A Robust Shewhart Control Chart Adjustment Strategy

by

Xueli Zou

Dissertation submitted to the Faculty of the

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirement for the degree of

Doctor of Philosophy

in

Industrial and Systems Engineering

APPROVED:

Dr. J. A. Nachlas, Chairman

Dr. H. D. Sherali

Dr. C. P. Koelling

Dr. J. C. Arnold

Dr. R. J. Beaton

December 1993

Blacksburg, Virginia

A Robust Shewhart Control Chart Adjustment Strategy

by

Xueli Zou

Dr. J. A. Nachlas, Chairman

Industrial and Systems Engineering

(ABSTRACT)

The standard Shewhart control chart for monitoring process stability is generalized by selecting a point in time at which the distance between the control limits is reduced. Three cost models are developed to describe the total cost per unit time of monitoring the mean of a process using both the standard and the generalized Shewhart control chart. The cost models are developed under the assumption that the quality characteristic of interest is normally distributed with known and constant variance. In the development of the first model, the negative exponential distribution is employed to model the time to process shift. Then, the uniform distribution and the Weibull distribution are used for the same purpose in the second and the third model, respectively. The motivation for this effort is to increase chart sensitivity to small but anticipated shifts in the process average.

Cost models are constructed to allow the optimal choice of change over time and the best values for the initial and adjusted control limit values. The cost models are analyzed to determine the optimal control chart parameters including those associated with both the standard and the generalized control chart. The models are also used to provide a comparison with conventional implementation of the control chart. It is shown that the proposed cost models are efficient and economical. Figures and tables are provided to aid in the design of models for both the standard and the generalized Shewhart control chart.

Acknowledgements

I would like to acknowledge my indebtedness to my advisor, Dr. J. A. Nachlas for suggesting this dissertation, and for his inspiring guidance. His wise advice has played a decisive role at many points during the completion of this dissertation.

I would also like to thank Dr. J. C. Arnold and Dr. M. R. Reynolds, Jr. who inspired and helped me to enter the field of statistics. The foundation of my statistical knowledge was obtained in their lucid, enthusiastic and stimulating lectures. Later, I was very much influenced by Dr. H. D. Sherali and Dr. R. J. Beaton, whose ideas are strongly reflected in this dissertation. I appreciate the support given by Dr. C. P. Koelling and Dr. R. D. Dryden through my stay at Virginia Tech.

I take this opportunity to express my gratitude to my parents Fuhui Zou and Fengzhen Yao, my old brother Xueming and my sister-in-law Houngxing for instilling in me a love for higher education and pursuit of knowledge. I sincerely thank my American mother Mrs.

George vonDubel (Williamsburg, VA) who has given me constant support and love. I also thank my daughter Mimi (6 years old) for her understanding and forgiving me for spending so little time with her. Thanks are due to all my close friends and relatives.

Last and most important I would like to thank my wife Nina for her wonderful support, encouragement, and active participation in an enterprise which at times seemed endless.

Table of Contents

List of Illustrations

List of Tables

CHAPTER 1

Introduction and Background

1.1 Introduction

Shewhart control charts are widely used to display sample data from a production process. They are used to indicate whether a process is in control. They have also been found valuable in evaluating process capability, in estimating process parameters, in determining a process control strategy, and in monitoring the behavior of a production process. A control chart is maintained by taking samples from a process and plotting in time order on the chart some statistic computed from the samples. Control limits on the chart represent the limits within which the plotted points would fall with high probability if the process is operating in control. A point outside the control limits is taken as an indication that something, sometimes called an assignable cause of variation, has happened to change the process. When the chart signals that an assignable cause is present, rectifying action is taken to remove the assignable cause and bring the process back into control.

In what follows, consider the situation in which the quality of the output of a process is defined by some quality characteristic or variable such as the strength or length of an item. In almost all cases there will be variation from item to item and from sample to sample in the observed values of this variable. Large variation in the variable usually corresponds to low quality. For example, if the variable is a dimension with a specified target value, then the closer the variable is to the target the higher the quality. Any variation above or below the target lowers the quality. As another example, if the quality variable is the level of impurities in a chemical, then lower values of this variable usually correspond to higher quality and thus variation above zero lowers the quality. As a third example, if the variable is the strength of a material, then high strength usually represents high quality and variation in the lower direction represents lower quality.

In addition to the common causes which produce random variation, assignable causes can individually produce a substantial amount of variation. When a special cause of variation is present the distribution of the quality variable is altered. In most cases it is assumed that the distribution of the quality metric is indexed by one or more parameters and the effect of the presence of a special cause is to change the values of these parameters. The purpose of a control chart is to detect special causes of variation so that these causes can be found and eliminated. Because a special cause is assumed to produce a parameter change, the problem for which a control chart is used can be formulated as the problem of monitoring a process to detect any change in the parameters of the distribution of the quality variable.

The usual practice in maintaining a control chart is to plot the sample statistic from the process relative to constant width control limits, say 3-sigma limits. In this dissertation, a modification to standard practice in which the sampling control limits are not fixed but instead can vary after the process has operated for a period of time is investigated. The basis of choice of control limit width is a model for the cost of operating the chart. Models are developed to describe the total cost per unit time of monitoring the mean of a process using both the standard and the generalized Shewhart control chart for each of three cases. The models are developed under the assumption that the quality characteristic of interest is normally distributed with known and constant variance. The cases correspond to different assumptions concerning the time to process shift.

In the development of the case 1, the negative exponential lifetime distribution is employed to describe the shift property of the process. In analyzing the resulting models, it is found that the negative exponential lifetime distribution can describe the process shift properly. In addition, the expected total cost per unit time functions constructed under this assumption for both the standard and the generalized Shewhart control charts are practical and analytical. The decision variables can be chosen to minimize the expected total cost per unit time functions for both cases. The motivation for the control limit adjustment strategy is to increase chart sensitivity to small but anticipated shifts in the process average.

In case 2, the uniform lifetime distribution is used to model the process shift behavior. The uniform lifetime distribution is used because it has a failure rate function which is the function of time t.

In case 3, the Weibull lifetime distribution is used to model the process shift behavior. The Weibull lifetime distribution is used because it can be defined to display increasing failure rate (IFR) or decreasing failure rate (DFR) with the shape parameter $b > 1$ or $b < 1$, respectively.

1.2 Description of the Approach

The models developed here depend on a set of assumed conditions. First, it is assumed that the control chart is applied to monitoring the mean quality characteristic of an item that is produced on an ongoing basis. The variance of the quality characteristic is assumed known and constant. When in control, the process generates units for which the quality characteristic is normally distributed with mean μ_1. The control chart target value is equal to this mean. Control limits are assumed to be fixed for the standard Shewhart control chart and to be adjustable for the generalized Shewhart control chart.

The definition of cost model I for the standard Shewhart control chart proceeds in two steps. First, the negative exponential lifetime distribution is employed to describe the random variable t, the time until a process shift. It is assumed that the process is subject to a shift from the in-control value of the process mean, μ_1, to an out-of-control value, μ_2, at a random point in time. Then, the cost of operating a standard Shewhart control chart is defined using four cost terms. They are, (1) Inspection cost; (2) False alarm cost; (3) True signal cost; (4) Cost of producing additional non-conforming items when the process is out-of-control. In addition, the expected cycle length is determined. Then the expected total cost per unit time is constructed as the inspection cost plus the ratio of the sum of the three expected costs to the expected cycle length.

The definition of the corresponding cost models for the generalized Shewhart control chart proceeds in a similar manner. Assume we plan to start the chart with one set of control limits and to change the control limits to be tighter after the process has operated for a period of time that is determined. Specifically, we assume the process is sampled every h hours and after the m^{th} sample the control limits are changed. (Figure 1.2.1.). The same four cost terms are constructed but the analysis is quite different because we must distinguish between events before and after m. The expected cycle length is also constructed for this case and has a closed form.

Comparable cost models are constructed for both the standard Shewhart control chart and the generalized Shewhart control chart under the assumptions of a uniform distribution and a Weibull distribution on the time to process shift. In each case, the same four cost terms and the expected cycle time are determined.

1.3 Problem Analysis

The objective of the dissertation is to study the relative effectiveness of the standard Shewhart control chart and the generalized Shewhart control chart. The motivation for this effort is to increase control chart sensitivity to anticipated shifts in the process average. The cost models for the standard Shewhart control chart are defined in terms of the sample size, n, the time between samples, h, and the width of the control limits k ($k = k_1$ for the standard case). No constraints are imposed on the minimization other than the requirement that n and h be integer. The analysis of the cost equations can be difficult as the functions are in general neither concave nor convex. The convexity of the objective function depends on the relationship among the cost parameters and the intensity parameter of the distribution on the time until a process shift.

The solution of the cost models for the generalized Shewhart control chart involves determining the optimal values for n and h as in the standard chart plus k_1 and k_2, the control limit widths, and m, the time at which the control limits are changed. The optimization in both cases is performed using GINO [25].

The analysis of the cost models for the generalized Shewhart control chart could also be complicated as the function is in general neither concave nor convex. The convexity of the objective function depends on the relationships among the cost parameters and the parameter of the distribution on the time until a process shift.

The objective of this research is to compare the costs associated with the standard Shewhart control chart and the generalized Shewhart control chart. The cost models for the standard and the generalized Shewhart control chart generate several interesting points. First, the cost models display invariant behavior relative to some of the process parameters. The second is that for proper choices of the rate parameters, the generalized Shewhart control chart is more economic than the standard Shewhart control chart. Finally, the type II error probability is found to be an important factor in analyzing the properties of the total cost functions.

1.4 Value of the work

Shewhart control charts are easy to construct and understand but they have the disadvantage that they can be very slow at detecting small changes in the process mean. For example, suppose that an $\bar{x}$ chart uses 3-sigma limits. If a shift from μ_1 to $\mu_2 = \mu_1 + 0.5\,\sigma/n^{1/2}$ occurs at time t then μ_2 is still well within the control limits and it will take an average of 155.2 samples to detect this shift. If a shift of this size is big enough to be of concern then the time corresponding to 155.2 samples would usually be unacceptably long. The intuitive reason why the $\bar{x}$ chart takes so long to detect small shift is that this chart uses only the information in the k^{th} sample mean at t_k. If there is a shift in m then all samples taken after this shift should contain information about the shift. If the shift is small then the information in a single sample, when looked at in isolation from other samples, may not be strong enough to produce a signal. Various modifications of the standard Shewhart control chart have been proposed. Some use past sample information to make this type of chart more sensitive to small changes in the parameters. Some of these modifications also make the chart more sensitive to other process irregularities such as drift or cycles in the process parameters. The basic idea of the modifications so far is to make it easier for the chart to signal by adding supplementary rules for signaling [48]. However, the simultaneous use of a large number of rules can substantially increase the false alarm rate. A high false alarm rate is undesirable even if the efficiency is high.

This research deals with a Shewhart control chart adjustment strategy. This strategy has the following features which provides a new way of studying and analyzing Shewhart control charts. First, there are no additional run rules and therefore the false alarm rate may be reduced. The cost models are constructed to allow the optimal choice of change-over time and the best values for the initial and adjusted control limits. This strategy can increase control chart sensitivity to small but anticipated shifts in the process average so that the chart is able to rapidly detect a special cause and bring the process into control. The models are also used to provide a comparison with conventional implementation of the control chart.

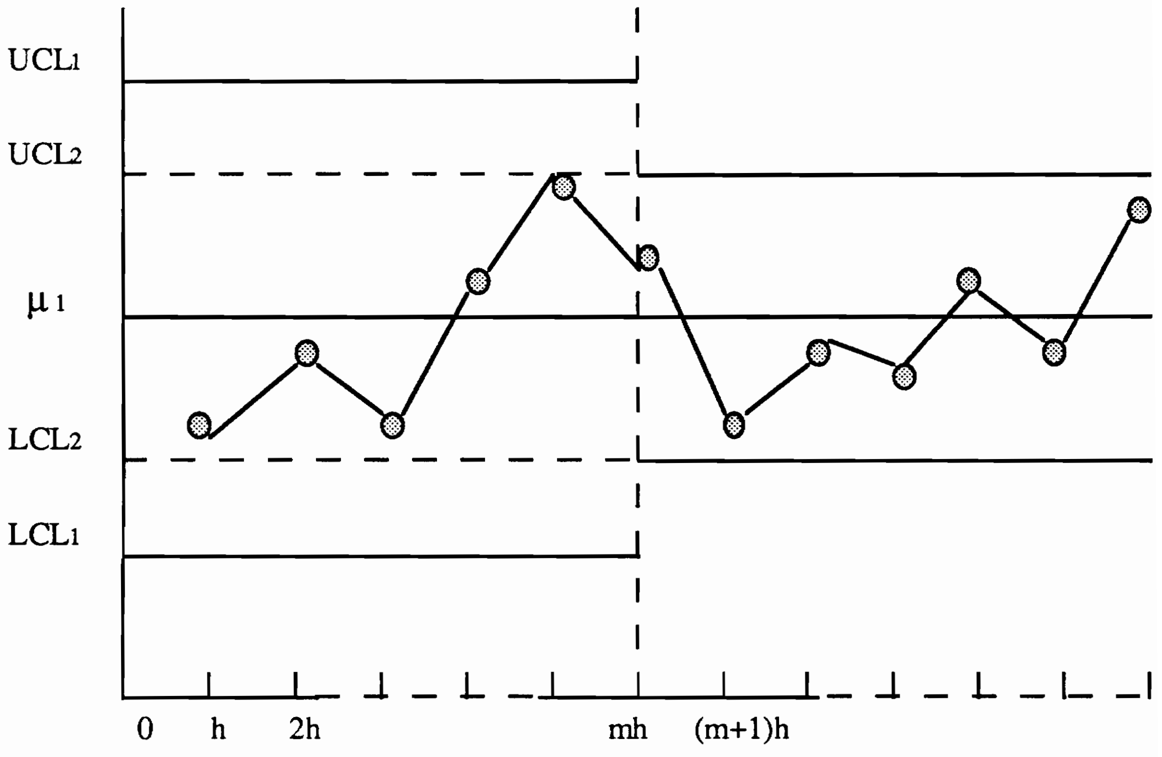

Figure 1.2.1. The Generalized Shewhart Control Chart

CHAPTER 2

Literature Review

Since Walter Shewhart introduced the control chart technique in 1924 [28], control schemes have found widespread application in improving the quality of manufacturing processes. Shewhart control charts are widely used to display sample data from a process for the purposes of determining whether a process is in control, for bringing an out-of-control process into control, and for monitoring a process to make sure that it stays in control. A control chart is maintained by taking samples from a process and plotting in time order on the chart some statistic computed from the samples. Control limits on the chart represent the limits within which the plotted points shouldfall with high probability if the process is operating in control. A point outside the control limits is taken as an indication that something, sometimes called an assignable cause of variation, has happened to change the process. When the chart signals that an assignable cause is present, rectifying action is taken to remove the assignable cause and bring the process back into control.

Duncan [13] establishes the foundation for the economic design of control charts. He defines a cost model that will support the choice of the optimal sample size and width for the control limits. Goel, Jain, and Wu [17] develop an algorithm for solving Duncan's model while Chiu and Wetherill [9] provide a simplified approach to obtaining an approximate solution to Duncan's model. Each of these analysis treats the conventional fixed sample interval control chart. Other authors offer alternate models. Gibra [16] used a cost model to focus upon the cost of detecting the cause in order to determine the sample size and the width of the control limits.

De Oliveira and Littauer [15] presented the first non-conventional idea with their development of warning limits. Montgomery [32] provides a thorough survey of the analyses of control charts and indicates that common practice is to select a sample size of five, a sampling interval of one hour, and three standard deviation control limits because of their ease of implementation.

The usual practice in maintaining a control chart is to take samples from the process at fixed - length sampling intervals. Reynolds, Amin, Arnold, and Nachlas [40] investigate the modification of the standard practice in which the sampling interval or time between samples is not fixed but instead can vary depending on what is observed from the data. The idea of using a variable sampling interval (VSI) control chart is intuitively reasonable. The proposed (VSI) control chart uses a short sampling interval if the sample mean is close to but not actually outside the control limits and a long sampling interval if the sample mean is close to the target. If the sample mean is actually outside the control limits, then the chart signals in the same way as the standard fixed sampling interval control chart. The problem of determining a sampling plan with variable time intervals between samples is investigated by Arnold [1], Hui and Jensen [21], and Reynolds and Arnold [41].

Nachlas, Clark, and Reynolds [31] develop a model to describe the total cost per unit time of monitoring the mean of a process using a variable sampling interval (VSI) Shewhart control chart. The model is developed under the assumption that the quality characteristic of interest is normally distributed with known and constant variance. A Markov model of the behavior of the sampling process is defined and used to construct the cost model. The cost model is then analyzed to determine the optimal control chart parameters including those associated with the variable sampling intervals. They show that the variable sampling interval (VSI) control charts are often more economical than standard Shewhart control charts.

The cumulative sum (CUSUM) control chart is introduced by Page (1954) and has been widely used for monitoring the mean of a quality characteristic or a production process. The CUSUM chart has been shown to be more efficient than the simpler Shewhart control chart in detecting small and moderate shifts in the process mean. The CUSUM chart is usually maintained by taking samples at fixed time intervals and plotting a cumulative sum of difference between the sample means and the target value in time order on the chart. The process mean is considered to be on target as long as the CUSUM statistic computed from the samples does not fall into the signal region of the chart. A value of the CUSUM statistic in the signal region is taken as an indication that the process mean has changed and that the possible causes of the change should be investigated.

The properties of the CUSUM control scheme are determined by the values of parameters. Bissel [5] studies a control scheme designed to detect a specific mean shift (unless the variance shifts with a shift in the mean level). Brook and Evans [8] investigate a standard CUSUM scheme and an associated head start method. Goel and Wu [18], evaluate ARL's for CUSUM charts using the ratio of two integral equations. Lucas and Crosier [27] study the property of fast initial response for CUSUM quality control schemes and they find that the fast initial response (FIR) feature for CUSUM quality control schemes permits a more rapid response to an initial out-of-control situation than does a standard CUSUM chart. This feature is especially valuable at start-up or after a CUSUM has given an out-of-control signal. They also present the average run length and the distribution of run length for CUSUM schemes with the FIR feature and compare FIR CUSUM schemes to standard CUSUM schemes. The comparisons show that if the process starts out in control, the fast initial response feature has little effect; however, if the process mean is not at the desired level, an out-of-control signal will be given faster when the FIR feature is used.

Roberts [44] introduces the exponentially weighted moving average (EWMA) control scheme. Using simulation to evaluate its properties, he shows that the EWMA is useful for detecting small shifts in the mean of a process. He developed nomograms of average run lengths (ARL's) for the case of normally distributed observations. In a subsequent article, Roberts [45] compares their performance to other procedures including CUSUM and Shewhart control schemes. More recently, Robinsion and Ho [46] numerically evaluate the

ARL's of EWMA control schemes using an Edgeworth series expansion. Although they consider a wider range of parameter values than does Roberts, their results are inaccurate for small values of the parameter. Crowder [10] evaluates the properties of EWMA's by formulating and solving a system of integral equations. Tables of the first and second moments of the run length distribution are given in his article. Hunter [22] suggests writing the current EWMA as the previous EWMA plus a fraction of the difference between the current observation and the previous EWMA. Lucas and Saccucci [28] evaluate the run length properties of EWMA control schemes by representing the EWMA statistic as a continuous-state Markov chain. Its properties can be approximated by a finite-state Markov chain following a procedure similar to that of Brook and Evans [8]. This allows the properties of EWMA's to be evaluated more easily and completely than has previously been done (Lucas and Saccucci [28]; Yashchin [53]).

In a short production run environment, data to estimate the process parameters and the limits for a standard Shewhart control chart are usually not available prior to the start of production. Since the number of parts needed to set the control limits could exceed the total number of parts produced in the run, the usual recommendation of gathering around twenty-five observations for setting valid control limits is also generally not appropriate in a short-run environment. In addition, as Hillier [19] points out, if a small number of sub groups is used to set the control limits, an inflated false alarm rate results.

Examples of manufacturing systems in which production runs are short, or equivalently, lots sizes are small, are numerous. Job shop manufacturing is usually characterized by manufacturing businesses have adopted shot production runs as a production strategy. Hillier [20] presents a method that gives valid control limits for Shewhart control charts regardless of the number of subgroups. His method, similar to classical control chart practice, is a two-stage procedure. The first m subgroups are used to assess control of the subsequent subgroups. This method gives the desired false alarm rate for any value of m, but the ARL while out of control may be too large if m is too small. This basic tradeoff also occurs in other control approaches. Therefore Hillier recommends using $m \geq 5$ and presents factors for setting control limits for the case of subgroups of size $n = 5$. Recently, Quesenberry [38] introduced Q charts for the short run problem when the quality variable follows a normal, binomial or Poisson distribution respectively. For the normal case he addresses both the case of grouped data and the case of individual measurements.

Much has appeared in the quality-control literature on incorporating economic considerations into the design of statistical process control (SPC) charts. In such approaches, a cost model is assumed and control-chart parameters are chosen to minimize expected cost. Montgomery [32] provides a review of the literature on this topic, and Saniga [47] discusses the use of economic design with the addition of statistical constraints on the design. Crowder [12] studies a finite-horizon or short-production-run version of an economic-process-control model. He derives an algorithm that allows implementation of this model and adjustment strategy for the short-production-run case. The solution to the control problem is consistent with traditional statistical process control philosophy in that process adjustment is called for only when the process mean is substantially off target. He also shows that the control or adjustment limits for this model are time-varying and depend on the break-even point between the quadratic cost for being off target and a fixed adjustment cost. It is shown that the length of the production run can greatly influence the control or adjustment strategy and the use of control limits based on the assumption of an infinte-run process can significantly increase total expected cost.

A process-control model is considered in which quadratic loss is associated with any deviation from target. Also it is assumed that a fixed cost is associated with any process adjustment. This is a special case of the cost structure assumed by Bather [3], who referred to the two costs as "running" costs and "overhaul" costs, respectively. A practical application of the short-run approach involves the problem of when to overhaul an expensive piece of manufacturing equipment. In rapidly changing industries,

such as semiconductor manufacturing, equipment may become obsolete and need to be retired in a relatively short number of years. In such cases, it cannot be assumed that the equipment will be used over an infinite period of time. In this context, a process adjustment means an overhaul of the equipment, restoring it to an on-target condition. The fixed adjustment cost is then the cost of performing the overhaul. More generally, the fixed adjustment cost could reflect the cost of manual adjustment, maintenance, or occurrence of process downtime.

In summary, many relevant results have been developed in different areas. Other needed results, particularly in the area of Shewhart control chart adjustment strategy, do not exist. Some are developed in this dissertation.

CHAPTER 3

Development of Cost Model I

3.1 Cost Model I for the Standard Shewhart Chart

3.1.1 Introduction and Assumptions

Assume a process is monitored using an $\overline{X}$ chart and the process is subject to a shift from the in control value of the process mean μ_1 to a single out of control value μ_2 at a random point in time. Assume the time until a process shift is a random variable with distribution F(t). Assume also that we plan to use the strategy of starting with a set of control limits and to change the limits to be tighter after the process has operated for a period of time that is to be determined. Specifically, the process is sampled every h hours and after the m^{th} sample the control limits are changed.

The question is to select the control limits to use before and after sample m and to choose a value for m. The basis of choice is a model for the cost of operating the chart. To start, the cost of operating a conventional (standard) chart is defined. The resulting model is then modified to reflect the consequences of the strategy of changing the control limits.

The cost categories considered are:

(1) C_i = sampling and inspection cost, unit cost per item = c_i

(2) C_f = false alarm cost, unit cost per event = c_f

(3) C_t = true signal and process correction cost, unit cost per event = c_t

(4) C_d= cost of producing substandard product while out-of-control, unit cost/item= c_d

(5) C_T = total cost per unit time

The expected total cost per unit time function is defined as:

$$E[C_T] = C_i + \frac{C_f + C_t + C_d}{E[t]} \tag{3.1.1}$$

Where E[t] is the expected cycle length (time to signal) and the following notation is used:

μ_1 = in-control value of the process mean

μ_2 = out-of-control value of the process mean

σ_x = the known and constant population standard deviation

UCL = upper control limit, UCL = $\mu_1 + k\sigma_x/n^{1/2}$

LCL = lower control limit, LCL = $\mu_1 - k\sigma_x/n^{1/2}$

U_x = upper specification limit

L_x = lower specification limit

p_1 = proportion non-conforming when $\mu = \mu_1$, then

$$p_1 = 1 - \Phi\left[\frac{U_x - \mu_1}{\sigma_x}\right] + \Phi\left[\frac{L_x - \mu_1}{\sigma_x}\right]$$

p_2 = proportion non-conforming when $\mu = \mu_2$, then

$$p_2 = 1 - \Phi\left[\frac{U_x - \mu_2}{\sigma_x}\right] + \Phi\left[\frac{L_x - \mu_2}{\sigma_x}\right]$$

$p = p_2 - p_1$

h = time between samples

r = production rate in units/hour

n = number items inspected per sample

m = number samples before changing the control limits

δ = number of units of σ_x from μ_1 to μ_2, so $\delta = (\mu_2 - \mu_1)/\sigma_x$

k_1= number of $\sigma_x/n^{1/2}$ from μ_1 to UCL before sample mh

k_2= number of $\sigma_x/n^{1/2}$ from μ_1 to UCL after sample mh

α = the Type I error probability, then

$$\begin{aligned}\alpha &= 1 - P\left(\mu_1 - k\sigma_x/\sqrt{n} \le \bar{x} \le \mu_1 + k\sigma_x/\sqrt{n} \,\middle|\, \mu = \mu_1\right) \\ &= 1 - [\Phi(k) - \Phi(-k)] = 2[1 - \Phi(k)]\end{aligned} \tag{3.1.2}$$

β = the Type II error probability, then

$$\beta = P\left(\mu_1 - k\sigma_x/\sqrt{n} \le \bar{x} \le \mu_1 + k\sigma_x/\sqrt{n} \,\middle|\, \mu = \mu_2\right) = \Phi\left(k - \delta\sqrt{n}\right) - \Phi\left(-k - \delta\sqrt{n}\right) \tag{3.1.3}$$

There are five decision variables in this dissertation, namely, n, h, m, k_1 and k_2. The optimal values for the five decision variables need to be chosen to minimize the expected total cost per unit time function as defined in (3.1.1).

3.1.2 Model Development

Suppose $F(t) = 1 - e^{-\lambda t}$, where λ is the rate parameter for the distribution on the time until a shift in the process mean. Then:

(1) Inspection cost = C_i= {fixed cost + (unit cost)(number inspected)}/ {time between samples}

$$C_i = \{c_0 + nc_i\}/h \tag{3.1.4}$$

(2) False alarm cost = C_f = (unit cost)(probability of false alarm)= c_fP[false alarm]. Let A="false alarm," A_1 = "false alarm on sample i," A_2 ="no process shift before sample i," then

$$\begin{aligned} P[A] &= \sum_i P[A_1]\,P[A_2] \\ &= \sum_{i=1}^{\infty} \alpha(1-\alpha)^{i-1}(1-F(ih)) = \alpha\sum_{i=1}^{\infty}(1-\alpha)^{i-1}e^{-\lambda ih} = \frac{\alpha e^{-\lambda h}}{1-(1-\alpha)e^{-\lambda h}} \end{aligned} \tag{3.1.5}$$

$$C_f = c_f P[A] = \frac{c_f \alpha e^{-\lambda h}}{1-(1-\alpha)e^{-\lambda h}} \tag{3.1.6}$$

(3) True signal cost = C_t = (unit cost)(probability of a true signal)= c_tP[true signal] (note that once a shift has occurred, the probability is 1.0 that a true signal will occur) Let B="true signal," B_1="process shift in interval j," B_2="no false alarm on proceeding j-1 samples," then the probability of the process terminating with a true signal can be constructed as:

$$P[B] = \sum_j P[B_1]P[B_2] = \sum_{j=1}^{\infty}\left[F(jh) - F((j-1)h)\right](1-\alpha)^{j-1}$$
$$= \sum_{j=1}^{\infty}\left[e^{-\lambda(j-1)h} - e^{-\lambda jh}\right](1-\alpha)^{j-1} = \frac{\left(1-e^{-\lambda h}\right)}{1-(1-\alpha)e^{-\lambda h}} \qquad (3.1.7)$$

$$C_t = c_t P[B] = \frac{c_t\left(1-e^{-\lambda h}\right)}{1-(1-\alpha)e^{-\lambda h}} \qquad (3.1.8)$$

(4) Cost of producing non-conforming items when the process is out of control = C_d= (unit cost)(production rate)(increase in proportion non-conforming)(expected time out of control).

Assume the process shifts after the k^{th} sample. Then the time that the process is out of control is comprised of the partial sampling interval during which the shift occurs and the full intervals that elapse before a signal. To determine the expected value of the partial sampling interval, t_p, during which the process is out of control, let T be the part of interval before process shift. Then t_p= h-T and E[t_p]= h- E[T]. Figure 3.1.1. The problem is therefore to construct E[T]. If the shift occurs in interval (jh,(j+1)h), then the construction of E[T] is:

$$T = t - jh \Rightarrow E[T] = E[t - jh] = \int (t - jh) f\left(t \middle| jh \le t \le (j+1)h\right) dt \tag{3.1.9}$$

And we know that:

$$F\left(t \middle| jh \le t \le (j+1)h\right) = \frac{F(t) - F(jh)}{F((j+1)h) - F(jh)}$$

$$f\left(t \middle| jh \le t \le (j+1)h\right) = \frac{f(t)}{F((j+1)h) - F(jh)} \tag{3.1.10}$$

This is the conditional probability density function of the random variable t given that the random variable t falls in the interval (jh, (j+1)h). Since the random variable t denotes the time that the process goes out of control, the above conditional density function represents the probability that the process goes out of control at time t given the shift occurs in the time interval (jh, (j+1)h). Therefore:

$$\begin{aligned} E[T] &= E[t - jh] = \int (t - jh) f\left(t \middle| jh \le t \le (j+1)h\right) dt \\ &= \frac{\int_{jh}^{(j+1)h} (t - jh) f(t) dt}{[F(j+1)h - F(jh)]} = \frac{\int_{jh}^{(j+1)h} \lambda (t - jh) e^{-\lambda t} dt}{[F(j+1)h - F(jh)]} \\ &= \frac{\left(\int_{jh}^{(j+1)h} \lambda t e^{-\lambda t} dt - \int_{jh}^{(j+1)h} \lambda jh e^{-\lambda t} dt\right)}{\left(1 - e^{-\lambda(j+1)h} - 1 + e^{\lambda jh}\right)} = \frac{1 - (1 + \lambda h) e^{-\lambda h}}{\lambda\left(1 - e^{-\lambda h}\right)} \end{aligned} \tag{3.1.11}$$

and finally:

$$E\left[t_p\right] = h - E[T] = h - \frac{1 - (1 + \lambda h) e^{-\lambda h}}{\lambda\left(1 - e^{-\lambda h}\right)} = \frac{\lambda h - \left(1 - e^{-\lambda h}\right)}{\lambda\left(1 - e^{-\lambda h}\right)} \tag{3.1.12}$$

This is the expected length of partial interval during which the process is out of control. Note that the above equation is a function of control interval h and rate parameter λ.

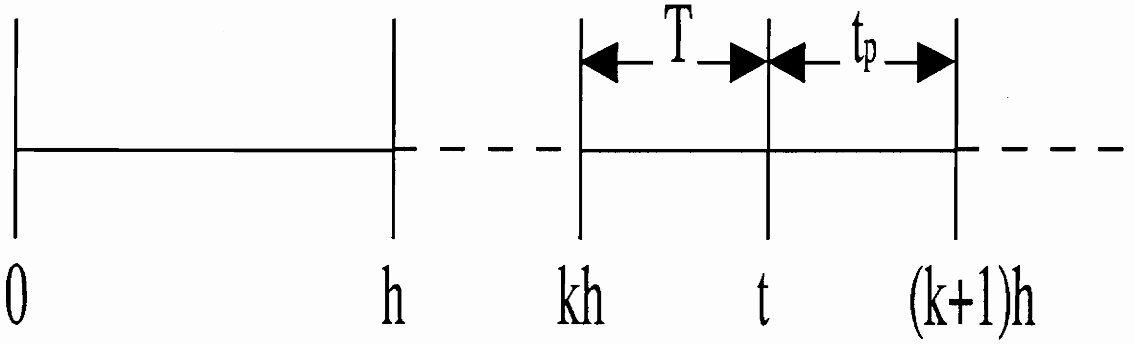

Figure 3.1.1 Time Intervals Involving T and t_p

E[time in full interval until a true signal] = $E[t_s]$ = h P[no signal at (j+1)h] ARL(μ_2), and so $E[t_s] = h\,\beta\,(1-\beta)^{-1}$, thus if D represents the time out of control, the expected time out of control is

$$E[D] = E\left[t_p\right] + E\left[t_s\right] = \frac{\lambda h - \left(1 - e^{-\lambda h}\right)}{\lambda\left(1 - e^{-\lambda h}\right)} + \frac{h\beta}{1-\beta} \tag{3.1.13}$$

Thus the cost of producing non-conforming items when the process is out of control, that is C_d, may also be obtained and it has the following form:

$$C_d = c_d r p\left[\frac{\lambda h - \left(1 - e^{-\lambda h}\right)}{\lambda\left(1 - e^{-\lambda h}\right)} + \frac{h\beta}{(1-\beta)}\right] \tag{3.1.14}$$

(5) To determine the expected cycle length, let E_1="false alarm on sample j and no process shift before sample j," E_2="process shift during interval l, no false alarm before interval l, and true signal on sample j (j-l+1st after shift)," and E[t]= expected cycle length. Therefore we obtain:

$$E[t] = \sum_j jhP[E_1] + \sum_j jh \sum_l P[E_2] \tag{3.1.15}$$

P[false alarm on sample j and no process shift before sample j]

$= \alpha(1-\alpha)^{j-1}(1-F(jh)) = \alpha(1-\alpha)^{j-1}e^{-\lambda jh}$

P[process shift during interval l]

$= F(lh)-F((l-1)h) = e^{-\lambda(l-1)h} - e^{-\lambda lh}$

P[no false alarm before interval l] = $(1-\alpha)^{l-1}$

P[true signal on sample j] = $\beta^{j-1}(1-\beta)$

Now, the expected cycle length, E[t], has the following form which is obtained by using the above information:

$$E[t] = \sum_{j=1}^{\infty} jh\left(\alpha(1-\alpha)^{j-1}e^{-\lambda jh} + \sum_{l=1}^{j}\left(e^{-\lambda(l-1)h} - e^{-\lambda lh}\right)(1-\alpha)^{l-1}\beta^{j-1}(1-\beta)\right)$$

$$= \alpha he^{-\lambda h}\sum_{j=1}^{\infty} j\left((1-\alpha)e^{-\lambda h}\right)^{j-1} + (1-\beta)\left(1-e^{-\lambda h}\right)h\sum_{j=1}^{\infty} j\sum_{l=1}^{j} e^{-\lambda(l-1)h}(1-\alpha)^{l-1}\beta^{j-1}$$

$$= \frac{\alpha he^{-\lambda h}}{\left[1-(1-\alpha)e^{-\lambda h}\right]^2} + \frac{\left(1-e^{-\lambda h}\right)h}{\left[\beta-(1-\alpha)e^{-\lambda h}\right]}\left[\frac{\beta}{1-\beta} - \frac{(1-\beta)(1-\alpha)e^{-\lambda h}}{\left[1-(1-\alpha)e^{-\lambda h}\right]^2}\right] \tag{3.1.16}$$

The objective is to choose the values of the decision variables to minimize equation (3.1.1).

3.2 Cost Model I for the Generalized Shewhart Chart

3.2.1 Introduction and Assumptions

Suppose that a control chart is used to monitor a process and that samples are taken every h units of time. Suppose further that the width of the control limits is changed after the m^{th} sample. Then the following facts may be used to describe control chart behavior:

Fact 3.1

(1) $\alpha(t) = \alpha_1$ if $t \leq mh$

$= \alpha_2$ if $t > mh$

(2) $\beta(t) = \beta_1$ if $t \leq mh$

$= \beta_2$ if $t > mh$

(3) $k(t) = k_1$ if $t \leq mh$

$= k_2$ if $t > mh$ (3.2.1)

Fact 3.2

The generalized Shewhart control chart is the same as the standard Shewhart control chart if and only if:

$\alpha_1 = \alpha_2 = \alpha;$

$\beta_1 = \beta_2 = \beta;$

$k_1 = k_2 = k.$ (3.2.2)

The question is again to select the control limits to use before and after sample m and to choose a value for m. The basis of choice is a model for the cost of operating the chart. The basic cost model I developed in section 3.1.2 is modified to reflect the consequences of the strategy of changing the control limits. The results is referred to as general cost model I.

3.2.2 Model Development

As discussed in the previous section, the construction of cost model I for the generalized Shewhart control chart can be based on the same cost categories. The development proceeds as:

(1) Inspection cost

Since the change of control limits does not change the form of the inspection cost, the inspection cost remains:

$$C_i = \{c_0 + nc_i\}/h \tag{3.2.3}$$

(2) False alarm cost = C_f = (unit cost)(probability of false alarm) = c_f P[false alarm] The expression for the probability of a false alarm in this case is quite different from that in the standard control chart since we have to consider $t \leq mh$ and $t > mh$ separately. The probability of false alarm can be constructed as:

$$\begin{aligned}
P[A] &= \sum_i P[A_1]\,P[A_2] \\
&= \sum_{i=1}^{m} \alpha_1(1-\alpha_1)^{i-1}(1-F(ih)) + (1-\alpha_1)^m \sum_{i=m+1}^{\infty} \alpha_2(1-\alpha_2)^{i-m-1}(1-F(ih)) \\
&= \alpha_1 \sum_{i=1}^{m} (1-\alpha_1)^{i-1} e^{-\lambda ih} + (1-\alpha_1)^m \alpha_2 \sum_{i-m=1}^{\infty} (1-\alpha_2)^{i-m-1} e^{-\lambda ih} \\
&= \frac{\alpha_1 e^{-\lambda h}\left\{1-\left[(1-\alpha_1)e^{-\lambda h}\right]^m\right\}}{1-(1-\alpha_1)e^{-\lambda h}} + \frac{(1-\alpha_1)^m e^{-\lambda(m+1)h}\alpha_2}{1-(1-\alpha_2)e^{-\lambda h}}
\end{aligned} \tag{3.2.4}$$

Therefore:

$$C_f = c_f \left\{ \frac{\alpha_1 e^{-\lambda h}\left\{1-\left[(1-\alpha_1)e^{-\lambda h}\right]^m\right\}}{1-(1-\alpha_1)e^{-\lambda h}} + \frac{(1-\alpha_1)^m e^{-\lambda(m+1)h}\alpha_2}{1-(1-\alpha_2)e^{-\lambda h}} \right\} \tag{3.2.5}$$

Note that the above expression is a nonlinear function of control parameters and it may become easier if bounds can be provided for C_f. The bounds should free from m, the number samples after changing the control limits. Experiments have shown that the lower and upper bounds are convenient for a small workshop. The result is shown below:

Lemma 3.1

The cost of false alarm has upper and lower bounds which are functions of type I error probabilities and control interval, and they are free from m. That is:

$$\frac{c_f \alpha_1^2 e^{-\lambda h}}{1-(1-\alpha_2)e^{-\lambda h}} \le C_f \le \frac{c_f(\alpha_1+\alpha_2)e^{-\lambda h}}{1-(1-\alpha_1)e^{-\lambda h}} \tag{3.2.6}$$

The proof is in Appendix B (1).

(3) True signal cost = C_t = (unit cost) (probability of true signal) = c_t P[true signal]. Note again that once a shift has occurred, the probability is 1.0 that a true signal will occur. Therefore the construction of the cost of true signal proceeds as follows:

(1) Construct the probability of the process shift in interval j;
(2) Construct the probability of no false alarm on proceeding j-1 samples;
(3) Construct the probability of true signal;
(4) Combine these results to obtain the true signal cost.

$$\begin{aligned}
P[B] &= \sum_j P[B_1]\,P[B_2] \\
&= \sum_{j=1}^{m+1}[F(jh)-F(j-1)h](1-\alpha_1)^{j-1} + \sum_{j=m+2}^{\infty}[F(jh)-F(j-1)h](1-\alpha_2)^{j-1} \\
&= \sum_{j=1}^{m+1}\left[e^{-\lambda(j-1)h}-e^{-\lambda jh}\right](1-\alpha_1)^{j-1} + \sum_{j=m+2}^{\infty}\left[e^{-\lambda(j-1)h}-e^{-\lambda jh}\right](1-\alpha_2)^{j-1} \\
&= \left(1-e^{-\lambda h}\right)\left\{\frac{1}{1-(1-\alpha_1)e^{-\lambda h}} - \frac{\left[(1-\alpha_1)e^{-\lambda h}\right]^{m+1}}{1-(1-\alpha_1)e^{-\lambda h}} + \frac{\left[(1-\alpha_2)e^{-\lambda h}\right]^{m+1}}{1-(1-\alpha_2)e^{-\lambda h}}\right\}
\end{aligned} \tag{3.2.7}$$

$$C_t = c_t\left(1-e^{-\lambda h}\right)\left\{\frac{1}{1-(1-\alpha_1)e^{-\lambda h}} - \frac{\left[(1-\alpha_1)e^{-\lambda h}\right]^{m+1}}{1-(1-\alpha_1)e^{-\lambda h}} + \frac{\left[(1-\alpha_2)e^{-\lambda h}\right]^{m+1}}{1-(1-\alpha_2)e^{-\lambda h}}\right\} \tag{3.2.8}$$

The above expression is a nonlinear function of the control parameters and useful bounds may be provided for C_t. The bounds should not depend on m. Experiments have shown that the lower and upper bounds are convenient for hand calculation and the result is:

Lemma 3.2

The cost of true signal has upper and lower bounds which are functions of the type I error probabilities and control interval, and they are free from m. That is:

$$\frac{c_t\alpha_1\left(1-e^{-\lambda h}\right)}{1-(1-\alpha_2)e^{-\lambda h}} \le C_t \le \frac{c_t\left(1-e^{-\lambda h}\right)}{1-(1-\alpha_1)e^{-\lambda h}} \tag{3.2.9}$$

The proof is in Appendix B (2).

(4) Cost of producing non-conforming items when the process is out of control = C_d. Let t = time that the process goes out of control, then E[time out of control] = $E[t_p]$ + $E[t_s]$. Since $E[t_p]$ = h - E[T]. The problem is again to construct E[T] but the analysis is the same as for the standard Shewhart control chart. Thus:

$$E\left[t_p\right] = h - E[T] = h - \frac{1-(1+\lambda h)e^{-\lambda h}}{\lambda\left(1-e^{-\lambda h}\right)} = \frac{\lambda h - \left(1-e^{-\lambda h}\right)}{\lambda\left(1-e^{-\lambda h}\right)} \tag{3.2.10}$$

The construction of $E[t_s]$ is a bit different as the identity of the interval in which the shift occurs affects the signal probability.

E[time in full intervals until a true signal] = $E[t_S]$ = h P[no signal at (j+1)h] ARL(μ_2). It is actually the ARL that depends on the time of shift and the signal probability. In our analysis, three cases are considered:

(a) Shift during $j \le m$ and signal at $i \le m$;

(b) Shift during $j \le m$ and signal at $i > m$;

(c) Shift during $j > m$.

For case (a), the equation of $E[t_S(a)]$ has the following form (Appendix B(3)):

$$E[t_s(a)] = \sum_{j=1}^{m}[F(jh) - F(j-1)h]\sum_{i=j}^{m}(i-j)h(1-\beta_1)\beta_1^{i-j}$$

$$E[t_s(a)] = \frac{h\beta_1(1-e^{-\lambda mh})}{(1-\beta_1)} - h\beta_1(1-e^{-\lambda h})\left[\frac{m\beta_1^m + e^{-\lambda mh}}{\beta_1 - e^{-\lambda h}} - \frac{\beta_1^{m+1} - e^{-\lambda(m+1)h}}{(\beta_1 - e^{-\lambda h})^2}\right]$$
$$-\frac{h\beta_1(1-e^{-\lambda mh})}{(1-\beta_1)}\frac{\beta_1^m - e^{-\lambda mh}}{\beta_1 - e^{-\lambda h}} \qquad (3.2.11)$$

For case (b), the equation of $E[t_S(b)]$ is as follows and the detail derivation can be found in Appendix B (4):

$$E[t_s(b)] = \sum_{j=1}^{m}[F(jh) - F((j-1)h)]\,\beta_1^{m-j+1}\sum_{i=m+1}^{\infty}(i-j)h\beta_2^{i-m-1}(1-\beta_2)$$

$$E[t_s(b)] = (1-e^{-\lambda h})h\beta_1\left\{\frac{m(\beta_1^m - e^{-\lambda mh})}{\beta_1 - e^{-\lambda h}} - \frac{\beta_1^{m+1} - e^{-\lambda(m+1)h}}{(\beta_1 - e^{-\lambda h})^2} + \frac{(m+1)e^{-\lambda mh}}{\beta_1 - e^{-\lambda h}}\right\}$$
$$+\frac{(1-e^{-\lambda h})h\beta_1}{(1-\beta_2)}\frac{\beta_1^m - e^{-\lambda mh}}{\beta_1 - e^{-\lambda h}} \qquad (3.2.12)$$

For case (c), it is obvious that:

$$E[t_s(c)] = [1 - F(mh)]h\beta_2 \frac{1}{1-\beta_2} = \frac{h\beta_2 e^{-\lambda mh}}{1-\beta_2} \tag{3.2.13}$$

Finally, combining the three cases yields:

$$E[t_s] = \frac{h\beta_1\left(1-e^{-\lambda mh}\right)}{1-\beta_1} + \frac{h\beta_2 e^{-\lambda mh}}{1-\beta_2} + \left[\frac{h\beta_1\left(1-e^{-\lambda h}\right)\left(\beta_1^m - e^{-\lambda mh}\right)}{\beta_1 - e^{-\lambda h}}\right]\left[\frac{1}{1-\beta_2} - \frac{1}{1-\beta_1}\right] \tag{3.2.14}$$

and then

$$C_d = c_d r(p_2 - p_1)\left(E\left(t_p\right) + E(t_s)\right) = c_d rp\left\{\frac{\lambda h - \left(1-e^{-\lambda h}\right)}{\lambda\left(1-e^{-\lambda h}\right)} + \frac{h\beta_1\left(1-e^{-\lambda mh}\right)}{1-\beta_1} + \frac{h\beta_2 e^{-\lambda mh}}{1-\beta_2}\right\}$$
$$+ c_d rp\left\{\left[\frac{h\beta_1\left(1-e^{-\lambda h}\right)\left(\beta_1^m - e^{-\lambda mh}\right)}{\beta_1 - e^{-\lambda h}}\right]\left[\frac{1}{1-\beta_2} - \frac{1}{1-\beta_1}\right]\right\} \tag{3.2.15}$$

(5) The expected cycle length is different from that of the cost model I for standard Shewhart control chart. We must again distinguish between events before and after m. The expected cycle length can be written as:

$$E[t] = \sum_j jhE[E_1] + \sum_j jh\sum_1 E[E_2] \tag{3.2.16}$$

To start, the following notations are employed:

$$E[t_f] = \sum_j jhE[E_1] \tag{3.2.17}$$

Then, $E[t_f]$ can be written as the following form:

$$E[t_f] = \sum_{j=1}^{m} jh(1-\alpha_1)^{j-1}\alpha_1[1-F(jh)] + (1-\alpha_1)^m \sum_{j=m+1}^{\infty} jh(1-\alpha_2)^{j-m-1}\alpha_2[1-F(jh)]$$

$$= \alpha_1 h \sum_{j=1}^{m} j(1-\alpha_1)^{j-1} e^{-\lambda jh} + (1-\alpha_1)^m \alpha_2 h \sum_{j=m+1}^{\infty} j(1-\alpha_2)^{j-m-1} e^{-\lambda jh}$$

$$= \alpha_1 h \left[\frac{e^{-\lambda h}\left\{1-\left[(1-\alpha_1)e^{-\lambda h}\right]^m\right\}}{\left[1-(1-\alpha_1)e^{-\lambda h}\right]^2} - \frac{m(1-\alpha_1)^m e^{-\lambda(m+1)h}}{1-(1-\alpha_1)e^{-\lambda h}} \right]$$

$$+ \alpha_2 h \left[\frac{(1-\alpha_1)^m e^{-\lambda(m+1)h}}{\left[1-(1-\alpha_2)e^{-\lambda h}\right]^2} + \frac{m(1-\alpha_1)^m e^{-\lambda(m+1)h}}{1-(1-\alpha_2)e^{-\lambda h}} \right] \tag{3.2.18}$$

Next, let

$$E[t_n] = \sum_j jh \sum_l E[E_2] \tag{3.2.19}$$

Three cases must be considered:

Case (a) $s \le m, j \le m$;

Case (b) $s \le m, j > m$;

Case (c) $s > m$.

For case (a), the derivation of $E[t_n(a)]$ is in Appendix B(5):

$$E[t_n(a)] = \sum_{s=1}^{m} [F(sh) - F((s-1)h)](1-\alpha_1)^{s-1} \sum_{j=s}^{m} jh(1-\beta_1)\beta_1^{j-s}$$

$$= h(1-e^{-\lambda h})\beta_1 \left\{ \frac{1-\left(e^{-\lambda h}(1-\alpha_1)\right)^m}{\left[1-e^{-\lambda h}(1-\alpha_1)\right]^2} - \frac{m\left[e^{-\lambda h}(1-\alpha_1)\right]^m}{1-e^{-\lambda h}(1-\alpha_1)} - \frac{m\beta_1\left[\beta_1^m - \left(e^{-\lambda h}(1-\alpha_1)\right)^m\right]}{\beta_1 - e^{-\lambda h}(1-\alpha_1)} \right\}$$

$$+h\left(1-e^{-\lambda h}\right)\beta_1\left\{\frac{\beta_1}{1-\beta_1}\left[\frac{1-\left[e^{-\lambda h}(1-\alpha_1)\right]^m}{1-e^{-\lambda h}(1-\alpha_1)}-\frac{\beta_1^m-\left(e^{-\lambda h}(1-\alpha_1)\right)^m}{\beta_1-e^{-\lambda h}(1-\alpha_1)}\right]\right\} \tag{3.2.20}$$

For case (b)

$$E[t_n(b)]=\sum_{s=1}^{m}[F(sh)-F(s-1)h](1-\alpha_1)^{s-1}\beta_1^{m-s+1}\sum_{j=m+1}^{\infty}jh(1-\beta_2)\beta_2^{j-m-1}$$

$$=(1-\beta_2)h\sum_{s=1}^{m}\left[e^{-\lambda(s-1)h}-e^{-\lambda sh}\right](1-\alpha_1)^{s-1}\beta_1^{m-s+1}\left[\frac{1}{(1-\beta_2)^2}+\frac{m}{1-\beta_2}\right]$$

$$=h\left(1-e^{-\lambda h}\right)\left(\frac{1}{1-\beta_2}+m\right)\left[\frac{\beta_1^{m+1}-\beta_1\left[e^{-\lambda h}(1-\alpha_1)\right]^m}{\beta_1-\left[e^{-\lambda h}(1-\alpha_1)\right]}\right] \tag{3.2.21}$$

For case (c), the equation of $E[t_n(c)]$ is (Appendix B(6)):

$$E[t_n(c)]=(1-\alpha_1)^m\sum_{l=m+1}^{\infty}\left[F(lh)-F((l-1)h)\right](1-\alpha_2)^{l-m-1}\sum_{j=1}^{\infty}jh(1-\beta_2)\beta_2^{j-1}$$

$$=\frac{(1-\alpha_1)^m\left(1-e^{-\lambda h}\right)h}{(1-\beta_2)}\left\{\frac{\left[\beta_2+m(1-\beta_2)\right]e^{-\lambda mh}}{1-(1-\alpha_2)e^{-\lambda h}}+\frac{(1-\beta_2)e^{-\lambda mh}}{\left[1-(1-\alpha_2)e^{-\lambda h}\right]^2}\right.$$

$$\left.-\frac{\left[\beta_2+m(1-\beta_2)\right]e^{-\lambda h}}{\alpha_2}-\frac{(1-\beta_2)e^{-\lambda h}}{\alpha_2^2}\right\} \tag{3.2.22}$$

Combining the results from the three cases we obtain the expression for $E[t_n]$ for the generalized Shewhart control chart under the the shift distribution of negative exponential. That is:

$$E[t_n]=h\left(1-e^{-\lambda h}\right)\beta_1\left\{\frac{1-\left(e^{-\lambda h}(1-\alpha_1)\right)^m}{\left[1-e^{-\lambda h}(1-\alpha_1)\right]^2}-\frac{m\left[e^{-\lambda h}(1-\alpha_1)\right]^m}{1-e^{-\lambda h}(1-\alpha_1)}-\frac{m\beta_1\left[\beta_1^m-\left(e^{-\lambda h}(1-\alpha_1)\right)^m\right]}{\beta_1-e^{-\lambda h}(1-\alpha_1)}\right\}$$

$$+h\left(1-e^{-\lambda h}\right)\beta_1\left\{\frac{\beta_1}{1-\beta_1}\left[\frac{1-\left[e^{-\lambda h}(1-\alpha_1)\right]^m}{1-e^{-\lambda h}(1-\alpha_1)}-\frac{\beta_1^m-\left(e^{-\lambda h}(1-\alpha_1)\right)^m}{\beta_1-e^{-\lambda h}(1-\alpha_1)}\right]\right\}$$

$$+h\left(1-e^{-\lambda h}\right)\left(\frac{1}{1-\beta_2}+m\right)\left[\frac{\beta_1^{m+1}-\beta_1\left[e^{-\lambda h}(1-\alpha_1)\right]^m}{\beta_1-\left[e^{-\lambda h}(1-\alpha_1)\right]}\right]$$

$$+\frac{(1-\alpha_1)^m\left(1-e^{-\lambda h}\right)h}{(1-\beta_2)}\left\{\frac{\left[\beta_2+m(1-\beta_2)\right]e^{-\lambda mh}}{1-(1-\alpha_2)e^{-\lambda h}}+\frac{(1-\beta_2)e^{-\lambda mh}}{\left[1-(1-\alpha_2)e^{-\lambda h}\right]^2}\right.$$

$$\left.-\frac{\left[\beta_2+m(1-\beta_2)\right]e^{-\lambda h}}{\alpha_2}-\frac{(1-\beta_2)e^{-\lambda h}}{\alpha_2^2}\right\} \qquad (3.2.23)$$

Now with the definition of E[t_f] of the same section, we obtain the expected cycle length for the generalized Shewhart control chart as follows:

$$E[t]=\alpha_1 h\left[\frac{e^{-\lambda h}\left\{1-\left[(1-\alpha_1)e^{-\lambda h}\right]^m\right\}}{\left[1-(1-\alpha_1)e^{-\lambda h}\right]^2}-\frac{m(1-\alpha_1)^m e^{-\lambda(m+1)h}}{1-(1-\alpha_1)e^{-\lambda h}}\right]$$

$$+\alpha_2 h\left[\frac{(1-\alpha_1)^m e^{-\lambda(m+1)h}}{\left[1-(1-\alpha_2)e^{-\lambda h}\right]^2}+\frac{m(1-\alpha_1)^m e^{-\lambda(m+1)h}}{1-(1-\alpha_2)e^{-\lambda h}}\right]$$

$$+h\left(1-e^{-\lambda h}\right)\beta_1\left\{\frac{1-\left(e^{-\lambda h}(1-\alpha_1)\right)^m}{\left[1-e^{-\lambda h}(1-\alpha_1)\right]^2}-\frac{m\left[e^{-\lambda h}(1-\alpha_1)\right]^m}{1-e^{-\lambda h}(1-\alpha_1)}-\frac{m\beta_1\left[\beta_1^m-\left(e^{-\lambda h}(1-\alpha_1)\right)^m\right]}{\beta_1-e^{-\lambda h}(1-\alpha_1)}\right\}$$

$$+h\left(1-e^{-\lambda h}\right)\beta_1\left\{\frac{\beta_1}{1-\beta_1}\left[\frac{1-\left[e^{-\lambda h}(1-\alpha_1)\right]^m}{1-e^{-\lambda h}(1-\alpha_1)}-\frac{\beta_1^m-\left(e^{-\lambda h}(1-\alpha_1)\right)^m}{\beta_1-e^{-\lambda h}(1-\alpha_1)}\right]\right\}$$

$$+h\left(1-e^{-\lambda h}\right)\left(\frac{1}{1-\beta_2}+m\right)\left[\frac{\beta_1^{m+1}-\beta_1\left[e^{-\lambda h}(1-\alpha_1)\right]^m}{\beta_1-\left[e^{-\lambda h}(1-\alpha_1)\right]}\right]$$

$$+\frac{(1-\alpha_1)^m\left(1-e^{-\lambda h}\right)h}{(1-\beta_2)}\left\{\frac{\left[\beta_2+m(1-\beta_2)\right]e^{-\lambda mh}}{1-(1-\alpha_2)e^{-\lambda h}}+\frac{(1-\beta_2)e^{-\lambda mh}}{\left[1-(1-\alpha_2)e^{-\lambda h}\right]^2}\right.$$

$$\left.-\frac{\left[\beta_2+m(1-\beta_2)\right]e^{-\lambda h}}{\alpha_2}-\frac{(1-\beta_2)e^{-\lambda h}}{\alpha_2^2}\right\} \qquad (3.2.24)$$

Then, the expected total cost per unit time function for the generalized Shewhart control chart is well defined and the objective is to choose the optimal values of the decision variables to minimize the expected total cost per unit time as defined in (3.1.1).

Lemma 3.3

The expected total cost per unit time function for the generalized Shewhart control chart is equivalent to that for the standard Shewhart control chart under Fact 3.2.

The proof of this lemma is in Appendix B(7).

CHAPTER 4

Development of Cost Model II

4.1 Cost Model II for the Standard Shewhart Chart

4.1.1 Introduction and Assumptions

Assume a process is monitored using the same control chart and the process is subject to a shift from the in control value of the process mean μ_1 to a single out of control value μ_2 at a random point in time. Assume the time until a process shift is a random variable with $F(t) = t/\theta$, $(0 < \theta < \infty)$. Assume also that we plan to use the strategy of starting with a set of control limits and to change the limits to be tighter after the process has operated for a period of time that is to be determined. Specifically, the process is sampled every h hours and after the m^{th} sample the control limits are changed.

The question is to select the control limits to use before and after sample m and to choose a value for m. The basis of choice is a model for the cost of operating the chart. To start, the cost of operating a standard chart is defined. The resulting model is then modified to reflect the consequences of the strategy of changing the control limits.

In this section, the cost categories considered are the same as those discussed in the previous chapter. The expected total cost per unit time function is defined as in (3.1.1). The decision variables are still n, h, m, k_1 and k_2 ($k=k_1$ for the standard Shewhart control chart). The optimal values for the decision variables need to be chosen to minimize the expected total cost per unit time as defined in (3.1.1). In addition, the time until the process shift is modeled using the uniform distribution because it has a failure rate function which is the function of time t. Since the parameter θ ($0 < \theta < \infty$) may take values widely, the performance of cost model II may depend upon the choice of the parameter θ.

4.1.2 Model Development

Referring to the previous chapter, the development of cost model II for a standard Shewhart control chart may be realized by first constructing the cost components. However, the range of the time is different here because of the use of uniform distribution. That is, since $0 < t < \theta < \infty$, let N be the maximum value of t, then $N = \theta/h$, and suppose that N is an integer.

(1) Inspection cost = C_i= {fixed cost + (unit cost)(number inspected)}/ {time between samples}, therefore:

$$C_i = \{c_0 + nc_i\}/h \tag{4.1.1}$$

(2) False alarm cost = C_f = (unit cost)(probability of false alarm)= c_fP[false alarm]. Let A="false alarm", A_1 = "false alarm on sample i", A_2 ="no process shift before sample i", then the construction of the cost of false alarm C_f proceeds as follows:

(i) Construct the probability of false alarm;

(ii) Formulate $C_f = c_f P[A]$.

Therefore:

$$\begin{aligned} P[A] &= \sum_i P[A_1]\,P[A_2] = \sum_{i=1}^{N} \alpha(1-\alpha)^{i-1}[1-F(ih)] = \alpha\sum_{i=1}^{N}(1-\alpha)^{i-1}\left(1-\frac{ih}{\theta}\right) \\ &= \alpha\sum_{i=1}^{N}(1-\alpha)^{i-1} - \frac{\alpha h}{\theta}\sum_{i=1}^{N} i\,(1-\alpha)^{i-1} = 1-\frac{h}{\alpha\theta}+\frac{h}{\alpha\theta}(1-\alpha)^{N} \end{aligned} \tag{4.1.2}$$

Thus, the false alarm cost can be constructed as:

$$C_f = c_f P[A] = c_f\left[1-\frac{h}{\alpha\theta}+\frac{h}{\alpha\theta}(1-\alpha)^{N}\right] \tag{4.1.3}$$

(3) True signal cost = C_t = (unit cost) (probability of a true signal) = c_tP[true signal]. (note that once a shift has occurred, the probability is 1.0 that a true signal will occur). Let B="true signal ", B_1 =" process shift in interval j", B_2 =" no false alarm on proceeding j-1

samples", then the expression for P[B] is:

$$P[B]=\sum_j P[B_1]P[B_2]=\sum_{j=1}^{N}\left[F(jh)-F((j-1)h)\right](1-\alpha)^{j-1}=\sum_{j=1}^{N}\left[\frac{jh}{\theta}-\frac{(j-1)h}{\theta}\right](1-\alpha)^{j-1}$$
$$=\frac{h}{\theta}\sum_{j=1}^{N}(1-\alpha)^{j-1}=\frac{h}{\theta}\left[\frac{1-(1-\alpha)^N}{\alpha}\right]=\frac{h}{\alpha\theta}-\frac{h}{\alpha\theta}(1-\alpha)^N \tag{4.1.4}$$

Thus, the true signal cost has the following form:

$$C_t=c_t\,P[B]=c_t\left[\frac{h}{\alpha\theta}-\frac{h}{\alpha\theta}(1-\alpha)^N\right] \tag{4.1.5}$$

(4) Cost of producing non-conforming items when the process is out of control = C_d= (unit cost)(production rate)(increase in proportion non-conforming)(expected time out of control). The time intervals at this step can be seen in Figure 3.1.1. The E[time out of control] = E[length of partial interval after shift and before sample] + E[time comprised of full intervals until a true signal] = $E[t_p]$ + $E[t_s]$.

Let t_p = length of partial interval during which the process is out of control and let T = part of interval before process shift, then t_p= h-T and this implies $E[t_p]$= h- E[T]. The problem is therefore how to construct E[T]. If the shift occurs in (jh,(j+1)h), then the construction of E[T] yields a interesting property of the uniform distribution:

Lemma 4.1

The expected part of interval before the process shift is equal to the expected length of partial interval during which the process is out of control and they are equal to half of length of the control chart interval. That is:

$$E[T] = E[t_p] = h/2 \tag{4.1.6}$$

The proof is in Appendix C. (1).

Let E[time in full intervals until a true signal] = $E[t_s]$ = h P[no signal at (j+1)h] ARL(μ_2). As ARL(μ_2) = $(1-\beta)^{-1}$, $E[t_s] = h\,\beta\,(1-\beta)^{-1}$. Let D="time out of control", then refer to Lemma 4.1, the expected time out of control is:

$$E[D] = E[t_p] + E[t_s] = \frac{1}{2}h + \frac{h\beta}{1-\beta} \tag{4.1.7}$$

Thus the cost of producing non-conforming items when the process is out of control, that is C_d, may also be obtained and it has the form:

$$C_d = c_d rpE[D] = c_d rph\left[\frac{1}{2} + \frac{\beta}{1-\beta}\right] \tag{4.1.8}$$

(5) Let E_1="false alarm on sample j and no process shift before sample j", E_2="process shift during interval l, no false alarm before interval l, and true signal on sample j (j-l+1st after shift)", and E[t]= expected cycle length. Therefore:

$$E[t] = \sum_j jhP[E_1] + \sum_j jh\sum_l P[E_2] \tag{4.1.9}$$

P[false alarm on sample j and no process shift before sample j] = $\alpha(1-\alpha)^{j-1}(1-F(jh)) = \alpha(1-\alpha)^{j-1}(1-jh/\theta)$

P[process shift during interval l] = $F(lh)-F((l-1)h) = [lh/\theta - (l-1)h/\theta] = h/\theta$

P[no false alarm before interval l] = $(1-\alpha)^{l-1}$

P[true signal on sample j] = $\beta^{j-1}(1-\beta)$

Now, the expected cycle length, E[t], has the following form:

$$
\begin{aligned}
E[t] &= \sum_{j=1}^{N} jh\alpha(1-\alpha)^{j-1}\left[1-\frac{jh}{\theta}\right] + \sum_{j=1}^{N} jh\left(\sum_{l=1}^{j}\left[\frac{lh}{\theta}-\frac{(l-1)h}{\theta}\right](1-\alpha)^{l-1}\beta^{j-1}(1-\beta)\right) \\
&= \frac{h}{\alpha}\left[1-(1-\alpha)^{N}(\alpha N+1)\right] \\
&\quad -\frac{\alpha h^2}{\theta}\left\{\frac{1}{\alpha^2}+\frac{2(1-\alpha)\left[1-(1-\alpha)^{N-1}\right]}{\alpha^3}-\frac{(N-1)^2(1-\alpha)^N}{\alpha^2}+\frac{N^2(1-\alpha)^{N+1}}{\alpha^2}\right\} \\
&\quad +\frac{h^2(1-\beta)}{[\beta-(1-\alpha)]\theta}\left\{\frac{\beta(1-\beta^N)}{(1-\beta)^2}-\frac{N\beta^{N+1}}{1-\beta}-\frac{(1-\alpha)\left[1-(1-\alpha)^N\right]}{\alpha^2}+\frac{N(1-\alpha)^{N+1}}{\alpha}\right\}
\end{aligned}
\tag{4.1.10}
$$

The reduction to this form is shown in Appendix C.(2).

The expected total cost per unit time function for the standard Shewhart control chart corresponds to equation (3.1.1) and the objective is to choose the optimal values of the decision variables to minimize the expected total cost per unit time function.

4.2 Cost Model II for the Generalized Shewhart Chart

4.2.1 Introduction and Assumptions

Suppose that a control chart is used to monitor a process and that samples are taken every h units of time. Suppose further that the width of the control limits are changed after the m^{th} sample. Then the following facts may be used to describe control chart behavior:

Fact 4.1

(1) $\alpha(t) = \alpha_1$ if $t \leq mh$

$= \alpha_2$ if $t > mh$

(2) $\beta(t) = \beta_1$ if $t \leq mh$

$= \beta_2$ if $t > mh$

(3) $k(t) = k_1$ if $t \leq mh$

$= k_2$ if $t > mh$ (4.2.1)

Fact 4.2

The generalized Shewhart control chart is the same as the standard Shewhart control chart if and only if:

$\alpha_1 = \alpha_2 = \alpha;$

$\beta_1 = \beta_2 = \beta;$

$k_1 = k_2 = k.$ (4.2.2)

4.2.2 Model Development

As discussed in the previous section, the construction of cost model II for the generalized Shewhart control chart may be achieved by analyzing the following costs:

(1) Inspection cost - Since the change of control limits does not alter the structure of the inspection cost, thus the inspection cost remains:

$$C_i = (c_0 + nc_i)/h \tag{4.2.3}$$

(2) False alarm cost = C_f = (unit cost)(probability of false alarm) = c_f P[false alarm] Then, the problem again is to construct the expression for the of probability of a false alarm. Again the expression for the of probability of a false alarm in this case is quite different from that for the standard Shewhart control chart because we have to consider $t \le mh$ or $t > mh$ separately. Therefore:

$$\begin{aligned}
P[A] &= \sum_i P[A_1]\,P[A_2] \\
&= \sum_{i=1}^{m} \alpha_1(1-\alpha_1)^{i-1}[1-F(ih)] + (1-\alpha_1)^m \sum_{i=m+1}^{N} \alpha_2(1-\alpha_2)^{i-m-1}[1-F(ih)] \\
&= 1-(1-\alpha_1)^m(1-\alpha_2)^{N-m} - \frac{h}{\theta}\left[\frac{1-(1-\alpha_1)^m}{\alpha_1} + \frac{(1-\alpha_1)^m}{\alpha_2}\right] \\
&\quad + \frac{h}{\theta}\left[\frac{(1-\alpha_1)^m(1-\alpha_2)^{N-m}(N\alpha_2+1)}{\alpha_2}\right]
\end{aligned} \tag{4.2.4}$$

The reduction of the above form is in Appendix C(3).

$$\begin{aligned}
C_f = c_f P[A] = c_f \Bigg\{ & 1-(1-\alpha_1)^m(1-\alpha_2)^{N-m} - \frac{h}{\theta}\left[\frac{1-(1-\alpha_1)^m}{\alpha_1} + \frac{(1-\alpha_1)^m}{\alpha_2}\right] \\
& + \frac{h}{\theta}\left[\frac{(1-\alpha_1)^m(1-\alpha_2)^{N-m}(N\alpha_2+1)}{\alpha_2}\right]\Bigg\}
\end{aligned} \tag{4.2.5}$$

It is an important fact that the above expression is a nonlinear function of the control parameters and therefore bounds may be provided for C_f and it is hoped that the bounds may free from m, the number samples after changing the control limits. In fact, this can be done and experiments have shown that the lower and upper bounds are convenient for hand calculation and the result is the following lemma:

Lemma 4.2

The cost of false alarm has upper and lower bounds which are functions of type I error probabilities and the control interval, and are free of m. That is:

$$c_f\left[\alpha_1 - \frac{h}{\theta}\left(\frac{1}{\alpha_1}\right)\right] \le C_f \le c_f\left[1 + \frac{h}{\theta}\left(\frac{1}{\alpha_2}\right)\right] \tag{4.2.6}$$

The proof is in Appendix C(4).

(3) True signal cost = C_t = (unit cost) (probability of true signal) = c_t P[true signal]

and the probability of a true signal is:

$$\begin{aligned}
P[B] &= \sum_j P[B_1]\,P[B_2] \\
&= \sum_{j=1}^{m+1}\left[F(jh) - F((j-1)h)\right](1-\alpha_1)^{j-1} + \sum_{j=m+2}^{N}\left[F(jh) - F((j-1)h)\right](1-\alpha_2)^{j-1} \\
&= \frac{h}{\theta}\sum_{j=1}^{m+1}(1-\alpha_1)^{j-1} + \frac{h}{\theta}\sum_{j=m+2}^{N}(1-\alpha_2)^{j-1} \\
&= \frac{h}{\theta}\left[\frac{1-(1-\alpha_1)^{m+1}}{\alpha_1} + \frac{(1-\alpha_2)^{m+1}\left[1-(1-\alpha_2)^{N-m-1}\right]}{\alpha_2}\right]
\end{aligned} \tag{4.2.7}$$

Therefore:

$$C_t = c_t P[B] = \frac{c_t h}{\theta}\left[\frac{1-(1-\alpha_1)^{m+1}}{\alpha_1} + \frac{(1-\alpha_2)^{m+1}\left[1-(1-\alpha_2)^{N-m-1}\right]}{\alpha_2}\right] \tag{4.2.8}$$

Note that equation (4.2.8) is a nonlinear function of control parameters and therefore bounds may be provided for C_t. It is hoped that the bounds may free from m, the number samples after changing the control limits. It has been shown that the so constructed lower and upper bounds are convenient for a small workshop and the result is the following lemma:

Lemma 4.3

The cost of true signal has upper and lower bounds which are functions of type I error probabilities and the control interval, and they are free from m. That is:

$$\frac{c_t h}{\theta}\left(\frac{\alpha_1}{\alpha_2^2}\right) \leq C_t \leq \frac{c_t h}{\theta}\left(\frac{1}{\alpha_1}\right) \tag{4.2.9}$$

The proof is in Appendix C(5).

(4) Cost of producing non-conforming items when the process is out of control = C_d. As described in Chapter 3, E[time out of control] = $E[t_p] + E[t_s]$. $E[t_p] = h - E[T]$. The problem is again to construct E[T] but the analysis is the same as for the conventional case. Thus $E[t_p] = h/2$.

The construction of $E[t_s]$ is a bit different as the identity of the interval in which the shift occurs affects the signal probability.

E[time in full intervals until a true signal] = $E[t_s]$ = h P[no signal at (j+1)h] ARL(μ_2). It is actually the ARL that depends on the time of shift and the signal probability. In our analysis, three cases are considered:

(a) Shift during $j \leq m$ and signal at $i \leq m$;

(b) Shift during $j \leq m$ and signal at $i > m$;

(c) Shift during $j > m$.

For case (a), the equation of $E[t_S(a)]$ has the form:

$$
\begin{aligned}
E[t_s(a)] &= \sum_{j=1}^{m}\left[F(jh) - F((j-1)h)\right]\sum_{i=j}^{m}(i-j)\,h(1-\beta_1)\beta_1^{i-j} \\
&= \frac{h^2\beta_1(1-\beta_1)}{\theta}\sum_{j=1}^{m}\left(\sum_{i=j}^{m}(i-j)\,\beta_1^{i-j-1}\right) \\
&= \frac{h^2\beta_1}{\theta}\left[\frac{m}{1-\beta_1} - \frac{\beta_1^m(1-m)}{1-\beta_1} - \frac{1-\beta_1-2\beta_1^m}{(1-\beta_1)^2}\right]
\end{aligned}
\qquad (4.2.10)
$$

The reduction to this form is in Appendix C(6).

For case (b), the method used for obtaining equation of $E[t_S(b)]$ is quite the same as that used in case (a) except that the range of the summation changes. The expression for $E[t_S(b)]$ is:

$$
\begin{aligned}
E[t_s(b)] &= \sum_{j=1}^{m}\left[F(jh) - F((j-1)h)\right]\beta_1^{m-j+1}\sum_{i=m+1}^{N}(i-j)\,h\beta_2^{i-m-1}(1-\beta_2) \\
&= \frac{h^2\beta_1}{\theta}\left\{\left[\frac{1-\beta_1^m}{1-\beta_1}\right]\left[m + \frac{1-\beta_2^{N-m}}{1-\beta_2} - N\beta_2^{N-m}\right] - \left[\frac{1-\beta_2^{N-m}}{1-\beta_1}\right]\left[m - \frac{\beta_1-\beta_1^m}{1-\beta_1} - \beta_1^m\right]\right\}
\end{aligned}
\qquad (4.2.11)
$$

The resolution of this form is shown in Appendix C(7).

For case (c), the type II error probability β_2 is the key factor of $E[t_s(c)]$ because the average run length after changing the control limits is determined by β_2. The form of $E[t_s(c)]$ is :

$$E[t_s(c)] = [1 - F(mh)]\, h\beta_2 \frac{1}{1-\beta_2} = h\left(\frac{\beta_2}{1-\beta_2}\right) - \frac{h^2}{\theta}\left(\frac{m\beta_2}{1-\beta_2}\right) \tag{4.2.12}$$

Finally, combining the three cases yields:

$$\begin{aligned} E[t_s] &= E[t_s(a)] + E[t_s(b)] + E[t_s(c)] \\ &= h\left(\frac{\beta_2}{1-\beta_2}\right) + \frac{h^2}{\theta}\left[\frac{m\beta_2}{1-\beta_2}\right] + \frac{h^2\beta_1}{\theta}\left[\frac{m}{1-\beta_1} - \frac{\beta_1^m(1-m)}{1-\beta_1} - \frac{1-\beta_1-2\beta_1^m}{(1-\beta_1)^2}\right] \\ &\quad + \frac{h^2\beta_1}{\theta}\left\{\left[\frac{1-\beta_1^m}{1-\beta_1}\right]\left[m + \frac{1-\beta_2^{N-m}}{1-\beta_2} - N\beta_2^{N-m}\right] - \left[\frac{1-\beta_2^{N-m}}{1-\beta_1}\right]\left[m - \frac{\beta_1-\beta_1^m}{1-\beta_1} - \beta_1^m\right]\right\} \end{aligned} \tag{4.2.13}$$

Therefore we obtain the expression for C_d below and this form is nonlinear function of the control parameters. That is:

$$\begin{aligned} C_d &= c_d rp\{E[t_p] + E[t_s]\} = c_d rph\left(\frac{1}{2} + \frac{\beta_2}{1-\beta_2}\right) + \frac{c_d rph^2}{\theta}\left[\frac{m\beta_2}{1-\beta_2}\right] \\ &\quad + \frac{c_d rph^2\beta_1}{\theta}\left\{\left[\frac{1-\beta_1^m}{1-\beta_1}\right]\left[m + \frac{1-\beta_2^{N-m}}{1-\beta_2} - N\beta_2^{N-m}\right] - \left[\frac{1-\beta_2^{N-m}}{1-\beta_1}\right]\left[m - \frac{\beta_1-\beta_1^m}{1-\beta_1} - \beta_1^m\right]\right\} \\ &\quad + \frac{c_d rph^2\beta_1}{\theta}\left[\frac{m}{1-\beta_1} - \frac{\beta_1^m(1-m)}{1-\beta_1} - \frac{1-\beta_1-2\beta_1^m}{(1-\beta_1)^2}\right] \end{aligned} \tag{4.2.14}$$

(5) The expected cycle length must also reflect differences in signal events before and after mh. The expected cycle length can be written as:

$$E[t] = \sum_{j} ihE[E_1] + \sum_{j} jh \sum_{l} E[E_2] \tag{4.2.15}$$

To start, the following notation is employed:

$$E[t_f] = \sum_{j} jhE[E_1] \tag{4.2.16}$$

and:

$$E[t_n] = \sum_{j} jhE[E_2] \tag{4.2.17}$$

Then, $E[t_f]$ can be written as the following form and the detail derivation of this expression is shown in Appendix C(8).

$$\begin{aligned} E[t_f] &= \sum_{i=1}^{m} ih\alpha_1(1-\alpha_1)^{i-1}[1-F(ih)] + (1-\alpha_1)^m \sum_{i=m+1}^{N} ih\alpha_2(1-\alpha_2)^{i-m-1}[1-F(ih)] \\ &= \alpha_1 h\left[\frac{1-(1-\alpha_1)^m}{\alpha_1^2} - \frac{m(1-\alpha_1)^m}{\alpha_1}\right] \\ &\quad - \frac{\alpha_1 h^2}{\theta}\left[\frac{1-(m-1)^2(1-\alpha_1)^m + m^2(1-\alpha_1)^{m+1}}{\alpha_1^2} + \frac{2(1-\alpha_1)\left[1-(1-\alpha_1)^{m-1}\right]}{\alpha_1^3}\right] \\ &\quad + \frac{\alpha_2 h^2(1-\alpha_1)^m}{\theta}\left\{\frac{1}{\alpha_2^2} + \frac{2(1-\alpha_2)\left[1-(1-\alpha_2)^{N-m-1}\right]}{\alpha_2^3}\right. \\ &\quad \left. - \frac{(N-m-1)^2(1-\alpha_2)^{N-m}}{\alpha_2^2} + \frac{(N-m)^2(1-\alpha_2)^{N-m+1}}{\alpha_2^2}\right. \end{aligned} \tag{4.2.18}$$

$$+2m\left[\frac{1-(1-\alpha_2)^{N-m}}{\alpha_2^2}-\frac{(N-m)(1-\alpha_2)^{N-m}}{\alpha_2}\right]+\frac{m^2\left[1-(1-\alpha_2)^{N-m}\right]}{\alpha_2}\Bigg\}$$

$$+\alpha_2 h(1-\alpha_1)^m\left[\frac{m}{\alpha_2}-\frac{N(1-\alpha_2)^{N-m}}{\alpha_2}+\frac{1-(1-\alpha_2)^{N-m}}{\alpha_2^2}\right] \qquad (4.2.19)$$

To evaluate $E[t_n]$, we must consider three cases:

Case (a) $s \le m, j \le m$;

Case (b) $s \le m, j > m$;

Case (c) $s > m$.

For case (a):

$$E[t_n(a)] = \sum_{s=1}^{m}[F(sh)-F((s-1)h)](1-\alpha_1)^{s-1}\sum_{j=s}^{m} jh(1-\beta_1)\beta_1^{j-s}$$

$$= \frac{(1-\beta_1)h^2}{\theta}\sum_{s=1}^{m}(1-\alpha_1)^{s-1}\sum_{j=s}^{m} j\,\beta_1^{j-s}$$

$$= \frac{h^2}{\theta}\left\{\frac{1-(1-\alpha_1)^m}{\alpha_1^2}-\frac{m(1-\alpha_1)^m}{\alpha_1}-\left[m+\frac{1}{1-\beta_1}\right]\left[\frac{\beta_1\left[(1-\alpha_1)^m-\beta_1^m\right]}{1-\alpha_1-\beta_1}\right]\right\}$$

$$+\frac{h^2}{\theta}\left\{\frac{\beta_1}{1-\beta_1}\left[\frac{(1-\alpha_1)^m}{\alpha_1}\right]\right\} \qquad (4.2.20)$$

where the details of the analysis are shown in Appendix C(9).

For case (b), the form of $E[t_n(b)]$ is:

$$E[t_n(b)] = \sum_{s=1}^{m}[F(sh)-F((s-1)h)](1-\alpha_1)^{s-1}\beta_1^{m-s+1}\sum_{j=m+1}^{N} jh(1-\beta_2)\beta_2^{j-m-1}$$

$$= \frac{h^2}{\theta}\left[m - N\beta_2^{N-m} + \frac{1-\beta_2^{N-m}}{1-\beta_2}\right]\left\{\frac{\beta_1\left[(1-\alpha_1)^m - \beta_1^m\right]}{1-\alpha_1-\beta_1}\right\} \tag{4.2.21}$$

For case (c), $E[t_n(c)]$ is:

$$E[t_n(c)] = (1-\alpha_1)^m \sum_{s=m+1}^{N}[F(sh)-F((s-1)h)](1-\alpha_2)^{s-m-1}\sum_{j=s}^{N} jh(1-\beta_2)\beta_2^{j-s}$$

$$= \frac{(1-\alpha_1)^m h^2}{\theta}\left\{\frac{m-N(1-\alpha_2)^{N-m}}{\alpha_2} + \frac{1-(1-\alpha_2)^{N-m}}{\alpha_2^2} - N\beta_2\left[\frac{(1-\alpha_2)^{N-m}-\beta_2^{N-m}}{1-\alpha_2-\beta_2}\right]\right.$$
$$\left. + \frac{1}{(1-\beta_2)}\left[\frac{\beta_2\left(1-(1-\alpha_2)^{N-m}\right)}{\alpha_2} - \frac{(1-\alpha_2)^{N-m}-\beta_2^{N-m}}{1-\alpha_2-\beta_2}\right]\right\} \tag{4.2.22}$$

The detail derivations of (4.2.21) and (4.2.22) are in Appendix (A.10) and (A.11) and combine the results from the three cases above, the expression for $E[t_n]$ is:

$$E[t_n] = \frac{h^2}{\theta}\left\{\frac{1-(1-\alpha_1)^m}{\alpha_1^2} - \frac{m(1-\alpha_1)^m}{\alpha_1} - \left[m + \frac{1}{1-\beta_1}\right]\left[\frac{\beta_1\left[(1-\alpha_1)^m-\beta_1^m\right]}{1-\alpha_1-\beta_1}\right]\right.$$
$$\left. + \frac{\beta_1}{1-\beta_1}\left[\frac{(1-\alpha_1)^m}{\alpha_1}\right]\right\} + \frac{h^2}{\theta}\left[m - N\beta_2^{N-m} + \frac{1-\beta_2^{N-m}}{1-\beta_2}\right]\left\{\frac{\beta_1\left[(1-\alpha_1)^m-\beta_1^m\right]}{1-\alpha_1-\beta_1}\right\}$$
$$+ \frac{(1-\alpha_1)^m h^2}{\theta}\left\{\frac{m-N(1-\alpha_2)^{N-m}}{\alpha_2} + \frac{1-(1-\alpha_2)^{N-m}}{\alpha_2^2} - N\beta_2\left[\frac{(1-\alpha_2)^{N-m}-\beta_2^{N-m}}{1-\alpha_2-\beta_2}\right]\right.$$
$$\left. + \frac{1}{(1-\beta_2)}\left[\frac{\beta_2\left(1-(1-\alpha_2)^{N-m}\right)}{\alpha_2} - \frac{(1-\alpha_2)^{N-m}-\beta_2^{N-m}}{1-\alpha_2-\beta_2}\right]\right\} \tag{4.2.23}$$

And with the definition of E[t_f] (4.2.19), the expected cycle length is:

$$
\begin{aligned}
E[t] = {} & \alpha_1 h\left[\frac{1-(1-\alpha_1)^m}{\alpha_1^2}-\frac{m(1-\alpha_1)^m}{\alpha_1}\right] \\
& -\frac{\alpha_1 h^2}{\theta}\left[\frac{1-(m-1)^2(1-\alpha_1)^m+m^2(1-\alpha_1)^{m+1}}{\alpha_1^2}+\frac{2(1-\alpha_1)\left[1-(1-\alpha_1)^{m-1}\right]}{\alpha_1^3}\right] \\
& +\alpha_2 h(1-\alpha_1)^m\left[\frac{m}{\alpha_2}-\frac{N(1-\alpha_2)^{N-m}}{\alpha_2}+\frac{1-(1-\alpha_2)^{N-m}}{\alpha_2^2}\right] \\
& +\frac{\alpha_2 h^2(1-\alpha_1)^m}{\theta}\Bigg\{\frac{1}{\alpha_2^2}+\frac{2(1-\alpha_2)\left[1-(1-\alpha_2)^{N-m-1}\right]}{\alpha_2^3}-\frac{(N-m-1)^2(1-\alpha_2)^{N-m}}{\alpha_2^2} \\
& +\frac{(N-m)^2(1-\alpha_2)^{N-m+1}}{\alpha_2^2} \\
& +2m\left[\frac{1-(1-\alpha_2)^{N-m}}{\alpha_2^2}-\frac{(N-m)(1-\alpha_2)^{N-m}}{\alpha_2}\right]+\frac{m^2\left[1-(1-\alpha_2)^{N-m}\right]}{\alpha_2}\Bigg\} \\
& +\frac{h^2}{\theta}\Bigg\{\frac{1-(1-\alpha_1)^m}{\alpha_1^2}-\frac{m(1-\alpha_1)^m}{\alpha_1}-\left[m+\frac{1}{1-\beta_1}\right]\left[\frac{\beta_1\left[(1-\alpha_1)^m-\beta_1^m\right]}{1-\alpha_1-\beta_1}\right] \\
& +\frac{\beta_1}{1-\beta_1}\left[\frac{(1-\alpha_1)^m}{\alpha_1}\right]\Bigg\}+\frac{h^2}{\theta}\left[m-N\beta_2^{N-m}+\frac{1-\beta_2^{N-m}}{1-\beta_2}\right]\left\{\frac{\beta_1\left[(1-\alpha_1)^m-\beta_1^m\right]}{1-\alpha_1-\beta_1}\right\} \\
& +\frac{(1-\alpha_1)^m h^2}{\theta}\Bigg\{\frac{m-N(1-\alpha_2)^{N-m}}{\alpha_2}+\frac{1-(1-\alpha_2)^{N-m}}{\alpha_2^2}-N\beta_2\left[\frac{(1-\alpha_2)^{N-m}-\beta_2^{N-m}}{1-\alpha_2-\beta_2}\right] \\
& +\frac{1}{(1-\beta_2)}\left[\frac{\beta_2\left(1-(1-\alpha_2)^{N-m}\right)}{\alpha_2}-\frac{(1-\alpha_2)^{N-m}-\beta_2^{N-m}}{1-\alpha_2-\beta_2}\right]\Bigg\}
\end{aligned}
\tag{4.2.24}
$$

Thus, the expected total cost per unit time for the generalized Shewhart control chart is well defined. The objective is to choose the values of the decision variables to minimize the expected total cost per unit time.

Lemma 4.4

The expected total cost per unit time function for the generalized Shewhart control chart is equivalent to that for the standard Shewhart control chart under Fact 4.2.

The proof of this lemma is in Appendix C(12).

CHAPTER 5

Development of Cost Model III

5.1 Cost Model III for the Standard Shewhart Chart

5.1.1 Introduction and Assumptions

Assume again a process is monitored using the same control chart and the process is subject to a shift from the in control value of the process mean μ_1 to a single out of control value μ_2 at a random point in time. Assume the time until a process shift is a random variable with distribution $F(t)$. Assume also that we plan to use the strategy of starting with a set of control limits and to change the limits to be tighter after the process has operated for a period of time that is to be determined. Specifically, the process is sampled every h hours and after the m^{th} sample the control limits are changed.

The question is to select the control limits to use before and after sample m and to choose a value for m. The basis of choice will be a model for the cost of operating the chart. To start, the cost of operating a conventional (standard) chart is defined. The resulting model is then modified to reflect the consequences of the strategy of changing the control limits.

The cost categories considered are the same as in chapter 3 and the expected total cost per unit time function is defined as in equation (3.1.1). The decision variables are n, h, m, k_1 and k_2. The objective is to choose the values of the decision variables to minimize the expected total cost per unit time.

5.1.2 Model Development

Suppose in this section that the distribution of the time to a process shift is Weibull:

$$F(t) = 1 - e^{-at^b} \tag{5.1.1}$$

Where a is the scale parameter, a > 0, b is the shape parameter, b>0.

(1) Inspection cost = C_i={fixed cost + (unit cost)(number inspected)}/{time between samples}. Then:

$$C_i = \{c_0 + nc_i\} / h \tag{5.1.2}$$

(2) False alarm cost = C_f = (unit cost)(probability of false alarm)= c_fP[false alarm] Let A="false alarm", A_1 = "false alarm on sample i", A_2 ="no process shift before sample i", then:

$$P[A] = \sum_i P[A_1]\,P[A_2] = \sum_{i=1}^{\infty} \alpha(1-\alpha)^{i-1}(1-F(ih)) = \alpha\sum_{i=1}^{\infty}(1-\alpha)^{i-1}e^{-a(ih)^b} \qquad (5.1.3)$$

Thus

$$C_f = c_f P[A] = c_f \alpha \sum_{i=1}^{\infty}(1-\alpha)^{i-1}e^{-a(ih)^b} \qquad (5.1.4)$$

(3) True signal cost = C_t = (unit cost)(probability of a true signal)= c_tP[true signal] (note that once a shift has occurred, the probability is 1.0 that a true signal will occur). Let B="true signal", B_1="process shift in interval j", B_2="no false alarm on proceeding j-1 samples", then the probability of the process terminating with a true signal can be constructed as:

$$\begin{aligned} P[B] &= \sum_j P[B_1]\,P[B_2] \\ &= \sum_{j=1}^{\infty}\left[F(jh) - F((j-1)h)\right](1-\alpha)^{j-1} = \sum_{j=1}^{\infty}\left[e^{-a(j-1)^b h^b} - e^{-aj^b h^b}\right](1-\alpha)^{j-1} \end{aligned} \qquad (5.1.5)$$

Therefore:

$$C_t = c_t\left\{\sum_{j=1}^{\infty}\left[e^{-a(j-1)^b h^b} - e^{-aj^b h^b}\right](1-\alpha)^{j-1}\right\} \qquad (5.1.6)$$

(4) Cost of producing non-conforming items when the process is out of control = C_d= (unit cost)(production rate)(increase in proportion non-conforming)(expected time out of control).

After a process shift, the time that the process is out of control is impressed of the sampling interval during which the shift occurs and the full interval that elapse before a signal. To determine the expected value of this period, let t_p represent the partial interval during which the process is out of control and let T be the part of interval before process shift. (Figure 3.1.1). Then t_p= h-T and $E[t_p]$= h- E[T]. The problem is therefore to construct E[T]. If the shift occurs in interval (jh,(j+1)h), then the construction of E[T] is:

$$T = t - jh \Rightarrow E[T] = E[t - jh] = \int (t - jh) f\left(t \middle| jh \le t \le (j+1)h\right) dt \tag{5.1.7}$$

And we know that:

$$F\left(t \middle| jh \le t \le (j+1)h\right) = \frac{F(t) - F(jh)}{F((j+1)h) - F(jh)}$$
$$f\left(t \middle| jh \le t \le (j+1)h\right) = \frac{f(t)}{F((j+1)h) - F(jh)} \tag{5.1.8}$$

This is the conditional probability density function of the random variable t given that the random variable t falls in the interval (jh, (j+1)h). Since the random variable t denotes the time that the process goes out of control, the above conditional density function represents the probability that the process goes out of control given the shift occurs in the time interval (jh, (j+1)h). Therefore:

$$E[T] = E[t - jh] = \int (t - jh) f(t | jh \le t \le (j+1)h) dt = \frac{\int_{jh}^{(j+1)h} (t - jh) f(t) dt}{[F(j+1)h - F(jh)]}$$

$$= \frac{\int_{jh}^{(j+1)h} abt^{b-1}(t - jh) e^{-at^b} dt}{[F(j+1)h - F(jh)]}$$

$$= \frac{ab}{e^{aj^b h^b} - e^{-a(j+1)^b h^b}} \left[\int_{jh}^{(j+1)h} t^b e^{-at^b} dt - jh \int_{jh}^{(j+1)h} t^{b-1} e^{-at^b} dt \right] \tag{5.1.9}$$

And finally:

$$E[t_p] = h - E[t] = h - \frac{ab}{e^{aj^b h^b} - e^{-a(j+1)^b h^b}} \left[\int_{jh}^{(j+1)h} t^b e^{-at^b} dt - jh \int_{jh}^{(j+1)h} t^{b-1} e^{-at^b} dt \right] \tag{5.1.10}$$

This is the expected length of partial interval during which the process is out of control. Note that the above equation is a function of control interval and parameter a and b.

E[time in full interval until a true signal] = $E[t_S]$ = h P[no signal at (j+1)h] ARL(μ_2), and so $E[t_S] = h\ \beta\ (1-\beta)^{-1}$, thus if D represents the time out of control, the expected time out of control is:

$$E[D] = E[t_p] + E[t_s]$$

$$= h + \frac{h\beta}{1-\beta} - \frac{ab}{e^{aj^b h^b} - e^{-a(j+1)^b h^b}} \left[\int_{jh}^{(j+1)h} t^b e^{-at^b} dt - jh \int_{jh}^{(j+1)h} t^{b-1} e^{-at^b} dt \right] \tag{5.1.11}$$

Thus the cost of producing non-conforming items when the process is out of control, that is C_d, may also be obtained and it has the following form:

$$C_d = c_d rph + c_d rph\left[\frac{\beta}{1-\beta}\right] - \frac{c_d rpab}{e^{aj^b h^b} - e^{-a(j+1)^b h^b}}\left[\int_{jh}^{(j+1)h} t^b e^{-at^b} dt - jh\int_{jh}^{(j+1)h} t^{b-1} e^{-at^b} dt\right] \tag{5.1.12}$$

(5) To determine the expected cycle length, let E_1="false alarm on sample j and no process shift before sample j", E_2="process shift during interval l, no false alarm before interval l, and true signal on sample j (j-l+1st after shift)", E[t]= expected cycle length. Therefore:

$$E[t] = \sum_j jhP[E_1] + \sum_j jh\sum_l P[E_2] \tag{5.1.13}$$

P[false alarm on sample j and no process shift before sample j] = $a(1-a)^{j-1}(1-F(jh))$

$$= \alpha(1-\alpha)^{j-1} e^{-aj^b h^b}$$

P[process shift during interval l] = F(lh)-F((l-1)h)

$$= e^{-a(l-1)^b h^b} - e^{-al^b h^b}$$

P[no false alarm before interval l] = $(1-\alpha)^{l-1}$

P[true signal on sample j] = $\beta^{j-l}(1-\beta)$

Now, the expected cycle length, E[t], has the following form and the reduction to this form is in Appendix D(1):

$$\begin{aligned} E[t] &= \sum_{j=1}^{\infty} jh\left(\alpha(1-\alpha)^{j-1} e^{-aj^b h^b} + \sum_{l=1}^{j}\left(e^{-a(l-1)^b h^b} - e^{-al^b h^b}\right)(1-\alpha)^{l-1}\beta^{j-l}(1-\beta)\right) \\ &= \alpha h\sum_{j=1}^{\infty}(1-\alpha)^{j-1} e^{-aj^b h^b} + h\sum_{l=1}^{\infty} l\, e^{-a(l-1)^b h^b}(1-\alpha)^{l-1} - h\sum_{l=1}^{\infty} l\, e^{-al^b h^b}(1-\alpha)^{l-1} \\ &\quad + h\left(\frac{\beta}{1-\beta}\right)\sum_{l=1}^{\infty} e^{-a(l-1)^b h^b}(1-\alpha)^{l-1} - h\left(\frac{\beta}{1-\beta}\right)\sum_{l=1}^{\infty} e^{-al^b h^b}(1-\alpha)^{l-1} \end{aligned} \tag{5.1.14}$$

The objective is to choose the values of the decision variables to minimize the expected total cost per unit time function (3.1.1).

5.2 Cost Model III for the Generalized Shewhart Chart

5.2.1 Introduction and Assumptions

Suppose that a control chart is used to monitor a process and that samples are taken every h units of time. Suppose further that the width of the control limits are changed after the m^{th} sample. Then the following facts may be used to describe control chart behavior:

Fact 5.1

(1) $\alpha(t) = \alpha_1$ if $t \leq mh$

$= \alpha_2$ if $t > mh$

(2) $\beta(t) = \beta_1$ if $t \leq mh$

$= \beta_2$ if $t > mh$

(3) $k(t) = k_1$ if $t \leq mh$

$= k_2$ if $t > mh$ (5.2.1)

Fact 5.2

The generalized Shewhart control chart is the same as the standard Shewhart control chart if and only if:

$\alpha_1 = \alpha_2 = \alpha;$

$\beta_1 = \beta_2 = \beta;$

$k_1 = k_2 = k.$ (5.2.2)

The question is again to select the control limits to use before and after sample m and to choose a value for m. The basis of choice is a model for the cost of operating the chart. The basic cost model III developed in section 5.1.2 is modified to reflect the consequences of the strategy of changing the control limits. The results is referred to a general cost model III.

5.2.2 Model Development

As discussed in the previous section, the construction of cost model III for generalized Shewhart control chart can be based on the same cost categories. The development proceeds as:

(1) Inspection cost: Since the change of control limits does not change the form of the inspection cost, the inspection cost remains:

$$C_i = \{c_0+nc_i\}/h \tag{5.2.3}$$

(2) False alarm cost = C_f = (unit cost)(probability of false alarm) = c_f P[false alarm]. Therefore the problem again is to construct the expression for the probability of a false alarm. The expression of the probability of a false alarm in this case is quite different from that in the standard control chart since we have to consider $t \leq mh$ or $t > mh$ separately. The probability of false alarm can be constructed as the following steps:

(1) Find the partial summation from $i = 1$ to $i = m$;

(2) Find the partial summation from $i = m+1$ to $i = \infty$;

(3) Construct the probability of false alarm by adding these two partial summations together.

$$
\begin{aligned}
P[A] &= \sum_i P[A_1]\,P[A_2] \\
&= \sum_{i=1}^{m} \alpha_1 (1-\alpha_1)^{i-1} (1 - F(ih)) + (1-\alpha_1)^m \sum_{i=m+1}^{\infty} \alpha_2 (1-\alpha_2)^{i-m-1} (1 - F(ih)) \\
&= \alpha_1 \sum_{i=1}^{m} (1-\alpha_1)^{i-1} e^{-ai^b h^b} + (1-\alpha_1)^m \alpha_2 \sum_{i-m=1}^{\infty} (1-\alpha_2)^{i-m-1} e^{-ai^b h^b}
\end{aligned} \tag{5.2.4}
$$

Therefore:

$$
C_f = c_f P[A] = c_f \left\{ \alpha_1 \sum_{i=1}^{m} (1-\alpha_1)^{i-1} e^{-ai^b h^b} + (1-\alpha_1)^m \alpha_2 \sum_{i-m=1}^{\infty} (1-\alpha_2)^{i-m-1} e^{-ai^b h^b} \right\} \tag{5.2.5}
$$

(3) True signal cost = C_t = (unit cost) (probability of true signal) = c_t P[true signal]. Note again that once a shift has occurred, the probability is 1.0 that a true signal will occur. Therefore the construction of the cost of true signal proceeds as follows:

$$
\begin{aligned}
P[B] &= \sum_j P[B_1]\,P[B_2] \\
&= \sum_{j=1}^{m+1} [F(jh) - F(j-1)h](1-\alpha_1)^{j-1} + \sum_{j=m+2}^{\infty} [F(jh) - F(j-1)h](1-\alpha_2)^{j-1} \\
&= \sum_{j=1}^{m+1} \left[e^{-a(j-1)^b h^b} - e^{-aj^b h^b} \right] (1-\alpha_1)^{j-1} + \sum_{j=m+2}^{\infty} \left[e^{-a(j-1)^b h^b} - e^{-aj^b h^b} \right] (1-\alpha_2)^{j-1}
\end{aligned} \tag{5.2.6}
$$

Therefore we obtain the expression of C_t:

$$
\begin{aligned}
C_t &= c_t P[B] \\
&= c_t \left\{ \sum_{j=1}^{m+1} \left[e^{-a(j-1)^b h^b} - e^{-aj^b h^b} \right] (1-\alpha_1)^{j-1} + \sum_{j=m+2}^{\infty} \left[e^{-a(j-1)^b h^b} - e^{-aj^b h^b} \right] (1-\alpha_2)^{j-1} \right\}
\end{aligned} \tag{5.2.7}
$$

(4) Cost of producing non-conforming items when the process is out of control = C_d. Let t = time that the process goes out of control, then E[time out of control] = $E[t_p] + E[t_S]$. Since $E[t_p] = h - E[T]$. The problem is again to construct E[T] but the analysis is the same as that for the standard Shewhart control chart. Thus:

$$E[t_p] = h - E[t] = h - \frac{ab}{e^{aj^b h^b} - e^{-a(j+1)^b h^b}}\left[\int_{jh}^{(j+1)h} t^b e^{-at^b}dt - jh\int_{jh}^{(j+1)h} t^{b-1}e^{-at^b}dt\right] \tag{5.2.8}$$

The construction of $E[t_S]$ is a bit different as the identity of the interval in which the shift occurs affects the signal probability. Let E[time in full intervals until a true signal] = $E[t_S]$ = h P[no signal at (j+1)h] ARL(μ_2). It is actually the ARL that depends on the time of shift and the signal probability. In our analysis, three cases are considered:

(1) Shift during $j \leq m$ and signal at $i \leq m$;

(2) Shift during $j \leq m$ and signal at $i > m$;

(3) Shift during $j > m$.

For case (1), the expression for $E[t_S(1)]$ is:

$$\begin{aligned} E[t_s(1)] &= \sum_{j=1}^{m}\left[F(jh) - F((j-1)h)\right]\sum_{i=j}^{m}(i-j)h(1-\beta_1)\beta_1^{i-j} \\ &= \frac{h\beta_1}{1-\beta_1}\left[1 - e^{-a(mh)^b}\right] - h\sum_{j=1}^{m}\left[e^{-a(jh-h)^b} - e^{-a(jh)^b}\right](m-j)\beta_1^{m-j+1} \\ &\quad - \frac{h}{1-\beta_1}\sum_{j=1}^{m}\left[e^{-a(jh-h)^b} - e^{-a(jh)^b}\right]\beta_1^{m-j+1} \end{aligned} \tag{5.2.9}$$

The detail derivation of this expression is in Appendix D(2).

For case (2), the expression for $E[t_S(2)]$ is below and the reduction to this form is in Appendix D(3).

$$
\begin{aligned}
E[t_s(2)] &= \sum_{j=1}^{m}\left[F(jh) - F((j-1)h)\right] \beta_1^{m-j+1} \sum_{i=m+1}^{\infty}(i-j)\, h\beta_2^{i-m-1}(1-\beta_2) \\
&= h\sum_{j=1}^{m}\left[e^{-a(jh-h)^b} - e^{-a(jh)^b}\right](m-j)\beta_1^{m-j+1} \\
&+ \frac{h}{1-\beta_2}\sum_{j=1}^{m}\left[e^{-a(jh-h)^b} - e^{-a(jh)^b}\right]\beta_1^{m-j+1}
\end{aligned}
\qquad (5.2.10)
$$

For case (3), the expression for $E[t_S(3)]$ is:

$$
E[t_s(3)] = [1 - F(mh)]h\beta_2 \frac{1}{1-\beta_2} = \frac{h\beta_2 e^{-am^b h^b}}{1-\beta_2} \qquad (5.2.11)
$$

Finally, combining the three cases yields:

$$
\begin{aligned}
E[t_s] &= \frac{h\beta_1}{1-\beta_1} + he^{-a(mh)^b}\left[\frac{\beta_2}{1-\beta_2} - \frac{\beta_1}{1-\beta_1}\right] \\
&+ h\left[\frac{1}{1-\beta_2} - \frac{1}{1-\beta_1}\right]\sum_{j=1}^{m}\left[e^{-a(jh-h)^b} - e^{-a(jh)^b}\right]\beta_1^{m-j+1}
\end{aligned}
\qquad (5.2.12)
$$

Therefore:

$$
\begin{aligned}
C_d &= c_d rp\left\{E[t_p] + E[t_s]\right\} = c_d rph\left[\frac{\beta_1}{1-\beta_1}\right] + c_d rph + c_d rphe^{-a(mh)^b}\left[\frac{\beta_2}{1-\beta_2} - \frac{\beta_1}{1-\beta_1}\right] \\
&- \frac{c_d rpab}{e^{aj^b h^b} - e^{-a(j+1)^b h^b}}\left[\int_{jh}^{(j+1)h} t^b e^{-at^b}dt - jh\int_{jh}^{(j+1)h} t^{b-1}e^{-at^b}dt\right] \\
&+ c_d rph\left\{\left[\frac{1}{1-\beta_2} - \frac{1}{1-\beta_1}\right]\sum_{j=1}^{m}\left[e^{-a(jh-h)^b} - e^{-a(jh)^b}\right]\beta_1^{m-j+1}\right\}
\end{aligned}
\qquad (5.2.13)
$$

(5) The expected cycle length is different from that of the cost model III for the standard Shewhart control chart. We must again distinguish between events before and after m. The expected cycle length can be written as:

$$E[t] = \sum_j jhE[E_1] + \sum_j jh \sum_l E[E_2] \tag{5.2.14}$$

To start, let:

$$E[t_f] = \sum_j jhE[E_1] \tag{5.2.15}$$

Then, $E[t_f]$ can be written as the following expression:

$$\begin{aligned} E[t_f] &= \sum_{j=1}^{m} (1-\alpha_1)^{j-1} \alpha_1 [1-F(jh)] + (1-\alpha_1)^m \sum_{j=m+1}^{\infty} (1-\alpha_2)^{j-m-1} \alpha_2 [1-F(jh)] \\ &= \alpha_1 \sum_{j=1}^{m} (1-\alpha_1)^{j-1} e^{-aj^b h^b} + (1-\alpha_1)^m \alpha_2 \sum_{j=m+1}^{\infty} (1-\alpha_2)^{j-m-1} e^{-aj^b h^b} \end{aligned} \tag{5.2.16}$$

Next, let:

$$E[t_n] = \sum_j jh \sum_l E[E_2] \tag{5.2.17}$$

Three cases must be considered:

Case (1) $s \le m, j \le m$;

Case (2) $s \le m, j > m$;

Case (3) $s > m$.

For case (1), $E[t_n(1)]$ has the form (Appendix D (4)):

$$
\begin{aligned}
E[t_n(1)] &= \sum_{s=1}^{m}\left[F(sh)-F((s-1)h)\right](1-\alpha_1)^{s-1}\sum_{j=s}^{m} jh(1-\beta_1)\beta_1^{j-s} \\
&= h\sum_{s=1}^{m} s\left[e^{-a((s-1)h)^b}-e^{-a(sh)^b}\right](1-\alpha_1)^{s-1} \\
&\quad -h\left[m+\frac{1}{1-\beta_1}\right]\sum_{s=1}^{m}\left[e^{-a((s-1)h)^b}-e^{-a(sh)^b}\right](1-\alpha_1)^{s-1}\beta_1^{m-s+1} \\
&\quad +\frac{h\beta_1}{1-\beta_1}\sum_{s=1}^{m}\left[e^{-a((s-1)h)^b}-e^{-a(sh)^b}\right](1-\alpha_1)^{s-1}
\end{aligned}
\tag{5.2.18}
$$

For case (2), the detail derivation of the following expression is in Appendix D (5):

$$
\begin{aligned}
E[t_n(2)] &= \sum_{s=1}^{m}\left[F(sh)-F(s-1)h\right](1-\alpha_1)^{s-1}\beta_1^{m-s+1}\sum_{j=m+1}^{\infty} jh(1-\beta_2)\beta_2^{j-m-1} \\
&= h\left[m+\frac{1}{1-\beta_2}\right]\sum_{s=1}^{m}\left[e^{-a((s-1)h)^b}-e^{-a(sh)^b}\right](1-\alpha_1)^{s-1}\beta_1^{m-s+1}
\end{aligned}
\tag{5.2.19}
$$

For case (3), the detail derivation of the following expression is in Appendix D (6):

$$
\begin{aligned}
E[t_n(3)] &= (1-\alpha_1)^m\sum_{s=m+1}^{\infty}\left[F(sh)-F((s-1)h)\right](1-\alpha_2)^{s-m-1}\sum_{j=s}^{\infty} jh(1-\beta_2)\beta_2^{j-s} \\
&= h\left(\frac{\beta_2}{1-\beta_2}\right)(1-\alpha_1)^m\sum_{s=m+1}^{\infty}\left[e^{-((s-1)h)^b}-e^{-a(sh)^b}\right](1-\alpha_2)^{s-m-1} \\
&\quad + h(1-\alpha_1)^m\sum_{s=m+1}^{\infty} s\left[e^{-((s-1)h)^b}-e^{-a(sh)^b}\right](1-\alpha_2)^{s-m-1}
\end{aligned}
\tag{5.2.20}
$$

Combine the results from the three cases we obtain the expression for $E[t_n]$ as:

$$
\begin{aligned}
E[t_n] = {} & h\sum_{s=1}^{m} s\left[e^{-a((s-1)h)^b} - e^{-a(sh)^b}\right](1-\alpha_1)^{s-1} \\
& + h(1-\alpha_1)^m \sum_{s=m+1}^{\infty} s\left[e^{-((s-1)h)^b} - e^{-a(sh)^b}\right](1-\alpha_2)^{s-m-1} \\
& + h\left[\frac{1}{1-\beta_2} - \frac{1}{1-\beta_1}\right]\sum_{s=1}^{m}\left[e^{-a((s-1)h)^b} - e^{-a(sh)^b}\right](1-\alpha_1)^{s-1}\beta_1^{m-s+1} \\
& + h\left(\frac{\beta_1}{1-\beta_1}\right)\sum_{s=1}^{m}\left[e^{-((s-1)h)^b} - e^{-a(sh)^b}\right](1-\alpha_2)^{s-1} \\
& + (1-\alpha_1)^m h\left(\frac{\beta_2}{1-\beta_2}\right)\sum_{s=m+1}^{\infty}\left[e^{-((s-1)h)^b} - e^{-a(sh)^b}\right](1-\alpha_2)^{s-m-1}
\end{aligned}
\tag{5.2.21}
$$

And together with the definition of $E[t_f]$, (5.2.16), the expected cycle length is:

$$
\begin{aligned}
E[t] = {} & \alpha_1\sum_{j=1}^{m}(1-\alpha_1)^{j-1}e^{-aj^b h^b} + (1-\alpha_1)^m\alpha_2\sum_{j=m+1}^{\infty}(1-\alpha_2)^{j-m-1}e^{-aj^b h^b} \\
& + h\sum_{s=1}^{m} s\left[e^{-a((s-1)h)^b} - e^{-a(sh)^b}\right](1-\alpha_1)^{s-1} \\
& + h(1-\alpha_1)^m \sum_{s=m+1}^{\infty} s\left[e^{-((s-1)h)^b} - e^{-a(sh)^b}\right](1-\alpha_2)^{s-m-1} \\
& + h\left[\frac{1}{1-\beta_2} - \frac{1}{1-\beta_1}\right]\sum_{s=1}^{m}\left[e^{-a((s-1)h)^b} - e^{-a(sh)^b}\right](1-\alpha_1)^{s-1}\beta_1^{m-s+1} \\
& + h\left(\frac{\beta_1}{1-\beta_1}\right)\sum_{s=1}^{m}\left[e^{-((s-1)h)^b} - e^{-a(sh)^b}\right](1-\alpha_2)^{s-1} \\
& + (1-\alpha_1)^m h\left(\frac{\beta_2}{1-\beta_2}\right)\sum_{s=m+1}^{\infty}\left[e^{-((s-1)h)^b} - e^{-a(sh)^b}\right](1-\alpha_2)^{s-m-1}
\end{aligned}
\tag{5.2.22}
$$

The objective is again to choose the values of the decision variables to minimize the the expected total cost per unit time function as defined in (3.1.1).

Lemma 5.1

The expected total cost per unit time function for the generalized Shewhart control chart is equivalent to that for the standard Shewhart control chart under Fact 5.2.

The proof is in Appendix D(7).

CHAPTER 6

Analysis and Results

The objective of this research is to compare the costs associated with the standard Shewhart control chart and the generalized Shewhart control chart. The motivation for this effort is to increase control chart sensitivity to a small but anticipated shift in the process average. Algebraic manipulation of the cost models is not trivial. The cost terms are functions of the decision variables, cost parameters and the distribution parameters. Two of the decision values of m and n are constrained to be integers, while k_1 and k_2 may take real values. As Montgomery [32] indicates that a sampling frequency of one hour is common for many charts, h = one unit of time is used through this research. The behavior of the expected total cost per unit time is analyzed algebraically and numerically. Mathematica [52] and GINO [25] are used to examine the behavior of the cost models over reasonable parameter sets and the generalized reduced gradient (GRG) algorithm is used to attempt to minimize the expected total cost per unit time function for those parameter sets. The parameter ranges evaluated are described below.

The first group of model parameters are λ, θ, a and b which govern the distribution of the time to shift. The range for λ is taken to be between 0.01 and 0.25. This range is commonly used in the literature, see Duncan [13]. The range for θ is between 8 and 200. This range provides the same range of the mean values of the time to shift for both the negative exponential and the uniform distributions. The range for a is between 0.01 and 0.25 and the values for b are 1.25 and 1.5.

The next model parameter considered is δ, the magnitude of the shift in the mean when a shift occurs. The value of δ is assumed to be 0.522. This value is selected because it corresponds to an increase in the proportion nonconforming from 0.01 to 0.02.

The range of values considered for c_i, the sampling and inspection cost is between 1.0 and 5.0. These represent small and large inspection costs relative to the other cost parameters.

The range for the cost of producing nonconforming product, c_d is between 1 and 10 which corresponds to a relatively small cost to a relatively large cost of producing nonconforming product. The value for r, the rate of production is 200.

The cost of investigating a false alarm, c_f, is arbitrarily assigned a value of 100. The cost of investigating and correcting a true signal, c_t, is assigned a value of 10 over the range of δ as there is no apparent reason that the cost of investigating a true signal should increase with the shift magnitude.

The above parameter ranges define the scenarios under which the economic performance of the standard and the generalized Shewhart control chart are investigated. The numerical analysis of the behavior of the expected total cost per unit time function with respect to the decision variables for a family of the parameter ranges is examined.

The expected total cost model is convex in k for all ranges of the other model parameters. Small values of k create large expected total cost because an excessive number of false alarms is given. This may dominate any cost savings due to rapid shift detection. Intermediate values of k produce the smallest expected total cost because they balance the costs of nonconforming production against the false alarm cost. Large values of k provide reduced shift detection probabilities and thus increasingly large nonconforming production cost. The total effect is that the expected total cost decrease to a minimum and then rises again as k increases.

The expected total cost is also convex in n for all ranges of the other model parameters. Small values of n imply low sampling costs but high nonconforming costs since shifts are not rapidly detected. Intermediate values of n balance the sampling cost against the nonconforming product cost to achieve the lowest expected total cost. Large values of n imply large sampling costs, which may dominate the savings in nonconforming product costs achieved through greater detection probabilities. These interpretations vary depending upon the relative importance of each of the cost categories but the overall effect is that the expected total cost is convex in n.

The above results for n and k are anticipated for the standard Shewhart control charts in general and confirmed for the generalized Shewhart control charts. The generalized

Shewhart control chart has features that the standard Shewhart chart does not. The properties that result from these additional features are now explored.

Model behavior in terms of the decision variables m, k_1 and k_2 is characterized by three cases. The relative magnitudes of the cost parameters in each case determine which behavior is observed.

In case one, the expected total cost per unit time function, C_T, displays convex behavior in each of the decision variables m, k_1 and k_2 and a minimum occurs in the interior of the convex feasible region. This means that the minimum cost control chart is some form of the generalized Shewhart chart. See figure 6.1.

In case two, C_T is still convex but it has a minimum corresponding to a boundary of $m = 0$ and $k_2 = k_1$ and it increases strictly in each of those variables. This means that the minimum cost control chart a standard chart with no control limit changes. See Figure 6.2.

In case three, C_T strictly decreases in both m and k_2 and has a minimum at the boundary $k_1 = k_2$ and m is unbounded. This means that the minimum cost control chart is a standard chart with no control limit changes. See Figure 6.3.

In order to determine why the C_T behaves this way, each of the four cost categories as well as the expected cycle length is examined for the nature of its contribution to the expected total cost. Note that the analysis below is for cost model I and it is also true for a uniform distribution since we can always choose the values of the rate parameter such that the mean value of the time to a shift is common for negative exponential and uniform distributions.

Note again that since h is defined as an arbitrary unit of time, its influence should be apparent in the effects of other parameters and variables so it need not to be examined specifically.

The inspection cost for a standard Shewhart control chart (3.1.4) is:

$$C_i = \{c_0 + nc_i\}/h \tag{6.1}$$

$$\frac{\partial C_i}{\partial c_i} = \frac{n}{h} \tag{6.2}$$

$$\frac{\partial C_i}{\partial n} = \frac{c_i}{h} \tag{6.3}$$

which is to say C_i is linear in c_i and n, respectively. Given the above algebraic results, numerical evaluation of the expressions should provide an indication of the relative magnitude of the impact of the values of the parameters. Consider the following values:

$$c_0 = 1.00,\ c_i = (1, 2, 3, 4, 5),\ n = (3, 5, 8, 10). \tag{6.4}$$

Then we obtain the values for C_i, see Table 6.1. These results are the same for the generalized Shewhart control chart.

The false alarm cost for a standard Shewhart control chart (3.1.6) is:

$$C_f = \frac{c_f \alpha e^{-\lambda h}}{1-(1-\alpha)e^{-\lambda h}} = \frac{c_f \alpha}{e^{\lambda h} - 1 + \alpha} \tag{6.5}$$

Thus C_f depends on the input parameters λ and c_f and the decision variable k. Therefore:

$$\frac{\partial C_f}{\partial c_f} = \frac{\alpha}{e^{\lambda h} - 1 + \alpha} \tag{6.6}$$

which is to say C_f is linear in c_f with a slope is less than 1. Consider the parameter λ:

$$\frac{\partial C_f}{\partial \lambda} = \frac{-c_f \alpha h e^{\lambda h}}{\left(e^{\lambda h} - 1 + \alpha\right)^2} < 0 \tag{6.7}$$

and:

$$\frac{\partial C_f}{\partial k} = \frac{e^{\lambda h} - 1}{\left(e^{\lambda h} - 1 + \alpha\right)^2} \frac{\partial \alpha}{\partial k} < 0 \tag{6.8}$$

since k is a standard normal coordinate, $\partial\alpha/\partial k < 0$. Consider the following example values which yield the values for P[A] that are shown in Table 6.2.

λ = (.01, .05, .1, .2)

k = (2.5, 2.75, 3.0) which imply α = (.0124, .006, .0026) (6.9)

Note that the numerical results conform to the behavior indicated by the derivatives. It appears that the influence of both λ and k is substantial. Finally as C_f is linear in c_f, increasing c_f will magnify the changes shown above.

Next consider the corresponding analysis for the generalized chart. In this case, the false alarm cost (3.2.5) is:

$$\begin{aligned} C_f &= c_f\left[\alpha_1 \sum_{i=1}^{m} (1-\alpha_1)^{i-1} e^{-\lambda i h} + (1-\alpha_1)^m \alpha_2 \sum_{i-m=1}^{\infty} (1-\alpha_2)^{i-m-1} e^{-\lambda i h}\right] \\ &= c_f\left\{\frac{\alpha_1\left[1-(1-\alpha_1)^m e^{-\lambda m h}\right]}{e^{\lambda h} - 1 + \alpha_1} + \frac{(1-\alpha_1)^m e^{-\lambda m h}\alpha_2}{e^{\lambda h} - 1 + \alpha_2}\right\} \end{aligned} \tag{6.10}$$

For this function, α_1 and α_2 replace α, k_1 and k_2 replace k, and m is added. The parameters λ, h, and c_f are the same as in the model for the standard chart. Observe that the derivative analysis for c_f is the same as C_f is linear in c_f. Thus c_f will tend to emphasize the effects of the other model quantities. The following inequalities are true:

$$
\begin{aligned}
&(1-\alpha_i)^m < \varepsilon_1(m),\ i=1,2 \\
&\beta_i^m < \varepsilon_2(m),\ i=1,2 \\
&e^{-\lambda mh} < \varepsilon_3(m) \\
&(1-\alpha_i)^m e^{-\lambda mh} < \varepsilon_4(m),\ i=1,2
\end{aligned}
\tag{6.11}
$$

Note that $\varepsilon_j(m)$, j = 1, 2, 3, 4 are small values depending on m. The analysis of the influence of λ can achieved most easily by analyzing the first equation of (6.10). Thus:

$$
\frac{\partial C_f}{\partial \lambda} = c_f\left[-h\alpha_1\sum_{i=1}^{m} i(1-\alpha_1)^{i-1}e^{-\lambda ih} - h(1-\alpha_1)^m\alpha_2\sum_{i-m=1}^{\infty} i(1-\alpha_2)^{i-m-1}e^{-\lambda ih}\right] < 0
\tag{6.12}
$$

The above summations are all positive with neagtive coefficents. In the case of k_1 and k_2, we study the third equation of (6.10) and obtain the derivatives:

$$
\begin{aligned}
\frac{\partial C_f}{\partial k_1} = c_f\Bigg\{&\frac{(e^{\lambda h}-1)\left[1-(1-\alpha_1)^m e^{-\lambda mh}\right]}{(e^{\lambda h}-1+\alpha_1)^2} + \frac{m\alpha_1(1-\alpha_1)^{m-1}e^{-\lambda mh}}{e^{\lambda h}-1+\alpha_1} \\
&- \frac{m\alpha_2(1-\alpha_1)^{m-1}e^{-\lambda mh}}{e^{\lambda h}-1+\alpha_2}\Bigg\}\frac{\partial \alpha_1}{\partial k_1}
\end{aligned}
\tag{6.13}
$$

which reduces to:

$$
\begin{aligned}
\frac{\partial C_f}{\partial k_1} = {}& c_f\left[\frac{(e^{\lambda h}-1)\left[1-(1-\alpha_1)^m e^{-\lambda mh}\right]}{(e^{\lambda h}-1+\alpha_1)^2}\right]\frac{\partial \alpha_1}{\partial k_1} \\
&+ c_f m(1-\alpha_1)^{m-1}e^{-\lambda mh}\left[\frac{\alpha_1}{e^{\lambda h}-1+\alpha_1} - \frac{\alpha_2}{e^{\lambda h}-1+\alpha_2}\right]\frac{\partial \alpha_1}{\partial k_1}
\end{aligned}
\tag{6.14}
$$

Note that the first term of the above equation is positive and can be large because of the form of the denominator. The second term is negative and small because $\alpha_1 < \alpha_2$ by construction and therefore:

$$\frac{\alpha_1}{e^{\lambda h}-1+\alpha_1} < \frac{\alpha_2}{e^{\lambda h}-1+\alpha_2} \tag{6.15}$$

Thus, the second term is negative but the magnitude is not great since the fourth equation of (6.11) is true and the difference between α_1 and α_2 is not too large. Therefore the first term dominates the second one. Then:

$$\frac{\partial C_f}{\partial k_2} = c_f \left\{ \frac{\left(e^{\lambda h}-1\right)\left[1-\left(1-\alpha_1\right)^m e^{-\lambda m h}\right]}{\left(e^{\lambda h}-1+\alpha_2\right)^2} \right\} \frac{\partial \alpha_2}{\partial k_2} < 0 \tag{6.16}$$

Because α_1 and α_2 are tail probabilities of a normal distribution, it is clear that $\partial\alpha_i/\partial k_i < 0$, $i = 1, 2$. As a consequence, both of the above derivatives are negative. Considering the influence of m. We study the third equation of (6.10) and obtain:

$$\begin{aligned}\frac{\partial C_f}{\partial m} &= c_f \frac{\alpha_1\left[-\left(1-\alpha_1\right)^m \log\left(1-\alpha_1\right)e^{-\lambda m h} + \lambda h\left(1-\alpha_1\right)^m e^{-\lambda m h}\right]}{e^{\lambda h}-1+\alpha_1} \\ &+ c_f \frac{\alpha_2\left[\left(1-\alpha_1\right)^m \log\left(1-\alpha_1\right)e^{-\lambda m h} - \lambda h\left(1-\alpha_1\right)^m e^{-\lambda m h}\right]}{e^{\lambda h}-1+\alpha_2}\end{aligned} \tag{6.17}$$

which reduces to:

$$\frac{\partial C_f}{\partial m} = c_f\left(1-\alpha_1\right)^m e^{-\lambda m h}\left[-\log\left(1-\alpha_1\right)+\lambda h\right]\left[\frac{\alpha_1}{e^{\lambda h}-1+\alpha_1} - \frac{\alpha_2}{e^{\lambda h}-1+\alpha_2}\right] \tag{6.18}$$

Now apply (6.15) and keep in mind that $\log(1-\alpha_1) < 0$. Thus the above derivative is negative. In order to obtain a sense of the trends indicated by all of the above derivatives, consider numerical cases in which:

$\lambda = (.01, .05, .1, .2)$
$k_1 = (2.5, 2.75, 3.0)$ which imply $\alpha_1 = (.0124, .006, .0026)$
$k_2 = (1.75, 2.0, 2.25, 2.5)$ which imply $\alpha_2 = (.0802, .0556, .0244, .0124)$
$m = (10, 20, 40)$ (6.19)

For these values of the model variables, the associated values of P[A] are listed Table 6.3. Observe that the numerical results illustrate the fact that C_f is decreasing in all λ, k_1, k_2 and m. Note also that increasing m enhances the effects of λ and k_1 and k_2. Furthermore, k_2 appears to have a greater effect on P[A] than does k_1. Note that setting $k_1 = k_2$ at any value should result in the same values of P[A] as occur in the standard case and that this is confirmed in the evaluation of the cases having $k_2 = k_1 = 2.5$. Finally, observe that the value of C_f is greater for the generalized chart than for the standard chart because the probability of a false alarm is greater for the generalized chart. This is a desirable feature of the generalized chart. The implied increment in cost should be offset by reduction in defect production costs.

The true signal cost for a standard Shewhart control chart (3.1.8) is:

$$C_t = \frac{c_t\left(1 - e^{-\lambda h}\right)}{1 - (1-\alpha)e^{-\lambda h}} = c_t \frac{e^{\lambda h} - 1}{e^{\lambda h} - 1 + \alpha} \tag{6.20}$$

Thus C_t depends on the input parameters λ and c_t and the decision variable k. Therefore:

$$\frac{\partial C_t}{\partial c_t} = \frac{e^{\lambda h} - 1}{e^{\lambda h} - 1 + \alpha} \tag{6.21}$$

which is to say C_t is linear in c_t with a slope that is less than 1. Consider the parameter λ:

$$\frac{\partial C_t}{\partial \lambda} = c_t \frac{\alpha h e^{\lambda h}}{\left(e^{\lambda h} - 1 + \alpha\right)^2} > 0 \tag{6.22}$$

This is to say that C_t is increasing in λ as should be expected in that $1/\lambda$ is the expected time to system shift so a large value of λ implies less time before a shift. Next, among the decision variables, C_t depends only on k. Thus:

$$\frac{\partial C_t}{\partial k} = c_t \frac{-[e^{\lambda t}-1]}{\left(e^{\lambda t}-1+\alpha\right)^2}\frac{\partial \alpha}{\partial k} > 0 \tag{6.23}$$

Since k is a standard normal coordinate, $\partial\alpha/\partial k < 0$ for any positive value of k.

Given the above algebraic results, numerical evaluation of the expressions should provide an indication of the relative magnitude of the impact of the values of the parameters. Considering the the same values, (6.9), the values for P[B] are listed in Table 6.4. Note that the numerical results conform to the behavior indicated by the derivatives. It appears that the influence of both λ and k is substantial. However, this should be considered relative to the other cost terms. It is also true that the effect of λ is diminished as k is increased. Finally as C_t is linear in c_t, increasing c_t will magnify the changes shown above.

Next consider the corresponding analysis for the generalized chart. In this case, the true signal cost (3.2.8) is:

$$\begin{aligned} C_t &= c_t\left(1-e^{-\lambda h}\right)\left\{\frac{1}{1-(1-\alpha_1)e^{-\lambda h}} - \frac{\left[(1-\alpha_1)e^{-\lambda h}\right]^{m+1}}{1-(1-\alpha_1)e^{-\lambda h}} + \frac{\left[(1-\alpha_2)e^{-\lambda h}\right]^{m+1}}{1-(1-\alpha_2)e^{-\lambda h}}\right\} \\ &= c_t\left(e^{\lambda h}-1\right)\left\{\frac{1}{e^{\lambda h}-1+\alpha_1} - \frac{(1-\alpha_1)^{m+1}/e^{\lambda(m+1)h}}{e^{\lambda h}-1+\alpha_1} + \frac{(1-\alpha_2)^{m+1}/e^{\lambda(m+1)h}}{e^{\lambda h}-1+\alpha_2}\right\} \end{aligned} \tag{6.24}$$

For this function, α_1 and α_2 replace α, k_1 and k_2 replace k, and m is added. The parameters λ, h, and c_t are the same as in the model for the standard chart. Observe that the derivative analysis for c_t is the same as C_t is linear in c_t. Thus c_t will tend to emphasize the effects of the other model quantities. The influence of λ is:

$$\begin{aligned} \frac{\partial C_t}{\partial \lambda} &= \frac{c_t\alpha_1 h e^{\lambda h}}{\left(e^{\lambda h}-1+\alpha_1\right)^2} - \frac{c_t h e^{\lambda h}}{e^{\lambda(m+1)h}}\left[\frac{\alpha_1(1-\alpha_1)^{m+1}}{\left(e^{\lambda h}-1+\alpha_1\right)^2} - \frac{\alpha_2(1-\alpha_2)^{m+1}}{\left(e^{\lambda h}-1+\alpha_2\right)^2}\right] \\ &\quad + \frac{c_t(m+1)h(e^{\lambda h}-1)}{e^{\lambda(m+1)h}}\left[\frac{(1-\alpha_1)^{m+1}}{e^{\lambda h}-1+\alpha_1} - \frac{(1-\alpha_2)^{m+1}}{e^{\lambda h}-1+\alpha_2}\right] \end{aligned} \tag{6.25}$$

While this form may appear complicated, it is not too cumbersome because the first term corresponds to the partial derivative constructed for the standard chart. In addition, since $\alpha_2 > \alpha_1$ by construction, then:

$$\frac{\alpha_2(1-\alpha_2)^{m+1}}{\left(e^{\lambda h}-1+\alpha_2\right)^2}-\frac{\alpha_1(1-\alpha_1)^{m+1}}{\left(e^{\lambda h}-1+\alpha_1\right)^2}\le\frac{\alpha_2(1-\alpha_2)^{m+1}-\alpha_1(1-\alpha_1)^{m+1}}{\left(e^{\lambda h}-1+\alpha_2\right)^2}$$
$$\le\frac{\alpha_2(1-\alpha_1)^{m+1}-\alpha_1(1-\alpha_1)^{m+1}}{\left(e^{\lambda h}-1+\alpha_2\right)^2}\le\frac{(1-\alpha_1)^{m+1}(\alpha_2-\alpha_1)}{\left(e^{\lambda h}-1+\alpha_2\right)^2}\le\frac{(1-\alpha_1)^{m}(\alpha_2-\alpha_1)}{\left(e^{\lambda h}-1+\alpha_2\right)^2} \tag{6.26}$$

and:

$$\frac{(1-\alpha_1)^{m+1}}{e^{\lambda h}-1+\alpha_1}-\frac{(1-\alpha_2)^{m+1}}{e^{\lambda h}-1+\alpha_2}\ge\frac{(1-\alpha_1)^{m+1}-(1-\alpha_2)^{m+1}}{e^{\lambda h}-1+\alpha_2}\ge\frac{(1-\alpha_1)^{m+1}-(1-\alpha_2)^{m+1}}{(e^{\lambda h}-1+\alpha_2)^2}$$
$$\ge\frac{(1-\alpha_1)^{m+1}-(1-\alpha_2)(1-\alpha_1)^{m}}{(e^{\lambda h}-1+\alpha_2)^2}\ge\frac{(1-\alpha_1)^{m}(\alpha_2-\alpha_1)}{\left(e^{\lambda h}-1+\alpha_2\right)^2} \tag{6.27}$$

The above inequalities mean that the magnitude of the third term of (6.24) is greater than that of the second term of (6.24), therefore the derivative is positive. In the case of k_1 and k_2, we obtain the derivatives:

$$\frac{\partial C_t}{\partial k_1}=\frac{c_t(e^{\lambda h}-1)}{\left(e^{\lambda h}-1+\alpha_1\right)^2 e^{\lambda(m+1)h}}\Big[-e^{\lambda(m+1)h}+(1-\alpha_1)^{m+1}$$
$$+(m+1)(1-\alpha_1)^{m}\left(e^{\lambda h}-1+\alpha_1\right)\Big]\frac{\partial\alpha_1}{\partial k_1} \tag{6.28}$$

and:

$$\frac{\partial C_t}{\partial k_2}=\frac{c_t(e^{\lambda h}-1)e^{-\lambda(m+1)h}}{\left(e^{\lambda h}-1+\alpha_1\right)^2}\left[-(1-\alpha_2)^{m+1}-(m+1)(1-\alpha_2)^{m}\left(e^{\lambda h}-1+\alpha_1\right)\right]\frac{\partial\alpha_2}{\partial k_2} \tag{6.29}$$

Because α_1 and α_2 are tail probabilities of a normal distribution, it is clear that $\partial\alpha_i/\partial k_i < 0$, $i = 1, 2$. In (6.28), both the second term and the third term are very small for suitably large values of m, therefore the first term dominates the others. As a consequence, both of the above derivatives are positive.

Next, consider the time of the change in the width of the control limits, m. We obtain:

$$\frac{\partial C_t}{\partial m} = c_t \frac{-\left(e^{\lambda h}-1\right)}{e^{\lambda h}-1+\alpha_1} \frac{\left(1-\alpha_1\right)^{m+1}\left[\log(1-\alpha_1)-(m+1)h\right]}{e^{\lambda(m+1)h}}$$
$$+ c_t \frac{\left(e^{\lambda h}-1\right)}{e^{\lambda h}-1+\alpha_2} \frac{\left(1-\alpha_2\right)^{m+1}\left[\log(1-\alpha_2)-(m+1)h\right]}{e^{\lambda(m+1)h}} \tag{6.30}$$

Factor this as:

$$\frac{\partial C_t}{\partial m} = c_t\left(e^{\lambda h}-1\right)\left[\frac{\left(1-\alpha_1\right)^{m+1}}{e^{\lambda h}-1+\alpha_1} \frac{\left[-\log(1-\alpha_1)+(m+1)h\right]}{e^{\lambda(m+1)h}}\right.$$
$$\left. - \frac{\left(1-\alpha_2\right)^{m+1}}{e^{\lambda h}-1+\alpha_2} \frac{\left[-\log(1-\alpha_2)+(m+1)h\right]}{e^{\lambda(m+1)h}}\right] \tag{6.31}$$

This quantity is small and positive. Since $\alpha_1 < \alpha_2$ by construction,

$$\frac{\left(1-\alpha_1\right)^{m+1}}{e^{\lambda h}-1+\alpha_1} \geq \frac{\left(1-\alpha_2\right)^{m+1}}{e^{\lambda h}-1+\alpha_2} \tag{6.32}$$

and since $-\log(1-\alpha_1) > -\log(1-\alpha_2) > 0$, the first term of (6.31) dominates the second one. This means that tha above derivative is positive. Refering to (6.27), we conclude that it is small. That is, increasing m increases C_t.

In order to obtain a sense of the trends indicated by all of the above derivatives, consider the same numerical cases (6.19). The associated values of P[B] are listed in Table 6.5. Observe that the numerical results illustrate the fact that C_t is increasing in all λ, k_1, k_2 and m. Note also that increasing m enhances the effects of λ and k_1 and k_2. Furthermore, k_2 appears to have a greater effect on P[B] than does k_1. Note that setting $k_1 = k_2$ at any value should result in the same values of P[B] as occur in the standard case and that this is confirmed in the evaluation of the cases having $k_2 = k_1 = 2.5$. Finally, observe that the value of C_t is greater for the generalized chart than for the standard chart because the probability of a true signal is greater for the generalized chart. This is a desirable feature of the generalized chart. The implied increment in cost should be offset by reduction in defect production costs.

The nonconforming production cost for a standard Shewhart control chart (3.1.14) is:

$$C_d = c_d rp\left[\frac{\lambda h-\left(1-e^{-\lambda h}\right)}{\lambda\left(1-e^{-\lambda h}\right)}+\frac{h\beta}{(1-\beta)}\right] = c_d rp\left[\frac{(\lambda h-1)e^{\lambda h}+1}{\lambda\left(e^{\lambda h}-1\right)}+\frac{h\beta}{(1-\beta)}\right] \tag{6.33}$$

Thus C_d depends on the input parameters λ, δ and c_t and the decision variables n and k. Therefore:

$$\frac{\partial C_d}{\partial(c_d rp)} = \frac{(\lambda h-1)e^{\lambda h}+1}{\lambda\left(e^{\lambda h}-1\right)}+\frac{h\beta}{(1-\beta)} \tag{6.34}$$

which is to say C_d is linear in $c_d rp$ with a slope that is greater than 0.

Consider the parameter λ:

$$\frac{\partial C_d}{\partial\lambda} = c_d rp\left\{\frac{1}{\lambda^2}-\frac{h^2e^{\lambda h}}{\left(e^{\lambda h}-1\right)^2}\right\} \tag{6.35}$$

This derivative is negative for very small values of λ and it is positive for all the values of λ in this analysis. This is to say that C_d is increasing in λ as should be expected in that $1/\lambda$ is the expected time to system shift so a large value of λ implies less time before a shift. Next, consider the decision variables. C_d depends on k. Thus:

$$\frac{\partial C_d}{\partial k} = c_d rp\frac{h}{(1-\beta)^2}\frac{\partial\beta}{\partial k} > 0 \tag{6.36}$$

Because k is a standard normal coordinate, $\partial\beta/\partial k > 0$ for any positive value of k.

C_d also depends on the sample size, n. Thus:

$$\frac{\partial C_d}{\partial n} = c_d rp \frac{h}{(1-\beta)^2} \frac{\partial \beta}{\partial n} < 0 \tag{6.37}$$

because $\partial\beta/\partial n < 0$ for any n.

Given the above algebraic results, numerical evaluation of the expressions should provide an indication of the relative magnitude of the impact of the values of the parameters. Consider the the following values:

$\delta = 0.522$

$n = (5, 10)$

$\lambda = (.01, .05, .1, .2)$

$k = (2.5, 2.75, 3.0)$ which imply $\alpha = (.0124, .006, .0026)$

$\beta = (.9087, .9433, .9667)$, when $n = 5$

$\beta = (.8021, .8644, .9113)$, when $n = 10$ (6.38)

Note that the numerical results confirm to the behavior indicated by the derivatives. See Table 6.6. It appears that the influences of both n and k are greater than that of λ. However, this should be considered relative to the other cost terms. Finally as C_d is linear in $c_d rp$, increase $c_d rp$ will magnify the changes shown above.

Next consider the corresponding analysis for the generalized chart. In this case, the cost of nonconforming production (3.2.15) is:

$$C_d = c_d rp \left\{ \frac{\lambda h - \left(1 - e^{-\lambda h}\right)}{\lambda\left(1 - e^{-\lambda h}\right)} + \frac{h\beta_1\left(1 - e^{-\lambda mh}\right)}{1-\beta_1} + \frac{h\beta_2 e^{-\lambda mh}}{1-\beta_2} \right\}$$
$$+ c_d rp \left\{ \left[\frac{h\beta_1\left(1 - e^{-\lambda h}\right)\left(\beta_1^m - e^{-\lambda mh}\right)}{\beta_1 - e^{-\lambda h}} \right] \left[\frac{1}{1-\beta_2} - \frac{1}{1-\beta_1} \right] \right\} \tag{6.39}$$

For this function, β_1 and β_2 replace β, k_1 and k_2 replace k, and m is added. The parameters λ, h, and $c_d rp$ are the same as in the model for the standard chart. Observe that the derivative analysis for $c_d rp$ is the same as C_d is linear in $c_d rp$. Thus $c_d rp$ will tend to emphasize the effects of the other model quantities. Next, consider the influence of λ:

$$\frac{\partial C_d}{\partial \lambda} = c_d rp\left[\frac{1}{\lambda^2} - \frac{h^2 e^{\lambda h}}{\left(e^{\lambda h}-1\right)^2}\right] + c_d rpmh^2 e^{-\lambda mh}\left[\frac{\beta_1}{1-\beta_1} - \frac{\beta_2}{1-\beta_2}\right]$$
$$+ c_d rp\beta_1\left[\frac{1}{1-\beta_2} - \frac{1}{1-\beta_1}\right]\frac{h^2 m}{e^{\lambda mh}}\frac{e^{\lambda h}-1}{\beta_1 e^{\lambda h}-1}$$
$$+ c_d rp\beta_1\left[-\frac{1}{1-\beta_2} + \frac{1}{1-\beta_1}\right]\frac{\beta_1^m e^{\lambda mh}-1}{e^{\lambda mh}}\frac{h^2(1-\beta_1)e^{\lambda h}}{(\beta_1 e^{\lambda h}-1)^2} \tag{6.40}$$

Since $0 < \beta_i < 1, i = 1, 2,$ and $\beta_1 > \beta_2$ by construction,

$$\frac{\beta_1}{1-\beta_1} > \frac{\beta_2}{1-\beta_2} \tag{6.41}$$

and:

$$\frac{1}{1-\beta_1} > \frac{1}{1-\beta_2} \tag{6.42}$$

While (6.40) may appear complicated, it is not too cumbersome because the first term corresponds to the partial derivate constructed for the standard chart. In addition, the second term is positive because (6.41) is true. Next, taking the third and the fourth term together we obtain:

$$c_d rp\beta_1\left[\frac{1}{1-\beta_2} - \frac{1}{1-\beta_1}\right]\frac{h^2 m}{e^{\lambda mh}}\frac{e^{\lambda h}-1}{\beta_1 e^{\lambda h}-1} + c_d rp\beta_1\left[-\frac{1}{1-\beta_2} + \frac{1}{1-\beta_1}\right]\frac{\beta_1^m e^{\lambda mh}-1}{e^{\lambda mh}}\frac{h^2(1-\beta_1)e^{\lambda h}}{(\beta_1 e^{\lambda h}-1)^2}$$
$$= c_d rp\beta_1\left[\frac{1}{1-\beta_2} - \frac{1}{1-\beta_1}\right]\frac{h^2}{e^{\lambda mh}\left(\beta_1 e^{\lambda h}-1\right)^2}[U+V] \tag{6.43}$$

where:

$$U = \left(e^{\lambda h}-1\right)\left[m\left(\beta_1 e^{\lambda h}-1\right) - \left[\left(\beta_1 e^{\lambda h}\right)^m - 1\right]\right], \quad V = \left(\beta_1 e^{\lambda h}-1\right)\left[\left(\beta_1 e^{\lambda h}\right)^m - 1\right] \tag{6.44}$$

Note that U is negative for the values of the parameters in our analysis and V is always positive because both components have same sign. Clearly:

$$|U| > |V| \tag{6.45}$$

Therefore U+V < 0 and (6.43) is positive since (6.42) is true. Finally, (6.40) is positive.

In the case of k_1 and k_2, we obtain the derivatives:

$$\frac{\partial C_d}{\partial k_1} = c_d rp \frac{h\left(1-e^{-\lambda mh}\right)}{\left(1-\beta_1\right)^2}\frac{\partial \beta_1}{\partial k_1} + c_d rph\left(1-e^{-\lambda h}\right)[W]\frac{\partial \beta_1}{\partial k_1} \tag{6.46}$$

where:

$$W = \frac{\beta_1^m - e^{-\lambda mh}}{\beta_1 - e^{-\lambda h}}\left[\frac{1}{1-\beta_2} - \frac{1}{1-\beta_1}\right] - \frac{\beta_1^m - e^{-\lambda mh}}{\beta_1 - e^{-\lambda h}}\frac{\beta_1}{\left(1-\beta_1\right)^2}$$
$$+ \frac{m\beta_1^m}{\beta_1 - e^{-\lambda h}}\left[\frac{1}{1-\beta_2} - \frac{1}{1-\beta_1}\right] - \frac{\beta_1^m - e^{-\lambda mh}}{\beta_1 - e^{-\lambda h}}\frac{\beta_1}{\beta_1 - e^{-\lambda h}} \tag{6.47}$$

It is clear that the first term of (6.46) is positive. And:

$$\frac{\beta_1^m - e^{-\lambda mh}}{\beta_1 - e^{-\lambda h}} > 0 \tag{6.48}$$

Note that (6.48) is true as numerator and denominator have the same sign. Thus the first term of (6.47) is positive and the second term is negative. The third and the fourth term always have different signs. Refering to (6.11) we conclude that the magnitude of (6.47) is small in any cases and therefore (6.46) is positive.

$$\frac{\partial C_d}{\partial k_2} = c_d rp\left[\frac{he^{-\lambda mh}}{\left(1-\beta_2\right)^2} + \frac{h\beta_1\left(1-e^{-\lambda h}\right)\left(e^{-\lambda mh} - \beta_1^m\right)}{\left(1-\beta_2\right)^2\left(e^{-\lambda h} - \beta_1\right)}\right]\frac{\partial \beta_2}{\partial k_2} \tag{6.49}$$

Because β_1 and β_2 are Type II error probabilities of a normal distribution, it is clear that $\partial\beta_i/\partial k_i > 0$, i = 1, 2. As a consequence, (6.49) is positive.

C_d depends on the sample size, n. However, the analysis is similar to the analysis of (6.46) and (6.49) except that the derivatives changed. In this case, $\partial\beta_i/\partial n < 0$, i = 1, 2. Thus the partial derivative with respect to n is negative. It means that C_d decreases as sample size increases which is expected.

Next, consider the time of the change in the width of the control limits, m. We obtain:

$$\frac{\partial C_d}{\partial m} = c_d r p h \beta_1 \left(1 - e^{-\lambda h}\right)\left[\frac{1}{1-\beta_2} - \frac{1}{1-\beta_1}\right]\frac{\beta_1^m \log\beta_1 + \lambda h e^{-\lambda m h}}{\beta_1 - e^{-\lambda h}}$$
$$+ c_d r p h^2 \lambda \left[\frac{\beta_1}{1-\beta_1} - \frac{\beta_2}{1-\beta_2}\right] e^{-\lambda m h} \qquad (6.50)$$

Refering to (6.41) it is apparent that the second term of (6.50) is posotive. The analysis of the following form is:

$$\frac{\beta_1^m \log\beta_1 + \lambda h e^{-\lambda m h}}{\beta_1 - e^{-\lambda h}} \qquad (6.51)$$

and let:

$$X(m) = \beta_1^m \log\beta_1\,, \quad Y(m) = \lambda h e^{-\lambda m h} \qquad (6.52)$$

Note that (6.51) is negative because:

(1) If the denominator of (6.51) is positive then X(m) dominates Y(m);

(2) If the denominator of (6.51) is negative then Y(m) dominates X(m).

Thus the first term of (6.50) is also positive since (6.42) is true. Therefore (6.50) is positive which implies increasing m increases C_d.

In order to obtain a sense of the trends indicated by all of the above derivatives, consider the following numerical cases:

$$
\begin{aligned}
&\delta = 0.522,\ n = (5, 10),\ m = (10, 20, 40),\ \lambda = (.01, .05, .1, .2)\\
&k_1 = (2.5, 2.75, 3.0) \text{ which imply } \alpha_1 = (.0124, .006, .0026)\\
&k_2 = (2.25, 2.5) \text{ which imply } \alpha_2 = (.0244, .0124)\\
&\beta_1 = (.9087, .9433, .9667) \text{ when } n = 5\\
&\beta_1 = (.8021, .8644, .9113) \text{ when } n = 10\\
&\beta_2 = (.8708, .9087) \text{ when } n = 5\\
&\beta_2 = (.7481, .8021) \text{ when } n = 10
\end{aligned}
\tag{6.53}
$$

The associated values of E[D] are listed in Table 6.7. Observe that the numerical results illustrate the fact that C_d is increasing in all λ, k_1, k_2 and m and C_d is decreasing in n. Note also that increasing m enhance the effect of λ and k_1 and k_2 and that increasing n reduce the effect of λ and k_1 and k_2. Furthermore, k_2 appears to have a greater effect on E[D] than does k_1. Note that setting $k_1 = k_2$ at any value should result in the same values of E[D] as occur in the standard case and that this is confirmed in the evaluation of the cases having $k_2 = k_1 = 2.5$. Finally, observe that the value of C_d is less for the generalized chart than for the standard chart because the expected time out of control is less for the generalized chart. This is a desirable feature of the generalized chart.

The expected cycle length for a standard Shewhart control chart (3.1.16) is:

$$
\begin{aligned}
E[t] &= \sum_{j=1}^{\infty} jh\left(\alpha(1-\alpha)^{j-1}e^{-\lambda jh} + \sum_{l=1}^{j}\left(e^{-\lambda(l-1)h} - e^{-\lambda lh}\right)(1-\alpha)^{l-1}\beta^{j-l}(1-\beta)\right)\\
&= \alpha h e^{-\lambda h}\sum_{j=1}^{\infty} j\left((1-\alpha)e^{-\lambda h}\right)^{j-1} + (1-\beta)\left(1-e^{-\lambda h}\right)h\sum_{j=1}^{\infty} j\sum_{l=1}^{j} e^{-\lambda(l-1)h}(1-\alpha)^{l-1}\beta^{j-l}\\
&= \frac{\alpha h e^{-\lambda h}}{\left[1-(1-\alpha)e^{-\lambda h}\right]^2} + \frac{\left(1-e^{-\lambda h}\right)h}{\left[\beta-(1-\alpha)e^{-\lambda h}\right]}\left[\frac{\beta}{1-\beta} - \frac{(1-\beta)(1-\alpha)e^{-\lambda h}}{\left[1-(1-\alpha)e^{-\lambda h}\right]^2}\right]
\end{aligned}
\tag{6.54}
$$

Thus E(t) depends on the input parameters λ and the decision variables n and k. Note that the influence of λ is analyzed by using the second equation of (6.54). Thus:

$$\frac{\partial E(t)}{\partial \lambda} = -\alpha h^2 \sum_{j=1}^{\infty} j^2\left((1-\alpha)e^{-\lambda h}\right)^{j-1} + (1-\beta)h\sum_{j=1}^{\infty} j \sum_{l=1}^{j} (1-\alpha)^{l-1}\beta^{j-1}[Z] \tag{6.55}$$

where:

$$Z = -(l-1)he^{-\lambda(l-1)h} + lhe^{-\lambda lh} = he^{-\lambda(l-1)h}\left[\left(1-e^{-\lambda h}\right)l - 1\right] \tag{6.56}$$

and:

$$\left(1-e^{-\lambda h}\right) l - 1 < 0 \tag{6.57}$$

It is clear that the first term of (6.55) is negative while Z may have either sign. However, for any λ, (6.57) is negative until l gets relatively large. By the time l gets large, the contribution of the second term will be very small since (6.11) is true. Thus (6.55) is negative in general. This means that E(t) is decreasing in λ as should be expected in that 1/λ is the expected time to system shift so a large value of λ implies less time before a shift. Next, consider the decision variables, E(t) depends on k. Refering to the second equation of (6.52), thus:

$$\frac{\partial E(t)}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left[\alpha h e^{-\lambda h}\sum_{j=1}^{\infty} j\left((1-\alpha)e^{-\lambda h}\right)^{j-1}\right]$$
$$-(1-\beta)\left(1-e^{-\lambda h}\right)h\sum_{j=1}^{\infty} j \sum_{l=1}^{j}(l-1)e^{-\lambda(l-1)h}(1-\alpha)^{l-2}\beta^{j-1} \tag{6.58}$$

It is clear that the second term is negative. Now we analyze the first term:

$$\alpha h e^{-\lambda h}\sum_{j=1}^{\infty} j\left((1-\alpha)e^{-\lambda h}\right)^{j-1} = \frac{\alpha h e^{-\lambda h}}{\left[1-(1-\alpha)e^{-\lambda h}\right]^2} \tag{6.59}$$

Therefore:

$$\frac{\partial}{\partial\alpha}\left[\alpha h e^{-\lambda h}\sum_{j=1}^{\infty} j\left((1-\alpha)e^{-\lambda h}\right)^{j-1}\right]=\frac{\partial}{\partial\alpha}\left[\frac{\alpha h e^{-\lambda h}}{\left[1-(1-\alpha)e^{-\lambda h}\right]^{2}}\right]$$

$$=\frac{he^{-\lambda h}\left[1-(1-\alpha)e^{-\lambda h}\right]\left\{\left[1-(1-\alpha)e^{-\lambda h}\right]-2\alpha e^{-\lambda h}\right\}}{\left[1-(1-\alpha)e^{-\lambda h}\right]^{4}}$$

$$=\frac{he^{-\lambda h}}{\left[1-(1-\alpha)e^{-\lambda h}\right]^{3}}\left[1-e^{-\lambda h}-\alpha e^{-\lambda h}\right] \tag{6.60}$$

This derivative is negative for most values of λ and α in our analysis. For large values of λ (6.60) is positive. Howevwe, by the time λ gets large, the second term of (6.58) dominates the first one because the second term represents the time of getting a true signal, large values of λ implies large values of P[B]. Therefore (6.60) is negative and the numerical results confirm this analysis. Refering to the third equation of (6.52) we obtain the following expressions:

$$\frac{\partial E(t)}{\partial\beta}=\frac{1}{(1-\beta)^{2}}\frac{\left(1-e^{-\lambda h}\right)h}{\left[\beta-(1-\alpha)e^{-\lambda h}\right]}-\frac{\beta}{1-\beta}\frac{\left(1-e^{-\lambda h}\right)h}{\left[\beta-(1-\alpha)e^{-\lambda h}\right]^{2}}$$
$$+\frac{\left(1-e^{-\lambda h}\right)h}{\left[\beta-(1-\alpha)e^{-\lambda h}\right]}\frac{(1-\alpha)e^{-\lambda h}}{\left[1-(1-\alpha)e^{-\lambda h}\right]^{2}}+\frac{\left(1-e^{-\lambda h}\right)h}{\left[\beta-(1-\alpha)e^{-\lambda h}\right]^{2}}\frac{(1-\beta)(1-\alpha)e^{-\lambda h}}{\left[1-(1-\alpha)e^{-\lambda h}\right]^{2}} \tag{6.61}$$

and with:

$$\beta-(1-\alpha)e^{-\lambda h} \tag{6.62}$$

and:

$$\left|\beta-(1-\alpha)e^{-\lambda h}\right|,\ \left|1-(1-\alpha)e^{-\lambda h}\right| \tag{6.63}$$

Note that if (6.62) is positive, then only the second term of (6.61) is negative. The fourth term of (6.61) donimates the second one because of the form of its denominator. For small values of β and λ, (6.62) is negative. In this case, both forms of (6.63) are very small and therefore the fourth term of (6.61) donimates the others because of the form of its denominator. In summary, the derivative (6.61) is positive. Because k is a standard normal coordinate, $\partial\alpha/\partial k < 0$ and $\partial\beta/\partial k > 0$ for any positive value of k. Finally:

$$\frac{\partial E(t)}{\partial k} = \frac{\partial E(t)}{\partial \alpha}\frac{\partial \alpha}{\partial k} + \frac{\partial E(t)}{\partial \beta}\frac{\partial \beta}{\partial k} > 0 \tag{6.64}$$

Since E(t) also depends on the sample size, n. Thus:

$$\frac{\partial E(t)}{\partial n} = \frac{\partial E(t)}{\partial \beta}\frac{\partial \beta}{\partial n} < 0 \tag{6.65}$$

This is obvious because (6.61) is true and $\partial\beta/\partial n < 0$ for any n.

Given the above algebraic results, numerical evaluation of the expressions should provide an indication of the relative magnitude of the impact of the values of the parameters. Consider the same values (6.38). We obtain the values for E(t). See Table 6.8. Note that the numerical results conform to the behavior indicated by the derivatives. It appears that the influence of n and k is greater than that of λ.

Next consider the corresponding analysis for the generalized Shewhart control chart. By defination, $E(t) = E(t_f) + E(t_n)$. Therefore the analysis of the monotone behavior of E(t) can be achieved by analyzing the monotone behaviors of $E(t_n)$ and $E(t_f)$. The expression for $E(t_n(b))$ (3.2.21) is:

$$\begin{aligned} E[t_n(b)] &= (1-\beta_2)h\sum_{s=1}^{m}\left[e^{-\lambda(s-1)h} - e^{-\lambda sh}\right](1-\alpha_1)^{s-1}\beta_1^{m-s+1}\sum_{j=m+1}^{\infty} j\,(1-\beta_2)\beta_2^{j-m-1} \\ &= h\left(1-e^{-\lambda h}\right)\left(\frac{1}{1-\beta_2} + m\left[\frac{\beta_1^{m+1} - \beta_1\left[e^{-\lambda h}(1-\alpha_1)\right]^m}{\beta_1 - \left[e^{-\lambda h}(1-\alpha_1)\right]}\right]\right) \end{aligned} \tag{6.66}$$

Using the second equation of (6.63) we obtain:

$$\frac{\partial E[t_n(b)]}{\partial \lambda} = -h^2 e^{-\lambda h}\left(\frac{1}{1-\beta_2}+m\right)\left[\frac{\beta_1^{m+1}-\beta_1\left[e^{-\lambda h}(1-\alpha_1)\right]^m}{\beta_1-\left[e^{-\lambda h}(1-\alpha_1)\right]}\right]$$

$$-h\left(1-e^{-\lambda h}\right)\left(\frac{1}{1-\beta_2}+m\right)\left[\frac{h(1-\alpha_1)e^{-\lambda h}(\beta_1^{m+1}-\beta_1\left[e^{-\lambda h}(1-\alpha_1)\right]^m)}{\left[\beta_1-\left[e^{-\lambda h}(1-\alpha_1)\right]\right]^2}\right]$$

$$+h\left(1-e^{-\lambda h}\right)\left(\frac{1}{1-\beta_2}+m\right)\left[\frac{mh\beta_1\left[e^{-\lambda h}(1-\alpha_1)\right]^m\left[\beta_1-\left[e^{-\lambda h}(1-\alpha_1)\right]\right]}{\left[\beta_1-\left[e^{-\lambda h}(1-\alpha_1)\right]\right]^2}\right] \tag{6.67}$$

Refering to (6.48), it is clear that:

$$\frac{\beta_1^{m+1}-\beta_1\left[e^{-\lambda h}(1-\alpha_1)\right]^m}{\beta_1-\left[e^{-\lambda h}(1-\alpha_1)\right]} = \beta_1\frac{\beta_1^{m}-\left[e^{-\lambda h}(1-\alpha_1)\right]^m}{\beta_1-\left[e^{-\lambda h}(1-\alpha_1)\right]} > 0 \tag{6.68}$$

Therefore the first term is negative and referring to (6.68) that the second term and the third term always have different coefficients. Thus the two terms balance together and therefore the first term donimates the others. In the case of k_1, analyzing the first equation of (6.66):

$$\frac{\partial E[t_n(b)]}{\partial \alpha_1}$$

$$= -(1-\beta_2)h\sum_{s=1}^{m}\left[e^{-\lambda(s-1)h}-e^{-\lambda sh}\right](s-1)(1-\alpha_1)^{s-2}\beta_1^{m-s+1}\sum_{j=m+1}^{\infty} j\,(1-\beta_2)\beta_2^{j-m-1} < 0 \tag{6.69}$$

and:

$$\frac{\partial E[t_n(b)]}{\partial \beta_1}$$

$$= (1-\beta_2)h\sum_{s=1}^{m}\left[e^{-\lambda(s-1)h}-e^{-\lambda sh}\right](m-s+1)(1-\alpha_1)^{s-1}\beta_1^{m-s}\sum_{j=m+1}^{\infty} j\,(1-\beta_2)\beta_2^{j-m-1} > 0 \tag{6.70}$$

Thus:

$$\frac{\partial E[t_n(b)]}{\partial k_1} = \frac{\partial E[t_n(b)]}{\partial \alpha_1}\frac{\partial \alpha_1}{\partial k_1} + \frac{\partial E[t_n(b)]}{\partial \beta_1}\frac{\partial \beta_1}{\partial k_1} > 0 \tag{6.71}$$

In the case of k_2, analyzing the second equation of (6.66) we obtain:

$$\frac{\partial E[t_n(b)]}{\partial \beta_2} = \frac{h(1-e^{-\lambda h})}{(1-\beta_2)^2}\left[\frac{\beta_1^{m+1} - \beta_1[e^{-\lambda h}(1-\alpha_1)]^m}{\beta_1 - [e^{-\lambda h}(1-\alpha_1)]}\right] > 0 \tag{6.72}$$

The above derivative is positive because (6.68) is true and therefore:

$$\frac{\partial E[t_n(b)]}{\partial k_2} = \frac{\partial E[t_n(b)]}{\partial \beta_2}\frac{\partial \beta_2}{\partial k_2} > 0 \tag{6.73}$$

In the case of the sample size n:

$$\frac{\partial E[t_n(b)]}{\partial n} = \frac{\partial E[t_n(b)]}{\partial \beta_1}\frac{\partial \beta_1}{\partial n} + \frac{\partial E[t_n(b)]}{\partial \beta_2}\frac{\partial \beta_2}{\partial n} \tag{6.74}$$

It is clear that the above derivative is negative because both (6.70) and (6.72) are true and $\partial\beta_i/\partial n < 0$, $i = 1, 2$. The influence of m is analyzed using the second equation of (6.66):

$$\frac{\partial E[t_n(b)]}{\partial m} = h(1-e^{-\lambda h})\left[\frac{\beta_1^{m+1} - \beta_1[e^{-\lambda h}(1-\alpha_1)]^m}{\beta_1 - [e^{-\lambda h}(1-\alpha_1)]}\right]$$
$$+ h(1-e^{-\lambda h})\left(\frac{1}{1-\beta_2} + m\right)\frac{\beta_1^{m+1}\log\beta_1 + (\lambda h - \log(1-\alpha_1))\beta_1[e^{-\lambda h}(1-\alpha_1)]^m}{\beta_1 - [e^{-\lambda h}(1-\alpha_1)]} \tag{6.75}$$

The first term of the above derivative is positive since (6.68) is true. Now consider:

$$\frac{\beta_1^{m+1}\log\beta_1 + (\lambda h - \log(1-\alpha_1))\beta_1[e^{-\lambda h}(1-\alpha_1)]^m}{\beta_1 - [e^{-\lambda h}(1-\alpha_1)]} \tag{6.76}$$

Note that the above form could be negative for some values of the model parameters since $\log\beta_1 < 0$. However, (6.76) is a very small value because (6.11) is true and the conclusion is that the derivative (6.75) is positive. Note that the above natures are also true for $E(t_n(a))$ and $E(t_n(c))$ since by defination that they have same properties except that the range of the summation changed. Note also that $E(t_n) = E(t_n(a)) + E(t_n(b)) + E(t_n(c))$ by defination, therefore we have studied the monotone behavior of $E(t_n)$. Next, the form of $E(t_f)$ (3.2.18) is:

$$
\begin{aligned}
E[t_f] &= \sum_{j=1}^{m} jh(1-\alpha_1)^{j-1}\alpha_1[1-F(jh)] + (1-\alpha_1)^m \sum_{j=m+1}^{\infty} jh(1-\alpha_2)^{j-m-1}\alpha_2[1-F(jh)] \\
&= \alpha_1 h \sum_{j=1}^{m} j(1-\alpha_1)^{j-1} e^{-\lambda jh} + (1-\alpha_1)^m \alpha_2 h \sum_{j=m+1}^{\infty} j(1-\alpha_2)^{j-m-1} e^{-\lambda jh} \\
&= \alpha_1 h \left[\frac{e^{-\lambda h}\left\{1-\left[(1-\alpha_1)e^{-\lambda h}\right]^m\right\}}{\left[1-(1-\alpha_1)e^{-\lambda h}\right]^2} - \frac{m(1-\alpha_1)^m e^{-\lambda(m+1)h}}{1-(1-\alpha_1)e^{-\lambda h}} \right] \\
&\quad + \alpha_2 h \left[\frac{(1-\alpha_1)^m e^{-\lambda(m+1)h}}{\left[1-(1-\alpha_2)e^{-\lambda h}\right]^2} + \frac{m(1-\alpha_1)^m e^{-\lambda(m+1)h}}{1-(1-\alpha_2)e^{-\lambda h}} \right]
\end{aligned}
\tag{6.77}
$$

Refering to the second equation of (6.74), we obtain:

$$
\frac{\partial E[t_f]}{\partial \lambda} = -\alpha_1 h^2 \sum_{j=1}^{m} j^2(1-\alpha_1)^{j-1} e^{-\lambda jh} - (1-\alpha_1)^m \alpha_2 h^2 \sum_{j=m+1}^{\infty} j^2(1-\alpha_2)^{j-m-1} e^{-\lambda jh}
\tag{6.78}
$$

The above derivative is negative since the two summations are all positive with negative coefficients. In the case of k_1:

$$
\frac{\partial E[t_f]}{\partial \alpha_1} = \frac{\partial}{\partial \alpha_1}\left[\alpha_1 h \sum_{j=1}^{m} j(1-\alpha_1)^{j-1} e^{-\lambda jh}\right] - m(1-\alpha_1)^{m-1}\alpha_2 h \sum_{j=m+1}^{\infty} j(1-\alpha_2)^{j-m-1} e^{-\lambda jh}
\tag{6.79}
$$

It is clear that the second term is negative, the first term is:

$$
\begin{aligned}
&\frac{\partial}{\partial \alpha_1}\left[\alpha_1 h \sum_{j=1}^{m} j(1-\alpha_1)^{j-1} e^{-\lambda jh}\right] \\
&= \frac{\partial}{\partial \alpha_1}\left[\frac{\alpha_1 h e^{-\lambda h}}{\left[1-(1-\alpha_1)e^{-\lambda h}\right]^2} - \frac{\alpha_1 h (1-\alpha_1)^m e^{-\lambda(m+1)h}}{\left[1-(1-\alpha_1)e^{-\lambda h}\right]^2} - \frac{m\alpha_1 h(1-\alpha_1)^m e^{-\lambda(m+1)h}}{1-(1-\alpha_1)e^{-\lambda h}} \right]
\end{aligned}
\tag{6.80}
$$

Note that the magnitudes of both the second term and the third term of (6.77) are small since (6.11) is true. Thus the first term dominates the others and we obtain:

$$\frac{\partial}{\partial \alpha_1}\left[\frac{\alpha_1 h e^{-\lambda h}}{\left[1-(1-\alpha_1)e^{-\lambda h}\right]^2}\right] = \frac{h e^{-\lambda h}\left[1-\alpha_1 e^{-\lambda h} - e^{-\lambda h}\right]}{\left[1-(1-\alpha_1)e^{-\lambda h}\right]^3} \tag{6.81}$$

Note that the above derivative is analyzed before, see (6.60). Thus (6.79) is negative for suitable values of the parameters. In the case of k_2, refering to the third equation of (6.77) we obtain:

$$\begin{aligned}\frac{\partial E[t_f]}{\partial \alpha_2} = &\left[\frac{mh\left(1-e^{-\lambda h}\right)\left(1-\alpha_1\right)^m e^{-\lambda(m+1)h}}{\left[1-(1-\alpha_1)e^{-\lambda h}\right]^2}\right] \\ &+ h\left(1-\alpha_1\right)^m e^{-\lambda(m+1)h}\left[\frac{1-\alpha_2 e^{-\lambda h} - e^{-\lambda h}}{\left[1-(1-\alpha_2)e^{-\lambda h}\right]^3}\right]\end{aligned} \tag{6.82}$$

The first term of (6.82) is positive. The second term is negative for small values of λ and it dominates the first term because of its denominator. Again, for large values of λ, the second term of (6.82) is positive and by the time λ gets large, the magnitude of (6.82) is very small because (6.11) is true. In summary, (6.82) is negative. Because α_1 and α_2 are tail probabilities of a normal distribution, it is clear that $\partial\alpha_i/\partial k_i < 0$, i = 1, 2. Thus:

$$\frac{\partial E[t_f]}{\partial k_1} = \frac{\partial E[t_f]}{\partial \alpha_1}\frac{\partial \alpha_1}{\partial k_1} > 0 \tag{6.83}$$

and:

$$\frac{\partial E[t_f]}{\partial k_2} = \frac{\partial E[t_f]}{\partial \alpha_2}\frac{\partial \alpha_2}{\partial k_2} > 0 \tag{6.84}$$

Next, consider the time of the change in the width of the control limits, m. Refering to the third equation of (6.77), we obtain:

$$\frac{\partial E[t_f]}{\partial m}$$

$$= \left[\lambda h^2 - h\log(1-\alpha_1)\right](1-\alpha_1)^m e^{-\lambda(m+1)h}\left[\frac{\alpha_1}{\left[1-(1-\alpha_1)e^{-\lambda h}\right]^2} - \frac{\alpha_2}{\left[1-(1-\alpha_2)e^{-\lambda h}\right]^2}\right] +$$

$$\left[\lambda mh^2 + h - mh\log(1-\alpha_1)\right](1-\alpha_1)^m e^{-\lambda(m+1)h}\left[\frac{\alpha_1}{1-(1-\alpha_1)e^{-\lambda h}} - \frac{\alpha_2}{1-(1-\alpha_2)e^{-\lambda h}}\right] \quad (6.85)$$

To determine wether the above derivative is positive or negative, the following forms are analyzed:

$$\frac{\alpha_1}{1-(1-\alpha_1)e^{-\lambda h}} - \frac{\alpha_2}{1-(1-\alpha_2)e^{-\lambda h}} = \frac{\left(e^{-\lambda h}-1\right)(\alpha_2-\alpha_1)}{\left(1-(1-\alpha_1)e^{-\lambda h}\right)\left(1-(1-\alpha_2)e^{-\lambda h}\right)} < 0 \quad (6.86)$$

and:

$$\frac{\alpha_1}{\left[1-(1-\alpha_1)e^{-\lambda h}\right]^2} - \frac{\alpha_2}{\left[1-(1-\alpha_2)e^{-\lambda h}\right]^2} = \frac{\alpha_1\left[1-(1-\alpha_2)e^{-\lambda h}\right]^2 - \alpha_2\left[1-(1-\alpha_1)e^{-\lambda h}\right]^2}{\left[1-(1-\alpha_1)e^{-\lambda h}\right]^2\left[1-(1-\alpha_2)e^{-\lambda h}\right]^2}$$

$$= \frac{(\alpha_2-\alpha_1)\left(\alpha_1\alpha_2 e^{-2\lambda h} + 2e^{-\lambda h} - e^{-2\lambda h} - 1\right)}{\left[1-(1-\alpha_1)e^{-\lambda h}\right]^2\left[1-(1-\alpha_2)e^{-\lambda h}\right]^2} \quad (6.87)$$

It is clear that (6.86) is true since $e^{-\lambda h} < 1$ and $\alpha_1 < \alpha_2$ by construction. While (6.87) is not that clear but for suitable values of the parameters it is positive. In addition, refering to (6.11) that the magnitudes of both terms of (6.85) are small, and since $E(t_n)$ is increasing in m this imply E(t) is increasing in m even though (6.85) could be negative for some values of the model parameters. Thus, the monotone behavior of E(t) is clear. In order to obtain a sense of the trends indicated by all of the above derivatives, consider the same numerical cases (6.53) and we obtain the associated values of E(t) are listed in Table 6.9. The numerical results conform to the above analysis and the conclusion is summarized below:

First observe that the numerical results illustrate the fact that E(t) is increasing in all k_1, k_2 and m and E(t) is decreasing in λ and n. Note also that increasing n or m enhance the effect of λ and k_1 and k_2. Furthermore, k_2 appears to have a greater effect on E(t) than does k_1. Note that setting $k_1 = k_2$ at any value should result in the same values of E(t) as occur in the standard case and that this is confirmed in the evaluation of the cases having $k_2 = k_1 = 2.5$. Finally, observe that the value of E(t) is less for the generalized chart than for the standard chart because the expected cycle length is less for the generalized chart. This is equivalent to say that 1/E(t) is greater for the generalized chart than for the standard chart because the inverse of the expected cycle length is greater for the generalized chart. This is a desirable feature of the generalized Shewhart control chart.

In order to determine if the generalized Shewhart control charts are economically attractive for any given set parameters, the conclusion is the following:

The convex behavior with an interior minimum that is characteristic of case one is apparent when the increasing behavior of the model terms and the decreasing behavior of the model terms are in balance. This means that the rate of increase of the increasing model terms will produce is offset by a similar rate of decreasing in the decreasing model terms to produce the minimum in the interior of the feasible region. The convex increasing behavior of the expected total cost per unit time that is characteristic of case two is apparent when the increasing behavior of the model terms dominate the decreasing behavior of the model terms. The convex decreasing behavior of the expected total cost per unit time that is characteristic of case three is apparent when the decreasing behavior of the model terms and the increasing behavior of the model terms. In both case two and case three above, the C_T curve appears to display strictly increasing or decreasing behavior over ranges of k_1, k_2, n, and m, the generalized Shewhart control charts are probably not advantageous. This analytical results can be seen in Table 6.10. Where the up arrow means that the corresponding term is increasing and the down arrow means that the corresponding term is decreasing. In addition, the GRG approach is used as direction may not yield integer values for m and n, and the following rounding method can be used to get integer values. This method functions as the following steps:

(1) Use GRG to obtain a continuous solution for m and n;

(2) Study the $2^3 = 8$ different combinations of the neighborhood integer values of (m, n);

(3) Choose the values of (m. n) which minimize the expected total cost.

Finally, the optimization problem for the general cost models can be solved using the two-step optimization technique and the following parameter values are used:

$$m = (10, 20, 30, 40)$$
$$n = (2, 4, 6, 8)$$
$$\theta = 20, \delta = 0.5, \lambda = 0.1, a = 0.05, b = 1.25$$
$$c_d = 1.0, r = 200, p = 0.01, c_0 = 1.0, c_i = 1.0, c_f = 100, c_t = 10 \tag{6.88}$$

For each (m, n) determine the optimal values of (k_1, k_2) and the optimal value of the expected total cost. At this step, the optimal value of the expected total cost occurs when $(k_1^*, k_2^*) = (2.79, 2.52)$, call this $C_T[1]$. As an example, the plot of $C_T(m = 20, n = 6, k_1, k_2)$ can be seen in Figure 6.4. Next, plot $C_T[1](m, n)$ and let $C_T[2]$ be the optimal value of $C_T[1](m, n)$. Then $C_T[2]$ occurs at $(m^*, n^*) = (19, 6)$. See Figure 6.5. Finally, the optimal value of the expected total cost per unit time function for the general cost model I, $C_T^* = C_T[2] = 8.6472$ and $(k_1^*, k_2^*, m^*, n^*) = (2.79, 2.52, 19, 6)$. The detail results of the cost models are listed in Table 6.11.

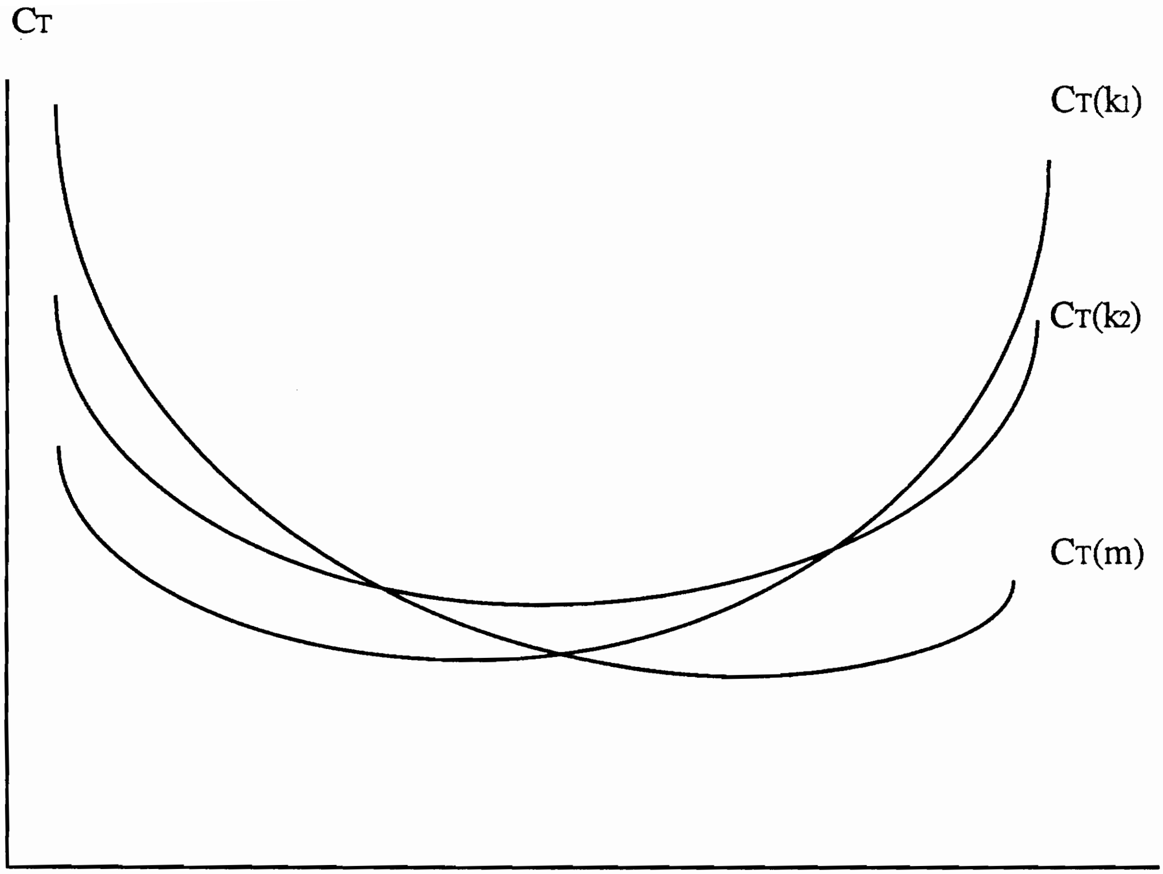

Figure 6.1 C_T as a Function of m, k_1, or k_2, Case 1.

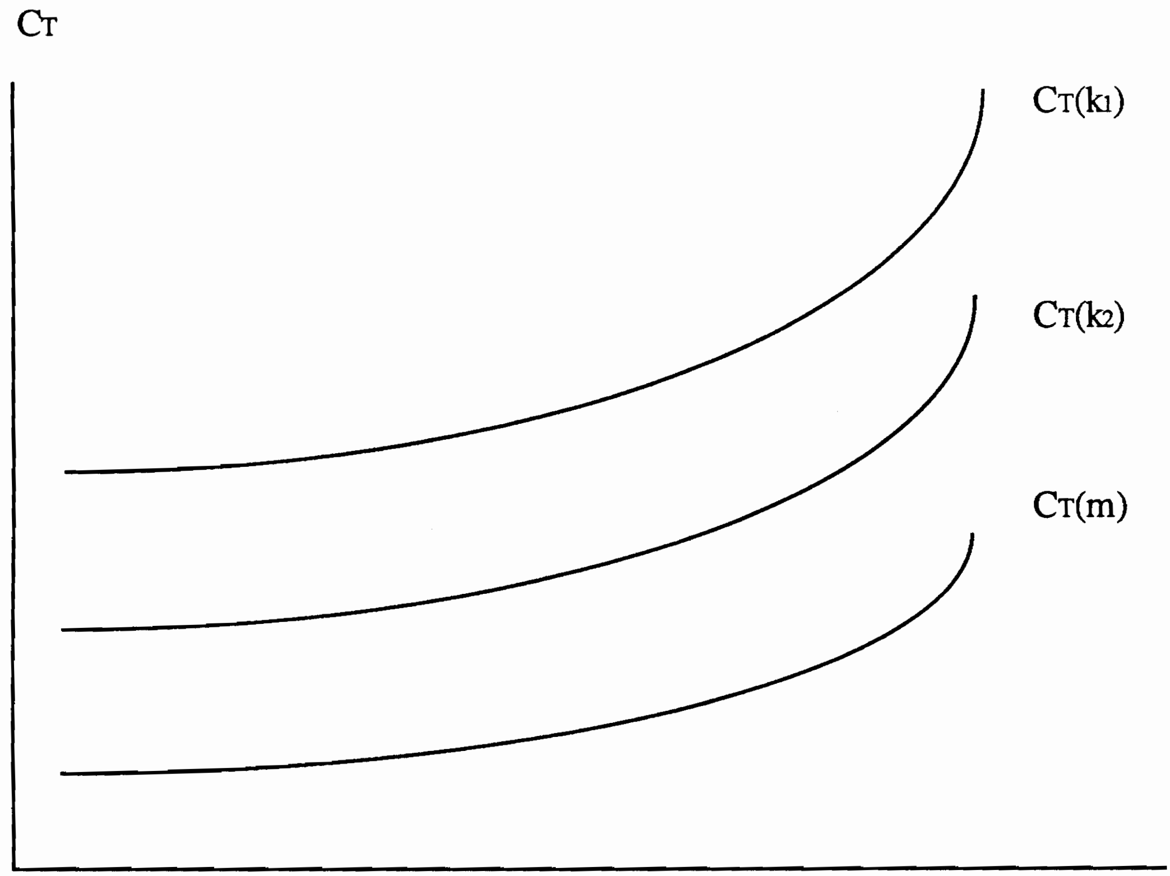

Figure 6.2 C_T as a Function of m, k_1 or k_2, Case 2.

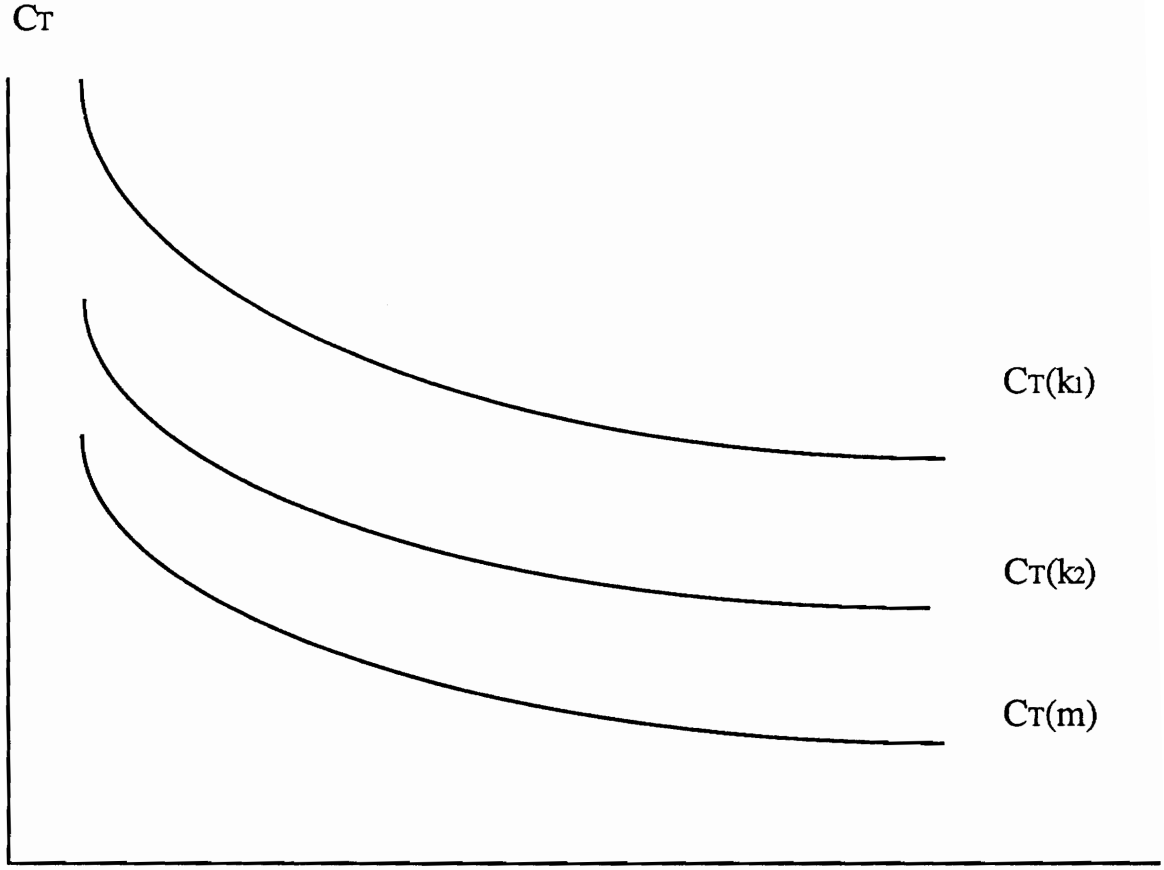

Figure 6.3 C_T as a Function of m, k_1 or k_2, Case 3.

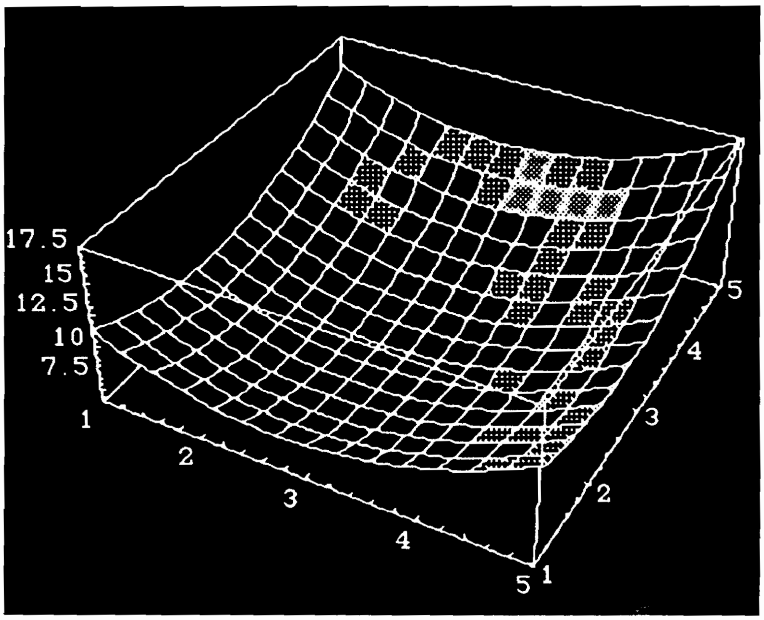

Figure 6.4 C_T as a Function of ($m=20$, $n=6$, k_1, k_2)

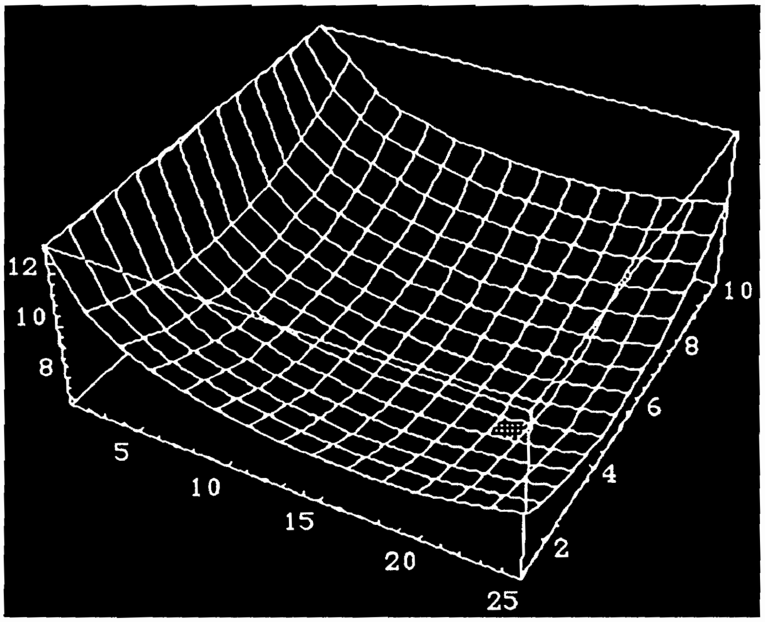

Figure 6.5 $C_T[1]$ as a Function of (m, n)

Table 6.1 Values for C_i

$n \backslash c_i$	1	2	3	4	5
3	4	7	10	13	16
5	6	11	16	21	26
8	9	17	25	33	41
10	11	21	31	41	51

Table 6.2 Values for P[A]. The Standard Case

λ\k	2.50	2.75	3.0
.01	.5527	.3738	.2055
.05	.1947	.1047	.0482
.10	.1055	.0570	.0421
.20	.0530	.0264	.0116

Table 6.3 Values of P[A]. The Generalized Case

k_2=1.75, m=10				k_2=1.75, m=20				k_2=1.75, m=40			
$\lambda \backslash k_1$	2.50	2.75	3.0	$\lambda \backslash k_1$	2.50	2.75	3.0	$\lambda \backslash k_1$	2.50	2.75	3.0
.01	.8208	.8123	.8078	.01	.7668	.7473	.7367	.01	.6892	.6447	.6189
.05	.4170	.3930	.3804	.05	.3138	.2691	.2451	.05	.2290	.1580	.1180
.10	.2117	.1848	.1708	.10	.1401	.0991	.0773	.10	.1092	.0591	.0317
.20	.0785	.0568	.0455	.20	.0562	.0301	.0164	.20	.0532	.0263	.0121

k_2=2.0, m=10				k_2=2.0, m=20				k_2=2.0, m=40			
$\lambda \backslash k_1$	2.50	2.75	3.0	$\lambda \backslash k_1$	2.50	2.75	3.0	$\lambda \backslash k_1$	2.50	2.75	3.0
.01	.7654	.7531	.7466	.01	.7226	.6969	.6828	.01	.6610	.6080	.5771
.05	.3423	.3133	.2981	.05	.2738	.2236	.1965	.05	.2176	.1431	.1011
.10	.1694	.1397	.1242	.10	.1263	.0835	.0605	.10	.1078	.0572	.0296
.20	.0671	.0446	.0329	.20	.0548	.0286	.0148	.20	.0531	.0263	.0121

k_2=2.25, m=10				k_2=2.25, m=20				k_2=2.25, m=40			
$\lambda \backslash k_1$	2.50	2.75	3.0	$\lambda \backslash k_1$	2.50	2.75	3.0	$\lambda \backslash k_1$	2.50	2.75	3.0
.01	.6772	.6590	.6494	.01	.6521	.6167	.5972	.01	.6610	.6080	.5771
.05	.2635	.2291	.2111	.05	.2316	.1756	.1451	.05	.2055	.1274	.0832
.10	.1325	.1004	.0836	.10	.1144	.0698	.0464	.10	.1078	.0572	.0296
.20	.0586	.0355	.0236	.20	.0538	.0274	.0136	.20	.0531	.0263	.0121

k_2=2.50, m=10				k_2=2.50, m=20				k_2=2.50, m=40			
$\lambda \backslash k_1$	2.50	2.75	3.0	$\lambda \backslash k_1$	2.50	2.75	3.0	$\lambda \backslash k_1$	2.50	2.75	3.0
.01	.5527	.5260	.5120	.01	.5527	.5034	.4762	.01	.5527	.4675	.4168
.05	.1947	.1560	.1356	.05	.1947	.1337	.1006	.05	.1947	.1138	.0676
.10	.1055	.0716	.0539	.10	.1055	.0599	.0354	.10	.1055	.0534	.0264
.20	.0530	.0296	.0175	.20	.0530	.0266	.0127	.20	.0530	.0262	.0121

Table 6.4 Values for P[B]. The Standard Case

$\lambda \backslash k$	2.50	2.75	3.0
.01	.4477	.6262	.7947
.05	.8053	.8952	.9517
.10	.8945	.9560	.9759
.20	.9470	.9736	.9884

Table 6.5 Values of P[B]. The Generalized Case

k_2=1.75, m=10

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	.1378	.1049	.1246
.05	.4900	.5015	.5078
.10	.7103	.7266	.7355
.20	.8879	.9051	.9144

k_2=1.75, m=20

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	.1840	.1945	.2004
.05	.6120	.6427	.6600
.10	.8223	.8559	.8747
.20	.9379	.9627	.9763

k_2=1.75, m=40

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	.2719	.3039	.3230
.05	.7447	.8068	.8433
.10	.8860	.9341	.9616
.20	.9468	.9734	.9882

k_2=2.0, m=10

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	.1712	.1743	.1759
.05	.5478	.5593	.5657
.10	.7510	.7674	.7763
.20	.9027	.9199	.9292

k_2=2.0, m=20

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	.2058	.2162	.2221
.05	.6389	.6697	.6869
.10	.8343	.8680	.8868
.20	.9396	.9688	.9780

k_2=2.0, m=40

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	.2793	.3112	.3303
.05	.7490	.8111	.8475
.10	.8867	.9348	.9624
.20	.9468	.9734	.9882

k_2=2.25, m=10

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	.2972	.3003	.3020
.05	.6981	.7097	.7160
.10	.8409	.8572	.8661
.20	.9316	.9484	.9580

k_2=2.25, m=20

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	.3092	.3193	.3255
.05	.7296	.7603	.7775
.10	.8694	.9031	.9219
.20	.9441	.9668	.9824

k_2=2.25, m=40

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	.3399	.3718	.3909
.05	.7748	.8369	.8733
.10	.8905	.9387	.9662
.20	.9468	.9734	.9882

k_2=2.50, m=10

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	.4477	.4508	.4525
.05	.8053	.8168	.8231
.10	.8945	.9109	.9198
.20	.9470	.9641	.9734

k_2=2.50, m=20

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	.4477	.4581	.4640
.05	.8053	.8360	.8532
.10	.8945	.9283	.9470
.20	.9470	.9717	.9853

k_2=2.50, m=40

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	.4477	.4796	.4987
.05	.8053	.8673	.9034
.10	.8945	.9427	.9702
.20	.9470	.9736	.9883

Table 6.6 Values for E[D]. The Standard Case

(n=5) $\lambda\backslash k$	2.50	2.75	3.0
.01	11.457	18.957	32.902
.05	11.460	18.960	32.905
.10	11.464	18.964	32.909
.20	11.473	18.973	32.918

(n=10) $\lambda\backslash k$	2.50	2.75	3.0
.01	5.0841	7.7491	12.326
.05	5.0874	7.7525	12.329
.10	5.0916	7.7566	12.333
.20	5.0999	7.7649	12.342

Table 6.7 Values of E[D]. The Generalized Case

n=5, k_2=2.25, m=10

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	6.0025	6.1022	6.1785
.05	6.7837	7.2214	7.5579
.10	7.4925	8.2454	8.8273
.20	8.3570	9.5122	10.413

n=5, k_2=2.25, m=20

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	6.4483	6.8205	7.1432
.05	8.3409	9.7950	11.074
.10	9.5867	11.818	13.811
.20	10.575	13.518	16.203

n=5, k_2=2.25, m=40

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	7.4445	8.6724	9.9652
.05	10.655	14.399	18.710
.10	11.731	16.695	22.319
.20	12.129	17.698	24.228

n=5, k_2=2.5, m=10

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	11.467	11.633	11.769
.05	11.460	12.226	12.821
.10	11.464	12.769	13.789
.20	11.473	13.443	15.001

n=5, k_2=2.5, m=20

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	11.467	12.013	12.504
.05	11.460	13.587	15.502
.10	11.464	14.658	17.588
.20	11.473	15.561	19.414

n=5, k_2=2.5, m=40

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	11.467	12.992	14.655
.05	11.460	16.073	21.322
.10	11.460	17.236	24.074
.20	11.473	17.770	25.532

n=10, k_2=2.25, m=10

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	3.6135	3.7023	3.7866
.05	4.1196	4.5069	4.8776
.10	4.5677	5.2293	5.8685
.20	5.0920	6.0950	7.0789

n=10, k_2=2.25, m=20

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	3.8351	4.1106	4.4222
.05	4.8415	5.8923	7.1071
.10	5.4534	7.0246	8.8839
.20	5.8728	7.8669	10.305

n=10, k_2=2.25, m=40

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	4.2409	4.9282	5.8731
.05	5.6237	7.6313	10.528
.10	5.9998	8.4456	12.117
.20	6.0923	8.6881	12.723

n=10, k_2=2.5, m=10

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	5.0841	5.2058	5.3232
.05	5.0874	5.6129	6.1250
.10	5.0916	5.6129	6.8535
.20	5.0999	6.4194	7.7443

n=10, k_2=2.5, m=20

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	5.0841	5.4115	5.7897
.05	5.0874	6.3110	7.7616
.10	5.0916	6.8838	9.0671
.20	5.0999	7.3124	10.113

n=10, k_2=2.5, m=40

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	5.0841	5.8235	6.8548
.05	5.0874	7.1874	10.274
.10	5.0916	7.5999	11.441
.20	5.0999	7.7262	11.888

Table 6.8 Values for E(t). The Standard Case

(n=5) λ\k	2.50	2.75	3.0
.01	49.852	74.675	104.75
.05	25.326	34.903	50.258
.10	19.197	27.411	41.835
.20	15.598	23.344	37.461

(n=10) λ\k	2.50	2.75	3.0
.01	47.002	52.536	88.540
.05	20.196	21.446	30.172
.10	13.498	14.320	21.774
.20	9.5631	10.243	17.132

Table 6.9 Values of E(t). The Generalized Case

$n=5, k_2=2.25, m=10$

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	18.955	19.099	20.636
.05	13.124	14.523	14.997
.10	10.314	11.332	11.967
.20	8.1170	8.5474	9.0523

$n=5, k_2=2.25, m=20$

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	20.931	21.957	22.241
.05	15.490	17.001	18.583
.10	12.053	13.638	15.244
.20	9.8860	11.629	14.156

$n=5, k_2=2.25, m=40$

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	22.174	23.077	26.104
.05	17.667	19.478	22.710
.10	13.731	15.421	22.366
.20	11.882	14.930	21.007

$n=5, k_2=2.5, m=10$

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	49.852	50.677	51.934
.05	25.326	25.503	26.079
.10	19.197	21.229	24.002
.20	15.598	17.322	19.140

$n=5, k_2=2.5, m=20$

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	49.852	51.766	52.685
.05	25.326	26.770	28.196
.10	19.197	21.651	25.535
.20	15.598	18.521	21.668

$n=5, k_2=2.5, m=40$

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	49.852	52.588	56.532
.05	25.326	27.658	33.920
.10	19.197	25.821	31.273
.20	15.598	19.916	27.061

$n=10, k_2=2.25, m=10$

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	18.537	18.921	19.059
.05	11.902	12.077	12.884
.10	8.7082	9.3277	9.9963
.20	6.2021	7.5992	8.2424

$n=10, k_2=2.25, m=20$

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	19.677	20.447	21.003
.05	12.763	13.099	15.788
.10	9.4889	11.353	13.269
.20	6.8839	9.0226	11.848

$n=10, k_2=2.25, m=40$

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	20.964	21.005	22.169
.05	13.443	15.723	18.206
.10	11.584	13.998	16.832
.20	9.9767	12.021	15.993

$n=10, k_2=2.5, m=10$

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	47.002	48.535	49.222
.05	20.196	21.723	22.274
.10	13.498	15.277	17.556
.20	9.5631	11.448	13.221

$n=10, k_2=2.5, m=20$

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	47.002	48.608	49.955
.05	20.196	22.008	24.533
.10	13.498	16.232	19.135
.20	9.5631	12.002	15.728

$n=10, k_2=2.5, m=40$

$\lambda \backslash k_1$	2.50	2.75	3.0
.01	47.002	49.992	51.194
.05	20.196	23.409	26.236
.10	13.498	17.207	21.778
.20	9.5631	12.311	17.650

Table 6.10 The Behavior of the Cost Terms

	λ	c_0	k_1	k_2	m	n	c_i	c_f	c_t	$c_{d,r}$	C_T
C_i		↑				↑	↑				↑
C_f	↓		↓	↓	↓			↑			↓
C_t	↑		↑	↑	↑				↑		↑
C_d	↑		↑	↑	↑	↓				↑	↑
$E(t)$	↓		↑	↑	↑	↓					↓

Table 6.11 A Set of Results of the Cost Models

	Model I		Model II		Model III	
	Standard	Generalized	Standard	Generalized	Standard	Generalized
n	6	6	5	6	5	5
m	0	19	0	16	0	14
k_1		2.79		2.78		2.78
k_2		2.52		2.54		2.53
k	2.78		2.79		2.77	
C_T	8.9655	8.6472	8.8242	8.6187	8.9122	8.7637

CHAPTER 7

Conclusions and Discussion

7.1 Conclusions

This dissertation develops cost models for the standard and the generalized Shewhart control chart. The models assume that the quality characteristic of interest is distributed normally with known variance and that the time between shifts are negative exponential, uniform and Weibull distributed. The analysis for the three cost models presented in this dissertation yield several interesting points. The first of these is that the analysis of the costs of operating any type of control chart should be treated very carefully as the cost function may not always have the commonly assumed regularity. The choice of cost coefficients, the time of shift distribution and time shift distribution parameters have a direct influence on the performance of the expected total cost per unit time function. The behavior of the models are analyzed algebraically and numerically using calculus and Mathematica [52] and GINO [25]. The important results of the analysis performed show that the generalized Shewhart charts for means may be economically attractive when the inspection cost, the true signal cost and the nonconforming product cost together balance the expected cycle length and the false alarm cost. When this is the case, the expected total cost per unit time function is convex with an interior minimum and an opportunity for optimization of the generalized Shewhart control chart. When one or more of the model terms dominates the others, the expected total cost per unit time will display the same increasing or decreasing behavior as the dominating factor and the generalized cost model as studied in this research will be unattractive.

The second conclusion is that all model parameters and variables are important to the expected total cost per unit time function. The control limits k_1 and k_2 have a greater effect than do the distribution parameters and k_2 has a greater effect than does k_1. It is also true that the sample size, n, and the time of the change in the width of the control limits, m, enhance the effect of the distribution parameters and k_1 and k_2.

The final conclusion is that there are control chart applications for which the cost models are useful. Values of the production process parameters that display more commonly encountered relationships leads to the generalized Shewhart control chart having lower cost than the corresponding standard Shewhart control chart. For the example case analyzed in the previous chapter, see Table 6.12, the savings is \$8.97 - \$8.65 = \$0.32 per item produced. Since the production rate assumed is 200/hour, the savings is \$64 per hour for cost model I. For cost model II, the savings is \$8.82 - \$8.62 = \$0.20 per item produced when θ = 1000. Since the production rate assumed is 200/hour, the savings is $40 per hour. For cost model III, the savings is $8.91 - $8.76 = $0.15. Since the production rate assumed is 200/hour, the savings is $ 30 per hour. These savings are dramatic and therefore the cost models are worth pursuing. Thus, the cost models can be appealing and the control chart adjustment strategy presented in this dissertation is robust.

7.2 Extensions

The Shewhart control chart adjustment strategy by its very structure creates an exhaustive range of possibilities for future research. Potential expansions are discussed below, practical and analytical.

(1) From the theoretical point of view, analysis of the cost models is interesting if some of the assumptions on distribution are relaxed. For example, the population distribution is not normal or the variance of the quality characteristic is not constant. In this case, the sample variance must be employed to estimate the population variance . In addition, both skewness and kurtosis should also be considered in the process average. One advantage of this analysis would be that a more general production process could be analyzed. Although more numerical methods will be involved, the model may be used to analyze a general production process as well.

(2) One extension of this dissertation is to change the distribution of the time until a process shift. Another candidate life time distribution may be Gamma. The cost model constructed

using the Gamma distribution should also be very interesting and it can represents the behavior of the time to process shift adequately. Cost categories can be constructed under the Gamma life distribution and if there is a cost term for which the formula has no closed form, numerical method and computer programming techniques may be used in obtaining the optimal solution for the corresponding cost model.

(3) The control chart adjustment strategy can be extended to the case of the variable sampling interval (VSI) Shewhart control chart. For VSI Shewhart control chart, the parameters that are optimized in order to specify a chart design are the width of the control limits, the sample size, the lengths of the delay intervals, the changed control limits, and the probabilities that the various delay intervals are selected. It is obvious that the corresponding cost model so constructed may be complicated. However, it is worth to study the control chart adjustment strategy for the VSI Shewhart control chart since it has been shown by Nachlas et. al. [31] that the VSI control charts are often more economical than fixed interval control charts.

(4) The control chart adjustment strategy can be extended to the other control charts, say CUSUM chart and EWMA chart, and the corresponding analysis could be interesting.

In summary, the Shewhart control chart adjustment strategy developed in this dissertation is found to be robust and economical. It has many extensions and applications for analyzing a production process. Under this strategy, the sensitivity to small but anticipated shifts can be increased in the process average and therefore the corresponding assignable cause can be detected.

References

[1] Arnold, J. C., (1970), "*A Markovian Sampling Policy Applied to Quality Monitoring of Systems*," Biometrics, 26, 739-747.

[2] Barlow, R. E. and Proschan, F., (1981), Statistical Theory of Reliability and Life Testing, McArdle Press, Inc.

[3] Bather, J. A., (1963),"*Control Charts and the Minimization of Cost*, " Journal of the Royal Statistical Society, Ser. B, 25, pp. 49-80.

[4] Bazaraa, M. S., Sherali, H. D. and Shetty, C. M., (1993), Nonlinear Programming: Theory and Algorithm, John Wiley & Sons, Inc.

[5] Bissel, A. F., (1969), "*CUSUM Techniques for Quality Control(with discussion)*," Applied Statistics, 18, pp. 1-30.

[6] Box, G. E. P., Jerkins, G. M., (1963), "*Further Contributions to Adaptive Quality Control: Simultaneous Estimation of Dynamics: Nonzero Cost*," Bulletin of the International Statistical Institute, 34, 943-974.

[7] Box, G. E. P., Jerkins, G. M. and MacGregor, J. F., (1974), "*Some Recent Advances in Forecasting and Control*, " Applied Statistics, 23, pp. 158-179.

[8] Brook, D. and Evans, D. A., (1972), "*An Approach to the Probability Distribution of CUSUM Run Length*," Biometrika, 59, pp. 539-549.

[9] Chui, W. K. and Wetherill, G. B., (1974), "*A Simplified Scheme for the Economic Design of $\overline{X}$ Charts*," Journal of Quality Technology, 6, 2, pp. 63-69.

[10] Croser, R. B., (1986), "*A New Two-Sided Cumulative Sum Quality Control Scheme*," Technometrics, 28, 187-194.

[11] Crowder, S. V., (1987), "*A Simple Method for Studying Run Length Distribution of Exponentially Weighted Moving Average Control Charts*," Technometrics, 29, 401-407.

[12] Crowder, S. V., (1992), "*An SPC Model for Short Production Runs: Minimizing Expected Cost*," Technometrics, 34, 1, pp. 64-73.

[13] Duncan, A. J., (1956), "*The Economic Design of $\overline{X}$ Charts Used to Maintain Current Control of a Process*," Journal of American Statistical Association, 51, 274, pp. 228-242.

[14] Duncan, A. J., (1985), Quality Control and Industrial Statistics, 5th ed., R. D. Irwin, Inc.,Homewood, IL.

[15] De Oliveira, J. and Littauer, S. B., (1965), "*Double Limit and Run Control Charts*," Revue de Statistique Appliques, 13, 2.

[16] Gibra, I. N., (1971), "*Economically Determination of the Parameters of $\overline{X}$ Control Charts*," Management Science, 17, 9, pp. 635-646.

[17] Goel, A. L., Jain, S. C. and Wu, S. M., (1968). "*An Algorithm for the Determination of the Economic Design of $\overline{X}$ Charts Based on Duncan's Model*," Journal of Am. Stat. Assoc.,
62, 321, pp. 304-320.

[18] Goel, A. L. and Wu. S. M., (1971), "*Determination of ARL and a Contour Nomogram for CUSUM Charts to Control Normal Mean*," Technometrics, 13, 221-230.

[19] Hiller, F. S., (1964), " $\overline{X}$ *Chart Control Limits Based on a Small Number of Subgroups*," Industrial Quality Control, 20, 8, 24-29.

[20] Hiller, F. S., (1969), " $\overline{X}$ *and R Chart Control Limits Based on a Small Number of Subgroups*," Journal of Quality Technology, 1, 1, 17-26.

[21] Hui, Y. V. and Jensen, D. R., (1980), "*Markovian Time-Delay Sampling Policies*," Technical Report Q-5, Virginia Polytechnic Institute and State University, Department of Statistics.

[22] Hunter, J. S., (1986), "*The Exponentially Weighted Moving Average*," Journal of Quality Technology, 18, 203-210.

[23] Jensen, K. L., (1989), "*Optimal Adjustment in the Presence of Process Drift and Adjustment Error*," Unpublished Ph. D. dissertation, Iowa State University, Dept. of Statistics.

[24] Knappenberger, H. A. and Granbage, A. H., (1969), "*Minimum Cost Quality Control Tests*," AIIE Transaction, 1, 1, pp. 24-32.

[25] Lasdon, L. and Warren A., (1985), General Interactive Optimizer-GINO, Scientific Press, Palo Alto.

[26] Lucas, J. M., (1976), "*The Design and Use of Cumulative Sum Quality Control Schemes*," Journal of Quality Technology, 8, 1-12.

[27] Lucas J. M. and Corsier, R. B., (1982), "*Fast Initial Response for CUSUM Quality Control Schemes*," Technometrics, 24, 199-205.

[28] Lucas, J. M. and Saccucci, M. S., (1990), "*Exponentially Weighted Moving Average Control Schemes: Properties and Enhancements*," Technometrics, 32, 1, pp. 1-12.

[29] Nachlas, J. A. and Kim, S. I., (1989), "*Generalized Attribute Acceptance Sampling Plans*," Journal of Quality Technology, Vol. 21, No. 1, pp 32-40.

[30] Nachlas, J. A. and Kumar, A., (1991), "*Reliability Estimation Using Doubly Censored Field Data*," Working Paper, Dept. of ISE, Virginia Tech.

[31] Nachlas, J. A., Clark, L. A. and Reynolds, Jr. M. R., (1991), *A Cost Analysis of Variable Time Delay Shewhart Control Charts*, Working Paper, Virginia Polytechnic Institute and State University, Dept. of ISE and Statistics.

[32] Montgomery, D. C., (1980), "*The Economic Design of Control Charts: A Review and Literature Survey*," Journal of Quality Technology, Vol. 12, No. 2, pp 75-87.

[33] Montgomery, D. C., Gardiner, J. S. and Pizzano, B. A., (1987), "*Statistical Process Control Methods for Detecting Small Process Shifts*," in Frontiers in Statistical Quality Control, eds. West Germany, Physica-Verlag, pp. 161-178.

[34] Muth, J. F., (1960), "*Optimal Properties of Exponentially Weighted Forecasts*," Journal of American Statistical Association, 55, 299-306.

[35] Page, E. S., (1954), "*Continuous Inspection Schemes*," Biometrika, 41, 100-114.

[36] Page, E. S., (1962), "*A Modified Control Chart With Warning Lines*," Biometrika, 49, pp. 171-176.

[37] Phillips, H. B., (1946), Analytical Geometry and Calculus, Addison-Wesley Press, Cambridge, MA.

[38] Quesenberry, C. P., (1991), "*SPC Q-Charts for Start-up Processes and Short or Long Runs*," Journal of Quality Technology, 23, 3, 213-224.

[39] Quesenberry, C. P., (1991), "*SPC Q-Charts for a Binomial Parameter p: Short or Long Runs*," Journal of Quality Technology, 23, 3, 239-246.

[40] Quesenberry, C. P., (1991), "*SPC Q-Charts for a Poisson Parameter l: Short or Long Runs*," Journal of Quality Technology, 23, 4, 296-303.

[41] Reynolds, Jr. M. R.; Amin, R. W.; Arnold, J. C. and Nachlas, J. A., (1988), "*$\bar{X}$ Charts With Variable Sampling Intervals,"* Technometrics, Vol. 30, 2, 181-192.

[42] Reynolds, Jr. M. R. and Arnold, J. C., (1989), "*Optimal One-Sided Shewhart Control Charts With Variable Sampling Intervals,"* Sequential Analysis, 8, pp. 51-77.

[43] Reynolds, Jr. M. R.; Amin, R. W.; Arnold, J. C., (1990), "*CUSUM Charts With Variable Sampling Intervals,"* Technometrics, Vol. 32, No. 4, pp. 371-384.

[44] Roberts, S. W., (1959), "*Control Chart Tests Based on Geometric Moving Averages,*" Technometrics, Vol. 1, pp. 239-250.

[45] Roberts, S. W., (1966), "*A Comparison of some Control Chart Procedures,*" Technometrics, Vol. 8, pp. 411-430.

[46] Robinson, P. B. and Ho, T. Y., (1978), "*Average Run Lengths of Geometric Moving Average Charts by Numerical Methods,*" Technometrics, Vol. 20, pp. 85-93.

[47] Saniga, E. M., (1989), " *Economic Statistical Control - Chart Designs With an Application to $\bar{X}$ and R Charts,*" Technometrics, 31, pp. 313-320.

[48] Waldmann, K. H., (1986), "*Bounds for the Distribution of the Run Length of Geometric Moving Average Charts,*" Applied Statistics, 35, 151-158.

[49] Weindling, J. I., Littauer, S. B. and de Oliveira, J. T., (1970), " *Mean Action Time of the $\bar{X}$ Control Chart With Warning Limits,*" J. of Quality. Tech., 2, pp. 79-85.

[50] Wheeler, D. J., (1983), "*Detecting a Shift in Process Average: Tables of the Power Function for $\bar{X}$ Charts,*" Journal of Quality Technology, 15, 155-170.

[51] Wilde, D. J. and Beighter, C. S., (1967), Foundations of Optimization, Prentice-Hall, Englewood Cliffs, N. J.

[52] Wolfram, S. et. al., (1988), Mathematica™, Addison-Wesley Publishing Company.

[53] Yashchin, E., (1987), "*Some Aspects of the Theory of Statistical Control Schemes*," IBM Journal of Research and Development, 31, 199-205.

[54] Zangwill, W. I., (1969), Nonlinear Programming: A Unified Approach, Prentice-Hall, Englewood Cliffs, N. J.

Appendix A.

Useful Formulations

In constructing the mathematical expressions for cost models of this dissertation, the following useful formulations are used frequently. Where $0 < x < 1$, $0 < y < 1$ and $x/y < 1$ for all equations with $N = \infty$:

$$\sum_{i=1}^{\infty} x^{i-1} = \frac{1}{1-x} \tag{A.1}$$

$$\sum_{i=1}^{N} x^{i-1} = \frac{1-x^N}{1-x} \tag{A.2}$$

$$\sum_{i=1}^{\infty} i\, x^{i-1} = \frac{1}{(1-x)^2} \tag{A.3}$$

$$\sum_{i=1}^{N} i\, x^{i-1} = \frac{1-x^N}{(1-x)^2} - \frac{Nx^N}{1-x} \tag{A.4}$$

$$\sum_{i=1}^{\infty} i^2\, x^{i-1} = \frac{1}{(1-x)^2} + \frac{2x}{(1-x)^3} \tag{A.5}$$

$$\sum_{i=1}^{N} i^2\, x^{i-1} = \frac{1}{(1-x)^2} + \frac{2x\left(1-x^N\right)}{(1-x)^3} - \frac{(N-1)^2 x^N}{(1-x)^2} + \frac{N^2 x^{N+1}}{(1-x)^2} \tag{A.6}$$

$$\sum_{i=m+1}^{\infty} i\, x^{i-m-1} = \frac{m+1}{1-x} + \frac{x}{(1-x)^2} \tag{A.7}$$

$$\sum_{i=m+1}^{N} i\, x^{i-m-1} = \frac{m+1}{1-x} + \frac{x\left(1-x^{N-m-1}\right)}{(1-x)^2} - \frac{Nx^{N-m}}{1-x} = \frac{m}{1-x} + \frac{1-x^{N-m}}{(1-x)^2} - \frac{Nx^{N-m}}{1-x} \tag{A.8}$$

$$\sum_{i=m+1}^{\infty} i^2\, x^{i-m-1} = \frac{1+2m}{(1-x)^2} + \frac{2x}{(1-x)^3} + \frac{m^2}{1-x} \tag{A.9}$$

$$\begin{aligned}\sum_{i=m+1}^{N} i^2\, x^{i-m-1} &= \frac{1}{(1-x)^2} + \frac{2x\left(1-x^{N-m-1}\right)}{(1-x)^3} - \frac{(N-m-1)^2 x^{N-m}}{(1-x)^2} + \frac{(N-m)^2 x^{N-m+1}}{(1-x)^2} \\ &\quad + 2m\left[\frac{1-x^{N-m}}{(1-x)^2} - \frac{(N-m)x^{N-m}}{1-x}\right] + \frac{m^2\left[1-x^{N-m}\right]}{1-x}\end{aligned} \tag{A.10}$$

$$\sum_{i=1}^{N} i\, x^{N-i} = \frac{N}{1-x} - \frac{x^N}{1-x} - \frac{x-x^N}{(1-x)^2} \tag{A.11}$$

$$\sum_{i=1}^{\infty} x^{i-1} y^{-i+1} = \frac{y}{y-x} \tag{A.12}$$

$$\sum_{i=1}^{N} x^{i-1} y^{N-i+1} = \frac{y\left(x^N - y^N\right)}{x-y} \tag{A.13}$$

$$\sum_{i=1}^{N} x^{i-1} y^{N-i} = \frac{x^N - y^N}{x-y} \tag{A.14}$$

$$\sum_{i=m+1}^{N} x^{i-m-1} y^{N-i+m+1} = \frac{y(x^{N-m} - y^{N-m})}{x-y} \tag{A.15}$$

$$\begin{aligned}\sum_{i=s}^{N} i^2 x^{i-s} &= \frac{1}{(1-x)^2} + \frac{2x\left(1-x^{N-s}\right)}{(1-x)^3} - \frac{(N-s)^2 x^{N-s+1}}{(1-x)^2} + \frac{(N-s+1)^2 x^{N-s+2}}{(1-x)^2} \\ &\quad + 2(s-1)\left[\frac{1-x^{N-s+1}}{(1-x)^2} - \frac{(N-s+1)x^{N-s+1}}{1-x}\right] + \frac{(s+1)^2\left(1-x^{N-s+1}\right)}{1-x}\end{aligned} \tag{A.16}$$

Appendix B.

Derivations and Proofs for Cost Model I

(1) Proof of Lemma 3.1

For any integer $m \geq 0$, the following inequalities are hold:

(1) $0 \leq \alpha_1 \leq \alpha_2 \leq 1$;

(2) $1-\alpha_1 \geq 1-\alpha_2$;

(3) $1-\alpha_1 \geq (1-\alpha_1)^m$, $1-\alpha_2 \geq (1-\alpha_2)^m$;

(4) $(1-\alpha_1)^m \geq (1-\alpha_2)^m$;

(5) $e^{-\lambda mh} \leq 1$. (B.1)

Then, the upper bound for P[A] can be constructed as:

$$P[A]=\frac{\alpha_1 e^{-\lambda h}\left\{1-\left[(1-\alpha_1)e^{-\lambda h}\right]^m\right\}}{1-(1-\alpha_1)e^{-\lambda h}}+\frac{(1-\alpha_1)^m e^{-\lambda(m+1)h}\alpha_2}{1-(1-\alpha_2)e^{-\lambda h}}$$
$$\leq\frac{\alpha_1 e^{-\lambda h}\left\{1-\left[(1-\alpha_1)e^{-\lambda h}\right]^m\right\}}{1-(1-\alpha_1)e^{-\lambda h}}+\frac{(1-\alpha_1)^m e^{-\lambda(m+1)h}\alpha_2}{1-(1-\alpha_1)e^{-\lambda h}}$$
$$\leq\frac{\alpha_1 e^{-\lambda h}+(1-\alpha_1)^m e^{-\lambda h}\alpha_2}{1-(1-\alpha_1)e^{-\lambda h}}\leq\frac{\alpha_1 e^{-\lambda h}+\alpha_2 e^{-\lambda h}}{1-(1-\alpha_1)e^{-\lambda h}} \tag{B.2}$$

Similarly, the lower bound for P[A] is:

$$P[A]=\frac{\alpha_1 e^{-\lambda h}\left\{1-\left[(1-\alpha_1)e^{-\lambda h}\right]^m\right\}}{1-(1-\alpha_1)e^{-\lambda h}}+\frac{(1-\alpha_1)^m e^{-\lambda(m+1)h}\alpha_2}{1-(1-\alpha_2)e^{-\lambda h}}$$
$$\geq\frac{\alpha_1 e^{-\lambda h}\left\{1-\left[(1-\alpha_1)e^{-\lambda h}\right]^m\right\}}{1-(1-\alpha_2)e^{-\lambda h}}+\frac{(1-\alpha_1)^m e^{-\lambda(m+1)h}\alpha_2}{1-(1-\alpha_2)e^{-\lambda h}}$$
$$\geq\frac{\alpha_1 e^{-\lambda h}\left[1-(1-\alpha_1)\right]+0}{1-(1-\alpha_2)e^{-\lambda h}}=\frac{\alpha_1^2 e^{-\lambda h}}{1-(1-\alpha_2)e^{-\lambda h}} \tag{B.3}$$

Therefore Lemma 3.1 is true.

(2) Proof of Lemma 3.2

Note that:

$$\left[(1-\alpha_i)e^{-\lambda h}\right]^{m+1}=(1-\alpha_i)^{m+1}e^{-\lambda h}e^{-\lambda mh}\leq(1-\alpha_i)^{m+1}e^{-\lambda h},\ i=1,2;\ \forall m\geq 0 \tag{B.4}$$

So that the upper bound for P[B] can be constructed as:

$$P[B] = \left(1-e^{-\lambda h}\right)\left\{\frac{1-\left[(1-\alpha_1)e^{-\lambda h}\right]^{m+1}}{1-(1-\alpha_1)e^{-\lambda h}} + \frac{\left[(1-\alpha_2)e^{-\lambda h}\right]^{m+1}}{1-(1-\alpha_2)e^{-\lambda h}}\right\}$$

$$\le \left(1-e^{-\lambda h}\right)\left\{\frac{1-\left[(1-\alpha_1)e^{-\lambda h}\right]^{m+1}}{1-(1-\alpha_1)e^{-\lambda h}} + \frac{\left[(1-\alpha_2)e^{-\lambda h}\right]^{m+1}}{1-(1-\alpha_1)e^{-\lambda h}}\right\} \le \left(1-e^{-\lambda h}\right)\left[\frac{1}{1-(1-\alpha_1)e^{-\lambda h}}\right]$$

(B.5)

Sinilarly, P[B] has the lower bound as:

$$P[B] = \left(1-e^{-\lambda h}\right)\left\{\frac{1-\left[(1-\alpha_1)e^{-\lambda h}\right]^{m+1}}{1-(1-\alpha_1)e^{-\lambda h}} + \frac{\left[(1-\alpha_2)e^{-\lambda h}\right]^{m+1}}{1-(1-\alpha_2)e^{-\lambda h}}\right\}$$

$$\ge \left(1-e^{-\lambda h}\right)\left\{\frac{1-\left[(1-\alpha_1)e^{-\lambda h}\right]^{m+1}}{1-(1-\alpha_2)e^{-\lambda h}} + \frac{\left[(1-\alpha_2)e^{-\lambda h}\right]^{m+1}}{1-(1-\alpha_1)e^{-\lambda h}}\right\} \ge \left(1-e^{-\lambda h}\right)\left[\frac{1-(1-\alpha_1)}{1-(1-\alpha_2)e^{-\lambda h}}\right]$$

$$\ge \left(1-e^{-\lambda h}\right)\left[\frac{\alpha_1}{1-(1-\alpha_2)e^{-\lambda h}}\right]$$

(B.6)

Thus Lemma 3.2 is true.

(3) Derivation of $E[t_S(a)]$ for Cost Model I

For case(a) of cost model I, the generalized Shewhart control chart, the derivation of $E[t_S(a)]$ can be achieved through the following steps:

(1) Evaluate the partial summation;

(2) Take derivative with respect to the partial summation;

(3) Exchange derivative and summation;

First we may write $E[t_S(a)]$ as the following form:

$$E[t_s(a)] = \sum_{j=1}^{m}[F(jh) - F(j-1)h]\sum_{i=j}^{m}(i-j)h(1-\beta_1)\beta_1^{i-j}$$

$$E[t_s(a)] = \sum_{j=1}^{m}\left[e^{-\lambda(j-1)h} - e^{-\lambda jh}\right]h(1-\beta_1)\sum_{i=j}^{m}(i-j)\beta_1^{i-j}$$

$$E[t_s(a)] = \sum_{j=1}^{m}\left[e^{-\lambda(j-1)h} - e^{-\lambda jh}\right]h\beta_1(1-\beta_1)\sum_{i-j=0}^{m-j}(i-j)\beta_1^{i-j-1} \tag{B.7}$$

Refer to equation (A.4), we have

$$\sum_{i-j=0}^{m-j}(i-j)\beta_1^{i-j-1} = \frac{1-\beta_1^{m-j}\left[(m-j)(1-\beta_1)+1\right]}{(1-\beta_1)^2} \tag{B.8}$$

Therefore we obtain:

$$E[t_s(a)] = \sum_{j=1}^{m}\left[e^{-\lambda(j-1)h} - e^{-\lambda jh}\right]h\beta_1(1-\beta_1)\frac{1-\beta_1^{m-j}\left[(m-j)(1-\beta_1)+1\right]}{(1-\beta_1)^2}$$

$$\begin{aligned} E[t_s(a)] = {} & \frac{h\beta_1}{(1-\beta_1)}F(mh) - h\beta_1\sum_{j=1}^{m}\left[e^{-\lambda(j-1)h} - e^{-\lambda jh}\right](m-j)\beta_1^{m-j} \\ & - \frac{h\beta_1}{(1-\beta_1)}\sum_{j=1}^{m}\left[e^{-\lambda(j-1)h} - e^{-\lambda jh}\right]\beta_1^{m-j} \end{aligned} \tag{B.9}$$

Now the summation terms can be resolved as follows:

$$\sum_{j=1}^{m}\left[e^{-\lambda(j-1)h} - e^{-\lambda jh}\right](m-j)\beta_1^{m-j} = \left(1-e^{-\lambda h}\right)\sum_{j=1}^{m}\left[e^{-\lambda(j-1)h}\right](m-j)\beta_1^{m-j}$$

$$\sum_{j=1}^{m}\left[e^{-\lambda(j-1)h}\right](m-j)\beta_1^{m-j} = m\sum_{j=1}^{m}\left(e^{-\lambda h}\right)^{j-1}\beta_1^{m-j} - \sum_{j=1}^{m}j\left(e^{-\lambda h}\right)^{j-1}\beta_1^{m-j} \tag{B.10}$$

Let:

$$u=e^{-\lambda h},\ v=\beta_1\ ,\quad S=\sum_{j=1}^{m} j\left(e^{-\lambda h}\right)^{j-1}\beta_1^{m-j}=\sum_{j=1}^{m} ju^{j-1}v^{m-j}$$

$$\Rightarrow\quad \frac{u}{v}S=\sum_{j=1}^{m} ju^{j}v^{m-j-1}$$

$$S-\frac{u}{v}S=\sum_{j=1}^{m} ju^{j-1}v^{m-j}-\sum_{j=1}^{m} ju^{j}v^{m-j-1}=v^{m-1}+\sum_{l=1}^{m-1}u^{l}v^{m-l-1}-\frac{mu^{m}}{v}$$

$$S-\frac{u}{v}S=\frac{1}{v}\sum_{l=0}^{m-1}u^{l}v^{m-l}-\frac{mu^{m}}{v}=\frac{1}{v}\sum_{l=0}^{m}u^{l}v^{m-l}-\frac{(m+1)u^{m}}{v} \tag{B.11}$$

Then

$$v^{m+1}-u^{m+1}=(v-u)\sum_{l=0}^{m}u^{l}v^{m-l}$$

$$S=\frac{\left(v^{m+1}-u^{m+1}\right)}{(v-u)^{2}}-\frac{(m+1)u^{m}}{v-u}=\frac{\beta_1^{m+1}-e^{-\lambda(m+1)h}}{\left(\beta_1-e^{-\lambda h}\right)^{2}}-\frac{(m+1)e^{-\lambda mh}}{\beta_1-e^{-\lambda h}} \tag{B.12}$$

Next, let

$$R=m\sum_{j=1}^{m}\left[e^{-\lambda h}\right]^{j-1}\beta_1^{m-j}=m\sum_{j=1}^{m}u^{j-1}v^{m-j}$$

$$\Rightarrow R=\frac{m}{u}\sum_{j=1}^{m}\left[e^{-\lambda h}\right]^{j-1}\beta_1^{m-j}=\frac{m}{u}\sum_{j=0}^{m}\left[e^{-\lambda h}\right]^{j-1}\beta_1^{m-j}-\frac{mv^{m}}{u}=\frac{m}{u}\frac{v^{m+1}-u^{m+1}}{v-u}-\frac{mv^{m}}{u}$$

$$R=\frac{m\left(v^{m+1}-u^{m+1}\right)}{v-u}=\frac{m\left(\beta_1^{m}-e^{-\lambda mh}\right)}{\beta_1-e^{-\lambda h}} \tag{B.13}$$

Then we obtain:

$$\sum_{j=1}^{m}\left[e^{-\lambda(j-1)h}\right](m-j)\beta_1^{m-j} = R - S = \frac{m\left(\beta_1^m - e^{-\lambda mh}\right)}{\beta_1 - e^{-\lambda h}} - \frac{\beta_1^{m+1} - e^{-\lambda(m+1)h}}{\left(\beta_1 - e^{-\lambda h}\right)^2} + \frac{(m+1)e^{-\lambda mh}}{\beta_1 - e^{-\lambda h}}$$

$$\sum_{j=1}^{m}\left[e^{-\lambda(j-1)h}\right](m-j)\beta_1^{m-j} = \frac{m\beta_1^m + e^{-\lambda mh}}{\beta_1 - e^{-\lambda h}} - \frac{\beta_1^{m+1} - e^{-\lambda(m+1)h}}{\left(\beta_1 - e^{-\lambda h}\right)^2} \tag{B.14}$$

Similarly:

$$\frac{h\beta_1}{(1-\beta_1)}\sum_{j=1}^{m}\left[e^{-\lambda(j-1)h} - e^{-\lambda jh}\right]\beta_1^{m-j} = \frac{h\beta_1}{(1-\beta_1)}\left(1-e^{-\lambda h}\right)\sum_{j=1}^{m}\left[e^{-\lambda h}\right]^{j-1}\beta_1^{m-j}$$

$$\Rightarrow \sum_{j=1}^{m}\left[e^{-\lambda h}\right]^{j-1}\beta_1^{m-j} = \frac{R}{m} = \frac{\left(\beta_1^m - e^{-\lambda mh}\right)}{\beta_1 - e^{-\lambda h}} \tag{B.15}$$

Finally we obtain:

$$E\left[t_s(a)\right] = \frac{h\beta_1}{(1-\beta_1)}\left(1-e^{-\lambda mh}\right) - h\beta_1\left(1-e^{-\lambda h}\right)\left[\frac{m\beta_1^m + e^{-\lambda mh}}{\beta_1 - e^{-\lambda h}} - \frac{\beta_1^{m+1} - e^{-\lambda(m+1)h}}{\left(\beta_1 - e^{-\lambda h}\right)^2}\right]$$
$$- \frac{h\beta_1}{(1-\beta_1)}\left(1-e^{-\lambda h}\right)\frac{\left(\beta_1^m - e^{-\lambda mh}\right)}{\beta_1 - e^{-\lambda h}} \tag{B.16}$$

(4) Derivation of $E[t_S(b)]$ for Cost Model I

For case(b) of cost model I, the generalized Shewhart control chart, the derivation of $E[t_S(b)]$ can be achieved through the following steps:

(1) Evaluate the partial summation;

(2) Construct the double summation;

(3) Add the partial summations together to obtain $E[t_S(b)]$.

Therefore:

$$
\begin{aligned}
E[t_s(b)] &= \sum_{j=1}^{m}\left[F(jh)-F((j-1)h)\right]\beta_1^{m-j+1}\sum_{i=m+1}^{\infty}(i-j)h\beta_2^{i-m-1}(1-\beta_2) \\
&= \sum_{j=1}^{m}\left[F(jh)-F((j-1)h)\right]\beta_1^{m-j+1}(1-\beta_2)h\left[(m-j)\sum_{i-m-1=0}^{\infty}\beta_2^{i-m-1}+\sum_{i-m=1}^{\infty}(i-m)\beta_2^{i-m-1}\right] \\
&= \sum_{j=1}^{m}\left[F(jh)-F((j-1)h)\right]\beta_1^{m-j+1}h\left[(m-j)+\frac{1}{(1-\beta_2)}\right] \\
&= h\sum_{j=1}^{m}\left[e^{-\lambda(j-1)h}-e^{-\lambda jh}\right](m-j)\beta_1^{m-j+1}+h\sum_{j=1}^{m}\left[e^{-\lambda(j-1)h}-e^{-\lambda jh}\right]\frac{\beta_1^{m-j+1}}{(1-\beta_2)}
\end{aligned}
\tag{B.17}
$$

and since

$$
\sum_{j=1}^{m}e^{-\lambda(j-1)h}(m-j)\beta_1^{m-j}=\frac{m\left(\beta_1^m-e^{-\lambda mh}\right)}{\beta_1-e^{-\lambda h}}-\frac{\beta_1^{m+1}-e^{-\lambda(m+1)h}}{\left(\beta_1-e^{-\lambda h}\right)^2}+\frac{(m+1)e^{-\lambda mh}}{\beta_1-e^{-\lambda h}}
\tag{B.18}
$$

Refer to equation (A.14), we obtain:

$$
\sum_{j=1}^{m}\left[e^{-\lambda h}\right]^{j-1}\beta_1^{m-j}=\frac{\beta_1^m-e^{-\lambda mh}}{\beta_1-e^{-\lambda h}}
\tag{B.19}
$$

it follows that:

$$
\begin{aligned}
E[t_s(b)] &= \left(1-e^{-\lambda h}\right)h\beta_1\left\{\frac{m\left(\beta_1^m-e^{-\lambda mh}\right)}{\beta_1-e^{-\lambda h}}-\frac{\beta_1^{m+1}-e^{-\lambda(m+1)h}}{\left(\beta_1-e^{-\lambda h}\right)^2}+\frac{(m+1)e^{-\lambda mh}}{\beta_1-e^{-\lambda h}}\right\} \\
&\quad+\frac{\left(1-e^{-\lambda h}\right)h\beta_1}{(1-\beta_2)}\frac{\beta_1^m-e^{-\lambda mh}}{\beta_1-e^{-\lambda h}}
\end{aligned}
\tag{B.20}
$$

(5) Derivation of $E[t_n(a)]$ for Cost Model I

For case(a) of cost model I, the generalized Shewhart control chart, the derivation of $E[t_n(a)]$ can be achieved through the following steps:

(1) Evaluate the partial summation;

(2) Construct the double summation;

(3) Add the partial summations together to obtain $E[t_n(a)]$.

First we know that:

$$\begin{aligned} E[t_n(a)] &= \sum_{s=1}^{m}[F(sh)-F((s-1)h)](1-\alpha_1)^{s-1}\sum_{j=s}^{m} jh(1-\beta_1)\beta_1^{j-s} \\ &= \sum_{s=1}^{m} e^{-\lambda(s-1)h}[1-e^{-\lambda h}](1-\alpha_1)^{s-1}\sum_{j=s}^{m} jh(1-\beta_1)\beta_1^{j-s} \\ &= h(1-\beta_1)[1-e^{-\lambda h}]\sum_{s=1}^{m} e^{-\lambda(s-1)h}(1-\alpha_1)^{s-1}\sum_{j=s}^{m} j\,\beta_1^{j-s} \end{aligned} \tag{B.21}$$

Next, let $x = \beta_1$, then refer to equation A.8 we obtain:

$$\sum_{j=s}^{m} j\,\beta_1^{j-s} = \frac{1}{1-\beta_1}\left[s - m\beta_1^{m-s+1} + \frac{\beta_1-\beta_1^{m-s+1}}{1-\beta_1}\right] \tag{B.22}$$

Thus

$$E[t_n(a)] = h(1-\beta_1)[1-e^{-\lambda h}]\sum_{s=1}^{m} e^{-\lambda(s-1)h}(1-\alpha_1)^{s-1}\left\{\frac{1}{1-\beta_1}\left[s-m\beta_1^{m-s+1}+\frac{\beta_1-\beta_1^{m-s+1}}{1-\beta_1}\right]\right\} \tag{B.23}$$

Finally we obtain:

$$E[t_n(a)] = \sum_{s=1}^{m}[F(sh)-F((s-1)h)](1-\alpha_1)^{s-1}\sum_{j=s}^{m} jh(1-\beta_1)\beta_1^{j-s}$$

$$= h(1-e^{-\lambda h})\beta_1\left\{\frac{1-(e^{-\lambda h}(1-\alpha_1))^m}{[1-e^{-\lambda h}(1-\alpha_1)]^2} - \frac{m[e^{-\lambda h}(1-\alpha_1)]^m}{1-e^{-\lambda h}(1-\alpha_1)} - \frac{m\beta_1[\beta_1^m-(e^{-\lambda h}(1-\alpha_1))^m]}{\beta_1-e^{-\lambda h}(1-\alpha_1)}\right\}$$

$$+ h(1-e^{-\lambda h})\beta_1\left\{\frac{\beta_1}{1-\beta_1}\left[\frac{1-[e^{-\lambda h}(1-\alpha_1)]^m}{1-e^{-\lambda h}(1-\alpha_1)} - \frac{\beta_1^m-(e^{-\lambda h}(1-\alpha_1))^m}{\beta_1-e^{-\lambda h}(1-\alpha_1)}\right]\right\}$$

(B.24)

(6) Derivation of $E[t_n(c)]$ for Cost Model I

For case(c) of cost model I, the generalized Shewhart control chart, the derivation of $E[t_n(c)]$ can be achieved through the following steps:

(1) Evaluate the partial summation;

(2) Add the partial summations together to obtain $E[t_n(c)]$.

By the definition of $E[t_n(c)]$ we have:

$$E[t_n(c)] = (1-\alpha_1)^m \sum_{l=m+1}^{\infty}[F(lh)-F((l-1)h)](1-\alpha_2)^{l-m-1}\sum_{j=l}^{\infty} jh(1-\beta_2)\beta_2^{j-l}$$

$$= (1-\beta_2)(1-\alpha_1)^m h \sum_{l=m+1}^{\infty}[e^{-\lambda(l-1)h}-e^{-\lambda h}](1-\alpha_2)^{l-m-1}\sum_{j=l}^{\infty} j\,\beta_2^{j-l}$$

$$= (1-\alpha_1)^m h \sum_{l=m+1}^{\infty}[e^{-\lambda(l-1)h}-e^{-\lambda h}](1-\alpha_2)^{l-m-1}\left[\frac{\beta_2+l(1-\beta_2)}{1-\beta_2}\right] \qquad \text{(B.25)}$$

Then:

$$E[t_n(c)]=\frac{(1-\alpha_1)^m(1-e^{-\lambda h})h}{1-\beta_2}[A+B]$$

$$A=\beta_2\left\{\sum_{l=m+1}^{\infty}e^{-\lambda(l+1)h}(1-\alpha_2)^{l-m-1}-\sum_{l=m+1}^{\infty}e^{-\lambda lh}(1-\alpha_2)^{l-m-1}\right\}$$
$$=\beta_2\left[\frac{e^{-\lambda mh}}{1-(1-\alpha_2)e^{-\lambda h}}-\frac{e^{-\lambda h}}{\alpha_2}\right] \tag{B.26}$$

Similarly:

$$B=(1-\beta_2)\left\{e^{-\lambda h}\sum_{l=m+1}^{\infty}e^{-\lambda lh}(1-\alpha_2)^{l-m-1}-\sum_{l=m+1}^{\infty}le^{-\lambda lh}(1-\alpha_2)^{l-m-1}\right\}$$
$$=(1-\beta_2)\left[\frac{me^{-\lambda mh}}{1-(1-\alpha_2)e^{-\lambda h}}+\frac{e^{-\lambda mh}}{\left[1-(1-\alpha_2)e^{-\lambda h}\right]^2}-\frac{me^{-\lambda h}}{\alpha_2}-\frac{e^{-\lambda h}}{\alpha_2^2}\right] \tag{B.27}$$

Therefore:

$$E[t_n(c)]=\frac{(1-\alpha_1)^m(1-e^{-\lambda h})h}{(1-\beta_2)}\left\{\frac{[\beta_2+m(1-\beta_2)]e^{-\lambda mh}}{1-(1-\alpha_2)e^{-\lambda h}}+\frac{(1-\beta_2)e^{-\lambda mh}}{\left[1-(1-\alpha_2)e^{-\lambda h}\right]^2}\right.$$
$$\left.-\frac{[\beta_2+m(1-\beta_2)]e^{-\lambda h}}{\alpha_2}-\frac{(1-\beta_2)e^{-\lambda h}}{\alpha_2^2}\right\} \tag{B.28}$$

(7) Proof of Lemma 3.3

To prove Lemma 3.3, the following steps are necessary:

(a) Show that equation (3.2.5) is the same as equation (3.1.6) under Fact 3.2;

(b) Show that equation (3.2.8) is the same as equation (3.1.8) under Fact 3.2;

(c) Show that equation (3.2.15) is the same as equation (3.1.14) under Fact 3.2;

(d) Show that equation (3.2.24) is the same as equation (3.1.16) under Fact 3.2.

For case (a):

$$C_f = c_f\left\{\frac{\alpha_1 e^{-\lambda h}\left\{1-\left[(1-\alpha_1)e^{-\lambda h}\right]^m\right\}}{1-(1-\alpha_1)e^{-\lambda h}} + \frac{(1-\alpha_1)^m e^{-\lambda(m+1)h}\alpha_2}{1-(1-\alpha_2)e^{-\lambda h}}\right\}$$

$$= c_f\left\{\frac{\alpha e^{-\lambda h}\left\{1-\left[(1-\alpha)e^{-\lambda h}\right]^m\right\} + (1-\alpha)^m e^{-\lambda(m+1)h}\alpha}{1-(1-\alpha)e^{-\lambda h}}\right\} = \frac{c_f \alpha e^{-\lambda h}}{1-(1-\alpha)e^{-\lambda h}} \tag{B.29}$$

For case (b):

$$C_t = c_t\left(1-e^{-\lambda h}\right)\left\{\frac{1}{1-(1-\alpha_1)e^{-\lambda h}} - \frac{\left[(1-\alpha_1)e^{-\lambda h}\right]^{m+1}}{1-(1-\alpha_1)e^{-\lambda h}} + \frac{\left[(1-\alpha_2)e^{-\lambda h}\right]^{m+1}}{1-(1-\alpha_2)e^{-\lambda h}}\right\}$$

$$= c_t\left(1-e^{-\lambda h}\right)\left\{\frac{1-\left[(1-\alpha)e^{-\lambda h}\right]^{m+1} + \left[(1-\alpha)e^{-\lambda h}\right]^{m+1}}{1-(1-\alpha)e^{-\lambda h}}\right\} = \frac{c_t\left(1-e^{-\lambda h}\right)}{1-(1-\alpha)e^{-\lambda h}} \tag{B.30}$$

For case (c):

$$C_d = c_d rp\left(E(t_p)+E(t_s)\right) = c_d rp\left\{\frac{\lambda h-\left(1-e^{-\lambda h}\right)}{\lambda\left(1-e^{-\lambda h}\right)} + \frac{h\beta_1\left(1-e^{-\lambda mh}\right)}{1-\beta_1} + \frac{h\beta_2 e^{-\lambda mh}}{1-\beta_2}\right\}$$

$$+ c_d rp\left\{\left[\frac{h\beta_1\left(1-e^{-\lambda h}\right)\left(\beta_1^m - e^{-\lambda mh}\right)}{\beta_1 - e^{-\lambda h}}\right]\left[\frac{1}{1-\beta_2} - \frac{1}{1-\beta_1}\right]\right\}$$

$$= c_d rp\left\{\frac{\lambda h-\left(1-e^{-\lambda h}\right)}{\lambda\left(1-e^{-\lambda h}\right)} + \frac{h\beta}{1-\beta}\right\} \tag{B.31}$$

For case (d):

$$E[t]=\alpha_1 h\left[\frac{e^{-\lambda h}\left\{1-\left[(1-\alpha_1)e^{-\lambda h}\right]^m\right\}}{\left[1-(1-\alpha_2)e^{-\lambda h}\right]^2}-\frac{m(1-\alpha_1)^m e^{-\lambda(m+1)h}}{1-(1-\alpha_2)e^{-\lambda h}}\right]$$

$$+\alpha_2 h\left[\frac{(1-\alpha_1)^m e^{-\lambda(m+1)h}}{\left[1-(1-\alpha_2)e^{-\lambda h}\right]^2}+\frac{m(1-\alpha_1)^m e^{-\lambda(m+1)h}}{1-(1-\alpha_2)e^{-\lambda h}}\right]$$

$$+h\left(1-e^{-\lambda h}\right)\beta_1\left\{\frac{1-\left(e^{-\lambda h}(1-\alpha_1)\right)^m}{\left[1-e^{-\lambda h}(1-\alpha_1)\right]^2}-\frac{m\left[e^{-\lambda h}(1-\alpha_1)\right]^m}{1-e^{-\lambda h}(1-\alpha_1)}-\frac{m\beta_1\left[\beta_1^m-\left(e^{-\lambda h}(1-\alpha_1)\right)^m\right]}{\beta_1-e^{-\lambda h}(1-\alpha_1)}\right\}$$

$$+h\left(1-e^{-\lambda h}\right)\beta_1\left\{\frac{\beta_1}{1-\beta_1}\left[\frac{1-\left[e^{-\lambda h}(1-\alpha_1)\right]^m}{1-e^{-\lambda h}(1-\alpha_1)}-\frac{\beta_1^m-\left(e^{-\lambda h}(1-\alpha_1)\right)^m}{\beta_1-e^{-\lambda h}(1-\alpha_1)}\right]\right\}$$

$$+h\left(1-e^{-\lambda h}\right)\left(\frac{1}{1-\beta_2}+m\right)\left[\frac{\beta_1^{m+1}-\beta_1\left[e^{-\lambda h}(1-\alpha_1)\right]^m}{\beta_1-\left[e^{-\lambda h}(1-\alpha_1)\right]}\right]$$

$$+\frac{(1-\alpha_1)^m\left(1-e^{-\lambda h}\right)h}{(1-\beta_2)}\left\{\frac{\left[\beta_2+m(1-\beta_2)\right]e^{-\lambda mh}}{1-(1-\alpha_2)e^{-\lambda h}}+\frac{(1-\beta_2)e^{-\lambda mh}}{\left[1-(1-\alpha_2)e^{-\lambda h}\right]^2}\right.$$

$$\left.-\frac{\left[\beta_2+m(1-\beta_2)\right]e^{-\lambda h}}{\alpha_2}-\frac{(1-\beta_2)e^{-\lambda h}}{\alpha_2^2}\right\}$$

(B.32)

The first two terms can be reduced to the following expression under Fact 3.2:

$$\frac{\alpha h e^{-\lambda h}}{\left[1-(1-\alpha)e^{-\lambda h}\right]^2} \tag{B.33}$$

The rest expressions can be reduced to:

$$\frac{\left(1-e^{-\lambda h}\right)h}{\left[\beta-(1-\alpha)e^{-\lambda h}\right]}\left[\frac{\beta}{1-\beta}-\frac{(1-\beta)(1-\alpha)e^{-\lambda h}}{\left[1-(1-\alpha)e^{-\lambda h}\right]^2}\right] \tag{B.34}$$

Therefore Lemma 3.3 is true.

Appendix C.

Derivations and Proofs for Cost Model II

(1) Proof of Lemma 4.1

Proof:

Note that the part of interval before the process shift can be written as:

$$T = t - jh \Rightarrow E[T] = E[t - jh] = \int (t - jh) f\big(t \,|\, jh \le t \le (j+1)h\big)dt \tag{C.1}$$

And we know that::

$$F\big(t \,|\, jh \le t \le (j+1)h\big) = \frac{F(t) - F(jh)}{F((j+1)h) - F(jh)} \tag{C.2}$$

Thus the condinitional probability density function of t given the shift occurs in (jh, (j+1))

can be constructed as:

$$f\left(t \middle| jh \le t \le (j+1)h\right) = \frac{f(t)}{F((j+1)h) - F(jh)} = \frac{\frac{1}{\theta}}{\frac{(j+1)h}{\theta} - \frac{jh}{\theta}} = \frac{1}{h} \tag{C.3}$$

Therefore:

$$E[T] = \int_{jh}^{(j+1)h} (t - jh) \frac{f(t)}{F((j+1)h) - F(jh)} dt = \int_{jh}^{(j+1)h} (t - jh) \frac{1}{h} dt = \frac{1}{2} h \tag{C.4}$$

And then as:

$$E\left[t_p\right] = h - E[T] = h - \frac{1}{2} h = \frac{1}{2} h \tag{C.5}$$

Thus Lemma 4.1 is true.

(2) Derivation of E[t] for Cost Model II

For the standard Shewhart control chart of cost model II, the derivation of E[t] can be achieved through the following steps:

(1) Evaluate the partial summation;

(2) Construct the double summation;

(3) Add the partial summations together to obtain E[t].

First, we may write E[t] as:

$$
\begin{aligned}
E[t] &= \sum_{j=1}^{N} jh\alpha(1-\alpha)^{j-1}\left[1-\frac{jh}{\theta}\right] + \sum_{j=1}^{N} jh\left(\sum_{l=1}^{j}\left[\frac{lh}{\theta}-\frac{(l-1)h}{\theta}\right](1-\alpha)^{l-1}\beta^{j-1}(1-\beta)\right) \\
&= \sum_{j=1}^{N} jh\alpha(1-\alpha)^{j-1} - \sum_{j=1}^{N} jh\alpha(1-\alpha)^{j-1}\left(\frac{jh}{\theta}\right) + \sum_{j=1}^{N} jh\left(\frac{h}{\theta}\right)\left(\sum_{l=1}^{j}(1-\alpha)^{l-1}\beta^{j-1}(1-\beta)\right) \\
&= \frac{h}{\alpha}\left[1-(1-\alpha)^{N}(\alpha N+1)\right] - \frac{\alpha h^{2}}{\theta}\sum_{j=1}^{N} j^{2}(1-\alpha)^{j-1} + \frac{h^{2}(1-\beta)}{\theta}\sum_{j=1}^{N} j\beta^{j-1}\left[\sum_{l=1}^{j}\left(\frac{1-\alpha}{\beta}\right)^{l-1}\right]
\end{aligned}
\tag{C.6}
$$

Then, let x = (1-α)/β in equation (A.2), we have:

$$
\sum_{l=1}^{j}\left(\frac{1-\alpha}{\beta}\right)^{l-1} = \frac{\beta}{\beta-(1-\alpha)} - \frac{(1-\alpha)\left(\frac{1-\alpha}{\beta}\right)^{j-1}}{\beta-(1-\alpha)} \tag{C.7}
$$

Similarly, let x = 1-α in equation (A.5), we obtain:

$$
\sum_{j=1}^{N} j^{2}(1-\alpha)^{j-1} = \frac{1}{\alpha^{2}} + \frac{2(1-\alpha)\left[1-(1-\alpha)^{N-1}\right]}{\alpha^{3}} - \frac{(N-1)^{2}(1-\alpha)^{N}}{\alpha^{2}} + \frac{N^{2}(1-\alpha)^{N+1}}{\alpha^{2}} \tag{C.8}
$$

So that:

$$
\begin{aligned}
&\sum_{j=1}^{N} j\beta^{j-1}\left[\sum_{l=1}^{j}\left(\frac{1-\alpha}{\beta}\right)^{l-1}\right] \\
&\qquad = \frac{1}{\beta-(1-\alpha)}\left\{\frac{\beta\left(1-\beta^{N}\right)}{(1-\beta)^{2}} - \frac{N\beta^{N+1}}{1-\beta} - \frac{(1-\alpha)\left[1-(1-\alpha)^{N}\right]}{\alpha^{2}} + \frac{N(1-\alpha)^{N+1}}{\alpha}\right\}
\end{aligned}
\tag{C.9}
$$

Therefore the form for E[t] is:

$$E[t]=\frac{h}{\alpha}\left[1-(1-\alpha)^N(\alpha N+1)\right]$$
$$-\frac{\alpha h^2}{\theta}\left\{\frac{1}{\alpha^2}+\frac{2(1-\alpha)\left[1-(1-\alpha)^{N-1}\right]}{\alpha^3}-\frac{(N-1)^2(1-\alpha)^N}{\alpha^2}+\frac{N^2(1-\alpha)^{N+1}}{\alpha^2}\right\}$$
$$+\frac{h^2(1-\beta)}{[\beta-(1-\alpha)]\theta}\left\{\frac{\beta(1-\beta^N)}{(1-\beta)^2}-\frac{N\beta^{N+1}}{1-\beta}-\frac{(1-\alpha)\left[1-(1-\alpha)^N\right]}{\alpha^2}+\frac{N(1-\alpha)^{N+1}}{\alpha}\right\} \quad (C.10)$$

(3) Derivation of P[A] for Cost Model II

We may write P[A] as:

$$P[A]=\sum_i P[A_1]\,P[A_2]$$
$$=\sum_{i=1}^{m}\alpha_1(1-\alpha_1)^{i-1}[1-F(ih)]+(1-\alpha_1)^m\sum_{i=m+1}^{N}\alpha_2(1-\alpha_2)^{i-m-1}[1-F(ih)]$$
$$=\sum_{i=1}^{m}\alpha_1(1-\alpha_1)^{i-1}\left[1-\frac{ih}{\theta}\right]+(1-\alpha_1)^m\sum_{i=m+1}^{N}\alpha_2(1-\alpha_2)^{i-m-1}\left[1-\frac{ih}{\theta}\right]$$
$$=\alpha_1\left[\frac{1-(1-\alpha_1)^m}{\alpha_1}-\frac{h}{\theta}\sum_{i=1}^{m}i\,(1-\alpha_1)^{i-1}\right]$$
$$+\alpha_2(1-\alpha_1)^m\left[\sum_{i=m+1}^{N}(1-\alpha_2)^{i-m-1}-\frac{h}{\theta}\sum_{i=m+1}^{N}i(1-\alpha_2)^{i-m-1}\right] \quad (C.10)$$

Let x = 1- α_1 in equation (A.3), we obtain:

$$\sum_{i=1}^{m}i\,(1-\alpha_1)^{i-1}=\frac{1-(1-\alpha_1)^m}{\alpha_1^2}-\frac{m(1-\alpha_1)^m}{\alpha_1} \quad (C.11)$$

Let x = 1- α_2 in equation (A.2), we obtain:

$$\sum_{i=m+1}^{N}(1-\alpha_2)^{i-m-1} = \frac{1-(1-\alpha_2)^{N-m}}{\alpha_2} \tag{C.12}$$

Let x = 1- α_2 in equation (A.4), we obtain:

$$\sum_{i=m+1}^{N} i(1-\alpha_2)^{i-m-1} = \frac{m+1}{\alpha_2} + \frac{(1-\alpha_2)\left[1-(1-\alpha_2)^{N-m-1}\right]}{\alpha_2^2} - \frac{N(1-\alpha_2)^{N-m}}{\alpha_2} \tag{C.13}$$

Thus we have:

$$P[A] = 1-(1-\alpha_1)^m(1-\alpha_2)^{N-m} - \frac{h}{\theta}\left[\frac{1-(1-\alpha_1)^m}{\alpha_1} + \frac{(1-\alpha_1)^m}{\alpha_2}\right] + \frac{h}{\theta}\left[\frac{(1-\alpha_1)^m(1-\alpha_2)^{N-m}(N\alpha_2+1)}{\alpha_2}\right] \tag{C.14}$$

(4) Proof of Lemma 4.2

Refer to the inequalities used in (B.1), the upper bound for P[A] can be constructed as the following steps:

$$\begin{aligned} P[A] &= 1-(1-\alpha_1)^m(1-\alpha_2)^{N-m} - \frac{h}{\theta}\left[\frac{1-(1-\alpha_1)^m}{\alpha_1} + \frac{(1-\alpha_1)^m}{\alpha_2}\right] \\ &\quad + \frac{h}{\theta}\left[\frac{(1-\alpha_1)^m(1-\alpha_2)^{N-m}(N\alpha_2+1)}{\alpha_2}\right] \le 1+\frac{h}{\theta}\left[\frac{(1-\alpha_1)^m}{\alpha_2}\right] \le 1+\frac{h}{\theta}\left(\frac{1}{\alpha_2}\right) \end{aligned} \tag{C.15}$$

Similerly, the lower bound for P[A] can be constructed as:

$$P[A]=1-(1-\alpha_1)^m(1-\alpha_2)^{N-m}-\frac{h}{\theta}\left[\frac{1-(1-\alpha_1)^m}{\alpha_1}+\frac{(1-\alpha_1)^m}{\alpha_2}\right]$$
$$+\frac{h}{\theta}\left[\frac{(1-\alpha_1)^m(1-\alpha_2)^{N-m}(N\alpha_2+1)}{\alpha_2}\right]\geq 1-(1-\alpha_1)^m-\frac{h}{\theta}\left[\frac{1-(1-\alpha_1)^m}{\alpha_1}+\frac{(1-\alpha_1)^m}{\alpha_1}\right]$$
$$\geq 1-(1-\alpha_1)-\frac{h}{\theta}\left[\frac{1-(1-\alpha_1)^m}{\alpha_1}+\frac{(1-\alpha_1)^m}{\alpha_1}\right]=\alpha_1-\frac{h}{\theta}\left(\frac{1}{\alpha_1}\right)$$
(C.16)

Therefore Lemma 4.2 is true.

(5) Proof of Lemma 4.3

To prove Lemma 4.3, the following steps are necessary:

(1) Construct the upper bound for P[B];

(2) Construct the lower bound for P[B].

First, the upper bound for P[B] can be constructed as:

$$P[B]=\frac{h}{\theta}\left[\frac{1-(1-\alpha_1)^{m+1}}{\alpha_1}+\frac{(1-\alpha_2)^{m+1}\left[1-(1-\alpha_2)^{N-m-1}\right]}{\alpha_2}\right]$$
$$\leq\frac{h}{\theta}\left[\frac{1-(1-\alpha_1)^{m+1}}{\alpha_1}+\frac{(1-\alpha_2)^{m+1}}{\alpha_2}\right]\leq\frac{h}{\theta}\left[\frac{1-(1-\alpha_1)^{m+1}}{\alpha_1}+\frac{(1-\alpha_2)^{m+1}}{\alpha_1}\right]$$
$$\leq\frac{h}{\theta}\left[\frac{1-(1-\alpha_1)^{m+1}}{\alpha_1}+\frac{(1-\alpha_1)^{m+1}}{\alpha_1}\right]=\frac{h}{\theta}\frac{1}{\alpha_1}$$
(C.17)

Similarly, the lower bound for P[B] is:

$$
\begin{aligned}
P[B] &= \frac{h}{\theta}\left[\frac{1-(1-\alpha_1)^{m+1}}{\alpha_1}+\frac{(1-\alpha_2)^{m+1}\left[1-(1-\alpha_2)^{N-m-1}\right]}{\alpha_2}\right] \\
&\geq \frac{h}{\theta}\left[\frac{1-(1-\alpha_1)^{m+1}}{\alpha_2}+\frac{(1-\alpha_2)^{m+1}}{\alpha_2}\right]=\frac{h}{\theta}\left[\frac{1-\left[(1-\alpha_1)^{m+1}-(1-\alpha_2)^{m+1}\right]}{\alpha_2}\right] \\
&= \left(\frac{h}{\theta}\right)\frac{1-(\alpha_2-\alpha_1)\left[(1-\alpha_1)^m+(1-\alpha_1)^{m-1}(1-\alpha_2)+\cdots+(1-\alpha_1)(1-\alpha_2)^{m-1}+(1-\alpha_2)^m\right]}{\alpha_2} \\
&\geq \left(\frac{h}{\theta}\right)\frac{1-(\alpha_2-\alpha_1)\left[1+(1-\alpha_2)+(1-\alpha_2)^2+\cdots+(1-\alpha_2)^{m-1}+(1-\alpha_2)^m\right]}{\alpha_2} \\
&= \left(\frac{h}{\theta}\right)\frac{1-(\alpha_2-\alpha_1)\frac{1}{\alpha_2}}{\alpha_2}=\frac{h}{\theta}\frac{\alpha_1}{\alpha_2^2}
\end{aligned}
\tag{C.18}
$$

Therefore Lemma 4.3 is true.

(6) Derivation of E[t_S(a)] for Cost Model II

For case(a) of cost model II, the generalized Shewhart control chart, the derivation of E[t_S(a)] can be achieved through the following steps:

(1) Evaluate the partial summation;

(2) Construct the double summation;

(3) Add the partial summations together to get E[t_S(a)];

First we may write E[t_S(a)] as the following form:

$$E[t_s(a)] = \sum_{j=1}^{m}\left[F(jh) - F((j-1)h)\right]\sum_{i=j}^{m}(i-j)\,h(1-\beta_1)\beta_1^{i-j}$$
$$= \frac{h^2\beta_1(1-\beta_1)}{\theta}\sum_{j=1}^{m}\left(\sum_{i=j}^{m}(i-j)\,\beta_1^{i-j-1}\right) \tag{C.19}$$

Then refer to equation (A.4) we obtain:

$$\sum_{i=j}^{m}(i-j)\,\beta_1^{i-j-1} = \frac{1-\beta_1^{m-j}}{(1-\beta_1)^2} - \frac{\beta_1^{m-j}(m-j)}{1-\beta_1} = \frac{1}{(1-\beta_1)^2} - \frac{\beta_1^{m-j}}{(1-\beta_1)^2} - \frac{m\beta_1^{m-j}}{1-\beta_1} + \frac{j\beta_1^{m-j}}{1-\beta_1} \tag{C.20}$$

Let x = β_1 in equation (A.11) we get:

$$\sum_{j=1}^{m} j\,\beta_1^{m-j} = \frac{m}{1-\beta_1} - \frac{\beta_1 - \beta_1^m}{(1-\beta_1)^2} - \frac{\beta_1^m}{1-\beta_1} \tag{C.21}$$

Thus we have:

$$\sum_{j=1}^{m}\left(\sum_{i=j}^{m}(i-j)\,\beta_1^{i-j-1}\right) = \frac{1}{1-\beta_1}\left[\frac{m}{1-\beta_1} - \frac{\beta_1^m(1-m)}{1-\beta_1} - \frac{1-\beta_1-2\beta_1^m}{(1-\beta_1)^2}\right] \tag{C.22}$$

Therefore:

$$E[t_s(a)] = \frac{h^2\beta_1}{\theta}\left[\frac{m}{1-\beta_1} - \frac{\beta_1^m(1-m)}{1-\beta_1} - \frac{1-\beta_1-2\beta_1^m}{(1-\beta_1)^2}\right] \tag{C.23}$$

(7) Derivation of $E[t_S(b)]$ for Cost Model II

For case(b) of cost model II, the generalized Shewhart control chart, the derivation of $E[t_S(b)]$ can be achieved through the following steps:

(1) Construct the partial summation from j = 1 to j = m;

(2) Construct the partial summation from j = m+1 to j = N;

(3) Construct the double summation;

(4) Add the partial summations together to obtain $E[t_S(b)]$.

First, we may write $E[t_S(b)]$ as:

$$
\begin{aligned}
E[t_s(b)] &= \sum_{j=1}^{m}\left[F(jh) - F((j-1)h)\right] \beta_1^{m-j+1} \sum_{i=m+1}^{N}(i-j)\, h\beta_2^{i-m-1}(1-\beta_2) \\
&= \frac{h^2(1-\beta_2)}{\theta}\sum_{j=1}^{m}\beta_1^{m-j+1}\sum_{i=m+1}^{N}(i-j)\,\beta_2^{i-m-1}
\end{aligned}
\tag{C.24}
$$

Therefore we obtain:

$$
\begin{aligned}
\sum_{i=m+1}^{N}(i-j)\,\beta_2^{i-m-1} &= \sum_{i=m+1}^{N} i\,\beta_2^{i-m-1} - j\sum_{i=m+1}^{N}\beta_2^{i-m-1} \\
&= \frac{1}{1-\beta_2}\left[m + \frac{1-\beta_2^{N-m}}{1-\beta_2} - N\beta_2^{N-m}\right] - j\left(\frac{1-\beta_2^{N-m}}{1-\beta_2}\right)
\end{aligned}
\tag{C.25}
$$

Refer to equation (A.11) we obtain:

$$
\sum_{j=1}^{m} j\beta_1^{m-j} = \frac{1}{1-\beta_1}\left[m - \frac{\beta_1-\beta_1^m}{1-\beta_1} - \beta_1^m\right]
\tag{C.26}
$$

To construct the double summstion below, we may first write it as the following expression:

$$\sum_{j=1}^{m}\beta_1^{m-j+1}\sum_{i=m+1}^{N}(i-j)\,\beta_2^{i-m-1}=\frac{1}{1-\beta_2}\left[m+\frac{1-\beta_2^{N-m}}{1-\beta_2}-N\beta_2^{N-m}\right]\sum_{j=1}^{m}\beta_1^{m-j+1}$$
$$-\beta_1\left(\frac{1-\beta_2^{N-m}}{1-\beta_2}\right)\sum_{j=1}^{m}j\beta_1^{m-j}$$
$$=\frac{\beta_1}{1-\beta_2}\left\{\left[\frac{1-\beta_1^m}{1-\beta_1}\right]\left[m+\frac{1-\beta_2^{N-m}}{1-\beta_2}-N\beta_2^{N-m}\right]-\left[\frac{1-\beta_2^{N-m}}{1-\beta_1}\right]\left[m-\frac{\beta_1-\beta_1^m}{1-\beta_1}-\beta_1^m\right]\right.$$

(C.27)

Finally:

$$E[t_s(b)]=\frac{h^2\beta_1}{\theta}\left\{\left[\frac{1-\beta_1^m}{1-\beta_1}\right]\left[m+\frac{1-\beta_2^{N-m}}{1-\beta_2}-N\beta_2^{N-m}\right]-\left[\frac{1-\beta_2^{N-m}}{1-\beta_1}\right]\left[m-\frac{\beta_1-\beta_1^m}{1-\beta_1}-\beta_1^m\right]\right\}$$

(C.28)

(8) Derivation of $E[t_f]$ for Cost Model II

To construct $E[t_f]$ for Cost Model II, the following steps are necessary:

(1) Express $E[t_f]$ as a combination of S_1, S_2, S_3 and S_4;

(2) Construct S_1, S_2, S_3 and S_4;

(3) Construct $E[t_f]$.

First, we may write $E[t_f]$ as:

$$E[t_f]=\sum_{i=1}^{m}ih\alpha_1(1-\alpha_1)^{i-1}[1-F(ih)]+(1-\alpha_1)^m\sum_{i=m+1}^{N}ih\alpha_2(1-\alpha_2)^{i-m-1}[1-F(ih)]$$
$$=\alpha_1hS_1-\frac{\alpha_1h^2}{\theta}S_2+\alpha_2h(1-\alpha_1)^mS_3-\frac{\alpha_2h^2(1-\alpha_1)^m}{\theta}S_4$$

(C.29)

Refer to equation (A.4) and (A.6), we have:

$$S_1 = \sum_{i=1}^{m} i(1-\alpha_1)^{i-1} = \frac{1-(1-\alpha_1)^m}{\alpha_1^2} - \frac{m(1-\alpha_1)^m}{\alpha_1} \tag{C.30}$$

and

$$S_2 = \sum_{i=1}^{m} i^2(1-\alpha_1)^{i-1} = \frac{1-(m-1)^2(1-\alpha_1)^m + m^2(1-\alpha_1)^{m+1}}{\alpha_1^2} + \frac{2(1-\alpha_1)\left[1-(1-\alpha_1)^{m-1}\right]}{\alpha_1^3} \tag{C.31}$$

Refer to equation (A.8) and (A.10), we obtain:

$$S_3 = \sum_{i=m+1}^{N} i(1-\alpha_1)^{i-m-1} = \frac{m}{\alpha_2} - \frac{N(1-\alpha_2)^{N-m}}{\alpha_2} + \frac{1-(1-\alpha_2)^{N-m}}{\alpha_2^2} \tag{C.32}$$

and

$$\begin{aligned} S_4 &= \sum_{i=m+1}^{N} i^2(1-\alpha_1)^{i-m-1} \\ &= \frac{1}{\alpha_2^2} + \frac{2(1-\alpha_2)\left[1-(1-\alpha_2)^{N-m-1}\right]}{\alpha_2^3} - \frac{(N-m-1)^2(1-\alpha_2)^{N-m}}{\alpha_2^2} + \frac{(N-m)^2(1-\alpha_2)^{N-m+1}}{\alpha_2^2} \\ &\quad + 2m\left[\frac{1-(1-\alpha_2)^{N-m}}{\alpha_2^2} - \frac{(N-m)(1-\alpha_2)^{N-m}}{\alpha_2}\right] + \frac{m^2\left[1-(1-\alpha_2)^{N-m}\right]}{\alpha_2} \end{aligned} \tag{C.33}$$

Therefore we obtain:

$$
\begin{aligned}
E[t_f] &= \alpha_1 h\left[\frac{1-(1-\alpha_1)^m}{\alpha_1^2} - \frac{m(1-\alpha_1)^m}{\alpha_1}\right] \\
&- \frac{\alpha_1 h^2}{\theta}\left[\frac{1-(m-1)^2(1-\alpha_1)^m + m^2(1-\alpha_1)^{m+1}}{\alpha_1^2} + \frac{2(1-\alpha_1)\left[1-(1-\alpha_1)^{m-1}\right]}{\alpha_1^3}\right] \\
&+ \alpha_2 h(1-\alpha_1)^m\left[\frac{m}{\alpha_2} - \frac{N(1-\alpha_2)^{N-m}}{\alpha_2} + \frac{1-(1-\alpha_2)^{N-m}}{\alpha_2^2}\right] \\
&+ \frac{\alpha_2 h^2(1-\alpha_1)^m}{\theta}\left\{\frac{1}{\alpha_2^2} + \frac{2(1-\alpha_2)\left[1-(1-\alpha_2)^{N-m-1}\right]}{\alpha_2^3} - \frac{(N-m-1)^2(1-\alpha_2)^{N-m}}{\alpha_2^2}\right. \\
&+ \frac{(N-m)^2(1-\alpha_2)^{N-m+1}}{\alpha_2^2} \\
&\left. + 2m\left[\frac{1-(1-\alpha_2)^{N-m}}{\alpha_2^2} - \frac{(N-m)(1-\alpha_2)^{N-m}}{\alpha_2}\right] + \frac{m^2\left[1-(1-\alpha_2)^{N-m}\right]}{\alpha_2}\right\}
\end{aligned}
\tag{C.34}
$$

(9) Derivation of $E[t_n(a)]$ for Cost Model II

For case(a) of cost model II, the generalized Shewhart control chart, the derivation of $E[t_n(a)]$ can be achieved through the following steps:

(1) Evaluate the partial summation;

(2) Construct the double summation;

(3) Add the partial summations together to obtain $E[t_n(a)]$.

First we know that:

$$E[t_n(a)] = \sum_{s=1}^{m}[F(sh) - F((s-1)h)](1-\alpha_1)^{s-1}\sum_{j=s}^{m} jh(1-\beta_1)\beta_1^{j-s}$$

$$= \frac{(1-\beta_1)h^2}{\theta}\sum_{s=1}^{m}(1-\alpha_1)^{s-1}\sum_{j=s}^{m} j\,\beta_1^{j-s} \tag{C.35}$$

Next, let $x = \beta_1$, then refer to equation (A.8) we obtain:

$$\sum_{j=s}^{m} j\,\beta_1^{j-s} = \frac{1}{1-\beta_1}\left\{s - \left[m + \frac{1}{1-\beta_1}\right]\beta_1^{m-s+1} + \frac{\beta_1}{1-\beta_1}\right\} \tag{C.36}$$

Thus:

$$\sum_{s=1}^{m}(1-\alpha_1)^{s-1}\sum_{j=s}^{m} j\,\beta_1^{j-s}$$

$$= \frac{1}{1-\beta_1}\left\{\sum_{s=1}^{m} s(1-\alpha_1)^{s-1} - \left[m + \frac{1}{1-\beta_1}\right]\sum_{s=1}^{m}(1-\alpha_1)^{s-1}\beta_1^{m-s+1} + \frac{1}{1-\beta_1}\sum_{s=1}^{m}(1-\alpha_1)^{s-1}\right\}$$

$$= \frac{1}{1-\beta_1}\left\{\frac{1-(1-\alpha_1)^m}{\alpha_1^2} - \frac{m(1-\alpha_1)^m}{\alpha_1} - \left[m + \frac{1}{1-\beta_1}\right]\left[\frac{\beta_1\left[(1-\alpha_1)^m - \beta_1^m\right]}{1-\alpha_1-\beta_1}\right]\right.$$

$$\left. + \frac{\beta_1}{1-\beta_1}\left[\frac{(1-\alpha_1)^m}{\alpha_1}\right]\right\} \tag{C.37}$$

Finally

$$E[t_n(a)] = \frac{h^2}{\theta}\left\{\frac{1-(1-\alpha_1)^m}{\alpha_1^2} - \frac{m(1-\alpha_1)^m}{\alpha_1} - \left[m + \frac{1}{1-\beta_1}\right]\left[\frac{\beta_1\left[(1-\alpha_1)^m - \beta_1^m\right]}{1-\alpha_1-\beta_1}\right]\right.$$

$$\left. + \frac{\beta_1}{1-\beta_1}\left[\frac{(1-\alpha_1)^m}{\alpha_1}\right]\right\} \tag{C.38}$$

(10) Derivation of $E[t_n(b)]$ for Cost Model II

First we may write $E[t_n(b)]$ as:

$$E[t_n(b)] = \sum_{s=1}^{m}\left[F(sh) - F((s-1)h)\right](1-\alpha_1)^{s-1}\beta_1^{m-s+1}\sum_{j=m+1}^{N} jh(1-\beta_2)\beta_2^{j-m-1}$$
$$= \frac{(1-\beta_2)h^2}{\theta}\sum_{s=1}^{m}(1-\alpha_1)^{s-1}\beta_1^{m-s+1}\sum_{j=m+1}^{N} j\beta_2^{j-m-1} \tag{C.39}$$

Refer to Equation (A.8), we obtain:

$$\sum_{j=m+1}^{N} j\beta_2^{j-m-1} = \frac{1}{1-\beta_2}\left[m - N\beta_2^{N-m} + \frac{1-\beta_2^{N-m}}{1-\beta_2}\right] \tag{C.40}$$

Refer to Equation (A.13), we obtain:

$$\sum_{s=1}^{m}(1-\alpha_1)^{s-1}\beta_1^{m-s+1} = \frac{\beta_1\left[(1-\alpha_1)^m - \beta_1^m\right]}{1-\alpha_1-\beta_1} \tag{C.41}$$

Finally:

$$E[t_n(b)] = \frac{(1-\beta_2)h^2}{\theta}\sum_{s=1}^{m}(1-\alpha_1)^{s-1}\beta_1^{m-s+1}\left[\frac{1}{1-\beta_2}\right]\left[m - N\beta_2^{N-m} + \frac{1-\beta_2^{N-m}}{1-\beta_2}\right]$$
$$= \frac{h^2}{\theta}\left[m - N\beta_2^{N-m} + \frac{1-\beta_2^{N-m}}{1-\beta_2}\right]\sum_{s=1}^{m}(1-\alpha_1)^{s-1}\beta_1^{m-s+1}$$
$$= \frac{h^2}{\theta}\left[m - N\beta_2^{N-m} + \frac{1-\beta_2^{N-m}}{1-\beta_2}\right]\left\{\frac{\beta_1\left[(1-\alpha_1)^m - \beta_1^m\right]}{1-\alpha_1-\beta_1}\right\} \tag{C.42}$$

(11) Derivation of $E[t_n(c)]$ for Cost Model II

For case(c) of cost model II, the generalized Shewhart control chart, the derivation of $E[t_n(c)]$ can be achieved through the following steps:

(1) Evaluate the partial summation;

(2) Construct the double summation;

(3) Add the partial summations together to obtain $E[t_n(c)]$.

By the definition of $E[t_n(c)]$ we have:

$$E[t_n(c)] = (1-\alpha_1)^m \sum_{s=m+1}^{N}\left[F(sh)-F((s-1)h)\right](1-\alpha_2)^{s-m-1}\sum_{j=s}^{N} jh(1-\beta_2)\beta_2^{j-s}$$
$$= \frac{(1-\alpha_1)^m(1-\beta_2)h^2}{\theta}\sum_{s=m+1}^{N}(1-\alpha_2)^{s-m-1}\sum_{j=s}^{N} j\,\beta_2^{j-s} \qquad \text{(C.43)}$$

And refre to equation (A.8), we obtain:

$$\sum_{j=s}^{N} j\,\beta_2^{j-s} = \frac{s}{1-\beta_2} - \frac{N\beta_2^{N-s+1}}{1-\beta_2} + \frac{\beta_2-\beta_2^{N-s+1}}{(1-\beta_2)^2} \qquad \text{(C.44)}$$

Refer to equation (A.13), we get:

$$\sum_{s=m+1}^{N}(1-\alpha_2)^{s-m-1}\beta_2^{N-s+1} = \beta_2\left[\frac{(1-\alpha_2)^{N-m}-\beta_2^{N-m}}{1-\alpha_2-\beta_2}\right] \qquad \text{(C.45)}$$

Therefore the double summation is:

$$\sum_{s=m+1}^{N}(1-\alpha_2)^{s-m-1}\sum_{j=s}^{N} j\,\beta_2^{j-s}$$
$$=\frac{1}{1-\beta_2}\left\{\frac{m-N(1-\alpha_2)^{N-m}}{\alpha_2}+\frac{1-(1-\alpha_2)^{N-m}}{\alpha_2^2}-N\beta_2\left[\frac{(1-\alpha_2)^{N-m}-\beta_2^{N-m}}{1-\alpha_2-\beta_2}\right]\right.$$
$$\left.+\frac{1}{(1-\beta_2)}\left[\frac{\beta_2\left(1-(1-\alpha_2)^{N-m}\right)}{\alpha_2}-\frac{(1-\alpha_2)^{N-m}-\beta_2^{N-m}}{1-\alpha_2-\beta_2}\right]\right\} \tag{C.46}$$

Finally:

$$E[t_n(c)]=\frac{(1-\alpha_1)^m(1-\beta_2)h^2}{\theta}\sum_{s=m+1}^{N}(1-\alpha_2)^{s-m-1}\sum_{j=s}^{N} j\,\beta_2^{j-s}$$
$$=\frac{(1-\alpha_1)^m h^2}{\theta}\left\{\frac{m-N(1-\alpha_2)^{N-m}}{\alpha_2}+\frac{1-(1-\alpha_2)^{N-m}}{\alpha_2^2}-N\beta_2\left[\frac{(1-\alpha_2)^{N-m}-\beta_2^{N-m}}{1-\alpha_2-\beta_2}\right]\right.$$
$$\left.+\frac{1}{(1-\beta_2)}\left[\frac{\beta_2\left(1-(1-\alpha_2)^{N-m}\right)}{\alpha_2}-\frac{(1-\alpha_2)^{N-m}-\beta_2^{N-m}}{1-\alpha_2-\beta_2}\right]\right\} \tag{C.47}$$

(12) Proof of Lemma 4.4

To prove of Lemma 4.4, the following steps are needed:

(a) Show that equation (4.2.5) is the same as equation (4.1.3) under Fact 4.2;

(b) Show that equation (4.2.8) is the same as equation (4.1.5) under Fact 4.2;

(c) Show that equation (4.2.14) is the same as equation (4.1.8) under Fact 4.2;

(d) Show that equation (4.2.24) is the same as equation (4.1.20) under Fact 4.2.

For case (a):

$$C_f = c_f P[A] = c_f \left\{ 1-(1-\alpha_1)^m (1-\alpha_2)^{N-m} - \frac{h}{\theta}\left[\frac{1-(1-\alpha_1)^m}{\alpha_1} + \frac{(1-\alpha_1)^m}{\alpha_2}\right] + \frac{h}{\theta}\left[\frac{(1-\alpha_1)^m (1-\alpha_2)^{N-m}(N\alpha_2+1)}{\alpha_2}\right]\right\}$$

$$= c_f\left[1-\frac{h}{\alpha\theta}+\frac{h}{\alpha\theta}(1-\alpha)^N\right] \tag{C.48}$$

For case (b):

$$C_t = c_t P[B] = \frac{c_t h}{\theta}\left[\frac{1-(1-\alpha_1)^{m+1}}{\alpha_1} + \frac{(1-\alpha_2)^{m+1}\left[1-(1-\alpha_2)^{N-m-1}\right]}{\alpha_2}\right]$$

$$= \frac{c_t h}{\theta}\left[\frac{1-(1-\alpha)^{m+1}+(1-\alpha)^{m+1}-(1-\alpha)^N}{\alpha}\right] = c_t\left[\frac{h}{\alpha\theta}-\frac{h}{\alpha\theta}(1-\alpha)^N\right] \tag{C.49}$$

For case (c):

$$C_d = c_d rp\{E[t_p]+E[t_s]\} = c_d rph\left(\frac{1}{2}+\frac{\beta_2}{1-\beta_2}\right) + \frac{c_d rph^2}{\theta}\left[\frac{m\beta_2}{1-\beta_2}\right]$$

$$+\frac{c_d rph^2\beta_1}{\theta}\left[\frac{m}{1-\beta_1}-\frac{\beta_1^m(1-m)}{1-\beta_1}-\frac{1-\beta_1-2\beta_1^m}{(1-\beta_1)^2}\right]$$

$$+\frac{c_d rph^2\beta_1}{\theta}\left\{\left[\frac{1-\beta_1^m}{1-\beta_1}\right]\left[m+\frac{1-\beta_2^{N-m}}{1-\beta_2}-N\beta_2^{N-m}\right]-\left[\frac{1-\beta_2^{N-m}}{1-\beta_1}\right]\left[m-\frac{\beta_1-\beta_1^m}{1-\beta_1}-\beta_1^m\right]\right\}$$

$$= c_d rph\left(\frac{1}{2}+\frac{\beta}{1-\beta}\right) \tag{C.50}$$

For case (d):

$$E[t] = \alpha_1 h\left[\frac{1-(1-\alpha_1)^m}{\alpha_1^2} - \frac{m(1-\alpha_1)^m}{\alpha_1}\right]$$

$$-\frac{\alpha_1 h^2}{\theta}\left[\frac{1-(m-1)^2(1-\alpha_1)^m + m^2(1-\alpha_1)^{m+1}}{\alpha_1^2} + \frac{2(1-\alpha_1)\left[1-(1-\alpha_1)^{m-1}\right]}{\alpha_1^3}\right]$$

$$+\alpha_2 h(1-\alpha_1)^m\left[\frac{m}{\alpha_2} - \frac{N(1-\alpha_2)^{N-m}}{\alpha_2} + \frac{1-(1-\alpha_2)^{N-m}}{\alpha_2^2}\right]$$

$$+\frac{\alpha_2 h^2(1-\alpha_1)^m}{\theta}\left\{\frac{1}{\alpha_2^2} + \frac{2(1-\alpha_2)\left[1-(1-\alpha_2)^{N-m-1}\right]}{\alpha_2^3} - \frac{(N-m-1)^2(1-\alpha_2)^{N-m}}{\alpha_2^2}\right.$$

$$+\frac{(N-m)^2(1-\alpha_2)^{N-m+1}}{\alpha_2^2}$$

$$\left.+2m\left[\frac{1-(1-\alpha_2)^{N-m}}{\alpha_2^2} - \frac{(N-m)(1-\alpha_2)^{N-m}}{\alpha_2}\right] + \frac{m^2\left[1-(1-\alpha_2)^{N-m}\right]}{\alpha_2}\right\}$$

$$+\frac{h^2}{\theta}\left\{\frac{1-(1-\alpha_1)^m}{\alpha_1^2} - \frac{m(1-\alpha_1)^m}{\alpha_1} - \left[m + \frac{1}{1-\beta_1}\right]\left[\frac{\beta_1\left[(1-\alpha_1)^m - \beta_1^m\right]}{1-\alpha_1-\beta_1}\right]\right.$$

$$\left.+\frac{\beta_1}{1-\beta_1}\left[\frac{(1-\alpha_1)^m}{\alpha_1}\right]\right\} + \frac{h^2}{\theta}\left[m - N\beta_2^{N-m} + \frac{1-\beta_2^{N-m}}{1-\beta_2}\right]\left\{\frac{\beta_1\left[(1-\alpha_1)^m - \beta_1^m\right]}{1-\alpha_1-\beta_1}\right\}$$

$$+\frac{(1-\alpha_1)^m h^2}{\theta}\left\{\frac{m - N(1-\alpha_2)^{N-m}}{\alpha_2} + \frac{1-(1-\alpha_2)^{N-m}}{\alpha_2^2} - N\beta_2\left[\frac{(1-\alpha_2)^{N-m} - \beta_2^{N-m}}{1-\alpha_2-\beta_2}\right]\right.$$

$$\left.+\frac{1}{(1-\beta_2)}\left[\frac{\beta_2\left(1-(1-\alpha_2)^{N-m}\right)}{\alpha_2} - \frac{(1-\alpha_2)^{N-m} - \beta_2^{N-m}}{1-\alpha_2-\beta_2}\right]\right\}$$

$$= \frac{h}{\alpha}\left[1-(1-\alpha)^N(\alpha N+1)\right]$$

$$-\frac{\alpha h^2}{\theta}\left\{\frac{1}{\alpha^2}+\frac{2(1-\alpha)\left[1-(1-\alpha)^{N-1}\right]}{\alpha^3}-\frac{(N-1)^2(1-\alpha)^N}{\alpha^2}+\frac{N^2(1-\alpha)^{N+1}}{\alpha^2}\right\}$$

$$+\frac{h^2(1-\beta)}{[\beta-(1-\alpha)]\theta}\left\{\frac{\beta\left(1-\beta^N\right)}{(1-\beta)^2}-\frac{N\beta^{N+1}}{1-\beta}-\frac{(1-\alpha)\left[1-(1-\alpha)^N\right]}{\alpha^2}+\frac{N(1-\alpha)^{N+1}}{\alpha}\right\} \quad \text{(C.51)}$$

Therefore Lemma 4.4 is true.

Appendix D.

Derivations and Proofs for Cost Model III

(1) Derevation of E[t] for Cost Model III

To construct the expression for E[t], we may write E[t] as:

$$
\begin{aligned}
E[t] &= \sum_{j=1}^{\infty} jh\left(\alpha(1-\alpha)^{j-1} e^{-aj^b h^b} + \sum_{l=1}^{j}\left(e^{-a(l-1)^b h^b} - e^{-al^b h^b}\right)(1-\alpha)^{l-1}\beta^{j-1}(1-\beta) \right) \\
&= \alpha h \sum_{j=1}^{\infty}(1-\alpha)^{j-1} e^{-aj^b h^b} + h(1-\beta)\sum_{j=1}^{\infty} j\,\beta^{j-1}\sum_{l=1}^{j}\left(e^{-a(l-1)^b h^b} - e^{-al^b h^b}\right)(1-\alpha)^{l-1}
\end{aligned}
\tag{D.1}
$$

Then refer to Equation (A.7), we obtain:

$$\sum_{j=1}^{\infty} j\,\beta^{j-1} = \frac{1}{1-\beta} + \frac{\beta}{1-\beta^2} = \frac{1}{1-\beta}\left[1 + \frac{\beta}{1-\beta}\right] \tag{D.2}$$

Therefore the expression for E[t] is:

$$\begin{aligned} E[t] = {} & \alpha h \sum_{j=1}^{\infty} (1-\alpha)^{j-1} e^{-aj^b h^b} + h \sum_{l=1}^{\infty} l\, e^{-a(l-1)^b h^b} (1-\alpha)^{l-1} - h \sum_{l=1}^{\infty} l\, e^{-al^b h^b} (1-\alpha)^{l-1} \\ & + h\left(\frac{\beta}{1-\beta}\right) \sum_{l=1}^{\infty} e^{-a(l-1)^b h^b} (1-\alpha)^{l-1} - h\left(\frac{\beta}{1-\beta}\right) \sum_{l=1}^{\infty} e^{-al^b h^b} (1-\alpha)^{l-1} \end{aligned} \tag{D.3}$$

(2) Derivation of $E[t_S(1)]$ for Cost Model III

To construct the expression for $E[t_S(1)]$, the following steps are necessary:

(1) Construct the partial summation;

(2) Construct the double summation;

(3) Construct the expression for $E[t_S(1)]$.

First, we may write $E[t_S(1)]$ as:

$$E[t_s(1)] = \sum_{j=1}^{m} \left[F(jh) - F((j-1)h)\right] \sum_{i=j}^{m} (i-j)h(1-\beta_1)\beta_1^{i-j} \tag{D.4}$$

Refer to equation (A.8) we can have:

$$\sum_{i-j}^{m}(i-j)\beta_1^{i-j}=\frac{1}{1-\beta_1}\left[\frac{\beta_1}{1-\beta_1}-\frac{\beta_1^{m-j+1}}{1-\beta_1}-(m-j)\beta_1^{m-j+1}\right] \tag{D.5}$$

Finally:

$$E[t_s(1)]=\frac{h\beta_1}{1-\beta_1}\left[1-e^{-a(mh)^b}\right]-h\sum_{j=1}^{m}\left[e^{-a(jh-h)^b}-e^{-a(jh)^b}\right](m-j)\beta_1^{m-j+1}$$
$$-\frac{h}{1-\beta_1}\sum_{j=1}^{m}\left[e^{-a(jh-h)^b}-e^{-a(jh)^b}\right]\beta_1^{m-j+1} \tag{D.6}$$

(3) Derivation of $E[t_S(2)]$ for Cost Model III

In order to find the expression for $E[t_S(2)]$, we may write it as:

$$E[t_s(2)]=\sum_{j=1}^{m}\left[F(jh)-F((j-1)h)\right]\beta_1^{m-j+1}\sum_{i=m+1}^{\infty}(i-j)\,h\beta_2^{i-m-1}(1-\beta_2)$$
$$=\sum_{j=1}^{m}\left[F(jh)-F((j-1)h)\right]\beta_1^{m-j+1}(1-\beta_2)h\left[(m-j)\sum_{i-m-1=0}^{\infty}\beta_2^{i-m-1}+\sum_{i-m=1}^{\infty}(i-m)\beta_2^{i-m-1}\right] \tag{D.7}$$

Then refer to equation (A.1), we obtain:

$$\sum_{i-m-1=0}^{\infty}\beta_2^{i-m-1}=\frac{1}{1-\beta_2} \tag{D.8}$$

Refer to equation (A.3), we have:

$$\sum_{i-m=1}^{\infty}(i-m)\beta_2^{i-m-1}=\frac{1}{(1-\beta_2)^2} \tag{D.9}$$

Therefore we obtain:

$$E[t_s(2)] = \sum_{j=1}^{m}\left[e^{-a(jh-h)^b} - e^{-a(jh)^b}\right]\beta_1^{m-j+1}h\left[(m-j)+\frac{1}{(1-\beta_2)}\right]$$
$$= h\sum_{j=1}^{m}\left[e^{-a(jh-h)^b} - e^{-a(jh)^b}\right](m-j)\beta_1^{m-j+1} + \frac{h}{1-\beta_2}\sum_{j=1}^{m}\left[e^{-a(jh-h)^b} - e^{-a(jh)^b}\right]\beta_1^{m-j+1} \tag{D.10}$$

(4) Derivation of $E[t_n(1)]$ for Cost Model III

First we know that:

$$E[t_n(1)] = \sum_{s=1}^{m}[F(sh) - F((s-1)h)](1-\alpha_1)^{s-1}\sum_{j=s}^{m} jh(1-\beta_1)\beta_1^{j-s}$$
$$= h(1-\beta_1)\sum_{s=1}^{m}[F(sh) - F((s-1)h)](1-\alpha_1)^{s-1}\sum_{j=s}^{m} j\,\beta_1^{j-s} \tag{D.11}$$

Then refer to equation (A.8), we obtain:

$$\sum_{j=s}^{m} j\,\beta_1^{j-s} = \frac{1}{1-\beta_1}\left[s - m\beta_1^{m-s+1} + \frac{\beta_1 - \beta_1^{m-s+1}}{1-\beta_1}\right] \tag{D.12}$$

Finally, the expression for $E[t_n(1)]$ is:

$$E[t_n(1)] = h\sum_{s=1}^{m}\left[e^{-a((s-1)h)^b} - e^{-a(sh)^b}\right](1-\alpha_1)^{s-1}\left[s - m\beta_1^{m-s+1} + \frac{\beta_1 - \beta_1^{m-s+1}}{1-\beta_1}\right]$$
$$= h\sum_{s=1}^{m} s\left[e^{-a((s-1)h)^b} - e^{-a(sh)^b}\right](1-\alpha_1)^{s-1}$$
$$-h\left[m + \frac{1}{1-\beta_1}\right]\sum_{s=1}^{m}\left[e^{-a((s-1)h)^b} - e^{-a(sh)^b}\right](1-\alpha_1)^{s-1}\beta_1^{m-s+1}$$
$$+\frac{h\beta_1}{1-\beta_1}\sum_{s=1}^{m}\left[e^{-a((s-1)h)^b} - e^{-a(sh)^b}\right](1-\alpha_1)^{s-1} \tag{D.13}$$

(5) Derivation of $E[t_n(2)]$ for Cost Model III

For case (2) of cost model III, we may first write $E[t_n(2)]$ as:

$$E[t_n(2)] = \sum_{s=1}^{m}[F(sh) - F(s-1)h](1-\alpha_1)^{s-1}\beta_1^{m-s+1}\sum_{j=m+1}^{\infty} jh(1-\beta_2)\beta_2^{j-m-1} \tag{D.14}$$

And we know from equation (A.7) that:

$$\sum_{j=m+1}^{\infty} j\beta_2^{j-m-1} = \frac{1}{1-\beta_2}\left[m + \frac{1}{1-\beta_2}\right] \tag{D.15}$$

Therefore:

$$E[t_n(2)] = h\left[m + \frac{1}{1-\beta_2}\right]\sum_{s=1}^{m}\left[e^{-a((s-1)h)^b} - e^{-a(sh)^b}\right](1-\alpha_1)^{s-1}\beta_1^{m-s+1} \tag{D.16}$$

(6) Derivation of $E[t_n(3)]$ for Cost Model III

For the case (3) of cost model III, we can express $E[t_n(3)]$ as:

$$\begin{aligned} E[t_n(3)] &= (1-\alpha_1)^m \sum_{s=m+1}^{\infty}[F(sh) - F((s-1)h)](1-\alpha_2)^{s-m-1}\sum_{j=s}^{\infty} jh(1-\beta_2)\beta_2^{j-s} \\ &= (1-\beta_2)(1-\alpha_1)^m h \sum_{s=m+1}^{\infty}\left[e^{-((s-1)h)^b} - e^{-a(sh)^b}\right](1-\alpha_2)^{s-m-1}\sum_{j=s}^{\infty} j\,\beta_2^{j-s} \end{aligned} \tag{D.17}$$

Then refer to equation (A.7) again, we obtain:

$$\sum_{j=s}^{\infty} j\,\beta_2^{j-s} = \frac{1}{1-\beta_2}\left[s+\frac{\beta_2}{1-\beta_2}\right] \tag{D.18}$$

Finally we obtain:

$$\begin{aligned} E[t_n(3)] &= h\left(\frac{\beta_2}{1-\beta_2}\right)(1-\alpha_1)^m \sum_{s=m+1}^{\infty}\left[e^{-((s-1)h)^b} - e^{-a(sh)^b}\right](1-\alpha_2)^{s-m-1} \\ &\quad + h(1-\alpha_1)^m \sum_{s=m+1}^{\infty} s\left[e^{-((s-1)h)^b} - e^{-a(sh)^b}\right](1-\alpha_2)^{s-m-1} \end{aligned} \tag{D.19}$$

(7) Proof of Lemma 5.1

To prove Lemma, the following steps are necessary:

(a) Show that equation (5.2.5) is the same as equation (5.1.4) under Fact 5.2;

(b) Show that equation (5.2.7) is the same as equation (5.1.6) under Fact 5.2;

(c) Show that equation (5.2.13) is the same as equation (5.1.12) under Fact 5.2;

(d) Show that equation (5.2.22) is the same as equation (5.1.14) under Fact 5.2.

For case (a):

$$\begin{aligned} C_f = c_f P[A] &= c_f\left\{\alpha_1 \sum_{i=1}^{m}(1-\alpha_1)^{i-1} e^{-ai^b h^b} + (1-\alpha_1)^m \alpha_2 \sum_{i-m=1}^{\infty}(1-\alpha_2)^{i-m-1} e^{-ai^b h^b}\right\} \\ &= c_f \alpha \sum_{i=1}^{\infty}(1-\alpha_1)^{i-1} e^{-ai^b h^b} \end{aligned} \tag{D.20}$$

For case (b):

$$
\begin{aligned}
C_t &= c_t P[B] \\
&= c_t \left\{ \sum_{j=1}^{m+1} \left[e^{-a(j-1)^b h^b} - e^{-aj^b h^b} \right] (1-\alpha_1)^{j-1} + \sum_{j=m+2}^{\infty} \left[e^{-a(j-1)^b h^b} - e^{-aj^b h^b} \right] (1-\alpha_2)^{j-1} \right\} \\
&= c_t \sum_{j=1}^{\infty} \left[e^{-a(j-1)^b h^b} - e^{-aj^b h^b} \right] (1-\alpha_1)^{j-1}
\end{aligned} \tag{D.21}
$$

For case (c):

$$
\begin{aligned}
C_d &= c_d rp\left\{E\left[t_p\right] + E\left[t_s\right]\right\} = c_d rph\left[\frac{\beta_1}{1-\beta_1}\right] + c_d rph + c_d rphe^{-a(mh)^b}\left[\frac{\beta_2}{1-\beta_2} - \frac{\beta_1}{1-\beta_1}\right] \\
&\quad - \frac{c_d rpab}{e^{aj^b h^b} - e^{-a(j+1)^b h^b}}\left[\int_{jh}^{(j+1)h} t^b e^{-at^b} dt - jh\int_{jh}^{(j+1)h} t^{b-1} e^{-at^b} dt\right] \\
&\quad + c_d rph\left\{\left[\frac{1}{1-\beta_2} - \frac{1}{1-\beta_1}\right]\sum_{j=1}^{m}\left[e^{-a(jh-h)^b} - e^{-a(jh)^b}\right]\beta_1^{m-j+1}\right\} = c_d rph \\
&\quad + c_d rph\left[\frac{\beta}{1-\beta}\right] - \frac{c_d rpab}{e^{aj^b h^b} - e^{-a(j+1)^b h^b}}\left[\int_{jh}^{(j+1)h} t^b e^{-at^b} dt - jh\int_{jh}^{(j+1)h} t^{b-1} e^{-at^b} dt\right]
\end{aligned} \tag{D.22}
$$

For case (d):

$$
\begin{aligned}
E[t] &= \alpha_1 \sum_{j=1}^{m} (1-\alpha_1)^{j-1} e^{-aj^b h^b} + (1-\alpha_1)^m \alpha_2 \sum_{j=m+1}^{\infty} (1-\alpha_2)^{j-m-1} e^{-aj^b h^b} \\
&\quad + h\sum_{s=1}^{m} s\left[e^{-a((s-1)h)^b} - e^{-a(sh)^b}\right](1-\alpha_1)^{s-1} \\
&\quad + h(1-\alpha_1)^m \sum_{s=m+1}^{\infty} s\left[e^{-((s-1)h)^b} - e^{-a(sh)^b}\right](1-\alpha_2)^{s-m-1} \\
&\quad + h\left[\frac{1}{1-\beta_2} - \frac{1}{1-\beta_1}\right]\sum_{s=1}^{m}\left[e^{-a((s-1)h)^b} - e^{-a(sh)^b}\right](1-\alpha_1)^{s-1}\beta_1^{m-s+1} \\
&\quad + h\left(\frac{\beta_1}{1-\beta_1}\right)\sum_{s=1}^{m}\left[e^{-((s-1)h)^b} - e^{-a(sh)^b}\right](1-\alpha_2)^{s-1} \\
&\quad + (1-\alpha_1)^m h\left(\frac{\beta_2}{1-\beta_2}\right)\sum_{s=m+1}^{\infty}\left[e^{-((s-1)h)^b} - e^{-a(sh)^b}\right](1-\alpha_2)^{s-m-1}
\end{aligned} \tag{D.23}
$$

$$= \alpha h \sum_{j=1}^{\infty} (1-\alpha)^{j-1} e^{-aj^b h^b} + h \sum_{l=1}^{\infty} l\, e^{-a(l-1)^b h^b} (1-\alpha)^{l-1} - h \sum_{l=1}^{\infty} l\, e^{-al^b h^b} (1-\alpha)^{l-1}$$

$$+ h\left(\frac{\beta}{1-\beta}\right) \sum_{l=1}^{\infty} e^{-a(l-1)^b h^b} (1-\alpha)^{l-1} - h\left(\frac{\beta}{1-\beta}\right) \sum_{l=1}^{\infty} e^{-al^b h^b} (1-\alpha)^{l-1} \qquad \text{(D.24)}$$

Therefore Lemma 5.1 is true.

Appendix E.

GINO and Mathematica Program Listing

```
! Generalized reduced gradient (GRG) algorithm.
! Minimize the expected total cost per unit time function.
! The standard cost model.
MODEL:
    MIN = COSTI + (COSTF + COSTT + CSOTD)/E(T);
    COSTI = (C0 + N*CI)/H;
    COSTF = CF*(ALPHA*(1-ALPHA)^(-1))*X;
    COSTT = CT*(1-ALPHA)^(-1)*Y;
    COSTD = CD*R*P*(E(TP)+E(TS));
```

```
E(TP) = H-(F((J+1)*H)-F(JH))^(-1)*Z;
E(TS) = H*BETA/(1-BETA);
E(T) = ALPHA*H*X/(1-ALPHA)+H*DOUBLE;
K ≥ 0;
H = 1;
ALPHA = 2*(1-PSN(k));
BETA = PSN(K - DELTA*(N^0.5)) - PSN(-K - DELTA*(N^0.5));
DELTA = 0.522;
N ≥ 0;
END;
```

```
! Generalized reduced gradient (GRG) algorithm.
! Minimize the expected total cost per unit time function.
! The generalized cost model.
MODEL:
   MIN = COSTI + (COSTF + COSTT + COSTD)/E(T);
   COSTI = (C0 + N*CI)/H;
   COSTF = CF*(ALPHA1*(1-ALPHA1)^(-1))*X1+ALPHA2*(1-ALPHA2)^M*X2;
   COSTT = CT*(1-ALPHA1)^(-1)*Y1+(1-ALPHA2)^(-1)*Y2;
   COSTD = CD*R*P*(E(TP)+E(TS));
   E(TP) = H-(F((J+1)*H)-F(JH))^(-1)*Z;
   E(TS) = H*(1-BETA1)*Z1+H*(1-BETA2)*Z2+H*BETA2*(1-F(MH))/(1-BETA2);
   E(T) = H*ALPHA1*Z3+H*ALPHA2*(1-ALPHA1)^M*Z4+H*(1-BETA1)*Z5+TT;
   TT = H*(1-BETA2)*Z6+H*(1-ALPHA1)^M*(1-BETA2)*Z7;
   K1 ≥ 0;
```

```
K2 ≥ 0;
H = 1;
ALPHA 1= 2*(1-PSN(K1));
ALPHA 2= 2*(1-PSN(K2));
BETA1 = PSN(K1 - DELTA*(N^0.5)) - PSN(-K1 - DELTA*(N^0.5));
BETA2 = PSN(K2 - DELTA*(N^0.5)) - PSN(-K2 - DELTA*(N^0.5));
DELTA = 0.522;
N ≥ 0;
M ≥ 0;
END;
!
[
X = SUM[((1-ALPHA)^I)*(1-F(IH)), {I,1,INF}];
Y = SUM[((1-ALPHA)^J)*(F(JH)-F((J-1)*H)), {J,1,INF}];
Z = INTEGRATE[(T-JH)*G(T), {T,JH,JH+H}];
G(T) = DERIVATIVE[F(T)];
DOUBLE = SUM[J*Y, {J,1,INF}];
N[%%, 7];
X1 = SUM[((1-ALPHA1)^I)*(1-F(IH)), {I,1,M}];
X2 = SUM[((1-ALPHA2)^(I-M-1)*(1-F(IH)), {I,1,INF}];
Y1 = SUM[((1-ALPHA1)^J)*(F(JH)-F((J-1)*H)), {J,1,M+1}];
Y2 = SUM[((1-ALPHA2)^J)*(F(JH)-F((J-1)*H)), {J,1,INF}];
Z1 = SUM[(F(JH)-F((J-1)*H))*SUM1, {J,1,M}];
SUM1 = SUM[(I-J)*BETA1^(I-J), {I,J,M}];
Z2 = SUM[SUM2*(F(JH)-F((J-1)*H)), {J,1,M}];
```

```
SUM2 = SUM[(I-J)*BETA2^(I-M-1), {I,M+1,INF}];
Z3 = SUM[J*((1-ALPHA1)^(J-1))*(1-F(IH)), {J,1,M}];
Z4 = SUM[J*((1-ALPHA2)^(J-M-1)*(1-F(IH)), {J,M+1,INF}];
Z5 = SUM[((1-ALPHA1)^(J-1)*(F(JH)-F((J-1)*H))*SUM3, {J,1,M}];
SUM3 = SUM[J*BETA1^(I-J), {I,J,M}];
Z6 = SUM[((1-ALPHA1)^(J-1)*(F(JH)-F((J-1)*H))*SUM4, {J,1,M}];
SUM4 = SUM[I*BETA2^(I-M-1), {I,M+1,INF}];
Z7 = SUM[((1-ALPHA2)^(J-M-1)*(F(JH)-F((J-1)*H))*SUM5, {J,1,INF}];
SUM5 = SUM[I*BETA2^(I-J), {I,J,INF}];
]
N[%%, 7];
[
LIST 1 = {};
F[W_] = 1-EXP[-LEMDA*W];
LIST = TABLE[N[%], {LEMDA, 0.01,0.02,0.05,0.1,0.2,0.5,0.75}];
F[W_] = W/THETA;
LIST1 = TABLE[N[%], {THETA, 8,20,40,80,100,200}];
F[W_] = 1-A*T^(A*(B-1))*EXP[-A*W^B];
LIST1 = TABLE[N[%], {A, 0.01,0.05,0.1,0.2},{B,1.05,1.25,1.5}];
W = I*H;
FOR[I = 1, I <= LENGTH[LIST1], ++I, APPENDTO[LIST2, F[W_]];
LIST2 = {};
SHOW[N[%], LIST1, LIST2, -> AUTOMATIC];
]
N[%%, 7];
```

Vita

Xueli Zou was bone in 1961 in Jilin, The Peoples Republic of China. After he finished his early schooling in Jilin, he joined Changchun Normal Institute in 1978 and received his B.S. degree in Mathematics in 1983. He had worked as an instructor for four years in Shulan Mining College, Jilin, PRC. He then came to The College of William and Mary in Virginia in 1988 and received his M. S. degree in Operations Research in 1990. He came to Virginia Polytechnic Institute and State University in 1990 and received another M. S. degree in Statistics in 1991. Since 1992 he has been in the Ph. D. program in Industrial and Systems Engineering at Virginia Tech where he recently completed the requirements for the Ph. D. degree. His research interests include statistical quality control, reliability theory and optimization.