to a statistically adequate empirical model.

In this discussion, we begin with the distribution, homogeneity and memory assumptions discussed in Chapter 3. While attributed to different sources, misspecification of multiple assumptions often occurs together. It is sometimes difficult to distinguish the source of error. However, working with all three assumptions, we can allow our residual plots to guide us to a new, better specification.

While we could and often do utilize such analysis on "raw" data series, I will concentrate on the graphical analysis of regression residuals which plays an important role in this dissertation. Very similar techniques could be utilized on data series themselves.

## A1.1 Memory Assumptions

We will first consider the autocorrelation which may remain in our residual series if dynamic terms are significant. If lagged values matter, then the error terms will display some degree of memory or persistence.

It may be the case that errors tend to take similar values as the errors around them. This is termed positive memory. Knowledge of the past of an error series, therefore, can significantly predict the future value.

Also, it may be the case that a large positive error value is followed by a large negative value and so on. If increases and decreases clearly alternate, knowledge of the past value of an error term is again a source of information and the series is not independent.

Ideally, we should not see long swings in the data series (an indicator of positive memory) or alternating observations (an indicator of negative memory). Some example graphs displaying these violations are shown below.

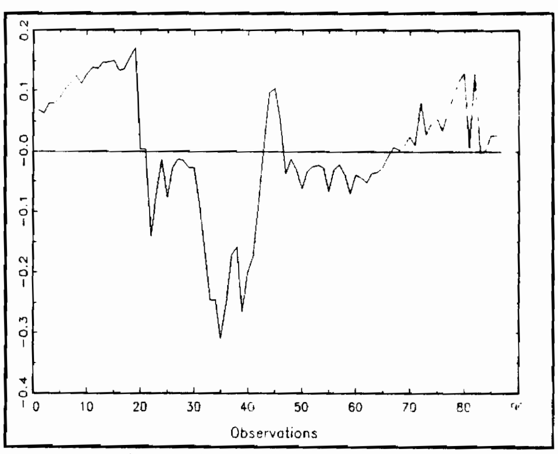


Figure 48 Residuals displaying positive memory

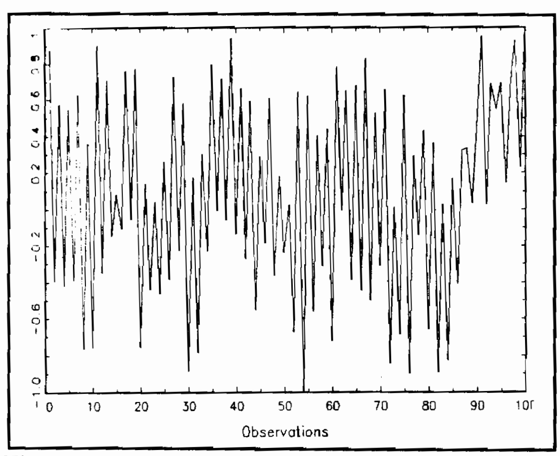


Figure 49 Residuals displaying negative memory

An independent residual series would be graphically characterized by random movement over the sample period. We should not be able to predict the direction or magnitude of upcoming observations based upon past behavior.

## A1.2 Distribution Assumptions

Another important assumption of the Linear Regression Model involves the distribution of the residual series. Based upon a normal conditional distribution existing between the independent and dependent variables, we require the residual (a linear formulation of Y and X) to also display a normal distribution.

We can examine our residual plot to see if it conforms to the requirements of a normal distribution. The normal distribution is symmetric. This is closely related to our assumption that the normal distribution has a linear mean. Also, the normal distribution is nicely bell shaped, not being platykurtic (flat topped) or leptokurtic (peaked). We can also examine the plot of the residual series to see that it displays a constant variance as required by the normal distribution.

To examine the symmetry of our distribution, we visualize all the observations piled up on the vertical axis. The more observations at a particular value, the higher the pile becomes at that value. The shape of this imaginary histogram or density will be clearly centered at zero for a symmetric distribution. This method also tells us of the linearity of the conditional mean leading to these residuals since violations of linearity often accompany this uneven pattern of skewness.

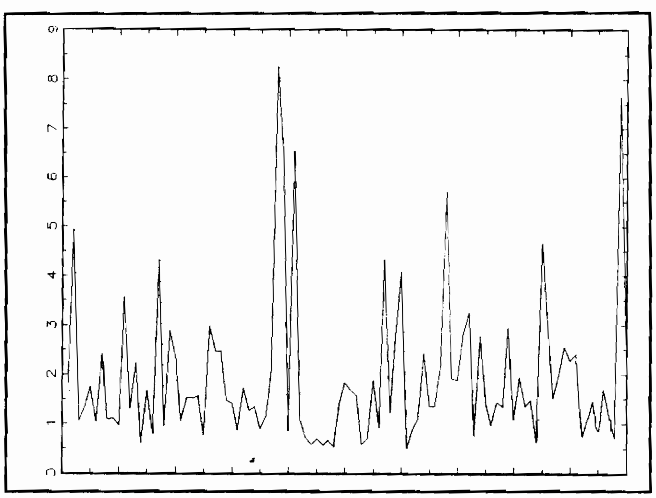

Figure 50 Residuals showing nonlinearity/skewness

We also may look to the residual plot for information about the dispersion of our series. The normal distribution assumption requires that our model have a constant conditional variance. We would like to see an equal variance across the entire sample of our residuals. If we notice regions of particularly large variation or small variation, then we have violated this assumption of a constant variance. Such a violation may arise through heteroskedasticity if the conditional variance depends on X or on homogeneity if the violation is a result of time dependence of some sort. This concentration of large or small changes in the residuals may also lead to a leptokurtic (too many values near zero) or platykurtic (too many values in the tails) shaped density function.

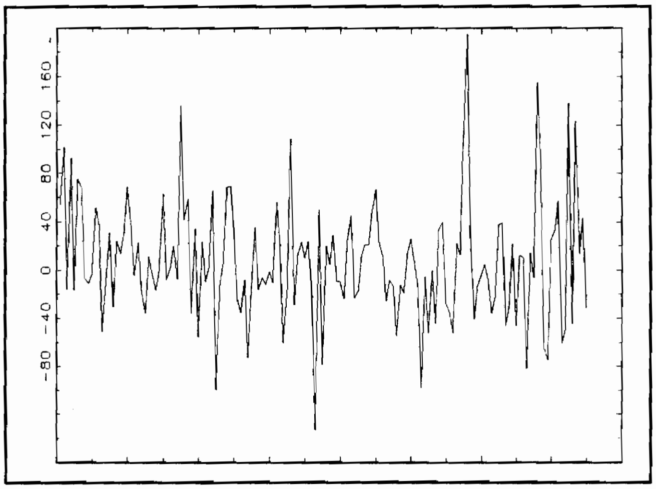


Figure 51 Residuals displaying a non-constant variance

## A1.3 Homogeneity Assumptions

Our final category of assumptions involves the constancy of the underlying distribution or moments of our data series over the sample period. Systematically higher than average and lower than average values of our residuals may indicate a violation of the time homogeneity assumption. However, this is difficult to untangle from memory.

Instead, we will concentrate on the behavior of our coefficients in the empirical model as the sample of data changes. In this way, we will observe the characteristics of the means, variances, and covariances underlying the data utilized. This topic will be discussed in detail in Chapter 7 including sample plots in Figures 31-33.

# *Appendix 2 : Distribution Plots with Specific Parameter Values*

Once we decide to move away from the Normal distribution assumption, we find ourselves with a very wide range of distribution families. These distributions are determined, not only be their first two moments as in the Normal case, but by higher moments as well.

Given the wide variety of distribution families to choose from it is useful to examine some examples of parametric densities. We may examine these density families utilizing their numerical characteristics as well as through the use of graphical tools.

I say families of distributions since each density "function" can take on a wide variety of shapes depending upon the value of parameters utilized. In many cases, changing parameter values will dramatically alter the shape of the curve and therefore the type of data distribution which could be supported by such a model.

## A2.1 Sources of Density Characteristics and Computer Work

The following are examples of very particular density families taken from Evans, Hastings and Peacock's excellent 1993 summary entitled *Statistical Distributions*. In this text, the authors provide distribution, density, characteristic functions, moments, and often additional information about a large number of univariate distributions. Random number generation for each distribution is also discussed. The reader is referred to this excellent text for information on the following univariate distributions.

Bivariate and multivariate distribution and density functions are more scattered in the literature. Good sources are *Families of Bivariate Distributions* by Mardia (1970) and *Continuous Multivariate Distributions* by Johnson and Kotz (1972). The reader is cautioned as well that there are many different functional forms for certain distributions, some with very different characteristics. This was the case with the bivariate exponential as discussed in Chapter 8.

The statistical package GAUSS (Aptech Industries) was utilized to create the following density plots. The parameter values were very specifically selected to illustrate the nature of the density family. Not all possible shapes have been captured as this would be an impossible task.

The program developed to accomplish this task is interactive and allows modification of parameter values (up to 6 on one plot), ranges of the X variable and plots 12 different univariate density functions. A bivariate extension of this program is planned.

This program serves two purposes. First, the program serves as an educational tool in understanding the nature of skewness and kurtosis. And, secondly, the program provides a graphical guide to the understanding of a density functions behavior, especially as particular parameters change.

## A2.2 Some Example Plots

### A. The Student's T Distribution.

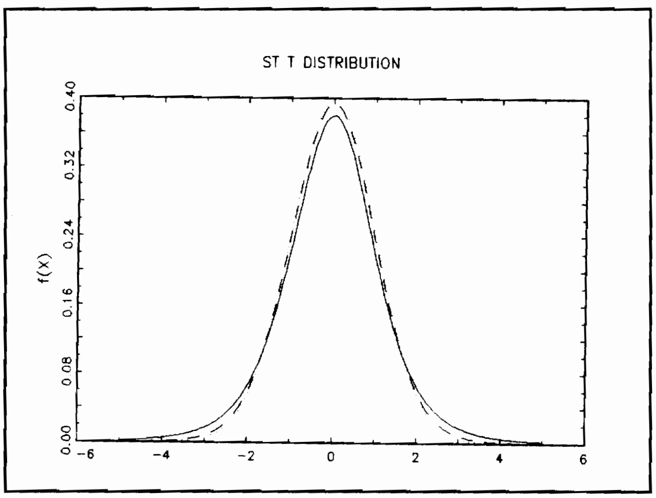

Figure 52 : Student's T Density Family (v= 5; 20; 100)

Manipulation of the degrees of freedom term $v$ affects the kurtosis seen in the density plot.

## B. The Exponential Distribution

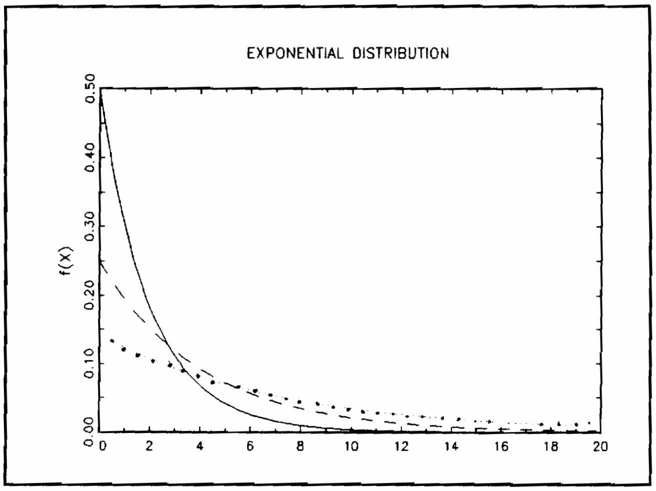


Figure 53 : The Exponential Density(s=2; 4; 7)

## C. The Gamma Distribution

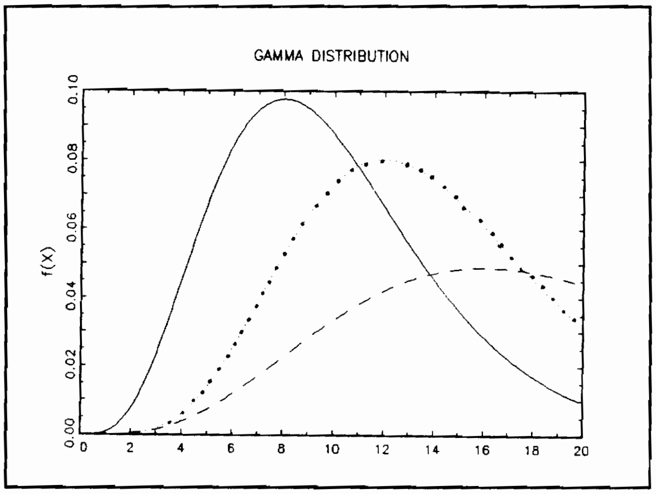


Figure 54 : Changing location of the Gamma Distribution ((a,b) =(2,1); (2,2); (2,5))

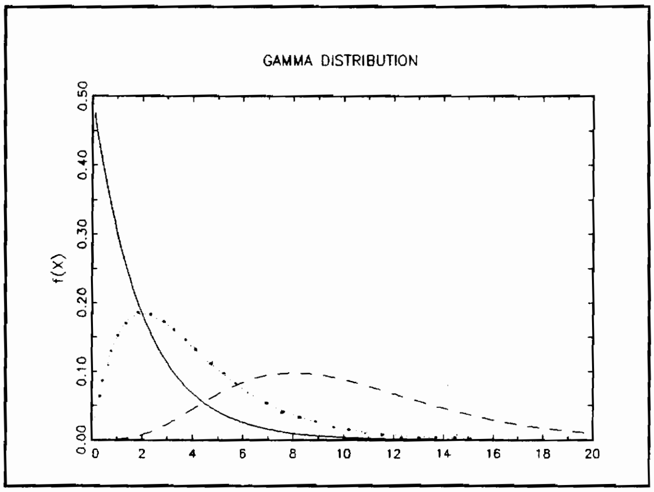


Figure 55 : Changing Dispersion of the Gamma Distribution ((a,b) = (2,1); (2,7); (4,5))

## D. The Weibull Distribution

Through manipulation of two parameters again, we control both the symmetry of the distribution and the dispersion making this also a very versatile density family.

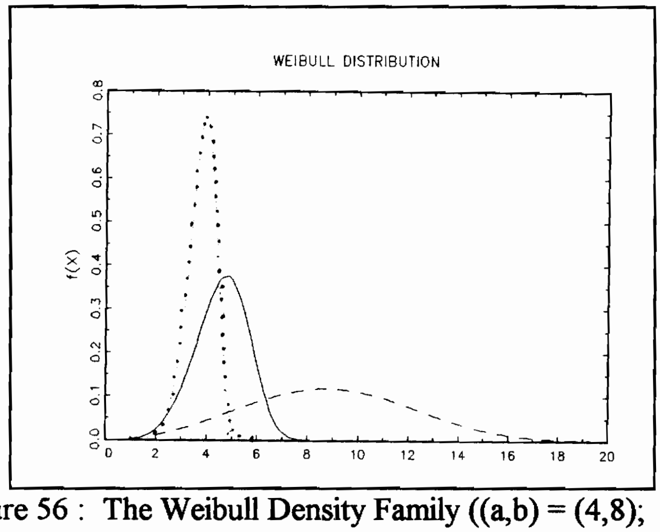


Figure 56 : The Weibull Density Family ((a,b) = (4,8); (5,10); (10,3))

## E. The Fisher's F Distribution

The F distribution is a skewed density which includes the very popular case of the chi-squared distribution as a special case. Both the amount of skewness and the kurtosis can be controlled through different parameter values.

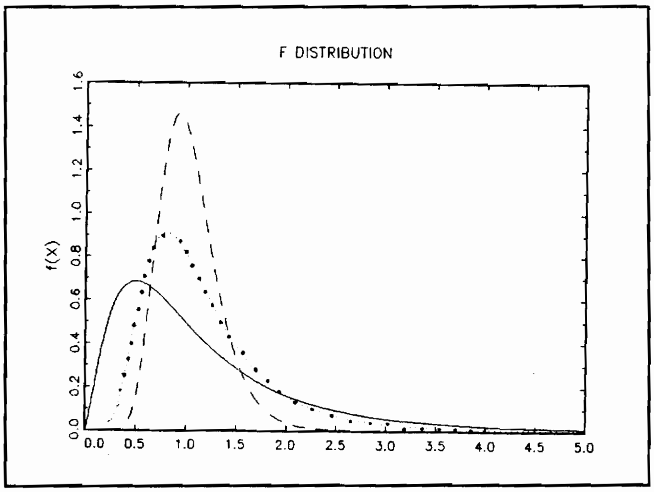

Figure 57 : The F Distribution ((a,b) = (5,10); (30,12); (50,50))

As you can see by the above examples, movement away from the assumption of normally distributed errors allows us great flexibility in modeling very different data generating processes. The use of very generalized density functions such as the Weibull and the Gamma distribution above hold great promise for a parametric method which conforms to the characteristics of the data set through flexible parameter combinations.

# *Vita*

Lisa Ann Rosenberry was born in Bethesda Maryland on February 23, 1967. The author grew up in Greencastle Pennsylvania and attended Shippensburg University where she graduated summa cum laude in 1989 with a dual BSBA in Management and Economics. Ms Rosenberry graduated with a Master's of Arts degree in Economics in 1991 from Virginia Polytechnic Institute and State University and earned her doctorate from Virginia Tech in 1995. In May 1995, she married Frank Wilder and serves as an Assistant Professor at Bowling Green State University in Bowling Green Ohio.

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