ON THE CAUCHY PROBLEM FOR THE LINEARIZED GPKdV AND GAUGE TRANSFORMATIONS FOR A QUADRATIC PENCIL AND AKNS SYSTEM

by

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(ABSTRACT)

The present work in the area of soliton theory studies two problems which arise when seeking analytic solutions to certain nonlinear partial differential equations.

In the first one, Lax pairs associated with prolonged eigenfunctions and prolonged squared eigenfunctions (prolonged squares) are derived for a Schrödinger equation with a potential depending polynomially on the spectral parameter (of degree N) and its respective hierarchy of nonlinear evolution equations (here named generalized polynomial Korteweg-de Vries equations or GPKdV).

It is shown that the prolonged squares satisfy the linearized GPKdV equations. On that basis, the Cauchy problem for the linearized GPKdV has been solved by finding a complete set of such prolonged squares and applying an expansion formula derived in another work by the author.


Moreover, a condition on the so-called recursion operator $\Lambda$ is derived which generates the whole hierarchy of Lax pairs associated with the prolonged squares.
As for the second part of the work, it develops a scheme for deriving gauge transformations between different linear spectral problems. Then the scheme is applied to obtain all known Darboux transformations for a quadratic pencil (the spectral problem considered in the first part at $N = 2$), Schrödinger equation ($N = 1$), Ablowitz-Kaup-Newell-Segur (AKNS) system and also derive the Jaulent-Miodek transformation. Moreover, the scheme yields a large family of new transformations of the above types. It also gives some insight on the structure of the transformations and emphasizes the symmetry with respect to inversion that they possess.
DEDICATION

To the memory of
my mother’s aunt
Nadka Stoyanova
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Chapter I

Introduction to Soliton Theory

Soliton theory is a comprehensive theory for analytical solution of a wide class of non-linear PDE’s (often called "nonlinear evolution equations" — NEE), many of them with important applications in physics, such as the nonlinear Schrödinger, Sine-Gordon, Boussinesq, three wave interaction, Korteweg-de Vries (KdV), Modified KdV, Kadomtsev-Petviaschwili, Benjamin-Ono and other equations.

Solitons are specific solutions of the NEE, usually representing solitary waves moving with constant speed and almost no change of shape. In addition, if two solitons interact they restore their speeds and shapes after the interaction. Mathematically, the so-called $N$-soliton solution is an analytical solution of the NEE corresponding to a reflectionless potential of the associated spectral problem with $N$ eigenvalues in the point spectrum.

The theory of solitons is a relatively new theory, considered to have begun with the numerical results of Zabusky and Kruskal 1 in 1965 who observed solitons when solving the KdV equation, and especially the classical paper of Gardner, Green, Kruskal and Miura 2 where the new Inverse Scattering Method was applied for the first time as a way to solve the Cauchy problem for the KdV equation. In 1968 Lax 3 put this result in a more general framework introducing what is now called Lax representation. In 1972 this framework was applied by Zakharov and Shabat 4 to solve successfully the nonlinear Schrödinger equation, thus showing the broad applicability of the method.

We will illustrate how the method works by using the KdV equation:

$$u_t = 6uu_x - u_{xxx}, \quad \int_{-\infty}^{\infty} (1 + |x|)u(x,0)dx < \infty. \quad (1.1)$$

For many problems, the method has been successfully applied with "decreasing" boundary conditions (such as (1.1)) or with periodic ones. However, even in these classes of functions,
only a dense set of the solutions can actually be expressed analytically by formulas. What the method does, in fact, is to reduce the solving of a certain NEE to solving the direct and inverse scattering problems for some linear spectral problem, associated with that NEE. For instance, KdV is associated with the Schrödinger equation

\[-\psi_{xx}(x, \lambda) + u(x)\psi(x, \lambda) = \lambda \psi(x, \lambda) \tag{1.2}\]

for which the direct and inverse problems have to be solved. Using e.g. Darboux transformations, we can do that analytically for an infinite set of potentials \(u(x)\).

The apparatus for solving the inverse problem is based on the so-called Gelfand-Levitan-Marchenko (GLM) equation derived in the early fifties \(^5\) of this century. It is an integral equation of Volterra type which restores the potential \(u(x)\) when some spectral data \(S(\lambda)\) for (1.2) is given. Here, \(S(\lambda)\) refers to the spectral function \(\rho(\lambda)\) in the case of Gelfand-Levitan equation, or a set of data consisting of the eigenvalues \(\lambda_l\), \(l = 1, M\), their multiplicity, a corresponding set of normalizing coefficients \(c_l\) and the reflection coefficient \(r(\lambda)\) in the case of Marchenko equation.

So, the Inverse Scattering Method solves the Cauchy problem for a certain NEE (e.g. KdV) in the following way:

\[
\begin{align*}
\begin{array}{c}
\text{nonlinear problem} \\
\downarrow \\
\text{linear problem}
\end{array} \quad \begin{array}{c}
\xrightarrow{\text{direct}} \\
\text{scattering} \\
\xrightarrow{\text{inverse}} \\
\text{scattering}
\end{array} \quad \begin{array}{c}
u(x, 0) \\
S(\lambda, 0) \\
u(x, t) \\
S(\lambda, t)
\end{array}
\end{align*}
\]

(All the functions here depend on the variable \(t\) as well.)

a) First, we find the scattering data \(S(\lambda, 0)\) for (1.2) with a potential \(u(x, 0)\) given by the initial condition for (1.1).

b) Secondly, we obtain the scattering data \(S(\lambda, t)\) for any moment \(t\) by solving simple linear equations for \(S(\lambda, t)\). It is crucial that the nonlinear equation becomes a set of linear
ones in the space of scattering data. For (1.1), (1.2) we will have, for instance,

$$\lambda_{l,t}(t) = 0, \quad r_l(\lambda, t) = 8\lambda^{3/2}r(\lambda, t), \quad \ldots$$

yielding

$$\lambda_l(t) = \text{const}, \quad r_l(\lambda, t) = r(\lambda, 0)e^{8\lambda^{3/2}t}, \quad \ldots$$

where $\lambda_l$ are the eigenvalues of (1.2), $r$ is the reflection coefficient, etc.

c) Thirdly, we restore the potential $u(x,t)$ knowing $S(\lambda, t)$ at any moment $t$ by using

the GLM equation.

The idea is not very different from that of the Fourier transform for solving linear
PDE's. Still, it is hard to understand how we can solve a nonlinear equation knowing that
linearity is essential in applying Fourier transform.

The answer is that we actually solve a linear equation. For KdV this would be the
Cauchy problem for the linearized KdV:

$$F_t = 6uF_x + 6u_xF - F_{xxx} \quad \text{(1.3)}$$

where $u(x,t)$ is a solution of (1.1). Equation (1.3) describes a "small" perturbation $F$ of a
specific solution $u(x,t)$ of (1.1) and is obtained by taking a variational derivative of (1.1)
at $\delta u = F$:

$$\frac{d}{dc}(u + \epsilon F)|_{\epsilon = 0} = \left. \frac{d}{dc} [6(u + \epsilon F)(u + \epsilon F)_x - (u + \epsilon F)_{xxx}] \right|_{\epsilon = 0}. $$

Here we pay the price of having two unknown independent (for the moment) functions in
order to have linearity (in $F$).

Due to a theorem (generalized in the present dissertation) we can find a complete set
of solutions of (1.3) in the form $F = (\psi_1 \psi_2)_x$ where $\psi_i, i = 1, 2$ are eigenfunctions of the
Schrödinger equation (1.2). We call a function of the form $\psi_1 \psi_2$ a squared eigenfunction
(square). (The word "product" is reserved for the case when $\psi_1$ and $\psi_2$ satisfy (1.2) for
different potentials $u_1$ and $u_2$.)
By differentiating (1.1) with respect to \( x \) we see that \( u_x \) is also a solution of (1.3). If we expand \( u_x(x, t) \) in terms of derivatives of squares \( [\psi_1(x, \lambda, t)\psi_2(x, \lambda, t)]_x \) (see section II.3 below) then the expansion coefficients happen to be exactly the scattering data \( S(\lambda, 0) \) in the commutative diagram above. So \( u \) is a superposition of squares which, unfortunately, depend on \( u \) itself.

When \( u \) is considered small (which it is for \( x \to \pm \infty \)) equations (1.1) and (1.3) can be approximated by linear PDE's, (1.2) has exponents for solutions and the whole scheme reduces to the usual Fourier transform.

The so-called soliton equations (completely integrable equations) for which the Inverse Scattering Method can be applied have some distinctive features such as Lax representation, an infinite set of first integrals (preserved quantities), Hamiltonian structure, Bäcklund transformations, etc. Maybe the most important of these is the Lax representation. This is a representation of the nonlinear equation in the form

\[
L_t = AL - LA \tag{1.4}
\]

where \( L \) and \( A \) are a pair of linear operators (Lax pair). For KdV equation,

\[
L = -\partial_{xx} + u, \quad A = -4\partial_{xxx} + 3(u\partial_x + \partial_x u).
\]

Notice that \( L \) determines the spectral problem (1.2) for KdV.

Actually (1.4) can be written as a commutation relation,

\[
[L, \partial_t - A] = 0
\]

and expresses compatibility between the two linear problems

\[
L\psi = \lambda \psi \quad \text{and} \quad \psi_t = A\psi. \tag{1.5}
\]

When \( u \) is small, these two equations are automatically compatible since \([\partial_x, \partial_t] = 0\).
The representation (1.4) is not unique. On one hand, we can use \( A + \Omega(L) \) instead of \( A \) where \( \Omega \) is a polynomial with constant coefficients. On the other, (1.4), being a compatibility condition, permits us to find solutions of (1.5) under appropriate conditions, so we can use (1.5) to derive analogous equations e.g. for the squares of these solutions,

\[
\Lambda^*(\psi_1\psi_2) = \lambda\psi_1\psi_2 \quad \text{and} \quad (\psi_1\psi_2)_t = B^*(\psi_1\psi_2)
\]
or their derivatives,

\[
\Lambda(\psi_1\psi_2)_x = \lambda(\psi_1\psi_2)_x \quad \text{and} \quad (\psi_1\psi_2)_{xt} = B(\psi_1\psi_2)_x
\]

thus obtaining the compatibility conditions

\[
\Lambda^*_t = B^*\Lambda^* - \Lambda^*B^*; \quad \Lambda_t = BA - \Lambda B
\]

where \( \Lambda, B, \Lambda^* \) and \( B^* \) are certain linear operators. We refer to these as Lax pairs for the squares. In the case of KdV equation,

\[
\Lambda^* = \partial_x^{-1}\left(-\frac{1}{4}\partial_{xxx} + u\partial_x + \frac{1}{2}u_x\right) \quad B^* = -\partial_{xxx} + 6u\partial_x
\]

\[
\Lambda = \left(-\frac{1}{4}\partial_{xxx} + u\partial_x + \frac{1}{2}u_x\right)\partial_x^{-1} \quad B = -\partial_{xxx} + 6\partial_x u
\]

\[
\partial_x^{-1} = \int_{-\infty}^x dx.
\]

In 1974 Ablowitz, Kaup, Newell and Segur (AKNS) developed a procedure for deriving soliton equations associated with a certain spectral problem and applied it to the 2x2 AKNS system. The reverse problem (finding a Lax pair for a certain soliton equation) is much more difficult and has not yet been solved entirely.

It appears that the Schrödinger equation (1.2) is associated with a whole class of NEE (called generalized KdV equations):

\[
u_t = \Omega(\Lambda)u_x, \quad \Omega(\mu) = a_0\mu^n + a_1\mu^{n-1} + \ldots + a_n.
\]

(1.6)

The simplest nonlinear equations in (1.6) are obtained when \( \Omega \) is linear, e.g. \( \Omega(\mu) = 4\mu \) yields the KdV equation.
In view of (1.4) it is strange that the equations (1.6) are determined by one operator only (for that reason \( \Lambda \) is called recursion operator). Even \( u_x \) here can be written formally as \( u_x = 2 \Lambda \cdot 0 \) assuming \( \partial_x^{-1}(0) = 1 \). The reason for this is that \( B \) can be expressed through \( \Lambda \), as we will see.

Here again we see the importance of \( \Lambda \) and the squared eigenfunctions.

One may ask if every operator \( \Lambda \) determines through (1.6) a soliton type of equations, or whether there are certain conditions on \( \Lambda \) in order for this to happen. We will answer this question partially later on (Theorem 2.4).

Often instead of (1.6) only the sequence

\[
  u_t = \Lambda^k u_x, \quad k = 0, 1, 2, \ldots \tag{1.7}
\]

is considered, known as the KdV hierarchy. Each equation in (1.7) has its own Lax pair \((L, A_k)\) with \( L \) defined above. Also, each one has infinitely many first integrals which are common for the whole hierarchy. They can be obtained as coefficients in the expansion of \( \ln a(\lambda) \) where \( a(\lambda) \) is reciprocal to the transmission coefficient for (1.2) and is part of the spectral data (the point spectrum of (1.2) is determined by the zeros of \( a(\lambda) \)). Here it is important that \( a_t(\lambda, t) = 0 \). The first integrals are

\[
c_k = -\int_{-\infty}^{\infty} \sigma_k(x, t) dx, \quad \sigma_0 = 0, \quad \sigma_1 = u(x, t), \quad \sigma_{k+1} = -\sigma_{k,x} - \sum_{i=1}^{k} \sigma_i \sigma_{k-i}.
\]

In connection with this, the equations (1.6), (1.7) can be represented as infinite-dimensional Hamiltonian systems. For instance,

\[
\text{KdV:} \quad u_t = \partial_x \frac{\delta H}{\delta u}, \quad \frac{\delta H}{\delta u} = -u_{xx} + 3u^2, \quad H = \int_{-\infty}^{\infty} \left( \frac{u_x^2}{2} + u^3 \right) dx.
\]

The first integrals above are, in fact, the set of Hamiltonians for the whole hierarchy (1.7). In addition, all of them are in involution, i.e. \( \{H_i, H_j\} = 0 \) for every two Hamiltonians \( H_i, H_j \) where \( \{,\} \) is an appropriately chosen Poisson bracket. In the framework of this
Hamiltonian formalism, the Inverse Scattering Transform has been shown \(^7\) to be a canonical transformation to angle-action variables.

Finally, Bäcklund transformations are usually associated with each soliton equation, transforming a solution into another solution. The spatial parts of these transformations are known as Darboux transformations and, as it was pointed out, can be used to find analytical solutions to the equation under consideration.

Here we will make a conjecture, namely, that Darboux transformations for a certain spectral problem exist if and only if there exists a GLM equation for it, as well as an \(N\)-soliton solution. It is based on the results of the present and other works about the polynomial Schrödinger equation.

First, in Ref. 8 and 9, asymptotics for the eigenfunctions were derived for \(|\lambda| \to \infty\) in order to obtain eigenfunction expansions. However, in the case \(N > 2\) (\(N\) is the degree of the polynomial in the polynomial Schrödinger equation) these asymptotics were proven only outside certain strips in the \(\lambda\)-plane containing the continuous spectrum.

In Ref. 10, the inverse scattering problem has been studied and GLM equation derived for a similar type of a problem which excludes, however, exactly the cases where the difficulties in Ref. 8 and 9 arise. This prompted the work in Ref. 11 where GLM equation was derived for the case \(N = 2\), in an attempt to extend the result for \(N > 2\), which was unsuccessful. Also, in Ref. 11 the obtained GLM equation was shown to be derivable by using Jaulent-Miodek transformation, which is a Darboux type of transformation (see chapter III). This suggests that different transformations could generate different GLM equations, in support of the above conjecture.

Finally, it is shown in chapter III that the method introduced for deriving Darboux transformations does not work for \(N > 2\) which once again confirms the conjecture made.

The results in chapters II and III are reflected in Ref. 12 and 13, respectively.
Chapter II

Cauchy Problem for the Linearized Version of the Generalized Polynomial KdV Equation

1. Introduction.

Gardner, Green, Kruskal and Miura\textsuperscript{14} observed in their classic paper that the squares of the eigenfunctions satisfy the formal adjoint of the linearized KdV equation, and the derivatives of these squares satisfy the linearized KdV equation itself. Sachs\textsuperscript{15} used this result and solved the Cauchy problem for the linearized KdV by applying an expansion formula for the squares of the eigenfunctions of the Schrödinger equation.

The present chapter is a generalization of these results for a wider class of equations, namely the "generalized polynomial KdV equation" or GPKdV (it reduces to KdV when \(n = 1, N = 1\)). Here we prove that the prolonged squared eigenfunctions of the associated spectral problem evolve according to the linearized GPKdV, also we derive the Lax pairs associated with the prolonged eigenfunctions as well as prolonged squared eigenfunctions and on the basis of the expansion formulae obtained in Ref. 8 and 9 we solve the Cauchy problem for the linearized GPKdV with a decreasing at infinity initial condition.

Analogous questions have been considered in Ref. 16 for the AKNS hierarchy.

The problem of deriving orthogonality relations and expansion formulae for the Schrödinger equation\textsuperscript{15,17,18} and other similar equations\textsuperscript{9,19} has been treated by many authors.

Let us consider the spectral problem

\[
l\psi = -\psi''(x, \lambda) + \sum_{r=0}^{N-1} \lambda^r v_r(x) \psi(x, \lambda) = \lambda^N \psi(x, \lambda), \quad \lambda \in \mathbb{C}, \quad \frac{\partial}{\partial x} (2.1)
\]

where \(v_r \in \mathcal{S}\) (\(\mathcal{S}\) is the Schwartz space of functions \(v : \mathbb{R} \to \mathbb{C}\)) and \(N\) is a natural number. As usual, the functions \(v_r, \psi\) are assumed to depend also on a "time variable" \(t\) in connection
with the evolution equations associated with (2.1).

Equation (2.1) is a generalization of the Schrödinger equation \((N = 1)\) and we will call it a polynomial Schrödinger equation. It is known to be the spectral problem associated with a certain class of nonlinear evolution equations (here referred to as GPKdV) solvable by the inverse scattering method. Strictly speaking, the associated linear problem is not (2.1) but, according to the Lax pair for GPKdV (theorem 2.1), the equivalent matrix equation

\[
L f = \lambda f
\]

with \(L\) independent of \(\lambda\), namely

\[
L = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
-\partial_{xx} + v_0 & v_1 & v_2 & \cdots & v_{N-1}
\end{pmatrix}, \quad f = \begin{pmatrix}
\psi \\
\lambda \psi \\
\vdots \\
\lambda^{N-1} \psi
\end{pmatrix}.
\]

Here \(f = \psi(x, \lambda)\sigma(\lambda), \quad \sigma(\lambda) = (1, \lambda, \ldots, \lambda^{N-1})^T, \quad T = \text{transposition}, \) is called a prolonged solution, \(\psi\) being a solution of (2.1). Also, if \(\phi, \psi\) are solutions of (2.1) then the product \(\phi \psi\) is a solution of

\[
\left[-\frac{1}{4} \partial_{xx} + \sum_{r=0}^{N-1} \lambda^r j(v_r)\right] (\phi \psi) = \lambda^N \partial_x (\phi \psi), \quad j(v_r) \equiv v_r \partial_x + \frac{1}{2} v_{r,x}, \quad (.)_x = \partial_x (.), \quad (2.3)
\]

or, equivalently, \(MF = \lambda JF\) where

\[
J = \begin{pmatrix}
-j(v_1) & \cdots & -j(v_{N-1}) & \partial_x \\
-j(v_2) & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
\partial_x & 0 & \cdots & 0
\end{pmatrix},
\]

\[
M = \begin{pmatrix}
-\frac{1}{4} \partial_{xx} + j(v_0) & 0 & 0 & \cdots & 0 \\
0 & -j(v_2) & -j(v_3) & \cdots & \partial_x \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & -j(v_{N-1}) & \partial_x & \ddots & 0 \\
0 & \partial_x & 0 & \cdots & 0
\end{pmatrix}
\]

and \(F(x, \lambda) = \phi(x, \lambda)\psi(x, \lambda)\sigma(\lambda)\) is the so-called prolonged squared solution. For convenience we define a product "\(f \circ g\)" of \(f = \psi \sigma\) and \(g = \phi \sigma\) as \(f \circ g = \phi \psi \sigma\), so that \(F = f \circ g\).
Also, \( M = \Lambda J \) with
\[
\Lambda = \begin{pmatrix}
0 & 0 & \ldots & 0 \\
1 & 0 & \ldots & 0 \\
0 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & 1
\end{pmatrix}
\begin{pmatrix}
(-\frac{1}{4}\partial_{xxx} + j(v_0))\partial_x^{-1} \\
j(v_1)\partial_x^{-1} \\
j(v_2)\partial_x^{-1} \\
\vdots \\
j(v_{N-1})\partial_x^{-1}
\end{pmatrix}, \quad \partial_x^{-1} = \int_{-\infty}^{x}.
\]
Therefore, \( MF = \lambda JF \) can be represented as
\[
\Lambda(JF) = \lambda(JF) \tag{2.4}
\]
where \( F = \phi \psi \sigma \) is a prolonged squared solution. (Here we also need \( F \xrightarrow[\lambda \to \pm \infty]{} 0 \) to ensure that \( \partial_x^{-1}\partial_x = \text{id} \).)

Moreover, the following is true:
\[
\Lambda^*F = \lambda F \tag{2.5}
\]
with
\[
\Lambda^* = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix}
\begin{pmatrix}
\partial_x^{-1}(-\frac{1}{4}\partial_{xxx} + j(v_0)) \\
\partial_x^{-1}j(v_1) \\
\partial_x^{-1}j(v_2) \\
\vdots \\
\partial_x^{-1}j(v_{N-1})
\end{pmatrix}.
\]

Using the operator \( \Lambda \) we can \( \text{write} \) the GP KdV equations in the form
\[
v_t = \Omega(\Lambda)v_x \tag{2.6}
\]
where \( v(x, t) = (v_0, v_1, \ldots, v_{N-1})^T \), \( \Omega(\mu) = a_0\mu^a + a_1\mu^{a-1} + \ldots + a_n \). Here \( \Omega \) is used for generality. Often only the case \( \Omega(\mu) = \mu^n \), \( n = 0, 1, 2, \ldots \) is considered, referred to as a hierarchy of evolution equations. For \( N = 1 \) this is the KdV hierarchy and for \( N = 2 \) it is the Jaulent-Miodek hierarchy.

Despite the presence of \( \partial_x^{-1} \) in \( \Lambda \), (2.6) represents a system of differential equations. Indeed, as a consequence of appendix B and formulae (3.12), (3.18) in Ref. 20 we conclude that the functions
\[
p_0 = 2, \quad p_m = \int_{-\infty}^{x} c_{m-1,N-1}(s)ds, \quad m \geq 1, \quad \text{where} \quad c_m \equiv \begin{pmatrix} c_{m,0} \\
\vdots \\
c_{m,N-1}
\end{pmatrix} = \Lambda^m v_x,
\]
depend polynomially on $v_r, \quad r = 1, N - 1$, and their derivatives only.


Let us introduce the matrix

$$C_n = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ c_{n,0} & \cdots & c_{n,N-1} \end{pmatrix}.$$

For simplicity the following theorem will consider only the case $a_0 = 1, a_1 = \ldots = a_n = 0$, the general result being written as a corollary.

**Theorem 2.1.** The GPKdV equations (2.6) for $a_0 = 1, a_1 = \ldots = a_n = 0$ have the Lax representation

$$L_t = A_n L - LA_n$$

where

$$A_n = \sum_{i=0}^{n} \left( -\frac{1}{4} p_{i,x} + \frac{1}{2} p_i \partial_x \right) L^{n-i}.$$

**Proof:** Using the approach in Ref. 21 (p.216) we will show first that

$$\left( -\frac{1}{4} p_{i,x} + \frac{1}{2} p_i \partial_x \right) L - L \left( -\frac{1}{4} p_{i,x} + \frac{1}{2} p_i \partial_x \right) = C_i - C_{i-1} L, \quad i \geq 0, \quad C_{-1} = 0.$$

Indeed, at $i = 0$ we get

$$\partial_x L - L \partial_x = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ v_{0,x} & \cdots & v_{N-1,x} \end{pmatrix} = C_0.$$

If $i > 0$ then

$$\left( -\frac{1}{4} p_{i,x} + \frac{1}{2} p_i \partial_x \right) L - L \left( -\frac{1}{4} p_{i,x} + \frac{1}{2} p_i \partial_x \right) = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ b_0 & \cdots & b_{N-1} \end{pmatrix},$$

where, for $r = 1, N - 1$,

$$b_r = \left( -\frac{1}{4} p_{i,x} + \frac{1}{2} p_i \partial_x \right) v_r - v_r \left( -\frac{1}{4} p_{i,x} + \frac{1}{2} p_i \partial_x \right) = \frac{1}{2} p_i v_{r,x}.$$
and (using the above at \( r = 0 \))

\[
b_0 = \left( -\frac{1}{4} p_{i,x} + \frac{1}{2} p_i \partial_x \right) (-\partial_{xx} v_0) + \left( -\frac{1}{4} p_{i,x} + \frac{1}{2} p_i \partial_x \right) \left( -\frac{1}{4} p_{i,x} + \frac{1}{2} p_i \partial_x \right) = \frac{1}{2} p_i v_{0,x} + \\
+ \left[ \frac{1}{4} p_{i,x} \partial_{xx} - \frac{1}{2} p_i \partial_{xx} - \frac{1}{4} \partial_{xx} p_{i,x} + \frac{1}{2} \partial_{xx} p_i \partial_x \right] = \frac{1}{2} p_i v_{0,x} + \left( -\frac{1}{4} p_{i,xx} + p_{i,x} \partial_{xx} \right).
\]

On the other hand

\[
C_i - C_{i-1} L = \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
d_0 & \cdots & d_{N-1}
\end{pmatrix}
\]

where due to \( c_i = \Lambda c_{i-1} \) we have for \( r = 1, N - 1, \)

\[
d_r = [c_{i-1,r-1} + j(v_r) \partial_x^{-1} c_{i-1,N-1}] - [c_{i-1,r-1} + c_{i-1,N-1} v_r] = \frac{1}{2} v_{r,x} \partial_x^{-1} c_{i-1,N-1} = b_r
\]

and for \( r = 0, \)

\[
d_0 = \left( -\frac{1}{4} \partial_{xx} v_0 + j(v_0) \right) \partial_x^{-1} c_{i-1,N-1} - c_{i-1,N-1} (-\partial_{xx} v_0) = \\
= \frac{1}{2} v_{0,x} (\partial_x^{-1} c_{i-1,N-1}) - \frac{1}{4} \partial_{xx} c_{i-1,N-1} + c_{i-1,N-1} = b_0.
\]

Finally, due to (2.7) we find \( A_n L - L A_n = \)

\[
\sum_{i=0}^{n} \left[ \left( -\frac{1}{4} p_{i,x} + \frac{1}{2} p_i \partial_x \right) L - L \left( -\frac{1}{4} p_{i,x} + \frac{1}{2} p_i \partial_x \right) \right] L^{n-i} = \sum_{i=0}^{n} (C_i - C_{i-1} L) L^{n-i} = C_n.
\]

**Corollary.** The nonlinear evolution equations (2.6) can be represented in the form

\[
L_t = AL - LA, \quad A = \sum_{i=0}^{n} a_i A_{n-i}.
\]

(2.8)

Now we will find the time-evolution of the solutions of (2.1), the prolonged and the prolonged squared solutions provided the potential \( v(x,t) \) evolves according to (2.6).

**Lemma 2.1.** If \( f(x, \lambda) \) and \( v(x) \) satisfy (2.2) and (2.6) respectively then \( f_t - Af \) is another solution of (2.2).

**Proof:** Differentiating \( Lf = \lambda f \) with respect to \( t \) yields \( Lf_t + Lf_t = \lambda f_t \). Using (2.8) and (2.2) we get \( L(f_t - Af) = \lambda (f_t - Af) \).

We will be especially interested in the partial case \( f_t - Af = 0 \).
Lemma 2.2. If \( f(x, \lambda) = \psi \sigma \) is a solution of (2.2) and \( f_t = Af \) then \( \psi \) evolves according to

\[
\psi_t = \sum_{i=0}^{n} \left( -\frac{1}{4} p_{i,x} + \frac{1}{2} p_{i} \partial_{x} \right) \left( a_{0} \lambda^{n-i} + \ldots + a_{n-i} \right) \psi.
\]

(This result was obtained independently of Ref. 22.)

Proof: Using change in order of summation we obtain

\[
f_t = Af = \left[ a_{0} \sum_{i=0}^{n} \left( -\frac{1}{4} p_{i,x} + \frac{1}{2} p_{i} \partial_{x} \right) L^{n-i} \right.
\]

\[
+ a_{1} \sum_{i=0}^{n-1} \left( -\frac{1}{4} p_{i,x} + \frac{1}{2} p_{i} \partial_{x} \right) L^{n-1-i} \ldots + a_{n} \sum_{i=0}^{0} \left( -\frac{1}{4} p_{i,x} + \frac{1}{2} p_{i} \partial_{x} \right) L^{0-i} \left. \right] f =
\]

\[
= \sum_{i=0}^{n} \left( -\frac{1}{4} p_{i,x} + \frac{1}{2} p_{i} \partial_{x} \right) \left[ a_{0} L^{n-i} + a_{1} L^{n-1-i} + \ldots + a_{n-i} \right] f =
\]

\[
= \sum_{i=0}^{n} \left( -\frac{1}{4} p_{i,x} + \frac{1}{2} p_{i} \partial_{x} \right) \left( a_{0} \lambda^{n-i} + \ldots + a_{n-i} \right) f.
\]

Theorem 2.2. If the solutions \( f = \psi \sigma, g = \phi \sigma \) of (2.2) satisfy the evolution equation \( \psi_t = Ay \) then:

(a) the prolonged squared solution \( F = f \circ g \) evolves according to

\[
F_t = \sum_{i=0}^{n} \left( -\frac{1}{2} p_{i,x} + \frac{1}{2} p_{i} \partial_{x} \right) \left( a_{0} \lambda^{n-i} + a_{1} \lambda^{n-i-1} + \ldots + a_{n-i} \right) F;
\]

(b) if moreover (2.6) holds then \( JF \) evolves according to

\[
(JF)_t = BJF \quad \text{where} \quad B = \sum_{i=0}^{n} a_{i} B_{n-i}, \quad B_{k} = \sum_{i=0}^{k} T_{i} \Lambda^{k-i}
\]

and \( T_{0} = \partial_{x} \),

\[
T_{m} = \left( p_{m,x} + \frac{1}{2} p_{m} \partial_{x} \right) + P_{m}, \quad P_{m} = \begin{pmatrix} 0 & \ldots & 0 & j(c_{m-1,0}) \partial_{x}^{-1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & j(c_{m-1,N-1}) \partial_{x}^{-1} \end{pmatrix}, \quad m > 0.
\]

Proof: (a) As a consequence of lemma 2.2,

\[
F_t = (\psi \phi_t + \phi \psi_t)\sigma = \sum_{i=0}^{n} \left[ \psi \left( -\frac{1}{4} p_{i,x} + \frac{1}{2} p_{i} \partial_{x} \right) \left( a_{0} \lambda^{n-i} + \ldots + a_{n-i} \right) \phi + \phi \left( -\frac{1}{4} p_{i,x} + \ldots + a_{n-i} \right) \psi \right].
\]
\[
+ \frac{1}{2} p_i \partial_x \right) (a_0 \lambda^{n-i} + \ldots + a_{n-i}) \psi \right] \sigma = \sum_{i=0}^{n} \left[ \left( -\frac{1}{2} p_{i,x} + \frac{1}{2} p_i \partial_x \right) \left( a_0 \lambda^{n-i} + \ldots + a_{n-i} \right) \phi \psi \right] \sigma.
\]

(b) We will prove the statement at \( a_0 = 1, a_1 = \ldots = a_n = 0 \). In the general case it follows from the linearity of the right-hand sides of the equalities used.

Let us denote by \( J'[w] \) the matrix operator \( \delta J|_{\delta w = w} \). (Here \( \delta \) is the variational derivative, \( \delta \Phi(v) = \left. \frac{d}{dt} \Phi(v + \epsilon(\delta v)) \right|_{t=0} \), where \( \Phi \) is a functional of \( v \) or an operator depending on \( v \).) Then for \( n = 0 \) we have

\[
(JF)_t = J'[v_x]F + JF_t = J'[v_x]F + J \partial_x F = \partial_x JF.
\]

Suppose it is true for some non-negative \( n \). Then

\[
B_n JF = J'[v]F + JF_t = J'[c_n]F + J \left( \sum_{i=0}^{n} \left( -\frac{1}{2} p_{i,x} + \frac{1}{2} p_i \partial_x \right) \lambda^{n-i} \right) F
\]

and therefore for \( n + 1 \) we obtain (using (2.4), (2.5) and \( c_{n+1} = \Lambda c_n \))

\[
(JF)_t = J'[c_{n+1}]F + J \left( \sum_{i=0}^{n+1} \left( -\frac{1}{2} p_{i,x} + \frac{1}{2} p_i \partial_x \right) \lambda^{n+1-i} \right) F =
\]

\[
= J'[c_{n+1}]F + J \left( -\frac{1}{2} p_{n+1,x} + \frac{1}{2} p_{n+1} \partial_x \right) F + \lambda \{ B_n JF - J'[\Lambda^n v_x]F \} =
\]

\[
= J'[\Lambda c_n]F + J \left( -\frac{1}{2} p_{n+1,x} + \frac{1}{2} p_{n+1} \partial_x \right) F + B_n \Lambda JF - J'[c_n] \Lambda^* JF.
\]

In order to calculate this we need to use the linearity of \( j \), namely,

\[
j(c_{n,r-1} + j(v_r)p_{n+1}) = j(c_{n,r-1}) + j(j(v_r)p_{n+1})
\]

as well as

\[
j(j(v_r)p_{n+1}) + j(v_r) \left( -\frac{1}{2} p_{n+1,x} + \frac{1}{2} p_{n+1} \partial_x \right) = \left( p_{n+1,x} + \frac{1}{2} p_{n+1} \partial_x \right) j(v_r).
\]

(2.9)

Now we find

\[
(JF)_t = B_n \Lambda JF + \left( p_{n+1,x} + \frac{1}{2} p_{n+1} \partial_x \right) JF - \begin{pmatrix}
  j(c_{n,0}) & 0 & \ldots & 0 \\
  \vdots & \ddots & \vdots & \vdots \\
  j(c_{n,N-1}) & 0 & \ldots & 0 
\end{pmatrix} F =
\]

\[
= (B_n \Lambda + T_{n+1}) JF = B_{n+1} JF.
\]
Lemma 2.3. The following identity holds:

\[ \Lambda T_m - T_m \Lambda = P_{m+1} - P_m \Lambda, \quad m \geq 0. \]  

(2.10)

Proof: We express the left-hand side as

\[ \Lambda T_m - T_m \Lambda = -P_m \Lambda + \left[ \Lambda T_m - \left( p_{m,xx} + \frac{1}{2} p_m \partial_x \right) \right] \Lambda = -P_m \Lambda + \left( \begin{array}{ccc} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{array} \right) \begin{array}{c} s_0 \\ \vdots \\ s_{N-1} \end{array} \]

where for \( m > 0, r = 1, N - 1 \) we have due to (2.9):

\[ s_r = j(v_r) \partial_x^{-1} \left( p_{m,xx} + \frac{1}{2} p_m \partial_x \right) - \left( p_{m,xx} + \frac{1}{2} p_m \partial_x \right) \partial_x^{-1} - \]

\[ -j(c_{m-1,r-1}) \partial_x^{-1} - j(v_r) \partial_x^{-1} j(c_{m-1,N-1}) \partial_x^{-1} = \]

\[ = j(v_r) \partial_x^{-1} \left( p_{m,xx} + \frac{1}{2} p_m \partial_x \right) - \left( j(v_r) p_m + \frac{1}{2} \partial_x p_m \right) \partial_x^{-1} - \]

\[ -j(c_{m-1,r-1}) \partial_x^{-1} - j(v_r) \partial_x^{-1} j(p_{m,xx}) \partial_x^{-1} = \]

\[ = -j(v_r) p_m + j(c_{m-1,r-1}) \partial_x^{-1} = -j(c_m) \partial_x^{-1}. \]  

(2.11)

Now we use (2.11) for \( r = 0 \) to obtain

\[ s_0 = \left[ -\frac{1}{4} \partial_{xxx} + j(v_0) \right] \partial_x^{-1} \left( p_{m,xx} + \frac{1}{2} p_m \partial_x \right) - \left( p_{m,xx} + \frac{1}{2} p_m \partial_x \right) \left[ -\frac{1}{4} \partial_{xxx} + j(v_0) \right] \partial_x^{-1} - \]

\[ - \left[ -\frac{1}{4} \partial_{xxx} + j(v_0) \right] \partial_x^{-1} j(c_{m-1,N-1}) \partial_x^{-1} = -j(v_0) p_m \partial_x^{-1} - \]

\[ -\frac{1}{4} \left[ \partial_{xx} \left( p_{m,xx} + \frac{1}{2} p_m \partial_x \right) - \left( p_{m,xx} + \frac{1}{2} p_m \partial_x \right) \partial_{xx} - \partial_{xx} \left( \frac{1}{2} p_{m,xx} + p_m \partial_x \right) \right] \partial_x^{-1} \]

\[ = -j \left( j(v_0) p_m - \frac{1}{4} p_{m,xxx} \right) \partial_x^{-1} - \frac{1}{4} \left[ j(p_{m,xxx}) \partial_x^{-1} + \frac{1}{2} p_{m,xx} \partial_x - \frac{1}{2} \partial_{xx} p_{m,xx} \partial_x^{-1} \right] \]

\[ = -j(c_{0,0}) \partial_x^{-1}. \]

At \( m = 0 \) we have \( T_0 = \partial_x, \quad P_0 = 0, \quad s_r = j(v_r) \partial_x^{-1} \partial_x - \partial_x j(v_r) \partial_x^{-1} = \]

\[ = [j(v_r) \partial_x - \partial_x j(v_r)] \partial_x^{-1} = -j(v_r) \partial_x^{-1} = -j(c_{0,0}) \partial_x^{-1}, \quad r = 1, N - 1, \]

\[ s_0 = \left[ -\frac{1}{4} \partial_{xxx} + j(v_0) \right] \partial_x^{-1} \partial_x - \partial_x \left[ -\frac{1}{4} \partial_{xxx} + j(v_0) \right] \partial_x^{-1} = -j(v_0) \partial_x^{-1} = -j(c_{0,0}) \partial_x^{-1}. \]

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Corollary 1. For each $n \geq 0$ we have

$$\Lambda B_n - B_n \Lambda = P_{n+1}.$$  

**Proof:** By changing the order of summation we obtain

$$\Lambda B_n - B_n \Lambda = \Lambda \left( \sum_{i=0}^{n} T_i \Lambda^{n-i} \right) - \left( \sum_{i=0}^{n} T_i \Lambda^{n-i} \right) \Lambda =$$

$$= \sum_{i=0}^{n} (\Lambda T_i - T_i \Lambda) \Lambda^{n-i} = \sum_{i=0}^{n} (P_{i+1} - P_i \Lambda) \Lambda^{n-i} = P_{n+1}.$$  

**Corollary 2.** If (2.6) is satisfied then the following Lax representation holds:

$$\Lambda_t = B \Lambda - \Lambda B.$$  

**Proof:** Now we use corollary 1 to obtain

$$B \Lambda - \Lambda B = \sum_{i=0}^{n} a_i (B_{n-i} \Lambda - \Lambda B_{n-i}) = -\sum_{i=0}^{n} a_i P_{n-i+1} =$$

$$= \sum_{i=0}^{n} a_i (\delta \Lambda)_{\delta \omega = A^{n-i} v_{x}} = (\delta \Lambda)_{\delta \omega = \Omega(A)v_{x}} = \Lambda_t.$$  

**Corollary 3.** The operator $B_n$ can also be represented as

$$B_n = \sum_{i=0}^{n} A^{n-i} (T_i - P_i).$$  

**Proof:** For $n = 0$ it is trivial. Let it be true for some $n$. Then

$$B_{n+1} = B_n \Lambda + T_{n+1} =$$

$$= \left\{ \sum_{i=0}^{n} A^{n-i} (T_i - P_i) \Lambda \right\} + T_{n+1} = \left\{ \sum_{i=0}^{n} A^{n-i} [\Lambda (T_i - P_i) + (\Lambda P_i - P_{i+1})] \right\} + T_{n+1} =$$

$$= \left\{ \sum_{i=0}^{n} A^{n-i} [\Lambda (T_i - P_i)] \right\} + (T_{n+1} - P_{n+1}) = \sum_{i=0}^{n+1} A^{n+1-i} (T_i - P_i).$$
Theorem 2.3. Let the conditions in Theorem 2 hold (including the one in (b)). Then $JF$ satisfies the linearized GP$KdV$ equation

$$h_t = \delta(\Omega(\Lambda) v_x)\big|_{\delta v = h}. \quad (2.13)$$

Proof: Again we will prove the statement for $a_0 = 1, a_1 = \ldots = a_n = 0$ only. We have to show that

$$\delta(\Lambda^n v_x) = B_n(\delta v). \quad (2.14)$$

First we will prove (2.14) in the case when $\delta v \in \mathcal{G}$. Then it will follow for the general case as well because $\Lambda^n v_x$ depends on $v, v_x, \ldots$ polynomially and therefore does not contain $\partial_x^{-1}$.

As a byproduct we get that $B_n$ is also a differential operator, not containing $\partial_x^{-1}$.

For $n = 0$ we have

$$\delta(\Lambda^n v_x) = \partial_x(\delta v) = T_0(\delta v) = B_0(\delta v).$$

If it is true for some $n \geq 0$ then $\delta(\Lambda^{n+1} v_x) =$

$$= (\delta \Lambda) \Lambda^n v_x + \Lambda \delta(\Lambda^n v_x) = \begin{pmatrix} 0 & \ldots & 0 & j(\delta v_0) \partial_x^{-1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & j(\delta v_{N-1}) \partial_x^{-1} \end{pmatrix} \begin{pmatrix} c_{n,0} \\ \vdots \\ c_{n,N-2} \\ p_{n+1,0} \end{pmatrix} + \Lambda B_n(\delta v) =$$

$$= (p_{n+1,0} + \frac{1}{2} p_{n+1,0} \partial_x)(\delta v) + (B_n \Lambda + p_{n+1})(\delta v) = (B_n \Lambda + T_{n+1})(\delta v) = B_{n+1}(\delta v).$$

Now we will try to clarify the idea behind the results obtained so far and to simplify some proofs as a consequence.

Notice that

$$P_m = -(\delta \Lambda) \Lambda^{m-1} v_x.$$  

(Here $\Lambda^{m-1} v_x$ is the value of the vector-function on which $\delta \Lambda$ depends, namely $\delta v$.)

Also,

$$T_m = (\delta \Lambda - \delta \Lambda) \Lambda^{m-1} v_x \quad (2.15)$$
where we define \( \overline{\Phi}_w \) for any linear operator \( \Phi_u \) which depends linearly on \( u \), by

\[
\overline{\Phi}_w u = \Phi_u w.
\]

Indeed, (2.15) follows from

\[
(\delta \Lambda)_w = \begin{pmatrix}
0 & \ldots & 0 & j(u_0)\partial_x^{-1} \\
\vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & j(u_{N-1})\partial_x^{-1}
\end{pmatrix}
\begin{pmatrix}
w_0 \\
\vdots \\
w_{N-1}
\end{pmatrix}
=
\begin{pmatrix}
0 w_{N-1} + \frac{1}{2} u_{0,x} (\partial_x^{-1} w_{N-1}) \\
\vdots \\
0 w_{N-1} + \frac{1}{2} u_{N-1,x} (\partial_x^{-1} w_{N-1})
\end{pmatrix}
= \left[ w_{N-1} + \frac{1}{2} (\partial_x^{-1} w_{N-1}) \partial_x \right]
\begin{pmatrix}
u_0 \\
\vdots \\
u_{N-1}
\end{pmatrix}
\]

and therefore

\[
(\delta \Lambda)_w = w_{N-1} + \frac{1}{2} (\partial_x^{-1} w_{N-1}) \partial_x.
\]

So (2.10) can be written as

\[
\Lambda (\overline{\delta \Lambda} - \delta \Lambda) \Lambda_m^{-1} v = (\overline{\delta \Lambda} - \delta \Lambda) \Lambda_m^{-1} v, \quad m \geq 1.
\]

This is a partial case of the next result.

**Theorem 2.4.** The following operator identity holds:

\[
(\delta \Lambda)_{w} \Lambda - \Lambda (\delta \Lambda)_{w} = \overline{\delta \Lambda}_w \Lambda - \Lambda \overline{\delta \Lambda}_w. \tag{2.16}
\]

**Proof:** Let the vector-function \( A(w, u) \) be defined as

\[
A(w, u) = (\overline{\delta \Lambda}_w \Lambda - \Lambda \overline{\delta \Lambda}_w) u.
\]

Then (2.16) is equivalent to \( A(w, u) = A(u, w) \). In order to show it we calculate the \( r \)-th component \( (r = 0, N-1) \) of \( A(w, u) \). For \( r > 0 \),

\[
\begin{aligned}
&\left[ w_{N-1} + \frac{1}{2} (\partial_x^{-1} w_{N-1}) \partial_x \right] [u_{r-1} + j(v_r) (\partial_x^{-1} u_{N-1})] - \\
&- \left\{ \left[ w_{N-1} + \frac{1}{2} (\partial_x^{-1} w_{N-1}) \partial_x \right] u_{r-1} + j(v_r) \partial_x^{-1} \left[ w_{N-1} + \frac{1}{2} (\partial_x^{-1} w_{N-1}) \partial_x \right] u_{N-1} \right\}
\end{aligned}
\]
\[ w_{N-1} + \frac{1}{2} (\partial_x^{-1} w_{N-1}) \partial_x \left[ v_r u_{N-1} + \frac{1}{2} v_{r,z} (\partial_x^{-1} u_{N-1}) \right] - \left[ v_r + \frac{1}{2} v_{r,z} \partial_x^{-1} \right] \times \]
\[ \times \left[ w_{N-1} + \frac{1}{2} (\partial_x^{-1} w_{N-1}) \partial_x \right] u_{N-1} = \frac{1}{4} v_{r,xx} (\partial_x^{-1} u_{N-1}) (\partial_x^{-1} w_{N-1}) + \]
\[ + \frac{1}{2} v_{r,x} \left[ u_{N-1} (\partial_x^{-1} w_{N-1}) + w_{N-1} (\partial_x^{-1} u_{N-1}) + 2 \partial_x^{-1} (u_{N-1} w_{N-1}) \right] \]

which is obviously symmetric with respect to the interchange \( u \leftrightarrow w \).

For \( r = 0 \) we have the expression above plus an additional term:
\[ \left[ w_{N-1} + \frac{1}{2} (\partial_x^{-1} w_{N-1}) \partial_x \right] \left( -\frac{1}{4} u_{N-1,xx} \right) - \left( -\frac{1}{4} \partial_{xx} \right) \left[ w_{N-1} + \frac{1}{2} (\partial_x^{-1} w_{N-1}) \partial_x \right] u_{N-1} = \]
\[ = \frac{1}{4} (u_{N-1} w_{N-1})_{xx} + \frac{1}{8} u_{N-1,xx} w_{N-1,xx}. \]

This is also symmetric. Therefore \( A(w, u) = A(u, w) \).

Using theorem 2.4 we can easily derive the Lax pair (2.12). Consider the sequence of identities
\[ (\delta \Lambda)_v = \partial_x \Lambda - \Lambda \partial_x \]
\[ (\delta \Lambda)_{v_x} - \Lambda (\delta \Lambda)_v = \delta \Lambda_{v_x} \Lambda - \Lambda \delta \Lambda_{v_x} \]
\[ (\delta \Lambda)_{v_x^2} - \Lambda (\delta \Lambda)_{v_x} = \delta \Lambda_{v_x^2} \Lambda - \Lambda \delta \Lambda_{v_x^2} \]
\[ (\delta \Lambda)_{v_x^n} - \Lambda (\delta \Lambda)_{v_x^{n-1}} = \delta \Lambda_{v_x^n} \Lambda - \Lambda \delta \Lambda_{v_x^n} \]
\[ \vdots \]
\[ (\delta \Lambda)_{v_x^{n-1}} - \Lambda (\delta \Lambda)_{v_x} = \delta \Lambda_{v_x^{n-1}} \Lambda - \Lambda \delta \Lambda_{v_x^{n-1}} \]
which follows from (2.16) with an exception of the first one. We add them after multiplying the \( k \)-th identity by \( \Lambda^{n+1-k} \) on the left:
\[ (\delta \Lambda)_{v_x} = D_n \Lambda - \Lambda D_n \quad \text{with} \quad D_n = \delta \Lambda_{v_x^1} \Lambda + \Lambda \delta \Lambda_{v_x^{n-1}} \Lambda + \ldots + \Lambda^{n-1} \delta \Lambda_{v_x} \Lambda + \Lambda^n \partial_x. \]

If (2.6) holds then (2.18) implies
\[ \Lambda_t = D \Lambda - \Lambda D \quad \text{with} \quad D = \sum_{i=0}^n a_i D_{n-i}. \]

Now let us show that \( D_n = B_n \) (hence \( D = B \)) where according to (2.15),
\[ B_n = (\delta \Lambda - \Lambda)_{v_x^1} + (\delta \Lambda - \Lambda)_{v_x^2} \Lambda + \ldots + (\delta \Lambda - \Lambda)_{v_x^n} \Lambda^{n-1} + \partial_x \Lambda^n. \]
For \( n = 1 \) it follows from the first identity in (2.17). Suppose that \( D_n = B_n \) for some integer \( n > 0 \). Then (see (2.18), (2.19)):

\[
B_{n+1} = (\overline{\delta A} - \delta A)\Lambda_n v_u + B_n \Lambda = \overline{\delta A} \Lambda_n v_u - (D_n \Lambda - \Lambda D_n) + B_n \Lambda = \overline{\delta A} \Lambda_n v_u + \Lambda D_n = D_{n+1}.
\]

We can see the symmetry for the evolution operator \( D = B \) of the prolonged squares \( JF \) (\( \Lambda \) being on the left, right, respectively).

So (2.16) answers to some extent the question that we asked in chapter I. It is a condition satisfied by \( \Lambda \) which generates a whole hierarchy of Lax pairs and therefore ensures the soliton type of the corresponding hierarchy of nonlinear evolution equations.

### 3. Cauchy Problem for the Linearized GPKdV Equations.

Now we will find specific solutions of (2.1)-(2.4) for which the statements above apply. Equation (2.1) has Jost solutions

\[
\psi_{\pm}(x, \lambda) \sim e^{\pm ix} \quad (x \to \pm \infty), \quad k = \lambda^{N/2}, \quad \text{Im} \ k \geq 0, \quad (2.20)
\]

analytic \(^8,9\) in the sectors \( \Omega_s = \{ \lambda : -(s-1) \frac{2\pi}{N} < \arg \lambda < s \frac{2\pi}{N} \}, \ s = \overline{1,N} \) and continuous up to the rays \( l_s = \{ \lambda : \arg \lambda = s \frac{2\pi}{N} \}, \ s = \overline{0,N} (l_0 \equiv l_N) \). The same is true for the function

\[
a(\lambda) = W(\psi_-(x, \lambda), \psi_+(x, \lambda)) \equiv (2ik)^{-1} [\psi_-(x, \lambda)\psi'_+(x, \lambda) - \psi'_-(x, \lambda)\psi_+(x, \lambda)].
\]

In addition, on the rays \( l_s \) we have

\[
\psi_+(x, \lambda_{\pm}) = b_{\pm}(\lambda)\psi_-(x, \lambda_{\pm}) + a_{\pm}(\lambda)\psi_-(x, \lambda_{\mp}), \quad \lambda \in l_s \quad (2.21)
\]

where for any function \( \phi \) we define \( \phi(\lambda_{\pm}) = \lim_{\epsilon \to 0} \phi(\lambda \pm i\epsilon) \), \( \lambda \in l_s \). Notice that \( k_{\pm} = \lambda_{\pm}^{N/2} \gg 0 \) and \( a_{\pm}(\lambda) = a(\lambda_{\pm}) \). Also

\[
a_{+}(\lambda)a_{-}(\lambda) - b_{+}(\lambda)b_{-}(\lambda) = 1 \quad (2.22)
\]
which, together with (2.21), implies

\[ \psi_-(x, \lambda) = -b_\mp(\lambda) \psi_+(x, \lambda) + a_\pm(\lambda) \psi_+(x, \lambda), \quad \lambda \in I_s. \]  

(2.23)

We suppose that \( a(\lambda) \) has a finite number of simple zeros \( \lambda_l, l = 1, M \). Then

\[ \psi_+(x, \lambda_l) = b_l \psi_-(x, \lambda_l). \]  

(2.24)

Define \( f_\pm(x, \lambda) = \psi_\pm(x, \lambda) \sigma(\lambda) \). Then \( f_\pm(x, \lambda) \) also satisfy (2.21), (2.23), (2.24) as well as \( f_\pm(x, \lambda) \sim e^{\pm i k x} \sigma(\lambda) \) for \( x \to \pm \infty \).

**Lemma 2.4.** If the solutions \( f = \psi \sigma, g = \phi \sigma \) of (2.2) satisfy \( y_t = Ay \) then \([W(\psi, \phi)]_t = 0.\)

**Proof:** According to lemma 2.2,

\[ [W(\psi, \phi)]_t = W(\psi_t, \phi) + W(\psi, \phi_t) = \sum_{i=0}^{n} (a_0 \lambda^n i + \ldots + a_{n-i}) \left\{ W \left( -\frac{1}{4} p_i x + \frac{1}{2} p_i \partial x \right) \psi, \phi \right\} \]

\[ + W \left( \psi, \left( -\frac{1}{4} p_i x + \frac{1}{2} p_i \partial x \right) \phi \right) \]  

\[ = 0. \]

**Lemma 2.5.** If (2.6) holds then:

(a) \( f_\pm, t(x, \lambda) = A f_\pm(x, \lambda) \mp ik \Omega(\lambda) f_\pm(x, \lambda); \)

(b) \( e^{\pm ik \Omega(\lambda) t} f_\pm(x, \lambda) \) satisfies \( y_t = Ay; \)

(c) \( a_t(\lambda) = 0; \)

(d) \( b_\pm, t(\lambda) = -2ik_\pm \Omega(\lambda) b_\pm(\lambda), \quad \lambda \in I_s; \)

(e) \( b_{lt}(\lambda) = -2ik_\pm \Omega(\lambda) b_l, \) where \( k_l = \lambda^{N/2}, \) \( I m k_l \geq 0; \)

(f) \( \) the vector-function \( g_\pm(x) = f_+(x, \lambda_\pm) - b_l f_-(x, \lambda_l), \) where \( \lambda = \partial x, \) is a solution of (2.2) and \( g_\pm(x) = Ag_\pm(x) - ik_\pm \Omega(\lambda) g(x) - (2ik_\pm \Omega(\lambda)) f_+(x, \lambda_\pm); \)

(g) \( e^{ik_\pm \Omega(\lambda) t} [g_\pm(x) + (2ik_\pm \Omega(\lambda)) f_+(x, \lambda_\pm)] \) satisfies \( y_t = Ay.\)

**Proof:** Statement (a) follows from lemma 2.1, the asymptotics (2.20) and the fact that \( f_\pm, t(x, \lambda) \sim o(e^{\pm i k x}) \sigma(\lambda) \) as \( x \to \pm \infty, \) as well as \( p_i(x) \in \mathcal{S} \) for \( i > 0 \) and \( p_0(x) = 2.\)
Statement (b) is a corollary of (a), and (c) and (d) follow from (b), lemma 2.4 and the formulae

\[ a(\lambda) = (2ik)^{-1} \mathcal{W} \left( \psi_-(x, \lambda)e^{-ik\Omega(x)}t, \psi_+(x, \lambda)e^{ik\Omega(x)}t \right), \]

\[ b_\pm(\lambda)e^{\pm2ik\Omega(x)}t \mathcal{W} \left( \psi_-(x, \lambda)e^{-\pm ik\Omega(x)}t, \psi_+(x, \lambda)e^{\pm ik\Omega(x)}t \right). \]

Next, (e) can be obtained by differentiating (2.24) with respect to \( t \) and using (a).

Notice that \( \lambda_{t,t} = 0 \) due to (c).

About (f), in order to show that \( g_1(x) \) solves (2.2) we differentiate (2.2) with respect to \( \lambda \) and use (2.24) again. This fact is not surprising since \( g_1(x) = \left[ \psi_+(x, \lambda_{t}) - b_1 \psi_-(x, \lambda_{t}) \right] \sigma(\lambda_{t}). \)

Concerning the second part of (f),

\[ g_{1,t}(x) = \left( \dot{f}_+ - b_1 \dot{f}_- \right) \bigg|_{\lambda = \lambda_{t}} = \left( \dot{f}_+ - b_1 \dot{f}_- \right) \bigg|_{\lambda = \lambda_{t}} = \]

\[ = \left( A(\dot{f}_+ - b_1 \dot{f}_-) - [ik\Omega(\lambda)](f_+ + b_1 f_-) - ik\Omega(\lambda)(\dot{f}_+ + b_1 \dot{f}_-) \right) \bigg|_{\lambda = \lambda_{t}} + 2ik\Omega(\lambda) b_1 \dot{f}_-(x, \lambda_{t}). \]

Now we use (2.24).

Finally, (g) is a direct consequence of (f) and (a).

**Remark 1.** According to (g) of lemma 2.5 the formula for \( \tilde{g}_1(x,t) \) in Ref. 15 is incorrect.

In order to solve the linearized GPKdV equation we need an expansion formula. We introduce the bilinear form

\[ < f_1, f_2 > = \int_{-\infty}^{\infty} \sum_{r=0}^{N-1} f_1^{(r)}(x)f_2^{(r)}(x)dx, \quad f_i = \left( f_i^{(0)}(x), f_i^{(1)}(x), \ldots, f_i^{(N-1)}(x) \right)^T, \quad i = 1, 2. \]

Then \(^8,^9\) for any vector-function \( h(x) \in L_1^N(-\infty, \infty) \equiv L_1(-\infty, \infty) \times \ldots \times L_1(-\infty, \infty) \) we have

\[ h(x) = (2\pi)^{-1} \sum_{s=0}^{N-1} \int_{\lambda} \left[ \frac{JF_\pm(x, \lambda_+)}{2\lambda^N a_\pm^2(\lambda)} < F_\pm(\lambda_+), h > - \frac{JF_\pm(x, \lambda_-)}{2\lambda^N a_\pm^2(\lambda)} < F_\pm(\lambda_-), h > \right] d\lambda \]

\[ = \sum_{s=1}^{M} (2\lambda^N \dot{a}_\pm^2(\lambda))^{-1} \left\{ \left[ \frac{N}{\lambda} + \frac{\dot{a}(\lambda_1)}{a(\lambda_1)} \right] JF_\pm(x, \lambda_1) - \frac{JF_\pm(x, \lambda_1)}{\dot{a}(\lambda_1)} \right\} < F_\pm(\lambda_1), h > - \]

\[ - JF_\pm(x, \lambda_1) < \dot{F}_\pm(\lambda_1), h > \right\} \quad \text{(2.25)} \]

with \( F_\pm(x, \lambda) = f_\pm(x, \lambda) \circ f_\pm(x, \lambda) \), provided \( \lambda_1 \in I_s, \quad l = \overline{1,M}, \quad s = \overline{1,N}. \)
Remark 2. According to Ref. 8 and 9 for $N > 2$ the expansions above are formal when some $v_r \neq 0, \ r = \left[ \frac{N}{2} + 1 \right], N - 1$ due to lack of results on the asymptotics of $\psi_{\pm}(x, \lambda)$ as $|\lambda| \to \infty$ for $\lambda$ close to the rays $l_s$. Let us not forget also about the assumption that $a(\lambda)$ has a finite number of simple zeros.

By adding the two formulae in (2.25) we will obtain a new expansion formula containing prolonged squared solutions only (and not their $\lambda$-derivatives as well), namely

$$
P(x, \lambda) = f_+(x, \lambda_+) \circ f_-(x, \lambda_-), \quad Q(x, \lambda) = f_+(x, \lambda_-) \circ f_-(x, \lambda_+), \quad \lambda \in l_s$$

$$
P_l(x) = F_+(x, \lambda_l), \quad Q_l(x) = f_+(x, \lambda_l) \circ g_l(x), \quad l = 1, M. \tag{2.26}$$

In order to do that we need the following

Lemma 2.6. The functions in (2.26) satisfy the identities

\begin{align*}
(a) \quad & \left[ P(x, \lambda)Q^T(y, \lambda) - Q(x, \lambda)P^T(y, \lambda) \right] \frac{2}{a_+(\lambda)a_-(\lambda)} \frac{F_-(x, \lambda_-)F_+^T(y, \lambda_-)}{a_+^2(\lambda)} - \frac{F_-(x, \lambda_+)F_+^T(y, \lambda_+)}{a_+^2(\lambda)} - \frac{F_+(x, \lambda_-)F_+^T(y, \lambda_-)}{a_+^2(\lambda)} + \frac{F_+(x, \lambda_+)F_+^T(y, \lambda_+)}{a_+^2(\lambda)}; \\
(b) \quad & \left[ P_l(x)Q_l^T(y) - Q_l(x)P_l^T(y) \right] \frac{2}{b_l^2} = \left[ \left( \frac{N}{\lambda_l} + \frac{\tilde{a}(\lambda_l)}{\tilde{a}(\lambda_l)} \right) F_+(x, \lambda_l) - \hat{F}_+(x, \lambda_l) \right] F_+^T(y, \lambda_l) - F_+(x, \lambda_l)\hat{F}_+^T(y, \lambda_l) - \left[ \left( \frac{N}{\lambda_l} + \frac{\tilde{a}(\lambda_l)}{\tilde{a}(\lambda_l)} \right) F_-(x, \lambda_l) - \hat{F}_-(x, \lambda_l) \right] F_+^T(y, \lambda_l) + F_-(x, \lambda_l)\hat{F}_+^T(y, \lambda_l). \tag{2.27}
\end{align*}

Proof: (a) Let $p(x, \lambda) = \psi_+(x, \lambda_+)\psi_-(x, \lambda_-)$, $q(x, \lambda) = \psi_+(x, \lambda_-)\psi_-(x, \lambda_+)$. Using (2.21), (2.23) several times we obtain

$$
P(x, \lambda)Q^T(y, \lambda) - Q(z, \lambda)P^T(y, \lambda) =$$

$$= \sigma(\lambda)\sigma^T(\lambda)\{\psi_-(x, \lambda_-)[b_+\psi_-(x, \lambda_+) + a_+\psi_-(x, \lambda_-)]\psi_+(y, \lambda_-)[-b_-\psi_+(y, \lambda_+) + a_+\psi_+(y, \lambda_-)]$$

$$-\psi_-(x, \lambda_+)[b_-\psi_-(x, \lambda_-) + a_-\psi_-(x, \lambda_+)]\psi_+(y, \lambda_+)[-b_+\psi_+(y, \lambda_-) + a_-\psi_+(y, \lambda_+)]\} =$$

$$= \sigma\sigma^T\{a_+^2\psi_+(y, \lambda_-)\psi_+^2(x, \lambda_+) - a_-^2\psi_-(x, \lambda_+)\psi_+^2(y, \lambda_+) + \psi_-(x, \lambda_-)\psi_-(x, \lambda_+)[b_+a_+\psi_+^2(y, \lambda_-)$$

$$-b_-a_-\psi_+^2(y, \lambda_+)] + [-b_+a_-\psi_+^2(x, \lambda_-) + b_+a_-\psi_+^2(x, \lambda_+)]\psi_+(y, \lambda_-)\psi_+(y, \lambda_+).$$

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Because of the identities

\[
b_+ a_+ \psi^2_+(y, \lambda_+) - b_- a_- \psi^2_+(y, \lambda_+) = b_+ \psi_+(y, \lambda_-) \psi_-(y, \lambda_+) - b_- \psi_+(y, \lambda_+) \psi_-(y, \lambda_-);
\]

\[
-b_+ a_+ \psi^2_-(x, \lambda_-) + b_- a_- \psi^2_-(x, \lambda_+) = b_+ \psi_-(x, \lambda_+) \psi_+(x, \lambda_-) - b_- \psi_-(x, \lambda_-) \psi_+(x, \lambda_+);
\]

\[
\psi_+(y, \lambda_-) \psi_+(y, \lambda_-) = [b_- \psi_-(y, \lambda_-) + a_- \psi_-(y, \lambda_+)] \psi_+(y, \lambda_+);
\]

\[
= b_- \psi_-(y, \lambda_-) \psi_+(y, \lambda_+) + \psi_-(y, \lambda_+)[\psi_-(y, \lambda_-) + b_+ \psi_+(y, \lambda_-)]
\]

The following equality holds:

\[
\sigma \sigma^T[p(x, \lambda)q(y, \lambda) - q(x, \lambda)p(y, \lambda)] = a_+^2 F_-(x, \lambda_-)F_+^T(y, \lambda_-) - a_-^2 F_-(x, \lambda_+)F_+^T(y, \lambda_+)
\]

\[+ \sigma \sigma^T \psi_-(x, \lambda_-) \psi_-(x, \lambda_+) [b_+ q(y, \lambda) - b_- p(y, \lambda)] +
\]

\[+ \sigma \sigma^T [b_+ q(x, \lambda) - b_- p(x, \lambda)] [\psi_-(y, \lambda_+ \psi_-(y, \lambda_-) + b_- p(y, \lambda) + b_+ q(y, \lambda)]. \tag{2.29}
\]

After interchanging \( x \leftrightarrow y \) in (2.29) and subtracting it out of (2.29) we obtain after cancellation

\[2[P(x, \lambda)Q^T(y, \lambda) - Q(x, \lambda)P^T(y, \lambda)] =
\]

\[= a_+^2 F_-(x, \lambda_-)F_+^T(y, \lambda_-) - a_-^2 F_-(x, \lambda_+)F_+^T(y, \lambda_+) - a_+^2 F_+(x, \lambda_-)F_+^T(y, \lambda_-) +
\]

\[+ a_+^2 F_+(x, \lambda_+)F_+^T(y, \lambda_+) - \sigma(\lambda)\sigma^T(\lambda)2b_- b_+ [p(x, \lambda)q(y, \lambda) - q(x, \lambda)p(y, \lambda)].
\]

Now we use (2.22) and obtain (2.27).

(b) Let \( A_l(x, y) \) be the right-hand side of (2.28). Then due to (2.24) we have

\[A_l(x, y) = - \dot{F}_+(x, \lambda_1)F_-(y, \lambda_1) - F_+(x, \lambda_1) \dot{F}_-(y, \lambda_1) + \dot{F}_-(x, \lambda_1)F_+^T(y, \lambda_1) + F_-(x, \lambda_1) \dot{F}_+^T(y, \lambda_1).
\]

On the other hand

\[2Q_l(x) = 2\sigma(\lambda)f_+(x, \lambda_1)[\dot{f}_+(x, \lambda_1) - b_1 \dot{f}_-(x, \lambda_1)] =
\]

\[= \sigma(\lambda)[f_+^2(x, \lambda_1) - b_1^2 f_-(x, \lambda_1)]\]

\[= \dot{F}_+(x, \lambda_1) - b_1^2 \dot{F}_-(x, \lambda_1).
\]

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Therefore

\[ A_l(x, y) = -2Q_l(y)F^T_{-}(y, \lambda_l) + F_{-}(x, \lambda_l)2Q^T_{l}(y) = [-Q_l(x)P^T_{l}(y) + P_l(x)Q^T_{l}(y)]^{2}_{b_l^2}. \]

Using lemma 2.6 we find, after adding the formulae in (2.25) and dividing by 2:

\[ h(x) = (2\pi i)^{-1} \sum_{s=0}^{N-1} \int_{I_s} [JP(x, \lambda) < Q(\lambda), h > -JQ(x, \lambda) < P(\lambda), h >] \frac{d\lambda}{2\lambda^N a_+^{(\lambda)} a_-^{(\lambda)}} + \]

\[ + \sum_{l=1}^{M} (2\lambda_l^2 a^2(\lambda_l) b_l^2)^{-1} [JP_l(x) < Q_l, h > -JQ_l(x) < P_l, h >]. \]  \hspace{1cm} (2.30)

As a corollary of theorem 2.3 and lemma 2.5 ((b) and (g)) we find that

\[ e^{2ik_l \Omega(\lambda)t} JP(x, \lambda), \quad -e^{-2ik_l \Omega(\lambda)t} JQ(x, \lambda), \quad \lambda \in I_l \]

\[ e^{2ik_l \Omega(\lambda_l)t} JP_l(x), \quad e^{2ik_l \Omega(\lambda_l)t} [JQ_l(x) + (2ik_l \Omega(\lambda_l)) JP_l(x)t], \quad l = 1, M \]

are solutions of (2.13). Therefore, applying the expansion formula (2.30) we obtain the following

**Theorem 2.5.** The Cauchy problem for the linearized GPKdV equation (2.13) (subject to the restrictions in Remark 2) where \( v(x, t) \) evolves according to (2.6), with initial condition

\[ h(x, t = 0) = h_0(x) \in L^N_1(-\infty, \infty), \]

has a solution

\[ h(x, t) = (2\pi i)^{-1} \sum_{s=0}^{N-1} \int_{I_s} \left[ e^{2ik_l \Omega(\lambda)t} JP(x, \lambda) < Q(\lambda, t = 0), h_0 > - \right. \]

\[ - e^{-2ik_l \Omega(\lambda)t} JQ(x, \lambda) < P(\lambda, t = 0), h_0 > \right] \frac{d\lambda}{2\lambda^N a_+^{(\lambda)} a_-^{(\lambda)}} + \sum_{l=1}^{M} [2\lambda_l^2 a^2(\lambda_l) b_l^2(t = 0)]^{-1} \times \]

\[ \times e^{2ik_l \Omega(\lambda_l)t} \{ JP_l(x) < Q_l(t = 0), h_0 > - [JQ_l(x) + (2ik_l \Omega(\lambda_l)) JP_l(x)t] < P_l(t = 0), h_0 > \}. \]

(Here \( P, Q, P_l, Q_l \) defined in (2.26) depend on \( t \) implicitly via \( v(x, t) \).)
Chapter III

Gauge Transformations for a Quadratic Pencil and AKNS System

1. Introduction.

The gauge transformations (GT) between different linear spectral problems play an important role in the theory of solitons. An example is the Jaulent-Miodek (JM) transformation, which is a generalization of the well known Miura transformation (see e.g. Ref. 23). JM transforms the eigenfunctions of the AKNS system (also called Zakharov-Shabat system):

\[ n'_1(x, \lambda) + i \lambda n_1(x, \lambda) = q(x)n_2(x, \lambda) \]
\[ n'_2(x, \lambda) - i \lambda n_2(x, \lambda) = r(x)n_1(x, \lambda) \]  \hspace{1cm} (3.1)

into the eigenfunctions of a quadratic pencil:

\[ y''(x, \lambda) = [u_0(x) + u_1(x)\lambda - \lambda^2]y(x, \lambda), \quad ' = \frac{\partial}{\partial x} \]  \hspace{1cm} (3.2)

with potentials \( u_0(x), u_1(x) \) depending on the potentials \( q(x), r(x) \) of (1). This transformation provides also a very simple relation between the spectral data of (3.1) and (3.2) which is the reason that (3.1) and (3.2) are often considered equivalent in the context of solving the nonlinear evolution equations (NEE) associated with (3.1) and (3.2) (the AKNS hierarchy and the Jaulent-Miodek hierarchy, respectively), via the inverse scattering method. JM allows for transferring features from (3.1) to (3.2) and vice versa. For instance, in Ref. 11 it was used to derive Marchenko equation for (3.2) from the one for the AKNS system (3.1).

When a gauge transformation \( \tau \) is between spectral problems of the same type (with different potentials \( U \) and \( \tau(U) \)) it is often called Darboux transformation (see e.g. Ref. 24). It expresses the eigenfunctions of the problem with a potential \( \tau(U) \) through the
eigenfunctions of the other one. So we get a whole family of linear problems with different potentials $\tau^n(U), n = 1, 2, \ldots$, for which we can solve analytically the direct and inverse scattering problems provided we can do it for the first one with a potential $U$ (usually $U = 0$). This is a way to generate $N$-soliton solutions (represented by the potentials $\tau^n(U)$) for the NEE associated with this spectral problem.

In this chapter we present a natural way of deriving all GT of certain class between QP and QP, QP and AKNS, and AKNS and AKNS. No boundary conditions are imposed. As a result we obtain all known Darboux transformations for QP and AKNS, the Miura and Jaulent-Miodek transformations, as well as new ones.

The proposed scheme classifies the transformations into four different types and also stresses the similarities between different transformations (e.g. it shows the common nature of JM with Darboux transformations). Since the inverses of these GT are in the same class and types of transformations, that symmetry with respect to inversion is incorporated into the form of the transformations. Some explanation for the symmetry in the QP$\leftrightarrow$QP case can be found on p. 39.

In addition, even though we do not show the transforming of the spectral data (which are defined only in the presence of boundary conditions) we do show the connection between Wronskians of eigenfunctions (through which spectral data are being defined).

Moreover, we derive some previously unknown GT, mainly between QP and AKNS, and also most of the ones of type 2 and 4.

QP$\leftrightarrow$QP transformations with decreasing boundary conditions (for $u_0, u_1, v_0, v_1$) have been derived in Ref. 25. They cover the types 1 and 2 only and are expressed in terms of the potentials solely, thus making the two types indistinguishable.

Ref. 26 obtains QP$\leftrightarrow$QP transformations without imposing boundary conditions by
applying the transformation

\[
\begin{pmatrix}
y' - i\lambda y \\
y
\end{pmatrix} = \begin{pmatrix}
U_1 & \frac{i}{2} w_1 U_1 \\
0 & U_1
\end{pmatrix}
\begin{pmatrix}
n_1 \\
n_2
\end{pmatrix}
\text{ with } U_1 = r^{-1/2}(x)
\]

(a QP→AKNS transformation of type 4 with \( T = 0 \) — cited from Ref. 27) to the three kinds of AKNS→AKNS transformations in Ref. 28. It covers type 3 (being labeled as I and II kind for \( \beta = \pm i \) respectively) and type 1 (III kind). A shortcoming of the paper is the difficult to use form of the transformations involving a lot of dependent on each other variables.

The three kinds of AKNS→AKNS transformations considered in Ref. 28 (without boundary conditions) are derived in an elegant form and correspond to type 3 (\( \beta = \mp i \) respectively) and type 1. A result analogous to lemma 3.4 is obtained.

Finally, the Jaulent-Miodek transformation 23 appears to be a QP→AKNS transformation of type 3 with \( \beta = 0 \) and \( d = 0 \), as will be seen later on.

2. GT for a Schrödinger Equation with an Energy-Dependent Potential.

QP (see e.g. Ref. 29 and the references there) is a partial case (\( N = 2 \)) of a Schrödinger equation with a potential depending polynomially on the spectral parameter \( \lambda \):

\[
y''(x, \lambda) = U(x, \lambda)y(x, \lambda),
\]

where

\[
U(x, \lambda) = \left( \sum_{r=0}^{N-1} u_r(x)\lambda^r \right) - \lambda^N \quad \text{(cf. (2.1))}.
\]

As was mentioned in Chapter I, there are not known \( N \)-soliton solutions to these NEE for \( N > 2 \) as well as a Gelfand-Levitan-Marchenko equation. In addition, there are problems with finding asymptotes of the eigenfunctions and expansion formulae when \( N > 2 \).

Now we will attempt to find Darboux transformations for that linear problem and will see that for \( N > 2 \) the method fails.
For simplicity, let us start with an arbitrary potential function $U(x, \lambda)$. We look for a transformation $\tau: y \to z$ of the type

$$z(x, \lambda) = a(x, \lambda)y(x, \lambda) + b(x, \lambda)y'(x, \lambda) \quad (3.5)$$

where $z(x, \lambda)$ satisfies

$$z''(x, \lambda) = V(x, \lambda)z(x, \lambda) \quad (3.6)$$

for some potential $V(x, \lambda)$. Since (3.3) and (3.6) are linear problems it is natural that $\tau$ be linear as well. Substituting (3.5) into (3.6) and using (3.3) yields:

$$[a'' + (bU)' + (a + b')U - aV]y + [2a' + bU' + b'' - bV]y' = 0. \quad (3.7)$$

Because we want (3.5) to give a solution to (3.6) for any solution $y(x, \lambda)$ of (3.3), we may put in (3.7) two linearly independent solutions $y_1, y_2(x, \lambda)$ and the $2 \times 2$ system obtained will have a nonzero determinant, $\det = y_1y_2' - y_1'y_2 = W(y_1, y_2) \neq 0$. Therefore

$$a'' + 2b'U + bU' + a(U - V) = 0 \quad (3.8)$$

$$b'' + 2a' + b(U - V) = 0.$$

From (3.8) we can exclude $V$:

$$ab'' - a''b - b(2b'U + bU') + 2aa' = 0.$$ 

After integrating with respect to $x$ we get

$$ab' - a'b - b^2U + a^2 = p \equiv W(\tau y_1, \tau y_2)/W(y_1, y_2), \quad p = p(\lambda). \quad (3.9)$$

Indeed, for any two solutions $y_1, y_2$ of (3.3) we have

$$W(\tau y_1, \tau y_2) = \begin{vmatrix} \tau y_1 & (\tau y_1)' \\ \tau y_2 & (\tau y_2)' \end{vmatrix} = \begin{vmatrix} ay_1 + by_1' & (a' + bU)y_1 + (a + b'y'_1) \\ ay_2 + by_2' & (a' + bU)y_2 + (a + b')y'_2 \end{vmatrix} =$$

$$[a(a + b') - b(a' + bU)] \begin{vmatrix} y_1 & y'_1 \\ y_2 & y'_2 \end{vmatrix} = (a^2 + ab' - a'b - b^2U)W(y_1, y_2).$$

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We may use (3.9) to introduce only one independent function \( T(x, \lambda) \) in place of the two functions \( a(x, \lambda) \) and \( b(x, \lambda) \):

\[
T' + T^2 - U = \frac{p}{b^2}, \quad \text{where } T(x, \lambda) = -\frac{a(x, \lambda)}{b(x, \lambda)}. \tag{3.10}
\]

So the system (3.8) is equivalent to (3.10) coupled with

\[
V = U + \left( \frac{b'}{b} \right)'' + \left( \frac{b'}{b} \right)^2 - 2T' \frac{b'}{b} - 2T'. \tag{3.11}
\]

We can give that system a more symmetrical look:

\[
T' + T^2 - U = \frac{p}{b^2} = \left( \frac{b'}{b} - T \right)' + \left( \frac{b'}{b} - T \right)^2 - V. \tag{3.12}
\]

It is easy to check that every function \( T(x, \lambda) \) defines a transformation \( \tau: y \rightarrow z \) of the form

\[
z(x, \lambda) = b(x, \lambda)[y'(x, \lambda) - T(x, \lambda)y(x, \lambda)] \tag{3.13}
\]

between the eigenfunctions \( y, z \) of (3.3) and (3.6), respectively, where \( b(x, \lambda) \) and \( V(x, \lambda) \) are determined by (3.12). Note that \( p(\lambda) \) and \( b(x, \lambda) \) are defined up to a multiplication by a function of \( \lambda \) which, however, does not affect \( \tau \) (due to its linearity) or \( V(x, \lambda) \). So we could choose \( p = 1 \) for simplicity. Unfortunately, \( p \) can be zero for some \( \lambda \).

The symmetry in (3.12) can be seen in (3.13) as well if we invert it (provided \( p \neq 0 \)):

\[
\begin{pmatrix} z \\ z' \end{pmatrix} = b \begin{pmatrix} -T \frac{b'}{b} - T \frac{b'}{b} & 1 \\ -\frac{b'}{b} - T \frac{b'}{b} & T \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix},
\]

\[
\begin{pmatrix} y \\ y' \end{pmatrix} = \frac{b}{p} \begin{pmatrix} -\left( \frac{b'}{b} - T \right) & 1 \\ -\frac{b'}{b} - T \left( \frac{b'}{b} - T \right) & T \end{pmatrix} \begin{pmatrix} z \\ z' \end{pmatrix}. \tag{3.14}
\]

In addition, since (3.3) and (3.6) are linear, it is actually \( \frac{y'}{y} \) being transformed into \( \frac{z'}{z} \):

\[
\left[ \frac{z'}{z} - \left( \frac{b'}{b} - T \right) \right] \left[ \frac{y'}{y} - T \right] = -\frac{p}{b^2}. \tag{3.15}
\]

The following lemma discusses a composition of transformations.
Lemma 3.1. The composition of two transformations $\tau_1(T_1, b_1, p_1)$ and $\tau_2(T_2, b_2, p_2)$ such that $y \xrightarrow{\tau_1} z \xrightarrow{\tau_2} w$ is again a transformation of the same kind, $\tau_3(T_3, b_3, p_3)$, $y \xrightarrow{\tau_3 = \tau_2 \tau_1} w$, where

$$T_3 = T_1 + \frac{p_1 b_1^{-2}}{\left(\frac{b_1'}{b_1} - T_1 - T_2\right)}, \quad b_3 = b_1 b_2 \left(\frac{b_1'}{b_1} - T_1 - T_2\right), \quad p_3 = p_1 p_2. \quad (3.16)$$

In the case when

$$\frac{b_1'}{b_1} - T_1 - T_2 = 0, \quad (3.17)$$

$\tau_3$ is essentially the identity transformation, $\tau_3 = id \sqrt{p_1 p_2}$.

Proof: Indeed, (3.14) implies

$$z' = b_1 \left[\left(\frac{b_1'}{b_1} - T_1\right) y' + \left(-\frac{p_1}{b_1^2} - \frac{b_1'}{b_1} T_1 + T_1^2\right) y\right]$$

and therefore

$$w = b_2 (z' - T_2 z) = b_2 b_1 \left[\left(\frac{b_1'}{b_1} - T_1 - T_2\right) y' + \left(-\frac{p_1}{b_1^2} - \frac{b_1'}{b_1} T_1 + T_1^2 + T_2 T_1\right) y\right] \quad (3.18)$$

yielding (3.16). Concerning $p_3$,

$$p_3 = \frac{W(\tau_2 \tau_1 y_1, \tau_2 \tau_1 y_2)}{W(y_1, y_2)} = \frac{W(\tau_1 y_1, \tau_1 y_2) W(\tau_2 \tau_1 y_1, \tau_2 \tau_1 y_2)}{W(y_1, y_2)} W(\tau_1 y_1, \tau_1 y_2) = p_1 p_2.$$

Also, $\tau_3$ satisfies a system of the kind (3.12) because $\tau_1$ and $\tau_2$ do. Here we use the equivalence between (3.12) and the statement that $\tau$ transforms a fundamental system of solutions of (3.3) into solutions of (3.6).

If (3.17) holds then (3.16) is inapplicable and (3.18) yields $w = -(p_1 b_2 / b_1) y$. However, due to $T_2 = \frac{b_1'}{b_1} - T_1$ we have

$$\frac{p_2}{b_2^2} = T_2' + T_2^2 - V = \left(\frac{b_1'}{b_1} - T_1\right)' + \left(\frac{b_1'^2}{b_1^2} - T_1\right)^2 - V = \frac{p_1}{b_1^2},$$

where $V$ is the potential of $z(x, \lambda)$. Hence $w = y \sqrt{p_1 p_2}$ which means that $\tau_3 = id \sqrt{p_1 p_2}$.

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3. Darboux Transformations for a Quadratic Pencil.

When the potentials $U(x, \lambda), V(x, \lambda)$ are of the form (3.4), according to (3.11) we get a restriction on $T(x, \lambda)$, namely that $(\frac{h'}{b})' + (\frac{h'}{b})^2 - 2T'\frac{k'}{b} - 2T'$ must be an $(N - 1)$-degree polynomial in $\lambda$. One way to do this is to choose:

$$T(x, \lambda) = \sum_{r=0}^{N-1} \alpha_r(x)\lambda^r, \quad \frac{y}{b} \equiv -\frac{1}{2} \left(\frac{T' + T^2 - U'}{T^2 + T^2 - U}\right)' \quad \text{independent of } \lambda. \quad (3.19)$$

**Lemma 3.2.** If $S(x, \lambda) = \sum_{r=0}^{M} s_r(x)\lambda^r$ then $S'/S$ is independent of $\lambda$ if and only if $s_r(x) = c_r h(x)$ for every $r = 0, M$. Then $S'/S = h'(x)/h(x)$.

**Proof:** Let $S'/S = \omega(x)$. After integrating with respect to $x$ we get $S(x, \lambda) = P(\lambda)e^{Q(x)}$ where $Q'(x) = \omega(x)$. Therefore

$$P(\lambda) = \sum_{r=0}^{M} s_r(x)e^{-Q(x)}\lambda^r.$$ 

Now we have $s_r(x)e^{-Q(x)} = \frac{P^{(r)}(0)}{r!} = c_r$ and so $s_r(x) = c_r h(x)$ with $h(x) = e^{Q(x)}$.

Applying lemma 3.2 for $S(x, \lambda) = T' + T^2 - U$ we obtain from (3.12) for $N = 2$:

$$(T' + T^2 - U) = h(x)(c_0 + c_1\lambda + c_2\lambda^2) = \left(-\frac{h'}{2h} - T\right)' + \left(-\frac{h'}{2h} - T\right)^2 - V \quad (3.20)$$

or:

$$\alpha' + \alpha^2 - u_0 = c_0 h(x) = -\frac{h'}{2h} + \alpha)' + \left(\frac{h'}{2h} + \alpha\right)^2 - v_0$$

$$\beta' + 2\alpha\beta - u_1 = c_1 h(x) = -\beta' + 2\left(\frac{h'}{2h} + \alpha\right)\beta - v_1 \quad (3.21)$$

$$1 + \beta^2 = c_2 h(x) = 1 + \beta^2$$

where $\alpha, \beta$ stand for $\alpha_0, \alpha_1$. Actually the left-hand side is a system of two equations for $\alpha, \beta$ provided $(c_0, c_1, c_2) \neq (0, 0, 0)$ (we assume $h(x) \neq 0$). When $N = 1$ we have one equation with one unknown ($\alpha_0$). When $N > 2$, however, the equations $(2N - 2)$ are more than the unknowns ($N$) and cannot be satisfied for arbitrary potentials $u_0, u_1, \ldots, u_{N-1}$. 

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THEOREM 3.1. For any choice of the constants \(c_0, c_1, c_2\) in (3.21) such that \((c_0, c_1, c_2) \neq (0, 0, 0)\) the system (3.21) has a solution \((\alpha, \beta)\) for arbitrary potentials \(u_0, u_1\). It defines a transformation \(\tau (y \rightarrow z)\):

\[
z(x, \lambda) = h^{-1/2}(x)[y'(x, \lambda) - T(x, \lambda)y(x, \lambda)], \quad T(x, \lambda) = \alpha(x) + \beta(x)\lambda
\]

between the linear problems (3.3) and (3.6) with potentials \(U = u_0(x) + u_1(x)\lambda - \lambda^2\) and \(V = v_0(x) + v_1(x)\lambda - \lambda^2\) where \(h(x)\) and \(V(x, \lambda)\) are determined by (3.20).

At the moment we can think of \(p(\lambda)\) as \(p(\lambda) = c_0 + c_1\lambda + c_2\lambda^2, \quad h(x, \lambda) = h^{-1/2}(x)\).

Now, using theorem 3.1 we will derive different types of Darboux transformations \(\tau\) by choosing the constants \(c_0, c_1, c_2\) differently. But first we need the following

LEMMA 3.3. If \(d \in \mathbb{C}\) is a zero of the quadratic equation

\[
c_0 + c_1\lambda + c_2\lambda^2 = 0 \quad (3.22)
\]

then

\[
\alpha(x) + \beta(x)d = \frac{\tilde{y}'(x, d)}{\tilde{y}(x, d)}, \quad \frac{-h'(x)}{2h(x)} - \alpha(x) - \beta(x)d = \frac{\tilde{z}'(x, d)}{\tilde{z}(x, d)} \quad (3.23)
\]

where \(\tilde{y}, \tilde{z}\) are some solutions of (3.3) and (3.6) respectively, related through \(\tilde{y}(x, d)\tilde{z}(x, d) = h^{-1/2}(x)\).

Lemma 3.3 is an easy consequence of (3.20). The relation between \(\tilde{y}\) and \(\tilde{z}\) follows from (3.23).

TYPE 1. Equation (3.22) has two different zeros \(d_{1,2}\). Then (3.23) yields

\[
\alpha(x) + \beta(x)d_s = \frac{\tilde{y}'(x, d_s)}{\tilde{y}(x, d_s)} = \frac{-h'(x)}{2h(x)} - \frac{\tilde{z}'(x, d_s)}{\tilde{z}(x, d_s)}, \quad s = 1, 2, \quad (3.24)
\]

so the transformation is determined by:

\[
\alpha(x) = \frac{1}{d_1 - d_2} \left[ -d_2 \frac{\tilde{y}'(x, d_1)}{\tilde{y}(x, d_1)} + d_1 \frac{\tilde{y}'(x, d_2)}{\tilde{y}(x, d_2)} \right], \quad \beta(x) = \frac{1}{d_1 - d_2} \left[ \frac{\tilde{y}'(x, d_1)}{\tilde{y}(x, d_1)} - \frac{\tilde{y}'(x, d_2)}{\tilde{y}(x, d_2)} \right]. \quad (3.25)
\]
If we denote \( \phi_s = \tilde{y}'(x, d_s)/\tilde{y}(x, d_s), \psi_s = \tilde{z}'(x, d_s)/\tilde{z}(x, d_s), s = 1, 2 \), then due to (3.24) the potentials transform according to

\[
\begin{align*}
&u_0 + u_1d_1 - d_1^2 = \phi'_1 + \phi_1^2, \quad \phi_1 - \phi_2 = -(\psi_1 - \psi_2), \quad v_0 + v_1d_1 - d_1^2 = \psi'_1 + \psi_1^2, \\
&u_0 + u_1d_2 - d_2^2 = \phi'_2 + \phi_2^2, \quad \phi_1 + \phi_2 = -\frac{h'}{h} (\psi_1 + \psi_2), \quad v_0 + v_1d_2 - d_2^2 = \psi'_2 + \psi_2^2
\end{align*}
\]

(3.26)

where because of (3.21), (3.25) and (3.26) we have

\[
\frac{h'}{h} = \frac{c_2h'(x)}{c_2h(x)} = \frac{2\beta' + \beta^2}{1 + \beta^2} = \frac{2(\phi_1 - \phi_2)(\phi_1 - \phi_2)'}{(d_1 - d_2)^2 + (\phi_1 - \phi_2)^2} = \frac{2(\psi_1 - \psi_2)(\psi_1 - \psi_2)'}{(d_1 - d_2)^2 + (\psi_1 - \psi_2)^2}.
\]

(3.27)

TYPE 2. Equation (3.22) has a multiple zero at \( x = d \), i.e. \( c_2 \neq 0, c_1^2 - 4c_0c_2 = 0, d = -c_1/2c_2 \). Then we take the limit \( d_1 \to d, d_2 \to d \) in (3.24) - (3.27) and choosing for simplicity the case when \( \lim_{d_i \to d} \tilde{y}(x, d_i) = \lim_{d_2 \to d} \tilde{y}(x, d_2) = \tilde{y}(x, d) \) we obtain:

\[
\beta = \dot{\phi}, \quad \alpha = \phi - \dot{\phi}d, \quad \text{where } \phi(x) = \frac{\partial}{\partial \lambda} \left( \tilde{y}'(x, \lambda) \right)_{\lambda=d},
\]

\[
\begin{align*}
u_0 + u_1d - d^2 &= \phi' + \phi^2, \quad \phi = -\psi, \quad \psi' + \psi^2 = v_0 + v_1d - d^2, \\
u_1 - 2d &= \dot{\phi}' + 2\dot{\phi}d, \quad \phi = -\frac{h'}{2h} - \psi, \quad \psi' + 2\psi\dot{\psi} = v_1 - 2d
\end{align*}
\]

with

\[
\frac{h'}{h} = \frac{2\dot{\phi}\phi'}{1 + \dot{\phi}^2} = \frac{2\dot{\psi}\psi'}{1 + \dot{\psi}^2}, \quad \text{where } \psi(x) = \frac{\tilde{z}'(x, d)}{\tilde{z}(x, d)}, \quad \dot{\psi}(x) = \frac{\partial}{\partial \lambda} \left( \tilde{z}'(x, \lambda) \right)_{\lambda=d}.
\]

TYPE 3. \( c_2 = 0, c_1 \neq 0 \). Then (3.21) implies \( \beta = \pm i \) and (3.22) has a simple zero \( d = -c_0/c_1 \) so that:

\[
\alpha(x) \pm id = \phi(x) = -\frac{h'(x)}{2h(x)} - \psi(x), \quad \text{where } \phi = \frac{\tilde{y}'(x, d)}{\tilde{y}(x, d)}, \quad \psi = \frac{\tilde{z}'(x, d)}{\tilde{z}(x, d)}.
\]

(3.28)

Another relation can be obtained from (3.21):

\[
\pm 2i\alpha(x) - u_1(x) = c_1h(x) = \pm 2i \left[ \frac{h'(x)}{2h(x)} + \alpha(x) \right] - v_1(x).
\]

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From here we exclude $\alpha(x)$ using (3.28):

$$\pm 2i\phi(x) + 2d - u_1(x) = c_1 h(x) = \mp 2i\psi(x) + 2d - v_1(x).$$

So the transformation is:

$$u_1 = u_1, \quad u_1 \mp 2i\phi = v_1 \pm 2i\psi, \quad v_1 = v_1$$

$$u_0 + u_1 d - d^2 = \phi' + \phi^2 \quad \phi = -\frac{h'}{2h} - \psi \quad v_0 + v_1 d - d^2 = \psi' + \psi^2$$

with

$$\frac{h'}{h} = \frac{(c_1 h(x))'}{c_1 h(x)} = \frac{(u_1 \mp 2i\phi - 2d)'}{u_1 \mp 2i\phi - 2d} \equiv \frac{(v_1 \pm 2i\psi - 2d)'}{v_1 \pm 2i\psi - 2d}.$$

TYPE 4. $c_2 = c_1 = 0$, $c_0 \neq 0$. Then $\beta(x) = \pm i$, $\alpha(x) = \mp \frac{i}{2} u_1(x)$. In addition, the first two equations of (3.21) yield after substituting for $\alpha$ and $h'/2h + \alpha$:

$$u_1 = v_1 \mp i\frac{h'}{h}, \quad u_0 \pm \frac{i}{2} u_1' + \frac{1}{4} u_1^2 = v_0 \mp \frac{i}{2} v_1' + \frac{1}{4} v_1^2$$

with

$$h' = \frac{(u_0 \pm \frac{i}{2} u_1' + \frac{1}{4} u_1^2)'}{u_0 \pm \frac{i}{2} u_1' + \frac{1}{4} u_1^2} \equiv \frac{(v_0 \mp \frac{i}{2} v_1' + \frac{1}{4} v_1^2)'}{v_0 \mp \frac{i}{2} v_1' + \frac{1}{4} v_1^2}.$$

4. Features of the QP$\to$QP Transformations.

Obviously the inverse of a transformation of the class (3.19) is a transformation of the same class (see (3.14)), the same type and with the same constants $d_{1,2}$ (when applicable).

A transformation of type 3 or 4 with $\beta = i$ has an inverse with $\beta = -i$ and vice versa.

On the other hand, the product of two transformations of the class (3.19) is generally outside that class. For instance, in lemma 3.1, if we choose $\tau_1$ and $\tau_2$ of type 4 with $\beta_1 = \beta_2 = i$ then according to (3.16) $b'_2/b_3$ depends on $\lambda$ in general.

The following lemmas show cases when $\tau_3 = \tau_2 \tau_1$ is still in the class (3.19).

**Lemma 3.4.** The product of two transformations of type 3 with different $\beta$ and different $d$ is a transformation of type 1.

**Proof:** Let the two transformations $\tau_1$, $\tau_2$ ($y \overset{\tau_1}{\to} z \overset{\tau_2}{\to} w$) be determined by $T_1 = \alpha_1 + \beta_1 \lambda$, $T_2 = \alpha_2 + \beta_2 \lambda$ where $\beta_1(x) = \pm i = -\beta_2(x)$ and

$$\alpha_1(x) + \beta_1(x)d_1 = \frac{\gamma'(x, d_1)}{\gamma(x, d_1)} \equiv \phi_1(x), \quad \alpha_2(x) + \beta_2(x)d_2 = \frac{\gamma'(x, d_2)}{\gamma(x, d_2)} \equiv \phi_2(x) \quad \text{(see (3.28))}. $$

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Then

\[ T_1(x, \lambda) = \beta_1(\lambda - d_1) + \phi_1(x), \quad T_2(x, \lambda) = \beta_2(\lambda - d_2) + \phi_2(x). \]

On the other hand (3.15) yields:

\[ \phi_2(x) \equiv \frac{\tilde{z}'(x, d_2)}{\tilde{z}(x, d_2)} = \frac{b_1'(x)}{b_1(x)} - T_1(x, d_2) - \frac{p_1(d_2)b_1^{-2}(x)}{\tilde{\phi}_2(x) - T_1(x, d_2)} \]

for some \( \tilde{\phi}_2(x) \equiv \frac{\tilde{y}'(x, d_2)}{\tilde{y}(x, d_2)}. \)

So

\[ \frac{b_1'(x)}{b_1(x)} - T_1(x, \lambda) - T_2(x, \lambda) = \]

\[ = \frac{b_1'(x)}{b_1(x)} - T_1(x, \lambda) - \beta_2(\lambda - d_2) - \frac{b_1'(x)}{b_1(x)} + T_1(x, d_2) + \frac{p_1(d_2)b_1^{-2}(x)}{\tilde{\phi}_2(x) - T_1(x, d_2)} = \]

\[ = -(\beta_1 + \beta_2)(\lambda - d_2) + \frac{p_1(d_2)b_1^{-2}(x)}{\beta_1(\lambda - d_2) - \phi_1(x) + \tilde{\phi}_2(x)} = \frac{p_1(d_2)b_1^{-2}(x)}{\beta_1(\lambda - d_2) - \phi_1(x) + \tilde{\phi}_2(x)}. \]

For \( T_3 \) we find according to (3.16):

\[ T_3(x, \lambda) = T_1(x, \lambda) + \frac{p_1(\lambda)b_1^{-2}(x)}{p_1(d_2)b_1^{-2}(x)} \left[ \beta_1(\lambda - d_2) - \phi_1(x) + \tilde{\phi}_2(x) \right] = \]

\[ = [\beta_1(\lambda - d_1) + \phi_1(x)] + \frac{\lambda - d_1}{d_2 - d_1} [\beta_1(\lambda - d_2) - \phi_1(x) + \tilde{\phi}_2(x)] = \]

\[ = \frac{1}{d_2 - d_1} [\lambda(\tilde{\phi}_2(x) - \phi_1(x)) + (d_2\tilde{\phi}_2(x) + d_2\phi_1(x))] \equiv \beta_3(x)\lambda + \alpha_3(x) \quad (\text{cf. (3.25)}). \]

The only exception would be when (3.17) holds and then \( \tau_2 \tau_1 \sim id. \) According to (3.23) this means:

\[ 0 = -\frac{h_1'(x)}{2h_1(x)} - \alpha_1(x) - \beta_1(\lambda - \alpha_2(x) - \beta_2(x)) = [\psi_1 - \beta_1(\lambda - d_1)] - [\phi_2 + \beta_2(\lambda - d_2)] \]

where

\[ \psi_1 = \frac{\tilde{z}'(x, d_1)}{\tilde{z}(x, d_1)}, \quad \phi_2 = \frac{\tilde{z}'(x, d_2)}{\tilde{z}(x, d_2)}. \]

However, this implies

\[ v_0 + v_1d_2 - d_2^2 = \phi_2' + \phi_2^2 = \]

\[ = \psi_1' + [\psi_1 - \beta_2(d_1 - d_2)]^2 = \beta_2^2(d_1 - d_2)^2 - 2\beta_2(d_1 - d_2)\psi_1 + (v_0 + v_1d_1 - d_1^2) \]

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or:

\[(d_2 - d_1) [v_1 - (d_2 + d_1) - 2\beta_2 \psi_1 - \beta_2^3 (d_2 - d_1)] = 0.\]

If the second multiplier in this expression is zero then there exists a relation between \(v_0(x)\) and \(v_1(x)\) which through (3.29) induces a relation between \(u_0(x)\) and \(u_1(x)\). This type of potentials were initially excluded from our considerations.

If \(d_1 = d_2\) then (3.17) holds only if \(\psi_1 \xrightarrow{d_1 \to d_2} \phi_2\). So we get the following

**Lemma 3.5.** The product of two transformations of type 3 with different \(\beta\) and the same \(d\) is a transformation of type 2 if \(\psi_1 \neq \phi_2\) in (3.29) (index \(k\) corresponds to the \(k\)-th transformation, \(k=1, 2\)) and it is the identity transformation otherwise.

**Lemma 3.6.** The product \(\tilde{\tau} \tau\) of two (type 1)-transformations

\[
\tau \left( \phi_1 = \frac{y'(x, p)}{\tilde{y}(x, p)}, \phi_2 = \frac{y'(x, q)}{\tilde{y}(x, q)} \right), \quad \tilde{\tau} \left( \tilde{\phi}_1 = \frac{z'(x, q)}{\tilde{z}(x, q)}, \tilde{\phi}_2 = \frac{z'(x, r)}{\tilde{z}(x, r)} \right)
\]

where

\[
y \to z \to w, \quad \tilde{\phi}_1 = \frac{z'(x, q)}{\tilde{z}(x, q)} = -\frac{h_1^r(x)}{2h_1(x)} \frac{y'(x, q)}{\tilde{y}(x, q)} = \psi_2
\]

is a transformation of type 1 as well (with \(d_1 = p, d_2 = r\)).

**Proof:** Using (3.24) for \(d_s = q\) and (3.30) we get:

\[
\frac{b_1'(x)}{b_1(x)} - T_1(x, \lambda) - T_2(x, \lambda) = -\frac{h_1^r(x)}{2h_1(x)} - [\phi_2 + \beta_1(\lambda - q)] - [\tilde{\phi}_1 + \beta_2(\lambda - q)] = -(\beta_1 + \beta_2)(\lambda - q).
\]

Here we have due to (3.15):

\[
\beta_2(x) = \frac{1}{q - r} (\tilde{\phi}_1 - \phi_2) = \frac{1}{q - r} \left[ \tilde{\phi}_1 - \frac{b_1'(x)}{b_1(x)} + T_1(x, r) + \frac{(r - p)(r - q)(1 + \beta_1^2)}{\tilde{y}(x, r) - T_1(x, r)} \right]
\]

\[
= \frac{1}{q - r} \left[ -\phi_2 + T_1(x, r) + \frac{(r - p)(r - q)(1 + \beta_1^2)}{\phi_3 - T_1(x, r)} \right] = -\beta_1 + \frac{(p - r)(1 + \beta_1^2)}{\phi_3 - T_1(x, r)}.
\]

Therefore, according to (3.16), (3.31):

\[
T_3 = T_1(x, \lambda) - \frac{(\lambda - p)(\lambda - q)(1 + \beta_1^2)}{(\beta_1 + \beta_2)(\lambda - q)} = T_1(x, \lambda) - (\lambda - p) \frac{1 + \beta_1^2}{(p - r)(1 + \beta_1^2)} [\phi_3 - T_1(x, r)] =
\]

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\[
= \frac{1}{p-r} \left\{ \frac{p-r}{p-q} (\lambda - q) \phi_1 - (\lambda - p) \phi_2 \right\} - \frac{\lambda - p}{p-q} ((p - q) \phi_3 - (r - q) \phi_1 + (r - p) \phi_2) \right\} = \\
= \frac{1}{p-r} [(\lambda - r) \phi_1 - (\lambda - p) \phi_3] = \alpha_3(x) + \beta_3(x) \lambda.
\]

As we could expect, the Darboux transformations for the standard Schrödinger equation

\[(u_1 = v_1 = 0)\]

are a partial case of the ones above. Namely, they can be obtained from type

1 when:

\[
d_2 = -d_1, \quad \tilde{y}(x, d_2) = \tilde{y}(x, d_1), \quad \text{implying} \quad \beta = 0, \quad \alpha(x) = \frac{\tilde{y}'(x, d_1)}{\tilde{y}(x, d_1)}', \quad b(x) = 1.
\]

(Naturally, at \(d_1 \to 0\) the above becomes a partial case of type 2.)

The transformation itself reduces to the simple form:

\[
\left[ \frac{z'(x, \lambda)}{z(x, \lambda)} - \frac{\tilde{z}'(x, d)}{\tilde{z}(x, d)} \right] \left[ \frac{y'(x, \lambda)}{y(x, \lambda)} - \frac{\tilde{y}'(x, d)}{\tilde{y}(x, d)} \right] = -(\lambda^2 - d^2), \quad \text{where} \quad \tilde{z}(x, d) = \frac{1}{\tilde{y}(x, d)}.
\]

The transformation QP \(\to\) QP of type 1 written also as

\[
\left[ \frac{z'(x, \lambda)}{z(x, \lambda)} - \left( \frac{h'(x)}{2h(x)} - T(x, \lambda) \right) \right] \left[ \frac{y'(x, \lambda)}{y(x, \lambda)} - T(x, \lambda) \right] = -(\lambda - d_1)(\lambda - d_2)[1 + \beta^2(x)]
\]

(3.32)

can similarly be represented in a simpler form. Indeed, if

\[
\phi = \frac{y'(x, \lambda)}{y(x, \lambda)}', \quad \psi = \frac{z'(x, \lambda)}{z(x, \lambda)}
\]

then because of (3.24) – (3.27) the equality (3.32) becomes

\[
\left[ \frac{\lambda - d_2}{d_1 - d_2} (\psi - \psi_1) + \frac{\lambda - d_1}{d_2 - d_1} (\psi - \psi_2) \right] \left[ \frac{\lambda - d_2}{d_1 - d_2} (\phi - \phi_1) + \frac{\lambda - d_1}{d_2 - d_1} (\phi - \phi_2) \right] = \\
= (\lambda - d_1)(\lambda - d_2) \left[ \frac{\psi_1 - \psi_2}{d_1 - d_2} + i \right] \left[ \frac{\phi_1 - \phi_2}{d_1 - d_2} + i \right] .
\]

Now we multiply by \((d_1 - d_2)^2\):

\[
[(\lambda - d_2)(\psi - \psi_1) - (\lambda - d_1)(\psi - \psi_2)][(\lambda - d_2)(\phi - \phi_1) - (\lambda - d_1)(\phi - \phi_2)] = \\
= (\lambda - d_1)(\lambda - d_2) \left\{ [(\psi - \psi_1) - (\psi - \psi_2)][(\phi - \phi_1) - (\phi - \phi_2)] - (d_1 - d_2)^2 \right\}.
\]

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After cancellation we get

\[(\lambda - d_1)(\lambda - d_2)(d_1 - d_2) = -(\lambda - d_2)(\psi - \psi_1)(\phi - \phi_1) + (\lambda - d_1)(\psi - \psi_2)(\phi - \phi_2)\]

or:

\[(\lambda - d_2) \left[ \frac{z'(x, \lambda)}{z(x, \lambda)} - \frac{\tilde{z}'(x, d_1)}{\tilde{z}(x, d_1)} \right] \left[ \frac{y'(x, \lambda)}{y(x, \lambda)} - \frac{\tilde{y}'(x, d_1)}{\tilde{y}(x, d_1)} \right] +
+(d_1 - \lambda) \left[ \frac{z'(x, \lambda)}{z(x, \lambda)} - \frac{\tilde{z}'(x, d_2)}{\tilde{z}(x, d_2)} \right] \left[ \frac{y'(x, \lambda)}{y(x, \lambda)} - \frac{\tilde{y}'(x, d_2)}{\tilde{y}(x, d_2)} \right] = -(\lambda - d_2)(d_2 - d_1)(d_1 - \lambda).

In the end, we will give a simple picture of how the Darboux transformations work and explain why we are able to represent analytically \(z(x, \lambda)\) for every \(\lambda\) through \(y(x, \lambda)\) and two specific solutions \(y(x, d_s)\), \(s = 1, 2\) (type 1).

First, for the Schrödinger equation case \((u_1 = v_1 = 0)\) we have

\[K y(x, \lambda) \equiv \left[ \partial_x + \frac{\tilde{y}'(x, d)}{\tilde{y}(x, d)} \right] \left[ \partial_x - \frac{\tilde{y}'(x, d)}{\tilde{y}(x, d)} \right] y(x, \lambda) = (d^2 - \lambda^2) y(x, \lambda)\]

Now, if we apply \(\left[ \partial_x - \frac{\tilde{y}'(x, d)}{\tilde{y}(x, d)} \right]\) to both sides we obtain

\[\tilde{K} z(x, \lambda) \equiv \left[ \partial_x - \frac{\tilde{y}'(x, d)}{\tilde{y}(x, d)} \right] \left[ \partial_x + \frac{\tilde{y}'(x, d)}{\tilde{y}(x, d)} \right] z(x, \lambda) = (d^2 - \lambda^2) z(x, \lambda)\]

where

\[z(x, \lambda) = \left[ \partial_x - \frac{\tilde{y}'(x, d)}{\tilde{y}(x, d)} \right] y(x, \lambda)\]

This is a Schrödinger equation for \(z(x, \lambda)\) with a different potential.

In the general case the picture is similar. Then

\[K = b \left[ \partial_x - \left( \frac{b'}{b} - T \right) \right] b (\partial_x - T) \equiv b^2 \left( \partial_{xx} - U \right) - b^2 \left( T' + T^2 - U \right)\]

where

\[T = \left( \frac{\lambda - \lambda_2}{\lambda_1 - \lambda_2} \right) \frac{\tilde{y}'(x, d_1)}{\tilde{y}(x, d_1)} + \left( \frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1} \right) \frac{\tilde{y}'(x, d_2)}{\tilde{y}(x, d_2)}\]

\[z(x, \lambda) = b (\partial_x - T) y(x, \lambda)\]
5. GT between a Quadratic Pencil and AKNS System.

Consider the AKNS system (3.1). We use the same approach as above. Let the transformation \( \tau \) have the form

\[
\begin{align*}
  n_1(x, \lambda) &= a(x, \lambda)y(x, \lambda) + b(x, \lambda)y'(x, \lambda) \\
  n_2(x, \lambda) &= c(x, \lambda)y(x, \lambda) + d(x, \lambda)y'(x, \lambda)
\end{align*}
\]  

(3.33)

where \( y(x, \lambda) \) is the general solution of the quadratic pencil (3.2). Then we substitute (3.33) into (3.1) and obtain due to (3.2):

\[
\begin{align*}
  (b' + a + i\lambda\delta - q\delta)y' + [a' + i\lambda a - qc + b(u_0 + u_1 \lambda - \lambda^2)]y &= 0 \\
  (d' + c - i\lambda d - rb)y' + [c' - i\lambda c - ra + d(u_0 + u_1 \lambda - \lambda^2)]y &= 0
\end{align*}
\]

As before we conclude that the coefficients in front of \( y, y' \) are zeros. So:

\[
a = q\delta - b' - i\lambda b, \quad c = rb - d' + i\lambda d
\]

and hence for the other two coefficients we get

\[
\begin{align*}
  -b'' - qrb + q'd + 2qd' - 2i\lambda b' + b(u_0 + u_1 \lambda) &= 0 \\
  -d'' - qrd + rb' + 2rb' + 2i\lambda d' + d(u_0 + u_1 \lambda) &= 0
\end{align*}
\]  

(3.34)

We exclude \( u_0 + u_1 \lambda \) and then integrate with respect to \( x \):

\[

\begin{align*}
  (-b'' - 2i\lambda b' + q'd + 2qd')d + (d'' - 2i\lambda d' - rb' - 2rb')b &= 0 \\
  (d'b - db') - 2i\lambda bd + qd^2 - rb^2 &= p \equiv W(\tau y_1, \tau y_2)/W(y_1, y_2), \quad p = p(\lambda).
\end{align*}
\]  

(3.35)

Again, (3.35) allows for an introduction of a single function \( T(x, \lambda) \):

\[
T' - 2i\lambda T - r + qT^2 = \frac{p}{b^2}, \quad T(x, \lambda) = \frac{d(x, \lambda)}{b(x, \lambda)}.
\]

Then (3.34) yields

\[
u_0 + u_1 \lambda = \left( \frac{b'}{b} \right)' + \left( \frac{b'}{b} \right)^2 + 2i\lambda \frac{b'}{b} + qr - (q'T + 2qT' + 2qT \frac{b'}{b}).
\]
This can be written as
\[ q(T' - 2i\lambda T - r + qT^2) = p \frac{q}{b^2} = \left( \frac{b'}{b} - qT + i\lambda \right)' + \left( \frac{b'}{b} - qT + i\lambda \right)^2 -(u_0 + u_1\lambda - \lambda^2). \] (3.36)

The equations (3.33) become
\[
\begin{pmatrix}
  n_1 \\
  n_2 
\end{pmatrix} = b \begin{pmatrix}
  - \left( \frac{b'}{b} - qT + i\lambda \right) & 1 \\
  - \frac{p}{b^2} - T & \left( \frac{b'}{b} - qT + i\lambda \right)
\end{pmatrix} \begin{pmatrix}
  y \\
  y'
\end{pmatrix} \tag{3.37}
\]
and the inverted formulae are
\[
\begin{pmatrix}
  y \\
  y'
\end{pmatrix} = -\frac{b}{p} \begin{pmatrix}
  - \left( \frac{b'}{b} - qT + i\lambda \right) & 1 \\
  - \frac{p}{b^2} - T & \left( \frac{b'}{b} - qT + i\lambda \right)
\end{pmatrix} \begin{pmatrix}
  n_1 \\
  n_2
\end{pmatrix}. \tag{3.38}
\]

Equations (3.37) can be written in the form
\[
\begin{bmatrix}
  n_2 - n_1 - T \\
  n_1 + (qT - i\lambda)
\end{bmatrix}
\begin{bmatrix}
  y' \\
  y
\end{bmatrix} - \left( \frac{b'}{b} - qT + i\lambda \right) = -\frac{p}{b^2}
\]
or:
\[
\begin{bmatrix}
  n_1' - (qT - i\lambda) \\
  n_1 + (qT - i\lambda)
\end{bmatrix}
\begin{bmatrix}
  y' \\
  y
\end{bmatrix} - \left( \frac{b'}{b} - qT + i\lambda \right) = -\frac{p}{b^2} q.
\]

Again we choose
\[ T(x, \lambda) = \alpha(x) + \beta(x)\lambda, \quad \frac{b'}{b} \equiv -\frac{1}{2} \frac{(T' - 2i\lambda T - r + qT^2)'}{T' - 2i\lambda T - r + qT^2} \text{ independent of } \lambda \]
which due to (3.36) and
\[ T' - 2i\lambda T - r + qT^2 = (\alpha' - r + q\alpha^2) + (\beta' - 2i\alpha + 2q\alpha\beta)\lambda + (q\beta^2 - 2i\beta)\lambda^2 \]
implies:
\[ q\alpha' - qr + q^2\alpha^2 = c_0h(x) = \left( \frac{b'}{b} - q\alpha \right)' + \left( \frac{b'}{b} - q\alpha \right)^2 - u_0 \]
\[ q\beta' + 2q\alpha(q\beta - i) = c_1h(x) = -(q\beta - i)' - 2(q\beta - i) \left( \frac{b'}{b} - q\alpha \right) - u_1 \tag{3.39} \]
\[ 1 + (q\beta - i)^2 = c_2h(x) = 1 + (q\beta - i)^2 \]
where we could choose:
\[ b = b(x) = [q(x)/h(x)]^{1/2}, \quad p = c_0 + c_1\lambda + c_2\lambda^2, \quad h(x) \neq 0. \tag{3.40} \]

Just as in the case of transformations between QP andQP the left-hand side of (3.39) allows us to find \( \alpha, \beta \) as functions of \( q, r, c_0, c_1, c_2 \) when \( q(x) \neq 0 \) and \( (c_0, c_1, c_2) \neq (0, 0, 0) \).
THEOREM 3.2. For any choice of the constants \( c_0, c_1, c_2 \) in (3.39) such that \((c_0, c_1, c_2) \neq (0, 0, 0)\) and any potentials \( q(x) \neq 0, r(x) \) there exist functions \( \alpha(x), \beta(x) \) satisfying (3.39) and defining a transformation \( \tau: (n_1, n_2) \to y \) (see (3.38)) between the linear problems (3.1) and (3.2) where the potentials \( u_0, u_1 \) are determined by (3.39).

LEMMA 3.7. If \( d \in \mathbb{C} \) is a zero of the quadratic equation

\[
c_0 + c_1 \lambda + c_2 \lambda^2 = 0
\]

then

\[
T(x, d) = \frac{\bar{n}_2(x, d)}{\bar{n}_1(x, d)}, \quad \frac{b'(x)}{b(x)} - q(x)T(x, d) + id = \frac{\bar{y}'(x, d)}{\bar{y}(x, d)}
\]

or:

\[
q(x)T(x, d) - id = \frac{1}{2} \left( \frac{q'(x)}{q(x)} - \frac{h'(x)}{h(x)} \right) - \psi(x) = \rho(x)
\]

where

\[
\psi(x) = \frac{\bar{y}'(x, d)}{\bar{y}(x, d)}, \quad \rho(x) = \frac{\bar{n}'_2(x, d)}{\bar{n}_1(x, d)}.
\]

TYPE 1. Let (3.41) have two different zeros \( d_{1,2} \). Then

\[
q(\alpha + \beta d_{1,2}) - id_{1,2} = \frac{1}{2} \left( \frac{q'}{q} - \frac{h'}{h} \right) - \psi_{1,2} = \rho_{1,2}.
\]

So the transformation is defined by:

\[
\alpha(x) = \frac{1}{d_1 - d_2} \left[ -d_2 \frac{\bar{n}_2(x, d_1)}{\bar{n}_1(x, d_1)} + d_1 \frac{\bar{n}_2(x, d_2)}{\bar{n}_1(x, d_2)} \right], \quad \beta(x) = \frac{1}{d_1 - d_2} \left[ \frac{\bar{n}_2(x, d_1)}{\bar{n}_1(x, d_1)} - \frac{\bar{n}_2(x, d_2)}{\bar{n}_1(x, d_2)} \right]
\]

and has the form

\[
qr + \frac{q'}{q} (\rho_s + id_s) - d_s^2 = \rho'_s + \rho_s^2, \quad s = 1, 2 \quad \rho_1 - \rho_2 = -(\psi_1 - \psi_2)
\]

\[
u_0 + u_1 d_s - d_s^2 = \psi'_s + \psi_s^2, \quad s = 1, 2 \quad \rho_1 + \rho_2 - \frac{q'}{q} = -\frac{h'}{h} - (\psi_1 + \psi_2)
\]

where due to (3.39) and the equality \( q\beta - i = (\rho_1 - \rho_2)/(d_1 - d_2) \) we have

\[
\frac{h'}{h} = \frac{[c_2 h(x)]'}{c_2 h(x)} = \frac{2(\rho_1 - \rho_2)(\rho_1 - \rho_2)'}{(d_1 - d_2)^2 + (\rho_1 - \rho_2)^2} \equiv \frac{2(\psi_1 - \psi_2)(\psi_1 - \psi_2)'}{(d_1 - d_2)^2 + (\psi_1 - \psi_2)^2}.
\]
If we use the transformation in the direction QP → AKNS then we first recover \( q'/q \) through

\[
i \frac{q'}{q} (d_1 - d_2) = (\rho_1 - \rho_2)' + (d_1^2 - d_2^2) + (\rho_1 + \rho_2 - \frac{q'}{q})(\rho_1 - \rho_2)
\]

which is the difference of the first pair of formulae in (3.43).

**TYPE 2.** Let (3.41) have a multiple zero \( \lambda = d \). Then the transformation can be obtained from type 1 when \( d_1 \to d \), \( d_2 \to d \):

\[
\alpha(x) = \theta - \dot{\theta} d, \quad \beta(x) = \dot{\theta} \quad \text{where} \quad \theta = \frac{\tilde{n}_2(x, d)}{\tilde{n}_1(x, d)}, \quad \dot{\theta} = \frac{\partial}{\partial \lambda} \left[ \frac{\tilde{n}_2(x, \lambda)}{\tilde{n}_1(x, \lambda)} \right]_{\lambda = d}.
\]

The transformation becomes:

\[
q r + \frac{q'}{q} (\rho + id) - d^2 = \rho' + \rho^2 \quad \quad \dot{\rho} = -\dot{\psi} \quad \quad \psi' + \psi^2 = u_0 + u_1d - d^2
\]

\[
\frac{q'}{q} (\dot{\rho} + i) - 2d = \dot{\rho}' + 2\rho \dot{\psi} \quad \quad \rho - \frac{q'}{2q} = -\frac{h'}{2h} - \psi \quad \quad \dot{\psi}' + 2\dot{\psi} \dot{\psi} = u_1 - 2d
\]

with

\[
\frac{h'}{h} = \frac{2\dot{\rho} \dot{\psi}'}{1 + \dot{\rho}^2} \equiv \frac{2\dot{\psi} \dot{\psi}'}{1 + \dot{\psi}^2}.
\]

**TYPE 3.** Let \( c_2 = 0 \), \( c_1 \neq 0 \). Then there exists one solution \( d = -c_0/c_1 \) to (3.41) and so (3.42) holds. Also, (3.39) yields \( q\beta - i = \pm i \) and (after excluding \( \alpha \) from the second equation using (3.42)):

\[
-(i \pm i) \frac{q'}{q} \pm 2i \dot{\rho} + 2d = c_1 h(x) = \mp 2i \psi + 2d - u_1. \quad (3.44)
\]

So the transformation becomes

\[
\frac{q'}{q} = \frac{q'}{q} + \mp 2i \left( \rho - \frac{q'}{2q} \right) + i \frac{q'}{q} = \pm 2i \psi + u_1 \quad \quad u_1 = u_1
\]

\[
q r + \frac{q'}{q} (\rho + id) - d^2 = \rho' + \rho^2 \quad \quad \rho - \frac{q'}{2q} = -\frac{h'}{2h} - \psi \quad \quad \psi' + \psi^2 = u_0 + u_1d - d^2
\]

with

\[
\frac{h'}{h} = \frac{(c_1 h(x))'}{c_1 h(x)} = \frac{(u_1 \pm 2i \psi - 2d)'}{u_1 \pm 2i \psi - 2d} = \frac{(i \pm i) \frac{q}{q} + 2i \rho - 2d'}{(i \pm i) \frac{q}{q} + 2i \rho - 2d}.
\]
Example. In the case $\beta = 0$ we get a generalization of the Jaulent-Miodek transformation.

Indeed,

$$T = \alpha = \frac{\rho + id}{q} \equiv \frac{\tilde{n}_2(x, d)}{\tilde{n}_1(x, d)} \quad \text{(from lemma 3.7)},$$

$$\frac{p}{b^2} = (\lambda - d)\frac{\tilde{c}_1\tilde{\eta}(x)}{q(x)} = -2i\alpha(\lambda - d) \quad \text{(from (3.39))},$$

so $b = \alpha^{-1/2}$ (up to a constant in $x$),

$$\frac{b'}{b} - qT = -\frac{\alpha'}{2\alpha} - q\alpha = -\left[\frac{(\alpha\tilde{n}_1^2)' + i\alpha\tilde{c}_1}{2(\alpha\tilde{n}_1^2)} + id\right] \equiv -\left[\frac{(\tilde{n}_1\tilde{n}_2)' + i\alpha\tilde{c}_1}{2(\tilde{n}_1\tilde{n}_2)} + id\right]$$

and therefore (3.38) becomes

$$\begin{pmatrix} y \\ y' \end{pmatrix} = \alpha^{-1/2} \begin{pmatrix} \alpha & \frac{\alpha}{\tilde{n}_1\tilde{n}_2} - i\frac{1}{2(\tilde{n}_1\tilde{n}_2)} - id \\ 2i\alpha(\lambda - d) + \alpha \left( i\lambda - \frac{(\tilde{n}_1\tilde{n}_2)'}{2(\tilde{n}_1\tilde{n}_2)} - id \right) \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}.$$ 

Also,

$$\frac{\alpha'}{\alpha} = -2\frac{b'}{b} = -2(\rho + \psi) = iu_1 \quad \text{(from (3.44))}$$

which is the above-mentioned transformation (at $d = 0$):

$$\frac{\alpha'}{\alpha} = \frac{r}{\alpha} - q\alpha + 2id \quad \frac{\alpha'}{\alpha} = iu_1 \quad u_1 = u_1 \quad (\tilde{n}_1\tilde{n}_2)' = \frac{r}{\alpha} + q\alpha \quad (\tilde{n}_1\tilde{n}_2)' = -2\psi \quad \psi' + \psi^2 = u_0 + u_1 d - d^2.$$ 

**TYPE 4.** Let $c_2 = c_1 = 0$, $c_0 \neq 0$. Then from (3.39) we obtain:

$$\beta(x) = \frac{i \pm i}{q}, \quad \alpha(x) = \frac{1 \pm 1}{q^2}, \quad \frac{b'}{b} - q\alpha = \pm \frac{i}{2}u_1, \quad -q\beta' = u_1 \pm \frac{2i}{b}$$

which, together with (3.39), leads to the transformation

$$qr - \frac{1 \pm 1}{2} \left( \frac{q'}{q} \right)' = u_0 \mp \frac{i}{2}u_1' + \frac{u_1^2}{4} \quad \text{with} \quad \frac{h'}{h} = \frac{(u_0 \mp \frac{i}{2}u_1' + \frac{u_1^2}{4})'}{uv_0 \mp \frac{i}{2}v_1 + \frac{u_1^2}{4}} = \frac{\left[ qr - \frac{1 \pm 1}{2} \left( \frac{q'}{q} \right) \right]'}{qr - \frac{1 \pm 1}{2} \left( \frac{x'}{x} \right)}.$$

In order to derive Darboux transformations for AKNS we need to use the following transformation:

\[ y(x, \lambda) = N_1(x, \lambda)[Q(x)]^{-1/2} \]  

(3.45)

where \((N_1, N_2)\) satisfy the AKNS system

\[ N'_1(x, \lambda) + i\lambda N_1(x, \lambda) = Q(x)N_2(x, \lambda) \]

\[ N'_2(x, \lambda) - i\lambda N_2(x, \lambda) = R(x)N_1(x, \lambda). \]

(3.46)

Then \(y(x, \lambda)\) is a solution of a quadratic pencil with potentials:

\[ u_0 = QR - \left( \frac{Q'}{2Q} \right)' + \left( \frac{Q'}{2Q} \right)^2, \quad u_1 = i\frac{Q'}{Q}. \]

The transformation (3.45) can be written in an invertible form:

\[
\begin{pmatrix}
  y' \\
  y
\end{pmatrix}
= Q^{-1/2}
\begin{pmatrix}
  1 & 0 \\
  -i\lambda + \frac{Q'}{2Q} & Q
\end{pmatrix}
\begin{pmatrix}
  N_1 \\
  N_2
\end{pmatrix}
\iff
\begin{pmatrix}
  N_1 \\
  N_2
\end{pmatrix}
= Q^{-1/2}
\begin{pmatrix}
  i\lambda + \frac{Q'}{2Q} & 0 \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  y' \\
  y
\end{pmatrix}
\]

(3.47)

and is of type 4.

Now we will derive AKNS\leftrightarrow AKNS transformations out of the QP\leftrightarrow AKNS ones by substituting (3.47) into (3.37):

\[
\begin{pmatrix}
  n_1 \\
  n_2
\end{pmatrix}
= bQ^{-1/2}
\begin{pmatrix}
  0 & 0 \\
  -\frac{2i}{b} & 1
\end{pmatrix}
\begin{pmatrix}
  1 & 0 \\
  \left( \frac{1}{b} - qT + 2i\lambda + \frac{Q'}{2Q} \right) & Q
\end{pmatrix}
\begin{pmatrix}
  N_1 \\
  N_2
\end{pmatrix}
\]  

(3.48)

In order to present (3.36) in a symmetrical form we need to make the substitution

\[
\frac{b'}{b} - qT + i\lambda = -\frac{Q'}{2Q} + QK - i\lambda
\]

(3.49)

where \(K(x, \lambda)\) is obviously of the form \(K = A(x) + B(x)\lambda\). Then:

\[
-\left( \frac{Q'}{2Q} \right)' + (QK)' + \left( QK - \frac{Q'}{2Q} - i\lambda \right)^2
- \left[ QR - \left( \frac{Q'}{2Q} \right)' + \left( \frac{Q'}{2Q} \right)^2 + i\lambda \frac{Q'}{Q} - \lambda^2 \right] =
\]

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\[ = Q(K' + QK^2 - 2i\lambda K - R). \]

So (3.36) takes the form

\[ q(T' - 2i\lambda T - r + qT^2) = p(\lambda)h(x) = Q(K' - 2i\lambda K - R + QK^2). \quad p = c_0 + c_1\lambda + c_2\lambda^2 \quad \text{(3.50)} \]

where \( T \) and \( K \) are connected through (see (3.40), (3.49)):

\[ \frac{h'}{2h} = \left( \frac{q'}{2q} + i\lambda - qT \right) + \left( \frac{Q'}{2Q} - i\lambda - QK \right). \quad \text{(3.51)} \]

For (3.48) we get due to (3.40), (3.49):

\[ \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \left( \begin{array}{c} h \\ q \end{array} \right)^{-1/2} \begin{pmatrix} -K & 1 \\ -p\frac{h}{qQ} - TK & T \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} \quad \text{(3.52)} \]

\[ \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = -\frac{1}{p} \left( \begin{array}{c} h \\ q \end{array} \right)^{-1/2} \begin{pmatrix} -T & 1 \\ -p\frac{h}{qQ} - TK & K \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}. \quad \text{(3.53)} \]

Again, (3.52), (3.53) can be written as

\[ \begin{bmatrix} n_2(x, \lambda) \\ n_1(x, \lambda) - T \end{bmatrix} \begin{bmatrix} N_2(x, \lambda) \\ N_1(x, \lambda) - K \end{bmatrix} = -p(\lambda) \frac{h(x)}{q(x)Q(x)}. \]

**Theorem 3.3.** For any choice of the constants \( c_0, c_1, c_2 \) in (3.50) such that \((c_0, c_1, c_2) \neq (0, 0, 0)\) and any potentials \( q(x) \neq 0, r(x) \) there exist functions \( \alpha(x), \beta(x) \) satisfying the left-hand side of

\[ q(\alpha' - r + q\alpha^2) = c_0 h(x) = Q(A' - R + QA^2) \]

\[ q\beta' + 2q\alpha(\beta - i) = c_1 h(x) = QB' + 2QA(QB - i) \] (see (3.50)). \quad \text{(3.54)}

\[ 1 + (q\beta - i)^2 = c_2 h(x) = 1 + (QB - i)^2 \]

The functions \( \alpha(x), \beta(x) \) define a transformation \( \tau: (n_1, n_2) \rightarrow (N_1, N_2) \) ((3.52), (3.53)) between the eigenfunctions of (3.1) and (3.46) where the potentials \( Q, R \) and the function \( K = A(x) + B(x)\lambda \) are determined by the right-hand side of (3.54) and

\[ \frac{h'}{2h} = \left( \frac{q'}{2q} - q\alpha \right) + \left( \frac{Q'}{2Q} - QA \right) \quad \text{(see (3.51))} \]

\[ 0 = (q\beta - i) + (QB - i) \]

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(we first substitute $A$, $B$ from (3.55) into (3.54) and then find $Q$, $R$ from (3.54)).

**Lemma 3.8.** If $d \in \mathbb{C}$ is a zero of the quadratic equation

$$c_0 + c_1 \lambda + c_2 \lambda^2 = 0 \quad (3.56)$$

then, according to (3.50),

$$T(x, d) = \frac{\tilde{n}_2(x, d)}{\tilde{n}_1(x, d)}, \quad K(x, d) = \frac{\tilde{N}_2(x, d)}{\tilde{N}_1(x, d)}.$$

Now we will derive four types of Darboux transformations for AKNS in a form similar to the one in Ref. 28 so that we can compare the results.

**Type 1.** Let (3.56) have two different zeros $d_{1,2}$. Then, by lemma 3.8,

$$\alpha(x) = \frac{-d_2 \theta_1 + d_1 \theta_2}{d_1 - d_2}, \quad \beta(x) = \frac{\theta_1 - \theta_2}{d_1 - d_2}, \quad \text{with} \quad \theta_s = \frac{\tilde{n}_2(x, d_s)}{\tilde{n}_1(x, d_s)}, \quad s = 1, 2,$$

and

$$A(x) = \frac{-d_2 \Theta_1 + d_1 \Theta_2}{d_1 - d_2}, \quad B(x) = \frac{\Theta_1 - \Theta_2}{d_1 - d_2}, \quad \text{with} \quad \Theta_s = \frac{\tilde{N}_2(x, d_s)}{\tilde{N}_1(x, d_s)}, \quad s = 1, 2.$$

We substitute $A$, $B$ from (3.55) into the second line of (3.54) and find for $Q(x)$:

$$i \frac{Q'}{Q} = -q \beta' - (q \beta)' - (q \beta - i) \left( \frac{q'}{q} - \frac{h'}{h} \right) \quad (3.57)$$

Since

$$\frac{h'}{h} = \left( \frac{(c_2 h)'}{c_2 h} \right) = \frac{[q \beta (q \beta - 2i)]'}{q \beta (q \beta - 2i)}$$

according to (3.54), we obtain for (3.57):

$$i \frac{Q'}{Q} = -q \beta' - (q \beta)' - (q \beta - i) \left[ \frac{\beta'}{\beta} - \frac{(q \beta - 2i)'}{q \beta - 2i} \right]$$

$$i \frac{Q'}{Q} = i \left[ -\frac{\beta'}{\beta} + \frac{(q \beta - 2i)'}{q \beta - 2i} \right].$$

Therefore

$$Q(x) = \frac{q(x) \beta(x) - 2i e^{2w_0}}{\beta(x)}, \quad w_0 = \text{const.} \quad (3.58)$$
The symmetric form for (3.58) is

\[
\frac{q}{Q} e^{2u_0} = \frac{q \beta}{q \beta - 2i} = -\frac{q \beta}{QB} = \frac{Q B - 2i}{QB}
\]

(see (3.55)).

This implies also:

\[
B = -\beta e^{-2u_0}.
\]

In addition we have

\[
\frac{c_2 h}{qQ} = \frac{q \beta (q \beta - 2i)}{qQ} = \frac{q \beta (-QB)}{qQ} = -\beta B = \beta^2 e^{-2u_n}
\]

(3.59)

which simplifies (3.55):

\[
q \alpha + QA = -\frac{\beta'}{\beta}.
\]

(3.60)

We will give (3.50) an even more convenient form by excluding the potentials \( q, Q \). The relations

\[
\theta_s' - 2id_s \theta_s - r + q \theta_s^2 = 0, \quad s = 1, 2
\]

imply

\[
\beta' + 2\alpha (q \beta - i) + (d_1 + d_2) \beta (q \beta - 2i) = 0,
\]

so (3.60) becomes:

\[
q \beta \left( \frac{\alpha}{\beta} \right) + QB \left( \frac{A}{B} \right) = 2 \frac{\alpha}{\beta} (q \beta - i) + (d_1 + d_2) (q \beta - 2i)
\]

or (using (3.55)):

\[
(q \beta - 2i) \left[ \frac{\alpha}{\beta} + \frac{A}{B} + (d_1 + d_2) \right] = 0 \quad \Rightarrow \quad \frac{\theta_1 + \theta_2}{\theta_1 - \theta_2} + \frac{\Theta_1 + \Theta_2}{\Theta_1 - \Theta_2} = 0
\]

(3.61)

since \( q \beta (q \beta - 2i) = c_2 h \neq 0 \).

Finally, we get the transformation

\[
\begin{align*}
\theta'_1 - 2id_1 \theta_1 - r + q \theta_1^2 &= 0 \quad -e^{2u_0} = \frac{\theta_1 - \theta_2}{\Theta_1 - \Theta_2} \quad \theta'_1 - 2id_1 \Theta_1 - R + Q \Theta_1^2 &= 0 \\
\theta'_2 - 2id_2 \theta_2 - r + q \theta_2^2 &= 0 \quad 0 = \frac{\theta_1 + \theta_2}{\theta_1 - \theta_2} + \frac{\Theta_1 + \Theta_2}{\Theta_1 - \Theta_2} \quad \theta'_2 - 2id_2 \Theta_2 - R + Q \Theta_2^2 &= 0.
\end{align*}
\]
For (3.53) we obtain through (3.59), (3.61):

$$-p\frac{h}{qQ} - TK = \beta B(\lambda - d_1)(\lambda - d_2) - (\alpha + \beta \lambda)(A + B\lambda) = -\lambda \beta B \left( d_1 + d_2 + \frac{\alpha}{\beta} + \frac{A}{B} \right) +$$

$$+ \beta B \left[ d_1 d_2 - \frac{\alpha}{\beta} \frac{A}{B} \right] = \beta B \left( d_1 + \frac{\alpha}{\beta} \right) \left( d_2 + \frac{\alpha}{\beta} \right) = \frac{B}{\beta} \theta_1 \theta_2 = -e^{-2\omega_0} \theta_1 \theta_2,$$

so (3.53) takes the form:

$$\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \text{const} \frac{e^{\omega_0}}{\beta} \begin{pmatrix} -\beta \left( \lambda + \frac{-d_2 \theta_1 + d_1 \theta_2}{\theta_1 - \theta_2} \right) & 1 \\ -e^{-2\omega_0} \theta_1 \theta_2 & B \left( \lambda - \left( d_1 + d_2 - \frac{-d_2 \theta_1 + d_1 \theta_2}{\theta_1 - \theta_2} \right) \right) \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix},$$

or:

$$\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \begin{pmatrix} -e^{\omega_0} \left( \lambda + \frac{-d_2 \theta_1 + d_1 \theta_2}{\theta_1 - \theta_2} \right) & e^{\omega_0} \frac{d_1 - d_2}{\theta_1 - \theta_2} \\ -e^{-\omega_0} \frac{(d_1 - d_2) \theta_1 \theta_2}{\theta_1 - \theta_2} & -e^{-\omega_0} \left( \lambda - \frac{d_2 \theta_1 - d_1 \theta_2}{\theta_1 - \theta_2} \right) \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}.$$

When $\omega_0 = 0$ this coincides with the formula obtained in Ref. 28.

TYPE 2. Letting $d_1 \rightarrow d$, $d_2 \rightarrow d$ yields:

$$\theta' - 2id\theta - r + q\theta^2 = 0 \quad -e^{2\omega_0} = \frac{\dot{\theta}}{\theta} \quad \Theta' - 2id\Theta - R + Q\Theta^2 = 0$$

$$\dot{\theta}' - 2i(\theta + d\theta) + 2q\theta\dot{\theta} = 0 \quad 0 = \frac{\theta}{\dot{\theta}} + \frac{\Theta}{\dot{\Theta}} \quad \dot{\theta}' - 2i(\Theta + d\dot{\Theta}) + 2Q\Theta\dot{\Theta} = 0,$$

$$\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \begin{pmatrix} -e^{\omega_0} \left( \lambda + \frac{-\dot{d} \theta}{\dot{\theta}} \right) & e^{\omega_0} \frac{\dot{d} \theta}{\dot{\theta}} \\ -e^{-\omega_0} \frac{\dot{d} \theta^2}{\dot{\theta}} & -e^{-\omega_0} \left( \lambda - \frac{\dot{d} \theta}{\dot{\theta}} \right) \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}.$$

TYPE 3. $c_2 = 0$, $c_1 \neq 0$. Then ((3.54), (3.55)):

$$(q\beta - i)^\pm i = 0 = -(QB - i)^\pm i \quad \Rightarrow \quad \beta = \frac{i \mp i}{q}, \quad B = \frac{i \pm i}{Q}. \quad (3.62)$$

Also, lemma 3.8 implies

$$\alpha + d\beta = \theta \equiv \frac{\tilde{n}_2(x, d)}{\tilde{n}_1(x, d)}, \quad A + dB = \Theta \equiv \frac{\tilde{N}_2(x, d)}{\tilde{N}_1(x, d)}, \quad d = -\frac{c_0}{c_1},$$

so

$$\alpha = \theta - id \left( \frac{1 \mp 1}{q} \right), \quad A = \Theta - id \left( \frac{1 \pm 1}{Q} \right).$$

The second line in (3.54) yields

$$q \left( \frac{i \mp i}{q} \right)' + 2iQ\alpha = c_1 h = Q \left( \frac{i \pm i}{Q} \right)' \pm 2iQA. \quad (3.63)$$
Now we substitute $q\alpha + QA$ from (3.55) to get

\[ \frac{h'}{h} = \pm \left( \frac{q'}{q} - \frac{Q'}{Q} \right) \Rightarrow c_1 h = \left( \frac{Q}{q} \right)^{\pm 1} e^{2u_0}, \quad w_0 = \text{const.} \quad (3.64) \]

Therefore the potentials transform according to

\[ (1 \mp 1) \left( 2d - i \frac{q'}{q} \right) \mp 2i q \theta = \left( \frac{q}{Q} \right)^{\pm 1} e^{2u_0} = (1 \pm 1) \left( 2d - i \frac{Q'}{Q} \right) \pm 2i Q \Theta, \]

\[ \theta' - 2i d \theta - r + q \theta^2 = 0, \quad \Theta' - 2i d \Theta - R + Q \Theta^2 = 0. \]

For the transformation of the eigenfunctions we obtain:

a) upper sign

\[ Q = \frac{\varepsilon^{2u_0}}{-2i \theta}, \quad \Theta = \frac{i}{Q} \left( i q \theta + 2d - i \frac{Q'}{Q} \right) = 2\theta e^{-2u_0} \left( i q \theta + 2d + i \frac{\theta'}{\theta} \right) = 2i \varepsilon^{-2u_0}, \]

\[ \beta = 0, \quad B = \frac{2i}{Q}, \quad \alpha = \theta, \quad A = \Theta - \frac{2id}{Q}, \quad c_1 h = \frac{e^{2u_0}}{Q^2}, \]

\[ \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \text{const.} \frac{Q}{e^{u_0}} \left( - (\lambda - d) \frac{e^{2u_0}}{Q^2} - \Theta + \frac{2i(\lambda - d)}{Q} \right) \left( 1 + \frac{2i(\lambda - d)}{Q} \right) \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \]

or:

\[ \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \left( \frac{\varepsilon^{u_0}}{2i} \right)^{\frac{1}{2}} e^{-u_0} \left( - \frac{\varepsilon^{u_0}}{2i} \right) \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} ; \]

b) lower sign

\[ 2 \left( 2d - i \frac{q'}{q} \right) + 2i q \theta = \frac{Q}{q} e^{2u_0} = -2i Q \Theta, \]

\[ \beta = \frac{2i}{q}, \quad B = 0, \quad \alpha = \theta - \frac{2id}{q}, \quad A = \Theta = -\frac{e^{2u_0}}{2i}, \quad c_1 h = \frac{e^{2u_0}}{qQ} = \frac{e^{2u_0}}{q^2}, \]

\[ \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \text{const.} \frac{Q}{e^{u_0}} \left( - (\lambda - d) \frac{e^{2u_0}}{q^2} - \left( \Theta + \frac{2i(\lambda - d)}{q} \right) \left( - \frac{e^{2u_0}}{2i} \right) e^{-u_0} \right) \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \]

or:

\[ \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \left( - \frac{[q \theta + 2i(\lambda - d)] e^{-u_0} q e^{-u_0}}{2i} \right) \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}. \quad (3.65) \]

When $e^{u_0} = -2i$ (3.55) yields the expression derived in Ref. 28.
TYPE 4. \( c_2 = c_1 = 0, c_0 \neq 0 \). Still (3.62) and (3.63) hold. Instead of (3.64) we have

\[
\frac{h'}{h} = \pm \left( \frac{q'}{q} - \frac{Q'}{Q} \right) \quad \Rightarrow \quad c_0 h = \left( \frac{q}{Q} \right)^{\pm 1} e^{2w_0}, \quad w_0 = \text{const}.
\]

Since \( c_1 = 0 \) (3.63) implies

\[
\alpha = \left( \frac{1 \mp 1}{2} \right) \frac{q'}{q}, \quad A = \left( \frac{1 \pm 1}{2} \right) \frac{Q'}{Q^2
\]

and therefore the first line in (3.54) yields

\[
-q' + \frac{1 \mp 1}{2} \left( \frac{q'}{q} \right)' = c_0 h \equiv \left( \frac{q}{Q} \right)^{\pm 1} e^{2w_0} = -QR + \frac{1 \pm 1}{2} \left( \frac{Q'}{Q} \right)'.
\]

The eigenfunctions transform in the following way:

a) upper sign

\[
-q' = \frac{q}{Q} e^{2w_0} = -QR + \left( \frac{Q'}{Q} \right)' \quad \Rightarrow \quad Q = -\frac{e^{2w_0}}{r},
\]

\[
\beta = 0, \quad B = \frac{2i}{Q}, \quad \alpha = 0, \quad A = \frac{Q'}{Q^2}, \quad \frac{c_0 h}{qQ} = \frac{e^{2w_0}}{Q^2},
\]

\[
\begin{pmatrix}
N_1 \\
N_2
\end{pmatrix}
= \text{const} \frac{Q}{e^{w_0}} \left( \begin{array}{cc}
0 & -\frac{e^{2w_0}}{r} \\
q'^2 & \frac{Q'}{Q^2} + \frac{2i\lambda}{Q}
\end{array} \right)
\begin{pmatrix}
n_1 \\
n_2
\end{pmatrix}
\]

or:

\[
\begin{pmatrix}
N_1 \\
N_2
\end{pmatrix}
= \begin{pmatrix}
0 & -\frac{e^{2w_0}}{r} \\
se^{-w_0} & (r \alpha - 2i \lambda) e^{-w_0}
\end{pmatrix}
\begin{pmatrix}
n_1 \\
n_2
\end{pmatrix}
\]

b) lower sign

\[
-q' + \left( \frac{q'}{q} \right)' = \frac{Q}{q} e^{2w_0} = -QR,
\]

\[
\beta = \frac{2i}{q}, \quad B = 0, \quad \alpha = \frac{q'}{q^2}, \quad A = 0, \quad \frac{c_0 h}{qQ} = \frac{e^{2w_0}}{q^2},
\]

\[
\begin{pmatrix}
N_1 \\
N_2
\end{pmatrix}
= \text{const} \frac{q}{e^{w_0}} \left( \begin{array}{cc}
-\frac{q'}{q^2} & 1 \\
-\frac{e^{2w_0}}{q} & 0
\end{array} \right)
\begin{pmatrix}
n_1 \\
n_2
\end{pmatrix}
\]

or:

\[
\begin{pmatrix}
N_1 \\
N_2
\end{pmatrix}
= \begin{pmatrix}
-\left( \frac{q'}{q} + 2i \lambda \right) e^{-w_0} & q e^{-w_0} \\
-\frac{e^{2w_0}}{q} & 0
\end{pmatrix}
\begin{pmatrix}
n_1 \\
n_2
\end{pmatrix}
\]
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VITA

Russi Georgiev Yordanov was born on May 23, 1969 in Sofia, Bulgaria. In 1985 he received his M.S. and B.S. degrees in Mathematics, Sofia University, Bulgaria. His diploma work was in the area of solitons and was done under the supervision of Dr. Evgeni Kh. Khristov.

After graduation he worked for eight months as a mathematician/programmer in the Central Institute for Computers, Sofia.

In April 1986 he enrolled as a graduate student in the Department of Mathematics at Sofia University, specialty Analytical Mechanics, under the same advisor, Dr. E. Khristov.

In the fall of 1987 R. Yordanov became a student at Virginia Polytechnic Institute and State University, in the Ph.D. program of Mathematical Physics. In December 1990 he received M.S. degree in Physics, and in May 1992 he graduated with a Ph.D. in Mathematics from the University.

At the time of graduation he had 10 published articles (2 of them in press) and one submitted for publication.

Here is a list of his more significant achievements and awards:

— First place in the Annual Competition Sponsored by "Matematika", the leading Bulgarian mathematical journal for high school students (1976/77 & 1977/78).

— Represented Bulgaria at the XIX and XX International Mathematical Olympiads for High School Students, Belgrade, Yugoslavia (1977) and Bucharest, Romania (1978). Third Prize each year.

— First place at the National Mathematical Olympiad for High School Students, Sofia, Bulgaria (1978).

— First place at the National Mathematical Olympiad for University Students, Sofia, Bulgaria (1983).

— Society of Sigma Xi, National Research Award for a (funded) proposal entitled "Analysis of Resonances for Wigner-von Neumann Potential in the Half-line Dirac Case" (1990).

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