MULTIVARIATE NONPARAMETRIC CONTROL CHARTS USING SMALL SAMPLES

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(ABSTRACT)

The problem under consideration is simultaneous monitoring of the means of two or more correlated variables of a process, by collecting a small fixed random sample at fixed time intervals. The target values are considered known, whereas the variance-covariance matrix of the data must be estimated. A typical parametric chart to monitor this process would involve the assumption that the data follow a multivariate normal distribution. If this assumption is not reasonable or if it is difficult to verify, for example in a short production run, a multivariate control chart based on classical nonparametric statistics could be used. Control charts based on the sign and signed rank statistics are explored.

Past sample information for each variable is retained through an exponentially weighted moving average statistic (EWMA) in order to increase the sensitivity of the charts to detect small shifts from the target. The properties of the charts are evaluated using simulation. Such charts are not distribution-free in the nonparametric sense, but they are more robust than the parametric equivalent chart because, among other reasons, they require only covariance estimates. Nonparametric charts are less efficient than the parametric equivalent chart if the measurements follow a normal distribution, but they improve significantly if the measurements follow a distribution with heavier tails.

To my parents,

who taught me to dream and dare,

to persevere against all odds,

and to stand up for my principles,

and to all my teachers,

who taught me to love mathematics.

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CHAPTER 1

INTRODUCTION

1.1 Historical perspective

After the World War II, the American industry flourished without any serious foreign competition. Europe was in shambles and eager to buy anything made in USA. It was during that time that productivity was emphasized over quality. In another country, namely Japan, this ideal was brought to such an extreme that the Japanese industry acquired the reputation of manufacturers of junk. The quality transformation of the Japanese industry that took place later on, partly with the help of W. Edwards Deming, often called the "Japanese miracle", changed Japan's image and competitiveness in the international market. This event and the strengthening of the economies of the European countries, as well as countries in other parts of the world, drastically increased the competition and brought many, previously complacent, US manufacturers into a stage of crisis. Today, it is widely recognized that in order to avert the crisis a series of changes must be made. Old ideas are being abandoned and a new philosophy is being established: *productivity and quality must go hand in hand.*

Statistical quality control consists of all statistical methods that can be used in a manufacturing process in order to improve the quality of the outcome of the process. As such, statistical quality control is a step in the right direction. There are many statistical methods available for quality control. These methods can be loosely grouped into four categories:

Acceptance sampling

Continuous sampling inspection

Process control

Experimental design

The first category includes methods such as *single sample inspection, sequential inspection,* and the *sequential probability ratio test* (SPRT). For all these methods the assumption is that a lot (or lots) has been produced and, upon inspection, it is either accepted or rejected. The second category includes plans where items are inspected as they are produced and the amount of future inspection depends on the outcome of the current inspection, such as the *Dodge-type plans*. The third category includes the various *control charts.* Among these three categories, it is the third one that monitors the process itself, from its onset, and not just the product after it has been produced. In other words, the control charts are designed to improve

quality through monitoring of the production process, and

productivity through reduction in waste.

Finally, the category of experimental design consists of all statistical techniques, including response surface methodology, used in determining the optimal conditions necessary to achieve the best quality output for a particular process. Examples of such techniques are the experimental designs attributed to G. Taguchi.

A few control charts are developed in this dissertation. In order to monitor a process using a control chart, random samples are collected from the process and certain variables are measured. These variables are considered important contributors to the quality of the process. The measurements are, then, summarized with statistics which are plotted on the control chart. Depending upon the positioning of the points on the chart we make a judgment about the quality of the process. If we judge that the quality is not good, we stop the process and take corrective action. Otherwise, we allow the process to continue.

1.2 Problem motivation and other related research

When two or more correlated characteristics of a process are monitored simultaneously, one has the option of using a separate univariate control chart to monitor each characteristic or one multivariate control chart to monitor the whole process at once. The approach taken here is that, in general, a multivariate control chart is more appropriate than separate univariate control charts in handling multivariate data. Moreover, the presence of computers today makes the computing difficulties usually associated with multivariate statistical techniques a problem of the past. Several types of multivariate control charts are available for process monitoring. For example, Hotelling (1947) proposed a multivariate extension of a simple Shewhart chart; Crosier (1988) proposed two types of multivariate CUSUM charts; Lowry et al (1992) proposed a multivariate exponentially weighted moving average (MEWMA) chart. The drawback of all these approaches is that they require the assumption that the data follow a multivariate normal distribution. Of course, not all data follow a multivariate normal distribution, and in the case of small samples (quite common in quality control) collected from a non-normal population we may not be able to find a reasonable approximation to the sampling distribution of a parametric statistic. Now, if the monitored process continues for a very long time the normality assumption could be verified by collecting data. Another kind of problem may occur if the production run is short and the normality assumption cannot be verified, or if there is not enough data to provide good estimates of all the parameters required in a parametric multivariate chart.

In all the above cases the attractiveness of a nonparametric monitoring approach becomes apparent. The advantages of nonparametric charts over parametric charts are many: they require fewer restrictive assumptions on the data; they are not sensitive to outliers if the process is expected to have a few; and they are easier to use because only rankings of the observations, and not accurate measurements, are needed. Several univariate nonparametric control charts have been developed. For example, Park and Reynolds (1987) developed nonparametric charts for monitoring a location parameter when the target value is not specified. But beyond the univariate nonparametric approach there is very little else. One has to look in other areas of statistics, beyond quality control, to find any literature in multivariate nonparametric theory and applications. The lack of research in this area and its potential usefulness in today's customized and computerized world, suggested it for further development.

For this research problem we assume that the process we want to monitor involves a number of correlated variables. We want to monitor the means of these correlated variables in the case where they all have known, predetermined, target values. All the other parameters of the process that are not monitored, for example variances and covariances, are considered unknown. Small samples are collected at fixed time intervals and used to monitor the means and to estimate the parameters that are necessary for the monitoring of the means. The objective is to start the monitoring early, before a lot of data is collected to estimate the parameters. Estimates based on small samples are not very reliable, so they are updated with each new sample as the monitoring of the process continues. After the first few samples, the updated estimates are much more reliable than the individual sample estimates because they are based on a larger amount of data. These updated estimates are used to construct a control chart.

The proposed control charts for the monitoring of the mean vector of the process are multivariate nonparametric. The nonparametric statistics used are classical nonparametric statistics, namely the *sign* statistic and the *signed rank* statistic, because they are easy to compute and they are well known to users of statistics. Exponentially weighted moving averages (EWMA) of the nonparametric statistics are used for increased efficiency. The resulting control charts are not distribution-free in the nonparametric sense, but they are more robust than the corresponding parametric multivariate charts, partly because they involve fewer nuisance parameters. These multivariate nonparametric control charts will be compared with their corresponding multivariate parametric charts for a number of distributions. Their efficiencies will be compared using the *average run length* criterion (ARL) computed with simulation.

In the following chapters several issues are discussed. Specifically, Chapter 2 contains a detailed description of general control charts and discussion on Shewhart, CUSUM, EWMA, multivariate, and nonparametric control charts. In the same chapter, there is also a discussion of the ARL criterion for chart evaluation. Chapter 3 contains a literature review of recent work in the area of nonparametric and multivariate control charts. Chapter 4 contains the description of the research problem, the algorithm of the control procedure proposed, and discussion about the sign and signed rank statistics. In Chapter 5 there is a discussion of the property of affine invariance and the proof that, when the process is in control, the control statistics of the proposed nonparametric charts asymptotically have a chi-squared distribution. Chapter 6 contains a discussion on the computational issues of the charts and how they were resolved. Chapter 7 contains tables and a discussion on the simulated performance of the nonparametric charts compared with the corresponding parametric chart for the case of known parameters. Chapter 8 contains tables and a discussion on the simulated performance of the nonparametric charts compared with the corresponding parametric chart for the case of unknown parameters and for a variety of possible estimators. Finally, Chapter 9 contains a summary of the research done in this dissertation, the major conclusions, a discussion on how to design the proposed charts, and a few topics for future research. A few side issues are discussed in the appendices.

CHAPTER 2

CONTROL CHARTS

2.1 Description of a control chart

The first step in the construction of a control chart is to define the meaning of "quality" for the specific process or product. This usually translates to specifying one or more measurable characteristics which, in the general statistical terminology, are called *variables.* If there is a required value for a variable, it is called the *target value.* Since variables have distributions, a target value is usually a required value for a parameter of the distribution, for example the mean. In this case, we say that the control chart *monitors the mean of the process.* Of course, besides the mean, other parameters could be monitored, such as the median, the variance, etc. individually or in combination.

The next step involves sampling considerations. As soon as the production begins, a few items are randomly selected and measured. Generally speaking, n observations are taken at time τ_k, where τ_k is the time at which the kth sample is taken. Both the sample size n and the length of the time interval $\tau_k - \tau_{k-1}$ between two consecutive samples can be fixed or variable. The n observations are summarized with a statistic S_k, which is, usually, a complete and sufficient statistic for the parameter to be monitored. In turn, the values of the statistic S_k, themselves, may be summarized over time using another statistic Y_k. The final statistic to be computed, either S_k or Y_k, is called the *control statistic.*

The construction of a control chart may include a *target value,* an *upper control limit* (UCL) and/or a *lower control limit* (LCL) (see figure 2.1.1). The values of the control statistic are plotted on the chart versus time. As long as the points remain between the control limits the process is said to be *in control,* otherwise it is a *signal* that the quality of the process may have deteriorated. If, for example, the control chart monitors the mean μ of a process, a signal above the UCL may indicate that the mean of the process has shifted from the target value t to another value μ_1, where w.l.o.g. $\mu_1 > t$. If, despite the signal, the mean is still on target, the signal is called a *false alarm*. In either case, the process is usually halted in order to determine and remove the cause of the shift.

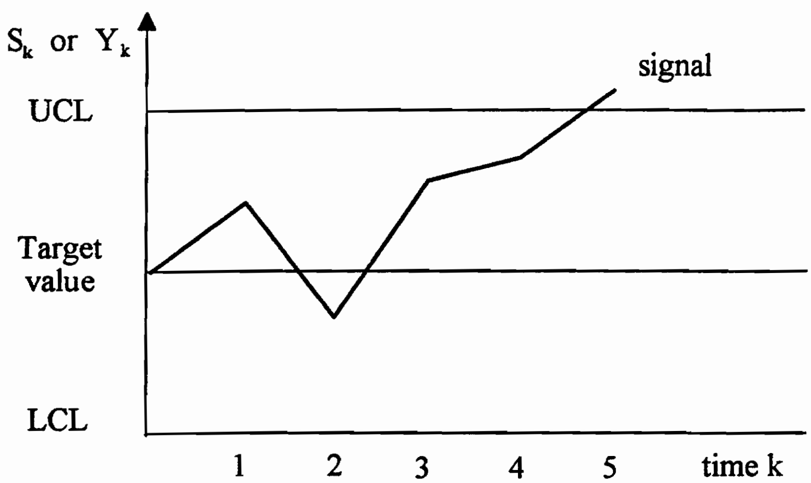

Figure 2.1.1: A control chart

The control limits of a chart are determined based either on probability considerations or on tradition, e.g. 3-sigma limits (defined in section 2.2). Assuming that the process is in control, the probability of false alarm or, in general, the false alarm rate is fixed to a small value (similar to the α level of a hypothesis test). Then the UCL and the LCL are determined according to the specified false alarm rate. Assuming that the observations are iid and follow a normal distribution, the distributions of the usual sample statistics, i.e. average, range, standard deviation, etc. are known, and the UCL and LCL can usually be determined. This task becomes much harder when these assumptions on the data cannot be made, especially for small samples. Then, approximate techniques or simulation are implemented to estimate the appropriate control limits. The importance of specifying accurate control limits cannot be overemphasized. A high false alarm rate would increase the cost of the product associated with stopping and starting the production, investigating for the cause of a signal and not finding it, and leaving everyone in doubt whether indeed something has gone wrong or not. In the long run, a high false alarm rate would affect the way operators respond to a true signal by cultivating a casual atmosphere. On the other hand, the false alarm rate should not be unnecessarily small. This would result in a larger type II error, in the sense of a hypothesis test, which means that too much defective output may be generated by the process before the control chart detects the change in the parameter and the process is halted.

2.2 To retain or not to retain past information?

In the construction of a control chart, one has the option of combining the value of the statistic S_k, computed from the current sample *k*, with previous sample statistics S_{k-1}, $S_{k-2}, \ldots$, loosely called *past information*. The decision about whether to retain or not to retain past information depends on what each option has to offer in terms of ability to detect a shift of the parameter from its target value. Three types of charts, which differ in the way past information is used, will be discussed here: the Shewhart chart, the cumulative sum (CUSUM) chart, and the exponentially weighted moving average (EWMA) chart.

The *Shewhart chart* (Shewhart, 1931) is the simplest type of control chart. It uses only the current sample information. For example, in a Shewhart chart for monitoring the process mean μ (see Figure 2.2.1), the average of the n observations taken at time τ_k, $\overline{X}_k$, is plotted against time. Here, $\overline{X}_k$ is the control statistic. If the target value is t and the variance σ^2 of the observations is considered known, the control limits are determined as follows:

$$\begin{aligned} \text{UCL} &= t + \varsigma\frac{\sigma}{\sqrt{n}} \\ \text{LCL} &= t - \varsigma\frac{\sigma}{\sqrt{n}} \end{aligned} \tag{2.2.1}$$

where ς is chosen to balance the false alarm rate with the failure to signal if a shift has taken place. A typical value for ς is 3, in which case the UCL and LCL are called *3-sigma limits*. This chart is two-sided, as it allows detection of a shift in the process mean both above and below the target value. A Shewhart chart quickly detects a large shift in the monitored parameter but it is inefficient in detecting small or moderate size shifts.

Shewhart charts with *runs rules*, a modification of a regular Shewhart chart, make use of past information to a limited extent. For example, in addition to the regular rule for a signal, a signal may be given if "*2 out of the last 3 values of $\overline{X}_k$, are in the warning region C*" (see Figure 2.2.1). A *warning region* is a subset of the acceptance region, adjacent to a control limit. *Acceptance region* is the area of the graph between the UCL and the LCL. In this case, the last 3 values of the sample statistic are retained at each time k instead of the current value only. A Shewhart chart with one or more runs rules is usually more efficient in detecting small shifts in the parameter than a simple Shewhart chart (Champ and Woodall, 1987). The improvement in efficiency, though, is limited by the number of additional rules one can use, the degree of simplicity or complexity of the rules, and the increase in false alarm rate these rules may cause.

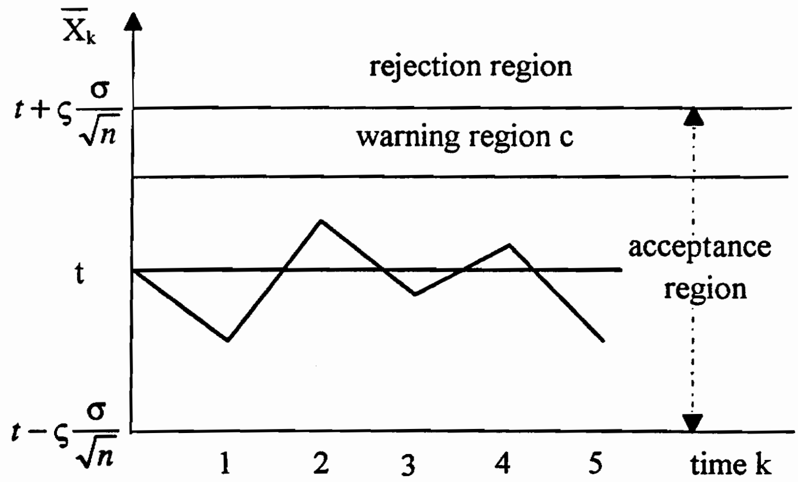

Figure 2.2.1: Shewhart $\overline{X}$ chart

In a Shewhart chart with runs rules the number of past samples used is fixed. Another possibility is that the number of past samples retained is a variable depending on the data. If a small shift in the parameter has occurred, the values of the control statistic would still be near the target value. In this case, more past samples would be necessary to detect the shift. The opposite would happen if a large shift has occurred and the values of the control statistic are near one of the control limits: fewer past samples would be necessary to detect the shift. A chart with this property would be much more efficient than a Shewhart chart in detecting smaller shifts, and, potentially, as efficient in detecting larger shifts.

The *Cumulative Sum* (CUSUM) *chart*, originally proposed by Page (1954), has the above property. It retains the information from past samples by adding the value of the current statistic to the previous sum. Since this action alone would ultimately "drown" the current sample information, when there is no indication of a shift, the CUSUM statistic resets itself to zero. Suppose that the parameter to be monitored is the mean of a process with target value t . The one-sided CUSUM statistic Z_k^* for detecting a positive shift in the mean is defined as:

$$Z_k^* = \max\{0, Z_{k-1}^*\} + Z_k\ , \quad k = 1,2,.... \qquad (2.2.2)$$

where

$$Z_0^* = 0 \quad \text{and} \quad Z_k = \overline{X}_k - t - d\,.$$

$\overline{X}_k$ is the current sample average, and d is one half of the size of the shift that the chart is suppose to detect quickly, in standard deviations of the statistic $\overline{X}_k$. This chart has an upper control limit h only (see Figure 2.2.2). It signals whenever $Z_k^* > h$, where h is chosen according to a prespecified false alarm rate.

In order to monitor the mean for both positive and negative shifts two such CUSUM charts, satisfying certain conditions, are often used. In this case, one chart is for positive shifts, denoted by Z_k^*, and the other is for negative shifts, denoted by Z_k^{**}. They are defined as

$$Z_k^* = max\{0, Z_{k-1}^*\} + Z_k^+\ , \quad k = 1,2,....$$

where

$$Z_0^* = 0 \quad \text{and} \quad Z_k^+ = \overline{X}_k - t - d^+,$$

and

$$Z_k^{**} = max\{0, Z_{k-1}^{**}\} + Z_k^- , \quad k = 1,2,....$$

where

$$Z_0^{**} = 0 \qquad \text{and} \qquad Z_k^- = \overline{X}_k - t - d^- .$$

The combined chart signals as soon as one CUSUM chart signals, i.e. when either $Z_k^* \geq h^+$ or $Z_k^{**} \geq h^-$.

Alternatively, a single CUSUM chart (Crosier, 1986) could be used to monitor both kinds of shifts simultaneously. The two-sided CUSUM statistic C_k^* developed by Crosier is defined as follows.

Let

$$C_k = \left| C_{k-1}^* + \overline{X}_k - t \right|$$

then

$$C_k^* = \begin{cases} 0 & \text{if } C_k \leq a\sigma_{\overline{X}} \\ (C_{k-1}^* + \overline{X}_k - t)(1 - \dfrac{a\sigma_{\overline{X}}}{C_k}) & \text{if } C_k > a\sigma_{\overline{X}} \end{cases} \tag{2.2.3}$$

where

$$C_0^* = 0,\ \ a > 0, \text{ and } k = 1,2,...$$

In the above definition, μ_0 is the target value and $\sigma_{\overline{X}}$ is the standard deviation of $\overline{X}_k$.The chart signals if $\left|C_k^*\right| > h\sigma_{\overline{X}}$.

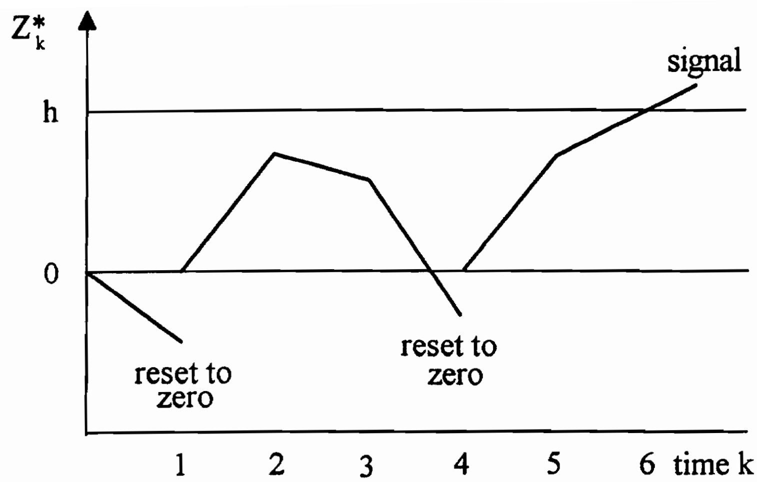

Figure 2.2.2: CUSUM chart

In a nutshell, the CUSUM chart uses past information in the form of a variable number of past sample means that are added together. The result is that the CUSUM chart is much more efficient than a Shewhart chart in detecting small shifts from the target. An alternative idea to retaining a variable number of past samples would be to retain them all by, for example, adding or averaging the sample averages from all the previous steps. In order to avoid the problem of "drowning" the current sample information (mentioned also earlier in this section), one could use a weighted average of all the sample statistics, instead of a simple average.

The *Exponentially Weighted Moving Average* (EWMA) statistic Y_k for monitoring the process mean, first introduced in quality control by Roberts (1959), is defined as follows.

$$Y_k = (1-r)^k Y_0 + (1-r)^{k-1} r\overline{X}_1 + \ldots + (1-r) r\overline{X}_{k-1} + r\overline{X}_k$$

$$= (1-r)^k Y_0 + \sum_{j=0}^{k-1} (1-r)^j r\overline{X}_{k-j}\ , \quad k \geq 0 \tag{2.2.4}$$

where

$\overline{X}_k$ is the current sample average

r is the EWMA parameter, $0 < r \leq 1$

Y_0 is usually the target value.

An alternative expression for Y_k is the following recurrence relationship:

$$Y_k = (1-r) Y_{k-1} + r\overline{X}_k \tag{2.2.5}$$

The EWMA chart is two-sided (see Figure 2.2.3) and signals when $Y_k > t + c$ or $Y_k < t - c$, where c is chosen to be a multiple of the asymptotic standard deviation of Y_k.

Since the topic of this dissertation involves the EWMA approach in retaining past information, more details on this subject are left for development in later chapters of this dissertation. In general, the EWMA chart is efficient in detecting small shifts in the monitored parameter when r is chosen to be small, and efficient in detecting large shifts when r is chosen to be large. Different studies comparing the CUSUM chart with the EWMA chart indicate that they are quite comparable in terms of efficiency in detecting a shift of any size (Lucas and Saccucci, 1990). Both schemes, CUSUM and EWMA, include the simple Shewhart chart as a special case. CUSUM and EWMA schemes use past information whereas the standard Shewhart scheme does not. The result is that CUSUM and EWMA charts are less sensitive to outliers and more efficient in detecting small shifts than the standard Shewhart chart.

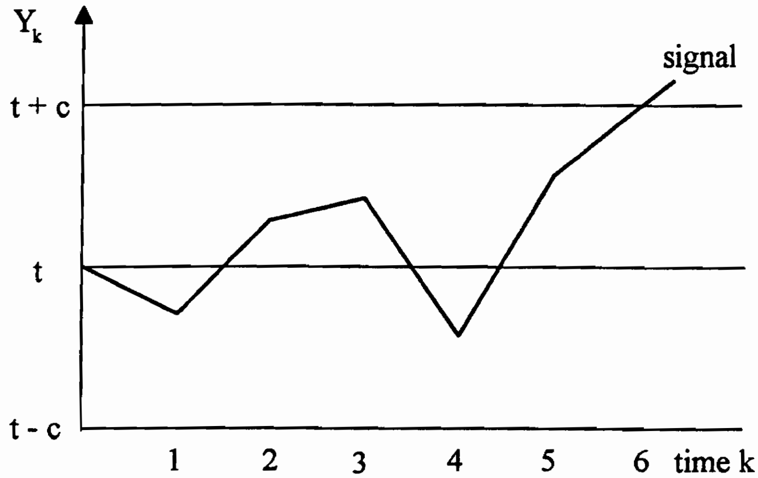

Figure 2.2.3: EWMA chart

2.3 Multivariate control charts

When p characteristics of a process are monitored we are faced with a new question. How do we take advantage of the dependence structure between the measurements of each observation and, in doing so, how do we still keep the measurements separate in order to help with the identification of the cause of a shift.

There are three different approaches that could be taken: using p separate univariate control charts, following the combine-accumulate approach, or following the accumulate-combine approach, as explained below. Let

$$\mathbf{X}_k = \begin{bmatrix} X_{k11} & . & . & . & X_{k1n} \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ X_{kp1} & . & . & . & X_{kpn} \end{bmatrix}$$

be a sample of n observations on p characteristics at time τ_k. Suppose we are interested in monitoring the means of the p characteristics $\boldsymbol{\mu}' = [\mu_1, \ldots, \mu_p]$. Without loss of generality, let us assume that the target $\mathbf{t}' = [t_1, \ldots, t_p] = [0, \ldots, 0]$. Then, the summary statistics for the n observations at time k could be $\bar{\mathbf{x}}_k' = [\bar{X}_{k1}, \ldots, \bar{X}_{kp}]$ with variance-covariance matrix $\Sigma_{\bar{\mathbf{x}}}$.

(I) *Using p separate univariate control charts*

This approach is simple and self-explanatory. One keeps p separate records, one for each type of measurement. A Shewhart, CUSUM or EWMA chart may be used to monitor each parameter. This way it is relatively easy to identify the source of trouble when a signal occurs. The problem is that we don't make use of the covariance structure between the variables, and it would be difficult to evaluate the joint properties of the p charts.

The acceptance regions of two different approaches in monitoring two parameters are depicted in Figure 2.3.1 below. Approach (a) uses two separate univariate control charts and has a rectangular acceptance region. Approach (b) uses one multivariate control chart and has an elliptical acceptance region. Assuming that the monitored parameters are means, let the points x and y, plotted on the graph, represent bivariate sample averages. Each point is a signal according to one approach but not a signal according to the other approach. So, for example, if a shift has taken place in the vector of parameters μ' that has not affected any of the individual μ_i's by very much, then it is possible that neither of the two separate charts will signal (see point y in Figure 2.3.1). This results in a decrease in power, i.e. more defective items will be produced before a signal occurs. In addition, the two charts will signal if any one of them signals (see point x in Figure 2.3.1).

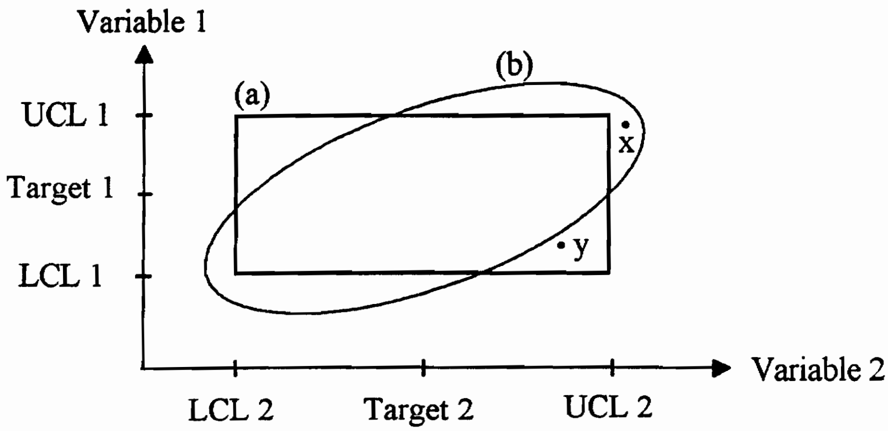

Figure 2.3.1: Rectangular versus elliptical acceptance region
(a) Acceptance region using two independent univariate control charts.
(b) Acceptance region using one multivariate control chart.
x: a signal point according to region (a), but not according to region (b).
y: a signal point according to region (b), but not according to region (a).

(II) *The combine-accumulate approach*

A Shewhart, CUSUM or EWMA chart could be used in this approach, except that a standard Shewhart chart would combine but not accumulate. According to this approach all measurements in a sample are combined into a univariate statistic at each time τ_k which is then summed or averaged over time. If $\overline{\mathbf{x}}_k' = [\overline{X}_{k1},\ldots..,\overline{X}_{kp}]$ is the vector of averages for the n observations at each time k, the data could be combined into the statistic V_k, where

$$V_k = \overline{\mathbf{x}}_k' \Sigma_{\overline{\mathbf{x}}}^{-1} \overline{\mathbf{x}}_k \tag{2.3.1}$$

A Shewhart chart would use V_k as the control statistic. Alternatively, V_k could be accumulated over time with, for example, an EWMA scheme as follows:

$$Y_k = (1-r)Y_{k-1} + rV_k \tag{2.3.2}$$

where $Y_0 = 0$. A signal is given if $Y_k >$ UCL (some appropriately chosen upper control limit).

This method makes use of the covariance structure between the variables but a signal would be hard to interpret. The measurements would be so hopelessly entangled in the control statistic, that it would be almost impossible to determine which characteristic has shifted from the target using the control statistic only. Diagnostics would have to be done using separate statistics kept for this purpose.

(III) *The accumulate-combine approach*

According to this approach, the sample averages would be accumulated over time separately for each variable but they would be combined at each time τ_k to form a univariate statistic. A CUSUM or EWMA scheme could be used to accumulate the information on each variable over time. A standard Shewhart chart would not be appropriate in this case since it would not accumulate past information. If $\overline{\mathbf{x}}_k' = [\overline{X}_{k1},\ldots..,\overline{X}_{kp}]$ is the vector of averages for the n observations at time k, as before, and the EWMA scheme is used, we accumulate according to

$$\mathbf{y}_k = (\mathbf{I} - \mathbf{R})\mathbf{y}_{k-1} + \mathbf{R}\overline{\mathbf{x}}_k \tag{2.3.3}$$

where

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & . & . & 0 \\ 0 & 1 & & & . \\ . & & . & & . \\ . & & & . & 0 \\ 0 & . & . & 0 & 1 \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} r_1 & 0 & . & . & 0 \\ 0 & r_2 & & & . \\ . & & . & & . \\ . & & & . & 0 \\ 0 & . & . & 0 & r_p \end{bmatrix}$$

$0 < r_j \leq 1$, and $j = 1,2,\ldots,p$. At each time *k*, the EWMA statistics are combined into the statistic

$$W_k = \mathbf{y}_k' \boldsymbol{\Sigma}_\mathbf{y}^{-1} \mathbf{y}_k \tag{2.3.4}$$

where $\boldsymbol{\Sigma}_\mathbf{y}$ is the variance-covariance matrix of $\mathbf{y}_k$. A signal is given if W_k> UCL (some appropriately chosen upper control limit). This chart has been explored by Lowry et al (1992).

The accumulate-combine approach makes use of the covariance structure between the variables and it does not combine the information on each variable until the very end of the computation. So, if a signal occurs, the vector statistic $\mathbf{y}_k$ can be used to identify the parameter or parameters that appear to have shifted from target. Since this is the approach taken in this dissertation, more details are left to be discussed in Chapter 4 (The proposed control chart).

2.4 Nonparametric control charts

Most standard procedures for monitoring one or more parameters of a process are designed on the assumption that the measurements follow a normal distribution. This is an important assumption for the quality control setting because, usually, only a few observations are taken at each sampling time k, and thus the central limit theorem cannot be used to justify the asymptotic normality of $\overline{X}$. Often, the normality assumption is reasonable and based on experience with the process to be monitored. Occasionally, it is based on pure faith. If the production run is going to be long, one could verify the normality assumption after a significant amount of data has been collected. In the case of a short production run, though, there will not be enough data to make such a verification. So, in a short production run or when it is not reasonable to assume that the observations follow a normal distribution, the standard parametric quality control procedures will fall short of expectations. A solution to this problem would be to use a nonparametric approach. Nonparametric control charts are less efficient than their parametric counterparts when the observations follow the normal distribution, but, usually, they are significantly more efficient when the data follow a distribution with heavier tails.

A statistic S is said to be *nonparametric distribution-free* if the class A of joint distributions for which S has the same distribution includes more than one distributional form (Randles and Wolfe, 1979). In the case of a univariate control chart to monitor the mean of a process, an assumption we make for a nonparametric statistic is that the data follow a continuous symmetric distribution. In the case of a multivariate control chart to monitor the multivariate mean of the process, the corresponding assumption for a nonparametric statistic computed for each variable is that the data follow a p-variate cdf F which is absolutely continuous with absolutely continuous and symmetric marginal cdfs $F_1,\ldots,F_p$, and absolutely continuous bivariate marginal cdfs F_{ij}, $i,j=1,\ldots,p$, $i\neq j$. Let S_k denote a univariate nonparametric statistic at time k, and $\mathbf{s}'_k = (S_{k1},\ldots,S_{kp})$ denote a p-variate nonparametric statistic at time k. In the univariate model, S_k is distribution-free. In the multivariate model, the vector $\mathbf{s}_k$ is not distribution-free because of the $\mathrm{Cov}(S_{ki}, S_{kj})$

which depends on the joint distribution of the observations. In a hypothesis test set-up, the solution is to estimate the covariances and construct a test that is asymptotically distribution-free. For finite samples, it is possible to construct a conditional permutation or randomization test which is distribution-free (Puri and Sen, 1971). These tests are only practical for small samples, but that would not pose a serious difficulty in the quality control area where small samples are the norm and not the exception. This approach, though, will not be considered here because it may be too time consuming to use during an actual production run.

In this research, classical nonparametric statistics are computed for each variable and although this does not result in a completely distribution-free procedure, the conjecture is made that this procedure is more robust than the parametric equivalent. One reason is the following. Suppose we consider the case where the variance-covariance matrix Σ of the multivariate observations is unknown and needs to be estimated to monitor the process mean. If the number of variables is p, then for a fully parametric multivariate chart, such as the MEWMA, p variances and $\binom{p}{2}$ covariances need to be estimated. For a multivariate nonparametric chart only the $\binom{p}{2}$ covariances need to be estimated, whereas the p variances are always known from nonparametric theory. If the number of variables is relatively small, this leads to a significant reduction in the number of parameters to be estimated, and, hence, the nonparametric chart will be subject to smaller overall sampling variability. Therefore, it should be more robust than the parametric chart. Another, more obvious, reason for our conjecture is, of course, the fact that nonparametric statistics are not sensitive to outliers. In this dissertation, we attempt to show, among other things, that the above conjecture is, in fact, true.

2.5 Chart evaluation

The most commonly used measure of performance of the different control charts is the *average run length* (ARL). The *run length* is the number of samples collected until a signal occurs and it is a random variable. In evaluating a control chart, it is important to know the distribution of the run length in order to compute how soon the chart will signal for different shifts from the target value. Since, quite often, it is very difficult to determine the distribution of the run length, the usual compromise is to compute its first and second moments. The expected value of the run length is the ARL. The ARL should be large when the process is in control and small when the process has shifted, which reflects the false alarm rate and the power of the chart, respectively.

The usual way the ARL is used as a criterion of relative performance between two charts is to set the in-control ARL of both charts to be the same. Then, the ARL values of the two charts for different shifts are compared. The chart with the smaller ARL for a particular size shift is more efficient than the other chart in detecting a shift of this size, because, on the average, it signals sooner. Very often, no chart is uniformly more efficient than the other. Instead, one chart may be more efficient for some range of shifts, whereas the other is more efficient for the rest. In this case, the best chart to use is the one which is more efficient for the range of likely shifts of the monitored process. This requires a good understanding of how the particular process works.

CHAPTER 3

LITERATURE REVIEW

3.1 Overview

The literature review given in this section is concentrated mainly on recent work in the area of Statistical Process Control. The discussed papers are grouped as univariate nonparametric and multivariate parametric. There is a gap in the quality control literature as far as the multivariate nonparametric approach is concerned, perhaps because of the inherent computational difficulties of the approach. Discussion of multivariate nonparametric papers not related to quality control is deferred until the last section of the literature review (Section 3.4), as this work is rather theoretical in nature. It is also somewhat difficult to apply in process control since it has not been developed for it. Additional papers may be mentioned in the following chapters, as the need arises.

3.2 Univariate nonparametric control charts

Bakir and Reynolds (1979) developed a nonparametric procedure based on within-group ranking, in order to monitor the mean of a process when the target value is known. They used a CUSUM chart type stopping rule with the Wilcoxon signed rank statistic, and found that their procedure is only slightly less efficient than the parametric procedures for normal data but considerably more efficient for non-normal data.

Park and Reynolds (1987) developed nonparametric procedures for monitoring the location parameter of a continuous distribution when the control value is not specified. They based their procedures on linear placement statistics, which is a class of two-sample nonparametric statistics, and compared current samples collected from the process with an initial, standard, sample. They developed versions of Shewhart and CUSUM charts, and considered the median placement statistic and the Wilcoxon-Mann-Whitney statistic. These charts, when compared with parametric charts, showed the same behavior as the charts developed by Bakir and Reynolds.

Hackl and Ledolter (1991) defined the rank of an observation at time t relative to the in-control distribution or relative to an initial, standard, sample. They proposed a technique that uses an EWMA of these ranks and extended Park's and Reynolds' work to the case of single observation charts. They concluded that their chart is much more efficient than the observation-based EWMA chart in detecting small to moderate size shifts in location for data from heavy-tailed distributions.

Hackl and Ledolter (1992) also developed a nonparametric control chart that uses the EWMA of the sequential ranks of the observations. The *sequential rank* is defined as the rank of an observation among the most recent g observations. They studied the performance of this chart under drift and shift alternatives and found that the EWMA chart based on sequential ranks behaves very similarly to the EWMA chart based on the actual observations; the rank-based EWMA performs better than the observation-based EWMA for heavy-tailed data distributions under drift, but worse under shift alternatives.

King and Longnecker (1990) computed the distribution of the Wilcoxon statistic for different shifts and a variety of data distributions. The computations were done with an algorithm King and Longnecker developed, and were used in combination with the standard CUSUM chart.

Amin and Searcy (1991) proposed a nonparametric EWMA procedure based on the Wilcoxon signed rank statistic in order to monitor the process mean when the data are not normal and/or the variance is not known. They compared their procedure with the $\overline{X}$-EWMA procedure using simulation. The simulation showed that the nonparametric chart is less efficient than the parametric chart when the data are normal but it is considerably more efficient for data from heavy tailed-distributions. The effect of autocorrelation and fast initial response (FIR) feature on the nonparametric procedure were also considered.

Alloway and Raghavachari (1991) proposed a nonparametric control chart based on the Hodges-Lehmann estimator associated with the Wilcoxon signed rank statistic, in order to monitor location. The control limits were calculated using order statistics for a variety of sample sizes. This chart performs as well as the Shewhart chart when the data are normal, and better than the Shewhart chart when the data have a heavier-tailed distribution.

3.3 Multivariate parametric control charts

Perhaps the most notable early work on multivariate analysis that was later used in quality control is due to Hotelling. Hotelling (1947), using a wartime case study, proposed constructing a control chart based on a χ^2 random variable. He did not actually use this control chart, though, because the variance-covariance matrix was unknown. Other early ideas, typically proposing Shewhart type charts, are summarized by Alt and Smith. Alt and Smith (1988), in the Handbook of Statistics, give an overview of control charts used to monitor two or more correlated quality characteristics. There, they discuss the problems of monitoring the process mean vector and dispersion matrix when the parameter values are known or unknown. The approach taken is that when the parameters are unknown, for example at the early stages of a process, they are replaced by their unbiased estimates. All control charts discussed in detail by Alt and Smith are Shewhart charts; other approaches are only mentioned briefly. These techniques were developed over a large period of time from 1947 to 1987, and they include the work of H. Hotelling, J.E. Jackson, A.J. Duncan, F.B. Alt, and others.

Page (1954) proposed using simultaneous one-sided CUSUM procedures to monitor two or more correlated quality characteristics. Other researchers, later, expanded on this idea and proposed several one-sided or two-sided CUSUM procedures that, used together, could monitor several correlated characteristics. Woodall and Ncube (1985) proposed a multivariate CUSUM (MCUSUM) procedure. They compared MCUSUM with Hotelling's T^2, and found that MCUSUM performs better than Hotelling's T^2 but it is less sensitive to changes in the correlation structure of the data.

Crosier (1988) developed two multivariate CUSUM procedures, one following the combine-accumulate approach and the other following the accumulate-combine approach (both approaches are described in Section 2.3). He compared these two procedures with the multivariate Shewhart chart. Crosier considered the case where the variance-covariance matrix of the data is known, and proved that the performance of all three charts depends on the mean vector and correlation structure of the data only through the noncentrality parameter. The comparisons showed that the accumulate-combine CUSUM is better than the combine-accumulate CUSUM, and both of them are better than the multivariate Shewhart chart in detecting small shifts in the mean vector.

The univariate EWMA scheme, first introduced in quality control by Roberts (1959), was not studied extensively or widely used until recent years. Recent work includes, among others, Crowder (1987) and Lucas and Saccucci (1990). It has been shown that the EWMA scheme is useful in detecting small shifts in the mean of the process. Comparisons of EWMA with CUSUM charts have revealed that the two schemes are comparable in terms of efficiency.

Lowry et al (1992) extended the EWMA scheme to the multivariate case in order to monitor the mean vector of the process according to the accumulate-combine approach. They, also, assumed that the variance-covariance matrix of the data is known. Comparisons with the multivariate CUSUM showed, just like the comparisons of the univariate charts done by Lucas and Saccucci, that the two schemes have approximately the same efficiency in detecting a shift in the mean vector. The performance of the multivariate EWMA chart depends on the mean vector and the correlation structure of the data only through the noncentrality parameter, which is the same property Crosier showed for the two multivariate CUSUM charts and the Hotelling's T^2 chart. Lowry et al showed that the multivariate EWMA is better than the multivariate CUSUM when the process is initially out of control but it performs roughly the same if the shift in the mean vector occurs later in the production run. Both types of charts have inertia problems that can delay reaction to a shift, so they suggest that these charts should be used in conjunction with Hotelling's T^2.

3.4 Multivariate nonparametric techniques from other areas of statistics

The techniques presented in this section are characterized and even motivated by the presence or absence of a property called affine invariance. Although the exact definition of this property will be given in Chapter 5 as part of the discussion of chart properties, a working definition we could use here is as follows.

A statistic is called *affine invariant* if it is invariant under rotations, reflections, and rescalings of the data. As we will explain in Chapter 5, this property becomes an issue in a multivariate set-up where parametric and nonparametric techniques can behave quite differently. In general, most parametric multivariate techniques possess affine invariance, whereas most multivariate extensions of univariate nonparametric techniques, such as the ones developed here, do not possess it. The quest for affine invariant nonparametric techniques that can be useful not only in the bivariate but also in the multivariate situation has been going on for a long time.

Hodges (1955) developed a bivariate analog of a two-sided sign test for paired observations (or before-after type of measurements) based on the projections of the data points in a certain direction. The directions are determined by drawing lines through each data point and the origin. He proposed finding the line with the maximum number M of data points on one side of it (or the minimum number of data points on the other side), and then applying a univariate sign test using M. He computed the distribution of this test statistic based on a combinatorial argument, similar to the gambler's ruin problem, and produced partial tables for different combinations of the sample size n and the statistic $k=n-m$. Hodges' test is affine invariant.

A few years later, Blumen (1958) proposed another bivariate sign test for paired observations based on the slopes of the vectors from the bivariate median to the n sample points. The sines and cosines of these vectors are computed and then combined into a statistic that has the chi-square distribution for large samples. Blumen's test was shown to be more efficient than Hodges' test. Like Hodges' test, Blumen's test is also affine invariant. Furthermore, both tests require that the bivariate distribution of the data under the null hypothesis be diagonally symmetric about the origin (Chatterjee, 1966). Oja and Nyblom later showed (1989) that, among invariant sign tests, Blumen's procedure is optimal against elliptic alternatives.

Bennett (1962) developed a multivariate sign test using certain properties of the multivariate normal distribution. The test statistic is a chi-square type statistic. Closed form solutions are given for 2 and 3 variables, and a few results are stated for 4 variables. Bennett (1964) also developed a bivariate signed rank test using some of the results from the bivariate sign test. Neither the multivariate sign nor the bivariate signed rank statistic described above are affine invariant.

Bickel (1964) investigated the vector of medians and the vector of medians of pairs as competitors of the vector mean. These estimates were found to be asymptotically normal and unbiased. In the bivariate case, both statistics showed excellent efficiency over the vector mean for zero correlation. Both statistics, though, showed significant loss of efficiency as the correlation increased, especially when it was close to 1. This behavior was attributed by Bickel, in part, to the lack of affine invariance of the statistics. Bickel (1965) extended his previous work to vectors of univariate rank statistics. He states that "as far as the 'multivariate portion' of the efficiency is concerned, there always exists a direction in which one does worse than in 1 dimension whatever estimate one chooses (and necessarily one in which one does better)."

Chatterjee (1966) proposed a strictly distribution-free bivariate sign test for location. The test statistic is based on the number of concordances and discordances in the sample, and it has an asymptotically chi-square distribution under the null hypothesis. This test is not affine invariant, but it is invariant under transformations in each variable which are zero preserving and monotonic increasing.

Barnett (1976) presented a classification of all the proposed methods of ordering or ranking multivariate data and discussed the various ranking concepts (and the lack of a natural one) in the multivariate case. In the discussion section of Barnett's paper, Loynes described some of the difficulties in constructing affine invariant tests.

Utts and Hettmansperger (1980) proposed a class of tests and estimates based on a vector of Winsorized rank statistics. The test statistic, a quadratic form, was shown to be more robust to outliers and, in many cases, more efficient than Hotelling's T^2. The proposed statistic is not rotation invariant and, hence, it is not affine invariant either.

Dietz (1982) modified Bickel's bivariate sign test and signed rank test to make them affine invariant. The modification involves transforming and rotating the observations according to certain rules. Then, either test is performed using the transformed and rotated observations. Dietz also shows that the efficiency and small sample power of the tests seem unaffected by the new procedure.

Gower (1974) defined *mediancentre* as the bivariate location measure to minimize the sum of absolute distances to observations. Brown (1983) renamed the mediancentre to *spatial median*, i.e. a statistic appropriate for spatial data, and explained that it is *rotation invariant* but not affine invariant. Comparisons show that the asymptotic efficiency of the spatial median relative to the bivariate mean is often superior to the univariate efficiency of the estimators. Brown also developed certain spatial analogues of sign tests, called "angle tests", using the estimating equations of the spatial median.

Oja (1983) proposed a measure of location for multivariate distributions, as well as measures of scatter, skewness, and kurtosis. Oja's generalized multivariate median is affine invariant and is defined as the point $\hat{\vartheta}$ that minimizes the sum of areas of the triangles formed by ϑ and all pairs of data points. Oja and Niinimaa (1985) showed that, in the case of multivariate normality, the generalized median is asymptotically at least as efficient as Brown's spatial median.

Brown and Hettmansperger (1987) used Oja's generalized multivariate median concept to define multivariate quantiles, or ranks, and multivariate scatter. Based on these quantiles, they developed affine invariant tests and estimates in one- and two-sample bivariate location models. The tests are analogues of the Wilcoxon signed rank test and the Mann-Whitney-Wilcoxon rank sum test. The estimates are analogues of the R-estimates. Brown and Hettmansperger (1989) proceeded to reformulate Oja's affine invariant bivariate quantile and showed that, if it is used as a test statistic, it is an analogue of the univariate sign test. Concentrating on bivariate symmetric distributions, they were able to compute the null covariance matrix of the statistic. Moreover, they showed that this test is robust and, compared to the t test, as efficient as the normal estimation efficiency of the Oja median.

Randles (1989), extending Blumen's (1958) idea to more than two directions, described another affine invariant multivariate analogue of the sign test based on counts called *interdirections*. This test statistic is shown to have a small-sample distribution-free property for distributions with elliptical directions, and it has a limiting χ_p^2 distribution under the null hypothesis. It performs better than Hotelling's T^2 in heavy-tailed or skewed data distributions and it is consistently more powerful than the component sign test.

Peters and Randles (1990) suggested another multivariate nonparametric test based on interdirections, namely a multivariate analogue of the signed rank test. This test statistic is also affine invariant. It performs better than other tests for light-tailed distributions, equally well as Hotelling's T^2 for the multivariate normal distribution, but worse than Randles's multivariate sign test for heavy-tailed distributions.

Peters (1991) proposed a modified adaptive test for the one-sample test for location by incorporating Randles's multivariate sign test, Peters and Randles's multivariate signed rank test, and a light-tailed version of the signed rank procedure. The appropriate statistic is chosen with a selection statistic based on Mahalanobis distances. Since each one of the incorporated statistics is affine invariant, the adaptive statistic is also affine invariant. The adaptive procedure is powerful and nearly optimal for distributions such as Caushy, Double Exponential, Normal, and Uniform, as shown with Monte Carlo results, provided the sample sizes are sufficiently large.

As it is indicated in the title of this section, all the above papers refer to tests and estimators that have been developed in the realm of nonparametric theory. They have not been developed for process control. Hence, issues important in quality control, such as sequential testing, computational facility, etc., have not been dealt with. Of course, quality control may also benefit from affine invariant multivariate nonparametric techniques. In multivariate quality control applications, the affine invariance property means that data manipulations, such as rotations, reflections, and rescalings, do not affect the chart efficiency. In other words, the efficiency of an affine invariant chart for a particular shift does not depend on the direction of the shift. Hence, affine invariance would be a useful property in some quality control applications. Albeit their good statistical properties, the affine invariant techniques described in this section are very complicated. Since ease of computation is important in quality control, none of these affine invariant techniques are as practical as, for example, the componentwise sign test. Nevertheless, many of these techniques are certainly good ideas and may lead to further development of techniques appropriate for quality control.

CHAPTER 4

THE PROPOSED CONTROL CHARTS

4.1 Introduction

The problem under consideration is the simultaneous monitoring of the means of two or more correlated variables, by collecting small fixed-size random samples at equally spaced time intervals. The means are assumed to have known, predetermined, target values. All parameters of the process that are not monitored, for example variances and covariances, are considered unknown, which is most often the case in real situations. These parameters will be estimated from the samples, if needed. Estimates based on small samples are not very reliable, so they will be updated with each new sample as the process monitoring continues.

We assume that the samples collected from the process are independent and identically distributed, both when the process is in control and out of control. The process is in control when the means of the variables equal their target values, and out of control when one or more means have shifted from the target values. We assume that the shift occurs abruptly and we want to detect it as soon as possible.

The process will be monitored using a multivariate nonparametric chart. For each sample, a nonparametric statistic will be computed separately for each variable measured, these statistics will be accumulated over time with the EWMA scheme and, finally, the EWMA statistics will be combined into a quadratic form, according to the accumulate-combine approach described in section 2.3. Since the EWMA statistics are not combined into a quadratic form until the very end of the computation, if a signal occurs, we can use them to identify the parameter or parameters that appear to have shifted from target. This procedure is described in more detail in section 4.2.

A class of nonparametric statistics that could be used for the construction of the proposed charts is the class of the linear rank statistics which are typically used in hypothesis tests for location. Randles and Wolfe (1979, p. 252) give the following definition for the univariate case:

> Let $\mathbf{R} = (R_1, \ldots, R_n)$ denote a vector of ranks, and let $a(1),\ldots,a(n)$ and $c(1),\ldots,c(n)$ be two sets of n constants such that the numbers within each set are not all the same. A statistic of the form
>
> $$S = \sum_{j=1}^{n} c(j)a(R_j) \tag{4.1.1}$$
>
> is called a *linear rank statistic*. The constants $a(1),\ldots,a(n)$ are called the *scores*, and $c(1),\ldots,c(n)$ are termed the *regression constants.*

For this research, we use a vector of p univariate linear rank statistics computed separately on each variable measured. From all the linear rank statistics, the vector *sign* and *signed rank* will be investigated in detail because they are the simplest in their class and they are familiar to many users of statistics.

For almost all multivariate linear rank statistics, an assumption on the data is that they have an absolutely continuous multivariate distribution, with absolutely continuous and symmetric marginal distributions, and absolutely continuous bivariate marginal distributions. The symmetry assumption on the marginal distributions is essential for the signed rank statistic (Hettmansperger, 1984, p. 32), but it is important for the sign statistic only if we want to monitor the marginal means. The same chart based on the sign statistic could be used to monitor the marginal medians if the symmetry assumption cannot be made or if the means do not exist (for example, if the data follow a Cauchy distribution). Detailed derivations for the sign-based and signed-rank-based control statistics are described in sections 4.3 and 4.4. A few final comments on the construction of the charts are given in section 4.5.

4.2 The control statistic

This section contains the assumptions we make and the steps we take to construct the control statistic. Some of this material has been given previously but it is repeated here for completeness.

At equal time intervals a small sample is collected, denoted as $\mathbf{X}_k = \left[X_{kij}\right]$, where k indicates the time the sample was collected, k=1,2,..., i indicates the ith variable measured, i = 1,...,p, and j indicates the jth observation of the sample, j = 1,...,n, where n is the sample size. We assume that $\mathbf{X}_k$ is a random sample from a p-variate cdf F which is absolutely continuous with absolutely continuous and symmetric marginal cdfs $\mathrm{F}_1,\ldots,\mathrm{F}_p$, and absolutely continuous bivariate marginal cdfs $\mathrm{F}_{ii'}$, $i,i' = 1,\ldots,p$, $i \neq i'$. The monitored parameters are the marginal means (or medians) of $\mathrm{F}_1,\ldots,\mathrm{F}_p$, $\mu_1,\ldots,\mu_p$, denoted as the vector $\boldsymbol{\mu}$.

Let $\mathbf{t}$ be the vector of the target values for $\boldsymbol{\mu}$. Then t_i is the target value for the parameter μ_i, i = 1,...,p. If the process is in control

$$\boldsymbol{\mu} = \mathbf{t}. \tag{4.2.1}$$

Let $\mathbf{m}$ be the vector of the marginal medians. The *median* m_i of an absolutely continuous distribution F_i is unique and defined as $\mathrm{F}_i(m_i) = \frac{1}{2}$ (Hettmansperger, 1984), or, equivalently, $\mathrm{P}(X_i \geq m_i) = \mathrm{P}(X_i \leq m_i) = \frac{1}{2}$. Because of the assumption that the marginal distributions are symmetric,

$$\mathbf{m} = \boldsymbol{\mu} \tag{4.2.2}$$

and, by equation (4.2.1), if the process is in control

$$\mathbf{m} = \mathbf{t}. \tag{4.2.3}$$

Now we describe the construction of the control statistic. Using sample $\mathbf{X}_k$, a nonparametric statistic is computed for each variable i across all observations. Let $\mathbf{s}_k' = \begin{bmatrix} S_{k1} & \dots & S_{kp} \end{bmatrix}$ be the vector of p nonparametric statistics at time k.

The EWMA of all such past statistics together with the present statistic is computed for each variable separately. For simplicity, we assume that there is no need to weigh the p variables differently, and therefore they all have the same EWMA parameter r. Then, at time k, the EWMA statistics are

$$Y_{ki} = (1-r)Y_{(k-1)i} + rS_{ki}, \quad i = 1,\dots,p \tag{4.2.4}$$

where the starting value Y_{0i} is taken to be

$$Y_{0i} = \mathrm{E}(S_{ki} | \boldsymbol{\mu} = \mathbf{t})$$

Let

$$Y_{ki}^{*} = Y_{ki} - \mathrm{E}(Y_{ki}), \quad i = 1,\dots,p \tag{4.2.5}$$

At each time k, the statistics Y_{ki}^*, $i=1,\ldots,p$, are combined into a quadratic form

$$W_k = \begin{bmatrix} Y_{k1}^* & \ldots & Y_{kp}^* \end{bmatrix} \Sigma_y^{-1} \begin{bmatrix} Y_{k1}^* \\ \vdots \\ Y_{kp}^* \end{bmatrix} \tag{4.2.6}$$

where Σ_y is the variance-covariance matrix of the vector $\mathbf{y}_k' = \begin{bmatrix} Y_{k1} & \ldots & Y_{kp} \end{bmatrix}$ and it will be estimated from the data as explained in section 4.5. The statistic W_k is the control statistic at time k.

4.3 The sign based statistic

For the sign based statistic, the assumption that the marginal cdfs must be symmetric is needed only in order to monitor the marginal means (to be precise, the assumption is sufficient but not necessary). If this assumption is not made, the following discussion would still hold true for monitoring the marginal medians but not for the marginal means. Using the notation given in section 4.2, we define the indicator variable

$$I_{kij} = \begin{cases} 1 & \text{if } X_{kij} > t_i \\ 0 & \text{otherwise} \end{cases} \tag{4.3.1}$$

and the *sign statistic*

$$S_{ki} = \sum_{j=1}^{n} I_{kij} = \#(X_{kij} > t_i) \tag{4.3.2}$$

for $i = 1,\ldots,p$, $j = 1,\ldots,n$, and $k = 1,2,\ldots$.

In the following discussion, the case $p = 2$ will be considered first in order to explain the ideas in a simple context. The results will, then, be generalized to the case

$p \geq 2$. When the means of only two variables are monitored, i.e. $p = 2$, the observations at time k could be plotted on a graph (see Figure 4.3.1) where, without loss of generality, the target value is taken to be $(0,0)$, the origin of the two axes.

For $j = 1,\ldots,n$, let

$$p_1 = \mathrm{P}(X_{k1j} > t_1) = \mathrm{P}\{\text{the point } (x_{k1j},\, x_{k2j}) \in \text{ quadrants I or IV}\} \qquad (4.3.3)$$

$$p_2 = \mathrm{P}(X_{k2j} > t_2) = \mathrm{P}\{\text{the point } (x_{k1j},\, x_{k2j}) \in \text{ quadrants I or II}\} \qquad (4.3.4)$$

$$p_{12} = \mathrm{P}(X_{k1j} > t_1 \text{ and } X_{k2j} > t_2) = \mathrm{P}\{\text{the point } (x_{k1j},\, x_{k2j}) \in \text{ quadrant I}\} \qquad (4.3.5)$$

If the process is in control, from the definitions of p_1 and p_2 and equation (4.2.3), it follows that

$$p_1 = \mathrm{P}(X_{k1j} > m_1)$$

$$p_2 = \mathrm{P}(X_{k2j} > m_2)$$

and hence, by the definition of the median,

$$p_1 = p_2 = \frac{1}{2} \qquad (4.3.6)$$

Now, since the target value is taken to be $(0,0)$,

$$p_1 = \mathrm{P}(X_{k1j} > 0) = \mathrm{P}(X_{k1j} > 0 \text{ and } X_{k2j} > 0) + \mathrm{P}(X_{k1j} > 0 \text{ and } X_{k2j} \leq 0) \qquad (4.3.7)$$

$$p_2 = \mathrm{P}(X_{k2j} > 0) = \mathrm{P}(X_{k1j} > 0 \text{ and } X_{k2j} > 0) + \mathrm{P}(X_{k1j} \leq 0 \text{ and } X_{k2j} > 0) \qquad (4.3.8)$$

Since the left-hand sides of equations (4.3.7) and (4.3.8) are equal, from the right-hand sides we conclude that, if the process is in control,

$$P(X_{k1j} > 0 \text{ and } X_{k2j} \le 0) = P(X_{k1j} \le 0 \text{ and } X_{k2j} > 0) \qquad (4.3.9)$$

which can be written in terms of quadrants as

$$P\{\text{the point } (x_{k1j},\ x_{k2j}) \in \text{ quadrant IV}\} = P\{\text{the point } (x_{k1j},\ x_{k2j}) \in \text{ quadrant II}\}$$

Similarly,

$$P(X_{k1j} \le 0) = P(X_{k1j} \le 0 \text{ and } X_{k2j} > 0) + P(X_{k1j} \le 0 \text{ and } X_{k2j} \le 0) \qquad (4.3.10)$$

If the process is in control, the left-hand sides of equations (4.3.7) and (4.3.10) are also equal. From their right-hand sides, using equation (4.3.9), we conclude that

$$p_{12} = P(X_{k1j} > 0 \text{ and } X_{k2j} > 0) = P(X_{k1j} \le 0 \text{ and } X_{k2j} \le 0) \qquad (4.3.11)$$

or, equivalently, in terms of quadrants

$$P\{\text{the point } (x_{k1j},\ x_{k2j}) \in \text{ quadrant I}\} = P\{\text{the point } (x_{k1j},\ x_{k2j}) \in \text{ quadrant III}\}$$

Here we can make the observation that, if the process we monitor is in control, we know the probabilities of all four quadrants if we know the probability of anyone of the four quadrants.

The magnitude of p_{12} depends on the underlying bivariate distribution F and it measures the degree of correlation between the two variables X_1 and X_2. From equations (4.3.6) and (4.3.7) and the definition (4.3.5), if the process is in control, we obtain the following result

$$0 \le p_{12} \le \frac{1}{2} \qquad (4.3.12)$$

The two extremes are achieved at perfect negative and perfect positive correlation between the two variables, respectively. If the two variables are independent

$$p_{12} = \mathrm{P}(X_{k1j} > 0 \text{ and } X_{k2j} > 0) = \mathrm{P}(X_{k1j} > 0) \cdot \mathrm{P}(X_{k2j} > 0) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \quad (4.3.13)$$

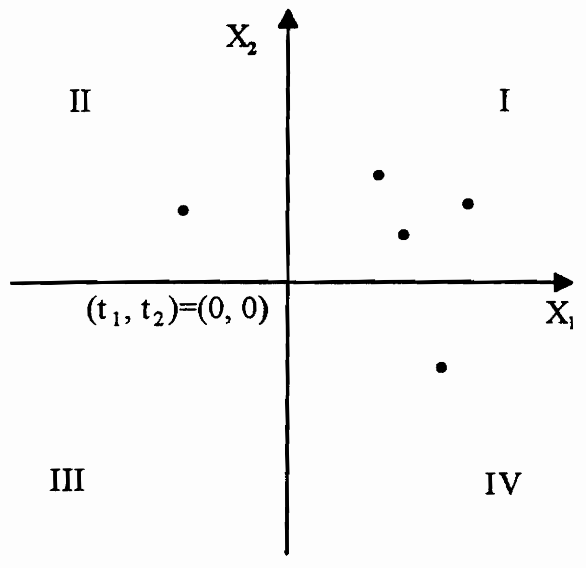

Figure 4.3.1: Bivariate sign statistic

Now, we generalize the discussion to the case where the number of variables p is greater than 2. In a way similar to a variance-covariance matrix in which the elements are the covariances of all possible pairs of variables, the multivariate geometric problem breaks down to the set of all possible bivariate geometric problems. For each pair of variables, the points on the graph (Figure 4.3.1) are the projections of the observations onto the plane determined by the axes of the two variables. For any pair of variables

$X_i, X_{i'}$ $i,i' = 1,\ldots,p$, $i \neq i'$, we generalize definitions (4.3.3), (4.3.4) and (4.3.5) as follows:

$$p_{\mathrm{i}} = \mathrm{P}(X_{kij} > t_i) \tag{4.3.14}$$

$$p_{\mathrm{i}'} = \mathrm{P}(X_{ki'j} > t_{i'}) \tag{4.3.15}$$

$$p_{\mathrm{ii}'} = \mathrm{P}(X_{kij} > t_i \text{ and } X_{ki'j} > t_{i'}) \tag{4.3.16}$$

Then, if the process is in control

$$p_i = p_{i'} = \frac{1}{2} \tag{4.3.17}$$

$$0 \leq p_{ii'} \leq \frac{1}{2} \tag{4.3.18}$$

and if the variables are independent

$$p_{ii'} = \frac{1}{4}. \tag{4.3.19}$$

The following are the derivations of means, variances and covariances necessary for the computation of the control statistic W_k. They are mostly standard derivations given here for completeness. For all k=1,2,..., $i,i' = 1,\ldots,p$, $i \neq i'$, and j=1,...,n, from definition (4.3.1) we compute

$$\mathrm{E}(I_{kij}) = \mathrm{P}(X_{kij} > t_i) = p_i \tag{4.3.20}$$

$$\mathrm{Var}(I_{kij}) = \mathrm{E}(I_{kij}^2) - \left[\mathrm{E}(I_{kij})\right]^2 = p_i - p_i^2 = p_i(1-p_i) \tag{4.3.21}$$

$$\begin{aligned}\mathrm{Cov}(I_{kij}, I_{ki'j}) &= \mathrm{E}(I_{kij}, I_{ki'j}) - \mathrm{E}(I_{kij})\mathrm{E}(I_{ki'j}) \\ &= \mathrm{P}(X_{kij} > t_i \text{ and } X_{ki'j} > t_{i'}) - p_i p_{i'} = p_{ii'} - p_i p_{i'}\end{aligned} \tag{4.3.22}$$

From the definition of the sign statistic (4.3.2) and the independence of all X_{kij}, j=1,...,n, observations of the ith variable at time k, we compute the following using the results in (4.3.20), (4.3.21) and (4.3.22) :

$$\mathrm{E}(S_{ki}) = \mathrm{E}(\sum_{j=1}^{n} I_{kij}) = \sum_{j=1}^{n} \mathrm{E}(I_{kij}) = n\,p_i \tag{4.3.23}$$

$$\mathrm{Var}(S_{ki}) = \mathrm{Var}(\sum_{j=1}^{n} I_{kij}) = \sum_{j=1}^{n} \mathrm{Var}(I_{kij}) = n\,p_i(1-p_i) \tag{4.3.24}$$

$$\mathrm{Cov}(S_{ki}, S_{ki'}) = \mathrm{Cov}(\sum_{j=1}^{n} I_{kij}, \sum_{j=1}^{n} I_{ki'j}) \tag{4.3.25}$$

Since all observations are independent, for $j \neq j'$

$$\mathrm{Cov}(I_{kij}, I_{ki'j'}) = 0$$

Hence, from (4.3.25) and (4.3.22)

$$\mathrm{Cov}(S_{ki}, S_{ki'}) = \sum_{j=1}^{n} \mathrm{Cov}(I_{kij}, I_{ki'j}) = n\ (p_{ii'} - p_i p_{i'}) \tag{4.3.26}$$

It remains to compute the means, variances and covariances of the EWMA statistics Y_{ki} and $Y_{ki'}$. The EWMA statistic defined in (4.2.4) for any nonparametric statistic, is rewritten for the sign statistic as follows

$$Y_{ki} = (1-r)Y_{(k-1)i} + rS_{ki}, \quad i = 1,...,p$$

or, equivalently,

$$Y_{ki} = (1-r)^k\ Y_{0i} + \sum_{l=0}^{k-1} (1-r)^l rS_{(k-l)i} \tag{4.3.27}$$

where the starting value Y_{0i} is taken to be

$$Y_{0i} = \mathrm{E}(S_{ki} | \mu = \mathbf{t}),$$

i.e. computed assuming the process is in control.

The following derivations apply for any multivariate EWMA statistic as long as the samples are independent. For completeness, the derivations are given here in detail but they will be omitted in the discussion of the signed rank based statistic. Then, from equations (4.3.27), (4.3.23), and (4.3.17), we compute the following expected value:

$$\begin{aligned}
\mathrm{E}(Y_{ki}) &= (1-r)^k \,\mathrm{E}(S_{ki} | \mu = \mathbf{t}) + \sum_{l=0}^{k-1} (1-r)^l \, r \,\mathrm{E}(S_{(k-l)i}) \\
&= (1-r)^k \frac{n}{2} + \sum_{l=0}^{k-1} (1-r)^l \, r \, np_i \\
&= (1-r)^k \frac{n}{2} + r \, np_i \frac{1-(1-r)^k}{1-(1-r)} \\
&= (1-r)^k \frac{n}{2} + \left[1-(1-r)^k\right] np_i \qquad (4.3.28)
\end{aligned}$$

Since the samples are independent for all k, $k = 1, 2, \ldots$, the sign statistics S_{ki} and $S_{k'i'}$ are independent for all $k \neq k'$. All covariances between such statistics are, then, zero and they drop out in the following computations:

$$\begin{aligned}
\mathrm{Var}(Y_{ki}) &= (1-r)^{2k}\, \mathrm{Var}(Y_{0i}) + \sum_{l=0}^{k-1} [(1-r)^l r]^2 \,\mathrm{Var}(S_{(k-l)i}) \\
&= (1-r)^{2k}\, 0 + r^2 \frac{1-(1-r)^{2k}}{1-(1-r)^2} np_i (1-p_i) \\
&= \left(\frac{r}{2-r}\right) [1-(1-r)^{2k}]\; np_i (1-p_i) \qquad (4.3.29)
\end{aligned}$$

$$\mathrm{Cov}(Y_{ki}, Y_{ki'}) = (1-r)^{2k}\, \mathrm{Cov}(Y_{0i}, Y_{0i'}) + \sum_{l=0}^{k-1} [(1-r)^l r]^2 \,\mathrm{Cov}(S_{(k-l)i}, S_{(k-l)i'})$$

$$= (1-r)^{2k} 0 + r^2 \frac{1-(1-r)^{2k}}{1-(1-r)^2} \, n(p_{ii'} - p_i p_{i'})$$

$$= \left(\frac{r}{2-r}\right)[1-(1-r)^{2k}] \, n(p_{ii'} - p_i p_{i'}) \tag{4.3.30}$$

Note that, as $k \to \infty$, and for $i, i' = 1, \ldots, p$, $i \neq i'$

$$\mathrm{Var}(Y_{ki}) \to \left(\frac{r}{2-r}\right) np_i(1-p_i) \tag{4.3.31}$$

$$\mathrm{Cov}(Y_{ki}, Y_{ki'}) \to \left(\frac{r}{2-r}\right) n(p_{ii'} - p_i p_{i'}) \tag{4.3.32}$$

These asymptotic results can be used in place of the exact results in (4.3.29) and (4.3.30). For this dissertation, though, we will use the exact results.

Finally, if the process is in control, using the result (4.3.17), the equations (4.3.28), (4.3.29) and (4.3.30) are simplified to:

$$\mathrm{E}(Y_{ki}) = (1-r)^k \frac{n}{2} + \left[1-(1-r)^k\right]\frac{n}{2}$$

$$= \frac{n}{2}\left[(1-r)^k + 1 - (1-r)^k\right]$$

$$= \frac{n}{2} \tag{4.3.33}$$

$$\mathrm{Var}(Y_{ki}) = \left(\frac{r}{2-\mathrm{r}}\right)[1-(1-r)^{2k}] \, \frac{n}{4} \tag{4.3.34}$$

$$\mathrm{Cov}(Y_{ki}, Y_{ki'}) = \left(\frac{r}{2-r}\right)[1-(1-r)^{2k}] \, n(p_{ii'} - \frac{1}{4}) \tag{4.3.35}$$

for $i, i' = 1, \ldots, p$, $i \neq i'$. These last three results are, then, used in the computation of the control statistic W_k, as described in (4.2.5) and (4.2.6).

4.4 The signed rank based statistic

Using the indicator variable defined in (4.3.1), we define the *signed rank statistic*

$$T_{ki}^{+} = \sum_{j=1}^{n} I_{kij} R_{kij} \tag{4.4.1}$$

where R_{kij} is the rank of $|X_{kij} - t_i|$ among $|X_{ki1} - t_i|, \ldots, |X_{kin} - t_i|$, and t_i is the target value of the ith variable, i=1,...,p.

Following similar steps as for the sign statistic, we can compute the EWMA separately for each variable. Then, from equation (4.2.4) we have

$$Y_{ki} = (1-r)Y_{(k-1)i} + rT_{ki}^{+}$$

or

$$Y_{ki} = (1-r)^k Y_{0i} + \sum_{j=0}^{k-1} (1-r)^j r T_{(k-j)i}^{+} \tag{4.4.2}$$

where the starting value Y_{0i} is taken to be

$$Y_{0i} = \mathrm{E}(T_{ki}^{+} | \mu = \mathbf{t}),$$

i.e. computed assuming the process is in control. When the process is in control, i.e. when $p_i = 1/2$, the standard derivation of the mean and variance of the signed rank statistic is as follows:

$$\mathrm{E}(T_{ki}^{+}) = \sum_{j=1}^{n} \mathrm{E}\left(I_{kij} R_{kij}\right) = \left(\sum_{j=1}^{n} j\right) \mathrm{E}\left(I_{kij}\right) = \frac{n(n+1)}{2} p_i = \frac{n(n+1)}{4} \tag{4.4.3}$$

$$\mathrm{Var}(T_{ki}^{+}) = \sum_{j=1}^{n} \mathrm{Var}\left(I_{kij} R_{kij}\right) = \left(\sum_{j=1}^{n} j^2\right) \mathrm{Var}\left(I_{kij}\right)$$

$$= \frac{n(n+1)(2n+1)}{6} p_i(1-p_i)$$

$$= \frac{n(n+1)(2n+1)}{24} \tag{4.4.4}$$

Then, similar to the derivations of the equations (4.3.33) and (4.3.34), we have

$$\mathrm{E}(Y_{ki}) = (1-r)^k \frac{n(n+1)}{4} + \left[1-(1-r)^k\right]\frac{n(n+1)}{4}$$

$$= \frac{n(n+1)}{4} \tag{4.4.5}$$

$$\mathrm{Var}(Y_{ki}) = \left(\frac{r}{2-r}\right)[1-(1-r)^{2k}]\frac{n(n+1)(2n+1)}{24} \tag{4.4.6}$$

For the general case, when the process may be in or out of control, there exist results for the mean and variance of the signed rank statistic, which have quite complicated expressions (Hettmansperger, 1984, p. 47). These results could be used with the EWMA and we would obtain results more general than (4.4.5) and (4.4.6). But this is not necessary, since to construct a control chart we only need results for the in-control case.

Now we will derive the expression for the in-control covariance. The following discussion will be temporarily restricted to two variables in order to simplify the notation. At the end, the result will be generalized to the multivariate case. Let p=2, and

$$T_{k1}^{+} = \sum_{j=1}^{n} I_{k1j} R_{k1j}$$

$$T_{k2}^{+} = \sum_{j'=1}^{n} I_{k2j'} R_{k2j'}$$

be the signed rank statistics computed for the two variables, respectively. Then

$$\mathrm{Cov}(T_{k1}^{+}, T_{k2}^{+}) = \mathrm{Cov}\left(\sum_{j=1}^{n} I_{k1j} R_{k1j}, \sum_{j'=1}^{n} I_{k2j'} R_{k2j'}\right)$$

$$= \sum_{j=1}^{n} \sum_{j'=1}^{n} jj' \mathrm{Cov}(I_{k1j}, I_{k2j'})$$

$$= \sum_{j=j'=1}^{n} j^2 \mathrm{Cov}(I_{k1j}, I_{k2j}) + \sum_{j \neq j'}^{n} jj' \mathrm{Cov}(I_{k1j}, I_{k2j'}) \tag{4.4.7}$$

Now, for $j = j'$ we can use equation (4.3.22), whereas for $j \neq j'$ we have $\mathrm{Cov}(I_{k1j}, I_{k2j'}) = 0$ because the n observations are independent. Then, equation (4.4.7) becomes

$$\mathrm{Cov}(T_{k1}^{+}, T_{k2}^{+}) = (p_{12} - p_1 p_2) \sum_{j=1}^{n} j^2$$

$$= (p_{12} - \frac{1}{4}) \frac{n(n+1)(2n+1)}{6} \tag{4.4.8}$$

Finally, the covariance between the two EWMA statistics, computed in a way similar to (4.3.30), is as follows:

$$\mathrm{Cov}(Y_{k1}, Y_{k2}) = \left(\frac{r}{2-r}\right)[1-(1-r)^{2k}]\ \mathrm{Cov}(T_{k1}^{+}, T_{k2}^{+})$$

$$= \left(\frac{r}{2-r}\right)[1-(1-r)^{2k}]\ \frac{n(n+1)(2n+1)}{6}(p_{12} - \frac{1}{4}) \tag{4.4.9}$$

The same result holds true for any pair of variables and, therefore, in the case $p > 2$ it can be generalized to:

$$\mathrm{Cov}(Y_{ki}, Y_{ki'}) = \left(\frac{r}{2-r}\right)[1-(1-r)^{2k}]\ \frac{n(n+1)(2n+1)}{6}(p_{ii'} - \frac{1}{4}) \tag{4.4.10}$$

Note that, as $k \to \infty$, and for $i, i' = 1, \ldots, p$, $i \neq i'$, the variances and covariances in (4.4.6) and (4.4.10) become

$$\mathrm{Var}(Y_{ki}) \to \left(\frac{r}{2-r}\right) \frac{n(n+1)(2n+1)}{24} \tag{4.4.11}$$

$$\operatorname{Cov}(Y_{ki}, Y_{ki'}) \to \left(\frac{r}{2-r}\right) \frac{n(n+1)(2n+1)}{6}(p_{ii'} - \frac{1}{4}) \qquad (4.4.12)$$

These asymptotic results can be used in place of the exact results, i.e. equations (4.4.6) and (4.4.10). In this dissertation, though, we will use the exact results. The means, variances, and covariances of the signed rank based EWMA statistics, i.e. results (4.4.5), (4.4.6), and (4.4.10), are then used in the computation of the control statistic W_k, as shown in (4.2.5) and (4.2.6).

4.5 Chart construction

In section 4.2 the steps necessary for the computation of the control statistic W_k were listed. In order to construct a control chart the following are needed:

- a target value
- control limit(s)

For the proposed chart the target value, also used in the computation of the control statistic W_k for the sign and signed rank, is assumed to be known. Since W_k is a quadratic form, only one control limit will have to be determined, the UCL, and this will be done with simulation (Chapter 6). Finally, a few more points need clarification.

(i) In the statement of the problem, section 4.1, it was mentioned that the variance-covariance matrix $\mathbf{\Sigma}_{\mathbf{y}}$, in a realistic situation, is unknown and it will have to be estimated. In fact, we will consider two cases. In the first case, $\mathbf{\Sigma}_{\mathbf{y}}$ is assumed to be known. In the second case, $\mathbf{\Sigma}_{\mathbf{y}}$ must be estimated as process data is obtained. The first case is being considered so that comparisons can be made with the corresponding parametric chart. It also presents the charts under ideal conditions that cannot be surpassed. The second case is being considered because it will be more useful in applications.

(ii)From the equations (4.3.34) and (4.3.35) for the sign based statistic, as well as the equations (4.4.6) and (4.4.10) for the signed rank based statistic, we observe that in order to estimate Σ_y, in the in-control case, we only need to estimate the probability $p_{ii'}$. The probability $p_{ii'}$ was defined in (4.3.16). This probability is estimated at each sampling time k as the proportion

$$\hat{p}_{ii',k} = \frac{\#\text{ of points}(x_i, x_{i'}) \text{ where } (x_i > t_i,\ x_{i'} > t_{i'}) \text{ or } (x_i < t_i,\ x_{i'} < t_{i'})}{2n} \qquad (4.5.1)$$

where n is the sample size, and the estimate is updated as follows

$$\bar{\hat{p}}_{ii',k} = \begin{cases} \hat{p}_{ii',k} & \text{for } k = 1 \\ \dfrac{(k-1)\bar{\hat{p}}_{ii',k-1} + \hat{p}_{ii',k}}{k} & \text{for } k > 1. \end{cases} \qquad (4.5.2)$$

The motivation for the estimator in (4.5.1) is equation (4.3.11), i.e. both quadrants I and III should be used to estimate the probability $p_{ii'}$. It is expected that the variance of an estimator based on quadrants I and III will be lower than the variance of an estimator based just on quadrant I. With minor modification of equation (4.5.2), the proposed estimator can also be given an initial value; for example, the initial value could be 1/4 which is the midpoint of the range of possible values of $p_{ii'}$ (see equation (4.3.18)).

For small k, the estimate $\bar{\hat{p}}_{ii',k}$ of $p_{ii'}$ may not be very good because it will be based on a few observations, but our objective is to start monitoring the process as early as possible before a lot of data has been collected. After the first few samples have been collected, these updated estimates of $p_{ii'}$ will improve significantly. These issues are also mentioned in Chapter 6, where the performed simulations are discussed.

CHAPTER 5

PROPERTIES OF THE CHARTS

5.1 The ARL as a performance criterion for long production runs

As it was mentioned earlier, control chart comparisons are usually based on their respective average run lengths (ARL) for different shifts in the monitored parameter(s). The distribution of the run length depends on the joint distribution of the control statistics W_k, W_{k-1}, W_{k-2}, ...,W_1. In the case of a simple Shewhart $\overline{X}$-chart, where the control statistic is $S_k = \overline{X}_k$ and the variance is assumed known, the distribution of the run length is geometric. If

$$\mathrm{P(signal)} = \mathrm{P}(\overline{X}_k > \mathrm{UCL} \text{ or } \overline{X}_k < \mathrm{LCL}) \tag{5.1.1}$$

then, the ARL is simply

$$\mathrm{ARL} = \frac{1}{\mathrm{P(signal)}} \tag{5.1.2}$$

From equations (5.1.1) and (5.1.2), we can see that, if the distribution of the control statistic is known, one can determine the UCL and LCL of the chart necessary to achieve a prespecified in-control ARL. In the control charts proposed in this dissertation, the statistics W_k, $k = 1,2,...$ are not i.i.d, so determining the control limits is more difficult.

We will now discuss the corresponding parametric version of the proposed charts, i.e. the procedure where the sample average $\overline{\mathbf{x}}_k$ is used in place of the sign statistic or the signed rank statistic. Assuming that there is no reason to weigh past observations differently for each of the p variables, we can make the following simplification: $r_1 = r_2 = ... = r_p = r$. Then, if the observations $\mathbf{x}_1,...,\mathbf{x}_n$ collected at time k are i.i.d.

MVN(μ,Σ), where Σ is assumed known and the target value for μ is $\mathbf{t}$, it can be shown that the control statistic W_k based on $\bar{\mathbf{x}}_k$ has a $\chi^2_{(p,\lambda^0)}$ distribution, with noncentrality parameter

$$\lambda^0 = \left[n\left(\frac{2-r}{r}\right)\frac{\left(1-(1-r)^k\right)^2}{1-(1-r)^{2k}} \right](\mu-\mathbf{t})'\Sigma^{-1}(\mu-\mathbf{t}) \qquad (5.1.3)$$

and degrees of freedom p = # of variables monitored. For the proof, see appendix A.

Lowry et al (1992) investigated the above chart, which they call MEWMA chart. If the in-control W_k, k=1,2,..., based on $\bar{\mathbf{x}}$ were independent random variables, in order to maintain a specified in-control ARL one would only need to look up a table for the appropriate percentile of the χ^2 distribution, namely $\chi^2_{(1-\alpha,p)}$, where α=1/ARL. For example, in order to maintain in-control ARL=200 when p=2, we could use the table value $\chi^2_{(0.995,2)} = 10.5966$ as the upper control limit (UCL) of the chart. Since the in-control W_k's of the MEWMA chart are identically distributed but not independent random variables, it is not possible to use such a direct method to find the UCL for the overall chart. In fact, it turns out that the UCL of the MEWMA chart is usually smaller than the chi-square value corresponding to the same false alarm rate (Lowry, 1989, p. 14). Lowry used simulation to estimate the UCLs of the MEWMA chart. For all values of the EWMA parameter r listed (Lowry, 1989, table on p. 49), the estimated UCLs of the MEWMA chart from simulated data are less than the chi-square value 10.5966. Alternatively, the UCLs of the MEWMA chart can be computed using the integral equation method, as shown by Rigdon (1994a, 1994b).

Lowry et al (1992) showed that the ARL of the MEWMA chart depends on the mean vector and the variance-covariance matrix of the data only through the noncentrality parameter λ. They define λ as follows:

$$\lambda = (\boldsymbol{\delta}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\delta})^{\frac{1}{2}} \tag{5.1.4}$$

where $\boldsymbol{\delta} = \boldsymbol{\mu} - \mathbf{t}$ is the shift of the mean from target.

Let us consider the case of two variables only (p=2). For a given matrix $\boldsymbol{\Sigma}$ and a specific value of λ, the equation (5.1.4) is the equation of an ellipse (see figure 5.1.1). For different values of λ the graph becomes a family of concentric ellipses. From the graph, it is obvious that an ellipse represents a variety of shifts from the target such that, if $\boldsymbol{\delta}_1$ and $\boldsymbol{\delta}_2$ are two shifts, then they will be detected equally fast as long as

$$\boldsymbol{\delta}_1' \boldsymbol{\Sigma}^{-1} \boldsymbol{\delta}_1 = \boldsymbol{\delta}_2' \boldsymbol{\Sigma}^{-1} \boldsymbol{\delta}_2 \tag{5.1.5}$$

The importance of the magnitude of the shift depends on the direction of the shift. In most of the cases, shifts in one parameter will be detected faster than same size shifts in both parameters. This may not be reasonable in all situations, although it helps in comparing charts that have the same property. Naturally, shifts in certain directions may be of particular interest, because they are more likely or more costly, and shifts in other directions may be of less interest. But, for some processes, it may be more appropriate to have contours of equal ARL forming circles instead of ellipses. In this case, shifts should be detected equally fast as long as

$$\boldsymbol{\delta}_1' \boldsymbol{\delta}_1 = \boldsymbol{\delta}_2' \boldsymbol{\delta}_2 \tag{5.1.6}$$

In the special case where $\boldsymbol{\Sigma} = \sigma^2 \mathbf{I}$, i.e. when the monitored variables are independent and have equal variances, equations (5.1.5) and (5.1.6) are equivalent. Then the contours of equal ARL values always form circles. If the monitored variables are dependent or have unequal variances, the points of equal ARL values form ellipses. If we standardize the two axes for variance, then the ellipses are more eccentric (elongated) when the variables are more correlated.

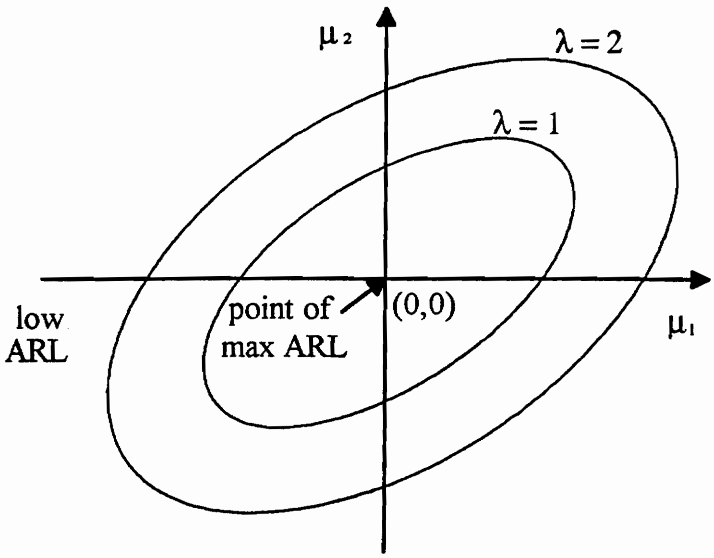

Figure 5.1.1: Contours of equal ARL

5.2 The affine invariance property

The proof of the result that Lowry (1989) showed for the MEWMA chart, i.e. that the ARL is a function of λ only, is based on the affine invariance property of the control statistic. Let $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ denote observations from a p-dimensional absolutely continuous population. A statistic T is said to be *affine invariant*, if

$$T(\mathbf{Dy}_1, \mathbf{Dy}_2, \ldots, \mathbf{Dy}_n) = T(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n) \tag{5.2.1}$$

for any nonsingular $p \times p$ matrix **D**. Another way of saying that a statistic is affine invariant is to say that the statistic is invariant under nonsingular linear transformations of the data. Such transformations include rotations, reflections, and rescalings.

One aspect of this property is that an affine invariant statistic maintains the same value, and hence the same distribution, under any change in the units of measurement of the data that involves *centering* and *scaling*. Another aspect is that, if one wishes to rotate the data and use the *principal components* in the analysis instead of the data themselves, the result of the analysis will be the same as if the original data were used. The parametric MEWMA statistic is affine invariant, as shown by Lowry et al (1992). The Hotelling's T^2 statistic is also affine invariant, as shown in Johnson and Wichern (1982, p. 183). The relationship between these two statistics becomes obvious if one makes the observation that Hotelling's T^2 is a special case of the parametric MEWMA statistic, when the MEWMA parameter r is equal to 1.

Unlike the two parametric statistics mentioned above, the nonparametric MEWMA statistics proposed in this dissertation are not affine invariant. This is partly due to the component sign statistic, which plays a role in both the sign based and the signed-rank based MEWMA statistics. Many researchers have looked at the component sign test. Bickel (1965) demonstrated that, although the component sign test is significantly more efficient than Hotelling's T^2 when the data distribution has heavy tails, it may not perform well when there are substantial correlations among the variables. Randles (1989, p. 104) states that the performance of the component sign test depends on Σ and the direction of the shift from the mean under the null hypothesis. Dietz (1982, 1984) actually proposed transforming the data before performing the test in order to create an invariant procedure, the power of which is comparable to the regular component sign test when $\mathbf{\Sigma} = \mathbf{I}$.

Here, it should be noted that the univariate sign statistic is "affine invariant" also. Of course, in the univariate case, the notion of affine invariance is reduced to simple centering and scaling of the observations. This can be proved as follows. Let X be a continuous random variable with mean μ. Suppose we monitor the mean of X using the sign statistic, and the target value for μ is t. We use the definition of the sign statistic given in equations (4.3.1) and (4.3.2). Then the sign of X is

$$\mathrm{S}(X) = \begin{cases} 1 & \text{if } X \geq t \\ 0 & \text{otherwise} \end{cases}$$

Let $Z = a + bX$ be a transformation of X that involves centering and scaling, a and $b \in \Re$, $b > 0$. Then the mean of Z is $\mu_Z = a + b\mu$ and the target value of Z is $t_Z = a + bt$. Now

$$\begin{aligned} X \geq t &\Leftrightarrow bX \geq bt \\ &\Leftrightarrow a + bX \geq a + bt \\ &\Leftrightarrow Z \geq t_Z \end{aligned}$$

Hence,

$$S(Z) = S(X) \qquad (5.2.2)$$

Now, we consider the multivariate case. Centering and scaling transformations of the data performed on each variable separately are simple extensions of the univariate case discussed above. For example, suppose we monitor a process by measuring two variables: one variable is height measured in inches and the other variable is temperature measured in degrees Fahrenheit. The control statistic we use is the componentwise sign statistic. Now, if we wanted to express the height in centimeters and the temperature in degrees Celsius, we would center and scale both variables of the data. This action would have no effect on the componentwise sign statistic, which would remain the same. As it was mentioned earlier, the componentwise sign statistic is not affine invariant, which refers to transformations that involve all variables together. For example, if we wanted to use the principal components of the data to monitor the process, the componentwise sign statistic would not remain the same. With a simple numerical example, we will illustrate the difference between the $\overline{X}$ based and the sign based MEWMA charts in terms of affine invariance.

Example 5.2.1

Let $\mathbf{x}$ be $\text{BVN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, with mean $\boldsymbol{\mu} = \mathbf{0}$ and variance-covariance matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0.75 \\ 0.75 & 1 \end{bmatrix}. \tag{5.2.3}$$

Then, the correlation ρ between X_1 and X_2 is 0.75. Let $\mathbf{z}$ be a nonsingular linear transformation of $\mathbf{x}$

$$\mathbf{z} = \mathbf{A}\mathbf{x}, \tag{5.2.4}$$

where $\mathbf{A}$ is the following nonsingular matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \tag{5.2.5}$$

From theorem 3.6 stated in Arnold (1981, p. 46), it follows that $\mathbf{z}$ is $\text{BVN}(\boldsymbol{\mu}^*, \boldsymbol{\Sigma}^*)$, where $\boldsymbol{\mu}^* = \mathbf{A}\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}^* = \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'$. Then, in our example, $\boldsymbol{\mu}^* = \mathbf{0}$ and

$$\Sigma^* = \begin{bmatrix} 3.5 & 1.75 \\ 1.75 & 1 \end{bmatrix}. \tag{5.2.6}$$

Now the correlation between Z_1 and Z_2 is as follows:

$$\rho^* = \frac{1.75}{\sqrt{3.5} \cdot \sqrt{1}} = 0.935 \tag{5.2.7}$$

Suppose that we want to monitor the mean vector. Then, if the target value of μ is $\mathbf{t} = \mathbf{0}$, the target value of μ^* is $\mathbf{t}^* = \mathbf{A}\mathbf{t} = \mathbf{0}$. Also, let the EWMA parameter r = 0.2, and the sample size n = 5.

Consider the following random sample in the form of a matrix:

$$\mathbf{X} = \begin{bmatrix} 1.4507 & 0.7660 & 0.0584 & 0.9035 & -0.8669 \\ 1.2463 & -0.0429 & -0.6692 & 0.4628 & -0.9334 \end{bmatrix}.$$

Using the transformation defined in equation (5.2.4), we obtain the transformed data

$$\mathbf{Z} = \begin{bmatrix} 2.6970 & 0.7231 & -0.6108 & 1.3663 & -1.8003 \\ 1.4507 & 0.7660 & 0.0584 & 0.9035 & -0.8669 \end{bmatrix}.$$

The vector of the averages for the original data is

$$\bar{\mathbf{x}} = \begin{bmatrix} 0.4623 \\ 0.0127 \end{bmatrix},$$

whereas, for the transformed data, it is

$$\bar{\mathbf{z}} = \begin{bmatrix} 0.4751 \\ 0.4623 \end{bmatrix}.$$

The vector of the signs for the original data, using definitions (4.3.1) and (4.3.2) with target vector $\mathbf{t}$, is computed as

$$\mathbf{s}(\mathbf{x}) = \begin{bmatrix} 4 \\ 2 \end{bmatrix},$$

whereas the vector of the signs of the transformed data, using also definitions (4.3.1) and (4.3.2) with target vector $\mathbf{t}^*$, is computed as

$$\mathbf{s}(\mathbf{z}) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

We observe that

$$\overline{\mathbf{z}} = \mathbf{A}\overline{\mathbf{x}}, \tag{5.2.8}$$

whereas

$$\mathbf{s}(\mathbf{z}) \neq \mathbf{A}\mathbf{s}(\mathbf{x}). \tag{5.2.9}$$

As we have explained above, (5.2.9) is due to the lack of affine invariance of the component sign.

Using the above computations, and the results from Lowry's dissertation and from this dissertation (Section 4.3), we can compute the first stage of the $\overline{X}$ based and the sign based MEWMA statistics for the original and the transformed data. Detailed computations are given in Appendix B. The results are summarized in Table (5.2.1). The nonparametric control statistic W_1 computed from the transformed data is different from the one computed from the original data, as expected. On the other hand, the parametric control statistic W_1 remains the same in the two cases.

Since these are only the first step results, i.e. for k = 1, using simulation we can obtain a more realistic picture of how much non-affine invariant the sign based chart is for this particular transformation. The simulations produced the results listed on Table (5.2.2), in terms of the upper control limits (UCL) of the charts. The UCLs were determined by specifying the in-control ARL at approximately 200. We observe that the UCL of the nonparametric control chart for the original data is 9.45, whereas for the transformed data it is 9.70. For example, suppose we want to use the principal components of the data to monitor a process, instead of the actual data. If we used the $\overline{X}$ based chart to monitor the process, we would have the same UCL for the original data as well as for the principal components of the data. So, the control chart would remain the same in both cases. If we used the sign based chart to monitor the process, we would have to compute a different UCL for the original data and a different UCL for the principal components. So, we would need two different charts. Now, the question is how much difference there is between the two UCLs for the sign based chart, due to the data transformation. Although the UCLs are different, if the case shown in Table (5.2.2) serves as an indication of the general behavior of the sign based chart, we may conclude that they are not very different. Yet, since the lack of affine invariance may have a significant effect on the chart's effectiveness, it will be carefully investigated in the following chapters.

Table 5.2.1: First step results for the value of the MEWMA control statistic.

W_1	original data	transformed data
$\overline{X}$ based MEWMA	2.3436	2.3436
sign based MEWMA	3.7380	2.6291

Table 5.2.2: Upper control limits of the MEWMA control charts based on *k* steps and 1000 iterations.

UCL	original data	transformed data
$\overline{X}$ based chart	9.55	9.55
sign based chart	9.45	9.70

We will now discuss the importance of the affine invariance property in general and, in particular, with respect to control charts. In the overall statistical methodology, there is a variety of invariance concepts, some more popular than others. A number of such concepts are listed in the Encyclopedia of Statistical Sciences (1983). According to Narayan Giri (Encyclopedia of Statistical Sciences, 1983, p.219), the main reason invariance concepts have an intuitive appeal is because of the belief that there should be a *unique best way* of analyzing statistical information. "However," he continues, "in cases where the use of an invariant decision rule conflicts violently with the desire to make a correct decision or have a small risk, it must be abandoned."

Brown and Hettmansperger (1987, p. 302) recommend the usage of affine invariant methods if there is a compelling reason, for example because of the measurement scales in the model. Chatterjee (1966, p. 1781) states that affine invariance is meaningful when the choice of co-ordinate axes is rather arbitrary. Alternatively, if one or more variables are measured on a calibrated instrument or on an indirect scale, invariance under zero-preserving monotonic transformations may be more important. Utts and Hettmansperger (1980, p. 939) note that it is reasonable to consider componentwise procedures that are not rotation invariant, a key element of affine invariance, if the components represent experimentally meaningful dimensions. They also make the observation that procedures that require rotational invariance have very poor robustness in higher dimensions "since protection against outliers must be accomplished jointly in all directions." Of course, there are cases where affine invariance seems not just desirable but necessary. Brown (1983) states that rotation invariance is a *must* when analyzing spatial data due to the fact that spatial data possess the property of isometry.

There are several issues involved with affine invariance, one of the most important ones being the difficulty of defining natural affine invariant test statistics. Loynes, in the discussion of a paper written by Barnett (1976, p. 347), states that although in many situations we would want to regard the axes as arbitrary we may have to agree that the axes may not be rotated. The reason is that, unlike the median point on a line, no such point exists (with the same property) on a plane. Such a point would have to have an equal number of points on either side of an arbitrary line through it. The point that comes close to possessing this property is the point of the marginal medians when the lines drawn through it are parallel to the two axes.

Now, how important is the property of affine invariance with respect to multivariate control charts? There are three main issues concerning multivariate control charts. The first two issues are directly connected with the issue of affine invariance. Namely, the effect high correlations in the data may have on the charts and whether we may want to use a data transformation during process monitoring. We have mentioned that the component sign test seems to deteriorate when the variables are highly correlated (Bickel, 1965). On the issue of high correlations, one may say that, if highly correlated variables monitored together do not provide additional information about a process, it would be more economical to simultaneously monitor only moderately correlated or uncorrelated variables. Moreover, data transformations will probably not be useful in quality control. Instead, any rotation of the axes would probably make it more difficult to identify the source of a shift.

The third issue, not to be overlooked in quality control, is one of practicality and easiness of computation. Having decided that a nonparametric technique is more appropriate for a process we wish to monitor, we are left with the decision whether an affine invariant or a non-affine invariant technique is needed. In comparing the component sign test statistic, which is not affine invariant, with other proposed multivariate sign statistics, which are affine invariant, we see that, unlike the component sign, most of the nonparametric affine invariant statistics require very complicated computations. The most ideal chart that satisfies all desirable statistical properties is of little use to the quality control engineer, if one cannot use it quickly and effectively. The use of computers will not alleviate the difficulty in understanding what exactly one measures and, if a signal occurs, what information it gives us about the process except that it is out of control. In conclusion, it seems that the affine invariance property is not very important for a control chart.

One last comparison we should make is between the nonparametric MEWMA charts, namely the sign based and the signed-rank based chart, and the parametric MEWMA chart, based on $\overline{X}$. The latter is affine invariant, whereas the former are not. One may consider using the affine invariant parametric MEWMA chart even if it doesn't comply with the distributional assumptions. In this case, we will show with simulations that affine invariance is not a fair trade-off for violating the statistical assumptions. Namely, when the data are not normal and the samples are small, the parametric chart is almost always slower than the nonparametric charts in detecting small to medium size shifts in the mean vector. Going back to Giri (1983, p.219), if "the use of an invariant decision rule conflicts violently with the desire to make a correct decision or have a small risk, it must be abandoned." Here, we would prefer the chart which would signal sooner that the process is out of control, even if this chart is not affine invariant. If, in addition, the nonparametric technique is also natural and easy to compute, the choice should be clear. In this dissertation, I intend to demonstrate the advantages the proposed multivariate nonparametric charts have to offer.

5.3 Proof that the control statistic of the sign based chart asymptotically has a chi-square distribution

In this section, we will show that the control statistic W_k of the sign based chart, when the process is in control and as $n \to \infty$, is asymptotically distributed as a χ_p^2 random variable, where p is the number of monitored parameters. Section 5.4 contains a similar proof for the control statistic of the signed rank based chart. In both cases, we make use of known theorems about the component sign and the component signed rank statistics.

Theorem 5.3.1 is taken from Hettmansperger (1984, p. 281). The proof of the theorem is omitted since it can be found in Hettmansperger's book. In order to show the asymptotic normality of the component sign statistic, Hettmansperger uses an alternative definition of the vector sign statistic, a form that has zero expectation under the null hypothesis. For $i = 1,\ldots,p$ and $j = 1,\ldots,n$, let

$$S_i^* = \sum_{j=1}^{n} I_{ij}^* = \#(X_{ij} > 0) - \#(X_{ij} < 0) \qquad (5.3.1)$$

where

$$I_{ij}^* = \begin{cases} +1 & \text{if } X_{ij} > 0 \\ 0 & \text{if } X_{ij} = 0 \\ -1 & \text{if } X_{ij} < 0 \end{cases} \tag{5.3.2}$$

Here the assumption is that no observations equal zero (true with probability one). We also need some additional notation. Let $\mathbf{x}' = (X_1, \dots, X_p)$ be a random vector with joint cdf $F(\cdot)$. Also, let

$$\begin{aligned} p_{ij}(0,0) &= P(X_i \le 0, X_j \le 0) \\ p_{ij}(0,1) &= P(X_i \le 0, X_j > 0) \\ p_{ij}(1,0) &= P(X_i > 0, X_j \le 0) \\ p_{ij}(1,1) &= P(X_i > 0, X_j > 0). \end{aligned} \tag{5.3.3}$$

Theorem 5.3.1

Under the null hypothesis $H_0 : \boldsymbol{\mu} = \mathbf{0}$, as $n \to \infty$

$$\frac{1}{\sqrt{n}} \mathbf{s}^* \xrightarrow{D} \mathbf{z} \sim MVN_p(\mathbf{0}, \mathbf{V}),$$

where $\mathbf{V} = ((v_{ij}))$, $i, j = 1, \dots, p$, and

$$v_{ii} = 1, \tag{5.3.4}$$

$$v_{ij} = p_{ij}(0,0) + p_{ij}(1,1) - p_{ij}(0,1) - p_{ij}(1,0) \blacklozenge \tag{5.3.5}$$

It is easy to show the relationship between the definition (5.3.1) of the sign statistic S_i^* used in Hettmansperger and the definition (4.3.2) of the sign statistic S_{ki} we use in this dissertation, where k denotes a sequence of statistics, k=1,2,.... It is as follows:

$$\begin{aligned} S_i^* &= \#(X_{ij} > 0) - \#(X_{ij} < 0) \\ &= \#(X_{ij} > 0) - \#(X_{ij} < 0) + \#(X_{ij} > 0) - \#(X_{ij} > 0) \\ &= 2 \cdot \#(X_{ij} > 0) - n \end{aligned}$$

$$= 2S_{ki} - n \tag{5.3.6}$$

Using result (5.3.6) and theorem 5.3.1, we conclude that, when the process is in control

$$\frac{1}{\sqrt{n}}(2\mathbf{s}_k - n\mathbf{1}) \xrightarrow{D} \mathbf{z} \sim MVN_p(\mathbf{0}, \mathbf{V}),$$

or

$$\frac{1}{\sqrt{n}}2\left(\mathbf{s}_k - \frac{n}{2}\mathbf{1}\right) \xrightarrow{D} \mathbf{z} \sim MVN_p(\mathbf{0}, \mathbf{V}).$$

Therefore,

$$\frac{1}{\sqrt{n}}\left(\mathbf{s}_k - \frac{n}{2}\mathbf{1}\right) \xrightarrow{D} \mathbf{z} \sim MVN_p(\mathbf{0}, \tfrac{1}{4}\mathbf{V}). \tag{5.3.7}$$

We observe that the term $n/2$ which appears on the left-hand side of (5.3.7) is the expected value of the sign statistic (4.3.2) under the null hypothesis. Also, under the null hypothesis, the expressions for the variances and covariances given in (5.3.4) and (5.3.5) in combination with (5.3.7) agree with our previous results given as (4.3.24) and (4.3.26). Moreover, if we form our control statistic W_k based on the vector

$$\mathbf{s}_k^0 = \frac{1}{\sqrt{n}}\left(\mathbf{s}_k - \frac{n}{2}\mathbf{1}\right), \tag{5.3.8}$$

instead of just the vector of signs $\mathbf{s}_k$, the result is exactly the same. To show this, we start by deriving the EWMA statistic using equation (4.3.27). Let $\mathbf{s}_k = \left[S_{k1}, \ldots, S_{kp}\right]$ and $\mathbf{s}_k^0 = \left[S_{k1}^0, \ldots, S_{kp}^0\right]$. Also, let Y_{ki} be the EWMA statistic based on the sign S_{ki} and Y_{ki}^0 be the EWMA statistic based on S_{ki}^0. By definition, for $k = 1,2,\ldots$ and $i = 1,\ldots,p$,

$$Y_{ki}^0 = (1-r)^k \; Y_{0i}^0 + \sum_{l=0}^{k-1}(1-r)^l \, r S_{(k-l)i}^0$$

where $$Y_{0i}^0 = \mathrm{E}(S_{ki}^0 | \mu_i = t_i).$$

Then,

$$Y_{ki}^0 = \frac{1}{\sqrt{n}}(1-r)^k \mathrm{E}\left(S_{ki} - \frac{n}{2} | \mu_i = t_i \right) + \frac{1}{\sqrt{n}} \sum_{l=0}^{k-1} (1-r)^l r \left(S_{(k-l)i} - \frac{n}{2} \right)$$

$$= \frac{1}{\sqrt{n}} \left\{ (1-r)^k \mathrm{E}\left(S_{ki} | \mu_i = t_i \right) - (1-r)^k \frac{n}{2} + \sum_{l=0}^{k-1} (1-r)^l r S_{(k-l)i} - \sum_{l=0}^{k-1} (1-r)^l r \frac{n}{2} \right\}$$

$$= \frac{1}{\sqrt{n}} \left\{ (1-r)^k \mathrm{E}\left(S_{ki} | \mu_i = t_i \right) + \sum_{l=0}^{k-1} (1-r)^l r S_{(k-l)i} - (1-r)^k \frac{n}{2} - \sum_{l=0}^{k-1} (1-r)^l r \frac{n}{2} \right\}$$

$$= \frac{1}{\sqrt{n}} \left\{ Y_{ki} - \left[(1-r)^k + r \frac{1-(1-r)^k}{1-(1-r)} \right] \frac{n}{2} \right\}$$

$$= \frac{1}{\sqrt{n}} \left\{ Y_{ki} - \frac{n}{2} \right\} \tag{5.3.9}$$

Now, using equations (4.3.33), (4.3.34), and (4.3.35) we derive the following:

$$\mathrm{E}(Y_{ki}^0) = \frac{1}{\sqrt{n}} \left\{ \mathrm{E}(Y_{ki}) - \frac{n}{2} \right\}$$

$$= \frac{1}{\sqrt{n}} \left\{ \frac{n}{2} - \frac{n}{2} \right\} = 0 \tag{5.3.10}$$

$$\mathrm{Var}(Y_{ki}^0) = \frac{1}{n} \{ \mathrm{Var}(Y_{ki}) \}$$

$$= \left(\frac{r}{2 - \mathrm{r}} \right) [1-(1-r)^{2k}] \ \frac{1}{4} \tag{5.3.11}$$

$$\mathrm{Cov}(Y_{ki}^0, Y_{ki'}^0) = \frac{1}{n} \{ \mathrm{Cov}(Y_{ki}, Y_{ki'}) \}$$

$$= \left(\frac{r}{2 - r} \right) [1-(1-r)^{2k}] (p_{ii'} - \frac{1}{4}). \tag{5.3.12}$$

Finally, following the same type of notation, we derive the control statistic W_k^0 as a quadratic form based on S_{ki}^0. Let $\mathbf{y}_k^0 = \left[Y_{k1}^0, \ldots, Y_{kp}^0\right]$, then:

$$W_k^0 = \mathbf{y}_k^{0\prime} \Sigma_{\mathbf{y}_k^0}^{-1} \mathbf{y}_k^0$$

$$= \frac{1}{\sqrt{n}}\left\{\mathbf{y}_k - \frac{n}{2}\right\}' \left(\frac{1}{n}\Sigma_{\mathbf{y}_k}\right)^{-1} \frac{1}{\sqrt{n}}\left\{\mathbf{y}_k - \frac{n}{2}\right\}$$

$$= \left\{\mathbf{y}_k - \frac{n}{2}\right\}' \Sigma_{\mathbf{y}_k}^{-1}\left\{\mathbf{y}_k - \frac{n}{2}\right\}$$

$$= W_k \tag{5.3.13}$$

So, the control statistic computed based on $\mathbf{s}_k^0$ is identical to the one computed based on the regular sign vector $\mathbf{s}_k$. Since $\mathbf{s}_k^0$ converges to a multivariate normal distribution, as given in (5.3.7), then the sign based control statistic W_k, as a quadratic form of $\mathbf{s}_k^0$, also converges in distribution . Using the theorem stated in Serfling (1980, p. 128) we conclude that, under the null hypothesis or if the process is in control, as $n \to \infty$

$$W_k \xrightarrow{D} \chi_p^2 \tag{5.3.14}$$

5.4 Proof that the control statistic of the signed rank based chart asymptotically has a chi-square distribution

For the component signed rank vector there exists a theorem similar to the one for the component sign (Hettmasperger, 1984, p. 283). The theorem is stated below, Theorem 5.4.1, again without proof. The symmetry assumption Hettmansperger imposes on the distribution of the data is the following: It the joint pdf $f(t_1, \ldots, t_p)$ satisfies

$$f(t_1,\ldots,t_p) = f(-t_1,\ldots,-t_p)$$

we say that f is *diagonally symmetric* about $\mathbf{0}$. This property results in symmetric marginal densities which is the assumption we stated in Chapter 4. Then the mean vector μ consists of the centers of symmetry of the marginal distributions. Suppose we want to test the hypothesis $H_0: \mu = \mathbf{0}$ versus $H_1: \mu \neq \mathbf{0}$. Let $\mathbf{t}' = (T_1,\ldots,T_p)$, where

$$T_i = \frac{2}{n+1}\left\{T_i^+ - \frac{n(n+1)}{4}\right\} \tag{5.4.1}$$

and T_i^+ is the signed rank statistic defined in (4.4.1). Observe that the term $n(n+1)/4$ that appears in (5.4.1) is the expected value of T_i^+ under the null hypothesis.

__Theorem 5.4.1__

Suppose $F(t_1,\ldots,t_p)$ has a density $f(t_1,\ldots,t_p)$, diagonally symmetric about $\mathbf{0}$. Then, as $n \to \infty$

$$\frac{1}{\sqrt{n}}\mathbf{t} \xrightarrow{D} \mathbf{z} \sim MVN_p(\mathbf{0}, \mathbf{V}),$$

where $\mathbf{V} = ((v_{ij}))$, $i, j = 1,\ldots,p$, and

$$v_{ii} = \frac{1}{3}, \tag{5.4.2}$$

$$v_{ij} = 4\int_{-\infty}^{\infty} F_i(u)F_j(v)\,dF_{ij}(u,v) - 1 \; \blacklozenge \tag{5.4.3}$$

Now, to all the above notation we must add the subscript k, since our test is sequential over time $k = 1,2,\ldots$. Let

$$\mathbf{t}_k^0 = \frac{1}{\sqrt{n}}\mathbf{t}_k \tag{5.4.4}$$

where $\mathbf{t}_k^0 = \left[T_{k1}^0,\ldots,T_{kp}^0\right]$ and $\mathbf{t}_k = \left[T_{k1},\ldots,T_{kp}\right]$. Then, using equation (5.4.1), we have:

$$\mathbf{t}_k^0 = \frac{1}{\sqrt{n}} \frac{2}{n+1} \left\{ \mathbf{t}_k^+ - \frac{n(n+1)}{4} \mathbf{1} \right\} \tag{5.4.5}$$

where $\mathbf{t}_k^+ = \left[T_{k1}^+, \ldots, T_{kp}^+ \right]$. Next we will show that if we form the control statistic W_k^0 based on the vector $\mathbf{t}_k^0$, the result is the same as forming W_k based on the vector $\mathbf{t}_k^+$. First we derive the EWMA statistic using equation (4.4.2). Let Y_{ki} be the EWMA statistic based on the signed rank T_{ki}^+ and Y_{ki}^0 be the EWMA statistic based on T_{ki}^0. By definition, for k=1,2,... and i=1,...,p,

$$Y_{ki}^0 = (1-r)^k Y_{0i}^0 + \sum_{j=0}^{k-1} (1-r)^j r T_{(k-j)i}^0$$

where
$$Y_{0i}^0 = \mathrm{E}(T_{ki}^+ | \mu_i = t_i)$$

Then,

$$\begin{aligned}
Y_{ki}^0 &= (1-r)^k \frac{1}{\sqrt{n}} \frac{2}{n+1} \left(\mathrm{E}\left(T_{ki}^+ | \mu_i = t_i\right) - \frac{n(n+1)}{4} \right) + \\
&\qquad + \sum_{j=0}^{k-1} (1-r)^j r \frac{1}{\sqrt{n}} \frac{2}{n+1} \left(T_{(k-j)i}^+ - \frac{n(n+1)}{4} \right) \\
&= \frac{1}{\sqrt{n}} \frac{2}{n+1} \left\{ (1-r)^k \mathrm{E}\left(T_{ki}^+ | \mu_i = t_i\right) - \frac{n(n+1)}{4} (1-r)^k + \right. \\
&\qquad \left. + \sum_{j=0}^{k-1} (1-r)^j r T_{(k-j)i}^+ - \frac{n(n+1)}{4} \sum_{j=0}^{k-1} (1-r)^j r \right\} \\
&= \frac{1}{\sqrt{n}} \frac{2}{n+1} \left\{ Y_{ki} - \frac{n(n+1)}{4} \left[(1-r)^k + r \frac{1-(1-r)^k}{1-(1-r)} \right] \right\} \\
&= \frac{1}{\sqrt{n}} \frac{2}{n+1} \left\{ Y_{ki} - \frac{n(n+1)}{4} \right\}
\end{aligned} \tag{5.4.6}$$

Now, using equations (4.4.5) and (5.4.6) we derive the following:

$$\mathrm{E}(Y_{ki}^0) = \frac{1}{\sqrt{n}}\frac{2}{n+1}\left\{\mathrm{E}(Y_{ki}) - \frac{n(n+1)}{4}\right\}$$

$$= \frac{1}{\sqrt{n}}\frac{2}{n+1}\left\{\frac{n(n+1)}{4} - \frac{n(n+1)}{4}\right\} = 0 \tag{5.4.7}$$

$$\mathrm{Var}(Y_{ki}^0) = \frac{1}{n}\left(\frac{2}{n+1}\right)^2 \mathrm{Var}(Y_{ki}) \tag{5.4.8}$$

$$\mathrm{Cov}(Y_{ki}^0, Y_{ki'}^0) = \frac{1}{n}\left(\frac{2}{n+1}\right)^2 \mathrm{Cov}(Y_{ki}, Y_{ki'}) \tag{5.4.9}$$

Finally, we calculate the control statistic based on T_{ki}^0:

$$W_k^0 = \mathbf{y}_k^{0\,\prime} \boldsymbol{\Sigma}_{\mathbf{y}_k^0}^{-1} \mathbf{y}_k^0$$

$$= \frac{1}{\sqrt{n}}\frac{2}{n+1}\left\{\mathbf{y}_k - \frac{n(n+1)}{4}\right\}' \left(\frac{1}{n}\left(\frac{2}{n+1}\right)^2 \boldsymbol{\Sigma}_{\mathbf{y}_k}\right)^{-1} \frac{1}{\sqrt{n}}\frac{2}{n+1}\left\{\mathbf{y}_k - \frac{n(n+1)}{4}\right\}$$

$$= \frac{1}{n}\left(\frac{2}{n+1}\right)^2 \left(\frac{1}{n}\left(\frac{2}{n+1}\right)^2\right)^{-1} \left\{\mathbf{y}_k - \frac{n(n+1)}{4}\right\}' \boldsymbol{\Sigma}_{\mathbf{y}_k}^{-1} \left\{\mathbf{y}_k - \frac{n(n+1)}{4}\right\}$$

$$= \left\{\mathbf{y}_k - \frac{n(n+1)}{4}\right\}' \boldsymbol{\Sigma}_{\mathbf{y}_k}^{-1} \left\{\mathbf{y}_k - \frac{n(n+1)}{4}\right\}$$

$$= W_k \tag{5.4.10}$$

So, the control statistic computed based on $\mathbf{t}_k^0$ is equal to the one computed based on the signed rank vector $\mathbf{t}_k^+$. Since $\mathbf{t}_k^0$ converges to a multivariate normal distribution, given in theorem 5.4.1, the signed rank based control statistic W_k, as a quadratic form of $\mathbf{t}_k^0$, also converges in distribution. Using the same theorem we used before from Serfling (1980, p. 128) we conclude that under the null hypothesis, as $n \to \infty$

$$W_k \xrightarrow{D} \chi_p^2. \tag{5.4.11}$$

CHAPTER 6

CHART COMPUTATIONS

6.1 Available computational methods

The following discussion pertains to the choice of a computational method that can be used to estimate the ARL of the nonparametric charts. Recall that successive values of the control statistic W_k, k=1,2,..., are not independent. When Σ_y is known, the distribution of W_k is a function of the stage k, similar to the parametric W_k described earlier (see equation (5.1.4)). When Σ_y is estimated from the data, the distribution of W_k changes also according to the estimate from stage to stage. Therefore, the exact distribution of the run length, being a function of W_k, is difficult to find. The usual methods for estimating the ARL are the *Markov chain method*, the *integral equation method*, and *simulation*. The Markov chain approach and the integral equation approach are not appropriate for the type of control chart we propose, because the process is not Markovian. Therefore, by elimination of the other techniques, simulation will be used for evaluation of the charts. Simulation is also the method chosen by Lowry et al (1992), which will facilitate comparisons with their MEWMA chart. Recently, Rigdon (1994a, 1994b) has shown that, besides simulation, an integral equation approach could be used for the MEWMA, but no similar result exists for the proposed nonparametric charts. Process simulation can be used as follows: the process is simulated a large number of times in order to determine an UCL for each chart, such that a prespecified in-control ARL is achieved. Using this UCL, the ARL is estimated for different shifts of the monitored parameter from its target value.

In the remainder of this chapter the general principle of simulation is stated and the performed simulations are described in detail. Only the case of two correlated variables was actually simulated but the results could be extended to the general multivariate case. The simulation results appear in Chapters 7 and 8.

6.2 Definition of simulation

Simulation, also called *Monte Carlo* method, in its simplest form is a way of computing an integral without knowing the exact form of the function to be integrated. The most common problem in statistics is finding the distribution of a statistic. If this problem is intractable, the next best thing is to estimate the moments of the unknown distribution. In order to use the simulation method, one needs an *algorithm* for computing the statistic from the sample and *random numbers*. What simulation amounts to is taking a very large sample from the distribution of the statistic of interest. It is known that the *empirical cumulative distribution function* (ecdf) converges to the true cdf (Thisted, 1988). The same holds true for any function of ecdf and cdf.

Suppose we want to compute E[g(X)], where X is a random variable with density f(x) on an interval [a, b]. The exact expectation requires the computation:

$$\theta = \mathrm{E}[g(X)] = \int_a^b g(x)\, f(x)\, dx \qquad (6.2.1)$$

If the exact form of f(x) is known but too complicated to directly perform the integration, a *quadrature* method (i.e. a method based on the Riemann sum) could be used. If the form of f(x) is unknown, one can estimate the E[g(X)] as follows:

$$\hat{\theta} = \hat{\mathrm{E}}[g(X)] = \frac{1}{R}\sum_{i=1}^{R} g(X_i) \qquad (6.2.2)$$

where $X_1,\ldots,X_R$ is a random sample of R observations from the distribution f(x). The estimated standard error of the estimate is obtained as

$$\hat{\mathrm{SE}}(\hat{\theta}) = \frac{1}{\sqrt{R}}\left(\frac{\sum_{i=1}^{R}[\mathrm{g}(X_i)-\hat{\theta}]^2}{R-1}\right)^{\frac{1}{2}} \tag{6.2.3}$$

Specifically, the quantity needed to be computed in quality control for chart evaluation is the ARL. Let

N = # of samples to signal for a particular shift

Then

$$\mathrm{ARL} \ = \ \mathrm{E}(N) = \sum_{j=0}^{\infty} j\mathrm{P}(N=j) \tag{6.2.4}$$

and the estimated ARL is given by

$$\mathrm{A\hat{R}L} \ = \frac{1}{R}\sum_{i=1}^{R} N_i \tag{6.2.5}$$

where $N_1,\ldots,N_R$ is a random sample from the distribution of the run length. If the standard deviation of the run length is estimated by

$$\mathrm{S\hat{R}L} = \left(\frac{\Sigma N_i^2 - \frac{(\Sigma N_i)^2}{R}}{R-1}\right)^{\frac{1}{2}}, \tag{6.2.6}$$

an estimate of the standard deviation of the $\mathrm{A\hat{R}L}$ due to simulation is

$$\mathrm{SA\hat{R}L} = \frac{1}{\sqrt{R}}\mathrm{S\hat{R}L} \tag{6.2.7}$$

Equations (6.2.5) and (6.2.7) were used in the final stages of the computations of the results listed in the tables of chapters 7 and 8.

6.3 Simulated distributions

The following three distributions were simulated: the multivariate normal distribution, a mixture of multivariate normal distributions, and the multivariate t(df) with independent variables. The multivariate normal distribution was used as the worst case scenario for the nonparametric charts, a case where a parametric technique is expected to outperform a corresponding nonparametric one. The other two distributions, namely the multivariate mixture normal and the multivariate t(df) distribution, were used to demonstrate cases where the proposed nonparametric charts outperform their parametric competitor.

Now, we introduce some notation. First, data were generated from a *multivariate normal distribution*:

$$\mathrm{MVN}_p(\mu, \Sigma) \qquad (6.3.1)$$

where p is the number of variables of the distribution.

In order to generate data from a distribution with tails heavier than the tails of a normal distribution, a mixture of two multivariate normal distributions with the same mean vector μ and the same correlation structure was used. The variance-covariance matrices differed by a factor b. The *multivariate mixture normal distribution* is usually denoted as:

$$\mathrm{MVN}_p^{\mathrm{m}} = (1-a)\,\mathrm{MVN}_p(\mu, \Sigma_1) + a\,\mathrm{MVN}_p(\mu, \Sigma_2) \qquad (6.3.2)$$

where a is a constant between 0 and 1. Moreover, the variance-covariance matrices Σ_1 and Σ_2 have similar structure, i.e.

$$\Sigma_2 = b \times \Sigma_1, \tag{6.3.3}$$

where b is a positive real number.

Finally, another distribution used to generate data was a *multivariate central t distribution* with df degrees of freedom and independent variables. This t distribution is symmetric around the mean vector $\boldsymbol{\mu} = \mathbf{0}$. Since the variables are independent, the variance of each variable of the multivariate t(df) distribution can be computed using the following expression (Hogg and Craig, 1978, p. 176), provided it exists (i.e., df >2).

$$\sigma^2 = \frac{df}{df-2}. \tag{6.3.4}$$

The above three distributions satisfy the statistical assumptions stated in Sections 4.1 and 4.2. Namely, they are absolutely continuous and have absolutely continuous and symmetric marginal distributions, and absolutely continuous bivariate marginal distributions. The marginal distributions of the multivariate normal distribution are univariate normal and, hence, symmetric. A marginal distribution of a mixture of two multivariate normal distributions with the same mean vector is the mixture of their respective marginals with a common marginal mean. Therefore the marginal distributions of a multivariate mixture normal distribution are symmetric. Finally, the marginal distributions of a multivariate central t(df) with uncorrelated variables are, by construction, univariate central t(df) distributions which are symmetric around zero.

6.4 Random number generation

Random samples from a *multivariate normal distribution* were generated using the FORTRAN subroutine RNMVN (IMSL STAT/LIBRARY, 1991, p. 1339). This subroutine is based on the *Cholesky factorization method*, also called *square root method*, which is done with the subroutine CHFAC (IMSL STAT/LIBRARY, 1991, p. 1456). According to this technique, p standard normal random variables are generated forming the vector **z** with distribution $\mathrm{MVN}_p(\mathbf{0},\mathbf{I})$. Using the transformation

$$\mathbf{x} = \mathbf{Az} + \boldsymbol{\mu} \qquad (6.4.1)$$

where **A** is a lower triangular matrix such that **AA'=Σ**, results in the vector of normal random variables

$$\mathbf{x} \sim \mathrm{MVN}_p(\boldsymbol{\mu},\boldsymbol{\Sigma}) \qquad (6.4.2)$$

Without loss of generality, the random samples generated by the FORTRAN subroutine RNMVN have mean vector $\boldsymbol{\mu} = \mathbf{0}$. Random samples from a multivariate normal distribution with mean other than zero can be generated by adding the vector of the desired means to the random samples generated by the subroutine.

Data from a *multivariate mixture normal distribution* were generated as follows: n samples were generated from a $\mathrm{MVN}_p(\boldsymbol{\mu},\boldsymbol{\Sigma}_1)$, n samples were generated from a $\mathrm{MVN}_p(\boldsymbol{\mu},\boldsymbol{\Sigma}_2)$, and n samples were generated from a uniform $u \sim \mathrm{U}(0,1)$. Each observation u was compared with the constant a of the mixture.

If $u \le a$, the observation from $\mathrm{MVN}_p(\boldsymbol{\mu},\boldsymbol{\Sigma}_2)$ was kept.

If $u > a$, the observation from $\mathrm{MVN}_p(\boldsymbol{\mu},\boldsymbol{\Sigma}_1)$ was kept.

This method was chosen because it results in fewer function calls during the execution of the simulation than generating the normal variables one-at-a-time. The multivariate normal random samples were generated using the FORTRAN subroutine RNMVN twice and the uniform random samples were generated using the FORTRAN subroutine RNUN (IMSL STAT/LIBRARY, 1991, p. 1269). As explained above, the resulting data have mean vector $\mu = \mathbf{0}$. Random samples from a mixture normal distribution with mean other than zero can be generated by adding the vector of the desired means to the random samples.

Random samples from a *multivariate t(df) distribution* were generated using the subroutine RNSTT (IMSL STAT/LIBRARY, 1991, p. 1322). This subroutine uses a method developed by Kinderman, Monahan, and Ramage (1977) which involves a representation of the t density as the sum of a triangular density over (-2, 2) and the difference of the triangular density and the t density. The mixing probabilities depend on the degrees of freedom of the t distribution. The subroutine RNSTT generates i.i.d. random vectors from a central univariate t(*df*) distribution which can be put together to construct a sample from a multivariate t(*df*) distribution with independent variables. The mean vector of this distribution is zero.

6.5 Simulation specifics

Simulation programs were written to compute the ARL of our proposed nonparametric charts based on the sign and the signed rank statistics, and to compute the ARL of the corresponding parametric chart based on the sample average (MEWMA). The following cases were explored with each statistic:

(i) Normal data, Σ is assumed known.

(ii) Mixture normal data, Σ is assumed known.

(iii) Data from a t(*df*) distribution, Σ is assumed known.

(iv) Normal data, Σ is not known.

(v) Mixture normal data, Σ is not known

All distributions listed above are multivariate, but only bivariate data were simulated. So, now we will restrict our discussion to the bivariate case.

Simulated data

The variance-covariance matrix Σ used to generate bivariate normal data was of the form

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \qquad (6.5.1)$$

where ρ is the correlation between the variables.

The variance-covariance matrices Σ_1 and Σ_2 needed to generate bivariate mixture normal data, with a prespecified pooled variance-covariance matrix Σ^m for the mixture, were determined as follows.

$$\begin{aligned} \Sigma^m &= (1-a)\,\Sigma_1 + a\,\Sigma_2 \\ &= (1-a)\,\Sigma_1 + ab\,\Sigma_1 \\ &= (1-a+ab)\,\Sigma_1 \end{aligned} \qquad (6.5.2)$$

Then,

$$\Sigma_1 = \frac{1}{(1-a+ab)}\Sigma^m \qquad (6.5.3)$$

and

$$\Sigma_2 = \frac{b}{(1-a+ab)}\Sigma^m. \qquad (6.5.4)$$

Moreover, in order to make the results comparable with the results from the bivariate normal distribution, the variance covariance matrix of the mixture was taken to be

$$\Sigma^m = \Sigma. \qquad (6.5.5)$$

Then, equations (6.5.3) and (6.5.4) become

$$\Sigma_1 = \frac{1}{(1-a+ab)}\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \qquad (6.5.6)$$

and

$$\Sigma_2 = \frac{b}{(1-a+ab)}\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}. \qquad (6.5.7)$$

So, mixture normal data were generated by mixing two bivariate normal distributions with variance-covariance matrices given by (6.5.6) and (6.5.7), respectively.

In order to generate data from a bivariate t(*df*) distribution with independent variables, two random samples were generated from a univariate t(*df*) distribution and were put together. The resulting data were divided by

$$\sigma = \sqrt{\frac{df}{df-2}}$$

because of equation (6.3.4). This standardization makes the results from the t distribution comparable with the results from a bivariate normal distribution with variance-covariance matrix

$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Parametric charts

For the parametric charts based on the vector sample average, the parameters of interest, besides the monitored mean vector, are the variances of the two variables and their covariance. When the variance-covariance matrix Σ was assumed to be known, it was used in the computations of the variance-covariance matrix of the EWMA vector $\mathbf{y}'_k = [\mathrm{Y}_{k1} \quad \mathrm{Y}_{k2}]$. When Σ was considered unknown, an estimate $\hat{\Sigma}_k$ based on the sample variances and covariance was computed at each sampling point k and the estimate was updated, element by element, as follows:

$$\overline{\hat{\Sigma}}_k = \frac{(k-1)\ \overline{\hat{\Sigma}}_{k-1} + \hat{\Sigma}_k}{k} \qquad (6.5.8)$$

Bivariate normal data and mixture bivariate normal data were simulated for both cases, i.e. known Σ and unknown Σ. For the t(*df*) distribution, we simulated only the known Σ case.

Nonparametric charts

For both nonparametric charts, based on the sign and the signed rank statistics, in addition to the monitored parameters the only other quantity of interest is the proportion

p_{12} = P(X_1>0 and X_2>0). Recall that we construct a chart assuming the process is in control. When the process is in control the proportions p_1 and p_2 are always equal to 1/2. When the multivariate normal distribution was simulated and the variance-covariance matrix $\boldsymbol{\Sigma}$ was assumed known, the probability p_{12} was computed using the FORTRAN subroutine BNRDF (IMSL STAT/LIBRARY, 1991, p. 1216), which computes probabilities for the bivariate normal distribution. When a mixture of normal distributions was used and the two variance-covariance matrices were considered known, the value of p_{12} was computed using BNRDF, since both distributions $\mathrm{MVN}_2(\mathbf{0},\boldsymbol{\Sigma}_1)$ and $\mathrm{MVN}_2(\mathbf{0},\boldsymbol{\Sigma}_2)$ have the same probability P(X_1>0 and X_2>0). When the t(*df*) distribution was used and the parameters were assumed known, the probability P(X_1>0 and X_2>0) was always 1/4 by construction, because of the independence of the two variables.

A bivariate normal distribution and a mixture bivariate normal distribution were used to illustrate the case of unknown $\boldsymbol{\Sigma}$. When the variance-covariance matrix of either distribution was unknown, p_{12} was initially estimated as described in section 4.5: by averaging the proportion of points (x_1,x_2) where $\{x_1 > 0 \text{ and } x_2 > 0\}$ with the proportion of points (x_1,x_2) where $\{x_1 < 0 \text{ and } x_2 < 0\}$, since both proportions estimate p_{12} (see equation (4.3.14)). This estimator is defined in equations (4.5.1) and (4.5.2). For both distributions, when p_{12} was estimated from the data, the variance-covariance matrix of the EWMA vector was often becoming ill-conditioned, which could result in unstable ARL values. The problem was solved by assigning an initial value to the proportion p_{12}, namely $p_{12,0}$ = $1/4$ (which is true when X_1 and X_2 are uncorrelated). This did not completely solve the problem of ill-conditioning, so a somewhat ad hoc constraint was used as follows:

If $\bar{\hat{p}}_{12,k} > 0.48$ and $\hat{p}_{12,k} > 0.45$

then set $\hat{p}_{12,k} = 0.45$ and compute $\bar{\hat{p}}_{12,k}$ again. (6.5.9)

The reason for this constraint, with some modification, is based on equation (4.3.15): although a sample estimate of p_{12} could be any number in the interval [0, 1], a reasonable estimate must be constrained by equation (4.3.15). Constraint (6.5.9) prevents the estimator from being too close to 0.5. This constraint is enough, because we simulate distributions with positive correlations. To simulate distributions with negative correlations, we need a constraint to prevent the estimator from being too close to 0. Since positive and negative correlations lead to symmetric results, we only need results for one of the two cases.

In addition to our proposed estimator, which was modified and named Two-quadrant estimator, three other estimators were used for comparison. All four estimators are defined below, for k =1,2,... and sample size n.

The *One-quadrant estimator* is defined as

$$\hat{p}^{o}_{12,k} = \frac{\#\text{ of points}(x_1, x_2)\text{ where }(x_1 > t_1,\, x_2 > t_2)}{n} \tag{6.5.10}$$

where the estimate is updated as follows

$$\bar{\hat{p}}^{o}_{12,k} = \begin{cases} \dfrac{1}{4} & \text{for } k = 0 \\ \dfrac{k\,\bar{\hat{p}}^{o}_{12,k-1} + \hat{p}^{o}_{12,k}}{k+1} & \text{for } k \geq 1 \end{cases} \tag{6.5.11}$$

and is constrained by condition (6.5.9).

The *Two-quadrant estimator*, which is our proposed estimator, is defined as

$$\hat{p}^{T}_{12,k} = \frac{\#\text{ of points}(x_1, x_2)\text{ where }(x_1 > t_1,\, x_2 > t_2)\text{ or }(x_1 < t_1,\, x_2 < t_2)}{2n} \tag{6.5.12}$$

and the estimate is updated as follows

$$\bar{\hat{p}}^{T}_{12,k} = \begin{cases} \dfrac{1}{4} & \text{for } k = 0 \\ \dfrac{k\,\bar{\hat{p}}^{T}_{112,k-1} + \hat{p}^{T}_{12,k}}{k+1} & \text{for } k \geq 1. \end{cases} \tag{6.5.13}$$

The estimate is constrained by condition (6.5.9).

The *Four-quadrant estimator*, based on equation (5.3.5) given in Hettmansperger (1984), estimates not just p_{12} but $(p_{12} - 1/4)$ and is defined as

$$\hat{p}^{F}_{12,k} = \frac{1}{4n}\big[\{\#\text{ of points}(x_1, x_2)\text{ where }(x_1 > t_1,\, x_2 > t_2)\text{ or }(x_1 < t_1,\, x_2 < t_2)\}$$

$$-\{\#\text{ of points}(x_1, x_2)\text{ where }(x_1 > t_1,\, x_2 < t_2)\text{ or }(x_1 < t_1,\, x_2 > t_2)\}\big] \tag{6.5.14}$$

The estimate is updated as follows

$$\bar{\hat{p}}^{F}_{12,k} = \begin{cases} 0 & \text{for } k = 0 \\ \dfrac{k\,\bar{\hat{p}}^{F}_{12,k-1} + \hat{p}^{F}_{12,k}}{k+1} & \text{for } k \geq 1 \end{cases} \tag{6.5.15}$$

and is constrained by the following rule:

If $\bar{\hat{p}}^{F}_{12,k} > 0.24$ and $\hat{p}^{F}_{12,k} > 0.23$

then set $\hat{p}^{F}_{12,k} = 0.23$ and compute $\bar{\hat{p}}^{F}_{12,k}$ again. (6.5.16)

The *Constant estimator* is defined as

$$\hat{p}^{C}_{12,k} = \frac{1}{4} \quad \text{for all } k, \tag{6.5.17}$$

i.e. it uses the value of p_{12} that corresponds to the case of independent variables.

In short, the four estimators use a different degree of sample information. The Constant estimator uses no sample information at all. Among the other three, the One-quadrant estimator uses the least amount of sample information, whereas the Four-quadrant estimator uses the most. The Two-quadrant estimator and the Four-quadrant estimator are expected to be equivalent.

Control limit determination

Since the control statistic in all these charts is of a quadratic form, only one control limit, namely the UCL, was determined with simulation. There is no LCL. Now, let ARL(UCL) denote the ARL as a function of UCL. The goal was to find an UCL such that $\hat{\text{ARL}}(\text{UCL}) \approx 200$ when the process is in control, because all charts have to have approximately equal in-control ARL to be comparable. For most performed simulations the estimated standard deviation of the run length, $\hat{\text{SRL}}$, was very close to 200. All simulations are based on 1000 runs. This means that the estimated standard error of the estimate of the in-control ARL, $\hat{\text{SARL}}$, in most cases is about $200/(\sqrt{1000}) \cong 200/32 = 6.3$. A UCL value is considered satisfactory if

$$198 \leq \hat{\text{ARL}}(\text{UCL}) \leq 202 \qquad (6.5.18)$$

In order to determine the UCL, the UCL was assigned an initial value c_1. Random samples were generated, as described earlier, and the control statistic was computed for k=1,2,... until the first k for which $W_k > c_1$. This k was the value of the run length N for this run. The process was repeated 1000 times and the average run length when UCL=c_1,

$\hat{ARL}(c_1)$, was computed. If $\hat{ARL}(c_1)$ was not between 198 and 202, UCL was assigned another value c_2, and following exactly the same procedure the $\hat{ARL}(c_2)$ was computed. The procedure was repeated by adjusting the values until, for some c_j, an $\hat{ARL}(c_j)$ was found that satisfied the inequality (6.5.18). Then, the conclusion was that UCL=c_j. The UCL was determined with the help of a graph (see figure 6.5.1). The sequence of constants $c_1, c_2, \ldots, c_j$ was determined initially by incrementing or decrementing the previous constant and later by linearly interpolating the previous two constants, according to the following algorithm.

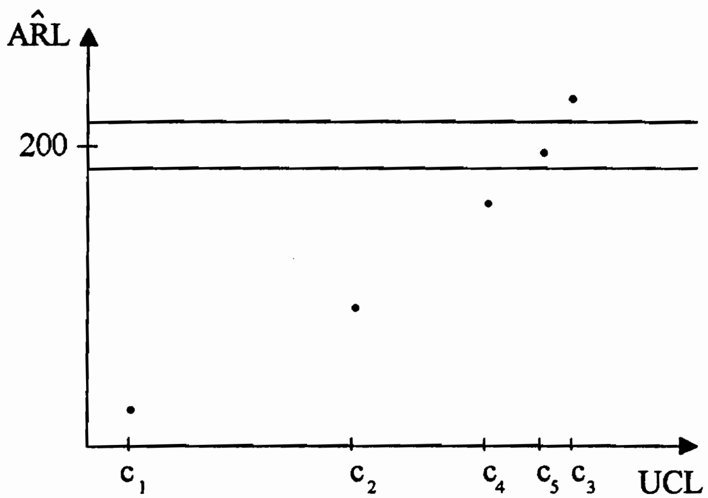

Figure 6.5.1: Determination of UCL

Algorithm

Find two values c_{i-1}, c_i that bracket the UCL.

$\hat{ARL}(c_0) := 200$

choose c_1

compute $\hat{\text{ARL}}(c_1)$

for i**:=1 to** ∞ **until** $\left[\hat{\text{ARL}}(c_{i-1}) - 200\right] \cdot \left[\hat{\text{ARL}}(c_i) - 200\right] < 0$

  if $\hat{\text{ARL}}(c_i)$<200 **then** $c_{i+1} = c_i + 1$

  else $c_{i+1} = c_i - 1$

  compute $\hat{\text{ARL}}(c_{i+1})$

Determine whether $c_{i-1} < c_i$ *or* $c_i < c_{i-1}$.

if $\hat{\text{ARL}}(c_i)$<200 **then**

  L:= c_i

  U:= c_{i-1}

else L:= c_{i-1}

  U:= c_i

FL:= $\hat{\text{ARL}}(\text{L})$

FU:= $\hat{\text{ARL}}(\text{U})$

Find the UCL using linear interpolation.

for j**:=1 to** ∞ **until** $198 \leq \hat{\text{ARL}}(c_j) \leq 202$

  $c_{j+1} := \left[\text{L} \cdot \text{FU} - \text{U} \cdot \text{FL} + (\text{U} - \text{L}) \cdot 200\right] / (\text{FU} - \text{FL})$

  compute $\hat{\text{ARL}}(c_{j+1})$

  if $\left[\hat{\text{ARL}}(c_{j+1}) - 200\right] \cdot \left[\text{FL} - 200\right] < 0$ **then**

    U:= c_{j+1}

    FU:= $\hat{\text{ARL}}(\text{U})$

  else L:= c_{j+1}

    FL:= $\hat{\text{ARL}}(\text{L})$

Out-of-control ARL

In order to compute the ARL for different shifts, a vector δ was added to the stream of the simulated data. To make the charts easy to compare, all charts were parameterized with respect to the noncentrality parameter λ. For normal data, the ARL of the parametric chart (MEWMA) is a function of λ only, as discussed in Section 5.1. The ARLs of our proposed nonparametric charts are functions of λ as well as the direction a shift occurs. The result is that the ARL of the nonparametric charts changes for different directions of shift, whereas the ARL of the $\overline{X}$-based chart remains the same.

Let

δ_1 = shift in the mean of the variable X_1,

δ_2 = shift in the mean of the variable X_2.

According to the definition of λ given in Section 5.1 for normal data, we can write equation (5.1.4) as follows:

$$\lambda^2 = \delta' \Sigma^{-1} \delta \qquad (6.5.19)$$

or, for bivariate normal data,

$$\lambda^2 = \begin{bmatrix} \delta_1 & \delta_2 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_1\sigma_2\rho & \sigma_2^2 \end{bmatrix}^{-1} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$

$$= \frac{\delta_1^2}{\sigma_1^2(1-\rho^2)} + \frac{\delta_2^2}{\sigma_2^2(1-\rho^2)} - \frac{2\delta_1\delta_2\rho}{\sigma_1\sigma_2(1-\rho^2)}. \qquad (6.5.20)$$

For a given λ there is a variety of combinations of shifts in the two means, for example δ and δ_o, that satisfy the condition

$$\delta'\Sigma^{-1}\delta = \delta_o'\Sigma^{-1}\delta_o \tag{6.5.21}$$

but only three types of shifts deemed most meaningful are described below.

Now, let us express the shifts in the means of the two variables in terms of their respective standard deviations. Let

$$\delta_1 = c_1\sigma_1 \tag{6.5.22}$$

$$\delta_2 = c_2\sigma_2, \tag{6.5.23}$$

where c_1 and c_2 represent standardized shifts in the means of variables X_1 and X_2, respectively. By substituting $c_1\sigma_1$ for δ_1 and $c_2\sigma_2$ for δ_2 from equations (6.5.22) and (6.5.23) in equation (6.5.20), we have

$$\lambda^2 = \frac{c_1^2}{(1-\rho^2)} + \frac{c_2^2}{(1-\rho^2)} - \frac{2c_1c_2\rho}{(1-\rho^2)}, \tag{6.5.24}$$

or

$$c_1^2 + c_2^2 - 2\rho c_1 c_2 = \lambda^2(1-\rho^2). \tag{6.5.25}$$

We will consider a shift from target in one variable only, an equal size shift (in standard deviations) from target in both variables, and an equal size shift (in standard deviations) above target in one variable and below target in the other variable. These three types of shift are shown in Figure (6.5.2). For simplicity, the following shortcut expressions were used:

Shift along direction (1): $c_1 = 0,\ c_2 = c$ $\Leftrightarrow$ *one variable* shift.

Shift along direction (2): $c_1 = c_2 = c^*$ $\Leftrightarrow$ *equal* shift

Shift along direction (3): $c_1 = -c_2 = c^{**}$ $\Leftrightarrow$ *opposite* shift

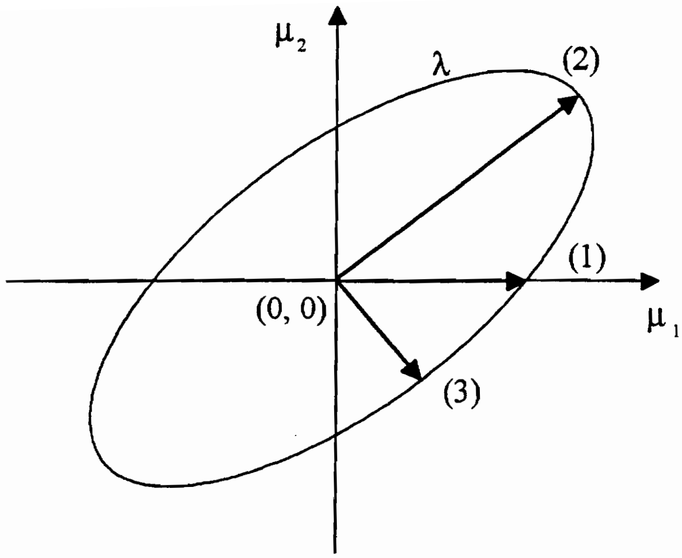

Figure 6.5.2: Simulated directions of shift (1) one variable shift, (2) equal shift, (3) opposite shift.

Using equation (6.5.25) we can compute the standardized shifts in each case:

(1) $c_1 = 0$ $\quad c_2 = \lambda\sqrt{1-\rho^2}$

(2) $c_1 = \lambda\sqrt{\frac{1+\rho}{2}}$ $\quad c_2 = \lambda\sqrt{\frac{1+\rho}{2}}$

(3) $c_1 = \lambda\sqrt{\frac{1-\rho}{2}}$ $\quad c_2 = -\lambda\sqrt{\frac{1-\rho}{2}}$

The three types of shifts $\mathbf{c} = (c_1, \ c_2)$, corresponding to the same value of λ, are listed in Table (6.5.1). The 18 pairs of shifts listed, one set for each value of the correlation ρ, were added to the stream of the data and the ARL was computed for each pair. The same sets of shifts were used with each simulated distribution, because each distribution was standardized with respect to variance. Since all variance-covariance matrices Σ of the simulated data are of the form shown in equation (6.5.1), i.e. the variables have variance 1, the standardized shifts $\mathbf{c} = (c_1, \quad c_2)$ listed in Table (6.5.1) coincide with the actual shifts $\delta = (\delta_1, \quad \delta_2)$, because of equations (6.5.22) and (6.5.23).

Table 6.5.1: List of simulated shifts.

$\rho = 0$	one var shift		equal shift		opposite shift	
λ	c_1	c_2	c_1	c_2	c_1	c_2
0.5	0	0.5	0.35355	0.35355	0.35355	-0.35355
1.0	0	1.0	0.70711	0.70711	0.70711	-0.70711
1.5	0	1.5	1.06066	1.06066	1.06066	-1.06066
2.0	0	2.0	1.41421	1.41421	1.41421	-1.41421
2.5	0	2.5	1.76777	1.76777	1.76777	-1.76777
3.0	0	3.0	2.12132	2.12132	2.12132	-2.12132

$\rho = 0.5$	one var shift		equal shift		opposite shift	
λ	c_1	c_2	c_1	c_2	c_1	c_2
0.5	0	0.43301	0.43301	0.43301	0.25	-0.25
1.0	0	0.86603	0.86603	0.86603	0.50	-0.50
1.5	0	1.29904	1.29904	1.29904	0.75	-0.75
2.0	0	1.73205	1.73205	1.73205	1.00	-1.00
2.5	0	2.16506	2.16506	2.16506	1.25	-1.25
3.0	0	2.59808	2.59808	2.59808	1.50	-1.50

$\rho = 0.9$	one var shift		equal shift		opposite shift	
λ	c_1	c_2	c_1	c_2	c_1	c_2
0.5	0	0.21794	0.48734	0.48734	0.11180	-0.11180
1.0	0	0.43589	0.97468	0.97468	0.22361	-0.22361
1.5	0	0.65383	1.46202	1.46202	0.33541	-0.33541
2.0	0	0.87178	1.94936	1.94936	0.44722	-0.44722
2.5	0	1.08972	2.43670	2.43670	0.55902	-0.55902
3.0	0	1.30767	2.92404	2.92404	0.67082	-0.67082

In literature, it is fairly common to present results according to the distribution of the statistic $\overline{X}$, instead of the distribution of the data. Let $\tilde{\lambda}$ be the noncentrality parameter of the distribution of $\overline{X}$. If we standardize $\tilde{\lambda}$ as we did with λ, we will arrive at equation (6.5.24) and the same sets of values of c_1 and c_2 listed in Table (6.5.1). The actual shifts, added to the data in order to compute the out-of-control ARL values, would now be given by the following equations:

$$\tilde{\delta}_1 = c_1\left(\sigma_1 / \sqrt{n}\right) \tag{6.5.26}$$

$$\tilde{\delta}_2 = c_2\left(\sigma_2 / \sqrt{n}\right) \tag{6.5.27}$$

Since $\tilde{\delta}_1$ and $\tilde{\delta}_2$ are smaller shifts than δ_1 and δ_2 for the same value of the noncentrality parameter, by using $\tilde{\delta}_1$ and $\tilde{\delta}_2$ we would, in effect, change the scale of shifts we study. But more than that, if we use δ_1 and δ_2 we can determine the effect of the sample size n on the charts, for a given shift. If we use $\tilde{\delta}_1$ and $\tilde{\delta}_2$, we can determine the effect of n relative to $(\sigma/\sqrt{n})$. Both approaches are advantageous. In the following chapters, we follow the first approach in Tables 7.1, 7.2, and 7.3, and the second approach in Table 7.4 and all tables of Chapter 8.

CHAPTER 7

NUMERICAL RESULTS FOR THE CASE OF KNOWN PARAMETERS

In this chapter we study the behavior of the bivariate EWMA charts based on the sign and signed rank statistics, as compared with the parametric competitor based on the sample average (MEWMA). The simulation results are presented in Tables 7.1, 7.2, and 7.3 for bivariate normal data, bivariate mixture normal data, and bivariate data from a t distribution, respectively. They use the noncentrality parameter λ of the data distribution and shifts δ_1 and δ_2, as defined in equations (6.5.22) and (6.5.23). Table 7.4 shows a few cases of the bivariate mixture normal distribution, also depicted in Table 7.2, using the noncentrality parameter $\tilde{\lambda}$ of the distribution of $\overline{X}$ and shifts $\tilde{\delta}_1$ and $\tilde{\delta}_2$, defined in equations (6.5.26) and (6.5.27). As it was mentioned earlier, this change in scale allows comparisons of the charts in smaller shift sizes and helps determine the effect of n relative to $(\sigma/\sqrt{n})$.

The values of the different parameters used in the computations were as follows.

For the normal distribution:	$\rho = 0, 0.5$, and 0.9
For the mixture normal distribution :	$a = 10\%$ and $b = 25$,
	$\rho = 0, 0.5$, and 0.9
For the t distribution:	$df = 3$ and $\rho = 0$.

For all distributions, the sample size n and the EWMA parameter r were given the values:

$$n = 5, 10, \text{ and } 15$$

$$r = 0.1, 0.2, \text{ and } 0.3.$$

All combinations of the above parameters were used in the computations, except for sample size 15 with the t distribution.

Now we will discuss the tables, first with respect to the behavior of the average run length function and then with respect to the upper control limits of the charts.

Average Run Length (ARL)

Table 7.1 depicts the worst case scenario for a nonparametric procedure, i.e. when the data are normal. Both nonparametric charts are slower than the parametric chart, from one and a half times to three or four times slower. The nonparametric out-of-control ARL values tend to increase as ρ increases, particularly for correlation 0.9. A quick glance at the charts reveals the following: all out-of-control ARL values of all the charts increase as the EWMA parameter r increases and decrease as n increases. Now, we will look at the charts in detail. First of all, the $\overline{X}$ based chart appears more stable than the signed rank based chart and the sign based chart, because the out-of-control ARL is the same for all shifts corresponding to the same λ and is independent of ρ. The ARL values of the nonparametric charts do not. Both nonparametric charts show less variability in the ARLs in different directions when the correlation is small. When the correlation is zero, the nonparametric charts have almost the same out-of-control ARL values for the three directions of shift studied. As the correlation increases, the ARL values become somewhat different at $\rho = 0.5$ and quite different at $\rho = 0.9$. The differences become larger as r increases, and smaller as n increases.

Table 7.2 shows the ARL values of mixture normal data with 10% contamination and known variances and covariances. In general, we make the same observation we made in the normal distribution, that is all out-of-control ARL values increase as r increases and decrease as n increases. The signed rank based chart is now faster than the $\overline{X}$ based chart in detecting small to moderate size shifts (up to $\lambda \cong 1$), depending on the correlation (larger correlation makes the signed rank chart a little bit slower) and on the EWMA parameter r for fixed n (larger r makes the signed rank chart much faster than the $\overline{X}$ based chart). The sign based chart is faster than the $\overline{X}$ based chart in detecting shifts up to $\lambda \cong 1.5$ when the correlation is low, up to $\lambda \cong 1$ when the correlation is moderate, and only up to $\lambda \cong 0.5$ when the correlation is high. So, the performance of the sign based chart depends on the correlation. Overall, the out-of-control ARL values of the nonparametric charts for the mixture normal data do not vary a lot with the direction of the shift.

Table 7.3 contains the ARL values of data from a t(3) distribution. The nonparametric charts are faster than the parametric chart in detecting shifts in the mean up to $\lambda \cong 1$. For fixed n, all charts are faster when $r = 0.1$. As r increases, the out-of-control ARL's of the nonparametric charts increase moderately (for example 5.4 becomes 6.0), but the out-of-control ARLs of the parametric chart increase up to two and a half times. As n increases all ARL values decrease. Since the correlation is always zero, no comparisons can be made with respect to ρ.

Table 7.4 contains a few of the cases shown in Table 7.2, for a 10% mixture normal distribution, but with different parameterization. Again, all charts are faster when $r = 0.1$. Now, because of the change in scale we observe that for $r = 0.1$ the signed rank based chart is faster than the $\overline{X}$ based chart up to $\tilde{\lambda} \cong 2.0$, when n is 5, and up to $\tilde{\lambda} \cong 2.5$, when n is 10. The sign based chart is faster than the $\overline{X}$ based chart up to $\tilde{\lambda} \cong 2.5$, when n is 5, and up to $\tilde{\lambda} \cong 3.0$, when n is 10. In Table 7.2 we observed that all ARL values of the charts decrease significantly as n increases, for all shift sizes. In Table 7.4, by comparison, when n increases from 5 to 10, the ARLs of the $\overline{X}$ based chart decrease moderately for shifts up to $\tilde{\lambda} \cong 1.0$, whereas the ARLs of the nonparametric charts decrease much less but for a wider range of shifts. In general, we can say that the effect of n on the chart efficiency with respect to σ is much more pronounced than the effect of n with respect to $\sigma/\sqrt{n}$, as expected.

Upper Control Limits (UCL)

All UCLs shown in Tables 7.1, 7.2, and 7.3 are also listed in Tables 7.5, 7.6, and 7.7 by chart type, for easier inspection. A quick glance at the UCL tables reveals the following: all UCL values of all the charts increase as the EWMA parameter r increases, especially the UCL of the $\overline{X}$ based chart.

For the normal data, as n increases, the UCLs for the $\overline{X}$ based chart are the same except for simulation error, but they may increase or decrease for the nonparametric charts depending on ρ and r. When r is 0.1, which is of particular interest since all charts are most efficient at that value, the UCLs of all charts remain about the same for different sample sizes. The UCL values remain the same for all correlations for the $\overline{X}$ based chart, as expected from theory. The UCLs of the signed rank based chart decrease slightly from $\rho = 0.0$ to $\rho = 0.5$, but more so at $\rho = 0.9$. The UCL values of the sign based chart increase slightly as ρ increases.

For the mixture normal data, the UCL values of the $\overline{X}$ based chart remain the same as ρ increases and decrease as n increases. The UCL values of the signed rank based chart decrease at high correlations and slightly increase as n increases. When r is 0.1, n has the least effect on the UCLs of the signed rank based chart. The UCL values of the sign based chart increase slightly as the correlation increases and remain about the same as n increases.

For the data from the t(3) distribution, the UCL values of the $\overline{X}$ based chart decrease as n increases, whereas the UCL values of the nonparametric charts tend to increase. Since we have only values for ρ=0.0, we cannot make comparisons with respect to correlation.

Another helpful comparison is of the UCL behavior of each chart when the distribution changes. The $\overline{X}$ based chart is obviously sensitive to changes in the distribution of the data. So, the UCLs of the $\overline{X}$ based chart increase significantly for the two nonnormal distributions, more so when n is small. On the other hand, the UCLs of both nonparametric charts change slightly or not at all for the various distributions, exhibiting robustness as one would expect. It seems that the UCLs of the nonparametric charts depend on the data distribution only through the correlation ρ.

Conclusions

The overall conclusions that can be drawn from these simulations are the following: the parametric chart ($\overline{X}$ based) is faster than the nonparametric charts (the sign and the signed rank based) in detecting shifts of any size when the data have a normal distribution. When the data have a 10% contaminated normal distribution or a t(3) distribution, the nonparametric charts are faster than the parametric chart in detecting small to medium size shifts of the means from their target values. These results show that, for nonnormal data distributions similar to the ones we studied, a nonparametric chart would be more efficient than a parametric chart in detecting smaller size shifts. The nonparametric charts, as well as the parametric competitor, are more efficient in detecting shifts of all sizes when r is 0.1 than when r is larger.

A comparison between the two nonparametric charts concerns the differences in the ARL values among the three types of shifts presented. Both nonparametric charts exhibit this behavior at high correlations, but the sign based chart seems to be affected the most. This behavior is more pronounced when the correlation is near 1. This suggests that, if we want to have the same ARL for different directions of shift and the variables may be highly correlated, then it would be more appropriate to use the signed rank based chart instead of the sign based chart. Of course, if ρ is large, depending on whether the means move together or independently, it may be sufficient and more economical to monitor only one variable.

Finally, we observed that the UCLs of the nonparametric charts are not affected by changes in the data distribution, whereas they are somewhat affected by changes in correlation and, to a lesser degree, by changes in the sample size.

Table 7.1: ARL values of EWMA charts for bivariate normal data with known Σ

$n = 5, \quad r = 0.1$

ρ = 0.0		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.655	6.5354	199.370	6.4400	198.623	6.3077
0.5	one var	6.513	0.1315	8.246	0.1555	10.126	0.2016
	equal	6.516	0.1350	7.768	0.1405	9.813	0.1942
	opposite	6.288	0.1300	7.764	0.1519	9.584	0.1837
1.0	one var	2.145	0.0358	3.565	0.0379	3.784	0.0533
	equal	2.204	0.0352	3.098	0.0356	3.509	0.0536
	opposite	2.195	0.0346	3.106	0.0370	3.606	0.0552
1.5	one var	1.293	0.0158	2.717	0.0185	2.614	0.0276
	equal	1.300	0.0162	2.188	0.0131	2.203	0.0275
	opposite	1.325	0.0165	2.192	0.0139	2.189	0.0275
2.0	one var	1.058	0.0074	2.576	0.0159	2.147	0.0173
	equal	1.054	0.0072	2.021	0.0045	1.618	0.0197
	opposite	1.052	0.0070	2.017	0.0041	1.635	0.0199
2.5	one var	1.004	0.0020	2.504	0.0158	1.978	0.0101
	equal	1.000	0.0000	2.001	0.0010	1.321	0.0152
	opposite	1.005	0.0022	2.000	0.0000	1.351	0.0155
3.0	one var	1.000	0.0000	2.480	0.0158	1.942	0.0088
	equal	1.000	0.0000	2.000	0.0000	1.149	0.0113
	opposite	1.000	0.0000	2.000	0.0000	1.158	0.0115
		UCL = 8.756		UCL = 8.535		UCL = 8.735	

$n = 5, \quad r = 0.1$

ρ = 0.5		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.655	6.5354	199.619	6.8232	200.924	6.6586
0.5	one var	6.513	0.1315	9.048	0.1703	11.546	0.2471
	equal	6.371	0.1286	7.425	0.1353	8.941	0.1741
	opposite	6.263	0.1218	9.315	0.1774	12.003	0.2642
1.0	one var	2.162	0.0357	3.623	0.0459	4.118	0.0648
	equal	2.154	0.0340	2.920	0.0312	3.455	0.0449
	opposite	2.184	0.0340	3.334	0.0502	4.084	0.0758
1.5	one var	1.307	0.0161	2.552	0.0244	2.643	0.0353
	equal	1.308	0.0165	2.208	0.0134	2.365	0.0197
	opposite	1.326	0.0168	2.065	0.0253	2.391	0.0393
2.0	one var	1.062	0.0076	2.285	0.0198	2.143	0.0214
	equal	1.059	0.0075	2.016	0.0040	2.058	0.0078
	opposite	1.062	0.0078	1.605	0.0181	1.652	0.0237
2.5	one var	1.004	0.0020	2.173	0.0182	1.914	0.0160
	equal	1.001	0.0010	2.000	0.0000	2.007	0.0026
	opposite	1.000	0.0000	1.334	0.0152	1.315	0.0175
3.0	one var	1.000	0.0000	2.127	0.0172	1.861	0.0135
	equal	1.000	0.0000	2.000	0.0000	2.000	0.0000
	opposite	1.000	0.0000	1.166	0.0119	1.132	0.0111
		UCL = 8.756		UCL = 8.342		UCL = 8.683	

Table 7.1(continued): ARL values of EWMA charts for bivariate normal data with known Σ

$n = 5, \quad r = 0.1$

$\rho = 0.9$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.655	6.5354	199.520	6.9650	199.047	6.1697
0.5	one var	6.513	0.1315	15.239	0.3268	21.929	0.5421
	equal	6.298	0.1238	6.873	0.1284	9.506	0.1776
	opposite	6.152	0.1232	16.036	0.3604	22.935	0.5963
1.0	one var	2.173	0.0354	5.068	0.0897	7.198	0.1464
	equal	2.142	0.0332	2.981	0.0327	3.747	0.0511
	opposite	2.206	0.0355	5.108	0.0888	7.527	0.1587
1.5	one var	1.309	0.0161	2.931	0.0445	3.897	0.0716
	equal	1.328	0.0170	2.238	0.0155	2.523	0.0228
	opposite	1.317	0.0163	2.835	0.0469	4.149	0.0836
2.0	one var	1.057	0.0073	2.127	0.0275	2.726	0.0451
	equal	1.059	0.0076	2.042	0.0063	2.154	0.0123
	opposite	1.059	0.0075	1.999	0.0310	2.718	0.0503
2.5	one var	1.006	0.0024	1.770	0.0211	2.219	0.0343
	equal	1.002	0.0014	2.004	0.0020	2.035	0.0058
	opposite	1.002	0.0014	1.509	0.0200	2.061	0.0350
3.0	one var	1.000	0.0000	1.624	0.0186	1.901	0.0253
	equal	1.000	0.0000	2.000	0.0000	2.006	0.0024
	opposite	1.000	0.0000	1.303	0.0161	1.689	0.0256
		UCL = 8.756		UCL = 7.417		UCL = 8.852	

$n = 5, \quad r = 0.2$

$\rho = 0.0$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.591	6.4355	201.130	6.7151	198.568	6.3671
0.5	one var	7.342	0.1594	9.595	0.1853	12.280	0.2726
	equal	7.336	0.1568	8.984	0.1952	11.260	0.2507
	opposite	7.391	0.1456	8.894	0.1770	11.783	0.2618
1.0	one var	2.416	0.0376	3.811	0.0415	4.088	0.0625
	equal	2.332	0.0369	3.307	0.0395	3.775	0.0604
	opposite	2.365	0.0374	3.313	0.0397	3.782	0.0587
1.5	one var	1.397	0.0179	2.924	0.0185	2.669	0.0306
	equal	1.388	0.0177	2.278	0.0159	2.152	0.0286
	opposite	1.396	0.0177	2.276	0.0167	2.232	0.0292
2.0	one var	1.068	0.0080	2.741	0.0153	2.112	0.0166
	equal	1.060	0.0075	2.023	0.0047	1.619	0.0188
	opposite	1.059	0.0076	2.040	0.0062	1.648	0.0194
2.5	one var	1.005	0.0022	2.690	0.0146	1.995	0.0114
	equal	1.007	0.0026	2.003	0.0017	1.306	0.0147
	opposite	1.002	0.0014	2.001	0.0010	1.320	0.0154
3.0	one var	1.000	0.0000	2.653	0.0151	1.953	0.0085
	equal	1.000	0.0000	2.000	0.0000	1.158	0.0115
	opposite	1.000	0.0000	2.000	0.0000	1.164	0.0117
		UCL = 9.645		UCL = 9.116		UCL = 9.387	

Table 7.1 (continued): ARL values of EWMA charts for bivariate normal data with known Σ

$n = 5, \quad r = 0.2$

$\rho = 0.5$		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.591	6.4355	200.813	6.4253	200.852	6.0228
0.5	one var	7.342	0.1594	10.527	0.2134	13.699	0.3338
	equal	7.332	0.1583	8.239	0.1634	11.021	0.2232
	opposite	7.428	0.1467	10.673	0.2373	15.014	0.3626
1.0	one var	2.424	0.0377	3.962	0.0492	4.561	0.0786
	equal	2.335	0.0370	3.251	0.0372	3.737	0.0506
	opposite	2.382	0.0373	3.471	0.0527	4.535	0.0813
1.5	one var	1.400	0.0180	2.747	0.0250	2.820	0.0377
	equal	1.357	0.0172	2.285	0.0161	2.438	0.0225
	opposite	1.385	0.0175	2.177	0.0278	2.481	0.0417
2.0	one var	1.069	0.0080	2.437	0.0209	2.217	0.0235
	equal	1.068	0.0080	2.038	0.0060	2.095	0.0097
	opposite	1.059	0.0075	1.594	0.0189	1.726	0.0270
2.5	one var	1.005	0.0022	2.395	0.0199	1.903	0.0164
	equal	1.009	0.0030	2.003	0.0017	2.019	0.0043
	opposite	1.002	0.0014	1.317	0.0154	1.332	0.0169
3.0	one var	1.000	0.0000	2.345	0.0198	1.853	0.0140
	equal	1.000	0.0000	2.001	0.0010	2.002	0.0014
	opposite	1.000	0.0000	1.151	0.0113	1.148	0.0117
		UCL = 9.645		UCL = 8.913		UCL = 9.413	

$n = 5, \quad r = 0.2$

$\rho = 0.9$		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.591	6.4355	200.373	6.5059	198.863	6.2099
0.5	one var	7.342	0.1594	19.758	0.5009	25.753	0.6899
	equal	7.328	0.1570	7.868	0.1529	11.186	0.2449
	opposite	7.485	0.1545	21.826	0.5683	30.586	0.8896
1.0	one var	2.420	0.0376	5.353	0.0951	8.085	0.1776
	equal	2.353	0.0378	3.223	0.0381	4.049	0.0585
	opposite	2.386	0.0370	5.745	0.1106	8.459	0.2003
1.5	one var	1.407	0.0180	3.078	0.0470	4.275	0.0832
	equal	1.364	0.0172	2.326	0.0165	2.608	0.0267
	opposite	1.362	0.0174	3.077	0.0504	4.402	0.0913
2.0	one var	1.062	0.0076	2.251	0.0322	2.925	0.0501
	equal	1.057	0.0073	2.062	0.0079	2.161	0.0131
	opposite	1.074	0.0084	2.062	0.0304	2.905	0.0567
2.5	one var	1.005	0.0022	1.865	0.0234	2.334	0.0379
	equal	1.009	0.0030	2.008	0.0028	2.054	0.0072
	opposite	1.001	0.0010	1.624	0.0234	2.173	0.0379
3.0	one var	1.000	0.0000	1.693	0.0196	2.000	0.0289
	equal	1.001	0.0010	2.001	0.0010	2.010	0.0031
	opposite	1.000	0.0000	1.344	0.0166	1.717	0.0290
		UCL = 9.645		UCL = 8.070		UCL = 9.692	

Table 7.1 (continued): ARL values of EWMA charts for bivariate normal data with known Σ

$n = 5, \quad r = 0.3$

ρ = 0.0		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.384	6.2212	201.548	6.7287	200.780	6.3567
0.5	one var	8.685	0.2020	10.834	0.2521	14.902	0.3973
	equal	9.096	0.2108	10.668	0.2514	13.943	0.3545
	opposite	8.223	0.1809	10.356	0.2333	14.665	0.3710
1.0	one var	2.462	0.0395	4.010	0.0455	4.478	0.0731
	equal	2.557	0.0418	3.411	0.0432	3.929	0.0639
	opposite	2.493	0.0407	3.529	0.0460	3.931	0.0634
1.5	one var	1.409	0.0185	2.988	0.0217	2.672	0.0323
	equal	1.406	0.0183	2.309	0.0174	2.298	0.0316
	opposite	1.424	0.0184	2.265	0.0168	2.301	0.0319
2.0	one var	1.074	0.0085	2.766	0.0149	2.135	0.0178
	equal	1.070	0.0081	2.031	0.0057	1.692	0.0217
	opposite	1.079	0.0087	2.022	0.0049	1.703	0.0223
2.5	one var	1.009	0.0030	2.729	0.0142	1.998	0.0097
	equal	1.006	0.0024	2.001	0.0010	1.357	0.0165
	opposite	1.010	0.0031	2.001	0.0010	1.328	0.0163
3.0	one var	1.000	0.0000	2.704	0.0144	1.942	0.0089
	equal	1.000	0.0000	2.000	0.0000	1.191	0.0126
	opposite	1.000	0.0000	2.000	0.0000	1.166	0.0119
		UCL = 10.152		UCL = 9.133		UCL = 9.537	

$n = 5, \quad r = 0.3$

ρ = 0.5		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.384	6.2212	200.132	6.2803	201.831	6.4419
0.5	one var	8.685	0.2020	12.643	0.3168	16.256	0.4254
	equal	9.096	0.2040	9.244	0.1978	12.546	0.3030
	opposite	8.294	0.1908	13.010	0.3173	16.683	0.4538
1.0	one var	2.473	0.0396	4.082	0.0576	4.889	0.0901
	equal	2.535	0.0412	3.399	0.0443	3.985	0.0568
	opposite	2.512	0.0407	3.701	0.0597	4.626	0.0937
1.5	one var	1.409	0.0185	2.845	0.0279	2.839	0.0409
	equal	1.427	0.0182	2.326	0.0169	2.517	0.0247
	opposite	1.424	0.0189	2.238	0.0287	2.503	0.0424
2.0	one var	1.075	0.0086	2.505	0.0210	2.222	0.0256
	equal	1.073	0.0084	2.040	0.0062	2.145	0.0121
	opposite	1.073	0.0084	1.558	0.0182	1.715	0.0267
2.5	one var	1.007	0.0026	2.411	0.0205	1.954	0.0181
	equal	1.004	0.0020	2.003	0.0017	2.032	0.0056
	opposite	1.005	0.0022	1.315	0.0150	1.306	0.0172
3.0	one var	1.000	0.0000	2.367	0.0208	1.823	0.0140
	equal	1.000	0.0000	2.000	0.0000	2.005	0.0022
	opposite	1.000	0.0000	1.129	0.0106	1.135	0.0111
		UCL = 10.152		UCL = 8.975		UCL = 9.551	

Table 7.1 (continued): ARL values of EWMA charts for bivariate normal data with known Σ

$n = 5, \quad r = 0.3$

$\rho = 0.9$		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.384	6.2212	198.375	6.5481	200.677	6.2528
0.5	one var	8.685	0.2020	25.561	0.7289	31.948	0.9307
	equal	9.055	0.2043	9.135	0.1964	14.754	0.3731
	opposite	8.314	0.2056	27.360	0.7605	34.399	1.0080
1.0	one var	2.474	0.0396	6.009	0.1126	9.026	0.2256
	equal	2.541	0.0405	3.391	0.0418	4.410	0.0643
	opposite	2.519	0.0407	6.220	0.1327	9.644	0.2399
1.5	one var	1.412	0.0187	3.296	0.0505	4.478	0.0857
	equal	1.424	0.0181	2.415	0.0186	2.735	0.0283
	opposite	1.416	0.0186	3.204	0.0527	4.576	0.0972
2.0	one var	1.076	0.0086	2.330	0.0316	3.025	0.0543
	equal	1.072	0.0082	2.106	0.0097	2.274	0.0163
	opposite	1.068	0.0080	2.163	0.0337	2.933	0.0574
2.5	one var	1.008	0.0028	1.979	0.0259	2.382	0.0389
	equal	1.003	0.0017	2.026	0.0050	2.068	0.0080
	opposite	1.005	0.0022	1.643	0.0227	2.174	0.0398
3.0	one var	1.000	0.0000	1.760	0.0220	2.068	0.0303
	equal	1.000	0.0000	2.008	0.0028	2.012	0.0034
	opposite	1.000	0.0000	1.397	0.0183	1.738	0.0289
		UCL = 10.152		UCL = 8.197		UCL = 9.889	

$n = 10, \quad r = 0.1$

$\rho = 0.0$		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.301	6.3991	198.778	6.2306	200.128	6.1811
0.5	one var	3.850	0.0697	4.619	0.0754	5.587	0.0991
	equal	3.650	0.0695	4.227	0.0731	5.492	0.1026
	opposite	3.689	0.0672	4.253	0.0732	5.710	0.1102
1.0	one var	1.429	0.0197	2.055	0.0201	2.207	0.0302
	equal	1.407	0.0187	1.771	0.0224	2.079	0.0300
	opposite	1.429	0.0193	1.799	0.0231	2.071	0.0299
1.5	one var	1.034	0.0057	1.740	0.0144	1.469	0.0178
	equal	1.026	0.0050	1.156	0.0116	1.322	0.0157
	opposite	1.032	0.0056	1.144	0.0111	1.348	0.0162
2.0	one var	1.001	0.0010	1.655	0.0150	1.177	0.0121
	equal	1.000	0.0000	1.010	0.0031	1.076	0.0084
	opposite	1.000	0.0000	1.009	0.0030	1.059	0.0075
2.5	one var	1.000	0.0000	1.644	0.0151	1.055	0.0072
	equal	1.000	0.0000	1.000	0.0000	1.010	0.0031
	opposite	1.000	0.0000	1.000	0.0000	1.006	0.0024
3.0	one var	1.000	0.0000	1.634	0.0152	1.011	0.0033
	equal	1.000	0.0000	1.000	0.0000	1.000	0.0000
	opposite	1.000	0.0000	1.000	0.0000	1.000	0.0000
		UCL = 8.800		UCL = 8.706		UCL = 8.703	

Table 7.1 (continued): ARL values of EWMA charts for bivariate normal data with known Σ

$n = 10, \quad r = 0.1$

$\rho = 0.5$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.301	6.3991	198.514	6.3331	199.097	6.6464
0.5	one var	3.850	0.0697	4.863	0.0849	6.367	0.1199
	equal	3.650	0.0682	3.933	0.0612	5.172	0.0924
	opposite	3.645	0.0663	4.966	0.0853	6.602	0.1288
1.0	one var	1.433	0.0197	2.113	0.0242	2.403	0.0357
	equal	1.401	0.0188	1.707	0.0201	1.974	0.0275
	opposite	1.418	0.0189	1.937	0.0265	2.421	0.0391
1.5	one var	1.035	0.0058	1.590	0.0159	1.508	0.0193
	equal	1.029	0.0053	1.181	0.0123	1.332	0.0152
	opposite	1.029	0.0053	1.244	0.0138	1.495	0.0199
2.0	one var	1.001	0.0010	1.411	0.0156	1.238	0.0138
	equal	1.000	0.0000	1.009	0.0030	1.047	0.0067
	opposite	1.001	0.0010	1.026	0.0050	1.138	0.0110
2.5	one var	1.000	0.0000	1.349	0.0151	1.080	0.0086
	equal	1.000	0.0000	1.000	0.0000	1.006	0.0024
	opposite	1.000	0.0000	1.002	0.0014	1.029	0.0053
3.0	one var	1.000	0.0000	1.327	0.0148	1.034	0.0057
	equal	1.000	0.0000	1.000	0.0000	1.001	0.0010
	opposite	1.000	0.0000	1.000	0.0000	1.002	0.0014
		UCL = 8.800		UCL = 8.372		UCL = 8.760	

$n = 10, \quad r = 0.1$

$\rho = 0.9$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.301	6.3991	200.710	6.5611	199.155	6.5545
0.5	one var	3.850	0.0697	7.835	0.1410	11.870	0.2629
	equal	3.603	0.0678	3.784	0.0608	5.203	0.0962
	opposite	3.646	0.0645	8.399	0.1493	12.811	0.2915
1.0	one var	1.439	0.0198	2.651	0.0386	3.903	0.0728
	equal	1.399	0.0186	1.680	0.0206	2.166	0.0286
	opposite	1.394	0.0188	2.787	0.0421	4.112	0.0832
1.5	one var	1.035	0.0058	1.664	0.0207	2.348	0.0392
	equal	1.027	0.0051	1.204	0.0129	1.456	0.0172
	opposite	1.030	0.0054	1.640	0.0222	2.339	0.0398
2.0	one var	1.001	0.0010	1.238	0.0136	1.656	0.0245
	equal	1.000	0.0000	1.023	0.0047	1.146	0.0112
	opposite	1.002	0.0014	1.207	0.0135	1.678	0.0259
2.5	one var	1.000	0.0000	1.090	0.0091	1.354	0.0172
	equal	1.000	0.0000	1.002	0.0014	1.036	0.0059
	opposite	1.000	0.0000	1.040	0.0062	1.397	0.0187
3.0	one var	1.000	0.0000	1.028	0.0052	1.174	0.0126
	equal	1.000	0.0000	1.000	0.0000	1.007	0.0026
	opposite	1.000	0.0000	1.011	0.0033	1.145	0.0119
		UCL = 8.800		UCL = 7.405		UCL = 8.738	

Table 7.1 (continued): ARL values of EWMA charts for bivariate normal data with known Σ

$n = 10, \quad r = 0.2$

$\rho = 0.0$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.689	6.1285	198.557	5.9613	198.023	6.1844
0.5	one var	4.173	0.0780	5.142	0.0898	6.319	0.1209
	equal	4.276	0.0773	4.818	0.0834	6.301	0.1187
	opposite	3.991	0.0729	4.691	0.0823	6.200	0.1168
1.0	one var	1.468	0.0197	2.197	0.0208	2.282	0.0342
	equal	1.480	0.0200	1.851	0.0219	2.125	0.0311
	opposite	1.491	0.0197	1.871	0.0244	2.118	0.0318
1.5	one var	1.042	0.0063	1.836	0.0125	1.518	0.0191
	equal	1.036	0.0059	1.201	0.0127	1.298	0.0155
	opposite	1.035	0.0060	1.243	0.0138	1.375	0.0170
2.0	one var	1.000	0.0000	1.786	0.0130	1.193	0.0128
	equal	1.000	0.0000	1.024	0.0048	1.070	0.0081
	opposite	1.000	0.0000	1.020	0.0044	1.071	0.0081
2.5	one var	1.000	0.0000	1.775	0.0132	1.063	0.0077
	equal	1.000	0.0000	1.001	0.0010	1.006	0.0024
	opposite	1.000	0.0000	1.000	0.0000	1.006	0.0024
3.0	one var	1.000	0.0000	1.782	0.0131	1.012	0.0034
	equal	1.000	0.0000	1.000	0.0000	1.000	0.0000
	opposite	1.000	0.0000	1.000	0.0000	1.000	0.0000
		UCL = 9.736		UCL = 9.399		UCL = 9.566	

$n = 10, \quad r = 0.2$

$\rho = 0.5$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.689	6.1285	200.209	6.0745	201.625	6.0418
0.5	one var	4.173	0.0780	5.365	0.0926	7.526	0.1525
	equal	4.255	0.0753	4.449	0.0720	5.671	0.1052
	opposite	4.007	0.0701	5.759	0.1008	7.968	0.1732
1.0	one var	1.469	0.0197	2.242	0.0243	2.665	0.0391
	equal	1.474	0.0199	1.858	0.0208	2.189	0.0270
	opposite	1.469	0.0201	2.121	0.0271	2.585	0.0417
1.5	one var	1.042	0.0063	1.738	0.0150	1.659	0.0211
	equal	1.033	0.0057	1.266	0.0140	1.468	0.0171
	opposite	1.030	0.0056	1.268	0.0142	1.550	0.0204
2.0	one var	1.000	0.0000	1.597	0.0155	1.303	0.0152
	equal	1.000	0.0000	1.039	0.0061	1.187	0.0123
	opposite	1.000	0.0000	1.040	0.0062	1.170	0.0120
2.5	one var	1.000	0.0000	1.557	0.0157	1.104	0.0097
	equal	1.000	0.0000	1.002	0.0014	1.038	0.0060
	opposite	1.000	0.0000	1.002	0.0014	1.036	0.0059
3.0	one var	1.000	0.0000	1.497	0.0158	1.032	0.0056
	equal	1.000	0.0000	1.000	0.0000	1.006	0.0024
	opposite	1.000	0.0000	1.000	0.0000	1.014	0.0037
		UCL = 9.736		UCL = 9.168		UCL = 9.606	

Table 7.1 (continued): ARL values of EWMA charts for bivariate normal data with known Σ

$n = 10, \quad r = 0.2$

$\rho = 0.9$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.689	6.1285	200.342	6.4243	199.028	6.3252
0.5	one var	4.173	0.0780	9.610	0.1927	13.902	0.3393
	equal	4.256	0.0756	4.055	0.0685	6.035	0.1135
	opposite	3.993	0.0734	10.510	0.2084	15.852	0.4087
1.0	one var	1.468	0.0197	2.903	0.0418	4.417	0.0844
	equal	1.477	0.0197	1.849	0.0201	2.223	0.0298
	opposite	1.448	0.0194	3.031	0.0459	4.756	0.0958
1.5	one var	1.042	0.0063	1.740	0.0214	2.496	0.0407
	equal	1.035	0.0058	1.293	0.0144	1.435	0.0171
	opposite	1.031	0.0057	1.673	0.0222	2.545	0.0456
2.0	one var	1.000	0.0000	1.332	0.0156	1.797	0.0257
	equal	1.000	0.0000	1.050	0.0069	1.139	0.0109
	opposite	1.000	0.0000	1.275	0.0144	1.734	0.0267
2.5	one var	1.000	0.0000	1.166	0.0118	1.474	0.0191
	equal	1.000	0.0000	1.006	0.0024	1.034	0.0057
	opposite	1.000	0.0000	1.045	0.0066	1.403	0.0205
3.0	one var	1.000	0.0000	1.080	0.0086	1.288	0.0155
	equal	1.000	0.0000	1.001	0.0010	1.008	0.0028
	opposite	1.000	0.0000	1.007	0.0026	1.204	0.0140
		UCL = 9.736		UCL = 8.000		UCL = 9.683	

$n = 10, \quad r = 0.3$

$\rho = 0.0$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.633	6.1955	200.712	6.3752	199.574	6.2694
0.5	one var	4.413	0.0810	5.268	0.0892	6.998	0.1506
	equal	4.613	0.0954	5.325	0.0975	6.927	0.1489
	opposite	4.581	0.0854	5.057	0.0897	6.830	0.1504
1.0	one var	1.553	0.0217	2.239	0.0214	2.453	0.0356
	equal	1.543	0.0210	1.968	0.0244	2.286	0.0337
	opposite	1.550	0.0214	1.894	0.0229	2.224	0.0343
1.5	one var	1.044	0.0065	1.877	0.0114	1.488	0.0188
	equal	1.036	0.0059	1.274	0.0143	1.342	0.0167
	opposite	1.044	0.0065	1.248	0.0137	1.342	0.0168
2.0	one var	1.001	0.0010	1.825	0.0120	1.193	0.0128
	equal	1.001	0.0010	1.032	0.0056	1.053	0.0071
	opposite	1.001	0.0010	1.027	0.0051	1.058	0.0075
2.5	one var	1.000	0.0000	1.821	0.0121	1.057	0.0073
	equal	1.000	0.0000	1.001	0.0010	1.006	0.0024
	opposite	1.000	0.0000	1.003	0.0017	1.008	0.0028
3.0	one var	1.000	0.0000	1.817	0.0122	1.014	0.0037
	equal	1.000	0.0000	1.000	0.0000	1.001	0.0010
	opposite	1.000	0.0000	1.000	0.0000	1.000	0.0000
		UCL = 10.176		UCL = 9.639		UCL = 9.865	

Table 7.1 (continued): ARL values of EWMA charts for bivariate normal data with known Σ

$n = 10, \quad r = 0.3$

ρ = 0.5		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.633	6.1955	201.869	6.2302	200.180	6.2774
0.5	one var	4.413	0.0810	6.062	0.1200	8.176	0.1879
	equal	4.572	0.0918	4.790	0.0805	6.442	0.1351
	opposite	4.591	0.0858	6.372	0.1236	8.913	0.2121
1.0	one var	1.556	0.0218	2.337	0.0257	2.673	0.0415
	equal	1.552	0.0216	1.878	0.0208	2.221	0.0287
	opposite	1.549	0.0210	2.141	0.0283	2.690	0.0437
1.5	one var	1.045	0.0066	1.759	0.0151	1.658	0.0215
	equal	1.043	0.0064	1.275	0.0144	1.467	0.0170
	opposite	1.040	0.0062	1.315	0.0152	1.615	0.0221
2.0	one var	1.001	0.0010	1.631	0.0153	1.280	0.0150
	equal	1.001	0.0010	1.040	0.0062	1.139	0.0109
	opposite	1.000	0.0000	1.043	0.0064	1.191	0.0129
2.5	one var	1.000	0.0000	1.597	0.0155	1.123	0.0105
	equal	1.000	0.0000	1.001	0.0010	1.044	0.0065
	opposite	1.000	0.0000	1.004	0.0020	1.043	0.0064
3.0	one var	1.000	0.0000	1.547	0.0157	1.040	0.0062
	equal	1.000	0.0000	1.000	0.0000	1.005	0.0022
	opposite	1.000	0.0000	1.000	0.0000	1.003	0.0017
		UCL = 10.176		UCL = 9.312		UCL = 9.929	

$n = 10, \quad r = 0.3$

ρ = 0.9		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.633	6.1955	200.323	6.2237	198.008	6.0370
0.5	one var	4.413	0.0810	11.556	0.2743	17.821	0.4975
	equal	4.574	0.0895	4.325	0.0747	6.687	0.1303
	opposite	4.563	0.0852	12.837	0.3172	19.490	0.5155
1.0	one var	1.556	0.0218	3.221	0.0466	4.761	0.0985
	equal	1.573	0.0230	1.972	0.0214	2.264	0.0316
	opposite	1.543	0.0214	3.176	0.0478	5.023	0.1087
1.5	one var	1.045	0.0066	1.841	0.0225	2.477	0.0437
	equal	1.046	0.0066	1.401	0.0157	1.441	0.0182
	opposite	1.046	0.0066	1.779	0.0241	2.671	0.0498
2.0	one var	1.001	0.0010	1.427	0.0162	1.851	0.0275
	equal	1.001	0.0010	1.095	0.0093	1.158	0.0115
	opposite	1.000	0.0000	1.264	0.0146	1.797	0.0301
2.5	one var	1.000	0.0000	1.223	0.0132	1.457	0.0196
	equal	1.000	0.0000	1.012	0.0034	1.037	0.0060
	opposite	1.000	0.0000	1.076	0.0084	1.419	0.0203
3.0	one var	1.000	0.0000	1.141	0.0110	1.277	0.0153
	equal	1.000	0.0000	1.000	0.0000	1.005	0.0022
	opposite	1.000	0.0000	1.017	0.0041	1.209	0.0145
		UCL = 10.176		UCL = 8.411		UCL = 10.124	

Table 7.1 (continued): ARL values of EWMA charts for bivariate normal data with known Σ

$n = 15,\ r = 0.1$

ρ = 0.0		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.041	6.5080	200.654	6.1504	201.220	6.2470
0.5	one var	2.646	0.0440	3.216	0.0530	4.077	0.0699
	equal	2.700	0.0435	3.037	0.0482	3.930	0.0673
	opposite	2.787	0.0460	3.119	0.0491	4.015	0.0718
1.0	one var	1.145	0.0116	1.441	0.0173	1.689	0.0204
	equal	1.129	0.0108	1.291	0.0154	1.629	0.0212
	opposite	1.182	0.0128	1.306	0.0160	1.613	0.0202
1.5	one var	1.001	0.0010	1.046	0.0066	1.189	0.0125
	equal	1.002	0.0014	1.012	0.0034	1.099	0.0094
	opposite	1.001	0.0010	1.009	0.0030	1.079	0.0085
2.0	one var	1.000	0.0000	1.001	0.0010	1.027	0.0051
	equal	1.000	0.0000	1.000	0.0000	1.003	0.0017
	opposite	1.000	0.0000	1.000	0.0000	1.001	0.0010
2.5	one var	1.000	0.0000	1.000	0.0000	1.004	0.0020
	equal	1.000	0.0000	1.000	0.0000	1.000	0.0000
	opposite	1.000	0.0000	1.000	0.0000	1.000	0.0000
3.0	one var	1.000	0.0000	1.000	0.0000	1.000	0.0000
	equal	1.000	0.0000	1.000	0.0000	1.000	0.0000
	opposite	1.000	0.0000	1.000	0.0000	1.000	0.0000
		UCL = 8.727		UCL = 8.697		UCL = 8.715	

$n = 15,\ r = 0.1$

ρ = 0.5		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.041	6.5080	198.113	6.3029	198.151	6.4601
0.5	one var	2.646	0.0440	3.481	0.0547	4.836	0.0850
	equal	2.650	0.0424	2.914	0.0450	3.678	0.0619
	opposite	2.752	0.0463	3.489	0.0560	5.141	0.0938
1.0	one var	1.140	0.0113	1.483	0.0179	1.816	0.0238
	equal	1.145	0.0112	1.274	0.0147	1.491	0.0191
	opposite	1.176	0.0123	1.425	0.0186	1.849	0.0279
1.5	one var	1.001	0.0010	1.067	0.0079	1.231	0.0136
	equal	1.003	0.0017	1.011	0.0033	1.060	0.0075
	opposite	1.001	0.0010	1.032	0.0056	1.198	0.0133
2.0	one var	1.000	0.0000	1.000	0.0000	1.061	0.0076
	equal	1.000	0.0000	1.000	0.0000	1.004	0.0020
	opposite	1.000	0.0000	1.001	0.0010	1.020	0.0044
2.5	one var	1.000	0.0000	1.000	0.0000	1.013	0.0036
	equal	1.000	0.0000	1.000	0.0000	1.000	0.0000
	opposite	1.000	0.0000	1.000	0.0000	1.002	0.0014
3.0	one var	1.000	0.0000	1.000	0.0000	1.000	0.0000
	equal	1.000	0.0000	1.000	0.0000	1.000	0.0000
	opposite	1.000	0.0000	1.000	0.0000	1.000	0.0000
		UCL = 8.727		UCL = 8.379		UCL = 8.821	

Table 7.1 (continued): ARL values of EWMA charts for bivariate normal data with known Σ

$n = 15, \quad r = 0.1$

ρ = 0.9		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.041	6.5080	199.955	6.5408	198.742	6.2864
0.5	one var	2.646	0.0440	5.412	0.0936	8.759	0.1853
	equal	2.650	0.0438	2.671	0.0428	3.748	0.0624
	opposite	2.717	0.0461	5.788	0.0942	9.340	0.1967
1.0	one var	1.145	0.0117	1.916	0.0257	3.008	0.0537
	equal	1.141	0.0113	1.224	0.0136	1.545	0.0188
	opposite	1.159	0.0121	1.950	0.0260	3.137	0.0550
1.5	one var	1.001	0.0010	1.202	0.0129	1.737	0.0255
	equal	1.002	0.0014	1.013	0.0036	1.110	0.0099
	opposite	1.002	0.0014	1.221	0.0133	1.813	0.0292
2.0	one var	1.000	0.0000	1.028	0.0052	1.350	0.0170
	equal	1.000	0.0000	1.000	0.0000	1.008	0.0028
	opposite	1.000	0.0000	1.025	0.0049	1.353	0.0173
2.5	one var	1.000	0.0000	1.001	0.0010	1.137	0.0112
	equal	1.000	0.0000	1.000	0.0000	1.000	0.0000
	opposite	1.000	0.0000	1.001	0.0010	1.135	0.0111
3.0	one var	1.000	0.0000	1.001	0.0010	1.042	0.0063
	equal	1.000	0.0000	1.000	0.0000	1.000	0.0000
	opposite	1.000	0.0000	1.000	0.0000	1.056	0.0074
		UCL = 8.727		UCL = 7.280		UCL = 8.773	

$n = 15, \quad r = 0.2$

ρ = 0.0		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.027	6.4180	198.648	5.8353	201.452	6.3675
0.5	one var	3.045	0.0494	3.541	0.0563	4.588	0.0832
	equal	2.915	0.0497	3.401	0.0584	4.355	0.0766
	opposite	2.994	0.0504	3.305	0.0526	4.447	0.0775
1.0	one var	1.177	0.0126	1.553	0.0178	1.780	0.0229
	equal	1.203	0.0136	1.405	0.0173	1.625	0.0214
	opposite	1.205	0.0135	1.351	0.0163	1.645	0.0224
1.5	one var	1.001	0.0010	1.099	0.0094	1.217	0.0133
	equal	1.002	0.0014	1.017	0.0041	1.092	0.0093
	opposite	1.002	0.0014	1.013	0.0036	1.103	0.0097
2.0	one var	1.000	0.0000	1.005	0.0022	1.038	0.0060
	equal	1.000	0.0000	1.000	0.0000	1.008	0.0028
	opposite	1.000	0.0000	1.000	0.0000	1.005	0.0022
2.5	one var	1.000	0.0000	1.000	0.0000	1.001	0.0010
	equal	1.000	0.0000	1.000	0.0000	1.000	0.0000
	opposite	1.000	0.0000	1.000	0.0000	1.000	0.0000
3.0	one var	1.000	0.0000	1.000	0.0000	1.000	0.0000
	equal	1.000	0.0000	1.000	0.0000	1.000	0.0000
	opposite	1.000	0.0000	1.000	0.0000	1.000	0.0000
		UCL = 9.724		UCL = 9.521		UCL = 9.588	

Table 7.1 (continued): ARL values of EWMA charts for bivariate normal data with known Σ

$n = 15, \quad r = 0.2$

ρ = 0.5		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.027	6.4180	201.871	6.1477	200.395	6.2709
0.5	one var	3.045	0.0494	3.766	0.0575	5.131	0.0936
	equal	2.926	0.0491	3.037	0.0462	4.007	0.0688
	opposite	2.983	0.0503	4.055	0.0695	5.537	0.1103
1.0	one var	1.177	0.0126	1.638	0.0194	1.899	0.0263
	equal	1.192	0.0131	1.339	0.0156	1.567	0.0208
	opposite	1.207	0.0137	1.514	0.0187	1.909	0.0294
1.5	one var	1.001	0.0010	1.097	0.0094	1.260	0.0140
	equal	1.003	0.0017	1.016	0.0040	1.070	0.0082
	opposite	1.002	0.0014	1.041	0.0063	1.171	0.0122
2.0	one var	1.000	0.0000	1.008	0.0028	1.052	0.0070
	equal	1.000	0.0000	1.000	0.0000	1.000	0.0000
	opposite	1.000	0.0000	1.001	0.0010	1.026	0.0050
2.5	one var	1.000	0.0000	1.000	0.0000	1.007	0.0026
	equal	1.000	0.0000	1.000	0.0000	1.000	0.0000
	opposite	1.000	0.0000	1.000	0.0000	1.002	0.0014
3.0	one var	1.000	0.0000	1.000	0.0000	1.000	0.0000
	equal	1.000	0.0000	1.000	0.0000	1.000	0.0000
	opposite	1.000	0.0000	1.000	0.0000	1.000	0.0000
		UCL = 9.724		UCL = 9.204		UCL = 9.677	

$n = 15, \quad r = 0.2$

ρ = 0.9		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.045	6.4179	201.948	6.6053	201.519	6.4590
0.5	one var	3.027	0.0490	6.047	0.1065	9.668	0.2185
	equal	2.944	0.0472	3.020	0.0489	4.243	0.0719
	opposite	3.008	0.0507	6.662	0.1170	10.925	0.2506
1.0	one var	1.186	0.0129	2.147	0.0275	3.255	0.0545
	equal	1.200	0.0133	1.369	0.0160	1.766	0.0212
	opposite	1.211	0.0139	2.158	0.0282	3.466	0.0603
1.5	one var	1.002	0.0014	1.308	0.0153	1.927	0.0273
	equal	1.003	0.0017	1.015	0.0038	1.196	0.0126
	opposite	1.003	0.0017	1.292	0.0151	1.960	0.0291
2.0	one var	1.000	0.0000	1.044	0.0065	1.440	0.0187
	equal	1.000	0.0000	1.000	0.0000	1.025	0.0049
	opposite	1.000	0.0000	1.053	0.0072	1.450	0.0202
2.5	one var	1.000	0.0000	1.005	0.0022	1.182	0.0125
	equal	1.000	0.0000	1.000	0.0000	1.003	0.0017
	opposite	1.000	0.0000	1.000	0.0000	1.174	0.0128
3.0	one var	1.000	0.0000	1.000	0.0000	1.084	0.0088
	equal	1.000	0.0000	1.000	0.0000	1.000	0.0000
	opposite	1.000	0.0000	1.000	0.0000	1.057	0.0076
		UCL = 9.724		UCL = 8.149		UCL = 9.762	

Table 7.1 (continued): ARL values of EWMA charts for bivariate normal data with known Σ

$n = 15, \quad r = 0.3$

$\rho = 0.0$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.926	6.0124	199.857	6.2552	201.244	6.6436
0.5	one var	3.089	0.0508	3.603	0.0565	4.946	0.0900
	equal	3.205	0.0561	3.401	0.0571	4.859	0.0983
	opposite	3.122	0.0550	3.564	0.0595	4.718	0.0863
1.0	one var	1.219	0.0136	1.616	0.0181	1.890	0.0245
	equal	1.219	0.0136	1.426	0.0175	1.694	0.0226
	opposite	1.211	0.0140	1.396	0.0170	1.710	0.0226
1.5	one var	1.002	0.0014	1.129	0.0106	1.228	0.0133
	equal	1.004	0.0020	1.018	0.0042	1.113	0.0100
	opposite	1.002	0.0014	1.020	0.0044	1.127	0.0105
2.0	one var	1.000	0.0000	1.015	0.0038	1.038	0.0060
	equal	1.000	0.0000	1.000	0.0000	1.005	0.0022
	opposite	1.000	0.0000	1.000	0.0000	1.005	0.0022
2.5	one var	1.000	0.0000	1.000	0.0000	1.002	0.0014
	equal	1.000	0.0000	1.000	0.0000	1.000	0.0000
	opposite	1.000	0.0000	1.000	0.0000	1.000	0.0000
3.0	one var	1.000	0.0000	1.000	0.0000	1.000	0.0000
	equal	1.000	0.0000	1.000	0.0000	1.000	0.0000
	opposite	1.000	0.0000	1.000	0.0000	1.000	0.0000
		UCL = 10.167		UCL = 9.774		UCL = 9.880	

$n = 15, \quad r = 0.3$

$\rho = 0.5$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.926	6.0124	199.642	6.2113	199.705	6.4651
0.5	one var	3.089	0.0508	4.014	0.0641	5.528	0.1135
	equal	3.155	0.0557	3.236	0.0534	4.337	0.0761
	opposite	3.137	0.0564	4.358	0.0756	5.888	0.1192
1.0	one var	1.217	0.0136	1.641	0.0191	2.055	0.0278
	equal	1.208	0.0137	1.382	0.0165	1.612	0.0215
	opposite	1.213	0.0140	1.496	0.0189	2.036	0.0307
1.5	one var	1.002	0.0014	1.125	0.0105	1.328	0.0159
	equal	1.003	0.0017	1.023	0.0047	1.080	0.0086
	opposite	1.004	0.0020	1.053	0.0072	1.260	0.0144
2.0	one var	1.000	0.0000	1.014	0.0037	1.082	0.0087
	equal	1.000	0.0000	1.000	0.0000	1.006	0.0024
	opposite	1.000	0.0000	1.001	0.0010	1.030	0.0054
2.5	one var	1.000	0.0000	1.000	0.0000	1.013	0.0036
	equal	1.000	0.0000	1.000	0.0000	1.001	0.0010
	opposite	1.000	0.0000	1.000	0.0000	1.003	0.0017
3.0	one var	1.000	0.0000	1.000	0.0000	1.003	0.0017
	equal	1.000	0.0000	1.000	0.0000	1.000	0.0000
	opposite	1.000	0.0000	1.000	0.0000	1.000	0.0000
		UCL = 10.167		UCL = 9.478		UCL = 10.024	

Table 7.1 (continued): ARL values of EWMA charts for bivariate normal data with known Σ

$n = 15, \quad r = 0.3$

$\rho = 0.9$		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.926	6.0124	200.021	6.4019	198.167	6.2831
0.5	one var	3.089	0.0508	7.323	0.1570	11.930	0.2848
	equal	3.160	0.0557	3.043	0.0482	4.546	0.0810
	opposite	3.203	0.0571	7.837	0.1579	12.654	0.3044
1.0	one var	1.213	0.0135	2.207	0.0300	3.455	0.0637
	equal	1.215	0.0140	1.346	0.0156	1.798	0.0220
	opposite	1.211	0.0136	2.212	0.0298	3.663	0.0669
1.5	one var	1.002	0.0014	1.333	0.0155	1.936	0.0299
	equal	1.002	0.0014	1.021	0.0045	1.173	0.0120
	opposite	1.007	0.0026	1.294	0.0152	1.964	0.0327
2.0	one var	1.000	0.0000	1.056	0.0073	1.430	0.0186
	equal	1.000	0.0000	1.001	0.0010	1.022	0.0046
	opposite	1.000	0.0000	1.031	0.0055	1.413	0.0197
2.5	one var	1.000	0.0000	1.004	0.0020	1.178	0.0125
	equal	1.000	0.0000	1.000	0.0000	1.002	0.0014
	opposite	1.000	0.0000	1.002	0.0014	1.154	0.0120
3.0	one var	1.000	0.0000	1.002	0.0014	1.093	0.0093
	equal	1.000	0.0000	1.000	0.0000	1.000	0.0000
	opposite	1.000	0.0000	1.000	0.0000	1.063	0.0077
		UCL = 10.167		UCL = 8.402		UCL = 10.164	

Table 7.2: ARL values of EWMA charts for 10% mixture bivariate normal data with known Σ

$n = 5, \quad r = 0.1$

$\rho = 0.0$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.028	6.7915	201.444	6.0244	199.516	6.2239
0.5	one var	7.582	0.1492	4.708	0.0672	4.846	0.0774
	equal	7.604	0.1465	4.293	0.0639	4.585	0.0765
	opposite	7.190	0.1355	4.107	0.0614	4.458	0.0752
1.0	one var	2.441	0.0350	2.941	0.0287	2.569	0.0274
	equal	2.495	0.0347	2.285	0.0178	2.055	0.0271
	opposite	2.433	0.0337	2.303	0.0184	1.998	0.0248
1.5	one var	1.369	0.0179	2.715	0.0225	2.227	0.0199
	equal	1.348	0.0168	2.089	0.0098	1.469	0.0181
	opposite	1.392	0.0174	2.089	0.0095	1.461	0.0177
2.0	one var	1.061	0.0078	2.599	0.0188	2.133	0.0165
	equal	1.062	0.0079	2.043	0.0066	1.258	0.0143
	opposite	1.056	0.0074	2.041	0.0067	1.299	0.0154
2.5	one var	1.011	0.0033	2.557	0.0175	2.067	0.0152
	equal	1.012	0.0034	2.024	0.0048	1.223	0.0137
	opposite	1.012	0.0034	2.031	0.0058	1.220	0.0134
3.0	one var	1.000	0.0000	2.524	0.0167	2.090	0.0146
	equal	1.001	0.0010	2.008	0.0028	1.199	0.0128
	opposite	1.003	0.0017	2.007	0.0026	1.196	0.0128
		UCL = 10.000		UCL = 8.582		UCL = 8.693	

$n = 5, \quad r = 0.1$

$\rho = 0.5$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.028	6.7915	200.921	6.1625	199.439	6.6282
0.5	one var	7.582	0.1492	4.868	0.0746	5.286	0.0940
	equal	7.789	0.1499	4.029	0.0602	4.522	0.0680
	opposite	7.327	0.1406	4.747	0.0808	5.145	0.0963
1.0	one var	2.468	0.0359	2.789	0.0315	2.592	0.0327
	equal	2.560	0.0347	2.348	0.0199	2.290	0.0174
	opposite	2.484	0.0340	2.004	0.0260	2.093	0.0329
1.5	one var	1.364	0.0173	2.406	0.0246	2.113	0.0225
	equal	1.369	0.0176	2.139	0.0117	2.088	0.0095
	opposite	1.391	0.0177	1.476	0.0185	1.370	0.0186
2.0	one var	1.054	0.0073	2.353	0.0221	2.005	0.0195
	equal	1.061	0.0076	2.099	0.0101	2.034	0.0059
	opposite	1.057	0.0073	1.244	0.0143	1.140	0.0115
2.5	one var	1.016	0.0040	2.281	0.0203	1.967	0.0181
	equal	1.012	0.0034	2.041	0.0063	2.017	0.0041
	opposite	1.013	0.0039	1.201	0.0128	1.089	0.0091
3.0	one var	1.002	0.0014	2.253	0.0193	1.913	0.0162
	equal	1.001	0.0010	2.024	0.0048	2.014	0.0037
	opposite	1.002	0.0014	1.145	0.0112	1.043	0.0066
		UCL = 10.000		UCL = 8.442		UCL = 8.736	

Table 7.2 (continued): ARL values of EWMA charts for 10% mixture normal data with known Σ

$n = 5, \quad r = 0.1$

$\rho = 0.9$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.028	6.7915	201.924	6.7909	198.663	6.2262
0.5	one var	7.582	0.1492	6.966	0.1263	8.709	0.1849
	equal	7.728	0.1478	4.057	0.0598	4.585	0.0671
	opposite	7.311	0.1355	7.319	0.1439	9.918	0.2300
1.0	one var	2.454	0.0353	2.725	0.0435	3.375	0.0616
	equal	2.552	0.0346	2.376	0.0216	2.536	0.0226
	opposite	2.488	0.0350	2.646	0.0434	3.420	0.0650
1.5	one var	1.353	0.0172	1.945	0.0280	2.240	0.0332
	equal	1.384	0.0177	2.195	0.0142	2.236	0.0148
	opposite	1.382	0.0175	1.646	0.0250	2.069	0.0361
2.0	one var	1.054	0.0073	1.723	0.0236	1.854	0.0248
	equal	1.059	0.0075	2.115	0.0106	2.156	0.0117
	opposite	1.065	0.0079	1.305	0.0167	1.557	0.0232
2.5	one var	1.013	0.0036	1.623	0.0211	1.678	0.0201
	equal	1.015	0.0038	2.061	0.0080	2.142	0.0113
	opposite	1.015	0.0038	1.144	0.0118	1.288	0.0161
3.0	one var	1.002	0.0014	1.577	0.0208	1.629	0.0191
	equal	1.001	0.0010	2.037	0.0060	2.099	0.0096
	opposite	1.002	0.0014	1.089	0.0090	1.132	0.0110
		UCL = 10.000		UCL = 7.540		UCL = 8.874	

$n = 5, \quad r = 0.2$

$\rho = 0.0$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.457	6.1901	201.850	6.0263	198.504	6.1954
0.5	one var	10.135	0.2191	5.129	0.0722	5.285	0.0854
	equal	10.025	0.2181	4.459	0.0681	4.715	0.0789
	opposite	10.469	0.2181	4.385	0.0656	4.915	0.0908
1.0	one var	2.922	0.0417	3.129	0.0276	2.621	0.0303
	equal	2.905	0.0415	2.356	0.0196	1.980	0.0265
	opposite	2.884	0.0415	2.329	0.0196	2.060	0.0276
1.5	one var	1.575	0.0199	2.924	0.0212	2.264	0.0208
	equal	1.607	0.0211	2.106	0.0108	1.461	0.0183
	opposite	1.601	0.0198	2.129	0.0121	1.452	0.0179
2.0	one var	1.083	0.0088	2.788	0.0181	2.151	0.0168
	equal	1.103	0.0100	2.056	0.0075	1.286	0.0152
	opposite	1.105	0.0107	2.057	0.0077	1.277	0.0149
2.5	one var	1.013	0.0036	2.778	0.0155	2.134	0.0163
	equal	1.021	0.0045	2.027	0.0053	1.233	0.0138
	opposite	1.021	0.0045	2.022	0.0046	1.228	0.0135
3.0	one var	1.003	0.0017	2.752	0.0154	2.070	0.0143
	equal	1.002	0.0014	2.012	0.0034	1.173	0.0121
	opposite	1.006	0.0024	2.018	0.0042	1.165	0.0119
		UCL = 12.227		UCL = 9.189		UCL = 9.427	

Table 7.2 (continued): ARL values of EWMA charts for 10% mixture normal data with known Σ

$n = 5, \quad r = 0.2$

$\rho = 0.5$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.457	6.1901	199.525	6.7065	200.942	6.5494
0.5	one var	10.135	0.2191	5.270	0.0841	6.010	0.1123
	equal	10.013	0.2137	4.416	0.0646	4.840	0.0769
	opposite	10.522	0.2240	5.016	0.0878	5.928	0.1180
1.0	one var	2.922	0.0415	2.982	0.0322	2.799	0.0379
	equal	2.885	0.0417	2.447	0.0225	2.446	0.0219
	opposite	2.910	0.0407	2.069	0.0294	2.118	0.0353
1.5	one var	1.579	0.0200	2.711	0.0262	2.128	0.0239
	equal	1.615	0.0211	2.205	0.0146	2.116	0.0113
	opposite	1.597	0.0203	1.477	0.0191	1.366	0.0185
2.0	one var	1.085	0.0089	2.580	0.0241	2.039	0.0211
	equal	1.107	0.0100	2.163	0.0126	2.091	0.0094
	opposite	1.103	0.0103	1.290	0.0156	1.151	0.0119
2.5	one var	1.013	0.0036	2.479	0.0231	2.020	0.0191
	equal	1.019	0.0043	2.081	0.0089	2.072	0.0083
	opposite	1.019	0.0043	1.197	0.0130	1.063	0.0078
3.0	one var	1.003	0.0017	2.462	0.0206	1.974	0.0171
	equal	1.003	0.0017	2.040	0.0064	2.042	0.0063
	opposite	1.008	0.0028	1.176	0.0121	1.035	0.0058
		UCL = 12.227		UCL = 9.056		UCL = 9.506	

$n = 5, \quad r = 0.2$

$\rho = 0.9$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.457	6.1901	198.190	6.3260	200.715	6.5536
0.5	one var	10.135	0.2191	8.352	0.1751	10.772	0.2501
	equal	10.124	0.2192	4.435	0.0648	5.014	0.0783
	opposite	10.628	0.2298	8.594	0.1743	10.857	0.2561
1.0	one var	2.911	0.0413	2.994	0.0468	3.755	0.0680
	equal	2.844	0.0405	2.618	0.0276	2.577	0.0259
	opposite	2.923	0.0400	2.911	0.0499	3.592	0.0760
1.5	one var	1.578	0.0201	2.098	0.0312	2.428	0.0374
	equal	1.616	0.0206	2.284	0.0176	2.261	0.0168
	opposite	1.596	0.0204	1.782	0.0257	2.107	0.0379
2.0	one var	1.090	0.0092	1.857	0.0268	1.976	0.0278
	equal	1.107	0.0100	2.169	0.0132	2.193	0.0144
	opposite	1.103	0.0100	1.348	0.0178	1.537	0.0230
2.5	one var	1.016	0.0040	1.748	0.0241	1.752	0.0230
	equal	1.022	0.0049	2.123	0.0114	2.152	0.0127
	opposite	1.016	0.0040	1.177	0.0129	1.277	0.0163
3.0	one var	1.003	0.0017	1.692	0.0217	1.706	0.0211
	equal	1.001	0.0010	2.068	0.0081	2.120	0.0109
	opposite	1.006	0.0024	1.088	0.0091	1.130	0.0112
		UCL = 12.227		UCL = 8.288		UCL = 9.767	

Table 7.2 (continued): ARL values of EWMA charts for 10% mixture normal data with known Σ

$n = 5, \quad r = 0.3$

$\rho = 0.0$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.974	6.4448	199.434	6.1879	201.749	6.2519
0.5	one var	15.845	0.4377	5.470	0.0872	6.032	0.1127
	equal	15.438	0.3855	4.723	0.0759	5.376	0.1023
	opposite	15.965	0.4059	4.693	0.0776	5.404	0.1031
1.0	one var	3.399	0.0486	3.228	0.0294	2.718	0.0322
	equal	3.434	0.0494	2.364	0.0205	2.126	0.0291
	opposite	3.402	0.0480	2.351	0.0190	2.078	0.0290
1.5	one var	1.788	0.0214	2.938	0.0205	2.259	0.0218
	equal	1.757	0.0213	2.113	0.0112	1.473	0.0193
	opposite	1.768	0.0216	2.123	0.0111	1.480	0.0193
2.0	one var	1.154	0.0120	2.868	0.0181	2.170	0.0177
	equal	1.124	0.0106	2.050	0.0072	1.319	0.0166
	opposite	1.133	0.0111	2.062	0.0081	1.284	0.0156
2.5	one var	1.028	0.0052	2.821	0.0160	2.112	0.0159
	equal	1.033	0.0057	2.025	0.0051	1.240	0.0144
	opposite	1.024	0.0048	2.028	0.0056	1.212	0.0143
3.0	one var	1.004	0.0020	2.779	0.0143	2.069	0.0142
	equal	1.005	0.0022	2.012	0.0034	1.173	0.0122
	opposite	1.006	0.0024	2.017	0.0043	1.181	0.0123
		UCL = 13.942		UCL = 9.190		UCL = 9.557	

$n = 5, \quad r = 0.3$

$\rho = 0.5$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.974	6.4448	200.939	6.3461	201.807	6.2911
0.5	one var	15.845	0.4377	5.850	0.0998	6.495	0.1326
	equal	15.846	0.4054	4.611	0.0705	5.266	0.0923
	opposite	16.044	0.4110	5.355	0.1008	6.545	0.1331
1.0	one var	3.389	0.0487	3.060	0.0352	2.690	0.0392
	equal	3.474	0.0544	2.541	0.0245	2.446	0.0216
	opposite	3.394	0.0490	2.134	0.0305	2.124	0.0372
1.5	one var	1.788	0.0212	2.685	0.0271	2.162	0.0242
	equal	1.789	0.0215	2.244	0.0158	2.128	0.0117
	opposite	1.770	0.0219	1.467	0.0194	1.345	0.0186
2.0	one var	1.152	0.0119	2.561	0.0255	2.022	0.0197
	equal	1.135	0.0113	2.142	0.0118	2.107	0.0104
	opposite	1.166	0.0123	1.270	0.0155	1.128	0.0110
2.5	one var	1.033	0.0057	2.518	0.0220	1.974	0.0195
	equal	1.024	0.0048	2.095	0.0095	2.068	0.0081
	opposite	1.018	0.0042	1.180	0.0126	1.064	0.0077
3.0	one var	1.003	0.0017	2.456	0.0216	1.956	0.0183
	equal	1.007	0.0026	2.051	0.0074	2.044	0.0065
	opposite	1.004	0.0020	1.141	0.0110	1.053	0.0071
		UCL = 13.942		UCL = 9.071		UCL = 9.670	

Table 7.2 (continued): ARL values of EWMA charts for 10% mixture normal data with known Σ

$n = 5,\quad r = 0.3$

ρ = 0.9		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.974	6.4448	199.681	6.0223	199.023	6.3716
0.5	one var	15.845	0.4377	9.205	0.2086	12.736	0.3433
	equal	15.621	0.3887	4.734	0.0725	5.912	0.1004
	opposite	16.028	0.4139	9.625	0.2195	12.977	0.3525
1.0	one var	3.405	0.0484	3.126	0.0517	3.887	0.0732
	equal	3.496	0.0532	2.640	0.0265	2.869	0.0292
	opposite	3.429	0.0486	2.995	0.0522	3.632	0.0782
1.5	one var	1.779	0.0210	2.160	0.0308	2.465	0.0407
	equal	1.758	0.0216	2.303	0.0186	2.394	0.0196
	opposite	1.776	0.0210	1.774	0.0267	2.167	0.0373
2.0	one var	1.155	0.0120	1.889	0.0252	1.959	0.0288
	equal	1.131	0.0110	2.192	0.0145	2.291	0.0170
	opposite	1.157	0.0120	1.392	0.0180	1.548	0.0237
2.5	one var	1.032	0.0056	1.808	0.0238	1.761	0.0247
	equal	1.028	0.0052	2.147	0.0121	2.247	0.0156
	opposite	1.025	0.0049	1.199	0.0137	1.279	0.0159
3.0	one var	1.003	0.0017	1.698	0.0220	1.697	0.0234
	equal	1.006	0.0024	2.113	0.0104	2.199	0.0138
	opposite	1.010	0.0031	1.109	0.0103	1.137	0.0112
		UCL = 13.942		UCL = 8.398		UCL = 10.054	

$n = 10,\quad r = 0.1$

ρ = 0.0		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.824	6.3655	198.581	6.4266	201.331	6.3850
0.5	one var	3.876	0.0680	2.652	0.0350	2.867	0.0409
	equal	4.033	0.0673	2.397	0.0342	2.553	0.0390
	opposite	4.046	0.0693	2.388	0.0348	2.542	0.0397
1.0	one var	1.434	0.0193	1.846	0.0174	1.483	0.0177
	equal	1.449	0.0190	1.226	0.0139	1.260	0.0148
	opposite	1.416	0.0186	1.228	0.0139	1.251	0.0144
1.5	one var	1.031	0.0057	1.703	0.0154	1.247	0.0139
	equal	1.034	0.0057	1.088	0.0092	1.039	0.0061
	opposite	1.044	0.0065	1.086	0.0089	1.028	0.0052
2.0	one var	1.000	0.0000	1.686	0.0152	1.156	0.0115
	equal	1.002	0.0014	1.036	0.0059	1.013	0.0036
	opposite	1.001	0.0010	1.040	0.0062	1.012	0.0034
2.5	one var	1.000	0.0000	1.676	0.0148	1.158	0.0115
	equal	1.000	0.0000	1.022	0.0046	1.004	0.0020
	opposite	1.000	0.0000	1.017	0.0041	1.008	0.0028
3.0	one var	1.000	0.0000	1.638	0.0153	1.110	0.0099
	equal	1.000	0.0000	1.006	0.0024	1.003	0.0017
	opposite	1.000	0.0000	1.007	0.0026	1.002	0.0014
		UCL = 9.298		UCL = 8.676		UCL = 8.719	

Table 7.2 (continued): ARL values of EWMA charts for 10% mixture normal data with known Σ

$n = 10, \quad r = 0.1$

$\rho = 0.5$		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.824	6.3655	200.107	6.2900	200.046	6.4043
0.5	one var	3.876	0.0680	2.764	0.0397	3.039	0.0487
	equal	3.979	0.0662	2.319	0.0300	2.547	0.0376
	opposite	3.999	0.0688	2.605	0.0408	3.157	0.0563
1.0	one var	1.434	0.0192	1.639	0.0185	1.508	0.0189
	equal	1.452	0.0193	1.305	0.0158	1.240	0.0137
	opposite	1.414	0.0185	1.225	0.0142	1.327	0.0161
1.5	one var	1.029	0.0055	1.471	0.0171	1.181	0.0123
	equal	1.028	0.0052	1.155	0.0118	1.063	0.0077
	opposite	1.038	0.0060	1.049	0.0068	1.042	0.0063
2.0	one var	1.000	0.0000	1.424	0.0161	1.123	0.0105
	equal	1.000	0.0000	1.077	0.0086	1.021	0.0045
	opposite	1.002	0.0014	1.012	0.0034	1.009	0.0030
2.5	one var	1.000	0.0000	1.406	0.0156	1.104	0.0097
	equal	1.000	0.0000	1.048	0.0068	1.027	0.0051
	opposite	1.000	0.0000	1.004	0.0020	1.001	0.0010
3.0	one var	1.000	0.0000	1.375	0.0153	1.105	0.0097
	equal	1.000	0.0000	1.020	0.0044	1.018	0.0042
	opposite	1.000	0.0000	1.004	0.0020	1.000	0.0000
		UCL = 9.298		UCL = 8.368		UCL = 8.655	

$n = 10, \quad r = 0.1$

$\rho = 0.9$		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.824	6.3655	198.119	6.1575	201.728	6.7835
0.5	one var	3.876	0.0680	3.631	0.0606	5.025	0.0982
	equal	3.993	0.0674	2.190	0.0307	2.633	0.0366
	opposite	3.957	0.0671	3.841	0.0652	5.424	0.1141
1.0	one var	1.429	0.0192	1.621	0.0213	2.013	0.0313
	equal	1.448	0.0188	1.313	0.0165	1.470	0.0176
	opposite	1.436	0.0182	1.578	0.0218	2.084	0.0343
1.5	one var	1.029	0.0055	1.228	0.0140	1.415	0.0191
	equal	1.029	0.0053	1.172	0.0122	1.219	0.0134
	opposite	1.030	0.0054	1.114	0.0103	1.340	0.0186
2.0	one var	1.000	0.0000	1.107	0.0099	1.168	0.0121
	equal	1.000	0.0000	1.113	0.0100	1.169	0.0119
	opposite	1.002	0.0014	1.030	0.0054	1.103	0.0098
2.5	one var	1.000	0.0000	1.072	0.0084	1.091	0.0093
	equal	1.000	0.0000	1.061	0.0076	1.137	0.0109
	opposite	1.000	0.0000	1.006	0.0024	1.020	0.0044
3.0	one var	1.000	0.0000	1.053	0.0071	1.074	0.0084
	equal	1.000	0.0000	1.033	0.0057	1.113	0.0100
	opposite	1.000	0.0000	1.002	0.0014	1.005	0.0022
		UCL = 9.298		UCL = 7.395		UCL = 8.750	

Table 7.2 (continued): ARL values of EWMA charts for 10% mixture normal data with known Σ

$n = 10, \quad r = 0.2$

$\rho = 0.0$		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.586	6.4455	201.486	6.0722	198.800	5.7765
0.5	one var	4.788	0.0849	2.873	0.0359	2.894	0.0441
	equal	4.849	0.0840	2.529	0.0354	2.729	0.0439
	opposite	4.653	0.0837	2.543	0.0353	2.786	0.0444
1.0	one var	1.633	0.0209	1.965	0.0159	1.509	0.0186
	equal	1.580	0.0210	1.283	0.0150	1.239	0.0143
	opposite	1.584	0.0212	1.304	0.0155	1.239	0.0144
1.5	one var	1.064	0.0077	1.849	0.0144	1.255	0.0141
	equal	1.061	0.0076	1.111	0.0101	1.033	0.0057
	opposite	1.056	0.0073	1.126	0.0107	1.032	0.0056
2.0	one var	1.004	0.0020	1.796	0.0134	1.174	0.0121
	equal	1.002	0.0014	1.054	0.0072	1.013	0.0036
	opposite	1.001	0.0010	1.064	0.0077	1.008	0.0028
2.5	one var	1.000	0.0000	1.809	0.0128	1.135	0.0110
	equal	1.000	0.0000	1.028	0.0052	1.009	0.0030
	opposite	1.000	0.0000	1.029	0.0053	1.003	0.0017
3.0	one var	1.000	0.0000	1.762	0.0135	1.137	0.0109
	equal	1.000	0.0000	1.015	0.0038	1.003	0.0017
	opposite	1.000	0.0000	1.009	0.0030	1.000	0.0000
		UCL = 11.030		UCL = 9.417		UCL = 9.521	

$n = 10, \quad r = 0.2$

$\rho = 0.5$		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.586	6.4455	200.096	6.3091	198.535	6.0130
0.5	one var	4.788	0.0849	2.970	0.0418	3.282	0.0506
	equal	4.788	0.0822	2.495	0.0335	2.593	0.0391
	opposite	4.689	0.0805	2.807	0.0436	3.403	0.0580
1.0	one var	1.632	0.0210	1.841	0.0188	1.565	0.0200
	equal	1.598	0.0214	1.421	0.0182	1.262	0.0146
	opposite	1.586	0.0211	1.288	0.0151	1.387	0.0172
1.5	one var	1.065	0.0078	1.658	0.0165	1.251	0.0144
	equal	1.047	0.0067	1.236	0.0137	1.074	0.0083
	opposite	1.046	0.0066	1.077	0.0084	1.080	0.0087
2.0	one var	1.004	0.0020	1.618	0.0159	1.173	0.0121
	equal	1.002	0.0014	1.129	0.0106	1.043	0.0064
	opposite	1.003	0.0017	1.023	0.0047	1.010	0.0031
2.5	one var	1.000	0.0000	1.546	0.0158	1.157	0.0115
	equal	1.000	0.0000	1.091	0.0091	1.021	0.0045
	opposite	1.000	0.0000	1.008	0.0028	1.001	0.0010
3.0	one var	1.000	0.0000	1.544	0.0158	1.124	0.0104
	equal	1.000	0.0000	1.047	0.0067	1.012	0.0034
	opposite	1.000	0.0000	1.004	0.0020	1.000	0.0000
		UCL = 11.030		UCL = 9.136		UCL = 9.560	

Table 7.2 (continued): ARL values of EWMA charts for 10% mixture normal data with known Σ

$n = 10, \quad r = 0.2$

$\rho = 0.9$		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.586	6.4455	199.727	6.4883	199.460	6.3672
0.5	one var	4.788	0.0849	4.223	0.0670	5.560	0.1103
	equal	4.783	0.0847	2.404	0.0324	2.740	0.0424
	opposite	4.686	0.0794	4.411	0.0748	6.271	0.1410
1.0	one var	1.632	0.0208	1.805	0.0231	2.188	0.0354
	equal	1.581	0.0207	1.403	0.0185	1.440	0.0172
	opposite	1.617	0.0215	1.680	0.0239	2.104	0.0370
1.5	one var	1.064	0.0077	1.319	0.0155	1.476	0.0195
	equal	1.054	0.0073	1.222	0.0144	1.239	0.0136
	opposite	1.049	0.0070	1.148	0.0117	1.387	0.0190
2.0	one var	1.004	0.0020	1.212	0.0132	1.258	0.0140
	equal	1.003	0.0017	1.132	0.0110	1.167	0.0119
	opposite	1.004	0.0020	1.030	0.0054	1.114	0.0105
2.5	one var	1.000	0.0000	1.145	0.0112	1.134	0.0109
	equal	1.000	0.0000	1.075	0.0083	1.106	0.0098
	opposite	1.000	0.0000	1.007	0.0026	1.034	0.0057
3.0	one var	1.000	0.0000	1.113	0.0102	1.114	0.0103
	equal	1.000	0.0000	1.049	0.0068	1.104	0.0097
	opposite	1.000	0.0000	1.001	0.0010	1.005	0.0022
		UCL = 11.030		UCL = 8.168		UCL = 9.659	

$n = 10, \quad r = 0.3$

$\rho = 0.0$		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.157	6.5921	200.838	6.0673	200.816	6.6810
0.5	one var	5.657	0.1059	3.064	0.0418	3.079	0.0500
	equal	5.818	0.1122	2.633	0.0371	2.983	0.0496
	opposite	5.604	0.1091	2.587	0.0364	2.791	0.0469
1.0	one var	1.741	0.0229	2.014	0.0172	1.495	0.0197
	equal	1.742	0.0228	1.346	0.0162	1.257	0.0153
	opposite	1.735	0.0236	1.361	0.0167	1.247	0.0146
1.5	one var	1.077	0.0086	1.885	0.0148	1.286	0.0149
	equal	1.067	0.0079	1.139	0.0111	1.028	0.0052
	opposite	1.071	0.0081	1.131	0.0109	1.030	0.0056
2.0	one var	1.009	0.0030	1.847	0.0130	1.210	0.0130
	equal	1.003	0.0017	1.069	0.0081	1.009	0.0030
	opposite	1.001	0.0010	1.067	0.0079	1.011	0.0033
2.5	one var	1.000	0.0000	1.820	0.0125	1.145	0.0111
	equal	1.001	0.0010	1.027	0.0051	1.001	0.0010
	opposite	1.000	0.0000	1.020	0.0044	1.004	0.0020
3.0	one var	1.000	0.0000	1.810	0.0125	1.132	0.0107
	equal	1.000	0.0000	1.014	0.0037	1.002	0.0014
	opposite	1.001	0.0010	1.016	0.0040	1.004	0.0020
		UCL = 12.287		UCL = 9.649		UCL = 9.844	

Table 7.2 (continued): ARL values of EWMA charts for 10% mixture normal data with known Σ

$n = 10, \quad r = 0.3$

ρ = 0.5		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.157	6.5921	201.015	6.5650	198.816	6.1383
0.5	one var	5.657	0.1059	3.106	0.0448	3.463	0.0583
	equal	5.799	0.1139	2.605	0.0363	2.791	0.0416
	opposite	5.730	0.1191	2.988	0.0477	3.491	0.0558
1.0	one var	1.748	0.0231	1.891	0.0198	1.603	0.0197
	equal	1.731	0.0231	1.422	0.0177	1.410	0.0163
	opposite	1.745	0.0225	1.330	0.0160	1.429	0.0190
1.5	one var	1.077	0.0086	1.699	0.0168	1.277	0.0146
	equal	1.060	0.0075	1.244	0.0139	1.207	0.0128
	opposite	1.072	0.0084	1.077	0.0086	1.075	0.0086
2.0	one var	1.008	0.0028	1.642	0.0158	1.197	0.0127
	equal	1.003	0.0017	1.171	0.0119	1.150	0.0113
	opposite	1.002	0.0014	1.019	0.0043	1.007	0.0026
2.5	one var	1.000	0.0000	1.634	0.0155	1.167	0.0118
	equal	1.001	0.0010	1.090	0.0091	1.106	0.0097
	opposite	1.000	0.0000	1.007	0.0026	1.000	0.0000
3.0	one var	1.000	0.0000	1.598	0.0156	1.129	0.0107
	equal	1.000	0.0000	1.049	0.0068	1.075	0.0083
	opposite	1.001	0.0010	1.000	0.0000	1.002	0.0014
		UCL = 12.287		UCL = 9.415		UCL = 9.843	

$n = 10, \quad r = 0.3$

ρ = 0.9		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.157	6.5921	200.011	6.5219	200.800	6.7334
0.5	one var	5.657	0.1059	4.531	0.0803	5.920	0.1300
	equal	5.802	0.1091	2.562	0.0341	2.958	0.0453
	opposite	5.754	0.1175	4.833	0.0900	7.035	0.1597
1.0	one var	1.750	0.0231	1.843	0.0241	2.295	0.0375
	equal	1.752	0.0229	1.456	0.0185	1.462	0.0182
	opposite	1.749	0.0220	1.703	0.0235	2.148	0.0389
1.5	one var	1.079	0.0087	1.354	0.0165	1.512	0.0214
	equal	1.056	0.0073	1.249	0.0150	1.205	0.0130
	opposite	1.076	0.0086	1.168	0.0121	1.386	0.0200
2.0	one var	1.008	0.0028	1.242	0.0142	1.247	0.0144
	equal	1.008	0.0028	1.154	0.0117	1.176	0.0120
	opposite	1.003	0.0017	1.035	0.0058	1.124	0.0109
2.5	one var	1.001	0.0010	1.201	0.0128	1.137	0.0112
	equal	1.000	0.0000	1.104	0.0097	1.124	0.0104
	opposite	1.000	0.0000	1.006	0.0024	1.022	0.0046
3.0	one var	1.000	0.0000	1.146	0.0113	1.086	0.0089
	equal	1.000	0.0000	1.056	0.0073	1.094	0.0092
	opposite	1.000	0.0000	1.002	0.0014	1.005	0.0022
		UCL = 12.287		UCL = 8.477		UCL = 10.085	

Table 7.2 (continued): ARL values of EWMA charts for 10% mixture normal data with known Σ

$n = 15, \quad r = 0.1$

$\rho = 0.0$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.915	6.5862	198.716	6.1492	200.560	6.3717
0.5	one var	2.862	0.0452	1.926	0.0272	2.135	0.0296
	equal	2.892	0.0475	1.746	0.0239	1.969	0.0278
	opposite	2.925	0.0476	1.749	0.0244	1.942	0.0261
1.0	one var	1.152	0.0119	1.281	0.0144	1.202	0.0127
	equal	1.126	0.0112	1.067	0.0079	1.072	0.0082
	opposite	1.146	0.0119	1.068	0.0080	1.069	0.0080
1.5	one var	1.005	0.0022	1.149	0.0114	1.062	0.0076
	equal	1.006	0.0024	1.013	0.0036	1.003	0.0017
	opposite	1.004	0.0020	1.018	0.0042	1.002	0.0014
2.0	one var	1.000	0.0000	1.086	0.0089	1.036	0.0059
	equal	1.000	0.0000	1.003	0.0017	1.000	0.0000
	opposite	1.000	0.0000	1.001	0.0010	1.000	0.0000
2.5	one var	1.000	0.0000	1.040	0.0062	1.022	0.0046
	equal	1.000	0.0000	1.001	0.0010	1.000	0.0000
	opposite	1.000	0.0000	1.000	0.0000	1.001	0.0010
3.0	one var	1.000	0.0000	1.015	0.0038	1.010	0.0031
	equal	1.000	0.0000	1.000	0.0000	1.000	0.0000
	opposite	1.000	0.0000	1.000	0.0000	1.000	0.0000
		UCL = 9.239		UCL = 8.774		UCL = 8.750	

$n = 15, \quad r = 0.1$

$\rho = 0.5$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.915	6.5862	198.988	6.1228	200.655	6.5647
0.5	one var	2.862	0.0452	1.990	0.0265	2.315	0.0334
	equal	2.902	0.0459	1.672	0.0238	1.851	0.0264
	opposite	2.946	0.0493	1.954	0.0274	2.283	0.0366
1.0	one var	1.153	0.0119	1.230	0.0137	1.191	0.0127
	equal	1.142	0.0115	1.097	0.0095	1.041	0.0063
	opposite	1.156	0.0120	1.074	0.0083	1.106	0.0097
1.5	one var	1.006	0.0024	1.114	0.0103	1.048	0.0068
	equal	1.004	0.0020	1.021	0.0045	1.002	0.0014
	opposite	1.004	0.0020	1.005	0.0022	1.000	0.0000
2.0	one var	1.000	0.0000	1.064	0.0077	1.030	0.0054
	equal	1.000	0.0000	1.009	0.0030	1.002	0.0014
	opposite	1.000	0.0000	1.003	0.0017	1.000	0.0000
2.5	one var	1.000	0.0000	1.048	0.0068	1.022	0.0046
	equal	1.000	0.0000	1.003	0.0017	1.000	0.0000
	opposite	1.000	0.0000	1.000	0.0000	1.000	0.0000
3.0	one var	1.000	0.0000	1.013	0.0036	1.007	0.0026
	equal	1.000	0.0000	1.001	0.0010	1.000	0.0000
	opposite	1.000	0.0000	1.000	0.0000	1.000	0.0000
		UCL = 9.239		UCL = 8.411		UCL = 8.751	

Table 7.2 (continued): ARL values of EWMA charts for 10% mixture normal data with known Σ

$n = 15, \quad r = 0.1$

ρ = 0.9		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.915	6.5862	198.841	6.3506	199.800	6.1559
0.5	one var	2.862	0.0452	2.640	0.0409	3.826	0.0695
	equal	2.884	0.0452	1.632	0.0222	1.944	0.0277
	opposite	2.908	0.0493	2.772	0.0428	3.991	0.0761
1.0	one var	1.153	0.0119	1.241	0.0138	1.595	0.0231
	equal	1.162	0.0122	1.131	0.0110	1.095	0.0093
	opposite	1.150	0.0117	1.225	0.0135	1.619	0.0240
1.5	one var	1.005	0.0022	1.056	0.0073	1.124	0.0105
	equal	1.004	0.0020	1.037	0.0060	1.011	0.0033
	opposite	1.001	0.0010	1.014	0.0037	1.147	0.0116
2.0	one var	1.000	0.0000	1.023	0.0047	1.026	0.0050
	equal	1.000	0.0000	1.011	0.0033	1.006	0.0024
	opposite	1.000	0.0000	1.000	0.0000	1.034	0.0059
2.5	one var	1.000	0.0000	1.013	0.0036	1.009	0.0030
	equal	1.000	0.0000	1.007	0.0026	1.002	0.0014
	opposite	1.000	0.0000	1.000	0.0000	1.005	0.0022
3.0	one var	1.000	0.0000	1.007	0.0026	1.001	0.0010
	equal	1.000	0.0000	1.001	0.0010	1.003	0.0017
	opposite	1.000	0.0000	1.000	0.0000	1.000	0.0000
		UCL = 9.239		UCL = 7.370		UCL = 8.753	

$n = 15, \quad r = 0.2$

ρ = 0.0		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.407	6.4190	200.957	6.6796	201.344	6.3997
0.5	one var	3.287	0.0533	2.122	0.0276	2.160	0.0292
	equal	3.328	0.0522	1.829	0.0243	2.097	0.0309
	opposite	3.211	0.0490	1.899	0.0265	2.057	0.0293
1.0	one var	1.228	0.0144	1.288	0.0150	1.169	0.0119
	equal	1.233	0.0141	1.084	0.0088	1.066	0.0079
	opposite	1.214	0.0138	1.075	0.0087	1.080	0.0086
1.5	one var	1.008	0.0028	1.189	0.0125	1.063	0.0077
	equal	1.004	0.0020	1.015	0.0038	1.002	0.0014
	opposite	1.005	0.0022	1.007	0.0026	1.001	0.0010
2.0	one var	1.000	0.0000	1.091	0.0091	1.030	0.0054
	equal	1.000	0.0000	1.005	0.0022	1.000	0.0000
	opposite	1.000	0.0000	1.005	0.0022	1.001	0.0010
2.5	one var	1.000	0.0000	1.057	0.0073	1.021	0.0045
	equal	1.000	0.0000	1.001	0.0010	1.001	0.0010
	opposite	1.000	0.0000	1.000	0.0000	1.000	0.0000
3.0	one var	1.000	0.0000	1.022	0.0046	1.014	0.0037
	equal	1.000	0.0000	1.000	0.0000	1.000	0.0000
	opposite	1.000	0.0000	1.000	0.0000	1.000	0.0000
		UCL = 10.709		UCL = 9.598		UCL = 9.584	

Table 7.2 (continued): ARL values of EWMA charts for 10% mixture normal data with known Σ

$n = 15,\ r = 0.2$

ρ = 0.5		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.407	6.4190	200.200	6.6022	200.146	6.1620
0.5	one var	3.287	0.0533	2.179	0.0284	2.339	0.0349
	equal	3.301	0.0510	1.799	0.0242	1.948	0.0264
	opposite	3.185	0.0506	2.040	0.0293	2.393	0.0397
1.0	one var	1.218	0.0142	1.287	0.0151	1.217	0.0131
	equal	1.222	0.0140	1.135	0.0109	1.068	0.0080
	opposite	1.231	0.0139	1.090	0.0093	1.099	0.0096
1.5	one var	1.008	0.0028	1.174	0.0121	1.054	0.0072
	equal	1.006	0.0024	1.037	0.0060	1.009	0.0030
	opposite	1.008	0.0028	1.005	0.0022	1.003	0.0017
2.0	one var	1.000	0.0000	1.080	0.0086	1.032	0.0056
	equal	1.000	0.0000	1.013	0.0036	1.003	0.0017
	opposite	1.001	0.0010	1.001	0.0010	1.002	0.0014
2.5	one var	1.000	0.0000	1.045	0.0066	1.017	0.0041
	equal	1.000	0.0000	1.006	0.0024	1.001	0.0010
	opposite	1.000	0.0000	1.000	0.0000	1.000	0.0000
3.0	one var	1.000	0.0000	1.020	0.0044	1.010	0.0031
	equal	1.000	0.0000	1.002	0.0014	1.000	0.0000
	opposite	1.000	0.0000	1.000	0.0000	1.000	0.0000
		UCL = 10.709		UCL = 9.217		UCL = 9.604	

$n = 15,\ r = 0.2$

ρ = 0.9		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.407	6.4190	201.284	6.4450	198.615	6.3475
0.5	one var	3.287	0.0533	2.914	0.0431	4.133	0.0751
	equal	3.307	0.0514	1.759	0.0226	2.126	0.0264
	opposite	3.187	0.0512	3.120	0.0482	4.397	0.0827
1.0	one var	1.224	0.0144	1.320	0.0157	1.723	0.0235
	equal	1.217	0.0138	1.240	0.0139	1.187	0.0123
	opposite	1.237	0.0140	1.278	0.0151	1.730	0.0251
1.5	one var	1.008	0.0028	1.096	0.0094	1.220	0.0134
	equal	1.004	0.0020	1.124	0.0104	1.057	0.0073
	opposite	1.005	0.0022	1.020	0.0044	1.144	0.0118
2.0	one var	1.000	0.0000	1.029	0.0053	1.074	0.0083
	equal	1.000	0.0000	1.055	0.0072	1.032	0.0056
	opposite	1.001	0.0010	1.002	0.0014	1.030	0.0054
2.5	one var	1.000	0.0000	1.025	0.0049	1.019	0.0043
	equal	1.000	0.0000	1.031	0.0055	1.014	0.0037
	opposite	1.000	0.0000	1.000	0.0000	1.005	0.0022
3.0	one var	1.000	0.0000	1.012	0.0034	1.017	0.0041
	equal	1.000	0.0000	1.005	0.0022	1.016	0.0040
	opposite	1.000	0.0000	1.000	0.0000	1.001	0.0010
		UCL = 10.709		UCL = 8.160		UCL = 9.653	

Table 7.2 (continued): ARL values of EWMA charts for 10% mixture normal data with known Σ

$n = 15, \quad r = 0.3$

ρ = 0.0		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.227	6.7748	199.387	6.4043	199.704	6.1020
0.5	one var	3.545	0.0582	2.156	0.0267	2.308	0.0326
	equal	3.619	0.0635	1.900	0.0266	2.162	0.0301
	opposite	3.608	0.0641	1.907	0.0261	2.151	0.0321
1.0	one var	1.266	0.0153	1.326	0.0157	1.252	0.0139
	equal	1.266	0.0151	1.100	0.0095	1.084	0.0088
	opposite	1.283	0.0151	1.112	0.0101	1.076	0.0084
1.5	one var	1.007	0.0026	1.191	0.0126	1.065	0.0078
	equal	1.013	0.0036	1.018	0.0042	1.004	0.0020
	opposite	1.006	0.0024	1.022	0.0046	1.003	0.0017
2.0	one var	1.000	0.0000	1.089	0.0090	1.030	0.0054
	equal	1.000	0.0000	1.005	0.0022	1.001	0.0010
	opposite	1.001	0.0010	1.004	0.0020	1.000	0.0000
2.5	one var	1.000	0.0000	1.063	0.0077	1.027	0.0051
	equal	1.000	0.0000	1.000	0.0000	1.000	0.0000
	opposite	1.000	0.0000	1.003	0.0017	1.000	0.0000
3.0	one var	1.000	0.0000	1.026	0.0050	1.011	0.0033
	equal	1.000	0.0000	1.000	0.0000	1.000	0.0000
	opposite	1.000	0.0000	1.001	0.0010	1.000	0.0000
		UCL = 11.618		UCL = 9.797		UCL = 9.916	

$n = 15, \quad r = 0.3$

ρ = 0.5		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.227	6.7748	201.854	6.1265	201.638	6.1656
0.5	one var	3.545	0.0582	2.260	0.0293	2.554	0.0384
	equal	3.629	0.0617	1.867	0.0251	1.973	0.0283
	opposite	3.588	0.0635	2.096	0.0308	2.490	0.0392
1.0	one var	1.266	0.0153	1.339	0.0161	1.293	0.0147
	equal	1.272	0.0150	1.145	0.0111	1.065	0.0078
	opposite	1.274	0.0153	1.088	0.0090	1.164	0.0120
1.5	one var	1.007	0.0026	1.169	0.0120	1.070	0.0081
	equal	1.011	0.0033	1.040	0.0062	1.010	0.0031
	opposite	1.008	0.0028	1.009	0.0030	1.007	0.0026
2.0	one var	1.000	0.0000	1.112	0.0101	1.029	0.0053
	equal	1.000	0.0000	1.018	0.0042	1.001	0.0010
	opposite	1.001	0.0010	1.002	0.0014	1.000	0.0000
2.5	one var	1.000	0.0000	1.048	0.0068	1.022	0.0046
	equal	1.000	0.0000	1.004	0.0020	1.000	0.0000
	opposite	1.000	0.0000	1.001	0.0010	1.000	0.0000
3.0	one var	1.000	0.0000	1.030	0.0054	1.011	0.0033
	equal	1.000	0.0000	1.000	0.0000	1.000	0.0000
	opposite	1.000	0.0000	1.000	0.0000	1.000	0.0000
		UCL = 11.618		UCL = 9.493		UCL = 9.916	

Table 7.2 (continued): ARL values of EWMA charts for 10% mixture normal data with known Σ

$n = 15, \quad r = 0.3$

$\rho = 0.9$		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.227	6.7748	198.535	6.2716	200.308	6.2437
0.5	one var	3.545	0.0582	3.129	0.0488	4.451	0.0879
	equal	3.629	0.0616	1.841	0.0238	2.186	0.0300
	opposite	3.548	0.0618	3.267	0.0514	4.757	0.0946
1.0	one var	1.269	0.0155	1.353	0.0164	1.729	0.0251
	equal	1.264	0.0147	1.239	0.0138	1.195	0.0125
	opposite	1.268	0.0148	1.252	0.0151	1.701	0.0259
1.5	one var	1.007	0.0026	1.095	0.0094	1.225	0.0141
	equal	1.009	0.0030	1.128	0.0107	1.055	0.0072
	opposite	1.009	0.0030	1.022	0.0046	1.164	0.0123
2.0	one var	1.000	0.0000	1.042	0.0063	1.067	0.0080
	equal	1.000	0.0000	1.064	0.0077	1.035	0.0058
	opposite	1.000	0.0000	1.002	0.0014	1.026	0.0050
2.5	one var	1.000	0.0000	1.031	0.0055	1.023	0.0047
	equal	1.000	0.0000	1.023	0.0047	1.015	0.0038
	opposite	1.000	0.0000	1.000	0.0000	1.002	0.0014
3.0	one var	1.000	0.0000	1.018	0.0042	1.016	0.0040
	equal	1.000	0.0000	1.013	0.0036	1.012	0.0034
	opposite	1.000	0.0000	1.000	0.0000	1.001	0.0010
		UCL = 11.618		UCL = 8.458		UCL = 10.127	

Table 7.3: ARL values of EWMA charts for bivariate data from a t(3) distribution with known Σ

$n = 5, \quad r = 0.1$

ρ = 0.0		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.563	6.5024	199.110	6.1573	201.721	6.3965
0.5	one var	7.303	0.1361	5.388	0.0819	5.542	0.0890
	equal	7.310	0.1402	4.775	0.0800	4.919	0.0836
	opposite	7.116	0.1355	4.794	0.0768	5.064	0.0902
1.0	one var	2.276	0.0334	3.025	0.0283	2.831	0.0330
	equal	2.449	0.0358	2.395	0.0209	2.298	0.0306
	opposite	2.374	0.0369	2.400	0.0214	2.232	0.0283
1.5	one var	1.315	0.0170	2.661	0.0201	2.304	0.0206
	equal	1.361	0.0171	2.056	0.0074	1.625	0.0192
	opposite	1.349	0.0165	2.065	0.0078	1.610	0.0192
2.0	one var	1.042	0.0065	2.530	0.0176	2.113	0.0150
	equal	1.041	0.0063	2.006	0.0024	1.375	0.0160
	opposite	1.043	0.0064	2.009	0.0033	1.410	0.0164
2.5	one var	1.009	0.0030	2.450	0.0162	2.049	0.0127
	equal	1.003	0.0017	2.004	0.0020	1.231	0.0133
	opposite	1.002	0.0014	2.004	0.0020	1.235	0.0137
3.0	one var	1.005	0.0022	2.455	0.0163	2.003	0.0113
	equal	1.000	0.0000	2.002	0.0014	1.175	0.0121
	opposite	1.000	0.0000	2.002	0.0014	1.181	0.0123
		UCL = 9.441		UCL = 8.524		UCL = 8.595	

$n = 5, \quad r = 0.2$

ρ = 0.0		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.040	6.1223	198.290	6.5534	200.313	6.2393
0.5	one var	10.037	0.2045	5.962	0.0977	6.004	0.1043
	equal	10.358	0.2294	5.230	0.0863	5.564	0.1015
	opposite	10.291	0.2099	5.220	0.0849	5.443	0.1034
1.0	one var	2.816	0.0388	3.276	0.0289	2.970	0.0370
	equal	2.898	0.0422	2.501	0.0236	2.318	0.0315
	opposite	2.937	0.0446	2.488	0.0218	2.270	0.0307
1.5	one var	1.580	0.0198	2.893	0.0181	2.352	0.0232
	equal	1.556	0.0196	2.094	0.0098	1.632	0.0195
	opposite	1.546	0.0192	2.088	0.0093	1.658	0.0189
2.0	one var	1.106	0.0098	2.761	0.0164	2.131	0.0171
	equal	1.093	0.0093	2.011	0.0033	1.379	0.0159
	opposite	1.104	0.0098	2.025	0.0049	1.400	0.0165
2.5	one var	1.007	0.0026	2.736	0.0153	2.027	0.0135
	equal	1.009	0.0030	2.002	0.0014	1.265	0.0142
	opposite	1.014	0.0037	2.006	0.0024	1.248	0.0137
3.0	one var	1.001	0.0010	2.703	0.0151	2.009	0.0114
	equal	1.000	0.0000	2.000	0.0000	1.184	0.0123
	opposite	1.000	0.0000	2.003	0.0017	1.166	0.0118
		UCL = 11.914		UCL = 9.178		UCL = 9.385	

Table 7.3 (continued): ARL values of EWMA charts for data from a t(3) distribution with known Σ

$n = 5, \quad r = 0.3$

$\rho = 0.0$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.597	6.0814	201.337	6.1696	198.258	6.5123
0.5	one var	17.347	0.4746	6.437	0.1161	6.862	0.1371
	equal	17.719	0.4486	5.735	0.0975	6.031	0.1142
	opposite	18.334	0.4999	5.606	0.1053	5.819	0.1158
1.0	one var	3.417	0.0492	3.438	0.0331	3.131	0.0427
	equal	3.489	0.0513	2.560	0.0249	2.375	0.0337
	opposite	3.426	0.0516	2.532	0.0236	2.339	0.0348
1.5	one var	1.738	0.0208	2.944	0.0191	2.319	0.0242
	equal	1.735	0.0219	2.133	0.0116	1.656	0.0207
	opposite	1.783	0.0224	2.101	0.0103	1.703	0.0216
2.0	one var	1.161	0.0122	2.819	0.0166	2.123	0.0170
	equal	1.174	0.0121	2.022	0.0046	1.385	0.0167
	opposite	1.193	0.0129	2.025	0.0049	1.418	0.0167
2.5	one var	1.017	0.0041	2.751	0.0151	2.049	0.0130
	equal	1.010	0.0031	2.008	0.0028	1.232	0.0137
	opposite	1.019	0.0043	2.002	0.0014	1.252	0.0140
3.0	one var	1.002	0.0014	2.749	0.0150	1.981	0.0114
	equal	1.001	0.0010	2.001	0.0010	1.164	0.0120
	opposite	1.000	0.0000	2.002	0.0014	1.155	0.0115
		UCL = 14.013		UCL = 9.212		UCL = 9.464	

$n = 10, \quad r = 0.1$

$\rho = 0.0$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.009	6.8853	199.975	6.2839	198.055	5.9886
0.5	one var	3.856	0.0644	2.988	0.0415	3.163	0.0489
	equal	4.022	0.0653	2.673	0.0395	2.887	0.0443
	opposite	4.026	0.0722	2.705	0.0418	2.866	0.0452
1.0	one var	1.400	0.0183	1.858	0.0167	1.670	0.0203
	equal	1.437	0.0189	1.316	0.0156	1.385	0.0172
	opposite	1.484	0.0193	1.312	0.0152	1.359	0.0163
1.5	one var	1.031	0.0057	1.711	0.0150	1.312	0.0147
	equal	1.018	0.0042	1.058	0.0074	1.074	0.0083
	opposite	1.035	0.0058	1.074	0.0083	1.072	0.0082
2.0	one var	1.004	0.0020	1.654	0.0152	1.164	0.0117
	equal	1.000	0.0000	1.016	0.0040	1.010	0.0031
	opposite	1.000	0.0000	1.011	0.0033	1.016	0.0040
2.5	one var	1.000	0.0000	1.644	0.0151	1.093	0.0092
	equal	1.000	0.0000	1.004	0.0020	1.001	0.0010
	opposite	1.000	0.0000	1.002	0.0014	1.003	0.0017
3.0	one var	1.000	0.0000	1.638	0.0152	1.060	0.0075
	equal	1.000	0.0000	1.002	0.0014	1.000	0.0000
	opposite	1.000	0.0000	1.002	0.0014	1.001	0.0010
		UCL = 9.283		UCL = 8.730		UCL = 8.754	

Table 7.3 (continued): ARL values of EWMA charts for data from a t(3) distribution with known Σ

$n = 10, \quad r = 0.2$

ρ = 0.0		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.051	6.3677	200.803	6.3456	200.889	6.6399
0.5	one var	4.970	0.0902	3.338	0.0464	3.259	0.0526
	equal	4.962	0.0878	3.031	0.0467	3.046	0.0508
	opposite	4.856	0.0801	2.975	0.0447	3.005	0.0478
1.0	one var	1.659	0.0219	1.997	0.0162	1.683	0.0219
	equal	1.648	0.0208	1.424	0.0173	1.380	0.0172
	opposite	1.666	0.0212	1.397	0.0168	1.379	0.0171
1.5	one var	1.051	0.0070	1.838	0.0129	1.328	0.0156
	equal	1.059	0.0075	1.104	0.0097	1.085	0.0089
	opposite	1.050	0.0069	1.099	0.0094	1.066	0.0079
2.0	one var	1.005	0.0022	1.771	0.0134	1.162	0.0118
	equal	1.002	0.0014	1.020	0.0044	1.014	0.0037
	opposite	1.000	0.0000	1.020	0.0044	1.013	0.0036
2.5	one var	1.000	0.0000	1.797	0.0127	1.091	0.0091
	equal	1.000	0.0000	1.008	0.0028	1.003	0.0017
	opposite	1.000	0.0000	1.004	0.0020	1.001	0.0010
3.0	one var	1.000	0.0000	1.758	0.0136	1.056	0.0073
	equal	1.000	0.0000	1.001	0.0010	1.000	0.0000
	opposite	1.000	0.0000	1.001	0.0010	1.000	0.0000
		UCL = 11.295		UCL = 9.471		UCL = 9.559	

$n = 10, \quad r = 0.3$

ρ = 0.0		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
λ	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.962	6.6037	200.695	6.2573	200.745	6.2317
0.5	one var	6.142	0.1077	3.456	0.0479	3.552	0.0588
	equal	6.115	0.1133	3.074	0.0467	3.151	0.0528
	opposite	6.365	0.1235	3.059	0.0472	3.295	0.0587
1.0	one var	1.794	0.0231	2.066	0.0172	1.750	0.0229
	equal	1.813	0.0237	1.418	0.0170	1.425	0.0180
	opposite	1.803	0.0238	1.493	0.0177	1.376	0.0173
1.5	one var	1.077	0.0086	1.879	0.0130	1.328	0.0163
	equal	1.063	0.0077	1.098	0.0094	1.077	0.0084
	opposite	1.071	0.0082	1.109	0.0099	1.084	0.0088
2.0	one var	1.013	0.0036	1.845	0.0124	1.174	0.0124
	equal	1.003	0.0017	1.027	0.0051	1.009	0.0030
	opposite	1.003	0.0017	1.015	0.0038	1.008	0.0028
2.5	one var	1.001	0.0010	1.842	0.0120	1.084	0.0088
	equal	1.000	0.0000	1.007	0.0026	1.000	0.0000
	opposite	1.000	0.0000	1.008	0.0028	1.002	0.0014
3.0	one var	1.001	0.0010	1.802	0.0128	1.074	0.0083
	equal	1.000	0.0000	1.002	0.0014	1.000	0.0000
	opposite	1.000	0.0000	1.001	0.0010	1.000	0.0000
		UCL = 12.838		UCL = 9.683		UCL = 9.849	

Table 7.4: ARL values of EWMA charts for 10% mixture bivariate normal data with known Σ

$n = 5, \quad r = 0.1$

ρ = 0.0		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.028	6.7915	201.444	6.0244	199.516	6.2239
0.5	one var	32.596	0.9355	13.991	0.3124	15.445	0.3346
	equal	32.111	0.8135	13.200	0.2747	15.548	0.3409
	opposite	31.026	0.8545	13.496	0.2787	15.949	0.3596
1.0	one var	9.214	0.1896	5.331	0.0830	5.436	0.0912
	equal	9.097	0.1790	4.847	0.0806	5.292	0.0948
	opposite	9.194	0.1821	5.002	0.0829	5.293	0.0948
1.5	one var	4.644	0.0792	3.581	0.0430	3.524	0.0485
	equal	4.657	0.0841	2.945	0.0360	3.028	0.0452
	opposite	4.631	0.0755	3.072	0.0379	3.088	0.0453
2.0	one var	2.866	0.0430	3.116	0.0301	2.772	0.0308
	equal	2.929	0.0441	2.384	0.0213	2.289	0.0293
	opposite	2.931	0.0445	2.388	0.0202	2.223	0.0298
2.5	one var	2.117	0.0289	2.856	0.0260	2.388	0.0237
	equal	2.088	0.0304	2.216	0.0150	1.835	0.0227
	opposite	2.070	0.0279	2.168	0.0139	1.787	0.0223
3.0	one var	1.646	0.0224	2.751	0.0226	2.282	0.0206
	equal	1.655	0.0211	2.114	0.0112	1.609	0.0200
	opposite	1.642	0.0217	2.139	0.0126	1.547	0.0189
		UCL = 10.000		UCL = 8.582		UCL = 8.693	

$n = 5, \quad r = 0.1$

ρ = 0.5		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.028	6.7915	200.921	6.1625	199.439	6.6282
0.5	one var	32.596	0.9355	15.033	0.3415	17.965	0.4075
	equal	31.848	0.8177	11.987	0.2682	14.832	0.3225
	opposite	31.214	0.8530	16.236	0.3727	19.764	0.4622
1.0	one var	9.239	0.1896	5.619	0.0896	6.220	0.1093
	equal	9.180	0.1823	4.686	0.0717	5.098	0.0827
	opposite	9.337	0.1808	5.455	0.0938	6.356	0.1244
1.5	one var	4.595	0.0787	3.596	0.0504	3.674	0.0572
	equal	4.667	0.0812	2.982	0.0357	3.158	0.0415
	opposite	4.630	0.0766	3.127	0.0469	3.476	0.0617
2.0	one var	2.876	0.0430	2.985	0.0355	2.845	0.0383
	equal	2.941	0.0432	2.522	0.0245	2.482	0.0231
	opposite	2.935	0.0438	2.300	0.0325	2.351	0.0383
2.5	one var	2.115	0.0292	2.652	0.0293	2.333	0.0298
	equal	2.089	0.0303	2.295	0.0183	2.199	0.0143
	opposite	2.057	0.0277	1.806	0.0236	1.813	0.0287
3.0	one var	1.644	0.0224	2.518	0.0264	2.140	0.0237
	equal	1.643	0.0214	2.199	0.0150	2.112	0.0110
	opposite	1.644	0.0215	1.578	0.0201	1.504	0.0213
		UCL = 10.000		UCL = 8.442		UCL = 8.736	

Table 7.4 (continued): ARL values of EWMA charts for 10% mixture normal data with known Σ

$n = 5, \quad r = 0.1$

$\rho = 0.9$		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.028	6.7915	201.924	6.7909	198.663	6.2262
0.5	one var	32.596	0.9355	25.451	0.6135	34.401	0.8992
	equal	30.934	0.8134	11.671	0.2521	14.993	0.3452
	opposite	30.926	0.8382	28.835	0.7246	38.057	1.0462
1.0	one var	9.232	0.1906	8.546	0.1612	10.947	0.2531
	equal	9.080	0.1748	4.596	0.0737	5.425	0.0862
	opposite	9.249	0.1823	8.511	0.1723	11.190	0.2567
1.5	one var	4.695	0.0832	4.519	0.0789	6.004	0.1140
	equal	4.650	0.0806	2.984	0.0355	3.373	0.0405
	opposite	4.652	0.0786	4.594	0.0818	6.120	0.1275
2.0	one var	2.932	0.0433	3.227	0.0518	3.961	0.0761
	equal	2.974	0.0461	2.526	0.0245	2.711	0.0278
	opposite	2.882	0.0420	2.997	0.0517	4.048	0.0820
2.5	one var	2.097	0.0283	2.459	0.0375	3.102	0.0523
	equal	2.081	0.0288	2.320	0.0193	2.430	0.0189
	opposite	2.067	0.0285	2.263	0.0374	3.085	0.0579
3.0	one var	1.646	0.0217	2.099	0.0293	2.501	0.0398
	equal	1.668	0.0217	2.232	0.0161	2.293	0.0162
	opposite	1.647	0.0209	1.829	0.0286	2.410	0.0423
		UCL = 10.000		UCL = 7.540		UCL = 8.874	

$n = 5, \quad r = 0.2$

$\rho = 0.0$		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.457	6.1901	201.850	6.0263	198.504	6.1954
0.5	one var	50.344	1.4677	16.973	0.4132	19.256	0.4649
	equal	53.778	1.6227	16.957	0.4032	18.897	0.4828
	opposite	55.588	1.7076	16.121	0.3846	19.476	0.5021
1.0	one var	13.533	0.3146	5.847	0.0915	6.211	0.1091
	equal	13.269	0.2946	5.288	0.0921	5.940	0.1060
	opposite	13.382	0.3221	5.229	0.0844	5.714	0.1027
1.5	one var	5.904	0.1000	3.918	0.0450	3.775	0.0542
	equal	5.839	0.1057	3.132	0.0386	3.230	0.0513
	opposite	5.856	0.1050	3.214	0.0401	3.154	0.0476
2.0	one var	3.482	0.0509	3.301	0.0321	2.867	0.0349
	equal	3.456	0.0535	2.492	0.0243	2.290	0.0311
	opposite	3.473	0.0506	2.541	0.0254	2.291	0.0333
2.5	one var	2.426	0.0321	3.052	0.0251	2.439	0.0254
	equal	2.487	0.0333	2.242	0.0161	1.840	0.0237
	opposite	2.471	0.0326	2.238	0.0161	1.838	0.0233
3.0	one var	1.882	0.0233	2.989	0.0234	2.324	0.0227
	equal	1.885	0.0238	2.158	0.0129	1.552	0.0200
	opposite	1.868	0.0223	2.158	0.0125	1.528	0.0195
		UCL = 12.227		UCL = 9.189		UCL = 9.427	

Table 7.4 (continued): ARL values of EWMA charts for 10% mixture normal data with known Σ

$n = 5, \quad r = 0.2$

$\rho = 0.5$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.457	6.1901	199.525	6.7065	200.942	6.5494
0.5	one var	50.344	1.4677	19.162	0.4956	23.274	0.6050
	equal	54.199	1.6496	14.446	0.3300	18.451	0.4416
	opposite	54.786	1.6161	21.611	0.5793	24.711	0.6640
1.0	one var	13.458	0.3149	6.231	0.1035	6.985	0.1379
	equal	13.027	0.2883	5.054	0.0773	5.627	0.0972
	opposite	13.165	0.3130	5.946	0.1135	7.339	0.1518
1.5	one var	5.908	0.1028	3.885	0.0520	3.977	0.0626
	equal	5.858	0.1006	3.133	0.0367	3.398	0.0440
	opposite	5.754	0.1003	3.333	0.0505	3.687	0.0655
2.0	one var	3.499	0.0530	3.117	0.0362	2.928	0.0412
	equal	3.459	0.0537	2.588	0.0254	2.600	0.0244
	opposite	3.513	0.0515	2.378	0.0340	2.529	0.0429
2.5	one var	2.424	0.0323	2.866	0.0301	2.518	0.0316
	equal	2.465	0.0332	2.359	0.0198	2.310	0.0174
	opposite	2.512	0.0339	1.860	0.0262	1.910	0.0312
3.0	one var	1.904	0.0239	2.746	0.0292	2.271	0.0261
	equal	1.920	0.0236	2.261	0.0170	2.190	0.0134
	opposite	1.900	0.0229	1.603	0.0213	1.549	0.0228
		UCL = 12.227		UCL = 9.056		UCL = 9.506	

$n = 5, \quad r = 0.2$

$\rho = 0.9$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.457	6.1901	198.190	6.3260	200.715	6.5536
0.5	one var	50.344	1.4677	38.155	1.0920	48.565	1.4981
	equal	53.795	1.6059	14.150	0.3271	19.641	0.4857
	opposite	53.598	1.5216	41.997	1.1916	46.722	1.4315
1.0	one var	13.313	0.3118	10.233	0.2298	13.584	0.3259
	equal	13.283	0.2986	4.966	0.0755	6.056	0.1050
	opposite	13.117	0.3138	10.315	0.2326	13.779	0.3390
1.5	one var	5.859	0.1068	5.015	0.0926	6.716	0.1482
	equal	5.923	0.1040	3.260	0.0370	3.558	0.0496
	opposite	5.737	0.1004	5.129	0.0979	7.086	0.1569
2.0	one var	3.521	0.0530	3.352	0.0556	4.195	0.0778
	equal	3.543	0.0517	2.757	0.0311	2.850	0.0321
	opposite	3.553	0.0539	3.294	0.0584	4.397	0.0883
2.5	one var	2.473	0.0334	2.738	0.0404	3.136	0.0524
	equal	2.407	0.0317	2.415	0.0218	2.442	0.0227
	opposite	2.453	0.0324	2.371	0.0373	3.192	0.0594
3.0	one var	1.855	0.0236	2.289	0.0337	2.704	0.0447
	equal	1.880	0.0232	2.322	0.0186	2.324	0.0184
	opposite	1.931	0.0237	1.952	0.0299	2.437	0.0440
		UCL = 12.227		UCL = 8.288		UCL = 9.767	

Table 7.4 (continued): ARL values of EWMA charts for 10% mixture normal data with known Σ

$n = 5, \quad r = 0.3$

ρ = 0.0		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.974	6.4448	199.434	6.1879	201.749	6.2519
0.5	one var	79.443	2.5424	20.500	0.5269	26.895	0.8002
	equal	74.019	2.2886	19.055	0.5245	24.446	0.6417
	opposite	76.657	2.2404	19.479	0.5039	23.866	0.6434
1.0	one var	20.311	0.5828	6.486	0.1111	6.828	0.1274
	equal	20.124	0.5659	5.498	0.0950	6.428	0.1359
	opposite	20.319	0.5356	5.662	0.1024	6.330	0.1296
1.5	one var	7.975	0.1675	4.060	0.0504	3.976	0.0643
	equal	7.784	0.1575	3.203	0.0412	3.447	0.0551
	opposite	7.851	0.1684	3.363	0.0459	3.444	0.0529
2.0	one var	4.318	0.0707	3.378	0.0343	2.930	0.0384
	equal	4.199	0.0680	2.529	0.0250	2.380	0.0343
	opposite	4.357	0.0727	2.555	0.0268	2.324	0.0328
2.5	one var	2.871	0.0386	3.154	0.0273	2.528	0.0281
	equal	2.748	0.0388	2.279	0.0173	1.938	0.0265
	opposite	2.803	0.0390	2.252	0.0166	1.902	0.0265
3.0	one var	2.073	0.0256	3.004	0.0242	2.292	0.0230
	equal	2.117	0.0282	2.157	0.0127	1.606	0.0213
	opposite	2.105	0.0252	2.177	0.0137	1.597	0.0216
		UCL = 13.942		UCL = 9.190		UCL = 9.557	

$n = 5, \quad r = 0.3$

ρ = 0.5		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.974	6.4448	200.939	6.3461	201.807	6.2911
0.5	one var	79.443	2.5424	24.420	0.6068	29.036	0.8420
	equal	73.874	2.2372	18.678	0.4831	23.679	0.6605
	opposite	76.802	2.2477	26.574	0.6951	31.978	0.9025
1.0	one var	20.311	0.5828	6.929	0.1291	8.097	0.1736
	equal	20.374	0.5711	5.477	0.0948	6.368	0.1203
	opposite	20.753	0.5600	6.596	0.1318	7.733	0.1792
1.5	one var	7.995	0.1698	4.104	0.0644	4.112	0.0697
	equal	7.729	0.1563	3.317	0.0416	3.468	0.0473
	opposite	7.670	0.1589	3.349	0.0533	3.889	0.0741
2.0	one var	4.279	0.0703	3.205	0.0379	3.018	0.0440
	equal	4.171	0.0642	2.664	0.0278	2.656	0.0272
	opposite	4.294	0.0688	2.377	0.0354	2.541	0.0439
2.5	one var	2.857	0.0388	2.930	0.0318	2.564	0.0348
	equal	2.742	0.0380	2.404	0.0216	2.329	0.0190
	opposite	2.759	0.0383	1.913	0.0266	1.911	0.0310
3.0	one var	2.059	0.0258	2.719	0.0278	2.272	0.0272
	equal	2.143	0.0278	2.314	0.0183	2.189	0.0150
	opposite	2.046	0.0246	1.581	0.0214	1.564	0.0241
		UCL = 13.942		UCL = 9.071		UCL = 9.670	

Table 7.4 (continued): ARL values of EWMA charts for 10% mixture normal data with known Σ

$n = 5, \quad r = 0.3$

$\rho = 0.9$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.974	6.4448	199.681	6.0223	199.023	6.3716
0.5	one var	79.443	2.5424	46.596	1.4264	54.186	1.7239
	equal	72.469	2.2358	18.205	0.4522	27.318	0.7734
	opposite	77.964	2.3156	51.623	1.5012	58.133	1.7201
1.0	one var	20.471	0.5894	11.745	0.2680	15.397	0.3947
	equal	20.371	0.5705	5.403	0.0867	7.118	0.1362
	opposite	20.484	0.5640	12.200	0.3122	15.766	0.4380
1.5	one var	7.958	0.1681	5.549	0.1084	7.252	0.1597
	equal	7.635	0.1544	3.454	0.0427	4.075	0.0540
	opposite	7.700	0.1589	5.632	0.1203	7.278	0.1746
2.0	one var	4.311	0.0706	3.600	0.0594	4.704	0.0951
	equal	4.153	0.0669	2.815	0.0301	3.103	0.0332
	opposite	4.367	0.0700	3.319	0.0590	4.397	0.0952
2.5	one var	2.863	0.0383	2.805	0.0457	3.397	0.0651
	equal	2.785	0.0384	2.526	0.0244	2.651	0.0248
	opposite	2.827	0.0397	2.558	0.0441	3.151	0.0648
3.0	one var	2.077	0.0258	2.414	0.0349	2.728	0.0472
	equal	2.149	0.0278	2.398	0.0211	2.471	0.0214
	opposite	2.057	0.0253	2.023	0.0314	2.453	0.0458
		UCL = 13.942		UCL = 8.398		UCL = 10.054	

$n = 10, \quad r = 0.1$

$\rho = 0.0$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.824	6.3655	198.581	6.4266	201.331	6.3850
0.5	one var	28.533	0.7206	12.865	0.2856	15.721	0.3636
	equal	28.609	0.7861	12.452	0.2708	15.249	0.3466
	opposite	29.024	0.7559	12.355	0.2803	14.603	0.3418
1.0	one var	8.371	0.1759	4.560	0.0735	5.197	0.0921
	equal	8.070	0.1694	4.252	0.0722	4.990	0.0919
	opposite	8.614	0.1775	4.234	0.0714	4.914	0.0913
1.5	one var	4.316	0.0764	2.890	0.0391	2.906	0.0458
	equal	4.311	0.0759	2.464	0.0354	2.830	0.0431
	opposite	4.314	0.0777	2.500	0.0362	2.832	0.0452
2.0	one var	2.721	0.0427	2.209	0.0243	2.121	0.0279
	equal	2.796	0.0446	1.819	0.0240	1.913	0.0263
	opposite	2.775	0.0431	1.817	0.0245	1.968	0.0272
2.5	one var	1.964	0.0285	1.987	0.0192	1.725	0.0218
	equal	1.957	0.0286	1.467	0.0180	1.515	0.0198
	opposite	2.015	0.0281	1.502	0.0187	1.554	0.0203
3.0	one var	1.544	0.0207	1.842	0.0168	1.522	0.0184
	equal	1.557	0.0209	1.305	0.0154	1.306	0.0152
	opposite	1.530	0.0206	1.291	0.0158	1.270	0.0147
		UCL = 9.298		UCL = 8.676		UCL = 8.719	

Table 7.4 (continued): ARL values of EWMA charts for 10% mixture normal data with known Σ

$n = 10,\ r = 0.1$

ρ = 0.5		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.824	6.3655	200.107	6.2900	200.046	6.4043
0.5	one var	28.533	0.7206	14.861	0.3325	17.339	0.4306
	equal	28.419	0.8013	11.112	0.2498	13.644	0.3088
	opposite	28.764	0.7565	15.451	0.3420	18.675	0.4571
1.0	one var	8.421	0.1772	4.909	0.0854	5.858	0.1048
	equal	8.140	0.1684	3.995	0.0676	4.601	0.0800
	opposite	8.743	0.1797	5.090	0.0901	6.273	0.1261
1.5	one var	4.323	0.0758	2.842	0.0420	3.269	0.0518
	equal	4.258	0.0755	2.376	0.0326	2.590	0.0396
	opposite	4.286	0.0758	2.791	0.0438	3.376	0.0582
2.0	one var	2.758	0.0431	2.201	0.0264	2.275	0.0353
	equal	2.747	0.0431	1.750	0.0239	1.904	0.0247
	opposite	2.704	0.0419	1.934	0.0278	2.214	0.0353
2.5	one var	1.992	0.0286	1.863	0.0219	1.756	0.0242
	equal	1.996	0.0283	1.427	0.0189	1.488	0.0190
	opposite	2.012	0.0287	1.505	0.0199	1.673	0.0240
3.0	one var	1.535	0.0206	1.712	0.0193	1.577	0.0202
	equal	1.577	0.0220	1.305	0.0157	1.323	0.0153
	opposite	1.530	0.0211	1.299	0.0162	1.384	0.0184
		UCL = 9.298		UCL = 8.368		UCL = 8.655	

$n = 10,\ r = 0.1$

ρ = 0.9		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.824	6.3655	198.119	6.1575	201.728	6.7835
0.5	one var	28.533	0.7206	25.757	0.6445	33.196	0.8697
	equal	28.328	0.7893	10.409	0.2193	14.174	0.3154
	opposite	28.787	0.7370	28.507	0.7369	37.835	1.0401
1.0	one var	8.409	0.1772	7.784	0.1434	10.375	0.2366
	equal	8.184	0.1676	3.877	0.0671	4.942	0.0829
	opposite	8.687	0.1807	8.144	0.1499	11.503	0.2660
1.5	one var	4.340	0.0766	3.996	0.0691	5.356	0.1046
	equal	4.257	0.0744	2.347	0.0338	2.770	0.0387
	opposite	4.242	0.0774	4.377	0.0736	5.673	0.1194
2.0	one var	2.773	0.0432	2.623	0.0396	3.646	0.0690
	equal	2.753	0.0417	1.774	0.0253	2.009	0.0259
	opposite	2.741	0.0419	2.819	0.0449	3.660	0.0698
2.5	one var	1.991	0.0286	2.045	0.0292	2.753	0.0486
	equal	1.996	0.0277	1.484	0.0199	1.661	0.0203
	opposite	2.014	0.0282	2.068	0.0305	2.755	0.0510
3.0	one var	1.529	0.0204	1.718	0.0240	2.160	0.0342
	equal	1.570	0.0213	1.353	0.0170	1.488	0.0175
	opposite	1.509	0.0206	1.605	0.0221	2.206	0.0373
		UCL = 9.298		UCL = 7.395		UCL = 8.750	

Table 7.5 Upper Control Limits for the $\overline{X}$ based MEWMA chart

UCL values for bivariate normal data with known Σ

		r		
ρ	n	0.1	0.2	0.3
0.0	5	8.756	9.645	10.152
	10	8.800	9.736	10.176
	15	8.727	9.724	10.167
0.5	5	8.756	9.645	10.152
	10	8.800	9.736	10.176
	15	8.727	9.724	10.167
0.9	5	8.756	9.645	10.152
	10	8.800	9.736	10.176
	15	8.727	9.724	10.167

UCL values for 10% mixture bivariate normal data with known Σ

		r		
ρ	n	0.1	0.2	0.3
0.0	5	10.000	12.227	13.942
	10	9.298	11.030	12.287
	15	9.239	10.709	11.618
0.5	5	10.000	12.227	13.942
	10	9.298	11.030	12.287
	15	9.239	10.709	11.618
0.9	5	10.000	12.227	13.942
	10	9.298	11.030	12.287
	15	9.239	10.709	11.618

UCL values for data from a t(3) distribution with known Σ

		r		
ρ	n	0.1	0.2	0.3
0.0	5	9.441	11.914	14.013
	10	9.283	11.295	12.838
	15			
0.5	5			
	10			
	15			
0.9	5			
	10			
	15			

Table 7.6 Upper Control Limits for the signed rank based MEWMA chart

UCL values for bivariate normal data with known Σ

		r		
ρ	*n*	0.1	0.2	0.3
0.0	5	8.535	9.116	9.133
	10	8.706	9.399	9.639
	15	8.697	9.521	9.774
0.5	5	8.342	8.913	8.975
	10	8.372	9.168	9.312
	15	8.379	9.204	9.478
0.9	5	7.417	8.070	8.197
	10	7.405	8.000	8.411
	15	7.280	8.149	8.402

UCL values for 10% mixture bivariate normal data with known Σ

		r		
ρ	*n*	0.1	0.2	0.3
0.0	5	8.582	9.189	9.190
	10	8.676	9.417	9.649
	15	8.774	9.598	9.797
0.5	5	8.442	9.056	9.071
	10	8.368	9.136	9.415
	15	8.411	9.217	9.493
0.9	5	7.540	8.288	8.398
	10	7.395	8.168	8.477
	15	7.370	8.160	8.458

UCL values for data from a t(3) distribution with known Σ

		r		
ρ	*n*	0.1	0.2	0.3
0.0	5	8.524	9.178	9.212
	10	8.730	9.471	9.683
	15			
0.5	5			
	10			
	15			
0.9	5			
	10			
	15			

Table 7.7 Upper Control Limits for the sign based MEWMA chart

UCL values for bivariate normal data with known Σ

		r		
ρ	n	0.1	0.2	0.3
0.0	5	8.735	9.387	9.537
	10	8.703	9.566	9.865
	15	8.715	9.588	9.880
0.5	5	8.683	9.413	9.551
	10	8.760	9.606	9.929
	15	8.821	9.677	10.024
0.9	5	8.852	9.692	9.889
	10	8.738	9.683	10.124
	15	8.773	9.762	10.164

UCL values for 10% mixture bivariate normal data with known Σ

		r		
ρ	n	0.1	0.2	0.3
0.0	5	8.693	9.427	9.557
	10	8.719	9.521	9.844
	15	8.750	9.584	9.916
0.5	5	8.736	9.506	9.670
	10	8.655	9.560	9.843
	15	8.751	9.604	9.916
0.9	5	8.874	9.767	10.054
	10	8.750	9.659	10.085
	15	8.753	9.653	10.127

UCL values for data from a t(3) distribution with known Σ

		r		
ρ	n	0.1	0.2	0.3
0.0	5	8.595	9.385	9.464
	10	8.754	9.559	9.849
	15			
0.5	5			
	10			
	15			
0.9	5			
	10			
	15			

CHAPTER 8

NUMERICAL RESULTS FOR THE CASE OF UNKNOWN PARAMETERS

In Chapter 7 it was observed that, in the particular nonnormal distributions studied, the nonparametric charts are faster than the MEWMA in detecting small to medium size shifts up to $\lambda \cong 1.0$ or 1.5. All tables in Chapter 7, except Table 7.4, were computed expressing the shift in the mean vector in terms of standard deviations of the population. In this chapter it is expected that, due to estimation of the unknown parameters, the nonparametric charts will be faster than the MEWMA in detecting shifts of sizes $\lambda<1.0$. In order to show smaller size shifts in more detail and determine the effect of n relative to $\sigma/\sqrt{n}$, the results presented in this chapter were computed by expressing the shifts in the mean vector in standard deviations of the statistic $\overline{X}$, using equations (6.5.26) and (6.5.27). All results were obtained using simulated data from a bivariate normal distribution and a 10% mixture bivariate normal distribution, as described in Section 6.5. The nonparametric charts are expected to perform better than the parametric chart, if the data distribution is mixture normal. The parametric chart is expected to be more efficient, if the data distribution is normal. In this chapter we will discuss the simulation results for the case where variances and covariances need to be estimated.

8.1 Chart comparisons using various estimators

For the parametric chart MEWMA, when the variances and covariances of the data are unknown, the variance-covariance matrix of $\overline{X}$ is estimated using sample variances and covariances. For the two proposed nonparametric charts, when the variances and covariances of the data are unknown, the variance-covariance matrix of the sign and signed rank is only partially unknown. As it was shown in Chapter 4, the variances of the components of the sign vector or the signed rank vector are known from nonparametric theory. Only, the covariances of the components need to be estimated from the data.

We consider four different estimators of the probability $p_{ii'}$: the One-quadrant estimator, the Two-quadrant estimator, the Four-quadrant estimator, an estimator based on a result taken from Hettmansperger, and the Constant estimator. The Two-quadrant estimator, defined in Section 4.5, is the estimator we propose in this research. All four estimators are described in Section 6.5 as equations (6.5.10) through (6.5.17). The first three estimators, the One-quadrant, the Two-quadrant, and the Four-quadrant estimator, have the restrictive features stated in Section 6.5. They were used with and without the initial value corresponding to the case of independent variables. When the estimators had no initial value, all charts gave a little better results, i.e. smaller out-of-control ARL values, than charts resulting from estimators with initial values. Unfortunately, despite the restrictive features named above, all tested cases without initial values generated at least a few warning messages about potentially unreliable inverse matrices. So, the decision was made that all estimators will be given an initial value. All results presented in this chapter were computed in this fashion. Of course, the Constant estimator, since it is always fixed, it has neither an initial value nor a restrictive feature.

First we will compare the four estimators, we will select the best one among them, and then we will use the selected estimator to compare the nonparametric charts with their parametric counterpart. In comparing estimators, one approach would be to use a criterion such as MSE to see how close an estimator is to the parameter being estimated. Our approach is to evaluate an estimator by its effect on the properties of the charts in which it is used. This is appropriate, because it is the effect that is of importance here.

Table 8.1.1 contains the ARL values of the sign based charts. The ARL values for the four estimators are given in the four columns of the table and several cases of sample size n, EWMA parameter r, and correlation ρ are examined. In summary, we observe the following. The Constant estimator does reasonably well in terms of in-control variability. Naturally, it is the best estimator when $\rho = 0$, but it deteriorates rapidly with increasing correlation between the variables. For example, when $n = 5$, $r = 0.2$, and $\rho = 0.9$, the sign based chart becomes extremely slow in detecting *opposite* and *one variable* type shifts. In addition, the UCL values computed using the Constant estimator vary with correlation, unlike the other three estimators which produce about the same UCLs regardless of correlation. So, the Constant estimator is deemed overall the worst one among the four estimators considered.

The One-quadrant estimator has the largest in-control SARLs and the highest UCLs among all the estimators, in all cases considered. Although it often has the lowest out-of-control ARL (especially for high correlations), the remaining two estimators, Two-quadrant and Four-quadrant, produce tighter UCLs and have lower in-control variability. For these reasons, the One-quadrant estimator is considered inferior to them. Lastly, we must choose between the Two-quadrant estimator and the Four-quadrant estimator. Both estimators perform equally well, as expected from our discussion in section 6.5. They produce results that are either very similar or, in some cases, exactly the same. Since the Four-quadrant estimator seems to offer no advantage over the Two-quadrant estimator, we choose to use the latter in the computations of the sign based chart when compared with its parametric counterpart.

Table 8.1.2 contains the ARL values of the signed rank based charts. Again, the four estimators are given in four columns and the same cases are examined as for the sign based chart. The observations we made for the sign based chart (Table 8.1.1) hold true also for the signed rank based chart. In a nutshell, the Constant estimator performs the worst overall, the One-quadrant estimator is reasonably good, whereas the Two-quadrant estimator and the Four-quadrant estimator are the best ones and perform equally well. Again we choose our proposed estimator, the Two-quadrant estimator, for further comparisons between the signed rank based chart and its parametric counterpart.

Table 8.1.3 shows the ARL values of the $\overline{X}$ based, the signed rank based, and the sign based charts when the data are mixture normal and the variance-covariance matrix of the data is unknown. The estimators used are the sample variances and covariances for the $\overline{X}$ based chart, and the Two-quadrant estimator for the signed rank based and the sign based charts. As for the case of known parameters, here, also, we observe that the nonparametric charts are faster than the $\overline{X}$ based chart in detecting smaller shifts in the mean vector. This is true for correlations 0 and 0.5, and for the *equal* shift of correlation 0.9.

As r increases, all three charts become slower in detecting shifts of any size, but the $\overline{X}$ based chart is affected a lot more than the nonparametric charts. There are two issues here. The first issue is which value of r makes the nonparametric charts most efficient in detecting shifts; they become most efficient when r is 0.1. The second issue is when the parametric chart would be particularly inefficient relative to the nonparametric charts. When $r = 0.1$, the nonparametric charts are faster than the $\overline{X}$ based chart for shifts up to $\tilde{\lambda} \cong 0.9$, but as r increases the range increases also. At $r = 0.3$, the nonparametric charts are faster than the $\overline{X}$ based chart for shifts up to $\tilde{\lambda} \cong 1.5$. So, higher values of r

benefit the nonparametric charts over the parametric competitor, but they also slow them down somewhat in detecting shifts of any size.

Finally, we look at the effect of sample size on the three charts. As n increases, all charts become faster in detecting shifts of any size. The nonparametric charts improve somewhat but the $\overline{X}$ based chart improves a lot more.

Now, we will look at the behavior of the nonparametric charts when the data are normal and Σ is unknown. Table 8.1.4 contains the ARL values of the three charts for this case. The estimators used are the sample variances and covariances for the $\overline{X}$ based chart, and the Two-quadrant estimator for the signed rank based and the sign based charts. Here we expect the $\overline{X}$ based chart to detect shifts faster than the nonparametric charts. For example, looking at r=0.1 and n=5, we see that the $\overline{X}$ based chart is one and a half to three times faster than the signed rank based chart in detecting various size shifts, depending on ρ. Comparing the $\overline{X}$ based chart with the sign based chart, we see that it is two to four times faster, depending on ρ and shift size.

In general, when Σ is unknown, the nonparametric charts seem to lose some ground to the $\overline{X}$ based chart, especially as n increases. For all shift sizes, the nonparametric charts perform the best when the EWMA parameter r is 0.1, and they improve but they don't gain very much by increases in n. On the other hand, the parametric chart becomes a lot faster as n increases. We also observe that the $\overline{X}$ based chart seems to have smaller out-of-control ARL values when the data are mixture normal than when the data are normal, and smaller out-of-control ARL values when Σ is unknown than when Σ is known. Both observations are counterintuitive. The UCL values for the $\overline{X}$ based chart, on the other hand, are in agreement with what we would expect for the different cases.

In comparing the two nonparametric charts only, the signed rank based chart, with a few exceptions, is almost uniformly faster than the sign based chart for small to moderate size shifts. For large size shifts, the signed rank based chart is a little slower than the sign based chart, unless the correlation is very high or the sample size is moderately large. Then, the signed rank based chart is faster than the sign based chart, with a few exceptions, for shifts of all sizes.

Tables 8.1.5, 8.1.6, and 8.1.7 contain the UCL values of the $\overline{X}$ based chart, the signed rank based chart, and the sign based chart, respectively. Overall, we have fewer UCL values to compare than we had in the known Σ case, but we still observe the same behaviors of the UCLs of the charts with respect to r, ρ, and n. The most striking observation, though, is that the UCL values of the nonparametric charts for the unknown Σ case (Tables 8.1.6 and 8.1.7) follow very closely the UCL values of the charts for the known Σ case (Tables 7.6 and 7.7). This suggests that, if Σ is unknown, we could use Table 7.6 to construct a signed rank based chart, but the UCL values depend on correlation. It also suggests that, if Σ is unknown, we could use Table 7.7 to construct a sign based chart, where the UCL values are independent of correlation. The UCL values of the $\overline{X}$ based chart are about the same for the known Σ and unknown Σ cases when the distribution is mixture normal, but they are different when the distribution is normal (Tables 7.5 and 8.1.5).

8.2 When the shift occurs after *m* good samples

In all the previous results presented in this dissertation, when we wanted to compute an out-of-control ARL for a particular shift size, we simulated data with the shift added to the mean right from the beginning of the process. In other words, we made the assumption that, when a shift occurs, it occurs at the onset of a process. Of course, this is a simplification of what may happen during a real process, where a shift may occur at any point during the run. For example, we may consider a process that starts out in-control and becomes out-of-control after a few good samples were collected. We then may assume that the estimates of the unknown parameters obtained from the good samples will help detect a shift faster when it occurs. This is especially true for the nonparametric charts, where the distribution of $p_{ii'}$ (defined in equation (4.5.1)) may depend on the size of the shift in the mean vector. So, now we will consider a case scenario where a shift occurs at stage $k = m + 1$, where m is the number of good samples collected from a process before the shift occurred.

Tables 8.2.1, 8.2.2, and 8.2.3 contain ARL results for $m = 10$. When there is no shift, the ARL values should be the same as in Tables 8.1.1, 8.1.2, and 8.1.3 for all cases shown. When there is a shift, the ARL values in Tables 8.2.1, 8.2.2, and 8.2.3 represent the average run length to signal after a shift occurred, i.e. m has been subtracted from the total run length. All ARL values were computed by including the false alarms. This explains the presence of a few ARL values less than 1 for large shifts in the tables.

Table 8.2.1 contains the results for the sign based charts using the One-quadrant estimator, the Two-quadrant estimator, and the Four-quadrant estimator. The Constant estimator was not used because it will not be affected by the good samples, since it always estimates $p_{ii'}$ as $\hat{p}_{ii'} = 0.25$. Comparing the out-of-control ARL values in Tables 8.2.1 and 8.1.1 for the different estimators we observe that, with 10 good samples at the beginning of the process, the One-quadrant estimator detects small shifts faster than without any good samples, and it benefits the most when ρ is 0.9. The Two-quadrant estimator and the Four-quadrant estimator, though, became a little slower in detecting small to medium size shifts when $\rho = 0$, and faster when ρ is high. So, they benefit mostly when the correlation is high. Comparing the three estimators we observe that, similar to the results of Table 8.1.1, the Two-quadrant estimator has smaller SARL than the One-quadrant estimator and about the same as the Four-quadrant estimator. Table 8.2.2 contains a few results for the signed rank based charts and, if we compare it with Table 8.1.2, we draw the same conclusions as for the sign based chart.

Table 8.2.3 depicts the ARLs of the three types of charts, i.e. $\overline{X}$ based, signed rank based, and sign based, for comparison. This table shows that, whereas the out-of-control ARL values of the nonparametric charts have changed little compared to Table 8.1.3, the out-of-control ARL values of the $\overline{X}$ based chart have increased more dramatically. Comparisons of the three types of control charts in Table 8.2.3 show that the nonparametric charts have gained ground and they are faster than their parametric counterpart over a wider range of shift sizes than the cases depicted in table 8.1.3. For example, for n=5, r=0.3, and ρ=0.5, the sign based chart is faster than the $\overline{X}$ based chart up to $\lambda \cong 1.5$ in Table 8.1.3, and up to $\lambda \cong 2.0$ in Table 8.2.3. For small r though, i.e. r = 0.1, the range of values of λ where the nonparametric charts are faster than the parametric chart remains about the same with and without 10 good samples. So, under the conditions in the case scenario described in this section, the nonparametric charts perform better than expected from our discussion in section 8.1, although for r = 0.1, at which all charts are most efficient, the benefit from the good samples at the beginning of the process is not clear.

Table 8.1.1: ARL values of the sign based chart for 10% mixture bivariate normal data with unknown Σ using various estimators

$n = 5, \quad r = 0.1$

$\rho = 0.0$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator		Constant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.247	7.4770	201.378	6.4094	201.330	6.3590	199.516	6.2239
0.5	one var	15.656	0.3771	15.854	0.3573	15.854	0.3575	15.445	0.3346
	equal	19.797	0.4677	16.308	0.3447	16.289	0.3450	15.548	0.3409
	opposite	15.438	0.3668	15.933	0.3464	15.929	0.3464	15.949	0.3596
1.0	one var	5.287	0.0866	5.890	0.0922	5.889	0.0919	5.436	0.0912
	equal	6.757	0.1149	5.919	0.0906	5.923	0.0909	5.292	0.0948
	opposite	5.699	0.0983	5.840	0.0955	5.842	0.0954	5.293	0.0948
1.5	one var	3.294	0.0427	3.714	0.0454	3.704	0.0453	3.524	0.0485
	equal	4.168	0.0574	3.630	0.0455	3.636	0.0454	3.028	0.0452
	opposite	3.599	0.0470	3.599	0.0451	3.595	0.0450	3.088	0.0453
2.0	one var	2.647	0.0282	2.955	0.0331	2.952	0.0331	2.772	0.0308
	equal	3.100	0.0347	2.756	0.0286	2.760	0.0286	2.289	0.0293
	opposite	2.772	0.0293	2.812	0.0290	2.811	0.0290	2.223	0.0298
2.5	one var	2.393	0.0203	2.500	0.0232	2.505	0.0232	2.388	0.0237
	equal	2.610	0.0246	2.417	0.0199	2.416	0.0199	1.835	0.0227
	opposite	2.410	0.0198	2.392	0.0189	2.396	0.0190	1.787	0.0223
3.0	one var	2.289	0.0167	2.395	0.0200	2.396	0.0200	2.282	0.0206
	equal	2.375	0.0195	2.228	0.0156	2.231	0.0156	1.609	0.0200
	opposite	2.264	0.0158	2.217	0.0143	2.217	0.0143	1.547	0.0189
		UCL = 9.009		UCL = 8.652		UCL = 8.651		UCL = 8.693	

Table 8.1.1 (continued): ARL values of the sign based chart for 10% mixture bivariate normal data with unknown Σ using various estimators

$n = 5, \quad r = 0.1$

$\rho = 0.5$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator		Constant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.813	6.8450	199.663	6.1900	199.663	6.1900	200.123	6.5480
0.5	one var	16.003	0.4118	18.903	0.4231	18.903	0.4231	22.099	0.5222
	equal	14.331	0.3485	14.689	0.3237	14.689	0.3237	11.868	0.2812
	opposite	21.231	0.5085	21.480	0.5104	21.480	0.5104	32.014	0.7730
1.0	one var	5.684	0.0981	6.907	0.1075	6.907	0.1075	7.347	0.1315
	equal	5.620	0.0901	5.278	0.0885	5.278	0.0885	4.327	0.0762
	opposite	7.416	0.1412	7.768	0.1209	7.768	0.1209	9.508	0.1726
1.5	one var	3.540	0.0488	4.198	0.0551	4.198	0.0551	4.325	0.0631
	equal	3.568	0.0456	3.171	0.0390	3.171	0.0390	2.527	0.0399
	opposite	4.609	0.0660	4.773	0.0600	4.773	0.0600	4.984	0.0759
2.0	one var	2.777	0.0293	3.216	0.0360	3.216	0.0360	3.137	0.0396
	equal	2.745	0.0288	2.552	0.0240	2.552	0.0240	1.845	0.0254
	opposite	3.576	0.0408	3.509	0.0369	3.509	0.0369	3.366	0.0458
2.5	one var	2.457	0.0210	2.705	0.0271	2.705	0.0271	2.678	0.0299
	equal	2.392	0.0198	2.369	0.0192	2.369	0.0192	1.498	0.0200
	opposite	3.010	0.0296	2.938	0.0283	2.938	0.0283	2.544	0.0327
3.0	one var	2.301	0.0170	2.501	0.0231	2.501	0.0231	2.438	0.0247
	equal	2.257	0.0162	2.222	0.0148	2.222	0.0148	1.340	0.0170
	opposite	2.675	0.0229	2.576	0.0226	2.576	0.0226	2.103	0.0266
		UCL = 9.137		UCL = 8.665		UCL = 8.665		UCL = 9.264	

Table 8.1.1 (continued): ARL values of the sign based chart for 10% mixture bivariate normal data with unknown Σ using various estimators

$n = 5, \quad r = 0.1$

$\rho = 0.9$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator		Constant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.406	7.1682	198.307	5.5239	198.382	5.4721	198.724	6.5975
0.5	one var	31.896	0.8600	41.072	0.9851	40.289	0.9576	86.366	2.4638
	equal	13.878	0.2914	14.765	0.3258	14.743	0.3255	12.123	0.2902
	opposite	44.533	1.3293	46.829	1.1093	46.510	1.1076	153.698	5.0099
1.0	one var	11.134	0.2306	14.819	0.2593	14.738	0.2579	28.729	0.7345
	equal	5.438	0.0906	5.207	0.0826	5.205	0.0817	4.244	0.0687
	opposite	16.688	0.3752	17.096	0.3163	16.924	0.3097	63.791	1.6821
1.5	one var	6.776	0.1211	8.783	0.1311	8.644	0.1290	13.450	0.2619
	equal	3.373	0.0433	3.221	0.0400	3.307	0.0408	2.808	0.0318
	opposite	9.363	0.1699	10.249	0.1487	10.239	0.1477	25.842	0.5452
2.0	one var	4.881	0.0730	6.306	0.0788	6.225	0.0800	8.630	0.1505
	equal	2.682	0.0255	2.661	0.0254	2.603	0.0241	2.294	0.0177
	opposite	7.099	0.1223	7.434	0.0903	7.366	0.0922	13.806	0.2279
2.5	one var	3.885	0.0529	4.786	0.0597	4.849	0.0575	6.384	0.0956
	equal	2.410	0.0193	2.398	0.0188	2.383	0.0193	2.131	0.0113
	opposite	5.596	0.0854	5.699	0.0635	5.771	0.0658	9.511	0.1402
3.0	one var	3.241	0.0405	4.027	0.0470	4.003	0.0472	4.941	0.0648
	equal	2.281	0.0163	2.247	0.0146	2.264	0.0153	2.087	0.0101
	opposite	4.606	0.0637	4.851	0.0514	4.884	0.0521	7.034	0.0971
		UCL = 9.128		UCL = 8.647		UCL = 8.640		UCL = 10.723	

Table 8.1.1 (continued): ARL values of the sign based chart for 10% mixture bivariate normal data with unknown Σ using various estimators

$n = 5, \quad r = 0.2$

$\rho = 0.0$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator		Constant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.272	7.2011	199.170	6.0763	199.170	6.0763	198.504	6.1954
0.5	one var	19.636	0.5248	20.506	0.5111	20.506	0.5111	19.256	0.4649
	equal	25.417	0.6868	21.896	0.5331	21.896	0.5331	18.897	0.4828
	opposite	20.389	0.5500	20.736	0.4925	20.736	0.4925	19.476	0.5021
1.0	one var	5.655	0.1047	6.570	0.1143	6.570	0.1143	6.211	0.1091
	equal	7.621	0.1491	6.643	0.1121	6.643	0.1121	5.940	0.1060
	opposite	6.299	0.1130	6.339	0.1085	6.339	0.1085	5.714	0.1027
1.5	one var	3.421	0.0449	3.903	0.0515	3.903	0.0515	3.775	0.0542
	equal	4.408	0.0661	4.039	0.0500	4.039	0.0500	3.230	0.0513
	opposite	3.913	0.0520	3.983	0.0528	3.983	0.0528	3.154	0.0476
2.0	one var	2.769	0.0300	2.979	0.0354	2.979	0.0354	2.867	0.0349
	equal	3.271	0.0408	2.950	0.0330	2.950	0.0330	2.290	0.0311
	opposite	2.996	0.0314	2.989	0.0313	2.989	0.0313	2.291	0.0333
2.5	one var	2.425	0.0219	2.601	0.0267	2.601	0.0267	2.439	0.0254
	equal	2.668	0.0278	2.584	0.0229	2.584	0.0229	1.840	0.0237
	opposite	2.551	0.0224	2.513	0.0223	2.513	0.0223	1.838	0.0233
3.0	one var	2.300	0.0175	2.376	0.0200	2.376	0.0200	2.324	0.0227
	equal	2.380	0.0204	2.336	0.0180	2.336	0.0180	1.552	0.0200
	opposite	2.346	0.0183	2.324	0.0173	2.324	0.0173	1.528	0.0195
		UCL = 9.662		UCL = 9.400		UCL = 9.400		UCL = 9.427	

Table 8.1.1 (continued): ARL values of the sign based chart for 10% mixture bivariate normal data with unknown Σ using various estimators

$n = 5, \quad r = 0.2$

$\rho = 0.5$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator		Constant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.727	6.8091	200.838	6.4145	200.838	6.4145	200.481	6.3807
0.5	one var	18.996	0.5255	25.791	0.6717	25.791	0.6717	31.346	0.8238
	equal	16.891	0.4632	19.204	0.4803	19.204	0.4803	14.117	0.3430
	opposite	28.605	0.8179	29.497	0.7371	29.497	0.7371	52.029	1.4066
1.0	one var	6.159	0.1177	7.809	0.1434	7.809	0.1434	8.425	0.1607
	equal	5.887	0.1040	5.805	0.0920	5.805	0.0920	4.681	0.0779
	opposite	8.981	0.1905	9.402	0.1615	9.402	0.1615	11.848	0.2497
1.5	one var	3.705	0.0511	4.551	0.0620	4.551	0.0620	4.798	0.0719
	equal	3.701	0.0502	3.487	0.0458	3.487	0.0458	2.880	0.0350
	opposite	5.056	0.0763	5.377	0.0730	5.377	0.0730	5.700	0.0934
2.0	one var	2.914	0.0324	3.309	0.0408	3.309	0.0408	3.484	0.0424
	equal	2.892	0.0321	2.723	0.0287	2.723	0.0287	2.347	0.0203
	opposite	3.848	0.0459	3.898	0.0442	3.898	0.0442	3.753	0.0501
2.5	one var	2.551	0.0235	2.792	0.0292	2.792	0.0292	2.946	0.0316
	equal	2.430	0.0223	2.420	0.0209	2.420	0.0209	2.112	0.0105
	opposite	3.171	0.0330	3.079	0.0300	3.079	0.0300	2.874	0.0315
3.0	one var	2.347	0.0175	2.534	0.0244	2.534	0.0244	2.706	0.0255
	equal	2.274	0.0173	2.243	0.0164	2.243	0.0164	2.072	0.0088
	opposite	2.798	0.0257	2.691	0.0238	2.691	0.0238	2.490	0.0229
		UCL = 9.786		UCL = 9.442		UCL = 9.442		UCL = 10.075	

Table 8.1.1 (continued): ARL values of the sign based chart for 10% mixture bivariate normal data with unknown Σ using various estimators

$n = 5, \quad r = 0.2$

$\rho = 0.9$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator		Constant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.341	7.4531	199.120	5.8322	199.046	5.8341	201.304	6.3092
0.5	one var	38.316	1.1061	57.228	1.4221	57.272	1.4222	109.475	3.3393
	equal	15.929	0.3691	19.225	0.4761	19.225	0.4758	14.340	0.3526
	opposite	61.965	2.0538	65.013	1.6192	65.022	1.6184	179.846	5.2698
1.0	one var	12.986	0.2954	19.368	0.4161	19.399	0.4165	43.000	1.2775
	equal	6.138	0.1052	5.865	0.1030	5.863	0.1027	4.739	0.0805
	opposite	22.990	0.6153	23.075	0.4755	23.108	0.4760	119.964	3.5903
1.5	one var	7.550	0.1365	10.711	0.1823	10.728	0.1825	19.092	0.4501
	equal	3.642	0.0487	3.587	0.0488	3.592	0.0488	3.079	0.0385
	opposite	12.716	0.2789	13.185	0.2240	13.200	0.2240	58.866	1.6800
2.0	one var	5.270	0.0847	7.084	0.1037	7.086	0.1038	10.708	0.2006
	equal	2.845	0.0313	2.810	0.0303	2.810	0.0303	2.466	0.0230
	opposite	8.421	0.1616	8.794	0.1256	8.794	0.1256	23.510	0.5430
2.5	one var	4.194	0.0543	5.437	0.0696	5.437	0.0696	7.683	0.1229
	equal	2.461	0.0225	2.413	0.0207	2.413	0.0207	2.253	0.0155
	opposite	6.602	0.1104	6.899	0.0869	6.899	0.0869	12.853	0.2474
3.0	one var	3.551	0.0436	4.428	0.0581	4.428	0.0581	5.914	0.0823
	equal	2.332	0.0188	2.298	0.0184	2.298	0.0184	2.155	0.0122
	opposite	5.428	0.0794	5.643	0.0656	5.643	0.0656	8.769	0.1434
		UCL = 9.977		UCL = 9.526		UCL = 9.526		UCL = 12.000	

Table 8.1.1 (continued): ARL values of the sign based chart for 10% mixture bivariate normal data with unknown Σ using various estimators

$n = 10, \quad r = 0.1$

$\rho = 0.0$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator		Constant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.316	6.8981	200.452	6.2575	200.452	6.2575	201.331	6.3850
0.5	one var	15.880	0.3907	15.578	0.3502	15.578	0.3502	15.721	0.3636
	equal	19.061	0.4499	15.829	0.3487	15.829	0.3487	15.249	0.3466
	opposite	15.291	0.3385	15.276	0.3558	15.276	0.3558	14.603	0.3418
1.0	one var	5.180	0.0926	5.393	0.0935	5.393	0.0935	5.197	0.0921
	equal	6.344	0.1118	5.240	0.0902	5.240	0.0902	4.990	0.0919
	opposite	5.215	0.0947	5.307	0.0922	5.307	0.0922	4.914	0.0913
1.5	one var	2.937	0.0443	3.096	0.0460	3.096	0.0460	2.906	0.0458
	equal	3.474	0.0564	3.057	0.0432	3.057	0.0432	2.830	0.0431
	opposite	2.966	0.0463	3.005	0.0438	3.005	0.0438	2.832	0.0452
2.0	one var	2.055	0.0274	2.173	0.0292	2.173	0.0292	2.121	0.0279
	equal	2.381	0.0364	2.225	0.0275	2.225	0.0275	1.913	0.0263
	opposite	2.156	0.0281	2.226	0.0290	2.226	0.0290	1.968	0.0272
2.5	one var	1.709	0.0204	1.792	0.0214	1.792	0.0214	1.725	0.0218
	equal	1.808	0.0260	1.768	0.0199	1.768	0.0199	1.515	0.0198
	opposite	1.670	0.0209	1.771	0.0210	1.771	0.0210	1.554	0.0203
3.0	one var	1.496	0.0173	1.515	0.0186	1.515	0.0186	1.522	0.0184
	equal	1.483	0.0201	1.519	0.0178	1.519	0.0178	1.306	0.0152
	opposite	1.449	0.0177	1.521	0.0175	1.521	0.0175	1.270	0.0147
		UCL = 8.873		UCL = 8.696		UCL = 8.696		UCL = 8.719	

Table 8.1.1 (continued): ARL values of the sign based chart for 10% mixture bivariate normal data with unknown Σ using various estimators

$n = 10, \quad r = 0.1$

$\rho = 0.5$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator		Constant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.821	6.7983	200.059	6.2746	200.471	6.2832	199.819	6.5536
0.5	one var	16.051	0.3900	17.787	0.4384	17.874	0.4391	21.316	0.5262
	equal	14.598	0.3596	12.824	0.2918	12.916	0.2911	11.643	0.2677
	opposite	19.777	0.4927	20.109	0.4534	20.188	0.4542	32.196	0.8536
1.0	one var	5.198	0.1012	6.219	0.1117	6.182	0.1111	6.664	0.1241
	equal	4.768	0.0921	4.601	0.0784	4.608	0.0788	3.972	0.0740
	opposite	6.585	0.1240	6.962	0.1195	6.978	0.1197	8.917	0.1641
1.5	one var	2.920	0.0455	3.493	0.0506	3.507	0.0512	3.710	0.0582
	equal	2.692	0.0477	2.648	0.0389	2.639	0.0389	2.276	0.0368
	opposite	3.993	0.0618	4.034	0.0587	4.047	0.0583	4.716	0.0747
2.0	one var	2.173	0.0298	2.542	0.0345	2.524	0.0348	2.654	0.0401
	equal	1.939	0.0317	1.962	0.0243	1.960	0.0246	1.633	0.0225
	opposite	2.802	0.0380	2.810	0.0353	2.784	0.0349	3.019	0.0442
2.5	one var	1.771	0.0208	1.992	0.0241	1.982	0.0242	1.997	0.0270
	equal	1.477	0.0208	1.588	0.0192	1.597	0.0188	1.318	0.0170
	opposite	2.222	0.0264	2.233	0.0242	2.251	0.0243	2.232	0.0299
3.0	one var	1.593	0.0187	1.729	0.0202	1.727	0.0198	1.719	0.0213
	equal	1.272	0.0159	1.416	0.0165	1.415	0.0169	1.149	0.0117
	opposite	1.907	0.0223	1.956	0.0203	1.954	0.0202	1.785	0.0239
		UCL = 8.823		UCL = 8.612		UCL = 8.614		UCL = 9.177	

Table 8.1.1 (continued): ARL values of the sign based chart for 10% mixture bivariate normal data with unknown Σ using various estimators

$n = 10, \quad r = 0.1$

$\rho = 0.9$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator		Constant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.496	7.1011	201.630	6.5316	201.630	6.5316	199.704	6.7510
0.5	one var	30.721	0.8375	38.384	0.9739	38.384	0.9739	82.195	2.6322
	equal	12.472	0.2808	14.320	0.3263	14.320	0.3263	11.300	0.2625
	opposite	40.296	1.2005	44.156	1.1479	44.156	1.1479	144.315	4.5780
1.0	one var	10.613	0.2345	13.191	0.2466	13.191	0.2466	26.913	0.6761
	equal	4.609	0.0867	4.498	0.0809	4.498	0.0809	3.931	0.0708
	opposite	15.142	0.3494	15.228	0.2639	15.228	0.2639	61.457	1.6757
1.5	one var	5.825	0.1075	7.468	0.1151	7.468	0.1151	12.846	0.2631
	equal	2.638	0.0438	2.447	0.0413	2.447	0.0413	2.351	0.0367
	opposite	8.288	0.1549	8.479	0.1255	8.479	0.1255	24.574	0.5261
2.0	one var	4.075	0.0665	5.220	0.0722	5.220	0.0722	8.028	0.1401
	equal	1.779	0.0290	1.792	0.0266	1.792	0.0266	1.732	0.0236
	opposite	5.734	0.0952	6.020	0.0812	6.020	0.0812	12.873	0.2123
2.5	one var	3.190	0.0479	3.982	0.0514	3.982	0.0514	5.600	0.0880
	equal	1.406	0.0200	1.400	0.0188	1.400	0.0188	1.378	0.0173
	opposite	4.610	0.0666	4.700	0.0584	4.700	0.0584	8.704	0.1237
3.0	one var	2.577	0.0361	3.235	0.0418	3.235	0.0418	4.480	0.0636
	equal	1.250	0.0151	1.240	0.0146	1.240	0.0146	1.207	0.0133
	opposite	3.774	0.0516	3.837	0.0440	3.837	0.0440	6.430	0.0876
		UCL = 8.910		UCL = 8.528		UCL = 8.528		UCL = 10.685	

Table 8.1.2: ARL values of the signed rank based chart for 10% mixture bivariate normal data with unknown Σ using various estimators

$n = 5, \quad r = 0.1$

$\rho = 0.0$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator		Constant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.600	6.7320	199.435	6.1785	199.435	6.1785	201.444	6.0244
0.5	one var	13.816	0.3185	13.922	0.2967	13.922	0.2967	13.991	0.3124
	equal	17.473	0.3744	14.121	0.2963	14.121	0.2963	13.200	0.2747
	opposite	13.653	0.3087	13.930	0.2881	13.930	0.2881	13.496	0.2787
1.0	one var	4.901	0.0817	5.458	0.0804	5.458	0.0803	5.331	0.0830
	equal	6.486	0.1047	5.371	0.0820	5.371	0.0820	4.847	0.0806
	opposite	5.085	0.0820	5.272	0.0780	5.272	0.0780	5.002	0.0829
1.5	one var	3.213	0.0413	3.732	0.0405	3.732	0.0405	3.581	0.0430
	equal	4.114	0.0519	3.441	0.0413	3.441	0.0413	2.945	0.0360
	opposite	3.374	0.0417	3.420	0.0393	3.420	0.0393	3.072	0.0379
2.0	one var	2.660	0.0287	3.253	0.0272	3.253	0.0272	3.116	0.0301
	equal	3.265	0.0328	2.799	0.0279	2.799	0.0279	2.384	0.0213
	opposite	2.884	0.0300	2.773	0.0271	2.773	0.0271	2.388	0.0202
2.5	one var	2.503	0.0241	3.082	0.0228	3.082	0.0228	2.856	0.0260
	equal	2.799	0.0280	2.477	0.0215	2.477	0.0215	2.216	0.0150
	opposite	2.606	0.0239	2.506	0.0222	2.506	0.0222	2.168	0.0139
3.0	one var	2.415	0.0198	3.002	0.0202	3.002	0.0202	2.751	0.0226
	equal	2.538	0.0225	2.336	0.0186	2.336	0.0186	2.114	0.0112
	opposite	2.471	0.0208	2.360	0.0187	2.360	0.0187	2.139	0.0126
		UCL = 8.862		UCL = 8.551		UCL = 8.551		UCL = 8.582	

Table 8.1.2 (continued): ARL values of the signed rank based chart for 10% mixture bivariate normal data with unknown Σ using various estimators

$n = 5,\ r = 0.1$

$\rho = 0.5$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator		Constant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.700	7.1758	200.350	6.3460	200.746	6.3478	199.422	6.4026
0.5	one var	14.250	0.3353	15.692	0.3156	15.706	0.3153	20.276	0.4645
	equal	13.564	0.2976	12.245	0.2635	12.245	0.2635	11.089	0.2357
	opposite	17.363	0.4172	17.898	0.3793	17.886	0.3792	28.668	0.6859
1.0	one var	4.981	0.0844	6.121	0.0897	6.115	0.0894	7.043	0.1139
	equal	5.380	0.0789	4.633	0.0690	4.629	0.0690	4.240	0.0675
	opposite	6.263	0.1066	6.586	0.0990	6.588	0.0992	8.685	0.1359
1.5	one var	3.422	0.0425	4.058	0.0486	4.046	0.0482	4.388	0.0569
	equal	3.535	0.0407	3.044	0.0365	3.062	0.0370	2.784	0.0329
	opposite	4.026	0.0572	4.177	0.0491	4.164	0.0491	4.689	0.0607
2.0	one var	2.787	0.0315	3.358	0.0316	3.362	0.0314	3.496	0.0348
	equal	2.858	0.0297	2.670	0.0291	2.662	0.0290	2.303	0.0191
	opposite	3.209	0.0374	3.283	0.0343	3.282	0.0343	3.302	0.0364
2.5	one var	2.543	0.0256	3.135	0.0269	3.140	0.0270	3.223	0.0283
	equal	2.529	0.0226	2.377	0.0218	2.375	0.0217	2.198	0.0152
	opposite	2.834	0.0271	2.843	0.0271	2.844	0.0271	2.802	0.0281
3.0	one var	2.400	0.0219	2.968	0.0238	2.965	0.0236	3.094	0.0247
	equal	2.331	0.0188	2.263	0.0179	2.265	0.0179	2.098	0.0097
	opposite	2.608	0.0226	2.529	0.0221	2.516	0.0217	2.404	0.0207
		UCL = 8.746		UCL = 8.401		UCL = 8.401		UCL = 9.395	

Table 8.1.2 (continued): ARL values of the signed rank based chart for 10% mixture bivariate normal data with unknown Σ using various estimators

$n = 5, \quad r = 0.1$

$\rho = 0.9$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator		Constant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.211	7.3829	200.511	6.0952	200.396	6.0949	200.209	6.1114
0.5	one var	27.067	0.7357	31.733	0.7155	31.538	0.7085	84.655	2.5535
	equal	12.237	0.2642	11.253	0.2465	11.279	0.2451	11.085	0.2322
	opposite	34.509	0.9953	36.622	0.8470	36.783	0.8466	151.626	4.9453
1.0	one var	9.299	0.1799	10.829	0.1767	10.801	0.1780	26.123	0.6109
	equal	4.706	0.0739	4.305	0.0700	4.332	0.0703	4.220	0.0694
	opposite	11.976	0.2535	12.638	0.2054	12.689	0.2062	58.803	1.5836
1.5	one var	5.400	0.0862	6.614	0.0916	6.606	0.0911	12.798	0.2455
	equal	3.138	0.0440	2.981	0.0385	2.975	0.0383	2.791	0.0319
	opposite	7.043	0.1287	7.459	0.0995	7.464	0.0998	23.952	0.4176
2.0	one var	4.074	0.0561	4.796	0.0576	4.809	0.0578	8.654	0.1227
	equal	2.524	0.0259	2.449	0.0245	2.454	0.0248	2.431	0.0219
	opposite	5.273	0.0821	5.469	0.0619	5.448	0.0622	12.938	0.1719
2.5	one var	3.382	0.0419	3.957	0.0463	3.979	0.0469	6.340	0.0838
	equal	2.330	0.0210	2.311	0.0193	2.309	0.0188	2.263	0.0174
	opposite	4.332	0.0605	4.437	0.0462	4.445	0.0465	8.737	0.1040
3.0	one var	2.933	0.0333	3.434	0.0355	3.434	0.0356	5.075	0.0613
	equal	2.223	0.0162	2.234	0.0166	2.236	0.0168	2.241	0.0164
	opposite	3.734	0.0446	3.689	0.0384	3.702	0.0383	6.414	0.0706
		UCL = 7.934		UCL = 7.500		UCL = 7.500		UCL = 11.212	

Table 8.1.2 (continued): ARL values of the signed rank based chart for 10% mixture bivariate normal data with unknown Σ using various estimators

$n = 5,\ r = 0.2$

$\rho = 0.0$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator		Constant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.306	6.8099	201.449	6.4393	201.449	6.4393	201.850	6.0263
0.5	one var	15.853	0.4277	18.073	0.4334	18.073	0.4334	16.973	0.4132
	equal	22.050	0.5947	17.256	0.4025	17.256	0.4025	16.957	0.4032
	opposite	15.555	0.4076	17.676	0.4045	17.676	0.4045	16.121	0.3846
1.0	one var	5.331	0.0951	6.076	0.0917	6.076	0.0917	5.847	0.0915
	equal	6.976	0.1116	5.811	0.0905	5.811	0.0905	5.288	0.0921
	opposite	5.592	0.0925	5.873	0.0884	5.873	0.0884	5.229	0.0844
1.5	one var	3.343	0.0435	4.151	0.0445	4.151	0.0445	3.918	0.0450
	equal	4.392	0.0532	3.767	0.0427	3.767	0.0427	3.132	0.0386
	opposite	3.649	0.0471	3.806	0.0455	3.806	0.0455	3.214	0.0401
2.0	one var	2.909	0.0326	3.519	0.0277	3.519	0.0277	3.301	0.0321
	equal	3.453	0.0337	3.059	0.0292	3.059	0.0292	2.492	0.0243
	opposite	3.064	0.0312	3.021	0.0301	3.021	0.0301	2.541	0.0254
2.5	one var	2.581	0.0261	3.278	0.0223	3.278	0.0223	3.052	0.0251
	equal	2.955	0.0262	2.722	0.0248	2.722	0.0248	2.242	0.0161
	opposite	2.765	0.0248	2.683	0.0244	2.683	0.0244	2.238	0.0161
3.0	one var	2.533	0.0235	3.204	0.0194	3.204	0.0194	2.989	0.0234
	equal	2.733	0.0252	2.499	0.0223	2.499	0.0223	2.158	0.0129
	opposite	2.679	0.0229	2.492	0.0224	2.492	0.0224	2.158	0.0125
		UCL = 9.377		UCL = 9.209		UCL = 9.209		UCL = 9.189	

Table 8.1.2 (continued): ARL values of the signed rank based chart for 10% mixture bivariate normal data with unknown Σ using various estimators

$n = 5, \quad r = 0.2$

$\rho = 0.5$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator		Constant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.692	7.4182	199.055	6.1654	199.171	6.1683	200.746	6.7807
0.5	one var	16.984	0.4208	20.587	0.4918	20.587	0.4918	27.513	0.7246
	equal	15.493	0.3728	15.023	0.3507	15.023	0.3507	12.607	0.2848
	opposite	22.897	0.6628	22.623	0.5641	22.623	0.5641	50.530	1.4345
1.0	one var	5.606	0.1027	6.708	0.1042	6.708	0.1042	7.829	0.1317
	equal	5.779	0.0896	5.143	0.0786	5.143	0.0786	4.396	0.0718
	opposite	7.110	0.1269	7.641	0.1152	7.641	0.1152	10.773	0.1992
1.5	one var	3.520	0.0452	4.404	0.0472	4.404	0.0472	4.776	0.0593
	equal	3.905	0.0456	3.409	0.0431	3.409	0.0431	2.876	0.0324
	opposite	4.419	0.0629	4.514	0.0523	4.514	0.0523	5.264	0.0691
2.0	one var	2.879	0.0305	3.665	0.0313	3.665	0.0313	3.856	0.0401
	equal	3.020	0.0318	2.797	0.0294	2.797	0.0294	2.455	0.0238
	opposite	3.420	0.0393	3.620	0.0365	3.620	0.0365	3.706	0.0420
2.5	one var	2.678	0.0264	3.394	0.0267	3.394	0.0267	3.474	0.0325
	equal	2.617	0.0248	2.551	0.0257	2.551	0.0257	2.224	0.0160
	opposite	3.060	0.0305	3.050	0.0267	3.050	0.0267	2.988	0.0301
3.0	one var	2.548	0.0233	3.261	0.0212	3.261	0.0212	3.269	0.0252
	equal	2.441	0.0214	2.364	0.0205	2.364	0.0205	2.174	0.0138
	opposite	2.887	0.0253	2.801	0.0232	2.801	0.0232	2.570	0.0234
		UCL = 9.311		UCL = 9.000		UCL = 9.000		UCL = 10.237	

Table 8.1.2 (continued): ARL values of the signed rank based chart for 10% mixture bivariate normal data with unknown Σ using various estimators

$n = 5, \quad r = 0.2$

$\rho = 0.9$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator		Constant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.075	7.2519	200.881	6.1047	200.551	6.0947	198.862	5.6951
0.5	one var	35.151	1.0530	44.686	1.1931	44.744	1.2009	111.510	3.4002
	equal	14.398	0.3345	14.010	0.3426	13.943	0.3355	12.251	0.2827
	opposite	47.899	1.7355	54.146	1.4271	54.044	1.4149	185.650	5.8601
1.0	one var	10.749	0.2469	13.811	0.2565	13.839	0.2557	39.411	1.1238
	equal	5.125	0.0849	4.795	0.0821	4.822	0.0834	4.566	0.0753
	opposite	16.539	0.4297	16.763	0.3181	16.745	0.3160	136.540	4.0178
1.5	one var	6.203	0.1097	7.803	0.1239	7.841	0.1250	18.007	0.4095
	equal	3.362	0.0444	3.158	0.0420	3.169	0.0415	3.086	0.0384
	opposite	8.900	0.1912	9.220	0.1397	9.162	0.1393	59.864	1.7659
2.0	one var	4.517	0.0649	5.616	0.0709	5.584	0.0709	10.840	0.1912
	equal	2.692	0.0309	2.615	0.0283	2.617	0.0280	2.490	0.0234
	opposite	6.091	0.1034	6.360	0.0801	6.370	0.0832	21.909	0.4447
2.5	one var	3.652	0.0441	4.384	0.0513	4.411	0.0515	7.449	0.1089
	equal	2.422	0.0241	2.364	0.0222	2.360	0.0213	2.342	0.0194
	opposite	4.851	0.0722	4.917	0.0550	4.922	0.0548	11.943	0.1859
3.0	one var	3.239	0.0363	3.732	0.0401	3.732	0.0403	6.103	0.0759
	equal	2.274	0.0186	2.242	0.0162	2.245	0.0168	2.253	0.0168
	opposite	4.037	0.0539	4.201	0.0434	4.210	0.0432	8.317	0.1052
		UCL = 8.698		UCL = 8.191		UCL = 8.191		UCL = 12.249	

Table 8.1.2 (continued): ARL values of the signed rank based chart for 10% mixture bivariate normal data with unknown Σ using various estimators

$n = 10, \quad r = 0.1$

$\rho = 0.0$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator		Constant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.930	7.1428	199.157	6.4116	199.157	6.4116	198.581	6.4266
0.5	one var	13.418	0.2934	13.092	0.2831	13.092	0.2831	12.865	0.2856
	equal	15.247	0.3177	13.144	0.2893	13.144	0.2893	12.452	0.2708
	opposite	12.892	0.3120	12.690	0.2716	12.690	0.2716	12.355	0.2803
1.0	one var	4.437	0.0798	4.661	0.0730	4.661	0.0730	4.560	0.0735
	equal	5.610	0.0956	4.492	0.0721	4.492	0.0721	4.252	0.0722
	opposite	4.453	0.0761	4.388	0.0733	4.388	0.0733	4.234	0.0714
1.5	one var	2.601	0.0387	2.896	0.0371	2.896	0.0371	2.890	0.0391
	equal	3.205	0.0475	2.696	0.0370	2.696	0.0370	2.464	0.0354
	opposite	2.662	0.0387	2.699	0.0370	2.699	0.0370	2.500	0.0362
2.0	one var	1.969	0.0254	2.343	0.0243	2.343	0.0243	2.209	0.0243
	equal	2.388	0.0314	2.049	0.0248	2.049	0.0248	1.819	0.0240
	opposite	2.068	0.0256	2.046	0.0253	2.046	0.0253	1.817	0.0245
2.5	one var	1.694	0.0219	2.071	0.0164	2.071	0.0164	1.987	0.0192
	equal	1.919	0.0247	1.711	0.0210	1.711	0.0210	1.467	0.0180
	opposite	1.764	0.0212	1.703	0.0205	1.703	0.0205	1.502	0.0187
3.0	one var	1.548	0.0198	2.008	0.0145	2.008	0.0145	1.842	0.0168
	equal	1.659	0.0211	1.466	0.0181	1.466	0.0181	1.305	0.0154
	opposite	1.516	0.0180	1.477	0.0179	1.477	0.0179	1.291	0.0158
		UCL = 8.796		UCL = 8.675		UCL = 8.675		UCL = 8.676	

Table 8.1.2 (continued): ARL values of the signed rank based chart for 10% mixture bivariate normal data with unknown Σ using various estimators

$n = 10, \quad r = 0.1$

$\rho = 0.5$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator		Constant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.320	7.2596	201.853	6.3743	201.853	6.3743	201.448	6.5616
0.5	one var	13.175	0.3213	14.811	0.3233	14.811	0.3233	18.085	0.4157
	equal	12.396	0.2888	11.281	0.2405	11.281	0.2405	10.170	0.2241
	opposite	15.815	0.3939	16.055	0.3451	16.055	0.3451	28.131	0.6588
1.0	one var	4.552	0.0827	5.214	0.0866	5.214	0.0866	5.963	0.1014
	equal	4.359	0.0763	3.855	0.0675	3.855	0.0675	3.607	0.0627
	opposite	5.489	0.0975	5.700	0.0905	5.700	0.0905	7.841	0.1258
1.5	one var	2.728	0.0399	3.101	0.0411	3.101	0.0411	3.613	0.0522
	equal	2.617	0.0394	2.331	0.0366	2.331	0.0366	2.196	0.0325
	opposite	3.279	0.0475	3.383	0.0439	3.383	0.0439	4.231	0.0572
2.0	one var	2.021	0.0262	2.403	0.0264	2.403	0.0264	2.606	0.0294
	equal	1.991	0.0287	1.710	0.0239	1.710	0.0239	1.612	0.0220
	opposite	2.455	0.0318	2.452	0.0255	2.452	0.0255	2.848	0.0358
2.5	one var	1.702	0.0205	2.133	0.0204	2.133	0.0204	2.284	0.0228
	equal	1.618	0.0214	1.482	0.0204	1.482	0.0204	1.370	0.0178
	opposite	1.997	0.0231	2.017	0.0213	2.017	0.0213	2.153	0.0248
3.0	one var	1.536	0.0186	1.931	0.0166	1.931	0.0166	2.117	0.0190
	equal	1.432	0.0190	1.328	0.0171	1.328	0.0171	1.225	0.0137
	opposite	1.711	0.0195	1.725	0.0185	1.725	0.0185	1.831	0.0203
		UCL = 8.541		UCL = 8.370		UCL = 8.370		UCL = 9.515	

Table 8.1.2 (continued): ARL values of the signed rank based chart for 10% mixture bivariate normal data with unknown Σ using various estimators

$n = 10, \quad r = 0.1$

$\rho = 0.9$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator		Constant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.623	7.1744	199.976	6.2614	199.816	6.2620	201.608	6.4385
0.5	one var	23.745	0.6561	30.331	0.7076	30.382	0.7042	78.474	2.4182
	equal	10.597	0.2388	10.348	0.2374	10.378	0.2380	9.914	0.2131
	opposite	31.682	0.8649	32.831	0.7703	32.852	0.7703	144.540	4.4615
1.0	one var	8.052	0.1555	9.626	0.1623	9.561	0.1619	25.956	0.6174
	equal	3.803	0.0699	3.630	0.0680	3.594	0.0674	3.733	0.0650
	opposite	10.042	0.1957	10.644	0.1692	10.640	0.1677	59.652	1.5432
1.5	one var	4.562	0.0730	5.553	0.0773	5.526	0.0767	11.874	0.2146
	equal	2.344	0.0397	2.147	0.0348	2.192	0.0355	2.197	0.0340
	opposite	5.699	0.0938	6.217	0.0801	6.213	0.0802	21.985	0.3973
2.0	one var	3.291	0.0493	3.861	0.0522	3.851	0.0517	7.374	0.1190
	equal	1.656	0.0251	1.580	0.0230	1.571	0.0231	1.635	0.0229
	opposite	4.161	0.0614	4.460	0.0513	4.435	0.0503	11.638	0.1567
2.5	one var	2.556	0.0347	3.057	0.0383	3.083	0.0386	5.180	0.0728
	equal	1.407	0.0193	1.362	0.0181	1.357	0.0173	1.422	0.0183
	opposite	3.439	0.0450	3.458	0.0380	3.438	0.0386	7.682	0.0885
3.0	one var	2.203	0.0289	2.540	0.0300	2.544	0.0303	4.127	0.0525
	equal	1.290	0.0166	1.282	0.0162	1.261	0.0159	1.304	0.0162
	opposite	2.837	0.0363	2.866	0.0305	2.876	0.0301	5.832	0.0617
		UCL = 7.616		UCL = 7.401		UCL = 7.401		UCL = 11.327	

Table 8.1.3: ARL values of EWMA charts for 10% mixture bivariate normal data with unknown Σ using the Two-quadrant estimator

$n = 5, \ \ r = 0.1$

$\rho = 0.0$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.241	7.3408	199.435	6.1785	201.378	6.4094
0.5	one var	20.883	0.8198	13.922	0.2967	15.854	0.3573
	equal	20.962	0.8219	14.121	0.2963	16.308	0.3447
	opposite	23.657	0.9742	13.930	0.2881	15.933	0.3464
1.0	one var	5.150	0.1831	5.458	0.0804	5.890	0.0922
	equal	4.929	0.1634	5.371	0.0820	5.919	0.0906
	opposite	4.889	0.1779	5.272	0.0780	5.840	0.0955
1.5	one var	2.249	0.0690	3.732	0.0405	3.714	0.0454
	equal	2.368	0.0740	3.441	0.0413	3.630	0.0455
	opposite	2.313	0.0697	3.420	0.0393	3.599	0.0451
2.0	one var	1.638	0.0424	3.253	0.0272	2.955	0.0331
	equal	1.622	0.0427	2.799	0.0279	2.756	0.0286
	opposite	1.662	0.0422	2.773	0.0271	2.812	0.0290
2.5	one var	1.330	0.0282	3.082	0.0228	2.500	0.0232
	equal	1.353	0.0270	2.477	0.0215	2.417	0.0199
	opposite	1.341	0.0282	2.506	0.0222	2.392	0.0189
3.0	one var	1.218	0.0222	3.002	0.0202	2.395	0.0200
	equal	1.178	0.0185	2.336	0.0186	2.228	0.0156
	opposite	1.175	0.0163	2.360	0.0187	2.217	0.0143
		UCL = 10.003		UCL = 8.551		UCL = 8.652	

$n = 5, \ \ r = 0.1$

$\rho = 0.5$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.241	7.3408	200.350	6.3460	199.663	6.1900
0.5	one var	20.883	0.8198	15.692	0.3156	18.903	0.4231
	equal	21.488	0.8448	12.245	0.2635	14.689	0.3237
	opposite	22.848	0.9524	17.898	0.3793	21.480	0.5104
1.0	one var	5.074	0.1802	6.121	0.0897	6.907	0.1075
	equal	4.988	0.1749	4.633	0.0690	5.278	0.0885
	opposite	4.836	0.1820	6.586	0.0990	7.768	0.1209
1.5	one var	2.185	0.0659	4.058	0.0486	4.198	0.0551
	equal	2.382	0.0766	3.044	0.0365	3.171	0.0390
	opposite	2.254	0.0651	4.177	0.0491	4.773	0.0600
2.0	one var	1.634	0.0429	3.358	0.0316	3.216	0.0360
	equal	1.633	0.0454	2.670	0.0291	2.552	0.0240
	opposite	1.627	0.0395	3.283	0.0343	3.509	0.0369
2.5	one var	1.323	0.0275	3.135	0.0269	2.705	0.0271
	equal	1.331	0.0285	2.377	0.0218	2.369	0.0192
	opposite	1.341	0.0280	2.843	0.0271	2.938	0.0283
3.0	one var	1.201	0.0212	2.968	0.0238	2.501	0.0231
	equal	1.209	0.0201	2.263	0.0179	2.222	0.0148
	opposite	1.207	0.0202	2.529	0.0221	2.576	0.0226
		UCL = 10.003		UCL = 8.401		UCL = 8.665	

Table 8.1.3 (cont.): ARL values of EWMA charts for 10% mixture normal data with unknown Σ using the Two-quadrant estimator

$n = 5, \quad r = 0.1$

$\rho = 0.9$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.704	7.3441	200.511	6.0952	198.307	5.5239
0.5	one var	20.904	0.8205	31.733	0.7155	41.072	0.9851
	equal	22.052	0.8455	11.253	0.2465	14.765	0.3258
	opposite	22.284	0.8837	36.622	0.8470	46.829	1.1093
1.0	one var	5.074	0.1802	10.829	0.1767	14.819	0.2593
	equal	4.868	0.1709	4.305	0.0700	5.207	0.0826
	opposite	4.668	0.1794	12.638	0.2054	17.096	0.3163
1.5	one var	2.266	0.0713	6.614	0.0916	8.783	0.1311
	equal	2.242	0.0678	2.981	0.0385	3.221	0.0400
	opposite	2.223	0.0647	7.459	0.0995	10.249	0.1487
2.0	one var	1.575	0.0402	4.796	0.0576	6.306	0.0788
	equal	1.621	0.0467	2.449	0.0245	2.661	0.0254
	opposite	1.644	0.0412	5.469	0.0619	7.434	0.0903
2.5	one var	1.341	0.0270	3.957	0.0463	4.786	0.0597
	equal	1.343	0.0285	2.311	0.0193	2.398	0.0188
	opposite	1.311	0.0261	4.437	0.0462	5.699	0.0635
3.0	one var	1.203	0.0206	3.434	0.0355	4.027	0.0470
	equal	1.206	0.0210	2.234	0.0166	2.247	0.0146
	opposite	1.183	0.0201	3.689	0.0384	4.851	0.0514
		UCL = 10.003		UCL = 7.500		UCL = 8.647	

$n = 5, \quad r = 0.2$

$\rho = 0.0$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.764	6.6690	201.449	6.4393	199.170	6.0763
0.5	one var	40.552	1.5325	18.073	0.4334	20.506	0.5111
	equal	43.137	1.7893	17.256	0.4025	21.896	0.5331
	opposite	42.899	1.6382	17.676	0.4045	20.736	0.4925
1.0	one var	7.505	0.2989	6.076	0.0917	6.570	0.1143
	equal	6.957	0.2991	5.811	0.0905	6.643	0.1121
	opposite	7.004	0.2786	5.873	0.0884	6.339	0.1085
1.5	one var	2.742	0.0967	4.151	0.0445	3.903	0.0515
	equal	2.913	0.0988	3.767	0.0427	4.039	0.0500
	opposite	2.964	0.1018	3.806	0.0455	3.983	0.0528
2.0	one var	1.864	0.0532	3.519	0.0277	2.979	0.0354
	equal	1.772	0.0526	3.059	0.0292	2.950	0.0330
	opposite	1.859	0.0529	3.021	0.0301	2.989	0.0313
2.5	one var	1.407	0.0315	3.278	0.0223	2.601	0.0267
	equal	1.441	0.0370	2.722	0.0248	2.584	0.0229
	opposite	1.413	0.0313	2.683	0.0244	2.513	0.0223
3.0	one var	1.264	0.0243	3.204	0.0194	2.376	0.0200
	equal	1.222	0.0217	2.499	0.0223	2.336	0.0180
	opposite	1.221	0.0222	2.492	0.0224	2.324	0.0173
		UCL = 12.208		UCL = 9.209		UCL = 9.400	

Table 8.1.3 (cont.): ARL values of EWMA charts for 10% mixture normal data with unknown Σ using the Two-quadrant estimator

$n = 5, \quad r = 0.2$

$\rho = 0.5$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.247	6.6672	199.055	6.1654	200.838	6.4145
0.5	one var	40.679	1.5431	20.587	0.4918	25.791	0.6717
	equal	43.701	1.6705	15.023	0.3507	19.204	0.4803
	opposite	42.243	1.6989	22.623	0.5641	29.497	0.7371
1.0	one var	7.458	0.2961	6.708	0.1042	7.809	0.1434
	equal	7.089	0.2969	5.143	0.0786	5.805	0.0920
	opposite	6.895	0.2778	7.641	0.1152	9.402	0.1615
1.5	one var	2.764	0.0999	4.404	0.0472	4.551	0.0620
	equal	2.977	0.1002	3.409	0.0431	3.487	0.0458
	opposite	2.948	0.1047	4.514	0.0523	5.377	0.0730
2.0	one var	1.880	0.0537	3.665	0.0313	3.309	0.0408
	equal	1.768	0.0500	2.797	0.0294	2.723	0.0287
	opposite	1.849	0.0549	3.620	0.0365	3.898	0.0442
2.5	one var	1.387	0.0303	3.394	0.0267	2.792	0.0292
	equal	1.463	0.0374	2.551	0.0257	2.420	0.0209
	opposite	1.400	0.0325	3.050	0.0267	3.079	0.0300
3.0	one var	1.260	0.0238	3.261	0.0212	2.534	0.0244
	equal	1.250	0.0228	2.364	0.0205	2.243	0.0164
	opposite	1.224	0.0221	2.801	0.0232	2.691	0.0238
		UCL = 12.206		UCL = 9.000		UCL = 9.442	

$n = 5, \quad r = 0.2$

$\rho = 0.9$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.764	6.6690	200.881	6.1047	199.120	5.8322
0.5	one var	40.552	1.5325	44.686	1.1931	57.228	1.4221
	equal	42.223	1.6754	14.010	0.3426	19.225	0.4761
	opposite	42.112	1.6779	54.146	1.4271	65.013	1.6192
1.0	one var	7.409	0.3009	13.811	0.2565	19.368	0.4161
	equal	7.597	0.3327	4.795	0.0821	5.865	0.1030
	opposite	7.105	0.2956	16.763	0.3181	23.075	0.4755
1.5	one var	2.790	0.0985	7.803	0.1239	10.711	0.1823
	equal	2.897	0.1018	3.158	0.0420	3.587	0.0488
	opposite	2.987	0.1047	9.220	0.1397	13.185	0.2240
2.0	one var	1.861	0.0538	5.616	0.0709	7.084	0.1037
	equal	1.855	0.0529	2.615	0.0283	2.810	0.0303
	opposite	1.907	0.0578	6.360	0.0801	8.794	0.1256
2.5	one var	1.384	0.0300	4.384	0.0513	5.437	0.0696
	equal	1.446	0.0334	2.364	0.0222	2.413	0.0207
	opposite	1.393	0.0321	4.917	0.0550	6.899	0.0869
3.0	one var	1.253	0.0242	3.732	0.0401	4.428	0.0581
	equal	1.278	0.0255	2.242	0.0162	2.298	0.0184
	opposite	1.286	0.0259	4.201	0.0434	5.643	0.0656
		UCL = 12.208		UCL = 8.191		UCL = 9.526	

Table 8.1.3 (cont.): ARL values of EWMA charts for 10% mixture normal data with unknown Σ using the Two-quadrant estimator

$n = 5, \quad r = 0.3$

$\rho = 0.0$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.745	6.5385	201.695	6.2276	198.455	5.9667
0.5	one var	71.799	2.7982	21.194	0.5536	26.490	0.6962
	equal	60.680	2.3974	22.037	0.5830	26.131	0.7126
	opposite	66.655	2.4223	20.982	0.5374	25.838	0.7218
1.0	one var	11.898	0.5457	6.746	0.1142	7.350	0.1467
	equal	12.523	0.5882	6.355	0.1064	7.394	0.1365
	opposite	11.225	0.5221	6.467	0.1128	7.315	0.1369
1.5	one var	3.689	0.1605	4.255	0.0479	4.124	0.0600
	equal	3.869	0.1785	3.921	0.0458	4.127	0.0614
	opposite	3.812	0.1814	3.870	0.0490	4.179	0.0604
2.0	one var	2.060	0.0695	3.586	0.0314	3.066	0.0388
	equal	2.052	0.0668	3.141	0.0316	3.099	0.0358
	opposite	2.181	0.0789	3.174	0.0346	3.042	0.0348
2.5	one var	1.665	0.0487	3.322	0.0237	2.602	0.0272
	equal	1.594	0.0455	2.770	0.0263	2.551	0.0233
	opposite	1.606	0.0483	2.735	0.0261	2.589	0.0241
3.0	one var	1.280	0.0250	3.241	0.0193	2.364	0.0204
	equal	1.330	0.0272	2.591	0.0232	2.324	0.0180
	opposite	1.253	0.0233	2.548	0.0231	2.364	0.0198
		UCL = 13.849		UCL = 9.179		UCL = 9.493	

$n = 5, \quad r = 0.3$

$\rho = 0.5$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.745	6.5385	201.697	6.2688	199.723	6.0256
0.5	one var	71.705	2.7989	26.719	0.7220	34.060	0.8877
	equal	58.384	2.3186	18.182	0.4854	24.142	0.6473
	opposite	63.201	2.3096	30.273	0.7147	36.744	0.9337
1.0	one var	12.971	0.6049	7.800	0.1331	9.179	0.1779
	equal	11.554	0.5642	5.723	0.1012	6.338	0.1169
	opposite	10.851	0.5022	8.857	0.1658	11.058	0.2245
1.5	one var	3.957	0.1904	4.712	0.0599	5.003	0.0850
	equal	3.903	0.1748	3.548	0.0485	3.543	0.0509
	opposite	3.791	0.1544	5.075	0.0705	5.907	0.0918
2.0	one var	2.017	0.0683	3.835	0.0375	3.540	0.0467
	equal	2.087	0.0654	2.841	0.0296	2.715	0.0295
	opposite	2.058	0.0672	3.783	0.0384	4.071	0.0522
2.5	one var	1.487	0.0360	3.506	0.0280	2.848	0.0328
	equal	1.504	0.0373	2.556	0.0242	2.416	0.0213
	opposite	1.611	0.0413	3.177	0.0275	3.200	0.0341
3.0	one var	1.299	0.0260	3.330	0.0240	2.559	0.0241
	equal	1.309	0.0280	2.436	0.0234	2.267	0.0169
	opposite	1.304	0.0280	2.930	0.0258	2.736	0.0263
		UCL = 13.849		UCL = 9.039		UCL = 9.581	

Table 8.1.3 (cont.): ARL values of EWMA charts for 10% mixture normal data with unknown Σ using the Two-quadrant estimator

$n = 5, \quad r = 0.3$

$\rho = 0.9$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.745	6.5385	198.329	5.9019	201.967	5.8301
0.5	one var	71.799	2.7982	57.793	1.5258	70.855	1.8385
	equal	59.744	2.3411	16.564	0.4397	26.143	0.6752
	opposite	64.753	2.4240	69.107	1.8409	72.270	1.8256
1.0	one var	12.578	0.5817	17.642	0.3760	25.505	0.5831
	equal	12.298	0.5770	5.128	0.0968	6.587	0.1333
	opposite	11.647	0.5320	22.403	0.5016	28.883	0.6456
1.5	one var	3.521	0.1465	9.344	0.1759	12.936	0.2525
	equal	3.808	0.1859	3.248	0.0445	3.686	0.0535
	opposite	3.805	0.1682	11.158	0.2056	16.753	0.3270
2.0	one var	2.114	0.0699	6.009	0.0858	8.239	0.1501
	equal	2.123	0.0811	2.658	0.0296	2.906	0.0355
	opposite	2.002	0.0624	7.273	0.1094	10.514	0.1786
2.5	one var	1.557	0.0418	4.622	0.0570	6.179	0.0917
	equal	1.547	0.0455	2.406	0.0222	2.441	0.0230
	opposite	1.550	0.0409	5.631	0.0726	8.112	0.1245
3.0	one var	1.326	0.0295	4.065	0.0472	4.987	0.0703
	equal	1.286	0.0249	2.354	0.0223	2.337	0.0196
	opposite	1.326	0.0279	4.603	0.0534	6.371	0.0863
		UCL = 13.849		UCL = 8.297		UCL = 9.778	

$n = 10, \quad r = 0.1$

$\rho = 0.0$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.339	5.6839	199.157	6.4116	200.452	6.2575
0.5	one var	9.807	0.3533	13.092	0.2831	15.578	0.3502
	equal	9.385	0.3411	13.144	0.2893	15.829	0.3487
	opposite	8.835	0.3114	12.690	0.2716	15.276	0.3558
1.0	one var	2.622	0.0823	4.661	0.0730	5.393	0.0935
	equal	2.527	0.0790	4.492	0.0721	5.240	0.0902
	opposite	2.512	0.0838	4.388	0.0733	5.307	0.0922
1.5	one var	1.462	0.0331	2.896	0.0371	3.096	0.0460
	equal	1.506	0.0357	2.696	0.0370	3.057	0.0432
	opposite	1.478	0.0334	2.699	0.0370	3.005	0.0438
2.0	one var	1.201	0.0202	2.343	0.0243	2.173	0.0292
	equal	1.215	0.0228	2.049	0.0248	2.225	0.0275
	opposite	1.197	0.0199	2.046	0.0253	2.226	0.0290
2.5	one var	1.100	0.0133	2.071	0.0164	1.792	0.0214
	equal	1.111	0.0142	1.711	0.0210	1.768	0.0199
	opposite	1.122	0.0146	1.703	0.0205	1.771	0.0210
3.0	one var	1.071	0.0110	2.008	0.0145	1.515	0.0186
	equal	1.065	0.0098	1.466	0.0181	1.519	0.0178
	opposite	1.058	0.0090	1.477	0.0179	1.521	0.0175
		UCL = 9.307		UCL = 8.675		UCL = 8.696	

Table 8.1.3 (cont.): ARL values of EWMA charts for 10% mixture normal data with unknown Σ using the Two-quadrant estimator

$n = 10, \quad r = 0.1$

ρ = 0.5		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.339	5.6839	201.853	6.3743	200.059	6.2746
0.5	one var	9.807	0.3533	14.811	0.3233	17.787	0.4384
	equal	9.299	0.3213	11.281	0.2405	12.824	0.2918
	opposite	9.203	0.3262	16.055	0.3451	20.109	0.4534
1.0	one var	2.597	0.0811	5.214	0.0866	6.219	0.1117
	equal	2.483	0.0749	3.855	0.0675	4.601	0.0784
	opposite	2.468	0.0824	5.700	0.0905	6.962	0.1195
1.5	one var	1.476	0.0357	3.101	0.0411	3.493	0.0506
	equal	1.519	0.0357	2.331	0.0366	2.648	0.0389
	opposite	1.529	0.0379	3.383	0.0439	4.034	0.0587
2.0	one var	1.188	0.0197	2.403	0.0264	2.542	0.0345
	equal	1.200	0.0224	1.710	0.0239	1.962	0.0243
	opposite	1.164	0.0164	2.452	0.0255	2.810	0.0353
2.5	one var	1.097	0.0130	2.133	0.0204	1.992	0.0241
	equal	1.108	0.0137	1.482	0.0204	1.588	0.0192
	opposite	1.118	0.0156	2.017	0.0213	2.233	0.0242
3.0	one var	1.075	0.0111	1.931	0.0166	1.729	0.0202
	equal	1.056	0.0084	1.328	0.0171	1.416	0.0165
	opposite	1.070	0.0104	1.725	0.0185	1.956	0.0203
		UCL = 9.307		UCL = 8.370		UCL = 8.612	

$n = 10, \quad r = 0.1$

ρ = 0.9		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.198	5.6855	199.976	6.2614	201.630	6.5316
0.5	one var	9.847	0.3547	30.331	0.7076	38.384	0.9739
	equal	9.412	0.3272	10.348	0.2374	14.320	0.3263
	opposite	9.222	0.3273	32.831	0.7703	44.156	1.1479
1.0	one var	2.539	0.0785	9.626	0.1623	13.191	0.2466
	equal	2.496	0.0790	3.630	0.0680	4.498	0.0809
	opposite	2.470	0.0797	10.644	0.1692	15.228	0.2639
1.5	one var	1.476	0.0351	5.553	0.0773	7.468	0.1151
	equal	1.537	0.0370	2.147	0.0348	2.447	0.0413
	opposite	1.516	0.0376	6.217	0.0801	8.479	0.1255
2.0	one var	1.196	0.0198	3.861	0.0522	5.220	0.0722
	equal	1.203	0.0218	1.580	0.0230	1.792	0.0266
	opposite	1.154	0.0167	4.460	0.0513	6.020	0.0812
2.5	one var	1.088	0.0116	3.057	0.0383	3.982	0.0514
	equal	1.116	0.0139	1.362	0.0181	1.400	0.0188
	opposite	1.100	0.0150	3.458	0.0380	4.700	0.0584
3.0	one var	1.077	0.0113	2.540	0.0300	3.235	0.0418
	equal	1.050	0.0082	1.282	0.0162	1.240	0.0146
	opposite	1.056	0.0091	2.866	0.0305	3.837	0.0440
		UCL = 9.307		UCL = 7.401		UCL = 8.528	

Table 8.1.4: ARL values of EWMA charts for bivariate normal data with unknown Σ using the Two-quadrant estimator

$n = 5, \quad r = 0.1$

$\rho = 0.0$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.224	8.1788	198.007	6.2694	198.722	6.2294
0.5	one var	20.793	0.7452	30.606	0.7616	38.850	1.0497
	equal	22.333	0.7715	31.609	0.8058	37.724	1.0054
	opposite	21.887	0.7294	30.301	0.7673	39.722	1.0218
1.0	one var	5.996	0.1626	9.971	0.1929	12.640	0.2639
	equal	6.448	0.1853	10.139	0.1900	12.603	0.2521
	opposite	6.257	0.1751	10.036	0.1885	12.416	0.2581
1.5	one var	3.357	0.0809	5.704	0.0809	6.836	0.1130
	equal	3.242	0.0785	5.509	0.0826	7.018	0.1200
	opposite	3.217	0.0802	5.452	0.0821	6.826	0.1149
2.0	one var	2.176	0.0464	4.092	0.0452	4.581	0.0640
	equal	2.181	0.0467	3.939	0.0487	4.757	0.0693
	opposite	2.189	0.0461	3.894	0.0486	4.669	0.0664
2.5	one var	1.664	0.0302	3.339	0.0281	3.624	0.0454
	equal	1.678	0.0286	3.133	0.0311	3.534	0.0418
	opposite	1.661	0.0305	3.139	0.0313	3.606	0.0438
3.0	one var	1.407	0.0225	3.085	0.0205	3.060	0.0338
	equal	1.406	0.0220	2.666	0.0235	2.964	0.0299
	opposite	1.377	0.0213	2.654	0.0236	2.978	0.0307
		UCL = 9.329		UCL = 8.539		UCL = 8.678	

$n = 5, \quad r = 0.1$

$\rho = 0.5$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.224	8.1788	199.719	6.2814	198.622	6.1750
0.5	one var	20.793	0.7452	35.545	0.8699	44.602	1.2170
	equal	22.308	0.7700	26.539	0.6824	35.108	0.9277
	opposite	21.634	0.7407	39.753	1.0414	49.216	1.2840
1.0	one var	6.119	0.1670	11.367	0.2130	14.166	0.3007
	equal	6.568	0.1800	8.685	0.1617	10.990	0.2193
	opposite	6.272	0.1749	12.395	0.2360	16.893	0.3443
1.5	one var	3.354	0.0809	6.215	0.0914	7.796	0.1324
	equal	3.131	0.0767	4.866	0.0739	6.106	0.1017
	opposite	3.222	0.0793	6.920	0.1045	9.286	0.1514
2.0	one var	2.178	0.0463	4.501	0.0536	5.292	0.0769
	equal	2.217	0.0481	3.517	0.0444	4.112	0.0568
	opposite	2.228	0.0474	4.918	0.0613	6.271	0.0925
2.5	one var	1.680	0.0304	3.520	0.0351	4.127	0.0513
	equal	1.651	0.0282	2.791	0.0284	3.134	0.0357
	opposite	1.645	0.0300	3.855	0.0410	4.734	0.0600
3.0	one var	1.421	0.0230	3.059	0.0247	3.366	0.0406
	equal	1.399	0.0222	2.397	0.0192	2.678	0.0279
	opposite	1.392	0.0211	3.204	0.0295	3.878	0.0409
		UCL = 9.329		UCL = 8.283		UCL = 8.619	

Table 8.1.4 (continued): ARL values of EWMA charts for bivariate normal data with unknown Σ using the Two-quadrant estimator

$n = 5, \quad r = 0.1$

$\rho = 0.9$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.224	8.1788	201.688	6.9735	201.950	6.1642
0.5	one var	20.793	0.7452	68.620	1.9563	82.849	2.3801
	equal	22.485	0.7649	24.798	0.6136	36.523	0.9609
	opposite	22.561	0.7557	73.007	1.9913	91.764	2.4656
1.0	one var	6.220	0.1683	22.027	0.4610	31.903	0.6724
	equal	6.313	0.1760	8.093	0.1591	10.925	0.2343
	opposite	6.081	0.1692	24.881	0.5283	35.372	0.7546
1.5	one var	3.340	0.0798	11.864	0.1964	17.598	0.3267
	equal	3.264	0.0805	4.468	0.0730	6.024	0.1065
	opposite	3.181	0.0796	13.238	0.2302	20.362	0.3552
2.0	one var	2.158	0.0469	8.015	0.1166	11.368	0.1849
	equal	2.155	0.0479	3.255	0.0409	4.056	0.0583
	opposite	2.283	0.0467	8.973	0.1261	13.359	0.2018
2.5	one var	1.642	0.0298	6.085	0.0792	8.833	0.1292
	equal	1.617	0.0288	2.592	0.0261	3.153	0.0374
	opposite	1.619	0.0301	6.902	0.0829	10.132	0.1537
3.0	one var	1.374	0.0218	5.036	0.0586	6.967	0.0906
	equal	1.383	0.0218	2.240	0.0154	2.674	0.0265
	opposite	1.389	0.0207	5.592	0.0617	8.215	0.0994
		UCL = 9.329		UCL = 7.411		UCL = 8.584	

$n = 5, \quad r = 0.2$

$\rho = 0.0$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.834	8.4923	200.471	6.2590	200.192	6.3710
0.5	one var	28.304	1.0412	40.646	1.1144	49.749	1.4072
	equal	29.508	1.1034	42.988	1.1880	51.060	1.4575
	opposite	30.373	1.1749	41.957	1.1193	53.678	1.5475
1.0	one var	7.462	0.2204	11.880	0.2364	15.286	0.3635
	equal	7.884	0.2429	11.419	0.2345	14.907	0.3507
	opposite	7.834	0.2357	11.532	0.2392	15.773	0.3784
1.5	one var	3.654	0.0909	6.323	0.0970	8.016	0.1452
	equal	3.643	0.0937	6.295	0.0993	7.744	0.1386
	opposite	3.559	0.0950	6.099	0.0940	7.676	0.1374
2.0	one var	2.328	0.0518	4.383	0.0497	4.964	0.0763
	equal	2.338	0.0542	4.241	0.0500	5.115	0.0777
	opposite	2.397	0.0561	4.376	0.0551	5.138	0.0782
2.5	one var	1.787	0.0339	3.656	0.0301	3.942	0.0531
	equal	1.795	0.0367	3.361	0.0327	3.724	0.0462
	opposite	1.740	0.0327	3.391	0.0356	3.955	0.0497
3.0	one var	1.438	0.0223	3.264	0.0198	3.107	0.0368
	equal	1.462	0.0241	2.852	0.0236	3.202	0.0346
	opposite	1.413	0.0222	2.908	0.0252	3.164	0.0330
		UCL = 10.351		UCL = 9.086		UCL = 9.341	

Table 8.1.4 (continued): ARL values of EWMA charts for bivariate normal data with unknown Σ using the Two-quadrant estimator

$n = 5,\ r = 0.2$

$\rho = 0.5$		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.386	7.7575	199.555	6.8538	199.622	6.1737
0.5	one var	27.780	1.0203	47.838	1.2916	60.942	1.7166
	equal	29.697	1.1242	34.372	0.9451	44.599	1.2784
	opposite	31.316	1.1403	52.442	1.4460	68.323	1.9617
1.0	one var	7.364	0.2342	13.385	0.2873	19.009	0.4597
	equal	7.724	0.2263	9.912	0.2090	13.107	0.2981
	opposite	7.512	0.2410	16.040	0.3678	21.939	0.5377
1.5	one var	3.654	0.0903	7.097	0.1085	9.092	0.1688
	equal	3.679	0.0974	5.505	0.0886	6.580	0.1141
	opposite	3.558	0.0894	8.121	0.1285	10.634	0.2135
2.0	one var	2.414	0.0531	4.935	0.0618	6.000	0.0948
	equal	2.350	0.0506	3.744	0.0475	4.527	0.0654
	opposite	2.331	0.0510	5.440	0.0702	7.095	0.1116
2.5	one var	1.771	0.0336	3.978	0.0385	4.511	0.0623
	equal	1.759	0.0330	2.968	0.0307	3.487	0.0449
	opposite	1.723	0.0321	4.185	0.0426	5.318	0.0744
3.0	one var	1.473	0.0242	3.468	0.0256	3.534	0.0424
	equal	1.428	0.0227	2.575	0.0219	2.805	0.0296
	opposite	1.459	0.0228	3.516	0.0312	4.220	0.0498
		UCL = 10.350		UCL = 8.896		UCL = 9.303	

$n = 5,\ r = 0.2$

$\rho = 0.9$		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.699	7.7577	200.730	6.5327	198.446	5.6504
0.5	one var	28.067	1.0273	86.995	2.3979	106.484	2.9053
	equal	29.684	1.0951	30.570	0.8543	50.746	1.4328
	opposite	31.506	1.1123	100.882	3.0676	111.279	3.1283
1.0	one var	7.386	0.2337	32.127	0.7806	43.014	1.0602
	equal	7.469	0.2174	9.114	0.1883	13.742	0.2944
	opposite	7.505	0.2440	38.623	0.9763	51.155	1.3127
1.5	one var	3.635	0.0895	15.589	0.3073	23.748	0.5122
	equal	3.666	0.0942	4.942	0.0806	6.902	0.1249
	opposite	3.558	0.0899	18.371	0.3834	28.062	0.5839
2.0	one var	2.434	0.0536	9.888	0.1653	15.245	0.2915
	equal	2.282	0.0480	3.381	0.0446	4.351	0.0634
	opposite	2.362	0.0507	11.335	0.1818	17.894	0.3276
2.5	one var	1.763	0.0334	6.969	0.0979	10.626	0.1807
	equal	1.797	0.0341	2.815	0.0295	3.464	0.0460
	opposite	1.740	0.0329	8.197	0.1200	12.861	0.2094
3.0	one var	1.452	0.0239	5.526	0.0677	7.989	0.1204
	equal	1.434	0.0226	2.400	0.0193	2.857	0.0310
	opposite	1.437	0.0215	6.583	0.0791	10.229	0.1539
		UCL = 10.350		UCL = 8.062		UCL = 9.461	

Table 8.1.4 (continued): ARL values of EWMA charts for bivariate normal data with unknown Σ using the Two-quadrant estimator

$n = 5, \quad r = 0.3$

$\rho = 0.0$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.383	7.9884	198.295	6.2159	201.209	6.3863
0.5	one var	37.714	1.4761	49.346	1.4445	68.288	1.9396
	equal	36.741	1.4624	50.199	1.4816	63.591	1.9436
	opposite	35.381	1.3904	49.892	1.3915	67.912	2.0537
1.0	one var	8.609	0.3060	14.139	0.3183	18.433	0.4651
	equal	8.201	0.2799	14.293	0.3508	19.458	0.4991
	opposite	8.583	0.2903	13.940	0.3260	19.033	0.4744
1.5	one var	3.828	0.1107	7.064	0.1246	8.754	0.1752
	equal	3.779	0.1039	6.883	0.1233	8.921	0.1800
	opposite	3.798	0.1064	6.940	0.1157	9.045	0.1858
2.0	one var	2.419	0.0573	4.677	0.0603	5.557	0.0974
	equal	2.427	0.0568	4.580	0.0595	5.549	0.0934
	opposite	2.482	0.0596	4.666	0.0601	5.517	0.0900
2.5	one var	1.805	0.0333	3.756	0.0340	3.905	0.0575
	equal	1.825	0.0336	3.464	0.0357	4.042	0.0561
	opposite	1.813	0.0346	3.606	0.0402	4.160	0.0592
3.0	one var	1.485	0.0240	3.345	0.0225	3.264	0.0422
	equal	1.500	0.0251	2.984	0.0254	3.269	0.0390
	opposite	1.458	0.0241	3.038	0.0271	3.223	0.0371
		UCL = 10.599		UCL = 9.110		UCL = 9.487	

$n = 5, \quad r = 0.3$

$\rho = 0.5$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.383	7.9884	201.598	6.2860	198.918	5.9471
0.5	one var	37.714	1.4761	60.218	1.7815	73.320	2.0397
	equal	38.054	1.4782	42.894	1.1848	61.373	1.9285
	opposite	35.913	1.4334	68.065	2.0127	82.105	2.3495
1.0	one var	8.323	0.2851	17.253	0.4051	23.450	0.6290
	equal	9.119	0.3084	11.805	0.2700	16.487	0.4171
	opposite	8.406	0.2884	21.599	0.4921	28.259	0.7325
1.5	one var	3.747	0.1048	8.085	0.1401	10.735	0.2218
	equal	3.796	0.1037	5.841	0.0993	7.549	0.1592
	opposite	3.829	0.1027	9.455	0.1754	13.199	0.2935
2.0	one var	2.355	0.0565	5.422	0.0755	6.584	0.1176
	equal	2.391	0.0560	4.016	0.0538	4.767	0.0798
	opposite	2.419	0.0534	6.166	0.0894	8.190	0.1548
2.5	one var	1.804	0.0345	4.240	0.0453	4.757	0.0752
	equal	1.754	0.0326	3.091	0.0329	3.551	0.0485
	opposite	1.728	0.0338	4.535	0.0527	5.775	0.0907
3.0	one var	1.485	0.0249	3.527	0.0297	3.774	0.0521
	equal	1.506	0.0241	2.640	0.0231	2.890	0.0344
	opposite	1.491	0.0255	3.731	0.0374	4.474	0.0595
		UCL = 10.599		UCL = 8.957		UCL = 9.482	

Table 8.1.4 (continued): ARL values of EWMA charts for bivariate normal data with unknown Σ using the Two-quadrant estimator

$n = 5, \quad r = 0.3$

$\rho = 0.9$		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.382	7.9882	201.163	6.3825	198.058	5.6646
0.5	one var	37.557	1.4716	109.773	3.3216	109.589	3.0275
	equal	38.071	1.4834	38.695	1.0972	64.640	1.8851
	opposite	35.353	1.4492	114.757	3.3408	115.551	3.1342
1.0	one var	8.505	0.2891	40.925	1.0919	54.343	1.3552
	equal	8.611	0.2952	10.936	0.2539	17.092	0.4394
	opposite	8.387	0.2968	48.560	1.2744	63.430	1.5200
1.5	one var	3.799	0.1061	21.013	0.4738	29.682	0.7022
	equal	3.774	0.1006	5.459	0.0999	7.681	0.1613
	opposite	3.899	0.1026	25.141	0.5698	33.791	0.7637
2.0	one var	2.370	0.0545	12.156	0.2191	18.992	0.4075
	equal	2.410	0.0560	3.649	0.0511	4.871	0.0845
	opposite	2.501	0.0571	15.097	0.3054	22.702	0.4431
2.5	one var	1.813	0.0343	8.185	0.1433	13.017	0.2566
	equal	1.818	0.0352	2.866	0.0311	3.568	0.0507
	opposite	1.792	0.0363	9.994	0.1707	16.143	0.2961
3.0	one var	1.509	0.0250	6.119	0.0811	9.640	0.1744
	equal	1.472	0.0247	2.499	0.0223	2.836	0.0333
	opposite	1.450	0.0236	7.583	0.1171	12.480	0.2213
		UCL = 10.598		UCL = 8.185		UCL = 9.688	

$n = 10, \quad r = 0.1$

$\rho = 0.0$		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.982	7.0005	198.853	6.1935	200.923	6.5036
0.5	one var	25.832	0.7342	29.407	0.7477	34.844	0.9516
	equal	23.019	0.6690	27.973	0.7141	37.387	1.0233
	opposite	24.295	0.6985	27.919	0.7558	36.730	0.9701
1.0	one var	7.226	0.1679	9.249	0.1773	11.786	0.2586
	equal	7.254	0.1660	8.966	0.1801	11.872	0.2470
	opposite	7.070	0.1773	9.061	0.1764	11.566	0.2348
1.5	one var	3.855	0.0805	4.997	0.0829	6.236	0.1087
	equal	3.833	0.0809	4.882	0.0809	6.207	0.1138
	opposite	3.684	0.0805	4.809	0.0790	6.234	0.1154
2.0	one var	2.416	0.0470	3.352	0.0457	4.046	0.0677
	equal	2.445	0.0460	3.184	0.0435	4.063	0.0640
	opposite	2.419	0.0455	3.282	0.0451	3.991	0.0622
2.5	one var	1.814	0.0305	2.663	0.0299	2.958	0.0411
	equal	1.807	0.0308	2.448	0.0300	3.033	0.0451
	opposite	1.775	0.0296	2.523	0.0317	3.053	0.0447
3.0	one var	1.475	0.0219	2.255	0.0203	2.409	0.0340
	equal	1.427	0.0211	2.059	0.0233	2.513	0.0340
	opposite	1.408	0.0196	2.078	0.0238	2.442	0.0309
		UCL = 9.034		UCL = 8.682		UCL = 8.683	

Table 8.1.4 (continued): ARL values of EWMA charts for bivariate normal data with unknown Σ using the Two-quadrant estimator

$n = 10, \quad r = 0.1$

$\rho = 0.5$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.982	7.0005	199.288	6.3739	201.082	6.5352
0.5	one var	25.832	0.7342	31.896	0.8340	44.566	1.2075
	equal	22.734	0.6578	24.079	0.6010	33.952	0.9285
	opposite	24.297	0.7020	37.977	1.0366	48.851	1.3517
1.0	one var	7.208	0.1696	10.106	0.1869	13.821	0.3041
	equal	7.158	0.1682	7.767	0.1610	10.588	0.2222
	opposite	7.160	0.1779	11.546	0.2239	16.114	0.3334
1.5	one var	3.844	0.0807	5.684	0.0941	7.483	0.1407
	equal	3.841	0.0795	4.177	0.0733	5.605	0.1071
	opposite	3.702	0.0792	6.296	0.1014	8.314	0.1489
2.0	one var	2.433	0.0473	3.705	0.0526	4.899	0.0820
	equal	2.486	0.0477	2.838	0.0430	3.663	0.0593
	opposite	2.419	0.0441	4.133	0.0583	5.502	0.0906
2.5	one var	1.791	0.0297	2.903	0.0353	3.579	0.0537
	equal	1.795	0.0298	2.106	0.0294	2.704	0.0388
	opposite	1.786	0.0300	3.133	0.0373	4.107	0.0566
3.0	one var	1.475	0.0216	2.414	0.0242	2.811	0.0377
	equal	1.435	0.0212	1.690	0.0215	2.165	0.0290
	opposite	1.448	0.0205	2.507	0.0263	3.227	0.0412
		UCL = 9.034		UCL = 8.384		UCL = 8.708	

$n = 10, \quad r = 0.1$

$\rho = 0.9$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.982	7.0005	198.662	6.1699	198.911	5.7516
0.5	one var	25.832	0.7342	66.811	1.9269	80.868	2.3149
	equal	22.311	0.6465	22.325	0.5889	33.313	0.9094
	opposite	24.062	0.7145	69.666	1.9780	82.841	2.2235
1.0	one var	7.136	0.1677	19.923	0.4120	29.312	0.6723
	equal	7.021	0.1659	7.160	0.1585	9.989	0.2362
	opposite	7.270	0.1729	22.882	0.5250	32.383	0.7318
1.5	one var	3.791	0.0806	10.482	0.1705	15.410	0.2982
	equal	3.892	0.0832	3.855	0.0706	5.507	0.1029
	opposite	3.777	0.0771	11.547	0.1847	18.289	0.3650
2.0	one var	2.417	0.0469	6.765	0.1007	10.224	0.1791
	equal	2.446	0.0458	2.528	0.0438	3.453	0.0612
	opposite	2.384	0.0432	7.737	0.1081	12.057	0.1992
2.5	one var	1.798	0.0298	5.098	0.0704	7.512	0.1168
	equal	1.829	0.0306	1.914	0.0290	2.552	0.0420
	opposite	1.778	0.0314	5.613	0.0684	8.898	0.1316
3.0	one var	1.458	0.0211	4.023	0.0500	5.994	0.0882
	equal	1.416	0.0207	1.605	0.0219	2.032	0.0306
	opposite	1.432	0.0200	4.632	0.0525	6.814	0.0935
		UCL = 9.034		UCL = 7.406		UCL = 8.519	

Table 8.1.5 Upper Control Limits for the $\overline{X}$ based MEWMA chart

UCL values for bivariate normal data with unknown Σ

ρ	n	r = 0.1	0.2	0.3
0.0	5	9.329	10.351	10.599
	10	9.034		
	15			
0.5	5	9.329	10.350	10.599
	10	9.034		
	15			
0.9	5	9.329	10.350	10.598
	10	9.034		
	15			

UCL values for 10% mixture bivariate normal data with unknown Σ

ρ	n	r = 0.1	0.2	0.3
0.0	5	10.003	12.208	13.849
	10	9.307		
	15			
0.5	5	10.003	12.206	13.849
	10	9.307		
	15			
0.9	5	10.003	12.208	13.849
	10	9.307		
	15			

Table 8.1.6 Upper Control Limits for the signed rank based MEWMA chart

UCL values for bivariate normal data with unknown Σ

		r		
ρ	n	0.1	0.2	0.3
0.0	5	8.539	9.086	9.110
	10	8.682		
	15			
0.5	5	8.283	8.896	8.957
	10	8.384		
	15			
0.9	5	7.411	8.062	8.185
	10	7.406		
	15			

UCL values for 10% mixture bivariate normal data with unknown Σ

		r		
ρ	n	0.1	0.2	0.3
0.0	5	8.551	9.209	9.179
	10	8.675		
	15			
0.5	5	8.401	9.000	9.039
	10	8.370		
	15			
0.9	5	7.500	8.191	8.297
	10	7.401		
	15			

Table 8.1.7 Upper Control Limits for the sign based MEWMA chart

UCL values for bivariate normal data with unknown Σ

		r		
ρ	*n*	0.1	0.2	0.3
0.0	5 10 15	8.678 8.683	9.341	9.487
0.5	5 10 15	8.619 8.708	9.303	9.482
0.9	5 10 15	8.584 8.519	9.461	9.688

UCL values for 10% mixture bivariate normal data with unknown Σ

		r		
ρ	*n*	0.1	0.2	0.3
0.0	5 10 15	8.652 8.696	9.400	9.493
0.5	5 10 15	8.665 8.612	9.442	9.581
0.9	5 10 15	8.647 8.528	9.526	9.778

Table 8.2.1: ARL values of the sign based chart for 10% mixture normal data with unknown Σ when a shift occurs after 10 good samples

$n = 5, \quad r = 0.1$

$\rho = 0.0$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.247	7.4770	201.378	6.4094	201.330	6.3590
0.5	one var	14.578	0.4471	16.128	0.3713	16.128	0.3714
	equal	18.619	0.5077	16.439	0.3687	16.439	0.3687
	opposite	14.131	0.4265	16.777	0.3828	16.777	0.3828
1.0	one var	6.060	0.1648	6.908	0.1235	6.908	0.1235
	equal	7.086	0.1787	6.729	0.1291	6.729	0.1291
	opposite	5.678	0.1801	6.509	0.1323	6.509	0.1321
1.5	one var	3.679	0.1305	4.782	0.0839	4.782	0.0839
	equal	4.252	0.1287	4.286	0.0780	4.286	0.0780
	opposite	3.454	0.1253	4.328	0.0779	4.328	0.0779
2.0	one var	3.183	0.1144	4.011	0.0724	4.011	0.0724
	equal	3.218	0.1181	3.542	0.0682	3.542	0.0682
	opposite	2.837	0.1032	3.496	0.0690	3.496	0.0690
2.5	one var	2.969	0.1069	3.608	0.0671	3.608	0.0671
	equal	2.765	0.1113	3.151	0.0570	3.151	0.0570
	opposite	2.276	0.1082	3.085	0.0620	3.085	0.0619
3.0	one var	2.609	0.1147	3.461	0.0693	3.461	0.0693
	equal	2.508	0.1037	2.832	0.0615	2.832	0.0615
	opposite	2.062	0.1030	2.781	0.0613	2.781	0.0613
		UCL = 9.009		UCL = 8.652		UCL = 8.651	

$n = 5, \quad r = 0.1$

$\rho = 0.5$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.813	6.8450	199.663	6.1900	199.663	6.1900
0.5	one var	15.897	0.4956	19.044	0.4503	19.044	0.4503
	equal	14.432	0.4013	14.955	0.3362	14.955	0.3362
	opposite	20.050	0.6094	20.692	0.4938	20.692	0.4938
1.0	one var	6.322	0.1796	7.580	0.1424	7.580	0.1424
	equal	5.595	0.1808	6.084	0.1134	6.084	0.1134
	opposite	7.039	0.2014	8.100	0.1570	8.100	0.1570
1.5	one var	3.784	0.1410	4.835	0.0954	4.835	0.0954
	equal	3.701	0.1344	4.164	0.0800	4.164	0.0800
	opposite	3.984	0.1485	5.096	0.0908	5.096	0.0908
2.0	one var	3.089	0.1183	4.168	0.0789	4.168	0.0789
	equal	2.838	0.1178	3.374	0.0759	3.374	0.0759
	opposite	3.164	0.1178	3.831	0.0750	3.831	0.0750
2.5	one var	2.640	0.1139	3.612	0.0672	3.612	0.0672
	equal	2.451	0.1196	3.155	0.0611	3.155	0.0611
	opposite	2.310	0.1173	3.145	0.0718	3.145	0.0718
3.0	one var	2.604	0.1147	3.495	0.0657	3.495	0.0657
	equal	2.408	0.1121	3.060	0.0590	3.060	0.0590
	opposite	1.799	0.1180	2.889	0.0609	2.889	0.0609
		UCL = 9.137		UCL = 8.665		UCL = 8.665	

Table 8.2.1 (cont.): ARL values of the sign based chart for 10% mixture normal data with unknown Σ when a shift occurs after 10 good samples

$n = 5, \quad r = 0.1$

$\rho = 0.9$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.406	7.1682	198.307	5.5239	198.382	5.4721
0.5	one var	28.821	0.8824	37.225	0.9838	36.993	0.9831
	equal	12.017	0.3570	15.551	0.3411	15.588	0.3358
	opposite	42.389	1.4482	43.296	1.1858	43.083	1.1875
1.0	one var	10.531	0.3043	13.928	0.2774	13.887	0.2783
	equal	5.057	0.1611	6.595	0.1165	6.552	0.1160
	opposite	14.687	0.4390	15.476	0.3292	15.803	0.3437
1.5	one var	5.852	0.1877	8.349	0.1436	8.266	0.1440
	equal	3.438	0.1336	4.553	0.0703	4.536	0.0705
	opposite	7.821	0.2388	8.757	0.1635	8.747	0.1624
2.0	one var	4.365	0.1549	6.232	0.1024	6.202	0.1033
	equal	2.806	0.1181	3.773	0.0621	3.758	0.0627
	opposite	5.741	0.1781	6.406	0.1213	6.384	0.1220
2.5	one var	3.557	0.1326	4.824	0.0843	4.809	0.0840
	equal	2.631	0.1150	3.462	0.0643	3.475	0.0619
	opposite	4.405	0.1551	5.141	0.0856	5.224	0.0817
3.0	one var	2.974	0.1264	4.336	0.0751	4.288	0.0766
	equal	2.470	0.1113	3.397	0.0575	3.427	0.0574
	opposite	3.627	0.1326	4.381	0.0777	4.335	0.0780
		UCL = 9.128		UCL = 8.647		UCL = 8.640	

$n = 5, \quad r = 0.2$

$\rho = 0.0$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.272	7.2011	199.170	6.0763	199.170	6.0763
0.5	one var	18.662	0.5640	19.863	0.5316	19.863	0.5316
	equal	22.916	0.7234	20.682	0.5493	20.682	0.5493
	opposite	19.420	0.6047	19.789	0.5113	19.789	0.5113
1.0	one var	5.923	0.1699	6.485	0.1302	6.485	0.1302
	equal	6.911	0.1946	6.398	0.1318	6.398	0.1318
	opposite	5.675	0.1645	6.137	0.1214	6.137	0.1214
1.5	one var	3.650	0.1130	4.239	0.0789	4.239	0.0789
	equal	3.542	0.1290	3.965	0.0754	3.965	0.0754
	opposite	2.963	0.1217	3.973	0.0732	3.973	0.0732
2.0	one var	2.712	0.1092	3.562	0.0604	3.562	0.0604
	equal	2.930	0.1038	3.067	0.0581	3.067	0.0581
	opposite	2.331	0.0994	3.030	0.0613	3.030	0.0613
2.5	one var	2.199	0.1109	3.128	0.0587	3.128	0.0587
	equal	2.550	0.0930	2.630	0.0517	2.630	0.0517
	opposite	2.135	0.0876	2.583	0.0485	2.583	0.0485
3.0	one var	2.442	0.0934	2.966	0.0601	2.966	0.0601
	equal	1.858	0.1009	2.497	0.0427	2.497	0.0427
	opposite	1.743	0.0959	2.450	0.0472	2.450	0.0472
		UCL = 9.662		UCL = 9.400		UCL = 9.400	

Table 8.2.1 (cont.): ARL values of the sign based chart for 10% mixture normal data with unknown Σ when a shift occurs after 10 good samples

$n = 5, \quad r = 0.2$

$\rho = 0.5$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.727	6.8091	200.838	6.4145	200.838	6.4145
0.5	one var	17.506	0.5921	25.000	0.6752	25.000	0.6752
	equal	17.002	0.5298	18.018	0.4567	18.018	0.4567
	opposite	24.744	0.8634	27.232	0.7299	27.232	0.7299
1.0	one var	5.679	0.1826	7.352	0.1528	7.352	0.1528
	equal	5.058	0.1828	6.031	0.1109	6.031	0.1109
	opposite	7.046	0.2236	8.290	0.1674	8.290	0.1674
1.5	one var	3.638	0.1208	4.719	0.0767	4.719	0.0767
	equal	3.399	0.1258	3.921	0.0649	3.921	0.0649
	opposite	3.612	0.1405	4.773	0.0923	4.773	0.0923
2.0	one var	2.425	0.1183	3.539	0.0708	3.539	0.0708
	equal	2.612	0.1072	3.058	0.0585	3.058	0.0585
	opposite	2.400	0.1188	3.386	0.0698	3.386	0.0698
2.5	one var	2.138	0.1105	3.122	0.0616	3.122	0.0616
	equal	2.262	0.1021	2.777	0.0516	2.777	0.0516
	opposite	2.050	0.1053	2.890	0.0564	2.890	0.0564
3.0	one var	2.048	0.1065	3.140	0.0490	3.140	0.0490
	equal	2.041	0.1008	2.651	0.0472	2.651	0.0472
	opposite	1.690	0.1029	2.459	0.0555	2.459	0.0555
		UCL = 9.786		UCL = 9.442		UCL = 9.442	

$n = 5, \quad r = 0.2$

$\rho = 0.9$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.341	7.4531	199.120	5.8322	199.046	5.8341
0.5	one var	35.369	1.1870	50.525	1.4234	50.581	1.4226
	equal	14.002	0.4158	18.797	0.4611	18.826	0.4611
	opposite	54.371	2.0301	57.751	1.4978	57.792	1.4973
1.0	one var	11.668	0.3628	16.173	0.3938	16.185	0.3938
	equal	4.636	0.1696	6.310	0.1196	6.326	0.1199
	opposite	18.385	0.6200	18.823	0.4383	18.823	0.4383
1.5	one var	6.380	0.2019	8.339	0.1753	8.339	0.1753
	equal	3.237	0.1291	4.100	0.0673	4.100	0.0673
	opposite	8.949	0.3004	9.478	0.2085	9.478	0.2085
2.0	one var	4.038	0.1506	5.935	0.1069	5.935	0.1069
	equal	2.480	0.1139	3.445	0.0515	3.445	0.0515
	opposite	5.574	0.1949	6.343	0.1199	6.343	0.1199
2.5	one var	3.060	0.1253	4.541	0.0854	4.541	0.0854
	equal	2.175	0.1117	3.127	0.0456	3.127	0.0456
	opposite	3.995	0.1538	4.839	0.0841	4.839	0.0841
3.0	one var	2.648	0.1143	3.980	0.0600	3.980	0.0600
	equal	2.164	0.1070	3.086	0.0377	3.086	0.0377
	opposite	2.990	0.1355	3.924	0.0707	3.924	0.0707
		UCL = 9.977		UCL = 9.526		UCL = 9.526	

Table 8.2.2: ARL values of the signed rank based chart for 10% mixture normal data with unknown Σ when a shift occurs after 10 good samples

$n = 5, \quad r = 0.1$

$\rho = 0.0$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	199.600	6.7320	199.435	6.1785	199.435	6.1785
0.5	one var	13.045	0.3811	14.330	0.3146	14.330	0.3146
	equal	15.994	0.4202	14.374	0.3241	14.374	0.3241
	opposite	12.719	0.3821	14.180	0.3153	14.180	0.3153
1.0	one var	5.427	0.1626	6.374	0.1167	6.374	0.1167
	equal	6.188	0.1706	6.179	0.1223	6.179	0.1223
	opposite	4.972	0.1600	6.020	0.1139	6.020	0.1139
1.5	one var	3.778	0.1373	4.905	0.0790	4.905	0.0790
	equal	3.778	0.1360	4.182	0.0835	4.182	0.0835
	opposite	3.330	0.1234	4.226	0.0775	4.226	0.0775
2.0	one var	3.358	0.1204	4.238	0.0726	4.238	0.0726
	equal	3.202	0.1192	3.540	0.0711	3.540	0.0711
	opposite	2.710	0.1130	3.412	0.0679	3.412	0.0679
2.5	one var	3.130	0.1171	3.930	0.0708	3.930	0.0708
	equal	2.808	0.1162	3.287	0.0631	3.287	0.0631
	opposite	2.488	0.1057	3.236	0.0631	3.236	0.0631
3.0	one var	3.018	0.1132	3.779	0.0736	3.779	0.0736
	equal	2.824	0.1070	2.962	0.0636	2.962	0.0636
	opposite	2.120	0.1148	3.025	0.0644	3.025	0.0644
		UCL = 8.862		UCL = 8.551		UCL = 8.551	

$n = 5, \quad r = 0.1$

$\rho = 0.5$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.700	7.1758	200.350	6.3460	200.746	6.3478
0.5	one var	12.843	0.4052	15.365	0.3347	15.351	0.3348
	equal	12.583	0.3669	12.841	0.2892	12.817	0.2895
	opposite	15.648	0.4922	17.326	0.3935	17.316	0.3941
1.0	one var	4.979	0.1915	6.834	0.1278	6.822	0.1288
	equal	4.717	0.1752	5.726	0.1085	5.718	0.1084
	opposite	5.345	0.1921	7.041	0.1333	7.057	0.1338
1.5	one var	3.229	0.1441	4.817	0.0873	4.809	0.0880
	equal	3.271	0.1389	4.187	0.0813	4.198	0.0818
	opposite	3.359	0.1477	4.784	0.0841	4.743	0.0860
2.0	one var	2.787	0.1329	3.911	0.0821	3.918	0.0808
	equal	2.596	0.1377	3.459	0.0732	3.463	0.0732
	opposite	2.486	0.1277	3.731	0.0742	3.743	0.0743
2.5	one var	2.541	0.1295	3.864	0.0701	3.859	0.0702
	equal	2.407	0.1302	3.377	0.0664	3.368	0.0672
	opposite	2.114	0.1197	3.203	0.0632	3.228	0.0627
3.0	one var	2.468	0.1237	3.833	0.0675	3.827	0.0676
	equal	2.460	0.1213	3.254	0.0654	3.251	0.0653
	opposite	1.665	0.1211	2.993	0.0603	2.993	0.0603
		UCL = 8.746		UCL = 8.401		UCL = 8.401	

Table 8.2.2 (cont.): ARL values of the signed rank based chart for 10% mixture normal data with unknown Σ when a shift occurs after 10 good samples

$n = 5, \quad r = 0.1$

$\rho = 0.9$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.211	7.3829	200.511	6.0952	200.396	6.0949
0.5	one var	25.028	0.7952	28.978	0.7234	28.926	0.7242
	equal	10.774	0.3278	11.932	0.2669	11.951	0.2674
	opposite	30.981	1.0612	32.098	0.8334	32.159	0.8339
1.0	one var	8.157	0.2527	10.493	0.1983	10.502	0.1996
	equal	4.143	0.1747	5.444	0.1152	5.499	0.1147
	opposite	10.269	0.3265	11.219	0.2227	11.230	0.2221
1.5	one var	4.685	0.1749	6.395	0.1328	6.398	0.1327
	equal	3.247	0.1319	3.899	0.0934	3.916	0.0928
	opposite	6.144	0.1971	6.856	0.1328	6.853	0.1320
2.0	one var	3.525	0.1410	4.817	0.1064	4.866	0.1044
	equal	2.518	0.1321	3.661	0.0778	3.672	0.0770
	opposite	4.288	0.1531	4.869	0.1041	4.866	0.1041
2.5	one var	2.616	0.1346	4.073	0.0864	4.085	0.0863
	equal	2.415	0.1272	3.393	0.0765	3.404	0.0766
	opposite	2.951	0.1474	3.944	0.0905	3.968	0.0895
3.0	one var	2.236	0.1309	3.633	0.0788	3.620	0.0796
	equal	2.349	0.1288	3.227	0.0794	3.225	0.0795
	opposite	2.614	0.1299	3.306	0.0794	3.298	0.0795
		UCL = 7.934		UCL = 7.500		UCL = 7.500	

$n = 5, \quad r = 0.2$

$\rho = 0.0$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.306	6.8099	201.449	6.4393	201.449	6.4393
0.5	one var	14.491	0.4597	18.350	0.4573	18.350	0.4573
	equal	19.892	0.6254	16.178	0.4102	16.178	0.4102
	opposite	15.809	0.5024	16.099	0.4116	16.099	0.4116
1.0	one var	5.133	0.1581	6.163	0.1061	6.163	0.1061
	equal	5.957	0.1699	5.822	0.1106	5.822	0.1106
	opposite	4.646	0.1543	5.874	0.1121	5.874	0.1121
1.5	one var	3.470	0.1218	4.405	0.0663	4.405	0.0663
	equal	3.459	0.1248	3.937	0.0665	3.937	0.0665
	opposite	3.012	0.1130	3.811	0.0690	3.811	0.0690
2.0	one var	3.184	0.0962	3.756	0.0599	3.756	0.0599
	equal	2.655	0.1128	3.038	0.0562	3.038	0.0562
	opposite	2.223	0.1055	3.115	0.0550	3.115	0.0550
2.5	one var	2.538	0.1106	3.465	0.0570	3.465	0.0570
	equal	2.428	0.1037	2.803	0.0494	2.803	0.0494
	opposite	2.090	0.0979	2.755	0.0517	2.755	0.0517
3.0	one var	2.505	0.1069	3.375	0.0525	3.375	0.0525
	equal	2.412	0.0941	2.670	0.0469	2.670	0.0469
	opposite	2.027	0.0918	2.617	0.0472	2.617	0.0472
		UCL = 9.377		UCL = 9.209		UCL = 9.209	

Table 8.2.2 (cont.): ARL values of the signed rank based chart for 10% mixture normal data with unknown Σ when a shift occurs after 10 good samples

$n = 5, \quad r = 0.2$

$\rho = 0.5$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.692	7.4182	199.055	6.1654	199.171	6.1683
0.5	one var	16.177	0.5059	20.387	0.5195	20.387	0.5195
	equal	14.770	0.4679	14.572	0.3640	14.572	0.3640
	opposite	19.402	0.6570	22.221	0.6160	22.221	0.6160
1.0	one var	5.041	0.1741	6.767	0.1358	6.767	0.1358
	equal	5.109	0.1604	5.198	0.0980	5.198	0.0980
	opposite	5.226	0.1955	6.913	0.1382	6.913	0.1382
1.5	one var	3.186	0.1312	4.503	0.0775	4.503	0.0775
	equal	3.248	0.1230	3.829	0.0655	3.829	0.0655
	opposite	3.297	0.1274	4.048	0.0782	4.048	0.0782
2.0	one var	2.648	0.1141	3.578	0.0705	3.578	0.0705
	equal	2.734	0.1122	3.247	0.0553	3.247	0.0553
	opposite	2.292	0.1157	3.151	0.0627	3.151	0.0627
2.5	one var	2.251	0.1141	3.454	0.0560	3.454	0.0560
	equal	2.390	0.1105	2.882	0.0570	2.882	0.0570
	opposite	1.935	0.1040	2.688	0.0533	2.688	0.0533
3.0	one var	1.975	0.1117	3.237	0.0586	3.237	0.0586
	equal	2.163	0.1091	2.745	0.0546	2.745	0.0546
	opposite	1.475	0.1063	2.451	0.0466	2.451	0.0466
		UCL = 9.311		UCL = 9.000		UCL = 9.000	

$n = 5, \quad r = 0.2$

$\rho = 0.9$		One-quadrant estimator		Two-quadrant estimator		Four-quadrant estimator	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	201.075	7.2519	200.881	6.1047	200.551	6.0947
0.5	one var	30.945	1.0912	40.726	1.1801	40.865	1.1832
	equal	11.463	0.4122	13.980	0.3595	13.979	0.3600
	opposite	46.845	1.7817	47.549	1.4119	47.435	1.4133
1.0	one var	8.959	0.3093	12.009	0.2839	11.965	0.2838
	equal	4.347	0.1625	5.125	0.1088	5.133	0.1082
	opposite	13.384	0.4574	13.117	0.3078	13.125	0.3089
1.5	one var	4.890	0.1846	6.362	0.1281	6.395	0.1284
	equal	2.865	0.1252	3.759	0.0707	3.806	0.0688
	opposite	6.401	0.2168	6.880	0.1396	6.882	0.1393
2.0	one var	3.115	0.1412	4.284	0.0925	4.276	0.0934
	equal	2.542	0.1142	3.305	0.0595	3.282	0.0604
	opposite	4.107	0.1512	4.631	0.0892	4.638	0.0903
2.5	one var	2.512	0.1238	3.792	0.0743	3.804	0.0733
	equal	2.326	0.1099	3.044	0.0566	3.041	0.0577
	opposite	2.849	0.1334	3.581	0.0750	3.602	0.0729
3.0	one var	2.086	0.1166	3.168	0.0708	3.144	0.0720
	equal	2.032	0.1166	2.935	0.0593	2.892	0.0608
	opposite	2.240	0.1203	2.936	0.0636	2.925	0.0641
		UCL = 8.698		UCL = 8.191		UCL = 8.191	

Table 8.2.3: ARL values of EWMA charts for 10% mixture normal data with unknown Σ using the Two-quadrant estimator, when a shift occurs after 10 good samples

$n = 5, \quad r = 0.1$

$\rho = 0.0$		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.241	7.3408	199.435	6.1785	201.378	6.4094
0.5	one var	25.124	0.9574	14.330	0.3146	16.128	0.3713
	equal	23.894	0.9233	14.374	0.3241	16.439	0.3687
	opposite	24.925	0.9833	14.180	0.3153	16.777	0.3828
1.0	one var	6.206	0.2908	6.374	0.1167	6.908	0.1235
	equal	6.282	0.2899	6.179	0.1223	6.729	0.1291
	opposite	6.655	0.2907	6.020	0.1139	6.509	0.1323
1.5	one var	3.409	0.1862	4.905	0.0790	4.782	0.0839
	equal	2.885	0.1921	4.182	0.0835	4.286	0.0780
	opposite	3.154	0.1856	4.226	0.0775	4.328	0.0779
2.0	one var	1.897	0.1606	4.238	0.0726	4.011	0.0724
	equal	1.790	0.1617	3.540	0.0711	3.542	0.0682
	opposite	1.507	0.1704	3.412	0.0679	3.496	0.0690
2.5	one var	1.123	0.1496	3.930	0.0708	3.608	0.0671
	equal	1.261	0.1438	3.287	0.0631	3.151	0.0570
	opposite	0.941	0.1519	3.236	0.0631	3.085	0.0620
3.0	one var	0.470	0.1444	3.779	0.0736	3.461	0.0693
	equal	0.678	0.1400	2.962	0.0636	2.832	0.0615
	opposite	0.595	0.1422	3.025	0.0644	2.781	0.0613
		UCL = 10.003		UCL = 8.551		UCL = 8.652	

$n = 5, \quad r = 0.1$

$\rho = 0.5$		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.241	7.3408	200.350	6.3460	199.663	6.1900
0.5	one var	25.124	0.9574	15.365	0.3347	19.044	0.4503
	equal	23.835	0.9305	12.841	0.2892	14.955	0.3362
	opposite	24.786	0.9954	17.326	0.3935	20.692	0.4938
1.0	one var	6.330	0.2927	6.834	0.1278	7.580	0.1424
	equal	5.911	0.2882	5.726	0.1085	6.084	0.1134
	opposite	6.570	0.2891	7.041	0.1333	8.100	0.1570
1.5	one var	3.386	0.1848	4.817	0.0873	4.835	0.0954
	equal	2.969	0.1907	4.187	0.0813	4.164	0.0800
	opposite	2.847	0.1955	4.784	0.0841	5.096	0.0908
2.0	one var	1.874	0.1626	3.911	0.0821	4.168	0.0789
	equal	1.863	0.1596	3.459	0.0732	3.374	0.0759
	opposite	1.494	0.1700	3.731	0.0742	3.831	0.0750
2.5	one var	1.106	0.1506	3.864	0.0701	3.612	0.0672
	equal	1.215	0.1453	3.377	0.0664	3.155	0.0611
	opposite	0.913	0.1524	3.203	0.0632	3.145	0.0718
3.0	one var	0.577	0.1427	3.833	0.0675	3.495	0.0657
	equal	0.775	0.1363	3.254	0.0654	3.060	0.0590
	opposite	0.592	0.1426	2.993	0.0603	2.889	0.0609
		UCL = 10.003		UCL = 8.401		UCL = 8.665	

Table 8.2.3 (cont.): ARL values of EWMA charts for 10% mixture normal data with unknown Σ using the Two-quadrant estimator, when a shift occurs after 10 good samples

$n = 5, \quad r = 0.1$

ρ = 0.9		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.704	7.3441	200.511	6.0952	198.307	5.5239
0.5	one var	25.145	0.9577	28.978	0.7234	37.225	0.9838
	equal	24.344	0.9620	11.932	0.2669	15.551	0.3411
	opposite	23.732	0.9572	32.098	0.8334	43.296	1.1858
1.0	one var	6.307	0.2932	10.493	0.1983	13.928	0.2774
	equal	6.312	0.2887	5.444	0.1152	6.595	0.1165
	opposite	6.737	0.2825	11.219	0.2227	15.476	0.3292
1.5	one var	3.386	0.1848	6.395	0.1328	8.349	0.1436
	equal	3.120	0.1918	3.899	0.0934	4.553	0.0703
	opposite	2.946	0.1903	6.856	0.1328	8.757	0.1635
2.0	one var	1.849	0.1628	4.817	0.1064	6.232	0.1024
	equal	2.066	0.1583	3.661	0.0778	3.773	0.0621
	opposite	1.445	0.1717	4.869	0.1041	6.406	0.1213
2.5	one var	1.121	0.1497	4.073	0.0864	4.824	0.0843
	equal	1.075	0.1460	3.393	0.0765	3.462	0.0643
	opposite	0.910	0.1523	3.944	0.0905	5.141	0.0856
3.0	one var	0.561	0.1425	3.633	0.0788	4.336	0.0751
	equal	0.485	0.1449	3.227	0.0794	3.397	0.0575
	opposite	0.574	0.1432	3.306	0.0794	4.381	0.0777
		UCL = 10.003		UCL = 7.500		UCL = 8.647	

$n = 5, \quad r = 0.2$

ρ = 0.0		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.764	6.6690	201.449	6.4393	199.170	6.0763
0.5	one var	43.278	1.6403	18.350	0.4573	19.863	0.5316
	equal	48.935	1.9539	16.178	0.4102	20.682	0.5493
	opposite	42.839	1.7361	16.099	0.4116	19.789	0.5113
1.0	one var	10.473	0.4330	6.163	0.1061	6.485	0.1302
	equal	10.693	0.4447	5.822	0.1106	6.398	0.1318
	opposite	9.692	0.4088	5.874	0.1121	6.137	0.1214
1.5	one var	3.969	0.2013	4.405	0.0663	4.239	0.0789
	equal	3.940	0.1957	3.937	0.0665	3.965	0.0754
	opposite	3.989	0.2054	3.811	0.0690	3.973	0.0732
2.0	one var	2.065	0.1499	3.756	0.0599	3.562	0.0604
	equal	1.959	0.1538	3.038	0.0562	3.067	0.0581
	opposite	2.309	0.1450	3.115	0.0550	3.030	0.0613
2.5	one var	1.294	0.1358	3.465	0.0570	3.128	0.0587
	equal	1.297	0.1328	2.803	0.0494	2.630	0.0517
	opposite	1.423	0.1352	2.755	0.0517	2.583	0.0485
3.0	one var	0.877	0.1237	3.375	0.0525	2.966	0.0601
	equal	0.852	0.1232	2.670	0.0469	2.497	0.0427
	opposite	0.662	0.1313	2.617	0.0472	2.450	0.0472
		UCL = 12.208		UCL = 9.209		UCL = 9.400	

Table 8.2.3 (cont.): ARL values of EWMA charts for 10% mixture normal data with unknown Σ using the Two-quadrant estimator, when a shift occurs after 10 good samples

$n = 5, \quad r = 0.2$

$\rho = 0.5$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.247	6.6672	199.055	6.1654	200.838	6.4145
0.5	one var	43.318	1.6462	20.387	0.5195	25.000	0.6752
	equal	50.648	1.9422	14.572	0.3640	18.018	0.4567
	opposite	42.930	1.6939	22.221	0.6160	27.232	0.7299
1.0	one var	10.727	0.4478	6.767	0.1358	7.352	0.1528
	equal	10.303	0.4290	5.198	0.0980	6.031	0.1109
	opposite	10.546	0.4259	6.913	0.1382	8.290	0.1674
1.5	one var	3.953	0.1964	4.503	0.0775	4.719	0.0767
	equal	3.865	0.1992	3.829	0.0655	3.921	0.0649
	opposite	3.939	0.2089	4.048	0.0782	4.773	0.0923
2.0	one var	2.006	0.1530	3.578	0.0705	3.539	0.0708
	equal	1.890	0.1553	3.247	0.0553	3.058	0.0585
	opposite	2.158	0.1506	3.151	0.0627	3.386	0.0698
2.5	one var	1.352	0.1358	3.454	0.0560	3.122	0.0616
	equal	1.362	0.1316	2.882	0.0570	2.777	0.0516
	opposite	1.475	0.1330	2.688	0.0533	2.890	0.0564
3.0	one var	0.870	0.1259	3.237	0.0586	3.140	0.0490
	equal	0.971	0.1209	2.745	0.0546	2.651	0.0472
	opposite	0.755	0.1269	2.451	0.0466	2.459	0.0555
		UCL = 12.206		UCL = 9.000		UCL = 9.442	

$n = 5, \quad r = 0.2$

$\rho = 0.9$		$\bar{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	200.764	6.6690	200.881	6.1047	199.120	5.8322
0.5	one var	43.278	1.6403	40.726	1.1801	50.525	1.4234
	equal	50.047	1.8204	13.980	0.3595	18.797	0.4611
	opposite	44.224	1.7050	47.549	1.4119	57.751	1.4978
1.0	one var	10.476	0.4384	12.009	0.2839	16.173	0.3938
	equal	10.141	0.4233	5.125	0.1088	6.310	0.1196
	opposite	10.090	0.4281	13.117	0.3078	18.823	0.4383
1.5	one var	3.947	0.1961	6.362	0.1281	8.339	0.1753
	equal	3.783	0.1941	3.759	0.0707	4.100	0.0673
	opposite	3.941	0.2047	6.880	0.1396	9.478	0.2085
2.0	one var	1.996	0.1525	4.284	0.0925	5.935	0.1069
	equal	2.318	0.1473	3.305	0.0595	3.445	0.0515
	opposite	2.200	0.1475	4.631	0.0892	6.343	0.1199
2.5	one var	1.385	0.1344	3.792	0.0743	4.541	0.0854
	equal	1.189	0.1379	3.044	0.0566	3.127	0.0456
	opposite	1.388	0.1324	3.581	0.0750	4.839	0.0841
3.0	one var	0.869	0.1264	3.168	0.0708	3.980	0.0600
	equal	0.811	0.1269	2.935	0.0593	3.086	0.0377
	opposite	0.927	0.1232	2.936	0.0636	3.924	0.0707
		UCL = 12.208		UCL = 8.191		UCL = 9.526	

Table 8.2.3 (cont.): ARL values of EWMA charts for 10% mixture normal data with unknown Σ using the Two-quadrant estimator, when a shift occurs after 10 good samples

$n = 5, \quad r = 0.3$

$\rho = 0.0$		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.745	6.5385	201.695	6.2276	198.455	5.9667
0.5	one var	73.731	2.7805	19.957	0.5942	25.000	0.6926
	equal	63.971	2.4222	20.371	0.5459	23.103	0.6946
	opposite	71.413	2.4771	20.228	0.5842	25.486	0.7533
1.0	one var	16.738	0.6803	6.189	0.1225	6.988	0.1492
	equal	15.790	0.6367	5.978	0.1242	6.391	0.1469
	opposite	16.377	0.6521	5.657	0.1202	6.702	0.1450
1.5	one var	5.315	0.2739	4.087	0.0699	4.246	0.0760
	equal	5.234	0.2491	3.501	0.0657	3.815	0.0703
	opposite	5.829	0.2560	3.479	0.0633	3.688	0.0862
2.0	one var	2.793	0.1494	3.479	0.0549	3.248	0.0639
	equal	2.697	0.1590	2.766	0.0546	2.859	0.0544
	opposite	2.672	0.1586	2.819	0.0496	2.843	0.0527
2.5	one var	1.672	0.1272	3.223	0.0537	2.903	0.0559
	equal	1.656	0.1278	2.509	0.0509	2.423	0.0468
	opposite	1.780	0.1270	2.354	0.0513	2.383	0.0513
3.0	one var	1.163	0.1140	2.963	0.0589	2.757	0.0550
	equal	1.127	0.1167	2.309	0.0483	2.229	0.0421
	opposite	1.033	0.1181	2.316	0.0455	2.191	0.0479
		UCL = 13.849		UCL = 9.179		UCL = 9.493	

$n = 5, \quad r = 0.3$

$\rho = 0.5$		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.745	6.5385	201.697	6.2688	199.723	6.0256
0.5	one var	73.561	2.7788	24.528	0.7102	31.720	0.8916
	equal	61.558	2.3735	17.866	0.5137	22.811	0.6302
	opposite	70.126	2.4006	27.300	0.7534	32.914	0.9788
1.0	one var	16.864	0.6881	6.759	0.1488	8.401	0.1887
	equal	14.986	0.5841	5.589	0.1156	6.203	0.1317
	opposite	15.617	0.6791	7.137	0.1613	9.050	0.2260
1.5	one var	5.101	0.2453	4.123	0.0800	4.485	0.0887
	equal	5.434	0.2527	3.503	0.0642	3.687	0.0654
	opposite	5.117	0.2394	4.017	0.0815	4.669	0.0906
2.0	one var	2.583	0.1572	3.384	0.0650	3.507	0.0677
	equal	2.895	0.1515	2.927	0.0531	2.806	0.0583
	opposite	2.519	0.1656	2.869	0.0590	3.216	0.0603
2.5	one var	1.409	0.1354	3.284	0.0506	3.017	0.0530
	equal	2.025	0.1176	2.645	0.0556	2.499	0.0526
	opposite	1.460	0.1333	2.418	0.0515	2.600	0.0521
3.0	one var	1.140	0.1136	3.007	0.0527	2.773	0.0470
	equal	1.069	0.1178	2.609	0.0452	2.380	0.0445
	opposite	1.008	0.1179	2.174	0.0452	2.169	0.0557
		UCL = 13.849		UCL = 9.039		UCL = 9.581	

Table 8.2.3 (cont.): ARL values of EWMA charts for 10% mixture normal data with unknown Σ using the Two-quadrant estimator, when a shift occurs after 10 good samples

$n = 5, \quad r = 0.3$

$\rho = 0.9$		$\overline{X}$ based chart		Signed rank based chart		Sign based chart	
$\tilde{\lambda}$	SHIFT	ARL	SARL	ARL	SARL	ARL	SARL
0.0	none	198.745	6.5385	198.329	5.9019	201.967	5.8301
0.5	one var	73.731	2.7805	50.250	1.4823	63.997	1.8438
	equal	62.906	2.3836	16.221	0.4335	24.727	0.6685
	opposite	69.236	2.3384	60.587	1.8001	63.456	1.7797
1.0	one var	16.996	0.6957	13.523	0.3652	19.282	0.5069
	equal	15.030	0.5674	5.131	0.1167	6.761	0.1314
	opposite	14.928	0.6061	16.290	0.4565	22.543	0.5726
1.5	one var	5.144	0.2462	7.126	0.1719	9.210	0.2129
	equal	5.422	0.2494	3.534	0.0702	4.083	0.0670
	opposite	5.597	0.2474	7.540	0.1741	10.279	0.2518
2.0	one var	2.625	0.1573	4.269	0.0964	6.112	0.1242
	equal	2.985	0.1488	2.980	0.0569	3.154	0.0522
	opposite	2.631	0.1580	4.531	0.0992	6.583	0.1412
2.5	one var	1.426	0.1333	3.503	0.0773	4.631	0.0873
	equal	1.894	0.1197	2.797	0.0509	2.956	0.0435
	opposite	1.502	0.1312	3.295	0.0702	4.798	0.1003
3.0	one var	1.099	0.1133	2.910	0.0689	3.590	0.0629
	equal	1.197	0.1129	2.638	0.0592	2.774	0.0413
	opposite	1.044	0.1155	2.724	0.0624	3.711	0.0715
		UCL = 13.849		UCL = 8.297		UCL = 9.778	

CHAPTER 9

CONCLUSIONS AND FUTURE RESEARCH

9.1 Description of the study

The objective of this dissertation was to develop nonparametric control charts to monitor processes that involve a number of correlated variables, by collecting small samples at regular time intervals. Our goal is to start monitoring as early as possible, before a lot of data have been collected. The proposed charts are appropriate for processes where either the data do not follow a normal distribution or the production run is too short to allow verification of the normality assumption and too short to allow estimation of all the parameters required for a multivariate parametric chart. This dissertation dealt with the problem of long production runs, when the data do not follow a normal distribution.

The proposed nonparametric charts are based on the vector sign and the vector signed rank statistics. They were evaluated using the ARL criterion and were compared with a parametric competitor chart, the MEWMA, which is based on the sample average statistic. The computations were done with simulation. In order to thoroughly compare the three charts, the ARL values of the charts were computed for many parameter combinations. This was necessary because the vector sign and signed rank statistics are not affine invariant. Thus, the ARL of neither nonparametric chart is a function of the noncentrality parameter λ only.

The charts were studied for the case of known parameters and for the case of unknown parameters. By automating the technique of finding the UCL of a chart with the algorithm described in Section 6.5, we were able to compute many cases of parameter combinations, i.e. combinations of n, r, and ρ. For all these cases, we simulated the $\overline{X}$ based chart, the signed rank based chart, and the sign based chart for bivariate normal data, 10% mixture bivariate normal data, and data from a $t(3)$ distribution, for known parameters. For unknown parameters, the same three charts were simulated for bivariate normal data and for 10% mixture bivariate normal data. Also, four estimators were compared: the One-quadrant estimator, the Two-quadrant estimator, the Four-quadrant estimator, and the Constant estimator. Our proposed estimator, which is the Two-quadrant estimator, was compared with the other three estimators and was shown to be a good estimator. Finally, we looked at the case where a process starts in control but becomes out-of-control soon afterwards.

9.2 Summary of results and recommendations

On the analytical aspect, for the sign based chart and the signed rank based chart we showed that, when the process is in control, the asymptotic distribution of the control statistic W_k is χ_p^2, where p is the number of variables we measure.

From all the simulations, we have observed the following. Both multivariate nonparametric charts are more efficient than the corresponding $\overline{X}$-based multivariate charts in detecting smaller shifts when the data are not normal, of the type studied. This is true for both known and unknown parameters, although in the unknown parameter case it holds only for small n (n=5).

The sign based and the signed rank based charts are not affine invariant and, therefore, their ARLs are not functions of the noncentrality parameter λ only. This behavior is exhibited in differences of the ARL values computed in different directions of shift for the same value of λ. Although these differences in the ARL values are not large for moderate correlations, they become larger for correlations near 1. Then the nonparametric charts may be faster than the parametric chart in some directions and slower in other directions. The differences in the ARLs are more pronounced in the sign based chart than in the signed rank based chart.

When Σ was unknown, the nonparametric charts did not perform as well as we hoped. We think the reason is that any estimator of $p_{ii'}$, because of its definition, is a biased estimator of the in-control $p_{ii'}$. This makes the nonparametric charts very inefficient if the data are normal. Among the four estimators we tested, we recommend the estimator we propose in this research, i.e. the Two-quadrant estimator, because it has low variability. Now, if we are interested primarily in large shifts or we may have normal data, we recommend using the $\overline{X}$ based chart. If we are interested in small shifts and have nonnormal data, then we recommend using either the sign based chart or the signed rank based chart.

Both nonparametric charts are at their best when the EWMA parameter r is 0.1. The reason is that we use the exact variance-covariance matrix and we assume that the shift occurs at the onset of the process. When the shift occurs after 10 in-control samples, we observe that larger values of r are better for large shifts. Both nonparametric charts improve slightly when the sample size increases. The signed rank based chart is faster than the sign based chart, in most cases, for small to moderate size shifts. Therefore, we would recommend the signed rank based chart most of the time, making the note that the sign based chart is a very close second.

9.3 How to design a chart

From the simulation results, we observed that the UCLs of the nonparametric charts for unknown Σ (Tables 8.1.6 and 8.1.7) are very close to the UCLs for known Σ (Tables 7.6 and 7.7). Moreover, the UCLs of the nonparametric charts remain about the same for the distributions we simulated. So, in order to construct a control chart, we could use Tables 7.6 or 7.7 to select a control limit and we would only need to look up the tables for the normal distribution. Since r=0.1 is the EWMA parameter value that makes the charts most efficient, we would choose it for the construction of our chart. Now, for the sign based chart, it seems that the value UCL=8.7, in combination with r=0.1, should work very well. For the signed rank based chart, we would also have to have an idea of the magnitude of the correlation between the variables we will be measuring. If the correlation is expected to be low, we recommend using r=0.1 and UCL=8.7; if the correlation is expected to be moderate, we recommend using r=0.1 and UCL=8.4; if the correlation is expected to be high, we recommend using r=0.1 and UCL=7.4. The UCL value, along with r and n, would then be used to construct a chart.

In the process of this research, we also studied the $\overline{X}$ based chart. When Σ was unknown, we used the estimator given in equation (6.5.8). We observed that the UCLs of the $\overline{X}$ based chart depend on the data distribution and they increase somewhat from the case of known Σ to the case of unknown Σ. The $\overline{X}$ based chart is best to use when the data are normal. In order to design an $\overline{X}$ based chart, if the data are normal, we could use UCL=9.3 when n=5, UCL=9.0 when n=10, and UCL=8.7 when n=15; but, if the data are not normal, then the UCLs will not be close to the UCLs computed for the normal distribution.

9.4 Topics for future research

For the long production run case, which was investigated and developed in this dissertation, there are two main issues that could be studied further. One issue is the estimator of the probability $p_{ii'}$, needed for the nonparametric charts. From definition (4.3.16), it is clear that the probability $p_{ii'}$ and the proposed estimator are sensitive to the direction and the size of the shift in the mean vector. This does not happen, for example, when we estimate variances and covariances for the $\overline{X}$ based chart. So, it would help improve the performance of the charts, if an estimator was found that could estimate $p_{ii'}$ irrespectively of shift. The second issue is to extend the results from bivariate data to multivariate.

The proposed nonparametric charts are also useful when monitoring of a process must start as early as possible, because there are fewer parameters to estimate in one of the proposed nonparametric charts than in a parametric chart. Quesenberry (1991a, b, c) proposed estimators that allow an early start in process monitoring. Unfortunately, we couldn't use these estimators in our study, because he assumes that the data distribution is known. Moreover, Quesenberry's results have not been extended to the multivariate case.

If the production run is short, a situation where a nonparametric chart would be useful as monitoring must start early, the ARL computed for an infinite production run may not be meaningful and the simulation results based on infinite runs may not be particularly useful. A performance criterion, that could be used instead of the ARL, is a *probability criterion*. Wade and Woodall (1992) used the following probability criterion in evaluating the performance of the Q charts proposed by Quesenberry (1991a). If the values of the control statistic are independent, then the probability that a chart will signal in m stages is

$$P(\text{the chart signals}) = 1 - (1 - p)^m$$

where p is the probability that a point falls outside the control limits for a given shift, and m is the number of samples that can be collected (determined by the length of the production run). If the values of the control statistic are not independent, the probability that the chart will signal in m stages can only be estimated, for example with simulation, as the proportion of time the chart signals for a given shift.

We can use a similar probability criterion to evaluate multivariate nonparametric control charts for short production runs. Since the values of the control statistic are not independent, the process can be simulated to achieve a particular probability that each chart will signal within m stages when the process is, in fact, in control. Based on this probability, we determine an UCL for each chart. This UCL can then be used for subsequent simulations to compute the probability that the charts will signal within a certain number of steps after a shift has occurred, depending upon the stage the shift occurred within the run and the size of the shift.

We could also develop multivariate nonparametric EWMA control charts based on permutation tests, instead of the componentwise sign and signed rank tests. Moreover, we could construct charts based on an affine invariant bivariate sign test, for example the Oja sign test that according to Brown and Hettmansperger (1989) is generally more efficient than the componentwise sign test.

REFERENCES

Alloway, J. A., Jr., and Raghavachari, M. (1991), "Control Chart Based on the Hodges-Lehmann Estimator," *Journal of Quality Technology*, 23, 336-347.

Alloway, J. A., Jr., and Raghavachari, M. (1991), "An Introduction to Multivariate Control Charts," *45th Annual Quality Congress Transactions, American Society for Quality Control*, Milwaukee, Wisconsin, pp. 773-783.

Alt, F. B., and Smith, N. D. (1988), "Multivariate Process Control," in *Quality Control and Reliability (Handbook of Statistics, 7)*, eds. P. R. Krishnaiah and C. R. Rao, Amsterdam: North-Holland, pp. 333-351.

Amin, R. W., and Searcy, A. J. (1991), "A Nonparametric Exponentially Weighted Moving Average Control Scheme," *Communications in Statistics, Part B: Computation and Simulation*, 20, 1049-1072.

Arnold, S. F. (1981), *The Theory of Linear Models and Multivariate Analysis*, New York: John Wiley & Sons.

Bakir, S. T., and Reynolds, M. R., Jr. (1979), "A Nonparametric Procedure for Process Control Based on Within-Group Ranking," *Technometrics*, 21, 175-183.

Barnett, V. (1976), "The Ordering of Multivariate Data," *Journal of the Royal Statistical Society, Series A: General*, 139(3), 318-355.

Bennett, B. M. (1962), "On Multivariate Sign Tests," *Journal of the Royal Statistical Society, Series B: Methodological*, 24, 159-161.

____________ (1964), "A Bivariate Signed Rank Test," *Journal of the Royal Statistical Society, Series B: Methodological*, 26(3), 457-461.

Bhattacharyya, G. K., and Johnson, R. A. (1969), "On Hodges's Bivariate Sign Test and a Test for Uniformity of a Circular Distribution," *Biometrica*, 56, 446-449.

Bickel, P. J. (1964), "On Some Alternative Estimates for Shift in the *P*-Variate One Sample Problem," *Annals of Mathematical Statistics*, 35, 1079-1090.

___________ (1965), "On Some Asymptotically Nonparametric Competitors of Hotelling's T^2," *Annals of Mathematical Statistics*, 36, 160-173.

Blumen, I. (1958), "A New Bivariate Sign Test," *Journal of the American Statistical Association*, 53, 448-456.

Bradley, R. A., Patel, K. M., and Wackerly, D. D. (1971), "Approximate Small-Sample Distributions for Multivariate Two-Sample Nonparametric Tests," *Biometrics*, 27, 515-530.

Brown, B. M. (1983), "Statistical Uses of the Spatial Median," *Journal of the Royal Statistical Society, Series B: Methodological*, 45(1), 25-30.

Brown, B. M., and Hettmansperger, T. P. (1987), "Affine Invariant Rank Methods in the Bivariate Location Model," *Journal of the Royal Statistical Society, Series B: Methodological*, 49(3), 301-310.

________________________________ (1989), "An Affine Invariant Bivariate Version of the Sign Test," *Journal of the Royal Statistical Society, Series B: Methodological*, 51(1), 117-125.

Champ, C. W., and Woodall, W. H. (1987), "Exact Results for Shewhart Control Charts with Supplementary Runs Rules," *Technometrics*, 29, 393-399.

Chatterjee, S. K. (1966), "A Bivariate Sign Test for Location," *Annals of Mathematical Statistics*, 37, 1771-1782.

CMS Primer (1984), *Virtual Machine / System Product*, IBM, Release 4.

Crosier, R. B. (1986), "A New Two-Sided Cumulative Sum Quality Control Scheme," *Technometrics*, 28, 187-194.

__________ (1988), "Multivariate Generalizations of Cumulative Sum Quality-Control Schemes," *Technometrics*, 30, 291-303.

Crowder, S. V. (1987), "A Simple Method for Studying Run-Length Distributions of Exponentially Weighted Moving Average Charts," *Technometrics*, 29, 401-407.

Csörgõ, M., and Horváth, L. (1988), "Nonparametric Methods for Changepoint Problems," in *Quality Control and Reliability (Handbook of Statistics, 7)*, eds. P. R. Krishnaiah and C. R. Rao, Amsterdam: North-Holland, pp. 403-425.

Devroy, L. (1986), *Non-Uniform Random Variate Generation*, New York: Springer-Verlag.

Dietz, E. J. (1982), "Bivariate Nonparametric Tests for the One-Sample Location Problem," *Journal of the American Statistical Association*, 77, 163-169.

Edwards, C. H., Jr., and Penney, D. E. (1982), *Calculus and Analytic Geometry*, ed. Eleanor Henshaw Hiatt, Englewood Cliffs, New Jersey: Prentice-Hall.

Friedman, F. L., and Koffman, E. B. (1982), *Problem Solving and Structured Programming in WATFIV*, Reading, Massachusetts: Addison-Wesley Publishing Company.

Gentleman, W. M. (1965), *Robust Estimation of Multivariate Location by Minimizing Pth Power Deviations*, Ph.D. Thesis, Princeton University.

Ghosh, M. (1983), "Sequential Rank Estimators," *Encyclopedia of Statistical Sciences, Volume 8*, eds. S. Kotz and N. L. Johnson, New York: John Wiley & Sons.

Giri, N. C. (1983), "Invariance Concepts in Statistics," *Encyclopedia od Statistical Sciences, Volume 4*, eds. S. Kotz and N. L. Johnson, New York: John Wiley & Sons.

Graybill, F. A. (1976), *Theory and Application of the Linear Model,* Boston, Massachusetts: Duxbury Press.

Hackl, P., and Ledolter, J. (1991), "A Control Chart Based on Ranks," *Journal of Quality Technology*, 23, 117-124.

____________________ (1992), "A New Nonparametric Quality Control Technique," *Communications in Statistics, Part B: Computation and Simulation*, 21, 423-443.

Hawkins, D. M. (1993), "Robustification of Cumulative Sum Charts by Winsorization," *Journal of Quality Technology*, 25, 248-261.

Hettmansperger, T. P. (1984), *Statistical Inference Based on Ranks*, *(Wiley Series in Probability and Mathematical Statistics)*, eds. R. A. Bradley, J. S. Hunter, D. G. Kendall, and G. S. Watson, New York: John Wiley & Sons.

Hodges, J. L., Jr. (1955), " A Bivariate Sign Test," *Annals of Mathematical Statistics*, 26, 523-527.

Hogg, R. V., and Craig, A. T. (1978), *Introduction to Mathematical Statistics, Fourth Edition,* New York: Macmillan Publishing Company.

Hollander, M., and Wolfe, D. A. (1973), *Nonparametric Statistical Methods*, *(Wiley Series in Probability and Mathematical Statistics)*, eds. R. A. Bradley, J. S. Hunter, D. G, Kendall, and G. S. Watson, New York: John Wiley & Sons.

Hotelling, H. (1947), "Multivariate Quality Control," in *Techniques of Statistical Analysis,* eds. C. Eisenhart, M. W. Hasty, and W. A. Wallis, New York: McDraw Hill, pp. 111-184.

Huber, P. J. (1967), "The Behaviour of Maximum Likelihood Estimates under Nonstandard Conditions," in *Proceedings Fifth Berkeley Symposium on Mathematical Statistics and Probability, Vol. 1*, Berkeley: University of California Press.

Huber, P. J. (1977), "Robust Statistical Procedures," *CBMS-NSF Regional Conference Series in Applied Mathematics*, Philadelpia: Society for Industrial and Applied Mathematics, 27, 1-54.

IMSL STAT/LIBRARY (1991), *User's Manual*, Version 2.0.

Jackson, J. E. (1959), "Quality Control Methods for Several Related Variables," *Technometrics*, 1, 359-377.

Johnson, N. L., and Kotz, S. (1972), *Distributions in Statistics: Continuous Multivariate Distributions, (Wiley Series in Probability and Mathematical Statistics)*, eds. R. A. Bradley, J. S. Hunter, D. G, Kendall, and G. S. Watson, New York: John Wiley & Sons.

____________________ (1969), *Distributions in Statistics: Discrete Distributions, (Wiley Series in Probability and Mathematical Statistics)*, eds. R. A. Bradley, J. S. Hunter, D. G, Kendall, and G. S. Watson, New York: John Wiley & Sons.

Johnson, R. A., and Wichern, D. W. (1982), *Applied Multivariate Statistical Analysis*, Englewood Cliffs, New Jersey: Prentice-Hall.

Juang, S.-H., and Wolfe, D. A. (1990), "Nonparametric Changepoint Procedures for Repeated Measures Data," *Communications in Statistics, Part B: Computation and Simulation*, 19, 263-286.

Kassam, S. A., and Thomas, J. B., eds., (1980), *Nonparametric Detection: Theory and Applications (Benchmark Papers in Electrical Engineering and Computer Science, 23)*, Stroudsburg, Pennsylvania: Dowden, Hutchinson & Ross.

Kendall, M. G., and Stuart, A. (1958), *The Advanced Theory of Statistics, Volume 1: Distribution Theory,* New York: Hafner Publishing Company.

Kennedy, W. J., Jr., and Gentle, J. E. (1980), *Statistical Computing (Statistics: Textbooks and Monographs, 33)*, ed. D. B. Owen, New York: Marcel Dekker.

King, D. W., and Longnecker, M. T. (1990), "Computing the Distribution of the Wilcoxon Statistic with Applications to Process Control," *Communications in Statistics, Part B: Computation and Simulation*, 19, 381-399.

Krishnaiah, P. R., and Miao, B. Q. (1988), "Review about Estimation of Change Points," in *Quality Control and Reliability (Handbook of Statistics, 7)*, eds. P. R. Krishnaiah and C. R. Rao, Amsterdam: North-Holland, pp. 375-402.

Lowry, C. A. (1989), *A Multivariate Exponentially Weighted Moving Average Control Chart*, unpublished Ph.D. dissertation, University of Southwestern Louisiana.

Lowry, C. A., Woodall, W. H., Champ, C. W., and Rigdon, S. E. (1992), "A Multivariate Exponentially Weighted Moving Average Control Chart," *Technometrics*, 34, 46-53.

Lucas, J. M., and Saccucci, M. S. (1990), "Exponentially Weighted Moving Average Control Schemes: Properties and Enhancements," *Technometrics*, 32, 1-12.

Maritz, J. S. (1981), *Distribution-Free Statistical Methods (Monographs on Applied Probability and Statistics)*, ed. D. R. Cox, F.R.S., New York: Chapman and Hall.

Maronna, R. A. (1976), "Robust *M*-Estimators of Multivariate Location and Scatter," *The Annals of Statistics*, 4(1), 51-67.

Müller-Funk, U. (1984), "Sequential Nonparametric Tests," in *Nonparametric Methods (Handbook of Statistics, 4)*, eds. P. R. Krishnaiah and P. K. Sen, Amsterdam: North-Holland, pp. 657-698.

Oja, H. (1983), " Descriptive Statistics for Multivariate Distributions," *Statistics & Probability Letters*, 1, 327-332.

Oja, H., and Niinimaa, A. (1985), "Asymptotic Properties of the Generalized Median in the Case of Multivariate Normality," *Journal of the Royal Statistical Society, Series B: Methodological*, 47(2), 372-377.

Page, E. S. (1954), "Continuous Inspection Schemes," *Biometrica*, 41, 100-114.

Park, C., and Reynolds, M. R., Jr. (1987), "Nonparametric Procedures for Monitoring a Location Parameter Based on Linear Placement Statistics," *Sequential Analysis*, 6, 303-323.

Peters, D. (1991), "An Adaptive Multivariate Signed-Rank Test for the One-Sample Location Problem," *Nonparametric Statistics*, 1, 157-163.

Peters, D., and Randles, R. H. (1990), "A Multivariate Signed-Rank Test for the One-Sample Location Problem," *Journal of the American Statistical Association*, 85, 552-557.

Press, W. H., Teukolsky, S. A., Vetterling, W. T., and Flannery, B. P. (1992), *Numerical Recipes in FORTRAN (The Art of Scientific Computing), Second Edition*, New York: Cambridge University Press.

Puri, M. L. and Sen, P. K. (1971), *Nonparametric Methods in Multivariate Analysis, (Wiley Series in Probability and Mathematical Statistics)*, eds. R. A. Bradley, J. S. Hunter, D. G, Kendall, and G. S. Watson, New York: John Wiley & Sons.

Quesenberry, C. P. (1991a), "SPC Q Charts for Start-up Processes and Short or Long Runs," *Journal of Quality Technology*, 23, 213-224.

_______________ (1991b), "SPC Q Charts for a Binomial Parameter p: Short or Long Runs," *Journal of Quality Technology*, 23, 239-246.

_______________ (1991c), "SPC Q Charts for a Poisson Parameter λ: Short or Long Runs," *Journal of Quality Technology*, 23, 296-303.

Randles, R. H. (1989), "A Distribution-Free Multivariate Sign Test Based on Interdirections," *Journal of the American Statistical Association*, 84, 1045-1050.

Randles, R. H. and Wolfe, D. A. (1979), *Introduction to the Theory of Nonparametric Statistics, (Wiley Series in Probability and Mathematical Statistics)*, eds. R. A. Bradley, J. S. Hunter, D. G, Kendall, and G. S. Watson, New York: John Wiley & Sons.

Rigdon, S. E. (1994a), "An Integral Equation for the In-Control Average Run Length of a Multivariate Exponentially Weighted Moving Average Control Chart," unpublished manuscript.

Rigdon, S. E. (1994b), "A Double Integral Equation for the Average Run Length of a Multivariate Exponentially Weighted Moving Average Control Chart," unpublished manuscript.

Roberts, S. W. (1959), "Control Chart Tests Based on Geometric Moving Averages," *Technometrics*, 1, 239-250.

Rocke, D. M. (1992), "$\overline{X}_Q$ and R_Q Charts: Robust Control Charts," *The Statistician*, 41, 97-104.

Ross, S. (1984), *A First Course in Probability, Second Edition,* New York: Macmillan Publishing Company.

Rubinstein, R. Y. (1981), *Simulation and the Monte Carlo Method (Wiley Series in Probability and Mathematical Statistics)*, eds. R. A. Bradley, J. S. Hunter, D. G, Kendall, and G. S. Watson, New York: John Wiley & Sons.

Searle, S. R. (1971), *Linear Models, (Wiley Series in Probability and Mathematical Statistics)*, eds. R. A. Bradley, J. S. Hunter, D. G. Kendall, and G. S. Watson, New York: John Wiley & Sons.

Sen, P. K. (1981), *Sequential Nonparametrics: Invariance Principles and Statistical Inference, (Wiley Series in Probability and Mathematical Statistics)*, eds. R. A. Bradley, J. S. Hunter, D. G. Kendall, and G. S. Watson, New York: John Wiley & Sons.

_________ (1984), "Nonparametric Sequential Estimation," in *Nonparametric Methods (Handbook of Statistics, 4)*, eds. P. R. Krishnaiah and P. K. Sen, Amsterdam: North-Holland, pp. 487-514.

Sen, P. K., and Puri, M. L. (1967), "On the Theory of Rank Order Tests for Location in the Multivariate One Sample Problem," *Annals of Mathematical Statistics*, 38, 1216-1228.

Serfling, R. J. (1980), *Approximation Theorems of Mathematical Statistics (Wiley Series in Probability and Mathematical Statistics)*, eds. R. A. Bradley, J. S. Hunter, D. G. Kendall, and G. S. Watson, New York: John Wiley & Sons.

Shewhart, W. A. (1931), *Economic Control of Quality of Manufactured Product*, New York: D. Van Nostrand Co.

Sturgul, J. R., and Merchant, M. J. (1976), *Applied FORTRAN IV Programming*, eds. D. Dellen and P. Niklas, Belmont, California: Wadsworth Publishing Company.

Thisted, R. A. (1988), *Elements of Statistical Computing*, New York: Chapman and Hall.

Utts, J. M., and Hettmansperger, T. P. (1980), "A Robust Class of Tests and Estimates for Multivariate Location," *Journal of the American Statistical Association*, 75, 939-946.

Wade, M. R., and Woodall, W. H. (1992), "Performance of Q Charts for Short Production Runs," unpublished manuscript.

Wasan M. T. (1969), *Stochastic Approximation (Cambridge Tracts in Mathematics and Mathematical Physics, 58)*, eds. J. F. C. Kingman, F. Smithies, J. A. Todd, F.R.S., and C. T. C. Wall, New York: Cambridge University Press.

Wheeler, D. J. (1991), *Short Run SPC*, Knoxville, Tennessee: SPS Press.

Woodall, W. H., and Ncube, M. M. (1985), "Multivariate CUSUM Quality-Control Procedures," *Technometrics*, 30, 285-292.

APPENDIX A

The distribution of W_k based on the sample average

Let $\mathbf{x}_{1k},\ldots,\mathbf{x}_{nk}$ be i.i.d. MVN$(\boldsymbol{\mu},\Sigma)$ random vectors collected at time k, where Σ is assumed known and the mean vector $\boldsymbol{\mu}$ is monitored. The target value of $\boldsymbol{\mu}$ is $\mathbf{t}$. Let us consider the multivariate EWMA statistics based on the sample average $\overline{\mathbf{x}}_k$ of the n vectors:

$$\mathbf{y}_k = (1-r)\mathbf{y}_{k-1} + r\overline{\mathbf{x}}_k, \tag{A.1}$$

with initial value $\mathbf{y}_0$ equal to the target value $\mathbf{t}$. Since $\mathbf{x}_{jk}$, j=1,...,n, k=1,2,..., is MVN$(\boldsymbol{\mu},\Sigma)$, then $\overline{\mathbf{x}}_k$, k=1,2,..., is MVN$(\boldsymbol{\mu},\frac{1}{n}\Sigma)$. Now, the vector of the expected values is

$$\mathrm{E}(\mathbf{y}_k) = (1-r)^k\mathbf{t} + (1-(1-r)^k)\boldsymbol{\mu}, \tag{A.2}$$

and the variance-covariance matrix is

$$\mathrm{Var}(\mathbf{y}_k) = \left(\frac{r}{2-\mathrm{r}}\right)[1-(1-r)^{2k}]\ \frac{1}{n}\Sigma. \tag{A.3}$$

Since $\mathbf{y}_k$ is a linear combination of normal random vectors, we conclude that

$$\mathbf{y}_k \text{ is MVN}[\mathrm{E}(\mathbf{y}_k),\mathrm{Var}(\mathbf{y}_k)] \tag{A.4}$$

where $\mathrm{E}(\mathbf{y}_k)$ and $\mathrm{Var}(\mathbf{y}_k)$ are given in equations (A.2) and (A.3) above.

The control statistic is defined as a quadratic form of the EWMA vector $\mathbf{y}_k$, as follows:

$$W_k = [\mathbf{y}_k - \mathbf{t}]'\ \Sigma_{\mathbf{y}_k}^{-1}[\mathbf{y}_k - \mathbf{t}]. \tag{A.5}$$

From equation (A.2) we have

$$\mathrm{E}(\mathbf{y}_k - \mu_0) = (1-r)^k \mathbf{t} + (1-(1-r)^k)\mu - \mathbf{t}$$

$$= (1-(1-r)^k)\mu - (1-(1-r)^k)\mathbf{t}$$

$$= (1-(1-r)^k)(\mu - \mathbf{t}) \qquad \text{(A.6)}$$

and from equation (A.3) we have

$$\mathrm{Var}(\mathbf{y}_k - \mathbf{t}) = \mathrm{Var}(\mathbf{y}_k)$$

$$= \left(\frac{r}{2-\mathrm{r}}\right)[1-(1-r)^{2k}]\ \frac{1}{n}\Sigma. \qquad \text{(A.7)}$$

Moreover, because of (A.4), $(\mathbf{y}_k - \mathbf{t})$ is MVN with mean vector and variance-covariance matrix as given in (A.6) and (A.7) above.

A well known theorem, given also by Arnold (1981, p. 49, Theorem 3.10), states that if $\mathbf{y}$ is $\mathrm{MVN}_p(\mu,\Sigma)$, then $\mathbf{y}'\Sigma^{-1}\mathbf{y}$ is $\chi^2_p(\mu'\Sigma^{-1}\mu)$. Applying this theorem to the vector $(\mathbf{y}_k - \mathbf{t})$, we conclude that the control statistic

$$W_k \text{ is } \chi^2_p(\lambda),$$

with noncentrality parameter

$$\lambda = [(1-(1-r)^k)(\mu-\mathbf{t})]'\left[\left(\frac{r}{2-\mathrm{r}}\right)[1-(1-r)^{2k}]\ \frac{1}{n}\Sigma\right]^{-1}[(1-(1-r)^k)(\mu-\mathbf{t})]$$

$$= \left\{n\left(\frac{2-r}{r}\right)\frac{\left(1-(1-r)^k\right)^2}{\left(1-(1-r)^{2k}\right)}\right\}(\mu-\mathbf{t})'\Sigma^{-1}(\mu-\mathbf{t}).$$

When the process is in control, i.e. when $\boldsymbol{\mu} = \mathbf{t}$, λ is zero and the control statistic W_k follows a central χ_p^2 distribution.

APPENDIX B

Example 5.2.1 (continued)

This appendix accompanies Example 5.2.1. Here, we show in detail the computations of the control statistic W_k at $k = 1$, i.e. W_1, using the $\overline{X}$ based MEWMA statistic and the sign based MEWMA statistic. We assume that all variances and covariances are known. For the computations we use the values given in the example, so the reader is advised to refer to section 5.2 as necessary.

We start by computing W_1 for the $\overline{X}$ based MEWMA, using the original data **X** and the transformed data **Z**. Besides the letter **z**, all symbols with a star $(*)$ refer to the transformed data. We have to compute the sample vector, the mean vector, and the variance-covariance matrix of the EWMA vector, and for all these we use the equations provided by Lowry (1989). Using the original data **X**, we obtain the following:

$$\mathbf{y}_1 = (1-r)\mathbf{y}_0 + r\overline{\mathbf{x}} = \begin{bmatrix} 0.0925 \\ 0.0025 \end{bmatrix}$$

where, assuming the process is in control, the starting value of the EWMA statistic is taken to be

$$\mathbf{y}_0 = \mathrm{E}(\overline{\mathbf{x}}) = \boldsymbol{\mu} = \mathbf{0}.$$

Now, when the process is in control,

$$\mathrm{E}(\mathbf{y}_1) = \boldsymbol{\mu} = \mathbf{0}$$

and, using equation (5.2.3),

$$\mathbf{V}(\mathbf{y}_1) = \left(\frac{r}{2-r}\right)\left[1-(1-r)^2\right]\frac{1}{n}\Sigma$$

$$= \begin{bmatrix} 0.008 & 0.006 \\ 0.006 & 0.008 \end{bmatrix}.$$

Hence, using equations (4.2.5) and (4.2.6), we obtain

$$W_1 = 2.3436. \qquad \text{(B.1)}$$

Now, we compute W_1 using the transformed data **Z**. First, the EWMA vector is

$$\mathbf{y}_1^* = (1-r)\mathbf{y}_0^* + r\bar{\mathbf{z}} = \begin{bmatrix} 0.0950 \\ 0.0925 \end{bmatrix}$$

where, assuming the process is in control, the starting value of the EWMA statistic is taken to be

$$\mathbf{y}_0^* = \mathrm{E}(\bar{\mathbf{z}}) = \mu^* = \mathbf{0}.$$

Now, when the process is in control,

$$\mathrm{E}(\mathbf{y}_1^*) = \mu^* = \mathbf{0}$$

and, using equation (5.2.6),

$$\mathbf{V}(\mathbf{y}_1^*) = \left(\frac{r}{2-r}\right)\left[1-(1-r)^2\right]\frac{1}{n}\Sigma^*$$

$$= \begin{bmatrix} 0.028 & 0.014 \\ 0.014 & 0.008 \end{bmatrix}.$$

Then, using equations (4.2.5) and (4.2.6), we obtain

$$W_1^* = 2.3436. \qquad \text{(B.2)}$$

Now we will compute W_1 for the sign based MEWMA, using the original data **X** and the transformed data **Z**. As above, besides the letter **z**, all symbols with a star (*) refer to the transformed data. We have to compute the sample vector, the mean vector, and the variance-covariance matrix of the EWMA vector, and for all these we use equations (4.2.4), (4.3.33), (4.3.34), and (4.3.35). In addition, we use Sheppard's 1898 result for bivariate normal data (Kendall and Stuart, 1958, p. 351), which states the following:

$$p_{12} = \frac{1}{4} + \arcsin(\rho)/(2\pi) \qquad \text{(B.3)}$$

where p_{12} is defined as in (4.3.5) and ρ is the correlation between the two variables.

Using the sign vector of the original data **X**, we have

$$\mathbf{y}_1 = (1-r)\mathbf{y}_0 + r\mathbf{s}(\mathbf{x}) = \begin{bmatrix} 2.8 \\ 2.4 \end{bmatrix}$$

where the starting value of the EWMA statistic is taken to be

$$\mathbf{y}_0 = \mathrm{E}(\mathbf{s}(\mathbf{x})) = n\mathbf{p} = \begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix},$$

assuming the process is in control. Using Sheppard's result, we compute that

$$\rho = 0.75 \quad \Rightarrow \quad p_{12} = 0.385$$

Now, also for an in-control process,

$$\mathrm{E}(\mathbf{y}_1) = n\mathbf{p} = \begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix}$$

and

$$\mathbf{V}(\mathbf{y}_1) = \left(\frac{r}{2-r}\right)\left[1-(1-r)^2\right]n\begin{bmatrix} 1/4 & (p_{12}-1/4) \\ (p_{12}-1/4) & 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.050 & 0.027 \\ 0.006 & 0.050 \end{bmatrix}.$$

Finally, using equations (4.2.5) and (4.2.6), we obtain

$$W_1 = 3.7380. \tag{B.4}$$

Now, we compute W_1 using the transformed data $\mathbf{Z}$. First, the EWMA vector is

$$\mathbf{y}_1^* = (1-r)\mathbf{y}_0^* + r\mathbf{s}(\mathbf{z}) = \begin{bmatrix} 2.6 \\ 2.8 \end{bmatrix}$$

where the starting value of the EWMA is taken to be

$$\mathbf{y}_0^* = \mathrm{E}(\mathbf{s}(\mathbf{z})) = n\mathbf{p}^* = \begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix},$$

assuming the process is in control. Using Sheppard's result, equation (B.3), we compute that

$$\rho^* = 0.935 \;\Rightarrow\; p_{12}^* = 0.442$$

Now, also for an in-control process,

$$\mathrm{E}(\mathbf{y}_1^*) = n\mathbf{p}^* = \begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix}$$

and

$$\mathbf{V}(\mathbf{y}_1^*) = \left(\frac{r}{2-r}\right)\left[1-(1-r)^2\right]n\begin{bmatrix} 1/4 & (p_{12}^*-1/4) \\ (p_{12}^*-1/4) & 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.050 & 0.038 \\ 0.038 & 0.050 \end{bmatrix}.$$

Finally, using equations (4.2.5) and (4.2.6), we obtain

$$W_1^* = 2.6291. \tag{B.5}$$

Results (B.1), (B.2), (B.4), and (B.5) are listed in Table 5.2.1.

VITA

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