MATHEMATICAL MODELLING, FINITE DIMENSIONAL APPROXIMATIONS AND SENSITIVITY ANALYSIS FOR PHASE TRANSITIONS IN SHAPE MEMORY ALLOYS

by

Ruben Daniel Spies

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APPROVED:

John A. Burns

John A. Burns, Chairman

Eugene M. Cliff

Eugene M. Cliff

Terry L. Herdman

Terry L. Herdman

Kenneth B. Hannsgen

Kenneth B. Hannsgen

Robert L. Wheeler

Robert L. Wheeler

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Abstract

Shape Memory Alloys (SMA's) are intermetallic materials (chemical compounds of two or more elements) that are able to sustain a residual deformation after the application of a large stress, but they "remember" the original shape to which they creep back, without the application of any external force, after they are heated above a certain critical temperature.

A general one-dimensional dynamic mathematical model is presented which accounts for thermal coupling, time-dependent distributed and boundary inputs and internal variables. Well-posedness is obtained using an abstract formulation in an appropriate Hilbert space and explicit decay rates for the associated linear semigroup are derived. Numerical experiments using finite-dimensional approximations are performed for the case in which the thermodynamic potential is given in the Landau-Devonshire form. The sensitivity of the solutions with respect to the model parameters is studied. Finally, an alternative approach to the stress-strain laws is presented which is able to capture the dependence on the strain history.

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CHAPTER I Introduction

1.1 Introduction

During the past few years considerable attention has been devoted to the developement of smart materials and structures. Due to their unique characteristics, Shape Memory Alloys (SMA's) have been considered among the materials with potential for applications in this area. In particular, these materials are being tested as actuators and sensors in various control systems.

The name "Shape Memory Alloys" comes from the fact that at low temperatures these intermetallic materials (chemical compounds of two or more elements) may sustain a residual deformation after the application of a large stress. However, their original shape can be completely restored simply by heating above a certain critical temperature.

The behavior of these materials at low temperatures is elasto-plastic (Figure 1.1.1a). It is also called ferroelastic because of the similarity of the stress-strain relations with the field-magnetization curves of a ferromagnet. In this range of temperatures the load-deformation curves exhibit an elastic region at small loads, a plastic yield and a second elastic branch corresponding to large loads. This second elastic branch permits the body to withstand loads beyond the plastic yield, after which, subsequent unloading produces a residual deformation.

In the intermediate range of temperatures a plastic yield can still be observed

(Figure 1.1.1b). Nevertheless, loading beyond the plastic yield followed by complete unloading does not lead to a residual deformation because of the existence of an intermediate elastic branch which the body reaches by creeping back after the load falls below a certain critical value. This type of conduct is called pseudoelasticity and for this reason, SMA's have also been termed pseudoelastic materials.

Finally, in the high-temperature range the behavior is almost linearly elastic for loads between certain bounds. However, application of large loads can produce a permanent deformation and the stress-strain curves may show a mixture of pseudoelasticity and shape recovery after unloading (Figure 1.1.1c). Some of the alloys which exhibit the above phenomena are AgCd, AuCd, CuAlNi, CuAuZn, CuSn, NiAl, NiTi to name only a few (see [**57**] for a complete list).

Although there is much to be studied and discovered in order to take full advantage of all their capabilities, SMA's have already found a broad variety of applications in aircraft, heat engines, orthodontic and other dental devices ([**8**]), robotic devices and actuators ([**57**], [**93**]), deployable antennas for spacecraft, pipe coupling devices, air conditioners, temperature switches and fuses ([**57**]), SMA Hybrid Composites ([**117**]), in medicine, as a substitute for the Harrington Rod in the treatment of scoliosis ([**121**]) and as boneplates ([**23**], [**57**]).

The first observations of the Shape Memory Effect (SME) go back to the 1930's. In 1938, Alen B. Greninger of Harvard University and G. Mooradian of the Massachusetts Institute of Technology showed that temperature changes could produce and make disappear the martensite phase in brass. However it was not until 1962, with the discovery of the NiTinol by Buehler ([**93**]), that rigorous in-depth studies were completed. Most of these initial efforts concentrated on metallurgical aspects and led to the publication of several books and papers related to the microscopic and mechanical properties ([**35**], [**36**], [**57**], [**114**], [**135**]). Between 1968 and 1986 several mathematical models were proposed and studied ([**3**], [**4**], [**5**], [**44**], [**45**], [**92**], [**97**], [**101**], [**102**], [**138**]). Most of these SMA models did not take into account the strong coupling between the thermal and the mechanical properties which characterize their behavior. Some of these static models dealt merely with the problem of finding a simple and appropriate fitting for the stress-strain relations that was able to describe and explain some of the properties observed experimentally. However, either temperature or stress was assumed constant everywhere, the processes were treated spatially pointwise, and time-dependent actions were excluded. These and other additional limitations make these models difficult to use in practical control design.

The design, modelling and control of intelligent systems and structures is an inmensely complex problem and the subject of many ongoing interdisciplinary programs. There is a rapidly growing need for a unified theory in this area. This theory must be able to capture the unusual and complex characteristics of SMA's and at the same time it must be practical enough to allow for the development of computational algorithms for the design of controllers.

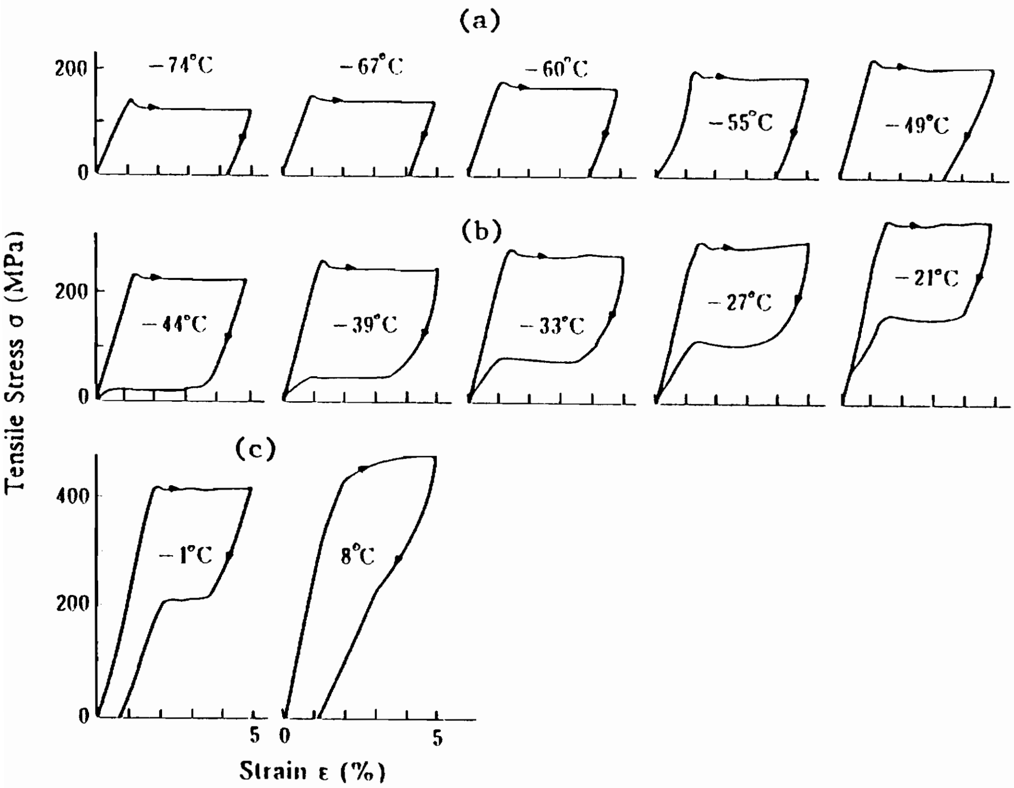


Figure 1.1.1: Stress-Strain curves obtained experimentally for Ti-51 at % Ni for different temperatures ([**57**]).

1.2 Notation

Most of the notation we will use is standard. If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed linear spaces, then $\mathcal{L}(X,Y)$ will denote the space of all linear bounded operators from X to Y. For any $A \in \mathcal{L}(X,Y)$, $\|A\|_{\mathcal{L}(X,Y)}$ or simply $\|A\|$ will denote the operator norm on the space $\mathcal{L}(X,Y)$. If $X = Y$ we denote $\mathcal{L}(X,Y)$ by $\mathcal{L}(X)$. For a Hilbert Space X, the inner product on $X \times X$ will be denoted by $\langle\cdot,\cdot\rangle_X$. Sometimes the Hilbert space X will depend on a certain parameter q. In those cases we will use $\langle\cdot,\cdot\rangle_q$ instead of $\langle\cdot,\cdot\rangle_X$, to emphasize the dependence on q. If A is an operator from X into X, then $dom(A)$, $\sigma(A)$, $\sigma_p(A)$, $\rho(A)$ and A^* will denote the domain, spectrum, point spectrum, resolvent and adjoint of A, respectively. For $\Omega \subset \mathbb{R}$ and $1 < p < \infty$, $L_p(\Omega; X)$ will denote the space of all Lebesgue measurable functions f from Ω into X such that $\|f\|_{L_p(\Omega;X)} = \left(\int_\Omega \|f(x)\|_X^p \, dx\right)^{1/p} < \infty$. In addition, if ρ is a positive constant, $L_{p,\rho}(\Omega; X)$ will denote the Banach space of all functions f in $L_p(\Omega; X)$ with the weighted norm $\|f\|^p_{L_{p,\rho}(\Omega:X)} = \rho\|f\|^p_{L_p(\Omega;X)}$. In the event that $X = \mathbb{R}$, $L_p(\Omega; X)$, $L_{p,\rho}(\Omega; X)$ will simply be denoted by $L_p(\Omega)$, $L_{p,\rho}(\Omega)$, respectively. From time to time we will use $\|\cdot\|_{L_p}$, $\|\cdot\|_{L_{p,\rho}}$ instead of $\|\cdot\|_{L_p(\Omega)}$, $\|\cdot\|_{L_{p,\rho}(\Omega)}$. In all such cases the appropriate set Ω will be clearly understood from the context. For a nonnegative integer k, $C^k(\Omega; X)$ will denote the space of all functions from Ω into X having k continuous derivatives in Ω. The spaces $H^k(\Omega)$ and $H_0^k(\Omega)$ will denote the standard Sobolev spaces defined by $H^k(\Omega) = \left\{f \in L_2(\Omega) \mid f^{(j)} \in L_2(\Omega),\, j = 1,2,\ldots,k\right\}$ and $H_0^k(\Omega) = \left\{f \in H^k(\Omega) \mid f^{(j)}|_{\partial\Omega} = 0,\, j = 1,2,\ldots,k-1\right\}$, respectively.

CHAPTER II Analysis of the Different Models. Literature Review

2.1 Müller and Villaggio's Model

In [**101**] Müller and Villaggio use an ingenious technique to construct an elasto-plastic body as a stack of snap-springs. The simple model obtained for the body is used to simulate and describe the main attributes possessed by elasto-plastic bodies, such as stress-strain curves, creep and yield, elastic unloading, strain hardening, residual deformation, etc. The unit element is depicted in Figure 2.1.1. Each snap-spring has two stable force-free configurations, one of which ("tip-down") is shown in Figure 2.1.1(a). The other equilibrium state corresponds to the "tip-up" position.

In the tip-down equilibrium position an upward force π is applied to the tip, which is displaced a distance Δ_2. At the same time the spring is compressed by Δ_1 (see Figure 2.1.1(b)). When the load reaches a certain critical value π_{cr}, the element suddenly flips over assuming the tip-up equilibrium configuration.

The energy stored in the spring corresponding to a certain displacement Δ_1 is given by

$$\frac{k}{2}\Delta_1^2 \tag{2.1.1}$$

where k is the stiffness of the spring. Thus, the force π needed to produce a displacement Δ_2 is given by

$$\pi(\Delta_2) = \frac{\partial}{\partial \Delta_2}\left(\frac{k}{2}\Delta_1^2\right) \tag{2.1.2}$$

where Δ_1 is regarded as a function of Δ_2.

Now, Δ_1 and Δ_2 are related to the angular displacement ν by the equations

$$\Delta_1(\nu) = 2L\,[cos(\alpha - \nu) - cos\alpha] \tag{2.1.3a}$$

$$\Delta_2(\nu) = L\,[sin\alpha - sin(\alpha - \nu)] \tag{2.1.3b}$$

where α is the angle between the horizontal line and any of the force-free configurations. From (2.1.2) and (2.1.3) one obtains

$$\pi(\nu) = \frac{\partial}{\partial \nu}\left(\frac{k}{2}\Delta_1^2\right)\frac{1}{\frac{\partial \Delta_2}{\partial \nu}} = 4kL\,[cos(\alpha - \nu) - cos\alpha]\,tan(\alpha - \nu) \tag{2.1.4}$$

Figure 2.1.2 shows a graph of $\pi(\nu)$ and $\pi(\Delta_2)$ for $\alpha = \frac{\pi}{8}$. The dotted parts in the figures (negative slope) correspond to unstable equilibria. The extrema of the $\pi(\Delta_2)$-curve correspond to the critical loads π_{cr}.

When a load increases past the positive load π_{cr}, Δ_2 jumps to reach the point on the right hand side of the curve corresponding to the same load. A similar situation occurs under unloading and subsequent compressive load.

Construction of the body. The body is constructed using four stacks of n snap-springs each, as shown in Figure 2.1.3 for $n = 11$. The stacks are cramped together at one end while at the other they are left free and can slide over each other. A normal stress σ on the surfaces F produces a shear stress $\tau = \dfrac{\sigma}{2}$ on any surface forming a 45^o angle with F. The shear force on each stack is

$$P_s = \tau\frac{F}{\sqrt{2}} = \frac{1}{2^{3/2}}P \tag{2.1.5}$$

where $P = \sigma F$ is the total load on F. We further assume that each snap-spring receives a quota π_i of P_s so that

$$\sum_{i=1}^{n} \pi_i = P_s \tag{2.1.6}$$

and that the total displacement of the body consists of the sum of the displacements of each snap-spring as shown in Figure 2.1.4. The lateral displacements are compensated by opposite stacks. Thus, the total displacement of the body is given by

$$D = 2\frac{1}{\sqrt{2}}\sum_{i=1}^{n} \Delta_2(\nu_i) + D_0 \tag{2.1.7}$$

where the factor "2" accounts for the upper and lower stacks, ν_i is the angular displacement of the i^{th} spring and D_0 is chosen so that $D = 0$ in the load-free configuration.

One postulates first that all the π_i's are equal so that $\pi_i = \frac{1}{n}P_s = \frac{P}{2^{3/2}n}$. Thus the $\Delta_2(\nu_i)$'s are equal between the elements in the upper position (U). The same happens with the elements in the lower (D) position. Under this assumption, (2.1.7) can be written as

$$D = \sqrt{2}\{nx\Delta_2(\nu_U) + n(1-x)\Delta_2(\nu_D) - nXJ\} \tag{2.1.8}$$

where x is the proportion of elements of type U (in the sequel x = phase factor), X is the initial proportion in the load-free configuration, $\Delta_2(\nu_U)$ and $\Delta_2(\nu_D)$ correspond to the displacements of the snap-springs of types U and D respectively and $J = 2L\, sin\alpha$ is the distance between the tips of a snap-spring in the two different stable load-free configurations (see Figure 2.1.1).

In Figure 2.1.4-(b) $0 < \dfrac{P}{n2^{3/2}} < \pi_{cr}$ and in Figure 2.1.4-(c) $\dfrac{P}{n2^{3/2}} > \pi_{cr}$. Note in the latter case that the type-D elements have flipped over producing a marked increase in length. Unloading produces now a "plastic" residual displacement because the elements that have changed from type D to type U, do not flip back under unloading (Figure 2.1.4-(d)). Roughly speaking, the types U and D represent two phases of the body. In the process depicted in Figures 2.1.4(a)-(c) the body has changed from an heterogeneous ($x = \frac{5}{11}$) to an homogeneous ($x = 1$) phase composition.

Load-Displacement Curves. The load displacement curves can be constructed as the "superposition" of n curves of the type displayed in Figure 2.1.2, as follows. The body can follow different load displacement curves depending on the phase factor x. For a given load P, the displacement of nx elements is read off from the right solid part of the curve 2.1.2-(b) corresponding to the load $\pi = \frac{P}{2^{3/2}n}$, while the displacement of $n(1-x)$ elements is read off from the left solid branch, corresponding to the same load. Figure 2.1.5 shows the different load-displacement relations for the case $n = 30$ and $X = \frac{1}{2}$. The region enclosed by the two elastic branches corresponding to homogeneous phase compositions, is called the "region of coexistence of phases" or simply the "coexistence region".

If a plastic body contains elements of both types U and D, under dead loading one moves up or down (depending on the character of the load) in the interior of the coexistence region along the curve corresponding to the given phase factor x. When the critical load is exceeded, one jumps to one of the curves where $x = 0$ or $x = 1$.

Strain Hardening. Assume now that the snap-springs with tips in the same direction of the load receive a bigger share than those with tips in the opposite direction. Let us denote them by $\pi_{||}$ and $\pi_{\uparrow\downarrow}$, respectively, and postulate

$$\pi_{||} = (1+\delta)\pi_{\uparrow\downarrow} \tag{2.1.9}$$

where $\delta > 0$. If P is tensile we then have $\frac{P}{2^{3/2}} = nx\pi_{||} + n(1-x)\pi_{\uparrow\downarrow}$ while $\frac{P}{2^{3/2}} = nx\pi_{\uparrow\downarrow} + n(1-x)\pi_{||}$ if P is compressive. The critical loads depend now upon the phase factor and we have $P_{cr}(x) = 2^{3/2}n(1+\delta x)\pi_{cr}$ The load-deformation curves are obtained as in the homogeneous load-shearing case, i.e., by superposition of the individual $\pi(\Delta_2)$-curves. The only difference is that now, if P is tensile and the phase factor is x, then the corresponding displacement is obtained as

$$n(1-x)\Delta_2^L(\pi_{\uparrow\downarrow}) + nx\Delta_2^R(\pi_{||}) \tag{2.1.10}$$

where $\Delta_2^L(\pi_{\uparrow\downarrow})$ is the displacement Δ_2 corresponding to the load $\dfrac{P}{2^{3/2}n(1+\delta x)}$ obtained from the left solid part of the $\pi(\Delta_2)$-curves in Figure 2.1.2-(b) and, similarly, $\Delta_2^R(\pi_{||})$ corresponds to the right solid part for the load $\dfrac{(1+\delta)P}{2^{3/2}n(1+\delta x)}$. The load-deformation curves obtained in this way are shown in Figure 2.1.6 for the case $n = 11$. The interpretation of these curves is similar to that corresponding to the homogeneous load-shearing. However, when a load reaches $P_{cr}(x)$, the body increases x by $\frac{1}{n}$ and a higher load, namely $P_{cr}(x+\frac{1}{n})$, is now needed to reach the next phase factor. This phenomenon is called "strain-hardening". Under unloading one moves down (or up if the original load was compressive) along the branch corresponding to the phase factor with which we started to unload.

The internal energy as a function of D and x. From equation (2.1.1) we know that the energy stored in each spring is $\frac{k}{2}\Delta_1^2$. Therefore, the total energy of the body is given by

$$e(x,\nu_U,\nu_D) = 4\left[nx\frac{k}{2}\{\Delta_1(\nu_U)\}^2 + n(1-x)\frac{k}{2}\{\Delta_1(\nu_D)\}^2\right] \tag{2.1.11}$$

where the factor "4" comes from the four stacks that compose the body. Furthermore, from equations (2.1.4) and (2.1.9) we obtain (for the tensile case)

$$\begin{aligned}[cos(\alpha-\nu_U) - cos\alpha]tan(\alpha-\nu_U)& \\ &= (1+\delta)\,[cos(\alpha-\nu_D) - cos\alpha]\,tan(\alpha-\nu_D)\end{aligned} \tag{2.1.12}$$

Finally $e(x,D)$ can be obtained from equations (2.1.8), (2.1.11) and (2.1.12) in the following way:

i) from (2.1.12) obtain $\nu_D = f(\nu_U)$;

ii) from (2.1.8) $D = D(x,\nu_U,\nu_D) = D(x,\nu_U,f(\nu_U))$ and therefore $\nu_U = g(x,D)$;

iii) finally, from (2.1.11) $e = e(x,g(x,D),f(g(x,D))) = e(x,D)$.

The functions f and g above, cannot be explicitly found. However we can simplify the problem by postulating $\nu_U = 2\alpha$ under tension and $\nu_D = 0$ under compression. This assumption is equivalent to replacing the positive-valued part of the "U"- branch and the negative-valued part of the "D"-branch in the $\pi(\Delta_2)$-curves shown in Figure 2.1.2 by vertical lines. One then obtains

$$e(D,x) = 8n(1-x)kL^2\left\{\sqrt{1-\left[sin\alpha - \frac{\frac{D}{\sqrt{2}} - n(x-X)J}{n(1-x)L}\right]^2} - cos\alpha\right\} \tag{2.1.13}$$

In a later work ([**97**]) Müller modified this model by accounting for possible fluctuations of the snap-springs around their stable equilibrium configurations. These fluctuations can be the result of a "thermal motion" due to energy exchange between the individual elements that compose the body or between the elements and their surroundings. Assuming that the mass m of a snap-spring is concentrated in its tip and by using the equation of motion of that tip, one finds that the potential energy $\phi(\Delta_2;\pi)$ of a snap-spring under a load π is given by

$$\phi(\Delta_2;\pi) = \frac{k}{2}\Delta_1^2 - \pi\Delta_2 \tag{2.1.14}$$

Figure 2.1.7 depicts this function for different values of π. There are three equilibrium positions provided that $\pi < \pi_{cr}$, two stable ones corresponding to the D and U positions and an unstable one related to the dotted part of the $\pi(\Delta_2)$-curves.

Now a small kinetic energy δ will cause the snap-springs to bounce back and forth between the two values of Δ_2 corresponding to the intersection of the curve $\phi(\Delta_2;\pi)$ with the horizontal line $\phi(\Delta_2^*;\pi) + \delta$ where Δ_2^* is the displacement corresponding to the initial stable position.

Using Statistical Mechanics, Müller derived expressions for the probability of finding a snap spring with velocity between $\dot{\Delta}_2$ and $\dot{\Delta}_2 + d\dot{\Delta}_2$ and displacement between Δ_2 and $\Delta_2 + d\Delta_2$, for the phase factor x, for the expected values of the displacements, etc.. The phase factor x, for instance, is given as

$$x = \frac{\int_{\Delta_2^I|_E}^{L(1+sin(\alpha))} e^{-\frac{1}{KT}\left(\frac{k}{2}\Delta_1^2 - \pi\Delta_2\right)} d\Delta_2}{\int_{-L(1-sin(\alpha))}^{L(1+sin(\alpha))} e^{-\frac{1}{KT}\left(\frac{k}{2}\Delta_1^2 - \pi\Delta_2\right)} d\Delta_2}$$

where π is the applied load, T is the temperature of the body, K is the Boltzman constant and $\Delta_2^I|_E$ is the displacement Δ_2 corresponding to the unstable equilibrium in the $\phi(\Delta_2;\pi)$-curve.

This approach followed by Müller and Villaggio is very ingenious and its "limiting process" certainly provides interesting qualitative insights into the behavior of pseudoelastic materials. However, it cannot be used as a tool for practical control design since it is not a dynamic model and it does not allow for the inclusion of any type of external conditions, boundary or distributed.

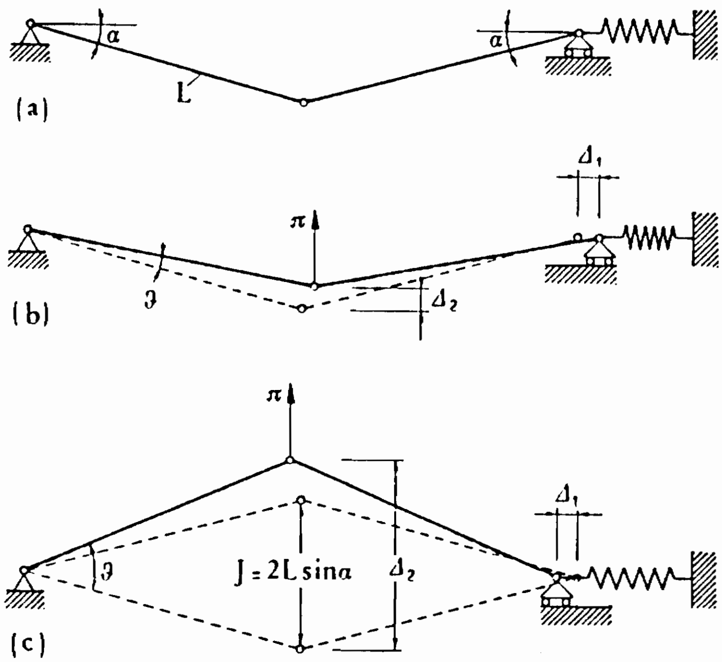


Figure 2.1.1: Different positions of the basic element (Snap-Spring)([**101**]).

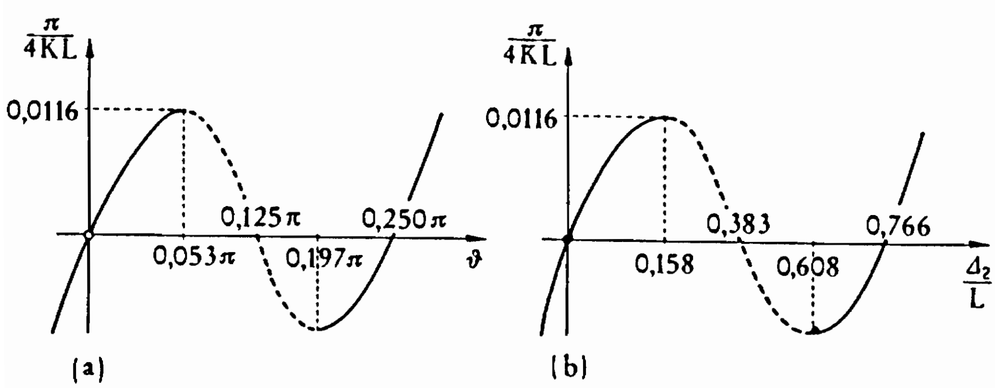


Figure 2.1.2: $\pi(\nu)$ and $\pi(\Delta_2)$-curves for $\alpha = \pi/8$ ([**101**]).

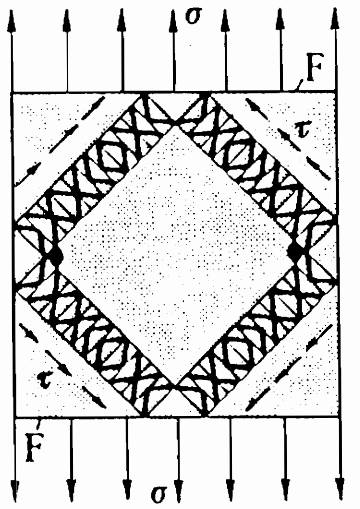


Figure 2.1.3: Body built using four stacks of eleven snaps-springs each ([**101**]).

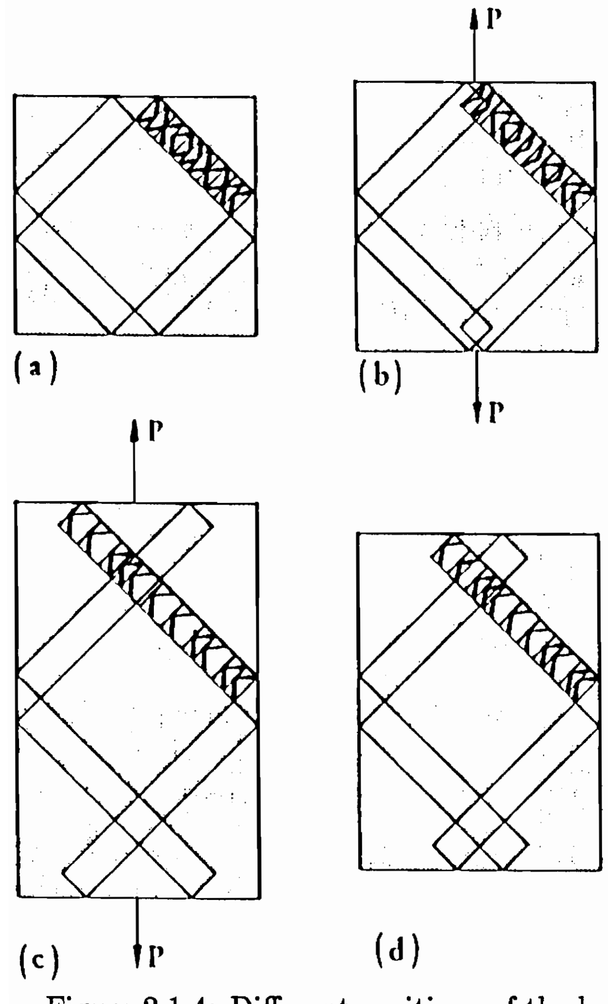


Figure 2.1.4: Different positions of the body ([**101**]).

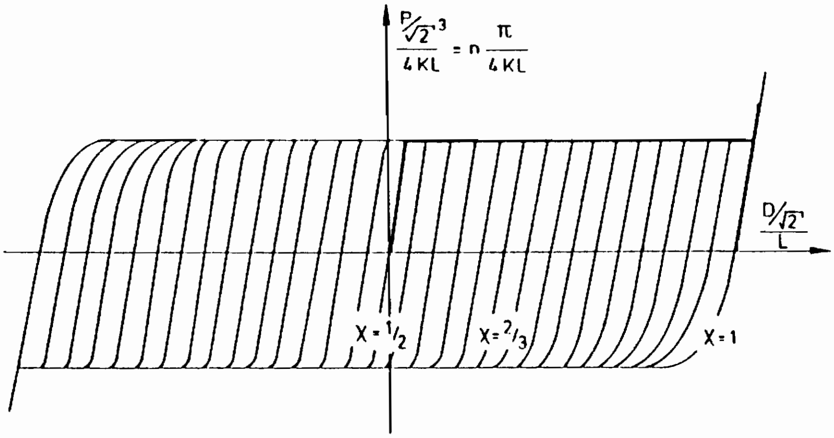


Figure 2.1.5: Load-displacement relations ($n = 30, X = 1/2$)([**97**]).

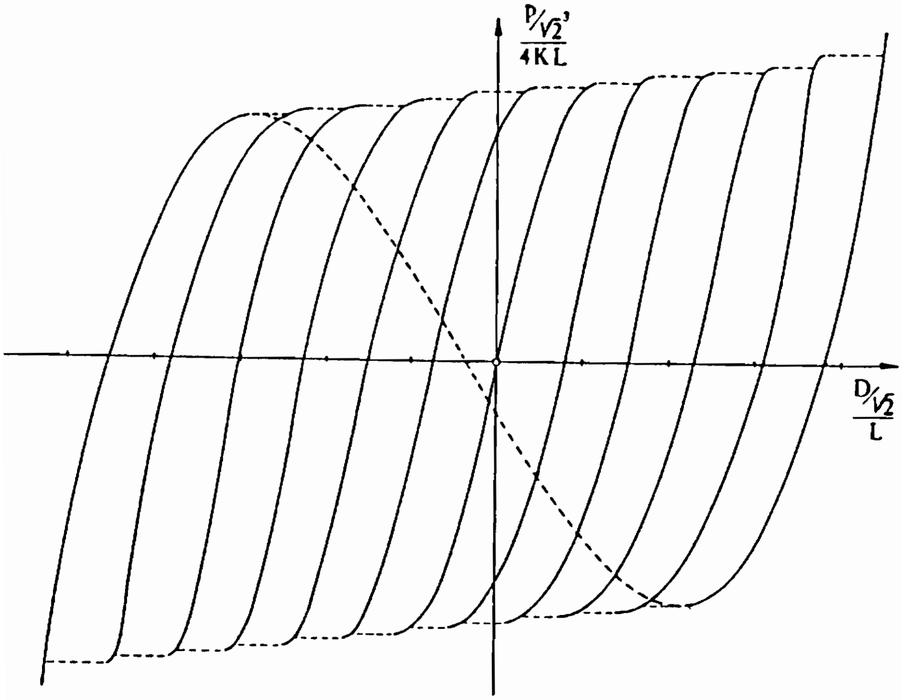


Figure 2.1.6: Load-Deformation curves for the non-homogeneous load-shearing case ([**101**]).

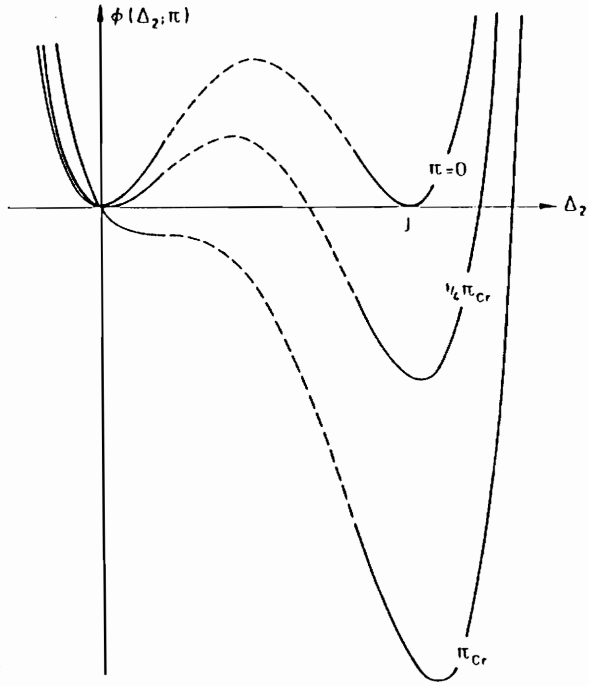


Figure 2.1.7: Potential energy as a function of Δ_2 for different values of π ([**97**]).

2.2 Falk's Model

The purpose of the model presented by Falk ([**44**],[**45**],[**47**]) is to reflect part of the qualitative behavior of Shape Memory Alloys. This model is constructed within the framework of the Landau theory of phase transitions ([**42**]) and it is restricted to the case of single crystals. The thermodynamic potential is given in the form of a Hemholtz free energy density Ψ, which is considered to be a function of the strain ϵ and the temperature θ. A simple form of this function Ψ is chosen ([**47**]) so as to satisfy all the underlying hypotheses of the Landau Theory:

$$\Psi(\epsilon,\theta) = \Psi_0(\theta) + \alpha_2(\theta-\theta_1)\epsilon^2 - \alpha_4\epsilon^4 + \alpha_6\epsilon^6 \tag{2.2.1}$$

where $\Psi_0(\theta)$ is some smooth function of the temperature and α_2, α_4, α_6, θ_1 are positive constants which depend on the specific material being considered. To avoid this dependence on the material, dimensionless quantities are introduced by means of the following normalization

$$\begin{aligned} f &= \frac{\alpha_6^2}{\alpha_4^3}\Psi & \qquad e &= \left(\frac{\alpha_6}{\alpha_4}\right)^{1/2}\epsilon \\ t &= \frac{\alpha_6\alpha_2}{\alpha_4^2}(\theta-\theta_1) - \frac{1}{4} & \qquad f_0(t) &= \frac{\alpha_6^2}{\alpha_4^3}\Psi_0(\theta) \end{aligned} \tag{2.2.2}$$

The normalized free energy is then independent of any particular parameter and takes the form

$$f(e,t) = f_0(t) + (t-\frac{1}{4})e^2 - e^4 + e^6 \tag{2.2.3}$$

(see Figure 2.2.1).

Using this representation for the free energy, several thermodynamical functions are derived. The stress-strain relation, for instance, is given by

$$\sigma = \frac{\partial f}{\partial e} = 2(t + \frac{1}{4})e - 4e^3 + 6e^5 \tag{2.2.4}$$

(see Figure 2.2.2). Several other characteristics of the material like elastic constant, Curie point, critical temperatures, entropy, Gibbs free energy, internal energy, specific heat, temperature-induced phase transitions, latent heat and shape memory effects are also considered and studied.

For the specific case of the alloy $Au_{23}Cu_{30}Zn_{47}$, the four parameters α_2, α_4, α_6 and θ_1 are approximated by using a set of data published by Y. Murakami ([**104**]) in 1972. The estimates are

$$\alpha_2 = 24 \quad cm^{-3}K^{-1}$$

$$\alpha_4 = 1.5 \times 10^5 \quad Jcm^{-3}$$

$$\alpha_6 = 7.5 \times 10^6 \quad Jcm^{-3}$$

$$\theta_1 = 208 \quad K$$

The Ginzburg-Landau theory considers the thermodynamic potential not only as a function of the strain ϵ and the temperature θ, but also as a function of the gradient of the strain, $\nabla\epsilon$. A similar analysis is made for this case. In Section 3.5 we shall discuss some of the problems derived from the use of this type of potential and we present an alternative approach to the stress-strain laws. In particular, this new approach will allows us to take into account non-uniform configurations and local memories.

Falk's model provides many interesting qualitative properties of SMA's. However it has several limitations among which one finds the following ones:

(1) The properties derived from the model are only valid in an isothermal context since no thermal fluctuations are taken into account;

(2) The model is only valid pointwise in space;

(3) It cannot be used for control purposes since it is not dynamic and no external actions are allowed.

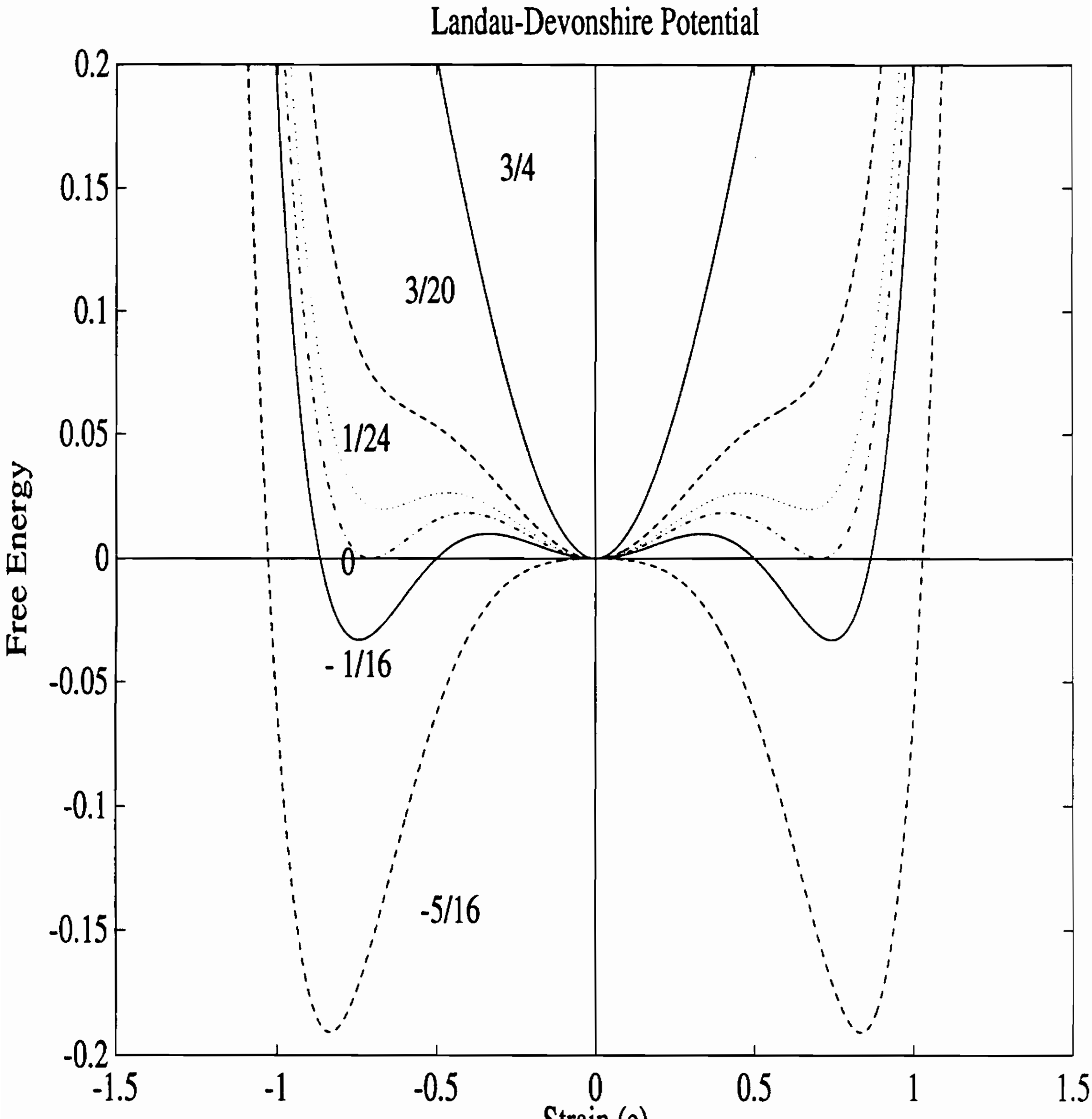


Figure 2.2.1: Normalized Hemholtz free energy as a function of the (normalized) strain for different (normalized) temperatures (fo(t)=0)

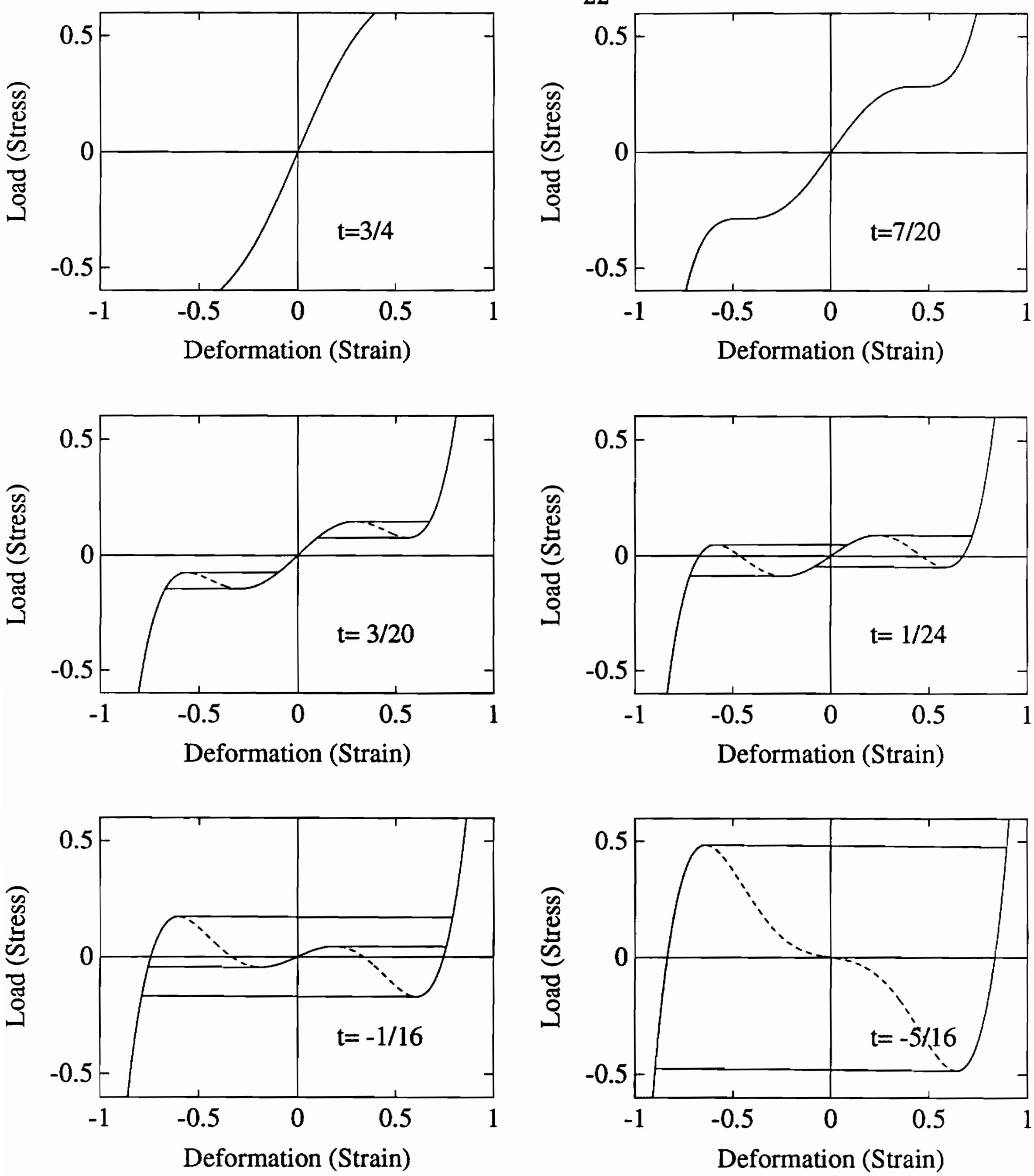


Figure 2.2.2: Stress-Strain relations for different temperatures obtained with the normalized Landau-Devonshire potential; the dotted lines correspond to the unstable parts of the curves.

2.3 Müller and Wilmanski's Model

In [**102**], I. Müller and K. Wilmanski use tools from Statistical Mechanics to derive explicit functions for the free energy. The body is modelled as a stack of lattice layers arranged at an angle of 45° with respect to the loading surfaces (see Figure 2.3.1). The potential energy of a lattice layer is postulated as

$$\Phi(\delta) = \begin{cases} \varphi_0 + \varphi_1\delta^2 - (3\varphi_0 + 2\varphi_1)\delta^4 + (2\varphi_0 + \varphi_1)\delta^6, & \text{for } |\delta| \leq 1 \\ \infty, & \text{for } |\delta| > 1 \end{cases} \tag{2.3.1}$$

(see Figure 2.3.2) where $\varphi_0 = 0.3125$, $\varphi_1 = 11.875$ and $\delta = \frac{\Delta}{J}$ is a dimensionless measure of the displacement with respect to the parent configuration or austenite. Here Δ is the actual displacement of the layer and J is the absolute value of the difference between the length of the layer in the austenitic phase and the length of the layer in any of the martensitic phases (see Figure 2.3.3).

Suppose the body is constructed by stacking N lattice layers and let D be its deformation with respect to the parent configuration. Then

$$D = \frac{1}{\sqrt{2}} \sum_{i=1}^{N} \Delta_i \tag{2.3.2}$$

where Δ_i is the deformation of the i^{th} layer. The deformation is measured as the difference between the length of the body and its length when all the lattice layers are in the austenitic configuration. The length is measured as the distance between the lower left and the upper right corners. Several methods and laws of statistical mechanics are then applied to determine the thermodynamic functions. The free energy Ψ_{pot}, for instance, is found to be

$$\Psi_{pot}(D,T) = -kT\left[Nln(2zJ) + \beta\sqrt{2}D + Nln\left(\frac{1}{J}\int_0^J e^{-\frac{\Phi(\Delta)}{KT}} cosh(\beta\Delta)d\Delta\right)\right] \tag{2.3.3}$$

where D, N and J are as defined above, T is the temperature of the heat bath in which the body is inmersed, z is a certain constant of proportionality, K is the Boltzman constant and β is the solution of the equation

$$\mathcal{L}(\beta J) = -\sqrt{2}\frac{D}{NJ} \tag{2.3.4}$$

where $\mathcal{L}(x)$ is the Langevin function $\mathcal{L}(x) = ctgh(x) - \dfrac{1}{x}$ ($ctgh$ denotes the hyperbolic cotangent). Note that β is a one-to-one function of D.

Figure 2.3.4 shows the graphical representation of $\frac{1}{N\varphi_0}\left[\Psi_{pot} - \Psi_{pot}(0,T)\right]$ as a function of $d = \frac{\sqrt{2}D}{NJ}$ for different values of $\theta = \frac{kT}{\varphi_0}$ presented in [**102**]. When we plotted this function (see Figure 2.3.5) we observed a different behavior as temperature increased from low to high. In fact, we did not obtained the two lateral minima in the intermediate temperature range claimed by Müller and Wilmanski in their article ([**102**]). This gives rise to rather different load-deformation relations. Figure 2.3.6 shows the load-deformation curves presented in [**102**]. The normalized temperature ranges are as follows: $\theta_I < 1.2143$, $1.2143 < \theta_{II} < 1.2553$, $1.2553 < \theta_{III} < 1.2760$, $1.2760 < \theta_{IV} < 1.4195$, $1.4195 < \theta_V$. Figure 2.3.7 shows our version of load-deformation curves corresponding to the potential depicted in Figure 2.3.5. We do not yet understand the reason for the differences between our plot of the function and the plot given in [**102**].

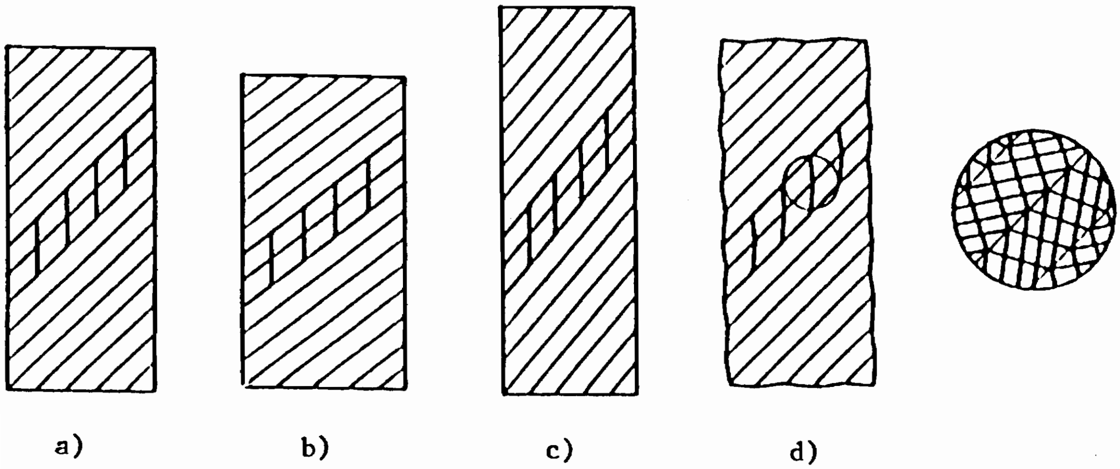


Figure 2.3.1: Different positions of the body; a)load-free configuration; b) under compressive loading (M_-configuration); c) under tensile loading (M_+configuration); d) $x = \frac{1}{2}$ ([**102**])

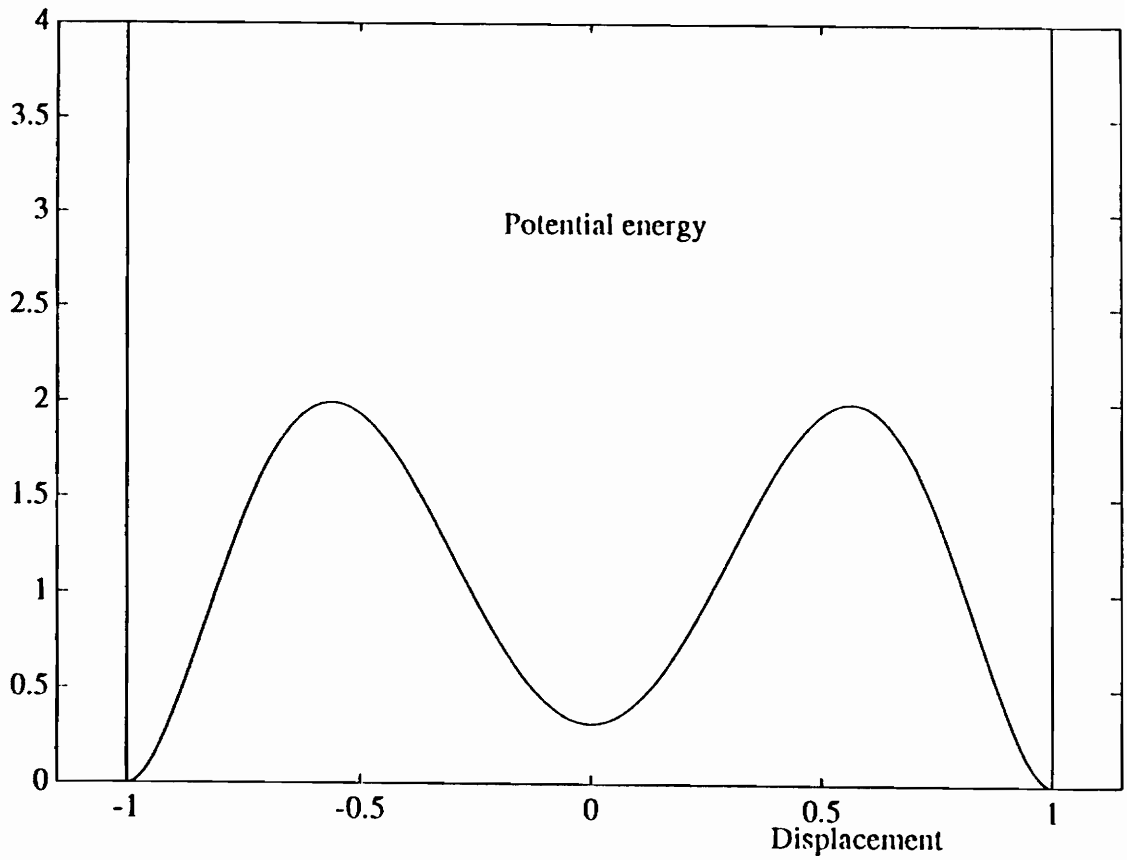


Figure 2.3.2: Postulated Potential Energy of a Lattice Layer

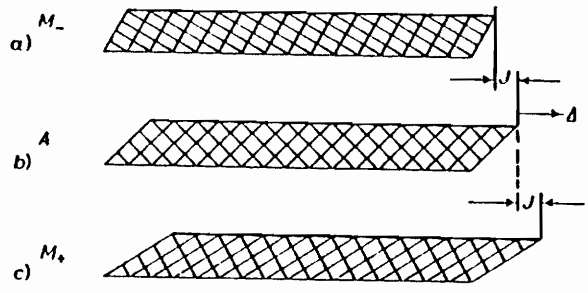


Figure 2.3.3: Lattice layers; (a) and (c) are the martensitic configurations; (b) is the austenitic or parent configuration ([**102**]).

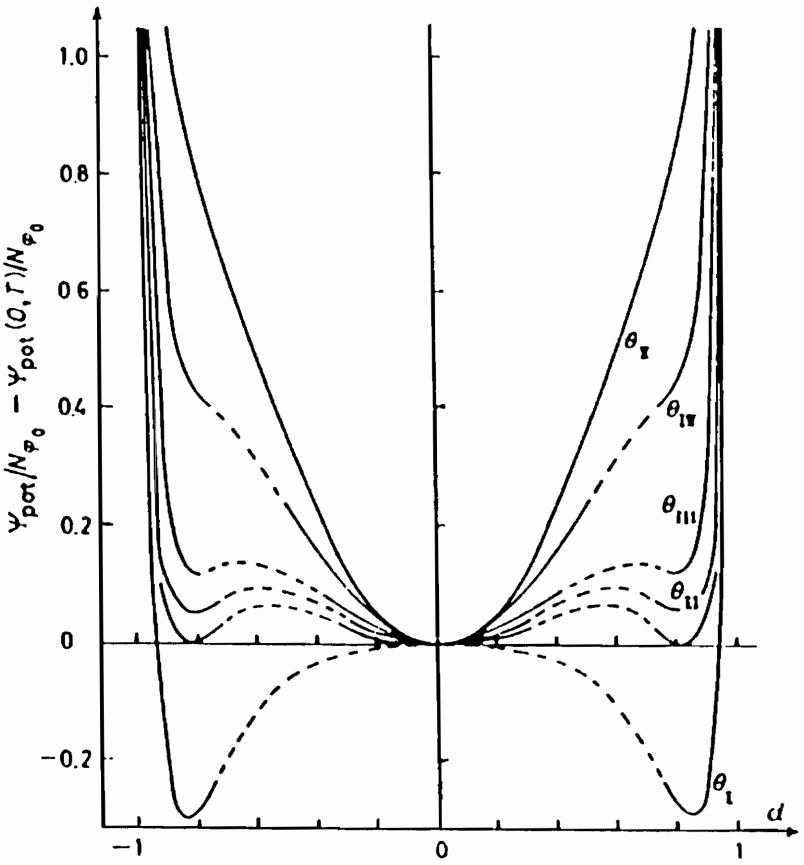


Figure 2.3.4: Normalized free energy as a function of $d = \frac{\sqrt{2}D}{NJ}$ for different temperatures obtained by Müller and Wilmanski in [**102**].

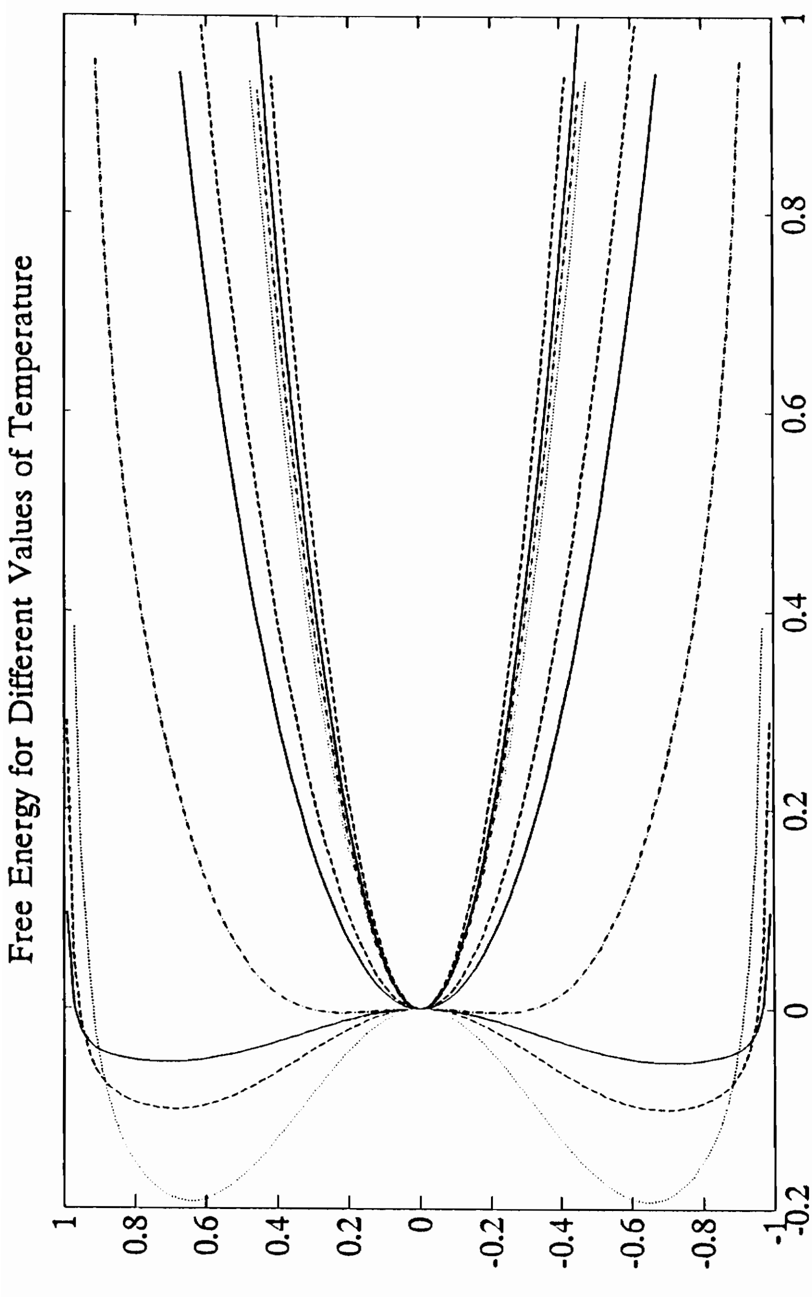


Figure 2.3.5: The function $\frac{1}{N\varphi_0}\left[\Psi_{pot} - \Psi_{pot}(0,T)\right]$ as a function of $d = \frac{\sqrt{2}D}{NJ}$ for different values of $\theta = \frac{kT}{\varphi_0}$.

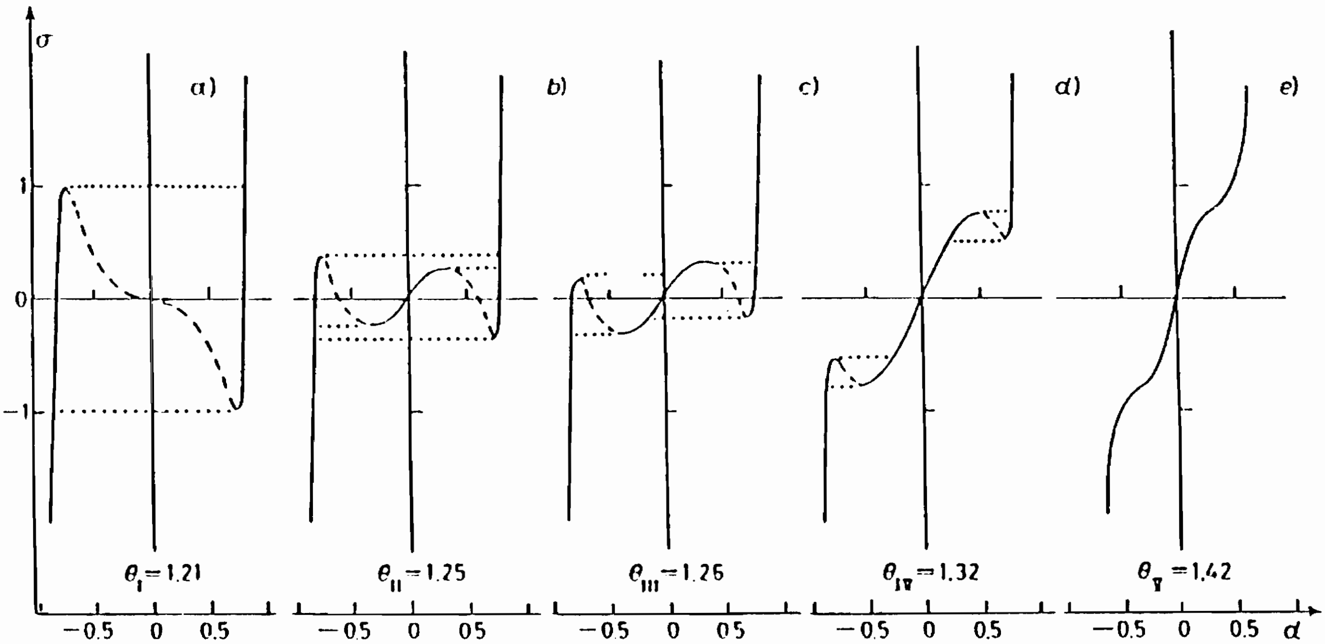


Figure 2.3.6: Load-deformation curves for the different temperature ranges obtained by Müller and Wilmanski in [**102**].

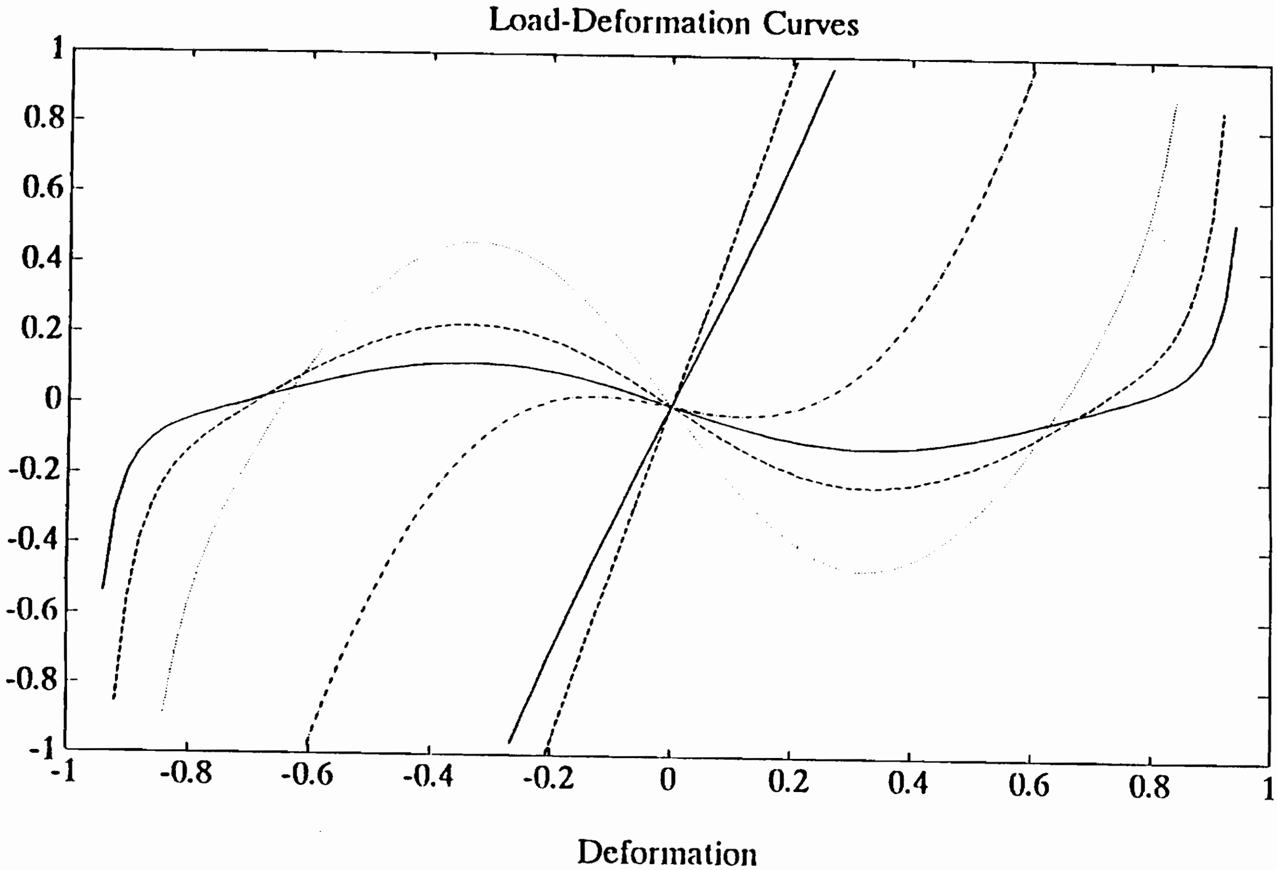


Figure 2.3.7: Load-deformation curves for different temperatures corresponding to the free energy depicted in Figure 2.3.5.

2.4 Fremond's Model

The model introduced by M. Fremond in ([**54**]) in 1987 is different from all the other models in several aspects. The main difference is that it deals with the three dimensional case (all the other models are valid only in one space dimension). Secondly, the free energy function is assumed to be different for each one of the three phases. The total free energy is obtained as the sum of the three weighted free energies plus a mixture part which ensures that the phase proportions are admissible.

Another very important difference with respect to all the other models is the fact that the phase proportions are introduced in the free energy as thermodynamic variables together with the absolute temperature and the strain tensor.

The model is built within the framework of the second gradient theory ([**58**]) to allow surface double forces and surface couples as part of the mechanical actions that are applied to the body. Other assumptions are that the phases coexist at each point (macroscopic model) and that the mass density ρ is the same for the three phases.

Existence and uniqueness of solutions to this model were later proved by P. Colli, M. Fremond and A. Visintin ([**31**]) using a variational formulation in Sobolev spaces and fixed-point arguments.

The free energies Ψ_i for the three different phases are assumed to be of the form:

$$\Psi_1(T, \epsilon, \nabla tr\epsilon) = \frac{1}{2}\epsilon K\epsilon + \frac{\nu}{2}|\nabla tr\epsilon|^2 - \tilde{\alpha}(T)tr\epsilon - C_0 T log T \qquad (2.4.1a)$$

$$\Psi_2(T, \epsilon, \nabla tr\epsilon) = \frac{1}{2}\epsilon K\epsilon + \frac{\nu}{2}|\nabla tr\epsilon|^2 + \tilde{\alpha}(T)tr\epsilon - C_0 T log T \qquad (2.4.1b)$$

$$\Psi_3(T, \epsilon, \nabla tr\epsilon) = \frac{1}{2}\epsilon K \epsilon + \frac{\nu}{2}|\nabla tr\epsilon|^2 - \frac{l}{T_0}(T - T_0) - C_0 T log T \tag{2.4.1c}$$

Here, Ψ_1 and Ψ_2 are the free energies of the two martensites and Ψ_3 is that of the austenite; T is the absolute temperature, ϵ is the strain tensor, K is the rigidity matrix, l is the latent heat of the austenite-martensite phase transition, ν is a positive constant, C_0 is the heat capacity, T_0 is a critical temperature and $\tilde{\alpha}(T)$ is a function proportional to the thermal expansion coefficient which vanishes for T bigger than T_C ($T_C > T_0$ is another critical temperature).

The total free energy is then

$$\tilde{\Psi}(T, \epsilon, \nabla tr\epsilon, \beta_1, \beta_2, \beta_3) = \rho \sum_{i=1}^{3} \beta_i \Psi_i + \rho T \tilde{I}(\beta_1, \beta_2, \beta_3) \tag{2.4.2}$$

where β_i, $i = 1, 2, 3$ are the phase proportions and $\tilde{I}$ is the indicator function of the admissible phase proportions set

$$\mathcal{S} = \{ (\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3 \mid 0 \leq \beta_i \leq 1 \text{ and } \sum_{i=1}^{3} \beta_i = 1 \} \tag{2.4.3}$$

Using the equilibrium equations, the constitutive laws and the balance equations, the mathematical problem is formulated as follows:

FREMOND'S PROBLEM. *Let Ω be a bounded domain in $\mathbb{R}^3$ and assume that $\partial\Omega$ is $C^{1,1}$-regular (see [**139**], §2). Let $\{\Gamma_0, \Gamma_1\}$ be a nontrivial partition of $\partial\Omega$ and let $\overrightarrow{n}$ be the outward normal unit vector to $\partial\Omega$. Fix any positive time τ and define*

$$Q \doteq \Omega \times (0, \tau), \tag{2.4.4a}$$

$$\Sigma \doteq \partial\Omega \times (0, \tau), \tag{2.4.4b}$$

$$\Sigma_j \doteq \Gamma_j \times (0, \tau), \quad j = 0, 1. \tag{2.4.4c}$$

Given

(1) *the external traction* $\vec{g} : \Sigma_1 \to \mathbb{R}^3$ *applied to* $\partial\Omega$*;*

(2) *the (Celsius) temperature of the surrounding medium* $\theta : \Sigma \to \mathbb{R}$*;*

(3) *the initial temperature* $\theta^0(x) : \Omega \to \mathbb{R}$*, and*

(4) *the initial martensite fractions* $\beta_i(x) : \Omega \to \mathbb{R}$*,* $i = 1, 2$*,*

find functions $\vec{u} = (u_1, u_2, u_3) : Q \to \mathbb{R}^3$ *and* θ*,* β_1*,* $\beta_2 : Q \to \mathbb{R}$ *(representing the displacement, temperature and martensite fractions on* Q*, respectively) such that setting*

$$\epsilon_{ij}(\vec{u}) \doteq \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad \text{in } Q,\ i = 1, 2, 3, \tag{2.4.5}$$

$$\sigma(\vec{u}) \doteq \lambda div(\vec{u})I + 2\mu\epsilon(\vec{u}) + \alpha(\theta)(\beta_2 - \beta_1)I \qquad \text{in } Q, \tag{2.4.6}$$

(equation (2.4.6) reflects the fact that the material is assumed to be homogeneous and isotropic; λ *and* μ *are the Lamé constants)*

the following equations hold:

$$C_0\frac{\partial\theta}{\partial t} - l\frac{\partial}{\partial t}(\beta_1 + \beta_2) - h\Delta\theta = 0 \qquad \text{in } Q, \tag{2.4.7}$$

$$div\left(\nu\Delta(div\,\vec{u})I + \sigma\right) = 0 \quad \text{in } Q, \tag{2.4.8}$$

$$k\frac{\partial}{\partial t}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} + \partial I_{\bar{K}}(\beta_1, \beta_2) \ni \begin{pmatrix}1\\ -1\end{pmatrix}\alpha(\theta)div(\vec{u}) - \frac{l}{T_0}\begin{pmatrix}1\\ 1\end{pmatrix}(\theta - \theta^*) \qquad \text{in } Q, \tag{2.4.9}$$

and the following initial and boundary conditions are satisfied

$$\vec{u} = 0 \qquad \text{on } \Sigma_0, \tag{2.4.10}$$

$$\left(\nu\Delta(div\,\vec{u})I + \sigma\right) \cdot \vec{n} = \vec{g} \qquad \text{on } \Sigma_1, \tag{2.4.11}$$

$$\frac{\partial}{\partial n}\left(div\,\overrightarrow{u}\right) = 0 \qquad on\ \Sigma \tag{2.4.12}$$

$$h\frac{\partial\theta}{\partial n} + a(\theta - \overline{\theta}) = 0 \qquad on\ \Sigma \tag{2.4.13}$$

$$\theta(x, 0) = \theta^0(x) \qquad in\ \Omega \tag{2.4.14}$$

$$\beta_i(x, 0) = \beta_i(x), \quad i = 1, 2 \qquad in\ \Omega \tag{2.4.15}$$

*where $\theta = T - 273$ is the Celsius temperature, σ is the stress tensor, h is the thermal conductivity coefficient, $\theta^* = T_0 - 273$, k is the viscosity constant, $\alpha(\theta) = \tilde{\alpha}(T)$, I is the identity matrix, a is a positive parameter representing the coefficient of thermal exchange at the boundary, $\overline{\theta}$ is the temperature of the surrounding medium, $\tilde{K}$ is the projection of the set $\mathcal{S}$ into $\mathbb{R}^2$, $I_{\tilde{K}}$ is the indicator function of $\tilde{K}$ and $\partial I_{\tilde{K}}$ is the subdifferential of $I_{\tilde{K}}$ (see [**38**], §5.1).*

As we mentioned earlier, existence and uniqueness of solutions for this problem were proved using a variational formulation in Sobolev spaces and the Banach fixed point theorem. The first numerical experiments using this model are due to Tiihonen ([**132**]) but are confined to the one dimensional case.

CHAPTER III A Generalized One-Dimensional Model

3.1 Derivation of the General Equations - Introduction

In this section we present a general mathematical model for the dynamics of phase transitions in Shape Memory Alloys. Some of the advantages of this model are that it accounts for time-dependent distributed and boundary inputs, viscosity effects, fading thermal memory, local curvature effects and internal variables. Also, it is able to capture all the interesting qualitative isothermal properties of Falk's model through the inclusion of the thermodynamic potential in the constitutive equations. This approach has its origins in a work by M. Niezgodka and J. Sprekels, published in 1988 ([**107**]).

The thermomechanical processes in a connected one-dimensional body $\Omega \subset \mathbb{R}$ are governed by the balance laws of mass, linear momentum and energy:

$$\rho_t = -\rho\epsilon_t \qquad \text{in } \Omega \tag{3.1.1a}$$

$$\rho u_{tt} = \sigma_x - \mu_{xx} + f \qquad \text{in } \Omega \tag{3.1.1b}$$

$$\rho e_t = -q_x + \sigma\epsilon_t + \mu\epsilon_{xt} + g \qquad \text{in } \Omega \tag{3.1.1c}$$

where ρ = mass density, u = displacement, σ = shear stress, μ = couple stress, f = distributed loads, e = specific internal energy, q = heat flux, ϵ = linearized shear strain, g = density of heat sources or sinks.

The model must also satisfy the second principle of thermodynamics for the production of entropy, i.e., the Clausius-Duhem inequality

$$\rho s_t \geq -\left(\frac{q}{\theta}\right)_x + \frac{g}{\theta} \tag{3.1.1d}$$

where s = specific entropy and θ = absolute temperature. We assume that the mass ρ is independent of the temperature θ. Therefore, (3.1.1a) implies that

$$\rho = \rho_0 e^{-\epsilon} \tag{3.1.2}$$

where ρ_0 is the mass density of the reference phase corresponding to the undeformed state. Provided that ϵ remains "small", equation (3.1.2) justifies the assumption that the mass density is constant. We shall maintain this assumption from now on. We should point out here that actual experiments show only a very small volume change. Therefore, only equations (3.1.1b-d) will be considered to construct our model.

In equations (3.1.1b-c) we find five unknowns, namely u, σ, μ, e and q. However, the number of unknowns can be reduced by proposing an expression for the specific Helmholtz free energy density $\Psi = \Psi(\epsilon, \epsilon_x, \theta, \vec{p})$, where $\vec{p} = (p_1, \ldots, p_L)$ is an L-dimensional vector which accounts for the possible internal variables of the model such as (martensite and austenite) phase fractions, dummy variables, etc.

The relations between the specific internal energy, the free energy density and the specific entropy are given by the well known thermomechanical equations

$$\rho s = -\Psi_\theta, \qquad \rho e = \Psi + \rho\theta s = \Psi - \theta\Psi_\theta \tag{3.1.3}$$

For the heat flux, it is standard to postulate

$$q = -k\theta_x - \alpha k\theta_{xt} \tag{3.1.4}$$

instead of the classical Fourier law, where k is the coefficient of thermal conductivity and α is a nonnegative parameter . This equation characterizes the heat conduction with short thermal memory ([**26**], [**61**], [**110**]). There are several articles devoted to mathematical models which consider the heat flux in the form (3.1.4) with α strictly positive ([**67**], [**68**], [**107**], [**108**]). Although this assumption provides sufficient smoothness for the existence of the solutions of such models, the physical meaning of $\alpha > 0$ is controversial since, as we shall see later, the second law of thermodynamics fails to hold in this case. For this reason, we shall concentrate on the case $\alpha = 0$. However, for completness we temporarily leave this term in the model equations.

The stress is the sum of a quasiconservative part and a dissipative part:

$$\sigma = \sigma_q + \sigma_d \tag{3.1.5}$$

with

$$\sigma_q = \Psi_\epsilon \qquad \text{and} \qquad \sigma_d = \beta\rho\epsilon_t \tag{3.1.6}$$

where β is a nonnegative constant representing the viscosity.

The couple stress μ is given by

$$\mu = \Psi_{\epsilon_x} \tag{3.1.7}$$

Using equations (3.1.3-7), equations (3.1.b-c) can be written as

$$\rho u_{tt} - (\Psi_\epsilon)_x - \beta\rho\,(\epsilon_t)_x + (\Psi_{\epsilon_x})_{xx} = f$$

$$(\Psi - \theta\Psi_\theta)_t - (k\theta_x + \alpha k\theta_{xt})_x - (\Psi_\epsilon + \beta\rho\epsilon_t)\,\epsilon_t - \Psi_{\epsilon_x}\epsilon_{xt} = g$$

and using the relations $\epsilon = u_x$ and $\Psi_t = \Psi_\epsilon \epsilon_t + \Psi_{\epsilon_x} \epsilon_{xt} + \Psi_\theta \theta_t + \sum_{i=1}^{L} \Psi_{p_i} p_{i,t}$, we find that

$$\rho u_{tt} - \beta \rho u_{xxt} - (\Psi_\epsilon)_x + (\Psi_{\epsilon_x})_{xx} = f \tag{3.1.8a}$$

$$-\theta (\Psi_\theta)_t - k\theta_{xx} - \alpha k \theta_{xxt} + \sum_{i=1}^{L} \Psi_{p_i} p_{i,t} - \beta \rho u_{xt}^2 = g \tag{3.1.8b}$$

for $x \in \Omega$ and $0 \leq t \leq T$ where T is a certain prescribed final time.

Equation (3.1.8a) is a nonlinear pseudohyperbolic (viscoelastoplasticity) equation in $u(x,t)$ and (3.1.8b) is a pseudoparabolic equation in $\theta(x,t)$. Both equations are strongly coupled due to the presence of the terms involving the partial derivatives of the function Ψ. The model must be complemented by prescribing appropriate laws for the evolution of the internal variables and coherent initial and boundary conditions.

The function Ψ, which must be provided, is the subject of several discussions. Based on experimentally observed stress-strain relations, Falk (**[44]**,**[45]**, see also Section 2.2) proposed the use of the so called Landau-Devonshire potential

$$\Psi(\epsilon, \theta) = \Psi_0(\theta) + \alpha_2(\theta - \theta_1)\epsilon^2 - \alpha_4 \epsilon^4 + \alpha_6 \epsilon^6 \tag{3.1.9}$$

where α_2, α_4, α_6 are positive constants, θ_1 is a critical temperature and $\Psi_0(\theta)$ is a certain smooth function of θ, all depending on the material being considered. Niezgodka and Sprekels (**[107]**) introduced a generalization of this function in order to correct for improper behavior at extremely low and high temperatures and for large strains. The generalized Landau-Devonshire function was then defined by

$$\Psi(\epsilon, \theta) = \Psi_0(\theta) + \Psi_1(\theta)\epsilon^2 + \Psi(\epsilon) \tag{3.1.10}$$

where Ψ_0, Ψ_1 and Ψ_2 satisfy certain general hypotheses (see [**107**] for details). Further generalizations gave rise to the Landau-Ginzburg potential, which includes a term depending on the first derivative of the strain, ϵ_x ([**56**], [**125**], [**126**]). Other approaches such as the use of statistical mechanics ([**102**], see also Section 2.3) and "snap-springs" ([**97**], [**101**], see also Section 2.1) have produced other expressions for this function. In Section 3.5 we discuss some of the advantages and disavantages of the use of this type of potential.

3.2 A Brief Review on the Existence and Uniqueness of Solutions

We give here a brief summary on the existence and uniqueness results for system (3.1.8a-b).

Using a special type of Galerkin approximation, Niezgodka and Sprekels ([**107**]) first proved local existence of solutions to the system (3.1.8a-b) when the Hemholtz free energy density is given in the generalized Landau-Devonshire form (3.1.10) and Ψ_0, Ψ_1, Ψ_2 satisfy certain specific growth restrictions. The initial and boundary conditions in this case were taken as follows:

$$\theta(x,0) = \theta_0(x), \qquad u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) \qquad \text{for } x \in \Omega \qquad (3.2.1\text{a})$$

with $u_0 \in C^2(\bar{\Omega})$, $u_0|_{\partial\Omega} = u_0''|_{\partial\Omega} = 0$, $\|u_0\|_{H^2(\Omega)} \leq E_0$, $u_1 \in H_0^1(\Omega)$ and $\theta_0 \in H^1(\Omega)$, $\theta_0(x) > \theta_s > 0$ for all $x \in \bar{\Omega}$, where E_0 and θ_s are two positive constants depending on the free energy, and

$$k\frac{\partial\theta}{\partial\nu} = k_1(\theta_\Gamma - \theta), \qquad u \equiv 0 \qquad \text{on } \partial\Omega \times (0,T) \tag{3.2.1b}$$

where ν is the outward normal unit vector to $\partial\Omega$, k_1 is a positive constant and θ_Γ is the temperature of the surrounding medium. The uniqueness of solutions in this case was later proved by Hoffman and Songmu ([**67**], [**68**]). The first results on global existence are due to Dafermos and Hsiao ([**32**], [**33**]) and Niezgodka, Songmu and Sprekels ([**108**]). The assumptions $\beta > 0$ (existence of viscous stress) and $\alpha > 0$ (existence of short thermal memory) played a very important role in all of the above mentioned articles. However, the physical meaning of the case $\alpha > 0$ is questionable since (3.1.1d) is not satisfied in this case.

The non-viscous case ($\beta = 0$) with no thermal memory ($\alpha = 0$) was treated later by Sprekels ([**127**]). Here Sprekels used a Landau-Ginzburg potential of the form

$$\Psi(\epsilon, \epsilon_x, \theta) = \Psi_0(\theta) + \Psi_1(\theta)\Psi_2(\epsilon) + \frac{\gamma}{2}\epsilon_x^2 \tag{3.2.2}$$

where γ is a positive constant . In this case the term $(\Psi_{\epsilon_x})_{xx}$ in (3.1.8a) takes the form $\gamma\epsilon_{xxx} = \gamma u_{xxxx}$. This term provides sufficient smoothness for the existence of solutions. Even so, very strong growth restrictions on Ψ were needed. One of these conditions, namely

$$|\Psi_1(\theta)| + |\Psi_1'(\theta)| + |\theta\Psi_1'(\theta)| \le C \qquad \text{for all } \theta \ge 0$$

excluded the physically relevant case in which $\Psi_1(\theta) = \alpha_2(\theta - \theta_1)$, $\Psi_2(\epsilon) = \epsilon^2$, $\Psi_3(\epsilon) = -\alpha_4\epsilon^4 + \alpha_6\epsilon^6$, which corresponds to the Landau-Devonshire potential (3.1.9) with the additional term $\frac{\gamma}{2}\epsilon_x^2$.

In [**124**], Songmu claims to have proved global existence for the non-viscous case with no thermal memory ($\beta = \alpha = 0$) and potentials of the form

$$\Psi(\epsilon, \epsilon_x, \theta) = \Psi_0(\theta) + \alpha_2(\theta - \theta_1)\epsilon^2 - \alpha_4\epsilon^4 + \alpha_6\epsilon^6 + \frac{\gamma}{2}\epsilon_x^2 \tag{3.2.3}$$

Here, Songmu derives certain a-priori estimates from which he concludes that any local solution can be extended globally in time. However, he gave no rigorous proof of the existence of local solutions. Also, $-\Psi_0(\theta)$ was assumed to grow at least quadratically in θ which now excludes the physically relevant case

$$\Psi_0(\theta) = -C_v\theta ln\left(\frac{\theta}{\theta_2}\right) + C_v\theta + C \tag{3.2.4}$$

where C_v is the specific heat, θ_2 is a critical temperature and C is a constant.

Finally, in 1988 Songmu and Sprekels ([**125**]) derived a-priori estimates for the case $\beta = 0$, $\alpha = 0$ and Landau-Ginzburg potentials of the type

$$\Psi(\epsilon, \epsilon_x, \theta) = \Psi_0(\theta) + \alpha_1\theta\Psi_1(\epsilon) + \Psi_2(\epsilon) + \frac{\gamma}{2}\epsilon_x^2 \tag{3.2.5}$$

where Ψ_0, Ψ_1, Ψ_2 satisfy certain weaker growth conditions than those imposed in [**124**] and [**127**]. These conditions are satisfied in particular by potentials of the form (3.2.3) with Ψ_0 as in (3.2.4). Using those estimates they show that local solutions can be extended globally in time. However, again no rigorous proof of the existence of such local solutions was given and none are known for the case $\beta = 0$.

<u>**Remarks**</u>:

(1) By using (3.1.1c) and (3.1.3-7), one obtains

$$\rho s_t + \left(\frac{q}{\theta}\right)_x - \frac{g}{\theta} = -\frac{q\theta_x}{\theta^2} - \frac{1}{\theta}\left(\sum_{i=1}^{L} \Psi_{p_i} p_{i,t} - \beta\rho\epsilon_t^2\right)$$

Hence (3.1.1d) is satisfied if $\alpha = 0$ and Ψ does not depend on internal variables.

(2) There are three "smoothing" terms in the model being considered, namely: viscosity (β), thermal memory (α) and couple stress (Ψ_{ϵ_x}). Up to now, no result on local existence of solutions is known when all these terms are absent although, as we will show in Chapter IV, numerical results seem to indicate that even in this case solutions do exist.

(3) As noted before, the case $\alpha > 0$ has a questionable physical meaning since the second law of thermodynamics is not satisfied in this case.

(4) No proof on local existence for the case $\alpha = 0$ has been given before. The articles [**124**], [**125**] and [**127**] systematically avoid dealing with this issue.

In the next section we give a rigorous proof of local existence and uniqueness of solutions of the problem (3.1.8a-b) for the case $\alpha = 0$ and Landau-Ginzburg potentials of the form (3.2.3). We first transform the system of PDE's into an abstract formulation in an appropriate Hilbert space. Then we prove that the operator $A(q)$ corresponding to the linear part of the system is the infinitesimal generator of an analytic semigroup $T(t;q)$ and the nonlinear part is Lipschitz in the state space variable with respect to the graph-norm of the operator $A(q)$. This approach allows also to obtain explicit spectral decompositions for the operator $A(q)$ and the semigroup $T(t;q)$. We also prove that the semigroup $T(t;q)$ is exponentially stable with decay rate depending on k, k_1, β, ρ and γ.

3.3 Well-Posedness

Assume $\alpha = 0$ (no thermal memory), $\beta > 0, \Omega = (0,1)$ and the free energy density is given in a Landau-Ginzburg form like (3.2.3) with $\Psi_0(\theta)$ as in (3.2.4), i.e., we take

$$\Psi(\epsilon, \epsilon_x, \theta) = -C_v \theta ln\left(\frac{\theta}{\theta_2}\right) + C_v\theta + C + \alpha_2(\theta - \theta_1)\epsilon^2 - \alpha_4\epsilon^4 + \alpha_6\epsilon^6 + \frac{\gamma}{2}\epsilon_x^2 \quad (3.3.1)$$

Under this assumption, the system (3.1.8a-b) takes the form

$$\rho u_{tt} - \beta\rho u_{xxt} + \gamma u_{xxxx} = f(x,t) + \frac{\partial}{\partial x}\left[\frac{\partial}{\partial \epsilon}\Psi(u_x, u_{xx}, \theta)\right] \quad (3.3.2a)$$

$$C_v\theta_t - k\theta_{xx} = g(x,t) + 2\alpha_2\theta u_x u_{xt} + \beta\rho u_{xt}^2 \quad (3.3.2b)$$

for $x \in \Omega$, $0 \le t \le T$.

We prescribe the initial conditions

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad \theta(x,0) = \theta_0(x), \qquad x \in \Omega \quad (3.3.2c)$$

and for $0 \le t \le T$, the following boundary conditions

$$u(0,t) = u(1,t) = 0 = u_{xx}(0,t) = u_{xx}(1,t),$$

$$\theta_x(0,t) = 0, \quad k\theta_x(1,t) = k_1(\theta_\Gamma(t) - \theta(1,t)). \quad (3.3.2d)$$

where $\theta_\Gamma(t)$ is the temperature of the surrounding medium at time t and k_1 is a positive coefficient. Next define

$$L(x,t) \doteq \theta_\Gamma(t)cos(2\pi x) \quad (3.3.3)$$

so that $L_x(0,t) = L_x(1,t) = 0$ and $L(1,t) = \theta_\Gamma(t)$ for all t, and let us make the transformation

$$\tilde{\theta}(x,t) \doteq \theta(x,t) - L(x,t) \quad (3.3.4)$$

Then our IBVP takes the form

$$\rho u_{tt} - \beta\rho u_{xxt} + \gamma u_{xxxx} = f(x,t) + \frac{\partial}{\partial x}\left[\frac{\partial}{\partial \epsilon}\Psi(u_x, u_{xx}, \tilde{\theta} + L)\right] \tag{3.3.5a}$$

$$C_v\tilde{\theta}_t - k\tilde{\theta}_{xx} = g(x,t) + 2\alpha_2(\tilde{\theta} + L)u_x u_{xt} + \beta\rho u_{xt}^2 - C_v\theta_\Gamma'(t)cos(2\pi x)$$
$$- 4k\pi^2\theta_\Gamma(t)cos(2\pi x) \tag{3.3.5b}$$

for $x \in \Omega$, $0 \le t \le T$,

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x),$$
$$\tilde{\theta}(x,0) = \theta_0(x) - \theta_\Gamma(0)cos(2\pi x), \qquad x \in \Omega \tag{3.3.5c}$$

and

$$u(0,t) = u(1,t) = 0 = u_{xx}(0,t) = u_{xx}(1,t),$$
$$\tilde{\theta}_x(0,t) = 0, \quad k\tilde{\theta}_x(1,t) + k_1\tilde{\theta}(1,t) = 0, \qquad 0 \le t \le T. \tag{3.3.5d}$$

We assume that the functions $f(x,t)$ and $g(x,t)$ satisfy the following hypotheses

(H1) There exist functions K_g, $K_f \in L_2(0,1)$, $K_g(x) \ge 0$, $K_f(x) \ge 0$, such that

$$|f(x,t_1) - f(x,t_2)| \le K_f(x)|t_1 - t_2|,$$

and

$$|g(x,t_1) - g(x,t_2)| \le K_g(x)|t_1 - t_2|$$

for all $x \in (0,1)$, t_1, $t_2 \in [0,T]$.

(H2) $\theta_\Gamma \in H^1(0,T)$, θ_Γ and θ_Γ' are locally uniformly Lipschitz continuous, i.e., for each compact set $S \subset [0,T]$ there are constants K_S, $K_S' > 0$ such that

$$|\theta_\Gamma(t_1) - \theta_\Gamma(t_2)| \le K_S|t_1 - t_2|,$$

and

$$|\theta'_\Gamma(t_1) - \theta'_\Gamma(t_2)| \leq K'_S|t_1 - t_2|$$

for all $t_1, t_2 \in S$

In order to formulate system (3.3.5.a-d) as a Cauchy problem in an abstract space, we define the state space

$$Z \doteq H_0^1(0,1) \cap H^2(0,1) \times L_2(0,1) \times L_2(0,1),$$

$$z(t) \doteq \begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} \doteq \begin{pmatrix} u(\cdot,t) \\ u_t(\cdot,t) \\ \tilde{\theta}(\cdot,t) \end{pmatrix} \quad \in Z,$$

and the admissible parameter set

$$\mathcal{Q} \doteq \left\{ q = (\rho, C_v, \beta, \alpha_2, \alpha_4, \alpha_6, \theta_1, \gamma) \mid q \in \mathbb{R}_+^8 \right\}$$

. Note: we assume $k > 0$ and $k_1 > 0$ are known. Although this assumption is made mainly for simplicity reasons, it is also rooted in the fact that the heat conductivity is a physical parameter and can be estimated from laboratory experiments. In any case, only slight modifications are needed to consider the case in which k and k_1 are components of the parameter q.

We next define in Z an inner product $\langle \cdot, \cdot \rangle_q$ depending on the parameter q as follows

$$\left\langle \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \end{pmatrix} \right\rangle_q \doteq \gamma \int_\Omega u''\hat{u}'' + \rho \int_\Omega v\hat{v} + \frac{C_v}{k} \int_\Omega w\hat{w} \tag{3.3.6}$$

and we denote by Z_q the Hilbert space Z with the inner product $\langle \cdot, \cdot \rangle_q$. The corresponding norm in Z_q is denoted by $\| \cdot \|_q$.

Then the IBVP (3.3.5a-d) can be formally written as an abstract Cauchy problem in Z_q as follows

$$(\Sigma):\begin{cases}\dot z(t)=A(q)z(t)+F(q,t,z(t)) & 0\le t\le T\\ z(0)=z_0\end{cases}\tag{3.3.7}$$

where

$$dom\,(A(q))\doteq\left\{\begin{pmatrix}u\\v\\w\end{pmatrix}\in Z_q\;\middle|\;\begin{array}{l}u\in H^4(\Omega),\ u(0)=u(1)=0=u''(0)=u''(1),\\ v\in H_0^1(\Omega)\cap H^2(\Omega),\\ w\in H^2(\Omega),\quad w'(0)=0,\quad kw'(1)=-k_1w(1)\end{array}\right\}\tag{3.3.8}$$

and for $z=\begin{pmatrix}u\\v\\w\end{pmatrix}\in dom\,(A(q))$

$$A(q)z=A(q)\begin{pmatrix}u\\v\\w\end{pmatrix}\doteq\begin{pmatrix}v\\ \beta v''-\frac{\gamma}{\rho}u''''\\ \frac{k}{C_v}w''\end{pmatrix}=\begin{pmatrix}0 & I & 0\\ -\frac{\gamma}{\rho}\frac{\partial^4}{\partial x^4} & \beta\frac{\partial^2}{\partial x^2} & 0\\ 0 & 0 & \frac{k}{C_v}\frac{\partial^2}{\partial x^2}\end{pmatrix}\begin{pmatrix}u\\v\\w\end{pmatrix}\tag{3.3.9}$$

$$z_0(\cdot)=\begin{pmatrix}u_0(\cdot)\\ u_1(\cdot)\\ \theta_0(\cdot)-\theta_\Gamma(0)cos(2\pi\cdot)\end{pmatrix}$$

Note that $dom\,(A(q))$ is a subspace of Z_q independent of $q\in\mathcal{Q}$ since k and k_1 are supposed to be known (i.e., they are not components of q).

The function $F(q,t,z):\mathcal{Q}\times\mathbb{R}_0^+\times Z_q\to Z_q$ is defined by

$$F(q,t,z)=F\left(q,t,\begin{pmatrix}u\\v\\w\end{pmatrix}\right)\doteq\begin{pmatrix}0\\ f_2(q,t,z)\\ f_3(q,t,z)\end{pmatrix},\tag{3.3.10}$$

where

$$f_2(q,t,z)\doteq\rho^{-1}f(\cdot,t)+\rho^{-1}\frac{\partial}{\partial x}\left[2\alpha_2(w+L(\cdot,t)-\theta_1)u'\right.$$
$$\left.-4\alpha_4u'^3+6\alpha_6u'^5\right],\tag{3.3.11a}$$

$$f_3(q,t,z)\doteq C_v^{-1}g(\cdot,t)+2\alpha_2C_v^{-1}(w+L(\cdot,t))u'v'+\beta\rho C_v^{-1}(v')^2$$
$$-\theta_\Gamma'(t)cos(2\pi\cdot)-4k\pi^2C_v^{-1}L(\cdot,t).\tag{3.3.11b}$$

THEOREM 3.3.1. *If $A(q)$ and Z_q are as defined above, then the operator $A(q) : dom(A(q)) \subset Z_q \to Z_q$ is dissipative.*

PROOF: Let $z = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in dom\,(A(q))$ so that $u \in H^4$, $u(0) = u(1) = u''(0) = u''(1)$, $v \in H_0^1 \cap H^2$, $w \in H^2$, $w'(0) = 0$ and $kw'(1) + k_1 w(1) = 0$.

Then

$$
\begin{aligned}
\langle A(q)z, z\rangle_q &= \left\langle \begin{pmatrix} v \\ \beta v'' - \frac{\gamma}{\rho} u'''' \\ \frac{k}{C_v} w'' \end{pmatrix}, \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right\rangle_q \\
&= \gamma \int_\Omega v''u'' + \rho \int_\Omega (\beta v'' - \frac{\gamma}{\rho} u'''')v + \frac{C_v}{k} \int_\Omega \frac{k}{C_v} w''w \\
&= \gamma \int_\Omega v''u'' + \rho\beta \left[vv'|_{x=0}^{x=1} - \int_\Omega (v')^2 \right] - \gamma \left[vu'''|_{x=0}^{x=1} - \int_\Omega u'''v' \right] \\
&\quad + \left[ww'|_{x=0}^{x=1} - \int_\Omega (w')^2 \right] \\
&= \gamma \int_\Omega v''u'' - \rho\beta \|v'\|_{L_2}^2 + \gamma \left[v'u''|_{x=0}^{x=1} - \int_\Omega u''v'' \right] \\
&\quad + w(1)w'(1) - \|w'\|_{L_2}^2 \\
&= -\rho\beta \|v'\|_{L_2}^2 - \|w'\|_{L_2}^2 - \frac{k_1}{k} [w(1)]^2 \\
&\leq 0
\end{aligned}
$$

Hence $A(q)$ is dissipative. ∎

THEOREM 3.3.2. *If $q \in \mathcal{Q}$, Z_q and $A(q)$ are as above, then the adjoint of $A(q)$, $A^*(q)$ is given by $dom\,(A^*(q)) = dom\,(A(q))$, and for $z = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in dom\,(A^*(q))$*

$$
A^*(q)z = \begin{pmatrix} -v \\ \beta v'' + \frac{\gamma}{\rho} u'''' \\ \frac{k}{C_v} w'' \end{pmatrix} = \begin{pmatrix} 0 & -I & 0 \\ \frac{\gamma}{\rho} \frac{\partial^4}{\partial x^4} & \beta \frac{\partial^2}{\partial x^2} & 0 \\ 0 & 0 & \frac{k}{C_v} \frac{\partial^2}{\partial x^2} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \tag{3.3.12}
$$

Proof: We prove this theorem in two parts. First we show that $dom\,(A(q)) \subset dom\,(A^*(q))$ and $A^*(q)z$ is given by the formula above. In the second part we show that the opposite inclusion holds for the domains.

First part: Let $z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in dom\,(A(q))$. Then for any $\eta = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in dom\,(A(q))$ we have

$$\begin{aligned}
\langle A(q)\eta, z\rangle_q &= \left\langle A(q)\begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}\right\rangle_q = \left\langle \begin{pmatrix} v \\ \beta v'' - \frac{\gamma}{\rho}u'''' \\ \frac{k}{C_v}w'' \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}\right\rangle_q \\
&= \gamma\int_\Omega v''z_1'' + \rho\int_\Omega\left(\beta v'' - \frac{\gamma}{\rho}u''''\right)z_2 + \frac{C_v}{k}\int_\Omega \frac{k}{C_v}w''z_3 \\
&= \gamma\left[z_1''v'|_{x=0}^{x=1} - \int_\Omega v'z_1'''\right] + \rho\beta\left[z_2v'|_{x=0}^{x=1} - \int_\Omega v'z_2'\right] \\
&\quad - \gamma\left[z_2u'''|_{x=0}^{x=1} - \int_\Omega u'''z_2'\right] + \left[z_3w'|_{x=0}^{x=1} - \int_\Omega w'z_3'\right] \\
&= -\gamma\left[z_1'''v|_{x=0}^{x=1} - \int_\Omega vz_1''''\right] - \rho\beta\left[z_2'v|_{x=0}^{x=1} - \int_\Omega vz_2''\right] \\
&\quad + \gamma\left[z_2'u''|_{x=0}^{x=1} - \int_\Omega u''z_2''\right] + z_3(1)w'(1) - \left[z_3'w|_{x=0}^{x=1} - \int_\Omega wz_3''\right] \\
&= \gamma\int_\Omega vz_1'''' + \rho\beta\int_\Omega vz_2'' - \gamma\int_\Omega u''z_2'' + \int_\Omega wz_3''.
\end{aligned}$$

The last equality follows from the fact that $z_3(1)w'(1) = z_3'(1)w(1)$.

By rearranging terms it follows that

$$\begin{aligned}
\langle A(q)\eta, z\rangle_q &= \gamma\int_\Omega u''(-z_2'') + \rho\int_\Omega v\left(\beta z_2'' + \frac{\gamma}{\rho}z_1''''\right) + \frac{C_v}{k}\int_\Omega w\left(\frac{k}{C_v}z_3''\right) \\
&= \left\langle \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} -z_2 \\ \beta z_2'' + \frac{\gamma}{\rho}z_1'''' \\ \frac{k}{C_v}z_3'' \end{pmatrix}\right\rangle_q
\end{aligned}$$

Hence if $z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in dom\,(A(q))$ then $z \in dom\,(A^*(q))$ and

$$A^*(q)z = A^*(q)\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} -z_2 \\ \beta z_2'' + \frac{\gamma}{\rho} z_1'''' \\ \frac{k}{C_v} z_3'' \end{pmatrix} = \begin{pmatrix} 0 & -I & 0 \\ \frac{\gamma}{\rho}\frac{\partial^4}{\partial x^4} & \beta\frac{\partial^2}{\partial x^2} & 0 \\ 0 & 0 & \frac{k}{C_v}\frac{\partial^2}{\partial x^2} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$

Second part: Now let $z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in dom\,(A^*(q))$. Then there exists $\tilde{z} = \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{z}_3 \end{pmatrix} \in Z_q$ such that for all $\eta = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in dom\,(A(q))$

$$\begin{aligned}
0 &= \langle A(q)\eta, z\rangle_q - \langle \eta, \tilde{z}\rangle_q \\
&= \left\langle \begin{pmatrix} v \\ \beta v'' - \frac{\gamma}{\rho} u'''' \\ \frac{k}{C_v} w'' \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \right\rangle_q - \left\langle \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{z}_3 \end{pmatrix} \right\rangle_q \\
&= \gamma \int_\Omega (v'' z_1'' - u'' \tilde{z}_1'') + \rho \int_\Omega \left[\left(\beta v'' - \frac{\gamma}{\rho} u''''\right) z_2 - v\tilde{z}_2 \right] \\
&\quad + \frac{C_v}{k} \int_\Omega \left(\frac{k}{C_v} w'' z_3 - w\tilde{z}_3 \right) \\
&= \int_\Omega (\gamma v'' z_1'' + \rho\beta v'' z_2 - \rho v \tilde{z}_2) - \gamma \int_\Omega (u'' \tilde{z}_1'' + u'''' z_2) + \frac{C_v}{k} \int_\Omega \left(\frac{k}{C_v} w'' z_3 - w\tilde{z}_3 \right).
\end{aligned}$$

Since this equality must hold for all $\eta = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in dom\,(A(q))$, each one of the terms in the above expression must vanish, i.e.

$$(a) \qquad \int_\Omega \tilde{z}_1'' u'' + z_2 u'''' = 0 \qquad \text{for all } u \in H^4,\ u|_{\partial\Omega} = u''|_{\partial\Omega} = 0,$$

$$(b) \qquad \int_\Omega (-\rho\tilde{z}_2)\, v + (\gamma z_1'' + \rho\beta z_2)\, v'' = 0 \qquad \text{for all } v \in H_0^1 \cap H^2,$$

$$(c) \qquad \int_\Omega (-\tilde{z}_3)\, w + \left(\frac{k}{C_v} z_3\right) w'' = 0 \qquad \begin{array}{l}\text{for all } w \in H^2,\ w'(0) = 0, \\ kw'(1) + k_1 w(1) = 0.\end{array}$$

Now (a) implies that $\int_\Omega (\tilde{z}_1'' h + z_2 h'') = 0$ for all $h \in H_0^1 \cap H^2$. In particular, this equality must hold for all $h \in H_0^2$. Then, by the Fundamental Lemma of the Calculus of Variations ([**39**], p. 31-32) there exist constants a and b such that

$$z_2(x) = ax + b - \int_0^x \int_0^s \tilde{z}_1''(\xi)\, d\xi\, ds, \qquad \text{for } x \in \Omega.$$

Hence, $z_2 \in H^2$ and by differentiating twice the above expression we get

$$\tilde{z}_1'' = -z_2'', \quad z_2 \in H^2. \tag{3.3.13a}$$

Similarly, the Fundamental Lemma of the Calculus of Variations applied to (b) gives the existence of two constants c and d such that

$$\gamma z_1''(x) = -\rho\beta z_2(x) + cx + d + \rho \int_0^x \int_0^s \tilde{z}_2(\xi)\, d\xi\, ds, \qquad \text{for } x \in \Omega.$$

Now, since $z_2 \in H^2$, the RHS in the above expression is in H^2. Hence $z_1 \in H^4$ and, by differentiating twice we obtain

$$\tilde{z}_2 = \beta z_2'' + \frac{\gamma}{\rho} z_1'''', \quad z_1 \in H^4. \tag{3.3.13b}$$

Finally, observe that (c) must hold in particular for all $w \in H_0^2$. Again, the Fundamental Lemma of the Calculus of Variations yields the existence of two constants p and q such that

$$\frac{k}{C_v} z_3(x) = px + q - \int_0^x \int_0^s (-\tilde{z}_3(\xi))\, d\xi\, ds.$$

By differentiating twice the above expression we get

$$\tilde{z}_3 = \frac{k}{C_v} z_3'', \quad z_3 \in H^2. \tag{3.3.13c}$$

Therefore, if $z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in dom\,(A^*(q))$, then $z_1 \in H^4 \cap H_0^1$ (note $z_1 \in H_0^1$ because z must belong to Z_q) and $z_2,\ z_3 \in H^2$. To show that $dom\,(A^*(q)) = dom\,(A(q))$ it remains to show that $z_1''|_{\partial\Omega} = 0$, $z_2|_{\partial\Omega} = 0$, $z_3'(0) = 0$ and $kz_3'(1) + k_1 z_3(1) = 0$.

From (3.3.13a) and (a) we have that for all $u \in H_0^1 \cap H^4$ such that $u''|_{\partial\Omega} = 0$

$$\begin{aligned} 0 &= \int_\Omega (-z_2''u'' + z_2 u'''') \\ &= -\left[u''z_2'|_{\partial\Omega} - \int_\Omega z_2'u'''\right] + \left[z_2 u'''|_{\partial\Omega} - \int_\Omega u'''z_2'\right] \\ &= z_2 u'''|_{\partial\Omega}. \end{aligned}$$

Since this must hold for all $u \in H_0^1 \cap H^4$ with $u''|_{\partial\Omega} = 0$, we must have

$$z_2|_{\partial\Omega} = 0. \tag{3.3.14}$$

From(3.3.13b) and (b) we get that for all $v \in H_0^1 \cap H^2$

$$\begin{aligned} 0 &= \int_\Omega \left[-\rho\left(\beta z_2'' + \frac{\gamma}{\rho} z_1''''\right) v + (\gamma z_1'' + \rho\beta z_2)\, v''\right] \\ &= -\rho\left[v\left(\beta z_2' + \frac{\gamma}{\rho} z_1'''\right)\Big|_{\partial\Omega} - \int \left(\beta z_2' + \frac{\gamma}{\rho} z_1'''\right) v'\right] \\ &\quad + \left[(\gamma z_1'' + \rho\beta z_2)\, v'\Big|_{\partial\Omega} - \int_\Omega v'\,(\gamma z_1''' + \rho\beta z_2')\right] \\ &= (\gamma z_1'' + \rho\beta z_2) v'|_{\partial\Omega} \\ &= \gamma z_1'' v'|_{\partial\Omega}. \end{aligned}$$

The last equality follows from (3.3.14) since $z_2|_{\partial\Omega} = 0$.

Since this equality must hold for all $v \in H_0^1 \cap H^2$, we must have

$$z_1''|_{\partial\Omega} = 0.$$

Finally, from (3.3.13c) and (c) we get that for all $w \in H^2$ with $w'(0) = 0$ and $kw'(1) + k_1 w(1) = 0$

$$
\begin{aligned}
0 &= \int_\Omega (-z_3'' w + z_3 w'') \\
&= (-w z_3' + w' z_3)|_{\partial\Omega} \\
&= -\left[w(1) z_3'(1) - w(0) z_3'(0)\right] + w'(1) z_3(1) \\
&= -\left[w(1) z_3'(1) - w(0) z_3'(0)\right] - \frac{k_1}{k} w(1) z_3(1) \\
&= w(1)\left[-\frac{k_1}{k} z_3(1) - z_3'(1)\right] + w(0) z_3'(0).
\end{aligned}
$$

Therefore, we must have $-\frac{k_1}{k} z_3(1) - z_3'(1) = 0$ and $z_3'(0) = 0$, i.e.,

$$z_3'(0) = 0 \quad \text{and} \quad k z_3'(1) + k_1 z_3(1) = 0.$$

Hence $z \in dom\,(A(q))$ and therefore $dom\,(A^*(q)) = dom\,(A(q))$. This completes the proof of theorem 3.3.2. ∎

Now we prove that $A(q)$ is the infinitesimal generator of an analytic semigroup. We will achieve this in several steps.

Let $L_{2,\rho}(\Omega)$ denote the Hilbert space $L_2(\Omega)$ with inner product defined by $\langle u, v \rangle_{L_{2,\rho}} \doteq \rho \int_\Omega uv$ and let us define the operators $A_1(q)$ and $B_1(q)$ on $L_{2,\rho}(\Omega)$ as follows:

$$
\begin{aligned}
dom\,(A_1(q)) &\doteq \left\{u \in H^4(\Omega) \mid u(0) = u(1) = u''(0) = u''(1) = 0\right\}, \\
dom\,(B_1(q)) &\doteq H_0^1(\Omega) \cap H^2(\Omega), \\
A_1(q)f &\doteq \frac{\gamma}{\rho}\frac{\partial f}{\partial x^4}, \qquad f \in dom\,(A_1(q)) && (3.3.15) \\
B_1(q)h &\doteq -\beta\frac{\partial h}{\partial x^2}, \qquad h \in dom\,(B_1(q)). && (3.3.16)
\end{aligned}
$$

Note that both $dom\,(A_1(q))$ and $dom\,(B_1(q))$ are dense in $L_{2,\rho}(\Omega)$.

THEOREM 3.3.3. *Let $A_1(q) : dom\,(A_1(q)) \subset L_{2,\rho}(\Omega) \to L_{2,\rho}(\Omega)$, and $B_1(q) : dom\,(B_1(q)) \subset L_{2,\rho}(\Omega) \to L_{2,\rho}(\Omega)$ be as above. Then*

i) *$A_1(q)$ is strictly positive and self-adjoint;*

i) *$B_1(q)$ is strictly positive and self-adjoint;*

iii) *$B_1(q) = \frac{\beta\sqrt{\rho}}{\sqrt{\gamma}}A_1^{1/2}(q)$.*

PROOF:

i) If $u \in dom\,(A_1(q))$, then

$$
\begin{aligned}
\langle A_1(q)u, u\rangle_{L_{2,\rho}} &= \left\langle \frac{\gamma}{\rho}u'''', u\right\rangle_{L_{2,\rho}} \\
&= \gamma\int_\Omega u''''u \\
&= \gamma\left(uu'''\Big|_{\partial\Omega} - u'u''\Big|_{\partial\Omega} + \int_\Omega (u'')^2\right) \\
&= \gamma\|u''\|^2_{L_2(\Omega)} \qquad (\text{since } u|_{\partial\Omega} = u''|_{\partial\Omega} = 0) \\
&\geq 0.
\end{aligned}
$$

Moreover, $\langle A_1(q)u, u\rangle_{L_{2,\rho}} = 0$ implies $\|u''\|_{L_2} = 0$ which in terms implies $u = 0$ since $u|_{\partial\Omega} = 0$. Hence, $A_1(q)$ is strictly positive. Let us prove now that $A_1(q)$ is self-adjoint. If $v \in dom\,(A_1(q))$, then for any $u \in dom\,(A_1(q))$ we have

$$\begin{aligned}
\langle A_1(q)u, v\rangle_{L_{2,\rho}} &= \gamma \int_\Omega u''''v \\
&= \gamma \left(vu''' \Big|_{\partial\Omega} - v'u'' \Big|_{\partial\Omega} + v''u' \Big|_{\partial\Omega} - v'''u \Big|_{\partial\Omega} + \int_\Omega v''''u \right) \\
&= \gamma \int_\Omega v''''u \\
&= \rho \int_\Omega u \frac{\gamma}{\rho} v'' \\
&= \langle u, A_1(q)v\rangle_{L_{2,\rho}} .
\end{aligned}$$

Therefore, $v \in dom\,(A_1^*(q))$ and $A_1^*(q)v = A_1(q)v$, i.e., $A_1(q)$ is symmetric.

Now if $u \in dom\,(A_1^*(q))$, then there exists $v \in L_{2,\rho}(\Omega)$ such that for all $w \in dom\,(A_1(q))$

$$\begin{aligned}
0 &= \langle A_1(q)w, u\rangle_{L_{2,\rho}} - \langle w, v\rangle_{L_{2,\rho}} \\
&= \rho \int_\Omega \left(\frac{\gamma}{\rho} w''''u - wv \right) .
\end{aligned} \tag{3.3.17}$$

This equality must hold in particular for all $w \in H_0^4(\Omega)$. The Fundamental Lemma of the Calculus of Variations implies that there exist four constants a, b, c and d such that

$$\frac{\gamma}{\rho} u(x) = ax^3 + bx^2 + cx + d - \int_0^x \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} (-v(\xi))\, d\xi\, ds_3\, ds_2\, ds_1, \quad \text{for } x \in \Omega.$$

Hence, $u \in H^4(\Omega)$ and, by differentiating four times the above expression becomes

$\frac{\alpha}{\rho}u'''' = v$. Substituting this expression into (3.3.17) we get

$$
\begin{aligned}
0 &= \int_{\Omega} (w''''u - wu'''') \\
&= uw'''|_{\partial\Omega} - u'w''|_{\partial\Omega} - wu'''|_{\partial\Omega} + w'u''|_{\partial\Omega} \\
&= uw'''|_{\partial\Omega} + u''w'|_{\partial\Omega}.
\end{aligned}
$$

Since this equality must hold for all $w \in dom\,(A_1(q))$, we conclude that $u|_{\partial\Omega} = u''|_{\partial\Omega} = 0$. Hence, $dom\,(A_1^*(q)) = dom\,(A_1(q))$ and $A_1(q)$ is self-adjoint.

ii) If $u \in dom\,(B_1(q))$, then

$$
\begin{aligned}
\langle B_1(q)u, u\rangle_{L_{2,\rho}} &= \langle -\beta u'', u\rangle_{L_{2,\rho}} \\
&= -\beta\rho \int_{\Omega} u''u \\
&= \beta\rho \|u'\|_{L_2}^2 \\
&\geq 0.
\end{aligned}
$$

Moreover, $\langle B_1(q)u, u\rangle_{L_{2,\rho}} = 0$ implies $\|u'\|_{L_2} = 0$ and, therefore, $u = 0$ since $u|_{\partial\Omega} = 0$. Thus, $B_1(q)$ is strictly positive.

We will show now that $B_1(q)$ is self-adjoint. Let $v \in dom\,(B_1(q))$. Then for any $u \in dom\,(B_1(q))$

$$
\begin{aligned}
\langle B_1(q)u, v\rangle_{L_{2,\rho}} &= \langle -\beta u'', v\rangle_{L_{2,\rho}} \\
&= -\beta\rho \int_{\Omega} u''v.
\end{aligned}
$$

After integrating by parts twice and using $u|_{\partial\Omega} = v|_{\partial\Omega} = 0$, one obtains

$$\begin{aligned}\langle B_1(q)u, v\rangle_{L_{2,\rho}} &= -\beta\rho \int_\Omega uv'' \\ &= \langle u, -\beta v''\rangle_{L_{2,\rho}} \\ &= \langle u, B_1(q)v\rangle_{L_{2,\rho}}.\end{aligned}$$

Therefore, $v \in dom\,(B_1^*(q))$ and $B_1^*(q)v = B_1(q)v$, i.e., $B_1(q)$ is symmetric.

Now if $u \in dom\,(B_1^*(q))$, then there exists $v \in L_{2,\rho}(\Omega)$ such that for all $w \in dom\,(B_1(q))$

$$\begin{aligned}0 &= \langle B_1(q)w, u\rangle_{L_{2,\rho}} - \langle w, v\rangle_{L_{2,\rho}} \\ &= -\rho \int_\Omega (\beta w''u + wv).\end{aligned} \tag{3.3.18}$$

This equality must hold for all $w \in H_0^2(\Omega)$. The Fundamental Lemma of the Calculus of Variations implies that there exist two constants a and b such that

$$\beta u(x) = ax + b - \int_0^x \int_0^s v(\xi)\, d\xi\, ds, \qquad \text{for } x \in \Omega.$$

Thus, $u \in H^2(\Omega)$ and, by differentiating twice, it follows that $\beta u'' = -v$. Substituting into (3.3.18) we get

$$\begin{aligned}0 &= \int_\Omega (w''u - wu'') \\ &= uw'|_{\partial\Omega} - wu'|_{\partial\Omega} \\ &= uw'|_{\partial\Omega}.\end{aligned}$$

Since this equality must hold for all $w \in dom\,(B_1(q))$, we conclude that $u|_{\partial\Omega} = 0$. Hence, $dom\,(B_1^*(q)) = dom\,(B_1(q))$ and $B_1(q)$ is self-adjoint.

iii) Since $A_1(q)$ is positive and self-adjoint, it possesses a unique positive self-adjoint square root $A_1^{1/2}(q)$ (see [**80**], p. 281 or [**137**], p. 197). Moreover, any positive fractional power $A_1^{\alpha}(q)$ of $A_1(q)$ is well defined, positive and self-adjoint.

It is easy to see that $dom\,(B_1^2(q)) = dom\,(A_1(q))$ and $B_1^2(q)u = \frac{\beta^2\rho}{\gamma}A_1(q)u$ for all $u \in dom\,(A_1(q))$. Hence $B_1(q) = \frac{\beta\sqrt{\rho}}{\sqrt{\gamma}}A_1^{1/2}(q)$ and this completes the proof of Theorem 3.3.3. ∎

We now define the Hilbert space E_q by

$$E_q \doteq dom\left(A_1^{1/2}(q)\right) \times L_{2,\rho}(\Omega) = dom\,(B_1(q)) \times L_{2,\rho}(\Omega)$$

with inner product

$$\left\langle \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \right\rangle_{E_q} \doteq \left\langle A_1^{1/2}(q)f_1, A_1^{1/2}(q)f_2 \right\rangle_{L_{2,\rho}} + \langle g_1, g_2 \rangle_{L_{2,\rho}},$$

and the operator $C_1(q) : E_q \to E_q$ by $dom\,(C_1(q)) \doteq dom\,(A_1(q)) \times dom\left(A_1^{1/2}(q)\right) = dom\,(A_1(q)) \times dom\,(B_1(q))$ and

$$C_1(q) \doteq \begin{pmatrix} 0 & I \\ -A_1(q) & -B_1(q) \end{pmatrix}. \tag{3.3.19}$$

Note that $dom\,(C_1(q))$ is dense in E_q. The operator $C_1(q)$ corresponds to the elastic model $\ddot{x} + B_1(q)\dot{x} + A_1(q)x = 0$ written as a first order system. By Theorem 3.3.3, the elastic operator $A_1(q)$ is positive and self-adjoint on $L_{2,\rho}(\Omega)$. The same is true for the dissipation operator $B_1(q)$.

THEOREM 3.3.4. *Let $C_1(q) : dom\,(C_1(q)) \subset E_q \to E_q$ be as defined above. Then $C_1(q)$ is the infinitesimal generator of a strongly continuous semigroup of contractions $e^{C_1(q)t}$ on E_q.*

Proof: Let $\eta = \begin{pmatrix} u \\ v \end{pmatrix} \in dom\,(C_1(q))$. Then $u \in dom\,(A_1(q))$, $v \in dom\,(B_1(q))$ and

$$\begin{aligned}
\langle C_1(q)\eta,\ \eta\rangle_{E_q} &= \left\langle \begin{pmatrix} 0 & I \\ -A_1(q) & -B_1(q) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_{E_q} \\
&= \left\langle \begin{pmatrix} v \\ -A_1(q)u - B_1(q)v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_{E_q} \\
&= \left\langle A_1^{1/2}(q)v, A_1^{1/2}(q)u \right\rangle_{L_{2,\rho}} + \langle -A_1(q)u - B_1(q)v, v\rangle_{L_{2,\rho}}.
\end{aligned}$$

Also, since $A_1^{1/2}(q)$ is self-adjoint, $u \in dom\,(A_1(q))$ and $B_1(q)$ is positive, it follows that

$$\begin{aligned}
\langle C_1(q)\eta, \eta\rangle_{E_q} &= \langle v, A_1(q)u\rangle_{L_{2,\rho}} - \langle A_1(q)u, v\rangle_{L_{2,\rho}} - \langle B_1(q)v, v\rangle_{L_{2,\rho}} \\
&= - \langle B_1(q)v, v\rangle_{L_{2,\rho}} \\
&\leq 0.
\end{aligned}$$

Hence $C_1(q)$ is dissipative.

One can easily verify that the adjoint $C_1^*(q)$ of $C_1(q)$ is given by $dom\,(C_1^*(q)) = dom\,(C_1(q)) = dom\,(A_1(q)) \times dom\,(B_1(q))$ and

$$C_1^*(q) = \begin{pmatrix} 0 & -I \\ A_1(q) & -B_1(q) \end{pmatrix}.$$

If $\eta = \begin{pmatrix} u \\ v \end{pmatrix} \in dom\,(C_1^*(q))$, we have

$$\begin{aligned}
\langle C_1^*(q)\eta,\ \eta\rangle_{E_q} &= \left\langle \begin{pmatrix} 0 & -I \\ A_1(q) & -B_1(q) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_{E_q} \\
&= \left\langle \begin{pmatrix} -v \\ A_1(q)u - B_1(q)v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_{E_q} \\
&= \left\langle A_1^{1/2}(q)v, A_1^{1/2}(q)u \right\rangle_{L_{2,\rho}} + \langle A_1(q)u - B_1(q)v, v\rangle_{L_{2,\rho}}.
\end{aligned}$$

Again, since $A_1^{1/2}(q)$ is self-adjoint, $u \in dom\,(A_1(q))$ and $B_1(q)$ is positive, it follows that

$$\begin{aligned}\langle C_1^*(q)\eta, \eta\rangle_{E_q} &= -\langle v, A_1(q)u\rangle_{L_{2,\rho}} + \langle A_1(q)u, v\rangle_{L_{2,\rho}} - \langle B_1(q)v, v\rangle_{L_{2,\rho}} \\ &= -\langle B_1(q)v, v\rangle_{L_{2,\rho}} \\ &\leq 0.\end{aligned}$$

Hence, $C_1^*(q)$ is also dissipative. It now follows from the Lummer-Phillips theorem (see [**113**], p.15, cor. 4.4) that $C_1(q)$ is the infinitesimal generator of a strongly continuous semigroup of contractions $e^{C_1(q)t}$ on E_q. ∎

<u>**Comment**</u>: Operators of the form $\mathcal{A}_B = \begin{pmatrix} 0 & I \\ -A & -B \end{pmatrix}$ where both A and B are positive and self-adjoint, appear often in the theory of elastic systems. In fact, they correspond to the elastic model $\ddot{x} + B\dot{x} + Ax = 0$, written as a first order system. In 1982, D.L. Russell and G. Chen ([**25**]) first conjectured the analyticity of the C_0-semigroup of contractions $e^{\mathcal{A}_B t}$ generated by $\mathcal{A}_B$, when B is "related" to the α-power of A, $0 \leq \alpha \leq 1$. This conjecture was later proved by Triggiani and S. Chen ([**27**]). It turns out that if $\rho_1 A^\alpha \leq B \leq \rho_2 A^\alpha$ for some constants ρ_1, ρ_2, $0 < \rho_1 < \rho_2 < \infty$, then $e^{\mathcal{A}_B t}$ is analytic if $\frac{1}{2} \leq \alpha \leq 1$ and $e^{\mathcal{A}_B t}$ is not analytic if $0 < \alpha < \frac{1}{2}$.

For fixed $q \in \mathcal{Q}$ the eigenvalues of the operator $A_1(q)$ are easily found to be

$$\mu_n = \mu_n(q) = \frac{\gamma n^4 \pi^4}{\rho}, \qquad n = 1, 2, \ldots, \tag{3.3.20}$$

with corresponding normalized eigenfunctions in $L_{2,\rho}(\Omega)$ given by

$$h_n(x) = \sqrt{\frac{2}{\rho}}\, sin(\pi n x). \tag{3.3.21}$$

THEOREM 3.3.5. *Let $q \in \mathcal{Q}$, $C_1(q)$ as in (3.3.19) and $\{\mu_n\}_{n=1}^{\infty}$ the eigenvalues of $A_1(q)$. Then,* **a)** *the strongly continuous semigroup of contractions $e^{C_1(q)t}$ generated by $C_1(q)$ on E_q is also analytic.* **b)** *The spectrum $\sigma(C_1(q))$ of $C_1(q)$ consists only of eigenvalues $\{\lambda_n^{+,-}\}_{n=1}^{\infty}$, which are the solutions of the equation*

$$\lambda^2 + 2r(q)\mu_n^{1/2}\lambda + \mu_n = 0,$$

where $r(q) = \frac{\beta\sqrt{\rho}}{2\sqrt{\gamma}}$, and are given by

$$\lambda_n^{+,-} = \sqrt{\mu_n}\left(-r(q) \pm \sqrt{r^2(q)-1}\right).$$

The eigenvalues are real if and only if $r(q) \geq 1$, i.e. $\beta^2\rho \geq 4\gamma$. If $r(q) < 1$, the eigenvalues lay symmetrically with respect to the real axis on the two rays $\left\{xe^{\pm i\alpha(q)}, \quad 0 \leq x < \infty\right\}$ where $e^{\pm i\alpha(q)} = -r(q) \pm i\sqrt{1-r^2(q)}$ (note that $\alpha(q) > \frac{\pi}{2}$). In any case, $Re\,\lambda_n^{+,-} < 0$ for all n,

$$\left|\frac{Im\,\lambda_n^{+,-}}{Re\,\lambda_n^{+,-}}\right| \leq M(q) \doteq \begin{cases} 0 & \text{if } r(q) \geq 1, \\ \frac{\sqrt{1-r^2(q)}}{r(q)} & \text{if } r(q) < 1. \end{cases}$$

and the spectrum $\sigma(C_1(q))$ of $C_1(q)$ is contained in a triangular sector of the form

$$\Sigma \doteq \left\{\lambda \in \mathbb{C} \mid |arg(\lambda)| > \frac{\pi}{2} + \theta_0(q)\right\},$$

where $\theta_0(q)$ is any number satisfying $0 < \theta_0(q) < \frac{\pi}{2}$ if $r(q) \geq 1$, and $0 < \theta_0(q) < \alpha(q) - \frac{\pi}{2}$ if $r(q) < 1$. The corresponding family of normalized eigenvectors $\{\phi_n^{+,-}\}_{n=1}^{\infty}$ on E_q is given by

$$\phi_n^+ = \begin{pmatrix} e_n \\ \lambda_n^+ e_n \end{pmatrix}, \qquad \phi_n^- = k_n \begin{pmatrix} e_n \\ \lambda_n^- e_n \end{pmatrix},$$

where

$$e_n(x) = \sqrt{\frac{2}{\rho\,(\mu_n + |\lambda_n^+|^2)}}\, sin(\pi n x), \quad \text{and} \quad k_n^2 = \frac{\mu_n + |\lambda_n^+|^2}{\mu_n + |\lambda_n^-|^2}.$$

Note that μ_n, $\lambda_n^{+,-}$, θ_0, e_n, $\phi_n^{+,-}$, *all depend on* $q \in \mathcal{Q}$. **c)** *The eigenvectors* $\{\phi_n^{+,-}\}_{n=1}^{\infty}$ *satisfy:*

(i) $\{\phi_n^+\}_{n=1}^{\infty}$ *is an orthonormal family on* E_q*;*

(ii) $\{\phi_n^-\}_{n=1}^{\infty}$ *is an orthonormal family on* E_q *and*

(iii) $$\langle \phi_m^+, \phi_n^- \rangle_{E_q} = \begin{cases} 0 & \text{if } n \neq m, \\ k_n\left(\mu_n + \lambda_n^+ \overline{\lambda_n^-}\right) \|e_n\|^2_{L_{2,\rho}} & \text{if } m = n \text{ and } \lambda_n^+ \neq \lambda_n^-, \\ 1 & \text{if } m = n \text{ and } \lambda_n^+ = \lambda_n^-, \end{cases}$$

d) *The eigenvalues of* $C_1^*(q)$ *are* $\left\{\overline{\lambda_n^{+,-}}\right\}_{n=1}^{\infty}$*, the conjugates of the eigenvalues of* $C_1(q)$*, with corresponding normalized eigenvectors on* E_q *given by*

$$\phi_m^{*+} = \begin{pmatrix} e_m \\ -\overline{\lambda_m^+} e_m \end{pmatrix}, \qquad \phi_m^{*-} = k_m \begin{pmatrix} e_m \\ -\overline{\lambda_m^-} e_m \end{pmatrix}.$$

e) *Assume, in addition that* $\beta^2 \rho \neq 4\gamma$ *(or equivalently,* $r(q) \neq 1$*) and let*

$$\psi_m^{*+} \doteq \frac{1}{v_m^+} \phi_m^{*+}, \qquad \psi_m^{*-} \doteq \frac{1}{v_m^- k_m} \phi_m^{*-},$$

where

$$v_m^- \doteq \frac{\mu_m - \left(\overline{\lambda_m^-}\right)^2}{\mu_m + |\lambda_m^-|^2}, \qquad v_m^+ \doteq \frac{\mu_m - \left(\overline{\lambda_m^+}\right)^2}{\mu_m + |\lambda_m^+|^2}.$$

Then, the non-normalized eigenvectors $\{\psi_m^{*+,*-}\}_{m=1}^{\infty}$ *form a bi-orthogonal system with respect to the eigenvectors* $\{\phi_m^{+,-}\}_{m=1}^{\infty}$ *of* $C_1(q)$*, in the sense that*

$$\left\langle \psi_m^{*+}, \phi_n^+ \right\rangle_{E_q} = \left\langle \psi_m^{*-}, \phi_n^- \right\rangle_{E_q} = \delta_{nm} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

and

$$\left\langle \psi_m^{*-}, \phi_n^+ \right\rangle_{E_q} = \left\langle \psi_m^{*+}, \phi_n^- \right\rangle_{E_q} = 0$$

for all m, n.

f) *The operator* $C_1(q)$ *and the semigroup* $e^{C_1(q)t}$ *generated by* $C_1(q)$ *on* E_q *have the following spectral representations:*

$$C_1(q)\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{n=1}^{\infty} \lambda_n^+ \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \psi_n^{*+} \right\rangle_{E_q} \phi_n^+ + \sum_{n=1}^{\infty} \lambda_n^- \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \psi_n^{*-} \right\rangle_{E_q} \phi_n^-,$$

$$e^{C_1(q)t}\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{n=1}^{\infty} e^{\lambda_n^+ t} \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \psi_n^{*+} \right\rangle_{E_q} \phi_n^+ + \sum_{n=1}^{\infty} e^{\lambda_n^- t} \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \psi_n^{*-} \right\rangle_{E_q} \phi_n^-.$$

g) *Finally, the semigroup* $e^{C_1(q)t}$ *satisfies the stability condition*

$$\|e^{C_1(q)t}\|_{\mathcal{L}(E_q)} \leq e^{-\epsilon(q)t}$$

where $\epsilon(q) = -\sup_{\lambda \in \sigma(C_1(q))} Re\,\lambda = -Re\,\lambda_1^+$ *satisfies*

$$\epsilon(q) = \begin{cases} \sqrt{\mu_1}\, r(q) = \frac{\beta\pi^2}{2}, & \text{if } \beta^2\rho \leq 4\gamma \\ \sqrt{\mu_1}\left(r(q) - \sqrt{r^2(q) - 1}\right) = \frac{\pi^2}{2\sqrt{\rho}}\left(\beta\sqrt{\rho} - \sqrt{\beta^2\rho - 4\gamma}\right), & \text{if } \beta^2\rho > 4\gamma \end{cases}$$

Proof: The proof of this theorem follows immediately from Theorems 1.1, 1.2 and Lemmas A1 and A2 in [**27**], by noting that the operator $C_1(q)$ corresponds to the elastic system $\ddot{x} + 2r(q)A_1^{1/2}(q)\dot{x} + A_1(q)x = 0$ written as a first order system. ∎

Let $L_{2,\frac{C_v}{k}}(\Omega)$ denote the Hilbert space $L_2(\Omega)$ with inner product

$$\langle w, \hat{w} \rangle_{L_{2,\frac{C_v}{k}}} \doteq \frac{C_v}{k} \int_\Omega w\hat{w}$$

and define the operator $C_2(q)$ on $L_{2,\frac{C_v}{k}}(\Omega)$ by

$$dom\,(C_2(q)) \doteq \left\{w \in H^2(\Omega) \mid w'(0) = 0,\ kw'(1) + k_1 w(1) = 0\right\},$$

$$C_2(q) \doteq \frac{k}{C_v}\frac{\partial^2}{\partial x^2}. \tag{3.3.22}$$

THEOREM 3.3.6. *Let $C_2(q) : dom\,(C_2(q)) \subset L_{2,\frac{C_v}{k}}(\Omega) \to L_{2,\frac{C_v}{k}}(\Omega)$ be as defined above. Then*

i) *$C_2(q)$ is dissipative and self-adjoint;*

ii) *The spectrum $\sigma\,(C_2(q))$ consists only of eigenvalues $\{\alpha_n\}_{n=1}^{\infty}$ given by*

$$\alpha_n = -\frac{k\tau_n^2}{C_v}, \qquad n = 1, 2 \ldots,$$

where $\{\tau_n\}_{n=1}^{\infty}$ are all the positive solutions of the equation $tan\,\tau = \frac{k_1}{k\tau}$. The corresponding normalized eigenfunctions $\{\chi_n\}_{n=1}^{\infty}$ in $L_{2,\frac{C_v}{k}}(\Omega)$ are given by

$$\chi_n(x) = \left(\frac{k\tau_n}{C_v \int_0^{\tau_n} cos^2(\xi)\, d\xi}\right)^{1/2} cos(\tau_n x).$$

iii) *The operator $C_2(q)$ generates a strongly continuous semigroup of contractions $e^{C_2(q)t}$ on $L_{2,\frac{C_v}{k}}(\Omega)$ which is also analytic and satisfies the stability property*

$$\|e^{C_2(q)t}\|_{\mathcal{L}(L_{2,\frac{C_v}{k}}(\Omega))} \leq e^{-\frac{k\tau_1^2}{C_v}t}, \qquad t \geq 0.$$

iv) *The semigroup $e^{C_2(q)t}$ has the representation*

$$e^{C_2(q)t}w = \sum_{n=1}^{\infty} e^{\alpha_n t}\,\langle w, \chi_n\rangle_{L_{2,\frac{C_v}{k}}}\,\chi_n$$

PROOF:

i) Using integration by parts and the boundary conditions for $C_2(q)$, it follows that $\langle C_2(q)w, w\rangle_{L_{2,\frac{C_v}{k}}} = -\frac{k_1}{k}[w(1)]^2 - \|w'\|^2_{L_2}$ for any $w \in dom(C_2(q))$. Thus $C_2(q)$ is dissipative. Similarly, one can easily show that $C_2(q)$ is symmetric. Moreover, an application of the Fundamental Lemma of the Calculus of Variations yields that $C_2(q)$ is self-adjoint. Since the steps are exactly the same as those in the proof of Theorem 3.3.2, we do not give details here.

ii) The function f is an eigenvector of $C_2(q)$ corresponding to the eigenvalue ξ if and only if $f \neq 0$ satisfies $f''(x) = \frac{\xi k}{C_v} f(x)$, $f'(0) = 0$ and $kf'(1) + k_1 f(1) = 0$. This implies that $\xi < 0$ satisfies $\frac{k}{k_1}\sqrt{-\frac{C_v\xi}{k}}\, tan\left(\sqrt{-\frac{C_v\xi}{k}}\right) = 1$ and $f(x) = b\, cos\left(\sqrt{-\frac{C_v\xi}{k}}\right)$, $b \in \mathbb{R}$. Letting $\tau = \sqrt{-\frac{C_v\xi}{k}}$ we get that the eigenvalues $\{\alpha_n\}_{n=1}^{\infty}$ of $C_2(q)$ are given by $\alpha_n = -\frac{k\tau_n^2}{C_v}$ where $\{\tau_n\}_{n=1}^{\infty}$ are all the positive solutions of the equation $\frac{k}{k_1}\tau\, tan\tau = 1$. Moreover, normalization of $\chi_n(x) = b_n\, cos(\tau_n x)$ gives $b_n = \left(\frac{k\tau_n}{C_v \int_0^{\tau_n} cos^2(\xi)\, d\xi}\right)^{1/2}$.

iii), iv) Since $C_2(q)$ is self-adjoint with pure point spectrum it follows that (see [**137**])

$L_{2,\frac{C_v}{k}}(\Omega) = \overline{\text{span}}\,\{\chi_n\}_{n=1}^{\infty}$,

$$w = \sum_{n=1}^{\infty} \langle w, \chi_n\rangle_{L_{2,\frac{C_v}{k}}} \chi_n, \qquad \text{for any } w \in L_{2,\frac{C_v}{k}}(\Omega),$$

$$C_2(q)w = \sum_{n=1}^{\infty} \alpha_n \langle w, \chi_n\rangle_{L_{2,\frac{C_v}{k}}} \chi_n, \qquad \text{for any } w \in dom\,(C_2(q)),$$

$$e^{C_2(q)t}w = \sum_{n=1}^{\infty} e^{\alpha_n t} \langle w, \chi_n\rangle_{L_{2,\frac{C_v}{k}}} \chi_n, \qquad \text{for any } w \in L_{2,\frac{C_v}{k}}(\Omega).$$

From this representation for $e^{C_2(q)t}$ it is easy to see that

$$\|e^{C_2(q)t}\|_{\mathcal{L}(L_{2,\frac{C_v}{k}}(\Omega))} \le e^{\alpha_1 t} = e^{-\frac{k\tau_1^2}{C_v}t}, \qquad t \ge 0.$$

∎

We note now that $Z_q = E_q \times L_{2,\frac{C_v}{k}}(\Omega)$ not only set-theoretically, but also topologically. In fact, $\eta = \begin{pmatrix} u \\ v \end{pmatrix} \in E_q$ and $w \in L_{2,\frac{C_v}{k}}(\Omega)$ if and only if $\begin{pmatrix} u \\ v \\ w \end{pmatrix} \in Z_q$. Also note that for any $\eta = \begin{pmatrix} u \\ v \end{pmatrix}$, $\hat{\eta} = \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} \in E_q$, w, $\hat{w} \in L_{2,\frac{C_v}{k}}(\Omega)$ we have

$$\begin{aligned}
\left\langle \begin{pmatrix} \eta \\ w \end{pmatrix}, \begin{pmatrix} \hat{\eta} \\ \hat{w} \end{pmatrix} \right\rangle_{E_q \times L_{2,\frac{C_v}{k}}} &= \langle \eta, \hat{\eta} \rangle_{E_q} + \langle w, \hat{w} \rangle_{L_{2,\frac{C_v}{k}}} \\
&= \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} \right\rangle_{E_q} + \langle w, \hat{w} \rangle_{L_{2,\frac{C_v}{k}}} \\
&= \left\langle A_1^{1/2} u, A_1^{1/2} \hat{u} \right\rangle_{L_{2,\rho}} + \langle v, \hat{v} \rangle_{L_{2,\rho}} + \langle w, \hat{w} \rangle_{L_{2,\frac{C_v}{k}}} \\
&= \rho \int_\Omega \sqrt{\frac{\gamma}{\rho}} u'' \sqrt{\frac{\gamma}{\rho}} \hat{u}'' + \rho \int_\Omega v\hat{v} + \frac{C_v}{k} \int_\Omega w\hat{w} \\
&= \gamma \int_\Omega u'' \hat{u}'' + \rho \int_\Omega v\hat{v} + \frac{C_v}{k} \int_\Omega w\hat{w} \\
&= \left\langle \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \end{pmatrix} \right\rangle_{Z_q}.
\end{aligned}$$

Thus, the Hilbert space $E_q \times L_{2,\frac{C_v}{k}}(\Omega)$ endowed with the usual inner product is isometrically isomorphic to Z_q. From now on we will freely make use of this identification.

The following identity relating the operators $A(q)$, $C_1(q)$ and $C_2(q)$ can be immediately verified

$$A(q) = \begin{pmatrix} C_1(q) & 0 \\ 0 & C_2(q) \end{pmatrix},$$

where the equality should be interpreted in the sense of the isomorphism previously defined. This identity allows us to use Theorems 3.3.5 and 3.3.6 from which we obtain the following.

THEOREM 3.3.7. *Let $A(q) : dom\,(A(q)) \subset Z_q \to Z_q$ be as in (3.3.9). Then*

i) $0 \in \rho\,(A(q))$, *the resolvent set of $A(q)$;*

ii) *The spectrum $\sigma\,(A(q))$ of $A(q)$ consists only of eigenvalues,*

$$\sigma\,(A(q)) = \sigma_p\,(A(q)) = \sigma_p\,(C_1(q)) \cup \sigma_p\,(C_2(q)) = \left\{\lambda_n^{+,-},\, \alpha_n\right\}_{n=1}^{\infty}$$

where

$$\lambda_n^{+,-} = \sqrt{\mu_n}\left(-r(q) \pm \sqrt{r^2(q)-1}\right), \qquad \alpha_n = -\frac{k\tau_n^2}{\rho},$$

with $\mu_n = \dfrac{\gamma n^4 \pi^4}{\rho}$, $r(q) = \dfrac{\beta\sqrt{\rho}}{2\sqrt{\gamma}}$ *and* $\{\tau_n\}_{n=1}^{\infty}$ *are all the positive solutions of the equation* $tan\,\tau = \dfrac{k_1}{k\tau}$.

The corresponding set of normalized eigenvectors in Z_q is given by

$$\left\{ \begin{pmatrix} e_n \\ \lambda_n^{+} e_n \\ 0 \end{pmatrix}, \begin{pmatrix} k_n e_n \\ k_n \lambda_n^{-} e_n \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \chi_n \end{pmatrix} \right\}_{n=1}^{\infty},$$

where

$$e_n(x) = \left(\frac{2}{\rho\,(\mu_n + |\lambda_n^{+}|^2)}\right)^{1/2} sin(\pi n x),$$

$$\chi_n(x) = \left(\frac{k\tau_n}{C_v \int_0^{\tau_n} cos^2(\xi)\, d\xi}\right)^{1/2} cos(\tau_n x)$$

and

$$k_n^2 = \frac{\mu_n + |\lambda_n^{+}|^2}{\mu_n + |\lambda_n^{-}|^2}.$$

iii) *The operator $A(q)$ generates an exponentially stable analytic semigroup $T(t;q)$ which satisfies*

$$\|T(t;q)\|_{\mathcal{L}(Z_q)} \leq e^{-\omega(q)t}, \qquad \text{for } t \geq 0,$$

where $\omega(q)$ is given by

$$\omega(q) = \begin{cases} min\left(\frac{k\tau_1^2}{C_v}, \frac{\beta\pi^2}{2}\right), & \text{if } \beta^2\rho \leq 4\gamma \\ min\left(\frac{k\tau_1^2}{C_v}, \frac{\beta\pi^2}{2} - \frac{\pi^2}{2\sqrt{\rho}}\sqrt{\beta^2\rho - 4\gamma}\right), & \text{if } \beta^2\rho > 4\gamma. \end{cases}$$

Moreover, the semigroup $T(t;q)$ has the representation

$$\begin{aligned} T(t;q)z = &\sum_{n=1}^{\infty} e^{\lambda_n^+ t} \left\langle z, \frac{1}{v_n^+}\begin{pmatrix} \overline{e_n} \\ -\overline{\lambda_n^+ e_n} \\ 0 \end{pmatrix}\right\rangle_q \begin{pmatrix} e_n \\ \lambda_n^+ e_n \\ 0 \end{pmatrix} \\ &+ \sum_{n=1}^{\infty} e^{\lambda_n^- t} \left\langle z, \frac{1}{v_n^-}\begin{pmatrix} \overline{e_n} \\ -\overline{\lambda_n^- e_n} \\ 0 \end{pmatrix}\right\rangle_q \begin{pmatrix} k_n e_n \\ k_n\lambda_n^- e_n \\ 0 \end{pmatrix} \\ &+ \sum_{n=1}^{\infty} e^{\alpha_n t} \left\langle z, \begin{pmatrix} 0 \\ 0 \\ \chi_n \end{pmatrix}\right\rangle_q \begin{pmatrix} 0 \\ 0 \\ \chi_n \end{pmatrix}, \end{aligned}$$

where $v_n^{+,-} = \dfrac{\mu_n - \left(\overline{\lambda_n^{+,-}}\right)^2}{\mu_n + \left|\overline{\lambda_n^{+,-}}\right|^2}.$

Proof: It can be immediately verified that $A(q)$ is invertible and

$$A^{-1}(q) = \begin{pmatrix} C_1^{-1}(q) & 0 \\ 0 & C_2^{-1}(q) \end{pmatrix},$$

where

$$C_1^{-1}(q) = \begin{pmatrix} -B_1(q)A_1^{-1}(q) & -A_1^{-1}(q) \\ I & 0 \end{pmatrix} = \begin{pmatrix} -\frac{\beta\sqrt{\rho}}{\sqrt{\gamma}}A_1^{-1/2}(q) & -A_1^{-1}(q) \\ I & 0 \end{pmatrix}.$$

Parts (ii) and (iii) follow immediately from Theorems 3.3.5 and 3.3.6 and the fact that $C_1(q)$ and $C_2(q)$ "decouple" the operator $A(q)$. ∎

Since $A(q)$ is closed, *dom* $(A(q))$ endowed with the graph norm of $A(q)$,

$$|||z|||_{A(q)} \doteq \|z\|_q + \|A(q)z\|_q$$

is a Hilbert space which we denote by $Z_{A(q)}$, i.e.,

$$Z_{A(q)} \doteq \left(dom\,(A(q))\,;|||\cdot|||_{A(q)}\right).$$

Since $A(q)$ is invertible, $|||\cdot|||_{A(q)}$ is equivalent to $|||\cdot|||_G$ defined by $|||z|||_G = \|A(q)z\|_q$ on *dom* $(A(q))$.

Now, for $z = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in dom\,(A(q))$ we define

$$\|z\|^2_{H^2} \doteq \|u\|^2_{H^2} + \|v\|^2_{H^2} + \|w\|^2_{H^2},$$

while $\|z\|^2_{H^1}$ is defined in the analogous way. The following lemma relates the H^2-norm and the $|||\cdot|||_{A(q)}$-norm on *dom* $(A(q))$.

LEMMA 3.3.8. *Let $\mathcal{S}$ be a bounded subset of $Z_{A(q)}$. Then $\mathcal{S}$ is also bounded in the H^2-norm, i.e., there exists a constant C depending on q and $\mathcal{S}$ such that $\|z\|^2_{H^2} \le C$ for all $z \in \mathcal{S}$. Moreover, the constant C can be chosen independent of q, for q on compact subsets of $\mathcal{Q}$ not containing $q = 0$.*

PROOF: Since $\mathcal{S}$ is bounded in $Z_{A(q)}$, there exists $M > 0$ such that $|||z|||^2_{A(q)} \le M$ for all $z \in \mathcal{S}$. Let $z = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathcal{S}$. Then,

$$\|u\|_{L_2} \le \|u'\|_{L_2} \le \|u''\|_{L_2} \qquad \text{since } u \in H^1_0(0,1) \cap H^2(0,1).$$

The first inequality, known as Poincaré's first inequality (see [**139**], p. 116), follows immediately from Schwartz's inequality and the fact that $u(0) = 0$. For the second one, note that $\|u'\|_{L_2}^2 = \int_0^1 (u')^2 = -\int_0^1 uu'' \leq \|u\|_{L_2}\|u''\|_{L_2} \leq \|u'\|_{L_2}\|u''\|_{L_2}$. Hence, $\|u'\|_{L_2} \leq \|u''\|_{L_2}$.

Now

$$\|u''\|_{L_2}^2 \leq \frac{\|z\|_q^2}{\gamma} \leq \frac{|||z|||_{A(q)}^2}{\gamma} \leq \frac{M}{\gamma}.$$

Similarly we have

$$\|v\|_{L_2} \leq \|v'\|_{L_2} \leq \|v''\|_{L_2}, \qquad \text{since } v \in H_0^1(0,1) \cap H^2(0,1).$$

and

$$\|v''\|_{L_2}^2 \leq \frac{\|A(q)z\|_q^2}{\gamma} \leq \frac{|||z|||_{A(q)}^2}{\gamma} \leq \frac{M}{\gamma}.$$

Finally, we have

$$\|w\|_{L_2}^2 \leq \frac{\|z\|_q^2}{\frac{C_v}{k}} \leq \frac{k}{C_v}|||z|||_{A(q)}^2 \leq \frac{Mk}{C_v},$$

$$\|w''\|_{L_2}^2 = \frac{C_v}{k}\left(\frac{C_v}{k}\left\|\frac{k}{C_v}w''\right\|_{L_2}^2\right) \leq \frac{C_v}{k}\|A(q)z\|_q^2 \leq \frac{C_v}{k}|||z|||_{A(q)}^2 \leq \frac{C_vM}{k},$$

and

$$\begin{aligned}\|w'\|_{L_2}^2 &= w'w|_{\partial\Omega} - \int_\Omega ww'' \leq w'(1)w(1) + \|w\|_{L_2}\|w''\|_{L_2} \\ &= -\frac{k_1}{k}[w(1)]^2 + \|w\|_{L_2}\|w''\|_{L_2} \leq \|w\|_{L_2}\|w''\|_{L_2} \\ &\leq \sqrt{\frac{kM}{C_v}}\sqrt{\frac{C_vM}{k}} = M.\end{aligned}$$

Hence $\|z\|_{H^2}^2 \leq C$, where $C = M\left(\frac{6}{\gamma} + \frac{C_v}{k} + \frac{k}{C_v} + 1\right)$ can be chosen independent of q

on any subset $\mathcal{Q}_S$ of $\mathcal{Q}$ of the form

$$\mathcal{Q}_S = \{q = (\rho, C_v, \beta, \alpha_2, \alpha_4, \alpha_6, \theta_1, \gamma) \in \mathcal{Q} \mid 0 < a \le C_v \le b < \infty, \quad 0 < c \le \gamma\}$$

and therefore, on any compact subset of $\mathcal{Q}$ not containig $q = 0$. ∎

Remark: By Theorem 3.3.7, $A(q)$ is the infinitesimal generator of an analytic semigroup and $0 \in \rho\left(-A(q)\right)$. Hence, the fractional powers of $-A(q)$, $[-A(q)]^{\delta}$ are well defined, closed, linear, invertible operators for all $0 \le \delta \le 1$ (see [**113**], section 2.6). Therefore, *dom* $\left([-A(q)]^{\delta}\right)$ endowed with the graph norm $|||z|||_{A^{\delta}(q)} \doteq \|z\|_q$ $+\| [-A(q)]^{\delta} z\|_q$ is a Hilbert space, which we denote by $Z_{A^{\delta}(q)}$.

One way of showing that the initial value problem (3.3.7) is well-posed, involves proving that the nonlinear term $F(q,t,z)$ is sufficiently regular with respect to the operator $A(q)$. More precisely, one of the requirements is that $F(q,t,z)$ be locally uniformly Hölder continuous in z with respect to the graph norm of some fractional δ-power of $-A(q)$.

Definition (Condition (F)). *Let $-A$ be the infinitesimal generator of an analytic semigroup $T(t)$ on a Banach space X and assume $0 \in \rho(-A)$. For $0 \le \delta \le 1$, let X_δ denote the Banach space $\left(dom\left(A^{\delta}\right); \|\cdot\|_{\delta}\right)$, where $\|x\|_\delta \doteq \|A^{\delta}x\|$. Let U be an open subset of $\mathbb{R}^+ \times X_\delta$. We say that the function $f : U \to X$ satisfies the condition (F) on U if for every $(t,x) \in U$ there exists a neighborhood $V \subset U$ and constants $L \ge 0$,*

$0 < \nu \leq 1$ *such that*

$$\|f(t_1, x_1) - f(t_2, x_2)\| \leq L\left(|t_1 - t_2|^\nu + \|x_1 - x_2\|_\delta\right)$$

for all (t_1, x_1), $(t_2, x_2) \in V$, *i.e., if* f *is locally Hölder continuous in* t *and locally Lipschitzian in* x, *on* U.

The proof of the following theorem can be found in [**113**], p. 196.

THEOREM 3.3.9. *Let* $-A$ *be the infinitesimal generator of an analytic semigroup* $T(t)$ *satisfying* $\|T(t)\| \leq M$ *and assume further that* $0 \in \rho(-A)$. *If* f *satisfies the condition (F), then for every initial data* $(t_0, x_0) \in U$ *the initial value problem*

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = f\left(t, u(t)\right), & t > 0 \\ u(t_0) = x_0 \end{cases}$$

has a unique local solution $u \in C\left([t_0, t_1); X\right) \cap C^1\left((t_0, t_1); X\right)$ *where* $t_1 = t_1(t_0, x_0) > t_0$.

The following theorem states that the nonlinear part of system (3.3.7) satisfies the condition (F).

THEOREM 3.3.10. *let* $q \in \mathcal{Q}$, U *a bounded open neighborhood of the origin in* $\mathbb{R}_0^+ \times Z_{A(q)}$, *and define* $h_q : U \to Z_q$ *by* $h_q(t, z) = F(q, t, z)$. *Then* h_q *satisfies the condition (F) on* U *with constants* $\delta = 1$, $\nu = 1$ *and* $L > 0$, *uniformly in* U, *i.e.,*

$$\|F(q, t_1, z_1) - F(q, t_2, z_2)\|_q \leq L\left(|t_1 - t_2| + |||z_1 - z_2|||_{A(q)}\right) \tag{3.3.23}$$

for all (t_1, z_1), $(t_2, z_2) \in U$. *The constant* L *in general depends on* q *but it can be chosen independent of* q *on compact subsets of* $\mathcal{Q}$ *not containing* $q = 0$.

PROOF: Since U is bounded, there exist constants M_1, $M_2 > 0$ such that $|t| \leq M_1$ and $|||z|||_{A(q)} \leq M_2$ for all $(t,z) \in U$. Also, by Lemma 3.3.8, there exists a constant C_1 depending on q such that $\|z\|_{H^2} \leq C_1$ for all $(t,z) \in U$. More precisely, $C_1 = M_2\sqrt{\frac{6}{\gamma} + \frac{C_v}{k} + \frac{k}{C_v} + 1}$.

Let $(t_i, z_i) \in U$, $z_i = \begin{pmatrix} u_i \\ v_i \\ w_i \end{pmatrix}$, $i = 1, 2$. Then

$$
\begin{aligned}
\|h_q(t_1,z_1) - h_q(t_2,z_2)\|_q &= \|F(q,t_1,z_1) - F(q,t_2,z_2)\|_q \\
&= \left\| \begin{pmatrix} 0 \\ f_2(q,t_1,z_1) - f_2(q,t_2,z_2) \\ f_3(q,t_1,z_1) - f_3(q,t_2,z_2) \end{pmatrix} \right\|_q \\
&= \left\{ \rho \|f_2(q,t_1,z_1) - f_2(q,t_2,z_2)\|_{L_2}^2 + \frac{C_v}{k}\|f_3(q,t_1,z_1) - f_3(q,t_2,z_2)\|_{L_2}^2 \right\}^{\frac{1}{2}} \\
&\leq \sqrt{\rho}\|f_2(q,t_1,z_1) - f_2(q,t_2,z_2)\|_{L_2} + \sqrt{\frac{C_v}{k}}\|f_3(q,t_1,z_1) - f_3(q,t_2,z_2)\|_{L_2}.
\end{aligned}
$$

Therefore, $h_q(t,z)$ satisfies the condition (F) uniformly on U with $\nu = 1$ if both $f_2(q,t,z)$ and $f_3(q,t,z)$ are uniformly Lipschitz in t on U and uniformly Lipschitz in z on U with respect to the norm $|||\cdot|||_{A(q)}$.

Let us work first with $f_3(q,t,z)$. We have

$$
\begin{aligned}
f_3(q,t_1,z_1) - f_3(q,t_2,z_2) =& C_v^{-1}\left[g(\cdot,t_1) - g(\cdot,t_2)\right] \\
&+ 2\alpha_2 C_v^{-1}\left[(w_1 + L(\cdot,t_1))\, u_1' v_1' - (w_2 + L(\cdot,t_2))\, u_2' v_2'\right] \\
&+ \beta\rho C_v^{-1}(v_1'^2 - v_2'^2) \\
&- \cos(2\pi\cdot)\left[\theta_\Gamma'(t_1) - \theta_\Gamma'(t_2)\right] \\
&- 4k\pi^2 C_v^{-1}\left[L(\cdot,t_1) - L(\cdot,t_2)\right] \\
=& I_1 + I_2 + I_3 + I_4 + I_5. \qquad (3.3.24)
\end{aligned}
$$

Now, using hypothesis (H1) we obtain

$$\|I_1\|_{L_2} \leq C_v^{-1}\|K_g\|_{L_2}|t_1 - t_2|. \tag{3.3.25a}$$

Also,

$$\begin{aligned}\|I_3\|_{L_2} =& \beta\rho C_v^{-1}\|v_1'^2 - v_2'^2\|_{L_2} \\ \leq& \beta\rho C_v^{-1}\left(\|v_1'\|_{L_\infty} + \|v_2'\|_{L_\infty}\right)\|v_1' - v_2'\|_{L_2},\end{aligned}$$

and using the Sobolev imbedding $H^1(\Omega) \hookrightarrow L_\infty(\Omega)$ we have $\|v_1'\|_{L_\infty} \leq C_\infty\|v_1'\|_{H^1} \leq C_\infty\|v_1\|_{H^2} \leq C_\infty\|z_1\|_{H^2} \leq C_\infty C_1$, where C_∞ is a positive constant coming from the imbedding. Similarly, $\|v_2'\|_{L_\infty} \leq C_\infty C_1$ and, therefore

$$\|I_3\|_{L_2} \leq 2\beta\rho C_v^{-1}C_\infty C_1\|v_1' - v_2'\|_{L_2}.$$

Since $v_i \in H_0^1 \cap H^2(0,1)$, $i = 1, 2$, it follows that

$$\begin{aligned}\|I_3\|_{L_2} &\leq 2\beta\rho C_v^{-1}C_\infty C_1\|v_1'' - v_2''\|_{L_2} \\ &\leq 2\beta\rho C_v^{-1}C_\infty C_1\frac{1}{\sqrt{\gamma}}\|A(q)(z_1 - z_2)\|_q \\ &\leq \frac{2\beta\rho C_\infty C_1}{C_v\sqrt{\gamma}}|||z_1 - z_2|||_{A(q)}.\end{aligned} \tag{3.3.25b}$$

From hypothesis (H2) we get

$$\|I_4\|_{L_2} \leq |\theta_\Gamma'(t_1) - \theta_\Gamma'(t_2)| \leq K_{M_1}'|t_1 - t_2|, \tag{3.3.25c}$$

where K_{M_1}' is the (local) Lipschitz constant for θ_Γ' on the interval $[0, M_1]$.

Turning to I_5, we obtain the bound

$$\begin{aligned}\|I_5\|_{L_2} =& 4k\pi^2C_v^{-1}|\theta_\Gamma(t_1) - \theta_\Gamma(t_2)|\left(\int_\Omega cos^2(2\pi x)\,dx\right)^{1/2} \\ \leq& 4k\pi^2C_v^{-1}K_{M_1}|t_1 - t_2|,\end{aligned} \tag{3.3.25d}$$

where K_{M_1} is the (local) Lipschitz constant for θ_Γ corresponding to the interval $[0, M_1]$.

We now have to estimate $\|I_2\|_{L_2}$. We separate I_2 into two terms, as follows:

$$\begin{aligned} I_2 =& 2\alpha_2 C_v^{-1}\left[(w_1 u_1' v_1' - w_2 u_2' v_2') + (L(\cdot,t_1)u_1'v_1' - L(\cdot,t_2)u_2'v_2')\right] \\ =& 2\alpha_2 C_v^{-1}\left(I_{21} + I_{22}\right). \end{aligned} \tag{3.3.26}$$

Writing I_{21} in the form

$$I_{21} = w_1 v_1'(u_1' - u_2') + u_2' w_1(v_1' - v_2') + u_2' v_2'(w_1 - w_2),$$

we obtain the estimate

$$\begin{aligned} \|I_{21}\|_{L_2} \leq & \|w_1\|_{L_\infty}\|v_1'\|_{L_\infty}\|u_1' - u_2'\|_{L_2} + \|u_2'\|_{L_\infty}\|w_1\|_{L_\infty}\|v_1' - v_2'\|_{L_2} \\ & + \|u_2'\|_{L_\infty}\|v_2'\|_{L_\infty}\|w_1 - w_2\|_{L_2}. \end{aligned}$$

Again, the Sobolev imbedding $H^1(\Omega) \hookrightarrow L_\infty(\Omega)$ immediately yields $\|w_1\|_{L_\infty} \leq C_\infty C_1$, $\|v_1'\|_{L_\infty} \leq C_\infty C_1$, $\|v_2'\|_{L_\infty} \leq C_\infty C_1$, and $\|u_2'\|_{L_\infty} \leq C_\infty C_1$. Hence,

$$\|I_{21}\|_{L_2} \leq C_\infty^2 C_1^2\left(\|u_1' - u_2'\|_{L_2} + \|v_1' - v_2'\|_{L_2} + \|w_1 - w_2\|_{L_2}\right).$$

Since $u_i,\ v_i \in H_0^1 \cap H^2$, $i = 1, 2$, it follows that

$$\begin{aligned} \|I_{21}\|_{L_2} \leq & C_\infty^2 C_1^2\left(\|u_1'' - u_2''\|_{L_2} + \|v_1'' - v_2''\|_{L_2} + \|w_1 - w_2\|_{L_2}\right) \\ \leq & C_\infty^2 C_1^2\left(\frac{\|z_1 - z_2\|_q}{\sqrt{\gamma}} + \frac{\|A(q)(z_1 - z_2)\|_q}{\sqrt{\gamma}} + \frac{\|z_1 - z_2\|_q}{\sqrt{\frac{C_v}{k}}}\right) \\ \leq & C_\infty^2 C_1^2\left(\frac{1}{\sqrt{\gamma}} + \sqrt{\frac{k}{C_v}}\right)|||z_1 - z_2|||_{A(q)}. \end{aligned} \tag{3.3.27a}$$

Similarly, we have

$$\begin{aligned}
I_{22} =& L(\cdot,t_1)u_1'v_1' - L(\cdot,t_2)u_2'v_2' \\
=& L(\cdot,t_1)v_1'(u_1'-u_2') + L(\cdot,t_1)u_2'(v_1'-v_2') + u_2'v_2'\left(L(\cdot,t_1) - L(\cdot,t_2)\right) \\
=& \theta_\Gamma(t_1)cos(2\pi\cdot)v_1'(u_1'-u_2') + \theta_\Gamma(t_1)cos(2\pi\cdot)u_2'(v_1'-v_2') \\
& + u_2'v_2'cos(2\pi\cdot)\left(\theta_\Gamma(t_1) - \theta_\Gamma(t_2)\right).
\end{aligned}$$

Using hypothesis (H2) and the imbedding $H^1(\Omega) \hookrightarrow L_\infty(\Omega)$, it follows that

$$\begin{aligned}
\|I_{22}\|_{L_2} \leq& \|\theta_\Gamma\|_{L_\infty(0,M_1)} C_\infty C_1 \|u_1'-u_2'\|_{L_2} + \|\theta_\Gamma\|_{L_\infty(0,M_1)} C_\infty C_1 \|v_1'-v_2'\|_{L_2} \\
& + C_\infty^2 C_1^2 K_{M_1}|t_1-t_2| \\
\leq& \|\theta_\Gamma\|_{L_\infty(0,M_1)} C_\infty C_1 \left(\|u_1''-u_2''\|_{L_2} + \|v_1''-v_2''\|_{L_2}\right) \\
& + C_\infty^2 C_1^2 K_{M_1}|t_1-t_2| \\
\leq& \|\theta_\Gamma\|_{L_\infty(0,M_1)} C_\infty C_1 \left(\frac{\|z_1-z_2\|_q}{\sqrt{\gamma}} + \frac{\|A(q)(z_1-z_2)\|_q}{\sqrt{\gamma}}\right) \\
& + C_\infty^2 C_1^2 K_{M_1}|t_1-t_2| \\
\leq& \frac{C_\infty C_1 \|\theta_\Gamma\|_{L_\infty(0,M_1)}}{\sqrt{\gamma}} |||z_1-z_2|||_{A(q)} + C_\infty^2 C_1^2 K_{M_1}|t_1-t_2|.
\end{aligned} \tag{3.3.27b}$$

From (3.3.26) and (3.3.27a,b) we get

$$\|I_2\|_{L_2} \leq C_2|t_1-t_2| + C_3|||z_1-z_2|||_{A(q)}, \tag{3.3.28}$$

where

$$C_2 = \frac{2\alpha_2 C_\infty^2 C_1^2 K_{M_1}}{C_v},$$

and

$$C_3 = \frac{2\alpha_2}{C_v}\left\{C_\infty^2 C_1^2\left(\frac{1}{\sqrt{\gamma}} + \sqrt{\frac{k}{C_v}}\right) + \frac{C_\infty C_1\|\theta_\Gamma\|_{L_\infty(0,M_1)}}{\sqrt{\gamma}}\right\}.$$

Substituting (3.3.25a-d) and (3.3.28) into (3.3.24) we finally obtain the estimate

$$\|f_3(q,t_1,z_1)-f_3(q,t_2,z_2)\|_{L_2}\le C_4|t_1-t_2|+C_5|||z_1-z_2|||_{A(q)}, \tag{3.3.29}$$

where

$$C_4=\frac{\|K_g\|_{L_2}}{C_v}+C_2+K'_{M_1}+\frac{4k\pi^2K_{M_1}}{C_v},$$

and

$$C_5=C_3+\frac{2\beta\rho C_\infty C_1}{C_v\sqrt{\gamma}}.$$

We turn now to $f_2(q,t,z)$. We have that

$$\begin{aligned}
f_2(q,t_1,z_1)-f_2(q,t_2,z_2)&=\rho^{-1}\left[f(\cdot,t_1)-f(\cdot,t_2)\right]\\
&+\rho^{-1}\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial\epsilon}\Psi\left(u_1',u_1'',w_1+L(\cdot,t_1)\right)-\frac{\partial}{\partial\epsilon}\Psi\left(u_2',u_2'',w_2+L(\cdot,t_2)\right)\right\}\\
&=\rho^{-1}\left[f(\cdot,t_1)-f(\cdot,t_2)\right]\\
&+\rho^{-1}\frac{\partial}{\partial x}\{2\alpha_2\left[\left(w_1+L(\cdot,t_1)-\theta_1\right)u_1'-\left(w_2+L(\cdot,t_2)-\theta_1\right)u_2'\right]\\
&-4\alpha_4\left(u_1'^3-u_2'^3\right)+6\alpha_6\left(u_1'^5-u_2'^5\right)\}\\
&=\rho^{-1}\left[f(\cdot,t_1)-f(\cdot,t_2)\right]+2\alpha_2\rho^{-1}\left(w_1'u_1'-w_2'u_2'\right)\\
&+2\alpha_2\rho^{-1}\left(\frac{\partial}{\partial x}L(\cdot,t_1)u_1'-\frac{\partial}{\partial x}L(\cdot,t_2)u_2'\right)\\
&+2\alpha_2\rho^{-1}\left(w_1u_1''-w_2u_2''\right)+2\alpha_2\rho^{-1}\left[L(\cdot,t_1)u_1''-L(\cdot,t_2)u_2''\right]\\
&-2\alpha_2\rho^{-1}\theta_1\left(u_1''-u_2''\right)-12\alpha_4\rho^{-1}\left(u_1'^2u_1''-u_2'^2u_2''\right)\\
&+30\alpha_6\rho^{-1}\left(u_1'^4u_1''-u_2'^4u_2''\right)\\
&=\sum_{i=1}^{8}T_i,
\end{aligned} \tag{3.3.30}$$

where T_i, $i = 1, 2, \ldots, 8$, represent the above terms in their respective orders. We shall obtain estimates on each one of the T_i's.

Using the hypothesis (H1), it follows that

$$\|T_1\|_{L_2} \leq \rho^{-1} \|K_f\|_{L_2} |t_1 - t_2|. \tag{3.3.31a}$$

To estimate T_2 we write

$$\begin{aligned} T_2 &= 2\alpha_2 \rho^{-1} (w_1' u_1' - w_2' u_2') \\ &= 2\alpha_2 \rho^{-1} [w_1' (u_1' - u_2') + u_2' (w_1' - w_2')], \end{aligned}$$

which implies

$$\|T_2\|_{L_2} \leq 2\alpha_2 \rho^{-1} (\|w_1'\|_{L_\infty} \|u_1' - u_2'\|_{L_2} + \|u_2'\|_{L_\infty} \|w_1' - w_2'\|_{L_2}).$$

By virtue of the imbedding $H^1(\Omega) \hookrightarrow L_\infty(\Omega)$, and since $u_i \in H_0^1 \cap H^2$, $w_i \in H^2$, $w_i'(0) = 0$, $i = 1, 2$, it follows that

$$\begin{aligned} \|T_2\|_{L_2} &\leq 2\alpha_2 \rho^{-1} C_1 C_\infty (\|u_1'' - u_2''\|_{L_2} + \|w_1'' - w_2''\|_{L_2}) \\ &\leq 2\alpha_2 \rho^{-1} C_1 C_\infty \left(\frac{\|z_1 - z_2\|_q}{\sqrt{\gamma}} + \frac{\|A(q)(z_1 - z_2)\|_q}{\sqrt{\frac{k}{C_v}}} \right) \\ &\leq \frac{2\alpha_2 C_1 C_\infty}{\rho} \left(\frac{1}{\sqrt{\gamma}} + \sqrt{\frac{C_v}{k}} \right) |||z_1 - z_2|||_{A(q)}. \end{aligned} \tag{3.3.31b}$$

We have that

$$\begin{aligned} T_3 &= 2\alpha_2 \rho^{-1} \left[\frac{\partial}{\partial x} L(\cdot, t_1) u_1' - \frac{\partial}{\partial x} L(\cdot, t_2) u_2' \right] \\ &= 2\alpha_2 \rho^{-1} \left\{ \frac{\partial}{\partial x} L(\cdot, t_1)(u_1' - u_2') + u_2' \left(\frac{\partial}{\partial x} L(\cdot, t_1) - \frac{\partial}{\partial x} L(\cdot, t_2) \right) \right\} \\ &= 2\alpha_2 \rho^{-1} \{ -2\pi \theta_\Gamma(t_1) sin(2\pi \cdot) (u_1' - u_2') - 2\pi sin(2\pi \cdot) u_2' (\theta_\Gamma(t_1) - \theta_\Gamma(t_2)) \}. \end{aligned}$$

Applying Hypothesis (H2), the Sobolev imbedding $H^1(\Omega) \hookrightarrow L_\infty(\Omega)$ and the fact that $\|u_1' - u_2'\|_{L_2} \le \|u_1'' - u_2''\|_{L_2}$, we obtain the estimate

$$\|T_3\|_{L_2} \le 4\alpha_2\rho^{-1}\pi\|\theta_\Gamma\|_{L_\infty(0,M_1)}\frac{|||z_1 - z_2|||_{A(q)}}{\sqrt{\gamma}}$$
$$+ 4\alpha_2\rho^{-1}\pi C_1 C_\infty K_{M_1}|t_1 - t_2|. \tag{3.3.31c}$$

To obtain an estimate on $\|T_4\|_{L_2}$ we observe that

$$\begin{aligned}
\|T_4\|_{L_2} &= 2\alpha_2\rho^{-1}\|w_1u_1'' - w_2u_2''\|_{L_2} \\
&= 2\alpha_2\rho^{-1}\,\|w_1(u_1'' - u_2'') + u_2''(w_1 - w_2)\|_{L_2} \\
&\le 2\alpha_2\rho^{-1}\left(\|w_1\|_{L_\infty}\|u_1'' - u_2''\|_{L_2} + \|u_2''\|_{L_2}\|w_1 - w_2\|_{L_\infty}\right) \\
&\le 2\alpha_2\rho^{-1}\left(C_1C_\infty\frac{\|A(q)(z_1 - z_2)\|_q}{\sqrt{\gamma}} + \frac{\|z_2\|_q}{\sqrt{\gamma}}C_\infty\|w_1 - w_2\|_{H^1}\right) \\
&\le \frac{2\alpha_2C_\infty}{\rho\sqrt{\gamma}}\left(C_1|||z_1 - z_2|||_{A(q)} + |||z_2|||_{A(q)}\|w_1 - w_2\|_{H^1}\right) \\
&\le \frac{2\alpha_2C_\infty}{\rho\sqrt{\gamma}}\left\{C_1|||z_1 - z_2|||_{A(q)} + M_2\left(\|w_1 - w_2\|_{L_2} + \|w_1' - w_2'\|_{L_2}\right)\right\} \\
&\le \frac{2\alpha_2C_\infty}{\rho\sqrt{\gamma}}\left(C_1|||z_1 - z_2|||_{A(q)} + M_2\frac{\|z_1 - z_2\|_q}{\sqrt{\frac{C_v}{k}}} + M_2\|w_1'' - w_2''\|_{L_2}\right) \\
&\le \frac{2\alpha_2C_\infty}{\rho\sqrt{\gamma}}\left(C_1|||z_1 - z_2|||_{A(q)} + M_2\sqrt{\frac{k}{C_v}}\|z_1 - z_2\|_q + M_2\frac{\|A(q)(z_1 - z_2)\|_q}{\sqrt{\frac{k}{C_v}}}\right) \\
&\le \frac{2\alpha_2C_\infty}{\rho\sqrt{\gamma}}\left(C_1 + M_2\sqrt{\frac{k}{C_v}} + M_2\sqrt{\frac{C_v}{k}}\right)|||z_1 - z_2|||_{A(q)} \\
&= \frac{2\alpha_2C_\infty}{\rho\sqrt{\gamma}}\left(C_1 + \frac{M_2(k + C_v)}{\sqrt{kC_v}}\right)|||z_1 - z_2|||_{A(q)}.
\end{aligned} \tag{3.3.31d}$$

By writting T_5 in the form

$$T_5 = 2\alpha_2\rho^{-1}\left[L(\cdot,t_1)u_1'' - L(\cdot,t_2)u_2''\right]$$

$$= 2\alpha_2\rho^{-1}\left[L(\cdot,t_1)(u_1''-u_2'') + u_2''\left(L(\cdot,t_1) - L(\cdot,t_2)\right)\right],$$

we immediately obtain the estimate

$$\begin{aligned}
\|T_5\|_{L_2} &\le 2\alpha_2\rho^{-1}\left(\|\theta_\Gamma\|_{L_\infty(0,M_1)}\frac{\|z_1-z_2\|_q}{\sqrt{\gamma}} + \|u_2''\|_{L_2}K_{M_1}|t_1-t_2|\right)\\
&\le 2\alpha_2\rho^{-1}\left(\|\theta_\Gamma\|_{L_\infty(0,M_1)}\frac{|||z_1-z_2|||_{A(q)}}{\sqrt{\gamma}} + \frac{\|z_2\|_q}{\sqrt{\gamma}}K_{M_1}|t_1-t_2|\right)\\
&\le \frac{2\,\alpha_2\|\theta_\Gamma\|_{L_\infty(0,M_1)}}{\rho\sqrt{\gamma}}|||z_1-z_2|||_{A(q)} + \frac{2\,\alpha_2 M_2 K_{M_1}}{\rho\sqrt{\gamma}}|t_1-t_2|. \qquad (3.3.31e)
\end{aligned}$$

The term $\|T_6\|_{L_2}$ satisfies the inequality

$$\begin{aligned}
\|T_6\|_{L_2} &= \|-2\alpha_2\rho^{-1}\theta_1(u_1''-u_2'')\|_{L_2}\\
&\le \frac{2\alpha_2\theta_1}{\rho}\frac{\|z_1-z_2\|_q}{\sqrt{\gamma}}\\
&\le \frac{2\alpha_2\theta_1}{\rho\sqrt{\gamma}}|||z_1-z_2|||_{A(q)}. \qquad (3.3.31f)
\end{aligned}$$

Observing that

$$T_7 = -12\alpha_4\rho^{-1}\left(u_1'^2u_1'' - u_2'^2u_2''\right)$$

$$= -12\alpha_4\rho^{-1}\left[u_1'^2\left(u_1''-u_2''\right) + u_2''(u_1'+u_2')(u_1'-u_2')\right],$$

it follows that

$$\|T_7\|_{L_2} \le 12\alpha_4\rho^{-1}\left(\|u_1'\|_{L_\infty}^2\|u_1''-u_2''\|_{L_2} + \|u_2''\|_{L_2}\|u_1'+u_2'\|_{L_\infty}\|u_1'-u_2'\|_{L_\infty}\right)$$

$$\le 12\alpha_4\rho^{-1}\left(C_\infty^2C_1^2\|u_1''-u_2''\|_{L_2} + C_1(2C_1C_\infty)C_\infty\|u_1-u_2\|_{H^1}\right).$$

Since $\|u\|_{H^1}^2 = \|u\|_{L_2}^2 + \|u'\|_{L_2}^2 \le 2\|u''\|_{L_2}^2$ and $u \in H_0^1 \cap H^2$, it follows that

$$\begin{aligned}
\|T_7\|_{L_2} &\le \frac{12\alpha_4 C_1^2 C_\infty^2}{\rho}\left(\|u_1'' - u_2''\|_{L_2} + 2\sqrt{2}\,\|u_1'' - u_2''\|_{L_2}\right) \\
&= \frac{12\alpha_4 C_1^2 C_\infty^2}{\rho}\left(1 + 2\sqrt{2}\right)\|u_1'' - u_2''\|_{L_2} \\
&\le \frac{12\alpha_4 C_1^2 C_\infty^2}{\rho\sqrt{\gamma}}\left(1 + 2\sqrt{2}\right)|||z_1 - z_2|||_{A(q)}. \qquad (3.3.31g)
\end{aligned}$$

Similarly, we use the indentity

$$\begin{aligned}
T_8 &= 30\alpha_6\rho^{-1}\left(u_1'^4 u_1'' - u_2'^4 u_2''\right) \\
&= 30\alpha_6\rho^{-1}\left[u_1'^4\left(u_1'' - u_2''\right) - u_2''\left(u_1'^4 - u_2'^4\right)\right] \\
&= 30\alpha_6\rho^{-1}\left[u_1'^4\left(u_1'' - u_2''\right) - u_2''\left(u_1'^3 + u_1'^2 u_2' + u_1' u_2'^2 + u_2'^3\right)\left(u_1' - u_2'\right)\right]
\end{aligned}$$

to obtain the estimate

$$\begin{aligned}
\|T_8\|_{L_2} &\le 30\alpha_6\rho^{-1}\left\{\|u_1'\|_{L_\infty}^4\|u_1'' - u_2''\|_{L_2}\right. \\
&\qquad \left. + \|u_2''\|_{L_2}\|u_1'^3 + u_1'^2 u_2' + u_1' u_2'^2 + u_2'^3\|_{L_\infty}\|u_1' - u_2'\|_{L_\infty}\right\} \\
&\le 30\alpha_6\rho^{-1}\left[C_1^4 C_\infty^4\|u_1'' - u_2''\|_{L_2} + C_1\left(4C_1^3 C_\infty^3\right)C_\infty\|u_1 - u_2\|_{H^1}\right] \\
&\le \frac{30\alpha_6 C_1^4 C_\infty^4}{\rho}\left(\|u_1'' - u_2''\|_{L_2} + 4\|u_1 - u_2\|_{H^1}\right) \\
&\le \frac{30\alpha_6 C_1^4 C_\infty^4}{\rho}\left(\|u_1'' - u_2''\|_{L_2} + 4\sqrt{2}\,\|u_1'' - u_2''\|_{L_2}\right) \\
&\le \frac{30\alpha_6 C_1^4 C_\infty^4}{\rho\sqrt{\gamma}}\left(1 + 4\sqrt{2}\right)|||z_1 - z_2|||_{A(q)}. \qquad (3.3.31h)
\end{aligned}$$

Substituting (3.3.31a-h) into (3.3.30) we finally obtain

$$\begin{aligned}
\|f_3(q, t_1, z_1) - f_3(q, t_2, z_2\|_{L_2} &= \left\|\sum_{i=1}^{8} T_i\right\|_{L_2} \\
&\le C_6|t_1 - t_2| + C_7|||z_1 - z_2|||_{A(q)}, \qquad (3.3.32)
\end{aligned}$$

where

$$C_6 = \rho^{-1}\left(\|K_f\|_{L_2} + 4\alpha_2\pi C_1 C_\infty K_{M_1} + \frac{2\alpha_2 M_2 K_{M_1}}{\sqrt{\gamma}}\right),$$

and

$$C_7 = \frac{1}{\rho\sqrt{\gamma}}\left\{4\alpha_2 C_1 C_\infty + 2\alpha_2\left[(1+2\pi)\|\theta_\Gamma\|_{L_\infty(0,M_1)} + \theta_1 + \frac{C_\infty M_2(k+C_v)}{\sqrt{kC_v}}\right]\right.$$
$$\left. +12\alpha_4 C_1^2 C_\infty^2(1+2\sqrt{2}) + 30\alpha_6 C_1^4 C_\infty^4(1+4\sqrt{2})\right\} + \frac{2\alpha_2 C_1 C_\infty}{\rho}\sqrt{\frac{C_v}{k}}.$$

An analysis of the constants C_i, $i = 1, 2, \ldots, 7$, reveals that they are all bounded independently of q on subsets of $\mathcal{Q}$ of the form

$$\left\{(\rho, C_v, \beta, \alpha_2, \alpha_4, \alpha_6, \theta_1, \gamma) \in \mathcal{Q} \;\middle|\; \begin{array}{c} 0 \le a_1 \le \rho \le b_1 < \infty, \quad 0 < a_2 \le C_v \le b_2 < \infty, \\ 0 < a_3 \le \gamma, \quad \beta \le b_3 < \infty, \quad \alpha_2 \le b_4 < \infty, \\ \alpha_4 \le b_5 < \infty, \quad \alpha_6 \le b_6 < \infty, \quad \theta_1 \le b_7 < \infty \end{array}\right\},$$

i.e., on any subset of $\mathcal{Q}$ where ρ, C_v, β, α_2, α_4 α_6, θ_1 are bounded above and ρ, C_v and γ are bounded away from zero, and therefore on any compact subset of $\mathcal{Q}$ not containing $q = 0$. This completes the proof of Theorem 3.3.10. ∎

We are now ready to state and prove the main local existence and uniqueness theorem for the solutions of the initial value problem (3.3.7).

Theorem 3.3.11 (local existence and uniqueness). *Let $q \in \mathcal{Q}$ and $A(q) : dom\,(A(q)) \subset Z_q \to Z_q$ as defined in (3.3.8)-(3.3.9). Then, for any initial data $z_0 \in dom\,(A(q))$, there exists $t_1 = t_1(z_0)$ such that the initial value problem (3.3.7) has a unique (strong) solution $z(t;q) \in C([0,t_1) : Z_q) \cap C^1((0,t_1) : Z_q)$.*

Proof: : Let $z_0 \in dom\,(A(q))$ and let U be a bounded open neighborhood of the origin in $\mathbb{R}_0^+ \times Z_{A(q)}$ containing the point $(0, z_0)$. By Theorem 3.3.10, the function

$h_q : U \to Z_q$ defined by $h_q(t,z) \doteq F(q,t,z)$ satisfies the condition (F) on U. The local existence and uniqueness then follows from the fact that $A(q)$ is the infinitesimal generator of an analytic semigroup $T(t;q)$ on Z_q with $0 \in \rho(A(q))$ (Theorem 3.3.7) and Theorem 3.3.9. ∎

3.4 Continuous Dependence on the Parameter q

In this section we will show that the semigroup $T(t;q)$ generated by $A(q)$ on Z_q depends continuously on q, i.e., if $\{q^N\}_{N=1}^{\infty} \subset \mathcal{Q}$ is a sequence of admissible parameters and $q^N \to q \in \mathcal{Q}$ as $N \to \infty$, then the semigroup $T(t;q^N)$ generated by $A(q^N)$ on Z_{q^N} "approaches" $T(t;q)$ in some sense.

The following is a version of the well known Trotter-Kato theorem due to T.G. Kurtz. Its proof can be found in [**9**], p. 40-43.

THEOREM 3.4.1 (TROTTER-KATO). *Let A be the infinitesimal generator of a strongly continuous semigroup $T(t)$ on a Banach space X with norm $\|\cdot\|$. Let X^N, $N = 1, 2 \ldots,$ be a sequence of Banach spaces with norms $\|\cdot\|_N$, and A^N the infinitesimal generator of a C_0-semigroup $T^N(t)$ on X^N. Let $\Pi^N \in \mathcal{L}(X, X^N)$, $N = 1, 2, \ldots,$ be a sequence of bounded linear operators satisfying*

$$\lim_{N\to\infty} \|\Pi^N x\|_N = \|x\| \qquad \textit{for all } x \in X \tag{J1}$$

Assume also that the following conditions hold

(A) *(Stability) There exist constants $M_0 \geq 1$, $\omega_0 \in \mathbb{R}$ independent of N such that*

$\|T^N(t)\|_{\mathcal{L}(X^N)} \leq M_0 e^{\omega_0 t}$ *for all* $t \geq 0$ *and* $N = 1, 2, \ldots$.

(B) *(Consistency) There exists a set* $D \subset dom(A)$ *and a complex number* λ_0 *with* $Re\, \lambda_0 > \omega_0$ *such that*

(i) $\Pi^N D \subset dom(A^N)$ *for each* $N = 1, 2, \ldots,$

(ii) $\overline{(\lambda_0 - A)D} = X$, *and*

(iii) $\|A^N \Pi^N y - \Pi^N A y\|_N \to 0$ *as* $N \to \infty$ *for each* $y \in D$.

Then for each $x \in X$

$$\|T^N(t)\Pi^N x - \Pi^N T(t)x\|_N \to 0 \qquad \text{as } N \to \infty,$$

and the convergence is uniform on compact t-intervals.

Remarks:

(1) By using the uniform boundedness principle with minor modifications to take into account the varying spaces, it can be proved that condition (J1) implies $\|\Pi^N\|_{\mathcal{L}(X,X^N)} \leq C$ for some constant C independent of N.

(2) Condition (B) implies that $\lambda_0 \in \rho(A) \bigcap_{N=1}^{\infty} \rho(A^N)$ and

$$\|R(\lambda; A^N)\Pi^N x - \Pi^N R(\lambda; A)x\|_N \to 0$$

for each $x \in X$. This is often referred to as the stability condition (see [**9**], p. 43).

THEOREM 3.4.2 (CONTINUITY OF $T(t;q)$ WITH RESPECT TO q).

Let $\{q^N = (\rho^N, C_v^N, \beta^N, \alpha_2^N, \alpha_4^N, \alpha_6^N, \theta_1^N, \gamma^N)\}_{N=1}^{\infty} \subset \mathcal{Q}$ be a sequence of admissible parameters such that $q^N \to q = (\rho, C_v, \beta, \alpha_2, \alpha_4, \alpha_6, \theta_1, \gamma) \in \mathcal{Q}$ as $N \to \infty$. Let $T(t;q^N)$ and $T(t;q)$ be the analytic semigroups generated by $A(q^N)$ and $A(q)$ respectively. Then for each $z \in Z_q$ and $t \geq 0$,

$$\lim_{N\to\infty} \|T(t;q^N)z - T(t;q)z\|_{q^N} = 0$$

and

$$\lim_{N\to\infty} \|T(t;q^N)z - T(t;q)z\|_{q^*} = 0$$

for any $q^* \in \mathcal{Q}$. Moreover, the convergence is uniform on compact t-intervals.

PROOF: Define $\Pi^N : Z_q \to Z_{q^N}$ to be the identity operator and let $z = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in Z_q$. It follows that

$$\begin{aligned} \|\Pi^N z\|_{q^N}^2 &= \gamma^N \|u''\|_{L_2(\Omega)}^2 + \rho^N \|v\|_{L_2(\Omega)}^2 + \frac{C_v^N}{k} \|w\|_{L_2(\Omega)}^2 \\ &\overset{N\to\infty}{\longrightarrow} \gamma \|u''\|_{L_2(\Omega)}^2 + \rho \|v\|_{L_2(\Omega)}^2 + \frac{C_v}{k} \|w\|_{L_2(\Omega)}^2 \\ &= \|z\|_q^2 \end{aligned}$$

Thus, condition (J1) of the Trotter-Kato theorem is satisfied. By Theorem 3.3.7, the semigroups $T(t;q^N)$ satisfy $\|T(t;q^N)\|_{\mathcal{L}(Z_{q^N})} \leq e^{-w(q^N)t}$. For the stability condition (A) of Theorem 3.4.1 we need this bound to be independent of N. This is achieved by choosing $M_0 = 1$ and $\omega_0 = 0$. Now, letting $D = dom\,(A(q)) = dom\,(A(q^N))$, condition B-(i) is trivial since Π^N is the identity operator and B-(ii) holds for any choice of $\lambda_0 > 0$. Finally, for each $z = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in D$ we have

$$\begin{aligned}
&\left\|A(q^N)\Pi^N z - \Pi^N A(q)z\right\|_{q^N}^2 = \left\|A(q^N)z - A(q)z\right\|_{q^N}^2 \\
&= \left\| \begin{pmatrix} v \\ \beta^N v'' - \frac{\gamma^N}{\rho^N}u'''' \\ \frac{k}{C_v^N}w'' \end{pmatrix} - \begin{pmatrix} v \\ \beta v'' - \frac{\gamma}{\rho}u'''' \\ \frac{k}{C_v}w'' \end{pmatrix} \right\|_{q^N}^2 \\
&= \rho^N \left\|(\beta^N - \beta)v'' - \left(\frac{\gamma^N}{\rho^N} - \frac{\gamma}{\rho}\right)u''''\right\|_{L_2(\Omega)}^2 + \frac{C_v^N}{k}\left\|k\left(\frac{1}{C_v^N} - \frac{1}{C_v}\right)w''\right\|_{L_2(\Omega)}^2 \\
&\leq 2\rho^N \left[(\beta^N - \beta)^2 \|v''\|_{L_2(\Omega)}^2 + \left(\frac{\gamma^N}{\rho^N} - \frac{\gamma}{\rho}\right)^2 \|u''''\|_{L_2(\Omega)}^2\right] \\
&\quad + kC_v^N\left(\frac{1}{C_v^N} - \frac{1}{C_v}\right)^2 \|w''\|_{L_2(\Omega)}^2 \\
&\to 0 \qquad \text{as } N \to \infty,
\end{aligned}$$

since $\beta^N \to \beta$, $\gamma^N \to \gamma$, $\rho^N \to \rho \neq 0$ and $C_v^N \to C_v \neq 0$ (note that $\rho \neq 0$ and $C_v \neq 0$ because $q \in \mathcal{Q}$). Thus, condition B-(iii) of the Trotter-Kato theorem 3.4.1 is also satisfied. Hence, for each $z \in Z_q$ and $t \geq 0$

$$\lim_{N\to\infty} \left\|T(t;q^N)z - T(t;q)z\right\|_{q^N} = 0,$$

and the limit is uniform in t on compact intervals.

The second part of the theorem follows immediately by noting that for any $q^* \in \mathcal{Q}$, $z \in Z_q$ and $t \geq 0$ one has

$$\left\|T(t;q^N)z - T(t;q)z\right\|_{q^*}^2 \leq max\left(\frac{\gamma^*}{\gamma^N}, \frac{\rho^*}{\rho^N}, \frac{C_v^*}{C_v^N}\right)\left\|T(t;q^N)z - T(t;q)z\right\|_{q^N}^2,$$

$\gamma^N \to \gamma \neq 0$, $\rho^N \to \rho \neq 0$ and $C_v^N \to C_v \neq 0$ and therefore

$$\sup_{N=1,2,\ldots} max\left(\frac{\gamma^*}{\gamma^N}, \frac{\rho^*}{\rho^N}, \frac{C_v^*}{C_v^N}\right) < \infty.$$

Hence,

$$\left\|T(t;q^N)z - T(t;q)z\right\|_{q^*} \to 0 \qquad \text{as } N \to \infty,$$

uniformly on compact t-intervals. ∎

We note here that the continuity of the semigroup $T(t;q)$ with respect to the parameter q does not necessarily imply the continuity of the solutions $z(t;q)$ with respect to q. A stronger result than the one in Theorem 3.3.10 is needed to achieve this goal. In fact if $\delta < 1$ (i.e., if $F(q,t,z)$ satisfies the condition (F) with δ strictly less than 1), then one could use Gronwall's inequality together with the bound $\|A^\delta(q)T(t;q)z\|_q \leq Mt^{-\delta}$ (see [**113**], Theorem 2.6.13) to conclude that $\|z(t;q) - z(t;\hat{q})\|_{A^\delta(q)} \to 0$ as $q \to \hat{q}$. Although we believe that Theorem 3.3.10 remains true for $\delta = \frac{1}{2}$, we have not been able to prove this result. This is a very important problem and is a necessary step for the parameter identification problem. We plan to devote further efforts to it in the near future.

3.5 An Alternative Approach to the Stress-Strain Laws

Parametrization of the Transition Curves - The Dynamics - The Phase Fractions. The hysteresis in SMA's is characterized by the existence of local or global memories. This can be easily observed by the fact that a particular load can produce different deformations depending on the initial state which, in terms, depends on the past history (see Figure 1.1.1). For this reason, even in their most general forms, none of the previously mentioned potentials completely captures the characteristics of SMA's. In fact, it is not difficult to observe that the same stress-strain relation ($\sigma_\theta(\epsilon) = \frac{\partial}{\partial\epsilon}\Psi(\epsilon, \epsilon_x, \theta)$) is used independently of whether the body is under loading or unloading. For example, with the potential (3.1.9), in the intermediate temperature range, the stress-strain relation in the non-monotone region goes diagonally across the hysteresis loop which is observed experimentally (see Figure 3.5.2(b) and compare it with Figure 1.1.1(b)). One way of overcoming this problem for the "loading" case would be to use the relative maximum (minimum if the load is compressive) of $\frac{\partial}{\partial\epsilon}\Psi(\cdot, \theta)$ to estimate the elastic modulus (yield stress) and from there, considering strain hardening, pass to the next elastic branch. The studies referenced in Sections 3.1 and 3.2 did not consider this phenomenon. Moreover, even if $\frac{\partial}{\partial\epsilon}\Psi(\epsilon, \theta)$ is modified as we have noted above, nonuniform phase structures are still not allowed. Nonuniform configurations are possible and can be, for instance, the result of unloading after an incomplete austenite $\rightarrow$ martensite phase transition due to strain hardening, or the result of reloading after partial recovery. In fact, isothermal uniaxial stretching experiments under controlled deformation performed in CuZnAl alloys ([**103**]), show that in any of those two cases, states within the hysteresis loops are achieved (see Figure 3.5.1).

In the low temperature range and for small strains, the Landau-Devonshire potential does not provide a good description of the experimentally observed stress-strain relations. In fact, in this case, the model predicts a deformation opposite to the character of the load (see Figure 3.5.2(a)). This problem with the Landau-Devonshire potential is reflected in some of the numerical results that we present in chapter IV.

In this section we use a completely different approach to approximate this type of hysteretic dynamics. We introduce new parameters in the stress-strain relations in order to rectify the problems outlined above. In particular, these parameters provide a mechanism to account for local memories and phase fractions.

For a fixed temperature θ, we assume that the stress-strain curves are completely characterized by the eight values $\epsilon_1 < \epsilon_2 < \epsilon_3 < \epsilon_4$, $\sigma_1 \leq \sigma_2 < \sigma_3 \leq \sigma_4$, as shown in Figure 3.5.3. These eight critical values will strongly depend on the temperature θ. We will confine ourselves to the tensile case since, to our knowledge, no experimental data is available for compressive loads. Therefore we assume $\epsilon_i \geq 0$, $\sigma_i \geq 0$, for $1 \leq i \leq 4$. Note that σ_3 estimates the modulus of elasticity in tension corresponding to the temperature θ.

Let $\mathcal{U} = \{\Gamma_\beta^U(\cdot)\}_{0 \leq \beta \leq 1}$ be the family of curves parametrized by β shown in Figure 3.5.4 (the index "U" stands for "unloading"). The curve $\Gamma_\beta^U(\epsilon)$ is defined explicitly below.

Let

$$
\begin{aligned}
\epsilon_\beta^{0,U} &\doteq \beta\epsilon_1 + (1-\beta)\epsilon_2 \qquad & \sigma_\beta^{0,U} &\doteq \beta\sigma_1 + (1-\beta)\sigma_3 \\
\epsilon_\beta^{1,U} &\doteq \beta\epsilon_3 + (1-\beta)\epsilon_2 \qquad & \sigma_\beta^{1,U} &\doteq \beta\sigma_2 + (1-\beta)\sigma_3 \\
\epsilon_\beta^{2,U} &\doteq \beta\epsilon_4 + (1-\beta)\epsilon_2 \qquad & \sigma_\beta^{2,U} &\doteq \beta\sigma_4 + (1-\beta)\sigma_3
\end{aligned}
\tag{3.5.1}
$$

and define $\Gamma_\beta^U(\epsilon)$ by

$$\Gamma_\beta^U(\epsilon) \doteq \begin{cases} \frac{\sigma_1}{\epsilon_1}\epsilon, & \text{for } 0 \le \epsilon < \epsilon_1 \\ \sigma_1 + \frac{\sigma_\beta^{0,U}-\sigma_1}{\epsilon_\beta^{0,U}-\epsilon_1}(\epsilon-\epsilon_1), & \text{for } \epsilon_1 \le \epsilon < \epsilon_\beta^{0,U} \\ \sigma_\beta^{0,U} + \frac{\sigma_\beta^{1,U}-\sigma_\beta^{0,U}}{\epsilon_\beta^{1,U}-\epsilon_\beta^{0,U}}(\epsilon-\epsilon_\beta^{0,U}), & \text{for } \epsilon_\beta^{0,U} \le \epsilon < \epsilon_\beta^{1,U} \\ \sigma_\beta^{1,U} + \frac{\sigma_\beta^{2,U}-\sigma_\beta^{1,U}}{\epsilon_\beta^{2,U}-\epsilon_\beta^{1,U}}(\epsilon-\epsilon_\beta^{1,U}), & \text{for } \epsilon_\beta^{1,U} \le \epsilon < \epsilon_\beta^{2,U} \\ \sigma_\beta^{2,U} + \frac{\sigma_4-\sigma_\beta^{2,U}}{\epsilon_4-\epsilon_\beta^{2,U}}(\epsilon-\epsilon_\beta^{2,U}), & \text{for } \epsilon_\beta^{2,U} \le \epsilon < \epsilon_4 \\ \sigma_2 + \frac{\sigma_4-\sigma_2}{\epsilon_4-\epsilon_3}(\epsilon-\epsilon_3) & \text{for } \epsilon \ge \epsilon_4\ . \end{cases} \tag{3.5.2}$$

Similarly, let $\mathcal{L} = \{\Gamma_\beta^L(\cdot)\}_{0\le\beta\le1}$ be the family of curves parametrized by β shown in Figure 3.5.5 (the index "L" stands for "loading"). The curve $\Gamma_\beta^L(\epsilon)$ is defined by the parameters

$$\begin{aligned} \epsilon_\beta^{0,L} &\doteq \beta\epsilon_3 + (1-\beta)\epsilon_1 & \sigma_\beta^{0,L} &\doteq \beta\sigma_2 + (1-\beta)\sigma_1 \\ \epsilon_\beta^{1,L} &\doteq \beta\epsilon_3 + (1-\beta)\epsilon_2 & \sigma_\beta^{1,L} &\doteq \beta\sigma_2 + (1-\beta)\sigma_3 \\ \epsilon_\beta^{2,L} &\doteq \beta\epsilon_3 + (1-\beta)\epsilon_4 & \sigma_\beta^{2,L} &\doteq \beta\sigma_2 + (1-\beta)\sigma_4 \end{aligned} \tag{3.5.3}$$

and

$$\Gamma_\beta^L(\epsilon) \doteq \begin{cases} \frac{\sigma_1}{\epsilon_1}\epsilon, & \text{for } 0 \le \epsilon < \epsilon_1 \\ \sigma_1 + \frac{\sigma_\beta^{0,L}-\sigma_1}{\epsilon_\beta^{0,L}-\epsilon_1}(\epsilon-\epsilon_1), & \text{for } \epsilon_1 \le \epsilon < \epsilon_\beta^{0,L} \\ \sigma_\beta^{0,L} + \frac{\sigma_\beta^{1,L}-\sigma_\beta^{0,L}}{\epsilon_\beta^{1,L}-\epsilon_\beta^{0,L}}(\epsilon-\epsilon_\beta^{0,L}), & \text{for } \epsilon_\beta^{0,L} \le \epsilon < \epsilon_\beta^{1,L} \\ \sigma_\beta^{1,L} + \frac{\sigma_\beta^{2,L}-\sigma_\beta^{1,L}}{\epsilon_\beta^{2,L}-\epsilon_\beta^{1,L}}(\epsilon-\epsilon_\beta^{1,L}), & \text{for } \epsilon_\beta^{1,L} \le \epsilon < \epsilon_\beta^{2,L} \\ \sigma_2 + \frac{\sigma_4-\sigma_2}{\epsilon_4-\epsilon_3}(\epsilon-\epsilon_3) & \text{for } \epsilon \ge \epsilon_\beta^{2,L}\ . \end{cases} \tag{3.5.4}$$

Observe that $\Gamma_0^L(\cdot) \equiv \Gamma_0^U(\cdot)$, $\Gamma_1^L(\cdot) \equiv \Gamma_1^U(\cdot)$, $\epsilon_\beta^{1,L} = \epsilon_\beta^{1,U}$ and $\sigma_\beta^{1,L} = \sigma_\beta^{1,U}$.

We add dynamics to the model as follows. Let $\Gamma_{\beta_0}^{w_0}$ be our initial curve, $0 \le \beta_0 \le 1$, $w_0 = U$ or $w_0 = L$. Let $\beta(t=0) = \beta_0$, $w(t=0) = w_0$ and (ϵ_0, σ_0) a point on

$\Gamma^{w_0}_{\beta_0}$ (i.e. $\sigma_0 = \Gamma^{w_0}_{\beta_0}(\epsilon_0)$). Let $\epsilon(t)$, $0 \le t \le T$ be our strain input, where T denotes a certain final time. We assume that $\epsilon(\cdot) \in C(0,T)$ satisfies the initial condition $\epsilon(0) = \epsilon_0$.

The points $(\epsilon(t), \sigma(t))$ will remain on the curve $\Gamma^{w_0}_{\beta_0}$ until the next relative extremum of $\epsilon(t)$ occurs. In general $\beta(t)$ and $w(t)$ will remain constant between any two consecutive relative extrema of $\epsilon(t)$. Let t^* be the minimum positive time such that t^* is a relative extremum of $\epsilon(t)$. One now updates w and β as follows.

First case: Maximum. Suppose that $\epsilon(t)$ has a relative maximum at $t = t^*$. This means that $w_0 = L$ and the points $(\epsilon(t), \sigma(t))$ are on $\Gamma^L_{\beta_0}$ for $0 \le t < t^*$. Now define $w(t^*)$ to be U. To update β one must find β^* such that

$$\begin{cases} \Gamma^L_{\beta_0}(\epsilon(t^*)) = \Gamma^U_{\beta^*}(\epsilon(t^*)) \\ 0 \le \beta^* \le 1. \end{cases} \tag{3.5.5}$$

The appropriate value of β^* is given by the following algorithm:

(1) If $\epsilon(t^*) \ge \epsilon^{2,L}_{\beta_0}$ or $\epsilon(t^*) \le \epsilon^{0,L}_{\beta_0}$ then $\beta^* = 1$;

(2) If $\epsilon^{1,L}_{\beta_0} < \epsilon(t^*) < \epsilon^{2,L}_{\beta_0}$, then

$$\beta^* = \frac{\epsilon(t^*) - \epsilon^{0,U}_{\beta_0}}{\epsilon^{2,L}_{\beta_0} - \epsilon^{0,U}_{\beta_0}};$$

(3) Finally if $\epsilon^{0,L}_{\beta_0} < \epsilon(t^*) \le \epsilon^{1,L}_{\beta_0}$ then

$$\beta^* = \frac{\epsilon(t^*) - \epsilon^{2,U}_{\beta_0}}{\epsilon^{0,L}_{\beta_0} - \epsilon^{2,U}_{\beta_0}}.$$

In cases (2) and (3) the solution β^* is unique. Once w and β have been updated, $w(t) = U$, $\beta(t) = \beta^*$ and $\sigma(t) = \Gamma^{w(t)}_{\beta(t)}(\epsilon(t)) = \Gamma^U_{\beta^*}(\epsilon(t))$ for $t^* \le t < t^{**}$ where t^{**} is the next relative minimum of $\epsilon(t)$.

Second case: Minimum. Suppose that $\epsilon(t)$ has a relative minimum at $t = t^*$. This means that $w_0 = U$ and the points $(\epsilon(t), \sigma(t))$ are on $\Gamma^U_{\beta_0}$ for $0 \leq t < t^*$. Then, w is updated by changing it from U to L. To update β one needs to find β^* such that

$$\begin{cases} \Gamma^U_{\beta_0}(\epsilon(t^*)) = \Gamma^L_{\beta^*}(\epsilon(t^*)) \\ 0 \leq \beta^* \leq 1. \end{cases} \tag{3.5.6}$$

Again, the appropriate value of β^* is given by the following algorithm:

(1) If $\epsilon(t^*) \geq \epsilon^{2,U}_{\beta_0}$ or $\epsilon(t^*) \leq \epsilon^{0,U}_{\beta_0}$ then $\beta^* = 0$;

(2) If $\epsilon^{1,U}_{\beta_0} < \epsilon(t^*) < \epsilon^{2,U}_{\beta_0}$, then

$$\beta^* = \frac{\epsilon(t^*) - \epsilon^{2,U}_{\beta_0}}{\epsilon^{0,L}_{\beta_0} - \epsilon^{2,U}_{\beta_0}};$$

(3) Finally if $\epsilon^{0,U}_{\beta_0} < \epsilon(t^*) \leq \epsilon^{1,U}_{\beta_0}$ then

$$\beta^* = \frac{\epsilon(t^*) - \epsilon^{0,U}_{\beta_0}}{\epsilon^{2,L}_{\beta_0} - \epsilon^{0,U}_{\beta_0}}.$$

Again, in cases (2) and (3) the solution β^* is unique. Once w and β have been updated, $w(t) = L$, $\beta(t) = \beta^*$ and $\sigma(t) = \Gamma^{w(t)}_{\beta(t)}(\epsilon(t)) = \Gamma^L_{\beta^*}(\epsilon(t))$ for $t^* \leq t < t^{**}$ where t^{**} is the next relative maximum of $\epsilon(t)$. At $t = t^{**}$ the above procedure is repeated to update w and β.

An analysis of the dynamics imposed on this model reveals that if $0 < \beta < 1$ the definition of $\Gamma^L_\beta(\epsilon)$ for $\epsilon_1 < \epsilon < \epsilon^{0,L}_\beta$ is superfluous. In fact, that part of the curve will never be followed under loading and its definition is given only for the sake of completeness.

By assuming a linear relation across the yield lines, the martensite phase fraction $f_{M^+}(\epsilon(t), \beta(t), w(t))$ can be computed as follows:

$$f_{M^+}(\epsilon(t), \beta(t), w(t)) \doteq \begin{cases} f_U(\epsilon(t), \beta(t)), & \text{if } w(t) = U \\ f_L(\epsilon(t), \beta(t)), & \text{if } w(t) = L \end{cases} \tag{3.5.7}$$

where

$$f_U(\epsilon, \beta) \doteq \begin{cases} 0, & \text{for } \epsilon \leq \epsilon_\beta^{0,U} \\ \frac{(\epsilon_\beta^{2,U} - \epsilon_2)(\epsilon - \epsilon_\beta^{0,U})}{(\epsilon_4 - \epsilon_2)(\epsilon_\beta^{1,U} - \epsilon_\beta^{0,U})}, & \text{for } \epsilon_\beta^{0,U} < \epsilon < \epsilon_\beta^{1,U} \\ \frac{\epsilon_\beta^{2,U} - \epsilon_2}{\epsilon_4 - \epsilon_2}, & \text{for } \epsilon_\beta^{1,U} \leq \epsilon \leq \epsilon_\beta^{2,U} \\ \frac{\epsilon - \epsilon_2}{\epsilon_4 - \epsilon_2}, & \text{for } \epsilon_\beta^{2,U} < \epsilon < \epsilon_4 \\ 1, & \text{for } \epsilon \geq \epsilon_4, \end{cases} \tag{3.5.8}$$

and

$$f_L(\epsilon, \beta) \doteq \begin{cases} 0, & \text{for } \epsilon \leq \epsilon_1 \\ \frac{(\epsilon_\beta^{2,U} - \epsilon_2)(\epsilon - \epsilon_1)}{(\epsilon_4 - \epsilon_2)(\epsilon_\beta^{0,L} - \epsilon_1)}, & \text{for } \epsilon_1 < \epsilon < \epsilon_\beta^{0,L} \\ \frac{\epsilon_\beta^{2,U} - \epsilon_2}{\epsilon_4 - \epsilon_2}, & \text{for } \epsilon_\beta^{0,L} \leq \epsilon \leq \epsilon_\beta^{1,L} \\ \frac{\epsilon_\beta^{2,U} - \epsilon_2}{\epsilon_4 - \epsilon_2} + \frac{(\epsilon_4 - \epsilon_\beta^{2,U})(\epsilon - \epsilon_\beta^{1,L})}{(\epsilon_4 - \epsilon_2)(\epsilon_\beta^{2,L} - \epsilon_\beta^{1,L})}, & \text{for } \epsilon_\beta^{1,L} < \epsilon < \epsilon_\beta^{2,L} \\ 1, & \text{for } \epsilon \geq \epsilon_\beta^{2,L}. \end{cases} \tag{3.5.9}$$

An Example. Consider the particular case in which $\epsilon_1 = 1$, $\epsilon_2 = 2$, $\epsilon_3 = 6$, $\epsilon_4 = 7$, $\sigma_1 = 1$, $\sigma_2 = 2$, $\sigma_3 = 5$ and $\sigma_4 = 6$. Let $\epsilon(t)$ for $0 \leq t \leq 7$ be as shown in Figure 3.5.6.

The stress-strain curve obtained with the dynamics described above corresponding to the input $\epsilon(t)$ is shown in Figure 3.5.7. Note how the alternating series of dominant input extrema is stored by the model and results in "nested" hysteresis loops. Figure 3.5.8 shows a graphical representation of the martensite fraction f_{M^+}

as a function of the time t. The behavior of $\beta(t)$ is shown in Figure 3.5.9. Finally, Figure 3.5.10 shows the martensite fraction f_{M^+} as a function of the strain ϵ, parametrized by the time t. Note here that for $\epsilon_1 < \epsilon < \epsilon_4$, the strain does not determine uniquely the phase fraction. In fact, for those values of ϵ a whole interval of possible phase fractions is obtained.

The stress-strain relations incorporate the "strain history" through the dynamics described above. This model possesses three properties characteristic of the Preisach models of hysteresis ([**94**]), namely the wiping-out and congruency properties and the fading-memory property. The first one states that only the alternating series of dominant input extrema are stored by the model while all other input extrema are wiped out. The congruency property consists of the fact that all minor hysteresis loops corresponding to back-and-forth variations of inputs between the same two consecutive extremum values, are congruent. For these reasons, this representation can be viewed as a simplified Preisach model and it is possible to incorporate such stress-strain "laws" into our system of P.D.E's. In fact the stress $\sigma_q = \Psi_\epsilon$ in (3.1.6) can now be written as $\sigma_q = \mathcal{F}(\epsilon, \beta, w, \vec{z}(\theta))$, where

$$\vec{z}(\theta) = (\epsilon_1(\theta), \ldots, \epsilon_4(\theta), \sigma_1(\theta), \ldots, \sigma_4(\theta)),$$

$$\epsilon = \epsilon(x, t),$$

$$\beta = \beta(x, t, \vec{z}(\theta(x,t)), [\epsilon]^t(x, \cdot)),$$

$$w = w(x, t, \vec{z}(\theta(x,t)), [\epsilon]^t(x, \cdot)).$$

Here $[\epsilon]^t(x, \cdot)$ denotes the "past history" of the strain at the point x. The resulting system is now a partial functional differential equation. Although we can now write down this system of PFDE's, we have not yet analyzed the basic existence, uniqueness and continuous dependence properties of such systems. This is left for future effort. We turn now to some computational results for the model in section 3.1.

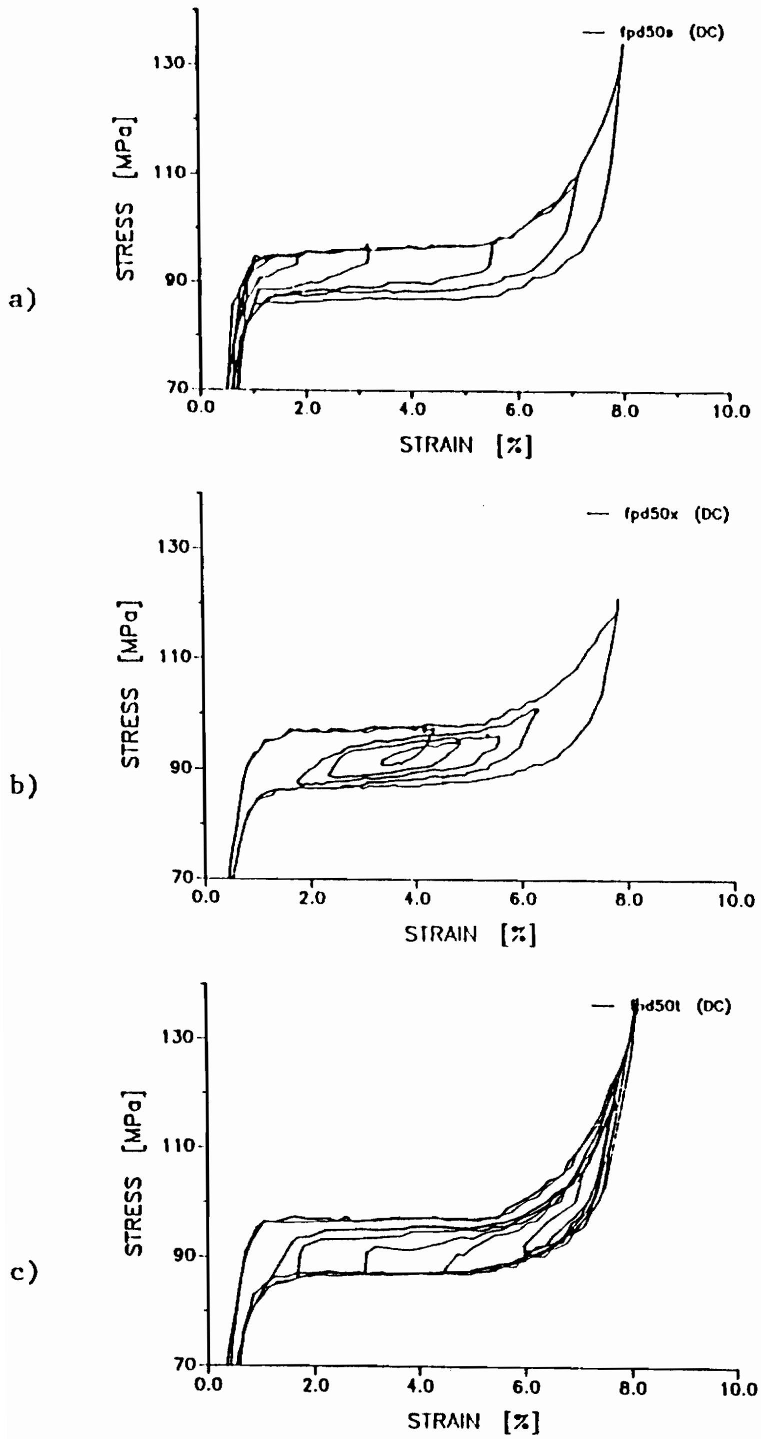


Figure 3.5.1: Isothermal stress-strain relations obtained from uniaxial stretching experiments under controlled deformation performed in CuZnAl alloys. a) unloading after partial yield; b) interior "nested" hysteresis loops; c) reloading after partial recovery ([**103**])

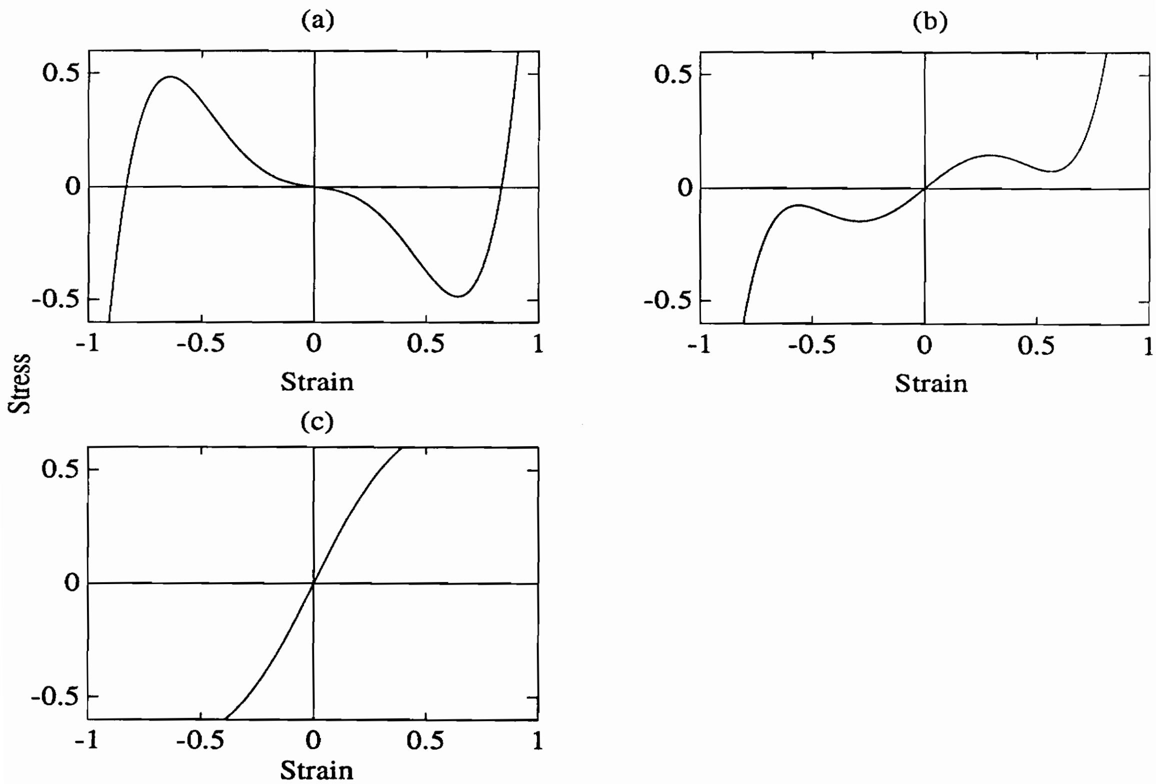


Figure 3.5.2: Stress-strain curves obtained with the Landau-Devonshire potential (3.1.9); (a) low temperature; (b) intermediate temperature; (c) high temperature.

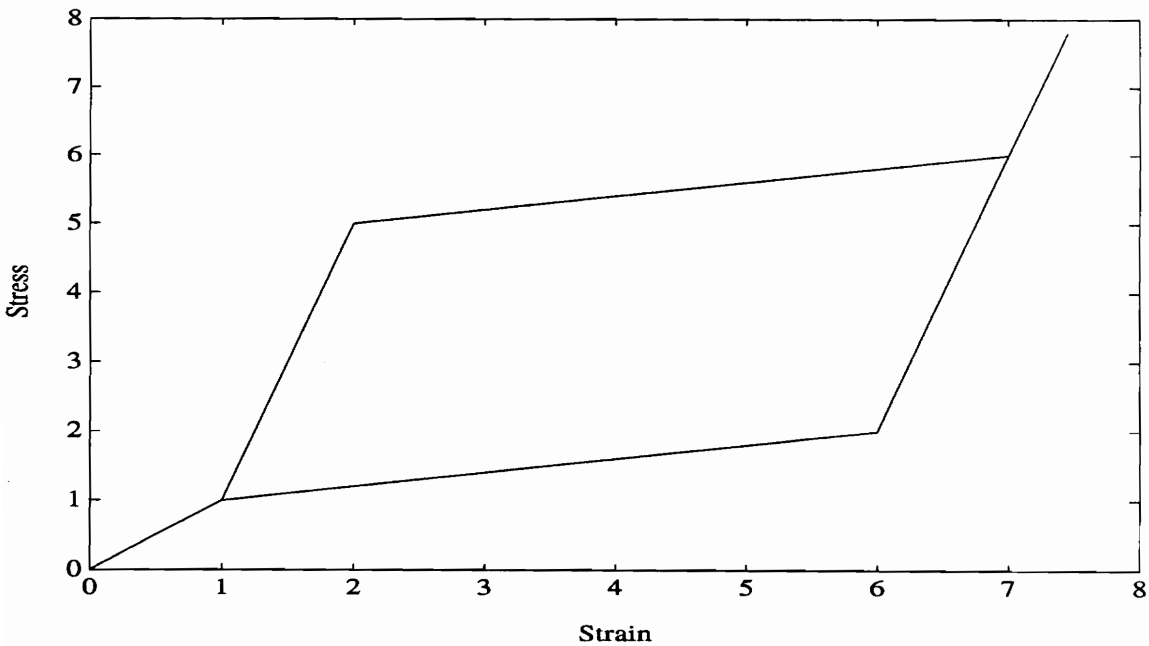


Figure 3.5.3: Characterization of the stress-strain curves by using the eight critical values ϵ_i, σ_i, $i = 1, \ldots, 4$.

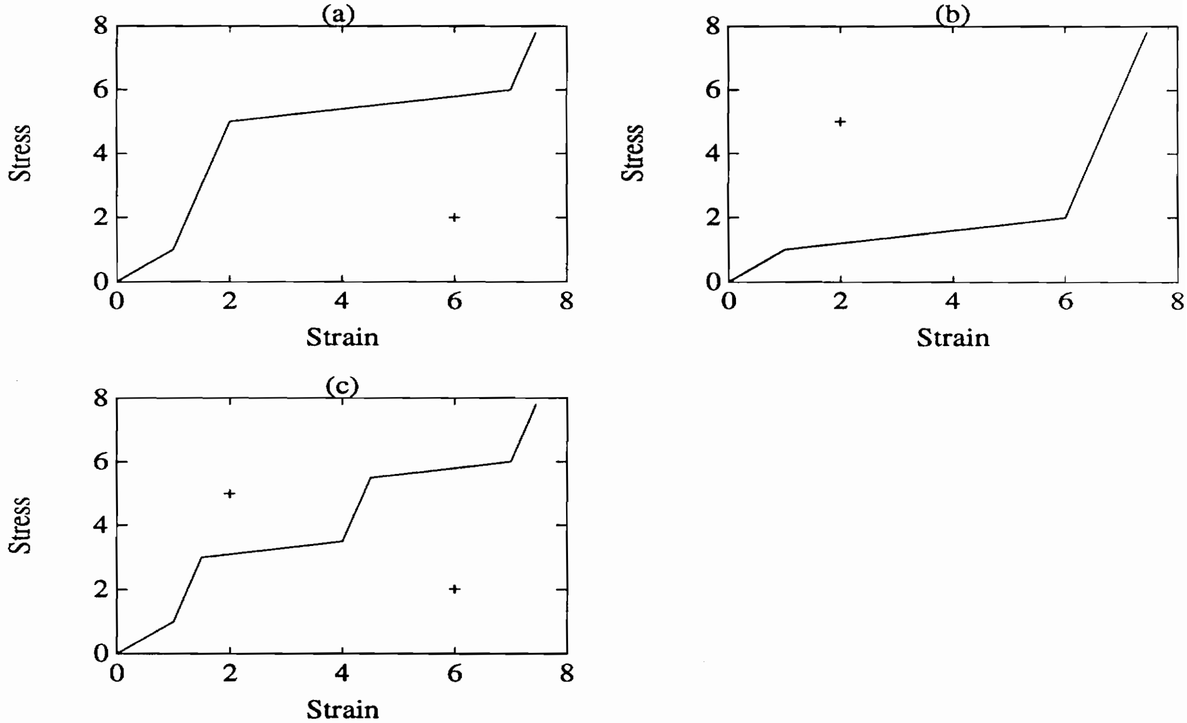


Figure 3.5.4: $\Gamma^U_\beta(\epsilon)$: (a) $\beta = 0$; (b) $\beta = 1$; (c) $0 < \beta < 1$

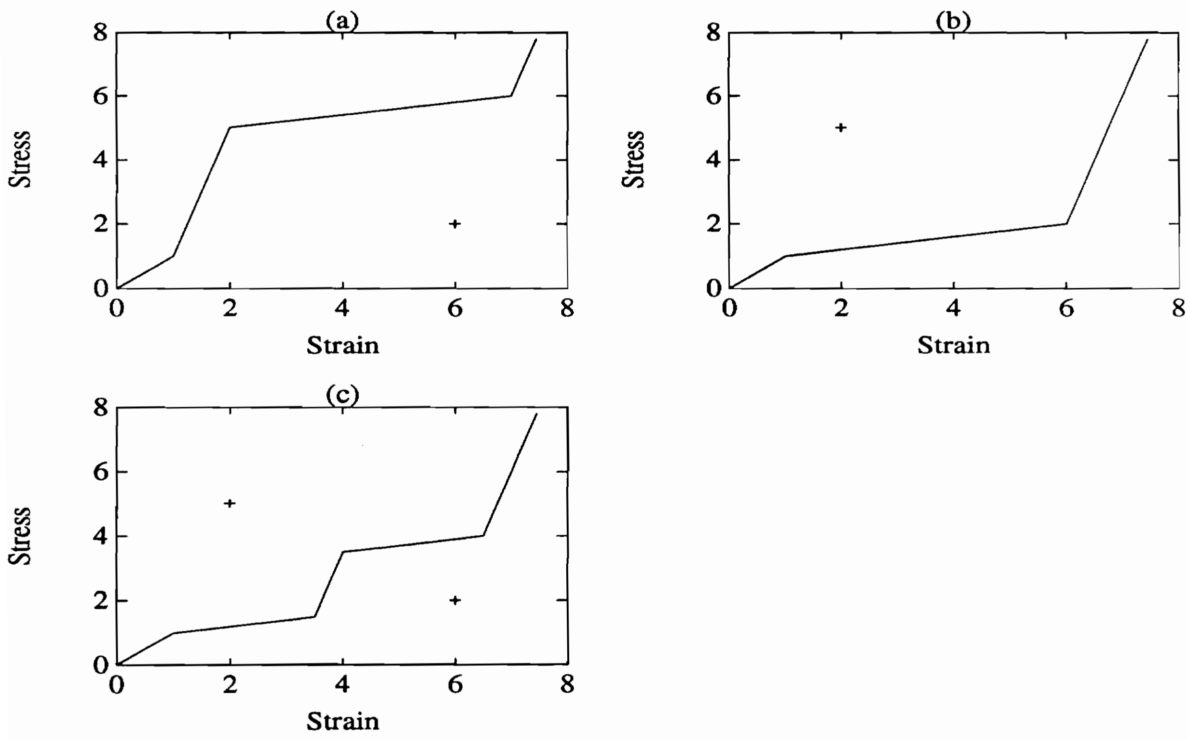


Figure 3.5.5: $\Gamma^L_\beta(\epsilon)$: (a) $\beta = 0$; (b) $\beta = 1$; (c) $0 < \beta < 1$

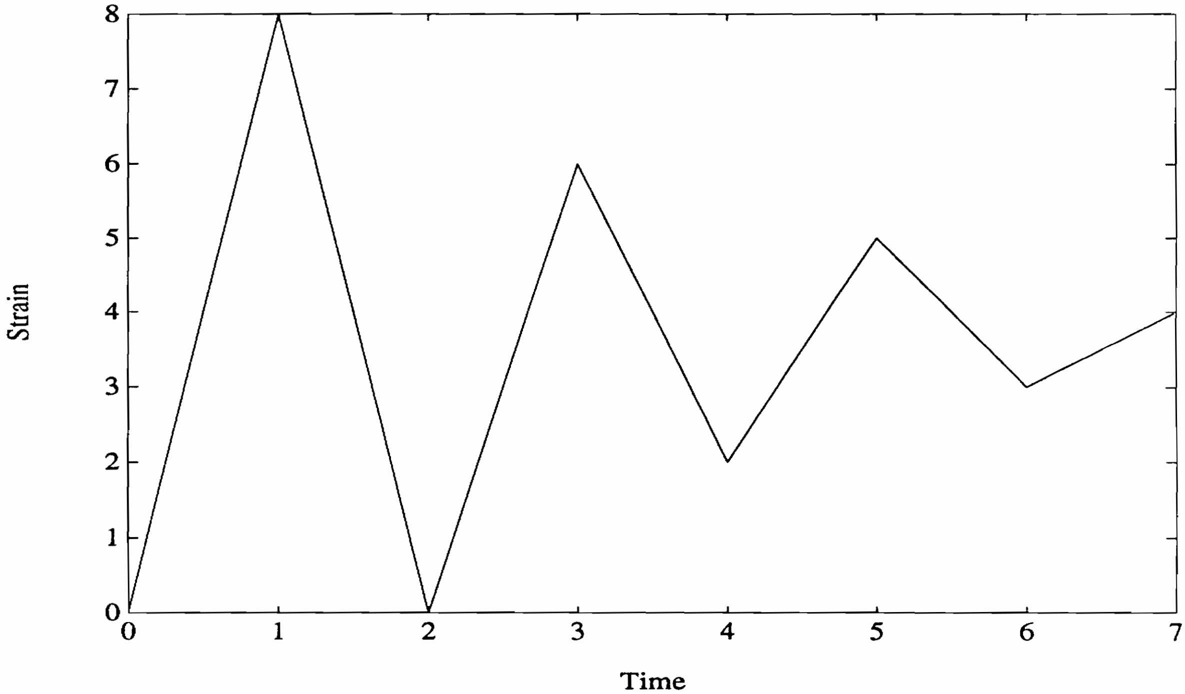


Figure 3.5.6: Strain input

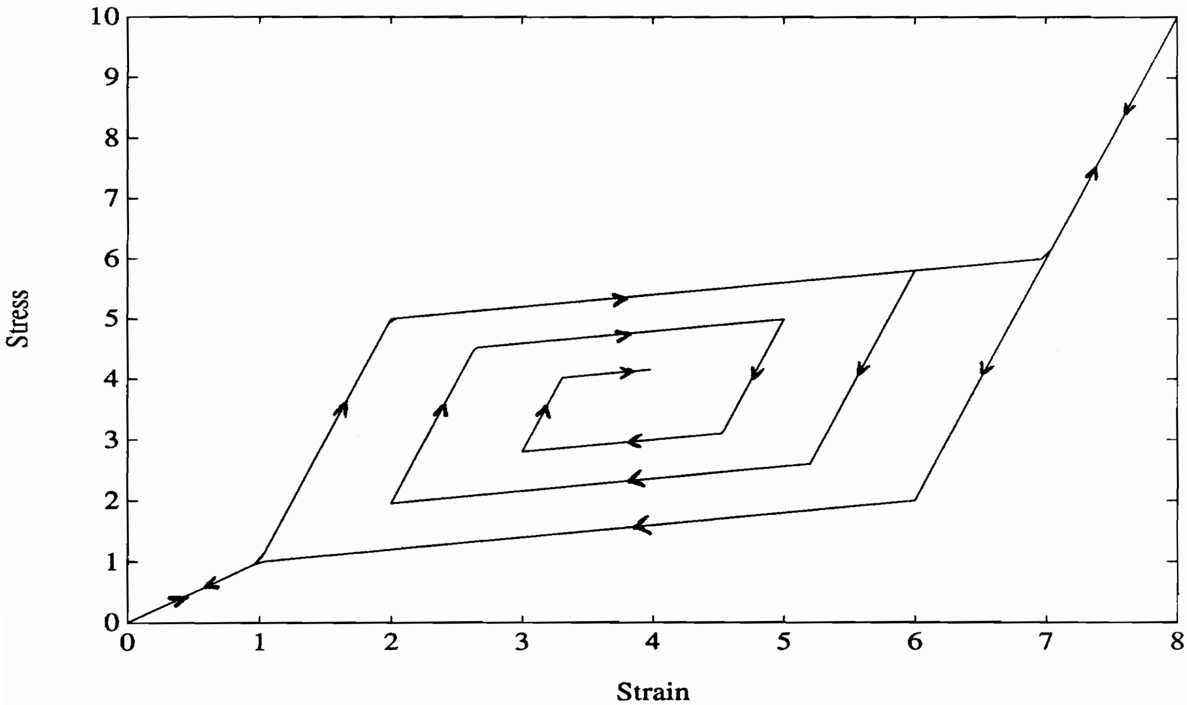


Figure 3.5.7: Stress (output) as a function of the strain (input) parametrized by the time t

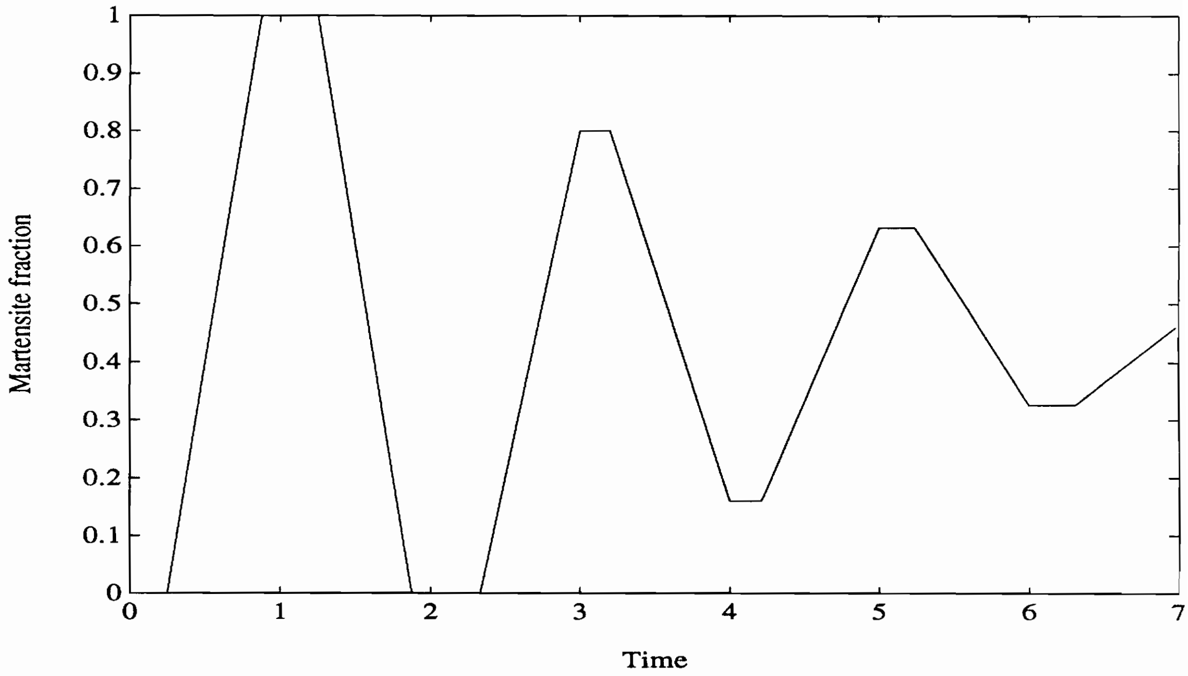


Figure 3.5.8: Martensite fraction f_{M^+} as a function of time t.

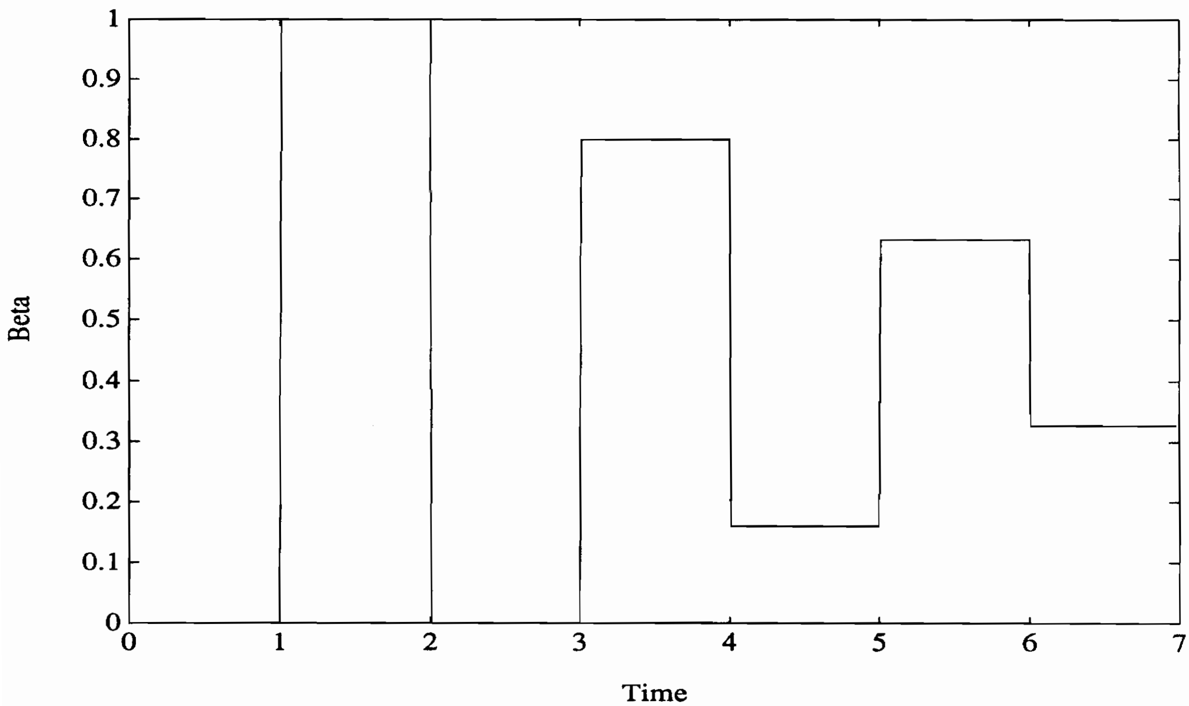


Figure 3.5.9: The parameter β as a function of time t.

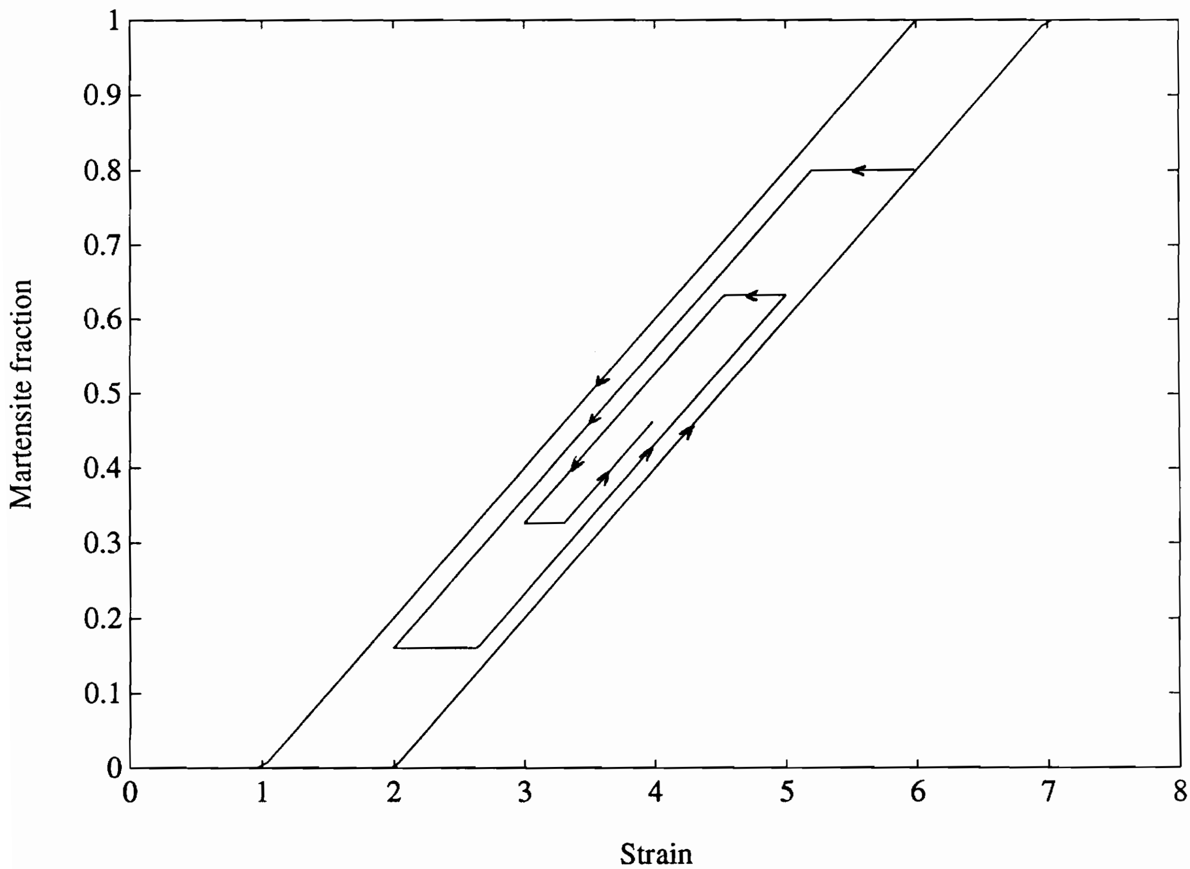


Figure 3.5.10: The martensite fraction as a function of the strain, parametrized by the time t

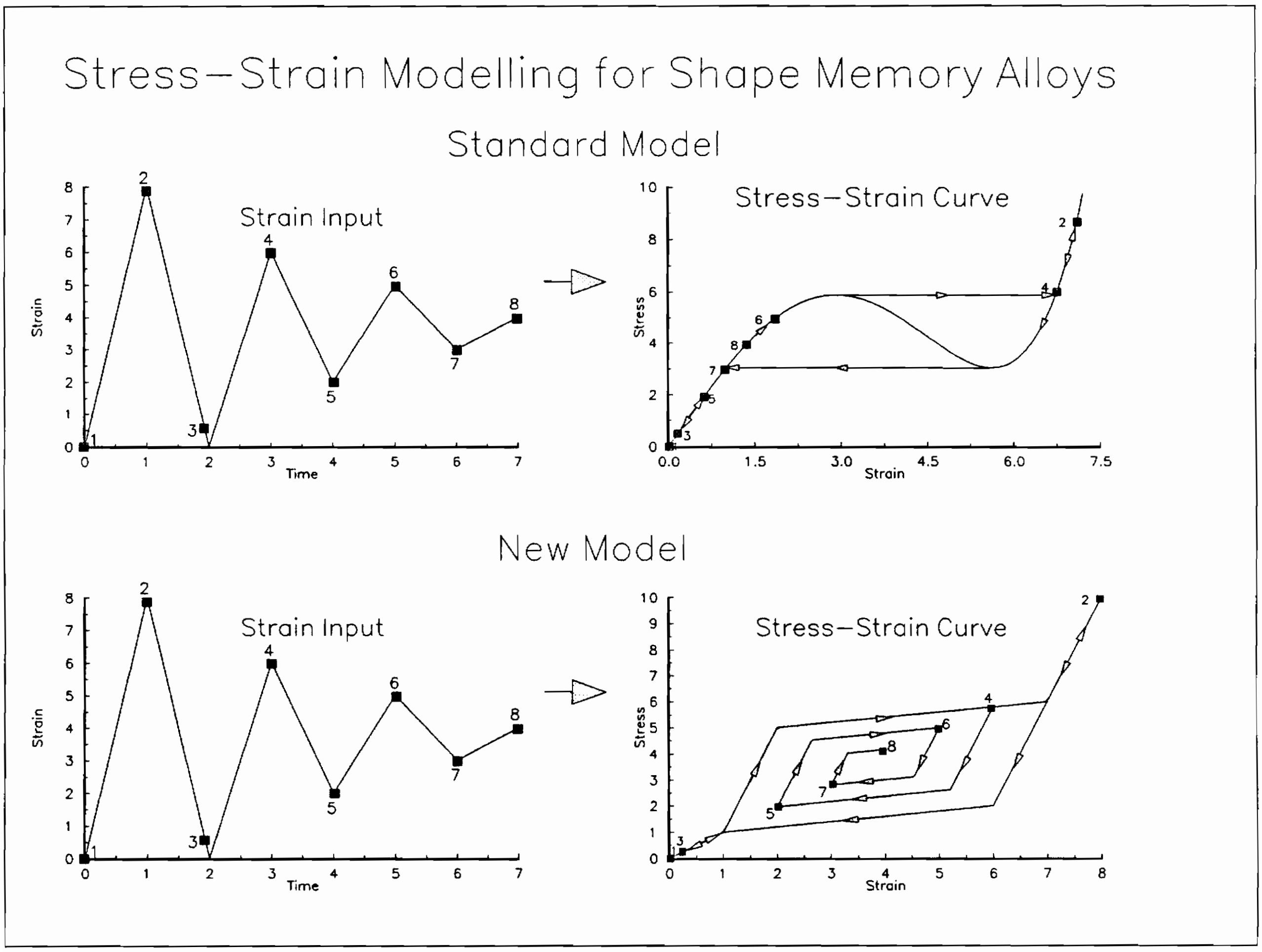


Figure 3.5.11: Comparisson between the stresses obtained with the standard Landau-Devonshire model and our new model for the same strain input.

CHAPTER IV Approximation and Numerical Results

4.1 Finite Dimensional Approximations

In this section we present a scheme to numerically approximate the solutions of problem (3.1.8a-b) for particular cases. The initial goal is to determine if an adequate choice of the potential Ψ and all the parameters involved, together with relevant initial and boundary conditions, can produce physically reasonable answers. If "reasonable" answers are produced, then an important second problem is to determine the model parameters that best fit experimental data. This is a particular type of inverse problem known as parameter estimation. The basic idea is to conduct an experiment (dynamic) to obtain data for a particular alloy and then to use a computational/optimization algorithm to estimate various parameters in the system (3.1.8a-b) that best fit observed data. The ultimate goal is to "automate" this process.

For the moment we confine ourselves to the problem of developing approximations for the solutions of (3.1.8a-b). With this in mind, let $\Omega = (0,1)$. The free energy density is chosen in the Landau-Devonshire form (3.3.1). Furthermore we set $\beta = \alpha = 0$, so that no viscosity or thermal memory effects are present. Under these assumptions, the system (3.1.8a-b) takes the form

$$(\Sigma)\begin{cases} \rho u_{tt} - \frac{\partial}{\partial x}(2\alpha_2(\theta-\theta_1)u_x - 4\alpha_4 u_x^3 + 6\alpha_6 u_x^5) = f(x,t) \\ C_v\theta_t - 2\alpha_2\theta u_x u_{xt} - k\theta_{xx} = g(x,t) \end{cases} \tag{4.1.1}$$

for $x \in \Omega = (0,1)$ and $0 \le t \le T$. Since $\alpha = 0$ and Ψ does not depend on internal variables, it can be easily proved that the second law of thermodynamics given by (3.1.1d) is satisfied. Temperature, displacement and velocity are given at time $t = 0$ by

$$\theta(x,0) = \theta_0(x), \qquad u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) \tag{4.1.2}$$

A symmetric stress and the rate of thermal exchange define the boundary conditions:

$$\Psi_\epsilon(u_x,\theta) = \sigma_\Gamma(t), \qquad k\frac{\partial\theta}{\partial\nu} = k_1(\theta_\Gamma - \theta), \qquad x = 0,1 \text{ and } 0 \le t \le T \tag{4.1.3}$$

where k_1 is a positive constant, θ_Γ is the temperature of the surrounding medium, ν is the outward normal unit vector to $\partial\Omega$ and $\sigma_\Gamma(t)$ is an applied stress.

Consider finite dimensional approximations for u and θ given by

$$u^N(x,t) = \sum_{i=-1}^{N+1} a_i^N(t)\phi_i^N(x), \qquad \theta^N(x,t) = \sum_{i=-1}^{N+1} b_i^N(t)\phi_i^N(x) \tag{4.1.4}$$

where the ϕ_i^N's are the usual cubic splines in $(0,1)$, $a_i^N(t)$ and $b_i^N(t)$, $-1 \le i \le N+1$, are coefficients to be computed and $N+3$ is the order of the approximation.

The weak formulation of the first equation in (4.1.1) is given by

$$\begin{aligned} 0 &= < \rho u_{tt} - (\Psi_\epsilon)_x - f, v > \\ &= < \rho u_{tt} - f, v > + < \Psi_\epsilon, v_x > - \Psi_\epsilon v \,|_{x=0}^{x=1} \end{aligned}$$

for all $v \in H^1(0,1)$, where $< \cdot,\cdot >$ denotes the inner product in $L^2(0,1)$. The corresponding finite dimensional approximating equation is given by (we suppress the index N whenever the context is clear enough)

$$\rho \sum_{i=-1}^{N+1} a_i''(t) < \phi_i, \phi_j > \quad = \quad < f(\cdot,t), \phi_j > + \sigma_\Gamma(t)\phi_j(x) \mid_{x=0}^{x=1} - < \Psi_\epsilon(u_x^N, \theta^N), \phi_j' > \tag{4.1.5}$$

for $j = -1, 0, 1, 2, \ldots, N+1$.

Similarly, the weak formulation of the second equation in (4.1.1)

$$\begin{aligned} 0 &= < C_v\theta_t - 2\alpha_2\theta u_x u_{xt} - k\theta_{xx} - g, v > \\ &= < C_v\theta_t - 2\alpha_2\theta u_x u_{xt} - g, v > + k < \theta_x, v_x > - k_1 \int_\Gamma (\theta_\Gamma - \theta) v dS \end{aligned}$$

for all $v \in H^1(0,1)$. Here $\Gamma = \partial\Omega = \{0,1\}$ and we have used (4.1.3) during the integration by parts. The corresponding finite dimensional approximating equation is given by

$$\begin{aligned} C_v \sum_{i=-1}^{N+1} b_i'(t) < \phi_i, \phi_j > \quad &= < g(\cdot,t), \phi_j > - k \sum_{i=-1}^{N+1} b_i(t) < \phi_i', \phi_j' > \\ &+ k_1 \left(\theta_\Gamma(0,t) - \theta^N(0,t)\right) \phi_j(0) \\ &+ k_1 \left(\theta_\Gamma(1,t) - \theta^N(1,t)\right) \phi_j(1) + 2\alpha_2 < \theta^N u_x^N u_{xt}^N, \phi_j > \end{aligned} \tag{4.1.6}$$

Since $\phi_i(0) = 0$ for $i = 2, 3, \ldots, N+1$, $\phi_{-1}(0) = 1$, $\phi_0(0) = 4$ and $\phi_1(0) = 1$ it follows that

$$\theta^N(0,t) = b_{-1}(t) + 4b_0(t) + b_1(t) \tag{4.1.7}$$

Similarly,

$$\theta^N(1,t) = b_{N-1}(t) + 4b_N(t) + b_{N+1}(t) \tag{4.1.8}$$

Consequently, equation (4.1.6) can be written as

$$\begin{aligned} C_v \sum_{i=-1}^{N+1} b_i'(t) < \phi_i, \phi_j > \quad &=< g(\cdot,t), \phi_j > -k \sum_{i=-1}^{N+1} b_i(t) < \phi_i', \phi_j' > \\ &+ k_1 \left(\theta_\Gamma(0,t) - b_{-1}(t) - 4b_0(t) - b_1(t)\right) \phi_j(0) \\ &+ k_1 \left(\theta_\Gamma(1,t) - b_{N-1}(t) - 4b_N(t) - b_{N+1}(t)\right) \phi_j(1) \\ &+ 2\alpha_2 < \theta^N u_x^N u_{xt}^N, \phi_j > . \end{aligned} \tag{4.1.9}$$

In order to write equations (4.1.5) and (4.1.9) in matrix form, we define

$$A(t) \doteq \begin{bmatrix} a_{-1}(t) & a_0(t) & a_1(t) & \dots & a_N(t) & a_{N+1}(t) \end{bmatrix}^T, \tag{4.1.10a}$$

$$B(t) \doteq \begin{bmatrix} b_{-1}(t) & b_0(t) & b_1(t) & \dots & b_N(t) & b_{N+1}(t) \end{bmatrix}^T, \tag{4.1.10b}$$

$$M \doteq [< \phi_j, \phi_i >]_{-1 \le j,i \le N+1} \qquad \text{(the mass matrix)}, \tag{4.1.10c}$$

$$S \doteq \left[< \phi_j', \phi_i' >\right]_{-1 \le j,i \le N+1} \qquad \text{(the stiffness matrix)}, \tag{4.1.10d}$$

$$F_1(t) \doteq [< f(\cdot,t), \phi_j >]_{-1 \le j \le N+1}^T, \tag{4.1.10e}$$

$$F_2(t) \doteq \left[\sigma_\Gamma(t)\phi_j(x) \mid_{x=0}^{x=1}\right]_{-1 \le j \le N+1}^T, \tag{4.1.10f}$$

$$P(A,B) \doteq \left[- < \Psi_\epsilon(u_x^N, \theta^N), \phi_j' >\right]_{-1 \le j \le N+1}^T, \tag{4.1.10g}$$

$$G_1(t) \doteq [< g(\cdot,t), \phi_j >]_{-1 \le j \le N+1}^T, \tag{4.1.10h}$$

$$G_2(t) \doteq [k_1\theta_\Gamma(0,t)\phi_j(0) + k_1\theta_\Gamma(1,t)\phi_j(1)]_{-1 \le j \le N+1}^T, \tag{4.1.10i}$$

$$\tilde{S} \doteq k_1 \begin{bmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T \end{bmatrix}_{(N+3)\times(N+3)}, \qquad \text{where} \qquad T \doteq \begin{bmatrix} -1 & -4 & -1 \\ -4 & -16 & -4 \\ -1 & -4 & -1 \end{bmatrix}, \tag{4.1.10j}$$

and

$$Q(A,B) \doteq \left[2\alpha_2 < \theta^N u_x^N u_{xt}^N, \phi_j >\right]_{-1 \le j \le N+1}^T . \tag{4.1.10k}$$

Here A, B, F_1, F_2, G_1, G_2, $P(A,B)$ and $Q(A,B)$ are all $(N+3)$–dimensional vectors. The matrices M, S and $\tilde{S}$ are $(N+3)\times(N+3)$ matrices.

Using the notation introduced in (4.1.10a-k), equations (4.1.5) and (4.1.9) can be written as the system

$$\begin{cases} \rho M A''(t) = F_1(t) + F_2(t) + P(A,B) \\ C_v M B'(t) = (-kS+\tilde{S})B(t) + G_1(t) + G_2(t) + Q(A,B), \end{cases} \tag{4.1.11}$$

Let

$$C(t) \doteq A'(t) \tag{4.1.12}$$

and

$$X(t) \doteq \begin{bmatrix} A(t)^T & C(t)^T & B(t)^T \end{bmatrix}^T. \tag{4.1.13}$$

Using (4.1.12) and (4.1.13), system (4.1.11) becomes

$$\begin{cases} A'(t) = C(t) \\ C'(t) = \rho^{-1}M^{-1}\left(F_1(t)+F_2(t)\right) + \rho^{-1}M^{-1}P\left(A(t),B(t)\right) \\ B'(t) = C_v^{-1}M^{-1}(-kS+\tilde{S})B(t) + C_v^{-1}M^{-1}\left(G_1(t)+G_2(t)\right) + C_v^{-1}M^{-1}Q(A,B). \end{cases} \tag{4.1.14}$$

We now define

$$\tilde{A}(t) \doteq \begin{bmatrix} 0 & I_{N+3} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C_v^{-1}M^{-1}(-kS+\tilde{S}) \end{bmatrix}_{3(N+3)\times 3(N+3)}, \tag{4.1.15}$$

where I_{N+3} is the identity matrix of order $N+3$,

$$F(t) \doteq \begin{bmatrix} 0 \\ \rho^{-1}M^{-1}\left(F_1(t)+F_2(t)\right) \\ C_v^{-1}M^{-1}\left(G_1(t)+G_2(t)\right) \end{bmatrix}_{3(N+3)\times 1} \tag{4.1.16}$$

and

$$NL\left(X(t)\right) \doteq \begin{bmatrix} 0 \\ \rho^{-1}M^{-1}P\left(A(t),B(t)\right) \\ C_v^{-1}M^{-1}Q\left(A(t),B(t)\right) \end{bmatrix}_{3(N+3)\times 1}. \tag{4.1.17}$$

Finally, using (4.1.15), (4.1.16) and (4.1.17) the system (4.1.14) can be written in matrix form as

$$\frac{d}{dt}X(t) = \tilde{A}X(t) + F(t) + NL\left(X(t)\right) \tag{4.1.18}$$

Initial conditions for (4.1.18) are obtained by orthogonal projections

$$< u_0(\cdot) - u^N(\cdot, 0), \phi_j(\cdot) >= 0 \tag{4.1.19a}$$

$$< u_1(\cdot) - u_t^N(\cdot, 0), \phi_j(\cdot) >= 0 \tag{4.1.19b}$$

$$< \theta_0(\cdot) - \theta^N(\cdot, 0), \phi_j(\cdot) >= 0 \tag{4.1.19c}$$

for $-1 \leq j \leq N+1$.

Equations (4.1.19a–c) imply that

$$A(0) = M^{-1}U_0, \qquad C(0) = M^{-1}U_1, \qquad B(0) = M^{-1}\Theta_0, \tag{4.1.20}$$

where

$$U_0 \doteq \left[< u_0, \phi_{-1} > \quad < u_0, \phi_0 > \quad \ldots \quad < u_0, \phi_{N+1} >\right]^T \tag{4.1.21a}$$

$$U_1 \doteq \left[< u_1, \phi_{-1} > \quad < u_1, \phi_0 > \quad \ldots \quad < u_1, \phi_{N+1} >\right]^T \tag{4.1.21b}$$

$$\Theta_0 \doteq \left[< \theta_0, \phi_{-1} > \quad < \theta_0, \phi_0 > \quad \ldots \quad < \theta_0, \phi_{N+1} >\right]^T. \tag{4.1.21c}$$

If

$$X_0 \doteq \left[A(0)^T \quad C(0)^T \quad B(0)^T\right]^T, \tag{4.1.22}$$

then the initial condition for (4.1.18) is given by

$$X(0) = X_0 \tag{4.1.23}$$

The resulting approximating finite dimensional system is given by

$$(\Sigma^N): \begin{cases} \frac{d}{dt}X(t) = \tilde{A}X(t) + F(t) + NL\left(X(t)\right) \\ X(0) = X_0 \end{cases}, \qquad (4.1.24)$$

for $0 \leq t \leq T$, where $\tilde{A}$, $F(t)$, $NL(X)$ and X_0 are as in (4.1.15), (4.1.16), (4.1.17) and (4.1.22), respectively.

4.2 Numerical Results

In order to determine the validity of this model and its possible use as an effective tool to predict the behavior of a given SMA under certain dynamical conditions, one must use physically meaningful values in our numerical approximations. However, some of the parameters in this model are not "physical" and therefore they need to be determined by parameter estimation techniques.

The numerical results we present below are based on estimates reported in [**44**] and [**45**] for the alloy $Au_{23}Cu_{30}Zn_{47}$: $\alpha_2 = 24$ J cm^{-3} K^{-1}, $\alpha_4 = 1.5 \times 10^5$ J cm^{-3}, $\alpha_6 = 7.5 \times 10^6$ J cm^{-3}, $\theta_1 = 208$ K, $C_v = 2.9$ J cm^{-3} K^{-1}, $k = 1.9$ W cm^{-1} K^{-1}, $\rho = 11.1$ g cm^{-3} (no precise information is given about how these estimates were obtained). We used a sixth order Runge–Kutta–Verner method to integrate $\left(\Sigma^N\right)$ for this data. The program was run in an IBM 3090 with vectorization facilities. For the first numerical experiments we took $N = 10$, $\theta_0(x) = 200$ K, $u_0(x) = 0$, $u_1(x) = 0$, $\theta_\Gamma(t) = 200$ K, $k_1 = 100$, $f = g = 0$, $T = 2 \times 10^{-3}$ sec. and $\sigma_\Gamma(t)$ as shown in Figure 4.2.1. Figure 4.2.2 shows the evolution of displacement (u), temperature (θ)

and deformation (u_x) up to $t = T$ obtained with this data. Note that the applied load produces a phase transition which results in large deformations near the boundary. Sharp gradients are also observed along the transition front due to the fact that no viscosity effects are being taken into account. At the same time, the temperature increases by 4^o K near the interfaces due to the kinetic energy released during the phase transition.

After this experiment was performed we noted that the deformation produced by the boundary stress does not propagate to the interior of the wire as one would intuitively expect. The same happened with the temperature. We performed a second experiment to investigate this issue. The final instant time was now taken to be $T = 40 \times 10^{-3}$ sec. The boundary stress consisted of a tensile load which was increased until it reached the value of 250 MPa at time $t = 1 \times 10^{-3}$ sec., after which it was held constant at 250 MPa until $t = T$, as shown in Figure 4.2.3. Figures 4.3.4-(A),(B),(C) show the evolution of displacement (u), temperature (θ) and deformation (u_x), respectively, obtained with this boundary input. Note now that the phase transition front moves towards the center which it reaches at approximately $t = 20 \times 10^{-3}$. The temperature in the meantime rose by more than 40^o K near the interfaces. We should note here that this type of experiment provides valuable data and information for the design of laboratory experiments to collect dynamic data for parameter identification purposes. For instance, the temperature evolution depicted in Figure 4.2.4-(B) provides important information about about the possible location of sensors to detect the stress-induced temperature increments. Figure 4.2.4-(C) shows that the deformation exhibits sharp edges along the phase boundaries due to the absence of viscosity effects. We also note here the so called "nucleation phenomena", which consists of the fact that the crystals begin to arrange in a random pattern prior to the phase transition itself.

We noted in Figure 4.2.4-(B) that the temperature increased by more than 40^o K due only to the applied stress. Experts in the field seem to agree on the fact that such an increment is to high for real materials. The coefficient that couples strain and temperature in our case is α_2. In the next experiment we decreased α_2 first by a factor of 10 and then by a factor of 100, keeping all the other parameters unchanged. When $\alpha_2 = 2.4$, the highest temperature increment was only of about 4^o K (Figure 4.2.5-(B)). This lower temperature resulted now in larger deformations near the boundaries, in agreement with the fact that SMA's are more ductile at lower temperatures. When $\alpha_2 = 0.24$ (Figures 4.2.6-(A),(B),(C)), the highest temperature increment observed was less than 0.5^o K. Figure 4.2.7 shows a comparison of the temperature evolution obtained with the different values of α_2. Assuming that all the other estimates in [**44**] and [**45**] are accurate, this experiment suggests that the value $\alpha_2 = 24$ for this alloy, is probably too high.

In the next experiment we studied the influence of k_1 and k. First, k_1 was increased by a factor of 5 keeping all the other parameters as those used for Figure 4.2.4. The influence of the external temperature (200^o K) due to the larger value of the boundary exchange coefficient $k_1 = 500$ results in much cooler boundaries (Figure 4.2.8-(B)) and larger deformations near the endpoints of the wire (Figure 4.2.8-(C)). When k was increased by a factor of 10 ($k = 19.0$, Figures 4.2.9-(A),(B),(C)) the thermal waves were damped much faster due to the larger heat conductivity coefficient (Figure 4.2.9-(B)). Figures 4.2.10-(A),(B) show a comparisson of the effects of changes in k and k_1 on temperature and deformation. Note that lower temperature regions near interfaces are always associated with larger deformations.

Finally we studied the effects of changes in the coefficient α_4. We first decreased α_4 by a factor of 10 ($\alpha_4 = 1.5 \times 10^4$, Figures 4.2.11-(A),(B),(C)). The greatest effects of this change were observed in the evolution of the temperature which is now dampen throughout the whole wire. This is somewhat contrary to what we expected, since α_4 is not associated with any thermal part of the free energy potential. A similar effect was observed when $\alpha_4 = 1.5 \times 10^3$.

In conclusion, we have studied the influence of some of the model parameters on the evolution of displacement, temperature and deformation. The numerical experiments show that the Landau-Devonshire free energy used in conjunction with the dynamic model previously described can provide interesting qualitative information on the dynamics of phase transitions in SMA's. The next step is to use dynamic experimental data to approximate the model parameters. This process is known as parameter identification and its implementation requires not only the design of adequate laboratory experiments but also the availability of efficient approximation schemes. This step will determine whether this model can also provide relevant quantitative information to be used as a tool for control design. We plan to devote further efforts into this area of research.

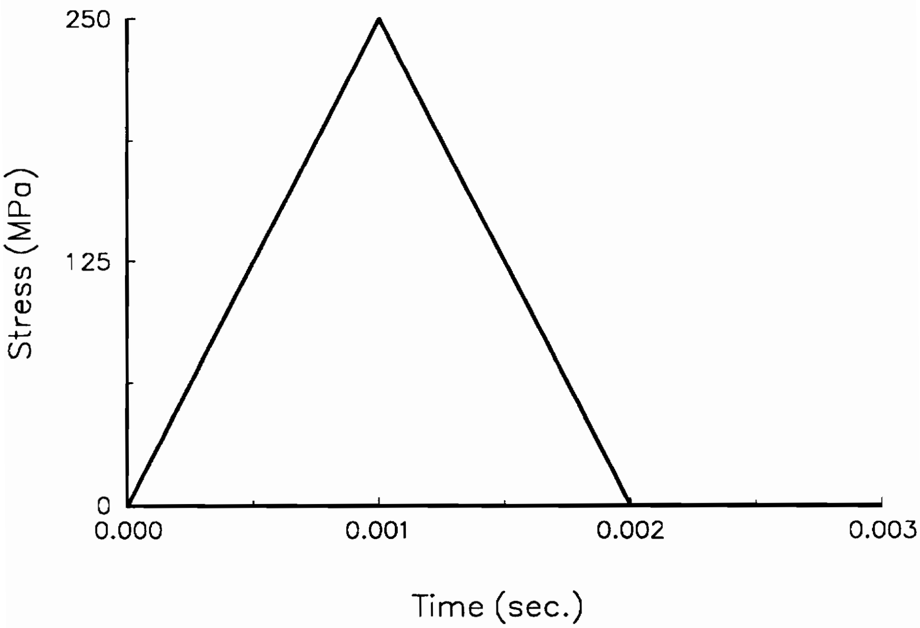


Figure 4.2.1: Boundary stress input, $T = 2 \times 10^{-3}$

$\alpha_2 = 24$	$\alpha_4 = 1.5 \times 10^5$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 2 \times 10^{-3}$	$f(x,t) = 0$	$g(x,t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

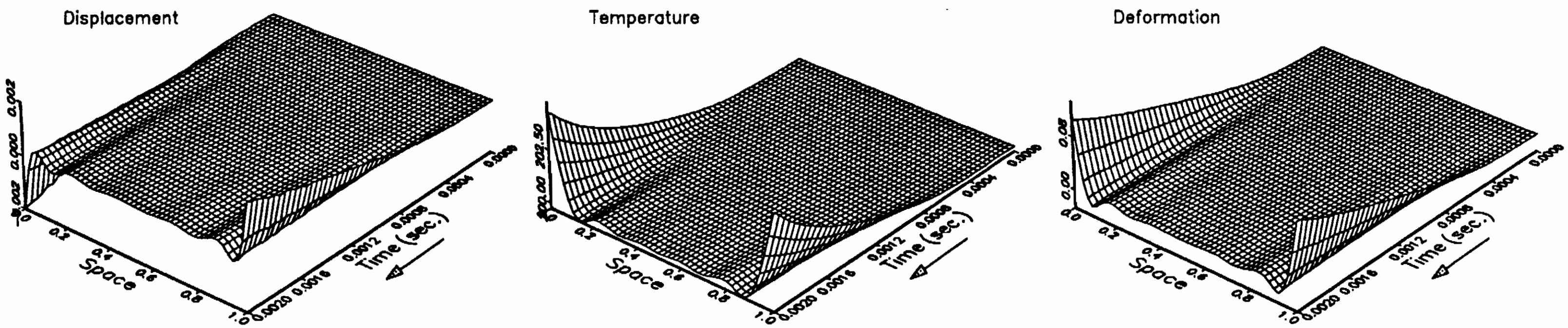


Figure 4.2.2: Evolution of displacement (u), temperature (θ) and deformation (u_x) obtained with the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.1

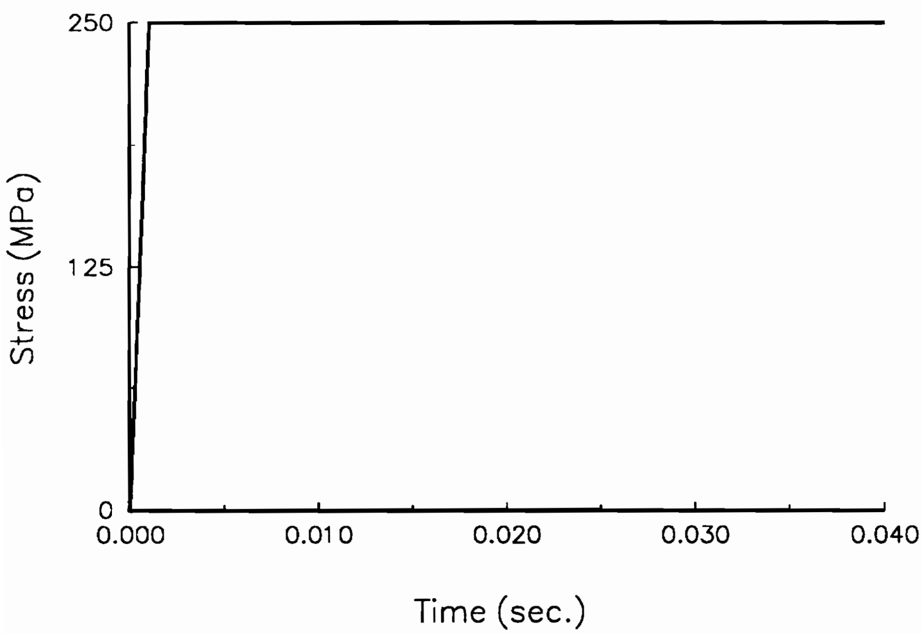


Figure 4.2.3: Boundary stress input, $T = 40 \times 10^{-3}$

$\alpha_2 = 24$	$\alpha_4 = 1.5 \times 10^5$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x,t) = 0$	$g(x,t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

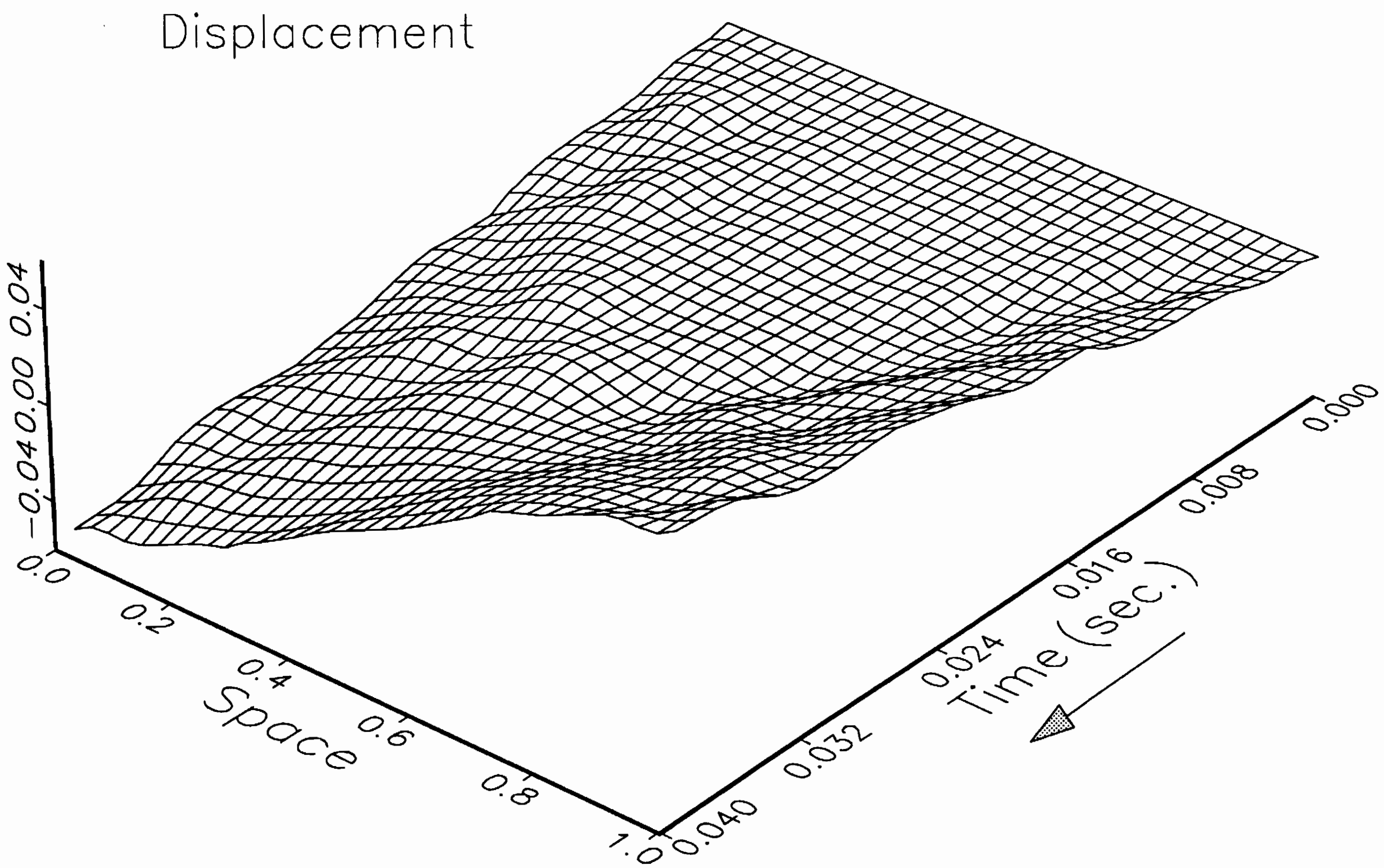


Figure 4.2.4-(A): Evolution of displacement (u), corresponding to the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.3 and $\alpha_2 = 24$

$\alpha_2 = 24$	$\alpha_4 = 1.5 \times 10^5$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x,t) = 0$	$g(x,t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

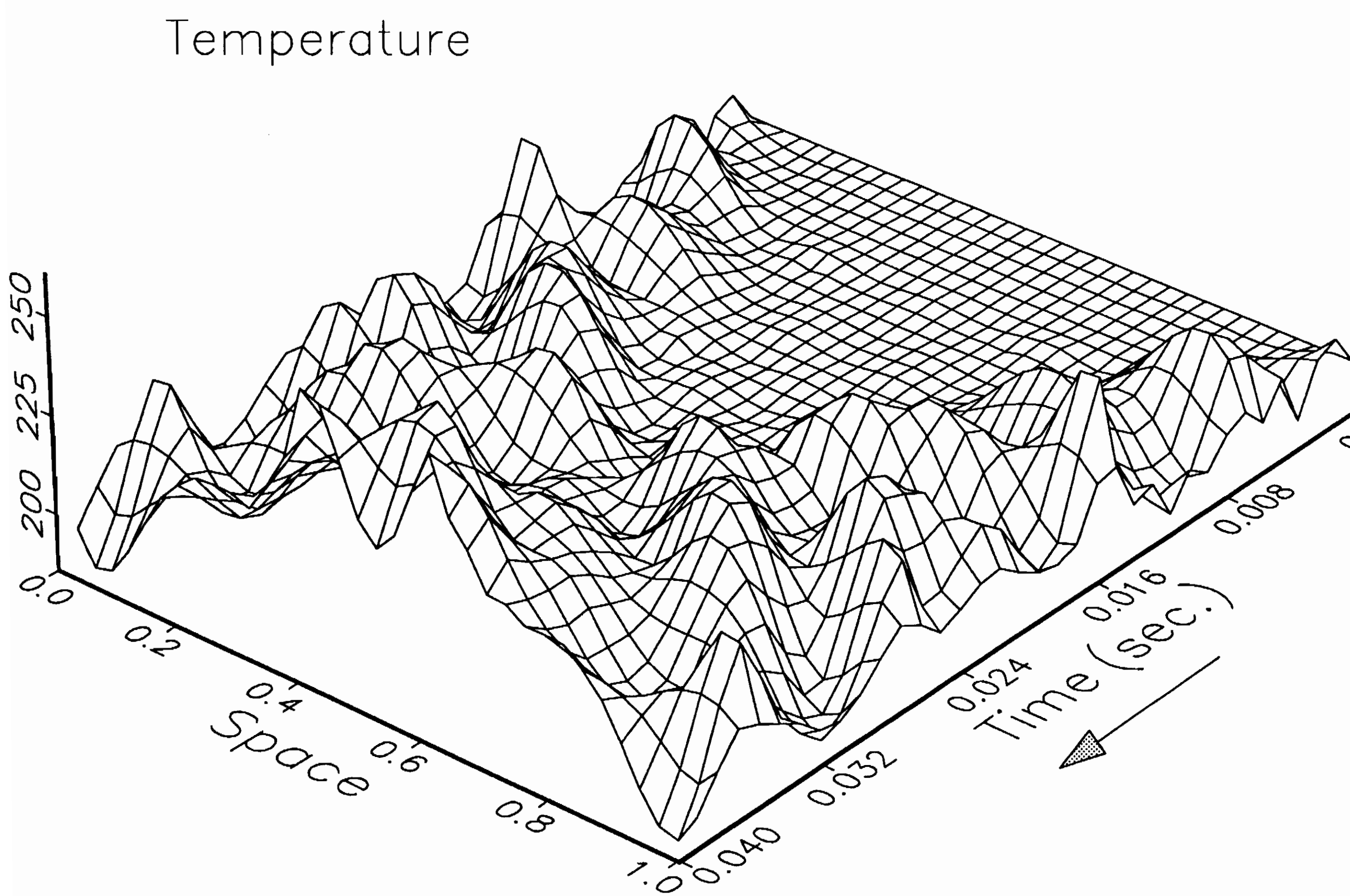


Figure 4.2.4-(B): Evolution of temperature (θ), corresponding to the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.3 and $\alpha_2 = 24$

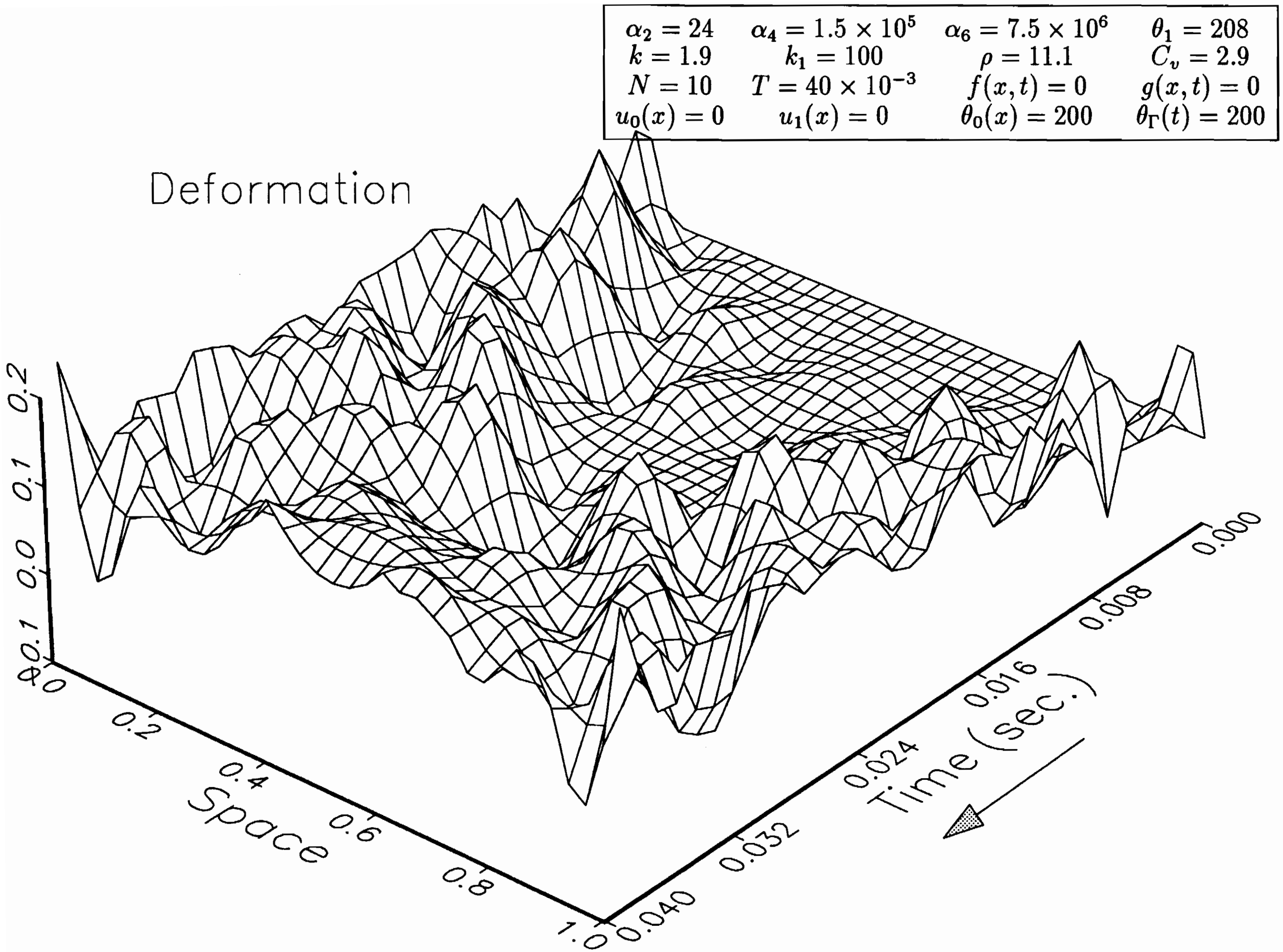


Figure 4.2.4-(C): Evolution of deformation (u_x), corresponding to the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.3 and $\alpha_2 = 24$

$\alpha_2 = 2.4$	$\alpha_4 = 1.5 \times 10^5$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x,t) = 0$	$g(x,t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

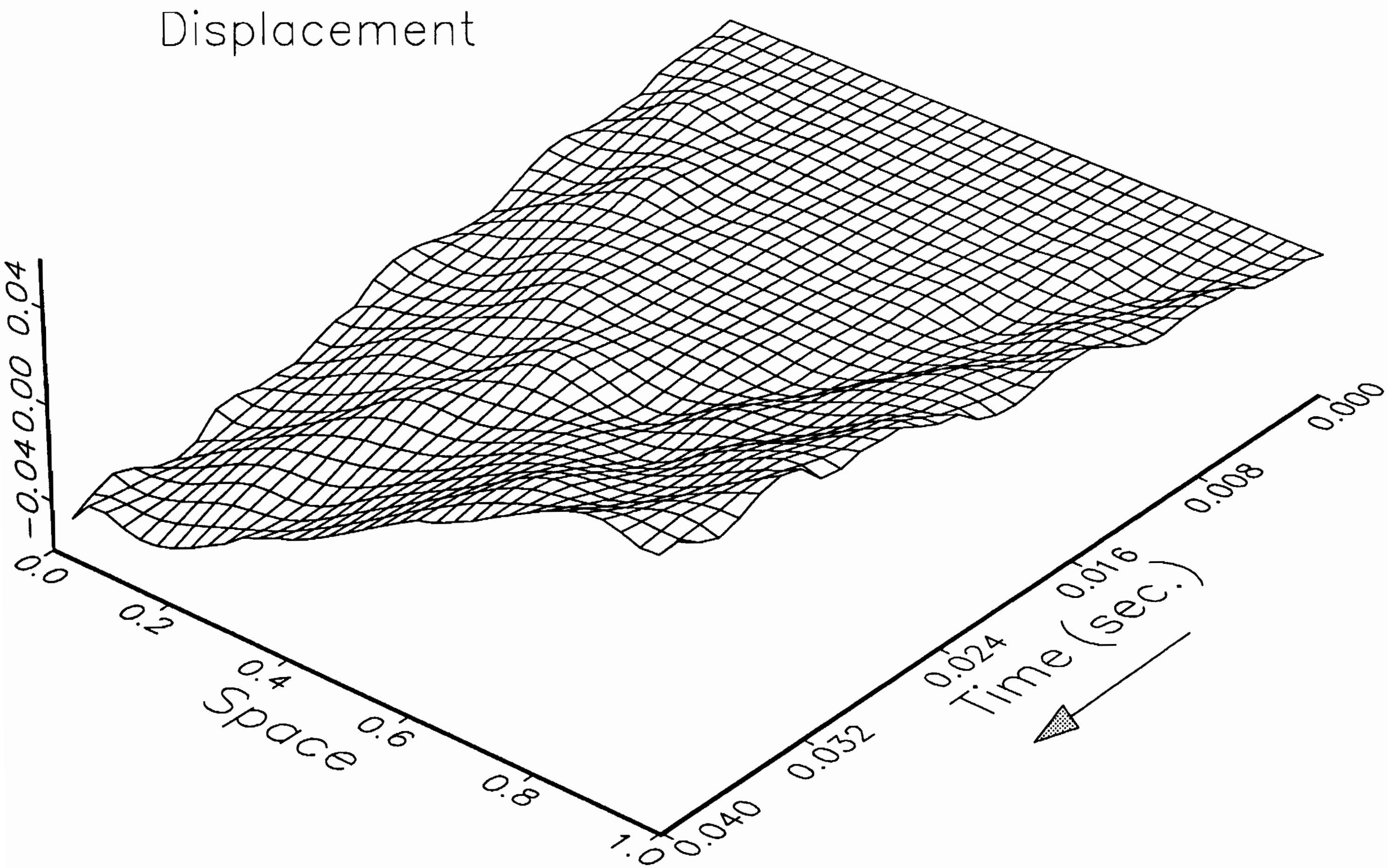


Figure 4.2.5-(A): Evolution of displacement (u), corresponding to the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.3 and $\alpha_2 = 2.4$

$\alpha_2 = 2.4$	$\alpha_4 = 1.5 \times 10^5$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x,t) = 0$	$g(x,t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

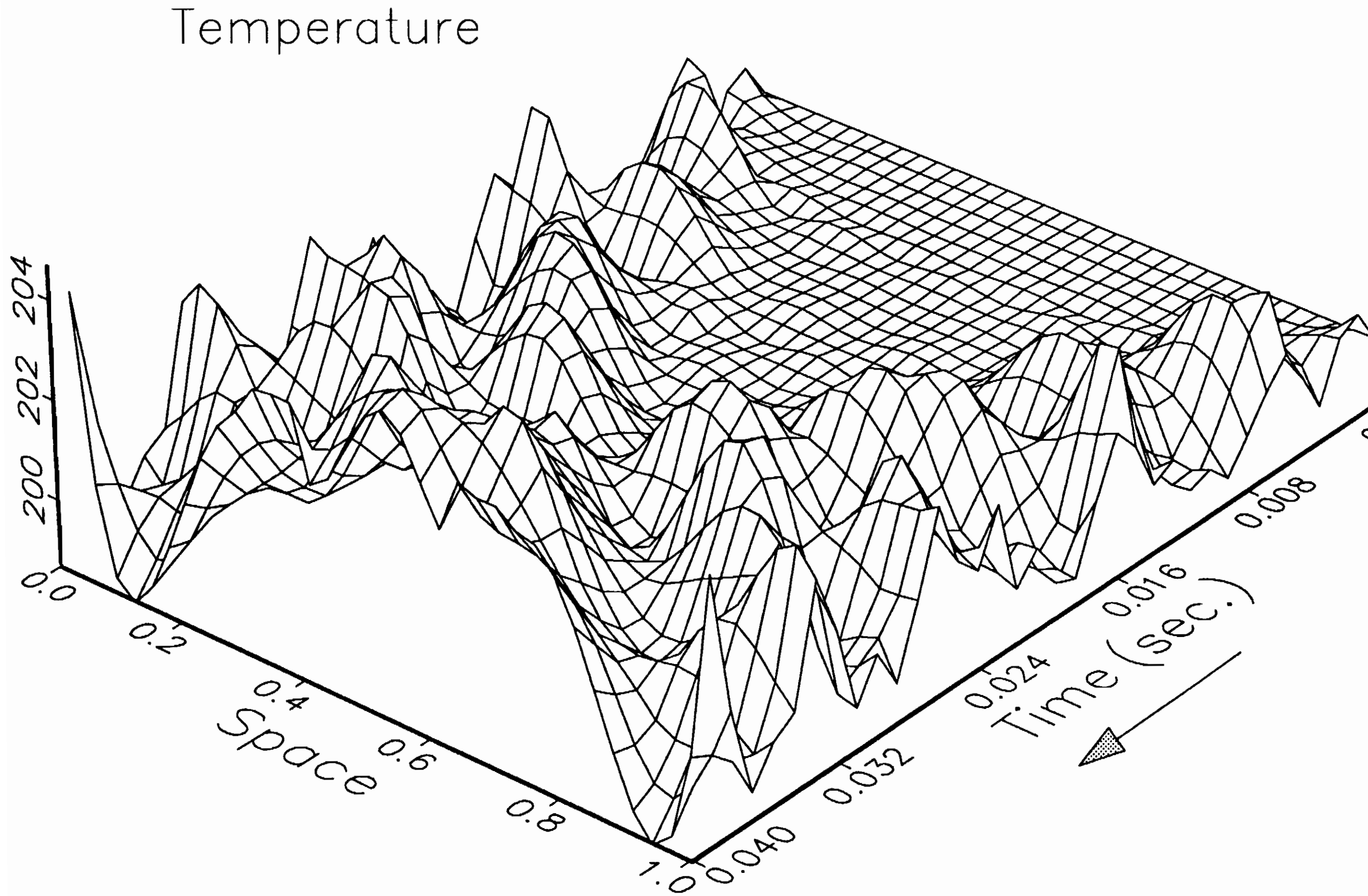


Figure 4.2.5-(B): Evolution of temperature (θ), corresponding to the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.3 and $\alpha_2 = 2.4$

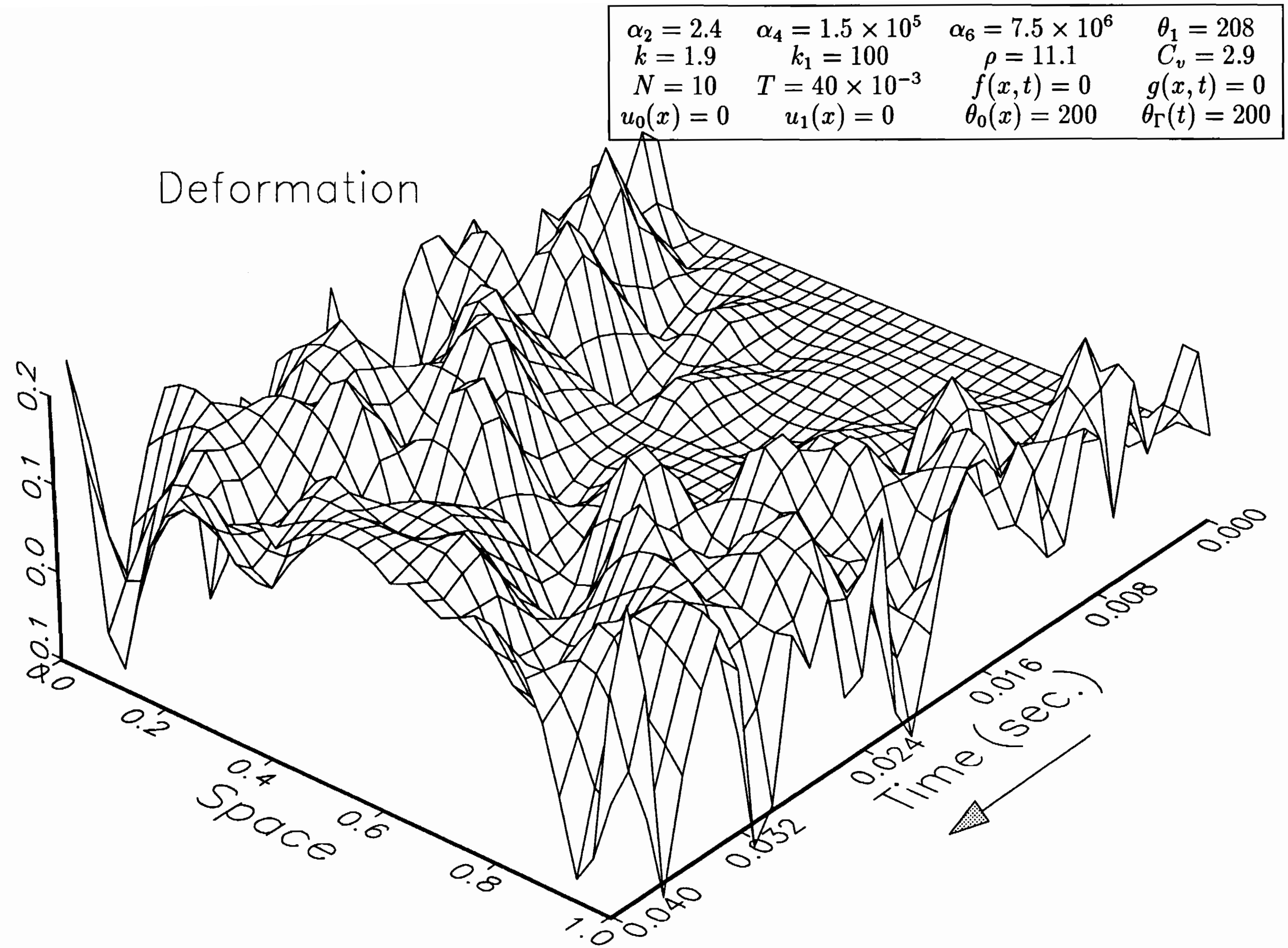


Figure 4.2.5-(C): Evolution of deformation (u_x), corresponding to the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.3 and $\alpha_2 = 2.4$

$\alpha_2 = 0.24$	$\alpha_4 = 1.5 \times 10^5$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x,t) = 0$	$g(x,t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

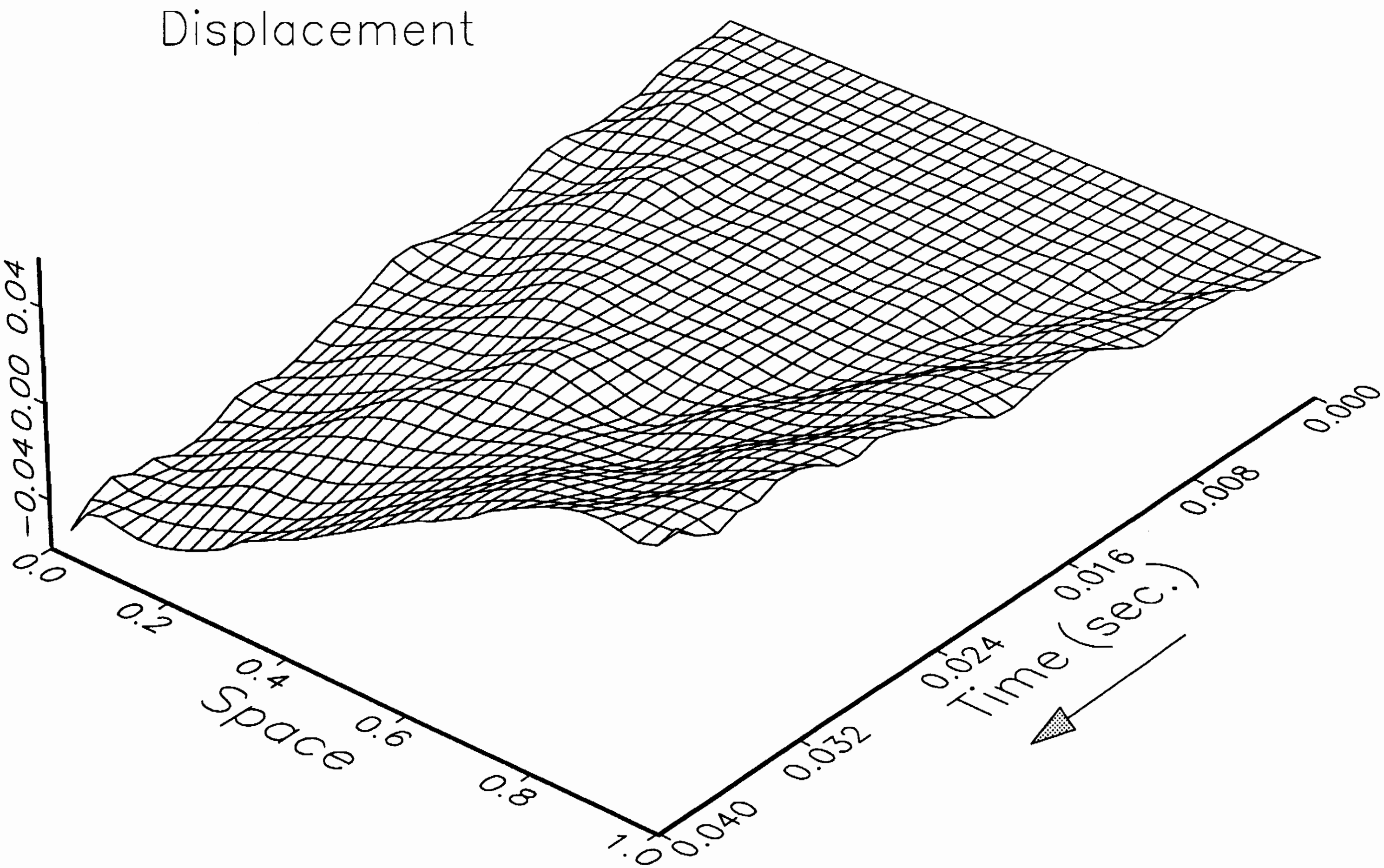


Figure 4.2.6-(A): Evolution of displacement (u), corresponding to the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.3 and $\alpha_2 = 0.24$

$\alpha_2 = 0.24$	$\alpha_4 = 1.5 \times 10^5$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x,t) = 0$	$g(x,t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

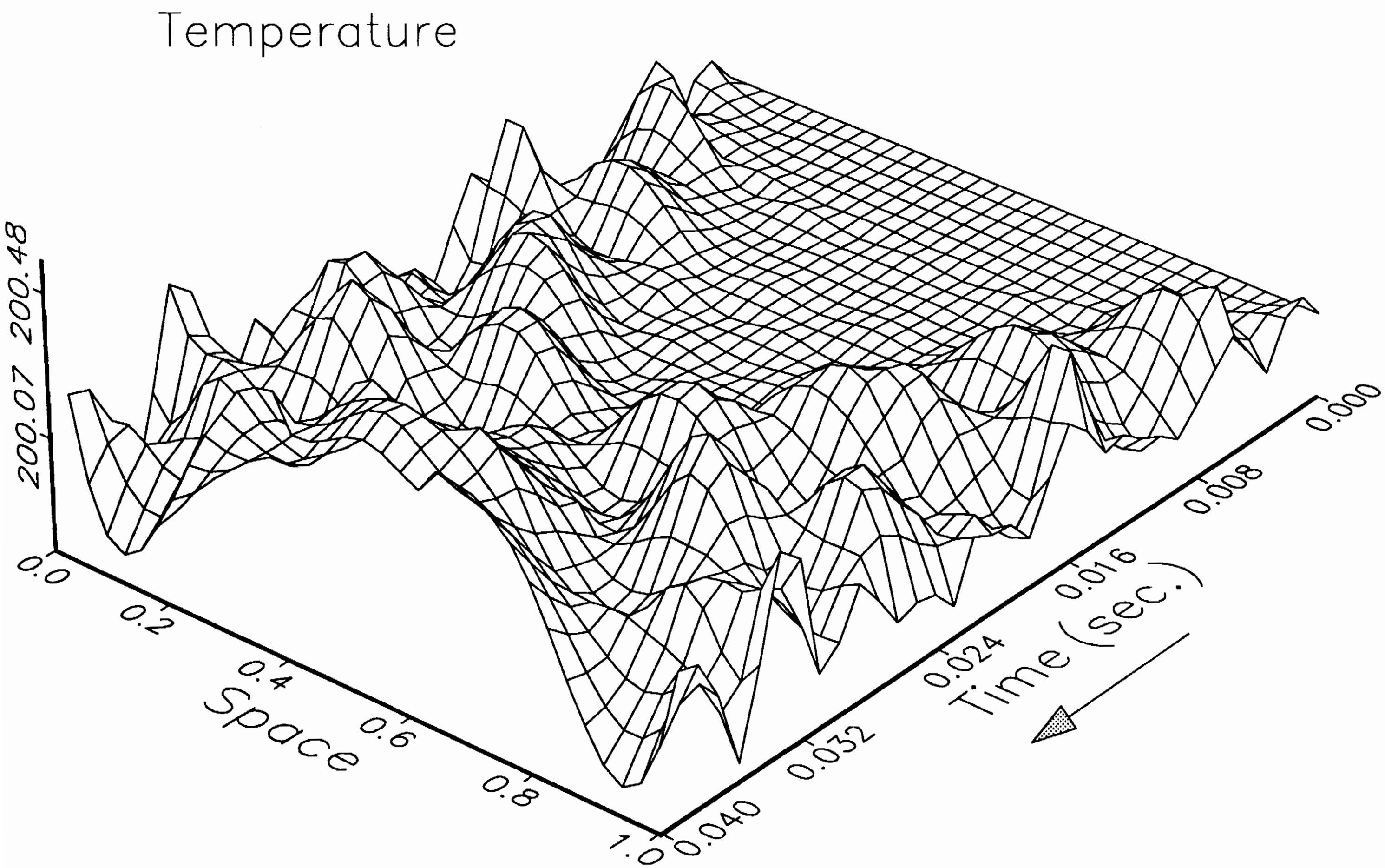


Figure 4.2.6-(B): Evolution of temperature (θ), corresponding to the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.3 and $\alpha_2 = 0.24$

$\alpha_2 = 0.24$	$\alpha_4 = 1.5 \times 10^5$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x,t) = 0$	$g(x,t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

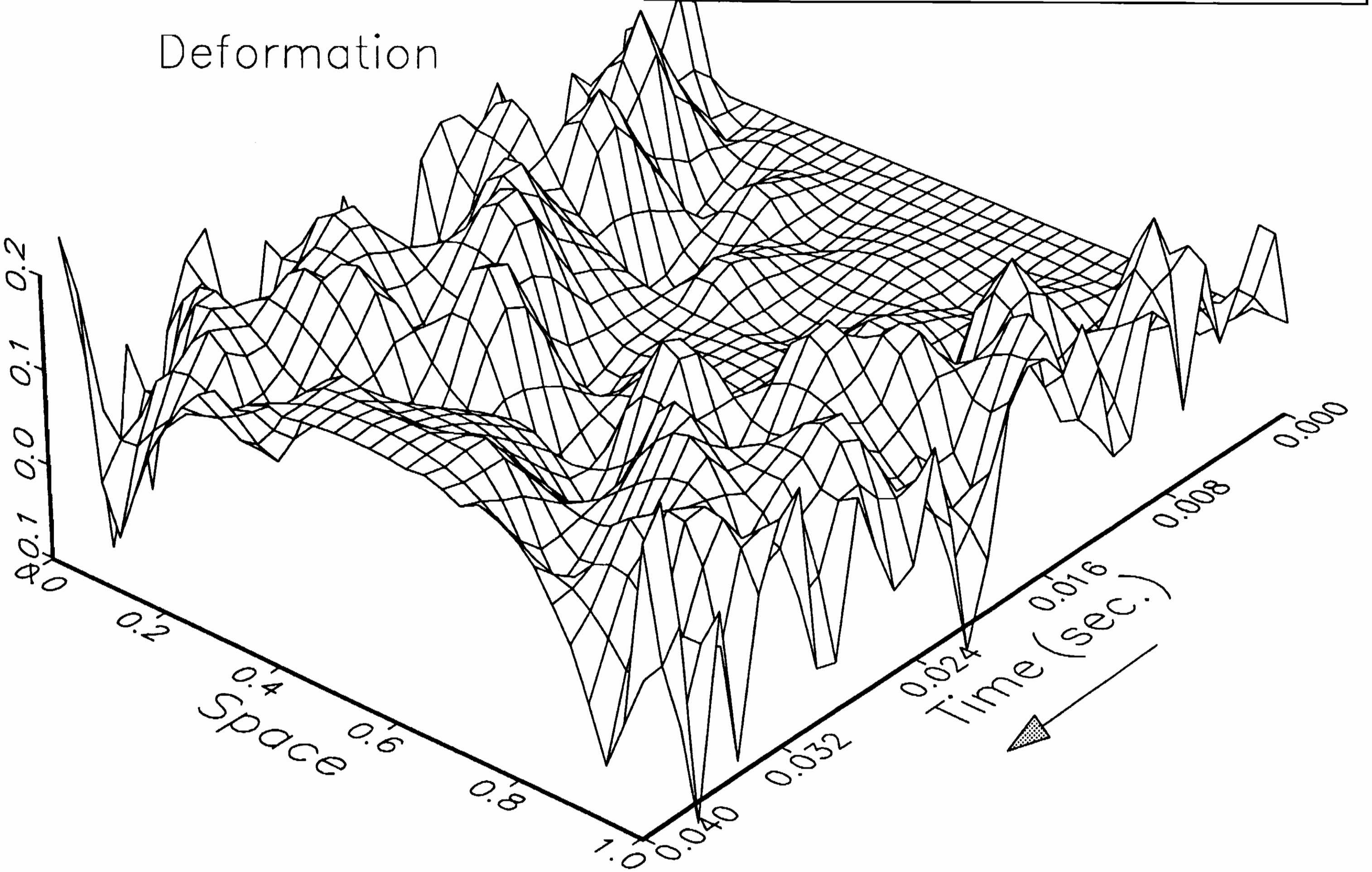


Figure 4.2.6-(C): Evolution of deformation (u_x), corresponding to the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.3 and $\alpha_2 = 0.24$

	$\alpha_4 = 1.5 \times 10^5$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x,t) = 0$	$g(x,t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

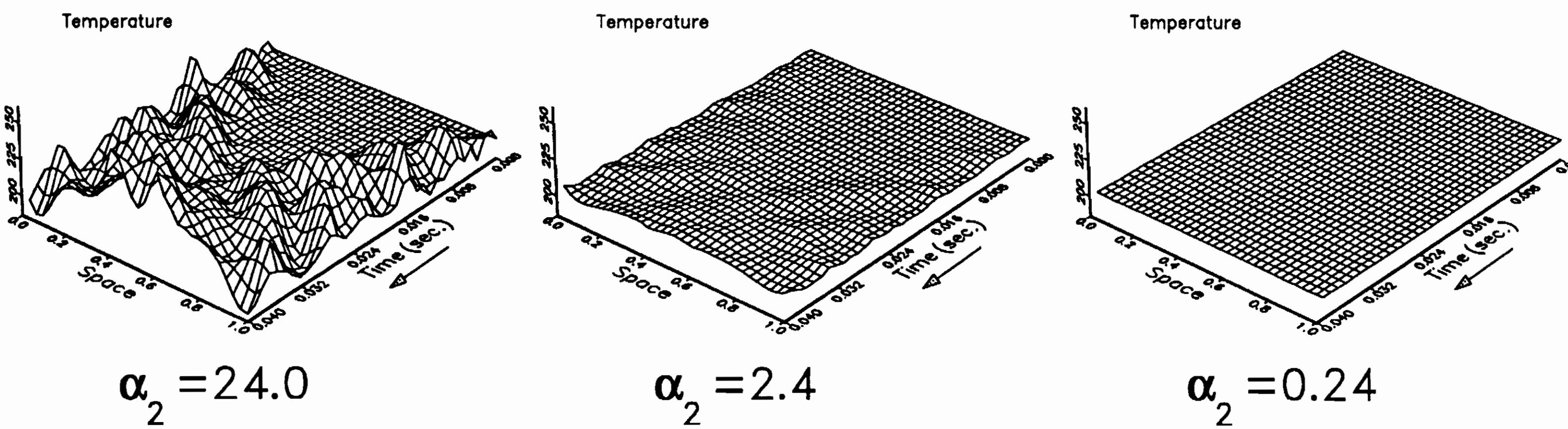


Figure 4.2.7: Comparisson of the temperature evolutions obtained with different values of α_2 and the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.3

$\alpha_2 = 24$	$\alpha_4 = 1.5 \times 10^5$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 500$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x,t) = 0$	$g(x,t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

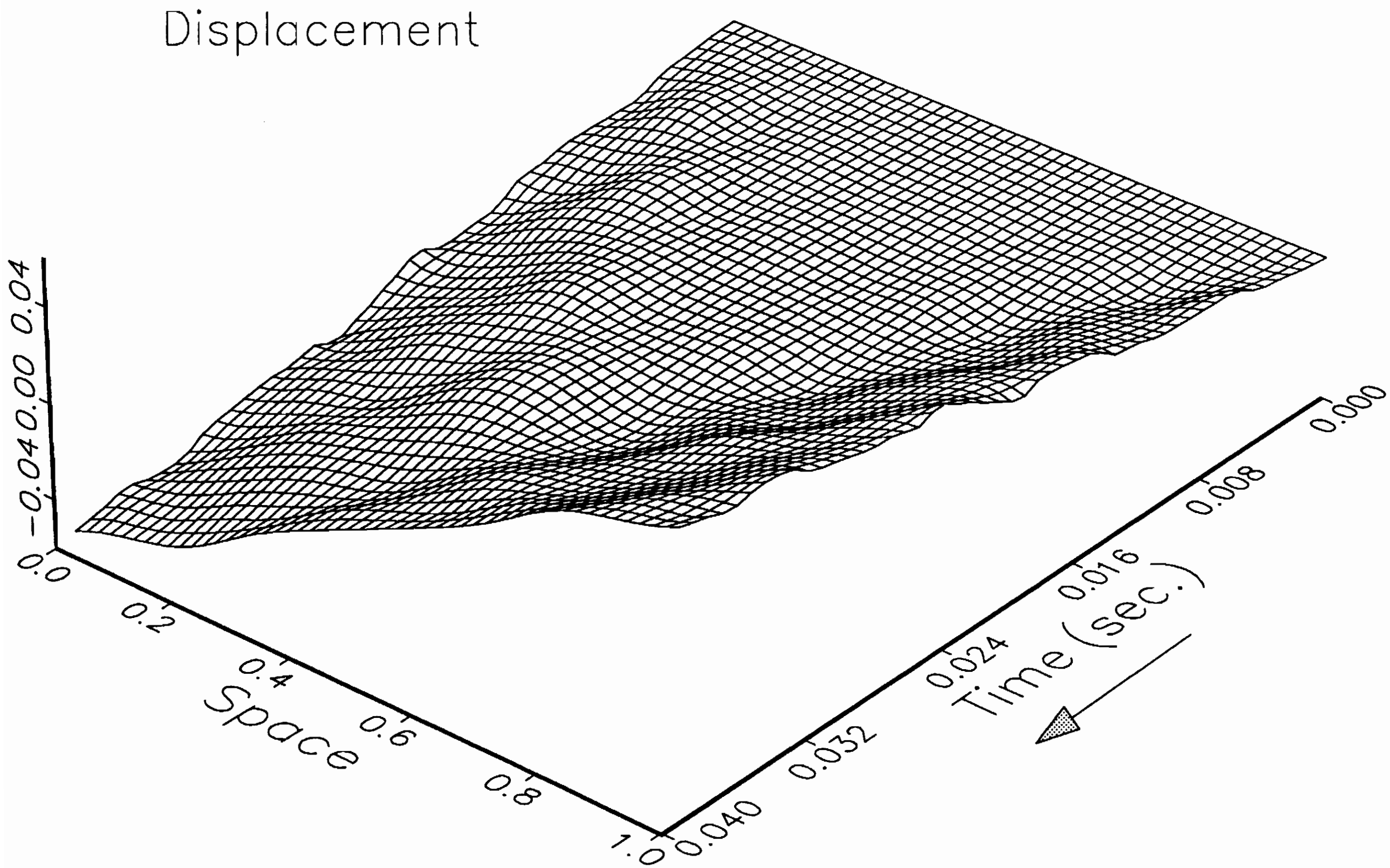


Figure 4.2.8-(A): Evolution of displacement (u), corresponding to the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.3 and $k_1 = 500$

$\alpha_2 = 24$	$\alpha_4 = 1.5 \times 10^5$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 500$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x,t) = 0$	$g(x,t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

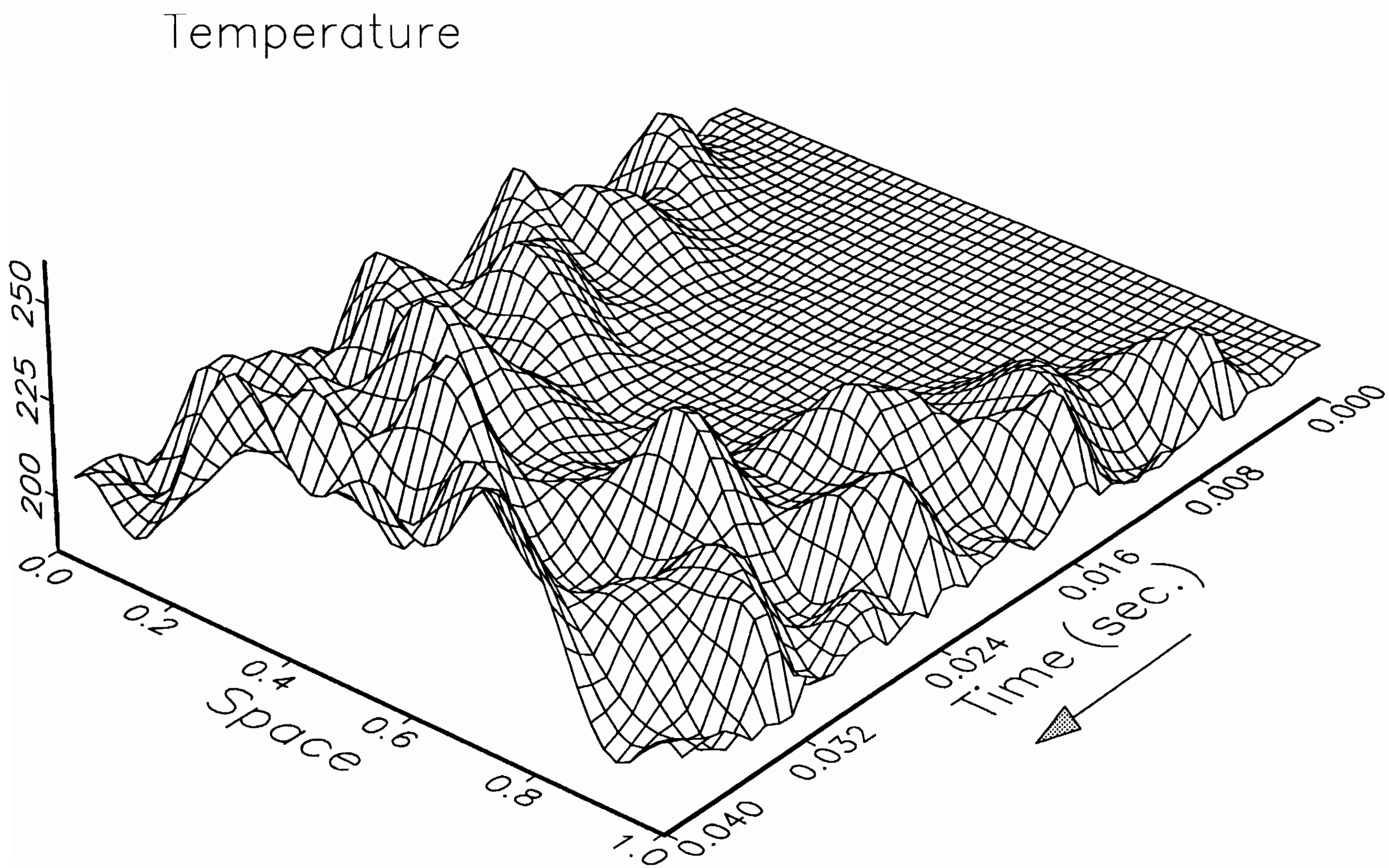


Figure 4.2.8-(B): Evolution of temperature (θ), corresponding to the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.3 and $k_1 = 500$

$\alpha_2 = 24$	$\alpha_4 = 1.5 \times 10^5$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 500$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x,t) = 0$	$g(x,t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

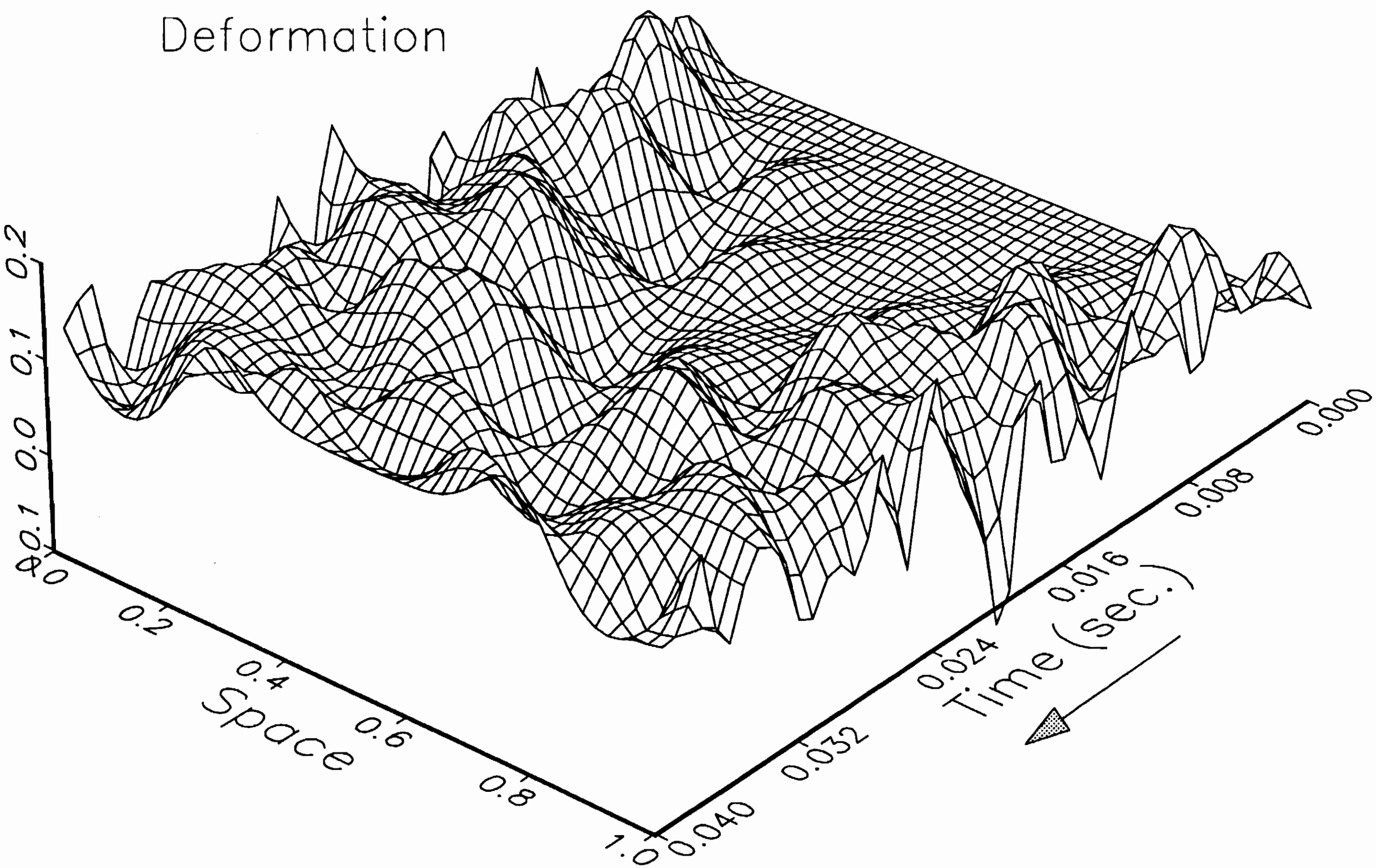


Figure 4.2.8-(C): Evolution of deformation (u_x), corresponding to the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.3 and $k_1 = 500$

$\alpha_2 = 24$	$\alpha_4 = 1.5 \times 10^5$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 19.0$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x,t) = 0$	$g(x,t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

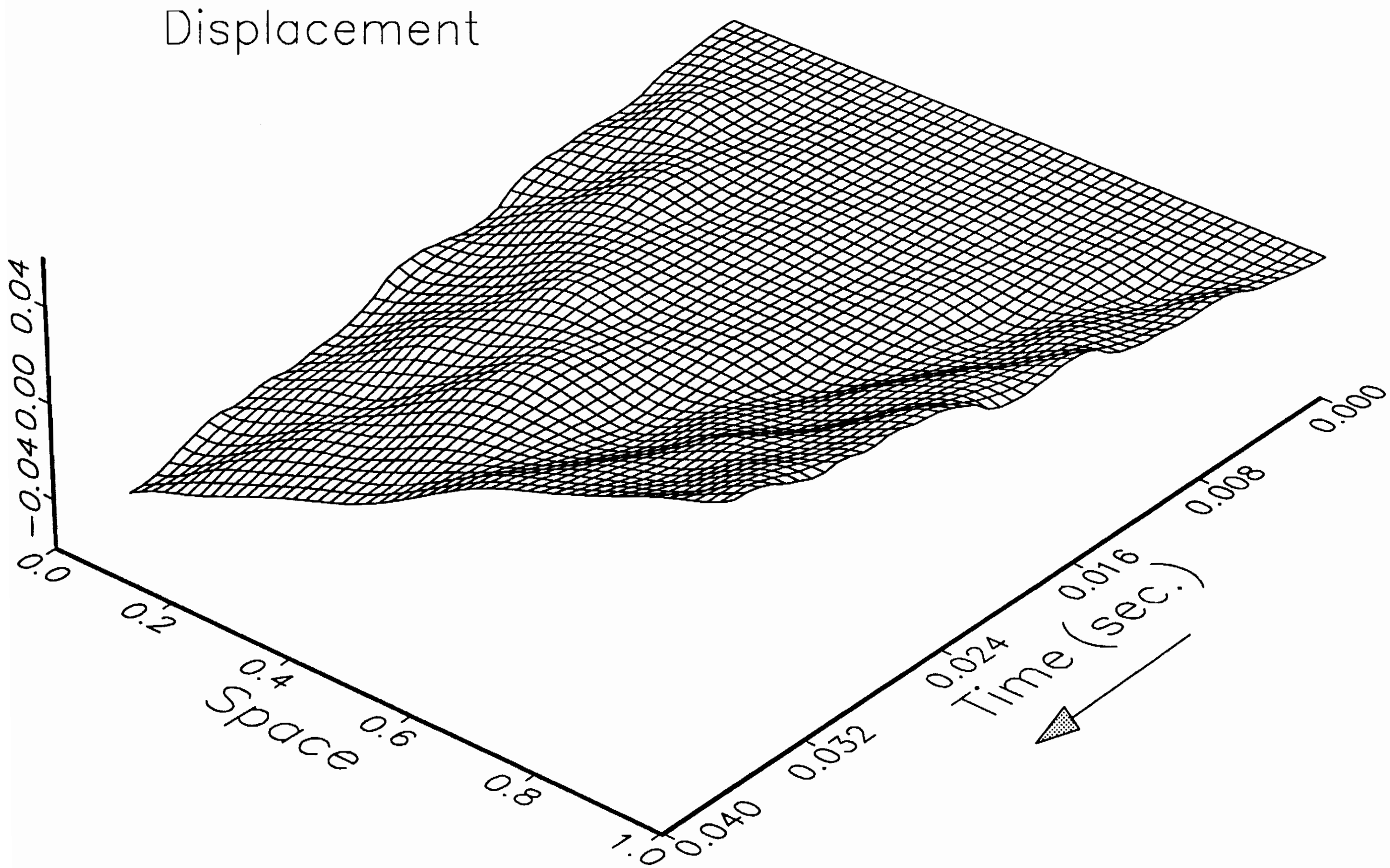


Figure 4.2.9-(A): Evolution of displacement (u), corresponding to the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.3 and $k = 19.0$

$\alpha_2 = 24$	$\alpha_4 = 1.5 \times 10^5$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 19.0$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x,t) = 0$	$g(x,t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

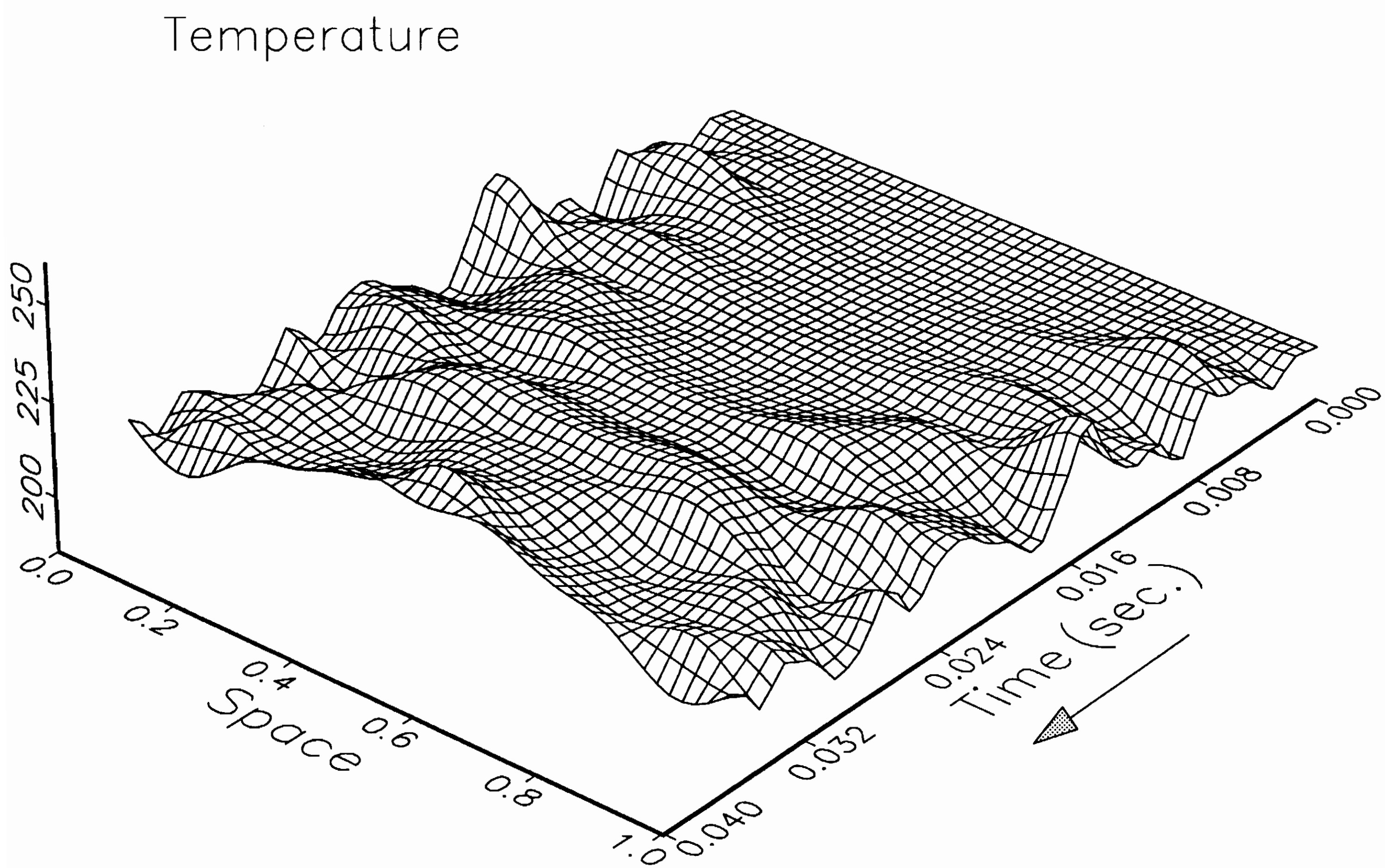


Figure 4.2.9-(B): Evolution of temperature (θ), corresponding to the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.3 and $k = 19.0$

$\alpha_2 = 24$	$\alpha_4 = 1.5 \times 10^5$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 19.0$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x,t) = 0$	$g(x,t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

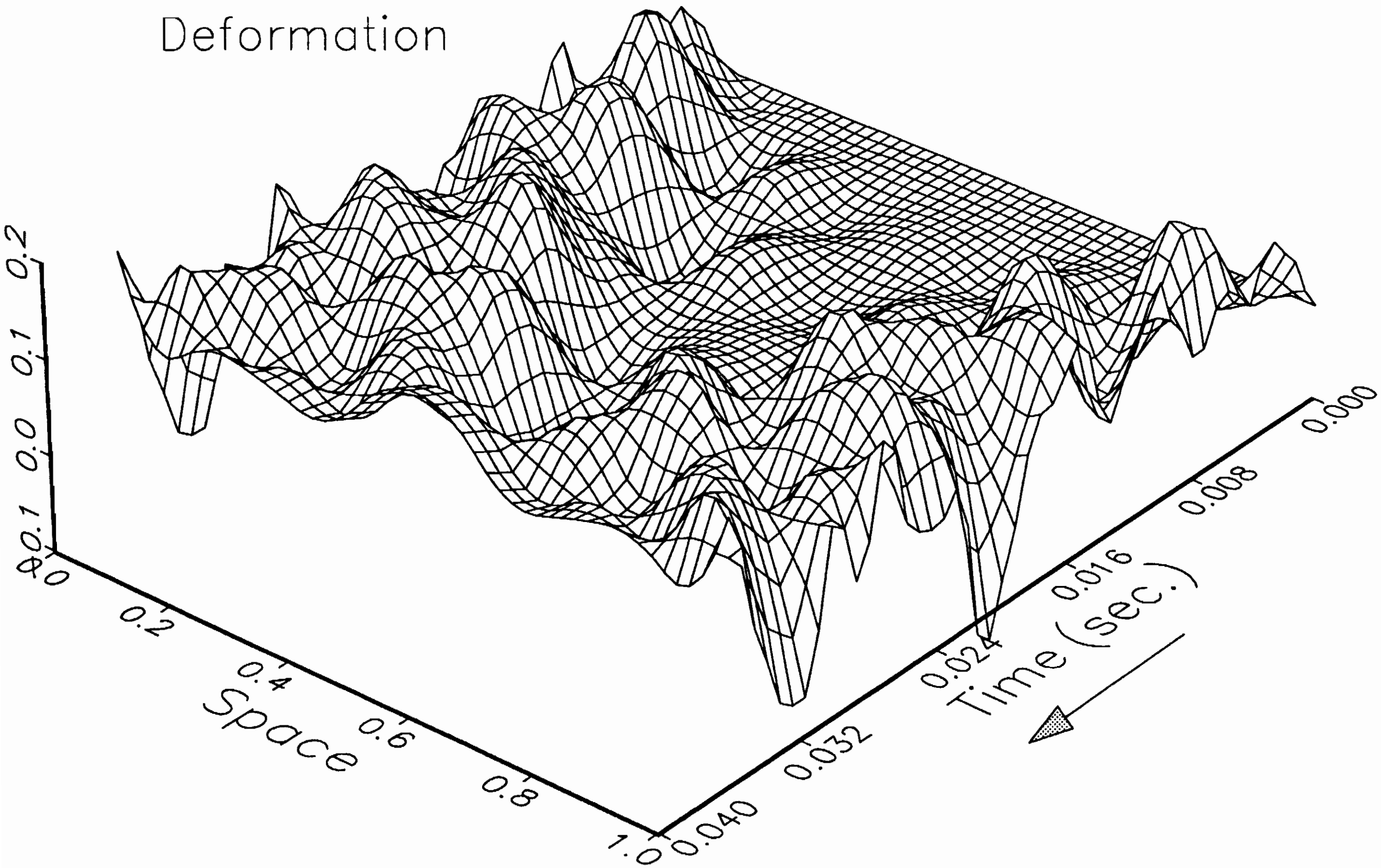
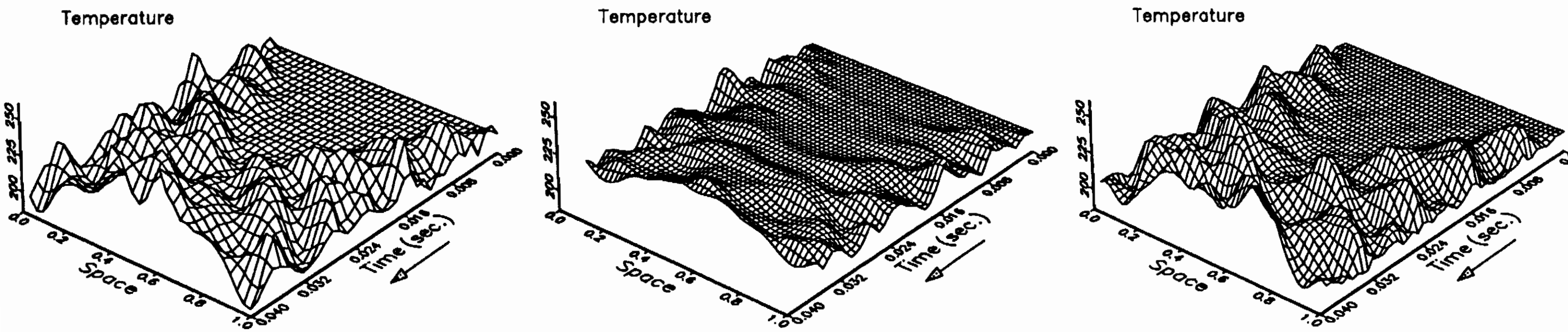


Figure 4.2.9-(C): Evolution of deformation (u_x), corresponding to the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.3 and $k = 19.0$

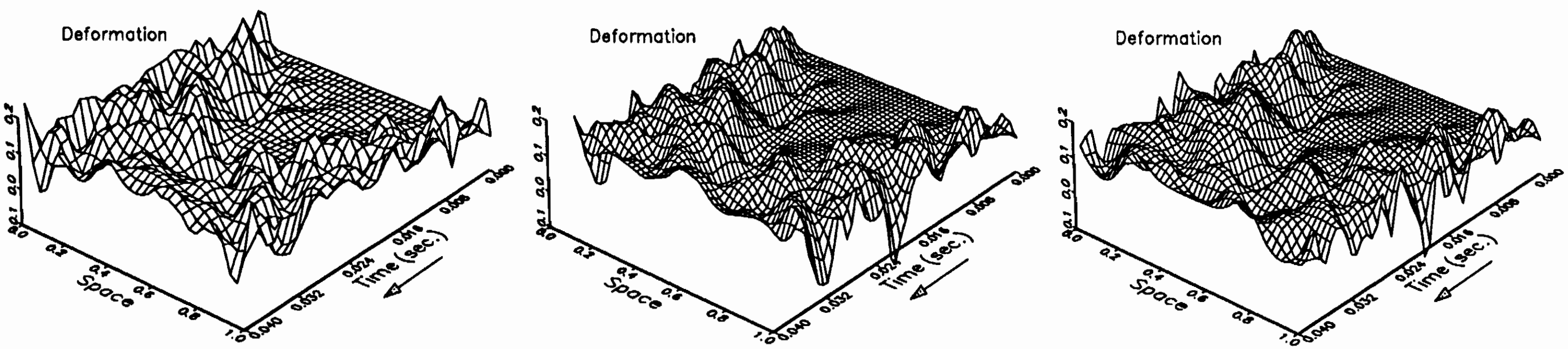
$\alpha_2 = 24$	$\alpha_4 = 1.5 \times 10^5$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
		$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x,t) = 0$	$g(x,t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$



$k_1 = 100$, $k = 1.9$ $\quad$ $k_1 = 100$, $k = 19.0$ $\quad$ $k_1 = 500$, $k = 1.9$

Figure 4.2.10-(A): Comparisson of the temperature evolutions obtained with different values of k and k_1 and the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.3

$\alpha_2 = 24$	$\alpha_4 = 1.5 \times 10^5$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
		$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x,t) = 0$	$g(x,t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$



$k_1 = 100,\ k = 1.9$ $\quad$ $k_1 = 100,\ k = 19.0$ $\quad$ $k_1 = 500,\ k = 1.9$

Figure 4.2.10-(B): Comparisson of the deformation evolutions obtained with different values of k and k_1 and the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.3 and $\alpha_2 = 24$

$\alpha_2 = 24$	$\alpha_4 = 1.5 \times 10^4$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x,t) = 0$	$g(x,t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

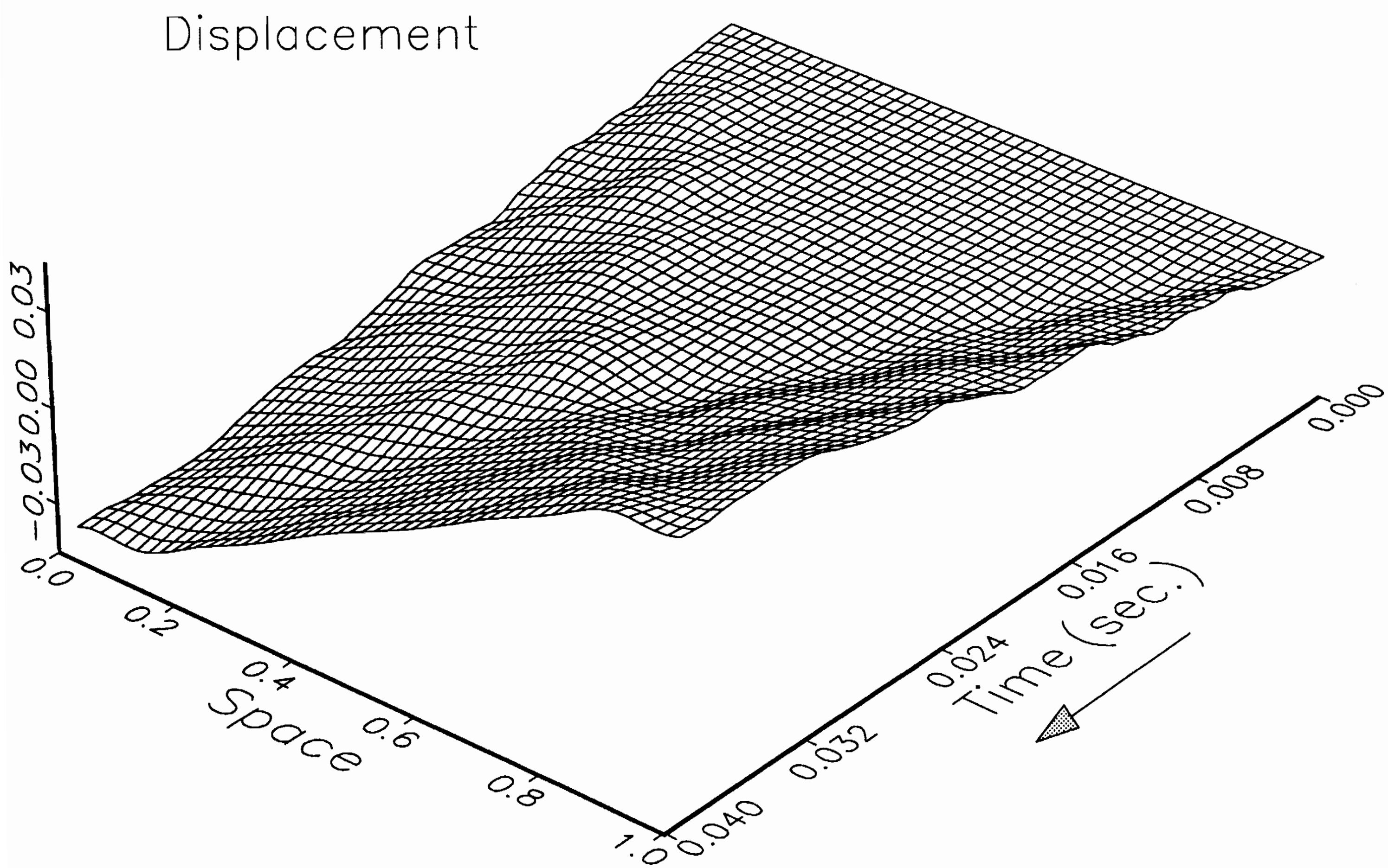


Figure 4.2.11-(A): Evolution of displacement (u), corresponding to the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.3 and $\alpha_4 = 1.5 \times 10^4$

$\alpha_2 = 24$	$\alpha_4 = 1.5 \times 10^4$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x,t) = 0$	$g(x,t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

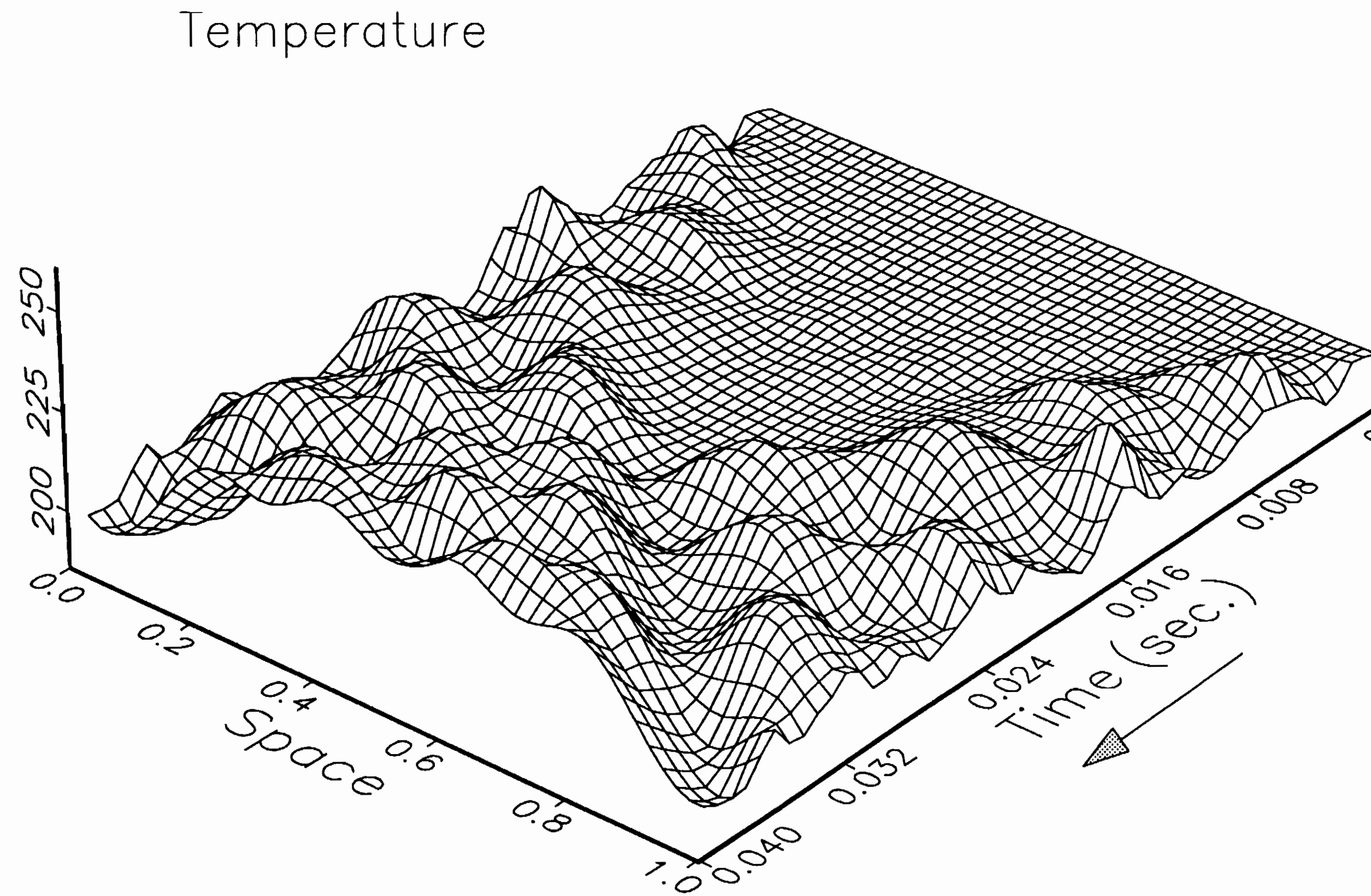


Figure 4.2.11-(B): Evolution of temperature (θ), corresponding to the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.3 and $\alpha_4 = 1.5 \times 10^4$

$\alpha_2 = 24$	$\alpha_4 = 1.5 \times 10^4$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x,t) = 0$	$g(x,t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

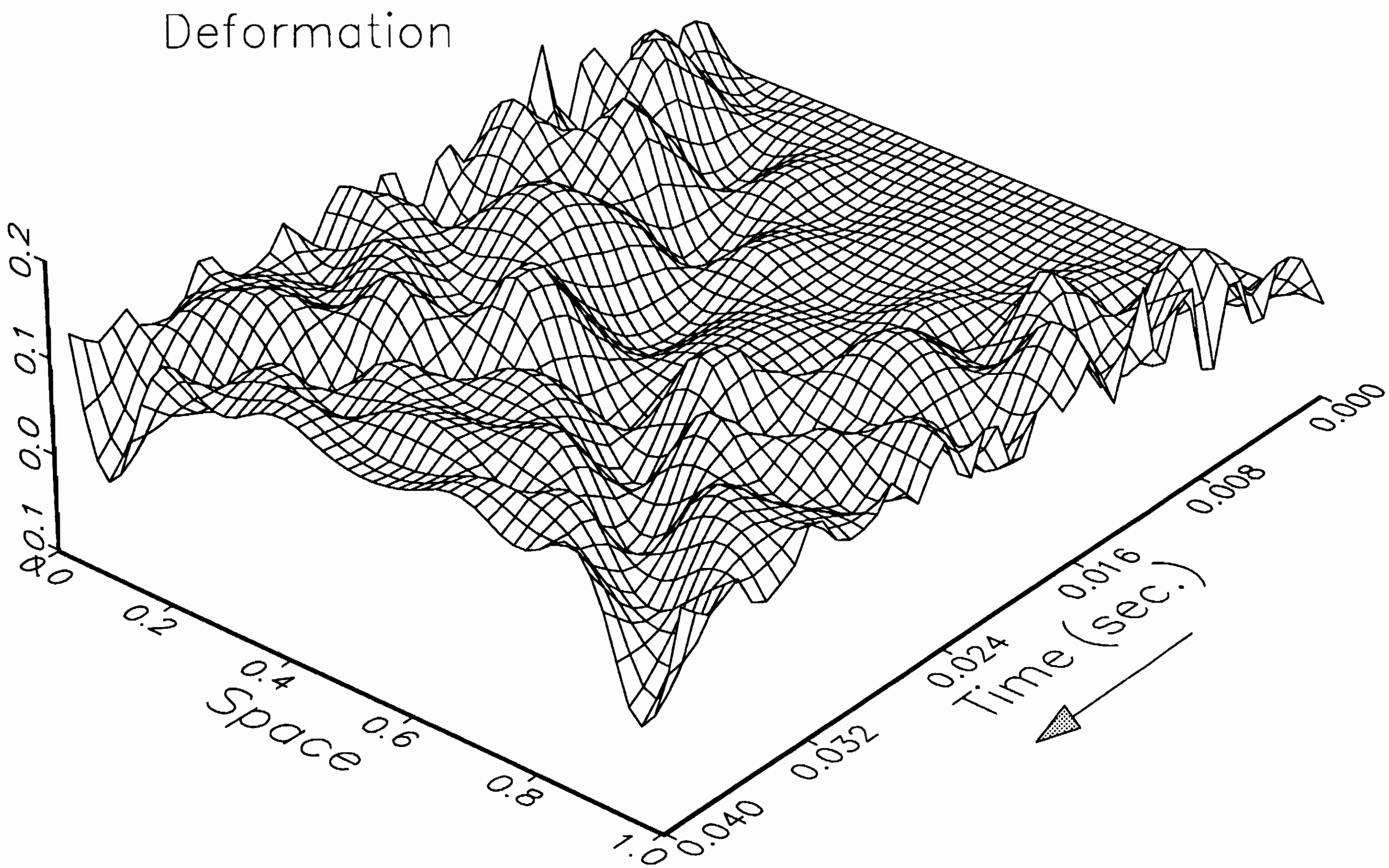


Figure 4.2.11-(C): Evolution of deformation (u_x), corresponding to the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.3 and $\alpha_4 = 1.5 \times 10^4$

$\alpha_2 = 24$	$\alpha_4 = 1.5 \times 10^3$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x,t) = 0$	$g(x,t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

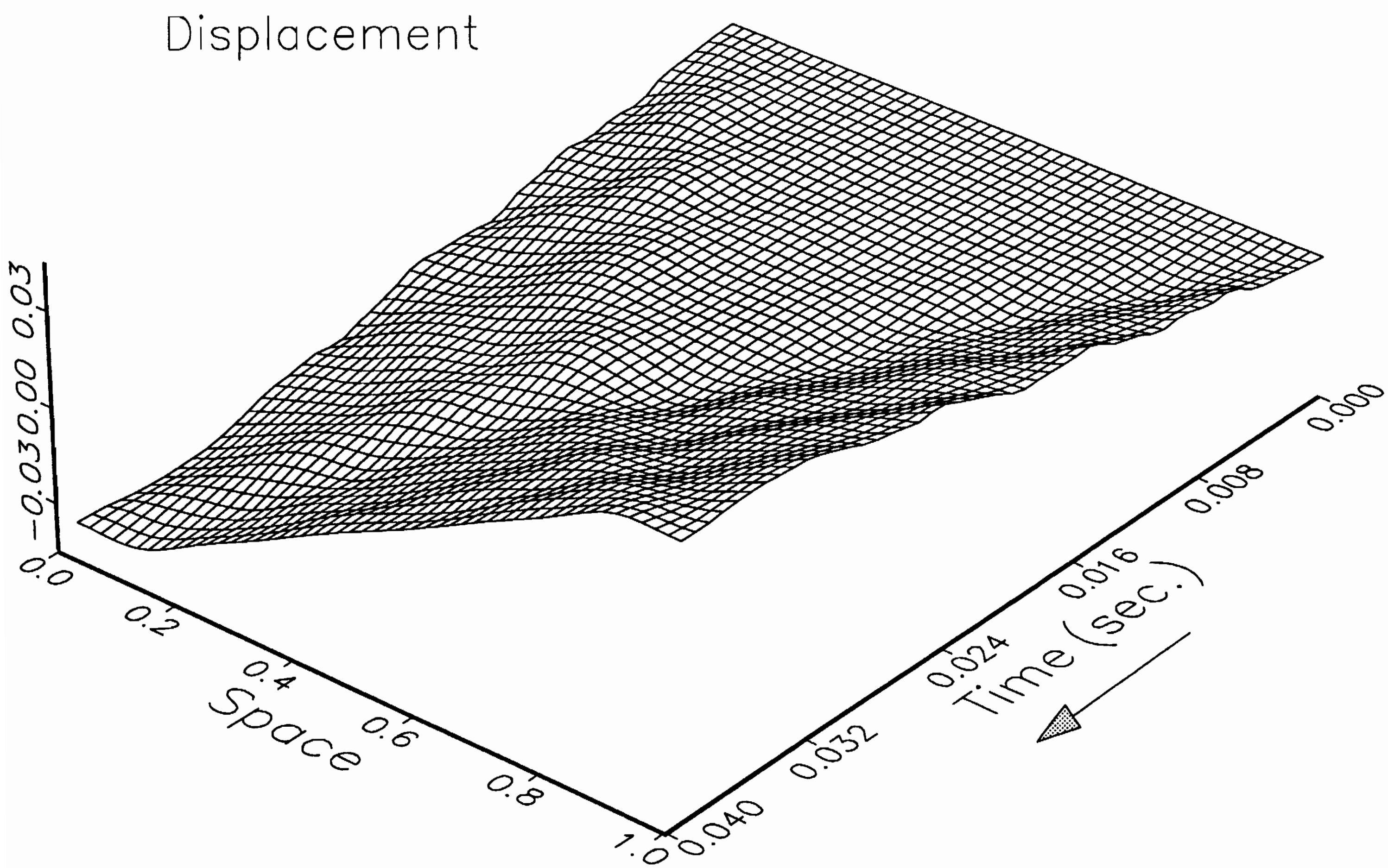


Figure 4.2.12-(A): Evolution of displacement (u), corresponding to the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.3 and $\alpha_4 = 1.5 \times 10^3$

$\alpha_2 = 24$	$\alpha_4 = 1.5 \times 10^3$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x,t) = 0$	$g(x,t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

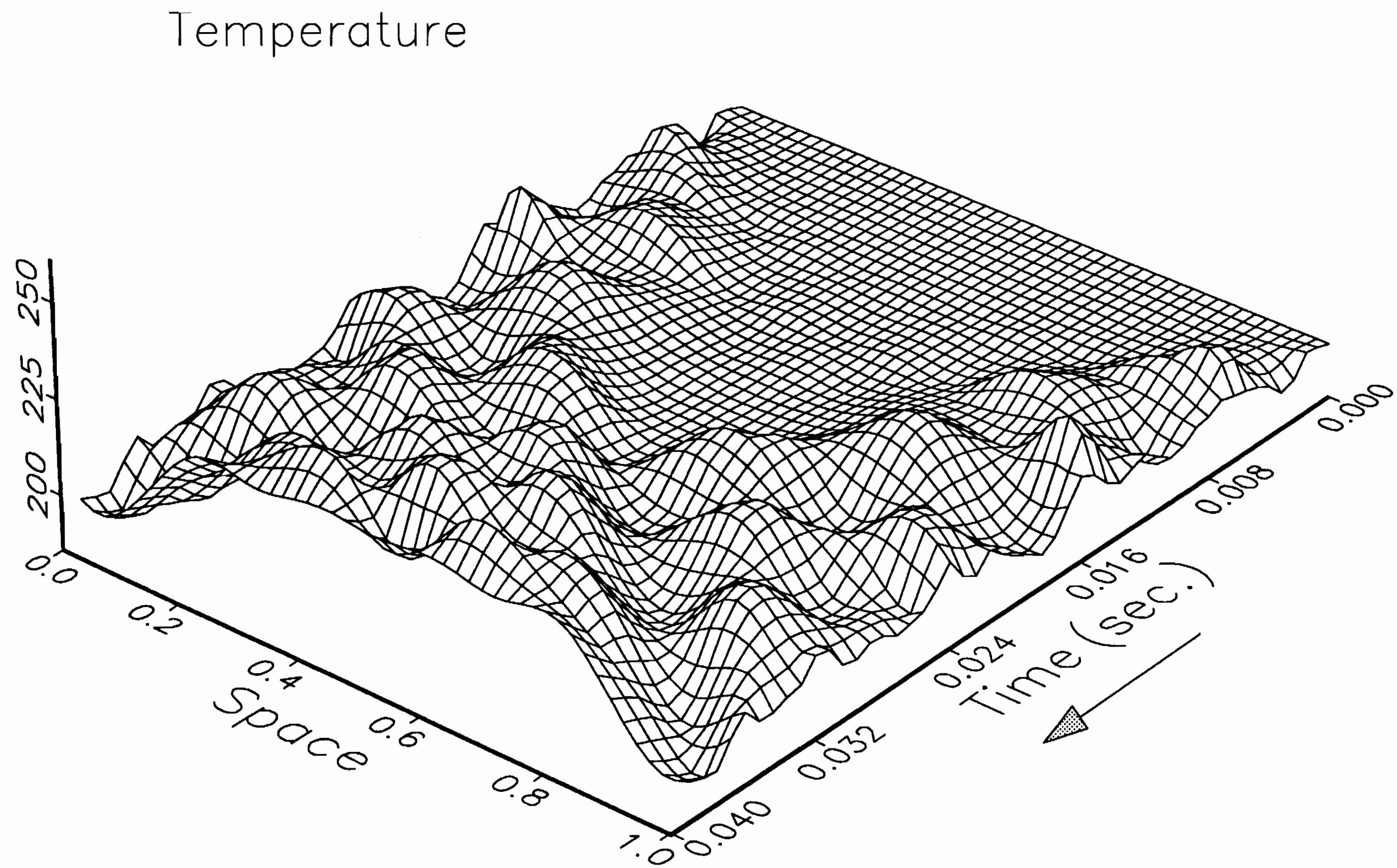


Figure 4.2.12-(B): Evolution of temperature (θ), corresponding to the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.3 and $\alpha_4 = 1.5 \times 10^3$

$\alpha_2 = 24$	$\alpha_4 = 1.5 \times 10^3$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x,t) = 0$	$g(x,t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

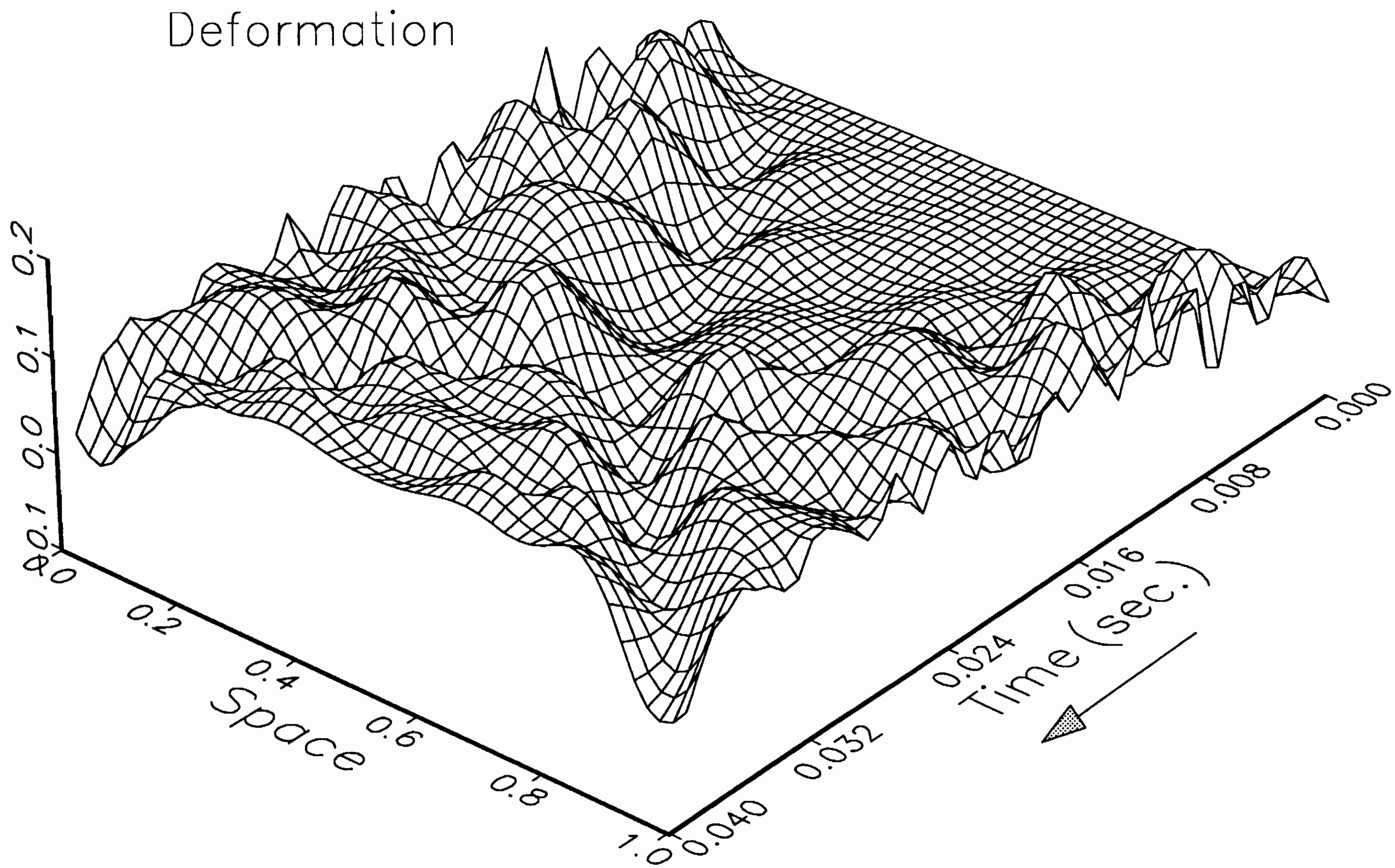
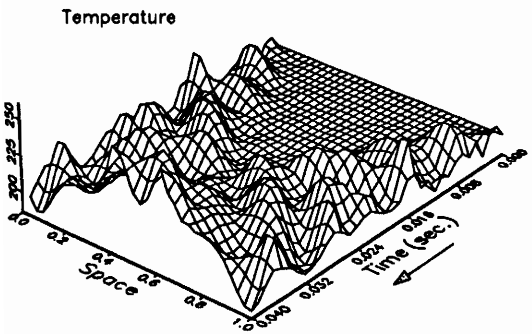
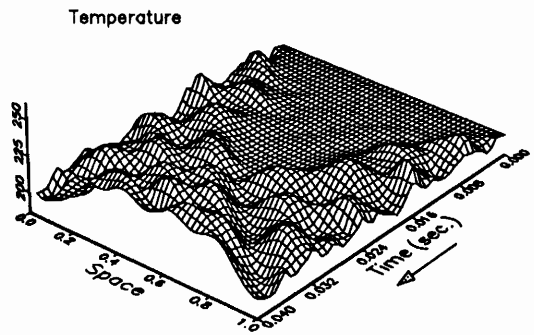


Figure 4.2.12-(C): Evolution of deformation (u_x), corresponding to the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.3 and $\alpha_4 = 1.5 \times 10^3$

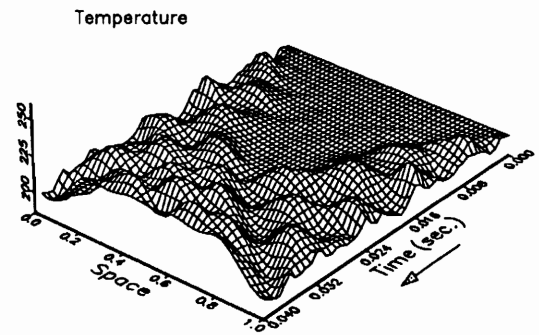
$\alpha_2 = 24$		$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x,t) = 0$	$g(x,t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$



$\alpha_4 = 1.5 \times 10^5$



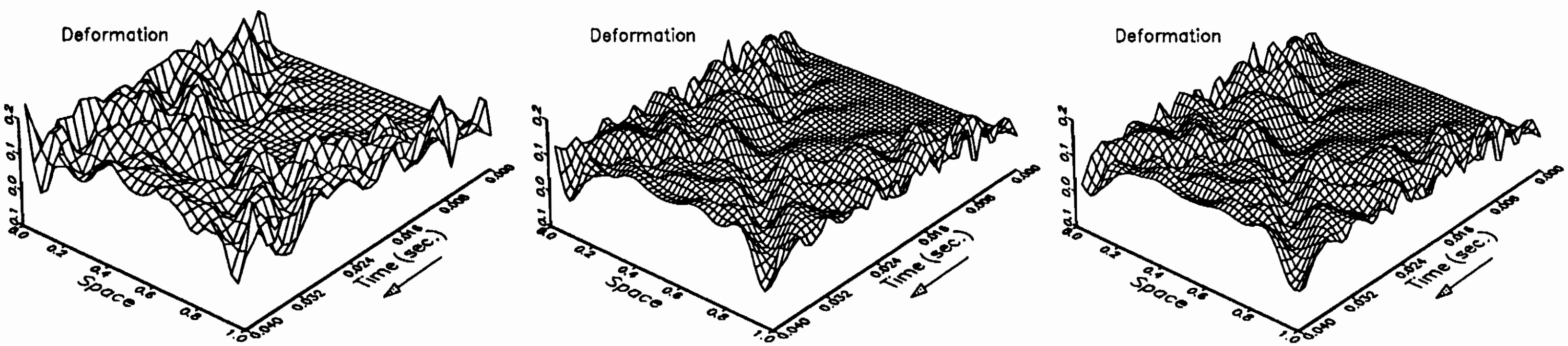
$\alpha_4 = 1.5 \times 10^4$



$\alpha_4 = 1.5 \times 10^3$

Figure 4.2.13-(A): Comparisson of the temperature evolutions obtained with different values of α_4 and the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.3

$\alpha_2 = 24$		$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x,t) = 0$	$g(x,t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$



$\alpha_4 = 1.5 \times 10^5$ $\qquad$ $\alpha_4 = 1.5 \times 10^4$ $\qquad$ $\alpha_4 = 1.5 \times 10^3$

Figure 4.2.13-(B): Comparisson of the deformation evolutions obtained with different values of α_4 and the boundary stress $\sigma_\Gamma(t)$ shown in Figure 4.2.3 and $\alpha_2 = 24$

4.3 Summary, Conclusions and Future Plans

In this paper we have developed an abstract framework for a state space formulation of a generalized one-dimensional dynamic mathematical model of phase transitions in materials with memory. We have proved the well-posedness for the physically relevant case in which no thermal memory is present ($\alpha = 0$) and for thermodynamic potentials of the Landau-Ginzburg type. We have obtained a spectral decomposition and explicit decay rates for the associated linear semigroup and we proved its continuous dependence on the model parameters. Numerical experiments using finite-dimensional approximations were performed and the sensitivity of the solutions with respect to some of the parameters in the free energy was studied. Finally, an alternative approach to the stress-strain laws was presented which captures the dependence on the strain history.

There is much room for further study. Some of the areas in which we plan to continue our research are described below.

1) The proof of the well-posedness that we have given, strongly depends on the fact that $\gamma > 0$. Although the numerical experiments seem to show that solutions exist even in the case $\gamma = 0$, no rigorous theoretical proof of this fact is known up to now. In this case, by leaving the term $2\alpha_2\theta_1 u_{xx}$ on the left-hand side of the equation (3.3.2a) we were able to show that the resulting linear operator $A(q)$ is quasidisipative and the generator of an analytic semigroup on the Hilbert space $H_0^1(\Omega) \times L_2(\Omega) \times L_2(\Omega)$

with the energy inner product $\left\langle \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \end{pmatrix} \right\rangle = 2\alpha_2\theta_1 \int_\Omega u'\hat{u}' + \rho \int_\Omega v\hat{v} + \frac{C_v}{k} \int_\Omega w\hat{w}$. However, we could not show that the nonlinear term is Lipschitz continuous in the state variable with respect to the graph norm of the operator $A(q)$. We plan to continue our efforts in this direction.

2) The operator $A(q)$ given by (3.3.8-9) is a fourth order differential operator in u and second order in v and the w, while the nonlinear term $F(q,t,z)$, $z = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$, given by (3.3.10) is only second order in u and first order in v and w. For this reason, we strongly believe that this nonlinear term is Lipschitz in the state variable with respect to the graph norm of the square root of $-A(q)$. If this were true, one could derive local existence for a much broader set of initial conditions, namely for $z_0 \in dom\left([-A(q)]^{1/2}\right)$. The problem is that the square root of a differential operator is explicitely known only for a special subset of the natural boundary conditions and even in simple cases is known only up to a multiplicative bounded operator ([**119**]). Although the proof of the conjecture described above does not necessarily imply finding $[-A(q)]^{1/2}$ explicitely, it does involve finding bounds for $\|u''\|_{L_2}$, $\|v'\|_{L_2}$ and $\|w'\|_{L_2}$ in terms of $\left|\left|\left| \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right|\right|\right|_{[-A(q)]^{1/2}}$ for $\begin{pmatrix} u \\ v \\ w \end{pmatrix} \in dom\left([-A(q)]^{1/2}\right)$. We also intent to devote efforts to solving this problem.

3) We plan to develop optimization algorithms for parameter estimation. This objective will involve the development of more efficient approximation schemes and the design of laboratory experiments to collect appropriate data for particular alloys. Estimation techniques will then be used to obtain accurate approximations to the model parameters. Success in the above step will provide us with the basic computational tools necessary to test and validate the proposed models.

4) Most of the data for SMA's come from uniaxial tensile stretching experiments. For this reason, it is important from a practical point of view that the mathematical models include time dependent stresses as possible boundary conditions. If $\gamma > 0$, an extra pair of boundary conditions is needed and it is not clear what would be appropriate for this case.

5) Other areas of interest in which we plan to pursue further results are the inclusion of history-dependent stress-strain relations into the dynamic model and the study of the effects of viscosity and couple stress.

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VITA

Ruben D. Spies was born in Calchaquí, Santa Fe, Argentina, on December 3, 1962. He graduated from the Escuela de Enseñanza Media 233 of Calchaquí in 1980. He received the B.S. degree in Mathematics from the Universidad Nacional del Litoral, Santa Fe, Argentina in 1986. During 1985 he was a graduate fellow at the Instituto Nacional de Investigaciones Estadísticas INIE, Tucumán, Argentina. During 1986-1987 he worked as a post-graduate fellow at the Instituto de Desarrollo Tecnológico para la Industria Química INTEC, Santa Fe, Argentina. He received the M.S. and the Ph.D. degrees in Mathematics from the Virginia Polytechnic Institute and State University in 1989 and 1992, respectively.

He is member of the American Mathematical Society and the Society for Industrial and Applied Mathematics.

Ruben D. Spies