 SOURCES OF LOCALIZED WAVES

by

Argyrios Alexandros Chatzipetros

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APPROVED

I. M. Besieris, Chairman

G. S. Brown

D. A. de Wolf

W. L. Stutzman

W. E. Kohler

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Argyrios Alexandros Chatzipetros

Committee Chairman: Ioannis M. Besieris

(ABSTRACT)

The synthesis of two types of Localized Wave (LW) pulses is considered; these are the 'Focus Wave Mode' (FWM) pulse and the X Wave pulse. First, we introduce the modified bidirectional representation where one can select new basis functions resulting in different representations for a solution to the scalar wave equation. Through this new representation, we find a new class of focused X Waves which can be extremely localized. The modified bidirectional decomposition is applied to the nonhomogeneous scalar wave equation, resulting in moving sources generating LW pulses. In this work, we also address the possibility of exciting LW pulses from dynamic apertures, or apertures the effective radius of which is varied with time. Ideal LW pulses cannot be realized because they require infinite time excitation. However, in the case of finite LW pulses, the aperture of excitation is finite and is varied from a time \(-T\) to \(T\). We show that the resulting LW pulses are more resistant to decay than classical monochromatic Gaussian pulses occupying the same beam waist. Both types of finite LW pulses, such as the FWM and X Wave pulse, can propagate without significant decay to much greater distances than classical monochromatic pulses. This desirable behavior is attributed to the superior aperture efficiency of the LW pulses, which in turn is attributed to their unique spectral structure.
In memory of my sister Mina
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pulse decay.
1.0 INTRODUCTION

The search for packet-like solutions to the homogeneous Maxwell's equations was initiated by Brittingham [1] ten years ago. He proposed solutions with the following properties: (1), they are continuous and nonsingular, (2) they have a three-dimensional pulse structure, (3) they are nondispersive for all time, (4) they move with the speed of light, and (5) they carry finite electromagnetic energy. These solutions have been termed focus wave modes (FWM). The original FWMs were found by Brittingham to satisfy the first four of the aforementioned properties but their energy content was infinitely large. Brittingham forced the FWMs to have finite energy by introducing two artificial surfaces of discontinuities infinitely extended along the direction of propagation, thus, dividing space into three regions. The fields between the two surfaces were chosen equal to the original FWMs, while the fields outside were set equal to zero. Wu and King [2] showed that Maxwell's equations could not be satisfied across these discontinuities and they established that the FWMs carried infinite energy. Wu and King's work was verified by Sezginer [3] and Wu and Lehman [4] who proved that any finite energy solution leads to dispersion and the spreading of energy. The work by Belanger [5], Sezginer [3] and Ziolkowski [6] showed that the original FWMs could be related to exact solutions of the three-dimensional scalar wave equation. Ziolkowski [6] pointed out that plane waves shared with the FWM's their infinite energy content, and he showed that a superposition of these modes could produce finite energy solutions. Such pulses, characterized by high directionality and slow energy decay, have been called electromagnetic directed energy pulse trains (EDEPTs) [7-10]. Besides the EDEPT solutions, several papers have been published over the past several years on a variety of LW pulses such as the 'splash modes' [11-15], 'electromagnetic
missiles' [16-21], the monochromatic 'Bessel beams' [22-24], 'transient beams' [25-28], 'EM bullets' [29,30], 'X-waves' or 'sling-shot pulses' [31,32], and 'Bessel-Gauss pulses' [33].

The above properties of these pulses are unique and very desirable. When compared with traditional monochromatic, continuous wave (CW) solutions, these localized wave (LW) solutions are characterized by extended regions of localization i.e., their spatial structure remains unchanged over much larger distances than their CW analogues. Such non separable space-time solutions represent pulses with highly localized transmission characteristics which may have potential applications in the areas of directed energy, secure communications and remote sensing. It is of great interest, then, to study the possibility of synthesizing these pulses.

Wu [16] was the first to address the issue of a source generating nondispersive wave packets or electromagnetic missiles. He argued that the electromagnetic energy density transmitted by a finite aperture under transient excitation does not have to decrease as $R^{-2}$ when $R \to \infty$. The energy reaching the receiver has to decay eventually to zero. He demonstrated that one can make the product of the missile's cross-sectional area and the average energy per unit area approach zero as slowly as one wished by choosing suitable frequency components of the exciting current. Wu deduced his results using the total received electromagnetic energy. Lee [19], on the other hand, used the Mellin transform to derive asymptotic expressions for the $E$- and $H$- field components of an instantaneously excited missile field. Later, Lee [20] rederived the field components for a source with a finite excitation time. Wu et al. [17] showed that electromagnetic missiles could be launched by a uniform dielectric sphere under transient excitation by a point source. They found that such pulses could be classified into strong and weak ones according to the corresponding critical points of

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differentiable maps in two dimensions. A more recent study of the launchability of EM missiles deals with a line source within a cylindrical dielectric lens [18]. Experimental studies of the launchability of EM missiles were reported by Shen [21].

On a completely different path Ziolkowski [6] made the important observation that the scalar FWMs describe fields that originate from moving complex sources. This observation linked the FWMs with earlier work by Deschamps [34] and Felsen [35] describing Gaussian beams as fields equivalent paraxially to spherical waves with centers at stationary complex locations. Ziolkowski et al. [36,37] performed an experiment investigating the feasibility of launching an acoustical directed energy pulse train (ADEPT). They established that an ADEPT pulse launched from a linear synthetic array retained its compact structure and did not spread out until a distance twice the Rayleigh length. They also demonstrated that an ADEPT field generated by a rectangular array spread out at a slower rate than a more conventional field generated by a Gaussian-driven array. Later, Ziolkowski [38] considered the possibility of generating a finite-energy EDEPT solution from a finite planar array. In particular, his approach involved an approximate Huygens reconstruction based on a discreet array of point sources arranged on a plane perpendicular to the direction of propagation.

Ideas similar to those underlying the work on E(A)DEPTs and EM missiles were contemplated by Durnin [22] when he introduced the diffraction-free 'Bessel beams' and he was able to demonstrate that such beams have a larger depth compared to Gaussian beams, even if their central spots have the same radii. The increase in the depth of the beam was achieved, however, at the expense of the power utilized [23]. This behavior which has been verified experimentally by Durnin et al. [24], can be attributed mainly to the differences in the energy distribution over identical apertures. Durnin's monochromatic beams are composed of different spatial spectral components.

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The depth of the monochromatic Bessel beams can be controlled by varying only their spatial spectral content or changing their energy distribution over the aperture. On the other hand, both temporal and spatial spectral components are required in synthesizing highly directional time-limited pulses, e.g., EDEPTs and EM missiles.

Shaarawi et al. [39] proposed the possibility of launching localized pulse trains from an open semi-infinite circular waveguide. The far fields outside the semi-infinite waveguide were computed using Kirchhoff's integral formula with a time-retarded Green's function. The resulting approximate solutions are causal, have finite-energy, and exhibit a slow energy decay behavior. A novel decomposition developed by Besieris et al. [40] was used in solving this initial boundary-value problem. It is a new decomposition of solutions to partial differential equations into bidirectional, forward and backward, traveling plane waves and it is a natural basis for synthesizing pulse solutions that can be tailored to give directed energy transfer in space. Vengsarkar et al. [41] continued this work extending this concept to nondispersive pulses launched in an optical fiber waveguide.

More recently Ziolkowski [42] considered the possibility of generating localized beams by ultra-wide bandwidth pulse-driven arrays. He addressed beam characteristics such as beam divergence, beam intensity and energy efficiency. He argued that new types of arrays are needed in order to generate LW pulses. Each array element, driven with broad bandwidth signals, must be independently addressable; that is, each element's time history is unique. Enhanced localization effects can be achieved by driving an array with properly designed spatial distribution of broad bandwidth signals; i.e., by controlling not only the amplitudes, but also the frequency spectra of the pulses driving the array. Along this framework a simulation study has been presented [43] that provides a method for calculating an optimum set of driving functions for an array of

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point sources given a scalar representation of the desired field. Comments on Ziolkowski's work were presented by Samaddar [44] who did not agree with Ziolkowski's approach.

A source generating FWM class of solutions was presented by Palmer and Donnelly [45]. They presented an infinite line source which propagates a field containing a FWM component and that exhibits a degree of localization on the transverse plane. This source is of infinite spatial and temporal extent.

Finally, Borisov and Utkin [60] obtained solutions to the inhomogeneous scalar wave equation that describe waves generated by moving sources whose front moves with speeds less and greater than the speed of light. Specifically, they concentrated on a source pulse that has an arbitrary shape and moves with the velocity of light. Using a special scalar potential, they constructed a LW pulse and discussed the possibility of launching it through a spike x-ray pulse.

One of our own methods of investigating different source structures producing LW pulses is based on the novel bidirectional representation developed by Besieris et al. [40]. The latter involves a product of plane waves propagating in opposite directions, while usual Fourier factorization techniques decompose the solutions into a sum of forward and backward traveling plane waves. This bidirectional decomposition, which was developed within the framework of a more general embedding procedure, allows the construction of general solutions by means of a superposition of elementary bidirectional basis functions. Such a novel superposition differs significantly from the more conventional ones, e.g., the Fourier synthesis. In particular, it is characterized by algebraic singularities that can be much easier to handle than the branch-cut singularities arising usually in the Fourier synthesis. In spite of these differences, there is a one-to-one correspondence between this new synthesis and the Fourier method.

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Several mathematical aspects of the new synthesis have been addressed. The elementary basis functions in this superposition are composed of exponential and Bessel functions which form complete sets of orthogonal functions. This has led to an inversion formula, from which different spectra can be calculated from the knowledge of exact solutions. The bidirectional decomposition has been applied to the three-dimensional scalar wave equation, the three-dimensional Klein-Gordon equation, the Dirac equation, the three dimensional dissipative wave equation and the telegraph equation, leading to numerous publications [32,39,41,46-52].

A modification to the bidirectional representation is introduced, in this thesis, according to which one can select new elementary basis functions resulting in different representations for a solution. The number of choices is infinite. This freedom facilitates the solution of the homogeneous, as well as the nonhomogeneous scalar wave equation. Different elementary basis functions lead to different degrees of difficulty in solving the forced wave equation. One can select the appropriate blocks in order to solve or simplify the problem. Through this new approach, a new class of focused X waves is presented. These extremely localized pulses are much more focused than X waves presented in the literature. The modified bidirectional representation is demonstrated by solving three classical source problems: point charge moving uniformly in a lossless dielectric, the time-dependent Green's function due to a line source, and the Green's function for the Poisson equation. The ordinary bidirectional method, as well as the aforementioned new superposition, can be applied to the forced wave equation in order to compute fields generated by sources. This new approach provides an easy way to derive the sources of LW pulses. For example, we show that LW solutions such as the FWM, the MPS, and the X waves or "sling-shot" pulses can be generated by real moving sources. In the FWM case, the real source has a

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Gaussian taper in the transverse plane and moves with the speed of light. This is in agreement with Hillion [14] who arrives at similar results using Goursat's theory [53] of boundary value problems with data on the characteristics of the wave equation. The source producing the MPS pulse has a complicated structure, moving with the speed of light in the medium. Finally, the X wave pulse can be generated by a source that has a $\rho^{-3}$ dependence in the transverse plane and moves with a speed greater than the speed of light in the medium. This makes sense since the X wave pulse is superluminal, i.e., moves with speed greater than the speed of light in the medium.

The possibility of exciting LW pulses from apertures is of great interest. Our approach to this problem is based on the proven possibility of exciting causal Bessel beams from infinite apertures [32]. It is well known that FWM-like solutions can be synthesized as a superposition of Bessel beams with appropriate choice of spectra. It has been shown by Ziolkowski [38] that a Bessel beam defined initially on an infinite plane (e.g. the $z = 0$ plane) will propagate under the effect of Huygen's operator in the positive z-direction and that the contributions from the acausal negative z-components sum up to zero. Such a proof has been applied directly to the FWM pulse to show that it has no acausal incoming wave contributions [54]. In fact, it has been shown, on the basis of recent work by Shaarawi [54] that the ideal FWM pulse can be launched from an aperture with radius that varies with time, i.e., it shrinks from infinity to a finite value and expands once more to infinity. Such a time-varying aperture requires an infinite time of excitation. This is the main reason why the ideal FWM pulse cannot be physically realized. In contradistinction to other solutions that could be excited from an infinite aperture, like plane waves and Bessel beams, the FWM and the X wave pulses do not require infinite power during excitation. This is the case because as the aperture becomes infinitely large, the power density of the field exciting the aperture

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decreases to zero at a rate \((ct)^{-2}\) while the area of the aperture increases as \((ct)^2\). These two effects balance each other and the energy on the aperture remains constant. Such an excitation acts like a temporal-focusing aperture, concentrating an extremely weak field distributed over a huge aperture onto a much smaller one, which in turn expands to infinity.

In the case of the FWM pulse we consider the case \(\beta a_1 \ll 1\), where \(\beta\) and \(a_1\) are the parameters that determine the physical characteristics of the FWM pulse. More specifically, it has been shown [32, 40, 62] that the product \(\beta a_1\) determines whether the field of the source-free FWM pulse is dominated by causal or acausal components. The case \(\beta a_1 > 1\) has been considered by Heyman and Felsen [28, 61] who proved that under such condition the FWM field is acausal. The condition \(\beta a_1 \ll 1\) has been considered by Shaarawi et al. [54, 61] and they have shown that the causal components of the field are the dominant ones while the acausal components are negligible and do not affect the pulse behavior. This is the main reason why we shall concentrate on the condition \(\beta a_1 \ll 1\). Our subsequent discussion will be based on this condition. In the case of the FWM pulse the aperture is shrinking and expanding at speeds greater than \(c\). Such aperture should consist of separately excitable elements, in order not to violate the theory of special relativity. In the case of the X wave pulse, the speed of expansion \(v_{ap}\) of the aperture field, depends on the speed of the pulse center \(v_p\) and can be greater or smaller than \(c\). The temporal and spatial frequency contents of LW pulses will be calculated in order to better understand basic characteristics such as bandwidth.

Finally, we shall also explore the possibility of using finite time excitation of time varying apertures to generate two types of approximate LW pulses: the FWM and the X wave pulse. The analysis will be carried out by introducing a Gaussian time window. The resulting fields will be calculated and their far fields will be compared to

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ordinary monochromatic pulses. It will be shown that finite versions of LW pulses can
propagate without significant decay much further than classical monochromatic pulses.
It should be noted that Shaarawi et al. [54] have studied the finite time excitation of
FWM pulses and agreement with the work presented here is generally good.

Chapter 2 of this thesis presents an overview of LW pulse synthesis. The LW
pulses considered, are the "electromagnetic missiles" [16], "electromagnetic directed
energy pulse trains" [38], and "X waves" [57]. Chapter 3 deals with the modified
bidirectional representation and its applications. Chapter 4 demonstrates how LW
pulses can be generated from moving sources. Chapter 5 addresses the aperture
excitation of LW pulses and, finally, Chapter 6 contains the conclusions and future
issues.
2.0 LOCALIZED PULSE SYNTHESIS: AN OVERVIEW

In this chapter an overview of the current research in LW pulse synthesis will be presented. More specifically, we shall discuss three different synthesized pulses: 1) Electromagnetic missiles, 2) EDEPTs, and 3) X waves.

2.1 ELECTROMAGNETIC MISSILES

Electromagnetic missiles [16] are electromagnetic pulses that decay more slowly than the usual $1/r$ law. This is achieved by using a broad-frequency spectrum. Referring to Fig. 2-1, the energy delivered to the screen $S$ is expressed as

$$E(S, r) = \int_{-\infty}^{\infty} dt \int_{S} dS \, \hat{a}_{n} \cdot \vec{E} \times \vec{B},$$  \hspace{1cm} (2.1.1)

where $\hat{a}_{n}$ is a unit vector normal to the screen, and $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields, respectively. The current density here is $\vec{J}(\vec{r}, t) = \delta(z) f(t) \, \hat{a}_{x}$ for $r < d$, which is confined to a disk of radius $d$. Then $\vec{J}(\omega) = \delta(z) F(\omega) \, \hat{a}_{x}$ and the vector potential can be found from

$$\vec{A}(\omega) = \int d^{3}\vec{r} \, \vec{J}(\omega) \, e^{i\omega t/c} \frac{1}{cr}.$$  \hspace{1cm} (2.1.2)

The Poynting flux along the $z$ axis integrated over all time is

$$P(z) = \int dt \, \hat{a}_{z} \cdot (\vec{E} \times \vec{B}).$$  \hspace{1cm} (2.1.3a)

The Poynting flux along the $z$ axis is now expressed as
\[ P(z) = \frac{1}{c} \left[ 1 + z(z^2 + d^2)^{-1/2} \right] \]

\[ \times \int_0^\infty d\omega |F(\omega)|^2 \left( 1 - \cos \left\{ \frac{\omega}{c} (z^2 + d^2)^{1/2} - z \right\} \right) \]. \quad (2.1.3b)

Now, for fixed \( \omega \) and \( z \to \infty \) we have

\[ \cos \left\{ \frac{\omega}{c} (z^2 + d^2)^{1/2} - z \right\} \to \cos \left( \frac{\omega d^2}{2cz} \right) \to 1, \]

making \( P = 0 \). Equation (2.1.3) is rewritten as

\[ P(z) \approx \frac{2}{c} \int_0^\infty d\omega |F(\omega)|^2 \left( 1 - \cos \left\{ \frac{\omega d^2}{2cz} \right\} \right) \]

\[ = \frac{2}{c} \left[ \int_0^\infty d\omega |F(\omega)|^2 - \int_0^\infty d\omega |F(\omega)|^2 \cos \left\{ \frac{\omega d^2}{2cz} \right\} \right]. \quad (2.1.4) \]

Wu [16] lets \( \int_0^\infty d\omega |F(\omega)|^2 = 1 \) and chooses \( |F(\omega)|^2 \) to satisfy

\[ \int_0^\infty d\omega |F(\omega)|^2 \cos \zeta = 1 - O(\zeta^\alpha), \quad \alpha < 2 \quad \text{for} \quad \zeta \to 0, \]

where \( \zeta = d^2/2cz \). Next, he considers the spectrum

\[ F(\omega) \propto \left\{ 1 + \left( \frac{\omega}{\omega_0} \right)^2 \right\}^{-(1+2\epsilon)/4}, \quad \epsilon > 0. \quad (2.1.5) \]
Using (2.1.5) he finds that $P$ falls off as $1/z^{2\varepsilon}$, which can be as slow as desired by taking the limit $\varepsilon \to 0$.

Hafizi and Sprangle [55] state that the spectrum (2.1.5) is not realistic since the spectrum $F(\omega)$ extends from $-\infty$ to $\infty$. So, a truncation of the spectrum is needed to yield realistic sources. Truncating, however, the spectrum at some frequency $f_{\text{max}}$ limits the maximum propagation distance to that of the Rayleigh range corresponding to $f_{\text{max}}$.

Experimental studies of the launchability of EM missiles are reported by Shen in [21]. The experiment involves an EM-missile launcher fed by a voltage pulse generator. The generated pulse, after traveling above a ground plane, is received by an EMP (ElectroMagnetic Pulse) sensor and recorded by a sampling oscilloscope. Two types of missile launchers are used: a parabolic dish fed by a point source at the focus, and an open circular waveguide. A special pulse antenna, the V-conical antenna, is used because of its ability to emit pure spherical waves. It is stated that the receiving system is good for up to 10 GHz. Shen claims that the EMP launched from the missile launcher has the property of slow decay. The conclusions are not complete, however, because of the many technical issues involved in the experimental setup, such as focal-point adjustments and receiving probe and antenna feed inaccuracies.

2.2 ELECTROMAGNETIC DIRECTED ENERGY PULSE TRAINS (EDEPTs)

EDEPTs are solutions to the three-dimensional scalar wave equation and are derived from the ideal FWM pulse. A specific EDEPT that has finite energy is the MPS pulse expressed as

2.0 LOCALIZED PULSE SYNTHESIS: AN OVERVIEW
\[ f(\rho, z, t) = [z_0 + i(z - ct)]^{-1} (s/\beta + \alpha)^{-\alpha} e^{-bs/\beta}, \quad (2.2.1a) \]

where
\[ s = \frac{\rho^2}{z_0 + i(z - ct)} - i(z + ct), \quad (2.2.1b) \]

and \( a, \alpha, \beta, b, z_0 \) are free spectral parameters controlling the characteristics of the pulse.

For a particular choice of the above parameters, the solution in (2.2.1) has the desired property of localization along the axis of propagation, \( z \), for very large distances. The real part of an ideal nondecaying MPS pulse is shown in Fig. 2-2. The backward plane wave portion of the solution can be made to be very small by "tweaking" the spectrum, setting \( \beta \) to be very large, thus, removing the physically pathological nature of the pulse. The behavior of the MPS pulse along \( z \) can be viewed from (2.2.1) by setting \( \rho = 0 \) and \( z = ct \):

\[ f(\rho = 0, z = ct) = \frac{\cos(2bz/\beta) - (2z/\beta a) \sin(2bz/\beta)}{[1 + (2z/\beta a)^2] az_0}, \]

\[ \frac{1}{(az_0)} \text{ when } 2bz/\beta \ll 1 \text{ and } 2z/\beta a < 1, \]

\[ \begin{cases} 
\cos(2bz/\beta)/az_0 \text{ when } 2z/\beta a < 1, \\
[\sin(2bz/\beta)/(2z_0/\beta)] (1/z) \text{ when } 2z/\beta a > 1.
\end{cases} \quad (2.2.2) \]

As seen in (2.2.2), for a distance \( z \ll \beta/2b \) and \( z < \beta a/2 \), the amplitude of the pulse at the pulse center is constant. Then, it becomes oscillatory in an intermediate region where \( \beta/2b < z < \beta a/2 \), recovering its initial amplitude every \( z = n(\pi\beta/b) \), \( n \) being

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a positive integer. Finally, very far from the origin, i.e., \( z > \beta a/2 \), the MPS pulse decays as \( 1/z \), satisfying the radiation condition at infinity.

The physical realization of the "tweaked" MPS pulse is addressed by Ziolkowski [38]. He uses the Huygen's reconstruction of the scalar MPS field, where the field is launched from a finite planar array of point sources with the causal, time-retarded Green's function. In the simulation he performed, each array element was driven with a unique time function specified by the exact field. According to the Huygen's representation an electric field to the right of an infinite plane \( z' = \text{const.} \) can be expressed as:

\[
f(\rho, t) = -\int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} dy' \Psi(x', y', z', t - R/c) \frac{1}{4\pi R}, \quad (2.2.3a)
\]

where

\[
\Psi(x', y', z', t - R/c) = [\partial_x f] - [\partial_{xy} f] \frac{(z - z')}{R} - [f] \frac{(z - z')}{R^2} \quad (2.2.3b)
\]

represents the driving functions, and \( R \) is the radial distance to the observation point. The \( z \)-plane constant is determined by the requirement that the field and its derivatives be zero on this plane and at \( t = 0 \). The aperture is made finite by truncating the limits of (2.2.3a) to some specified dimensions. Ziolkowski implements the Huygen's representation of the MPS pulse numerically using different configurations, i.e., rectangular, circular, and hexagonal arrays of equally spaced elements. For the case of rectangular arrays, the field in (2.2.3) is approximated as

\[
f(\rho, t) \simeq -\sum_{n=-N}^{+N} \sum_{m=-M}^{+M} \left[ \Psi(n\Delta x, m\Delta y, z', t - R_{nm}/c) \Delta x \Delta y \right] \frac{1}{4\pi R_{nm}}, \quad (2.2.4)
\]

where \( \Delta x \) and \( \Delta y \) are the spacings in the \( x \) and \( y \) directions, respectively, and \( R_{nm} = \sqrt{[(x - n\Delta x)^2 + (y - m\Delta y)^2 + (z - z')^2]}^{1/2} \).
The approximate MPS pulse is reconstructed at the observation distances: 
\((\rho = 0.0 \text{ m}, z = 1 \text{ Km})\) and \((\rho = 0.0 \text{ m}, z = 10 \text{ Km})\) along the \(z\) axis from a circular array of radius \(r_{\text{max}} = 0.5 \text{ m}\) and 5 m, respectively. The time history of the approximate MPS pulse is compared to the time history of the exact field. The arrays for the two cases above consist of 101 and 1001 elements, respectively and the array location is at \(z' = 0.001 \text{ m}\). Ziolkowski shows a quantitative measure of the size of a circular array needed to reconstruct the MPS pulse at increasingly larger observation distances. He concludes that if one wishes to reconstruct the pulse at a distance of 10 \(z_0 \text{ Km}\), then the array has to be increased by a factor of 10 \(\rho_0 \text{ cm}\), where \(\rho_0\) is the array size required for the reconstruction of the MPS pulse at the observation distance \(z_0 \text{ Km}\).

The important question here is the following: What is the Rayleigh distance in this case? Ziolkowski calculates the Rayleigh range of \(\pi \omega_0^2 / \lambda\), where \(\lambda\) is the wavelength corresponding to the highest frequency in the spectrum and \(\omega_0\) is the MPS pulse width at \(z = 0\). Haftiz and Sprangle [55] are using a scale length of \(2\pi \omega_0 d / \lambda\), where \(d\) is the antenna dimension which is much larger than \(\omega_0\). They indicate that the Rayleigh range \(\pi \omega_0^2 / \lambda\) is valid only at the pulse center \((z - ct = 0)\), arguing that, away from the pulse center the actual diffraction length will be longer than \(\pi \omega_0^2 / \lambda\) due to the increase of the effective waist. At \(z = 0\) the MPS pulse has a Gaussian profile with waist \(\omega_0\) and energy \(E_{\text{MPS}}\). Now, if one uses a standard Gaussian pulse with the same waist \(\omega_0\) and same energy \(E_G = E_{\text{MPS}}\), then, the MPS pulse will decay more slowly than the Gaussian pulse due to its unique spatial and spectral structure. So, Ziolkowski correctly points out that a Gaussian pulse decays more rapidly than the MPS pulse.

The feasibility of launching acoustic directed energy pulse trains (ADEPTs) was tested experimentally by Ziolkowski et al. [36]. The ADEPTs generated in the

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experiment had the same form as their electromagnetic counterparts (EDEPTs) expressed in (2.2.1), with the parameters: \( a = 1 \text{ m}, \alpha = 1.0, b = 600 \text{ m}^{-1}, \beta = 300.0, z_0 = 4.5 \times 10^{-4} \text{ m} \). The radiation experiment was based on ultrasonic pulses propagating in water. A one dimensional synthetic array was used, which consisted of 21 element positions (occupied by ultrasonic transducers) symmetrically arranged about \( y = 0 \) with a 3.0-mm separation. Eleven driving functions were employed and the total array width was 6.0 cm. Most of the frequency spectrum of the driving functions came from around 0.6 MHz and 98.0% of its energy was below 2.0 MHz. The field of this array was constructed by weighing and superimposing the generated 176 waveforms. Then, a Gaussian beam with the same transverse waist at \( z = ct = 0 \) as the ADEPT was constructed and compared to the ADEPT. Ziolkowski et al. [36] showed that the linear array produced fields that began to break up after 50.0 cm or approximately twice the Rayleigh distance corresponding to the experimental conditions.

### 2.3 X WAVES

Another interesting class of LW solutions are the X waves or "sling-shot" pulses. These are sub- or superluminal pulses representing interference patterns which have been constructed from basic building blocks each traveling at the speed of light \( c \).

#### 2.3.1 Rayleigh-Sommerfeld Formulation

The numerical synthesis of X waves has been addressed by Lu and Greenleaf [57]. They state that these pulses can be realized almost exactly over a finite depth of field with finite apertures by either broadband or band-limited radiators. In an
experiment, they have shown that a zeroth-order band-limited X wave is produced in water by their 10-element, 50 mm diameter, 2.5 MHz \( J_0 \) Bessel nondiffracting annular array transducer.

Lu and Greenleaf [57] express a general X wave pulse as:

\[
\Psi_n(\rho, \phi, z, t) = e^{i\phi} \int_0^\infty d\kappa \: B(\kappa) \: J_n(\kappa \rho \sin \lambda) \\
\exp\{-\kappa[\alpha_0 - i(z \cos \lambda - ct)]\},
\]

\((n = 0, 1, 2, \ldots)\), where \(0 < \lambda < \pi/2\), and \(\alpha_0\) is a free parameter.

The zeroth-order, axially symmetric, nondiffracting X-Wave is obtained by setting \(B(\kappa) = \alpha_0\) and \(n = 0\):

\[
\Psi_0(\rho, z, t) = \frac{\alpha_0}{\{\left(\rho \sin \lambda\right)^2 + \left[\alpha_0 - i(z \cos \lambda - ct)\right]^2\}^{1/2}}.
\]

(2.3.2)

The calculation of X waves from finite apertures is done using finite versions of the spectrum \(B(\kappa)\). Equation (2.3.1) is rewritten as:

\[
\Psi_n(\rho, \phi, z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \frac{2\pi}{c} \: e^{i\phi} \: B\left(\frac{\omega}{c}\right) \: J_n\left(\frac{\omega}{c} \rho \sin \lambda\right) \: H\left(\frac{\omega}{c}\right) \right\} \: e^{-i\omega t} d\omega,
\]

\((n = 0, 1, 2, \ldots)\), where \(H\left(\frac{\omega}{c}\right)\) is the Heaviside step function. As seen in (2.3.3), \(\Psi_n(r, \phi, z, t)\) is the inverse Fourier transform of the function

\[
\tilde{\Psi}_n(\rho, \phi, z, \frac{\omega}{c}) = \frac{2\pi}{c} \: e^{i\phi} \: B\left(\frac{\omega}{c}\right) \: J_n\left(\frac{\omega}{c} \rho \sin \lambda\right) \: H\left(\frac{\omega}{c}\right) \\
\times \exp\left[-\frac{\omega}{c} (\alpha_0 - iz \cos \lambda)\right],
\]

(2.3.4)
\(n = 0, 1, 2, \ldots\). Now, at the plane \(z = 0\), \(\tilde{\Psi}_n(\rho, \phi, z, \frac{\omega}{c})\) becomes

\[
\tilde{\Psi}_n(\rho, \phi, \frac{\omega}{c}, z = 0) = B\left(\frac{\omega}{c}\right) E_n(\rho, \phi, \frac{\omega}{c}), \quad (n = 0, 1, 2, \ldots)
\]  \hspace{1cm} (2.3.5a)

where

\[
E_n(\rho, \phi, \frac{\omega}{c}) = \frac{2\pi}{c} J_n\left(\frac{\omega}{c}\rho \sin \lambda\right) \exp\left[-\frac{\omega}{c}a_0 + i\omega \phi]\right]. \hspace{1cm} (2.3.5b)
\]

The temporal frequency spectrum in (2.3.5) is used for the calculation of X waves generated by radiators of finite apertures. For a circular radiator of diameter \(D\), the Rayleigh-Sommerfeld formulation of diffraction by a plane screen is given as [57]:

\[
\tilde{\Psi}_{Rn}(\rho, \kappa) = \frac{1}{i\lambda_0} \int_{0}^{2\pi} d\phi' \int_{0}^{D/2} d\rho' \rho' \tilde{\Psi}_n\left(\rho', \phi', \kappa, z = 0\right) \frac{e^{ikr_01}}{r_{01}^2} z
\]

\[
+ \frac{1}{2\pi} \int_{0}^{2\pi} d\phi' \int_{0}^{D/2} d\rho' \rho' \tilde{\Psi}_n\left(\rho', \phi', \kappa, z = 0\right) \frac{e^{ikr_01}}{r_{01}^3} z, \quad (n = 0, 1, 2, \ldots),
\]  \hspace{1cm} (2.3.6)

where \(\lambda_0\) is the wavelength, \(r_{01}\) is the distance between the observation point \((\rho, \phi)\) and the source point \((\rho', \phi')\), \(\kappa = \omega/c\), and \(\tilde{\Psi}_n\left(\rho', \phi', \kappa, z = 0\right)\) is given in (2.3.5). Lu and Greenleaf [57] perform numerical simulations using (2.3.6) and conclude that, even though X waves are superluminal, and have infinite total energy and apertures, they can be truncated to produce practical diffraction-resistant waves.

### 2.3.2 Huygen's Reconstruction

The Huygen's reconstruction of X waves has been addressed by Donnelly et al. [59]. It is basically the same work as Ziolkowski's [38], described in section 2.2 above, applied to X

\[2.0 \text{ LOCALIZED PULSE SYNTHESIS: AN OVERVIEW} \]
waves. The approximate field is launched from a finite planar array of point sources with the causal, time-retarded Green's function. The nondecaying X wave solution they present is

$$\Psi_\gamma(\rho, t) = \frac{[z_0 - i(z - ct/\gamma)]}{\left\{\rho^2 (1 - \gamma^2)/\gamma^2 + [z_0 - i(z - ct/\gamma)]^2 \right\}^{3/2}}, \quad (2.3.7)$$

which is a normalized first time derivative of the zeroth-order pulse in (2.3.2). The surface plot of such solution is shown in Fig. 2-3. The plane of the aperture is the $z' = 0$ plane. In general, the electric field to the right of an infinite plane $z' = 0$ can be expressed as

$$f(\vec{r}, t) = -\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \Psi(x', y', z' = 0, t - R/c) \frac{1}{4\pi R}, \quad (2.3.8a)$$

where

$$\Psi(x', y', z' = 0, t - R/c) = [\partial_x f] - [\partial_{x'} f] \frac{z}{R} - [f] \frac{z}{R^2} \quad (2.3.8b)$$

represents the driving functions and $R$ is the observation distance. The aperture is made finite by truncating the limits of (2.3.8a) to some specified distance. For a uniform array of equally spaced radiators the field in (2.3.7) is approximated as

$$\Psi_\gamma(\vec{r}, t) \simeq -\sum_{n=1}^{N} \left\{ [\partial_x \Psi_n] - [\partial_{x'} \Psi_n] \frac{z}{R_n} - [\Psi_n] \frac{z}{R_n^2} \right\} \frac{1}{4\pi R_n}, \quad (2.3.9)$$

where $\Psi_n = \Psi(\rho'_n, z', t - R_n/c)$ and $\rho'_n = n\Delta\rho'$, $R_n = [\rho'_n]^2 + (z - z')^2]^{1/2}$. Donnelly et al. [59] use (2.3.9) for the numerical simulations using a finite number of point sources located on the $z = 0$ plane. They attempt to reconstruct a pulse traveling with speed $2c$, at a point on the $z$ axis close to the array. The superluminal nature of the resulting pulse is maintained in the neighborhood of the chosen field point. However, further away from the array, the pulse slows down and loses localization.

2.0 LOCALIZED PULSE SYNTHESIS: AN OVERVIEW
2.4 CONCLUSION

An overview of the synthesis of LW pulses has been addressed in this chapter. More specifically, three LW pulses were considered: 1) Electromagnetic Missiles, 2) EDEPTs, and 3) X waves.

Electromagnetic missiles can be generated by a source with a very broad frequency spectrum. In practice, this is difficult to achieve today because extremely high-rise time pulses are needed to generate diffraction-resistant EM missiles.

The physical realization of the "tweaked" MPS pulse is addressed by Ziolkowski [38]. He uses the Huygen's reconstruction of the scalar MPS field, where the field is launched from a finite planar array of point sources with the causal, time-retarded Green's function. The feasibility of launching acoustic directed energy pulse trains (ADEPTs) was tested experimentally by Ziolkowski et al. [36]. They showed that a linear array produced ADEPT fields that outperformed Gaussian beams and began to break up after 50.0 cm, or approximately twice the Rayleigh distance corresponding to the experimental conditions.

The synthesis of X waves is addressed by Lu and Greenleaf [57] and Donnelly et al. [59]. Lu and Greenleaf [57] state that these pulses can be realized almost exactly over a finite depth of field with finite apertures by either broadband or band-limited radiators. In an experiment, they showed that a zeroth-order band-limited X-Wave is produced in water by their 10-element, 50 mm diameter, 2.5 MHz $J_0$ Bessel nondiffracting annular array transducer. Donnelly et al. [59] address the numerical synthesis of X waves using the Huygen's reconstruction of X waves, where the approximate field is launched from a finite planar array of point sources with the causal, time-retarded Green's function.

2.0 LOCALIZED PULSE SYNTHESIS: AN OVERVIEW
Finite-energy diffraction-free beams do not exist, as pointed out in [38] and [40]. However, an approximation of LW infinite-energy beams can result in diffraction-resistant pulses propagating without significant decay at distances larger than classical monochromatic pulses.
Fig. 2-1: Coordinate system for showing the energy delivered to the screen $S'$ due to a current density confined in the disk of radius $d$. 
Fig. 2-2: Surface plot of the real part of the MPS pulse corresponding to the following parameters: \( \alpha = 1 \text{ cm}, \ \beta = 1 \times 10^{10} \text{ cm}^{-1}, \ \beta = 6 \times 10^{15}, \ z_0 = 1.667 \times 10^{-3} \text{ cm}. \)
Fig. 2-3: Surface plot of the real part of the superluminal X wave pulse moving with speed $v_p = 4c$ with the parameters $\gamma = 0.25$ and $z_0 = 0.7 \text{ cm}$.
3.0 A MODIFICATION OF THE BIDIRECTIONAL SYNTHESIS; ELEMENTARY APPLICATIONS

This chapter begins with the bidirectional traveling plane wave decomposition described in Sec. 1. In Sec. 2, we introduce the modified bidirectional representation. Within this framework, we can select new basis functions resulting in different representations for a solution. This freedom facilitates the solutions of the homogeneous, as well as the nonhomogeneous scalar wave equation. In Sec. 3, we show how known LW solutions to the homogeneous scalar wave equation can be obtained through this new approach. In Sec. 4 a new class of focused X waves is presented which results in pulses that are extremely localized. This new class of superluminal X waves are much more localized than the X waves presented in the literature. Finally, in Sec. 5, this new approach is applied to the Fourier synthesis resulting in an alternative representation of solutions to the homogeneous scalar wave equation.

3.1 THE BIDIRECTIONAL SYNTHESIS

A complete method for obtaining LW solutions to the wave equation is the bidirectional plane wave decomposition developed by Besieris et al. [40]. According to this representation an azimuthally symmetric, zeroth order [40] solution to the three dimensional scalar wave equation can be expressed as

$$\Psi(\rho, \zeta, \eta) = \frac{1}{(2\pi)^2} \int_0^\infty d\kappa \int_0^\infty d\alpha \int_0^\infty d\beta \kappa C_0(\alpha, \beta, \kappa) J_0(\kappa \rho) e^{-i\alpha \zeta} e^{i\beta \eta}$$

$$\times \delta(\alpha \beta - k^2/4),$$ \hspace{1cm} (3.1.1)
where \( \zeta = z - ct, \eta = z + ct \) are the light-cone variables, \( C_0(\alpha, \beta, \kappa) \) is the spectrum, and the argument of the \( \delta \)-function equal to zero is the constraint relationship corresponding to the basis function

\[
\psi_b(\rho, \zeta, \eta) = J_0(\kappa \rho) \ e^{-i\alpha \zeta} \ e^{i\beta \eta}.
\] (3.1.2a)

The basis function \( \psi_b \) is an eigenfunction of the wave operator

\[
L = L_1 + L_2
\]

\[
L_1 = \partial_{\rho}^2 + \rho^1 \partial_{\rho}, \quad L_2 = 4 \partial_{\zeta}^2,
\] (3.1.2b)

with \( J_0(\kappa \rho) \) and \( e^{-i\alpha \zeta} e^{i\beta \eta} \) being the elementary eigenfunctions of \( L_1 \) and \( L_2 \), respectively. Eq. (3.1.1) is the ordinary bidirectional decomposition.

### 3.2 EXTENSION TO THE BIDIRECTIONAL SYNTHESIS

A more generalized form of the bidirectional representation (3.1.1) is given by

\[
\Psi(\rho, \zeta, \eta) = \frac{1}{(2\pi)^{3/2}} \int_0^\infty d\kappa \int_0^\infty d\alpha \int_0^\infty d\beta \ \kappa C_0(\alpha, \beta, \kappa) J_0[\rho f(\alpha, \beta, \kappa)] e^{-i\alpha \zeta} e^{i\beta \eta} \ 
\times \ e^{i\kappa h(\alpha, \beta, \kappa)} \delta(gh - f^2/4),
\] (3.2.1)

where \( f(\alpha, \beta, \kappa), \ g(\alpha, \beta, \kappa), \) and \( h(\alpha, \beta, \kappa) \) are nonsingular analytical functions and their units are \( m^{-1} \). The representation (3.2.1) is general. Choosing specific values for the functions \( f, \ g, \ h, \) results in different basis functions which, in turn, result in different integral representations for the solution \( \Psi(\rho, \zeta, \eta) \). In (3.1.1) we have only one basis function and can choose the spectrum \( C_0(\alpha, \beta, \kappa) \) to produce solutions. The superposition (3.2.1), however, allows one to select the basis function in addition to the

### 3.0 A MODIFICATION OF THE BIDIRECTIONAL SYNTHESIS: ELEMENTARY APPLICATIONS
spectrum. Note that (3.2.1) reduces to (3.1.1) for the special case \( f(\alpha, \beta, \kappa) = \kappa, \)
\( g(\alpha, \beta, \kappa) = \alpha \), and \( h(\alpha, \beta, \kappa) = \beta \).

Consider now the simple case

\[
\begin{align*}
  f(\alpha, \beta, \kappa) &= \kappa \\
  g(\alpha, \beta, \kappa) &= \alpha \kappa \\
  h(\alpha, \beta, \kappa) &= \beta \kappa,
\end{align*}
\]

where a scaling has been applied to the parameters \( \alpha \) and \( \beta \). Then (3.2.1) simplifies to

\[
\Psi(\rho, \zeta, \eta) = \frac{1}{(2\pi)^{3/2}} \int_0^\infty d\kappa \int_0^\infty d\alpha \int_0^\infty \kappa \mathcal{C}_0(\alpha, \beta, \kappa) J_0(\kappa \rho) e^{i\kappa \alpha \zeta} e^{i\kappa \beta \eta} \\
\, \times \delta(\alpha \beta - 1/4),
\]

where \( \kappa \mathcal{C}_0(\alpha, \beta, \kappa) = \kappa^{-1} \mathcal{C}_0(\alpha, \beta, \kappa) \). The choice (3.2.2) is just one of many different combinations resulting in an alternative representation. It should be noted that it is easier to obtain a particular solution by choosing the appropriate basis function, as will be shown in the next section.

**3.3 KNOWN SOLUTIONS**

The superluminal X wave pulse solution can be found by the choice of spectrum

\[
\mathcal{C}_0(\alpha, \beta, \kappa) = (2\pi)^2 \kappa^{-1} \delta(\beta - \beta') e^{-a_1 \kappa},
\]

where \( \beta' \) and \( a_1 \) are positive scalars. Substituting (3.3.1) into (3.2.3) and integrating over \( \alpha \) and \( \beta \) yields

**3.0 A MODIFICATION OF THE BIDIRECTIONAL SYNTHESIS: ELEMENTARY APPLICATIONS**
\[ \Psi(\rho, \zeta, \eta) = \int_0^\infty d\kappa J_0(\kappa \rho) \exp[-a_1 - i(\frac{\zeta}{4\beta'} - \beta' \eta)] \kappa. \quad (3.3.2) \]

The integration of (3.3.2) is carried out using (6.611.1) of [56] resulting in

\[ \Psi(\rho, \zeta, \eta) = \frac{1}{(\sigma^2 + \rho^2)^{1/2}}, \quad (3.3.3) \]

where \( \sigma = a_1 + i(\frac{\zeta}{4\beta'} - \beta' \eta) \). In terms of the variables \( z \) and \( t \), (3.3.3) is expressed as

\[ \Psi(\rho, \zeta, \eta) = \frac{z_0}{\{ (z_0 \rho)^2 + [a_1 z_0 - i(z - \nu t)]^2 \}^{1/2}}, \quad (3.3.4) \]

where \( z_0 = -\frac{4\beta'}{1-4(\beta')^2} \) and \( \nu = \frac{1+4(\beta')^2}{1-4(\beta')^2} c \), with \( \beta' > 0 \).

The X-wave solution can be obtained by the ordinary bidirectional synthesis with the spectrum [32]

\[ C_0(\alpha, \beta, \kappa) = (2\pi)^2 \frac{iz_0 a_3^2}{a_1 \kappa} \mid \alpha - i \frac{\kappa}{2a_3} \mid \exp\left(\frac{2z_0}{a_2}\right) \delta(\beta + \alpha a_3^2) \quad (3.3.5) \]

where \( a_1 = \sqrt{1-(\nu/c)^2}, a_2 = 1 + \nu/c, a_3 = \sqrt{1 + \nu/c} / \sqrt{1 + \nu/c} \), \( \nu \) is the speed of the pulse, and \( c \) is the speed of light in vacuum. The above spectrum results in the solution [32]

\[ \Psi(\rho, \zeta, \eta) = \frac{iz_0}{[\gamma^{-2} \rho^2 + (z - \nu t - iz_0)^2]^{1/2}}, \quad (3.3.6) \]

with \( \gamma = a_1^{-1} \), and \( \nu > c \).

Comparing the spectra in (3.3.1) and (3.3.5) it is obvious that the spectrum in (3.3.5) is much more complicated and it is very difficult to guess. In this case, using the representation in (3.2.3) one can arrive at the X-wave pulse much more easily than
using the ordinary bidirectional representation. Note that, only smart choices of the spectrum $C_0(\alpha, \beta, \kappa)$ result in LW solutions.

The FWM solution is obtained from (3.2.3) by introducing the spectrum

$$C_0(\alpha, \beta, \kappa) = 2\pi \kappa a_0^2 \delta(\alpha - a_0 \kappa) e^{-a_0 a_1 \kappa^2}, \quad (3.3.7)$$

where $a_0$ and $a_1$ are scalar parameters and $a_0 a_1 > 0$. Substituting (3.3.7) into (3.2.3) and integrating over $\alpha$, we obtain

$$\Psi(\rho, \zeta, \eta) = \frac{a_0^2}{2\pi} \int_0^\infty d\kappa \int_0^\infty d\beta \kappa \ k^2 J_0(\kappa \rho) e^{-i\zeta \kappa} e^{-a_0 a_1 \kappa^2} e^{i\kappa \eta \beta} \delta(a_0 \kappa \beta - 1/4)$$

$$= \frac{a_0}{2\pi} \int_0^\infty d\kappa \int_0^\infty d\beta \kappa J_0(\kappa \rho) e^{-i\zeta a_0 \kappa^2} e^{-a_0 a_1 \kappa^2} e^{i\kappa \eta \beta} \delta(\beta - \frac{1}{4a_0 \kappa}). \quad (3.3.8)$$

Integrating over $\beta$ yields

$$\Psi(\rho, \zeta, \eta) = \frac{a_0}{2\pi} e^{i\eta/(4a_0)} \int_0^\infty d\kappa \kappa J_0(\kappa \rho) e^{-a_0 V \kappa^2}, \quad (3.3.9)$$

where $V = a_1 + i\zeta$. The integration of (3.3.9) over $\kappa$ is carried out using (6.631.4) of [56], resulting in

$$\Psi(\rho, \zeta, \eta) = \frac{1}{4\pi V} e^{i\eta/(4a_0)} e^{-\frac{\rho^2}{4a_0 V}}, \quad (3.3.10)$$

which is the FWM solution.

Now, note that, for the same representation, the choice of a spectrum leading to a solution is not unique. For example, in the representation (3.2.3) we can also obtain the FWM solution using the spectrum

$$C_0(\alpha, \beta, \kappa) = 2\pi \delta(\alpha - a_0 \kappa) \delta(\beta - \frac{1}{4a_0 \kappa}) e^{-a_0 a_1 \kappa^2}. \quad (3.3.11)$$

Consider now the choice

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\[ f(\alpha, \beta, \kappa) = 2\kappa \]
\[ g(\alpha, \beta, \kappa) = \alpha \kappa \]
\[ h(\alpha, \beta, \kappa) = \beta, \] (3.3.12)

where a scaling has been applied to the parameters \( \alpha \) and \( \kappa \). Then (3.2.1) becomes

\[
\Psi(\rho, \zeta, \eta) = \frac{1}{(2\pi)^2} \int_0^\infty d\kappa \int_0^\infty d\alpha \int_0^\infty d\beta \kappa C_0(\alpha, \beta, \kappa) J_0(2\kappa \rho) e^{-i \kappa \alpha \zeta} e^{i \beta \eta} \times \delta(\alpha \beta - \kappa),
\] (3.3.13)

where \( \kappa C_0(\alpha, \beta, \kappa) = \overline{C_0}(\alpha, \beta, \kappa) \). Depending on how the \( \delta \)-function is written, one can arrive on the following two representations:

\[
\Psi_1(\rho, \zeta, \eta) = \frac{1}{(2\pi)^2} \int_0^\infty d\kappa \int_0^\infty d\alpha \int_0^\infty d\beta \kappa C_0(\alpha, \beta, \kappa) J_0(2\kappa \rho) e^{-i \kappa \alpha \zeta} e^{i \beta \eta} \frac{1}{\alpha} \delta(\beta - \frac{\kappa}{\alpha}).
\] (3.3.14a)

\[
\Psi_2(\rho, \zeta, \eta) = \frac{1}{(2\pi)^2} \int_0^\infty d\kappa \int_0^\infty d\alpha \int_0^\infty d\beta \kappa C_0(\alpha, \beta, \kappa) J_0(2\kappa \rho) e^{-i \kappa \alpha \zeta} e^{i \beta \eta} \frac{1}{\beta} \delta(\alpha - \frac{\kappa}{\beta}).
\] (3.3.14b)

Integrating (3.3.14a) over \( \beta \) yields

\[
\Psi_1(\rho, \zeta, \eta) = \frac{1}{(2\pi)^2} \int_0^\infty d\kappa \int_0^\infty d\alpha \frac{\kappa}{\alpha} C_0(\alpha, \frac{\kappa}{\alpha}, \kappa) J_0(2\kappa \rho).
\]

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\[ x e^{-i\kappa \zeta} e^{i\eta / \alpha} \] 

Now, choosing
\[ C_0(\alpha, \frac{\kappa}{\alpha}, \kappa) = (2\pi)^2 \frac{\alpha}{\kappa} \delta(\alpha - \alpha') e^{-\alpha_1 \kappa}, \] 

where \( \alpha' \) is a scalar, and integrating over \( \alpha \) yields
\[ \Psi_1(\rho, \zeta, \eta) = \int_0^\infty d\kappa J_0(2\kappa \rho) \exp\{ -a_1 - i(\alpha' \zeta - \frac{\eta}{\alpha'}) \} \kappa. \] 

The integration of (3.3.17) over \( \kappa \) is carried out using (6.611.1) of [56] resulting in
\[ \Psi_1(\rho, \zeta, \eta) = \frac{1}{[\sigma^2 + (2\rho)^2]^{1/2}}, \] 

where \( \sigma = a_1 + i(\alpha' \zeta - \frac{\eta}{\alpha'}) \). Equation (3.3.18) is identical (except for the factor of 2) to (3.3.3) which is the X-wave pulse.

Integrating (3.3.14b) over \( \alpha \) yields
\[ \Psi_2(\rho, \zeta, \eta) = \frac{1}{(2\pi)^2} \int_0^\infty d\kappa \int_0^\infty d\beta \frac{\kappa}{\beta} C_0(\frac{\kappa}{\beta}, \beta, \kappa) J_0(2\kappa \rho) e^{-i\zeta \kappa^2 / \beta} e^{i\eta \beta}. \] 

Choosing
\[ C_0(\frac{\kappa}{\beta}, \beta, \kappa) = 2\pi \delta(\beta - \beta') e^{-\alpha_1 \kappa^2 / \beta}, \] 

where \( \beta' \) is a scalar, and integrating over \( \beta \) yields
\[ \Psi_2(\rho, \zeta, \eta) = \frac{1}{2\pi} \frac{e^{i\eta \beta'}}{\beta'} \int_0^\infty d\kappa \kappa J_0(2\kappa \rho) e^{\frac{\kappa^2}{\beta'}}. \] 

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\[ V = a_1 + i \zeta. \] The integration of (3.3.21) over \( \kappa \) is carried out using (6.631.4) of [56], resulting in
\[ \Psi_2(\rho, \zeta, \eta) = \frac{1}{4\pi V} e^{i\eta \beta'} e^{-\frac{\rho^2}{V}}, \tag{3.3.22} \]
which is the FWM solution.

### 3.4 NEW FOCUSED X WAVES

We begin with representation (3.2.3), viz.,
\[ \Psi(\rho, \zeta, \eta) = \frac{1}{(2\pi)^2} \int_0^\infty d\kappa \int_0^\infty d\alpha \int_0^\infty d\beta \ \kappa \ C_0(\alpha, \beta, \kappa) J_0(\kappa \rho) e^{ik\alpha \zeta} e^{ik\beta \eta} \]
\[ \times \delta(\alpha \beta - 1/4), \tag{3.4.1} \]
and choose the spectrum
\[ C_0(\alpha, \beta, \kappa) = (2\pi)^2 \beta' \frac{1}{\kappa} \delta(\beta - \beta') e^{-\alpha_1 \alpha} e^{-\alpha_2 \kappa} J_0(2 \sqrt{\alpha_3 \kappa}), \tag{3.4.2} \]
where \( \beta', \alpha_1, \alpha_2 \) are positive scalars and \( \alpha_3 \) is a complex number. Substituting (3.4.2) into (3.4.1) and integrating over \( \beta \) yields
\[ \Psi(\rho, \zeta, \eta) = \int_0^\infty d\kappa \int_0^\infty d\alpha \ J_0(\kappa \rho) J_0(2 \sqrt{\alpha_3 \kappa}) e^{-\kappa (\alpha_2 - i\beta' \eta)} \]
\[ \times e^{-i\kappa \alpha \zeta} e^{-\alpha_1 \alpha} \delta(\alpha - \frac{1}{4\beta'}) \tag{3.4.3} \]
and carrying out the integration over \( \alpha \) yields

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\[ \Psi(\rho, \zeta, \eta) = e^{-a_1/4 \beta'} \int_0^\infty d\kappa \ J_0(\kappa \rho) \ J_0(2\sqrt{a_3 \kappa}) \]
\[ \times \ e^{-\kappa \{a_2+i \left( -\beta' \eta + \zeta/(4\beta') \right) \}}. \]  

(3.4.4)

Now, using formula (6.644) of reference [56] we integrate (3.4.4) and obtain the general X wave solution

\[ \Psi(\rho, \zeta, \eta) = \frac{1}{(\sigma^2 + \rho^2)^{1/2}} \exp \left[ -\frac{a_3 \sigma}{\sigma^2 + \rho^2} \right] \ J_0 \left[ -\frac{a_3 \rho}{\sigma^2 + \rho^2} \right], \]  

(3.4.5)

where \( \sigma = a_2+i \left( -\beta' \eta + \zeta/(4\beta') \right) \), \( \zeta = z - c \ t \), \( \eta = z + c \ t \), and we have omitted the multiplicative constant \( e^{-a_1/4 \beta'} \). It should be noted that (3.4.5) is a three-parameter X wave solution and represents an X wave pulse whose center moves with speed \( \nu = (1+4\beta^2)/(1-4\beta^2) \). The behavior of the new X wave expressed in (3.4.5) is shown in Fig. 3-1, where four surface plots of the real part of \( \Psi \) are shown corresponding to pulse center speeds of \( \nu = 2c \) and \( 4c \), and \( a_3 = 0 \) and \( -1 \). The degrees of focusing in the transverse plane \( (z = 0) \) and the \( z \) plane \( (\rho = 0) \) are shown in Figs 3-2 to 3-5 for speeds of \( 2c \) and \( 4c \). As seen in the plots, for \( a_3 = -1 \), the pulse is very focused and somewhat oscillatory. Note that \( a_3 \) can be a complex number. The solution (3.4.5) is a general form of X waves and reduces to the zeroth order solution by setting \( a_3 = 0+0i \). The most focused X wave reported to date is a two-parameter solution presented by Donnelly et al. [59] and is expressed as

\[ \Psi_D(\rho, z, t) = \frac{z_0 - i (z - c t/\gamma)}{\left[ \rho^2 (1 - \gamma^2/\gamma^2) + (z_0 - i (z - c t/\gamma))^2 \right]^{3/2}}, \]  

(3.4.6)

where \( z_0 > 0 \) is arbitrary and \( 0 < \gamma < 1 \) is a free parameter that controls the speed \( \nu = c/\gamma \) of the pulse center. The surface plots of the real part of this X wave solution are

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shown in Figs. 3-6 and 3-7 for \((z_0, \gamma) = (0.7, 0.5)\) and \((z_0, \gamma) = (0.7, 0.25)\), respectively. As seen from the plots, the transverse focusing of the pulse becomes more pronounced for smaller \(\gamma\) or higher speed \(v\). Note that the leading and trailing "Mach cones" corresponding to \(v = 4c\) have larger vertex angles than the ones corresponding to \(v = 2c\). In terms of our parameters, the solution (3.4.6) is obtained from our solution (3.4.5) by setting \(a_3 = 0 + j0\), naming \(\beta' = \beta\), and taking the derivative with respect to \(\sigma\),

\[
\Psi_\beta(\rho, \zeta, \eta) = -\frac{\sigma}{(\sigma^2 + \rho^2)^{3/2}}
\]

\[
= -\frac{a_2 - i [\beta \eta - \zeta/(4\beta)]}{\left\{a_2 - i [\beta \eta - \zeta/(4\beta)]\right\}^2 + \rho^2}^{3/2},
\]

(3.4.7)

where

\[
\gamma = (1 - 4\beta^2)/(1 + 4\beta^2), \text{ and } z_0 = 4\beta a_2/(4\beta^2 - 1).
\]

(3.4.8)

The relations in (3.4.8) establish the connection between the representations (3.4.6) and (3.4.7).

The comparison of our three-parameter general solution with that of reference [59] is shown in Figs 3-8 and 3-9. As seen from the figures, our solution is more localized and can be extremely localized through the appropriate choice of the parameter \(a_3\). For example, for \(a_3\) negative and large absolute value, we obtain highly localized but oscillatory solutions as shown in Fig. 3-10.
3.5 MODIFIED FOURIER REPRESENTATIONS

Consider the Fourier representation of the solution to the three-dimensional scalar homogeneous wave equation,

\[ \Psi(\vec{r},t) = \frac{1}{(2\pi)^2} \int_0^\infty d\kappa \int d\omega \int_{-\infty}^{\infty} dk_z \ \kappa \ A_0(\omega,k_z,\kappa) \ J_0(\kappa \rho) \ e^{-ik_z z} e^{i\omega t} \]

\[ \times \ \delta \left[ \frac{\omega^2}{c^2} - \kappa^2 - k_z^2 \right], \quad (3.5.1) \]

Let us now apply the following scaling

\[ \kappa' = \kappa \]

\[ \kappa' k_z' = k_z \]

\[ \kappa' \omega' = \frac{\omega}{c}. \quad (3.5.2) \]

where the units of \( \kappa' \) are \( m^{-1} \), and \( k_z', \omega' \) are dimensionless. We rewrite (3.5.1) in terms of the new variables \( \kappa' \), \( k_z' \), and \( \omega' \) and obtain the modified Fourier representation

\[ \Psi(\vec{r},t) = \frac{1}{(2\pi)^2} \int_0^\infty d\kappa' \int d\omega' \int_{-\infty}^{\infty} dk_z' \ c \ A_0(\omega',k_z',\kappa') \ \kappa' \ J_0(\kappa' \rho) \ e^{-ik'z'} e^{i\kappa'\omega' t} \]

\[ \times \ \delta \left[ \omega'^2 - k_z'^2 - 1 \right]. \quad (3.5.3) \]

Let us rename the parameters \( \kappa', k_z', \omega' \) to \( \kappa, k_z, \omega \), respectively, and obtain

\[ \Psi(\vec{r},t) = \frac{c}{(2\pi)^2} \int_0^\infty d\kappa \int d\omega \int_{-\infty}^{\infty} dk_z \ A_0(\omega,k_z,\kappa) \ \kappa \ J_0(\kappa \rho) \ e^{-ikz} e^{i\kappa \omega t} \]

\[ \times \ \delta \left[ \omega^2 - k_z^2 - 1 \right]. \quad (3.5.4) \]

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The general solution given in (3.5.4) can be decomposed into two parts, viz.,

$$
\Psi(\rho, z, t) = \Psi_+(\rho, z, t) + \Psi_-(\rho, z, t)
$$

(3.5.5)

corresponding to forward (with respect to z) and backward waves. The forward solution is given explicitly as follows:

$$
\Psi(\rho, z, t) = 2 \text{Re} \left\{ \frac{c}{(2\pi)^2} \int_0^\infty d\kappa \int_0^\infty dk_z A_0[\kappa, k_z, \sqrt{k_z^2 + 1}] \kappa J_0(\kappa \rho) \right.

\times e^{-i \kappa k_z} \frac{e^{-i \kappa \sqrt{k_z^2 + 1}}}{2\sqrt{k_z^2 + 1}} \left. \right\}.
$$

(3.5.6)

Now, let us choose the spectrum

$$
A_0(\kappa, k_z, \sqrt{k_z^2 + 1}) = 2 \sqrt{k_z^2 + 1} A_1(\kappa) \delta[k_z + i \sin \beta],
$$

(3.5.7)

where $\beta$ is a free parameter. Substituting (3.5.7) into (3.5.6) and integrating over $k_z$, we obtain

$$
\Psi_+(\rho, z, t) = 2 \text{Re} \left\{ \frac{c}{(2\pi)^2} \int_0^\infty d\kappa \ A_1(\kappa) \kappa J_0(\kappa \rho) e^{-\kappa \rho \sin \beta} e^{i \kappa t \cos \beta} \right\}.
$$

(3.5.8)

Choosing

$$
A_1(\kappa) = \frac{(2\pi)^2}{c \kappa} e^{-a_1 \kappa},
$$

(3.5.9)

where $a_1$ is a real and positive parameter, and substituting (3.5.9) into (3.5.8) yields

$$
\Psi_+(\rho, z, t) = 2 \text{Re} \left\{ \int_0^\infty d\kappa J_0(\kappa \rho) e^{-\kappa \rho \sin \beta + a_1 \rho \cos \beta} \right\}.
$$

(3.5.10)

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Using formula 6.611.1 of [56] we integrate (3.5.8) and obtain the following XW solution

$$\Psi_{+}(\rho, z, t) = 2 \Re \left\{ \frac{1}{\{ \rho^2 + \left[ a_1 + \sin \beta (z - ct \cot \beta) \right]^2 \}^{1/2}} \right\}. \quad (3.5.11)$$

Letting $\beta = b$, where $b$ is a real parameter, (3.5.11) results in a more familiar representation of the superluminal X wave pulse

$$\Psi(\vec{r}, t) = \frac{z_0}{\left( z_0^2 + \left[ a_1 z_0 + i (z - ct \coth b) \right]^2 \right)^{1/2}}, \quad (3.5.12)$$

where $z_0 = 1/\sinh b$ and $v = c \coth b$ is the speed of the pulse.

### 3.6 CONCLUSION

In this chapter we introduced the modified bidirectional representation. Within this framework, we can select new basis functions resulting in different representations for a solution. This freedom facilitates the solutions of the homogeneous, as well as the nonhomogeneous scalar wave equation. We showed how known LW solutions to the scalar wave equation can be obtained through this new approach. A new class of focused X waves was presented which results in pulses that are extremely localized. This new class of superluminal X waves are much more localized than the X waves presented in the literature. Finally, this new approach is applied to the Fourier synthesis resulting in an alternative representation of solutions to the homogeneous scalar wave equation.

### 3.6 A MODIFICATION OF THE BIDIRECTIONAL SYNTHESIS:

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Fig. 3-1a: Surface plot of the real part of the new focused X wave pulse moving with speed $v_p = 2c$ with the parameters $a_3 = 0$, $\beta = 0.2887$, and $a_2 = 0.404$ cm.
Fig. 3-1b: Surface plot of the real part of the new focused X wave pulse moving with speed \( v_p = 4c \) with the parameters \( a_3 = 0, \beta = 0.3873, \) and \( a_2 = 0.181 \) cm.
Fig. 3-1c: Surface plot of the real part of the new focused X wave pulse moving with speed $v_p = 2c$ with the parameters $\alpha_3 = -1$, $\beta = 0.2887$, and $a_2 = 0.404$ cm.
Fig. 3-1d: Surface plot of the real part of the new focused X wave pulse moving with speed \( v_p = 4c \) with the parameters \( a_3 = -1, \beta = 0.3873 \), and \( a_2 = 0.181 \) cm.
Fig. 3-2: Plots of the real part of the new focused X wave pulse at the $\rho = 0$ plane for (a): $\alpha_3 = 0$, $\beta = 0.289$, $\alpha_2 = 0.404$ cm, $\nu_p = 2c$ and (b): $\alpha_3 = 0$, $\beta = 0.387$, $\alpha_2 = 0.181$ cm, $\nu_p = 4c$. 

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Fig. 3-3: Plots of the real part of the new focused X wave pulse at the $\rho = 0$ plane for (a): $a_3 = -1$, $\beta = 0.289$, $a_2 = 0.404$ cm, $v_p = 2c$ and (b): $a_3 = -1$, $\beta = 0.387$, $a_2 = 0.181$ cm, $v_p = 4c$. 

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Fig. 3-4: Plots of the real part of the new focused X wave pulse at the $z = 0$ plane for (a): $a_3 = 0$, $\beta = 0.289$, $a_2 = 0.404$ cm, $v_p = 2c$ and (b): $a_3 = 0$, $\beta = 0.387$, $a_2 = 0.181$ cm, $v_p = 4c$. 

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Fig. 3-5: Plots of the real part of the new focused X wave pulse at the $z = 0$ plane for (a): $a_3 = -1$, $\beta = 0.289$, $a_2 = 0.404$ cm, $v_p = 2c$ and (b): $a_3 = -1$, $\beta = 0.387$, $a_2 = 0.181$ cm, $v_p = 4c$. 

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Fig. 3-6: Surface plot of the real part of the X wave pulse of reference [59] for the parameters $z_0 = 0.7 \text{ cm}$, and $\gamma = 0.5$. The speed of the pulse is $v_p = 2c$. 

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Fig. 3-7: Surface plot of the real part of the X wave pulse of reference [59] for the parameters $z_0 = 0.7$ cm, and $\gamma = 0.25$. The speed of the pulse is $v_\nu = 4c$.  

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Fig. 3-8:  (a) Surface plot of the real part of the $X$ wave pulse of reference [59] for the parameters $z_0 = 0.7$ cm, and $\gamma = 0.5$. (b) Surface plot of the real part of the new focused $X$ wave pulse corresponding to the same parameters as in (a) with the additional parameter $a_3 = -1$. The speed of the pulse is $v_p = 2c$. 

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Fig. 3-9:  (a) Surface plot of the real part of the X wave pulse of reference [59] for the parameters $z_0 = 0.7 \text{ cm}$, and $\gamma = 0.25$. (b) Surface plot of the real part of the new focused X wave pulse corresponding to the same parameters as in (a) with the additional parameter $\alpha_3 = -1$. The speed of the pulse is $v_p = 4c$. 

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Fig. 3-10: Surface plot of the real part of the new focused $X$ wave pulse:
(a) $a_3 = -2, \beta = .3873, a_2 = 0.1807$, (b) $a_3 = -10, \beta = .3873, a_2 = 0.1807$. The speed of the pulse is $v_p = 4c$. Note the different scales on both the $z - v_p t$ and $\rho$ axes, as compared to Fig. 3-9.
4.0 MOVING SOURCES

4.1 INTRODUCTION

It is well known that one of the LW solutions, the FWM pulse in free space, can be generated by a \textit{complex} source moving with the speed of light [6]. Other LW solutions can also be generated by \textit{complex} moving sources. Such sources, obviously, are not realistic and cannot be implemented. The ordinary bidirectional method, as well as the modified one discussed in the previous chapter, can be applied to the forced wave equation in order to compute fields generated by sources. In section 4.2, we shall demonstrate the modified bidirectional method by solving three well-known source problems: (1) a point charge moving uniformly in free space, (2) the time-dependent Green's function due to a line source, and (3) the Green's function for the Poisson equation. After this familiarization with the new method, we shall show that LW solutions such as the FWM, the MPS, and the "sling-shot" pulses, can be generated by \textit{real} moving planar sources. This will be illustrated in Sec. 4.3. In the FWM case, the real source has a Gaussian taper in the transverse plane and moves with the speed of light. The source producing the MPS pulse has a complicated structure, moving with the speed of light. In the case of the "sling-shot" pulse, the real source has a $\rho^3$ dependence in the transverse plane and moves with a speed greater than the speed of light. This makes sense since the sling-shot is superluminal, i.e., moving with speed greater than $c$. 
4.2 APPLICATION OF THE MODIFIED BIDIRECTIONAL DECOMPOSITION TO THREE KNOWN SOURCE PROBLEMS

4.2.1 Point Charge Moving Uniformly in Free Space

Consider a point charge $Q$ moving in free space with velocity $\vec{v} = v\hat{a}_z$. Within the framework of the Lorentz gauge, the scalar electromagnetic potential obeys the equation

$$[\nabla^2 - c^2 \frac{\partial^2}{\partial t^2}] \phi(\vec{r}, t) = -\frac{1}{\varepsilon_0} \rho_v(\vec{r}, t), \quad (4.2.1)$$

where $\rho_v(\vec{r}, t) = Q \delta(x) \delta(y) \delta(z - vt)$. The problem is axisymmetric. It is preferable, then, to work in cylindrical coordinates. Eq. (4.2.1) can be rewritten as

$$[L_1 + L_2] \phi(\rho, z, t) = -\frac{1}{\varepsilon_0} \frac{1}{2\pi} \frac{\delta(\rho)}{\rho} Q \delta(z - vt); \quad (4.2.2a)$$

$$L_1 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho}, \quad (4.2.2b)$$

$$L_2 = \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}. \quad (4.2.2c)$$

In terms of the variables $\zeta = z - ct$ and $\eta = z + ct$, Eq. (4.2.2) is rewritten as

$$[L_1 + L_2] \phi(\rho, \zeta, \eta) = -Q \frac{1}{\varepsilon_0} \frac{1}{2\pi} \frac{\delta(\rho)}{\rho} \frac{2}{(1 + v/c)} \delta \left[ \zeta + \eta \left( \frac{1 - v/c}{1 + v/c} \right) \right]; \quad (4.2.3a)$$

4.0 MOVING SOURCES

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\[ \mathcal{L}_2 = 4 \frac{\partial^2}{\partial \zeta \partial \eta} \]  \hspace{1cm} (4.2.3b)

We invoke, next, the modified bidirectional representation, \textit{viz.},

\[ \phi(\rho, \zeta, \eta) = \text{Re} \{ \Psi(\rho, \zeta, \eta) \} \]  \hspace{1cm} (4.2.4a)

\[ \Psi(\rho, \zeta, \eta) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\kappa \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \, C_0(\alpha, \beta, \kappa) \kappa J_0(\kappa \rho) \]

\[ \times e^{i\zeta \alpha \kappa} e^{i\eta \beta \kappa} . \]  \hspace{1cm} (4.2.4b)

We note that

\[ [L_1 + \bar{L}_2] \Psi(\rho, \zeta, \eta) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\kappa \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \, C_0(\alpha, \beta, \kappa) \kappa J_0(\kappa \rho) \]

\[ \times e^{i\zeta \alpha \kappa} e^{i\eta \beta \kappa} \delta(4\alpha\beta - 1) . \]  \hspace{1cm} (4.2.5)

The right-hand side of (4.2.3a) can be expressed as

\[ \frac{1}{2\pi} \frac{\delta(\rho)}{\rho} \delta \left[ \zeta + \eta \left( \frac{1 - \nu/c}{1 + \nu/c} \right) \right] = \]

\[ \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\kappa \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \kappa \delta \left[ \beta + \alpha \left( \frac{1 - \nu/c}{1 + \nu/c} \right) \right] \kappa J_0(\kappa \rho) e^{i\zeta \alpha \kappa} e^{i\eta \beta \kappa} . \]  \hspace{1cm} (4.2.6)

Now, from Eqs. (4.2.3), (4.2.5), and (4.2.6) we can find the spectrum \( C_0(\alpha, \beta, \kappa) \); specifically

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\[ C_0(\alpha, \beta, \kappa) = \frac{Q}{\varepsilon_0} \frac{2}{(1 + \nu/c) \kappa(1 - 4\alpha\beta)} \delta \left[ \beta + \alpha \left( \frac{1 - \nu/c}{1 + \nu/c} \right) \right]. \quad (4.2.7) \]

Substituting (4.2.7) in (4.2.4b) and integrating first over \( \beta \), and then over \( \alpha \), we obtain

\[ \Psi(\rho, \zeta, \eta) = \frac{Q \gamma}{4\pi \varepsilon_0} \int_0^\infty d\kappa J_0(\kappa \rho) \]
\[ \times \exp \left[ -\frac{\kappa}{2} \left( \zeta + \eta \left( \frac{1 - \nu/c}{1 + \nu/c} \right) \right) \sqrt{\frac{1 + \nu/c}{1 - \nu/c}} \right], \quad (4.2.8) \]

where we have used Gradshteyn and Ryzhik's formula 3.723.2 [56] under the condition
\[ \kappa \{ \zeta + \eta \left[ (1 - \nu/c)/(1 + \nu/c) \right] \} \geq 0. \]
Integration of (4.2.8), using the formula 6.611.1 of [56], yields the well-known solution

\[ \Psi(\rho, \zeta, \eta) = \frac{Q \gamma}{4\pi \varepsilon_0} \left[ \rho^2 + \gamma^2(z - \nu t)^2 \right]^{-1/2}, \quad (4.2.9) \]

where \( \gamma = [1 - (\nu/c)^2]^{-1/2} \).

### 4.2.2 Line Source; Time Dependent Green's Function

Consider the canonical problem

\[ [\nabla^2 - c^2 \frac{\partial^2}{\partial t^2}] g(\tau, t) = -\delta(x) \delta(y) \delta(t), \quad (4.2.10a) \]

and, assuming azimuthal symmetry, rewrite it as

\[ \left[ \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + 4 \frac{\partial^2}{\partial \zeta \partial \eta} \right] g(\rho, \zeta, \eta) = -\frac{c}{\tau} \frac{\delta(\rho)}{\rho} \delta(\zeta - \eta). \quad (4.2.10b) \]

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The right-hand side of (4.2.10b) can be expressed as

$$- \frac{c}{\pi} \frac{\delta(p)}{p} \delta(\zeta - \eta) =$$

$$- \frac{1}{(2\pi)^2} \int_0^\infty d\kappa \int_{-\infty}^\infty d\alpha \int_{-\infty}^\infty d\beta \, 2 \kappa \, \delta(\beta - \alpha) \kappa J_0(\kappa p) e^{i\zeta\alpha\kappa} e^{i\eta\beta\kappa}, \quad (4.2.11)$$

and the left-hand of (4.2.10b) can be represented as

$$[L_1 + L_2] g(\rho, \zeta, \eta) = \frac{1}{(2\pi)^2} \int_0^\infty d\kappa \int_{-\infty}^\infty d\alpha \int_{-\infty}^\infty d\beta \, C_0(\alpha, \beta, \kappa) \kappa J_0(\kappa p) e^{i\zeta\alpha\kappa} e^{i\eta\beta\kappa}$$

$$\times \delta(4\alpha\beta - 1). \quad (4.2.12)$$

Equating both sides of (4.2.10b) we determine the spectrum

$$C_0(\alpha, \beta, \kappa) = \frac{2 c \delta(\beta - \alpha)}{\kappa (1 - 4\alpha\beta)}. \quad (4.2.13)$$

Substituting (4.2.13) in (4.2.4b) and integrating over \( \beta \) we have

$$g(\rho, \zeta, \eta) = \frac{c}{(2\pi)^2} \int_0^\infty d\kappa \int_0^\infty d\alpha \, J_0(\kappa p) \frac{\cos[\alpha \kappa (\zeta - \eta)]}{(1/2)^2 - \alpha^2}. \quad (4.2.14)$$

To integrate over \( \alpha \) we use formula 3.723.9 of [56]; as a result, we obtain

$$g(\rho, \zeta, \eta) = \frac{c}{4\pi} \int_0^\infty d\kappa \, J_0(\kappa p) \sin \left[ \frac{\kappa}{2} (\zeta - \eta) \right]. \quad (4.2.15)$$

To integrate over \( \kappa \), we use formula 6.671.1 of Gradshteyn and Ryzhik [56]. This yields the expression

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\[ g(\rho, \zeta, \eta) = \begin{cases} \frac{c}{4\pi} \left[ (1/4) (\zeta - \eta)^2 - \rho^2 \right]^{-1/2}; & \rho < \frac{\zeta - \eta}{2} < \infty \\ 0; & 0 < \frac{\zeta - \eta}{2} < \rho \end{cases} \] (4.2.16)

with the condition \(\kappa(\zeta - \eta > 0\), or \(t > 0\). In terms of \(\rho\) and \(t\) the solution becomes

\[ g(\rho, \zeta, \eta) = \begin{cases} \frac{c}{4\pi} \left[ c^2 t^2 - \rho^2 \right]^{-1/2}; & t > 0, \rho > 0, \rho < ct < \infty \\ 0; & 0 < ct < \rho \end{cases} \] (4.2.17)

### 4.2.3 The Green's Function for the Poisson Equation

Consider the Poisson equation

\[ \nabla^2 \phi(\vec{r}) = -\frac{1}{\epsilon} \delta(x) \delta(y) \delta(z). \] (4.2.18)

We may add the operator \(c^2 \partial^2_{tt}\) on the left hand side of (4.2.18) without affecting the solution; specifically,

\[ [\nabla^2 - c^2 \partial^2_{tt}] \phi(\vec{r}) = -\frac{1}{\epsilon} \delta(x) \delta(y) \delta(z). \] (4.2.19)

Eq. (4.2.19) is a "pseudo-hyperbolic" equation. Assuming cylindrical symmetry, it can be rewritten as

\[ \left[ \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + 4 \frac{\partial^2}{\partial \zeta \partial \eta} \right] \phi(\rho, \zeta, \eta) = -\frac{1}{\epsilon} \frac{1}{2\pi} \frac{\delta(\rho)}{\rho} \delta \left( \frac{\zeta + \eta}{2} \right). \] (4.2.20)

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The right-hand side of (4.2.20) is expressed in terms of the modified bidirectional decomposition as

\[- \frac{1}{\epsilon} \frac{1}{2\pi} \frac{\delta(\rho)}{\rho} \delta \left( \frac{\zeta + \eta}{2} \right) =

- \frac{1}{(2\pi)^2} \int_0^\infty \int_{-\infty}^{\infty} d\kappa \int_{-\infty}^{\infty} d\alpha \frac{J_0(\kappa \rho)}{\kappa} e^{-i\zeta\alpha\kappa} e^{i\eta\beta\kappa}. \quad (4.2.21)\]

Using (4.2.5), we can solve for the spectrum \(C_0(\alpha, \beta, \kappa)\), viz.,

\[C_0(\alpha, \beta, \kappa) = \frac{2 \delta(\beta + \alpha)}{\epsilon \kappa (1 - 4\alpha\beta)}. \quad (4.2.22)\]

Substituting (4.2.22) in (4.2.4b) and integrating over \(\beta\) we have

\[\phi(\rho, \zeta, \eta) = \frac{2}{(2\pi)^2 \epsilon} \int_0^\infty \int_{-\infty}^{\infty} d\kappa \int_0^\infty \frac{J_0(\kappa \rho)}{\kappa} \frac{\cos[\alpha \kappa (\zeta + \eta)]}{(1/2)^2 + \alpha^2}, \quad (4.2.23)\]

To integrate over \(\alpha\) we use formula 3.723.2 of [56]. This yields

\[\phi(\rho, \zeta, \eta) = \frac{1}{4\pi\epsilon} \int_0^\infty d\kappa J_0(\kappa \rho) \exp \left[ -\frac{\kappa}{2} (\zeta + \eta) \right], \quad (4.2.24)\]

where \(\kappa(\zeta + \eta) \geq 0\) or \(z \geq 0\). To integrate over \(\kappa\) we use formula 6.611.1 of Gradshteyn and Ryzhik [56], resulting in

\[\phi(\rho, \zeta, \eta) = \frac{1}{4\pi\epsilon} \left[ \left(\frac{(\zeta + \eta)}{4}\right)^2 + \rho^2 \right]^{1/2}. \quad (4.2.25)\]
The solution in terms of $\rho$ and $z$ is given by

$$\phi(\rho, \zeta, \eta) = \frac{1}{4\pi \epsilon} \left( z^2 + \rho^2 \right)^{1/2} = \frac{1}{4\pi \epsilon} \frac{1}{r} = \phi(r). \quad (4.2.26)$$

### 4.3 MOVING SOURCES GENERATING LOCALIZED WAVES

In this section moving sources generating localized waves will be presented. In the first part we shall show that LW solutions can be generated by moving complex sources, while in the second part we shall show that these LW solutions can also be generated by real moving sources.

#### 4.3.1 Moving Complex Sources

Consider the following problem

$$\left[ \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + 4 \frac{\partial^2}{\partial \zeta \partial \eta} \right] \psi(\rho, \zeta, \eta) = -f(\rho, \zeta, \eta), \quad (4.3.1)$$

where $f(\rho, \zeta, \eta)$ is the source producing the localized solution $\psi(\rho, \zeta, \eta)$. We resort to the ordinary bidirectional representation

$$\psi(\rho, \zeta, \eta) = \frac{1}{(2\pi)^2} \int_0^\infty d\kappa \int_0^\infty d\alpha \int_{-\infty}^{\infty} d\beta \kappa C_0(\alpha, \beta, \kappa) J_0(\kappa \rho) e^{-i\alpha \zeta} e^{-i\beta \eta}. \quad (4.3.2)$$
For a spectrum of the form

\[ C_0(\alpha, \beta, \kappa) = C_1(\alpha, \kappa) \delta(\beta - \beta') , \]  

(4.3.3)

we find that

\[ \psi(\rho, \zeta, \eta) = e^{i\eta} G(\rho, \zeta) , \]  

(4.3.4)

where

\[ G(\rho, \zeta) = \frac{1}{(2\pi)^2} \int_0^\infty d\kappa \int_{-\infty}^{\infty} d\alpha \ \kappa J_0(\kappa \rho) e^{-i\alpha \zeta} . \]  

(4.3.5)

The FWM pulse can be generated by the moving complex source

\[ f(\rho, \zeta, \eta) = -\frac{A}{\varepsilon_0} \frac{\delta(\rho)}{2\pi \rho} \delta(\zeta - i\alpha_1) e^{i\eta} . \]  

(4.3.6)

Substituting (4.3.4) and (4.3.6) into (4.3.1) results in

\[ \left[ \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + 4i\beta \frac{\partial}{\partial \zeta} \right] G(\rho, \zeta) = -\frac{A}{\varepsilon_0} \frac{\delta(\rho)}{2\pi \rho} \delta(\zeta - i\alpha_1) . \]  

(4.3.7)

The right-hand side of (4.3.7) can be written as

\[ \cdot \ -\frac{A}{\varepsilon_0} \frac{\delta(\rho)}{2\pi \rho} \delta(\zeta - i\alpha_1) = \frac{1}{(2\pi)^2} \int_0^\infty d\kappa \int_{-\infty}^{\infty} d\alpha \ \kappa J_0(\kappa \rho) e^{-i\alpha \zeta} \left[ -\frac{A}{\varepsilon_0} e^{\alpha \alpha_1} \right] . \]  

(4.3.8)

Using (4.3.7) and (4.3.8), we have

\[ G(\rho, \zeta) = \frac{1}{(2\pi)^2} \int_0^\infty d\kappa \int_{-\infty}^{\infty} d\alpha \ \kappa J_0(\kappa \rho) \left[ -\frac{A}{\varepsilon_0} \frac{1}{4\beta} e^{-i\alpha(\zeta - \alpha_1)} \right] . \]  

(4.3.9)

Now, let \( \alpha = -i \omega \) and \( \widetilde{\zeta} = i \zeta \). Then,
\[ G(\rho, \zeta) = \frac{i}{(2\pi)^2} \frac{A}{\epsilon_0} \frac{1}{4\beta'} \int_0^\infty \kappa \int_{-\infty}^\infty dw \kappa J_0(\kappa \rho) \frac{e^{i\omega(\zeta + \alpha_1)}}{\omega + (\alpha^2 / 4\beta')} . \] (4.3.10)

We integrate (4.3.10) with respect to \( \omega \) using the identity

\[ \frac{1}{2\pi} \int_{-\infty}^\infty d\omega \frac{e^{i\omega \alpha}}{\omega + c} = e^{-c\alpha} u(\alpha) . \] (4.3.11)

As a result, we have

\[ G(\rho, \zeta) = \frac{i}{2\pi} \frac{A}{\epsilon_0} \frac{1}{4\beta'} \int_0^\infty \kappa \kappa J_0(\kappa \rho) e^{-\frac{\alpha^2}{4\beta}(\zeta + \alpha_1)} . \] (4.3.12)

Finally, we integrate over \( \kappa \) using formula 6.631.4 of [56] and use (4.3.4). The final expression is the FWM solution

\[ \psi_{FWM}(\rho, \zeta, \eta) = \frac{i}{4\pi} \frac{A}{\epsilon_0} \frac{1}{\alpha_1 + i\zeta} \exp\left[ -\frac{\beta \rho^2}{\alpha_1 + i\zeta} \right] . \] (4.3.13)

Now, consider the problem

\[ \left[ \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + 4 \frac{\partial^2}{\partial \zeta \partial \eta} \right] \psi(\rho, \zeta, \eta) = -\frac{\delta(\rho)}{2\pi \rho} \delta(\zeta - i\alpha_1) f(\eta) , \] (4.3.14)

and represent the right-hand side as follows:

\[ -\frac{\delta(\rho)}{2\pi \rho} \delta(\zeta - i\alpha_1) f(\eta) = \]

\[ \frac{1}{(2\pi)^2} \int_0^\infty d\kappa \int_{-\infty}^\infty d\alpha \kappa J_0(\kappa \rho) e^{-i\alpha \zeta} e^{i\beta \eta} \left[ -2\pi e^{-\alpha_1 \alpha} F(\beta) \right] . \] (4.3.15)

Then,
\[
\psi(\rho, \zeta, \eta) = \frac{1}{(2\pi)^3} \int_0^\infty d\kappa \int_{-\infty}^\infty d\alpha \int_{-\infty}^\infty d\beta \ \kappa J_0(\kappa \rho) e^{-i \alpha \zeta} e^{i \beta \eta} \frac{2\pi e^{-\alpha \alpha_1}}{\kappa^2 - 4\alpha \beta} F(\beta), \quad (4.3.16)
\]

or
\[
\psi(\rho, \zeta, \eta) = -\frac{1}{2\pi} \int_0^\infty d\kappa \int_{-\infty}^\infty d\alpha \int_{-\infty}^\infty d\beta \ \kappa J_0(\kappa \rho) e^{i \beta \eta} F(\beta) \frac{e^{-i \alpha (\zeta - i \alpha_1)}}{\alpha - \kappa^2/4\beta}. \quad (4.3.17)
\]

Now, let \( \alpha = -iw \) and \( F(\beta) = \mathcal{F}(\beta) u(\beta) \), where \( u(\beta) \) is the unit step function. Then,
\[
\psi(\rho, \zeta, \eta) = \frac{i}{2\pi} \int_0^\infty d\kappa \int_0^\infty d\beta \int_{-\infty}^\infty dw \ \kappa J_0(\kappa \rho) e^{i \beta \eta} \mathcal{F}(\beta) \frac{e^{iw(\zeta + \alpha_1)}}{4\beta (iw + \kappa^2/4\beta)}. \quad (4.3.18)
\]

Carrying out the integration over \( w \), we obtain
\[
\psi(\rho, \zeta, \eta) = \frac{i}{4} \int_0^\infty d\kappa \int_0^\infty d\beta \ \kappa J_0(\kappa \rho) e^{i \beta \eta} \frac{\mathcal{F}(\beta)}{\beta} e^{-\frac{2}{i\beta} (i\zeta + \alpha_1)}. \quad (4.3.19)
\]

Now, let
\[
\frac{-i}{2\pi} \frac{p(p \beta - b)^{q-1}}{\Gamma(q)} \exp[-(p \beta - b)a_2], \quad \beta > \frac{b}{p}
\]

\[
\mathcal{F}(\beta) = \begin{cases} 
- \frac{i}{2\pi} \frac{p(p \beta - b)^{q-1}}{\Gamma(q)} \exp[-(p \beta - b)a_2], & \beta > \frac{b}{p} \\
0, & \frac{b}{p} > \beta \geq 0.
\end{cases} \quad (4.3.20)
\]

Then,
\[
\psi(\rho, \zeta, \eta) = \frac{1}{8\pi} \int_0^\infty d\kappa \int_0^{b/p} d\beta \ \kappa J_0(\kappa \rho) e^{i \beta \eta} \frac{p(p \beta - b)^{q-1}}{\beta \Gamma(q)} e^{-(p \beta - b)a_2} e^{-\frac{4\pi^2}{b^2} (i\zeta + \alpha_1)}
\]
\[
= \frac{1}{4\pi(a_1 + i\zeta)} \frac{e^{-bs/p}}{[a_2 + s/p]}, \quad s = \frac{\rho^2}{(a_1 + i\zeta)} - i\eta, \quad (4.3.21)
\]

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where \( q = 1 \), and \( a_1, a_2, b, p \) are free parameters. The solution (4.3.21) is the "Modified Power Spectrum" (MPS) pulse corresponding to \( q = 1 \). Much effort has been concentrated on this MPS pulse because of its appealing analytical form and its finite energy content. In addition, its pulse shape can be tailored to a particular application with a change in parameters. Its source is found from the right-hand side of (4.3.14), using (4.3.20) and Fourier inverting \( F(\beta) = \tilde{F}(\beta) u(\beta) \); it is given specifically by

\[
 f(\rho, \zeta, \eta) = -\frac{\delta(\rho)}{2\pi \rho} \delta(\zeta - i\alpha_1) \left( \frac{-i}{2\pi} \right) \int_{b/p}^{\infty} d\beta e^{i\beta \eta} \frac{p(p\beta - b)^{q-1}}{\Gamma(q)} e^{-(p\beta - b)a_2}
\]

\[
 = -\frac{\delta(\rho)}{2\pi \rho} \frac{i}{(2\pi)^2} \delta(\zeta - i\alpha_1) \frac{1}{\Gamma(q)} e^{i\beta \eta} \int_0^\infty dw w^{q-1} e^{-w(a_2 + i\eta/p)}, \quad (4.3.22)
\]

where \( w = p\beta - b \). Integration of (4.3.22) over \( w \) results in the following moving complex source generating the MPS pulse

\[
 f_{\text{MPS}}(\rho, \zeta, \eta) = \frac{\delta(\rho)}{2\pi \rho} \frac{i}{(2\pi)^2} \delta(\zeta - i\alpha_1) \frac{e^{i\beta \eta}}{(a_2 - i\eta/p)}. \quad (4.3.23)
\]
4.3.2 Sources Generating LW Pulses

Consider the following source problem:

\[
\left[ \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + 4 \frac{\partial^2}{\partial \zeta \partial \eta} \right] \psi(\rho, \zeta, \eta) = -\frac{2a_3}{\pi a_1} e^{-a_3 \rho^2/a_1} e^{i a_3 \eta} \delta(\zeta). \tag{4.3.24}
\]

It is seen that \(Re\{\psi(\rho, \zeta, \eta)\}\) is generated by a real source moving with a speed of light \(c\). We can express this source in terms of the modified bidirectional representation as

\[
-\frac{2a_3}{\pi a_1} e^{-a_3 \rho^2/a_1} e^{i a_3 \eta} \delta(\zeta) =
\]

\[
\frac{1}{(2\pi)^2} \int_0^\infty d\kappa \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \kappa J_0(\kappa \rho) e^{-i \kappa \alpha \zeta} e^{i \kappa \beta \eta}
\]

\[
\times \left[ \kappa e^{-\left(\frac{\alpha}{4a_3}\right)^2} \delta(\beta - \frac{a_3}{\kappa}) \right]. \tag{4.3.25}
\]

Using the same procedure as in Sec. 4.2, we can solve for the spectrum and obtain

\[
C_0(\alpha, \beta, \kappa) = -\frac{2}{\kappa (1 - 4\alpha \beta)} e^{-\left(\frac{\alpha}{4a_3}\right)^2} \delta(\beta - \frac{a_3}{\kappa}). \tag{4.3.26}
\]

Then,

\[
\psi(\rho, \zeta, \eta) = \frac{1}{(2\pi)^2} \int_0^\infty d\kappa \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta J_0(\kappa \rho) e^{-i \kappa \alpha \zeta} e^{i \kappa \beta \eta}
\]
\[ \frac{2 e^{-(a_1/4a_3)\kappa^2} \delta(\beta - a_3/\kappa)}{(1 - 4\alpha\beta)}. \]  

(4.3.27)

Integration with respect to \( \beta \) and changing variables \( \alpha = -\alpha' \) results in

\[ \psi(\rho, \zeta, \eta) = \frac{-2 e^{ia_3\eta}}{(2\pi)^2 4a_3} \int_0^\infty d\kappa \ J_0(\kappa \rho) \ e^{-(a_1/4a_3)\kappa^2} \]

\[ \times \int_{-\infty}^\infty d\alpha' \ \frac{e^{i\kappa\alpha'} \zeta}{\alpha' - (-\kappa/4a_3)}. \]  

(4.3.28)

Using formula 3.352.7 of [56], the integration over \( \alpha' \) yields

\[ \psi(\rho, \zeta, \eta) = -\frac{2i e^{ia_3\eta}}{16\pi a_3} \int_0^\infty d\kappa \ \kappa \ J_0(\kappa \rho) \ e^{-\frac{1}{4a_3} (a_1 + i\zeta)^2}, \]  

(4.3.29)

for \( \kappa \zeta > 0 \). Finally, integrating over \( \kappa \) using 6.631.4 of [56] and setting \( a_3 = \beta \), we obtain

\[ \psi_{FWM}(\rho, \zeta, \eta) = \frac{1}{4\pi i} \ \frac{e^{i\eta}}{a_1 + i\zeta} \ \exp\left[ -\frac{\beta \rho^2}{(i\zeta + a_1)} \right]. \]  

(4.3.30)

The real source producing the MPS pulse is found from that generating the FWM pulse [cf. Eq. (4.3.24), with \( a_3 = \beta \)] as

\[ f_{MPS}(\rho, \zeta, \eta) = -\frac{2}{\pi a_1} \ \delta(\zeta) \ \int_0^\infty d\beta \ \beta \ \overline{F}(\beta) \ e^{-\beta \rho^2/a_1} \ e^{i\eta}. \]  

(4.3.31)

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where \( \widetilde{F}(\beta) \) is given in (4.3.20). Substituting \( \widetilde{F}(\beta) \) in (4.3.31), we obtain

\[
f(\rho, \zeta, \eta) = \frac{-2i}{\pi^2 a_1} \delta(\zeta) \int_{b/p}^{\infty} d\beta \beta \frac{p(\rho \beta - b)^{q-1}}{\Gamma(q)} e^{-(\rho \beta - b)\alpha_2} e^{-\beta(\xi^2_{a_1} - i\eta)}. \tag{4.3.32}
\]

Let \( \beta' = \beta - b/p \); then (4.3.32) becomes

\[
f(\rho, \zeta, \eta) = \frac{-2i}{\pi^2 a_1} \frac{p}{\Gamma(q)} \delta(\zeta) \int_{0}^{\infty} d\beta' (\beta' + b/p) (\rho \beta')^{q-1} \times e^{-\rho \beta' \alpha_2} e^{-(\rho \beta' + b/p)(\xi^2_{a_1} - i\eta)}. \tag{4.3.33}
\]

The expression in (4.3.33) can be split into two integrals

\[
f(\rho, \zeta, \eta) = \frac{-2i}{\pi^2 a_1} \frac{p^q}{\Gamma(q)} \delta(\zeta) \ e^{-(b/p)(\xi^2_{a_1} - i\eta)}
\times \left[ \int_{0}^{\infty} d\beta' \beta'^q e^{-(\xi^2_{a_1} - i\eta + p\alpha_2)\beta'} + \frac{b}{p} \int_{0}^{\infty} d\beta' \beta'^{q-1} e^{-(\xi^2_{a_1} - i\eta + p\alpha_2)\beta'} \right]. \tag{4.3.34}
\]

Integration over \( \beta' \) yields the source generating the MPS pulse; specifically,

\[
f_{MPS}(\rho, \zeta, \eta) = \frac{-2i}{\pi^2 a_1} \frac{p^q}{\Gamma(q)} \frac{e^{-(b/p)(\xi^2_{a_1} - i\eta)}}{\left[ \frac{\xi^2_{a_1} - i\eta + p\alpha_2}{\beta'} \right]^q}
\times \left[ \frac{\Gamma(q + 1)}{\left[ \frac{\xi^2_{a_1} - i\eta + p\alpha_2}{\beta'} \right]^q} + \frac{b}{p} \Gamma(q) \right] \delta(\zeta). \tag{4.3.35}
\]

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Another interesting class of LW solutions are the X-waves [57]. These are sub- or superluminal pulses which represent interference patterns that can be constructed from basic building blocks each traveling at the speed of light \( c \). We will consider the superluminal case below. One such solution can be expressed as

\[
\psi_{XW}(\rho, \zeta, \eta) = \frac{z_0}{\left[(z_0 \rho)^2 + [a_1 z_0 - i(z - \nu t)]^2\right]^{1/2}},
\]  

(4.3.36)

where \( a_1 \) and \( z_0 \) are real parameters and \( \nu > c \). Let the real source now be

\[
f(\rho, \zeta, \eta) = \frac{a_1 \Gamma(3/2)}{\pi^{3/2} (a_1^2 + \rho^2)^{3/2}} \delta(\zeta - z_0 \eta).
\]  

(4.3.37)

In terms of the modified bidirectional representation, (4.3.37) is expressed as

\[
f(\rho, \zeta, \eta) = \frac{1}{(2\pi)^3} \int_0^\infty d\kappa \int_{-\infty}^\infty d\alpha \int_{-\infty}^\infty d\beta \ \kappa \ J_0(\kappa \rho) \ e^{-i\kappa \alpha \zeta} \ e^{i\kappa \beta \eta} \\
\times \left[ \kappa \ e^{-a_1 \kappa} \delta(\beta - z_0 \alpha) \right].
\]  

(4.3.38)

This leads to a spectrum

\[
C_0(\alpha, \beta, \kappa) = \frac{e^{-a_1 \kappa} \delta(\beta - \alpha z_0)}{\kappa (1 - 4\alpha \beta)}.
\]  

(4.3.39)

Substitution of (4.3.39) in (4.2.4b) and integration over \( \beta \) yields

\[
\psi(\rho, \zeta, \eta) = \frac{1}{2(2\pi)^2 z_0} \int_0^\infty d\kappa \ J_0(\kappa \rho) \ e^{-a_1 \kappa} \int_0^\infty d\alpha \ \frac{\cos(\alpha \kappa(z - z_0 \eta))}{(1/2 \sqrt{z_0})^2 - \alpha^2}.
\]  

(4.3.40)

To integrate over \( \alpha \) we use 3.723.9 of [56]. This yields

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\[ \psi(\rho, \zeta, \eta) = \frac{1}{8\pi \sqrt{z_0}} \int_0^\infty d\kappa \ J_0(\kappa \rho) \ e^{-a_1 \kappa} \sin \left[ \frac{\kappa}{2} \left( \frac{\zeta}{\sqrt{z_0}} - \eta \sqrt{z_0} \right) \right], \]  
(4.3.41)

where \( \kappa (\zeta - z_0 \eta) > 0 \) and \( z_0 > 0 \). To integrate over \( \kappa \) we write the \( \sin \) function in terms of exponentials and use 3.723.2 of [56]; as a result, we obtain

\[ \psi(\rho, \zeta, \eta) = \frac{1}{16i\pi \sqrt{z_0}} \left[ \frac{1}{\left[ \rho^2 + \left( a_1 - i\sigma \right)^2 \right]^{1/2}} - \frac{1}{\left[ \rho^2 + \left( a_1 + i\sigma \right)^2 \right]^{1/2}} \right], \]  
(4.3.42)

where

\[ \sigma = \frac{1}{2} \left( \frac{\zeta}{\sqrt{z_0}} - \eta \sqrt{z_0} \right) \]  
and \( a_1 > 1 \).

Equation (4.3.42) can also be written as

\[ \psi(\rho, \zeta, \eta) = \frac{1}{16i\pi \sqrt{z_0}} \left[ \psi^*_{XW}(\rho, \zeta, \eta) - \psi_{XW}(\rho, \zeta, \eta) \right] \]

\[ = \frac{1}{8\pi \sqrt{z_0}} \text{Im}[\psi_{XW}(\rho, \zeta, \eta)], \]  
(4.3.43)

where the star denotes complex conjugate. So, the real source in (4.3.37) generates the imaginary part of the "sling-shot" solution. It should be noted that (4.3.43), just like its real part, is a localized pulse moving at a speed greater than the speed of light.

### 4.4 VARIABLE SPEEDS

In this section we shall solve the general problem of a localized field generated by a source moving with speed \( v_s \) not equal to \( c \).

Consider the following source moving with speed \( v_s \):

\[ f(\rho, z, t) = -\frac{2a_3}{\pi a_1} \frac{2}{1 + v_s/c} \frac{1}{|1 + \epsilon|} \ e^{-a_3 \rho^2/a_1} \ e^{ia_3 (z + v_s t)} \delta(z - v_s t), \]  
(4.4.1)

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where \( \epsilon = (1 - v_s/c)/(1 + v_s/c) \) or \( v_s = c (1 - \epsilon)/(1 + \epsilon) \), and \( c \) is the speed of light. The source \( f \) can be expressed in terms of \( \zeta \) and \( \eta \):

\[
f(\rho, \zeta, \eta) = - \frac{2a_3}{\pi a_1} \frac{2}{1 + v_s/c} e^{-a_3 \rho^2/a_1} e^{ia_3 \eta} \delta(\zeta + \epsilon \eta).
\]

The parameter \( \epsilon \), above, determines the character of the speed \( v \) of the source. For example, we have the following cases:

a) \( \epsilon = 0 \), source moving with speed \( c \) \( (v_s = c) \)

b) \( \epsilon = 1 \), stationary source \( (v_s = 0) \)

c) \( 0 < \epsilon < 1 \), slowly-moving source \( (v_s < c) \) \hspace{1cm} (4.4.3)

d) \( \epsilon < 0 \), superluminal source \( (v_s > c) \).

e) \( \epsilon > 1 \), backward moving source \( (|v_s| < c) \).

Now, let us use the following modified bidirectional representation

\[
f(\rho, \zeta, \eta) = \frac{1}{(2\pi)^3} \int_0^\infty d\kappa \int_{-\infty}^\infty d\alpha \int_{-\infty}^\infty d\beta \kappa C_f(\alpha, \beta, \kappa) J_0(\kappa \rho) e^{i\zeta \alpha \kappa} e^{i\eta \beta \kappa}, \hspace{1cm} (4.4.4a)
\]

\[
C_f(\alpha, \beta, \kappa) = \kappa^2 \int_0^\infty d\rho \int_{-\infty}^\infty d\zeta \int_{-\infty}^\infty d\eta \rho f(\rho, \zeta, \eta) J_0(\kappa \rho) e^{i\zeta \alpha \kappa} e^{-i\eta \beta \kappa}. \hspace{1cm} (4.4.4b)
\]

Substitution of (4.4.2) into (4.4.4b) yields

\[
C_f(\alpha, \beta, \kappa) = \kappa^2 \int_0^\infty d\rho \int_{-\infty}^\infty d\zeta \int_{-\infty}^\infty d\eta \rho J_0(\kappa \rho) e^{i\zeta \alpha \kappa} e^{-i\eta \beta \kappa}
\]

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\[
\times \left\{ - \frac{2a_3}{\pi a_1} \frac{2}{1 + v_s/c} e^{-a_3 \rho^2/a_1} e^{ia_3 \eta} \delta(\zeta + \epsilon \eta) \right\}. 
\]

(4.4.5)

Integrating over \( \zeta \), we obtain

\[
C_f(\alpha, \beta, \kappa) = - \frac{2a_3}{\pi a_1} \frac{2}{1 + v_s/c} \kappa^3 \int_0^\infty d\rho \rho J_0(\kappa \rho) e^{-a_3 \rho^2/a_1}
\]

\[
\times \int_{-\infty}^\infty d\eta e^{-i\eta(\alpha \epsilon + \beta \kappa - a_3)}. 
\]

(4.4.6)

The integration over \( \rho \) in (4.4.6) is carried out with the help of formula 6.631.4 of [56] and the integration over \( \eta \) is a delta function resulting in the source spectrum

\[
C_f(\alpha, \beta, \kappa) = - \frac{8 \kappa}{1 + v_s/c} \delta(\beta + \alpha \epsilon - a_3/\kappa) e^{-a_3 \kappa^2/4a_3}. 
\]

(4.4.7)

Let us now solve the nonhomogeneous three dimensional scalar wave equation

\[
[L_1 + L_2] \Psi(\rho, \zeta, \eta) = - f(\rho, \zeta, \eta), 
\]

(4.4.8)

where \( L_1 = \partial^2_\rho + \rho^{-1} \partial_\rho \) and \( L_2 = 4 \partial^2_\zeta \). We express \( \Psi \) as

\[
\Psi(\rho, \zeta, \eta) = \frac{1}{(2\pi)^6} \int_0^\infty d\kappa \int_{-\infty}^\infty d\alpha \int_{-\infty}^\infty d\beta \kappa J_0(\kappa \rho) e^{-i\kappa \alpha \zeta} e^{i\kappa \beta \eta} C_0(\alpha, \beta, \kappa). 
\]

(4.4.9)

Then, the application of the operator \( L_1 + L_2 \) yields

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\[ [L_1 + L_2] \Psi(\rho, \zeta, \eta) = \frac{1}{(2\pi)^2} \int_0^\infty d\kappa \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \kappa J_0(\kappa \rho) e^{-i\kappa \alpha \zeta} e^{i\kappa \beta \eta} \]
\[ \times C_0(\alpha, \beta, \kappa) [\kappa^2 (4\alpha \beta - 1)]. \] (4.4.10)

Using (4.4.4a), along with (4.4.7) and (4.4.10), we obtain
\[ C_0(\alpha, \beta, \kappa) [\kappa^2 (4\alpha \beta - 1)] + \frac{8 \kappa}{1 + v/c} \delta(\beta + \alpha \epsilon - a_3 / \kappa) e^{-a_1 \kappa^2 / 4a_3} = 0 \]
or
\[ C_0(\alpha, \beta, \kappa) = \frac{8 \delta(\beta + \alpha \epsilon - a_3 / \kappa) e^{-a_1 \kappa^2 / 4a_3}}{\kappa(1 - 4\alpha \beta)(1 + v_s / c)}. \] (4.4.11)

Now, substituting (4.4.11) into (4.4.9) yields
\[ \Psi(\rho, \zeta, \eta) = \frac{1}{(2\pi)^2} \frac{8}{1 + v_s / c} \int_0^\infty d\kappa J_0(\kappa \rho) e^{-a_1 \kappa^2 / 4a_3} \int_{-\infty}^{\infty} d\alpha e^{-i\kappa \alpha \zeta} \]
\[ \times \int_{-\infty}^{\infty} d\beta e^{i\kappa \beta \eta} \frac{\delta(\beta + \alpha \epsilon - a_3 / \kappa)}{1 - 4\alpha \beta}. \] (4.4.12)

Note that (4.4.12) reduces to (4.3.27) when \( \epsilon \to 0 \), ignoring the scalar factors, which is the FWM solution.

Integration of (4.4.12) over \( \beta \), results in
\[ \Psi(\rho, \zeta, \eta) = \frac{8}{(2\pi)^2} \frac{e^{i\alpha_3 \eta}}{1 + v_s / c} \int_0^\infty d\kappa J_0(\kappa \rho) e^{-a_1 \kappa^2 / 4a_3}. \]
\[ \times \int_{-\infty}^{\infty} d\alpha \frac{e^{-i\alpha(\eta \epsilon + \zeta)}}{4\epsilon \alpha^2 - 4\alpha \frac{a_3}{\kappa} + 1}. \quad (4.4.13) \]

To integrate (4.4.13) over \( \alpha \), we first express the denominator as

\[
4\epsilon \alpha^2 - 4\alpha \frac{a_3}{\kappa} + 1 = \alpha'^2 + a^2,
\]

where

\[
\alpha' = \alpha - \frac{a_3}{(2\epsilon \kappa)} \quad a = \sqrt{\frac{1}{4\epsilon} - \left(\frac{a_3}{2\epsilon \kappa}\right)^2}.
\]

The integration over \( \alpha \) becomes

\[
I_\alpha = \frac{1}{4\epsilon} e^{-ia_3(\eta \epsilon + \zeta)/2\epsilon} \int_{-\infty}^{\infty} d\alpha' \frac{e^{-i\alpha'(\eta \epsilon + \zeta)\kappa}}{\alpha'^2 + a^2}
\]

\[
= \frac{1}{2\epsilon} e^{-ia_3(\eta \epsilon + \zeta)/2\epsilon} \int_{0}^{\infty} d\alpha' \frac{\cos[\alpha'(\eta \epsilon + \zeta)\kappa]}{\alpha'^2 + a^2}. \quad (4.4.14)
\]

Using formula 3.723.2 of [56], the integration over \( \alpha' \) is carried out yielding

\[
I_\alpha = \frac{\pi}{2a} e^{-\kappa(\eta \epsilon + \zeta)a}
\]

or

\[
I_\alpha = \frac{\pi}{2\sqrt{\frac{1}{4\epsilon} - \left(\frac{a_3}{2\epsilon \kappa}\right)^2}} \exp \left(-\kappa(\eta \epsilon + \zeta) \sqrt{\frac{1}{4\epsilon} - \left(\frac{a_3}{2\epsilon \kappa}\right)^2}\right), \quad (4.4.15a)
\]

with the restrictions

\[
\frac{1}{4\epsilon} - \left(\frac{a_3}{2\epsilon \kappa}\right)^2 > 0 \quad \text{and} \quad \kappa(\eta \epsilon + \zeta) > 0. \quad (4.4.15b)
\]

The restrictions in (4.4.15b) imply that \( \kappa > a_3/\sqrt{\epsilon} \) and \( \epsilon > 0 \). Finally, \( \Psi \) can be expressed as

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\[
\Psi(\rho, \zeta, \eta) = \frac{1 + \varepsilon}{2\pi} e^{-i\alpha_3(-\eta \zeta + \zeta)} \int_{\alpha_3/\varepsilon}^{\infty} d\kappa \sqrt{\kappa} J_0(\kappa \rho) e^{-a_1 \kappa^2/4a_3} \times \exp \left( - \frac{\zeta + \eta \varepsilon}{2\varepsilon} \sqrt{\varepsilon \kappa^2 - a_3^2} \right) \sqrt{\varepsilon \kappa^2 - a_3^2} \right) .
\]

(4.4.16)

In terms of \( \rho, z, t \) and setting \( a_3 = \beta \), equation (4.4.16) becomes

\[
\Psi(\rho, z, t) = \frac{1 + \varepsilon}{2\pi} e^{-i\beta \frac{1-\varepsilon}{2\varepsilon} (z - \frac{1+\varepsilon}{1-\varepsilon} ct)} \int_{\alpha_3/\varepsilon}^{\infty} d\kappa \sqrt{\kappa} J_0(\kappa \rho) e^{-a_1 \kappa^2/4a_3} \times \exp \left( - \frac{1+\varepsilon}{2\varepsilon} [z - \frac{1-\varepsilon}{1+\varepsilon} ct] \sqrt{\varepsilon \kappa^2 - a_3^2} \right) .
\]

(4.4.17)

The field expressed in (4.4.17) represents a pulse generated by a real source moving with a speed \( v_s = [(1 - \varepsilon)/(1 + \varepsilon)]c \) with \( \varepsilon > 0 \) \( (v_s < c) \). The pulse moves with the speed \( v_p = v_s \). It is of interest to examine this pulse at its center \( z = v_p t \) and at \( \rho = 0 \).

The expression in (4.4.17) is greatly simplified by the introduction of the variable \( u = \sqrt{\varepsilon \kappa^2 - a_3^2} \). Setting \( \rho = 0 \) and \( a_3 = \beta \) in (4.4.16), we rewrite the expression in terms of the new variable \( u \):

\[
\Psi(\rho = 0, \zeta, \eta) = \frac{1 + \varepsilon}{2\pi\varepsilon} e^{-i\beta(-\eta \zeta + \zeta)/2\varepsilon} e^{a_1 \beta/4\varepsilon} \int_{0}^{\infty} du \, e^{-a_1 u^2/4\beta \varepsilon} \times e^{-u(\zeta + \eta \varepsilon)/2\varepsilon} .
\]

(4.4.18)

The above expression can be written in terms of the variables \( z, t \):

\[
\Psi(\rho = 0, z, t) = \frac{1 + \varepsilon}{2\pi\varepsilon} \exp \left\{ - \frac{i \beta(1 - \varepsilon)}{2\varepsilon} [z - \frac{1+\varepsilon}{1-\varepsilon} ct] \right\} e^{a_1 \beta/4\varepsilon} .
\]
\[
\times \int_{0}^{\infty} du \ e^{-a_1 u^2 / 4\beta}\exp\left\{-\frac{1+\epsilon}{2\epsilon} \left[z - \left(\frac{1-\epsilon}{1+\epsilon}\right)ct\right]u\right\}. \quad (4.4.19)
\]

The solution in (4.4.19) represents a pulse moving with speed \(v_p = c(1 - \epsilon)/(1 + \epsilon)\) and it is generated by a source moving with speed \(v_s = v_p\). At the pulse center \(z = v_p t\) and \(\rho = 0\) the solution becomes:

\[
\Psi(\rho = 0, z = v_p t, t) = \frac{1+\epsilon}{2\pi\epsilon} \ e^{i\frac{2\epsilon ct}{1+\epsilon}} \ e^{a_1\beta/4\epsilon} \int_{0}^{\infty} du \ e^{-a_1 u^2 / 4\beta}\epsilon. \quad (4.4.20)
\]

Integrating over \(u\), we obtain

\[
\Psi(\rho = 0, z = v_p t, t) = \frac{1+\epsilon}{2\sqrt{\pi\epsilon}} \ \sqrt{\beta/a_1}\ e^{i\frac{2\epsilon ct}{1+\epsilon}} \ e^{a_1\beta/4\epsilon}. \quad (4.4.21)
\]

Taking the real part of (4.4.21) results in the relation

\[
\Psi(\rho = 0, z = v_p t, t) \propto \cos\left\{\frac{2\epsilon ct}{1+\epsilon}\right\}. \quad (4.4.22)
\]

where the initial amplitude of the pulse is recovered for every \(z = [n \pi(1 - \epsilon)/\beta]\), \(n\) being a positive integer. Note that, for \(\epsilon = 0\), a condition that corresponds to a source moving with speed \(c\), leading to the FWM solution, the initial amplitude of the pulse is recovered for every \(z = n\pi/\beta\), as indicated also by Ziolkowski [38]. For \(0 < \epsilon < 1\), the source and the corresponding pulse move with speed less than the speed of light, or \(v_s = v_p < c\). This condition corresponds to a nondecaying subluminal pulse generated by a subluminal source.

4.0 MOVING SOURCES
4.5 CONCLUSION

In this chapter we have shown how interesting, exact pulse solutions of the three-dimensional scalar wave equation can be generated. As a tool, we have utilized the modified bidirectional decomposition, described in chapter 3, in solving the nonhomogeneous scalar wave equation, and have shown that two different types of moving sources, complex and real, generating these localized pulses. In the FWM case, the real source has a Gaussian taper in the transverse plane and moves with the speed of light. A more complicated source, moving with the speed c, is required to produce the MPS pulse. In the case of the superluminal X wave, the real source has a $\rho^{-3}$ dependence in the transverse plane and moves with a speed greater than c. In this chapter, we also showed that nondecaying subluminal pulses can be generated by subluminal sources.
5.0 APERTURE SYNTHESIS OF LW PULSES

5.1 INTRODUCTION

In this chapter, we shall study the possibility of exciting localized pulses from apertures. We shall consider two different LW pulses: the FWM pulse and the superluminal X wave pulse. Our approach sets mainly on the proven possibility of exciting causal Bessel beams from infinite apertures [32]. It is well known that FWM-like solutions can be synthesized as a superposition of Bessel beams with appropriate choice of spectra [40]. It has been shown by Ziolkowski [38] that a Bessel beam defined initially on an infinite plane (e.g. the \( z = 0 \) plane) will propagate under the effect of Huygen's operator in the positive \( z \)-direction and that the contributions from the acausal negative \( z \)-components sum up to zero. It will be shown that the FWM and X wave pulses can be launched from an aperture whose field occupies a region of radius that shrinks from infinity to a finite value and expands once more to infinity. Such a time-varying aperture requires an infinite time of excitation. This is the main reason why the ideal FWM and X wave pulses cannot be excited from practical apertures. In contradistinction to the other solutions excited from infinite apertures, such as plane waves and Bessel beams, the excitation of the FWM and X wave pulses does not require infinite power. This is the case because as the aperture of the field becomes infinitely large, the power density of the field exciting the aperture decreases to zero at a rate \( (ct)^{-2} \) while the area of the aperture increases as \( (ct)^{2} \). These two effects balance each other and the energy on the aperture remains constant.

In the case of the FWM pulse we consider the case \( \beta a_1 \ll 1 \), where \( \beta \) and \( a_1 \) are the parameters that determine the physical characteristics of the FWM pulse. More
specifically, it has been shown [32,40,62] that the product $\beta a_1$ determines whether the field of the source-free FWM pulse is dominated by causal or acausal components. The case $\beta a_1 > 1$ has been considered by Heyman and Felsen [28,61] who proved that under such condition the FWM field is acausal. The condition $\beta a_1 \ll 1$ has been considered by Shaarawi et al. [54,62] and they have shown that the causal components of the field are the dominant ones while the acausal components are negligible and do not affect the pulse behavior. This is the main reason why we shall concentrate on the condition $\beta a_1 \ll 1$. Our subsequent discussion will be based on this condition. In the case of the FWM pulse the aperture is shrinking and expanding at speeds greater than $c$. Such aperture should consist of separately excitable elements, in order not to violate the theory of special relativity. In the case of the X wave pulse, the speed of expansion $v_{ap}$ of the aperture field, depends on the speed of the pulse center $v_p$ and can be greater or smaller than $c$. The temporal and spatial frequency contents of LW pulses will be calculated in order to better understand basic characteristics such as bandwidth.

Finally, we shall also explore the possibility of using finite time excitation of time varying apertures to generate two types of approximate LW pulses: the FWM and the X wave pulse. The analysis will be carried out by introducing a Gaussian time window. The resulting fields will be calculated and their far fields will be compared to ordinary monochromatic pulses. It will be shown that finite versions of LW pulses can propagate without significant decay much further than classical monochromatic pulses. It should be noted that Shaarawi et al. [54] have studied the finite time excitation of FWM pulses and agreements of the work presented here are generally good.

5.0 APERTURE SYNTHESIS OF LW PULSES
5.2 LOCALIZED WAVE (LW) APERTURES

5.2.1 The FWM Aperture

We start by defining the FWM field on the infinite aperture situated at the $z = 0$ plane. The field is the real part of the azimuthally symmetric complex FWM pulse

$$\Psi(\rho, z, t) = \frac{1}{a_1 + i(z - ct)} \exp\left\{-\frac{\beta \rho^2}{a_1 + i(z - ct)}\right\} e^{i\beta(z + ct)}. \quad (5.2.1)$$

On the $z = 0$ plane, the field becomes

$$u(\rho, z, t) \big|_{z = 0} = Re \left| \Psi(\rho, 0, t) \right| = Re \left\{ \frac{1}{a_1 - ict} \right\} \right. \times \exp\left\{-\frac{\beta \rho^2}{a_1 - ict}\right\} e^{i\beta ct} \quad (5.2.2a)$$

$$u(\rho, 0, t) = \frac{1}{a_1^2 + (ct)^2} \exp\left\{-\frac{\beta a_1 \rho^2}{a_1^2 + (ct)^2}\right\} \left\{ a_1 \cos\left(\beta ct \left(\frac{\rho^2}{a_1^2 + (ct)^2} - 1\right)\right) \right. \right. \right.$$

$$+ ct \sin\left(\beta ct \left(\frac{\rho^2}{a_1^2 + (ct)^2} - 1\right)\right) \right\} \right}. \quad (5.2.2b)$$

The surface plot of the aperture field expressed in (5.2.2b) is shown in Fig. 5-1 for $\beta = 400 \text{ m}^{-1}$, and $a_1 = 4 \times 10^{-7} \text{ m}$.

The radius of expansion of the aperture field is expressed as

$$R(t_4) = v_{ap} t_4 + R(t = 0), \quad (5.2.3)$$

where $v_{ap}$ is the speed of expansion of the aperture field and $t_4$ is the time where the amplitude of the real electric field is $e^{-4}E_0$, $E_0$ being the amplitude at $(\rho, z,$

5.0 APERTURE SYNTHESIS OF LW PULSES
$t) = (0, 0, 0)$. $R(t = 0)$ is the initial radius of the aperture field. Here, we let $E_0 = 1$. The speed $v_{ap}$, at which the aperture field expands is found from (5.2.3) and is expressed as

$$v_{ap} = \frac{R(t_4) - R(t = 0)}{t_4}.$$  \hspace{1cm} (5.2.4)

For $t = 0$ sec, $R(t = 0) = 2\sqrt{a_1/\beta}$ as seen from the exponential of Eq. (5.2.2b). Note that the beam waist of the FWM pulse is $w_0 = \sqrt{a_1/\beta}$. For the real field expressed in (5.2.2b), $R(t_4)$ depends on the parameters $\beta$ and $a_1$. Our goal here is to examine the excitation of finite apertures and whether diffraction-resistant pulses can be generated from these apertures. The maximum radius of a finite aperture, $R(t_4)$, is the one corresponding to the region where the value of the amplitude of the real electric field is greater or equal to $e^{-4}E_0$. As mentioned in Sec. 1, the condition $\beta a_1 \ll 1$ is of practical interest. Shaarawi et al. [54] analyzed the FWM pulse corresponding to the parameters $\beta = 400$ m$^{-1}$, and $a_1 = 4 \times 10^{-7}$ m, resulting in $\beta a_1 = 0.00016 \ll 1$. In the numerical evaluation of the FWM pulse we shall use the same parameter values. Now, the aperture speed $v_{ap}$ corresponding to these parameters is found numerically from (5.2.2b) by varying $\rho$ and $t > 0$. More specifically, the normalized real electric field amplitude is maximum on the aperture $(z = 0)$ and $(\rho, t) = (0, 0)$, and $R(t = 0) = 2\sqrt{a_1/\beta} = 6.32 \times 10^{-5}$ m. Note, that the beam waist of the pulse is $w_0 = \sqrt{a_1/\beta} = 3.16 \times 10^{-5}$ m. The normalized amplitude of the real electric field is numerically found to be $e^{-4}$ at $R(t_4) = 0.000286$ m and $t_4 = 7.07 \times 10^{-14}$ sec. Applying Eq. (5.2.4) yields $v_{ap} = 3.15 \times 10^9$ m/sec or $v_{ap} = 10.5$ c.

It is obvious that the total aperture field is contained on an infinite aperture corresponding to $t \to \infty$. So, an ideal diffraction-free LW pulse can be generated from an infinite aperture. It is clear that such a radius depends on the time such that as $ct$ changes from $-\infty$ to $0$ to $+\infty$, the radius $R(t)$ decreases from $\infty$ to $2\sqrt{a_1/\beta}$ and

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then increases once more to \(\infty\). On the other hand, the field amplitude decreases as \((ct)^{-1}\) as \(ct \to \infty\). Now, since the energy density of the field is proportional to \(u^2\), the energy density is proportional to \((ct)^{-2}\). This means that as \(ct \to \infty\), the energy on the aperture \((=\text{energy density} \times \text{Area})\) remains constant. More rigorously, one can carry out the following calculation:

\[
E \left|_{z=0} \right. = |\Psi| \left|_{z=0} \right. ^2 = \frac{1}{a_1^2 + (ct)^2} \exp \left\{ - \frac{2\beta a_1 \rho^2}{a_1^2 + (ct)^2} \right\}, \tag{5.2.5}
\]

The total energy on the aperture is then

\[
E_T = 2\pi \int_0^{R(t)} E \rho \, d\rho = 2\pi \int_0^{R(t)} \frac{1}{a_1^2 + (ct)^2} \exp \left\{ - \frac{2\beta a_1 \rho^2}{a_1^2 + (ct)^2} \right\} \rho \, d\rho
\]

\[
= \frac{\pi}{2\beta a_1} \exp \left\{ 1 - e^{-R(t)} \right\}, \tag{5.2.6}
\]

and for \(ct \gg a_1 \to \infty\)

\[
E_T \ll \frac{\pi}{2\beta a_1}. \tag{5.2.7}
\]

This shows that the total energy on the aperture is finite and is proportional to \((\beta a_1)^{-1}\). Such a factor controls the speed of the shrinking and expanding of the aperture.

### 5.2.2 The X Wave Aperture

The X wave pulse, a solution to the three dimensional scalar wave equation, is expressed as

\[
\Psi_{XW}(\rho, z, t) = z_0 \left[ (z_0 \rho)^2 + (a_1 z_0 + i(z - ct \coth\beta))^2 \right]^{-1/2}, \tag{5.2.8}
\]
where \( a_1 \) and \( \beta \) are free real parameters, and \( z_0 = 1/\sinh \beta \). Note that \( c(\coth \beta) \) is the propagation speed of the pulse. To ensure that the field will propagate in the forward direction we restrict \( \beta \) to be positive. On the \( z = 0 \) plane

\[
\Psi_{XW}(\rho, z = 0, t) = \left[ \rho^2 + (a_1 - i y_0 t)^2 \right]^{-1/2}, \tag{5.2.9}
\]

where \( y_0 = c(\cosh \beta) \). The real part of \( \Psi_{XW} \) on the aperture \( (z = 0) \) is found by expanding the term inside the brackets in Eq. (5.2.9). More specifically,

\[
\Psi_{XW}(\rho, z = 0, t) = \frac{1}{(e_1 - i e_2)^{1/2}}, \tag{5.2.10}
\]

where \( e_1 = \rho^2 + a_1^2 - (y_0 t)^2 \) and \( e_2 = 2 a_1 y_0 t \). We express \( \Psi_{XW} \) as

\[
\Psi_{XW}(\rho, z = 0, t) = \frac{1}{(e_1^2 + e_2^2)^{1/2}} (e_1 + i e_2)^{1/2}
\]

\[
= \frac{1}{(e_1^2 + e_2^2)^{1/2}} \left[ (e_1^2 + e_2^2)^{1/2} \exp[i \tan^{-1}(e_2/e_1)] \right]^{1/2}. \tag{5.2.11}
\]

The real part of \( \Psi_{XW} \) on the aperture is

\[
u(\rho, z = 0, t) = \text{Re}[\Psi_{XW}(\rho, z = 0, t)]
\]

\[
= (e_1^2 + e_2^2)^{-1/4} \cos \left[ \frac{1}{2} \tan^{-1}(e_2/e_1) \right]. \tag{5.2.12}
\]

The surface plot of the aperture field expressed in (5.2.12) is shown in Fig. 5-2 for \( a_1 = .001 \text{ m} \) and \( \beta = 6 \).
The intensity of the ideal diffraction-free X wave pulse is expressed as

\[ I(\rho, 0, t) = \Psi(\rho, 0, t) \Psi^*(\rho, 0, t) \]

\[ = \frac{1}{\left[ \rho^2 + (a_1 - i\gamma_0 t)^2 \right]^{1/2} \left[ \rho^2 + (a_1 + i\gamma_0 t)^2 \right]^{1/2}} \]

or

\[ I(\rho, 0, t) = \frac{1}{\left[ (\rho^2 - (\gamma_0 t)^2)^2 + a_1^4 + 2a_1^2\rho^2 + 2a_1^2(\gamma_0 t)^2 \right]^{1/2}}. \quad (5.2.13) \]

After expanding, the denominator in (5.2.13) becomes

\[ D = [\rho^4 + (\gamma_0 t)^4 - 2\rho^2(\gamma_0 t)^2 + a_1^4 + 2a_1^2\rho^2 + 2a_1^2(\gamma_0 t)^2]^{1/2}, \quad (5.2.14) \]

and for \( \rho \gg a_1 \)

\[ D \approx [\rho^4 + (\gamma_0 t)^4 - 2\rho^2(\gamma_0 t)^2]^{1/2} = [\rho^2 - (\gamma_0 t)^2]. \quad (5.2.15) \]

Expressions (5.2.15) and (5.2.13) show how the intensity varies with respect to time and the radius \( \rho \) of the aperture. For \( t = 0 \) and \( \rho = 0 \) the intensity \( I \) is finite and equal to \( 1/a_1^2 \). For finite radius \( \rho \) and \( t \) very large (\( t \gg \rho \)) the intensity on the aperture \( (z = 0) \) is very small. Finally, for \( \rho = \gamma_0 t \) the intensity varies as \( 1/[a_1\sqrt{a_1^2 + 4\rho^2}] \) as can be deduced from (5.2.13). So, as \( ct \to \infty \), the intensity or the energy on the aperture \( (= \text{energy density} \times \text{Area}) \) remains constant.

The maximum radius that the X wave aperture will expand to, is found from (5.2.12) by varying \( \rho \) and \( t > 0 \). The amplitude of the real electric field is maximum on the aperture \( (z = 0) \) and \( (\rho, t) = (0, 0) \), and is normalized to 1. Then, the values of \( \rho \) and \( t \) for which the normalized amplitude of the real electric field equals \( e^{-4} \) determine

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the speed of expansion of the aperture field. So, the radius of expansion of the aperture field is expressed as

\[ R(t_4) = v_{ap} t_4 + R(t = 0), \quad (5.2.16) \]

where \( v_{ap} \) is the speed of expansion of the aperture field and \( t_4 \) is the time where the real electric field is \( e^{-4} E_0 \), \( E_0 \) being the value of the field at \( (\rho, z, t) = (0, 0, 0) \). \( R(t = 0) \) is the initial radius of the aperture field. The speed \( v_{ap} \), at which the aperture field expands is found numerically is expressed as

\[ v_{ap} = \frac{R(t_4) - R(t = 0)}{t_4} = y_0 - \frac{R(t = 0)}{t_4}, \quad (5.2.17) \]

where \( y_0 = c (\cosh \beta) \). Consider, now, the X wave pulse corresponding to the parameters \( a_1 = .001 \) m and \( \beta = 6 \). The speed of the pulse center is \( v_p = c (\coth \beta) = 1.000001 \) c. The beam waist of the pulse in this case is \( w_0 = 0.00253 \) m, corresponding to the 3 dB point of the electric field amplitude. It is found numerically that the normalized amplitude of the real electric field is \( e^{-4} \) at \( \rho_o \approx .908 \) m and \( t_0 = 1.5 \times 10^{-11} \) sec, and from (5.2.16) \( R(t = 0) = 5.46 \) cm. Then, \( v_{ap} \approx (.908 - .0546) / 1.5 \times 10^{-11} \) m/sec = \( 5.69 \times 10^{10} \) m/sec, or \( v_{ap} \approx 190 \) c. Note that the speed of expansion of the aperture of the FWM example was 10.5 c. Thus, the aperture field of the X wave pulse is expanding much faster in this case. The relation between \( v_{ap} \) and the speed of the center of the X wave pulse \( v_p \) is examined numerically and is shown in Fig. 5-3, where a table and a plot are provided that show values of \( v_{ap} \) corresponding to \( v_p \). It is concluded that the faster the pulse travels, the slower the expansion of the aperture will be.

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5.3 FREQUENCY SPECTRA OF LW PULSES

In this section we shall derive the temporal and spatial frequency spectra of the FWM and XW pulse.

5.3.1 FWM Pulse

In this section, we calculate the temporal and spatial frequency contents of the FWM field on the aperture. With no loss of generality this can be done using the complex FWM pulse given in Eq. (5.2.1), with \( z = 0 \), where

\[
\Phi(\chi, \omega) = \int_{-\infty}^{+\infty} dt \int_0^\infty d\rho \rho J_0(\chi \rho) e^{-i\omega t} \frac{1}{a_1 - ic} \exp \left\{ - \beta \frac{\rho^2}{a_1 - ic} \right\} e^{i\beta ct} \tag{5.3.1}
\]

Carrying out the integration over \( \rho \), using 6.631.4 of [56], we obtain

\[
\Phi(\chi, \omega) = \int_{-\infty}^{+\infty} dt \frac{1}{2\beta} \exp \left\{ - \frac{\chi^2}{4\beta} (a_1 - ic) + i(\beta ct - \omega t) \right\},
\]

\[
\Phi(\chi, \omega) = \pi \frac{1}{\beta} \delta \left\{ \omega - \left( \frac{\chi^2}{4\beta} + \beta \right) c \right\} \exp \left\{ - \frac{\chi^2}{4\beta} a_1 \right\}. \tag{5.3.2}
\]

The temporal frequency content can be calculated by integrating (5.3.2) over \( \chi \),

\[
\hat{\Phi}_t(\rho, \omega) = \int_0^\infty d\chi \chi J_0(\chi \rho) \Phi(\chi, \omega)
\]
\[
\frac{\pi}{\beta} \int_0^\infty d\chi \chi J_0(\chi \rho) \delta \left\{ \omega - \left( \frac{\chi^2}{4\beta} + \beta \right)c \right\} \exp \left\{ - \frac{\chi^2}{4\beta} a_1 \right\}.
\]  
(5.3.3)

The above integration yields

\[
\hat{\Phi}_t(\rho, \omega) = \int_0^\infty d\chi \chi J_0(\chi \rho) \delta \left\{ \chi - 2\sqrt{\left( \frac{\omega}{c} - \beta \right)\beta} \right\} \exp \left\{ - \frac{\chi^2}{4\beta} a_1 \right\} u\left( \frac{\omega}{c} - \beta \right)
\]

\[
= 2\pi J_0 \left\{ 2\rho \sqrt{\left( \frac{\omega}{c} - \beta \right)\beta} \right\} e^{-\omega a_1/c} e^{\beta a_1} u\left( \frac{\omega}{c} - \beta \right).
\]  
(5.3.4)

Notice that the resulting \( e^{\beta a_1} \) factor determines how large the contribution is of the spatial spectral components. The larger \( \beta \) is, the larger is the spatial bandwidth in \( \chi \) because of the presence of the \( \exp(-\chi^2 a_1/4\beta) \) factor in the integrand of (5.3.3). For \( \beta a_1 \ll 1 \), the factor \( e^{\beta a_1} \) is negligible in (5.3.4) and approximately equal to 1. The maximum frequency is determined from the exponential \( e^{-\omega a_1/c} \), \( \omega_{max} \approx 4\omega_{3dB} \), or

\[
\omega_{max} \approx \frac{4c}{a_1}.
\]  
(5.3.5)

Now, since \( \omega_{min} = \beta c \), we see that the temporal frequency bandwidth becomes very narrow and exists mainly in the tail of \( e^{-\omega a_1/c} \). For the parameter values \( \beta = 400 \, \text{m}^{-1} \), and \( a_1 = 4 \times 10^{-7} \, \text{m} \), we have: \( \omega_{min} = 1.2 \times 10^{11} \, \text{rad/sec} \) or \( f_{min} = 19.1 \, \text{GHz} \) and \( \omega_{max} = 3 \times 10^{15} \, \text{rad/sec} \) or \( f_{max} = 478 \, \text{THz} \). The minimum wavelength is \( \lambda_{min} = c/f_{max} = \pi a_1/2 = 6.28 \times 10^{-7} \, \text{m} \). The spectrum in (5.3.4) is shown in Fig. 5-4 for \( \beta = 400 \, \text{m}^{-1} \), and \( a_1 = 4 \times 10^{-7} \, \text{m} \) where plots versus \( \omega \) and \( \rho \) are shown.

It is also worthwhile to consider the spatial frequency content. This can be done by integrating (5.3.2) over \( \omega \):

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\[ \Phi_s(\chi, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \frac{\pi}{2\beta} \delta\{\omega - \left(\frac{\chi^2}{4\beta} + \beta\right) c\} \exp\left\{-\frac{\chi^2}{4\beta} a_1\right\} e^{-i\omega t} \]

\[ = \frac{1}{2\beta} \exp\left\{-\frac{\chi^2}{4\beta} a_1\right\} \exp\left\{i \left(\frac{\chi^2}{4\beta} + \beta\right) ct\right\}. \quad (5.3.6) \]

We notice that the spatial spectrum \( \Phi_s(\chi, t) \) contains a factor of \( 1/\beta \) which does not exist in the expression for the temporal spectrum \( \Phi_t(\rho, \omega) \). Therefore, for \( \Phi_s(\chi, t) \), the temporal frequency contributions (upon carrying out the integration) do not cancel out the \( \beta \) factor as in the case of the spatial frequency contributions. As seen in (5.3.6), for \( \sqrt{\beta/a_1} \ll 1/a_1 \) (or \( \beta a_1 \ll 1 \)) we can deduce that the temporal frequency content is dominant. This is shown in Fig. 5-5.

**5.3.2 X Wave Pulse**

We shall start with the complex X wave pulse on the aperture \((z = 0)\) as given in Eq. (5.2.15). The spatio-temporal spectrum is given by

\[ \Phi(\chi, \omega) = \int_{-\infty}^{+\infty} dt \int_{0}^{\infty} d\rho \, J_0(\chi\rho) \, e^{-i\omega t} \left[ \rho^2 + (a_1 - i\delta_0 t)^2 \right]^{-1/2} \]

\[ = \int_{-\infty}^{+\infty} dt \, e^{-i\omega t} \int_{0}^{\infty} d\rho \, \frac{\rho}{(\rho^2 + \alpha^2)^{1/2}} J_0(\chi\rho), \quad (5.3.7) \]
where $\alpha = a_1 - iy_0 t$, and $y_0 = c(\cosh \beta)$. The integration over $\rho$ is carried out using 6.554.1 of [56]; as a consequence,

$$
\Phi(\chi, \omega) = \int_{-\infty}^{+\infty} dt \, e^{-it\omega} \frac{e^{-\alpha t}}{\chi} = \int_{-\infty}^{+\infty}dt \, e^{-it(\omega - y_0 \chi)},
$$

or

$$
\Phi(\chi, \omega) = 2\pi \frac{e^{-a_1 \chi}}{\chi} \delta(\omega - \chi y_0), \quad (5.3.8)
$$

where $y_0 = c(\cosh \beta)$ with $y_0 \geq 0$, and since $\chi > 0$, it follows that $\omega \geq 0$ due to the $\delta$-function constraint.

The temporal frequency content can be calculated by inverting (5.3.8) over $\chi$,

$$
\Phi_t(\rho, \omega) = \int_0^{\infty} d\chi \chi J_0(\chi \rho) \Phi(\chi, \omega) = 2\pi \int_0^{\infty} d\chi \chi J_0(\chi \rho) \frac{e^{-a_1 \chi}}{\chi} \delta(\omega - \chi y_0)
$$

$$
= \frac{2\pi}{|y_0|} \int_0^{\infty} d\chi \, J_0(\chi \rho) \frac{e^{-a_1 \chi}}{y_0} \delta(\chi \frac{\omega}{y_0}). \quad (5.3.9)
$$

The above integration yields

$$
\Phi_t(\rho, \omega) = \begin{cases} 
0, & \omega < 0 \\
\frac{2\pi}{y_0} J_0(\frac{\omega}{y_0} \rho) e^{-a_1 \omega/y_0}, & \omega \geq 0
\end{cases}
$$

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or
\[
\Phi_t(\rho, \omega) = \frac{2\pi}{y_0} J_0\left(\frac{\omega}{y_0} \rho\right) e^{-a_1\omega/y_0} u(\omega).
\] (5.3.10)

As seen in (5.3.10), the highest contribution to the temporal frequency content comes from \(\omega = 0\) rad/sec. The maximum frequency in (5.3.10) is determined from the exponential and is
\[
\omega_{\text{max}} \approx \frac{4}{a_1} \frac{y_0}{c} = \frac{4}{a_1} \frac{c(\cosh\beta)}{a_1}.
\] (5.3.11)

For the parameters \(a_1 = .1\) cm and \(\beta = 6\), \(\omega_{\text{max}} = 2\pi f_{\text{max}} = 2.42 \times 10^{14}\) rad/sec, or \(f_{\text{max}} = 38.52\) GHz. This implies that, the minimum wavelength in this case is \(\lambda_{\text{min}} = 7.8\) \(\mu\)m. The spectrum in (5.3.10) is shown in Fig. 5-6 for \(\rho = 0\) m and \(\rho = 100\) m, where \(a_1 = .1\) cm and \(\beta = 6\).

The spatial frequency content is the inverse Fourier transform of (5.3.8) with respect to \(\omega:\)
\[
\Phi_s(\chi, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega 2\pi \frac{e^{-a_1\chi}}{\chi} \delta(\omega - \chi y_0) e^{i\omega t}
\]
or
\[
\Phi_s(\chi, t) = \frac{e^{-a_1\chi}}{\chi} e^{i\chi y_0 t}, \quad \chi > 0.
\] (5.3.12)

The spectrum in (5.3.12) is shown in Fig. 5-7 for \(t = 0\) sec and \(t = 10\) nsec, where \(a_1 = 0.1\) cm and \(\beta = 6\).

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5.4 FINITE TIME EXCITATION

In this section we consider the case of keeping the excitation time of LW pulses finite.

5.4.1 FWM Pulse

We already have an example that lends itself directly to this situation; namely the cylindrical parabolic function derived in [40]. This is easily achieved by imposing a Gaussian time window on the excitation field of the aperture. In this case, the frequency content of the aperture can be easily calculated as follows:

\[
\Phi(\chi, \omega) = \int_{-\infty}^{+\infty} dt \int_{0}^{\infty} d\rho \rho J_0(\chi \rho) e^{-i\omega t} \frac{A(\beta)}{a_1 - i\omega c} \exp\left\{ - \frac{\beta \rho^2}{a_1 - i\omega c} \right\} \\
\times e^{i\beta ct} e^{-t^2/(4T^2)}
\]

(5.4.1)

Here we have used the FWM excitation given in (5.3.1) together with the time window function \(e^{-t^2/(4T^2)}\), where \(2T\) represents the period of excitation from \(-T\) to \(+T\). Carrying out the integration over \(\rho\) in (5.4.1) we obtain

\[
\Phi(\chi, \omega) = \frac{1}{2\beta} \int_{-\infty}^{+\infty} dt e^{-i\omega t} \exp\left\{ - \frac{\chi^2}{4\beta} (a_1 - i\omega c) \right\} e^{i(\beta c - \omega)t} e^{-t^2/(4T^2)}
\]

\[
= \frac{1}{2\beta} e^{-a_1 \chi^2/(4\beta)} \int_{-\infty}^{+\infty} dt \exp\left\{ - i(\omega - (\chi^2/(4\beta) + \beta)c)t \right\} e^{-t^2/(4T^2)}.
\]

(5.4.2)

Completing the square in the exponential in (5.4.2) yields
\[
\Phi(\chi, \omega) = \frac{1}{2\beta} \ e^{-\alpha_1 \chi^2 / (4\beta)} \int_{-\infty}^{\infty} dt \exp\left\{ -T^2 (\omega - \left(\frac{\chi^2}{4\beta} + \beta\right)c)^2 \right\} \\
\times \exp\left\{ -\frac{1}{4T^2} \left( t^2 + i2(\omega - \left(\frac{\chi^2}{4\beta} + \beta\right)cT^2) \right)^2 \right\}.
\]

(5.4.3)

The integration over \( t \) in (5.5.3) is performed with the help of formula 3.461.2 \((n = 0)\) of reference [56]:

\[
\Phi(\chi, \omega) = \frac{\pi}{\beta} \ e^{-\alpha_1 \chi^2 / (4\beta)} \left\{ \frac{T}{\sqrt{\pi}} \exp\left( -T^2 (\omega - \left(\frac{\chi^2}{4\beta} + \beta\right)c)^2 \right) \right\}.
\]

(5.4.4)

It should be noted that in the limit \( T \to \infty \), the bracketed term on the right hand side of (5.4.4) reduces to the delta function \( \delta(\omega - \left(\frac{\chi^2}{4\beta} + \beta\right)c) \) which is characteristic of the FWM excitation spectrum [cf. (5.3.2)]. One can rewrite (5.4.4) as

\[
\Phi(\chi, \omega) = \frac{\pi A(\beta)}{\beta} \ e^{-\alpha_1 \chi^2 / (4\beta)} \ \delta \left( \omega - \left(\frac{\chi^2}{4\beta} + \beta\right)c ; T \right),
\]

(5.4.5a)

where

\[
\delta \left( \omega - \left(\frac{\chi^2}{4\beta} + \beta\right)c ; T \right) = \frac{T}{\sqrt{\pi}} \exp\left\{ -T^2 (\omega - \left(\frac{\chi^2}{4\beta} + \beta\right)c)^2 \right\}.
\]

(5.4.5b)

When \( T \) is large, the Gaussian in the spectrum (5.4.4) reduces to a narrow distribution with a very small bandwidth for which \( \omega \sim (\chi^2/4\beta + \beta)c \) yields most of the significant components of the spectrum. The field propagating in the \( z > 0 \) half plane can be expressed as

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\[ \Psi(\rho, z, t) = \frac{1}{2\pi} \frac{\pi}{\beta} \int_0^\infty d\chi \chi J_0(\chi \rho) \int_0^\infty d\omega \delta\{\omega - \left(\frac{\chi^2}{4\beta} + \beta\right)c;T\} \times \exp\left\{ - \frac{\chi^2}{4\beta} a_1 \right\} e^{-i\sqrt{(\omega/c)^2 - \chi^2} z - \omega t}\ (5.4.6) \]

or

\[ \Psi(\rho, z, t) = \frac{1}{2\beta} \int_0^\infty d\chi \chi J_0(\chi \rho) \int_0^\infty d\omega \exp\left\{ - \frac{\chi^2}{4\beta} a_1 \right\} e^{-i(\sqrt{(\omega/c)^2 - \chi^2} z - \omega t)} \times \frac{T}{\sqrt{\pi}} \exp\left\{ - T^0(\omega - \left(\frac{\chi^2}{4\beta} + \beta\right)c)^2 \right\}. \ (5.4.7) \]

In terms of \( \kappa = \omega/c \), equation (5.4.7) becomes

\[ \Psi(\rho, z, t) = \frac{1}{2\beta} \int_0^\infty d\chi \chi J_0(\chi \rho) \int_0^\infty d\kappa \exp\left\{ - \frac{\chi^2}{4\beta} a_1 \right\} \times \exp\left[ -i(\sqrt{\kappa^2 - \chi^2} z - \kappa ct) \right] \frac{cT}{\sqrt{\pi}} \exp\left\{ - c^2 T^0(\kappa - \frac{\chi^2}{4\beta} - \beta)^2 \right\}. \ (5.4.8) \]

Equation (5.4.8) is the finite FWM pulse excited from an aperture. We shall consider the real field of (5.4.8):

\[ \Psi(\rho, z, t) = \frac{cT}{2\sqrt{\pi} \beta} \int_0^\infty d\chi \chi J_0(\chi \rho) \int_0^\infty d\kappa \exp\left\{ - \frac{\chi^2}{4\beta} a_1 \right\} \cos\left( z\sqrt{\kappa^2 - \chi^2} - \kappa ct \right) \times \exp\left\{ - \beta^2 c^2 T^0 \left[ \frac{\kappa}{\beta} - \left(\frac{\chi}{2\beta}\right)^2 - 1 \right]^2 \right\}. \ (5.4.9) \]
Note again that in the limit $T \to \infty$, the last term (Gaussian) on the right hand side of (5.4.9) reduces to the delta function $\delta(\kappa - \chi^2/4\beta - \beta)$ which is characteristic of the FWM excitation spectrum [cf. (5.3.2)], and (5.4.9) in this case results in the real FWM pulse. When $\beta c T$ is large, this Gaussian reduces to a narrow distribution with very small bandwidth for which $\kappa \sim \chi^2/4\beta + \beta$ provides most of the significant contributions to the spectrum. The square root $\sqrt{\kappa^2 - \chi^2}$ acquires only positive values to ensure that all field components are diverging from the aperture. To study the decay characteristics of the real field given in (5.4.9), we shall concentrate on the pulse center $z = c t$ for $t > 0$. Then (5.4.9) becomes

$$
\Psi(\rho, z = ct) = \frac{c T}{2\sqrt{\pi} \beta} \int_0^\infty d\chi \chi J_0(\chi \rho) \int_0^\infty d\kappa \exp\left\{-\frac{\chi^2}{4\beta} a_1 \right\}
\times \cos\left[ z \left( \kappa - \sqrt{\kappa^2 - \chi^2} \right) \right] \exp\left\{-\beta^2 c^2 T^2 \left[ \frac{\kappa}{\beta} - \left( \frac{\chi}{2\beta} \right)^2 - 1 \right]^2 \right\}. \tag{5.4.10}
$$

For the field generated from an infinite aperture, i.e., the ideal FWM, the Gaussian in (5.4.10) is a Dirac delta function forcing $\kappa = \chi^2/4\beta + \beta$ and the argument of the $\cos$ function becomes $z (\kappa - \sqrt{\kappa^2 - \chi^2}) = 2\beta z$, independent of $\chi$. In this ideal case, the pulse center propagates to infinite distances from the aperture without decay recovering its initial amplitude every $z = n \pi / \beta$, $n$ being a positive integer. In the case of the finite FWM pulse, given in (5.4.10), even though $z (\kappa - \sqrt{\kappa^2 - \chi^2}) \sim 2\beta z$, the Gaussian introduces small deviations that are dependent on both $\chi$ and $z$. For large values of $z$, the $\cos$ term can become highly oscillatory over large portions of the $\chi$-spectrum. Upon integration, such high oscillations progressively chop out significant portions of the $\chi$-spectrum as $z$ is increased. Consequently, the amplitude of the real

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field of the generated pulse at its pulse center decreases as it propagates away from the aperture.

The numerical evaluation of (5.4.10) is shown in Fig. 5-8, in which the normalized amplitudes of the real electric field are plotted for various distances \( z = \frac{\pi}{\beta} = ct \geq 0 \), \( n \) being a positive integer. All values are normalized with respect to the maximum value at \( z = ct = 0 \). The values of the parameters \( a_1 \) and \( \beta \) have been chosen in such way as to comply with the condition \( \beta a_1 \ll 1 \), leading to causal FWM pulses. So, in Fig. 5-8, we have chosen \( \beta = 400 \) m\(^{-1} \), and \( a_1 = 4 \times 10^{-7} \) m, corresponding to a speed of expansion of the aperture field \( v_{ap} = 10.5 \) c. Then, the maximum radius of the aperture is \( R_{max} = v_{ap} T = 10.5 cT \), where \( T \) is the excitation time of the aperture. Three different cases are shown in Fig. 5-8, \( T_1 = 4.33 \times 10^{-10} \) sec, \( T_2 = 8.67 \times 10^{-10} \) sec, and \( T_3 = 1.73 \times 10^{-9} \) sec, corresponding to maximum aperture radii of \( R_{1max} = 1.36 \) m, \( R_{2max} = 2.73 \) m, and \( R_{3max} = 5.45 \) m, respectively. Keeping the parameters \( a_1 \) and \( \beta \) the same for all three cases above, implies that the three pulses formed have the same spectral characteristics. A comparison of the decay patterns of the pulses indicates that the amplitude of a pulse decays at a slower rate as \( R_{max} \) is increased. More specifically, the amplitude of the FWM pulse reaches half its maximum (or aperture) value at distances that double as \( R_{max} \) is increased to twice its original value. This is in agreement with the conclusions of Hafizi and Sprangle [55] as they attempt to define a diffraction length for LW pulses.

In our effort to better understand the behavior of dynamic apertures in relation to diffraction length, we shall concentrate on the efforts of Hafizi and Sprangle [55]. According to their analysis, the diffraction length, associated with a monochromatic beam, is given as

\[
Z_{HS} \approx \frac{2\pi \omega_0 d}{\lambda}, \tag{5.4.11}
\]
where \( w_0 \) is the beam waist, \( d \) is the dimension of the aperture or antenna, and \( \lambda \) is the wavelength. In the case of the wideband LW pulses we express the diffraction length as

\[
Z_{HS} \approx \frac{2\pi w_0 R_{\text{max}}}{\lambda_{\text{min}}},
\]

(5.4.12a)

where \( w_0 \) is the beam waist, \( R_{\text{max}} \) is the maximum dimension of the aperture or antenna, and \( \lambda_{\text{min}} \) is the minimum wavelength. As seen in (5.4.12a), the diffraction length of an LW pulse is directly proportional to the pulse's beam waist and maximum aperture dimension, and inversely proportional to the minimum wavelength. For the case of the FWM pulse, \( \lambda_{\text{min}} = \pi a_1/2, \ w_0 = \sqrt{a_1/\beta}, \) and \( R_{\text{max}} = v_{ap}T, \) where \( v_{ap} \) is the speed of expansion of the aperture field and \( T \) is the excitation time of the aperture. Then, the diffraction length defined by Hafizi and Sprangle is expressed as

\[
Z_{HS} \approx \frac{2\pi \sqrt{a_1/\beta} v_{ap}T}{\pi a_1/2} = \frac{4 v_{ap}T}{\sqrt{\beta} a_1}.
\]

(5.4.12b)

Substituting the parameters \( \beta = 400 \text{ m}^{-1} \), and \( a_1 = 4 \times 10^{-7} \text{ m}, \ v_{ap} = 10.5 \text{ c} \) and \( T \) from Fig. 5-8, yields \( Z_{HS} \approx 431.3 \text{ m}, 862.6 \text{ m}, \) and \( 1725.3 \text{ m}. \) Such values agree nicely with the far field distances of Fig. 5-8.

We shall now compare the performance of the finite FWM pulse with a classical pulse such as a Gaussian pulse. The beam waist, \( w_0 \), of the Gaussian pulse should be the same as that of the finite FWM pulse. Then, its far field will be

\[
Z_G \approx \frac{2\pi w_0^2}{\lambda_{\text{min}}},
\]

(5.4.13)

where \( \lambda_{\text{min}} \) is the minimum wavelength, corresponding to the highest frequency. For the parameter values above, \( \lambda_{\text{min}} = 6.28 \times 10^{-7} \text{ m} \) and \( w_0 = 3.16 \times 10^{-5} \text{ m} \), the far field distance is \( Z_G \approx 1 \text{ cm} \ll Z_{HS} \). So, the FWM pulse outperforms the Gaussian

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pulse. This is the case because the pulse is in the near-field region of a static aperture of radius \( R_{\text{max}} \) whose Rayleigh length is \( Z_R \approx 2\pi R_{\text{max}}^2 / \lambda_{\min} \gg Z_{\text{HS}} \). The behavior of a pulse depends mainly on its temporal and spatial frequency spectra. To be more specific, the spectral contents of a finite FWM pulse are much more complex than the those of a classical Gaussian pulse. The decay pattern of the monochromatic Gaussian pulse is exclusively dependent on its beam waist and wavelength, no matter where it is launched from. On the other hand, a finite FWM pulse makes a full use of the aperture it is excited from, and if the aperture is large enough, it will propagate without significant decay, to distances much larger than the ones traveled by ordinary monochromatic pulses.

### 5.4.2 X Wave Pulse

In a similar fashion to the FWM pulse, we start by imposing a Gaussian time window on the excitation field of the aperture. In this case, the frequency content of the aperture can be calculated as follows:

\[
\Phi(\chi, \omega) = \int_{-\infty}^{+\infty} dt \int_{0}^{\infty} d\rho \ J_0(\chi \rho) \ e^{-i \omega t} \ \frac{e^{-t^2/(4T^2)}}{\left[ \rho^2 + (a_1 - iy_0 t)^2 \right]^{1/2}},
\]

\[
= \int_{-\infty}^{+\infty} dt \ e^{-t^2/(4T^2)} \ e^{-i \omega t} \ \int_{0}^{\infty} d\rho \ \frac{\rho}{(\rho^2 + \alpha^2)^{1/2}} \ J_0(\chi \rho), \quad (5.4.13)
\]

where \( \alpha = a_1 - iy_0 t \), and \( y_0 = c \ \cosh \beta \). The integration over \( \rho \) is carried out using 6.554.1 of [56]:

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\[
\Phi(\chi, \omega) = \int_{-\infty}^{+\infty} dt \ e^{-i\omega t} \ e^{-t^2/(4T^2)} \ \frac{e^{-(a_1 - i\eta_0 t)\chi}}{\chi} = e^{-a_1 \chi} \int_{-\infty}^{+\infty} dt \ e^{-it(\omega - y_0 \chi)} \ e^{-t^2/(4T^2)}. \tag{5.4.14}
\]

The integration in (5.4.14) is simply the Fourier transform of \(e^{-t^2/(4T^2)}\) with \(t \leftrightarrow \omega'\) and \(\omega' = \omega - y_0 t\). Thus, the spectrum of the finite X-wave pulse is

\[
\Phi(\chi, \omega) = 2\sqrt{\pi} \ T \ \frac{e^{-a_1 \chi}}{\chi} \ \exp[-T^2(\omega - y_0 \chi)^2]
\]

or

\[
\Phi(\chi, \omega) = 2\sqrt{\pi} \ T \ \frac{e^{-a_1 \chi}}{\chi} \ \exp[-\frac{y_0^2 T^2}{y^2}(\frac{\omega}{y_0} - \chi)^2]. \tag{5.4.15}
\]

Note that, when \(y_0 T\) is large the Gaussian in the spectrum (5.4.15) reduces to a narrow distribution with a very small bandwidth for which \(\omega \sim y_0 \chi = \chi \ c \ \cosh \beta\) yields most of the significant components of the spectrum. The field propagating in the \(z > 0\) half plane can be expressed as

\[
\Psi(\rho, z, t) = \int_0^{\infty} d\chi \ \chi \ J_0(\chi \rho) \ \int_0^{\infty} d\omega \ \Phi(\chi, \omega) \ \exp[-i(\sqrt{(\omega/c)^2 - \chi^2} \ z - \omega t)]
\]

\[
= 2\sqrt{\pi} \ cT \ \int_0^{\infty} d\chi \ \chi \ J_0(\chi \rho) \ \int_0^{\infty} d\kappa \ \frac{e^{-a_1 \chi}}{\chi} \ \exp[-\frac{y_0^2 T^2}{y^2}(\frac{\kappa}{y_1} - \chi)^2]
\]

\[
\times \ \exp[-i(\sqrt{\kappa^2 - \chi^2} \ z - \kappa ct)], \tag{5.4.16}
\]

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where \( y_1 = \cosh \beta \), and \( \kappa = \omega / c \). For the field generated from an infinite aperture \( (T \to \infty) \), i.e., the ideal X wave pulse, the Gaussian in (5.4.17) becomes a Dirac delta function forcing \( \kappa = \chi y_1 \) and the argument of the second exponential function becomes
\[
- i \left[ \chi z \sqrt{\cosh^2 \beta - 1} - \chi c t \cosh \beta \right] = - i \left[ \chi \sinh \beta (z - ct \coth \beta) \right],
\]
leading to the diffraction-free X wave solution moving with speed \( \nu_p = c \coth \beta \). Equation (5.4.16) represents the X wave pulse generated by an aperture which is excited for a finite amount of time. The real field of (5.4.16) is expressed as
\[
\Psi(\rho, z, t) = 2 \sqrt{\frac{\pi}{3}} \frac{c}{T} \int_0^\infty d\chi \; J_0(\chi \rho) \; e^{-a_1 \chi} \int_0^\infty d\kappa \; \exp \left[ - y_0^2 T^2 \left( \frac{\kappa}{y_1} - \chi \right)^2 \right] 
\]
\[
\times \cos \left( z \sqrt{\kappa^2 - \chi^2} - \kappa c t \right).
\] (5.4.17)

The numerical evaluation of (5.4.17) is shown in Fig. 5-9, in which the normalized amplitudes of the real electric field are plotted for various distances corresponding to \( z = ct \coth \beta \). The parameter values used here are \( a_1 = 0.1 \text{ cm} \) and \( \beta = 6 \). This choice makes the product \( y_0^2 T^2 \) in the exponential a large number and the highest contribution of the integral is around \( \kappa = y_1 \chi = \chi \cosh \beta = 201.7 \chi \). The speed of the pulse is \( \nu_p = c \coth \beta = 1.00001 c \). The argument of the cosine function in this case becomes extremely small and invariant to relatively small distances. For large values of \( z \), the \( \cos \) term can becomes highly oscillatory over large portions of the \( \chi \)-spectrum and, upon integration, the amplitude of the real field of the generated pulse at its pulse center decreases as it propagates away from the aperture. All values are normalized with respect to the maximum value at \( z = ct \coth \beta = 0 \). The speed of expansion of the aperture field here is \( \nu_a \approx 190 c \). Then, the maximum radius of the aperture is \( R_{\text{max}} = \nu_a \tau \approx 190 c T \), where \( T \) is the excitation time of the aperture.

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Three different cases are shown in Fig. 5-9, $T_1 = 0.3$ nsec, $T_2 = 0.6$ nsec, and $T_3 = 1.2$ nsec, corresponding to maximum aperture radii of $R_{\text{max}} = 17.1$ m, $R_{\text{max}} = 34.2$ m, and $R_{\text{max}} = 68.4$ m, respectively. Keeping the parameters $a_1$ and $\beta$ the same for all three cases above, implies that the three pulses formed have the same spectral characteristics. A comparison of the decay patterns of the pulses indicates that the amplitude of a pulse decays at a slower rate as $R_{\text{max}}$ is increased.

The theoretical diffraction lengths corresponding to the three cases above are calculated using Eq. (5.4.11) and they are: $Z_1 \approx 34.8$ Km, $Z_2 \approx 69.7$ Km, and $Z_3 \approx 139.4$ Km, and they are approximately five times as much as the ones shown in Fig. 5-9. So, our pulse starts the far-field decay at distances smaller than the ones predicted by Hafizi and Sprangle [55]. The diffraction length associated with a monochromatic Gaussian pulse of beam waist, $w_0$, and wavelength $\lambda_{\text{min}}$ is $Z_G \approx 2\pi w_0^2/\lambda_{\text{min}}$, where $\lambda_{\text{min}} = 7.8 \times 10^{-6}$ m and $w_0 = 2.53 \times 10^{-3}$ m, resulting in $Z_G \approx 5.15$ m, which is much smaller than the far-field distances observed from Fig. 5-9. So, just as in the case of the FWM pulse, the finite X wave pulse outperforms the Gaussian pulse.

5.5 CONCLUSION

In this chapter, we have studied the possibility of exciting localized pulses from dynamic apertures, or apertures whose effective radius is varied with time. In the case of an ideal LW pulse, the aperture shrinks from an infinite radius to its smallest dimension and then it expands to its original infinite size, thus generating a diffraction-free pulse. Obviously, such ideal LW pulses cannot be realized.

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In the case of finite LW pulses, the aperture of excitation is varied from a time \(-T\) to \(T\). The resulting LW pulses are much more resistant than classical monochromatic Gaussian pulses occupying the same beam waist. In the case of the FWM pulse the numerical values of the parameters \(\beta\) and \(a_1\) were chosen to satisfy the condition \(\beta a_1 \ll 1\), leading to causal fields. The aperture of the finite FWM pulse is shrinking and expanding at speeds greater than \(c\). Such aperture should consist of separately excitable elements, in order not to violate the theory of special relativity. In the case of the X wave pulse, the speed of expansion \(v_{ap}\) of the aperture field, depends on the speed of the pulse center \(v_p\) and can be greater or smaller than \(c\).

Both types of finite LW pulses, such as the FWM and X wave pulse can propagate without significant decay much further than classical monochromatic pulses. This desirable behavior is attributed to the superior aperture efficiency of the LW pulses, which in turn is attributed to their unique spectral structure. A finite LW pulse makes a full use of the aperture it is excited from, and if the aperture is large enough, it will propagate without significant decay to distances much larger than the ones traveled by ordinary monochromatic pulses having the same beam waist. In the case of the monochromatic Gaussian pulse, the Rayleigh length is exclusively dependent on its beam waist and wavelength, regardless of the size of the aperture it is launched from.

Eventhough the finite LW pulses considered here perform much better than their monochromatic counterparts, their physical realization remains an important engineering issue. To actually generate an electromagnetic finite LW pulse, it will require an aperture that consists of separately excited elements whose bandwidth is very large for today's standards. However, recent advances in sources of ultra-wide bandwidth electromagnetic energy, such as optoelectronic switch technology [63,64], bring the implementation of these pulses closer to reality.

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Fig. 5-1: Surface plot of the real part of the FWM pulse on the aperture \((z = 0)\) for the parameters \(a_1 = 4 \times 10^{-7}\) m and \(\beta = 400\) 1/m.
Fig. 5-2: Surface plot of the real part of the X wave pulse on the aperture ($z = 0$) for the parameters $a_1 = 0.001$ m and $\beta = 6$. The speed of the pulse is $v_p = 1.00001c$. \\

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Fig. 5-3:  (a) Table showing the relation between the speed, $v_{ap}$, of expansion of the aperture field of the X wave pulse and the speed of the pulse center, $v_p$. (b) Plot of $v_{ap}$ (units of c) versus $v_p$.

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Fig. 5-4: The temporal frequency content, $\Phi_t(\rho, \omega)$, of the FWM pulse: (a) $\rho = 0$ m, and (b) $\rho = 2 \times 10^{-4}$ m. (Parameters: $a_1 = 4 \times 10^{-7}$ m, and $\beta = 400$ 1/m).
Fig. 5-5: The spatial frequency content, $\Phi_s(\chi, \ell)$, of the FWM pulse: (a) $t = 0$ sec, and (b) $t = 3 \times 10^{-14}$ sec. (Parameters: $a_1 = 4 \times 10^{-7}$ m, and $\beta = 400$ 1/m).

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Fig. 5-6: The temporal frequency content, $\Phi_t(\rho, \omega)$, of the X wave pulse:

(a) $\rho = 0$ m, and (b) $\rho = 1$ m. (Parameters: $a_1 = 0.1$ cm, and $\beta = 6$).
Fig. 5-7: The spatial frequency content, $\Phi_\lambda(\chi, t)$, of the X wave pulse: (a) $t = 0$ sec, and (b) $t = 1 \times 10^{-8}$ sec. (Parameters: $a_1 = 0.1$ cm, and $\beta = 6$).
Fig. 5-8: Decay pattern of the finite FWM pulse for three different values of the aperture excitation time $T$. Note that larger $T$ corresponds to a slower pulse decay.
Fig. 5-9: Decay pattern of the finite X wave pulse for three different values of the aperture excitation time $T$. Note that larger $T$ corresponds to a slower pulse decay.
6.0 CONCLUSIONS

The synthesis of LW pulses has been addressed in this thesis. More specifically, two different types of LW pulses were considered: the FWM pulse and the X wave pulse. We started by introducing the modified bidirectional representation in solving the three dimensional scalar wave equation. Within this framework, we can select new basis functions resulting in different representations for a solution. This freedom facilitates the solutions of the homogeneous, as well as the nonhomogeneous scalar wave equation. We showed how known LW solutions, such as the FWM pulse and the X wave pulse, can be obtained through this new approach. Then, a new class of focused X waves was presented that results in pulses that are extremely localized. This new class of superluminal X waves are much more localized than the X waves presented in the literature. Finally, this new approach was applied to the Fourier synthesis resulting in a simpler representation of solutions to the scalar wave equation. This also resulted in obtaining LW solutions through the Fourier representation directly.

The modified bidirectional decomposition, has been applied to the nonhomogeneous scalar wave equation, in order to moving sources, complex and real, generating these localized pulses. In the FWM case, the real source has a Gaussian taper in the transverse plane and moves with the speed of light. A more complicated source, moving with the speed \(c\), is required to produce the Modified Power Spectrum (MPS) pulse. In the case of the superluminal X wave, the real source has a \(\rho^3\) dependence in the transverse plane and moves with a speed greater than \(c\). In this chapter, we also showed that nondecaying subluminal pulses can be generated by subluminal sources.
Finally, in this thesis we addressed the possibility of exciting localized pulses from dynamic apertures, or apertures whose effective radius is varied with time. In the case of an ideal LW pulse, the aperture shrinks from an infinite radius to its smallest dimension and then it expands to its original infinite size, thus generating a diffraction-free pulse. Obviously, such ideal LW pulses cannot be realized. In the case of finite LW pulses, the aperture of excitation is varied from a time — $T'$ to $T$. The resulting LW pulses are much more resistant than classical monochromatic Gaussian pulses occupying the same beam waist. In the case of the FWM pulse the numerical values of the parameters $\beta$ and $a_1$ were chosen to satisfy the condition $\beta a_1 \ll 1$, leading to causal fields. The aperture of the finite FWM pulse is shrinking and expanding at speeds greater than $c$. Such aperture should consist of separately excitable elements, in order not to violate the theory of special relativity. In the case of the X wave pulse, the speed of expansion $v_{ap}$ of the aperture field, depends on the speed of the pulse center $v_p$ and can be greater or smaller than $c$. Both types of finite LW pulses, such as the FWM and X wave pulse can propagate without significant decay much further than classical monochromatic pulses. This desirable behavior is attributed to the superior aperture efficiency of the LW pulses, which in turn is attributed to their unique spectral structure. A finite LW pulse makes a full use of the aperture it is excited from, and if the aperture is large enough, it will propagate without significant decay, to distances much larger than the ones traveled by ordinary monochromatic pulses having the same waist. In the case of the monochromatic Gaussian pulse, the Rayleigh length is exclusively dependent on its beam waist and wavelength, regardless of the size of the aperture it is launched from. Even though the finite LW pulses considered here perform much better than their monochromatic counterparts, their physical realization remains an important engineering issue. To actually generate an electromagnetic finite LW
pulse, it will require an aperture that consists of separately excited elements whose bandwidth is very large for today's standards. However, recent advances in sources of ultra-wide bandwidth electromagnetic energy, such as optoelectronic switch technology [63,64], bring the implementation of these pulses closer to reality.
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VITA

Argyrios Chatzipetros was born in Nea Seleskia, Northwestern Greece, December 7, 1964. He graduated in 1982 from Igoumenitsa Highschool with honors and received the Hellenic Mathematics Society award after participating in a national competition.

After he came to the United States, he attended Central New England College in Worcester, Massachusetts where he received his Associate's in Electronic Engineering. There he also received the Outstanding Engineering Student Award.

In the Fall of 1984 he transferred to Virginia Polytechnic Institute and State University to continue his undergraduate studies and graduated in March of 1987 receiving his B.S. degree in Electrical Engineering.

In the Fall of 1987 Mr. Chatzipetros returned to Virginia Polytechnic Institute and State University to pursue graduate studies. His Master of Science in Electrical Engineering degree, obtained in 1990, involved modeling of intense laser beams. After the Master's degree he got involved in the propagation and synthesis of directed electromagnetic energy. In May of 1993 Mr. Chatzipetros joined the Personal Communication Systems (PCS) division of Motorola in Plantation, Florida, where he has been working on antenna design and propagation measurements. There, he has made numerous contributions and has four patents pending and one internal publication.

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