INCORPORATING DEFAULT RISK INTO THE BLACK-SCHOLES MODEL
USING STOCHASTIC BARRIER OPTION PRICING THEORY

by

Don R. Rich

Dissertation Submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirement for the degree of
Doctor of Philosophy in Finance
Department of Finance, Insurance, and Business Law

APPROVED:

Don M. Chance, Finance, Chairman

George E. Morgan, Finance
Marion R. Reynolds, Statistics
Gregory B. Kadlec, Finance

Royce K. Zia, Physics
David J. Demis, Finance

December 12, 1993
Blacksburg, Virginia
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(ABSTRACT)

The valuation of many types of financial contracts and contingent claim agreements is complicated by the possibility that one party will default on their contractual obligations. This dissertation develops a general model that prices Black-Scholes options subject to intertemporal default risk using stochastic barrier option pricing theory. The explicit closed-form solution is obtained by generalizing the reflection principle to k-space to determine the appropriate transition density function. The European analytical valuation formula has a straightforward economic interpretation and preserves much of the intuitive appeal of the traditional Black-Scholes model. The hedging properties of this model are compared and contrasted with the default-free model. The model is extended to include partial recoveries. In one situation, the option holder is assumed to recover $\alpha$ (a constant) percent of the value of the writer's assets at the time of default. This version of the partial recovery option leads to an analytical valuation formula for a first passage option - an option with a random payoff at a random time.
ACKNOWLEDGEMENTS

I am very thankful for the help and support of my committee members, Don Chance, George Morgan, Marion Reynolds, Dave Denis and Greg Kadlec. I am particularly grateful to my chairman, Don Chance, for his abundant assistance and patience in helping me reach this point in my career, and to Dave Denis for his guidance throughout the dissertation. Special thanks are also due to Martin Day, Peter Carr, and Phelim Boyle for several insightful conversations.

Most of all, I am indebted to my wife for her support (morally and financially), without which, this would not have been possible. Finally, I would like to thank Gene Weber for his unending encouragement.
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CHAPTER I

Overview and Statement of the Problem

1. A Introduction

For much of the 1980's, mergers, takeovers, and restructurings dominated the academic and trade literature. Today, the profession is confronted with understanding and solving a new problem: credit risk management. At the forefront of this issue is over-the-counter (OTC) derivative security credit risk. This has originated, in part, due to the recent explosion in the popularity of OTC instruments. From 1986 to 1991 the notional value of outstanding OTC contracts has increased 790 percent. Counterparty credit exposure has even led some to consider "whether [OTC] derivatives could cause the collapse of the global financial system." Clearly, as the popularity and volume of customized derivative securities continues to grow, it becomes increasingly important for institutions to accurately assess the creditworthiness of their counterparties. "Although not yet the case, it seems likely that in the future the credit rating of a writer will be an explicit determinant of an OTC stock index option's price" (Hudson, 1991).

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1 Gastineau (1992) defines any "security or other instrument that is not traded on an organized exchange or a market that is not part of an organized exchange" to be an OTC instrument (p. 166). There are many differences between the OTC market and an organized exchange. Due to the back-up system used by the Options Clearing Corporation, exchange-traded options are effectively insured against writer default (see Cox and Rubinstein (1985, pp. 69-71)). OTC options are not guaranteed against writer default. In the OTC market there are no standardized margin requirements, periodic settlements, or clearing firm. The industry is largely self-regulated and, hence, the creditworthiness of the option writer is important.

2 Falloon (1993a, p. 72).

3 Falloon (1993b, p. 8).

Chapter I
Perhaps the future is now.

In this thesis a new approach is proposed to incorporating and quantifying the economic importance of counterparty credit risk for customized option contracts. An explicit theoretical valuation formula is developed to price options subject to intertemporal writer default risk. The model, which is referred to as the intertemporal vulnerable Black-Scholes model, has a closed-form solution and preserves much of the intuitive appeal of the traditional Black-Scholes model.

Previous attempts to incorporate default risk into an option pricing framework (e.g., Johnson and Stulz (1987) and Hull and White (1993)) have concentrated on simultaneously determining the value of the writer's option to default and the holder's option to receive partial recoveries. This approach is arguably inappropriate for the problem at hand for at least two reasons. First, it can be difficult, if not impossible, to disentangle the value of the writer's option to default from the value of the holder's option to receive recoveries (particularly in the Hull and White model). Second, little is known and, in fact, no empirical evidence exists, as to how customized options are handled in the event of default. In the proposed model a clear distinction is made between the value of the writer's default option and the holder's recovery option. This approach allows one to easily quantify the economic importance of each option. In addition, as more is learned about how customized options are handled in the event of default, adjustments can quickly be made (if needed) to the valuation of the recovery option.

The Black-Scholes model assumes there is no possibility of writer default and heretofore, no closed-form solution exists for valuing Black-Scholes options in the presence of intertemporal default risk. The primary purpose of this dissertation is to provide a systematic theory for valuing Black-Scholes options under such conditions. However, since this is the first attempt at solving this problem, the theoretical foundation is established progressively. The approach taken within is to create a new family of option pricing models, called stochastic barrier options, from

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4 This terminology is consistent with previous research; see Johnson and Stulz (1987, p. 276) and Hull and White (1994, p. 1).

5 Johnson and Stulz (1987) explore how credit risk affects the value of an option when default can only occur at the option's expiration. Hull and White (1993) examine how intertemporal credit risk affects an option's value but were unable to arrive at a closed-form solution except when simplifying assumptions were made.
which the value of an intertemporal vulnerable Black-Scholes model can eventually be determined.

I.A.1 General Description of Stochastic Barrier Options

Barrier options, as referred to in the trade literature, are the most popular exotic option and are actively traded in the over-the-counter market (Chew, 1992). Hudson (1991) notes that barrier options have recently increased in popularity, particularly with Japanese investors. Kunitomo and Ikeda (1992) argue that "these contracts achieve particular risk management functions more efficiently than alternative strategies using simple options without this feature, which may give some economic rationale for the existence of these contracts" (p. 276). To date, the family of barrier options is restricted to those options which incorporate fixed or, at most, time-dependent deterministic upper and/or lower boundaries. This definition will be enlarged to include options with stochastic barriers.

Stochastic barrier options are path dependent options in which the payoff space is dependent on some aspect of the paths taken by the underlying asset, and at least one other asset, to reach specified states. Stochastic barrier options are categorized as "in" options or "out" options. Holders of "in" options are required to pay the option premium up-front but an "in" option does not "come into existence" until certain conditions are met. For an example of an "in" option, imagine an otherwise standard Black-Scholes option (written, on say, IBM common stock) with the additional clause that it cannot be exercised at expiration unless the optioned asset price has reached some other asset price at some point during the life of the option (for example, the price of IBM must, at some point, reach the price of General Electric).

"Out" options are the antithesis of "in" options. Holders of "out" options pay the premium up-front and receive the option immediately. However, "out" options are purchased with the understanding the option will expire prematurely, without the right to exercise the option, if certain conditions are met. The vulnerable Black-Scholes model is an example of an "out" option. This is an "out" option in which the option ceases to exist (expire prematurely) when the value of the writer’s assets are substantially depressed. The specific "in" and "out"

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6 Other names for barrier options include: knock-out or knock-in options and touch options.
requirements used in general are explicitly stated when the models are developed.

If a barrier option is "alive" at expiration, it has a payoff identical to a Black-Scholes option. Accordingly, the value of a barrier option can never exceed the value of an otherwise identical unrestricted option (i.e., a Black-Scholes option). This is because "in" barrier options may never "come into existence" and "out" options may prematurely expire. For this reason, the owners of some "out" ("in") barrier options receive a prespecified non-negative cash payment - called the rebate - if the option prematurely expires (never "comes into existence"). The "out" rebate is received the instant the barrier is touched and, thus, the rebate payoff is at a random time. The "in" rebate is never paid prior to expiration because at all previous points in time there is a positive probability that the option will "come into existence".

The standard Black-Scholes model can be considered an "out" option with a fixed barrier and a zero rebate. This is because the option has zero value if the underlying asset price ever touches zero (the fixed barrier). However, the Black-Scholes model is a trivial "out" option since there is a zero probability of it being "knocked-out". In this dissertation the framework is extended to include a stochastic barrier in which the "knock-out" or "knock-in" conditions are defined by a collection of states (i.e., there is a positive probability of the option being "knocked-in" or "knocked-out"). The economic significance of stochastic barriers causal to options "appearing" or "disappearing" is an important, and previously unexplored, question. This issue has direct theoretical implications to option pricing and, in turn, the applicability of the appropriate option pricing formula to the valuation of other assets with option-like payoffs that are complicated by the presence of a stochastic boundary.

The remainder of this dissertation is organized as follows. There are eight chapters. Much of the theory underlying the intertemporal vulnerable model is a natural extension of the original Black-Scholes formulation. However, since this is the first theoretical attempt to incorporate a stochastic boundary into an otherwise standard Black-Scholes framework, a number of interim results are developed to progressively establish the foundation for the vulnerable Black-Scholes model. The literature review in Chapter 2 introduces existing theoretical option pricing models that incorporate fixed or, at most, time-dependent deterministic upper and/or lower boundaries. In Chapter 3 most of the theoretical work discussed in Chapter 2 is integrated into one generalized model. The first stochastic barrier model is introduced in Chapter 4. It is a
model that prices options with a stochastic strike price and a stochastic barrier level. Exploiting the power of linear homogeneity, it is shown how this option can be valued as a particular standard barrier option. In this light, Chapter 4’s results are seen to be a natural extension to the model developed in Chapter 3. Chapter 5 considers the valuation of options with a non-stochastic strike price but stochastic barrier level. That is, the methodology developed in Chapter 4 is expanded to allow for a fixed exercise price. As one might expect, this model is more complex than the previously mentioned models but is also believed to be applicable to a wider range of valuation problems. A number of the interim results are then assembled in Chapter 6 with the end result being the first closed-form solution for Black-Scholes options subject to intertemporal default risk. Chapter 7 briefly discusses other applications of the stochastic barrier models and a summary is offered in Chapter 8.
CHAPTER II

Literature Review

II.A Introduction

Options restricted by a continuous underlying boundary are referred to as barrier options. Barrier options are not a recent innovation; they have been traded sporadically in the OTC market since 1967. In this chapter the existing theoretical literature on barrier options is reviewed to serve as the basis from which more complex barrier options can be valued. In Chapter VI, it is then shown how extensions to the existing barrier models can be applied to the valuation of Black-Scholes options subject to intertemporal default risk.

The existing literature on barrier options, although limited, can be divided into four main categories. First, Snyder (1969) and an article in Fortune (1971) discusses their use in investment strategies. Second, Merton (1973), Cox and Rubinstein (1985), Rubinstein and Reiner (1991a), Hudson (1991), Benson and Daniel (1991), and Rich (1994) discuss the pricing and characteristics of European options written on an underlying asset whose price is restricted by a single continuous non-stochastic barrier. Kunitomo and Ikeda (1992) extend the work on non-stochastic barriers to include two boundaries (i.e., the underlying asset price is restricted by an upper and lower boundary). Finally, Yu (1992) and Carr (1993a) consider the pricing and characteristics of American options written on an underlying asset whose price is restricted by a single continuous non-stochastic barrier.

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8 Further research, not directly related to barrier option pricing theory, is reviewed in later chapters.
II.B The Trade Literature

In reviewing the early trade articles on barrier options, it is important to keep in mind that an organized options market did not exist until 1973.\(^9\) This is important because the early trade articles described the increased efficiency in which a covered position could be liquidated as one of the principle advantages associated with using barrier options.

Snyder (1969) was the first to discuss "out" barrier options, although he does not refer to them as such; he called them special options.\(^{10}\) Special options are described as being favorable to both parties. The buyer of a call option, in exchange for a lower option premium, agrees to limit the risk of the writer. Snyder suggests that option writers cover themselves with a long stock position purchased at the striking price with a stop-loss order at the barrier price.\(^{11}\) If the stock price falls, both positions are liquidated automatically. Snyder describes three "distinct advantages" associated with writing special options (over standard options) to cover a position: (1) financial loss is minimized due to the "out" provision, (2) the "out" provision reduces a writer's risk exposure and, therefore, "the writer may find the use of margin highly desirable" (p. 96), and (3) the option writer can increase his/her rate of capital turnover with less risk of subsequent losses on the option position.

In a 1971 Fortune article, the down-and-out call option\(^{12}\) is discussed as a "contract geared to the needs of sophisticated investors, managers of hedge funds" (p. 213). The down-and-out call is said to be typically written on "more volatile stocks". The writer benefits from the reduction in downside risk and the buyer benefits from a more favorable price, and the ability to purchase a larger supply of calls from a single writer (an advantage due to the "notoriously thin" options market).

\(^9\) Over-the-counter options were traded for many years before 1973 through an organization called the Put and Call Brokers and Dealers Association.

\(^{10}\) Snyder (1969) discusses only "out" options.

\(^{11}\) Snyder (1969) refers to the fixed barrier level as the expiration price.

\(^{12}\) The down-and-out option is an "out" option in which the initial underlying asset price starts above the knock-out boundary.
II.C The Non-Stochastic European Barrier Model Literature: The Single Barrier Case

A closed-form valuation solution for a European barrier option is presented by Merton (1973, pp. 175-176). He values a European down-and-out call option with zero rebate written on a non-dividend paying asset. The barrier is allowed to increase with time to expiration at a known exponential rate. Merton also shows that: (1) the European down-and-out call formula is linear homogeneous (of degree one) in the underlying asset price and the strike price, (2) an American down-and-out call on a non-dividend paying asset should not be prematurely exercised and, thus, has the same value as its European counterpart and (3) a perpetual down-and-out call option (with zero rebate) is a levered security like a standard call option; however, the barrier call price is shown to be a concave function of the underlying security price.\(^\text{13}\)

Cox and Rubinstein (1985, pp. 408-412) extend Merton’s European down-and-out call formula to include a rebate.\(^\text{14}\) If the option hits the barrier, and thus, expires immediately, the option holder receives a prespecified rebate payment. They also discuss and provide an example of how the binomial model can be modified to value European barrier options.\(^\text{15}\) Cox and Rubinstein also suggest that bonds embedded with short down-and-out call options may serve to reduce the costs associated with writing and enforcing costly bond covenants.\(^\text{16}\)

In more recent work, Rubinstein and Reiner (1991a) greatly extend and generalize barrier

\(^\text{13}\) This is in striking contrast to the standard (Black-Scholes) option price which is well-known to be a convex function of the optioned asset price. See Merton (1973) and Jarrow and Rudd (1983, p. 107).

\(^\text{14}\) Additional textbook discussions of barrier options can be found in Ingersoll (1987, pp. 369-370), Stoll and Whaley (1993, pp. 403-404) and Hull (1993, pp. 419-420). In the absence of a rebate, it should be immediately clear (for European options) that an "in" option combined with an otherwise identical "out" option must be equivalent in value to a otherwise identical standard option. See Stoll and Whaley (1993, p. 404) for a numerical example using this parity relationship.

\(^\text{15}\) For a supplemental discussion on the efficiency and shortcomings of using the binomial model in this framework, see Hudson (1991).

\(^\text{16}\) This same example is also mentioned in Black and Cox (1976, p. 355).
option pricing theory (for a fixed barrier). Their generalization allows for an underlying asset with a constant payout rate and their extension includes closed-form solutions to all possible European put and call barrier options. Rich (1994) provides many of the unpublished derivations to the Rubinstein and Reiner (1991a) framework. In addition, he shows that (1) European barrier options with a fixed barrier are linear homogeneous of degree one not only with respect to the underlying asset price and the present value of the strike price, but also for the barrier level and the rebate, (2) down-and-in call options can, under certain conditions (discussed in the next chapter), be hedged using Black-Scholes put options, and (3) the perpetual down-and-out call option is the only perpetual option that is a levered security.

II.D The Non-Stochastic European Barrier Model Literature: The Dual Barrier Case

Kunitomo and Ikeda (1992) extend the "out" non-stochastic barrier option pricing literature (discussed above) by deriving a closed-form solution for "out" options restricted by an upper and lower boundary on the underlying asset price. Prior to their work, previous studies focused on the single, fixed, or at most, time-dependent deterministic boundary model. They allow both boundaries to change exponentially (at a predetermined rate) with time to expiration. It is shown that the dual barrier "out" European option cannot be created by combining an "out" option restricted by a lower underlying boundary with an "out" option restricted by an upper underlying boundary. While their valuation solution is closed-form, it contains an infinite number of univariate standard normal cumulative distribution functions. Their numerical analysis

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17 Benson and Daniel (1991) further advance the literature by deriving the premium for foreign currency up-and-out calls.

18 There are sixteen possible European barrier options. For each option there is the case in which the barrier lies above the strike price and visa versa. There are down-and-in calls and puts, up-and-in calls and puts, and eight otherwise identical "out" options.

19 This result is also presented (independently) in Carr (1993a).

20 Recall that Merton (1973) documents that the perpetual down-and-out call is a levered security (i.e., the option's price elasticity with respect to the stock price is strictly greater than one). Rich (1994) showed that this result only holds for this option, and not for any of the other fifteen fixed barrier options.
indicates, however, that convergence of the infinite series is quite rapid.

II.E The Non-Stochastic American Barrier Model Literature: The Single Barrier Case

Carr (1993a) and Yu (1992) consider the valuation of American barrier options restricted by a single underlying continuous boundary.

Carr (1993a) provides a detailed examination of the general theory of rational barrier option pricing (much like Merton (1973) did for Black-Scholes options). He examines the boundary relationships/restrictions between European and American barrier options. The theoretical value of European barrier options and the value of the underlying asset provides a foundation from which bounds on the American option value are determined. Carr also discusses and analyzes how barrier options can combined to create barrier forward contracts.

Yu (1992) generates closed-form solutions for American options written on underlying assets that are allowed to follow various diffusion processes.\(^{21}\) He did not specifically focus on the barrier model, however, in one case an underlying absorbing boundary is superimposed on an asset’s price path. For this case, changes in the underlying asset’s price are assumed to obey geometric Brownian motion and Yu is able to obtain closed-form solutions for American and European options.

II.F Chapter Summary

This chapter provides a review of the pricing, hedging, and potential use of barrier options restricted by underlying non-stochastic barriers and contrasts them with standard Black-Scholes options. While the barrier option literature is rather limited, our understanding of the non-stochastic barrier model is becoming fairly complete. To organize and unify most of the outstanding literature, a single generalized non-stochastic barrier model is developed in Chapter III. This model encompasses all outstanding literature with the exception of the dual barrier European model (Kunitomo and Ikeda, 1992) and the American non-stochastic barrier model (Yu 1992) and Carr (1993a)). The generalized non-stochastic barrier model establishes a

\(^{21}\) While Yu (1992) explores various diffusion processes, it should be noted that Carr (1993a) focuses solely on the lognormal diffusion process.
foundation from which a previously unexplored barrier model can be examined, one in which the barrier is specified to be stochastic. In Chapter VI, it is then shown how a stochastic barrier model can be used to value Black-Scholes options subject to intertemporal default risk.
CHAPTER III

The Valuation of Options With a Non-Stochastic Strike Price and a Time-Dependent Deterministic Boundary

III.A Introduction

In this chapter the objective is to integrate much of the outstanding work on European barrier options with a single, fixed or time-dependent deterministic, upper or lower barrier into one generalized framework. Subsequently, some hedging and potential differences in use between our generalized barrier model and standard Black-Scholes options are explored. However, no applications of the model are discussed. This is because the overall goal of this chapter is to establish a general foundation for the barrier option pricing literature, from which extensions to the literature can later be made. In addition, Rich (1994) is discusses various applications of barrier options with a single, fixed or time-dependent deterministic, upper or lower barrier.

III.B The Theoretical Model

Let $W_{STD}(S_1, K, H, R, \tau)$ denote the current value of a standard European barrier option with the arguments being defined as follows:

- $S_1$ is the price of a risky asset,
- $K$ is the (constant) exercise price,
- $H$ denotes the position of the continuous barrier level,
- $R$ is the rebate, and
- $\tau=T-t$ is the expiration time, in years, of the derivative security.

$S_1(t)$ is the price of the underlying asset at time $t$ ($t \in [t_0, T]$). When referring to the initial asset price, the time parameter is suppressed (i.e., $S_1(t_0) = S_1$). The rate of return for the underlying
asset is assumed to satisfy the time homogeneous stochastic differential equation,

$$dS_1(t)/S_1 = \alpha dt + \sigma dZ_1(t),$$

where $\alpha$ is the instantaneous drift, $\sigma$ is the instantaneous standard deviation, and $dZ_1$ is a Weiner process. The time homogeneous discrete riskless rate of interest is (r-1) and (d-1) is the payout rate of $S_1$. Barrier options are normally payout unprotected.

Barrier options have two mutually exclusive sources of value, the terminal payoff and the rebate. "Out" ("in") option holders receive a predetermined rebate of $\$R$ at hit (at $T$), i.e., when $S_1(t)=H$, if the underlying asset price ever (never) breaches the continuous barrier level $H$. The binary variables $\phi$ and $\eta$ are employed to write the terminal payoff for a European barrier option as $\max(\phi S_1(T)-\phi K,0)$ if $\eta S_1(t)>\eta H$ for all $t \in [t_0,T]$, where $\phi$ is defined to be 1 if the barrier option is a call option and -1 if it is a put and $\eta$ is defined to be 1 if the underlying asset price, $S_1$, is approaching the barrier, $H$, from above and -1 if the barrier is being approached from below.22

III.B.1 The Standard Barrier Model With a Non-Stochastic Strike Price and a Non-Stochastic Barrier Level

Consider the standard European barrier option pricing model when the barrier is assumed non-stochastic (i.e., $W_{STD}(S_1,K,H,R,\tau)$). Rubinstein and Reiner (1991a) show that when $H$ and $R$ are constant, the valuation solution for options following under this heading are determined as a linear combination of six equations.23 To generalize previous research, I begin by valuing options with a barrier level and a rebate that is a known function of time to expiration.24 Let

22 The first use of these types of binary variables to summarize the results is Rubinstein and Reiner (1991a and 1991b), and in many other related papers by them.


24 One should note that a few of the results presented in this chapter are not original. Merton (1973, pp. 175-176) values a European down-and-out call option $(K > H(\tau))$ with zero rebate and a barrier that increases exponentially with time to expiration. Rich (1994) extends Merton's work to include valuation solutions for all European options with zero rebate. However, the results presented in this chapter are more general with the inclusion of the rebate and with the barrier level and the rebate being allowed to increase or decrease (exponentially) over time.
$H(\tau) = bKe^{\theta \tau}$ denote a barrier level that changes exponentially with time to expiry at rate $\theta$, where $b$ and $\theta$ are non-negative constants.\footnote{It is important to note that, for convenience, the barrier level $H(\tau)$ is written as function of the exercise price $K$. This specification is chosen because the barrier level is NOT allowed to cross (but it can touch) the strike price at any time. That is, a simple inspection of $b$, being greater than or less than 1, tells one whether $K \leq H(\tau)$ or $K > H(\tau)$. The relationship between $H(\tau)$ and $K$ must be known to determine the appropriate valuation formula, which appears in Table III.B.1.a. Therefore, it should be clear that $bK$ is any arbitrary real number (as long as the relationship between $K$ and $H(\tau)$ is known).}

Define $\xi$ to be a binary variable that takes the value of one if the barrier is an increasing function of time to expiration, and negative one if the barrier is decreasing over time. Let $R(t) = Re^{\beta \tau}$ denote a rebate amount that changes exponentially with time to expiry at rate $\beta$, where $R$ and $\beta$ are non-negative constants. Define $\xi$ to be a binary variable that takes the value of one if the rebate is an increasing function of time to expiration, and negative one if it is decreasing over time.\footnote{At times it is clearer to denote the barrier level and the rebate at time $t$ as $H(t)$ and $R(t)$, respectively, where it is understood that $H(t) = bKe^{\theta t - \xi t}$ and $R(t) = Re^{\beta t - \xi t}$.}

The closed-form valuation solutions for options following under this heading can also be written as a linear combination of six equations.

\begin{align*}
\frac{e^{-T}}{\ln(H(\tau)/S_t)} \int_{\ln(H(\tau)/S_t)} (S_t e^{Z(\tau) + (\theta - K)Z(\tau)}) dZ(T) & = \phi d_1^{-\tau} N(\phi y_1) - \phi Kr^{-\tau} N(\phi (y_1 - \sigma_1 \sqrt{\tau})). \\
\frac{e^{-T}}{\ln(H(\tau)/S_t)} \int_{\ln(H(\tau)/S_t)} (S_t e^{Z(\tau) + (\theta - K)Z(\tau)}) dZ(T) & = \phi d_1^{-\tau} N(\phi y_2) - \phi Kr^{-\tau} N(\phi (y_2 - \sigma_1 \sqrt{\tau})). \\
\frac{e^{-T}}{\ln(H(\tau)/S_t)} \int_{\ln(H(\tau)/S_t)} (S_t e^{Z(\tau) + (\theta - K)Z(\tau)}) dZ(T) & = \phi (H(\tau)/S_t)^{\gamma - 2} [S_t d_1^{-\tau} (H(\tau)/S_t)^{\gamma/2} N(\eta y_3) - Kr^{-\tau} N(\eta (y_3 - \sigma_1 \sqrt{\tau}))].
\end{align*}
\[ \eta \phi r^{-1} \int_{\ln(\theta(x)/\theta_i)}^{\eta \ln(\theta(x)/\theta_i)} (S_1 e^{2\kappa \alpha - \kappa} - \kappa)g_{\alpha\eta}(z)dz(T) \]

\[ = \phi(H(\tau)/S_i)^{2r-2}[S_1 d_1^{-1}(H(\tau)/S_i)^2 N(\eta y_4) - Kr^{-1}N(\eta (y_4 - \sigma_1 \sqrt{\tau}))]. \]  

\[ \eta R(\tau)r^{-\tau} \int_{\ln(\theta(x)/\theta_i)}^{\eta \ln(\theta(x)/\theta_i)} [f_{\alpha\eta}(z) - g_{\alpha\eta}(z)]dz(T) \]

\[ = Rr^{-\tau}[N(\eta (y_2 - \sigma_1 \sqrt{\tau})) - (H(\tau)/S_i)^{2r-2}N(\eta (y_4 - \sigma_1 \sqrt{\tau}))]. \]  

\[ \int_0^{\tau-\eta} R(\tau^*)r^{-\eta}h(\tau^*)d\tau^* \]

\[ = R(\tau)(S_i/H(\tau))[\{(H(\tau)/S_i)^{2r-2}N(\eta y_3) + (H(\tau)/S_i)^{2r-2}N(\eta (y_2^2 - 2m \sigma_1 \sqrt{\tau}))]. \]

where

\[ f_{\alpha\eta}(z) = \frac{1}{\sigma_1 \sqrt{\tau}} n \left( \frac{z - (\mu \xi \theta \tau)}{\sigma_1 \sqrt{\tau}} \right) \]

\[ g_{\alpha\eta}(z) = \exp \left( \frac{2\ln(H(\tau)/S_i)(\mu \xi \theta \tau)}{\sigma_1^2} \right) \frac{1}{\sigma_1 \sqrt{\tau}} n \left( \frac{z - 2\ln(H(\tau)/S_i) - (\mu \xi \theta \tau)}{\sigma_1 \sqrt{\tau}} \right) \]

\[ h(\tau^*) = \frac{\eta \ln(S_i H(\tau)/\nu K^2)}{\sigma_1 \sqrt{\tau^*}} n(y_2(\tau^*) - \sigma_1 \sqrt{\tau^*}). \]

\[ Z(T) = \ln([S_i(T)/S_i(H(T)/\nu K)]), \]

\[ \mu = \ln(\eta/d_1) - 0.5 \sigma_1^2, \]

\[ \gamma = (\ln(\eta/d_1) + 0.5 \sigma_1^2 - \xi \theta)/\sigma_1^2, \]

\[ y_1 = \ln(S_i/k) - (\ln(\eta/d_1) + 0.5 \sigma_1^2 - \xi \theta)/\sigma_1 \sqrt{\tau}, \]

\[ y_2 = \ln(S_i/k) + (\ln(\eta/d_1) + 0.5 \sigma_1^2 - \xi \theta)/\sigma_1 \sqrt{\tau}, \]

\[ y_3 = \ln(H(\tau)/S_i) + (\ln(\eta/d_1) + 0.5 \sigma_1^2 - \xi \theta)/\sigma_1 \sqrt{\tau}, \]

\[ y_4 = \ln(H^2(\tau)/S_i k) + (\ln(\eta/d_1) + 0.5 \sigma_1^2 - \xi \theta)/\sigma_1 \sqrt{\tau}, \]

\[ y_5 = \ln(H(\tau)/S_i) + m \sigma_1^2/\sigma_1^2, \]

\[ m = \sqrt{\ln(\eta/d_1) + 0.5 \sigma_1^2 - \xi \theta^2 + 2(\ln(\eta - \xi \beta) \sigma_1)^2}/\sigma_1^2. \]

\[ N(\cdot) \] is the univariate standard cumulative normal distribution and \( \partial N(Z)/\partial z = n(Z) = \]

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(1/\sqrt{2\pi})\exp(-z^2/2) is the normal density function. Appendix IIIA sketches the derivation of equations [3.1]-[3.6] and the densities listed in equation [3.7].

From Appendix IIIA, f(z) is seen to be the continuously compounded risk-neutral relative return density where the relative price of S_1 is determined by dividing S_1(t) by H(r). g(z) is the density for the continuously compounded risk-neutral relative return when S_1(t)/H(r) breaches the barrier level (at some point during the life of the option) and the terminal relative asset price satisfies a time T condition. Hence, since f(z) represents the probability density of Z(T) for z at time T (path independent), and g(z) depicts the probability density of Z(T) for z at T, having crossed the barrier at some t ≤ T (path dependent), the "defective density" (f(z)-g(z)) represents the probability density of Z(T) for z at T, having not crossing the barrier level for any t ≤ T. \(^{27}\) h(\tau^*) is the first passage time density; \(\tau^*\) is the first passage time. h(\tau^*) should be interpreted as the probability density of the first time the continuously compounded risk-neutral relative return breaches the barrier.

The valuation solutions for standard European barrier options with a barrier level that depends exponentially on time to expiration are reported in Table III.B.1.a. The solutions are verified from the comparative statics presented in Appendix IIIB. It should be immediately clear that the results presented in Table III.B.1.a can also be used to value (1) foreign currency barrier options, by letting \(d_o\), the payout rate, be the foreign riskfree rate, or (2) barrier futures options, by letting \(d_o\) equal \(r\). It is also easy to verify that (1) the constant barrier model (i.e., \(\theta=0\)) is a special case of equations [3.1]-[3.6], and (2) in the absence of a rebate, the down-and-out call (H(\(r\)) < K) and put (H(\(r\)) < K) formulas converge in value as H(\(r\)) goes to zero to equation [3.1], which is the Black-Scholes model.\(^ {28}\) Of further interest, observe (from Appendix IIIB) that all

\(^{27}\) For an intuitive sketch of the derivation of the "defective density" note that at time zero a solution to the partial differential equations of motion and the initial conditions will have a superposition of sources of strength (one for each barrier and a source of unit strength at the origin) along the Z(T) axis. Then, imagine placing a mirror along the time axis and locating the image of the origin; the image source is seen to be 2ln[H(\(r\))/S_0]. Finally, the term \(\exp\{2ln[H(\(r\))/S_0(\mu-\theta)]\}\) premultiplying the normal density in g(z) is calculated by setting the "defective density" equal to zero when f(z) and g(z) are evaluated at \(z=ln[H(\(r\))/S_0]\); this is a boundary condition. See Cox and Miller (1965, pp. 220-221).

\(^{28}\) A proof of these two results are found in Appendix IIIIC.
### Table III.B.1.a
Valuation Formulas for European Barrier Options With a Barrier Level Which is a Known Exponential Function of Time to Maturity

<table>
<thead>
<tr>
<th>European Option Type</th>
<th>Valuation Equation</th>
<th>$\eta^*$</th>
<th>$\phi^{**}$</th>
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<tr>
<td>Up-and-In Call ($K &gt; H(t)$)</td>
<td>[3.1]+[3.5]</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>Up-and-In Call ($K \leq H(t)$)</td>
<td>[3.2]-[3.3]+[3.4]+[3.5]</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>Up-and-In Put ($K &gt; H(t)$)</td>
<td>[3.1]-[3.2]+[3.4]+[3.5]</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>Up-and-In Put ($K \leq H(t)$)</td>
<td>[3.3]+[3.5]</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>Down-and-In Call ($K &gt; H(t)$)</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Down-and-In Call ($K \leq H(t)$)</td>
<td>[3.1]-[3.2]+[3.4]+[3.5]</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Down-and-In Put ($K &gt; H(t)$)</td>
<td>[3.2]-[3.3]+[3.4]+[3.5]</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>Down-and-In Put ($K \leq H(t)$)</td>
<td>[3.1]+[3.5]</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>Up-and-Out Call ($K &gt; H(t)$)</td>
<td>[3.6]</td>
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<td>1</td>
</tr>
<tr>
<td>Up-and-Out Call ($K \leq H(t)$)</td>
<td>[3.1]-[3.2]+[3.3]-[3.4]+[3.6]</td>
<td>-1</td>
<td>1</td>
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<tr>
<td>Up-and-Out Put ($K &gt; H(t)$)</td>
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<td>-1</td>
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<tr>
<td>Up-and-Out Put ($K \leq H(t)$)</td>
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<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>Down-and-Out Call ($K &gt; H(t)$)</td>
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<td>1</td>
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<tr>
<td>Down-and-Out Call ($K \leq H(t)$)</td>
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<td>1</td>
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<tr>
<td>Down-and-Out Put ($K &gt; H(t)$)</td>
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<td>Down-and-Out Put ($K \leq H(t)$)</td>
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<td>-1</td>
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</tbody>
</table>

* $\eta$ is defined to be 1 if the barrier is being approached from above and -1 if the barrier is being approached from below.

** $\phi$ is defined to be 1 if the barrier option is a call and -1 if the option is a put.
the results presented in Table III.B.1.a are linearly homogeneous (of degree one) with respect to the underlying asset price, the present value of the strike price, the barrier level and the rebate. This is a key result that is in the next chapter.

Some numerical values for standard barrier options are presented in Table III.B.1.b. It is interesting to compare Black-Scholes option values with constant and changing barrier option values, the former being the case of $\theta=0$. When the barrier is an increasing (decreasing) function of time to expiry, $\xi=1$ ($\xi=-1$), "in" call options and "out" put options decrease (increase) in value as the exponential growth rate of the barrier level increases. A converse result holds for "out" call options and "in" put options. Intuitively this is because the larger the exponential growth rate the lower (higher) the barrier level is at time zero. Therefore, "in" ("out") call options are less likely to "come into existence" ("cease existing"). The converse holds for "in" ("out") put options. Since $S_t < K$ (i.e., at time zero put options are out-of-the-money), as the initial barrier level falls, "in" ("out") put holders are less (more) likely to receive the rebate.

III.C Parity Results

Rich (1994) and Carr (1993a) observe that under certain conditions, standard down-and-in call options with zero rebate and a constant barrier level are equivalent in value to a certain number of Black-Scholes put options. In this section a more general result is presented. It is demonstrated that under certain conditions all European options with a constant or time-dependent continuous barrier level can be hedged with path independent options. A critical assumption in the results that follow is the requirement of zero rebate. Before proceeding, a couple of useful identities are stated.

An "in" option with zero rebate combined with an otherwise identical "out" option must be equivalent in value to a Black-Scholes option. This parity relationship is well-known to hold for puts and calls, "outs" and "ins", whether the barrier is constant or changing exponentially with time to expiry and is stated formally as

$$W_{STD:out}(S_t,K,H(\tau),\tau) + W_{STD:in}(S_t,K,H(\tau),\tau) = W_{BS}(S_t,K,\tau)$$  \[3.8\]
Table III.B.1.b
Standard Barrier and Black-Scholes Option Values*

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<tr>
<th>$\xi$</th>
<th>$\theta$</th>
<th>$bK$</th>
<th>$\tau$</th>
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<th>$C_{oa}(.)$</th>
<th>$C_{bs}(.)$</th>
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</table>

* $C_a(.)$, $C_{oa}(.)$, and $C_{bs}(.)$ denotes, respectively, call option values for "in" standard barrier options, "out" standard barrier options, and Black-Scholes options. $P_{in}(.)$, $P_{out}(.)$, and $P_{bs}(.)$ represents otherwise identical put option values. The parameter values for each of the options, unless otherwise stated, are $S_i=$60, $K=$58, $\sigma=.2$, $r=1.1$, $d_1=1.05$, and for the barrier options, $R=$2 and $\xi=0$. 

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It will also be useful to recall put-call parity from standard option pricing theory.

\[ C_{BS}(S_t, K, \tau) - P_{BS}(S_t, K, \tau) = S_t d_1^{-r\tau} - K r^{-\tau} \quad [3.9] \]

III.C.1 Barrier Option Parity: Hedging Barrier Options With Black-Scholes Options

Down-and-in call \((K > H(\tau))\) options (with \(R = \theta = 0\)) written on an underlying asset with a payout rate equal to the riskless rate are seen to be equal in value to a certain number of Black-Scholes put options. Specifically, when \(r = d_1\), \(\theta = 0\), and \(\phi = \eta = 1\), equation [3.3] is rewritten as

\[ C_{STD-D}(S_t, K, bK, \tau : b \geq 1, r = d_1) = (1/b) P_{BS}(S_t, b^2 K, \tau : r = d_1) \quad [3.10] \]

where \(H = bK\) (when \(\theta = 0\)) or \(b = H/K\).

The generality of this result is restricted by the requirement that \(r = d_1\); this requirement is not be satisfied for many underlying assets. However, since a forward price can be considered a security with a payout rate equal to the riskless rate, equation [3.10] should be of interest to holders of options on forwards. Equations [3.8], [3.9], and [3.10] suggest that when \(r = d_1\) (and \(\theta = 0\)) down-and-out \((K > H(\tau))\) call options with a constant barrier level can also be hedged synthetically from a portfolio of standard put options.

\[ C_{STD-DC}(S_t, K, bK, \tau : r = d_1, b > 1) = C_{BS}(S_t, K, \tau) - (1/b) P_{BS}(S_t, b^2 K, \tau) \quad [3.11] \]

Equations [3.10] and [3.11] apply equally, of course, to foreign currency barrier options when the foreign riskless rate is equivalent to the domestic riskless rate (and the rebate is zero).

A more practical implication of the above results is found when the barrier is allowed to change exponentially over time (and certain other conditions are met).

Recall from Appendix IIIA that when the barrier changes exponentially with time to expiry, a change of variables is required to derive equations [3.1]-[3.6]. The change of variables specified the optioned asset price as the relative price \([S_t(t)/H(\tau)]\) of the underlying asset. ²⁹ With this in mind, consider equation [3.3] when \(\theta \geq 0\) and \(\ln(r/d_1) = \xi \theta\).

²⁹ Appendix IIIA and IIIC establish that when the barrier is constant, the optioned asset is the underlying asset. This is opposed to the time-dependent barrier model when the optioned asset price has to be transformed into the relative underlying asset price.
Analogous to equations [3.10] and [3.11] it is seen that

$$C_{STD,DO}(S_1, K, H(t), \tau : K \geq H(t), \ln(r/d_t) = \xi \theta) = \frac{1}{b} P_{BS}(S_1, b^2 K, \tau)$$

$$C_{STD,DO}(S_1, K, H(t), \tau : K \geq H(t), \ln(r/d_t) = \xi \theta) = C_{BS}(S_1, K, \tau) - \left(\frac{1}{b}\right) P_{BS}(S_1, b^2 K, \tau)$$

[3.12]

where $H(t) = b K e^{\theta t}$. In fact, equations [3.10] and [3.11] are simply corollaries of [3.12].

Equation [3.12] says that investors can always hedge down-and-in and down-and-out call options (with $K \geq H(t)$) with standard options if they are willing to allow the barrier level to change over time at a rate equal to the continuous domestic riskless rate minus the continuous dividend yield (or foreign riskless rate).

It is worth reiterating that [3.10] and [3.11], as well as, [3.12] transpire because the optioned asset has zero drift. However, in [3.10] and [3.11] the zero drift requirement implies $\ln(r) = \ln(d_t)$, whereas in [3.12] zero drift implies $\ln(r/d_t) = \xi \theta$. It should be clear that these are two quite different requirements. The former is quite restrictive and unlikely to hold in general (except for options on forwards where it holds by definition). The latter is also rather restrictive but it does say that regardless of the relationship between the payout rate (or foreign riskless rate) and the domestic riskless rate, a speculator can always statically hedge the barrier options listed [3.12] by choosing the appropriate growth rate for the barrier.

Mathematically, it is obvious from equation [3.3] that down-and-in call options with $K \geq H(t)$ are equivalent in value to a portfolio of put options when $\ln(r/d_t) = \xi \theta$ (as is done in [3.12]). A more scientific explanation for this finding is now offered.

Merton (1973) is the first to note that the Black-Scholes option pricing model is linearly homogeneous (of degree one) with respect to the optioned asset price and the strike price. Thus, note

$$C_{BS}(S_1, K, \tau) = K C_{BS}(S_1/K, 1, \tau) = K S_1 C_{BS}(1/K, 1/S_1, \tau).$$

It can readily be confirmed (Grabbe, 1983) that

$$K S_1 C_{BS}(1/K, 1/S_1, \tau) = P_{BS}(S_1, K, \tau),$$

and since

$$P_{BS}(S_1, K, \tau) = (K/S_1) P_{BS}(S_1, S^2/K, \tau),$$

it is concluded that:
\[ C_{BS}(S_t,K,\tau) = (K/S_t) \, P_{BS}(S_t,S_{t+}/K,\tau). \]

Now, if it is noted that at hit, time \( \tau^* \), an "in" call option is equivalent in value to a standard Black-Scholes call option (i.e., \( C_{STD-IN}(H,K,T-\tau^*) = C_{BS}(H,K,T-\tau^*) \) \( (H(\tau^*) = S(\tau^*)) \)), one can write
\[ C_{BS}(H,K,T-\tau^*) = C_{STD-IN}(H,K,T-\tau^*) = (K/H) \, P_{BS}(H,H^2/K,T-\tau^*). \]

Thus, prior to hit, in the absence of arbitrage it must be true that
\[ C_{STD-IN}(S_t,K,\tau) = (K/H) \, P_{BS}(S_t,H^2/K,\tau), \]
which agrees with our finding in equation [3.12]. Note that this finding is solely attributable to the linear homogeneity of the Black-Scholes option pricing model; knowledge of the terminal asset distribution or any characteristics of its moments (e.g., time homogeneous parameters) is not required. Before exploring the economic significance of this result in more detail, note the following.

The ability to hedge standard barrier options with path independent options, when the underlying asset has zero drift, is not limited to "down" call \( (K > H(\tau)) \) options. After a little manipulation, a like result is shown below for all of the options listed in Table III.B.1.b (when \( R = 0 \)). To this end, denote \( W_{BS-C-O-N}(\cdot) \) as a path independent binary option that pays out a predetermined amount of cash (the exercise price) at expiration if \( \phi S(T) \geq \phi K \) \( (H(T) = K) \) and zero otherwise.\(^{30}\) From equation [3.1] the value of such an option can quickly be determined as:
\[ W_{BS-C-O-N}(S_t,K,H(\tau),\tau) = K e^{-\tau N(\phi(y_1-\sigma\sqrt{\tau}})). \]

Equations [3.1]-[3.4] is solved in accordance with the combinations recorded in Table III.B.1.a to obtain
\[ C_{STD:U1}(S_t,K,H(\tau),\tau : K \leq H(\tau), \ln(r/d_t) = \xi \theta) = (1/b)C_{BS}(S_t,b^2K,\tau) + (1-(1/b))[C_{BS}(S_t,bK,\tau) + 2C_{BS-C-O-N}(S_t,bK,\tau)] \]  
\[ C_{STD:U0}(S_t,K,H(\tau),\tau : K \geq H(\tau), \ln(r/d_t) = \xi \theta) = - (1/b)C_{BS}(S_t,b^2K,\tau) + C_{BS}(S_t,K,\tau) - (1-(1/b))[C_{BS}(S_t,bK,\tau) + 2C_{BS-C-O-N}(S_t,bK,\tau)] \]

\(^{30}\) For a further discussion on binary options, see Rubinstein and Reiner (1991b). Rubinstein and Reiner refer to \( W_{BS-C-O-N}(\cdot) \) as a cash-or-nothing call or put (depending on whether \( \phi = 1 \) (call) or \( \phi = -1 \) (put)).
\[ C_{\text{STD: DO}}(S_1, K, H(t), \tau : K \leq H(t), \ln(\tau/d) = \xi \theta) = (1/b)[C_{BS}(S_1, bK, \tau) - P_{BS}(S_1, b^2K, \tau)] + (1 - (1/b))[C_{BS}(S_1, bK, \tau) - 2C_{BS: C-O-N}(S_1, bK, \tau)] \]

\[ C_{\text{STD: DO}}(S_1, K, H(t), \tau : K \leq H(t), \ln(\tau/d) = \xi \theta) = C_{BS}(S_1, K, \tau) \]

\[ (1/b)[P_{BS}(S_1, b^2K, \tau) - C_{BS}(S_1, bK, \tau)] \]

\[ - (1 - (1/b))[C_{BS}(S_1, bK, \tau) + 2C_{BS: C-O-N}(S_1, bK, \tau)] \]

\[ P_{\text{STD: UO}}(S_1, K, H(t), \tau : K \leq H(t), \ln(\tau/d) = \xi \theta) = (1/b)[P_{BS}(S_1, b^2K, \tau) - C_{BS}(S_1, bK, \tau)] + (1 - (1/b))[P_{BS}(S_1, bK, \tau) - 2P_{BS: C-O-N}(S_1, bK, \tau)] \]

\[ P_{\text{STD: UO}}(S_1, K, H(t), \tau : K \leq H(t), \ln(\tau/d) = \xi \theta) = P_{BS}(S_1, K, \tau) \]

\[ + (1/b)[C_{BS}(S_1, b^2K, \tau) - P_{BS}(S_1, bK, \tau)] \]

\[ - (1 - (1/b))[P_{BS}(S_1, bK, \tau) - 2P_{BS: C-O-N}(S_1, bK, \tau)] \]

\[ P_{\text{STD: DO}}(S_1, K, H(t), \tau : K \leq H(t), \ln(\tau/d) = \xi \theta) = (1/b)P_{BS}(S_1, b^2K, \tau) \]

\[ + (1 - (1/b))[P_{BS}(S_1, bK, \tau) - 2P_{BS: C-O-N}(S_1, bK, \tau)] \]

\[ P_{\text{STD: UO}}(S_1, K, H(t), \tau : K \leq H(t), \ln(\tau/d) = \xi \theta) = (1/b)P_{BS}(S_1, b^2K, \tau) \]

\[ + (1 - (1/b))[P_{BS}(S_1, bK, \tau) - 2P_{BS: C-O-N}(S_1, bK, \tau)] \]

\[ P_{\text{STD: UO}}(S_1, K, H(t), \tau : K \leq H(t), \ln(\tau/d) = \xi \theta) = (1/b)C_{BS}(S_1, b^2K, \tau) \]

\[ P_{\text{STD: UO}}(S_1, K, H(t), \tau : K \leq H(t), \ln(\tau/d) = \xi \theta) = P_{BS}(S_1, K, \tau) \]

\[ - (1/b)C_{BS}(S_1, b^2K, \tau) \]

\[ C_{\text{STD: UO}}(S_1, K, H(t), \tau : K \leq H(t), \ln(\tau/d) = \xi \theta) = C_{BS}(S_1, K, \tau) \]

\[ = (K/S_1(t_j))P_{BS}(S_1(t_j), S_1(K)/K, \tau) \]

\[ C_{\text{STD: UO}}(S_1, K, H(t), \tau : K \leq H(t), \ln(\tau/d) = \xi \theta) = 0 \]

Chapter III
\[ P_{\text{STD-Do}}(S_1, K, H(t), \tau : K \leq H(t), \ln(r/d_1) = \xi \theta) = P_{\text{Bd}}(S_1, K, \tau) \]
\[ = (K/S_1(t_e)) C_{\text{Bd}}(S_1(S_1^2(t_e)/K), \tau) \]
\[ P_{\text{STD-Do}}(S_1, K, H(t), \tau : K \leq H(t), \ln(r/d_1) = \xi \theta) = 0 \]

Equations [3.12]-[3.19] illustrate that when the underlying asset has zero drift, barrier options are hedged with path independent options. This result has two very important potential implications. First, and most importantly, it implies that barrier option price risk can, at times, be hedged using a static (as opposed to a dynamic) replicating portfolio strategy. To see this consider our finding in equation [3.12]:
\[ C_{\text{STD-Do}}(S_1, K, \tau) = (K/H) \ P_{\text{Bd}}(S_1, H^2/K, \tau). \]
A writer of \( C_{\text{STD-Do}}(S_1, K, \tau) \) can perfectly hedge his/her position by using the proceeds from the transaction to purchase \( (K/H) \) Black-Scholes put options written on \( S_1 \) and struck at \( (H^2/K) \). If the "in" option never comes into existence, it and the put expire worthless. If the "in" option touches the barrier, it immediately becomes a standard Black-Scholes call option which is identical in value to the portfolio of put options; hence, the value of the writer's portfolio at hit is zero.

The ability to avoid a dynamic hedging strategy is by no means trivial; it is of most value when delta-hedging barrier options near the barrier. To see this, note that the delta for barrier options is discontinuous at the barrier. Thus, when dynamically hedging near the barrier, unsatisfactory delta-hedgers are quite common.\(^{31}\)

Second, as an alternative to the derivations listed in Appendix IIIA, Cox and Rubinstein (1985) show that equations similar to [3.1]-[3.6] are derived as the limiting result of the Cox-Ross-Rubinstein (1979) binomial model. It is also well-known that many traders like the intuition and the flexibility offered by the binomial model. However, Hudson (1991) observes a potential problem when using the binomial model to value the barrier options listed in Table III.B.1.a. "...as the spot price approaches the knock-out boundary the accuracy of the binomial process (compared to the closed-form solution) becomes significantly lower...and the number of iterations

\(^{31}\) Carr (1993a) notes that the risk of hedging barrier options near the barrier is akin to hedging at-the-money Black-Scholes options just prior to expiration.
required for acceptable accuracy becomes very intense" (italicized term added). Hudson's finding suggests that if the binomial model is to be used to value barrier options, accuracy and computational efficiency may be improved (when the underlying has zero drift) by using the binomial model to value each of the path independent options as opposed to the barrier option. This is because the zero drift requirement allows one to transform the partial differential equation from a boundary value problem to an initial value problem.

### III.D Options With Random Maturity Dates

"Option problems that as yet have no known solution are ... options with random maturity dates" (Copeland and Weston (1988, p. 282)). Consider the problem of valuing an option with a random maturity date. Our attention is restricted to contracts in which the stopping decision is given exogenously by contract specifications.

Standard European "out" options are considered to have a random expiration date that is specified exogenously but not a random maturity date. This is seen by noting that the time at which an "out" option will expire is uncertain. In the event of premature expiration, however, the option becomes null and void and the holder is not entitled to exercise on this date. In contrast, I wish to allow a European barrier option holder to prematurely exercise the option if premature expiration occurs. Premature expiration will again be specified exogenously by the contractual provision that the derivative security immediately matures when the underlying asset price breaches a predetermined barrier level. Observe that this type of an option is quite different

32 See Cox and Rubinstein (1985, p. 409) for a discussion on how the binomial model is used to value European barrier options with a constant barrier level. Generalization of their approach to barrier options with a barrier level that changes exponentially with time to expiration is quite straightforward.

33 An exogenously specified stopping decision is defined here, similar to Black and Cox (1976), as a boundary on the underlying asset's price that is prespecified throughout the life of the derivative security. As an example of an endogenously determined stopping decision, consider the American put option. At every instant in time there is a positive probability of prematurely exercising the option. The critical asset value that triggers early exercise is a function of the strike price, interest rate, volatility, and time to expiry, though not the current asset price.
from an American option that can be exercised at any time based on the holder’s discretion. The optimal stopping problem for an American option is an example of an endogenously determined maturity date.

The type of option described above is not as difficult to value as one might initially think. Let \( W_{\text{out}}^*(S_t(t), K, H(t), \tau) \) represent the value of an option with the following payoffs:

\[
W_{\text{out}}^*(S_t(t), K, H(t)) = \begin{cases} 
\max(0, \phi S_t(t) - \phi K) & \text{if } \eta S_t(t) > \eta H(t) \forall t \\
\max(0, \phi S_t(t^*) - \phi K) & \text{if } \eta S_t(t^*) \leq \eta H(t) 
\end{cases}
\]  

where \( T \) is the nominal expiration date, \( t^* \) (a random variable) is the first time at which \( \eta S_t(t) \leq \eta H(t) \), and \( S_t, K, \) and \( H(\tau) \) are as previously defined. This problem initially appears to be unmanageable because the time at which the barrier is first hit and the value of \( S_t \) at this point in time each appear to be random variables. Closer inspection will reveal, however, that only the first passage time is a random variable. This is because at hit it is known that the value of the underlying asset must be equal to \( H(\tau) \). At time zero it is also known whether \( H(\tau) \) is greater than or less than \( K \) at hit. Thus, a rational holder will prematurely exercise this option at hit if and only if \( S_t(t^*) = H(t^*) > K \). The exercise value at this point in time is \( \max(bK e^{\tau(T-t^*)} - K, 0) \).

It is immediately concluded that the value of this option is equivalent to the value of a standard "out" option (see Table III.B.1.a) with two rebates. Such an option entitles a holder to receive a rebate of \( R(\tau) = H(\tau) \) (i.e., \( R = bK \) and \( e^{\tau r} = e^{\tau r} \) in equation [3.6]) and relinquish a rebate worth \( R(\tau) = K \) (i.e., \( R(\tau) = R = K (\beta = 0) \)) if \( H(\tau) > K \). Conversely, if \( H(\tau) \leq K \) the value of this option with a random maturity date is equivalent in value to the value of a standard "out" option with a rebate of 0.

III.E Chapter Summary

In this chapter a analytical valuation formula for European options with a non-stochastic strike price and a time-dependent deterministic continuous boundary and rebate is developed. This model integrates much of the outstanding research into one generalized framework. The formulation allows us to demonstrate that under certain circumstances, barrier options can be hedged with path independent options. Lastly, it is shown how the constant barrier model can be viewed as an option with a random maturity date.
While some previously unpublished results are uncovered, the overall objective of this chapter has been to consolidate and generalize previous research and lay the initial foundation from which advancements to the literature can later be made.

In the next two chapters the focus is on options with stochastic barriers. In Chapter IV a stochastic barrier option is discussed that is closely tied to the model developed in this chapter. In contrast, in Chapter V a stochastic barrier model is examined that is quite different from that presented here (but it will still be seen as a natural extension to the results developed in Chapter IV). Results from Chapters IV and V are then combined (in Chapter VI) to assist in the development of a general solution to the problem of valuing Black-Scholes options subject to intertemporal default risk.
III.F Appendix IIIA: Derivation of the Standard Barrier Model

In this appendix the European barrier option solutions are derived for the case of when the barrier level and the rebate are an exponential function of time to maturity.

Black and Scholes (1973) and Merton (1973) have shown that for an option whose price is dependent on one underlying state variable, $S$, the value of the option, $W_{STD}(\cdot)$, can be shown to be the solution to

$$
.5 \sigma^2 S^2 \frac{\partial^2 W_{STD}(\cdot)}{\partial S^2} + \ln \left( \frac{r}{d_1} \right) S \frac{\partial W_{STD}(\cdot)}{\partial S} - \frac{\partial W_{STD}(\cdot)}{\partial t} = \ln(\tau) W_{STD}(\cdot)
$$

subject to the standard boundary conditions. For barrier options the partial differential equation listed in [3.21] is defined over all $\eta S_1(t) > \eta H(t)$ $t \in [t_0, T]$ subject to the boundary conditions

$$
W_{STD}(S_1(T), K, H(T), R(T), 0) = \max(\phi S_1(T) - \phi K, 0)
$$

if $\eta S_1(t) > \eta H(t)$ $\forall t$,

$$
W_{STD,in}(S_1(T), K, H(T), R(T), 0) = R \text{ (at } T)
$$

if $\eta S_1(t) \leq \eta H(t)$ for some $t$,

$$
W_{STD,ext}(H(t^*), K, H(t^*), R(t^*), T-t^*) = R e^{-\beta (T-t^*)} \text{ (at } t^*)
$$

if $\eta S_1(t^*) \leq \eta H(t^*)$ and $\eta S_1(t) > \eta H(t)$ $\forall (T-t^*) \leq (T-t)$.

where $t^*$ is, again, the first time the barrier is violated.

When the barrier level is exogenously specified to be constant or changing exponentially with time to expiry this becomes a boundary condition; but such a barrier does not preclude the construction of an instantaneously riskless hedge that can be continuously maintained. This is why barrier options have to satisfy the same partial differential equation as standard Black-Scholes options but with different boundary conditions.

Since equation [3.21] does not contain any preference-dependent terms, one can proceed with preference-free valuation just as Rubinstein and Reiner (1991a) did with the constant barrier model. Following Rubinstein and Reiner, Rich (1994) has provided a detailed examination of the preference-free derivation of European barrier options with a constant barrier level. Rich's derivations are reproduced in a condensed form for options with a barrier level that changes exponentially with time to expiry. For a sketch of what follows, note that the constant barrier model is derived in continuous time; hence, only slight modifications are required to generalize
these results to include a barrier that depends exponentially on time to expiry (i.e., a change in variables is performed so the underlying asset price is specified as the relative price \([S_t(t)/H(t)]\) of \(S_t\).

III.F.1 Using the Reflection Principle When the Barrier Changes Exponentially With Time

The reflection principle is an effective way to evaluate complex probabilities when the barrier is constant and continuous over time. The major stipulations necessary to employ the reflection principle are that (1) the (arithmetic) Brownian motion must have zero drift and (2) the barrier be constant over the relevant parameter space. Therefore, it would first appear that the reflection principle cannot be used when the barrier is an exponential function of time to expiration because it changes over time. This is not the case however, and this is shown after considering a change in variables.

Recall that the barrier level is defined as \(H(t) = bKe^{-\theta(t)}\) (\(\tau = T-t\)) and changes in the underlying asset price are assumed to be governed by the stochastic differential equation: \(dS_t(s)/S_t = \alpha_t dt + \sigma_t dz_t\) (\(t \in [t_0, T]\)). The distributional properties of the relative price variable \(Y(t) = S_t(t)/H(t)\) are now explored. Notice that \(Y(t)\) is a function of \(S_t(t)\) and \(H(t)\) so by Itô’s lemma

\[
dY(t) = \frac{\partial Y(t)}{\partial S_t(t)} dS_t(t) + \frac{1}{2} \frac{\partial^2 Y(t)}{\partial S_t(t)^2} [dS_t(t)]^2 + \frac{\partial Y(t)}{\partial H(t)} dH(t) + \frac{1}{2} \frac{\partial^2 Y(t)}{\partial H(t)^2} [dH(t)]^2 \tag{3.23}
\]


\[35\] The requirement of zero drift is of less importance because by first considering the zero drift case, generalization is fairly straightforward using the change of measure theorem (Girsanov’s Theorem), see Harrison (1985, pp. 9-10).
where

\[
\frac{\partial Y(t)}{\partial S_1(t)} = \frac{1}{H(t)}, \quad \frac{\partial^2 Y(t)}{\partial S_1^2(t)} = 0, \quad \frac{\partial Y(t)}{\partial H(t)} = \frac{S_1(t)}{H(t)}, \quad \frac{\partial^2 Y(t)}{\partial H^2(t)} = \frac{2S_1(t)}{H^2(t)}
\]

[3.24]

\[
dH(t) = \frac{\partial H(t)}{\partial \tau} d\tau = -H(t) \xi \theta d\tau, \quad (dH(t))^2 = H^2(t) \xi^2 \theta^2 (d\tau)^2 = 0
\]

so

\[
dY(t) = \frac{S_1}{H(t)}(\alpha_1 - \xi \theta) dt + \frac{S_1}{H(t)} \sigma_1 dz_1
\]

[3.25]

\[
dY(t)/Y = (\alpha_1 - \xi \theta) dt + \sigma_1 dz_1 \quad (\text{note: } d\tau = -dt)
\]

In other words, changes in \(Y(t)\) also follows a time homogeneous Geometric Brownian motion process. Converting this process into arithmetic Brownian motion, Ito's lemma implies \(\ln(Y(t)) = \ln(S_1(t)/H(t))\) follows \(d(\ln(Y(t))) = (\alpha_1 - \xi \theta - 0.5 \sigma_1^2) dt + \sigma_1 dz_1\). That is, \(\ln(S_1(t)/H(t))\) is a normally distributed random variable with instantaneous mean \(\ln(S_1(t)/H(t)) + (\alpha_1 - \xi \theta - 0.5 \sigma_1^2) t\) and instantaneous variance \(\sigma_1^2 t\). Hence, the reflection principle can again be used by defining the continuously compounded risk-neutral relative return to be \(Z(t) = [\ln(S_1(t)/H(t))\ln(H/S_i)]\). Operationally, this simply means that when the barrier is an exponential function of time, one must work with relative prices \((S_1(t)/H(t))\) as opposed to actually prices, but once this change of variables is performed the valuation process is exactly the same manner as in Rich (1994). Next, a condensed version of Rich's derivations for the case in which the barrier is allowed to change exponentially with time to expiration is reproduced.

### III.F.2 Derivation of the Densities

In this section I consider the derivation of the three densities that are required to value barrier options.

The density of the continuously compounded risk-neutral relative return:
\[ Pr(Z(T) > z) = N \left( \frac{-z + (\mu - \xi \theta) \tau}{\sigma_1 \sqrt{\tau}} \right) = 1 - N \left( \frac{z - (\mu - \xi \theta) \tau}{\sigma_1 \sqrt{\tau}} \right) \]
\[ \Rightarrow f_{Z_{\tau,T}}(z) = \frac{1}{\sigma_1 \sqrt{\tau}} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{(z - (\mu - \xi \theta) \tau)^2}{2(\sigma_1 \sqrt{\tau})^2} \right) \]  

\[ Pr(Z(T) < z) = N \left( \frac{z - (\mu - \xi \theta) \tau}{\sigma_1 \sqrt{\tau}} \right) \Rightarrow f_{Z_{\tau,T}}(z) = \frac{1}{\sigma_1 \sqrt{\tau}} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{(z - (\mu - \xi \theta) \tau)^2}{2(\sigma_1 \sqrt{\tau})^2} \right) \]  

where all the variables are defined in the text of this chapter and in equation [3.7].

\( f(z) \), although slightly disguised, is the density used in the derivation of standard options.\(^{36} \)

For the second density, one has to evaluate complex probabilities of the form:

\[ Pr(Y(T) > y, \inf Y(t) < 1) \]
\[ Pr(Y(T) < y, \sup Y(t) > 1) \]

where \( \inf \) and \( \sup \) are defined, respectively, as the infimum and the supremum of the process. The reflection principle is applied for this task.\(^{37} \) Hence, below the continuously compounded risk-neutral relative return density when the underlying relative asset price breaches the barrier level (at some point during the life of the option) and the terminal asset price satisfies the time T condition is derived.

\(^{36} \) The density \( f(z) \) could have been derived without regard to the barrier. That is, \( f(z) \) could have been derived in terms of actual prices instead of relative prices. However, from footnote 6 one can see a definite advantage to stating \( f(z) \) in terms of relative prices.

\(^{37} \) See Karlin (1968, pp. 276-277).
Pr(Y(T)>y, \inf Y(t)<1) = \Pr(Y(T)<2-\gamma) = \Pr(Z(T)>z-2\ln(H(\tau)/S_0))
= \exp\left(\frac{2(\mu-\xi \theta)\ln(H(\tau)/S_0)}{\sigma_1^2}\right) \frac{1}{\sigma_1 \sqrt{\tau}} \exp\left(\frac{z-2\ln(H(\tau)/S_0)-(\mu-\xi \theta)\tau}{\sigma_1 \sqrt{\tau}}\right)

\Rightarrow g_{Z(T)}(z) = \frac{\partial [1-\Pr(Z(T)>z-2\ln(H(\tau)/S_0))]}{\partial \tau}
= \exp\left(\frac{2(\mu-\xi \theta)\ln(H(\tau)/S_0)}{\sigma_1^2}\right) \frac{1}{\sigma_1 \sqrt{\tau}} \exp\left(\frac{z-2\ln(H(\tau)/S_0)-(\mu-\xi \theta)\tau}{\sigma_1 \sqrt{\tau}}\right)

Pr(Y(T)<y, \sup Y(t)>1) = \Pr(Z(T)<z-2\ln(H(\tau)/S_0))
= \exp\left(\frac{2(\mu-\xi \theta)\ln(H(\tau)/S_0)}{\sigma_1^2}\right) \frac{1}{\sigma_1 \sqrt{\tau}} \exp\left(\frac{z-2\ln(H(\tau)/S_0)-(\mu-\xi \theta)\tau}{\sigma_1 \sqrt{\tau}}\right)

\Rightarrow g_{Z(T)}(z) = \exp\left(\frac{2(\mu-\xi \theta)\ln(H(\tau)/S_0)}{\sigma_1^2}\right) \frac{1}{\sigma_1 \sqrt{\tau}} \exp\left(\frac{z-2\ln(H(\tau)/S_0)-(\mu-\xi \theta)\tau}{\sigma_1 \sqrt{\tau}}\right)

Ingersoll (1987, p. 369) defines the density [f(z)-g(z)] as the defective density function. Since f(z) represents the probability density of Z(T) being at z at time T (path independent), and g(z) depicts the probability density of crossing the barrier and being at z at T (path dependent), the defective density represents the probability density of not crossing the barrier and being at z at T.

In addition to the two previously stated densities, when valuing "out" options the first passage time density will also be needed.\(^{38}\) "Out" option holders receive the rebate the first time the barrier is touched, whereas, holders of "in" options will never receive the rebate prior to time T. Thus, the time at which "out" option holders receive the rebate is uncertain. Therefore, to find the current value of the discounted expected rebate associated with "out" options, one has to evaluate the random time elapsed before the barrier is first breached.

\(^{38}\) Actually, this is not a new density. As shown, the first passage time density is derived by differentiating the integral of [f(z)+g(z)] with respect to \(\tau\).

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Define \( \tau^* = t^* - t_w \), a random variable, to be the random time elapsed until the barrier is first breached. Since the rebate is received only if the barrier is first breached before (or at) time \( T \), one must focus on the probability that \( \tau^* \) is less than or equal to \( (T - t_w) \). Some previous results can be used by restating this probability as: \( \text{Pr}(\tau^* \leq T - t_w) = 1 - \text{Pr}(\tau^* > T - t_w) \). That is, \( \text{Pr}(\tau^* > T - t_w) = \text{Pr}(S(T) > H(T), \inf S(t) > H(t)) = \text{Pr}(Y(T) > 1), \inf Y(t) > 1) \) when the barrier is approached from above, and \( \text{Pr}(\tau^* > T - t_w) = \text{Pr}(S(T) < H(T), \sup S(t) < H(t)) = \text{Pr}(Y(T) < 1), \sup Y(t) < 1) \) when the barrier is approached from below. To derive the first passage time densities, assume that the \( \text{Pr}(\tau^* \leq \infty) = 1 \). This ensures that \( \tau^* \) is a "proper" random variable.

The first passage time density when the barrier is being approached from below can be determined as

\[ \text{density} \]

\[ \text{density} \]

---

\(^{39}\) See Gillespie (1992, p. 175).
\[ h_1(\tau^*) = \frac{\partial \left[ 1 - Pr(S_1(T) > H(T), \inf S_1(t) < H(t)) \right]}{\partial \tau^*} \]
\[ = \frac{\partial \left[ 1 - \{ Pr(S_1(T) > H(t)) - Pr(S_1(T) > H(T), \inf S_1(t) < H(t)) \} \right]}{\partial \tau^*} \]
\[ = \frac{\partial Pr(S_1(T) > H(T), \inf S_1(t) = H(t))}{\partial \tau^*} + \frac{\partial Pr(S_1(T) < H(T))}{\partial \tau^*} \]
\[ = (H(\tau)/S_1)^{2(\mu - \xi \theta)\phi_1^2} \left( \frac{\partial N\left( \ln(H(\tau)/S_1) + (\mu - \xi \theta)\tau^* \right)}{\sigma_1 \sqrt{\tau^*}} \right) + \left( \frac{-\ln(S_1/H(\tau)) - (\mu - \xi \theta)\tau^*}{\sigma_1 \sqrt{\tau^*}} \right) \]
\[ = (H(\tau)/S_1)^{2(\mu - \xi \theta)\phi_1^2} \left[ \frac{\partial N(y_4 - \sigma_1 \sqrt{\tau^*})}{\partial \tau^*} + \frac{\partial N(-y_2 - \sigma_1 \sqrt{\tau^*})}{\partial \tau^*} \right] \]

\[ = \frac{\partial N(y_4 - \sigma_1 \sqrt{\tau^*})}{\partial \tau^*} \frac{\partial (y_4 - \sigma_1 \sqrt{\tau^*})}{\partial \tau^*} \frac{\partial N(-y_2 - \sigma_1 \sqrt{\tau^*})}{\partial \tau^*} \frac{\partial (-y_2 - \sigma_1 \sqrt{\tau^*})}{\partial \tau^*} \]
\[ = (H(\tau)/S_1)^{2(\mu - \xi \theta)\phi_1^2} \left[ \frac{\partial N(y_4 - \sigma_1 \sqrt{\tau^*})}{\partial \tau^*} - \frac{\partial N(-y_2 - \sigma_1 \sqrt{\tau^*})}{\partial \tau^*} \right] \]
\[ = n(y_2 - \sigma_1 \sqrt{\tau^*}) \left[ \frac{\ln(S_1/H(\tau)) - 2\xi \theta \tau}{\sigma_1 \sqrt{\tau^*}} \right] n(y_2 - \sigma_1 \sqrt{\tau^*}) \]
\[ = n(y_2 - \sigma_1 \sqrt{\tau^*}) \left[ \frac{\ln(S_1/H(\tau)) - 2\xi \theta \tau}{\sigma_1 \sqrt{\tau^*}} \right] n(y_2 - \sigma_1 \sqrt{\tau^*}) \]

where \( H(T) = bKe^{-\xi \theta (T-T)} = bK \) and \( H(\tau) = bKe^{-\xi \theta \tau} \) for any \( t \in [0, T) \).

For future reference, it is useful to highlight an interim step in this derivation. That is,
\[ h_1(\tau^*) = \frac{\partial Pr(S_1(T) > H(T), \inf S_1(t) < H(t))}{\partial \tau^*} + \frac{\partial Pr(S_1(T) < H(T))}{\partial \tau^*} \]

\[ \Rightarrow \int_0^{\tau^*} h_1(\tau^*)d\tau^* = Pr(S_1(T) > H(T), \inf S_1(t) < H(t)) + Pr(S_1(T) < H(T)) \]

This identity is useful when deriving the present value of the rebate for "out" options when the barrier is being approached from above.

When the barrier is being approached from below, the first passage time probability density can be written as

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\[ h_2(\tau^*) = \frac{\partial[1 - \text{Pr}(S_1(t) < H(T), \sup S_1(t) < H(t))]}{\partial \tau^*} \]
\[ = \frac{\partial}{\partial \tau^*} \left( 1 - \left( \text{Pr}(S_1(T) < H(T)) - \text{Pr}(S_1(t) < H(t), \sup S_1(t) > H(t)) \right) \right) \]
\[ = \frac{\partial \text{Pr}(S_1(T) < H(T), \sup S_1(t) > H(t))}{\partial \tau^*} + \frac{\partial \text{Pr}(S_1(T) > H(T))}{\partial \tau^*} \]
\[ = (H(T)/\sigma_1^2) \left[ \frac{\partial N(-y_4 - \sigma_1 \sqrt{\tau^*})}{\partial \tau^*} + \frac{\partial N(y_2 - \sigma_1 \sqrt{\tau^*})}{\partial \tau^*} \right] \]
\[ = n(y_2 - \sigma_1 \sqrt{\tau^*}) \left[ - \left( \frac{\partial N(-y_4 - \sigma_1 \sqrt{\tau^*})}{\partial \tau^*} - \frac{\partial N(y_2 - \sigma_1 \sqrt{\tau^*})}{\partial \tau^*} \right) \right] \]
\[ = - \frac{\ln(S_1 H(\tau)/(bK)^2)}{\sigma_1 \sqrt{\tau^*}^3} n(y_2 - \sigma_1 \sqrt{\tau^*}) = - h_2(\tau^*) \] [3.31]

For future reference, it will, again, be useful to note an interim step in the above derivation:

\[ h_2(\tau^*) = \frac{\partial \text{Pr}(S_1(T) > H(T), \inf S_1(t) < H(t))}{\partial \tau^*} + \frac{\partial \text{Pr}(S_1(T) > H(T))}{\partial \tau^*} \] [3.32]

\[ \Rightarrow \int_0^{\tau^*} h_2(\tau^*) d\tau^* = \text{Pr}(S_1(T) < H(T), \sup S_1(t) > H(t)) + \text{Pr}(S_1(T) > H(T)) \]

This identity is useful when deriving the present value of the rebate for "out" options when the barrier is being approached from below.

Notice, by utilizing the binary variable \( \eta \), the need for two first passage time density functions can be avoided. Namely,

\[ h(\tau^*) = \frac{\eta}{\sigma_1 \sqrt{\tau^*}^3} \frac{\ln(S_1 H(\tau)/(bK)^2)}{\frac{\partial N(-y_4 - \sigma_1 \sqrt{\tau^*})}{\partial \tau^*} - \frac{\partial N(y_2 - \sigma_1 \sqrt{\tau^*})}{\partial \tau^*}} n(y_2 - \sigma_1 \sqrt{\tau^*}) \] [3.33]

III.F.3 Valuing the Non-Rebate Payoff of European Barrier Options

In this section the valuation of European barrier options is commenced by determining...
the non-rebate portion of the option’s value. The current value of each barrier option can then be fully determined by combining the non-rebate value of the option (derived below) with the corresponding rebate value (derived in the next section of this appendix). The complete valuation solutions are reported in Table III.B.1.a.

III.F.3.a Up-and-In European Call Options (K > H)

\[
    r^{-T} E(S_1(T) - K) \Pr(S_1(T) > K) \\
    = r^{-T} b K E[Y(t) -(1/b)] \Pr(Y(T) > (1/b)) \\
    = r^{-T} b K E[e^{r T + X(T) - t} -(1/b)] \Pr(Y(T) > (1/b)) \\
    = r^{-T} E(S_1 e^{Z(T) + t \sigma_1} - K) \Pr(Z(T) > \ln(H(t)/S_1 b)) \\
    = r^{-T} \int_{\ln(H(t)/S_1 b)}^{\infty} (S_1 e^{Z(T) + t \sigma_1} - K) g_{Z(T)}(z) dZ(T) \\
    = S_1 d_{1}^{*} N(y_1) - r^{-T} K N(y_1 - \sigma_1 \sqrt{T}) \\
\]

III.F.3.b Up-and-In European Call Options (K < H)

\[
    r^{-T} E(S_1(T) - K) [\Pr(S_1(T) > H(T)) \\
    + \Pr(S_1(T) < H(T), \sup S_1(T) > H(t)) - \Pr(S_1(T) < K, \sup S_1(T) > H(t))] \\
    = r^{-T} E(S_1 e^{Z(T) + t \sigma_1} - K) [\Pr(Z(T) > \ln(H(t)/S_1)) + \Pr(Z(T) < \ln(H(t)/S_1)) \\
    \sup Z(T) > \ln(H(t)/S_1)] - \Pr(Z(T) < \ln(H(t)/S_1), \sup Z(T) > \ln(H(t)/S_1)] \\
    = r^{-T} \int_{\ln(H(t)/S_1)}^{\infty} (S_1 e^{Z(T) + t \sigma_1} - K) g_{Z(T)}(z) dZ(T) \\
    + r^{-T} (H(t)/S_1)^{2 \gamma - 2} \int_{\ln(H(t)/S_1)}^{\infty} (S_1 e^{Z(T) + t \sigma_1} - K) g_{Z(T)}(z) dZ(T) \\
    - r^{-T} (H(t)/S_1)^{2 \gamma - 2} \int_{\ln(H(t)/S_1)}^{\infty} (S_1 e^{Z(T) + t \sigma_1} - K) g_{Z(T)}(z) dZ(T) \\
    = S_1 d_{1}^{*} N(y_2) - r^{-T} K N(y_2 - \sigma_1 \sqrt{T}) \\
    + S_1 d_{1}^{*} (H(t)/S_1)^{2 \gamma - 2} N(-y_4) - r^{-T} K (H(t)/S_1)^{2 \gamma - 2} N(-y_4 - \sigma_1 \sqrt{T}) \\
    - S_1 d_{1}^{*} (H(t)/S_1)^{2 \gamma - 2} N(-y_3) + r^{-T} K (H(t)/S_1)^{2 \gamma - 2} N(-y_3 - \sigma_1 \sqrt{T})] \\
\]
III.F.3.c Up-and-In European Put Options (K > H)

\[ r^{-t} \ E(K-S_1(T)) \left[ \Pr(S_1(T)<K) 
+ \Pr(S_1(T)<H(T), \sup S_i(t)>H(t)) - \Pr(S_1(T)<H(T)) \right] \]
\[ r^{-t} E(K-S_1 e^{2\sigma \tau}) \left[ \Pr(Z(T)<\ln(H(\tau)/S_1)) \right. 
+ \Pr(Z(T)<\ln(H(\tau)/S_1), \sup Z(t)\to H(\tau)/S_1) - \Pr(Z(T)<\ln(H(\tau)/S_1)) \right] \]
\[ \ln(H(\tau)/S_1) \]
\[ = r^{-t} \int_{\ln(H(\tau)/S_1)}^{\ln(K-S_1 e^{2\sigma \tau}\tau^2)} \frac{(K-S_1 e^{2\sigma \tau}\tau^2)}{\sigma \sqrt{T}} \, dz \]  
\[ + r^{-t} \ln(H(\tau)/S_1) \int_{\ln(H(\tau)/S_1)}^{\ln(K-S_1 e^{2\sigma \tau}\tau^2)} \frac{(K-S_1 e^{2\sigma \tau}\tau^2)}{\sigma \sqrt{T}} \, dz \]  
\[ = K r^{-t} N(-y_1 - \sigma_1 \sqrt{\tau}) - S_1 \, d_1^{-t} \, N(-y_1) \]
\[ + K r^{-t} (H(\tau)/S_1)^{2\tau^2} N(-y_2 - \sigma_1 \sqrt{\tau}) - S_1 \, d_1^{-t} (H(\tau)/S_1)^{2\tau} N(-y_2) \]
\[ - K r^{-t} N(-y_2 - \sigma_1 \sqrt{\tau}) + S_1 \, d_1^{-t} N(-y_2) \]

III.F.3.d Up-and-In European Put Options (K < H)

\[ r^{-t} \ E(K-S_1(T)) \left[ \Pr(S_1(T)<K, \sup S_i(t)>H(t)) \right] \]
\[ \ln(H(\tau)/S_1) \]
\[ = r^{-t} \int_{\ln(H(\tau)/S_1)}^{\ln(K-S_1 e^{2\sigma \tau}\tau^2)} \frac{(K-S_1 e^{2\sigma \tau}\tau^2)}{\sigma \sqrt{T}} \, dz \]
\[ = K r^{-t} (H(\tau)/S_1)^{2\tau^2} N(-y_2 - \sigma_1 \sqrt{\tau}) - S_1 \, d_1^{-t} (H(\tau)/S_1)^{2\tau} N(-y_2) \]

III.F.3.e Down-and-In European Call Options (K > H)

\[ r^{-t} \ E(S_1(T)-K) \left[ \Pr(S_1(T)>K, \inf S_i(t)<H(t)) \right] \]
\[ \ln(H(\tau)/S_1) \]
\[ = r^{-t} (H(\tau)/S_1)^{2\tau^2} \int_{\ln(H(\tau)/S_1)}^{\ln(S_1 e^{2\sigma \tau}\tau^2-K)} \frac{(S_1 e^{2\sigma \tau}\tau^2-K)}{\sigma \sqrt{T}} \, dz \]
\[ = S_1 \, d_1^{-t} (H(\tau)/S_1)^{2\tau} N(y_3) - K r^{-t} (H(\tau)/S_1)^{2\tau^2} N(y_3 - \sigma_1 \sqrt{\tau}) \]
III.F.3.f Down-and-In European Call Options (K < H)

\[ r^{-\tau} E(S_1(T)-K) \left[ \Pr(S_1(T)>H) + \Pr(S_1(T)>H, \inf S_1(t)<H(t)) \right] = r^{-\tau} \int_{\text{ln}(H(T)/S_1)} (S_1 e^{zT} + t^0 - K)f_{Z(T)}(z)dZ(T) - r^{-\tau} \int_{\text{ln}(H(T)/S_1)} (S_1 e^{zT} - t^0 - K)g_{Z(T)}(z)dZ(T) + r^{-\tau}(H(T)/S_1)^{Y-2} \int_{\text{ln}(H(T)/S_1)} (S_1 e^{zT} + t^0 - K)g_{Z(T)}(z)dZ(T) \]  

III.F.3.g Down-and-In European Put Options (K < H)

\[ r^{-\tau} E(K-S_1(T)) \Pr(S_1(T)<K) = r^{-\tau} \int_{\text{ln}(H(T)/S_1)} (K-S_1 e^{zT} + t^0 - K)f_{Z(T)}(z)dZ(T) \]  

III.F.3.h Down-and-In European Put Options (K > H)

\[ r^{-\tau} E(K-S_1(T)) \left[ \Pr(S_1(T)<H) + \Pr(S_1(T)>H, \inf S_1(t)<H(t)) \right] = r^{-\tau} \int_{\text{ln}(H(T)/S_1)} (K-S_1 e^{zT} + t^0 - K)f_{Z(T)}(z)dZ(T) + r^{-\tau}(H(T)/S_1)^{Y-2} \int_{\text{ln}(H(T)/S_1)} (K-S_1 e^{zT} + t^0 - K)g_{Z(T)}(z)dZ(T) - r^{-\tau}(H(T)/S_1)^{Y-2} \int_{\text{ln}(H(T)/S_1)} (K-S_1 e^{zT} + t^0 - K)g_{Z(T)}(z)dZ(T) \]
III.F.3.i Down-and-Out European Call Options (K > H)

\[
E(S_T) - K \left[ \Pr(S_T > H) - \Pr(S_T > H, \inf S_t < H(t)) \right]
\]

\[
= r^{-\tau} \int_{\ln(H/S_0)} \left( S_T e^{Z(T) + t\theta - K} f_{Z(T)}(z) dZ(T) \right)
\]

\[
- r^{-\tau} (H/S_T)^{2\tau} \int_{\ln(H/S_0)} \left( S_T e^{Z(T) + t\theta - K} g_{Z(T)}(z) dZ(T) \right)
\]

\[
= S_T d_i^{-\tau} N(y_i) - K r^{-\tau} N(y_i - \sigma_1 \sqrt{\tau})
\]

\[
- S_T d_i^{-\tau} (H/S_T)^{2\tau} N(y_i) + K r^{-\tau} (H/S_T)^{2\tau} N(y_i - \sigma_1 \sqrt{\tau})
\]

III.F.3.j Down-and-Out European Call Options (K < H)

\[
E(S_T) - K \left[ \Pr(S_T > H(T)) - \Pr(S_T > H(T), \inf S_t < H(t)) \right]
\]

\[
= r^{-\tau} \int_{\ln(H/S_0)} \left( S_T e^{Z(T) + t\theta - K} f_{Z(T)}(z) dZ(T) \right)
\]

\[
- r^{-\tau} (H/S_T)^{2\tau} \int_{\ln(H/S_0)} \left( S_T e^{Z(T) + t\theta - K} g_{Z(T)}(z) dZ(T) \right)
\]

\[
= S_T d_i^{-\tau} N(y_i) - K r^{-\tau} N(y_i - \sigma_1 \sqrt{\tau})
\]

\[
- S_T d_i^{-\tau} (H/S_T)^{2\tau} N(y_i) + K r^{-\tau} (H/S_T)^{2\tau} N(y_i - \sigma_1 \sqrt{\tau})
\]

III.F.3.k Down-and-Out European Put Options (K < H)

\[
r^{-\tau} E(K - S_T) \Pr(S_T < K, \inf S_t > H(t)) = 0
\]
III.F.3.1 Down-and-Out European Put Options \((K > H)\)

\[
\begin{align*}
 r^{-t} & \quad E(K-S_1(T)) \quad [Pr(S_1(T) < K) - Pr(S_1(T) > H(T)) - Pr(S_1(T) < H(t)) + Pr(S_1(T) > K, inf S_1(t) < H(t))] \\
& = r^{-t} \int_{\ln(H(t)/S_1)}^{\infty} (K-S_1 e^{ZT} e^{-t\sigma_1}) g_{ZT}(z) dz(T) \\
& - r^{-t} (H(t)/S_1)^{2r-2} \int_{\ln(H(t)/S_1)}^{\infty} (K-S_1 e^{ZT} e^{-t\sigma_1}) g_{ZT}(z) dz(T) \\
& = S_1 d_{1r}^{-t} N(-y_1) + K r^{-t} N(-y_2 - \sigma_v \sqrt{t}) + S_1 d_{1r}^{-t} (H(t)/S_1)^{2r-2} N(y_4 - \sigma_v \sqrt{t}) \\
& + S_1 d_{1r}^{-t} N(-y_3 - \sigma_v \sqrt{t}) + K r^{-t} (H(t)/S_1)^{2r-2} N(y_3 - \sigma_v \sqrt{t}) \\
& + r^{-t} (H(t)/S_1)^{2r-2} \int_{\ln(H(t)/S_1)}^{\infty} (K-S_1 e^{ZT} e^{-t\sigma_1}) g_{ZT}(z) dz(T) \\
& = S_1 d_{1r}^{-t} N(-y_1) + K r^{-t} N(-y_2 - \sigma_v \sqrt{t}) + S_1 d_{1r}^{-t} (H(t)/S_1)^{2r-2} N(y_4 - \sigma_v \sqrt{t}) \\
& + S_1 d_{1r}^{-t} N(-y_3 - \sigma_v \sqrt{t}) + K r^{-t} (H(t)/S_1)^{2r-2} N(y_3 - \sigma_v \sqrt{t}) \\
& + r^{-t} (H(t)/S_1)^{2r-2} \int_{\ln(H(t)/S_1)}^{\infty} (K-S_1 e^{ZT} e^{-t\sigma_1}) g_{ZT}(z) dz(T) \\
& = K r^{-t} N(-y_2 - \sigma_v \sqrt{t}) - S_1 d_{1r}^{-t} N(-y_3) - K (H(t)/S_1)^{2r-2} r^{-t} N(-y_4 - \sigma_v \sqrt{t}) + S_1 (H(t)/S_1)^{2r} d_{1r}^{-t} N(-y_3)
\end{align*}
\]

III.F.3.m Up-and-Out European Put Options \((K > H)\)

\[
\begin{align*}
 r^{-t} & \quad E(K-S_1(T)) \quad [Pr(S_1(T) < H(T)), sup S_1(t) > H(t)] \\
& = r^{-t} \int_{\ln(H(t)/S_1)}^{\infty} (K-S_1 e^{ZT} e^{-t\sigma_1}) g_{ZT}(z) dz(T) \\
& - r^{-t} (H(t)/S_1)^{2r-2} \int_{\ln(H(t)/S_1)}^{\infty} (K-S_1 e^{ZT} e^{-t\sigma_1}) g_{ZT}(z) dz(T) \\
& = K r^{-t} N(-y_2 - \sigma_v \sqrt{t}) - S_1 d_{1r}^{-t} N(-y_3) - K (H(t)/S_1)^{2r-2} r^{-t} N(-y_4 - \sigma_v \sqrt{t}) + S_1 (H(t)/S_1)^{2r} d_{1r}^{-t} N(-y_3)
\end{align*}
\]
III.F.3.n  Up-and-Out European Put Options (K < H)

\[ r^{-\tau} E(K-S_1(T)) \left[ \Pr(S_1(T)\leq K) - \Pr(S_1(T)\leq K, \sup S_1(t)\geq H(t)) \right] \]

\[ = r^{-\tau} \int_{\ln(H(\tau)/S_i)}^{\ln(H(\tau)/S_i)} (K-S_1 e^{Z(T)+\theta_{t}}) f_{Z(T)}(z)dz + \int_{\ln(H(\tau)/S_i)}^{\ln(H(\tau)/S_i)} (K-S_1 e^{Z(T)+\theta_{t}}) g_{Z(T)}(z)dz \]

\[ - K \ r^{-\tau} N(-y_1) - S_1 \ d_1^{-\tau} N(-y_1) \]

\[ = K \ r^{-\tau} N(-y_1 - \sigma_1 \sqrt{\tau}) - S_1 \ d_1^{-\tau} N(-y_1) \]

\[ - K \ (H(\tau)/S_j) \ r^{-\tau} N(-y_3 - \sigma_1 \sqrt{\tau}) + S_1 \ (H(\tau)/S_j) \ r^{-\tau} d_1^{-\tau} N(-y_3) \]

III.F.3.o  Up-and-Out European Call Options (K > H)

\[ r^{-\tau} E(S_1(T) - K) \ Pr(S_1(T) > K, \sup S_1(t) < H(t)) = 0 \]

III.F.3.p  Up-and-Out European Call Options (K < H)

\[ r^{-\tau} E(S_1(T) - K) \left[ \Pr(S_1(T) > K) - \Pr(S_1(T) < H(T), \sup S_1(t) > H(t)) \right] \]

\[ = r^{-\tau} \int_{\ln(H(\tau)/S_i)}^{\ln(H(\tau)/S_i)} (S_1 e^{Z(T)+\theta_{t}} - K) f_{Z(T)}(z)dz \]

\[ - r^{-\tau} \ (H(\tau)/S_j) \ r^{-\tau} \int_{\ln(H(\tau)/S_i)}^{\ln(H(\tau)/S_i)} (S_1 e^{Z(T)+\theta_{t}} - K) g_{Z(T)}(z)dz \]

\[ - r^{-\tau} \int_{\ln(H(\tau)/S_i)}^{\ln(H(\tau)/S_i)} (S_1 e^{Z(T)+\theta_{t}} - K) f_{Z(T)}(z)dz + \int_{\ln(H(\tau)/S_i)}^{\ln(H(\tau)/S_i)} (S_1 e^{Z(T)+\theta_{t}} - K) g_{Z(T)}(z)dz \]

\[ + r^{-\tau} \ (H(\tau)/S_j) \ r^{-\tau} \int_{\ln(H(\tau)/S_i)}^{\ln(H(\tau)/S_i)} (S_1 e^{Z(T)+\theta_{t}} - K) g_{Z(T)}(z)dz \]

\[ = S_1 \ d_1^{-\tau} N(y_1) - K \ r^{-\tau} N(y_1 - \sigma_1 \sqrt{\tau}) \]

\[ - S_1 \ d_1^{-\tau} N(y_2) + K \ r^{-\tau} N(y_2 - \sigma_1 \sqrt{\tau}) \]

\[ - S_1 \ d_1^{-\tau} (H(\tau)/S_j) \ r^{-\tau} N(-y_3) - K \ r^{-\tau} (H(\tau)/S_j) \ r^{-\tau} N(-y_4 - \sigma_1 \sqrt{\tau}) \]

\[ + S_1 \ d_1^{-\tau} (H(\tau)/S_j) \ r^{-\tau} N(-y_3) - K \ r^{-\tau} (H(\tau)/S_j) \ r^{-\tau} N(-y_4 - \sigma_1 \sqrt{\tau}) \]

Chapter III
III.F.4 Valuing the Rebate Payoff of European Barrier Options

In this section the valuation of European barrier options is completed by determining the rebate portion of the option's value.

Holders of "in" options receive the rebate at expiration if the barrier is never touched (i.e., if the option never "starts"). Since the value of the rebate is deterministic and the timing of the payoff is known, the value of the "in" rebate is easily determined.

III.F.4.a Current Value of Down-and-In Rebate

\[ r^{-T} R \left[ Pr(S_1(T)>H(T)) - Pr(S_1(T)<H(T), \inf S_1(t)<H(t)) \right] \]

\[ = r^{-T} R \int_{H(T)<S_1} [f_{Z(T)}(z) - g_{Z(T)}(z)] dZ(T) \]

\[ = r^{-T} R \left[ N(y_2 - \sigma_1 \sqrt{T}) - \frac{(H(T)/S_1)^{2\tau-2}}{N(y_4 - \sigma_1 \sqrt{T})} \right] \]

III.F.4.b Current Value of Up-and-In Rebate

\[ r^{-T} R \left[ Pr(S_1(T)<H(T)) - Pr(S_1(T)<H(T), \sup S_1(t)>H(t)) \right] \]

\[ = r^{-T} R \int_{H(T)>S_1} [f_{Z(T)}(z) - g_{Z(T)}(z)] dZ(T) \]

\[ = r^{-T} R \left[ N(-y_2 - \sigma_1 \sqrt{T}) - \frac{(H(T)/S_1)^{2\tau-2}}{N(-y_4 - \sigma_1 \sqrt{T})} \right] \]

III.F.4.c Current Value of Down-and-Out Rebate

Unfortunately, the value of the rebate for "out" option holders cannot be determined in such a straightforward fashion because the timing of the rebate payoff is uncertain.

In addition to the above concern the notation is also somewhat confusing. Recall from Section III.F.2 of this appendix that \( \tau^* \) denoted the random time elapsed until the barrier is first hit. Let us more specifically write this as \( \tau^*=(t^*-t_0) \), where \( t^* \) is the instant the barrier is touched. At hit, an "out" option holder receives a rebate amount of \( R(t^*) = Re^{i\beta (T-t^*)} \). Notice that \( \tau^*=(t^*-t_0)=T-T+t^*-t_0=-(T-t^*)+(T-t_0) \), implying \( \tau^*-(T-t^*) \). Thus, an "out" option holder receives a rebate of \( R(t^*) = Re^{i\beta \tau}e^{i\beta t^*} \). With the rebate stated in this form, let \( A = \ln(bK)/S1H(\tau))Re^{i\beta \tau}/[\sqrt{(\sigma^2 \tau 2 \pi)}] \) and start by completing the square and simplifying to find the
value of the rebate for down-and-out holders as

\[
Re^{-\xi \beta} \int_0^{T-t_0} e^{-\xi \beta \tau} r^{-1} h_1(\tau; \mu; -\xi \theta) d\tau = Re^{-\xi \beta} \int_0^{T-t_0} \exp\left[-(\ln(r) - \xi \beta) \tau\right] h_1(\tau; \mu; -\xi \theta) d\tau
\]

\[
= \int_0^{T-t_0} -A \exp (\ln(H(\tau)/S_0)^2 + \ln(H(\tau)/S_0)(\mu - \xi \theta)) d\tau
\]

\[
= \int_0^{T-t_0} -A \exp \frac{-\ln(H(\tau)/S_0)^2}{2\sigma_1^2 \tau} + \ln(H(\tau)/S_0)(\gamma - 1) - \frac{\tau^2}{2\sigma_1^2} ((\mu - \xi \theta)^2 + 2\sigma_1^2 (\ln(r) - \xi \beta)) d\tau
\]

\[
= \int_0^{T-t_0} -A \exp \left[\frac{-\ln(H(\tau)/S_0)^2}{2\sigma_1^2 \tau} + \ln(H(\tau)/S_0)(\gamma - 1) - \frac{\tau^2}{2\sigma_1^2} m_1^2 \sigma_1^4\right] d\tau
\]

\[
= \int_0^{T-t_0} -A \exp \left[\frac{-\ln(S_0/H(\tau) - m_1^2 \sigma_1^2 \tau^2)^2}{2\sigma_1^2 \tau} + \ln(H(\tau)/S_0)(\gamma - 1 - m_1^2 \sigma_1^4)\right] d\tau
\]

\[
= R e^{-\xi \beta} \int_0^{T-t_0} \frac{\ln(H(\tau)/S_0)}{\sigma_1 \sqrt{\tau}} \left(\frac{\ln(S_0/H(\tau) - m_1^2 \sigma_1^2 \tau^2)^2}{\sigma_1 \sqrt{\tau}}\right) d\tau
\]

\[
= R e^{-\xi \beta} \int_0^{T-t_0} h_1(\tau; \mu; m_1^2 \sigma_1^2 \tau^2) d\tau
\]

The solution for the value of the down-and-out rebate can then be determined by substituting equation [3.30] into the final integral expression in [3.52], noting the change of mean in the probability density function. This gives

\[
R e^{-\xi \beta} \int_0^{T-t_0} h_1(\tau; \mu; m_1^2 \sigma_1^2 \tau^2) d\tau
\]

\[
= R e^{-\xi \beta} \int_0^{T-t_0} \left[Pr(S_1(T) > H(T), \inf S_1(t) < H(t)) + Pr(S_1(T) < H(T))\right]
\]

\[
= R(T) (H(\tau)/S_0)^{\gamma - 1 - m} N(y_2) + R(T) (H(\tau)/S_0)^{\gamma - 1 - m} N(y_2 - 2m_1 \sigma_1 \sqrt{\gamma})
\]

III.F.4.d Current Value of Up-and-Out Rebate

Proceeding in exactly the same manner, but substituting in equation [3.20] this time, the value of the up-and-out rebate can be written as

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\[ Re^{-\xi \theta t} \int_0^{T_1} h_2(\tau; \mu - \xi \theta) d\tau = Re^{-\xi \theta t} \int_0^{T_1} \exp[\left(-\frac{1}{2}\frac{(\ln(\tau) - \xi \theta)^2}{\sigma^2}\right)] h_2(\tau; \mu - \xi \theta) d\tau \]

\[ = \int_0^{T_1} A \exp \left[ -\frac{\ln(H(\tau)/S_1)^2}{2\sigma_1^2 \tau^*} - \frac{\ln(H(\tau)/S_1)(\mu - \xi \theta)}{\sigma^2} - \frac{\mu - \xi \theta)^2}{2\sigma^2} \right] d\tau^* \]

\[ = \int_0^{T_1} A \exp \left[ -\frac{\ln(H(\tau)/S_1)^2}{2\sigma_1^2 \tau^*} + \frac{\ln(H(\tau)/S_1)(\gamma - 1)\tau^*}{\sigma_1^2} - \frac{\tau^*}{2\sigma_1^2} \right] d\tau^* \quad [3.54] \]

\[ = \int_0^{T_1} A \exp \left[ -\frac{(\ln(H(\tau)/S_1))^2}{2\sigma_1^2 \tau^*} + \frac{\ln(H(\tau)/S_1)(\gamma - 1 - m)\tau^*}{\sigma_1^2} \right] d\tau^* \]

\[ = R e^{-\xi \theta t} (H(\tau)/S_1)^{\gamma - 1 - m} \int_0^{T_1} \frac{\ln(H(\tau)/S_1)(b(k)^2)}{\sigma_1^2 \tau^*} \frac{\left(\frac{\ln(S_1/H(\tau))^2 + m\sigma_1^2 \tau^*}{\sigma_1 \sqrt{\tau^*}}\right)}{\sigma_1^2 \tau^*} d\tau^* \]

\[ = R(\tau) (H(\tau)/S_1)^{\gamma - 1 - m} \int_0^{T_1} h_2(\tau^*; m\sigma_1^2 \tau^*) d\tau^* \]

\[ = R(\tau) (H(\tau)/S_1)^{\gamma - 1 - m} \left[ Pr(S_1(T) < H(T), sup S_1(t) > H(t)) + Pr(S_1(T) > H(T)) \right] \]

\[ = R(\tau) (H(\tau)/S_1)^{\gamma - 1 - m} N(-y_2) + R(\tau) (H(\tau)/S_1)^{\gamma - 1 - m} N(-y_2 - 2m\sigma_1 \sqrt{\tau^*}) \]

**III.G Appendix IIIIB: Comparative Statics and Verification of the Standard Barrier Model**

In this appendix the comparative statics are derived for the resulted presented in Table III.B.1.a. It is important to note that the \((\gamma)\) equations listed in equation [3.7] have been simplified. The comparative statics, and hence, verification of linear homogeneity, require the original equations. They are listed below with other helpful identities.
\[ \begin{align*}
\gamma_1 &= \frac{[\ln(S_1/K) - \ln(H(\tau)/bK) + (\mu + \sigma_1^2 - \xi \theta) \tau]}{\sigma_1 \sqrt{\tau}} \\
\gamma_2 &= \frac{[\ln(S_1/H(\tau)) + (\mu + \sigma_1^2 - \xi \theta) \tau]}{[\sigma_1 \sqrt{\tau}]} \\
\gamma_3 &= \frac{[\ln(H^2(\tau)/S_1 K) - \ln(H(\tau)/bK) + (\mu + \sigma_1^2 - \xi \theta) \tau]}{[\sigma_1 \sqrt{\tau}]} \\
\gamma_4 &= \frac{[\ln(H(\tau)/S_1) + (\mu + \sigma_1^2 - \xi \theta) \tau]}{[\sigma_1 \sqrt{\tau}]} \\
\gamma_5 &= \frac{[\ln(H(\tau)/S_1) + \mu \sigma_1^2 \tau]}{[\sigma_1 \sqrt{\tau}]} \\
\eta(y_1 - \sigma_1 \sqrt{\tau}) &= n(y_2) [S_1 d^{-\tau} / K r^{-\tau}] \\
\eta(y_2 - \sigma_1 \sqrt{\tau}) &= n(y_2) [S_1 d^{-\tau} / b K r^{-\tau}] \\
\eta(y_3 - \sigma_1 \sqrt{\tau}) &= n(y_2) [H^2(\tau) d^{-\tau} / S_1 K r^{-\tau}] \\
\eta(y_4 - \sigma_1 \sqrt{\tau}) &= n(y_2) [H(\tau) S_1]^{-\tau} / b K S_1 r^{-\tau}] \\
\eta(y_5 - \sigma_1 \sqrt{\tau}) &= n(y_2) (H(\tau) / S_1)^{\tau} \\
\eta(y_6 - 2 \mu \sigma_1 \sqrt{\tau}) &= n(y_2) (H(\tau) / S_1)^{2 \tau} \\
\eta(y_7 - 2 \mu \sigma_1 \sqrt{\tau}) &= n(y_2) (H(\tau) / S_1)^{2 \tau} \\
\eta(y_8 - 2 \mu \sigma_1 \sqrt{\tau}) &= n(y_2) (H(\tau) / S_1)^{2 \tau} \\
\eta(y_9 - 2 \mu \sigma_1 \sqrt{\tau}) &= n(y_2) (H(\tau) / S_1)^{2 \tau}
\end{align*} \]

It will also be useful to note that at hit:

\[ \begin{align*}
\gamma_1 &= \gamma_2 = [\ln(b) + (\ln(r/d_1) + 5 \sigma_1^2 - \xi \theta) \tau] / [\sigma_1 \sqrt{\tau}] \\
\gamma_2 &= \gamma_4 = [\ln(r/d_1) + 5 \sigma_1^2 - \xi \theta] \tau] / [\sigma_1 \sqrt{\tau}] \\
\gamma_3 &= \mu \sigma_1 \sqrt{\tau}
\end{align*} \]

From the following comparative statics, each valuation solution presented in Table III.B.1.a is verified to satisfy

\[ 5 \sigma_1^2 \frac{\partial^2 W_{\text{STD}}(.)}{\partial s_1^2} + \ln \left( \frac{r}{d_1} \right) S_1 \frac{\partial W_{\text{STD}}(.)}{\partial s_1} - \frac{\partial W_{\text{STD}}(.)}{\partial \tau} = \ln(r) W_{\text{STD}}(.) \]

for all \( \eta S_1(t) > \eta H(t) \) \( t \in [t_n, T] \), subject to

\[ W_{\text{STD}}(S_1(T), K, H(T), R(T), 0) = \max(\phi S_1(T) - \phi K, 0) \]

if \( \eta S_1(t) > \eta H(t) \) \( \forall t \),

\[ W_{\text{STD,in}}(S_1(T), K, H(T), R(T), 0) = R \) (at T)

if \( \eta S_1(t) \leq \eta H(t) \) for some \( t \),

\[ W_{\text{STD, out}}(H(t^*), K, H(t^*), R(t^*), T-t^*) = R e^{-\alpha(t^*-t^*)} \) (at \( t^* \))

if \( \eta S_1(t^*) \leq \eta H(t^*) \) and \( \eta S_1(t) > \eta H(t) \) \( \forall t < t^* \).

Each valuation solution presented in Table III.B.1.a will also be shown to be linearly
homogeneous of degree one with respect to $S_i$, $K$, $H$, and $R$. Linear homogeneity can be verified by showing the following identity is satisfied.

$$S_i(t) \frac{\partial W_{STD}(\cdot)}{\partial S_i(t)} + K \frac{\partial W_{STD}(\cdot)}{\partial K} + H(\tau) \frac{\partial W_{STD}(\cdot)}{\partial H(\tau)} + R(\tau) \frac{\partial W_{STD}(\cdot)}{\partial R(\tau)} = W_{STD}(\cdot). \quad [3.59]$$

The easiest way to confirm [3.24] and [3.26] is to confirm that each of the individual equations, [3.1]-[3.6], satisfy the conditions. That is, [3.24] and Euler's theorem would not hold if each of the six equations, [3.1]-[3.6], did not satisfy the Black-Scholes partial differential equation and were not linearly homogeneous with respect to the four parameters. Once it has been established that each of the six equations individually satisfy the Black-Scholes partial differential equation and are linearly homogeneous, and each valuation solution is a linear combination of the six equations, it follows that each barrier option is linearly homogeneous and satisfies [3.24]. Each of the individual six equations is examined next.
The comparative statics for equation [3.1] are

\[
\begin{align*}
\frac{\partial \text{[3.1]}}{\partial s_i} &= \phi d_i^{-\tau} N(\phi y_i) \\
\frac{\partial^2 \text{[3.1]}}{\partial s^2} &= d_i^{-\tau} n(y_i)/(S_i \sigma_i \sqrt{\tau}) \\
\frac{\partial \text{[3.1]}}{\partial H(\tau)} &= 0 \\
\frac{\partial \text{[3.1]}}{\partial K} &= -\phi r^{-\tau} N(\phi(y_i - \sigma_i \sqrt{\tau})) \\
\frac{\partial \text{[3.1]}}{\partial \tau} &= 0 \\
\frac{\partial \text{[3.1]}}{\partial \tau} &= \frac{.5 \sigma_i n(y_i)[1/\sqrt{\tau}] S_i d_i^{-\tau} - \phi S_i d_i^{-\tau} N(\phi y_i) \ln(d_i)}{} + \phi K r^{-\tau} N(\phi(y_i - \sigma_i \sqrt{\tau})) \ln(r)
\end{align*}
\]

[3.60]

If the solution is correct, these results will satisfy

\[
\frac{.5 \sigma_i^2 S_i d_i^{-\tau} \partial^2 \text{[3.1]}}{\partial s_i^2} + \ln\left(\frac{r}{d_i}\right) \frac{\partial \text{[3.1]}}{\partial s_i} - \frac{\partial \text{[3.1]}}{\partial \tau} = \ln(r) \text{[3.1]}. \tag{3.61}
\]

To see this, note that the left-hand side is equal to

\[
LHS = \ln(r/d_i)\phi S_i d_i^{-\tau} N(\phi y_i) + .5 d_i^{-\tau} S_i \sigma_i n(y_i)/(\sqrt{\tau}) - .5 \sigma_i n(y_i)[1/\sqrt{\tau}] S_i d_i^{-\tau} + \phi S_i d_i^{-\tau} N(\phi y_i) \ln(d_i) - \phi K r^{-\tau} N(\phi(y_i - \sigma_i \sqrt{\tau})) \ln(r) = \ln(r)\phi S_i d_i^{-\tau} N(\phi y_i) - \ln(r) K r^{-\tau} N(\phi(y_i - \sigma_i \sqrt{\tau})) = RHS. \tag{3.62}
\]

Equation [3.1] can be verified to be linearly homogeneous by noting

\[
S_i(t) \frac{\partial \text{[3.1]}}{\partial S_i(t)} + K \frac{\partial \text{[3.1]}}{\partial K} + H(\tau) \frac{\partial \text{[3.1]}}{\partial H(\tau)} + R(\tau) \frac{\partial \text{[3.1]}}{\partial R(\tau)} = [3.1]. \tag{3.63}
\]

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The comparative statics for equation [3.2] are
\[ \frac{\partial [3.2]}{\partial S_1} = \phi d_1^{-T} N(\phi y_2) + d_1^{-T} n(y_2)[1 - (1/b)] / [\sigma_1 \sqrt{\tau}] \]
\[ \frac{\partial^2 [3.2]}{\partial s_1^2} = d_1^{-T} n(y_2)[\sigma_1 \sqrt{\tau} - y_2(1 - (1/b))] / [s_1 \sigma_1^2 \tau] \]
\[ \frac{\partial [3.2]}{\partial H(\tau)} = -(S_1 / H(\tau)) d_1^{-T} n(y_2)[1 - (1/b)] / [\sigma_1 \sqrt{\tau}] \]
\[ \frac{\partial [3.2]}{\partial K} = -\phi r^{-T} N(\phi (y_2 - \sigma_1 \sqrt{\tau})) \]
\[ \frac{\partial [3.2]}{\partial R(\tau)} = 0 \]
\[ \frac{\partial [3.2]}{\partial \tau} = 0.5 \sigma_1 n(y_2)[1/\sqrt{\tau}] S_1 d_1^{-T}(1/b) \]
\[ - \phi S_1 d_1^{-T} N(\phi y_2) \ln(d_1) + \phi K r^{-T} N(\phi (y_2 - \sigma_1 \sqrt{\tau})) \ln(r) \]
\[ + S_1 d_1^{-T} n(y_2) \frac{\partial y_2}{\partial \tau}[1 - (1/b)] \]

If the solution is correct, these results will satisfy
\[ 0.5 \sigma_1^2 \frac{\partial^2 [3.2]}{\partial s_1^2} + \ln \left( \frac{r}{d_1} \right) S_1 \frac{\partial [3.2]}{\partial s_1} - \frac{\partial [3.2]}{\partial \tau} = \ln(r)[3.2]. \]  

To see this, note that the left-hand side is equal to
\[ LHS = \phi \ln(r/d_1) S_1 d_1^{-T} N(\phi y_2) \]
\[ + \ln(r/d_1) S_1 d_1^{-T} n(y_2)[1 - (1/b)] / [\sigma_1 \sqrt{\tau}] \]
\[ + 0.5 S_1 d_1^{-T} n(y_2)[\sigma_1 \sqrt{\tau} - y_2(1 - (1/b))] / [\tau] \]
\[ - 0.5 \sigma_1 n(y_2)[1/\sqrt{\tau}] S_1 d_1^{-T}(1/b) + \phi S_1 d_1^{-T} N(\phi y_2) \ln(d_1) \]
\[ - \phi K r^{-T} N(\phi (y_2 - \sigma_1 \sqrt{\tau})) \ln(r) - S_1 d_1^{-T} n(y_2) \frac{\partial y_2}{\partial \tau}[1 - (1/b)] \]
\[ = 0.5 S_1 d_1^{-T} n(y_2)(1 - (1/b))(1/\sqrt{\tau}) \left[ \sigma_1^2 - y_2 \sigma_1 \sqrt{\tau} \right] \]
\[ + 2 \ln(r/d_1) - 2 \sigma_1 \sqrt{\tau} \frac{\partial y_2}{\partial \tau} + \ln(r)[3.2] \]
\[ = \ln(r)[3.2] \text{ because } \frac{\partial y_2}{\partial \tau} = -(1/(2\sqrt{\tau}))y_2 - 2\ln(r/d_1) \sqrt{\tau} / \sigma_1 - \sigma_1 \sqrt{\tau} \]

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Equation [3.2] can be verified to be linearly homogeneous by noting

\[
S_i(t) \frac{\partial[3.2]}{\partial S_i(t)} + K \frac{\partial[3.2]}{\partial K} + H(\tau) \frac{\partial[3.2]}{\partial H(\tau)} + R(\tau) \frac{\partial[3.2]}{\partial R(\tau)} = [3.2]. \tag{3.67}
\]

The comparative statics for equation [3.3] are

\[
\frac{\partial[3.3]}{\partial S_i} = \phi d_i^{-1} (H(\tau)/S_i)^{2\gamma} N(\eta y_2)(1 - 2\gamma)
\]

\[+ \phi (2\gamma - 2)(H(\tau)/S_i)^{2\gamma - 2}(K/S_i) r^{-\tau} N(y_3 - \sigma_1 \sqrt{\tau}) \]

\[
\frac{\partial[3.3]}{\partial S_i^2} = -\phi (1 - 2\gamma) 2\gamma d_i^{-1} N(\eta y_2) (H(\tau)/S_i)^{2\gamma} (1/S_i)
\]

\[+ \eta \Phi (H(\tau)/S_i)^{2\gamma} n(y_3) d_i^{-1} / (S_i \sigma_1 \sqrt{\tau}) \]

\[+ \phi (2\gamma - 2)(2\gamma - 1)(H(\tau)/S_i)^{2\gamma - 2}(K/S_i^2) r^{-\tau} N(\eta y_3 - \sigma_1 \sqrt{\tau}) \]

\[
\frac{\partial[3.3]}{\partial H(\tau)} = \phi (S_i/H(\tau)) d_i^{-1} 2\gamma (H(\tau)/S_i)^{2\gamma} N(\eta y_2)
\]

\[+ \phi (2\gamma - 2)(H(\tau)/S_i)^{2\gamma - 2}(K/H(\tau)) r^{-\tau} N(\eta y_3 - \sigma_1 \sqrt{\tau}) \tag{3.68} \]

\[
\frac{\partial[3.3]}{\partial K} = -\phi (H(\tau)/S_i)^{2\gamma - 2} r^{-\tau} N(\eta y_3 - \sigma_1 \sqrt{\tau})
\]

\[
\frac{\partial[3.3]}{\partial R(\tau)} = 0
\]

\[
\frac{\partial[3.3]}{\partial \tau} = -\phi S_i d_i^{-1} \ln(d_i) (H(\tau)/S_i)^{2\gamma} N(\eta y_2)
\]

\[+ \phi (2\gamma - 2)(H(\tau)/S_i)^{2\gamma - 2} N(\eta y_3 - \sigma_1 \sqrt{\tau}) \]

\[+ \phi (H(\tau)/S_i)^{2\gamma - 2} K r^{-\tau} N(\eta y_3 - \sigma_1 \sqrt{\tau}) \]

\[+ \eta \Phi (H(\tau)/S_i)^{2\gamma} n(y_3) 5d_i^{-1} S_i \sigma_1 \sqrt{\tau} \]

If the solution is correct, these results will satisfy

\[
.5 \sigma_1^2 \delta^2 \frac{\partial[3.3]}{\partial S_i^2} + \ln \left( \frac{r}{d_i} \right) \frac{\partial[3.3]}{\partial S_i} - \frac{\partial[3.3]}{\partial \tau} = \ln(r)[3.3]. \tag{3.69}
\]
To see this is satisfied, note that the left-hand side is equal to

\[
LHS = \ln(r/d_i) S_1 \phi d_i^{1-\gamma}(H(\tau)/S_1)^{\gamma^2 N(\eta \gamma_3)(1-2\gamma)} \\
+ \phi(\gamma-2)K r^{-2}N(\eta \gamma_3)(H(\tau)/S_1)^{\gamma^2 r^{-2} \phi \frac{d_i}{\sqrt{\gamma}} N(\eta \gamma_3(1-2\gamma))} \\
- \phi(\gamma-1)(\gamma-2)K r^{-2}N(\eta \gamma_3(1-2\gamma)) \\
+ \phi S_1 d_i^{1-\gamma} \ln(r/d_i) (H(\tau)/S_1)^{\gamma^2 N(\eta \gamma_3)} \\
+ \phi S_1 d_i^{1-\gamma} N(\eta \gamma_3) 2 \gamma (H(\tau)/S_1)^{\gamma^2 r^{-2} \phi \frac{d_i}{\sqrt{\gamma}}} \\
- \phi(\gamma-1)(\gamma-2)K r^{-2}N(\eta \gamma_3(1-2\gamma)) \\
- \eta \phi (H(\tau)/S_1)^{\gamma^2 r^{-2} \phi \frac{d_i}{\sqrt{\gamma}}} N(\eta \gamma_3(1-2\gamma)) \\
= -2 \gamma \phi d_i^{1-\gamma}(H(\tau)/S_1)^{\gamma^2 N(\eta \gamma_3)} \ln(r/d_i) + \ln(r)[3] \\
+ \phi(2\gamma-2)K r^{-2}N(\eta \gamma_3(1-2\gamma)) \ln(r/d_i) - \phi \frac{d_i}{\sqrt{\gamma}} N(\eta \gamma_3(1-2\gamma)) \\
- \phi(2\gamma-1)(H(\tau)/S_1)^{\gamma^2 r^{-2} \phi \frac{d_i}{\sqrt{\gamma}}} N(\eta \gamma_3(1-2\gamma)) \\
= \phi S_1 d_i^{1-\gamma} N(\eta \gamma_3) (H(\tau)/S_1)^{\gamma^2 2 \gamma \sigma^2 + \phi \frac{d_i}{\sqrt{\gamma}} - \ln(r/d_i)} \\
+ \phi K r^{-2}N(\eta \gamma_3(1-2\gamma)) (H(\tau)/S_1)^{\gamma^2 r^{-2} [(2\gamma-2)(\ln(r/d_i) - \phi \frac{d_i}{\sqrt{\gamma}})} \\
- (2\gamma-1) \sigma^2 (\gamma-1)] + \ln(r)[3.3] \\
= \ln(r)[3.3] because \sigma^2 (\gamma-1) - \ln(r/d_i) and \\
(2\gamma-2)(\ln(r/d_i) - \phi \frac{d_i}{\sqrt{\gamma}}) = (2\gamma-1) \sigma^2 (\gamma-1).
\]

Equation [3.3] can be verified to be linearly homogeneous by noting

\[
S_i(t) \phi[3.3] + \frac{\partial H(\tau)}{\partial K} + \frac{\partial H(\tau)}{\partial R(\tau)} + R(\tau) \phi[3.3] = [3.3].
\]
The comparative statics for equation [3.4] are

\[
\frac{\partial[3.4]}{\partial S_1} = \phi d_1^{-1+1}(H(\tau)/S_1)^{2\gamma} N(\eta y_\gamma) (1 - 2\gamma)
\]

\[
+ \phi (2\gamma - 2)(H(\tau)/S_1)^{2\gamma - 2}(K/S_1) r^{-\gamma} N(\eta y_\gamma - \sigma_1 \sqrt{\tau})
\]

- \phi \eta (H(\tau)/S_1)^{2\gamma} d_1^{-1+1} (1/(\sigma_1 \sqrt{\tau}))(1 - (1/b)) n(y_\gamma)

\[
\frac{\partial[3.4]}{\partial S_1^2} = -\phi 2\gamma \eta d_1^{-1+1}(H(\tau)/S_1)^{2\gamma} (1/S_1) N(\eta y_\gamma) (1 - 2\gamma)
\]

- \phi \eta (H(\tau)/S_1)^{2\gamma} d_1^{-1+1} (1/(\sigma_1 \sqrt{\tau}))(1 - (1/b)) n(y_\gamma)

\[
- \phi (2\gamma - 2)(2\gamma - 1)(H(\tau)/S_1)^{2\gamma - 2}(K/S_1) r^{-\gamma} N(\eta y_\gamma - \sigma_1 \sqrt{\tau})
\]

+ \phi \eta (H(\tau)/S_1)^{2\gamma} 2\gamma d_1^{-1+1} (1/(\sigma_1 \sqrt{\tau}))(1 - (1/b)) n(y_\gamma)

+ \phi \eta (H(\tau)/S_1)^{2\gamma} (1/b) n(y_\gamma) d_1^{-1+1} (1/(\sigma_1 \sqrt{\tau}))

\[
\frac{\partial[3.4]}{\partial H(\tau)} = \phi 2\gamma (H(\tau)/S_1)^{2\gamma} (S_1/H(\tau)) d_1^{-1+1} N(\eta y_\gamma)
\]

+ \phi \eta (H(\tau)/S_1)^{2\gamma} (S_1/H(\tau)) (1/(\sigma_1 \sqrt{\tau})) d_1^{-1+1} n(y_\gamma)(1 - (1/b))

- \phi (2\gamma - 2)(H(\tau)/S_1)^{2\gamma - 2}(K/H(\tau)) r^{-\gamma} N(\eta y_\gamma - \sigma_1 \sqrt{\tau})

\[
\frac{\partial[3.4]}{\partial K} = -\phi (H(\tau)/S_1)^{2\gamma - 2} r^{-\gamma} N(\eta y_\gamma - \sigma_1 \sqrt{\tau})
\]

\[
\frac{\partial[3.4]}{\partial R(\tau)} = 0
\]

\[
\frac{\partial[3.4]}{\partial \tau} = -\phi S_1 d_1^{-1+1} \ln(d_1) (H(\tau)/S_1)^{2\gamma} N(\eta y_\gamma)
\]

- \phi S_1 d_1^{-1+1} \theta(H(\tau)/S_1)^{2\gamma} N(\eta y_\gamma)

\[
+ \phi \eta S_1 d_1^{-1+1} (H(\tau)/S_1)^{2\gamma} n(y_\gamma) \frac{\partial y_\gamma}{\partial \tau} (1 - (1/b))
\]

+ \phi (H(\tau)/S_1)^{2\gamma - 2} Kr^{-\gamma} \ln(r) N(\eta y_\gamma - \sigma_1 \sqrt{\tau})

+ \phi (2\gamma - 2) \theta(H(\tau)/S_1)^{2\gamma - 2} Kr^{-\gamma} N(\eta y_\gamma - \sigma_1 \sqrt{\tau})

+ \phi \eta (H(\tau)/S_1)^{2\gamma} (1/b) n(y_\gamma) S_1 d_1^{-1+1} \sigma_1 \sqrt{\tau}

If the solution is correct, these results will satisfy

\[
.5\sigma_1^2 S_1^2 \frac{\partial[3.4]}{\partial S_1^2} + \int \left( \frac{r}{d_1} \right) S_1 \frac{\partial[3.4]}{\partial S_1} - \frac{\partial[3.4]}{\partial \tau} = \ln(r)[3.4].
\]

[3.73]
To see this note that

\[ \text{LHS} = \phi Sd_r \gamma^2 (H(\tau)/S_r)^{2\gamma} N(y) [2\gamma(\xi \theta - \ln(\tau/d_r)) + \gamma a_r^2 + 2\gamma a_r^3] \\
+ \phi (2\gamma - 2)(H(\tau)/S_r)^{2\gamma - 1} K_r^{-\gamma} N(\eta(y - \sigma_1\sqrt{\tau}))/([\ln(r/d_r) - \gamma a_r^2 + 5a_r^2 - \xi \theta]) \]

\[ \phi \eta(H(\tau)/S_r)^{2\gamma} Sd_r^{-1} \gamma (1 - (1/b)) n(y)[\ln(r/d_r)/[\sigma_1/\sqrt{\tau}] \\
+ \frac{\partial y}{\partial \tau} + 5y/\tau - 2y \sigma_1/\sqrt{\tau} + 5\sigma_1/\sqrt{\tau} + \ln(\tau)[3.4] = \ln(\tau)[3.4] \]  

Equation [3.4] can be verified to be linearly homogeneous by noting

\[ S_r(\tau) \frac{\partial [3.4]}{\partial S_r(\tau)} + K \frac{\partial [3.4]}{\partial K} + H(\tau) \frac{\partial [3.4]}{\partial H(\tau)} + R(\tau) \frac{\partial [3.4]}{\partial R(\tau)} = [3.4]. \]
The comparative statics for equation [3.5] are

\[
\frac{\partial [3.5]}{\partial S_1} = 2\left( \frac{R(\tau)/bK}{\eta n(y_2)} \right) d^{-\gamma} \frac{1}{(\sigma_1 \sqrt{\tau})}
+ \left( \frac{H(\tau)/S_1}{S_1} \right)^{2\gamma - 2} R(\tau) r^{-\gamma} (2\gamma - 2) \left( \frac{1}{S_1} \right) N(\eta(y_4 - \sigma_1 \sqrt{\tau}))
\]

\[
\frac{\partial^2 [3.5]}{\partial S_1^2} = -y_2 \left( \frac{R(\tau)/bK}{\eta n(y_2)} \right) d^{-\gamma} \frac{1}{(S_1, \sigma_1^2 r)}
- \left( \frac{R(\tau)/bK}{\eta n(y_2)} \right) (2\gamma - 2) \left( \frac{1}{(S_1, \sigma_1^2 r)} \right) N(\eta(y_4 - \sigma_1 \sqrt{\tau}))
\]

\[
\frac{\partial [3.5]}{\partial H(\tau)} = -\frac{R(\tau)/H(\tau)}{(2\gamma - 2)} \left( \frac{H(\tau)/S_1}{S_1} \right)^{2\gamma - 2} N(\eta(y_4 - \sigma_1 \sqrt{\tau}))
- 2 \eta \left( \frac{R(\tau)/bK}{\eta n(y_2)} \right) d^{-\gamma} \left( \frac{S_1}{H(\tau)} \right)
\]

\[
\frac{\partial [3.5]}{\partial K} = 0
\]

\[
\frac{\partial [3.5]}{\partial R(\tau)} = \frac{[3.5]}{R(\tau)}
\]

\[
\frac{\partial [3.5]}{\partial \tau} = -\ln(r)[3.5]
\]

\[
+ \left( \frac{R(\tau)/bK}{\eta n(y_2)} \right) (S_1 d^{-\gamma}/(\sigma_1 \sqrt{\tau})) \left( \ln(H(\tau)/S_1)/\tau + 2 \xi \theta \right)
+ R(\tau) r^{-\gamma} (2\gamma - 2) \xi \theta N(\eta(y_4 - \sigma_1 \sqrt{\tau})) (H(\tau)/S_1)^{2\gamma - 2}
\]

If the solution is correct, these results will satisfy

\[
.5 \sigma_1^2 S_1^2 \frac{\partial^2 [3.5]}{\partial S_1^2} + \left( \frac{r}{d_1} \right) S_1 \frac{\partial [3.5]}{\partial S_1} - \frac{\partial [3.5]}{\partial \tau} = \ln(r)[3.5].
\]

To see this note that

\[
LHS = 2\left( \frac{R(\tau)/bK}{\eta n(y_2)} \right) d^{-\gamma} \left( \ln(r/d_1)/(\sigma_1 \sqrt{\tau}) \right)
- .5y_2 - (\gamma - 1) .5 \sigma_1 \sqrt{\tau} - .5 \ln(H(\tau)/S_1) \left( \frac{1}{S_1} \sigma_1 \sqrt{\tau} \right) - \xi \theta [\sigma_1 \sqrt{\tau}]
\]

\[
+ \left( \frac{H(\tau)/S_1}{S_1} \right)^{2\gamma - 2} R(\tau) r^{-\gamma} (2\gamma - 2) \left( \frac{1}{S_1} \right) N(\eta(y_4 - \sigma_1 \sqrt{\tau})) \left( \ln(r/d_1) - (\gamma - 5) \sigma_1^2 - \xi \theta \right)
+ \ln(r)[3.5] = \ln(r)[3.5]
\]

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Equation [3.5] can be verified to be linearly homogeneous by noting

\[ S_1(t) \frac{\partial [3.5]}{\partial S_1(t)} + K \frac{\partial [3.5]}{\partial K} + H(\tau) \frac{\partial [3.5]}{\partial H(\tau)} + R(\tau) \frac{\partial [3.5]}{\partial R(\tau)} = [3.5]. \]  

The comparative statics for equation [3.6] are

\[ \frac{\partial [3.6]}{\partial S_1} = -m(R(\tau)/S_1)(H(\tau)/S_1)^{\gamma - 1} N(\eta y_2) \]
\[ - 2\eta (R(\tau)/S_1)(H(\tau)/S_1)^{\gamma - 1} n(y_2) / [\sigma_1 \sqrt{\tau}] \]
\[ + m(R(\tau)/S_1)(H(\tau)/S_1)^{\gamma - 1 - m} N(\eta (y_2 - 2m\sigma_1 \sqrt{\tau})) - [3.6](\gamma - 1)/S_1 \]
\[ \frac{\partial^2 [3.6]}{\partial S_1^2} = (\gamma - 1 - m)(\gamma - m)(R(\tau)/S_1)^2 (H(\tau)/S_1)^{\gamma - 1} N(\eta y_2) \]
\[ + (\gamma - 1 - m)(\gamma - m)(R(\tau)/S_1)^2 (H(\tau)/S_1)^{\gamma - 1} N(\eta (y_2 - 2m\sigma_1 \sqrt{\tau})) \]
\[ + 2\eta (R(\tau)/S_1)^2 (H(\tau)/S_1)^{\gamma - 1 - m} n(y_2) (1/\sigma_1 \sqrt{\tau}) (2\gamma + m - 1 - y_2 / \sigma_1 \sqrt{\tau}) \]
\[ \frac{\partial [3.6]}{\partial H(\tau)} = m(R(\tau)/H(\tau))(H(\tau)/S_1)^{\gamma - 1} N(\eta y_2) \]
\[ + 2\eta (R(\tau)/H(\tau))(H(\tau)/S_1)^{\gamma - 1} n(y_2) / [\sigma_1 \sqrt{\tau}] \]
\[ - m(R(\tau)/H(\tau))(H(\tau)/S_1)^{\gamma - 1} N(\eta (y_2 - 2m\sigma_1 \sqrt{\tau})) + [3.6](\gamma - 1)/H(\tau) \]
\[ \frac{\partial [3.6]}{\partial K} = 0 \]
\[ \frac{\partial [3.6]}{\partial R(\tau)} = [3.6] \frac{\partial}{\partial R(\tau)} \]
\[ \frac{\partial [3.6]}{\partial \tau} = -m\xi \theta R(\tau)(H(\tau)/S_1)^{\gamma - 1} N(\eta y_2) \]
\[ - \eta R(\tau)(H(\tau)/S_1)^{\gamma - 1} n(y_2) [y_2 / \tau - m\sigma_1 \sqrt{\tau} + 2\xi \theta / \sigma_1 \sqrt{\tau}] \]
\[ + m\xi \theta R(\tau)(H(\tau)/S_1)^{\gamma - 1} N(\eta (y_2 - 2m\sigma_1 \sqrt{\tau})) - \zeta\beta[3.6] \]
If the solution is correct, these results will satisfy

\[0.5\sigma^2_1 \frac{\partial^2 [3.6]}{\partial r^2} + \ln \left( \frac{r}{d_1} \right) \frac{\partial [3.6]}{\partial r} - \frac{\partial^2 [3.6]}{\partial \tau} = \ln(r) [3.6]. \quad [3.81]\]

To see this note that

\[LHS = R(\tau)(H(\tau)/S_1)^{\gamma -1+m}N(\eta y_3) \left( (\gamma -1)(\xi \theta -\ln(r/d_1)) -m\ln(r/d_1) \right. \]
\[+ (\gamma -1+m)/(\gamma +m).5\sigma^2_1 +m\xi \theta \]
\[-\eta R(\tau)(H(\tau)/S_1)^{\gamma -1+m}(y_3)[2\ln(r/d_1)/ [\sigma_1^2 - y_3 + m\sigma_1^2 / \gamma \]
\[-2\xi \theta / [\sigma_1^2 - (2\gamma +m-1) - y_3 / \sigma_1^2 / \gamma)] \]
\[+ R(\tau)(H(\tau)/S_1)^{\gamma -1+m}N(\eta (y_3 - 2m\sigma_1^2 / \gamma))(\gamma -1)(\xi \theta -\ln(r/d_1)) \]
\[\left. + m[\ln(r/d_1) -\xi \theta] + .5\sigma^2_1 (\gamma -1-m)(\gamma -m) \right) = \ln(r) [3.6] \quad [3.82]\]

because, \(m^2 \sigma^2_1 = (\gamma -1)^2 \sigma^2_1 + 2m\ln(r)\).

Thus, \((\gamma -1)(\xi \theta -\ln(r/d_1)) -m\ln(r/d_1) + (\gamma -1+m)/(\gamma +m).5\sigma^2_1 +m\xi \theta \)
\[= (\gamma -1)(\xi \theta -\ln(r/d_1)) +m\ln(r/d_1) + (\gamma -1-m)(\gamma -m).5\sigma^2_1 -m\xi \theta = \ln(r) - \xi \beta. \]

Equation [3.6] can be verified to be linearly homogeneous by noting

\[S_1(0) \frac{\partial [3.6]}{\partial S_1(0)} + \frac{\partial [3.6]}{\partial K} + H(\tau) \frac{\partial [3.6]}{H(\tau)} + R(\tau) \frac{\partial [3.6]}{\partial R(\tau)} = [3.6]. \quad [3.83]\]
III.H Appendix IIIIC: Relationship Between the Standard Barrier Model and the Black-Scholes Model

In this appendix it is proven that (1) the constant barrier is simply a special case of equations [3.1]-[3.6], and (2) the Black-Scholes model is the limiting solution (when H goes to zero) of the down-and-out call (H ≤ K) and put (H ≤ K) formulas.

III.H.1 Constant Barrier Model

To make the results listed in equations [3.1]-[3.6] applicable to a barrier level that is constant over time set θ = 0, which implies that H = bK. This transformation implies that the results listed in equation [3.7] can then be stated in terms of actual prices instead of relative prices. That is,

\[
\begin{align*}
\phi_{\bar{Z}_T}(z) &= \frac{1}{\sigma_1 \sqrt{\tau}} \exp \left( \frac{z-\mu \tau}{\sigma_1 \sqrt{\tau}} \right), \\
\psi_{\bar{Z}_T}(z) &= \exp \left( \frac{2 \ln(H/S_0) \mu}{\sigma_1^2} \right) \frac{1}{\sigma_1 \sqrt{\tau}} n \left( \frac{z-2 \ln(H/S_0)-\mu \tau}{\sigma_1 \sqrt{\tau}} \right), \\
h(\tau^*) &= \frac{\eta \ln(S_0/H)}{\sigma_1 \sqrt{\tau^*}} n(y_2(\tau^*) - \sigma_1 \sqrt{\tau^*}),
\end{align*}
\]

\[Z(T) = \ln[S_0(T)/S_0], \quad \mu = \ln(r/d_p) - 0.5 \sigma_2^2, \quad \gamma = (\ln(r/d_p) + 0.5 \sigma_2^2)/\sigma_2^2, \]

\[
y_1 = \frac{\ln(S_0/K) + (\ln(r/d_p) + 0.5 \sigma_2^2) \tau}{\sigma_1 \sqrt{\tau}}, \\
y_2 = \frac{\ln(H/S_0) + (\ln(r/d_p) + 0.5 \sigma_2^2) \tau}{\sigma_1 \sqrt{\tau}}, \\
y_3 = \frac{\ln(H^2/S_0 K) + (\ln(r/d_p) + 0.5 \sigma_2^2) \tau}{\sigma_1 \sqrt{\tau}}, \\
y_4 = \frac{\ln(H/S_0) + (\ln(r/d_p) + 0.5 \sigma_2^2) \tau}{\sigma_1 \sqrt{\tau}}, \\
y_5 = \frac{\ln(H/\bar{S}_0) + m \sigma_2^2 \tau}{\sigma_1 \sqrt{\tau}}, \\
m = \frac{\sqrt{(\ln(r/d_p) - 0.5 \sigma_2^2)^2 + 2 \ln(\gamma - \beta) \sigma_2^2}}{\sigma_1^2}.
\]

In like manner, equations [3.1]-[3.6] can be simplified and rewritten as
\[ r^{-t} \int_{\ln(K/S)}^{\infty} (S_t e^{z-K}) f(z) dz = \phi S_t d_t^{-N} N(\phi y_1) - \phi K r^{-t} N(\phi (y_1 - \sigma_1 \sqrt{t})). \] [3.85]

\[ r^{-t} \int_{\ln(K/S)}^{\infty} (S_t e^{z-K}) g(z) dz = \phi (H/S_t)^{2r-2} \left[ S_t d_t^{-r} (H/S_t)^{2} N(\eta y_3) - K r^{-t} N(\eta (y_3 - \sigma_1 \sqrt{t})) \right]. \] [3.87]

\[ r^{-t} \int_{\ln(H/S_t)}^{\infty} (S_t e^{z-K}) g(z) dz = \phi (H/S_t)^{2r-2} \left[ S_t d_t^{-r} (H/S_t)^{2} N(\eta y_4) - K r^{-t} N(\eta (y_4 - \sigma_1 \sqrt{t})) \right]. \] [3.88]

\[ \eta R(\tau) r^{-t} \int_{\ln(H/S_t)}^{\infty} [f(z) - g(z)] dz = R(\tau) r^{-t} \left[ N(\eta (y_2 - \sigma_1 \sqrt{t})) - (H/S_t)^{2r-2} N(\eta (y_4 - \sigma_1 \sqrt{t})) \right]. \] [3.89]

\[ r^{-t} \int_{0}^{\tau} R(\tau^*) r^{-t} h(\tau^*) d\tau^* \] [3.90]

\[ = R(\tau) (S_t/H)[(H/S_t)^{1-w} N(\eta y_3) + (H/S_t)^{1-w} N(\eta (y_3 - 2m_\sigma \sqrt{t}))]. \]

which is consistent with the six equations derived by Rubinstein and Reiner (1991a).

III.H.2 Black-Scholes Model as the Limiting Solution

The Black-Scholes model can be considered as a down-and-out option with zero rebate. This result can be proven two ways.

First, recall the underlying asset price dynamics assumed in the Black-Scholes model (geometric Brownian motion): \( dS_t(t) = S_t \sigma dt + S_t \sigma dZ \). When \( S_t = H = 0 \) the process clearly ceases to exist. Any random variable whose movements can be described by geometric Brownian motion is, by definition, said to be restricted by a natural absorbing barrier at zero. This has two
widely cited, but seldom proven, implications. First, a random variable that follows geometric Brownian motion can take on only non-negative values. Second, in the absence of arbitrage possibilities, if the stock price ever enters the absorbing state it must stay there. If this were not true an investor could purchase the asset for $0 and, based on our first result, be assured that all future asset values are non-negative.

In the previous paragraph it is established that Black-Scholes options must be identical in value to down-and-out options (with zero rebate) when the barrier level set equal to zero. To confirm this finding, it is proven that the Black-Scholes formula can be obtained as the limit (as \( H \) goes to zero) of the down-and-out call \( (K \leq H) \) and put formulas.

Equations [3.3] and [3.4] are premultiplied by \( H \). Thus, as \( H \) goes to zero, equations [3.3] and [3.4] go to zero with probability one. Equation [3.2] also goes to zero as \( H \) goes to zero (when \( \phi = -1 \)). To see this note that as \( b \) goes to zero \( H \) goes to zero. As \( b \) goes to zero, \( y_1 \) and \( (y_1 - \sigma \sqrt{t}) \) go to infinity; hence, \( N(-y_1) \) and \( N(-(y_1 - \sigma \sqrt{t})) \) go to zero as \( H \) goes to zero. The down-and-out call and put formulas (when \( H \leq K \)), thus, converge in value to equation [3.1], which is the Black-Scholes model, as \( H \) goes to zero.
CHAPTER IV
The Valuation of Options With a Stochastic Strike Price and a Stochastic Boundary

IV.A Introduction

An option to exchange one asset for another has been found to have several applications. Margrabe (1978) and Fischer (1978) derive a valuation formula for such an option when the option is not restricted by any underlying boundaries. In this chapter, the exchange option pricing model literature is integrated with the barrier option literature and the first stochastic barrier model is presented. A valuation formula is developed for options with a stochastic strike price (like the exchange model) and a stochastic underlying boundary and an application of this model is considered.

It is important to note a key relationship between the non-stochastic barrier model developed in Chapter III and the model developed in this chapter. The non-stochastic barrier model can also be viewed as an exchange option; the holder has the option to exchange cash (the exercise price, K) for asset one, S₁, at expiration if the option is "alive". Based on this underlying connection between options with a non-stochastic strike price and those with a stochastic strike price, many of the results presented in this chapter parallel those presented in Chapter III.

\footnote{See Rubinstein (1991).}

\footnote{Margrabe (1978) is the to first point out this relationship between exchange options and Black-Scholes options.

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IV.B The Theoretical Model

Unless otherwise specified, the notation set forth in Chapter III continues to be used with the exception of $H(t)$ denoting the barrier level. In this chapter an option pricing formula is developed that allows the strike price to be stochastic and the state space to be demarcated by a continuous stochastic barrier level. $S_2(t)$ denotes the position of the barrier at time $t$ (instead of $H(t)$). $S_1(t)$ again represents the price of the optioned asset price at time $t$. When referring to the $i$th ($i=1,2$) initial asset price, the time parameter subscript is suppressed (i.e., $S_i(t_i) = S_i$). The rate of return for each underlying asset is assumed to follow the time homogeneous lognormal diffusion process, $dS_i(t)/S_i = \alpha_i dt + \sigma_i dZ_i(t)$, where $\alpha_i$ is the instantaneous drift, $\sigma_i$ is the instantaneous standard deviation, and $dZ_i(t)$ is a Weiner process. The correlation between $Z_i$ and $Z_j$ is $\rho_{ij}$. Let $d_i$ represent one plus the payout rate of the underlying asset $i$.

It is assumed that $S_2$ is the price of a traded asset. If this assumption is found to be inaccurate the stochastic barrier model derived in this chapter could be modified, in the spirit of Fischer (1978), to incorporate this feature.\footnote{Fischer (1978) assumed that the capital markets are perfect and complete in the sense that the price of the hedged security could always be determined, if it were not traded, from the full set of pure (Arrow-Debreu) securities.} However, it is presumed that "if speculators needed the hedged security it would be created" (Fischer (1978, p. 174)).

IV.B.1 The Stochastic Barrier Model With a Stochastic Strike Price and a Stochastic Boundary

Consider a stochastic barrier option of the form $W_i(S_1, XS_2, B(r)S_2, B(t)S_2, r)$, where $X$ and $\delta$ are prespecified non-negative constants, $B(r) = bXe^{\delta r}$ is a known exponential function of time to expiration, and all other notation is as previously defined.\footnote{The rebate is initially specified as $\delta S_2$. At the end of this section other possible specifications of the rebate are considered.} Due to the stochastic strike price, following Margrabe (1978), such an option is referred to as a stochastic barrier exchange option.

The payoffs for the "out" and "in" exchange options are, respectively,
\[
W_{\text{out}}(S_1, XS_2, B(\tau)S_2, \mathbb{R}S_2) = \begin{cases}
\max(0, \phi S_1(T) - \phi XS_2(T) : \tau = 0) \\
\mathbb{R}S_2(t) & \text{if } \eta S_1(t) > \eta B(t) S_2(t) \forall t \\
\text{for some } t
\end{cases}
\]

\[
W_{\text{in}}(S_1, XS_2, B(\tau)S_2, \mathbb{R}S_2) = \begin{cases}
\max(0, \phi S_1(T) - \phi XS_2(T) : \tau = 0) \\
\mathbb{R}S_2(t) & \text{if } \eta S_1(t) \leq \eta B(t) S_2(t) \text{ for some } t \\
\text{for some } t
\end{cases}
\]

In other words, the "out" option allows a holder to exchange \( X \) units of \( S_2 \) for a unit of \( S_1 \) at expiry if \( \eta S(t) = \eta[S_1(t)/S_2(t)] > \eta B(t) \) for all \( t \in [t_0, T] \). If for some \( t \in [t_0, T] \eta S(t) \leq \eta B(t) \), the "out" exchange option expires and the owner receives a rebate of \( \mathbb{R}S_2(t) \) at hit. The "in" option allows a holder to exchange \( X \) units of \( S_2 \) for a unit of \( S_1 \) at expiry if \( \eta S(t) \leq \eta B(t) \) for some \( t \in [t_0, T] \). If for all \( t \in [t_0, T] \eta S(t) > \eta B(t) \), the "in" exchange option expires and the owner receives a rebate of \( \mathbb{R}S_2(T) \) at expiration. Note that if \( X = 1 \) and \( B = \mathbb{R} = 0 \), the stochastic barrier exchange option would simply be the standard exchange option of Margrabe (1978). Further note that a stochastic barrier call option (with no rebate) written on \( S_1 \) with an exercise price of \( XS_2 \), is identical in value to a stochastic barrier put option written on \( XS_2 \) with an exercise price of \( S_1 \). This result holds for both "in" and "out" options and is consistent with the standard exchange model.

The option \( W_x(S_1, XS_2, B(\tau)S_2, \mathbb{R}S_2, \tau) \) is linearly homogeneous with respect to \( S_1 \) and \( S_2 \). From Euler's Theorem, the value of a stochastic barrier exchange option is shown to be the solution to

---

This is, of course, because both options have an identical terminal payoff of \( \max(S_1(T) - XS_2(T), 0) \).

A proof of this statement is presented in Appendix IVA.
\[
\begin{align*}
\ln(r/d_1)S_1 \frac{\partial W(f)(t)}{\partial S_1} + \ln(r/d_2)S_2 \frac{\partial W(f)(t)}{\partial S_2} - \frac{\partial W'(f)(t)}{\partial \tau} \\
+ \frac{\sigma_1^2 S_1^2}{2} \frac{\partial^2 W(f)(t)}{\partial S_1^2} + \frac{\sigma_2^2 S_2^2}{2} \frac{\partial^2 W(f)(t)}{\partial S_2^2} - \sigma_1 \sigma_2 \rho_{12} S_1 S_2 = \ln\left(\frac{S_1}{S_2}\right)
\end{align*}
\]

subject to [4.1], \(W_s(0, X, S_1, B(t), T_s, \mathfrak{R}, \mathfrak{R} | \sigma^2) = 0\), and \(0 \leq W_s(S_1, X, S_1, B(t), S_2, \mathfrak{R}, T_s, \mathfrak{R} | \sigma^2) \leq S_1\). Following Merton (1973) and Margrabe (1978), linear homogeneity allows one to quickly determine the solution to [4.2] by performing a change of numeraire. Let asset two be the numeraire. The current value of a stochastic barrier exchange option is \(S_2 W_{STO}(S, X, B(\tau), \mathfrak{R}, \mathfrak{R} | \sigma^2)\) where \(W_{STO}(S, X, B(\tau), \mathfrak{R}, \mathfrak{R} | \sigma^2)\) is determined using equations [3.1]-[3.6] with \(K = X\), \(H(\tau) = B(\tau)\), \(R(\tau) = \mathfrak{R}\) \((\beta = 0)\), \(r = d_2\), an underlying asset price of \(S = S_1/S_2\), and \(\sigma^2 = \text{Var}[d((S_1(t)/S_2(t))(S_2/S_1))] = \sigma_1^2 + \sigma_2^2 - 2\sigma_1 \sigma_2 \rho_{12}\). After simplifying, one can rewrite equations [3.1]-[3.6] and write the valuation solution for options falling under the heading of barrier exchange options as a linear combination of the following equations.

\[\phi S_1 d_1^{-1}N(\phi y_1) - \phi XS_2 d_2^{-1}N(\phi (y_2 - \sigma \sqrt{\tau})).\]  

\[\phi S_1 d_1^{-1}N(\phi y_1) - \phi XS_2 d_2^{-1}N(\phi (y_2 - \sigma \sqrt{\tau})).\]  

\[\phi (B(\tau) S_1 S_2)^{\frac{1}{2}} \left[ S_1 d_1^{-1}N(\eta y_2 (B(\tau) S_1 S_2)^{\frac{1}{2}} - XS_2 d_2^{-1}N(\eta (y_2 - \sigma \sqrt{\tau}))\right].\]  

\[\phi (B(\tau) S_1 S_2)^{\frac{1}{2}} \left[ S_1 d_1^{-1}N(\eta y_2 (B(\tau) S_1 S_2)^{\frac{1}{2}} - XS_2 d_2^{-1}N(\eta (y_2 - \sigma \sqrt{\tau}))\right].\]

\[\phi (B(\tau) S_1 S_2)^{\frac{1}{2}} \left[ S_1 d_1^{-1}N(\eta y_2 (B(\tau) S_1 S_2)^{\frac{1}{2}} - XS_2 d_2^{-1}N(\eta (y_2 - \sigma \sqrt{\tau}))\right].\]

---

46 That is, a change of variables is performed on the underlying asset converting \(S_1\) to \(S = S_1/S_2\).

47 In the Black-Scholes model, an instantaneous riskless hedge is constructed and investors are assumed to be able to continually update and maintain this hedge throughout the life of the derivative security. Hence, the investment required for this hedge is the riskless rate of return (per unit of time). In the Margrabe exchange model and the model under consideration here, a riskless hedge is formed by shorting asset two. Thus, the investment required to form this hedge is \(d_2\); this is why \(r = d_2\) in this model.

Chapter IV 62
\[85. f^\tau \left[ N(\eta (y_\gamma - \sigma \sqrt{\tau})) - (B(\tau)S_f / S_i)^{1/2} N(\eta (y_\zeta - \sigma \sqrt{\tau})) \right]. \tag{4.7} \]

\[85. f^\tau \left[ (B(\tau)S_f / S_i)^{1/2} \cdot N(\eta y_{10}) + (B(\tau)S_f / S_i)^{1/2} N(\eta (y_{10} - 2m \sigma \sqrt{\tau})) \right]. \tag{4.8} \]

where

\[\lambda = 2 \left[ \ln(d_2) - \xi - \frac{1}{2} \sigma^2 \right] / [\sigma^2],\]

\[y_6 = \left[ \ln(S_i / S_s) - \ln(X) + (\ln(d_2 / d_1) + 5\sigma^2 \tau) / [\sigma \sqrt{\tau}],\right.\]

\[y_7 = \left[ \ln(S_i / S_s) - \ln(bX) + (\ln(d_2 / d_1) + 5\sigma^2 \tau) / [\sigma \sqrt{\tau}],\right.\]

\[y_8 = \left[ \ln(S_i / S_s) - \ln(XB^2(\tau)) + (\ln(d_2 / d_1) + 5\sigma^2 \tau) / [\sigma \sqrt{\tau}],\right.\]

\[y_9 = \left[ \ln(S_i / S_s) - \ln(bX/B^2(\tau)) + (\ln(d_2 / d_1) + 5\sigma^2 \tau) / [\sigma \sqrt{\tau}],\right.\]

\[y_{10} = \left[ \ln(B(\tau)S_f / S_i) + 5m \sigma^2 \tau / [\sigma \sqrt{\tau}],\right.\]

\[m^* = \sqrt{5 \lambda \sigma^2} + 2 \ln(d_2) \sigma^2 / [\sigma^2],\]

and \(\eta\) is again defined to be 1 if the barrier is being approached from above (but now this implies \(S_i > B(\tau)S_f\)) and -1 if the barrier is being approached from below. The valuation solutions for stochastic barrier exchange options are displayed in Table IV.B.1.a and the comparative statics are presented in Appendix IVB. When \(X=1\) and \(R=0\), it is easy to verify that as \(B(\tau)\) goes to zero the down-and-out exchange call \((X > B(\tau))\) and put \((X > B(\tau))\) formulas converge in value to equation \([4.3]\), which is the Margrabe (1978) model.

Some numerical values for stochastic barrier exchange options and standard (Margrabe) exchange options are presented in Table IV.B.1.b. The results are quite intuitive. Observe that a negative correlation between \(S_i\) and \(S_s\) increases the variance \(\sigma^2\). This increases the probability that "in" ("out") options written on \(S = S_i / S_s\) will start (prematurely expire) and decreases (increases) the probability that the rebate is received. The farther that the initial stock price is away from the barrier, the greater the influence of the correlation coefficient.

Equations \([4.7]\) and \([4.8]\) are recognized, respectively, as the current value of the "in" rebate and the value of the "out" rebate. In the original specification of this model (i.e., equations \([4.7]\) and \([4.8]\)), \(85S_2\) is stated as the rebate. Other rebate specifications are now considered. To see that this is possible, note from equations \([3.5]\) and \([3.6]\) and \([4.7]\) and \([4.8]\), that the rebate value of a barrier option is completely detachable from the non-rebate value of the option. In fact, the rebate is, itself, an option. Hence, a barrier option with a rebate is
<table>
<thead>
<tr>
<th>EUROPEAN OPTION TYPE</th>
<th>VALUATION EQUATION</th>
<th>$\eta$</th>
<th>$\phi$*</th>
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<tr>
<td>Up-and-In Call ($X &gt; B(t)$)</td>
<td>[4.3]+[4.7]</td>
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<td>1</td>
</tr>
<tr>
<td>Up-and-In Call ($X \leq B(t)$)</td>
<td>[4.4]-[4.5]+[4.6]+[4.7]</td>
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<td>1</td>
</tr>
<tr>
<td>Up-and-In Put ($X &gt; B(t)$)</td>
<td>[4.3]-[4.4]+[4.6]+[4.7]</td>
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<td>-1</td>
</tr>
<tr>
<td>Up-and-In Put ($X \leq B(t)$)</td>
<td>[4.5]+[4.7]</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>Down-and-In Call ($X &gt; B(t)$)</td>
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<td>1</td>
</tr>
<tr>
<td>Down-and-In Call ($X \leq B(t)$)</td>
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<td>Down-and-In Put ($X &gt; B(t)$)</td>
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<td>-1</td>
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<tr>
<td>Down-and-In Put ($X \leq B(t)$)</td>
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<td>-1</td>
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<tr>
<td>Up-and-Out Call ($X &gt; B(t)$)</td>
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<td>1</td>
</tr>
<tr>
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<td>-1</td>
<td>1</td>
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<td>Up-and-Out Put ($X &gt; B(t)$)</td>
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<td>-1</td>
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<td>Up-and-Out Put ($X \leq B(t)$)</td>
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<td>-1</td>
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<tr>
<td>Down-and-Out Put ($X \leq B(t)$)</td>
<td>[4.8]</td>
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<td>-1</td>
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</table>

* $\eta$ is defined to be 1 if the barrier is being approached from above (i.e., $S_t > B(t)S_0$) and -1 if the barrier is being approached from below.

** $\phi$ is defined to be 1 if the barrier exchange option is a call and -1 if the option is a put.
Table IV.B.1.b
Values for Stochastic Barrier and Standard Exchange Options*

<table>
<thead>
<tr>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$\rho_{12}$</th>
<th>$\tau$</th>
<th>$C_{\text{in}}(\cdot)$</th>
<th>$C_{\text{out}}(\cdot)$</th>
<th>$C_{X}(\cdot)$</th>
<th>$P_{\text{in}}(\cdot)$</th>
<th>$P_{\text{out}}(\cdot)$</th>
<th>$P_X(\cdot)$</th>
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</table>

* $C_{\text{in}}(\cdot)$, $C_{\text{out}}(\cdot)$, and $C_{X}(\cdot)$ denote, respectively, call option values for "in" and "out" stochastic barrier exchange options, and standard (Margerbe) exchange options. $P_{\text{in}}(\cdot)$, $P_{\text{out}}(\cdot)$, and $P_X(\cdot)$ represents otherwise identical put option values. All values represent options to exchange $S_i(T)$ with $X S_x(T)$ with parameter values: $X=1$, $b=1.1$, $\theta=0$, $d_1=d_2=\xi=1$, $\sigma_1=.2$, $\sigma_2=.3$, and $\delta_k=0$; unless otherwise specified.
technically valued as a portfolio of options.

If the rebate is specified as $RS_1$, instead of $RS_2$, the value of the "in" and "out" rebates can, respectively, be written as

$$RS_1d_1^{-1}[N(\eta(y_7-\sigma_1\sqrt{t})) - (S_0/B\tau S_1)^{1/2}N(\eta(y_9-\sigma_1\sqrt{t}))].$$  \[4.10\]

$$RS_1[(S_0/B\tau S_2)^{1/2}N(\eta(y_{10}\lambda_1) + (S_0/B\tau S_2)^{1/2}N(\eta(y_{10}-2\sigma_1\sqrt{t}))].$$  \[4.11\]

where

$$\lambda = 2[\ln(d_1/d_2)-\xi\theta-0.5\sigma_1^2] / [\sigma_1^2],$$
$$y_7 = [\ln(S_0/S_1)+\ln(bX)+[(\ln(d_1/d_2)+0.5\sigma_1^2)t] / [\sigma_1\sqrt{t}]$$
$$y_9 = [\ln(S_0/S_2)+\ln(bX/B\tau(t))+[(\ln(d_1/d_2)+0.5\sigma_1^2)t] / [\sigma_1\sqrt{t}]$$
$$y_{10} = [\ln(S_0/B\tau S_2)+\ln(bX/B\tau(t))+[(\ln(d_1/d_2)+0.5\sigma_1^2)t] / [\sigma_1\sqrt{t}]$$
$$\bar{m} = \sqrt{(\lambda d_1^2 + 2\ln(d_1)/\sigma_1^2} / [\sigma_1^2].$$  \[4.12\]

In this case, the current value of the barrier exchange options is determined by replacing \[4.7\] with \[4.10\] and \[4.8\] with \[4.11\] in Table IV.B.1.a, leaving equations \[4.3\]-\[4.6\] unchanged. The rebate equations \[4.10\] and \[4.11\] are also verified in Appendix IVB.

To confirm equations \[4.10\] and \[4.11\] let $RS_1(B(r)S_2,r)$ denote the current value of this rebate. This rebate option is homogeneous of degree one with respect to $S_1$ and $S_2$. Therefore, $RS_1(B(r)S_2,r) = R(S_2,S_1) = R(B(r)S_2,r)$ where $S_2 = S_2/S_1$. The rebate value is now written in terms of one underlying asset, $B\bar{S}$, and the critical threshold determining the payoff of this option is unity. Accordingly, the current value of the "in" and "out" rebates can, respectively, be determined as $S_1$, multiplied by equations \[3.5\] and \[3.6\] when: (1) the underlying asset price is $B\bar{S}$, (2) whose payout rate is $d_2$, (3) $H(r)=1$, (4) $R(r)=R$, (5) $\sigma_1^2 = \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{1,2}$, and (6) $t=d_1$.

As a third possible recovery specification, consider the case found in Chapter III; suppose the rebate is specified as a deterministic function of time to expiry. Specifically, define the rebate as $\Re^{\text{dbr}}$, where $\beta$ and $\beta'$ are as defined in Chapter III, and recall that the critical threshold level is still defined as: $S_1(t)=B(t)S_2(t)$. Let $I(\Re^{\text{dbr}}, S_1, B(r)S_2, r)$ denote the current value of this...
rebate. Notice that in this case the rebate option is homogeneous of degree zero with respect to $S_1$ and $B(\tau)S_2$. Therefore, one can write $I(\Re^{+}\tau, S_1, B(\tau)S_2, \tau) = I(\Re^{+}\tau, S_1, \tau)$ where $S = B(\tau)S_2/S_1$. Now this valuation problem is extremely similar to the rebate valuation problem that is solved in Chapter III. In this case, the underlying asset is $S$, whose instantaneous continuously compounded mean and variance is determined from Itô's lemma as:

$[\ln(d_j/d_t) - \xi \theta - \frac{1}{2} \sigma_1^2] \text{ and } \sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\sigma_1 \sigma_2 \rho_{1,2}$, respectively. Thus,

\begin{equation}
\Re^{r-1}[N(\eta \bar{y}) - (B(\tau)S_2/S_1)^{\frac{1}{2}}N(\eta \bar{y}_0)]
\end{equation}

\begin{equation}
\Re^{r-1}[\Re^{(B(\tau)S_2/S_1)^{\frac{3}{2}}} N(\eta \bar{y}_1) \text{ and } (B(\tau)S_2/S_1)^{\frac{1}{2}} N(\eta (\bar{y}_0 - 2\lambda\sigma\sqrt{\tau}))].
\end{equation}

where

\begin{equation}
\begin{align*}
\lambda &= 2[\ln(d_j/d_t) - \xi \theta - \frac{1}{2} \sigma_1^2] / [\sigma^2], \\
\bar{y}_t &= \frac{[\ln(S_1/S_2) - \ln(BX_0) + \ln(d_j/d_t) - \frac{1}{2} \sigma_1^2] \tau} {\sigma \sqrt{\tau}}, \\
\bar{y}_0 &= \frac{[\ln(S_1/S_2) - \ln(BX_0) + \ln(d_j/d_t) - \frac{1}{2} \sigma_1^2] \tau} {\sigma \sqrt{\tau}}, \\
\bar{y}_0 &= \frac{[\ln(BX_0) + \lambda \sigma^2 \tau]} {\sigma \sqrt{\tau}}, \\
\bar{y}_1 &= \frac{[\ln(BX_0) + \lambda \sigma^2 \tau]} {\sigma \sqrt{\tau}}.
\end{align*}
\end{equation}

\begin{equation}
\begin{align*}
\Re^{r-1}[N(\eta \bar{y}_0) - (B(\tau)S_2/S_1)^{\frac{1}{2}}N(\eta \bar{y}_0)]
\end{align*}
\end{equation}

Equation [4.13] is the current value of the "in" rebate in this case and [4.14] is the corresponding "out" rebate value. Equations [4.13] and [4.14] are (also) verified in Appendix IVB.

IV.C Parity Results

In Chapter III it is noted that under certain conditions barrier options with a non-stochastic exercise price and a time-dependent deterministic barrier level can be hedged with Black-Scholes options. In the previous section a parity relationship between stochastic barrier down-and-in exchange call options and Margrabe exchange put options is also noted. This result is now generalized by demonstrating, similar to that in Chapter III, that under certain conditions all of the results listed in Tables IVA can be hedged with Margrabe exchange options. The
critical assumption in the results that follow is, again, the requirement of zero rebate. Before proceeding a couple of useful identities are stated.

An "in" barrier exchange option with zero rebate combined with an otherwise identical "out" option must be equivalent in value to a Margrabe exchange option. This parity relationship holds for puts and calls, "outs" and "ins", whether $B(t)$ is an increasing or decreasing over time and is stated formally as

$$W_{x,n}(S_1, X S_2, B(t)) S_2, \tau) + W_{x,b}(S_1, X S_2, B(t)) S_2, \tau) = W_x(S_1, X S_2, \tau) \tag{4.16}$$

It is also useful to recall put-call parity from exchange option pricing theory.

$$W_x(S_1, X S_2, \tau) - W_x(X S_2, S_1, \tau) = S_1 - X S_2 \tag{4.17}$$

With this, when $\theta = 0$ observe from equations [4.3]-[4.6] when combined in accordance with that stated in Table IV.B.1.a,

$$C_{x,Do}(S_1, S_2, X, B, \theta = 0) = (1/b) C_x(S_2 b X, S_1, \tau) \tag{4.18}$$

$$C_{x,Do}(S_1, S_2, X, B, \theta = 0) = C_x(S_1, S_2, X, \tau) - (1/b) C_x(S_2 b X, S_1, \tau)$$

$$C_{x,Up}(S_1, S_2, X, B, \theta = 0) = (1/b) C_x(S_1, S_2 b X, \tau)$$

$$+ (1 - (1/b)) [C_x(S_1, S_2 b X, \tau) + 2 S_2, C_{BS:C-O-N}(S_2 b X, \tau)] \tag{4.19}$$

$$C_{x,Up}(S_1, S_2, X, B, \theta = 0) = C_x(S_1, S_2 X, \tau) - (1/b) C_x(S_1, S_2 b X, \tau)$$

$$- (1 - (1/b)) [C_x(S_1, S_2 b X, \tau) + 2 S_2, C_{BS:C-O-N}(S_2 b X, \tau)]$$

$$C_{x,Do}(S_1, S_2, X, B, \theta = 0) = (1/b) [C_x(S_1, S_2 b X, \tau) - C_x(S_2 b X, S_1, \tau)]$$

$$+ (1 - (1/b)) [C_x(S_1, S_2 b X, \tau) + 2 S_2, C_{BS:C-O-N}(S_2 b X, \tau)] \tag{4.20}$$

$$C_{x,Do}(S_1, S_2, X, B, \theta = 0) = (1/b) [C_x(S_2 b X, S_1, \tau) - C_x(S_1, S_2 b X, \tau)]$$

$$+ C_x(S_1, S_2, X, \tau) - (1 - (1/b)) [C_x(S_1, S_2 b X, \tau) + 2 S_2, C_{BS:C-O-N}(S_2 b X, \tau)]$$
\[ P_{x:UO}(S_1, S_2, X, BS_2, \tau : X \geq B, \theta = 0) = (1/b)[C(S_2, b^2 X, S_1, \tau ) - C(S_1, S_2, b^2 X, \tau )] + (1/(b^2))[C(S_2, b X, S_1, \tau ) - 2S_2 P_{BS:C-O-N}(S_2, b X, \tau )] \]  
[4.21]

\[ P_{x:UO}(S_1, S_2, X, BS_2, \tau : X \geq B, \theta = 0) = (1/b)[C(S_1, S_2, b^2 X, \tau ) - C(S_2, b^2 X, S_1, \tau )] + C(S_2, X, S_1, \tau ) - (1/(b^2))[C(S_2, b X, S_1, \tau ) - 2S_2 P_{BS:C-O-N}(S_2, b X, \tau )] \]  
\[ \text{[4.22]} \]

\[ P_{x:DO}(S_1, S_2, X, BS_2, \tau : X \geq B, \theta = 0) = (1/b)[C(S_2, b X, S_1, \tau ) - 2S_2 P_{BS:C-O-N}(S_2, b X, \tau )] \]  
\[ \text{[4.23]} \]

\[ P_{x:DO}(S_1, S_2, X, BS_2, \tau : X \geq B, \theta = 0) = (1/b)[C(S_2, b X, S_1, \tau ) - 2S_2 P_{BS:C-O-N}(S_2, b X, \tau )] \]
\[ \text{[4.24]} \]

\[ C_{x:UO}(S_1, S_2, X, BS_2, \tau : X \geq B, \theta = 0) = C(S_1, S_2, X, \tau ) \]
\[ \text{[4.25]} \]

where \( W_{BS:C-O-N}(.) \), as defined in Chapter III, is the path independent cash-or-nothing at expiration option.

Equations [4.18]-[4.25] illustrate that when the underlying asset has zero drift, barrier exchange options can be hedged with path independent options. This, again, implies that, at times, option price risk can be hedged using a static (as opposed to a dynamic) replicating portfolio strategy.\(^{49}\)

\(^{49}\) As in Chapter III, this result has potential implications for traders using the binomial model to value barrier exchange options. See Cox and Rubinstein (1985, p. 409) for a discussion on how the binomial model is used to value an European barrier option with a constant barrier level.
IV.D Options With Random Maturity Dates

Consider an exchange option with an uncertain expiration date that has the following payoffs.

\[
W^*_{X,\text{out}}(S_1, X_S, BS_2) = \begin{cases} 
\max(0, \phi S_1(t) - \phi X_S(t)) & \text{if } \eta S_1(t) > \eta BS_2(t) \forall t \\
\max(0, \phi S_1(t^*) - \phi X_S(t^*)) & \text{if } \eta S_1(t^*) \leq \eta BS_2(t^*)
\end{cases}
\]

where \( T \) is the nominal expiration date, \( t^* \) (a random variable) is the first time at which \( \eta S_1(t) \leq \eta BS_2 \), and \( S_1, S_2, X \), and \( B = bX (\theta = 0) \) are as previously defined. This problem initially appears to be unmanageable because the time at which the barrier is first hit and the value of \( S_1 \) and \( S_2 \) at this point in time each appear to be random variables. Closer inspection will reveal, however, that only the first passage time is a random variable. This is because at hit \( S_1(t^*) = BS_2(t^*) \). A rational option holder will exercise this option at hit only if \( S_1(t^*) > X_S(t^*) \). Thus, a rational holder will exercise this option only if \( BS_2(t^*) > X_S(t^*) \), implying \( B \geq X \). The value of this option is, accordingly, equivalent to the value of a stochastic barrier "out" option (see Table IV.B.1.a) with \( \Gamma = (B - X) \) if \( B \geq X \) and \( \Gamma = 0 \) if \( B < X \).

IV.E Application

In Chapter V, two stochastic barrier option pricing models are developed and a detailed application of one of the models is offered in Chapters VI. Prior to closing this chapter, however, a brief application of the barrier exchange model is considered. The application that follows is not fully developed because this model is not the focus of the dissertation. As stated in Chapter I, the primary reason for developing the barrier exchange model is not to apply this model to a specific valuation problem per se, but to progressively establish the foundation for the general stochastic barrier models that are presented in the next chapter. Nevertheless, a potential application of the barrier exchange model to the pricing of deposit insurance is discussed next.

Generalization of their approach to barrier options with a barrier level that changes exponentially with time to expiration is quite straightforward after performing a change of variables to relative prices. Furthermore, generalization to exchange barrier options follows from the procedure outlined by Rubinstein (1991).
IV.E.1 Application to the Pricing of Deposit Insurance

The inefficiencies of the current uniformly priced Federal deposit insurance system are widely recognized (by regulators and academicians alike) and well documented.\textsuperscript{50} The fixed-rate system charges all banks an equal percentage of their total assessable deposits, regardless of individual riskiness. While critics of this antiquated approach have proposed numerous reforms, one ironic issue remains largely unresolved: actually determining the true cost/value of FDIC insurance.\textsuperscript{51} Due to federal regulation and the accompanying surveillance, pricing deposit insurance is far more complicated than simply accounting for the explicit rate charged. How then, can it be convincingly argued that an alternative system would be more efficient when it has yet to be determined how inefficient the current system is, if at all? I will focus on determining how deposit insurance affects the value of the equity holders’ call option.

Deposit insurance has historically been viewed as alleviating depositor risk by truncating a bank’s downside risk (Merton, 1977). However, the influence of deposit insurance on the value of the equity holder’s call option on the assets of the firm has been largely ignored. A complication arises when valuing equity as a call option for financial firms because deposit insurance automatically entails regulatory surveillance by the guarantor. In the interest of social welfare (i.e., to maintain the soundness of the banking industry), the FDIC can declare a financial institution technically insolvent when a non-financial institution, with otherwise identical financial status, would still be considered solvent. The barrier exchange model is used to determine the economic significance of this implicit indenture provision associated with the fixed-rate pricing system.\textsuperscript{52}

To value the bank equity holders’ call option, imagine a bank, solely owned and

\textsuperscript{50} Buser, Chen, and Kane (1981), Koehn and Santomero (1980), Kim and Santomero (1988), etc.

\textsuperscript{51} It should be noted that I do not attempt to refute the belief that the current pricing system is inefficient.

\textsuperscript{52} A common criticism leveled against fixed-rate system is that it encourages banks to alter their risk composition, in effect making deposit insurance underpriced (see, for example, Boyle and Lee (1993) and references cited within). It should be noted that this possibility is not considered in our analysis.

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managed, with no bonds outstanding.\textsuperscript{33} The value of the bank's assets, \( S_1 \), and the value of deposits, \( S_2 \), are assumed stochastic, with the price dynamics being described by geometric Brownian motion. The value of deposits is stochastic because the liability portfolio is assumed to consist of the deposits in place at time zero that are growing at the riskless rate of interest, combined with withdrawals and new deposits - both of which are uncertain.

For this bank, the guarantor is believed to conduct an in-depth examination of the bank every eighteen months to determine whether or not to dispose of the financial institution (i.e., to decide whether or not to extend the bank's charter). In addition, in the intervals between the detailed examinations, the guarantor is believed to maintain continuous surveillance of the financial health of the bank to ensure that a minimum net worth requirement is always satisfied. If the minimum net worth requirement is ever violated, the regulators are assumed to intervene immediately and shut the bank down with the remaining value of the firm, net of deposits, being equally distributed to the equity holders. Kumar and Morgan (1993) report that the current net worth requirement is 1.02.

For this scenario the equity holders' call option is analogous to a down-and-out barrier exchange call option written on the assets of the firm, with a stochastic strike price equal to the deposits of the firm, with the stochastic barrier level specified to be 1.02 times the value of the deposits, and a time to maturity of 1.5 years. Moreover, our results from Section IV.D are used to specify the stochastic rebate amount for this down-and-out barrier exchange call as .02 times the value of the deposits to incorporate the fact that the call option is exercised early if intervention occurs. In sum, the value of the equity holders' call option is written as \( C_{X}(S_1, S_2, 1.02S_2, 0.02S_2, 1.5) \), where the value of \( C_{X}(.) \) is calculated as \( [4.4] - [4.6] + [4.8] \) (see Table IV.B.1.a) with \( X = 1 \), \( B(r) = 1.02 \) (constant), and \( \beta = 0.02 \).

IV.F Chapter Summary

In this chapter, the theoretical principles developed in the Chapter III are expanded to

\textsuperscript{33} No loss of generality is incurred by assuming the debt holders are solely depositors; the framework can easily incorporate the addition of bank bond holders. Likewise, sole ownership allows us to abstract from any agency problems. The assumptions simply serve to simplify the discussion.

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include the first analytical valuation formula for European options restricted by a stochastic underlying boundary. The barrier exchange model is shown to be closely related to the non-stochastic barrier model discussed in Chapter III. In fact, similar to chapter III, it is found that stochastic barrier options can (1) be hedged under certain circumstances with a static portfolio strategy and (2) be viewed as an option with a random maturity date. In the next chapter, I continue to build on the theoretical foundation that has been established thus far by developing a stochastic barrier model with a non-stochastic strike price. Thus, in Chapter V the methodology developed in this chapter is expanded to allow for a static exercise price.
IV.G Appendix IVA: Linear Homogeneity Proof

In this appendix it will proven that the European stochastic barrier exchange option is linearly homogeneous of degree one with respect to each of the underlying risky assets.

This requires proving that for each dollar investment in the underlying assets, the distribution of returns on a stochastic barrier exchange option is independent of the stock price levels. That is,

$$W_x(\delta S_1, \delta X_2, \delta B_{S_2}, \delta R_{S_2}, \tau) = \delta W_x(S_1, X_2, B_{S_2}, R_{S_2}, \tau) \quad \forall \delta \neq 0.$$ \[4.27\]

To complete this proof a couple of interim results are presented. The rate of return for each underlying asset is assumed to follow the time homogeneous lognormal diffusion process, $dS_i(t)/S_i = \alpha dt + \sigma_i dZ_i(t)$. The rate of return for each underlying asset scaled by a factor of $\delta$ is found as follows using Itô's lemma:

$$d(\delta S_i(t)) = \frac{\partial[\delta S_i]}{\partial S_i} dS_i(t) = \delta dS_i(t).$$ \[4.28\]

From our interim results, it should be concluded that if both assets sell for $\delta$ times their original amount, the rate of return for each asset is simply $\delta$ times the original rate of return; the distribution of returns remains stationary.

To complete the proof note that the left-hand side of equation [4.27] is the value of a stochastic barrier exchange option when each asset sells for $\delta$ times its original amount. The right-hand side of equation [4.27] is the value of $\delta$ stochastic barrier exchange options. Given our interim result, the left-hand side must equal the right-hand side. That is, since the distribution of returns on the underlying assets remains stationary (i.e., is independent of the price level of that asset), the distribution of returns on the option must also remain stationary. To see this mathematically, Itô's lemma can once again be employed. Since $W_x(.)$ is function of $S_1$, $S_2$, and $t$, 

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\[ dW_x(. \cdot) = \frac{\partial W_x(\cdot)}{\partial s_1} ds_1(t) + \frac{\partial W_x(\cdot)}{\partial s_2} ds_2(t) \]
\[ + dt \left[ \frac{\partial W_x(\cdot)}{\partial t} + .5 \sigma_1^2 s_1 \frac{\partial^2 W_x(\cdot)}{\partial s_1^2} + .5 \sigma_2^2 s_2 \frac{\partial^2 W_x(\cdot)}{\partial s_2^2} + \sigma_1 \sigma_2 \rho_{12} s_1 s_2 \frac{\partial^2 W_x(\cdot)}{s_1 s_2} \right] \tag{4.29} \]

\( \delta \) times this distribution is then
\[ \delta dW_x(. \cdot) = \frac{\partial W_x(\cdot)}{\partial s_1} \delta ds_1(t) + \frac{\partial W_x(\cdot)}{\partial s_2} \delta ds_2(t) \]
\[ + \delta dt \left[ \frac{\partial W_x(\cdot)}{\partial t} + .5 \sigma_1^2 s_1 \frac{\partial^2 W_x(\cdot)}{\partial s_1^2} + .5 \sigma_2^2 s_2 \frac{\partial^2 W_x(\cdot)}{\partial s_2^2} + \sigma_1 \sigma_2 \rho_{12} s_1 s_2 \frac{\partial^2 W_x(\cdot)}{s_1 s_2} \right] \tag{4.30} \]

and [4.27] follows directly from our interim result.
IV.H Appendix IVB: Comparative Statics

In this appendix the comparative statics for equations [4.3]-[4.8] are derived. To this end, the following results will prove useful.

\[ n(y_6 - \sigma \sqrt{\tau}) = n(y_6) \frac{S_1d_1^{\tau}}{S_2d_2^{\tau}X} \]

\[ n(y_7 - \sigma \sqrt{\tau}) = n(y_7) \frac{S_1d_1^{\tau}}{S_2d_2^{\tau}bX} \]

\[ n(y_8 - \sigma \sqrt{\tau}) = n(y_8) \frac{S_2d_1^{\tau}B^2}{S_1d_2^{\tau}X} \]

\[ n(y_9 - \sigma \sqrt{\tau}) = n(y_9) \frac{S_2d_1^{\tau}B^2}{S_1d_2^{\tau}bX} \]  \[ [4.31] \]

\[ n(y_{10} - \sigma \sqrt{\tau}) \left( \frac{BS_1}{S_1} \right)^{1/2} = n(y_7 - \sigma \sqrt{\tau}) \]

\[ n(y_{10} - 2m^* \sigma \sqrt{\tau}) = \left( \frac{BS_2}{S_1} \right)^{2m} n(y_{10}) \]
The comparative statics for equation [4.3] are listed below.

\[ \frac{\partial [4.3]}{\partial s_1} = \phi d_1^{-}N(\phi y_d) \]
\[ \frac{\partial^2 [4.3]}{\partial s_1^2} = [d_1^{-}n(y_d)]/[S_1 \sigma \sqrt{\tau}] \]
\[ \frac{\partial^2 [4.3]}{\partial s_2} = -\phi X d_2^{-} N(\phi(y_d - \sigma \sqrt{\tau})) \]
\[ \frac{\partial^2 [4.3]}{\partial s_1 \partial s_2} = [S_1 d_1^{-}n(y_d)]/[S_2 \sigma \sqrt{\tau}] \]
\[ \frac{\partial^2 [4.3]}{\partial s_1 \partial s_2} = [-d_1^{-}n(y_d)]/[S_2 \sigma \sqrt{\tau}] \]
\[ \frac{\partial [4.3]}{\partial \tau} = S_1 d_1^{-} 5 \sigma \tau^{-5} n(y_d) - \phi S_1 d_1^{-} N(\phi y_d) \ln(d_1) + \phi X S_2 d_2^{-} N(\phi(y_d - \sigma \sqrt{\tau})) \ln(d_2) \]
The comparative statics for equation [4.4] are listed below.

\[
\frac{\partial [4.4]}{\partial s_1} = \phi d_1^{-1}N(\phi y_r) + (1-(1/b))[d_1^{-1}n(y_r)]/[\sigma \sqrt{\tau}]
\]

\[
\frac{\partial^2 [4.4]}{\partial s_1^2} = -[d_1^{-1}n(y_r)(y_r-\sigma \sqrt{\tau})]/[S_1 \sigma^2 \tau] + (1/b)[y_r n(y_r)d_1^{-1}]/[S_1 \sigma^2 \tau]
\]

\[
\frac{\partial [4.4]}{\partial s_2} = -\phi X d_2^{-1}N(\phi(y_r-\sigma \sqrt{\tau}))- (1-(1/b))[n(y_r)S_1d_1^{-1}]/[S_2 \sigma \sqrt{\tau}]
\]

\[
\frac{\partial^2 [4.4]}{\partial s_2^2} = -[S_1d_1^{-1}n(y_r)(y_r-\sigma \sqrt{\tau})]/[S_2^2 \sigma^2 \tau] + (1/b)[S_1d_1^{-1}n(y_r)y_r]/[S_2^2 \sigma^2 \tau]
\]

\[
\frac{\partial^2 [4.4]}{\partial s_1 \partial s_2} = [d_1^{-1}n(y_r)(y_r-\sigma \sqrt{\tau})]/[S_2 \sigma^2 \tau] - (1/b)[d_1^{-1}n(y_r)y_r]/[S_2 \sigma^2 \tau]
\]

\[
\frac{\partial [4.4]}{\partial \tau} = S_1 d_1^{-1} \cdot 5 \sigma \tau^{-5} n(y_r) y_r (1/b) - \phi S_1d_1^{-1}N(\phi y_r) \ln(d_1)
\]

\[
+ \phi X S_2 d_2^{-1}N(\phi(y_r-\sigma \sqrt{\tau})) \ln(d_2)
\]

\[
+ S_1 d_1^{-1}n(y_r) \ln(d_2/d_1)/[\sigma \sqrt{\tau}] (1-(1/b)) - S_1 d_1^{-1}n(y_r)(y_r-\sigma \sqrt{\tau})/[2 \tau]
\]

[4.33]
The comparative statics for equation [4.5] are listed below.

\[
\frac{\partial [4.5]}{\partial S_1} = \phi d_1^{-1}N(\eta y_0)(B/S_0/S_1)_{1^{-2}} - \phi (\lambda + 2)d_1^{-1}N(\eta y_0)(B/S_0/S_1)_{1^{-2}} \\
+ \phi \lambda (B/S_0/S_1)^{1^{-2}}[X_2 d_2^{-1}N(\eta (y_0 - \sigma \sqrt{\tau}))/[[S_1]
\frac{\partial [4.5]}{\partial S_1^2} = [\phi (\lambda + 1)(\lambda + 2)d_1^{-1}N(\eta y_0)(B/S_0/S_1)_{1^{-2}}/[[S_1]
- \phi \lambda (\lambda + 1)(B/S_0/S_1)^{1^{-2}}[X_2 d_2^{-1}N(\eta (y_0 - \sigma \sqrt{\tau}))/[[S_1]
+ \phi \eta (B/S_0/S_1)^{1^{-2}}[N(\eta y_0)]]/[[S_1][\sigma \sqrt{\tau}]
\frac{\partial [4.5]}{\partial S_2} = -\phi X d_2^{-1}N(\eta (y_0 - \sigma \sqrt{\tau}))(B/S_0/S_1)^{1^{-2}} - \phi \lambda X d_2^{-1}(B/S_0/S_1)^{1^{-2}}N(\eta (y_0 - \sigma \sqrt{\tau})) \\
+ \phi (\lambda + 2)N(\eta y_0)(B/S_0/S_1)^{1^{-2}}[S_1 d_1^{-1}]/[[S_2]
\frac{\partial [4.5]}{\partial S_2^2} = \phi \eta (B/S_0/S_1)^{1^{-2}}[S_1 d_1^{-1}n(y_0)]/[[S_2^2][\sigma \sqrt{\tau}]
+ \phi (\lambda + 1)(\lambda + 2)N(\eta y_0)(B/S_0/S_1)^{1^{-2}}d_1^{-1}S_1S_2^2 \\
- \phi \lambda (\lambda + 1)(B/S_0/S_1)^{1^{-2}}N(\eta (y_0 - \sigma \sqrt{\tau}))X d_2^{-1}/S_2
\frac{\partial [4.5]}{\partial S_1 \partial S_2} = -\phi \eta (B/S_0/S_1)^{1^{-2}}[S_1 d_1^{-1}n(y_0)]/[[S_2^3][\sigma \sqrt{\tau}]
- \phi (\lambda + 1)(\lambda + 2)(B/S_0/S_1)^{1^{-2}}[d_1^{-1}N(\eta y_0)]/[[S_1]
+ \phi \lambda (\lambda + 1)(B/S_0/S_1)^{1^{-2}}N(\eta (y_0 - \sigma \sqrt{\tau}))d_2^{-1}X/S_1
\frac{\partial [4.5]}{\partial \sigma} = \phi \eta d_1^{-1}5.5\sigma^{-5}n(y_0)(B/S_0/S_1)^{1^{-2}} \\
- \phi S_1 d_1^{-1}N(\eta y_0)ln(d_1) + \phi XS_2 d_2^{-1}N(\eta (y_0 - \sigma \sqrt{\tau}))ln(d_2) \\
+ [.5\sigma^2(\lambda + 1) - \ln(d_2/d_1)](\lambda + 2)\phi S_1 d_1^{-1}(B/S_0/S_1)^{1^{-2}}N(\eta y_0) \\
- [.5\sigma^2(\lambda + 1) - \ln(d_2/d_1)]\lambda \phi XS_2 d_2^{-1}(B/S_0/S_1)^{1^{-2}}N(\eta (y_0 - \sigma \sqrt{\tau}))
\]
The comparative statics for equation [4.6] are listed below.

\[
\frac{\partial[4.6]}{\partial S_1} = -\phi(\lambda+2)d_1^{-1}N(\eta y_0)(B_S/S_1)^{1/2} + \phi d_1^{-1}N(\eta y_0)(B_S/S_1)^{1/2} \\
+ \phi(\lambda(B_S/S_2)^{1/2}N(\eta(y_0-\sigma\sqrt{\tau}\sqrt{S_1}))/[S_1] - \phi(\lambda y_0)(1-(1/b))d_1^{-1}/[\sigma\sqrt{\tau}] \\
\frac{\partial^2[4.6]}{\partial S_1^2} = \phi(\lambda+1)(\lambda+2)(B_S/S_1)^{1/2}N(\eta y_0)d_1^{-1}/S_1 \\
-\phi(\lambda+1)(B_S/S_1)^{1/2}N(\eta(y_0-\sigma\sqrt{\tau}))X_2d_1^{-1}/S_1^2 + \phi(\lambda y_0)(1/b)[y_0]\sqrt{\tau}/[S_2^2\sigma^2\tau] \\
+\eta(\lambda+1)(\lambda+2)N(\lambda y_0)(1-(1/b))2\sigma\sqrt{\tau} - n(y_0)(\lambda+1)(\lambda+2)[S_1d_1^{-1}]/[S_2^2\sigma^2\tau] \\
\frac{\partial[4.6]}{\partial S_2} = -\phi(\lambda+2)N(\eta y_0)(B_S/S_1)^{1/2}[S_1d_1^{-1}]/[S_2] + \phi(\lambda y_0)(1-(1/b))[\eta y_0]d_1^{-1}/[S_2\sigma\sqrt{\tau}] \\
+ \phi(\lambda+2)(\lambda+2)(B_S/S_1)^{1/2}N(\eta y_0)S_1d_1^{-1}/S_2^2 \\
-\phi(\lambda+1)(\lambda+2)(B_S/S_1)^{1/2}N(\eta y_0)[X_2d_1^{-1}]/S_2 + \eta(\lambda y_0)(\lambda+2)\sigma\sqrt{\tau} - n(y_0)(\lambda+1)(\lambda+2)[S_1d_1^{-1}]/[\sigma\sqrt{\tau}] \\
+\phi(\lambda+2)(\lambda+2)N(\eta y_0)(1-(1/b))\sqrt{S_1}/[\sigma\sqrt{\tau}] \\
[4.35]
\]
The comparative statics for equation [4.6] are listed below.

\[ \frac{\partial^2[4.6]}{\partial S_1 \partial S_2} = -\phi(\lambda+1)(\lambda+2)(BS_2/S_1)^{1/2}d_1^{-1}N(\eta y_0)/S_2 \\
+ \phi \lambda(\lambda+1)(BS_2/S_1)^{1/2}N(\eta(y_0-\sigma\sqrt{\tau}))X d_2^{-1}/S_1 \\
- \phi \eta(BS_2/S_1)^{1/2}N(\eta y_0)(1/b)d_1^{-1}y_0/[S_2 \sigma^2 \tau] \\
- \phi \eta(BS_2/S_1)^{1/2}N(\eta y_0)(\lambda+1)(1-(1/b))[2d_1^{-1}y_0/[S_2 \sigma^2 \tau] \\
+ \eta \phi(BS_2/S_1)^{1/2}N(\eta y_0)(y_0-\sigma\sqrt{\tau})d_1^{-1}/[S_2 \sigma^2 \tau] \]

\[ \frac{\partial[4.6]}{\partial \tau} = -\phi(BS_2/S_1)^{1/2}S_1d_1^{-1}N(\eta y_0)\ln(d_1) \\
+ \phi X S_2 d_1^{-1}(BS_2/S_1)N(\eta(y_0-\sigma\sqrt{\tau}))\ln(d_2) \\
- \phi(BS_2/S_1)^{1/2}S_1 d_1^{-1}N(\eta y_0)\ln(d_1d_2)(\lambda+2) \\
+ \phi X S_2 d_2^{-1}(BS_2/S_1)N(\eta(y_0-\sigma\sqrt{\tau}))\ln(d_1d_2). \\
- \phi \eta(BS_2/S_1)^{1/2}S_1 d_1^{-1}N(\eta y_0)(1-(1/b))\ln(d_1d_2)/[\sigma \sqrt{\tau}] \\
- \eta N(\eta y_0)5\sigma^2(\lambda+1)(\lambda+2) \\
+ \phi \eta(BS_2/S_1)^{1/2}S_1 d_1^{-1}N(\eta y_0)5\tau^{-3}(1/b)y_0 \\
- (y_0-\sigma\sqrt{\tau}) \\
- \phi(BS_2/S_1)^{1/2}X S_2 d_2^{-1}N(\eta(y_0-\sigma\sqrt{\tau}))5\sigma^2(\lambda+1)/\sqrt{\tau} \\
+ \phi \eta(BS_2/S_1)^{1/2}S_1 d_1^{-1}N(\eta y_0)(1-(1/b))\sigma(\lambda+1)/\sqrt{\tau} \]
The comparative statics for equation [4.7] are listed below.

\[
\frac{\partial [4.7]}{\partial s_1} = 2\eta n(y_\gamma - \sigma \sqrt{\tau}) S_2 d_\tau^{-1} + (BS/S_1)^1 N(\eta(y_\gamma - \sigma \sqrt{\tau})) \lambda S_2 d_\tau^{-1} / [S_1]
\]

\[
\frac{\partial [4.7]}{\partial s_1^2} = -\eta \lambda n(y_\gamma - \sigma \sqrt{\tau}) S_2 d_\tau^{-1} / [S_1^2 \sigma \sqrt{\tau}] - 2\eta n(y_\gamma - \sigma \sqrt{\tau}) S_2 d_\tau^{-1} y_\gamma / [S_1^2 \sigma^2 \tau]

- \lambda(\lambda + 1)(BS/S_1)^1 N(\eta(y_\gamma - \sigma \sqrt{\tau})) S_2 d_\tau^{-1} / [S_1^2]
\]

\[
\frac{\partial [4.7]}{\partial s_2} = \mathcal{R} d_\tau^{-1} - n(y_\gamma - \sigma \sqrt{\tau}) 2\eta \mathcal{R} d_\tau^{-1} / [\sigma \sqrt{\tau}]

- \mathcal{R} d_\tau^{-1} (\lambda + 1)(BS/S_1)^1 N(\eta(y_\gamma - \sigma \sqrt{\tau}))
\]

\[
\frac{\partial^2 [4.7]}{\partial s_2^2} = -2\eta n(y_\gamma - \sigma \sqrt{\tau}) \mathcal{R} d_\tau^{-1} y_\gamma / [S_2 \sigma \tau] - \eta \lambda n(y_\gamma - \sigma \sqrt{\tau}) \mathcal{R} d_\tau^{-1} / [S_2 \sigma \sqrt{\tau}]

- \lambda(\lambda + 1)(BS/S_1)^1 N(\eta(y_\gamma - \sigma \sqrt{\tau})) \mathcal{R} d_\tau^{-1} / [S_2]
\]

\[
\frac{\partial^2 [4.7]}{\partial s_1 \partial s_2} = n(y_\gamma - \sigma \sqrt{\tau}) 2\eta \mathcal{R} d_\tau^{-1} y_\gamma / [S_1^2 \sigma^2 \tau] + n(y_\gamma - \sigma \sqrt{\tau}) \eta \lambda \mathcal{R} d_\tau^{-1} / [S_1 \sigma \sqrt{\tau}]

+ \lambda(\lambda + 1)(BS/S_1)^1 N(\eta(y_\gamma - \sigma \sqrt{\tau})) \mathcal{R} d_\tau^{-1} / [S_1]
\]

\[
\frac{\partial [4.7]}{\partial \tau} = -[4.7] \ln(d_\gamma) + \mathcal{R} S_2 d_\tau^{-1}(BS/S_1)^1 N(\eta(y_\gamma - \sigma \sqrt{\tau})) \lambda \ln(d_\gamma / d_\gamma) - .5 \sigma^2 \lambda (\lambda + 1)\)

+ \eta \mathcal{R} S_2 d_\tau^{-1} n(y_\gamma - \sigma \sqrt{\tau}) [2\ln(d_\gamma / d_\gamma) / [\sigma \sqrt{\tau}] - y_\gamma / [\tau] - .5 \sigma \lambda / [\sqrt{\tau}]}
\]
The comparative statics for equation \([4.8]\) are listed below.

\[
\frac{\partial^2[4.8]}{\partial S_1^2} = -\frac{(5\lambda + m^*)}{(5\lambda + m^* - 1)} \left( \frac{(BS_2/S_1')(BS_2/S_1)^{51-m^*}N(\eta y_{10})}{(BS_2/S_1)^{51-m^*}n(\eta y_{10})2\eta \theta S_2/[S_1 \sigma \sqrt{\tau}]}ight) \\
- \frac{(5\lambda - m^*)}{(5\lambda - m^* + 1)} \left( \frac{(BS_2/S_1')(BS_2/S_1)^{51-m^*}N(\eta y_{10})2m^* \sigma \sqrt{\tau})}{(BS_2/S_1)^{51-m^*}n(\eta y_{10})2\eta \theta S_2/[S_1 \sigma \sqrt{\tau}]}ight)
\]

\[
\frac{\partial^2[4.8]}{\partial S_2^2} = \frac{(5\lambda + m^* + 1)}{(5\lambda + m^* - 1)} \left( \frac{(BS_2/S_1')(BS_2/S_1)^{51-m^*}N(\eta y_{10})}{(BS_2/S_1)^{51-m^*}n(\eta y_{10})2\eta \theta j/[\sigma \sqrt{\tau}]}ight) \\
+ \frac{(5\lambda - m^* + 1)}{(5\lambda - m^* - 1)} \left( \frac{(BS_2/S_1')(BS_2/S_1)^{51-m^*}N(\eta y_{10})2m^* \sigma \sqrt{\tau})}{(BS_2/S_1)^{51-m^*}n(\eta y_{10})2\eta \theta j/[\sigma \sqrt{\tau}]}ight)
\]

\[
\frac{\partial^2[4.8]}{\partial S_1 \partial S_2} = -\frac{(5\lambda + m^*)}{(5\lambda + m^* - 1)} \left( \frac{(BS_2/S_1')(BS_2/S_1)^{51-m^*}N(\eta y_{10})}{(BS_2/S_1)^{51-m^*}n(\eta y_{10})2\eta \theta S_2/[S_1 \sigma \sqrt{\tau}]}ight) \\
- \frac{(5\lambda - m^*)}{(5\lambda - m^* + 1)} \left( \frac{(BS_2/S_1')(BS_2/S_1)^{51-m^*}N(\eta y_{10})2m^* \sigma \sqrt{\tau})}{(BS_2/S_1)^{51-m^*}n(\eta y_{10})2\eta \theta S_2/[S_1 \sigma \sqrt{\tau}]}ight)
\]

\[\text{[4.38]}\]
The comparative statics for equation [4.8] are listed below.

\[
\frac{\partial^2 [4.8]}{\partial S_1 \partial S_2} = - (5\lambda + m^*)(5\lambda + m^* + 1)(R/S_i)(BS_i/S_i)^{51+m^*}N(\eta y_{10}) - (5\lambda - m^*)(5\lambda - m^* + 1)(BS_i/S_i)^{51-m^*}(R/S_i)N(\eta (y_{10} - 2m^* \sigma \sqrt{\gamma})) - (\lambda + m^* + 1)(BS_i/S_i)^{51+m^*}n(y_{10})2\eta R/[S_i \sigma \sqrt{\gamma}] + (BS_i/S_i)^{51-m^*}n(y_{10})2\eta \delta y_{10}/[S_i \sigma^2 \gamma] \tag{4.39}
\]

\[
\frac{\partial [4.8]}{\partial \tau} = - \delta S_2(5\lambda + m^*)(\ln(d_S/d_i) - 5\sigma^2(\lambda + 1))(BS_i/S_i)^{51+m^*}N(\eta y_{10}) - \delta S_2(5\lambda - m^*)(\ln(d_S/d_i) - 5\sigma^2(\lambda + 1))(BS_i/S_i)^{51-m^*}N(\eta (y_{10} - 2m^* \sigma \sqrt{\gamma})) - \eta \delta S_2(BS_i/S_i)^{51+m^*}n(y_{10}) \left[ \frac{y_{10}}{\tau} + \frac{2\xi \theta}{\sigma \sqrt{\gamma}} - \frac{m \sigma}{\sqrt{\gamma}} \right]
\]
The comparative statics for equation [4.10] are listed below.

$$\frac{\partial[4.10]}{\partial s_2} = 2\eta n(\bar{y}_T - \sigma\sqrt{\bar{\tau}}) S_T d_1^{-\tau}/[S_2\sigma\sqrt{\bar{\tau}}] + \langle S_T/BS_T \rangle^2 N(\eta(\bar{y}_S - \sigma\sqrt{\bar{\tau}}))\hat{\lambda} S_T d_1^{-\tau}/[S_2]$$

$$\frac{\partial^2[4.10]}{\partial s_2^2} = -\eta \hat{\lambda} n(\bar{y}_T - \sigma\sqrt{\bar{\tau}}) S_T d_1^{-\tau}/[S_2^2\sigma\sqrt{\bar{\tau}}] - 2\eta n(\bar{y}_S - \sigma\sqrt{\bar{\tau}}) S_T d_1^{-\tau} \bar{y}_T/[S_2^2\sigma^2\bar{\tau}]$$

$$- \hat{\lambda}(\hat{\lambda} + 1)(S_T/BS_T)^2 N(\eta(\bar{y}_S - \sigma\sqrt{\bar{\tau}})) S_T d_1^{-\tau}/[S_2^2]$$

$$\frac{\partial[4.10]}{\partial s_1} = \eta d_1^{-\tau} N(\eta(\bar{y}_T - \sigma\sqrt{\bar{\tau}})) - n(\bar{y}_T - \sigma\sqrt{\bar{\tau}}) 2\eta \eta d_1^{-\tau}/[\sigma\sqrt{\bar{\tau}}]$$

$$- \eta d_1^{-\tau}(\hat{\lambda} + 1)(S_T/BS_T)^2 N(\eta(\bar{y}_S - \sigma\sqrt{\bar{\tau}}))$$

$$\frac{\partial^2[4.10]}{\partial s_1^2} = -2\eta n(\bar{y}_S - \sigma\sqrt{\bar{\tau}}) S_T d_1^{-\tau} \bar{y}_T/[S_1^2\sigma\sqrt{\bar{\tau}}] - \eta \hat{\lambda} n(\bar{y}_T - \sigma\sqrt{\bar{\tau}}) S_T d_1^{-\tau}/[S_1\sigma\sqrt{\bar{\tau}}]$$

$$- \hat{\lambda}(\hat{\lambda} + 1)(S_T/BS_T)^2 N(\eta(\bar{y}_S - \sigma\sqrt{\bar{\tau}})) S_T d_1^{-\tau}/[S_1]$$

$$\frac{\partial^2[4.10]}{\partial s_1 \partial s_2} = n(\bar{y}_S - \sigma\sqrt{\bar{\tau}}) 2\eta \eta d_1^{-\tau} \bar{y}_T/[S_2^2\sigma^2\bar{\tau}] + n(\bar{y}_S - \sigma\sqrt{\bar{\tau}}) \eta \eta d_1^{-\tau}/[S_2\sigma\sqrt{\bar{\tau}}]$$

$$+ \hat{\lambda}(\hat{\lambda} + 1)(S_T/BS_T)^2 N(\eta(\bar{y}_S - \sigma\sqrt{\bar{\tau}})) S_T d_1^{-\tau}/[S_2]$$

$$\frac{\partial[4.10]}{\partial \tau} = -[4.10]|\ln(d_T) + S_T d_1^{-\tau}(S_T/BS_T)^2 N(\eta(\bar{y}_S - \sigma\sqrt{\bar{\tau}}))[\hat{\lambda} \ln(d_T/d_T) - .5\sigma^2 \hat{\lambda}(\hat{\lambda} + 1)]$$

$$+ \eta S_T d_1^{-\tau} n(\bar{y}_T - \sigma\sqrt{\bar{\tau}})[2\ln(d_T/d_T)/[\sigma\sqrt{\bar{\tau}}] - \bar{y}_T/[\bar{\tau}] - .5\sigma \hat{\lambda}/[\sqrt{\bar{\tau}}]]$$
The comparative statics for equation [4.11] are listed below.

\[ \frac{\partial [4.11]}{\partial s_2} = -(\lambda + \nu) (\Re s_1 / s_2) (s_1 / B s_2)^{s_1 + s_2} N(\eta \bar{y}_{10}) \]
\[ - (s_1 / B s_2)^{s_1 + s_2} n(\bar{y}_{10}) 2 \eta \Re s_1/ [s_2^2 \sigma^2] \]
\[ - (\lambda + \nu) (\Re s_1 / s_2) (s_1 / B s_2)^{s_1 + s_2} N(\eta (y_{10} - 2 \nu \sigma \sqrt{\tau})) \]
\[ \frac{\partial^2 [4.11]}{\partial s_2^2} = (\lambda + \nu)(\lambda + \nu + 1) (\Re s_1 / s_2^2) (s_1 / B s_2)^{s_1 + s_2} N(\eta \bar{y}_{10}) \]
\[ + (\lambda + \nu)(\lambda + \nu + 1)(\Re s_1 / s_2^2) (s_1 / B s_2)^{s_1 + s_2} N(\eta (y_{10} - 2 \nu \sigma \sqrt{\tau})) \]
\[ + (\lambda + \nu)(\lambda + \nu + 1)(s_1 / B s_2)^{s_1 + s_2} n(\bar{y}_{10}) 2 \eta \Re s_1/ [s_2^2 \sigma^2] \]
\[ - (s_1 / B s_2)^{s_1 + s_2} n(\bar{y}_{10}) 2 \eta \Re \bar{y}_{10} / [s_2^2 \sigma^2 \tau] \]

[4.41]

\[ \frac{\partial [4.11]}{\partial s_1} = (\lambda + \nu + 1) (\Re s_1 / B s_2)^{s_1 + s_2} N(\eta \bar{y}_{10}) \]
\[ + (s_1 / B s_2)^{s_1 + s_2} n(\bar{y}_{10}) 2 \eta \Re / [\sigma \sqrt{\tau}] \]
\[ + (\lambda + \nu + 1)(\Re s_1 / B s_2)^{s_1 + s_2} N(\eta (y_{10} - 2 \nu \sigma \sqrt{\tau})) \]

\[ \frac{\partial^2 [4.11]}{\partial s_1^2} = (\lambda + \nu)(\lambda + \nu + 1)(\Re s_1 / s_2) (s_1 / B s_2)^{s_1 + s_2} N(\eta \bar{y}_{10}) \]
\[ + (\lambda + \nu)(\lambda + \nu + 1)(\Re s_1 / s_2) (s_1 / B s_2)^{s_1 + s_2} N(\eta (y_{10} - 2 \nu \sigma \sqrt{\tau})) \]
\[ + (\lambda + \nu + 1)(s_1 / B s_2)^{s_1 + s_2} n(\bar{y}_{10}) 2 \eta \Re / [s_2 \sigma \sqrt{\tau}] \]
\[ - (s_1 / B s_2)^{s_1 + s_2} n(\bar{y}_{10}) 2 \eta \Re \bar{y}_{10} / [s_2 \sigma^2 \tau] \]
The comparative statics for equation [4.11] are listed below.

\[ \frac{\partial^2 [4.11]}{\partial s_1 \partial s_2} = - (0.5\hat{\lambda} + \hat{m})(0.5\hat{\lambda} + \hat{m} + 1)(S_f/BS_2)(S_f/BS_2)^{\hat{s}_{1-\theta}}N(\eta \hat{y}_{10}) \]
\[ - (0.5\hat{\lambda} - \hat{m})(0.5\hat{\lambda} - \hat{m} + 1)(S_f/BS_2)(S_f/BS_2)^{\hat{s}_{1-\theta}}N(\eta \hat{y}_{10} - 2\hat{m}\sigma \sqrt{\tau}) \]
\[ + (\hat{\lambda} + \hat{m} + 1)(S_f/BS_2)(S_f/BS_2)^{\hat{s}_{1-\theta}}N(\eta \hat{y}_{10} - 2\hat{m}\sigma \sqrt{\tau}) \]
\[ + (S_f/BS_2)(S_f/BS_2)^{\hat{s}_{1-\theta}}N(\eta \hat{y}_{10} - 2\hat{m}\sigma \sqrt{\tau}) \]

\[ \frac{\partial [4.11]}{\partial \tau} = - \theta S_f(0.5\hat{\lambda} + \hat{m})[\ln(d_1/d_2) - 0.5\sigma^2(\hat{\lambda} + 1)](S_f/BS_2)^{\hat{s}_{1-\theta}}N(\eta \hat{y}_{10}) \]
\[ - \theta S_f(0.5\hat{\lambda} - \hat{m})[\ln(d_1/d_2) - 0.5\sigma^2(\hat{\lambda} + 1)](S_f/BS_2)^{\hat{s}_{1-\theta}}N(\eta \hat{y}_{10} - 2\hat{m}\sigma \sqrt{\tau}) \]
\[ - \eta \theta S_f(0.5\hat{\lambda} - \hat{m})[\ln(d_1/d_2) - 0.5\sigma^2(\hat{\lambda} + 1)](S_f/BS_2)^{\hat{s}_{1-\theta}}N(\eta \hat{y}_{10} - 2\hat{m}\sigma \sqrt{\tau}) \]

[4.42]
The comparative statics for equation (4.13) are listed below.

\[ \frac{\partial [4.13]}{\partial S_2} = -\eta \delta r^{-1}/[S_2^2 \sigma \sqrt{\tau}] [n(\dot{y}_t)+ (BS_2/S_1)^4 n(\dot{y}_o)] - (BS_2/S_1)^4 N(\eta \dot{y}_o) \lambda \delta r^{-1}/[S_2] \]

\[ \frac{\partial^2 [4.13]}{\partial S_2^2} = \eta \delta r^{-1}/[S_2^2 \sigma ^2 \tau] [n(\dot{y}_t)+ (BS_2/S_1)^4 n(\dot{y}_o)(1-2\lambda)] \]

\[ - \eta \delta r^{-1}/[S_2^2 \sigma ^2 \tau] [n(\dot{y}_t)\dot{y}_t - (BS_2/S_1)n(\dot{y}_o)\dot{y}_o] - \lambda (\lambda -1)(BS_2/S_1)^4 N(\eta \dot{y}_o) \delta r^{-1}/[S_2^2] \]

\[ \frac{\partial [4.13]}{\partial S_1} = \delta r^{-1}/(S_1)(BS_2/S_1)^4 N(\eta \dot{y}_o) + \eta \delta r^{-1}/[S_1 \sigma \sqrt{\tau}] [n(\dot{y}_t) + (BS_2/S_1)^4 n(\dot{y}_o)] \]

\[ \frac{\partial^2 [4.13]}{\partial S_1^2} = -\eta \delta r^{-1}/[S_1^2 \sigma ^2 \tau] [n(\dot{y}_t)\dot{y}_t - (BS_2/S_1)^4 n(\dot{y}_o)\dot{y}_o] \]

\[ - \eta \delta r^{-1}/[S_1^2 \sigma ^2 \tau] [n(\dot{y}_t) + (BS_2/S_1)^4 n(\dot{y}_o)(1+2\lambda)] \]

\[ \lambda (\lambda +1)(BS_2/S_1)^4 N(\eta \dot{y}_o) \delta r^{-1}/[S_1 S_2] \]  

\[ \frac{\partial [4.13]}{\partial S_1 \partial S_2} = \eta \delta r^{-1}/[S_1 S_2 \sigma ^2 \tau] [n(\dot{y}_t)\dot{y}_t - (BS_2/S_1)^4 n(\dot{y}_o)\dot{y}_o] \]

\[ + 2(BS_2/S_1)^4 n(\dot{y}_o) \eta \delta r^{-1}/[S_1 S_2 \sigma \sqrt{\tau}] + \lambda^2 (BS_2/S_1)^4 N(\eta \dot{y}_o) \delta r^{-1}/[S_1 S_2] \]

\[ \frac{\partial [4.13]}{\partial \tau} = -[4.13] \ln(\tau) + \delta r^{-1}(BS_2/S_1)^4 N(\eta \dot{y}_o) \lambda \xi \theta \]

\[ - \delta r^{-1}(BS_2/S_1)^4 n(\dot{y}_o)(1/\sigma \sqrt{\tau}) [\lambda \sigma ^2 - \ln(d_2/d_1) + .5(\sigma ^2 - \sigma ^2_x - \sigma ^2_y)2 \sigma ^2 (2\sigma ^2)] \]

\[ + \eta \delta r^{-1}n(\dot{y}_o)(1/\sigma \sqrt{\tau}) [\ln(d_2/d_1) - \dot{y}_t \sigma /2 \sigma ^2] - .5(\sigma ^2 - \sigma ^2_x) \]
The comparative statics for equation [4.14] are listed below.

\[
\frac{\partial \delta[4.14]}{\partial S_1} = -(5\lambda + \tilde{m})(e^{-\delta \tau/\bar{S}_1})(BS_2/S_1)N(\eta_\tilde{y}_1) \\
\quad - (BS_2/S_1)N(\eta_\tilde{y}_1)2\eta e^{-\delta \tau/\bar{S}_1} \frac{\partial \ln(S_1^{\tilde{m}})}{\partial \bar{S}_1} \\
\quad - (5\lambda - \tilde{m})(e^{-\delta \tau/\bar{S}_1})(BS_2/S_1)N(\eta(\tilde{y}_1 - 2\tilde{m}\sigma/\sqrt{\tau}))
\]

\[
\frac{\partial^2 \delta[4.14]}{\partial S_1^2} = -(5\lambda + \tilde{m})(5\lambda + \tilde{m} + 1)(e^{-\delta \tau/\bar{S}_1})(BS_2/S_1)N(\eta_\tilde{y}_1) \\
\quad + (5\lambda - \tilde{m})(5\lambda - \tilde{m} + 1)(e^{-\delta \tau/\bar{S}_1})(BS_2/S_1)N(\eta(\tilde{y}_1 - 2\tilde{m}\sigma/\sqrt{\tau})) \\
\quad + (\tilde{m} + 1)(BS_2/S_1)N(\eta_\tilde{y}_1)2\eta e^{-\delta \tau/\bar{S}_1} \frac{\partial \ln(S_1^{\tilde{m}})}{\partial \bar{S}_1} \\
\quad - (BS_2/S_1)N(\eta_\tilde{y}_1)2\eta e^{-\delta \tau/\bar{S}_1} \frac{\partial^2 \ln(S_1^{\tilde{m}})}{\partial \bar{S}_1^2}
\]

\[
\frac{\partial \delta[4.14]}{\partial S_2} = -(5\lambda + \tilde{m})(e^{-\delta \tau/\bar{S}_2})(BS_2/S_2)N(\eta_\tilde{y}_1) \\
\quad + (BS_2/S_2)N(\eta_\tilde{y}_1)2\eta e^{-\delta \tau/\bar{S}_2} \frac{\partial \ln(S_2^{\tilde{m}})}{\partial \bar{S}_2} \\
\quad - (5\lambda - \tilde{m})(e^{-\delta \tau/\bar{S}_2})(BS_2/S_2)N(\eta(\tilde{y}_1 - 2\tilde{m}\sigma/\sqrt{\tau}))
\]

\[
\frac{\partial^2 \delta[4.14]}{\partial S_2^2} = -(5\lambda + \tilde{m})(5\lambda + \tilde{m} - 1)(e^{-\delta \tau/\bar{S}_2})(BS_2/S_2)N(\eta_\tilde{y}_1) \\
\quad - (5\lambda - \tilde{m})(5\lambda - \tilde{m} - 1)(e^{-\delta \tau/\bar{S}_2})(BS_2/S_2)N(\eta(\tilde{y}_1 - 2\tilde{m}\sigma/\sqrt{\tau})) \\
\quad + (\tilde{m} - 1)(BS_2/S_2)N(\eta_\tilde{y}_1)2\eta e^{-\delta \tau/\bar{S}_2} \frac{\partial \ln(S_2^{\tilde{m}})}{\partial \bar{S}_2} \\
\quad - (BS_2/S_2)N(\eta_\tilde{y}_1)2\eta e^{-\delta \tau/\bar{S}_2} \frac{\partial^2 \ln(S_2^{\tilde{m}})}{\partial \bar{S}_2^2}
\]

[4.44]
The comparative statics for equation [4.14] are listed below.

\[
\frac{\partial \mathcal{F}[4.14]}{\partial S_1 \partial S_2} = - (\tilde{\mathcal{F}}_{[4.14]})^2 (\Re e^{-\iota \theta / S_1 S_2}) (BS/S_1 S_2)^{51 + \dagger} N(\eta \mathcal{Y}_{[4.14]})
\]

\[
- (\tilde{\mathcal{F}}_{[4.14]})^2 (BS/S_1 S_2)^{51 + \dagger} N(\eta \mathcal{Y}_{[4.14]}) N(\eta (\mathcal{Y}_{[4.14]} - 2 \tilde{\mathcal{F}}_{[4.14]} \sigma \sqrt{\tau}))
\]

\[
- (\tilde{\mathcal{F}}_{[4.14]})^2 (BS/S_1 S_2)^{51 + \dagger} N(\mathcal{Y}_{[4.14]}) 2 \eta \Re e^{-\iota \theta / S_1 S_2 \sigma \sqrt{\tau}}
\]

\[
+ (BS/S_1 S_2)^{51 + \dagger} N(\mathcal{Y}_{[4.14]}) 2 \eta \Re e^{-\iota \theta / S_1 S_2 \sigma \sqrt{\tau}}
\]

\[
[4.45]
\]

\[
\frac{\partial \mathcal{F}[4.14]}{\partial \tau} = - \Re e^{-\iota \theta / (S_1 + S_2)} [\ln(d/d_i) - 0.5 \sigma^2 \lambda - 0.5 (\sigma_1^2 - \sigma_2^2)] (BS/S_1 S_2)^{51 + \dagger} N(\eta \mathcal{Y}_{[4.14]})
\]

\[
- \Re e^{-\iota \theta / (S_1 + S_2)} [\ln(d/d_i) - 0.5 \sigma^2 \lambda - 0.5 (\sigma_1^2 + \sigma_2^2)] (BS/S_1 S_2)^{51 + \dagger} N(\eta (\mathcal{Y}_{[4.14]} - 2 \tilde{\mathcal{F}}_{[4.14]} \sigma \sqrt{\tau}))
\]

\[
- \eta \Re e^{-\iota \theta / (BS/S_1 S_2)} \mathcal{Y}_{[4.14]} \left[ \frac{\mathcal{Y}_{[4.14]} + 2 \tilde{\mathcal{F}}_{[4.14]} \sigma \sqrt{\tau}}{\mathcal{Y}_{[4.14]} - 2 \tilde{\mathcal{F}}_{[4.14]} \sigma \sqrt{\tau}} - \frac{\tilde{\mathcal{F}}_{[4.14]} \sigma \sqrt{\tau}}{\mathcal{Y}_{[4.14]} - 2 \tilde{\mathcal{F}}_{[4.14]} \sigma \sqrt{\tau}} \right] - \mathcal{Y}_{[4.14]} [4.14]
\]

Chapter IV
CHAPTER V
The Valuation of Barrier Options With a
Non-Stochastic Strike Price and a Stochastic Boundary

V.A Introduction

In Chapter III much of the existing work on barrier options is integrated into one
generalized model. This methodology is then further elaborated on in Chapter IV where the
exchange option pricing model is integrated with the barrier model. In this chapter an underlying
stochastic boundary is incorporated into an otherwise standard Black-Scholes framework in two
different ways. First, the methodology developed in Chapter IV is expanded to allow for a fixed
strike price. I refer to this model as the attached barrier model; the name becomes apparent later
in the chapter. Second, after having developed the attached barrier model, a second model is
considered that is conceptually very different from the attached model. However, the general
methodology that is developed to value the attached barrier model is shown to readily apply to
the valuation of the second model, which is referred to as the detached barrier model. That is,
while the two models developed in this chapter are conceptually different, similar methodology
apply to their derivations. Thus, the detached barrier model is shown to be an extension of the
methodology developed for the attached model. In Chapter VI, I show how the detached barrier
model is applied to the valuation of Black-Scholes options subject to intertemporal default risk.
An application of the attached barrier model is considered in Chapter VII.

The previous chapter presented the first analytical valuation formula for European options
restricted by a stochastic underlying boundary. However, the valuation of such an option is
greatly simplified by exploiting the power of linear homogeneity\textsuperscript{54}; such is not the case in this chapter. As a result, it turns out that the two models developed in this chapter are slightly more complex than the barrier exchange model (derived in Chapter IV). The valuation of the options considered in this chapter require the evaluation of cumulative bivariate standard normal distribution functions. The cost of this additional complexity is, however, more than offset by the wider applicability of these models.

The remainder of this chapter is organized as follows. In Section V.B, the attached barrier model is developed, followed by the detached barrier model, which is presented in Section V.C. As stated, these two models are shown to be conceptually quite different. Closer inspection reveals, however, a number of similarities in terms of their derivations. In fact, to emphasize this point, in Section V.D the value of each of these options is written in terms of one generalized formula. In this light, the generalized valuation formula developed in Chapter III can be considered to be further generalized by the results presented in Section V.D. Section V.E includes a summary of this chapter.

V.B Valuation of the Attached Barrier Model

Let \( W(S_1,X,S_2,C,R,r) \) denote the value of a European barrier option written on \( S_1 \) and struck at \( X \) (a constant). \( R \) is, again, the amount of the rebate. This option either comes into existence (an "in" option) or ceases to exist (an "out" option) when the random variable \( S_1(t) \) violates the stochastic boundary \( S_2(t)C \) (where \( C \) is some arbitrary constant). This model is referred to as the attached barrier model because the option is written on asset one and "knocks out" or "knocks in" depending on asset one (as well as asset two).\textsuperscript{55} That is, the "knock out" or "knock in" condition is directly tied or attached to the path of the underlying asset. This is opposed to the detached barrier model, which is discussed in the next section, in which the "knock out" or "knock in" condition is tied to another asset (which is not the underlying asset).

"Out" ("In") versions of the attached barrier option have a non-rebate payoff of

\textsuperscript{54} Linear homogeneity allowed the stochastic barrier exchange option to be valued as a particular standard barrier option.

\textsuperscript{55} This is not to imply anything about the correlation structure between the two assets. Attached does not imply dependence, nor independence, of the two asset prices.
\[ \max(\phi S_1(T) - \phi X, 0) \text{ at } T \text{ if } \eta S_1(t) > \eta S_2(t)C \forall t \in [t_0, T] \text{ (if } \eta S_1(t) \leq \eta S_2(t)C \text{ for some } t \in [t_0, T]) \text{ and zero otherwise}. \]

Changes in each asset value \((i=1,2)\) are assumed to obey geometric Brownian motion with time homogeneous parameters:

\[ dS_i(t) = \mu_i S_i dt + \sigma_i S_i dZ_i(t) \]  \hspace{1cm} \[5.1\]

where \([Z_i(t), t \in [t_0, T))\] is a standard Brownian motion defined on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Note the probability measures are equivalent. Also note that \(\mathbb{P}\) is defined on \(\mathcal{F}_i\) as an equivalent martingale measure - this is demonstrated below.\(^{57}\) Define \(\rho_{ij} = \text{corr} (Z_i, Z_j)\), where \(\text{corr}\) denotes the correlation operator. \(\rho_{ij}\) is assumed constant over the closed interval \([t_0, T]\).

It is also instructive to note that the attached barrier option is linearly homogeneous with respect to \(S_1, X, \text{ and } S_2C\).

Merton (1973, p. 164) has shown that for an option whose price is dependent on two underlying state variables, the value of the option is the solution to

\[ \ln(r)W(.) = \ln(r/d_i)S_1 \frac{\partial W(.)}{\partial S_1} + \ln(r/d_2)S_2 \frac{\partial W(.)}{\partial S_2} - \frac{\partial W(.)}{\partial \tau} \]

\[ + \frac{\sigma_1^2 S_1^2}{2} \frac{\partial^2 W(.)}{\partial S_1^2} + \frac{\sigma_2^2 S_2^2}{2} \frac{\partial^2 W(.)}{\partial S_2^2} + \frac{\partial^2 W(.)}{\partial S_1 \partial S_2} \rho_{1,2} \sigma_1 \sigma_2 + \frac{\partial^2 W(.)}{\partial \tau^2} \sigma_1 \sigma_2 \tau \]

\[5.2\]

Thus, the non-rebate portion of an attached barrier option’s price must satisfy \([5.2]\) subject to boundary conditions:

\[ W_{\text{att}}(S_1(T), X, S_2(T)C, 0) = \max(0, \phi S_1(T) - \phi X) \text{ if } \eta S_1(t) > \eta S_2(t)C \text{ for all } t \in [t_0, T], \]

\[ W_{\text{att}}(S_1(T), X, S_2(T)C, 0) = 0 \text{ if } \eta S_1(t) \leq \eta S_2(t)C \text{ for any } t \in [t_0, T], \]

\[ \]

\(^{56}\) Recall \(\phi\) equals 1 if the option is a call option and -1 if the option is a put. \(\eta\) equals 1 if the barrier is being approached from above (i.e., \(S_1 > S_2C\)) and -1 otherwise.

\(^{57}\) Let \(m_1 = [\beta \ln(r/d_i)]/\sigma_i\) be the market price of risk of asset \(i\), where \(\beta\) is the expected growth rate of a derivative security that is dependent on price of asset \(i\). An equivalent martingale measure is obtained by replacing \(\mu_i\) in the diffusion process defined in equation \([5.1]\) with \(\mu_i^* = [\mu_i - m_1 \sigma_i]\). For more details, see Hull (1993, p. 281) and the references cited within. Traded assets, of course, can be perfectly hedged and have a market price of risk of zero (i.e., \(\mu_i^* = \mu_i = \ln(r/d_i)\) and the drift remains unchanged).

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\[ W_w(S_1(T), X, S_2(T)|C, 0) = \max(0, \phi S_1(T) - \phi X) \text{ if } \eta S_1(t) \leq \eta S_2(t)|C \text{ for some } t \in [t_0, T], \]
\[ W_w(S_1(T), X, S_2(T)|C, 0) = 0 \text{ if } \eta S_1(t) > \eta S_2(t)|C \text{ for all } t \in [t_0, T], \]
\[ W(0, X, S_2(t)|C, T-t) = 0 \text{ if } S_1(t) = 0 \text{ for any } t \in [t_0, T], \]

and \[ 0 \leq W(S_1(t), X, S_2(t)|C, T-t) \leq S_1(t) \text{ for all } t \in [t_0, T]. \]

Neither equation [5.2] nor the boundary conditions contain any preference dependent terms. Hence, a necessary and sufficient restriction on the fundamental solution to [5.2] is that it cannot contain any preference dependent terms. [5.2] is solved using preference-free valuation.\(^{58}\) The solution for each of the eight possible attached barrier options is written as a linear combination of, at most, three equations.

\[
\phi S_1 d_1^{-1} N(\phi w_1) - \phi r^{-1} X N(\phi(w_1 - \sigma_1 \sqrt{t})) \tag{5.3}
\]
\[
\phi S_1 d_1^{-1} M(\phi w_1, \eta w_2, \phi \eta \rho_{1,12}) - \phi X r^{-1} M(\phi(w_1 - \sigma_1 \sqrt{t}), \eta(w_2 - \sigma_1 \rho_{1,12} \sqrt{t}), \phi \eta \rho_{1,12}) \tag{5.4}
\]
\[
\phi(S_2 C/S_1)^\kappa [S_1 d_1^{-1} (S_2 C/S_1)^{2 \rho_{1,12} \sqrt{t}} M(\phi w_3, \eta w_4, \phi \eta \rho_{1,12}) - X r^{-1} M(\phi(w_3 - \sigma_1 \sqrt{t}), \eta(w_4 - \sigma_1 \rho_{1,12} \sqrt{t}), \phi \eta \rho_{1,12})] \tag{5.5}
\]

where

\[
\kappa = \frac{[\ln(d_2/d_1) - 5(\sigma_1^2 - \sigma_2^2)]/[\sigma^2]}{[\sigma^2]}
\]
\[
w_1 = \frac{[\ln(S_1/X) + (\ln(r/d_1) + 5 \sigma_2^2) t]}{[\sigma_1 \sqrt{t}]}, \]
\[
w_2 = \frac{[\ln(S_1/S_2 C) + (\ln(d_2/d_1) + 5 \sigma_2^2) t]}{[\sigma_1 \sqrt{t}]},
\]
\[
w_3 = w_1 + 2 \rho_{1,12} \ln(S_2 C/S_1)/[\sigma_1 \sqrt{t}]
\]
\[
w_4 = w_2 + 2 \ln(S_2 C/S_1)/[\sigma_1 \sqrt{t}]
\]
\[
\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2 \rho_{1,12} \sigma_1 \sigma_2
\]
\[
\rho_{1,12} = \text{Corr}[\ln(S_1(T)), \ln(S_2(T))]
\]
\[
\rho_{1,12} = \text{Corr}[\ln[S_1(T)], \ln[S_1(T)/S_2(T)]] = [\sigma_1 - \sigma_2 \rho_{1,12}]/[\sigma]
\]

\(M(a, b, \rho)\) is the bivariate standard cumulative normal distribution with upper limits of integration \(a\) and \(b\) and a correlation coefficient of \(\rho\).

\(^{58}\) That is, the discounted expectations are taken under an equivalent martingale measure as mentioned above.
The rebate portion of an attached barrier option’s value can also be determined directly from previous results. Recall that the "out" rebate for this model is received when the underlying state variable \( S_1(t) \) violates the critical stochastic boundary level \( S_2(t) \). The "in" rebate is received at expiration if the barrier is never severed. Accordingly, after a change of numeraire equations [3.86] and [3.87] can be rewritten as

\[
Rr^{-1}
\left[
N(\eta(w_2-\sigma_1\rho_{1,12}\sqrt{\tau})) - (CS_2/S_1)^{2\kappa}N(\eta(w_4-\sigma_1\rho_{1,12}\sqrt{\tau}))\right]
\]

where

\[
R(S_1/S_2,C) \left[(CS_2/S_1)^{2\kappa}N(\eta w_3) + (CS_2/S_1)^{2\kappa}N(\eta (w_4-2\dot{m}\sigma\sqrt{\tau}))\right]
\]

\[
w_3 = \frac{\ln(S_2/C) + \dot{m}\sigma^2\tau}{\sigma\sqrt{\tau}},
\]

\[
\dot{m} = \sqrt{\frac{\ln(d_2/d_1) - 0.5(\sigma_1^2-\sigma_2^2)}{\ln(\tau) + 2\ln(\rho) / \sigma^2}}
\]

Equation [5.3] is recognized as the standard Black-Scholes equation for equity options written on dividend paying assets. Equation [5.7] is the current value of the "in" rebate and [5.8] is the current value of the "out" rebate. Equations [5.4] and [5.5] are considerably more difficult to derive. A sketch of their derivations are found in Appendix VA. Verification of equations [5.4] and [5.5] are found in Appendix VB. The valuation solutions for detached barrier options with a non-stochastic strike price are presented in Table V.B.a.

V.C Valuation of the Detached Barrier Model

In this section, a model is developed in which the "knock out" or "knock in" condition is not directly tied to the level of the underlying asset.

Let \( W(S_1,X,S_2,C,R,\tau) \) denote the value of a European barrier option written on \( S_1 \) and struck at \( X \) (a constant). \( R \) is, again, the amount of the rebate. This option either comes into existence (an "in" option) or ceases to exist (an "out" option) when the asset price \( S_2(t) \) violates the prespecified, static, boundary level \( C \). This model is referred to as the detached barrier

---

\( ^{39} \) Equation [5.3], as stated, is the Black-Scholes model and is verified in Appendix IIIB. The rebate equations, [5.7] and [5.8], are also verified in Appendix IIIB.

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### Table V.B.a
Valuation Formulas for European Attached Stochastic Barrier Options With a Non-Stochastic Strike Price: $W(S_t, X, S_0, C, R, \tau)$

<table>
<thead>
<tr>
<th>EUROPEAN OPTION TYPE (X=Exercise Price) (CS_2=Barrier Level)</th>
<th>VALUATION EQUATION</th>
<th>$\eta$</th>
<th>$\phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up-and-In Call</td>
<td>$[5.3]-[5.4]+[5.5]+[5.7]$</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>Up-and-In Put</td>
<td>$[5.3]-[5.4]+[5.5]+[5.7]$</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>Down-and-In Call</td>
<td>$[5.3]-[5.4]+[5.5]+[5.7]$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Down-and-In Put</td>
<td>$[5.3]-[5.4]+[5.5]+[5.7]$</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>Up-and-Out Call</td>
<td>$[5.4]-[5.5]+[5.8]$</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>Up-and-Out Put</td>
<td>$[5.4]-[5.5]+[5.8]$</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>Down-and-Out Call</td>
<td>$[5.4]-[5.5]+[5.8]$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Down-and-Out Put</td>
<td>$[5.4]-[5.5]+[5.8]$</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

* $\eta$ is defined to be 1 if the barrier is being approached from above (i.e., $CS_2 < S_0$) and -1 if the barrier is being approached from below.

* * $\phi$ is defined to be 1 if the stochastic barrier option is a call and -1 if the option is a put.
model because the option is written on asset one but "knocks out" or "knocks in" depending on asset two.\(^6\) "Out" ("In") versions of this option have a non-rebate payoff of \(\max(\phi S_1(T)-\phi X,0)\) at \(T\), if \(\eta S_3 > \eta C \lor t \in [t_0,T]\) (if \(\eta S_3(t) < \eta C\) for some \(t \in [t_0,T]\)) and zero otherwise.

The stock price dynamics for \(S_1(t)\) and \(S_3(t)\) are described by the stochastic differential equation listed in [5.1].

A detached barrier option's value is dependent on two underlying state variables. Accordingly, \(W(S_1,X,S_3,C,0,\tau)\) is the solution to the second order, linear, partial differential equation (see Merton (1973, p. 164):

\[
\ln(r)W(.) = \ln(r/d_1)S_1 \frac{\partial W(.)}{\partial S_1} + \ln(r/d_2)S_3 \frac{\partial W(.)}{\partial S_3} - \frac{\partial W(.)}{\partial \tau}
\]

\[
+ .5 \sigma_1^2 S_1^2 \frac{\partial^2 W(.)}{\partial S_1^2} + .5 \sigma_3^2 S_3^2 \frac{\partial^2 W(.)}{\partial S_3^2} + \frac{\partial^2 W(.)}{\partial S_1 \partial S_3} \sigma_1 \sigma_3 \rho_{1,3} S_1 S_3
\]

[5.10]

The non-rebate portion of a detached barrier option must satisfy [5.10] subject to boundary conditions:

\[
W_{\text{out}}(S_1(T),X,S_3(T),C,0) = \max(0,\phi S_1(T)-\phi X) \text{ if } \eta S_3(t) > \eta C \text{ for all } t \in [t_0,T],
\]

\[
W_{\text{out}}(S_1(T),X,S_3(T),C,0) = 0 \text{ if } \eta S_3(t) \leq \eta C \text{ for any } t \in [t_0,T],
\]

\[
W_{\text{in}}(S_1(T),X,S_3(T),C,0) = \max(0,\phi S_1(T)-\phi X) \text{ if } \eta S_3(t) \leq \eta C \text{ for some } t \in [t_0,T],
\]

\[
W_{\text{in}}(S_1(T),X,S_3(T),C,0) = 0 \text{ if } \eta S_3(t) > \eta C \text{ for all } t \in [t_0,T],
\]

\[
W(0,X,S_3(t),C,T-\tau) = 0 \text{ if } S_1(t) = 0 \text{ for any } t \in [t_0,T],
\]

and \(0 \leq W(S_1(t),X,S_3(t),C,T-\tau) \leq S_1(t) \text{ for all } t \in [t_0,T]\).

Similar to the attached barrier model, the non-rebate solution for each of the eight possible detached barrier options is written as a linear combination of, at most, three equations.

\[
\phi S_1 d_{1,3}^{-1} N(\phi w_2) - \phi x r^{-\tau} XN(\phi (w_1-\sigma_{1,3}\sqrt{\tau}))
\]

[5.11]

\[
\phi S_1 d_{1,3}^{-1} M(\phi w_1, \eta w_6, \phi \eta \rho_{1,3}) - \phi X r^{-\tau} M(\phi (w_1-\sigma_{1,3}\sqrt{\tau}), \eta w_6-\sigma_{1,3}\rho_{1,3}\sqrt{\tau}, \eta \phi \rho_{1,3})
\]

[5.12]

\(^6\) Again, this is not to imply anything about the correlation structure between the two assets. Detached does not imply independence.
\[
\phi(C/S_2)^{25-2} [S_2 d_1^{*\tau} (C/S_2)^{23\tau} \gamma^2 \rho_{1,3} \text{M}(\phi w_7, \eta w_8, \phi \eta \rho_{1,3})
- X^{\tau \eta} M(\phi(w_7 - \sigma_1 \sqrt{\tau}), \eta(w_8 - \sigma_1 \rho_{1,3} \sqrt{\tau}), \eta \phi \rho_{1,3})]
\]

where

\[
\begin{align*}
\delta &= \frac{[\ln(r/d) + 5\sigma_3]}{[\sigma_3]} \\
\omega_1 &= \frac{[\ln(S_2/X) + (\ln(r/d) + 5\sigma_3^2)\tau]}{[\sigma_3 \sqrt{\tau}]} \\
\omega_6 &= \frac{[\ln(S_2/C) - (\ln(r/d) - 5(\sigma_3^2 - 2\sigma_1 \sigma_3 \rho_{1,3}))\tau]}{[\sigma_3 \sqrt{\tau}]} \\
\omega_7 &= \omega_1 + 2\ln(C/S_2) \rho_{1,3} / [\sigma_3 \sqrt{\tau}] \\
\omega_8 &= \omega_6 + 2\ln(C/S_2) / [\sigma_3 \sqrt{\tau}] \\
\rho_{1,3} &= \text{Corr}(\ln(S_1(T)), \ln(S_2(T)))
\end{align*}
\]

The rebate portion of a detached barrier option's value is determined directly from previous results. Recall that the "out" rebate for this model is received when the underlying state variable \(S_2(t)\) violates the critical boundary level \(C\). The "in" rebate is received at expiration if the barrier is never severed. Accordingly, equations [3.86] and [3.87] is rewritten as

\[
R r^{-\tau} [N(\omega_6 - \sigma_1 \rho_{1,3} \sqrt{\tau})] - (C/S_2)^{25-2} N(\omega_8 - \sigma_1 \rho_{1,3} \sqrt{\tau})]
\]

\[
R(S_2/C)(C/S_2)^{4-A} N(\omega_9) + (C/S_2)^{4-A} N(\omega_9 - 2\hat{\nu} \sigma_3 \sqrt{\tau})]
\]

where

\[
\begin{align*}
\omega_9 &= \frac{[\ln(C/S_2) + \hat{\nu} \sigma_3^2 \tau]}{[\sigma_3 \sqrt{\tau}]} \\
\hat{\nu} &= \sqrt{(\ln(r/d) - 5\sigma_3^2 \tau^2 + 2\ln(r)} / [\sigma_3^2].
\end{align*}
\]

Equation [5.11] is again recognized as the standard Black-Scholes equation for options written on dividend paying assets. Equation [5.15] is the current value of the detached "in" rebate and [5.16] is the current value of the "out" rebate. A sketch of the derivations of equations [5.12] and [5.13] are found in Appendix VC. The comparative statics for equations [5.12] and [5.13] are found in Appendix VD. The valuation solutions for attached barrier options

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with a non-stochastic strike price are presented in Table V.C.a.\textsuperscript{61}

V.D The Generalized Stochastic Barrier Model

In this section the attached barrier model and the detached barrier model is written as one generalized formula to demonstrate the overall generality of this model.

Let $W(S_1,X,S_b,C,R,\tau)$ denote the value of a European barrier option written on $S_1$ and struck at $X$ (a constant). This option either comes into existence (an "in" option) or ceases to exist (an "out" option) when the barrier asset, $S_b(t)$, violates some critical constant level $C$.\textsuperscript{62} "Out" ("In") versions of this option have a non-rebate payoff of $\max(\phi S_1(T),-\phi X,0)$ at $T$ if $\eta S_b(t) > \eta C \land t \in [t, T]$ (if $\eta S_b(t) \leq \eta C$ for some $t \in [t, T]$) and zero otherwise.\textsuperscript{63} This model has a rebate payoff of $\mathcal{R}$.

The stock price dynamics for the optioned asset price, $S_1(t)$, are described by the stochastic differential equation listed in [5.1]. The stock price dynamics for the barrier asset price, $S_b(t)$, are either directly described by [5.1] or can be determined from [5.1] using Itô's lemma. To see this, consider simultaneously the attached barrier option, $W(S_1,X,S,C,0,\tau)$, and the detached barrier option, $W(S_1,X,S_b,C,0,\tau)$. For the detached model the barrier asset is $S_b = S_3$ and the stock price dynamics for $S_1(t)$ are given directly by the stochastic differential equation [5.1]. For the attached barrier model, a change of numéraire first has to be performed on the barrier asset. That is, define the barrier asset price as $S_b(t) = S(t) = S_1(t)/S_2(t)$ and rewrite the current value of this attached option as $W(S_1,X,S,C,0,\tau)$.\textsuperscript{64} The stock price dynamics for $S_1(t)$

\textsuperscript{61} In related research, Carr (1993b) has independently derived the valuation formula for the up-and-out detached barrier call option with zero rebate.

\textsuperscript{62} $S_b$ is referred to as the barrier asset, as opposed to the optioned asset, because this asset's position in relation to the barrier determines the status (i.e., "in" or "out") of the option. When the barrier asset and the optioned asset are identical, the standard constant barrier model results; see Chapter III.

\textsuperscript{63} Recall $\phi$ equals 1 if the option is a call option and -1 if the option is a put. $\eta$ equals 1 if the barrier is being approached from above (i.e., $S_b > C$) and -1 otherwise.

\textsuperscript{64} Stated in this manner, it is be clear that the derivation of the detached barrier model is very similar to the derivation of the attached model.
Table V.C.a
Valuation Formulas for European Detached Stochastic Barrier Options With a Non-Stochastic Strike Price: \( W(S_t, X, S_n, C, R, \tau) \)

<table>
<thead>
<tr>
<th>EUROPEAN OPTION TYPE (X = Exercise Price) (C = Barrier Level On ( S_t ))</th>
<th>VALUATION EQUATION</th>
<th>( \eta^* )</th>
<th>( \phi^{**} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up-and-In Call</td>
<td>( [5.11] - [5.12] + [5.13] + [5.15] )</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>Down-and-In Call</td>
<td>( [5.11] - [5.12] + [5.13] + [5.15] )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Up-and-Out Call</td>
<td>( [5.12] - [5.13] + [5.16] )</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>Down-and-Out Call</td>
<td>( [5.12] - [5.13] + [5.16] )</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

* \( \eta \) is defined to be 1 if the barrier is being approached from above (i.e., \( S_t > C \)) and -1 if the barrier is being approached from below.

** \( \phi \) is defined to be 1 if the stochastic barrier option is a call and -1 if the option is a put.
and $S_2(t)$ are given directly by [5.1] and, thus, the stock price dynamics for the relative asset price $S(t)$ is determined from Itô's lemma as follows.

$$
\frac{dS(t)}{S(t)} = \frac{\partial S(t)}{\partial S_1(t)} dS_1(t) + \frac{\partial S(t)}{\partial S_2(t)} dS_2(t) + \frac{1}{2} \frac{\partial^2 S(t)}{\partial S_1^2(t)} [dS_1(t)]^2 + \frac{1}{2} \frac{\partial^2 S(t)}{\partial S_2^2(t)} [dS_2(t)]^2 + \frac{\partial^2 S(t)}{\partial S_1 \partial S_2} dS_1(t) dS_2(t)
$$

$$
= [\mu_1 dt + \sigma_1 S dt dZ_2(t)] - [\mu_2 dt + \sigma_2 S dt dZ_2(t)] + [0] + \left[ \sigma_1^2 dt \right] - \left[ \sigma_3^2 dt \right] - [\sigma_1 \sigma_2 \rho_{12} dt].
$$

[5.18]

Assuming $S_1$ and $S_2$ are the prices of traded assets with respective payout rates of $d_1$ and $d_2$, [5.18] is as:

$$
dS(t)/S = \mu dt + \sigma dZ(t)
$$

where:

$$
\mu = \ln(d_2/d_1) - (\sigma_1 \sigma_2 \rho_{12} - \sigma_2^2),
$$

$$
\sigma dZ(t) = \sigma_1 dZ_1(t) - \sigma_2 dZ_2(t),
$$

$$
\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\sigma_1 \sigma_2 \rho_{12}.
$$

[5.19]

Thus, changes in the relative asset price also obey geometric Brownian motion.

Furthermore, note for future reference that Itô's lemma also implies that the continuously compounded return on the barrier asset (i.e., $\ln(S_B(t)/S_B)$) follows arithmetic Brownian motion:

$$
d \ln \left( \frac{S_1(t)}{S_2(t)} \right) = \hat{\mu} dt + \sigma dZ(t) \quad (\hat{\mu} = \ln(d_2/d_1) - \frac{1}{2}(\sigma_1^2 - \sigma_2^2))
$$

[5.20]

or

$$
d \ln \left( \frac{S_3(t)}{S_3} \right) = \hat{\mu}_3 dt + \sigma_3 dZ_3(t) \quad (\hat{\mu}_3 = \ln(n/d_3) - \frac{1}{2}(\sigma_3^2))
$$

[5.21]

depending on how the barrier asset is specified.

From the analysis above, regardless if $S_B = S_1(t)/S_2(t)$ or if $S_B(t) = S_3(t)$, the barrier option's value is dependent on only two underlying state variables: $S_1(t)$ and $S_2(t)$ or $S_1(t)$ and $S_3(t)$.
Accordingly, \( W(S_1, X, S_b, C, 0, \tau) \) must be the solution to the second order, linear, partial differential equation\textsuperscript{65}:

\[
\ln(r) W(t) = \ln(r/d) S_1 \frac{\partial W(t)}{\partial S_1} + \ln(r/d) S_j \frac{\partial W(t)}{\partial S_j} - \frac{\partial W(t)}{\partial \tau} \\
+ \frac{1}{2} \sigma_1^2 S_1 \frac{\partial^2 W(t)}{\partial S_1^2} + \frac{1}{2} \sigma_j^2 S_j \frac{\partial^2 W(t)}{\partial S_j^2} + \frac{\partial^2 W(t)}{\partial S_1 \partial S_j} \sigma_1 \sigma_j \rho_{1,j} S_1 S_j \quad (j = 2, 3)
\]  

\textbf{[5.22]}

The non-rebate portion of a barrier option must satisfy \textbf{[5.22]} subject to boundary conditions:

\begin{align*}
W_{\text{ac}}(S_1(T), X, S_b(T), C, 0) &= \max(0, S_1(T) - \phi X) \quad \text{if } \eta S_b(t) > \eta C \text{ for all } t \in [t_0, T], \\
W_{\text{ac}}(S_1(T), X, S_b(T), C, 0) &= 0 \quad \text{if } \eta S_b(t) \leq \eta C \text{ for any } t \in [t_0, T], \\
W_{\text{b}}(S_1(T), X, S_b(T), C, 0) &= \max(0, S_1(T) - \phi X) \quad \text{if } \eta S_b(t) \leq \eta C \text{ for some } t \in [t_0, T], \\
W_{\text{b}}(S_1(T), X, S_b(T), C, 0) &= 0 \quad \text{if } \eta S_b(t) > \eta C \text{ for all } t \in [t_0, T], \\
W(0, X, S_b(t), C, T-t) &= 0 \quad \text{if } S_1(t) = 0 \text{ for any } t \in [t_0, T], \\
\text{and } \quad 0 \leq W(S_1(t), X, S_b(t), C, T-t) \leq S_1(t) \quad \text{for all } t \in [t_0, T].
\end{align*}

The generalized (non-rebate) valuation solution for each of the eight possible barrier options is written as a linear combination of at most three equations.

\[
\phi S_1 d_1^{-\tau} N(\phi w_t) - \phi r^{-\tau} X N(\phi (w_1 - \sigma_1 \sqrt{\tau})) \tag{5.23}
\]

\[
\phi S_1 d_1^{-\tau} M(\phi w_t, \eta w_{10}, \phi \eta \rho_{1, b}) - \phi r^{-\tau} M(\phi (w_1 - \sigma_1 \sqrt{\tau}), \eta (w_{10} - \sigma_1 \rho_{1, b} \sqrt{\tau}), \eta \phi \rho_{1, b}) \tag{5.24}
\]

\[
\phi (C/S_b) \sigma_b^2 \left[ S_1 d_1^{-\tau} (C/S_b) \right]^{2 \sigma_e} P_L M(\phi w_{11}, \eta w_{12}, \phi \eta \rho_{1, b}) - X r^{-\tau} M(\phi (w_{11} - \sigma_1 \sqrt{\tau}), \eta (w_{12} - \sigma_1 \rho_{1, b} \sqrt{\tau}), \eta \phi \rho_{1, b}) \tag{5.25}
\]

\textsuperscript{65} For a sketch of the derivation of equation \textbf{[5.22]}, see Merton (1973, pp. 162-165).
where

\[ w_{10} = \frac{\ln(S_y/C) + \mu \tau}{\sigma_y \sqrt{\tau}} + \sigma_1 \rho_{1,9} \sqrt{\tau} \]

\[ w_{11} = w_1 + \frac{2 \rho_{1,9} \ln(C/S_y)}{\sigma_y \sqrt{\tau}} \]

\[ w_{12} = w_{10} + \frac{2 \ln(C/S_y)}{\sigma_y \sqrt{\tau}} \]

\[ \hat{\mu}_y = \begin{cases} 
\hat{\mu}_y = \ln(r/d_1) - 0.5 \sigma_y^2 & \text{if } S_y = S_3 \\
\hat{\mu}_y = \ln(d_0/d_1) - 0.5 (\sigma_1^2 - \sigma_2^2) & \text{if } S_y = S_1/S_2
\end{cases} \]

\[ \sigma_y^2 = \begin{cases} 
\sigma_3^2 & \text{if } S_y = S_3 \\
\sigma_1^2 + \sigma_2^2 - 2 \sigma_1 \sigma_2 \rho_{1,2} & \text{if } S_y = S_1/S_2
\end{cases} \]

\[ \rho_{1,9} = \begin{cases} 
\rho_{1,3} & \text{if } S_y = S_3 \\
\rho_{1,12} = [\sigma_1 - \sigma_2 \rho_{1,3}] / [\sigma] & \text{if } S_y = S_1/S_2
\end{cases} \]

The rebate portion of the stochastic barrier option's value is also written in a generalized framework.

\[ Rr^{-1} \left[ N(\eta(w_{10} - \sigma_1 \rho_{1,9} \sqrt{\tau})) - (C/S_y)^\sigma_3 N(\eta(w_{12} - \sigma_1 \rho_{1,9} \sqrt{\tau})) \right] \]

\[ R(S_y/C) \left[ (C/S_y)^\sigma_3 N(\eta w_{13}) + (C/S_y)^\sigma_3 N(\eta(w_{13} - 2 \bar{m} \sigma_y \sqrt{\tau})) \right] \]

where

\[ w_{13} = [\ln(C/S_y) + \bar{m} \sigma_3^2 \cdot \tau] / [\sigma_y \sqrt{\tau}] \]

\[ \bar{m} = \sqrt{\mu^2 + 2 \ln(r)} / [\sigma_y^2] \]

Equation [5.23] is again recognized as the standard Black-Scholes equation for equity options written on dividend paying assets. Equation [5.27] is the current value of the "in" rebate and [5.28] is the current value of the "out" rebate. A sketch of the derivations of equations
[5.24] and [5.25] is found in Appendix VA (when $S_b=S_1/S_2$) and VC (when $S_b=S_3$). The valuation solutions for the generalized model are presented in Table V.D.a.

V.E Chapter Summary

In this chapter a generalized option pricing formula is developed to value options restricted by an underlying stochastic boundary. The formula is shown to be applicable to attached and detached barrier models. In the next chapter, the detached barrier model is applied to the valuation of Black-Scholes options subject to intertemporal default risk.
<table>
<thead>
<tr>
<th>EUROPEAN OPTION TYPE (X = Exercise Price)</th>
<th>VALUATION EQUATION</th>
<th>$\eta^*$</th>
<th>$\phi^{**}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up-and-In Call</td>
<td>[5.23]-[5.24]+[5.25]+[5.27]</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>Up-and-In Put</td>
<td>[5.23]-[5.24]+[5.25]+[5.27]</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>Down-and-In Call</td>
<td>[5.23]-[5.24]+[5.25]+[5.27]</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Down-and-In Put</td>
<td>[5.23]-[5.24]+[5.25]+[5.27]</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>Up-and-Out Call</td>
<td>[5.24]-[5.25]+[5.28]</td>
<td>-1</td>
<td>1</td>
</tr>
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<td>[5.24]-[5.25]+[5.28]</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>Down-and-Out Call</td>
<td>[5.24]-[5.25]+[5.28]</td>
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<td>1</td>
</tr>
<tr>
<td>Down-and-Out Put</td>
<td>[5.24]-[5.25]+[5.28]</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

$^*$ $\eta$ is defined to be 1 if the barrier is being approached from above (i.e., $S_n > C$) and -1 if the barrier is being approached from below.

$^{**}$ $\phi$ is defined to be 1 if the stochastic barrier option is a call and -1 if the option is a put.
V.F Appendix VA: Derivation of the Attached Barrier Model

In this appendix the valuation formula for the European attached barrier model with a non-stochastic strike price for "out" attached options with zero rebate is developed. The derivation for "in" options then follows immediately from the result that an "out" option with zero rebate combined with an otherwise identical "in" option must be equivalent in value to a Black-Scholes option.

V.F.1 Derivation of the Densities

The reflection principle (see Rich (1994)) is a powerful graphic depiction of the strong Markov property (see Karlin (1968, pp. 230-232)). It is an intuitively appealing technique that can be used to simplify complex probabilities of the form

$$ Pr \left( S(T) \geq X, \sup_{t \in [t, T]} S(t) \leq C \right) $$

[5.30]

where the supremum extends over all \{S(t), t \in [t_0, T]\} of \Omega.

Unfortunately, the reflection principle as commonly presented (see Harrison (1985), Chapter 1) can be applied only in 2-space. I use "reflection principle intuition" to generalize this concept to k-space. A rigorous proof of the generalized reflection principle is beyond the purpose of this thesis. However, the bivariate normal density that is obtained from using the generalized reflection principle in 3-space is verified by contradiction. That is, the solution to [5.2] subject to the boundary conditions is unique. The solution derived by preference-free valuation that confirms [5.2] thereby, also confirms the densities used.

To evaluate the following complex probability, begin by performing a change of numeraire on the barrier asset (see footnote 9). Define \( S(t) = S_1(t)/S_2(t) \) and note from the law of total probability:
\[
\Pr\left(S_1(T) \geq X, \sup_{t \in [t, T]} S_2(t) \leq C\right) = \Pr\left(S_1(T) \geq X, \sup_{t \in [t, T]} S(t) \leq C\right) \\
= \Pr(S_1(T) \geq X, S(T) \leq C, \sup S(t) \leq C) \\
= \Pr(S_1(T) \geq X, S(T) \leq C) - \Pr(S_1(T) \geq X, S(T) \leq C, \sup S(t) \geq C)
\]

where C, some constant, is the absorbing continuous barrier level restricting the process of the relative asset [S_1/S_2], asset one is the optioned asset, and X is the strike price. Note that the region in which the supremum is extended over has been suppressed for notational convenience.

Recall that in 2-space, application of the reflection principle requires the Brownian motion to have zero drift. It is natural to assume that this requirement must also hold in k-space. Begin by assuming each drift coefficient, \( \mu_1 \) and \( \mu_2 \), is zero, where \( \mu_1 \) is defined in [5.1] and \( \mu_2 \) is defined in [5.19]. After considering the zero drift case, the results are generalized to include a drift.

Converting into continuously compounded returns by letting
\[
X_i(T) = \ln(S_i(T)/S_i), z_i = \ln(S_i/X), X(T) = \ln(S(T)/S), \text{ and } z = b = \ln(C/S),
\]
where \( S_i(0) = S_i \), equation [5.31] can be written as
\[
\Pr(X_1(T) \geq -z_1, X(T) \leq z) = \Pr(X_1(T) \geq -z_1, X(T) \leq z, \sup X(t) \geq b).
\]

Since \( S_1(T) \) and \( S(T) \) are lognormally distributed (by construction), \( X_1(T) \) and \( X(T) \) are normally distributed.

It is assumed that each distribution function in [5.32] is continuously differentiable. In which case, the density associated with the left-hand term in [5.32] can be determined directly by standardizing and differentiating as follows.

\[66\] To make the derivation as general as possible we use a different place holder for \( z \) and \( b \). Of course, in this case they are the same but that need not always be the case.
\[
Pr(X_i(T) \leq z_i, X(T) \leq z) = Pr \left( \frac{Z_1(T) - z_1}{\sigma_1 \sqrt{\tau}}, \frac{Z(T) - z}{\sigma \sqrt{\tau}} \right) = \frac{\partial M(.)}{\partial Z_1} = \frac{\partial}{\partial Z} \left( \frac{1}{\sigma_1 \sqrt{\tau}} \left( \frac{z - \rho_{1,12} z_1}{\sigma_1 \sqrt{\tau}} \right) \right)
\]

Thus,

\[
f_{Z_1(T), X(T)}(z_1, z) = \frac{\partial^2 M(.)}{\partial Z_1 \partial Z} = \frac{1}{\sigma_1 \sigma \sqrt{1 - \rho_{1,12}^2}} N \left( \frac{z - \rho_{1,12} z_1}{\sigma_1 \sqrt{\tau}} \right) N \left( \frac{z_1}{\sigma_1 \sqrt{\tau}} \right)
\]

where \( \text{Var}(Z_1(T)) = \sigma^2 \), \( Z_1(T) = X_i(T)/[\sigma_1 \sqrt{\tau}] \) and \( Z(T) = X(T)/[\sigma \sqrt{\tau}] \) are standard normal random variables with a correlation coefficient of \( \rho_{1,12} = (\sigma_1 - \sigma_2 \rho_{1,2})/\sigma \) (see equation [5.6]),

\[
n(q) = \frac{e^{-q^2}}{\sqrt{2\pi}} \text{ is the standard normal density function, } N(q) = \int_{-\infty}^{q} e^{-x^2/2} \, dx \text{ is the standard normal distribution function, } M(q_1, q_2, \rho) \text{ is the bivariate cumulative normal distribution with upper limits of integration } q_1 \text{ and } q_2 \text{ and a correlation coefficient of } \rho, \text{ and } f(z_1, z) \text{ is the standard bivariate normal density function (as defined above).}
\]

The difficult part in evaluating [5.32] is the right-hand term. First, suppose for a moment that \( S_1 \) and \( S \) are independent. From previous results (Rich, 1994), I have shown that in this case the right-hand term in [5.32] can be written as:

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\[ Pr(X_i(T) \geq -z_i, X(T) \leq z_i, \sup X(t) \geq b) = Pr\left( Z_i(T) \geq \frac{-z_i}{\sigma_i \sqrt{\tau}} \right) \right) Pr\left( Z(T) \geq \frac{2b - z}{\sigma \sqrt{\tau}} \right) \]  

[5.35]

In the more general case, when \( S_i \) and \( S \) are not independent the "correlation effect" resulting from the method of images has to be considered. This discussion can best be motivated by first briefly reviewing 2-state reflection principle intuition.

Consider only the arguments in [5.32] containing the random variable \( Z(t) = X(t)/[\sigma \sqrt{\tau}] \). The reflection principle results in respecifying the condition that

\[ Z(T) \leq z/[\sigma \sqrt{\tau}] \text{ and } \sup \{ Z(t) \} \geq b/[\sigma \sqrt{\tau}] \]

in terms of the sole condition:

\[ Z(T) \geq [2b - z]/[\sigma \sqrt{\tau}] \]

Intuitively, this conclusion is seen by considering a time zero solution to the partial differential equations of motion and the initial conditions (see Ingersoll (1987, p. 350)). At time zero, there must be a superposition of source of strength at the barrier along the \( Z \) axis. Then by placing a mirror along the time axis and locating the image of the origin along the \( Z \) axis, the image source can be seen to be \([2b]/[\sigma \sqrt{\tau}]\).

Now consider the more general case in which independence is not assumed. In the bivariate case there is an image source (from the mirror plane) along the \( Z_i \) axis as well as along the \( Z \) axis. The image source along the \( Z_i \) axis results from the correlation between \( Z_i \) and \( Z \) (i.e., in the independent case there is only one image source, excluding the source of unit strength at the origin). Thus, we conclude that the reflection principle results in the condition of \( Z(T) \leq z/[\sigma \sqrt{\tau}] \) being modified to \([-2b]/[\sigma \sqrt{\tau}] \) (because this is a one-to-one effect) and the condition of \( Z_i(T) \leq z_i/[\sigma \sqrt{\tau}] \) being modified by \([-\rho_{1z}2b]/[\sigma \sqrt{\tau}] \) (because this is the "correlation effect" of the reflection principle). Hence,

\[ Pr(X_i(T) \geq -z_i, X(T) \leq z_i, \sup X(t) \geq b) = Pr\left( Z_i(T) \leq \frac{-z_i}{\sigma_i \sqrt{\tau}} - \frac{\rho_{1z}2b}{\sigma \sqrt{\tau}}, Z(T) \leq \frac{-2b}{\sigma \sqrt{\tau}} \right) \]

[5.36]

\[ = Pr\left( Z_i(T) \leq \frac{-2b \rho_{1z} \sigma_i \sigma_i^{-1}}{\sigma_i \sqrt{\tau}}, Z(T) \leq \frac{-2b}{\sigma \sqrt{\tau}} \right) = M\left( \frac{z_i - 2b \rho_{1z} \sigma_i \sigma_i^{-1}}{\sigma_i \sqrt{\tau}}, \frac{z - 2b}{\sigma \sqrt{\tau}} \right) \]

It follows that the density associated with this term is:

\textit{Chapter V}
\[ g_{Z_1, Z_2}(z_1, z) = \frac{\partial^2 M(z)}{\partial z_1 \partial z_2} \]

\[ = \frac{1}{2\pi \sigma_1 \sigma \sqrt{1 - \rho_{1,12}^2}} e^{-\frac{1}{2\sigma_1^2 \sigma^2} \left[ (z - \mu_1)^2 - \frac{2 \rho_{1,12} (z - \mu_1) (z - \mu_2)}{\sigma_1 \sigma_2} + \frac{(z - \mu_1)^2 + (z - \mu_2)^2}{\sigma_1^2 + \sigma_2^2} \right]} \]

\[ \text{[5.37]} \]

Following Ingersoll (1987, p. 369), define the "defective density" as \( f(z_1, z) \) less \( g(z_1, z) \).

It is quickly seen that when \( Z(T) \) is evaluated at \( z = b \) and \( Z_1(T) \) is evaluated at \( z_1 = h \sigma_1 \sigma \), the "defective density" goes to zero (as it should, because this is a boundary condition on the "defective density", see Cox and Miller (1965, pp. 220-221)).

For Brownian motions with drift, \( \mu_1 \), equations [5.34] and [5.37] is rewritten as

\[ f_{Z_1, Z_2}(z_1, z) = \frac{1}{2\pi \sigma_1 \sigma \sqrt{1 - \rho_{1,12}^2}} e^{-\frac{1}{2\sigma_1^2 \sigma^2} \left[ (z - \mu_1)^2 - \frac{2 \rho_{1,12} (z - \mu_1) (z - \mu_2)}{\sigma_1 \sigma_2} + \frac{(z - \mu_1)^2 + (z - \mu_2)^2}{\sigma_1^2 + \sigma_2^2} \right]} \]

\[ \text{[5.38]} \]

and

\[ g_{Z_1, Z_2}(z_1, z) = \frac{2b}{\sigma^2} n(z_1, z) \]

\[ \text{[5.39]} \]

where

\[ n(z_1, z) = \frac{1}{2\pi} e^{-\frac{1}{2\sigma_1^2 \sigma^2} \left[ (z - \mu_1)^2 - \frac{2 \rho_{1,12} (z - \mu_1) (z - \mu_2)}{\sigma_1 \sigma_2} + \frac{(z - \mu_1)^2 + (z - \mu_2)^2}{\sigma_1^2 + \sigma_2^2} \right]} \]

\[ \text{[5.40]} \]

The constant term, \( \exp(2b / \sigma^2) \), premultiplying the \( n(z_1, z) \) density is calculated by setting the

---

\(^6\) \( \beta_1 \) and \( \bar{\mu} \) are, respectively, the continuously compounded expected return of asset one and the relative asset \( (S_1/S_2) \).

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"defective density" equal to zero when Z(T) is evaluated at \( z=b \) and \( Z_i(T) \) is concurrently evaluated at \( z_i=b \rho_{1,12}\sigma_i\sigma^{-1} \).

Proceeding in exactly the same manner, the complex probability

\[
\Pr(S_i(T) \geq X, \inf S(t) \geq C)
\]

is evaluated and differentiated to yield equations [5.38], [5.39] and [5.40].

In the next section of this appendix, detached barrier options are valued. To this end, it is useful to define \( \xi_1 = \frac{z_1-\mu_1}{\sigma_1\sqrt{\tau}} \) and \( \xi = \frac{z-\mu}{\sigma\sqrt{\tau}} \) and rewrite \( f(z_1,z) \) and \( g(z_1,z) \).

\[
f_{z_1(z_1,z)}(\xi_1,\xi) = \frac{1}{2\pi\tau\sqrt{1-\rho_{1,12}^2}} \left( 1 - \frac{1}{2(1-\rho_{1,12}^2)^2} \right)^{-\frac{1}{2}} \left[ \xi_1^2 - 2\rho_{1,12}\xi_1 + \xi^2 \right]
\]

[5.41]

and

\[
g_{z_1(z_1,z)}(\xi_1,\xi) = \frac{(S_2/C)^{2\sigma_1^2} \cdot 2\omega_{1,12}}{2\pi\tau\sqrt{1-\rho_{1,12}^2}} \left( 1 - \frac{1}{2(1-\rho_{1,12}^2)^2} \right)^{-\frac{1}{2}} \left[ \left( \xi_1 - \frac{2b}{\sigma_1\sqrt{\tau}} \right)^2 - 2\rho_{1,12}\xi_1 \left( \xi_1 - \frac{2b}{\sigma_1\sqrt{\tau}} \right) + \xi^2 \right]
\]

[5.42]

Finally, to standardize each of the densities define \( \tilde{\xi}_1 = \xi_1 - \frac{2b}{\sigma_1\sqrt{\tau}} \) and \( \tilde{\xi} = \xi - \frac{2b}{\sigma\sqrt{\tau}} \), which gives

\[
g_{z_1(z_1,z)}(\tilde{\xi}_1,\tilde{\xi}) = \left( \frac{S_2}{S_1} \right)^{2\omega_{1,12}(\tilde{\xi}_1^2 - \sigma_1^2)} \left( \frac{1}{2\pi\tau\sqrt{1-\rho_{1,12}^2}} \right)^{\frac{1}{2}} \left[ \tilde{\xi}_1^2 - 2\rho_{1,12}\tilde{\xi}_1 + \tilde{\xi}^2 \right]
\]

[5.43]

Before concluding this section, an economic interpretation of this "defective density" is given. Since \( f(z_1,z) \) represents the joint probability of \( Z_i(T) \) being at \( z_1 \) and \( Z(T) \) being at \( z \) at

\[\text{---}\]

\[68\] For a more rigorous, but equivalent, derivation of this constant term, Girsanov's Theorem could have been applied.

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time \( T \) (path independent), and \( g(z_1,z) \) depicts the joint probability of crossing of the barrier and \( Z_1(T) \) being at \( z_1 \) and \( Z(T) \) being at \( z \) at \( T \) (path dependent), the "defective density" represents the joint probability density of not crossing the barrier and \( Z_1(T) \) being at \( z_1 \) and \( Z(T) \) being at \( z \) at \( T \) (path dependent).

V.F.2 "Out" Attached Barrier Option Valuation - Non- Rebate Value

From the results in Section V.F.1 of this appendix, the non-rebate value of attached barrier options is determined as follows. Preference-free valuation is used to write the discounted expected terminal payoff as

\[
 r^{-1} E(\phi S_1(T) - \phi X | \phi S_1(T) \geq \phi X, \inf \eta S(t) > \eta C) \\
= r^{-1} E(\phi S_1(T) | \phi S_1(T) \geq \phi X, \eta S(T) > \eta C) \\
- \phi r^{-1} X Pr(\phi S_1(T) \geq \phi X, \eta S(T) > \eta C) \\
- r^{-1} E(\phi S_1(T) | \phi S_1(T) \geq \phi X, \eta S(T) > \eta C, \inf \eta S(t) > \eta C) \\
+ \phi r^{-1} X Pr(\phi S_1(T) \geq \phi X, \eta S(T) > \eta C, \inf \eta S(t) < \eta C) .
\]

\[\text{[5.44]}\]

\( E \) denotes the expectation operator which is taken under an equivalent martingale measure over all first passage paths (i.e. over all paths of \( \eta S(t) > \eta C \)) to the non-negative payoff space (i.e., \( S_1(T) \geq X \)). To conserve space, it is assumed that the reader understands that when \( \eta = -1 \), \( \inf \) (the infimum of the process) is replaced with sup (the supremum of the process). Each of the four terms in this valuation equation is now individually evaluated.

The easiest term to evaluate is \( \phi r^{-1} X Pr(\phi S_1(T) \geq \phi X, \eta S(T) > \eta C) \).

\[
\begin{align*}
\phi r^{-1} X Pr(\phi S_1(T) \geq \phi X, \eta S(T) > \eta C) \\
= \phi r^{-1} X \left( \int \int f(\xi,\tau)d\tilde{Z}(T)dZ_1(T) \right) \\
- \phi(\omega_1 - \sigma_1 \sqrt{\tau}) \left( \int \int f(\xi,\tau)d\tilde{Z}(T)dZ_1(T) \right) \\
= \phi r^{-1} X M(\phi(\omega_1 - \sigma_1 \sqrt{\tau}), \eta(\omega_2 - \sigma_1 \rho_{1,12} \sqrt{\tau})) \phi \eta \rho_{1,12} .
\end{align*}
\]

\[\text{[5.45]}\]

The following results are used in [5.45]. Recall \( Z_1(T) = X_1(T)/[\sigma \sqrt{\tau}] = \ln(S_1(T)/S_0)/[\sigma \sqrt{\tau}] \) and \( Z(T) = X(T)/[\sigma \sqrt{\tau}] = \ln(S(T)/S)/[\sigma \sqrt{\tau}] \) so \( X_1(0) = \ln(S_1(0)/S)/[\sigma \sqrt{\tau}] = 0 \) and \( X(0) = 0 \). \( X(t) \) is
a function of $S_t(t)$. Hence, Itô’s lemma is used to determine the arithmetic process followed by $X_t(t)$ and $X(t)$.

\[
\frac{dX_t(t)}{dS_t(t)} = \frac{\partial X_t(t)}{\partial S_t(t)} dS_t + \frac{\partial^2 X_t(t)}{\partial S_t(t)^2} (dS_t)^2, \tag{5.46}
\]

so

\[
dx_t(t) = (\mu - 0.5\sigma_1^2) dt + \sigma_1 dZ_1
= \mu dt + \sigma_1 dZ_1 \quad (\mu = \mu - 0.5\sigma_1^2) \tag{5.47}
\]

\[
dx(t) = (\mu - 0.5(\sigma_1^2 - \sigma_2^2)) dt + \sigma dZ
= \mu dt + \sigma dZ \quad (\mu = \mu - 0.5(\sigma_1^2 - \sigma_2^2))
\]

In the absence of a risk premium, $\mu = \ln(r/d_t)$ and $\mu = \ln(d_t/d_t)$ and [5.45] follows immediately.

The second term in [5.44], $\phi r^\gamma E(\phi S_1(T) | \phi S_1(T) \geq \phi X, \eta S(T) > \eta C)$, is evaluated as follows.

\[
r^{-\tau} E(\phi S_1(T) | \phi S_1(T) \geq \phi X, \eta S(T) > \eta C)
= \phi d_1^{-\tau} S_1 e^{-0.5\sigma_1^2} E(e^{\sigma_1 \sqrt{\tau} Z_1} | \phi S_1(T) \geq \phi X, \eta S(T) > \eta C)
= \phi S_1 d_1^{-\tau} e^{-0.5\sigma^2} \int \int e^{\sigma_1 \sqrt{\tau} Z_1} f(Z_1) dZ_1 dZ_2(T)
= \phi S_1 d_1^{-\tau} \int \int f(v_1, v_2) dv_1 dv_2 \int \int f(v_1, v_2) dv_1 \tag{5.48}
= \phi S_1 d_1^{-\tau} M(\phi w_1, \eta w_2, \phi \eta \rho_{1,12})
\]

To see this, note that

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\[ e^{\frac{\alpha_1\sqrt{\tau_1}f_1(x_1,\xi)}{2(1-\rho_{1,12}^2)}} = e^{\frac{2\alpha_1\sqrt{\tau_1}f_1(x_1,\xi)}{2(1-\rho_{1,12}^2)}} \cdot f_1(x_1,\xi) \]

\[ = f_1(x_1,\xi)e^{\frac{-1}{2(1-\rho_{1,12}^2)}\left[(\phi_1^2) - (2\rho_{1,12}\sigma_{1,12}\sqrt{\tau_1}) + (\xi_1^2 - 2\rho_{1,12}\rho_{1,12}^2)\right]} \]

\[ = e^{\frac{-1}{2(1-\rho_{1,12}^2)}\left[(\phi_1^2) - (2\rho_{1,12}\sigma_{1,12}\sqrt{\tau_1}) + (\xi_1^2 - 2\rho_{1,12}\rho_{1,12}^2)\right]} \]

\[ = e^{\frac{-1}{2(1-\rho_{1,12}^2)}\left[(\phi_1^2) - (2\rho_{1,12}\sigma_{1,12}\sqrt{\tau_1}) + (\xi_1^2 - 2\rho_{1,12}\rho_{1,12}^2)\right]} \]

where

\[ \phi_1 = \left(\xi_1 - \sigma_{1,12}\sqrt{\tau_1}\right)^2 - \sigma_{1,12}^2 + 2\rho_{1,12}\sigma_{1,12}\sqrt{\tau_1} \]

\[ \phi_2 = 2\rho_{1,12}(\xi_1 - \sigma_{1,12}\sqrt{\tau_1}) + 2\sigma_{1,12}^2 - 2\sigma_{1,12}^2 \]

\[ \phi_3 = (\xi_1 - \sigma_{1,12}\sqrt{\tau_1})^2 - \sigma_{1,12}^2 \]

\[ \phi_1 - \phi_2 + \phi_3 = (\xi_1 - \sigma_{1,12}\sqrt{\tau_1})^2 - (\xi_1 - \sigma_{1,12}\sqrt{\tau_1})(\xi_1 - \sigma_{1,12}\sqrt{\tau_1}) - \sigma_{1,12}^2(1-\rho_{1,12}^2) \]

Thus,

\[ e^{\frac{\alpha_1\sqrt{\tau_1}f_1(x_1,\xi)}{2(1-\rho_{1,12}^2)}} = e^{\frac{-1}{2(1-\rho_{1,12}^2)}\left[(\phi_1 - \phi_2 + \phi_3) - 2\rho_{1,12}(\xi_1 - \sigma_{1,12}\sqrt{\tau_1}) + (\xi_1 - \sigma_{1,12}\sqrt{\tau_1})^2\right]} \]

Combining [5.51] with the third-to-last equation in [5.48], yields the second-to-last equation in [5.48]. The final equation in [5.48] is derived by making the change of variables \( \nu_1 = (\xi_1 - \sigma_{1,12}\sqrt{\tau_1}) \), \( \nu = (\xi_1 - \sigma_{1,12}\sqrt{\tau_1}) \), and redefining the limits of integration.

Note that equation [5.48] less equation [5.45] yields equation [5.4].

To derive [5.5] we begin by evaluating the term: \( \phi^XPr(\phi S_1(T) \geq \phi X, \eta S(T) > \eta C, \text{inf} \eta S(T) \leq \eta C) \) of [5.44].
\[
\phi r^{-1} X Pr(\phi S_1(T) \geq \phi X, \eta S(T) > \eta C, \inf \eta S(t) \leq \eta C)
= \phi r^{-1} X Pr\left(\phi Z_1(T) \geq \phi \left[\frac{\ln(X/S_1) - \tilde{\mu}_1}{\sigma_1 \sqrt{\tau}} - \frac{2\ln(CS_2/S_1) \rho_{1,12}}{\sigma_1 \sqrt{\tau}}\right], \eta Z(T) \geq \eta \left[\frac{\ln(S_2/C/S_1) - \tilde{\mu}_1}{\sigma_1 \sqrt{\tau}} - \frac{2\ln(S_2/C/S_1)}{\sigma_1 \sqrt{\tau}}\right]\right)
\]

= \phi r^{-1} X \int_{-\Phi(w_4 - \sigma_1 \sqrt{\tau})}^{\eta w_4} \int_{-\Phi(w_1 - \sigma_1 \sqrt{\tau})}^{\eta w_1} g(\xi_1, z) dZ_1(T) dZ(T)

= \phi \left(\frac{S_C}{S_1}\right)^{2\ln(d/d_1 - S_1) \sigma_1^2} r^{-1} XM(\phi(w_3 - \sigma_1 \sqrt{\tau}), \eta(w_4 - \sigma_1 \rho_{1,12} \sqrt{\tau}))$

The final term to evaluate is: \(r^{-1} E(S_1(T) | Pr(\phi S_1(T) \geq \phi X, \eta S(T) > \eta C, \inf \eta S(t) \leq \eta C)\). If it is noted that

\[
r^{-1} E(\phi S_1(T) | \phi S_1(T) \geq \phi X, \eta S(T) > \eta C, \inf \eta S(t) \leq \eta C)
= \phi S_1 d^{-1} e^{-5\sigma_1^2 \eta \tau} E(e^{\sigma_1 \sqrt{\tau} Z_1} | \phi S_1(T) \geq \phi X, \eta S(T) > \eta C, \inf \eta S(t) \leq \eta C)
= \phi S_1 d^{-1} e^{-5\sigma_1^2 \eta \tau} 2\Phi(2\sigma_1 \rho_{1,12} \sqrt{\tau}) E(e^{\sigma_1 \sqrt{\tau} Z_1} | \phi S_1(T) \geq \phi X, \eta S(T) > \eta C, \inf \eta S(t) \leq \eta C)
\]

the solution is immediately determined. This is because the same steps can be followed of completing the squares and making a change of variables that are used to evaluate the previous conditional expectation of this section, [5.48]. Using these previous results it follows that

\[
r^{-1} E(\phi S_1(T) | \phi S_1(T) \geq \phi X, \eta S(T) > \eta C, \inf \eta S(t) \leq \eta C)
= \phi S_1 d^{-1} e^{-5\sigma_1^2 \eta \tau} 2\Phi(2\sigma_1 \rho_{1,12} \sqrt{\tau}) E(e^{\sigma_1 \sqrt{\tau} Z_1} | \phi S_1(T) \geq \phi X, \eta S(T) > \eta C, \inf \eta S(t) \leq \eta C)
\]

\[
= \phi S_1 d^{-1} e^{2\ln(S_2/C/S_1) \rho_{1,12} \sigma_1 \sqrt{\tau}} (S_2/C/S_1)^{2\ln(d/d_1 - S_1) \sigma_1^2} \Phi(2\sigma_1 \rho_{1,12} \sqrt{\tau}) M(\phi w_2, \eta w_4; \phi \eta \rho_{1,12})
\]

Note that equation [5.54] less equation [5.52] yields equation [5.5].
V.G Appendix VB: Comparative Statics for the Attached Barrier Model

In this appendix we verify equations [5.4] and [5.5] by deriving the comparative statics. Verification follows by substituting these results into the partial differential equation [5.2] (subject to the boundary conditions). To begin with, some preliminary checks of the model are performed.

V.G.1 Initial Checks

As an initial check on equations [5.4] and [5.5] note the following. If \( S_2 \) is a constant, the constant barrier model results, where the barrier level is specified to be \( H = S_2 C \). If \( S_2 \) is set to the specific constant zero the "out" model converges to the Black-Scholes option pricing model and all "in" option values go to zero.

Perhaps the most interesting case to examine (and the most rigorous initial check) is when \( X \) goes to zero. As \( X \) goes to zero, \( w_3 \), and \((w_3-\sigma \sqrt{T})\) both go to infinity and equation [5.4] minus equation [5.5] (when \( \phi = 1 \)) converge to

\[
S_1 d_1^{L} \left[ N(\eta w_2) - (H/S_1)^{2 \left( \text{ln}(d_2/d_1) + 5 \sigma^2 T / 2 \right)} N(\eta w_3) \right]
\]  

[5.55]

To confirm that this is the correct result, note that when \( X = 0 \) we have a stochastic barrier binary option. Specifically, this is an asset-or-nothing at expiry option. An "out" stochastic barrier asset-or-nothing at expiry option is a path dependent option that pays off \( S_t(T) \) at time \( T \) if the barrier is never violated (i.e., if \( S_t(t) > C S_t(t) \forall t \in [t_\tau, T] \)) and zero otherwise. Exploiting the power of linear homogeneity again, we restate this problem as \( S_2 \) stochastic barrier asset-or-nothing at expiry options that pay off \( S(T) = S_t(T)/S_2(T) \) at \( T \) if \( S(t) > C \forall t \in [T, t_\tau] \). The first term in equation [4.4] less the first term in [4.6] (when \( \theta = 0 \)) is used to find the value of such an option as [5.55].

---

\( ^69 \) For a further discussion of binary options see Rubinstein and Reiner (1991b).
V.G.2 Comparative Statics

In the comparative statics that follow note that since $\frac{\partial p_i}{\partial s_{i'}} = 0$,

$$\frac{\partial M(c_1(S), c_2(S), \rho)}{\partial s_i} = N\left(\frac{c_1(S) - \rho c_2(S)}{\sqrt{1 - \rho^2}}\right)n(c_2(S)) \frac{\partial c_2(S)}{\partial s_i}$$

$$+ N\left(\frac{c_2(S) - \rho c_1(S)}{\sqrt{1 - \rho^2}}\right)n(c_1(S)) \frac{\partial c_1(S)}{\partial s_i}.$$

[5.56]
To simplify the second-order derivatives it is also useful to note

\[
\begin{align*}
n \left( \frac{\eta w_2 - \rho_{1,12} \eta w_1}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_1) &= n \left( \frac{\phi w_1 - \rho_{1,12} \phi w_2}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_2), \\
n \left( \frac{\eta (w_2 - \sigma_1 \rho_{1,12} \sqrt{\tau}) - \rho_{1,12} \eta (w_1 - \sigma_1 \sqrt{\tau})}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_1 - \sigma_1 \sqrt{\tau}) &= n \left( \frac{\phi (w_1 - \sigma_1 \sqrt{\tau}) - \rho_{1,12} \phi (w_2 - \sigma_1 \rho_{1,12} \sqrt{\tau})}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_2 - \sigma_1 \rho_{1,12} \sqrt{\tau}). \\
n \left( \frac{\eta w_4 - \rho_{1,12} \eta w_3}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_3) &= n \left( \frac{\phi w_3 - \rho_{1,12} \phi w_4}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_4), \\
n \left( \frac{\eta (w_4 - \sigma_1 \rho_{1,12} \sqrt{\tau}) - \rho_{1,12} \eta (w_3 - \sigma_1 \sqrt{\tau})}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_3 - \sigma_1 \sqrt{\tau}) &= n \left( \frac{\phi (w_3 - \sigma_1 \sqrt{\tau}) - \rho_{1,12} \phi (w_4 - \sigma_1 \rho_{1,12} \sqrt{\tau})}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_4 - \sigma_1 \rho_{1,12} \sqrt{\tau}).
\end{align*}
\]
Other useful results include:

\[ \frac{\partial w_1}{\partial \tau} = \frac{-(w_1 - \sigma_1 \sqrt{\tau})}{2\tau} + \frac{\ln(r/d_1)}{\sigma_1 \sqrt{\tau}}, \quad \frac{\partial (w_1 - \sigma_1 \sqrt{\tau})}{\partial \tau} = \frac{-w_1}{2\tau} + \frac{\ln(r/d_1)}{\sigma_1 \sqrt{\tau}}, \]

\[ \frac{\partial w_2}{\partial \tau} = \frac{-(w_2 - \sigma_1 \sqrt{\tau})}{2\tau} + \frac{\ln(d_2/d_1)}{\sigma_1 \sqrt{\tau}}, \]

\[ \frac{\partial (w_2 - \sigma_1 \rho_{1,12} \sqrt{\tau})}{\partial \tau} = \frac{-(w_2 - \sigma_1 \rho_{1,12})}{2\tau} + \frac{\ln(d_2/d_1)}{\sigma_1 \sqrt{\tau}} - \frac{\sigma_1^2}{2\sigma_1 \sqrt{\tau}}, \quad [5.58] \]

\[ \frac{\partial w_3}{\partial \tau} = \frac{-(w_3 - \sigma_1 \sqrt{\tau})}{2\tau} + \frac{\ln(r/d_1)}{\sigma_1 \sqrt{\tau}}, \quad \frac{\partial (w_3 - \sigma_1 \sqrt{\tau})}{\partial \tau} = \frac{-w_3}{2\tau} + \frac{\ln(r/d_1)}{\sigma_1 \sqrt{\tau}}, \]

\[ \frac{\partial w_4}{\partial \tau} = \frac{\ln(d_2/d_1)}{\sigma_1 \sqrt{\tau}} - \frac{(w_4 - \sigma_1 \sqrt{\tau})}{2\tau}, \]

\[ \frac{\partial (w_4 - \sigma_1 \rho_{1,12} \sqrt{\tau})}{\partial \tau} = \frac{\ln(d_2/d_1)}{\sigma_1 \sqrt{\tau}} - \frac{(\sigma_1^2 - \sigma_2^2)}{2\sigma_1 \sqrt{\tau}} - \frac{(w_4 - \sigma_1 \rho_{1,12} \sqrt{\tau})}{2\tau}. \]
The comparative statics for the first term in [5.4], \( \partial S_1 \partial S_2 M(\phi w_1, \eta w_2, \eta \phi_{1,12}) \), are

\[
\frac{\partial (\cdot)}{\partial S_1} = \phi d_1^{-\tau} M(\phi w_1, \eta w_2, \eta \phi_{1,12}) + \left( \frac{\eta w_2 - \rho_{1,12} \eta w_1}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_1) d_1^{-\tau} \frac{1}{\sigma_1 \sqrt{\tau}}
\]

\[
+ \phi \eta M \left( \frac{\phi w_1 - \phi w_2 \rho_{1,12}}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_2) d_1^{-\tau} \frac{1}{\sigma_1 \sqrt{\tau}}
\]

\[
\frac{\partial (\cdot)}{\partial S_2} = -\phi \eta n \left( \frac{\phi w_1 - \phi w_2 \rho_{1,12}}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_2) \frac{S_1 d_1^{-\tau}}{S_2 \sigma_1 \sqrt{\tau}}
\]

\[
\frac{\partial^2 (\cdot)}{\partial S_1^2} = -\left( \frac{\eta w_2 - \rho_{1,12} \eta w_1}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_1) \frac{d_1^{-\tau}}{S_1 \sigma_1^2 \tau} \left( w_1 - \sigma_1 \sqrt{\tau} \right)
\]

\[
- \phi \eta M \left( \frac{\phi w_1 - \phi w_2 \rho_{1,12}}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_2) d_1^{-\tau} \frac{1}{S_1 \sigma_1^2 \tau} \left( w_2 - \sigma_1 \sqrt{\tau} \right)
\]

\[
- \frac{\eta}{\sqrt{1 - \rho_{1,12}^2}} \left( \frac{\eta w_2 - \rho_{1,12} \eta w_1}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_1) \frac{d_1^{-\tau}}{S_1 \sigma_1^2 \tau} \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_1^2} \right)
\]

\[
\frac{\partial^2 (\cdot)}{\partial S_2^2} = -\phi \eta n \left( \frac{\phi w_1 - \phi w_2 \rho_{1,12}}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_2) \frac{S_1 d_1^{-\tau}}{S_2 \sigma_2^2 \tau}
\]

\[
- \frac{\eta \rho_{1,12}}{\sqrt{1 - \rho_{1,12}^2}} \left( \frac{\eta w_2 - \rho_{1,12} \eta w_1}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_1) \frac{S_1 d_1^{-\tau}}{S_2 \sigma_2^2 \tau}
\]

\[
\frac{\partial^2 (\cdot)}{\partial S_1 \partial S_2} = \phi \eta n \left( \frac{\phi w_1 - \phi w_2 \rho_{1,12}}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_2) \frac{d_1^{-\tau}}{S_2 \sigma_2^2 \tau} \left( w_2 - \sigma_1 \sqrt{\tau} \right)
\]

\[
- \frac{\eta}{\sqrt{1 - \rho_{1,12}^2}} \left( \frac{\eta w_2 - \rho_{1,12} \eta w_1}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_1) \frac{d_1^{-\tau}}{S_2 \sigma_2^2 \tau} \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_1^2} \right)
\]
The remaining comparative statics for \( \phi S_1 d_1^{-\tau} M(\phi w_1, \eta w_2, \phi \eta \rho_{1,12}) \), are

\[
\frac{\partial (\cdot)}{\partial \tau} = - \phi S_1 d_1^{-\tau} M(\phi w_1, \eta w_2, \phi \eta \rho_{1,12}) \ln(d_1)
+ N \left( \frac{\eta w_2 - \rho_{1,12} \eta w_1}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_1) S_1 d_1^{-\tau} \left[ \frac{\ln(r d_1)}{\sigma_1 \sqrt{\tau}} - \frac{(w_1 - \sigma_1 \sqrt{\tau})}{2 \tau} \right]
+ \phi \eta N \left( \frac{\phi w_1 - \rho_{1,12} \phi w_2}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_2) S_1 d_1^{-\tau} \left[ \frac{\ln(d_2/d_1)}{\sigma_1 \sqrt{\tau}} - \frac{(w_2 - \sigma_1 \sqrt{\tau})}{2 \tau} \right]
\]  

[5.60]

The comparative statics for the second term in [5.4], \( \phi X r^{-\tau} M(\phi (w_1 - \sigma_1 \sqrt{\tau}), \eta (w_1 - \sigma_1 \sqrt{\tau}), \phi \eta \rho_{1,12}) \), are

\[
\frac{\partial (\cdot)}{\partial S_1} = N \left( \frac{n(w_2 - \sigma_1 \rho_{1,12} \sqrt{\tau}) - \rho_{1,12} n(w_1 - \sigma_1 \sqrt{\tau})}{S_1 \sigma_1 \sqrt{\tau}} \right) \frac{X r^{-\tau}}{S_1 \sigma_1 \sqrt{\tau}}
+ \phi \eta N \left( \frac{\phi (w_1 - \sigma_1 \sqrt{\tau}) - \rho_{1,12} \phi (w_2 - \sigma_1 \rho_{1,12} \sqrt{\tau})}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_2 - \sigma_1 \rho_{1,12} \sqrt{\tau}) \frac{X r^{-\tau}}{S_1 \sigma_1 \sqrt{\tau}}
\]

[5.61]

\[
\frac{\partial (\cdot)}{\partial S_2} = - \phi \eta N \left( \frac{\phi (w_1 - \sigma_1 \sqrt{\tau}) - \rho_{1,12} \phi (w_2 - \sigma_1 \rho_{1,12} \sqrt{\tau})}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_2 - \sigma_1 \rho_{1,12} \sqrt{\tau}) \frac{X r^{-\tau}}{S_2 \sigma_1 \sqrt{\tau}}
\]
Continuing, the comparative statics for the second term in \([5.4]\), \(\phi X r^{-\gamma} M(\phi(w_1 - \sigma_1 \sqrt{\tau}), \eta(w_2 - \sigma_2 \rho_{1,12} \sqrt{\tau}), \phi \eta \rho_{1,12})\), are

\[
\frac{\partial (\cdot)}{\partial \tau} = -\phi X r^{-\gamma} M(\phi(w_1 - \sigma_1 \sqrt{\tau}), \eta(w_2 - \sigma_2 \rho_{1,12} \sqrt{\tau}), \phi \eta \rho_{1,12}) \ln(r)
\]

\[
+ M \left( \frac{\eta(w_2 - \sigma_2 \rho_{1,12} \sqrt{\tau}) - \rho_{1,12} \eta(w_1 - \sigma_1 \sqrt{\tau})}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_1 - \sigma_1 \sqrt{\tau}) X r^{-\gamma} \left[ \frac{\ln(r/d_r)}{\sigma_1 \sqrt{\tau}} - \frac{w_1}{\sigma_1 \sqrt{\tau}} \right]
\]

\[
+ \phi \eta M \left( \frac{\phi(w_1 - \sigma_1 \sqrt{\tau}) - \rho_{1,12} \phi(w_2 - \sigma_2 \rho_{1,12} \sqrt{\tau})}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_2 - \sigma_2 \rho_{1,12} \sqrt{\tau}) X r^{-\gamma} \left[ \frac{\ln(d_2/d_r)}{\sigma_2 \sqrt{\tau}} - \frac{(w_2 - \sigma_2 \rho_{1,12} \sqrt{\tau})^2}{2 \tau} \right]
\]

\[
\frac{\partial (\cdot)}{\partial \tau} = -\frac{X r^{-\gamma} M(\phi(w_1 - \sigma_1 \sqrt{\tau}), \eta(w_2 - \sigma_2 \rho_{1,12} \sqrt{\tau}), \phi \eta \rho_{1,12})}{\phi X r^{-\gamma} M(\phi(w_1 - \sigma_1 \sqrt{\tau}), \eta(w_2 - \sigma_2 \rho_{1,12} \sqrt{\tau}), \phi \eta \rho_{1,12})} \ln(r)
\]

\[
+ M \left( \frac{\eta(w_2 - \sigma_2 \rho_{1,12} \sqrt{\tau}) - \rho_{1,12} \eta(w_1 - \sigma_1 \sqrt{\tau})}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_1 - \sigma_1 \sqrt{\tau}) X r^{-\gamma} \left[ \frac{\ln(r/d_r)}{\sigma_1 \sqrt{\tau}} - \frac{w_1}{\sigma_1 \sqrt{\tau}} \right]
\]

\[
+ \phi \eta M \left( \frac{\phi(w_1 - \sigma_1 \sqrt{\tau}) - \rho_{1,12} \phi(w_2 - \sigma_2 \rho_{1,12} \sqrt{\tau})}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_2 - \sigma_2 \rho_{1,12} \sqrt{\tau}) X r^{-\gamma} \left[ \frac{\ln(d_2/d_r)}{\sigma_2 \sqrt{\tau}} - \frac{(w_2 - \sigma_2 \rho_{1,12} \sqrt{\tau})^2}{2 \tau} \right]
\]
The remaining comparative statics for $\phi X_{1:\tau} M(\phi(w_1, \sigma; \sqrt{\tau}), \eta(w_2, \sigma, \rho_{1,12}, \sqrt{\tau}), \phi \eta \rho_{1,12})$, are

\[
\frac{\partial^2(\cdot)}{\partial s_2^2} = -\phi \eta n(w_2 - \sigma_2 \rho_{1,12} \sqrt{\tau}) \left[ N \left( \frac{\phi(w_1, \sigma; \sqrt{\tau}) - \rho_{1,12} \phi(w_2 - \sigma_1 \rho_{1,12} \sqrt{\tau})}{\sqrt{1 - \rho_{1,12}^2}} \right) \right] \frac{X_{1:\tau}}{S_2 \sigma^2 \tau} (w_2 - \sigma_1 \rho_{1,12} \sqrt{\tau})
\]

\[
- \frac{\eta}{\sqrt{1 - \rho_{1,12}^2}} \left[ \frac{\phi(w_1, \sigma; \sqrt{\tau}) - \rho_{1,12} \phi(w_2, \sigma; \sqrt{\tau})}{\sqrt{1 - \rho_{1,12}^2}} \right] n(w_1 - \sigma_1 \sqrt{\tau}) \rho_{1,12} X_{1:\tau} \frac{X_{1:\tau}}{S_2 \sigma^2 \tau} (w_2 - \sigma_1 \rho_{1,12} \sqrt{\tau})
\]

\[
+ \phi \eta N \left( \frac{\phi(w_1, \sigma; \sqrt{\tau}) - \rho_{1,12} \phi(w_2, \sigma; \sqrt{\tau})}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_2 - \sigma_1 \rho_{1,12} \sqrt{\tau}) \frac{X_{1:\tau}}{S_2 \sigma^2 \tau} (w_2 - \sigma_1 \rho_{1,12} \sqrt{\tau})
\]

\[\text{[5.64]}\]

\[
\frac{\partial^2(\cdot)}{\partial s_1 \partial s_2} = -\frac{\eta}{\sqrt{1 - \rho_{1,12}^2}} n(w_1 - \sigma_1 \sqrt{\tau}) \left[ N \left( \frac{\phi(w_1, \sigma; \sqrt{\tau}) - \rho_{1,12} \phi(w_2, \sigma; \sqrt{\tau})}{\sqrt{1 - \rho_{1,12}^2}} \right) \right] \frac{X_{1:\tau}}{S_1 S_2 \tau} \left( \frac{1}{\sigma^2} - \rho_{1,12} \frac{1}{\sigma^2} \right)
\]

\[
+ \phi \eta \left[ N \left( \frac{\phi(w_1, \sigma; \sqrt{\tau}) - \rho_{1,12} \phi(w_2, \sigma; \sqrt{\tau})}{\sqrt{1 - \rho_{1,12}^2}} \right) \right] \frac{X_{1:\tau}}{S_1 S_2 \sigma^2 \tau} (w_2 - \sigma_1 \rho_{1,12} \sqrt{\tau})
\]
The comparative statics for the first term in \([5.5]\), \(\phi(CS_j/S_j)^{X\tau}M(\phi w_3, \eta w_4, \phi \eta \rho_{1,12})\)

where \(a=[\ln(d_j/d_i)+ .5\sigma^2]/[\sigma^2]\), are

\[
\frac{\partial()}{\partial S_1} = - \frac{\phi \eta (S_2C/S_1)^{2a} \left( \frac{\phi w_3 - \phi \rho_{1,12} w_4}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_4) d_1^{-\frac{\tau}{2}}}{\sigma \sqrt{\tau}}
\]

\[\frac{\eta w_4 - \phi \rho_{1,12} \eta w_3}{\sqrt{1 - \rho_{1,12}^2}} n(w_3) d_1^{-\frac{\tau}{2}} \left( \frac{1}{\sigma_1} - \frac{2 \rho_{1,12}}{\sigma} \right) \]

\[\frac{2a - 1}{\phi(S_2 C/S_1)^{2a} d_1^{-\frac{\tau}{2}} M(\phi w_3, \eta w_4, \phi \eta \rho_{1,12})} \]

\[+ \left( \frac{S_2 C}{S_1} \right)^{2a} \left( \frac{\eta w_4 - \phi \rho_{1,12} \eta w_3}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_4) d_1^{-\frac{\tau}{2}} \left( \frac{1}{\sigma_1} - \frac{2 \rho_{1,12}}{\sigma} \right) \]

\[\frac{\partial()}{\partial S_2} = \frac{\phi \eta (S_2 C/S_1)^{2a} \left( \frac{\phi w_3 - \phi \rho_{1,12} w_4}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_4) S_1 d_1^{-\frac{\tau}{2}}}{\sigma \sqrt{\tau}}
\]

\[\frac{\eta w_4 - \phi \rho_{1,12} \eta w_3}{\sqrt{1 - \rho_{1,12}^2}} n(w_3) S_1 d_1^{-\frac{\tau}{2}} 2 \rho_{1,12} \]

\[\frac{2a}{\phi(S_2 C/S_1)^{2a} S_1 d_1^{-\frac{\tau}{2}} M(\phi w_3, \eta w_4, \phi \eta \rho_{1,12})} \]

\[+ \left( \frac{S_2 C}{S_1} \right)^{2a} \left( \frac{\eta w_4 - \phi \rho_{1,12} \eta w_3}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_3) S_1 d_1^{-\frac{\tau}{2}} 2 \rho_{1,12} \]

\[\frac{\partial()}{\partial \tau} = - \frac{\phi(CS_j/S_j)^{X\tau} M(\phi w_3, \eta w_4, \phi \eta \rho_{1,12})}{\phi w_3 - \rho_{1,12} \phi w_4} \]

\[\left[ \frac{\ln(r/d_1)}{\sigma_1 \sqrt{\tau}} - \frac{(w_3 - \sigma_1 \sqrt{\tau})}{2 \tau} \right] \]

\[+ \frac{\phi \eta (S_2 C/S_1)^{2a} \left( \frac{\phi w_3 - \phi \rho_{1,12} w_4}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_4) S_1 d_1^{-\frac{\tau}{2}}}{\sigma \sqrt{\tau}} \]

\[\left[ \frac{\ln(d_1/d_1)}{\sigma_1 \sqrt{\tau}} - \frac{(w_4 - \sigma_1 \sqrt{\tau})}{2 \tau} \right] \]

\[+ \phi \eta \left( \frac{CS_2}{S_1} \right)^{2a} \left( \frac{\phi w_3 - \phi \rho_{1,12} w_4}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_4) S_1 d_1^{-\frac{\tau}{2}} \left[ \frac{\ln(d_1/d_1)}{\sigma_1 \sqrt{\tau}} - \frac{(w_4 - \sigma_1 \sqrt{\tau})}{2 \tau} \right] \]
Continuing, the comparative statics for the first term in [5.5],
\[ \phi(C_{j}/S_{j})^{2a}X^{r}M(\phi w_{3}, \eta w_{4}, \phi \eta \rho_{1,12}) \text{ where } a = [\ln(d_{j}/d_{1}) + .5\sigma^{2}]/[\sigma], \text{ are} \]

\[
\frac{\partial^{2}}{\partial s_{1}^{2}} = (2a)(2a-1)\phi(S_{j}/S_{1})^{2a}(1/S_{1})d_{1}^{-\tau}M(\phi w_{3}, \eta w_{4}, \phi \eta \rho_{1,12})
\]

\[
- (4a-1)\left(\frac{C_{j}}{S_{1}}\right)^{2a}N\left(\frac{\eta w_{4}-\rho_{1,12}\eta w_{3}}{\sqrt{1-\rho_{1,12}^{2}}}\right)n(w_{2})\left(\frac{1}{\sigma} - \frac{2\rho_{1,12}}{\sigma}\right)
\]

\[
+ \phi \eta(4a-2)(C_{j}/S_{j})^{2a}N\left(\frac{\phi w_{3}-\phi \rho_{1,12}w_{4}}{\sqrt{1-\rho_{1,12}^{2}}}\right)n(w_{2})\frac{d_{1}^{-\tau}}{S_{1}\sigma^{\tau}}
\]

\[
- \phi \eta(S_{j}/S_{j})^{2a}N\left(\frac{\eta w_{4}-\rho_{1,12}\eta w_{3}}{\sqrt{1-\rho_{1,12}^{2}}}\right)n(w_{2})\frac{d_{1}^{-\tau}}{S_{1}\sigma^{\tau}}(w_{4} - \sigma\sqrt{\tau})
\]

\[
- (S_{j}/S_{1})^{2a}N\left(\frac{\eta w_{4}-\rho_{1,12}\eta w_{3}}{\sqrt{1-\rho_{1,12}^{2}}}\right)n(w_{2})\frac{d_{1}^{-\tau}}{S_{1}\sigma^{\tau}}\left(\frac{1}{\sigma} - \frac{2\rho_{1,12}}{\sigma}\right)^{2}
\]

\[
- \frac{\eta}{\sqrt{1-\rho_{1,12}^{2}}}\left(\frac{\eta w_{4}-\rho_{1,12}\eta w_{3}}{\sqrt{1-\rho_{1,12}^{2}}}\right)n(w_{2})\left(\frac{1}{\sigma} - \frac{2\rho_{1,12}}{\sigma}\right)^{2}
\]

\[
n(w_{3})\frac{d_{1}^{-\tau}}{S_{1}\sigma^{\tau}}\left[\frac{2}{\sigma} - \frac{4\rho_{1,12}}{\sigma^{2}}\right] + \rho_{1,12}\left[\frac{1}{\sigma} - \frac{2\rho_{1,12}}{\sigma}\right]^{2}
\]
Continuing, the comparative statics for the first term in [5.5], \( \phi(C_2/C_1)^{2a} \times \Gamma \times M(\phi w_3, S_1, S_2, \phi \eta \rho_{1,12}) \) where \( a = \text{ln}(d_i/d_i) + 0.5 \sigma^2/[\sigma^2] \), are

\[
\frac{\partial^2 \gamma}{\partial S_2^2} = (2a)(2a-1) \phi(C_2/C_1)^{2a}(S_1/S_2)^{2a} \left( \frac{\phi w_3 - \rho_{1,12} \eta w_3}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_3) \frac{S_1 d_1^{-\tau} 2 \rho_{1,12}}{S_2^2 \sqrt{\tau} \sigma^2} \\
+ \phi \eta (4a-1)(C_2/C_1)^{2a} \left( \frac{\phi w_3 - \phi \rho_{1,12} \eta w_3}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_3) \frac{S_1 d_1^{-\tau}}{S_2^2 \sigma \sqrt{\tau}} \\
- \phi \eta (S_2/C_1)^{2a} \left( \frac{\phi w_3 - \rho_{1,12} \phi w_4}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_3) \frac{S_1 d_1^{-\tau}}{S_2^2 \sigma \sqrt{\tau}} (w_3 - \sigma \sqrt{\tau}) \\
- \left( \frac{S_2}{S_1} \right)^{2a} \left( \frac{\eta w_4 - \rho_{1,12} \eta w_3}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_3) \frac{4 \rho_{1,12}^2 S_1 d_1^{-\tau}}{S_2^2 \sigma^2 \sqrt{\tau}} w_3 \\
+ \frac{\eta}{\sqrt{1 - \rho_{1,12}^2}} \left( \frac{\eta w_4 - \rho_{1,12} \eta w_3}{\sqrt{1 - \rho_{1,12}^2}} \right) n(w_3) \left( \frac{4 \rho_{1,12}^2 S_1 d_1^{-\tau}}{S_2^2 \sigma^2 \sqrt{\tau}} \right) \left( \frac{4 \rho_{1,12}^2}{\sigma^2} - \frac{1}{\sigma^2} \right)
\]

[5.69]
The remaining comparative static for the first term in [5.5],  
\( \phi(C_2/S_2)^{2a}Xr\varepsilon M(\phi w_2, \eta w_4, \phi \eta_{1,12}) \) where \( a = \ln(d_v/d_t) + .5\sigma^2/\sigma^2 \), is

\[
\frac{\partial \mathcal{X}}{\partial S_1 \partial S_2} = -2a(2a-1)\phi(C_2/S_2)^{2a}(1/S_2) d_1^{-\tau} M(\phi w_2, \eta w_4, \phi \eta_{1,12})
\]

\[- 2a \left( \frac{C_2}{S_1} \right)^{2a} N \left( \frac{\eta w_4 - \rho_{1,12} \eta w_3}{\sqrt{1 - \rho_{1,12}^2}} \right) m(w_3) \frac{d_1^{-\tau}}{S_2 \sigma \sqrt{\tau}} \left( \frac{4 \rho_{1,12} - 1}{\sigma} + \frac{\rho_{1,12}}{\sigma \tau} \right)
\]

\[- (S_2/C_2)^{2a} N \left( \frac{\phi w_3 - \phi \rho_{1,12} w_4}{\sqrt{1 - \rho_{1,12}^2}} \right) m(w_4) \frac{d_1^{-\tau}}{S_2 \sigma^2 \tau} \left( \frac{2 \rho_{1,12}}{\sigma^2} - \frac{4 \rho_{1,12}^2}{\sigma^2} \right)
\]

\[+ \frac{\eta}{\sqrt{1 - \rho_{1,12}^2}} (S_2/C_2)^{2a} N \left( \frac{\eta w_4 - \rho_{1,12} \eta w_3}{\sqrt{1 - \rho_{1,12}^2}} \right) \left( \frac{1}{\sigma^2} - \frac{4 \rho_{1,12}}{\sigma^2} \right) - \rho_{1,12} \left( \frac{2 \rho_{1,12}}{\sigma^2} - \frac{4 \rho_{1,12}^2}{\sigma^2} \right) \frac{1}{\sigma^2} \]
The comparative statics for the second term in [5.5], \( \phi(CS_2/S_1)^2X_{\tau}M(\phi(w_3-\sigma_1\sqrt{\tau}),\eta(w_4-\sigma_1\rho_{1,12}\sqrt{\tau}),\phi\eta\rho_{1,12}) \) are

\[
\frac{\partial(\cdot)}{\partial S_1} = -\phi\eta \left( \frac{S_2C}{S_1} \right)^2M \left( \frac{\phi(w_3-\sigma_1\sqrt{\tau})-\rho_{1,12}\phi(w_4-\sigma_1\rho_{1,12}\sqrt{\tau})}{\sqrt{1-\rho_{1,12}^2}} \right) n(w_4-\sigma_1\rho_{1,12}\sqrt{\tau}) \frac{X_{\tau}}{S_1\sigma_1} - \frac{2x\phi(S_2C/S_1)^2X_{\tau}M(\phi(w_3-\sigma_1\sqrt{\tau}),\eta(w_4-\sigma_1\rho_{1,12}\sqrt{\tau}),\phi\eta\rho_{1,12})}{S_1\sigma_1} + \left( \frac{S_2C}{S_1} \right)^2 \left( \frac{\eta(w_4-\sigma_1\rho_{1,12}\sqrt{\tau})-\rho_{1,12}\eta(w_3-\sigma_1\sqrt{\tau})}{\sqrt{1-\rho_{1,12}^2}} \right) n(w_3-\sigma_1\sqrt{\tau}) \frac{X_{\tau}}{S_1\sigma_1} \left( \frac{1}{\sigma_1} - \frac{2\rho_{1,12}}{\sigma} \right) \tag{5.71}
\]

\[
\frac{\partial(\cdot)}{\partial S_2} = \phi\eta(S_2C/S_1)^2M \left( \frac{\phi(w_3-\sigma_1\sqrt{\tau})-\rho_{1,12}\phi(w_4-\sigma_1\rho_{1,12}\sqrt{\tau})}{\sqrt{1-\rho_{1,12}^2}} \right) n(w_4-\sigma_1\rho_{1,12}\sqrt{\tau}) \frac{X_{\tau}}{S_2\sigma_1} + (2x)\phi(S_2C/S_1)^2X_{\tau}M(\phi(w_3-\sigma_1\sqrt{\tau}),\eta(w_4-\sigma_1\rho_{1,12}\sqrt{\tau}),\phi\eta\rho_{1,12}) + (S_2C/S_1)^2M \left( \frac{\eta(w_4-\sigma_1\rho_{1,12}\sqrt{\tau})-\rho_{1,12}\eta(w_3-\sigma_1\sqrt{\tau})}{\sqrt{1-\rho_{1,12}^2}} \right) n(w_3-\sigma_1\sqrt{\tau}) \frac{X_{\tau}}{S_2\sigma_1} \left( \frac{2\rho_{1,12}}{\sigma} \right) \tag{5.72}
\]

\[
\frac{\partial(\cdot)}{\partial \tau} = -\phi(CS_2/S_1)^2X_{\tau}M(\phi(w_3-\sigma_1\sqrt{\tau}),\eta(w_4-\sigma_1\rho_{1,12}\sqrt{\tau}),\phi\eta\rho_{1,12})\ln(\tau) + \frac{(CS_2^2/S_1)^2M \left( \frac{\eta(w_4-\sigma_1\rho_{1,12}\sqrt{\tau})-\rho_{1,12}\eta(w_3-\sigma_1\sqrt{\tau})}{\sqrt{1-\rho_{1,12}^2}} \right) n(w_3-\sigma_1\sqrt{\tau})X_{\tau} \left[ \frac{\ln(\tau/d_i)}{\sigma_1\sqrt{\tau}} - \frac{w_3}{2\tau} \right]}{\frac{\sigma_1\sqrt{\tau}}{2\tau}} \tag{5.73}
\]

\[
+ \phi\eta \left( \frac{CS_2}{S_1} \right)^2M \left( \frac{\phi(w_3-\sigma_1\sqrt{\tau})-\rho_{1,12}\phi(w_4-\sigma_1\rho_{1,12}\sqrt{\tau})}{\sqrt{1-\rho_{1,12}^2}} \right) n(w_4-\sigma_1\rho_{1,12}\sqrt{\tau})X_{\tau} \left[ \frac{\ln(d_i/d_j)}{\sigma_1\sqrt{\tau}} - \frac{(\sigma_1^2-\sigma_2^2)}{2\sigma_1\sqrt{\tau}} - \frac{(w_4-\sigma_1\rho_{1,12}\sqrt{\tau})}{2\tau} \right]
\]
Continuing, the comparative statics for the second term in [5.5], \( \phi(C_2/S_1)^{3x}X^{r^*}M(\phi(w_3-\sigma_1\sqrt{\tau}),\eta(w_4-\sigma_1\rho_{1,12}\sqrt{\tau}),\phi\eta\rho_{1,12}) \) are

\[
\frac{\partial \xi}{\partial S_1} = (2k)(2k+1)(\phi(S_2/C_1)^{3x}(X/S_1)^{3r^*}M(\phi(w_3-\sigma_1\sqrt{\tau}),\eta(w_4-\sigma_1\rho_{1,12}\sqrt{\tau}),\phi\eta\rho_{1,12}) \\
- (4k+1)(C_2/S_1)^{3x}M\left[\frac{\eta(w_4-\sigma_1\rho_{1,12}\sqrt{\tau})-\rho_{1,12}\eta(w_3-\sigma_1\sqrt{\tau})}{\sqrt{1-\rho_{1,12}^2}}\right]n(w_3-\sigma_1\sqrt{\tau})X^{r^*}\left(\frac{1}{\sigma_1} - \frac{2\rho_{1,12}}{\sigma}\right) \\
+ \phi\eta(4k+1)(C_2/S_1)^{3x}M\left[\frac{\phi(w_3-\sigma_1\sqrt{\tau})-\phi\rho_{1,12}(w_4-\sigma_1\rho_{1,12}\sqrt{\tau})}{\sqrt{1-\rho_{1,12}^2}}\right]n(w_4-\sigma_1\rho_{1,12}\sqrt{\tau})X^{r^*} \\
- \phi\eta(S_2/C_1)^{3x}M\left[\frac{\phi(w_3-\sigma_1\sqrt{\tau})}{\sqrt{1-\rho_{1,12}^2}}\right]n(w_4-\sigma_1\rho_{1,12}\sqrt{\tau}) \\
- (S_2/C_1)^{3x}M\left[\frac{\eta(w_4-\sigma_1\rho_{1,12}\sqrt{\tau})-\rho_{1,12}\eta(w_3-\sigma_1\sqrt{\tau})}{\sqrt{1-\rho_{1,12}^2}}\right]n(w_3-\sigma_1\sqrt{\tau})X^{r^*} \\
- \frac{\eta}{\sqrt{1-\rho_{1,12}^2}}\left[M\left(\frac{\eta(w_4-\sigma_1\rho_{1,12}\sqrt{\tau})-\rho_{1,12}\eta(w_3-\sigma_1\sqrt{\tau})}{\sqrt{1-\rho_{1,12}^2}}\right)\right]n(w_3-\sigma_1\sqrt{\tau})X^{r^*} \\
= [5.74]
\]

\[
\frac{\partial \xi}{\partial S_1} \\
= \left[\frac{\eta(w_4-\sigma_1\rho_{1,12}\sqrt{\tau})-\rho_{1,12}\eta(w_3-\sigma_1\sqrt{\tau})}{\sqrt{1-\rho_{1,12}^2}}\right]n(w_3-\sigma_1\sqrt{\tau})X^{r^*} \\
+ \phi\eta(4k+1)(C_2/S_1)^{3x}M\left[\frac{\phi(w_3-\sigma_1\sqrt{\tau})-\phi\rho_{1,12}(w_4-\sigma_1\rho_{1,12}\sqrt{\tau})}{\sqrt{1-\rho_{1,12}^2}}\right]n(w_4-\sigma_1\rho_{1,12}\sqrt{\tau})X^{r^*} \\
- \phi\eta(S_2/C_1)^{3x}M\left[\frac{\phi(w_3-\sigma_1\sqrt{\tau})}{\sqrt{1-\rho_{1,12}^2}}\right]n(w_4-\sigma_1\rho_{1,12}\sqrt{\tau}) \\
- (S_2/C_1)^{3x}M\left[\frac{\eta(w_4-\sigma_1\rho_{1,12}\sqrt{\tau})-\rho_{1,12}\eta(w_3-\sigma_1\sqrt{\tau})}{\sqrt{1-\rho_{1,12}^2}}\right]n(w_3-\sigma_1\sqrt{\tau})X^{r^*} \\
- \frac{\eta}{\sqrt{1-\rho_{1,12}^2}}\left[M\left(\frac{\eta(w_4-\sigma_1\rho_{1,12}\sqrt{\tau})-\rho_{1,12}\eta(w_3-\sigma_1\sqrt{\tau})}{\sqrt{1-\rho_{1,12}^2}}\right)\right]n(w_3-\sigma_1\sqrt{\tau})X^{r^*} \\
\]

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Continuing, the comparative statics for the second term in [5.5]. $\phi(CS_2/S_1)2^x Xr^{-\tau}M(\phi(w_3-\sigma_1, \eta(w_3-\sigma_1, 1/\tau), \phi\eta_{1,12})$ are

$$\frac{\partial (\cdot)}{\partial S^2_2} = (2k)(2k-1)\phi(S_2/C/S_1)2^x(X/S_2)^{-\tau}M(\phi(w_3-\sigma_1, \eta(w_4-\sigma_1, 1/\tau), \phi\eta_{1,12})$$

$$+ (4k+1)(CS_2/S_1)^{2k} N \left\{ \frac{\eta(w_4-\sigma_1, 1/\tau, \eta(w_4-\sigma_1, 1/\tau)}{\sqrt{1-\rho_{1,12}}} \right\}$$

$$\phi\eta(4k-1)(CS_2/S_1)^{2k} N \left\{ \frac{\phi(w_3-\sigma_1, 1/\tau, \phi\rho_{1,12}(w_4-\sigma_1, 1/\tau)}{\sqrt{1-\rho_{1,12}}} \right\}$$

$$\eta(w_4-\sigma_1, 1/\tau, \eta(w_4-\sigma_1, 1/\tau)}{\sqrt{1-\rho_{1,12}}} \right\}$$

$$- \phi\eta(S_2/C/S_1)^{2k} N \left\{ \frac{\eta(w_4-\sigma_1, 1/\tau, \eta(w_4-\sigma_1, 1/\tau)}{\sqrt{1-\rho_{1,12}}} \right\}$$

$$[5.75]$$

$$\frac{n(w_3-\sigma_1, 1/\tau, \frac{Xr^{-\tau}}{S_2^2 \sigma^2}}{\sigma^2} \frac{(w_4-\sigma_1, 1/\tau)}{S_2^2 \sigma^2}$$

$$- \left( \frac{S_2/C}{S_1} \right)^{2k} N \left\{ \frac{\eta(w_3-\sigma_1, 1/\tau, \eta(w_3-\sigma_1, 1/\tau)}{\sqrt{1-\rho_{1,12}}} \right\}$$

$$\frac{\eta(w_4-\sigma_1, 1/\tau, \eta(w_4-\sigma_1, 1/\tau)}{\sqrt{1-\rho_{1,12}}} \right\}$$

$$+ \frac{\eta}{\sqrt{1-\rho_{1,12}}} \left( \frac{4\rho_{1,12}Xr^{-\tau}}{S_2^2 \sigma^2} \right)$$

$$\frac{n(w_3-\sigma_1, 1/\tau, \frac{Xr^{-\tau}}{S_2^2 \sigma^2}}{\sigma^2} \frac{4\rho_{1,12}}{\sigma^2} \frac{\rho_{1,12} \left[ \frac{4\rho_{1,12}^2 + 1}{\sigma^2} \right]}{\rho_{1,12} \left[ \frac{4\rho_{1,12}^2 + 1}{\sigma^2} \right]}$$

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The remaining comparative static for the second term in [5.5], \( \phi(CS_2/S_1)^2 X_r \tau M(\phi(w_3 - \sigma \sqrt{r}), \eta(w_4 - \sigma \rho_{1,12} \sqrt{r}), \phi \eta \rho_{1,12}) \) is

\[
\frac{\partial}{\partial S_1 \partial S_2} = -(4\pi^2) \phi(S_2/C/S_1)^2 \times X_r \tau M(\phi(w_3 - \sigma \sqrt{r}), \eta(w_4 - \sigma \rho_{1,12} \sqrt{r}), \phi \eta \rho_{1,12}) \]

\[
- 4\pi \phi(S_2/C/S_1)^2 \left[ \frac{\eta(w_4 - \sigma \rho_{1,12} \sqrt{r}) - \rho_{1,12} \eta(w_3 - \sigma \sqrt{r})}{\sqrt{1 - \rho_{1,12}^2}} \right] \]

\[
\frac{n(w_3 - \sigma \sqrt{r}) X_r \tau}{S_1 S_2 \sigma \sqrt{r}} \left( \frac{2 \rho_{1,12}}{\sigma} - 1 \right) \]

\[
- \phi \eta 4\pi \phi(S_2/C/S_1)^2 \left[ \frac{\phi(w_3 - \sigma \sqrt{r}) - \phi \rho_{1,12} (w_4 - \sigma \rho_{1,12} \sqrt{r})}{\sqrt{1 - \rho_{1,12}^2}} \right] \]

\[
\frac{n(w_4 - \sigma \rho_{1,12} \sqrt{r}) X_r \tau}{S_1 S_2 \sigma \sqrt{r}} \]

\[
+ \phi \eta \left( S_2/C/S_1 \right)^2 \left[ \frac{\phi(w_3 - \sigma \sqrt{r}) - \phi \rho_{1,12} (w_4 - \sigma \rho_{1,12} \sqrt{r})}{\sqrt{1 - \rho_{1,12}^2}} \right] \]

\[
\frac{n(w_4 - \sigma \rho_{1,12} \sqrt{r}) X_r \tau}{S_1 S_2 \sigma \sqrt{r}} \left( \frac{2 \rho_{1,12}}{\sigma} - 1 \right) \]

\[
\left( S_2/C/S_1 \right)^2 \left[ \frac{\eta(w_4 - \sigma \rho_{1,12} \sqrt{r}) - \rho_{1,12} \eta(w_3 - \sigma \sqrt{r})}{\sqrt{1 - \rho_{1,12}^2}} \right] \]

\[
\frac{n(w_3 - \sigma \sqrt{r}) X_r \tau}{S_1 S_2 \sigma \sqrt{r}} \left( \frac{2 \rho_{1,12}}{\sigma} - 1 \right) \]

\[
+ \frac{\eta}{\sqrt{1 - \rho_{1,12}^2}} \left[ \eta(w_4 - \sigma \rho_{1,12} \sqrt{r}) - \rho_{1,12} \eta(w_3 - \sigma \sqrt{r}) \right] \]

\[
\frac{n(w_3 - \sigma \sqrt{r}) X_r \tau}{S_1 S_2 \sigma \sqrt{r}} \left( \frac{1}{\sigma} - \frac{4 \rho_{1,12}}{\sigma^2} \right) - \rho_{1,12} \left( \frac{2 \rho_{1,12}}{\sigma} - \frac{4 \rho_{1,12}}{\sigma^2} \right) \]

\[
\left[ \frac{\eta(w_4 - \sigma \rho_{1,12} \sqrt{r}) - \rho_{1,12} \eta(w_3 - \sigma \sqrt{r})}{\sqrt{1 - \rho_{1,12}^2}} \right] \]

\[
\frac{n(w_3 - \sigma \sqrt{r}) X_r \tau}{S_1 S_2 \sigma \sqrt{r}} \left( \frac{1}{\sigma} - \frac{4 \rho_{1,12}}{\sigma^2} \right) - \rho_{1,12} \left( \frac{2 \rho_{1,12}}{\sigma} - \frac{4 \rho_{1,12}}{\sigma^2} \right) \frac{1}{\sigma^2} \]

\[ [5.76] \]
Appendix VC: Derivation of the Detached Barrier Model

In this appendix the derivations of equations [5.12] and [5.13] are provided. I begin with the derivation of the densities that are needed.

Derivation of the Densities

The reflection principle (see Rich (1994)) is a powerful graphic depiction of the strong Markov property (see Karlin (1968, pp. 230-232)). It is an intuitively appealing technique that is used to simplify complex probabilities of the form

\[ P(\sup_{t \in [t_n, T]} S_t(t) \leq C) \]

where the supremum extends over all \( \{S_t(t), t \in [t_n, T]\} \) of \( \Omega \).

Unfortunately, the reflection principle as commonly presented (see Harrison, Chapter 1) can be applied in only 2-space. I use "reflection principle intuition" to generalize this concept to k-space. A rigorous proof of the generalized reflection principle is beyond the purpose of this thesis. However, the bivariate normal density that is obtained from using the generalized reflection principle in 3-space is verified by contradiction. That is, the solution to [5.10] subject to the boundary conditions is unique. The solution derived by preference-free valuation that confirms [5.10] thereby, also confirms the densities used.

Recall that in 2-space, application of the reflection principle requires the Brownian motion to have zero drift. It is natural to assume that this requirement must also hold in k-space. Begin by assuming each drift coefficient in [5.1], \( \mu_1 \) and \( \mu_2 \), is zero. After considering the zero drift case, the results are generalized to include a drift.

To evaluate the following complex probability, begin by employing the law of total probability.
\[ Pr( \sup_{t \in [T, T]} S_1(t) \geq X, \forall t \in [T, T] S_2(t) \leq C) \]

\[ = Pr(S_1(T) \geq X, S_2(T) \leq C, \sup S_3(t) \leq C) \]

\[ = Pr(S_1(T) \geq X, S_2(T) \leq C) - Pr(S_1(T) \geq X, S_2(T) \leq C, \sup S_3(t) > C) \]  \[ 5.78 \]

where \( C \), some constant, is the absorbing continuous barrier level restricting the process of asset three, asset one is the optioned asset, and \( X \) is the strike price. Note that the region in which the supremum is extended over has been suppressed for notational convenience. Converting into continuously compounded returns by letting

\[ X_1(T) = \ln(S_1(T)/S_1), \quad z_1 = \ln(S_1/X), \quad X_2(T) = \ln(S_2(T)/S_2), \quad \text{and} \quad z_3 = b = \ln(C/S_2), \]

where \( S_1(0) = S_1 \), equation \([5.31]\) is written as

\[ Pr(X_1(T) \geq -z_1, X_2(T) \leq z_2) - Pr(X_1(T) \geq -z_1, X_2(T) \leq z_2, \sup X_3(t) > b). \]  \[ 5.79 \]

Since \( S_1(T) \) is lognormally distributed (by construction), \( X_1(T) \) is normally distributed.

It is assumed that each distribution function in \([5.79]\) is continuously differentiable. The density associated with the left-hand term in \([5.79]\) is determined directly by standardizing and differentiating as follows.

---

\(^{70}\) To make the derivation as general as possible we use a different place holder for \( z_1 \) and \( b \). Of course, in this case they are the same but that need not always be the case.
\[
\Pr(X_1(T) > -z_1, X_2(T) < z_2) = \Pr\left(Z_1(T) > \frac{-z_1}{\sigma_1 \sqrt{\tau}}, Z_2(T) < \frac{z_2}{\sigma_2 \sqrt{\tau}}\right)
\]

\[
= \Pr\left(Z_1(T) < \frac{-z_1}{\sigma_1 \sqrt{\tau}}, Z_2(T) < \frac{z_2}{\sigma_2 \sqrt{\tau}}\right) = M\left(\frac{z_1}{\sigma_1 \sqrt{\tau}}, \frac{z_2}{\sigma_2 \sqrt{\tau}}, \rho_{1,2}\right)
\]

Thus,

\[
f_{Z_1(z_1), Z_2(z_2)}(z_1, z_2) = \frac{\partial M(z_1, Z_2)}{\partial z_1 \partial z_2} = \frac{\partial}{\partial z_1} \left[ \frac{1}{\sigma_1 \sqrt{\tau}} \right] N\left(\frac{z_1}{\sigma_1 \sqrt{\tau}}, \frac{z_2}{\sigma_2 \sqrt{\tau}}\right) N\left(\frac{z_2}{\sigma_2 \sqrt{\tau}}, \frac{z_1}{\sigma_1 \sqrt{\tau}}\right)
\]

\[
= \frac{1}{\sigma_1 \sigma_2 \sqrt{1 - \rho_{1,2}}} \left[ \frac{z_1}{\sigma_1 \sqrt{\tau}} - \frac{\rho_{1,2} z_2}{\sigma_2 \sqrt{\tau}} \right] \frac{1}{\sqrt{2\pi}} e^{-\frac{z_1^2}{2\sigma_1^2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z_2^2}{2\sigma_2^2}}
\]

where \(Z_1(T) = X_1(T) / (\sigma_1 \sqrt{\tau})\) is a standard normal random variable, \(N(q) = \frac{e^{-q^2}}{\sqrt{2\pi}}\) is the standard normal density function, \(M(q_1, q_2, \rho)\) is the bivariate cumulative normal distribution with upper limits of integration \(q_1\) and \(q_2\), and a correlation coefficient of \(\rho\), and \(f(z_1, z_2)\) is the standard bivariate normal density function (as defined above).

The difficult part in evaluating [5.79] is the right-hand term. As before, first consider the case in which \(S_1\) and \(S_2\) are independent. From previous results (Rich, 1994), I have shown that in this case the right-hand term in [5.79] is written as:
\[
Pr(X_1(T) \geq -z_1, X_3(T) \leq z_3, \sup X_3(t) \geq b) = Pr\left( \frac{Z_1(T) - z_1}{\sigma_1 \sqrt{\tau}} \right) \cdot Pr\left( \frac{Z_3(T) - z_3}{\sigma_3 \sqrt{\tau}} \right) \cdot Pr\left( Z_3(T) \geq \frac{2b - z_3}{\sigma_3 \sqrt{\tau}} \right)
\]  

[5.82]

When \( S_1 \) and \( S_2 \) are not independent the "correlation effect" resulting from the method of images has to be considered. This discussion can best be motivated by first briefly reviewing 2-state reflection principle intuition.

Consider only the arguments in [5.79] containing the random variable \( Z_3(t) = \frac{X_3(t)}{[\sigma_3 \sqrt{\tau}]} \). The reflection principle results in respecifying the condition that

\[
Z_3(T) \leq z_3/\sigma_3 \sqrt{\tau} \text{ and sup } \{Z_3(t)\} \geq b/\sigma_3 \sqrt{\tau}
\]

in terms of the sole condition:

\[
Z_3(T) \geq [2b - z_3]/\sigma_3 \sqrt{\tau}.
\]

Intuitively, this conclusion is seen by considering a time zero solution to the partial differential equations of motion and the initial conditions (see Ingersoll (1987, p. 350)). At time zero, there must be a superposition of source of strength at the barrier along the \( Z_3 \) axis. Then by placing a mirror along the time axis and locating the image of the origin along the \( Z_3 \) axis, the image source is seen to be \([2b]/\sigma_3 \sqrt{\tau}\).

Now consider the more general case in which independence is not assumed. In the bivariate case there is an image source (from the mirror plane) along the \( Z_3 \) axis as well as along the \( Z_3 \) axis. The image source along the \( Z_3 \) axis results from the correlation between \( Z_1 \) and \( Z_3 \) (i.e., in the independent case there is only one image source, excluding the source of unit strength at the origin). Thus, we conclude that the reflection principle results in the condition of \( Z_3(T) \leq z_3/\sigma_3 \sqrt{\tau} \) being modified to \(-[2b]/\sigma_3 \sqrt{\tau}\) (because this is a one-to-one effect) and the condition of \( Z_3(T) \leq z_3/\sigma_3 \sqrt{\tau} \) being modified by \(-[\rho_1, 2b]/\sigma_3 \sqrt{\tau}\) (because this is the "correlation effect" of the reflection principle). Hence,

\[
Pr(X_1(T) \geq -z_1, X_3(T) \leq z_3, \sup X_3(t) \geq b) = Pr\left( \frac{Z_1(T) - z_1}{\sigma_1 \sqrt{\tau}} \leq \frac{Z_3(T) - z_3}{\sigma_3 \sqrt{\tau}} \leq \frac{Z_3(T) - 2b}{\sigma_3 \sqrt{\tau}} \right)
= Pr\left( \frac{Z_1(T) - 2b \rho_{1,3} \sigma_1 \sigma_3^{-1}}{\sigma_1 \sqrt{\tau}} \leq \frac{Z_3(T) - 2b}{\sigma_3 \sqrt{\tau}} \right) = M\left( \frac{Z_1(T) - 2b \rho_{1,3} \sigma_1 \sigma_3^{-1}}{\sigma_1 \sqrt{\tau}}, \frac{Z_3(T) - 2b}{\sigma_3 \sqrt{\tau}} \right)
\]

[5.83]
It follows that the density associated with this term is:

\[
g_{z_1(z_2)} = \frac{\partial^2 M(\cdot)}{\partial z_1 \partial z_2}
\]

\[
= \frac{1}{2\pi \sigma_1 \sigma_3 \tau \sqrt{1 - \rho_{13}^2}} \exp\left\{-\frac{1}{2(1-\rho_{13}^2)} \left[ \frac{(\mu_2 - \mu_1)^2}{\sigma_2^2} - \frac{2\rho_{13}(\mu_2 - \mu_1)(\mu_3 - \mu_1)}{\sigma_1 \sigma_3} + \frac{(\mu_3 - \mu_1)^2}{\sigma_3^2} \right]\right\}
\]

[5.84]

Following Ingersoll (1987, p. 369), define the "defective density" as \(f(z_1, z_2) = g(z_1, z_2)\). It is quickly seen that when \(Z_2(T)\) is evaluated at \(z_2 = b\) and \(Z_1(T)\) is evaluated at \(z_1 = b\rho_{13} \sigma_1 \sigma_3^{-1}\), the "defective density" goes to zero (as it should, because this is a boundary condition on the "defective density", see Cox and Miller (1965, pp. 220-221)).

The above methodology can quickly be extended to \(k\)-space. If there are \(k\) random variables that follow the described diffusion process and \((k-1)\) of them are not restricted by an absorbing boundary, the generalized reflection principle says that each "correlation effect" must be considered. That is, every dependency with the barrier asset is seen in the mirror hyperplane as an image source.

For Brownian motions with drift, \(\mu_i\), equations [5.84] and [5.81] are rewritten as\(^7i\)

\[
f_{z_1(z_2)} = \frac{1}{2\pi \sigma_1 \sigma_3 \tau \sqrt{1 - \rho_{13}^2}} \exp\left\{-\frac{1}{2(1-\rho_{13}^2)} \left[ \frac{(\mu_2 - \mu_1)^2}{\sigma_2^2} - \frac{2\rho_{13}(\mu_2 - \mu_1)(\mu_3 - \mu_1)}{\sigma_1 \sigma_3} + \frac{(\mu_3 - \mu_1)^2}{\sigma_3^2} \right]\right\}
\]

[5.85]

\(^7i\) \(\mu_i\) is the continuously compounded expected return of asset \(i\).
\[ \delta_{z_1(x), x, z}(z_1 z_3) = \frac{e^{\frac{2\rho_{1,3}^2}{\sigma_1^2 \sigma_3^2 (1 - \rho_{1,3}^2)}}}{\sigma_1 \sigma_3 \sqrt{1 - \rho_{1,3}^2}} n(z_1, z_3) \]  

[5.86]

where

\[ n(z_1, z_3) = \frac{1}{2\pi} e^{-\frac{1}{2(1 - \rho_{1,3}^2)} \left[ \frac{(z_1 - 2\rho_{1,3} z_3)^2}{\sigma_1^2 \sigma_3^2} - \frac{2\rho_{1,3} (z_1 - 2\rho_{1,3} z_3) (z_1 - 2\rho_{1,3} z_3) - \rho_{1,3}}{\sigma_1 \sigma_3} \right]} \]  

[5.87]

The constant term, \( \exp(2\rho_{1,3}^2 / \sigma^2) \), premultiplying the \( n(z_1, z_3) \) density is calculated by setting the "defective density" equal to zero when \( Z_3(T) \) is evaluated at \( z_3 = b \) and \( Z_1(T) \) is concurrently evaluated at \( z_1 = b \rho_{1,3} \sigma_1 \sigma_3^{-1} \).\(^{72}\)

Proceeding in exactly the same manner, the complex probability

\[ \Pr(S_1(T) \geq X, \inf S_2(t) \geq C) \]

is evaluated and differentiated to yield equations [5.85], [5.86] and [5.87]. To intuitively see that [5.85], [5.86] and [5.87] must result, consider again the generalized reflection principle. The well-behaved time zero solution to the partial differential equations of motion must, again, be written a superposition of sources of strength (one for each absorbing barrier).\(^{73}\) There is, again, an image source along each axis; one resulting from the "one-to-one" effect and one resulting from the "correlation" effect. But these image points vary uniformly as the barrier is moved.

In the next section, detached barrier options are valued. To this end, it is useful to define

\(^{72}\) For a more rigorous, but equivalent, derivation of this constant term, Girsanov's Theorem could have been applied.

\(^{73}\) Note that the partial differential equations of motion remain unchanged as the barrier is moved. However, when the absorbing barrier is being approached from above (below) the differential equations are only defined over the space in which \( S_3(t) > C \) (\( S_3 < C \)).
\[ \xi_1 = \frac{z_1 - \bar{\xi}_1 \tau}{\sigma_1 \sqrt{\tau}} \quad \text{and} \quad \xi_3 = \frac{z_3 - \bar{\xi}_3 \tau}{\sigma_3 \sqrt{\tau}} \quad \text{and rewrite } f(z_1, z_3) \text{ and } g(z_1, z_3). \]

\[
f_{z_1(z_3)}(\xi_1, \xi_3) = \frac{1}{2\pi \tau \sqrt{1 - \rho_{1,3}^2}} e^{-\frac{1}{2(1 - \rho_{1,3}^2)} \left[ (\xi_1^2 - 2\rho_{1,3} \xi_1 \xi_3 + \xi_3^2) \right]} \quad [5.88]
\]

and

\[
g_{z_1(z_3)}(\xi_1, \xi_3) = \frac{(C/S_3)^{2b_{1,3}^2} \sigma_1^2}{2\pi \tau \sqrt{1 - \rho_{1,3}^2}} e^{-\frac{1}{2(1 - \rho_{1,3}^2)} \left[ (\xi_1 - \frac{2b_{1,3}}{\sigma_1 \sqrt{\tau}})^2 - 2\rho_{1,3} \frac{2b_{1,3}}{\sigma_1 \sqrt{\tau}} (\xi_1 \xi_3 - \frac{2b_{1,3}}{\sigma_1 \sqrt{\tau}}) + (\xi_3 - \frac{2b_{1,3}}{\sigma_3 \sqrt{\tau}})^2 \right]} \quad [5.89]
\]

Finally, to standardize each of the densities define \( \tilde{\xi}_1 = \frac{\xi_1 - 2b_{1,3}}{\sigma_1 \sqrt{\tau}} \) and \( \tilde{\xi}_3 = \frac{\xi_3 - 2b_{1,3}}{\sigma_3 \sqrt{\tau}} \), which gives

\[
g_{z_1(z_3)}(\tilde{\xi}_1, \tilde{\xi}_3) = \frac{2\alpha_{1,3}^2 - 5\tilde{\xi}_1^2}{\left( \frac{C}{S_3} \right)^{\frac{1}{2}} \sigma_1^2} \quad \text{and rewrite } f(z_1, z_3) \text{ and } g(z_1, z_3). \]

Before concluding this section, an economic interpretation of the "defective density" is given. Since \( f(z_1, z_3) \) represents the joint probability of \( Z_1(T) \) being at \( z_1 \) and \( Z_3(T) \) being at \( z_3 \) at time \( T \) (path independent), and \( g(z_1, z_3) \) depicts the joint probability of crossing of the barrier and \( Z_1(T) \) being at \( z_1 \) and \( Z_3(T) \) being at \( z_3 \) at \( T \) (path dependent), the "defective density" represents the joint probability density of not crossing the barrier and \( Z_1(T) \) being at \( z_1 \) and \( Z_3(T) \) being at \( z_3 \) at \( T \) (path dependent).

**V.H.2 "Out" Detached Barrier Option Valuation - Non-Rebate Value**

From the results in Section VA of this appendix, the non-rebate value of detached barrier options is determined as follows. Preference-free valuation is used to write the discounted
expected terminal payoff as

\[ r^{-\tau} E(\phi S_1(T) - \phi X | \phi S_1(T) \geq \phi X, \inf \eta S_3(t) > \eta C) \]

\[ = r^{-\tau} E(\phi S_1(T) | \phi S_1(T) \geq \phi X, \eta S_3(T) > \eta C) \]

\[- \phi r^{-\tau} X Pr(\phi S_1(T) \geq \phi X, \eta S_3(T) > \eta C) \]

\[ - r^{-\tau} E(\phi S_1(T) | \phi S_1(T) \geq \phi X, \eta S_3(T) > \eta C, \inf \eta S_3(t) \leq \eta C) \]

\[ + \phi r^{-\tau} X Pr(\phi S_1(T) \geq \phi X, \eta S_3(T) > \eta C, \inf \eta S_3(t) \leq \eta C). \]

\[ [5.91] \]

\( E \) denotes the expectation operator which is taken under an equivalent martingale measure over all first passage paths (i.e., over all paths of \( \eta S_3(t) > \eta C \)) to the non-negative payoff space (i.e., \( S_1(T) \geq X \)). To conserve space, it is assumed that the reader understands that when \( \eta = -1 \), \( \inf \) (the infimum of the process) is replaced with \( \sup \) (the supremum of the process). Each of the four terms in this valuation equation is now individually evaluated.

The easiest term to evaluate is \( \phi r^{-\tau} X Pr(\phi S_1(T) \geq \phi X, \eta S_3(T) > \eta C) \).

\[ \phi r^{-\tau} X Pr(\phi S_1(T) \geq \phi X, \eta S_3(T) > \eta C) \]

\[ = \phi r^{-\tau} X Pr\left( \phi Z_1(T) \geq \phi \left[ \frac{\ln(X/S_3) - \mu_1 \tau}{\sigma_1 \sqrt{\tau}} \right], \eta Z_3(T) > \eta \left[ \frac{\ln(C/S_3) - \mu_3 \tau}{\sigma_3 \sqrt{\tau}} \right] \right) \]

\[ = \phi r^{-\tau} X \int \int f(\xi_1, \xi_2) dZ_1(T) dZ_3(T) \]

\[ = \phi r^{-\tau} XM(\phi(w_1 - \sigma_1 \sqrt{\tau}), \eta(w_3 - \sigma_3 \sqrt{\tau})); \phi \eta \rho_{1,3} \]

\[ [5.92] \]

The following results are used in [5.92]. Recall \( Z(T) = X(T)/[\sigma \sqrt{T}] = \ln(S(T)/S_0)/[\sigma \sqrt{T}] \) so \( X_i(0) = \ln(S_i/S_0)/[\sigma_i \sqrt{T}] = 0 \). \( X_i(t) \) is a function of \( S_i(t) \). Hence, Itô's lemma can be used to determine the process followed by \( X_i(t) \).

\[ dX_1(t) = \frac{\partial X_1(t)}{\partial S_i(t)} dS_i + \frac{1}{2} \frac{\partial^2 X_1(t)}{\partial S_i(t)^2} (dS_i)^2, \]

\[ [5.93] \]

so

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\[dX_1(t) = (\mu_1 - 0.5\sigma_1)dt + \sigma_1dZ_1 = \mu_1 dt + \sigma_1dZ_1 \quad (\mu_1 = \mu_1 - 0.5\sigma_1)\]  
[5.94]

\[dX_2(t) = (\mu_2 - 0.5\sigma_2)dt + \sigma_2dZ_2 = \mu_2 dt + \sigma_2dZ_2 \quad (\mu_2 = \mu_2 - 0.5\sigma_2)\]

In the absence of a risk premium, \(\mu_1 = \ln(r/d_1)\) and \(\mu_2 = \ln(r/d_2)\) and [5.92] follows immediately. For future use, it is also useful to recall that the solution for each stochastic differential equation listed in [5.1], implies

\[S_i(T) = S_i e^{(\ln(r/d_i) - 0.5\sigma_i^2)\tau} + \sigma_i\sqrt{\tau} = r^\tau d_i^{\tau} \cdot e^{-0.5\sigma_i^2 \cdot \sigma_i\sqrt{\tau}}\]  
[5.95]

This result is useful when evaluating \(r^\tau E(\phi S_i(T) \mid \phi S_i(T) \geq \phi X, \eta S_i(T) > \eta C)\).

\[r^\tau E(\phi S_i(T) \mid \phi S_i(T) \geq \phi X, \eta S_i(T) > \eta C) = \phi d_i^{\tau} e^{-0.5\sigma_i^2 \tau} E(\phi e^{\sigma_i\sqrt{\tau}} \mid \phi S_i(T) \geq \phi X, \eta S_i(T) > \eta C)\]

\[= \phi S_i d_i^{\tau} e^{-0.5\sigma_i^2 \tau} \int \int e^{\sigma_i\sqrt{\tau}} f(\bar{S}_i, \bar{Z}_i) d\bar{S}_i d\bar{Z}_i \quad \text{[5.96]}\]

\[= \phi S_i d_i^{\tau} \int \int f(\nu_1, \nu_2) d\nu_1 d\nu_2\]

To see this, note that

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\[ e^{\sigma_1 \sqrt{\varepsilon_1} f(\xi_1, \xi_2)} = e^{\frac{\sigma_1 \sqrt{\varepsilon_1}}{1 - \rho_{1,3}} f(\xi_1, \xi_2)} f(\xi_1, \xi_2) e^{\frac{-1}{2(1 - \rho_{1,3})^2} \left[ (\xi_1^2 + 2\xi_1 \xi_3 \sigma_1 \sqrt{\varepsilon_1} \rho_{1,3} + \sigma_1^2 \rho_{1,3}^2 \right]} \]

\[ = \frac{1}{2\pi \sqrt{1 - \rho_{1,3}^2}} e^{\frac{-1}{2(1 - \rho_{1,3})} \left[ (\xi_1^2 + 2\xi_1 \xi_3 \sigma_1 \sqrt{\varepsilon_1} \rho_{1,3} + \sigma_1^2 \rho_{1,3}^2 \right]} \]

where

\[ q_1 = (\xi_3 - \sigma_1 \rho_{1,3} \sqrt{\varepsilon_1})^2 - \sigma_1^2 \rho_{1,3}^2 + 2\xi_3 \sigma_1 \rho_{1,3} \sqrt{\varepsilon_1} \]

\[ q_2 = (2\rho_{1,3} \xi_3 \sqrt{\varepsilon_1} \sqrt{\varepsilon_1}) + 2\sigma_1^2 \rho_{1,3} \sqrt{\varepsilon_1} - 2\sigma_1^2 \rho_{1,3}^2 \]

\[ q_3 = (\xi_1 - \sigma_1 \sqrt{\varepsilon_1})^2 - \sigma_1^2 \]

\[ q_1 - q_2^2 q_3 = (\xi_1 - \sigma_1 \sqrt{\varepsilon_1})^2 + (\xi_3 - \sigma_1 \rho_{1,3} \sqrt{\varepsilon_1})^2 - 2\rho_{1,3} (\xi_1 - \sigma_1 \sqrt{\varepsilon_1}) (\xi_3 - \sigma_1 \rho_{1,3} \sqrt{\varepsilon_1}) - \sigma_1^2 (1 - \rho_{1,3}^2) \]

Thus,

\[ e^{\sigma_1 \sqrt{\varepsilon_1} f(\xi_1, \xi_2)} = e^{\frac{\sigma_1 \sqrt{\varepsilon_1}}{1 - \rho_{1,3}} f(\xi_1, \xi_2)} f(\xi_1, \xi_2) e^{\frac{-1}{2(1 - \rho_{1,3})^2} \left[ (\xi_3 - \sigma_1 \rho_{1,3} \sqrt{\varepsilon_1})^2 - 2\rho_{1,3} (\xi_1 - \sigma_1 \sqrt{\varepsilon_1}) (\xi_3 - \sigma_1 \rho_{1,3} \sqrt{\varepsilon_1}) + \sigma_1^2 \sqrt{\varepsilon_1}^2 \right]} \]

[5.99]

Combining [5.99] with the third-to-last equation in [5.96], yields the second-to-last equation in [5.96]. The final equation in [5.96] is derived by making the change of variables

\[ \nu_1 = (\xi_1 - \sigma_1 \sqrt{\varepsilon_1}), \nu_3 = (\xi_3 - \sigma_1 \rho_{1,3} \sqrt{\varepsilon_1}), \]

and redefining the limits of integration.

Note that equation [5.96] less equation [5.92] yields equation [5.12].

To derive [5.13] we begin by evaluating the term: \( \phi^{\tau} \mathcal{X} \Pr(\phi S_1(T) \geq \phi X, \eta S_3(T) > \eta C, \inf \eta S_3(T) \leq \eta C) \) of [5.91].

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\[
\phi r^{-\tau}X \Pr(\phi S_1(T) \geq \phi X, \eta S_3(T) > \eta C, \inf \eta S_2(t) \leq \eta C) = \phi r^{-\tau}X \Pr \left( \phi Z_1(T) \geq \phi \left[ \frac{\ln(X/S_1) - \mu_1}{\sigma_1 \sqrt{\tau}} - \frac{2\ln(C/S_2) \rho_{1,3}}{\sigma_3 \sqrt{\tau}} \right] \right) \eta Z_3(T) \geq \eta \left[ \frac{\ln(C/S_3) - \mu_2}{\sigma_2 \sqrt{\tau}} - \frac{2\ln(C/S_3) \rho_{1,3}}{\sigma_3 \sqrt{\tau}} \right] \] [5.100]

\[
= \phi r^{-\tau} X \int \int g(\xi_1, \xi_2) dZ_1(T) dZ_2(T) \xrightarrow[\sim]{\phi(w_\tau - \sigma_1 \sqrt{\tau}) - \eta(w_\tau - \sigma_1 \rho_{1,3} \sqrt{\tau})} \phi \left( \frac{C}{S_3} \right)^{2\ln(\phi C/S_3) - \frac{\theta \rho_{1,3}^2}{2}} r^{-\tau}X M(\phi(w_\tau - \sigma_1 \sqrt{\tau}), \eta(w_\tau - \sigma_1 \rho_{1,3} \sqrt{\tau}); \phi \eta \rho_{1,3})
\]

The final term to evaluate is: \( r^{-\tau} E(S_1(T) | \phi S_1(T) \geq \phi X, \eta S_3(T) > \eta C, \inf \eta S_2(t) \leq \eta C) \). If it is noted that

\[
r^{-\tau} E(\phi S_1(T) | \phi S_1(T) \geq \phi X, \eta S_3(T) > \eta C, \inf \eta S_2(t) \leq \eta C)
= \phi S_1 d^{-\tau} e^{-\frac{\theta \rho_{1,3}^2}{2}} E(e^{a \sqrt{\tau} t}) \phi S_1(T) \geq \phi X, \eta S_3(T) > \eta C, \inf \eta S_2(t) \leq \eta C)
= \phi S_1 d^{-\tau} e^{-\frac{\theta \rho_{1,3}^2}{2}} e^{2\rho_{1,3} \sigma_{1} \xi_1} E(e^{a \sqrt{\tau} t}) | \phi S_1(T) \geq \phi X, \eta S_3(T) > \eta C, \inf \eta S_2(t) \leq \eta C) \] [5.101]

\[
= \phi S_1 d^{-\tau} e^{-\frac{\theta \rho_{1,3}^2}{2}} e^{2\rho_{1,3} \sigma_{1} \xi_1} \int \int e^{a \sqrt{\tau} t} g(\xi_1, \xi_2) dZ_1(T) dZ_2(T) \xrightarrow[\sim]{\phi(w_\tau - \sigma_1 \sqrt{\tau}) - \eta(w_\tau - \sigma_1 \rho_{1,3} \sqrt{\tau})}
\]

the solution is immediately determined. This is because the same steps can be followed of completing the squares and making a change of variables that are used to evaluate the previous conditional expectation, [5.96]. Using these previous results it follows that

\[
r^{-\tau} E(\phi S_1(T) | \phi S_1(T) \geq \phi X, \eta S_3(T) > \eta C, \inf \eta S_2(t) \leq \eta C)
= \phi S_1 d^{-\tau} e^{-\frac{\theta \rho_{1,3}^2}{2}} e^{2\rho_{1,3} \sigma_{1} \xi_1} \int \int e^{a \sqrt{\tau} t} g(\xi_1, \xi_2) dZ_1(T) dZ_2(T) \] [5.102]

\[
= \phi S_1 d^{-\tau} e^{2\ln(C/S_3) \rho_{1,3} \sigma_{1} \xi_1} (C/S_3)^{\frac{\theta \rho_{1,3}^2}{2}} \phi \left( \frac{C}{S_3} \right)^{2\ln(\phi C/S_3) - \frac{\theta \rho_{1,3}^2}{2}} M(\phi w_\tau, \eta w_\tau; \phi \eta \rho_{1,3})
\]

Note that equation [5.102] less equation [5.101] yields equation [5.13].
V.1 Appendix VD: Comparative Statics for the Detached Barrier Model

In this appendix equations [5.12] and [5.13] are verified by deriving the comparative statics. Verification follows by substituting these results into partial differential equation [5.10] (subject to the boundary conditions). To begin with, some preliminary checks of the model are carried out.

V.I.1 Initial Checks

As an initial check on equations [5.12] and [5.13] note the following. As $C$ goes to zero the "out" model converges to the Black-Scholes option pricing model and all "in" option values go to zero. If $S_i = S_3$ (implying, $\rho_{1,3} = 1$, $\sigma_i = \sigma_3$) the model converges to the constant barrier model.

V.I.2 Comparative Statics

In the comparative statics that follow note that since $\frac{\partial \rho_{1,3}}{\partial S_i} = 0$,

\[
\frac{\partial M(c_1(S_i), c_2(S_i), \rho_1)}{\partial S_i} = N \left( \frac{c_1(S_i) - \rho c_2(S_i)}{\sqrt{1 - \rho^2}} \right) n(c_2(S_i)) \frac{\partial c_2(S_i)}{\partial S_i} + N \left( \frac{c_2(S_i) - \rho c_1(S_i)}{\sqrt{1 - \rho^2}} \right) n(c_1(S_i)) \frac{\partial c_1(S_i)}{\partial S_i}.
\]

[5.103]
To simplify the second-order derivatives it is also useful to note

\[
\begin{align*}
\frac{n}{\sqrt{1 - \rho_{1,3}^2}} n(w_i) &= n \left( \frac{\phi w_i - \rho_{1,3} \phi w_5}{\sqrt{1 - \rho_{1,3}^2}} \right) n(w_5), \\
\frac{n}{\sqrt{1 - \rho_{1,3}^2}} (w_i - \sigma_1 \sqrt{\tau}) - \rho_{1,3} \eta (w_i - \sigma_1 \sqrt{\tau}) &= n \left( \frac{\phi(w_i - \sigma_1 \sqrt{\tau}) - \rho_{1,3} \phi(w_5 - \sigma_1 \sqrt{\tau})}{\sqrt{1 - \rho_{1,3}^2}} \right) n(w_5 - \sigma_1 \sqrt{\tau}), \\
\frac{n}{\sqrt{1 - \rho_{1,3}^2}} n(w_7) &= n \left( \frac{\phi w_7 - \rho_{1,3} \phi w_8}{\sqrt{1 - \rho_{1,3}^2}} \right) n(w_8), \\
\frac{n}{\sqrt{1 - \rho_{1,3}^2}} (w_7 - \sigma_1 \sqrt{\tau}) - \rho_{1,3} \eta (w_7 - \sigma_1 \sqrt{\tau}) &= n \left( \frac{\phi(w_7 - \sigma_1 \sqrt{\tau}) - \rho_{1,3} \phi(w_8 - \sigma_1 \sqrt{\tau})}{\sqrt{1 - \rho_{1,3}^2}} \right) n(w_8 - \sigma_1 \sqrt{\tau}).
\end{align*}
\]

[5.104]
Other useful results include:

\[
\frac{\partial w_1}{\partial \tau} = \frac{-w_1 - \sigma_1 \sqrt{\tau}}{2\tau} + \frac{\ln(r/d_\tau)}{\sigma_1 \sqrt{\tau}}, \quad \frac{\partial (w_1 - \sigma_1 \sqrt{\tau})}{\partial \tau} = -\frac{w_1}{2\tau} + \frac{\ln(r/d_\tau)}{\sigma_1 \sqrt{\tau}},
\]

\[
\frac{\partial w_6}{\partial \tau} = \frac{-w_6}{2\tau} + \frac{\ln(r/d_\tau)}{\sigma_3 \sqrt{\tau}} - \frac{\sigma_3^2 - 2\sigma_1 \sigma_3 \rho_{1,3}}{2\sigma_3 \sqrt{\tau}},
\]

\[
\frac{\partial (w_6 - \sigma_1 \rho_{1,3} \sqrt{\tau})}{\partial \tau} = -\frac{w_6}{2\tau} + \frac{\ln(r/d_\tau)}{\sigma_3 \sqrt{\tau}} - \frac{\sigma_3^2 - \sigma_1 \rho_{1,3}}{2\sqrt{\tau}},
\]

\[
\frac{\partial w_7}{\partial \tau} = \frac{-w_7 - \sigma_1 \sqrt{\tau}}{2\tau} + \frac{\ln(r/d_\tau)}{\sigma_1 \sqrt{\tau}}, \quad \frac{\partial (w_7 - \sigma_1 \sqrt{\tau})}{\partial \tau} = -\frac{w_7}{2\tau} + \frac{\ln(r/d_\tau)}{\sigma_1 \sqrt{\tau}},
\]

\[
\frac{\partial w_8}{\partial \tau} = \frac{a \sigma_3}{\sqrt{\tau}} - \frac{w_8}{2\tau}, \quad \frac{\partial (w_8 - \sigma_1 \rho_{1,3} \sqrt{\tau})}{\partial \tau} = \frac{a \sigma_3 (\delta - 1)}{\sqrt{\tau}} - \frac{(w_8 - \sigma_1 \rho_{1,3} \sqrt{\tau})}{2\tau},\]

\[
a = \frac{\ln(r/d_\tau)}{2\sigma_3^2 - 2\sigma_1 \sigma_3 \rho_{1,3}}. \quad \frac{a \sigma_3^2 (2\delta - 1)}{\delta} = 2\ln(r/d_\tau).
\]

\[
[-2a \ln(r/d_\tau) + a(2a+1)\sigma_3^2 - 2a \sigma_1 \sigma_3 \rho_{1,3}] = 0.
\]

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The comparative statics for the first term in [5.12], $\phi S_i d_i^{-\tau} M(\phi w_i, \eta w_6, \phi \rho_{i,3})$, are

\[
\frac{\partial c_i}{\partial S_1} = \phi d_i^{-\tau} M(\phi w_1, \eta w_6, \eta \phi \rho_{1,3}) + N \left( \frac{\eta w_6 - \rho_{1,3} \eta w_1}{\sqrt{1 - \rho_{1,3}^2}} \right) n(w_1) \frac{d_i^{-\tau}}{\sigma_1 \sqrt{\tau}}
\]

\[
\frac{\partial c_i}{\partial S_2} = \phi \eta \sqrt{\frac{\phi w_1 - \rho_{1,3} \phi w_6}{1 - \rho_{1,3}^2}} n(w_2) \frac{S_i d_i^{-\tau}}{S_2 \sqrt{\sigma_2 \sqrt{\tau}}}
\]

\[
\frac{\partial c_i}{\partial S_3^2} = - N \left( \frac{\eta w_6 - \rho_{1,3} \eta w_1}{\sqrt{1 - \rho_{1,3}^2}} \right) n(w_3) \frac{d_i^{-\tau}}{S_1 \sigma_1 \sqrt{\tau}} - \frac{\eta \rho_{1,3}}{\sqrt{1 - \rho_{1,3}^2}} \left( \frac{\eta w_6 - \rho_{1,3} \eta w_1}{\sqrt{1 - \rho_{1,3}^2}} \right) n(w_1) \frac{d_1^{-\tau}}{S_1 \sigma_1 \sqrt{\tau}}
\]

\[
\frac{\partial c_i}{\partial S_2^2} = - \phi \eta \sqrt{\frac{\phi w_1 - \rho_{1,3} \phi w_6}{1 - \rho_{1,3}^2}} n(w_2) \frac{S_i d_i^{-\tau}}{S_2^2 \sqrt{\sigma_2 \sqrt{\tau}}}
\]

\[
- \frac{\eta \rho_{1,3}}{\sqrt{1 - \rho_{1,3}^2}} n(w_3) \frac{d_i^{-\tau}}{S_2 \sigma_2 \sqrt{\tau}} - \phi \eta \sqrt{\frac{\phi w_1 - \rho_{1,3} \phi w_6}{1 - \rho_{1,3}^2}} n(w_3) \frac{S_i d_i^{-\tau}}{S_2^2 \sigma_2 \sqrt{\tau}} - \eta \rho_{1,3} n(w_3) \frac{S_i d_i^{-\tau}}{S_2^2 \sigma_2 \sqrt{\tau}} w_6
\]

\[
\frac{\partial c_i}{\partial S_1 \partial S_3} = \phi \eta \sqrt{\frac{\phi w_1 - \rho_{1,3} \phi w_6}{1 - \rho_{1,3}^2}} n(w_2) \frac{d_i^{-\tau}}{S_3 \sigma_3 \sqrt{\tau}} + \frac{\eta}{\sqrt{1 - \rho_{1,3}^2}} \left( \frac{\eta w_6 - \rho_{1,3} \eta w_1}{\sqrt{1 - \rho_{1,3}^2}} \right) n(w_1) \frac{d_i^{-\tau}}{S_3 \sigma_3 \sqrt{\tau}}
\]

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The remaining comparative statics for $\phi S_1 \tau M(\phi w_1, \eta w_6, \phi \eta \rho_{1,3})$, are

$$
\frac{\partial (\cdot)}{\partial \tau} = - \phi S_1 \tau M(\phi w_1, \eta w_6, \phi \eta \rho_{1,3}) \ln(d_1) + N \left( \frac{\eta w_6 - \rho_{1,3} \eta w_1}{\sqrt{1 - \rho_{1,3}^2}} \right) n(w_1) S_1 \tau^{-\tau} \left[ \frac{\ln(r/d_1)}{\sigma_1 \sqrt{\tau}} - \frac{(w_1 - \sigma_1 \sqrt{\tau})}{2 \tau} \right] \tag{5.107}
$$

$$
+ \phi \eta M \left( \frac{\phi w_1 - \rho_{1,3} \phi w_6}{\sqrt{1 - \rho_{1,3}^2}} \right) n(w_6) S_1 \tau^{-\tau} \left[ \frac{\ln(r/d_1)}{\sigma_3 \sqrt{\tau}} - \frac{w_6 - 5(\sigma_3^2 - 2 \sigma_1 \sigma_3 \rho_{1,3})}{2 \tau \sigma_3 \sqrt{\tau}} \right]
$$

The comparative statics for the second term in [5.12], $\phi X \tau M(\phi (w_1 - \sigma_1 \sqrt{\tau}), \eta (w_1 - \sigma_1 \sqrt{\tau}), \phi \eta \rho_{1,3})$, are

$$
\frac{\partial (\cdot)}{\partial S_1} = N \left( \frac{\eta (w_6 - \sigma_1 \rho_{1,3} \sqrt{\tau}) - \rho_{1,3} \eta (w_1 - \sigma_1 \sqrt{\tau})}{\sqrt{1 - \rho_{1,3}^2}} \right) n(w_1 - \sigma_1 \sqrt{\tau}) \frac{X \tau^{-\tau} \sigma_1 \sqrt{\tau}}{S_1 \sigma_1 \sqrt{\tau}}
$$

$$
\frac{\partial (\cdot)}{\partial S_3} = \phi \eta M \left( \frac{\phi (w_1 - \sigma_1 \sqrt{\tau}) - \rho_{1,3} \phi (w_6 - \sigma_1 \rho_{1,3} \sqrt{\tau})}{\sqrt{1 - \rho_{1,3}^2}} \right) n(w_6 - \sigma_1 \rho_{1,3} \sqrt{\tau}) \frac{X \tau^{-\tau} \sigma_3 \sqrt{\tau}}{S_3 \sigma_3 \sqrt{\tau}} \tag{5.108}
$$

$$
\frac{\partial^2 (\cdot)}{\partial S_1^2} = - N \left( \frac{\eta (w_6 - \sigma_1 \rho_{1,3} \sqrt{\tau}) - \rho_{1,3} \eta (w_1 - \sigma_1 \sqrt{\tau})}{\sqrt{1 - \rho_{1,3}^2}} \right) n(w_1 - \sigma_1 \sqrt{\tau}) \frac{X \tau^{-\tau} \sigma_1^2 \sqrt{\tau}}{S_1 \sigma_1 \sqrt{\tau}}
$$

$$
- \frac{\eta \rho_{1,3}}{\sqrt{1 - \rho_{1,3}^2}} \left( \frac{\eta (w_6 - \sigma_1 \rho_{1,3} \sqrt{\tau}) - \rho_{1,3} \eta (w_1 - \sigma_1 \sqrt{\tau})}{\sqrt{1 - \rho_{1,3}^2}} \right) n(w_1 - \sigma_1 \sqrt{\tau}) \frac{X \tau^{-\tau}}{S_1 \sigma_1 \sqrt{\tau}}
$$
The remaining comparative statics for \( \phi Xr^{-\tau}M(\phi(w_1-\sigma_1\sqrt{\tau}),\eta(w_6-\sigma_1\rho_{1,3}\sqrt{\tau}),\phi\eta_{1,3}) \), are

\[
\frac{\partial^2(\cdot)}{\partial s^2_3} = -\phi \eta n(w_6-\sigma_1\rho_{1,3}\sqrt{\tau}) \left[ N\left( \frac{\phi(w_1-\sigma_1\sqrt{\tau})-\rho_{1,3}\phi(w_6-\sigma_1\rho_{1,3}\sqrt{\tau})}{\sqrt{1-\rho_{1,3}^2}} \right) \frac{Xr^{-\tau}}{S^2_3 \sigma^2_3 \tau} (\sigma_3-\sigma_1\rho_{1,3}) \right] \\
- \frac{\eta \rho_{1,3}}{\sqrt{1-\rho_{1,3}^2}} n\left( \frac{\eta(w_6-\sigma_1\rho_{1,3}\sqrt{\tau})-\rho_{1,3}\eta(w_1-\sigma_1\sqrt{\tau})}{\sqrt{1-\rho_{1,3}^2}} \right) n(w_1-\sigma_1\sqrt{\tau}) \frac{Xr^{-\tau}}{S^2_3 \sigma^2_3 \tau} w_6 \\
- \phi \eta n\left( \frac{\phi(w_1-\sigma_1\sqrt{\tau})-\rho_{1,3}\phi(w_6-\sigma_1\rho_{1,3}\sqrt{\tau})}{\sqrt{1-\rho_{1,3}^2}} \right) n(w_6-\sigma_1\rho_{1,3}\sqrt{\tau}) \frac{Xr^{-\tau}}{S^2_3 \sigma^2_3 \tau} w_6
\]

[5.109]

\[
\frac{\partial^2(\cdot)}{\partial s^1\partial s_3} = \frac{\eta}{\sqrt{1-\rho_{1,3}^2}} n(w_1-\sigma_1\sqrt{\tau}) \left[ n\left( \frac{\eta(w_6-\sigma_1\rho_{1,3}\sqrt{\tau})-\rho_{1,3}\eta(w_1-\sigma_1\sqrt{\tau})}{\sqrt{1-\rho_{1,3}^2}} \right) \frac{Xr^{-\tau}}{S_3 \sigma_3 \sigma_1 \sigma_3 \tau} \right]
\]

\[
\frac{\partial(\cdot)}{\partial \tau} = -\phi Xr^{-\tau}M(\phi(w_1-\sigma_1\sqrt{\tau}),\eta(w_6-\sigma_1\rho_{1,3}\sqrt{\tau}),\phi\eta_{1,3})\ln(r)
\]

\[
+ N\left( \frac{\eta(w_6-\sigma_1\rho_{1,3}\sqrt{\tau})-\rho_{1,3}\eta(w_1-\sigma_1\sqrt{\tau})}{\sqrt{1-\rho_{1,3}^2}} \right) n(w_1-\sigma_1\sqrt{\tau}) Xr^{-\tau} \left[ \frac{\ln(rd_1)}{\sigma_1\sqrt{\tau}} - \frac{w_1}{2\tau} \right]
\]

\[
+ \phi \eta n\left( \frac{\phi(w_1-\sigma_1\sqrt{\tau})-\rho_{1,3}\phi(w_6-\sigma_1\rho_{1,3}\sqrt{\tau})}{\sqrt{1-\rho_{1,3}^2}} \right) n(w_6-\sigma_1\rho_{1,3}\sqrt{\tau}) Xr^{-\tau} \left[ \frac{\ln(rd_2)}{\sigma_3\sqrt{\tau}} - \frac{w_6}{2\tau} - \frac{5(\sigma_3-\sigma_1\rho_{1,3})}{\sqrt{\tau}} \right]
\]
The comparative statics for the first term in \([5.13]\), \( \phi(C/S_3)^{2a}S_1d_i^{-1}M(\phi w_{r_i, \eta w_{r_i, \eta} \rho_{13}}) \)

where \(a = \delta_1 + \rho_{13} \sigma_1 \sigma_3^{-1} \), are

\[
\frac{\partial()}{\partial S_1} = \phi(C/S_3)^{2a}d_i^{-1}M(\phi w_{r_i, \eta w_{r_i, \eta} \phi \rho_{13}}) \\
+ (C/S_3)^{2a} \left( \frac{\eta w_{r_i, \rho_{13} \eta w_{r_i}}}{\sqrt{1-\rho_{13}^2}} \right) n(w_{r_i}) \frac{d_i^*}{\sigma_1 \sqrt{\tau}}
\]

\[
\frac{\partial()}{\partial S_3} = -\phi \eta (C/S_3)^{2a} \left( \frac{\phi w_{r_i, \rho_{13, } \phi w_{r_i}}}{\sqrt{1-\rho_{13}^2}} \right) n(w_{r_i}) \frac{S_1 d_i^*}{S_3 \sigma_3 \sqrt{\tau}}
- 2\alpha (C/S_3)^{2a} \left( \frac{\eta w_{r_i, \rho_{13} \eta w_{r_i}}}{\sqrt{1-\rho_{13}^2}} \right) n(w_{r_i}) \frac{2S_1 d_i^* \rho_{13}}{S_3 \sigma_3 \sqrt{\tau}}
- (C/S_3)^{2a} \left( \frac{\eta w_{r_i, \rho_{13} \eta w_{r_i}}}{\sqrt{1-\rho_{13}^2}} \right) n(w_{r_i}) \frac{2S_1 d_i^* \rho_{13}}{S_3 \sigma_3 \sqrt{\tau}}
\]

\[
\frac{\partial()}{\partial S_1^2} = -(C/S_3)^{2a} \left( \frac{\eta w_{r_i, \rho_{13} \eta w_{r_i}}}{\sqrt{1-\rho_{13}^2}} \right) n(w_{r_i}) \frac{d_i^*}{S_1 \sigma_1 \sqrt{\tau}} (w_{r_i} - \sigma_1 \sqrt{\tau})
- \frac{-\eta \rho_{13}}{\sqrt{1-\rho_{13}^2}} (C/S_3)^{2a} \left( \frac{\eta w_{r_i, \rho_{13} \eta w_{r_i}}}{\sqrt{1-\rho_{13}^2}} \right) n(w_{r_i}) \frac{d_i^*}{S_1 \sigma_1 \sqrt{\tau}}
\]

\[
\frac{\partial()}{\partial \tau} = -\phi (C/S_3)^{2a} \left( \frac{\eta w_{r_i, \rho_{13} \eta w_{r_i}}}{\sqrt{1-\rho_{13}^2}} \right) n(w_{r_i}) \frac{d_i^*}{\sigma_1 \sqrt{\tau}} \left[ \frac{\ln(r/d_i)}{\sigma_1 \sqrt{\tau}} - \frac{(w_{r_i} - \sigma_1 \sqrt{\tau})}{2\tau} \right]
+ \phi \eta (C/S_3)^{2a} \left( \frac{\phi w_{r_i, \rho_{13, } \phi w_{r_i}}}{\sqrt{1-\rho_{13}^2}} \right) n(w_{r_i}) \frac{S_1 d_i^*}{\sqrt{\tau}} \left[ \frac{\sigma_3}{\sqrt{\tau}} - \frac{w_{r_i}}{2\tau} \right]
\]
Continuing, the comparative statics for the first term in [5.13],

\[ \frac{\partial \bar{\psi}}{\partial S_3} = 2\alpha(2\alpha + 1)\phi(C/S_3)2\alpha(S_3/S_3'2)\delta_1^{\tau}M(\phi \eta \varphi \eta \omega_3, \phi \eta \varphi_1, 3) \]

\[ + 2\alpha \eta \phi(C/S_3)^{2\alpha}(S_3/S_3')2\delta_1^{\tau}M(\phi \eta \varphi \eta \omega_3, \phi \eta \varphi_1, 3) \]

\[ + 2\alpha \eta \phi(C/S_3)^{2\alpha}(S_3/S_3')2\delta_1^{\tau}M(\phi \eta \varphi \eta \omega_3, \phi \eta \varphi_1, 3) \]

\[ + \phi \eta (2\alpha + 1)(C/S_3)^{2\alpha}M(\phi \eta \varphi \eta \omega_3, \phi \eta \varphi_1, 3) \]

\[ - \phi \eta (C/S_3)^{2\alpha}M(\phi \eta \varphi \eta \omega_3, \phi \eta \varphi_1, 3) \]

\[ + \frac{\eta \varphi_1}{\sqrt{1 - \rho_1^2}}(C/S_3)^{2\alpha}M(\phi \eta \varphi \eta \omega_3, \phi \eta \varphi_1, 3) \]

\[ + \frac{\eta \varphi_1}{\sqrt{1 - \rho_1^2}}(C/S_3)^{2\alpha}M(\phi \eta \varphi \eta \omega_3, \phi \eta \varphi_1, 3) \]

\[ - \frac{\eta \varphi_1}{\sqrt{1 - \rho_1^2}}(C/S_3)^{2\alpha}M(\phi \eta \varphi \eta \omega_3, \phi \eta \varphi_1, 3) \]

\[ + \frac{\eta \varphi_1}{\sqrt{1 - \rho_1^2}}(C/S_3)^{2\alpha}M(\phi \eta \varphi \eta \omega_3, \phi \eta \varphi_1, 3) \]

\[ - \frac{\eta \varphi_1}{\sqrt{1 - \rho_1^2}}(C/S_3)^{2\alpha}M(\phi \eta \varphi \eta \omega_3, \phi \eta \varphi_1, 3) \]

\[ = 2\alpha(2\alpha + 1)\phi(C/S_3)2\alpha(S_3/S_3')2\delta_1^{\tau}M(\phi \eta \varphi \eta \omega_3, \phi \eta \varphi_1, 3) \]

\[ + 2\alpha \eta \phi(C/S_3)^{2\alpha}(S_3/S_3')2\delta_1^{\tau}M(\phi \eta \varphi \eta \omega_3, \phi \eta \varphi_1, 3) \]

\[ + 2\alpha \eta \phi(C/S_3)^{2\alpha}(S_3/S_3')2\delta_1^{\tau}M(\phi \eta \varphi \eta \omega_3, \phi \eta \varphi_1, 3) \]

\[ + \phi \eta (2\alpha + 1)(C/S_3)^{2\alpha}M(\phi \eta \varphi \eta \omega_3, \phi \eta \varphi_1, 3) \]

\[ - \phi \eta (C/S_3)^{2\alpha}M(\phi \eta \varphi \eta \omega_3, \phi \eta \varphi_1, 3) \]

[5.111]
The remaining comparative statics for the first term in [5.13], 
\( \phi(C/S_3)\delta S_3 d_1^{-\tau} M(\phi w_{\gamma}, \eta w_{\delta}, \phi \eta \rho_{i,3}) \) where \( a = [\delta^{-1} + \rho_{i,3} \sigma_{i,3}^{-1}] \), are

\[
\frac{\partial(\cdot)^2}{\partial s_3 \partial s_1} = -2a(C/S_3)^{2a}(1/S_3) d_1^{-\tau} M(\phi w_{\gamma}, \eta w_{\delta}, \phi \eta \rho_{i,3})
\]

\[
- 2a(C/S_3)^{2a}(\eta w_{\delta} - \rho_{i,3} \eta w_{\gamma}) n(w_{\gamma}) \frac{d_1^{-\tau}}{S_3 \sigma_1 \tau}
\]

\[
- (C/S_3)^{2a} \left( \frac{\eta w_{\delta} - \rho_{i,3} \eta w_{\gamma}}{\sqrt{1 - \rho_{i,3}^2}} \right) n(w_{\gamma}) \frac{2d_1^{-\tau} \rho_{i,3}}{S_3 \sigma_3 \tau}
\]

\[
- \phi \eta (C/S_3)^{2a} \left( \frac{\phi w_{\gamma} - \phi \rho_{i,3} w_{\delta}}{\sqrt{1 - \rho_{i,3}^2}} \right) n(w_{\gamma}) \frac{d_1^{-\tau}}{S_3 \sigma_3 \tau}
\]

\[
+ (C/S_3)^{2a} \left( \frac{\eta w_{\delta} - \rho_{i,3} \eta w_{\gamma}}{\sqrt{1 - \rho_{i,3}^2}} \right) n(w_{\gamma}) \frac{2d_1^{-\tau} \rho_{i,3}}{S_3 \sigma_1 \sigma_3 \tau}
\]

\[
- \frac{\eta \rho_{i,3}}{\sqrt{1 - \rho_{i,3}^2}} (C/S_3)^{2a} \left( \frac{\eta w_{\delta} - \rho_{i,3} \eta w_{\gamma}}{\sqrt{1 - \rho_{i,3}^2}} \right) n(w_{\gamma}) \frac{2d_1^{-\tau} \rho_{i,3}}{S_3 \sigma_1 \sigma_3 \tau}
\]

\[
- \frac{\eta}{\sqrt{1 - \rho_{i,3}^2}} (C/S_3)^{2a} \left( \frac{\eta w_{\delta} - \rho_{i,3} \eta w_{\gamma}}{\sqrt{1 - \rho_{i,3}^2}} \right) n(w_{\gamma}) \frac{d_1^{-\tau}}{S_3 \sigma_1 \sigma_3 \tau}
\]
The comparative statics for the second term in [5.13], \( \phi(C/S_y)^{28-2}X \tau^{-1}M(\phi(w_y - \sigma_1 \sqrt{\tau}), \eta(w_8 - \sigma_1 \sqrt{\tau}), \phi \eta \rho_{13}, \phi \eta \rho_{13}) \) are

\[
\frac{\partial(\cdot)}{\partial S_1} = (C/S_y)^{28-2}X \tau^{-1}N \left( \frac{\eta(w_8 - \sigma_1 \rho_{13} \sqrt{\tau}) - \rho_{13} \eta(w_7 - \sigma_1 \sqrt{\tau})}{\sqrt{1 - \rho_{13}^2}} \right) \frac{n(w_7 - \sigma_1 \sqrt{\tau})}{S_1 \sigma_1 \sqrt{\tau}} \frac{1}{S_1 \sigma_1 \sqrt{\tau}}
\]

\[
\frac{\partial(\cdot)}{\partial S_3} = - \phi \eta (C/S_y)^{28-2}N \left( \frac{\eta(w_8 - \sigma_1 \sqrt{\tau}) - \rho_{13} \phi \eta \rho_{13} \sqrt{\tau}}{\sqrt{1 - \rho_{13}^2}} \right) \frac{n(w_7 - \sigma_1 \rho_{13} \sqrt{\tau})}{S_3 \sigma_3 \sqrt{\tau}} \frac{X \tau^{-1}}{S_3 \sigma_3 \sqrt{\tau}}
\]

\[
\frac{\partial^2(\cdot)}{\partial S_1^2} = (C/S_y)^{28-2}N \left( \frac{\eta(w_8 - \sigma_1 \rho_{13} \sqrt{\tau}) - \rho_{13} \eta(w_7 - \sigma_1 \sqrt{\tau})}{\sqrt{1 - \rho_{13}^2}} \right) \frac{n(w_7 - \sigma_1 \sqrt{\tau})}{S_1^2 \sigma_1 \sqrt{\tau}} \frac{X \tau^{-1}}{S_1 \sigma_1 \sqrt{\tau}}
\]

\[
\frac{\partial(\cdot)}{\partial \tau} = - \phi (C/S_y)^{28-2}X \tau^{-1}M(\phi(w_y - \sigma_1 \sqrt{\tau}), \eta(w_8 - \sigma_1 \rho_{13} \sqrt{\tau}), \phi \eta \rho_{13}, \phi \eta \rho_{13}) \ln(\tau)
\]

\[
+ (C/S_y)^{28-2}N \left( \frac{\eta(w_8 - \sigma_1 \rho_{13} \sqrt{\tau}) - \rho_{13} \eta(w_7 - \sigma_1 \sqrt{\tau})}{\sqrt{1 - \rho_{13}^2}} \right) \frac{n(w_7 - \sigma_1 \sqrt{\tau})}{\sigma_1 \sqrt{\tau}} \frac{X \tau^{-1}}{\sigma_1 \sqrt{\tau}} \left[ \frac{\ln(r/d_1)}{\sigma_1 \sqrt{\tau}} - \frac{w_7}{2 \tau} \right]
\]

\[
+ \phi \eta (C/S_y)^{28-2}N \left( \frac{\phi(w_y - \sigma_1 \sqrt{\tau}) - \rho_{13} \phi \eta \rho_{13} \sqrt{\tau}}{\sqrt{1 - \rho_{13}^2}} \right) \frac{n(w_8 - \sigma_1 \rho_{13} \sqrt{\tau})}{\sigma_1 \sqrt{\tau}} \frac{X \tau^{-1}}{\sqrt{\tau}} \left[ \frac{(\delta - 1) \sigma_3}{\sqrt{\tau}} - \frac{(w_8 - \sigma_1 \rho_{13} \sqrt{\tau})}{2 \tau} \right]
\]
Continuing, the comparative statics for the second term in [5.13], \( \phi(C/S_3)^{2\delta - 2}Xr^{-\tau}M(\phi(w_{\gamma} - \sigma_{1}V_\tau), \eta_{1}(w_{\gamma} - \sigma_{1}1_3V_\tau), \phi_{1}(w_{\gamma} - \sigma_{1}1_3V_\tau)) \) are

\[
\frac{\partial \mathcal{Y}^2}{\partial S_3^2} = (2\delta - 2)(2\delta - 1)\phi(C/S_3)^{2\delta - 2}(X/S_3^2)r^{-\tau}M(\phi(w_{\gamma} - \sigma_{1}1_3V_\tau), \eta_{1}(w_{\gamma} - \sigma_{1}1_3V_\tau), \phi_{1}(w_{\gamma} - \sigma_{1}1_3V_\tau))
\]

\[
+ (4\delta - 3)(C/S_3)^{2\delta - 2}M \left( \frac{\eta_{1}(w_{\gamma} - \sigma_{1}1_3V_\tau) - \phi_{1}(w_{\gamma} - \sigma_{1}1_3V_\tau)}{\sqrt{1 - \rho_{1,3}^2}} \right) \frac{Xr^{-\tau}P_{1,3}}{S_3^2\sigma_{3}V_\tau}
\]

\[
- \phi\eta(C/S_3)^{2\delta - 2}M \left( \frac{\eta_{1}(w_{\gamma} - \sigma_{1}1_3V_\tau) - \phi_{1}(w_{\gamma} - \sigma_{1}1_3V_\tau)}{\sqrt{1 - \rho_{1,3}^2}} \right) \frac{Xr^{-\tau}}{S_3^2\sigma_{3}V_\tau}
\]

\[
+ \frac{\eta_{1}(w_{\gamma} - \sigma_{1}1_3V_\tau) - \phi_{1}(w_{\gamma} - \sigma_{1}1_3V_\tau)}{\sqrt{1 - \rho_{1,3}^2}} \frac{Xr^{-\tau}}{S_3^2\sigma_{3}V_\tau}
\]

\[
- (C/S_3)^{2\delta - 2}M \left( \frac{\eta_{1}(w_{\gamma} - \sigma_{1}1_3V_\tau) - \phi_{1}(w_{\gamma} - \sigma_{1}1_3V_\tau)}{\sqrt{1 - \rho_{1,3}^2}} \right) \frac{Xr^{-\tau}P_{1,3}^2}{S_3^2\sigma_{3}V_\tau}
\]

\[
+ \frac{\eta_{1}(w_{\gamma} - \sigma_{1}1_3V_\tau) - \phi_{1}(w_{\gamma} - \sigma_{1}1_3V_\tau)}{\sqrt{1 - \rho_{1,3}^2}} \frac{Xr^{-\tau}P_{1,3}^2}{S_3^2\sigma_{3}V_\tau}
\]

\[
- \frac{\rho_{1,3}P_{1,3}}{\sqrt{1 - \rho_{1,3}^2}^2} \left( \frac{\eta_{1}(w_{\gamma} - \sigma_{1}1_3V_\tau) - \phi_{1}(w_{\gamma} - \sigma_{1}1_3V_\tau)}{\sqrt{1 - \rho_{1,3}^2}} \right) \frac{Xr^{-\tau}P_{1,3}^2}{S_3^2\sigma_{3}V_\tau}
\]

\[\text{[5.114]}\]
The remaining comparative statics for the second term in [5.13], \( \phi(C/S_3)^{2b-2}x_{1}^{-\gamma}M(\phi(w_{-}^{-}\sigma, r_{1}, \sqrt{\nu}, r), \eta, \rho_{1,3}) \) are

\[
\frac{\partial(\cdot)}{\partial S_3 \partial S_1} = - (2 \delta - 2)(C/S_3)^{2b-2} \frac{X_{r}^{-\gamma}}{S_1 S_3 \sigma_{1} \sigma_{r} \sqrt{\nu}} \\
\times \left[ \frac{\eta(w_0 - \sigma_1 \rho_{1,3} \sqrt{\nu} - \rho_{1,3} \eta(w_1 - \sigma_1 \sqrt{\nu}))}{\sqrt{1 - \rho_{1,3}^2}} \right] n(w_1 - \sigma_1 \sqrt{\nu}) \\
+ (C/S_3)^{2b-2} \frac{X_{r}^{-\gamma} \rho_{1,3}}{S_1 S_3 \sigma_{1} \sigma_{3} \sqrt{\nu}} (w_1 - \sigma_1 \sqrt{\nu}) \\
+ \frac{\eta \rho_{1,3}}{\sqrt{1 - \rho_{1,3}^2}} (C/S_3)^{2b-2} \frac{X_{r}^{-\gamma} \rho_{1,3}}{S_1 S_3 \sigma_{1} \sigma_{3} \sqrt{\nu}} \left[ \frac{\eta(w_0 - \sigma_1 \rho_{1,3} \sqrt{\nu} - \rho_{1,3} \eta(w_1 - \sigma_1 \sqrt{\nu}))}{\sqrt{1 - \rho_{1,3}^2}} \right] n(w_1 - \sigma_1 \sqrt{\nu}) \\
- \frac{\eta}{\sqrt{1 - \rho_{1,3}^2}} (C/S_3)^{2b-2} \frac{X_{r}^{-\gamma}}{S_1 S_3 \sigma_{1} \sigma_{3} \sqrt{\nu}} \left[ \frac{\eta(w_0 - \sigma_1 \rho_{1,3} \sqrt{\nu} - \rho_{1,3} \eta(w_1 - \sigma_1 \sqrt{\nu}))}{\sqrt{1 - \rho_{1,3}^2}} \right] n(w_1 - \sigma_1 \sqrt{\nu})
\]

\[\text{[5.115]}\]
CHAPTER 6
The Valuation of Black-Scholes Options Subject to
Intertemporal Default Risk: An Application of the
Down-and-Out Detached Barrier Model

VI.A Introduction

In their seminal paper, Black and Scholes (1973) assume there is no possibility of writer default. This assumption is certainly tenable for exchange-traded options because the back-up system used by the Options Clearing Corporation serves to insure that contracts are honored. However, customized or over-the-counter (OTC) options, which are widely used today, are not guaranteed against writer default. The creditworthiness of the writer is key to maintaining a successful hedging position; if the writer defaults, the hedge is worthless.

In this chapter, the down-and-out detached barrier model (see Chapter V) is used to price options subject to intertemporal default risk. The model, which is referred to as the intertemporal vulnerable Black-Scholes model, has a closed-form solution and preserves much of the intuitive appeal of the traditional Black-Scholes model. The intertemporal vulnerable Black-Scholes

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74 In fact, no holder of an exchange-traded option has ever experienced a financial loss due to writer default.

75 The OTC option's market is a "lucrative $50 billion-a-year business" (Taylor, 1993). Swap sales, another similar customized contract used to hedge interest rate and currency risks, "have rocketed in a few short years into a $4.8 trillion market" (Greising, 1993).

76 The term "vulnerable" is consistent with previous research; see Johnson and Stulz (1987, p. 267) and Hull and White (1994, p. 1). Our analysis is intertemporal in the sense that default risk is examined across time. However, the analysis is intratemporal in the sense that our examination period is confined to the life of the option.
model provides prices for options that become null and void when the value of the writer’s assets, A(t), which is stochastic, falls below some prespecified critical boundary - which hereinafter is called the default boundary, which may also be stochastic. Default is defined to occur with probability one if the writer ever experiences a state of "financial distress". "Financial distress" is denoted by the first default boundary-level crossing of A(t) to the right of A(t_0) (t ∈ [t_0, T]).

The valuation solution is fundamentally different from any previous model in that a distinction is made between the value of the writer’s option to default and the value of the holder’s option to receive proportional recoveries. This distinction has two principle advantages. First, by focusing solely on the possibility of writer default, the value of a vulnerable Black-Scholes option is written as the value of a default-free Black-Scholes option less the value of the writer’s option to default. Consequently, the problem and solution to incorporating intertemporal default risk into an otherwise standard Black-Scholes framework is a natural extension of the original Black-Scholes model. This allows the vulnerable Black-Scholes model to preserve two of the most attractive features of the default-free model - its ease of understanding and the fact that the input variables can be directly observed or readily estimated. Second, the model explicitly allows for potential deviations from strict absolute priority by analyzing two partial recovery scenarios. First, it is assumed that the vulnerable option is secured by a prespecified margin deposit. Second, deviations from absolute priority are considered by allowing the option holder to recover α (a constant) percent of the value of the writer’s assets at the time of default. This latter specification requires the development of a valuation formula for a first passage option - an option (the recovery option) with a random payoff at a random time.

The intertemporal vulnerable Black-Scholes model is complementary to the credit risk models of Johnson and Stulz (1987, pp. 277-279) and Hull and White (1994). Johnson and Stulz examine how credit risk affects the value of an option when there is a positive probability of default occurring at expiration. They obtain an analytical solution for European options that pay

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77 See Longstaff and Schwartz (1992, p. 4) and the references cited therein for evidence that absolute priority rules are not generally followed in cases of bond default. No evidence exists pertaining to the adherence of absolute priority rules on customized options.
off the minimum of the value of the writer’s assets and the intrinsic value. The implication of their solution is that options subject to expiration default risk can have completely different hedging properties than the Black-Scholes formulation.\textsuperscript{78} The intertemporal vulnerable model is also shown to have a closed-form solution with some atypical hedging properties. In this light, our model may be considered an intertemporal version of the Johnson and Stulz model.\textsuperscript{79} However, the Johnson and Stulz framework implicitly assumes that the option is the dominant security in the writer’s liability portfolio. The option is dominant in terms of economic significance because default can only occur due to, and at, the option’s expiration. The option is also dominant over all other claims in the event of default. The approach taken here is more general in that the writer is allowed to have many outstanding obligations, with some claims being senior in rank.

In more recent work, Hull and White (1994) propose a intertemporal default risk model in which partial recoveries are determined by endogenous factors. They explore how intertemporal default risk affects the value of a contingent claim when both the probability of default and the size of the recovery are random variables. They are unable, however, to arrive at a closed-form solution except when the simplifying assumption is made “that the variables concerned with defaults are independent of the variables underlying the value of the derivative security in a no-default world” (p. 20). Therefore, the model is implemented numerically. The analysis of default risk across time renders the emphasis of our model to be similar to the Hull and White model. In fact, the valuation formula developed in this chapter is a closed-form solution to the model estimated numerically by Hull and White. The Hull and White framework, however, implicitly assumes strict absolute priority rules are followed, whereas our model allows for deviations.

\textsuperscript{78} For instance, Johnson and Stulz (1987, p. 269) show that the value of a European option subject to expiration credit risk can decrease with time to expiration, the interest rate, and the volatility of the optioned asset.

\textsuperscript{79} In Section VI.C of this chapter, it is shown when the Johnson and Stulz model and our model are equivalent. However, in general, differences in the definitions and timing of default preclude the Johnson and Stulz model from simply being a special case of the intertemporal vulnerable model.
This chapter develops the general theory needed to price Black-Scholes options subject to intertemporal default risk. In Section VI.B, the vulnerable Black-Scholes model with a static default boundary is presented and the analytical properties of the valuation solution are compared and contrasted with the default-free model. The valuation solution is shown to be applicable to the valuation of vulnerable currency options, vulnerable futures options, and vulnerable forward contracts. In the general situation the writer’s assets are assumed to differ from the optioned asset. In the special case where the two are identical, for example, when equity in a levered firm is viewed as a vulnerable call option, the resulting solution is shown to be identical to the Cox and Rubinstein (1985) paradigm. In Section VI.C, an intertemporal vulnerable model is developed in which the default boundary is allowed to be stochastic. In Section VI.D, the models are extended to include partial recoveries. Concluding remarks are provided in Section VI.E.

VI.B The Vulnerable Black-Scholes Pricing Model When the Default Boundary is Static

In this section the framework of the analysis is presented, followed by the general valuation solution for options exposed to intertemporal credit risk with a static default boundary. The analytical properties of this vulnerable model are compared and contrasted with the Black-Scholes model. In addition, the relationship between this model to other option pricing results are considered.

VI.B.1 The General Framework

Throughout the chapter, the following assumptions and notation are used. It is assumed that markets are frictionless and complete, unlimited short sales are allowed, trading takes place continuously, the term structure of interest rates is flat and constant, \((r-1)\) is the discrete riskless rate of interest and \((d_i-1)\) is the payout rate of asset \(i\).\(^1\) The European option is assumed to be

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\(^0\) Cox and Rubinstein (1985, pp. 408-412) focus on determining a levered firm’s value when the zero-coupon bonds contained a safety covenant that give "the bondholders the right to force bankruptcy or reorganization of the firm if it is doing poorly according to some standard" (p. 408). The equity position is shown to be analogous to a down-and-out call option.

\(^1\) Typically, the payout rate of an asset is the dividend yield. When asset \(i\) is the value of the writer’s assets, however, care should be taken when interpreting \((d_i-1)\). This is because the
payout unprotected and matures at time $T$. Let $\tau = T - t_o$ denote the time to expiration, in years, of the derivative security.

Changes in asset values are assumed to obey geometric Brownian motion with time homogeneous parameters. Let $S(t)$ and $A(t)$ denote, respectively, the optioned asset price and the value of the writer's assets whose price dynamics are described by

$$dS(t) = \mu_S S dt + \sigma_S S dZ_S(t)$$  \hspace{1cm} \text{[6.1]}$$

and

$$dA(t) = \mu_A A dt + \sigma_A A dZ_A(t).$$  \hspace{1cm} \text{[6.2]}$$

$\{Z_i(t), t \in [t_o, T], i=S,A\}$ is a standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mu_i$ is the instantaneous expected return from holding asset $i$, and $\sigma_i$ is the instantaneous standard deviation rate on asset $i$. When referring to an initial asset price, for notational convenience, the time parameter is suppressed (i.e., $S(t_o) = S$ and $A(t_o) = A$). Note the probability measures, $\mathbb{P}_i$, are equivalent across assets and are defined on $\mathcal{F}_i$ as an equivalent martingale measure, as demonstrated below. Define $\rho_{S,A} = \text{corr}(Z_S(t), Z_A(t))$ and $\sigma_{S,A} = \rho_{S,A} \sigma_S \sigma_A$, where $\text{corr}$ denotes the correlation coefficient, $\sigma_{S,A}$ represents the covariance between the two asset prices, and $\rho_{S,A}$ and $\sigma_{S,A}$ are assumed constant. Additional notation is defined when needed.

VI.B.2 The Theoretical Model When the Default Boundary is Static

For ease of exposition, restructuring possibilities and bankruptcy litigation are ignored. Hence, if the writer ever experiences a state of "financial distress" in the interval $[t_o, T]$, default is defined to occur immediately with probability one. Assume that no partial recoveries are made in the event of default. Default does not necessarily imply that the option holder realizes a loss. However, our analysis of possible recoveries is delayed until Section VI.D of this chapter so we may focus on the value of the writer's option to default.

"Financial distress" is defined by the first default boundary-level crossing of $A(t)$ to the right of $A(t_o)$. To make the discussion as general as possible, define "financial distress" as

payout in this case is tied to the value of the writer's assets, not the equity value.
A(t) ≤ D for any t ∈ [t₀, T], where D is a prespecified non-negative constant representing the default states. The inclusion of D in this definition is general enough to accommodate such features as a minimum asset requirement (to prevent the writer from selling off physical assets).

Let Wₛ(S,X,A,D,τ) denote the value of a vulnerable European option written on S and struck at X (a constant). This option expires prematurely, without the right to exercise, if A(t) ≤ D for any t ∈ [t₀, T]. The price dynamics for S(t) and A(t) are given by the stochastic differential equations in [6.1] and [6.2]. Define φ, a binary variable, to be 1 if the vulnerable option is a call option and -1 if it is a put. Vulnerable options have a payoff of max(φS(T)−X,0) at T if A(t) > D ∀ t ∈ [t₀, T] and zero otherwise.

Merton (1973, p. 164) shows that for an option whose price H(P₁,P₂,τ) is dependent on two underlying state variables, P₁ and P₂, the value of the option is the solution to the following second order, linear, partial differential equation:

\[
\ln(r)H(t) = \ln(r/d_P₁)P₁ \frac{\partial H(t)}{\partial P₁} + \ln(r/d_P₂)P₂ \frac{\partial H(t)}{\partial P₂} - \frac{\partial H(t)}{\partial T} \\
+ \frac{\sigma₁²}{2}P₁² \frac{\partial² H(t)}{\partial P₁²} + \frac{\sigma₂²}{2}P₂² \frac{\partial² H(t)}{\partial P₂²} + \frac{\sigma₁ \sigma₂}{P₁P₂} \sigma₁ σ₂ \phi P₁P₂ P₁P₂.
\]

[6.3]

The value of the vulnerable option must satisfy [6.3], with H(·)=Wₛ(·), P₁=S, and P₂=A, subject to boundary conditions⁴²:

\[ Wₛ(S(T),X,A(T),D,0) = \max(0, φS(T)−X) \text{ if } A(t) > D \text{ for all } t ∈ [t₀, T], \]

\[ Wₛ(S(T),X,A(T),D,0) = 0 \text{ if } A(t) ≤ D \text{ for any } t ∈ [t₀, T], \]

\[ Wₛ(0,X,A(t),D,T−t) = 0 \text{ if } S(t)=0 \text{ for any } t ∈ [t₀, T], \]

and \[ 0 ≤ Wₛ(S(t),X,A(t),D,T−t) ≤ S(t) \text{ for all } t ∈ [t₀, T]. \]

Neither equation [6.3] nor the boundary conditions contain any preference-dependent terms. Accordingly, a necessary and sufficient restriction on the fundamental solution to [6.3] is that it cannot contain any preference dependent terms. Thus, equation [6.3] is solved using

⁴² It is instructive to note that the traditional Black-Scholes hedge is not riskless in the presence of credit risk and, therefore, unsatisfactory delta-hedges would be expected.
preference-free valuation. For this paradigm\textsuperscript{83}, the following solution for the value of Black-Scholes options subject to default risk can be found.

**Theorem 1:** Assume that economic agents require the same rate of return on perfect substitutes, that the term structure of interest rates is flat and constant, that markets are frictionless and complete, that trading is continuous, that non-satiation holds, that short sales are unrestricted, that the optioned asset price, \( S(t) \), and the value of the writer's assets, \( A(t) \), each follow time homogeneous geometric Brownian motion, that everyone agrees on \( \sigma \), \( \sigma' \), and \( \rho \), that \((r-1)\) is the discrete riskless rate of interest and \((d-1)\) is the payout rate of asset \( i (i=S,A) \), that no partial recoveries are made in the event of default, and that default is agreed upon by all investors to occur with probability one if \( A(t) \leq D \) for any \( t \in [t_0,T] \), then:

\[
W_n(S,X,A,D,t) = \Phi \left[ Sd_t^{-1}M(\phi x_1, x_2, \phi \rho S_A) - Xr^{-1}M(\phi (x_1-\sigma_x \sqrt{t}), (x_2-\sigma_x \rho S_A \sqrt{t}), \phi \rho S_A) \right]_{[0,4]} \\
- \Phi \left( \frac{D}{A} \right) [Sd_t^{-1} \frac{D}{A} \frac{\sigma^2}{\rho \sqrt{t}} M(\phi x_3, x_4, \phi \rho S_A) - Xr^{-1}M(\phi (x_3-\sigma_x \sqrt{t}), (x_4-\sigma_x \rho S_A \sqrt{t}), \phi \rho S_A)]
\]

where

\[
\Phi = \begin{cases} 
1 & \text{if a call option} \\
-1 & \text{if a put option}
\end{cases}
\]

\[
\gamma = 2[\ln(r/d_A)-.5\sigma_A^2]/[\sigma_A^2]
\]

\[
x_1 = [\ln(S/X)+\ln(r/d_A)+.5\sigma_A^2] / [\sigma_A \sqrt{t}]
\]

\[
x_2 = [\ln(A/D)+\ln(r/d_A)-.5(\sigma_A^2-2\sigma_x \rho S_A)^2] / [\sigma_A \sqrt{t}]
\]

\[
x_3 = x_1 + 2\ln(D/A)\rho_S A / [\sigma_A \sqrt{t}]
\]

\[
x_4 = x_2 + 2\ln(D/A) / [\sigma_A \sqrt{t}]
\]

and \( M(c_1,c_2,\rho) \) is the bivariate standard cumulative normal distribution function with upper limits of integration \( c_1 \) and \( c_2 \) and a correlation coefficient of \( \rho \).

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\textsuperscript{83} That is, the discounted expectations are taken under an equivalent martingale measure as mentioned above.

\textsuperscript{84} The formulation that follows assumes that \( A(t) \) is the price of a traded asset. If this assumption is found to be inaccurate, the model could be modified, in the spirit of Fischer (1978, p. 171), to incorporate this feature.
Proof: See Appendix VIA.\footnote{The valuation formula can also be verified from the down-and-out detached barrier option formula presented in Chapter V, Table V.C.a. Equation [6.4] can be seen to be equivalent to the down-and-out formula ([5.4]-[5.5]) listed in Table V.B.a when \( C=0 \), \( S_2(t)=A(t) \), \( \eta=1 \), and the rebate amount is zero.}

Equation [6.4] can be verified to satisfy the partial differential equation listed in [6.3]; see Appendix VIB. Moreover, observe that at the instant of default \( x_1=x_3 \) and \( x_2=x_4 \), which verifies a boundary condition.

VI.B.3 Relationship to the Black-Scholes Model and the Value of a Writer’s Option to Default

The impact of credit risk on the price of an option declines as the likelihood of default depreciates. That is, as the value of the writer’s assets goes to infinity and/or the default boundary goes to zero, it is easy to see that the vulnerable option value converges to the Black-Scholes formula.\footnote{As \( A\to\infty \) or \( D\to0 \), \( x_2\to\infty \), \( x_3\to\infty \), and \( x_4\to\infty \).} This can be seen more intuitively by decomposing the value of a vulnerable option into its respective parts.

If it is noted that

\[
\phi \left[ S d_s^{-1} M(\phi x_1, x_2, \phi \rho_{sA}) - X r^{-1} M(\phi(x_1-\sigma_s \sqrt{\tau}), (x_2-\sigma_s \rho_{sA} \sqrt{\tau}), \phi \rho_{sA}) \right] \\
= \phi \left[ S d_s^{-1} N(\phi x_1) - X r^{-1} N(\phi(x_1-\sigma_s \sqrt{\tau})) \right] \\
- \phi \left[ S d_s^{-1} M(\phi x_1, -x_2, -\phi \rho_{sA}) - X r^{-1} M(\phi(x_1-\sigma_s \sqrt{\tau}), -(x_2-\sigma_s \rho_{sA} \sqrt{\tau}), -\phi \rho_{sA}) \right],
\]

where \( N(q) \) is the standard normal cumulative distribution function with an upper limit of integration of \( q \), the value of a vulnerable Black-Scholes option is written as the value of an otherwise identical default-free Black-Scholes option less the value of the writer’s option to default. To see this, observe that the first bracketed term on the right-hand side of the equality in [6.5] is the standard Black-Scholes equation for an equity option. Combining equation [6.4] with equation [6.5] gives the value of the writer’s option to default as:
\[
\phi \left[ Sd_x^{-1} M(\phi x_1, -x_2, -\phi \rho_{SA}) - Xr^{-\tau} M(\phi (x_1 - \sigma S \sqrt{\tau}), -(x_2 - \sigma S \rho_{SA} \sqrt{\tau}), -\phi \rho_{SA}) \right] \\
+ \phi \left( \frac{D}{A} \right)^{\gamma} \left[ Sd_x^{-1} D^{\gamma} e^{\sigma^2 \tau^2} M(\phi x_3, x_4, \phi \rho_{SA}) - Xr^{-\tau} M(\phi (x_3 - \sigma S \sqrt{\tau}), (x_4 - \sigma S \rho_{SA} \sqrt{\tau}), \phi \rho_{SA}) \right] \tag{6.6}
\]

The economic interpretation of the writer's option to default is straightforward. The first (top) bracketed term in [6.6] is the current value of a path-independent option that pays off \( \max(\phi S(T) - \phi X, 0) \) if default occurs at \( T \) (i.e., \( A(T) \leq D \)). The second bracketed term is the current value of a path-dependent option that pays off \( \max(\phi S(T) - \phi X, 0) \) if default occurs prior to, but not at, \( T \) (i.e., \( A(t) \leq D \) for some \( t \in [t_n, T) \)).\(^7\) Hence, the current value of the default option is the discounted expected value of the option finishing in-the-money and default occurring in the closed interval \([t_n, T]\). Further note that the default option is guaranteed to have non-negative value since it is the sum of two non-negative terms.

Equation [6.6] clearly shows the "discount" embedded in vulnerable options due to the possibility of default. It should be clear that the current value of a vulnerable option can never exceed the value of an otherwise identical default-free option. In fact, since the default option cannot have negative value, the lower bound for the value of an intertemporal vulnerable option is the value of an otherwise identical default-free option. The upper bound for a vulnerable option coincides with the upper bound for a default-free option, the underlying asset price.

**VI.B.4 Properties of the Writer's Option to Default**

It is well-known that the Black-Scholes formulation is homogeneous of degree one with respect to \( S \) and \( X \). In similar spirit, the vulnerable formulation and the writer's option to default are linearly homogeneous with respect to \( S, X, A, \) and \( D \). It is also instructive to note that a Black-Scholes option written on \( S(t) \) is path independent. However, a vulnerable Black-Scholes option written on \( S(t) \) is path dependent with respect to \( A(t) \) and \( S(t) \). The path dependency of \( A(t) \) is obvious, whereas the path dependency of \( S(t) \) stems from the correlation between \( S(t) \) and \( A(t) \). Accordingly, a vulnerable Black-Scholes option is considered directly path dependent with

\(^7\) Path dependency means that the payoff space is dependent on some aspect of the paths taken by \( S(t) \) and \( A(t) \) to reach specified states.
respect to $A(t)$ and only indirectly path dependent with respect to $S(t)$ (i.e., this path dependency disappears if the two assets are uncorrelated).\textsuperscript{**}

Figures VI.B.4.a through VI.B.4.l plot vulnerable and default-free option prices against changes in the input values. Moreover, since the value of the default option is defined as the difference between the Black-Scholes value and the vulnerable value, Figures VI.B.4.a through VI.B.4.l illustrate how the value of the default option changes as the input values change. The initial input values, unless otherwise stated, are specified as $S = 50$, $X = 50$, $A = 100$, $D = 90$, $\tau = .40$ years, $\sigma_s = .30$ per annum, $\sigma_a = .15$ per annum, $\rho_{A,S} = .2$ per annum, $r = 1.06$ per annum, $d_s = 1.05$ per annum, and $d_a = 1.02$ per annum. Figures VI.B.4.a and VI.B.4.b show the deeper the option is initially in-the-money, the larger the default premium. Notice that the vulnerable option increases in value as the optioned asset price moves in-the-money. This is because the default-free option is increasing in value at a faster rate than the default option. Figures VI.B.4.c through VI.B.4.h reveal another interesting result. The default premium is larger the longer the time to maturity, the more volatile the spot price, and the more volatile the writer's assets. This behavior is explained by the limited liability feature of the default option. Also note that when Figure VI.B.4.e (VI.B.4.f) is compared with Figure VI.B.4.g (VI.B.4.h), the value of the at-the-money vulnerable option is nearly linear in $\sigma_s^2$ but non-linear in $\sigma_a^2$. Figure VI.B.4.i (VI.B.4.j) demonstrates that a call (put) default option increases in value as the correlation between the optioned asset and the writer's assets decrease (increase).\textsuperscript{**} The intuition for this result is that these are currently at-the-money options and a high correlation implies a strong relationship between the probability that the option will finish in-the-money and the probability that default will not occur in the interval $[t_w, T]$. Furthermore, as Johnson and Stulz (1978) argue, this implies a smaller default premium if the vulnerable option is written by a hedger as opposed to a speculator. This is because $\rho_{A,S}$ is more likely to be positive for hedgers than speculators. Figures VI.B.4.k and VI.B.4.l show that the value of the default option increases as the default

\textsuperscript{**} The path dependency with respect to $A(t)$ disappears, of course, when the default boundary is zero.

\textsuperscript{**} Johnson and Stulz (1987, p. 278) found a similar result for options subject to expiration default risk.
Figure VI.B.4.a
Value of Default Option as a Function of the Current Stock Price

Figure VI.B.4.b
Value of Default Option as a Function of the Current Stock Price

Figure VI.B.4.c
Value of Default Option as a Function of the Time to Expiration

Figure VI.B.4.d
Value of Default Option as a Function of the Time to Expiration
Figure VI.B.4.e
Value of Default Option as a Function of the Volatility of S

Figure VI.B.4.f
Value of Default Option as a Function of the Volatility of S

Figure VI.B.4.g
Value of Default Option as a Function of the Volatility of A

Figure VI.B.4.h
Value of Default Option as a Function of the Volatility of A
Figure VLB.4J
Value of Default Option as a Function of the Correlation Coefficient

Figure VLB.4J
Value of Default Option as a Function of the Correlation Coefficient

Figure VLB.4k
Value of Default Option as a Function of the Default Boundary

Figure VLB.4J
Value of Default Option as a Function of the Default Boundary
boundary decreases (and/or the value of the writer's assets increase).\textsuperscript{90} In other words, as the probability of default declines, so does the default premium; just as one would expect. This implies that since the acquisition of the underlying asset for covered writers increases the value of his/her assets, a covered vulnerable option is worth less than a default-free option but worth more than an uncovered vulnerable option.

To gain additional insight into the hedging differences between vulnerable options and default-free options, the comparative statics are examined. Unfortunately however, the derivatives cannot generally be unambiguously signed. This can lead to some atypical hedging results. For example, it is well-known that the delta and the gamma of a Black-Scholes call option are strictly non-negative. In contrast, it is theoretically possible, when default is imminent, for an otherwise identical vulnerable call option to have a negative delta and gamma.\textsuperscript{91} Thus, while traders of default-free options are accustomed to continually increasing the size of their hedge position, traders of vulnerable options may have to actually reverse their position at some time. It should be noted, however, that under most circumstances, vulnerable call options also have a positive delta and gamma.

To further highlight the hedging properties of the default option, numerical examples can again be used as benchmark from which the sensitivity of vulnerable model can be compared to the sensitivity of the default-free model. The same input values will continue to be used (i.e., unless otherwise stated, $S=50$, $X=50$, $A=100$, $D=90$, $\tau=.40$, $\sigma_s=.30$, $\epsilon_\alpha=.15$, $\rho_{\alpha,s}=.2$, $r=1.06$, $d_s=1.05$, and $d_A=1.02$).

Figures VI.B.4.m and VI.B.4.n compare the delta of a vulnerable option as a function of the current stock price with the delta of a default-free option. For deep-out-of-the-money options it can be seen that the default-free delta and the vulnerable delta are virtually identical (in the graphs they appear to be identical). For deep in-the-money options the default-free delta approaches one (in absolute value) but the vulnerable delta does not - it levels off at about .8 (in

\textsuperscript{90} Graphs of the value of the default option as a function of the writer's assets are virtually identical to Figures VI.B.4.k and VI.B.4.l; therefore, to conserve space, they are not included.

\textsuperscript{91} The easiest way to prove this is to let $A$ be extremely close to $D$, $\rho_{A,A}=0$, $r=d_A$, and $\sigma_\alpha$ to be near 1.0.
Figure VI.B.4.m
The Delta of a Default Option as a Function of the Current Stock Price

Figure VI.B.4.n
The Delta of a Default Option as a Function of the Current Stock Price

Figure VI.B.4.o
The Delta of a Default Option as a Function of the Time to Expiration

Figure VI.B.4.p
The Delta of a Default Option as a Function of the Time to Expiration
absolute value). This implies that the delta of the default option is approximately two (in absolute value) for deep in-the-money options. A comparison of the deltas as a function of the time to expiration is offered in Figures VI.B.4.o and VI.B.4.p and another unusual result is uncovered. Longer term default-free call options require investors to increase the initial size of their hedge position while longer term vulnerable call options allow investors to decrease the size of their initial position. This result is not completely counter-intuitive. It suggests that the delta of the call default option, similar to the default-free call delta, is larger for longer maturing options.

The gamma for Black-Scholes and vulnerable options are identical for calls and puts. Figure VI.B.4.q, therefore, compares the gammas as a function of the spot price. Observe that both gammas reach their highest value when $S = X(r/d_2)^{-1} \exp[-1.5e^2t]$. However, for the selected parameter values, the vulnerable gamma never exceeds the default-free gamma. Figure VI.B.4.r shows that the vulnerable and default-free gammas converge, and become very large, as expiration approaches.

The elasticity of the vulnerable option (with respect to the spot price) as a function of time to expiration is virtually indistinguishable from its default-free counterpart. This can be seen in Figures VI.B.4.s and VI.B.4.t. The options, in this case, are clearly levered securities. This does not imply, however, that vulnerable and default-free options are always equally sensitive (in percentage terms) to a small percentage change in the optioned asset price. This finding is purely a result of the input values used in this example.

Our numerical analysis reinforces and strengthens Johnson and Stulz’s (1987) finding that vulnerable options can have unusual hedging properties. In fact, unlike Johnson and Stulz, none of the comparative statics for the intertemporal model can be signed in general. This can create several atypical hedging strategies. As an extreme example, vulnerable calls need not be levered

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92 The elasticities as functions of the current stock price are not presented because no graphic differences existed with this example.
Figure VI.B.4.g
The Gamma of a Default Option as a Function of the Current Stock Price

Figure VI.B.4.f
The Gamma of a Default Option as a Function of the Time to Expiration

Figure VI.B.4.e
The Elasticity of a Default Option as a Function of the Time to Expiration

Figure VI.B.4.d
The Elasticity of a Default Option as a Function of the Time to Expiration
securities, nor their price a convex function of the optioned asset price.\footnote{Jarrow and Rudd (1983, p. 107) have proven that a sufficient condition for establishing that a call option is a levered security is to prove that its price is a convex function of the spot price. Merton (1973, p. 176) has shown that certain call options can be levered securities even though the call price is a concave function of the stock price.}

VI.B.5 Relation to the Hull and White (1994) Model

Without any loss of generality, the valuation solution presented in [6.4] is also used to value (1) foreign currency options, by letting $S$ denote the foreign currency exchange rate and $d_s$ be the foreign riskless interest rate, $r_f$, or (2) futures options, by letting $d_s$ equal $r$. Table VI.B.5.a provides some numerical values for intertemporal vulnerable European foreign currency call options when $S=X=100$, $A=100$, $\sigma_s=1.5$, $\sigma_h=.05$, $\ln(r) = .05$, $\ln(r_f)=\ln(d_s) = .05$, and $\ln(d_h)=0$. Observe that the default-free option value is never less than the vulnerable option value. Further, note that the call default premium increases as the default boundary rises or the correlation coefficient falls (i.e., becomes closer to -1). These results are consistent with the graphic analysis.

The input values for Table VI.B.5.a are chosen to coincide with those specified by Hull and White (1994, pp. 10-13). Using the binomial model, Hull and White numerically estimate the percentage reduction in option value resulting from the presence of intertemporal credit risk for European and American options. Equation [6.4] provides an exact solution to the European option values estimated by Hull and White. The final column in Table VI.B.5.a presents the actual percentage decrease in value versus the estimated decrease. Observe that the actual percentage decrease in value always exceeds the estimated percentage decrease. This finding is probably explained by the imprecision of the binomial method with only 100 time steps (as used by Hull and White).\footnote{Optimal accuracy for this problem is achieved by having one of the "down" nodes in the binomial tree for the writer’s assets rest exactly on the default boundary. If a node is not on the default boundary one of the "down" option prices may be non-zero when it should be zero. This will overestimate the foreign currency call price and underestimate the percentage reduction in value due to default risk.}

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Chapter VI
Table VI.B.5.a
Default-Free and Vulnerable Black-Scholes Options
Values for European Foreign Currency Call Options

This table contains default-free and vulnerable Black-Scholes option values (assuming no recoveries) for one year European foreign currency call options. In addition, the value of the writer's option to default is listed, as well as, the percent reduction in value arising from the presence of credit risk. The parameter values for each option, unless otherwise specified, are \( S=X=100, A=100, \tau=1, \sigma_S=.15, \sigma_X=.05, \ln(\tau)=-.05, \ln(\tau_2)=\ln(d_2)=.05, \) and \( \ln(d_4)=0. \) The parameter values are chosen to coincide with those specified by Hull and White (1994).

<table>
<thead>
<tr>
<th>( \rho_{s,A} )</th>
<th>D</th>
<th>( C_{BS}(S,X,\tau) )</th>
<th>( C_{V}(S,X,\tau) )</th>
<th>DIF*</th>
<th>% DIF*</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.80</td>
<td>$94.00</td>
<td>$5.69</td>
<td>$4.58</td>
<td>$1.11</td>
<td>19.52 (17.86)</td>
</tr>
<tr>
<td>-0.80</td>
<td>$96.00</td>
<td>$5.69</td>
<td>$3.29</td>
<td>$2.40</td>
<td>42.19 (40.87)</td>
</tr>
<tr>
<td>-0.40</td>
<td>$94.00</td>
<td>$5.69</td>
<td>$5.07</td>
<td>$0.62</td>
<td>10.86 (9.48)</td>
</tr>
<tr>
<td>-0.40</td>
<td>$96.00</td>
<td>$5.69</td>
<td>$4.16</td>
<td>$1.52</td>
<td>26.77 (23.88)</td>
</tr>
<tr>
<td>0.00</td>
<td>$94.00</td>
<td>$5.69</td>
<td>$5.41</td>
<td>$0.28</td>
<td>4.89 (4.44)</td>
</tr>
<tr>
<td>0.00</td>
<td>$96.00</td>
<td>$5.69</td>
<td>$4.83</td>
<td>$0.86</td>
<td>15.11 (13.95)</td>
</tr>
<tr>
<td>0.40</td>
<td>$94.00</td>
<td>$5.69</td>
<td>$5.68</td>
<td>$0.01</td>
<td>1.44 (1.21)</td>
</tr>
<tr>
<td>0.40</td>
<td>$96.00</td>
<td>$5.69</td>
<td>$5.30</td>
<td>$0.39</td>
<td>6.84 (5.69)</td>
</tr>
<tr>
<td>0.80</td>
<td>$94.00</td>
<td>$5.69</td>
<td>$5.69</td>
<td>$0.00</td>
<td>0.17 (0.13)</td>
</tr>
<tr>
<td>0.80</td>
<td>$96.00</td>
<td>$5.69</td>
<td>$5.57</td>
<td>$0.12</td>
<td>2.07 (1.81)</td>
</tr>
</tbody>
</table>

*DIF represents the difference between the default-free option value and vulnerable value. DIF, therefore, represents the value of the writer's option to default. % DIF denotes the percentage reduction in option value arising from default risk. The % DIF values estimated numerically by Hull and White (1994, p. 31) are reproduced in parentheses.
VI.B.6 Relation to the Cox and Rubinstein (1985) Paradigm

For many customized options the writer's assets will differ from the optioned asset. However, for many corporate liabilities with option-like payoffs the two are equivalent. The intertemporal vulnerable model developed in equation [6.4] has the flexibility to incorporate this special case. To illustrate, the Cox and Rubinstein (1985, p. 411) paradigm is considered.

Suppose a levered firm has zero coupon (risky) bonds outstanding, all of identical maturity, that have a safety covenant that gives the bondholders the right to reorganize the firm, under their ownership, if the value of the firm falls to a prespecified level, D (a constant). Cox and Rubinstein show that the value of this levered firm can be partitioned into two parts: the equity position is analogous to a European down-and-out call option written on the value of the firm (and struck at the par value of the bonds) and the bondholders' position is analogous to holding the risky asset, the levered firm, and writing the down-and-out call option granted to the shareholders. This formulation is true invariant to the number of other assets held by the bondholders. Therefore, consider the case in which the bondholders have no other holdings.

For this scenario the option writer's assets and the optioned asset are identical. This suggests that equation [6.4] should simplify to the down-and-out valuation formula when S = A, \( \sigma_s = \sigma_A = \sqrt{\sigma_{sA}} \), \( \rho_{sA} = 1 \), and \( d_s = d_A \). It is verified in Appendix V1C that when S = A, \( \sigma_s = \sigma_A = \sqrt{\sigma_{sA}} \), \( \rho_{sA} = 1 \), and \( d_s = d_A \), equation [6.4] agrees with the down-and-out valuation formula derived from first principles by Rubinstein and Reiner (1991a).

VI.B.7 One-Sided Vulnerable Forward Contracts

It is well-known that the payoff from a default-free forward contract can be replicated with a portfolio of Black-Scholes options. This parity result is used to determine the forward price in continuous time. The intertemporal vulnerable model developed in equation [6.4] is similarly used to determine the vulnerable forward price in continuous time.

One-sided vulnerable forward contracts are defined as forward contracts in which one party is exposed to credit risk. For convenience the contract under consideration is assumed to be a

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95 The original valuation of bonds embedded with a down-and-out call option is derived in Black and Cox (1976, p. 355).
long forward position.

Imagine a portfolio that contains a one-sided vulnerable forward contract that is written on $S$, matures at $T$, and has a forward price of $X$, where $X$ is chosen so the value of the contract would be zero when written. It is understood by both parties that this forward contract will terminate at $t$ (without any recoveries being made) if $A(t) \leq D$ for any $t \in [t_0, T]$, where $A(t)$ is the value of the counterparty's assets at time $t$. 66 Thus, a one-sided long vulnerable forward position either expires worthless (possibly prematurely) or pays off $[S(T) - X]$ at expiration. Also consider a second portfolio which contains one long European intertemporal vulnerable Black-Scholes call option, whose value is given by equation [6.4], and one otherwise identical short vulnerable put option. The options are written on the same asset and have identical expiration dates as the vulnerable forward contract.

Both portfolios pay off $[S(T) - X]$ at expiration if $A(t) > D$ for all $t \in [t_0, T]$ or simultaneously expire worthless otherwise. In the absence of arbitrage, two portfolios that have equivalent payoffs in all states of the world must have equivalent initial values. Using the fact that: $N(q_1) = M(q_1, q_2, \rho) + M(q_1, -q_2, -\rho)$, the vulnerable forward price in continuous time is explicitly determined as $X = \psi S(d_\rho/r) \gamma$, where

$$\psi = \frac{\left[N(y_2) - N(y_4) \frac{(D/A)^{\gamma}}{(\sigma^2 + \sigma^2 A)}\right]}{\left[N(y_2) - \sigma A \sqrt{\gamma} - N(y_4) - \sigma A \sqrt{\gamma} \frac{(D/A)^{\gamma}}{(\sigma^2 + \sigma^2 A)}\right]}.$$ 

This conclusion is interesting because in the absence of credit risk $X = S(d_\rho/r) \gamma$ (i.e., $\psi = 1$ because as $A \to \infty$ or $D \to 0$, $x_\gamma \to \infty$ and $x_\gamma \to \infty$), but in the presence of credit risk $0 < \psi < \infty$. It is quickly verified that when $\gamma \geq 1$, $\psi \geq 1$ if and only if $\rho_{S, A} \geq 0$ and $0 < \psi < 1$ if and only if $\rho_{S, A} < 0$. 97 In other words, the vulnerable forward price can be greater than, equal to, or less than the default-

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66 Practitioners often measure the default risk for a bank holding company (BHC) by the quality of their bond rating. Therefore, as an example of a one-sided vulnerable forward contract, consider a forward contract entered into by a AAA rated BHC with a AA rated counterparty. It may reasonably be assumed that the AAA rated BHC has a negligible probability of default and therefore, is assumed riskless.

97 The requirement that $\gamma \geq 1$ implies $ln(r/d_A) \geq \sigma^2_A$. This condition is likely to hold in general, especially because $d_A$ is equal to unity for many option writers.
free forward price. The intuition for this result is that if the underlying asset and the counterparty’s assets are positively (negatively) correlated, default is most (least) likely to occur when the contract has negative value to the long default-free party. To see this, observe that if, at some future date, the underlying asset price has fallen relative to its initial level, the vulnerable forward contract will have negative value to the long party. However, if $\rho_{S,A} \geq 0$, the possibility of the counterparty defaulting is greatest when the contract is a liability to the long party.

VI.C The Vulnerable Black-Scholes Pricing Model When the Default Boundary is Stochastic

The continuous default boundary, $D$, in equation [6.4], is assumed to be constant. It is shown in this section that due to the reproductive property of lognormally distributed random variables, only slight modifications are needed to generalize our results. The intertemporal vulnerable model is generalized to include a stochastic default boundary. The relationship between the intertemporal vulnerable model with a stochastic default boundary and the Johnson and Stulz (1987, p. 277) model is then discussed.

Unless otherwise specified, the general framework set forth in Section II.B.1 is continued to be used with the exception of $D$, the default boundary, being constant. In this section, $D(t)$ is allowed to be a stochastic variable whose price dynamics are characterized by

$$dD(t) = \mu_D D \Delta t + \sigma_D D \Delta Z_D(t),$$

where $\{Z_D(t), t \in [t_n, T]\}$ is a standard Brownian motion, $D(t_n) = D$, and $\mu_D$ and $\sigma_D$ are, respectively, the instantaneous expected return and standard deviation from holding asset $D$. Define $\rho_{S,D} = \text{corr}(Z_S(t), Z_D(t))$ and $\rho_{A,D} = \text{corr}(Z_A(t), Z_D(t))$, where $\rho_{S,D}$ and $\rho_{A,D}$, like $\rho_{S,A}$, are assumed constant. It is assumed that $D(t)$ is the price of a traded asset with a continuous payout rate of $(d_D-1)$. The notation $W_t(S,X,A,D,t)$ continues to be used to represent the current value of the intertemporal vulnerable pricing function.

The value of this contingent claim is dependent on three underlying state variables. Therefore, simple hedging arguments are used to show that the appropriate pricing equation for this function (for all $\infty > A(t) > D(t), t \in [t_n, T]$) is
\[
\ln(r)W(.) = \sum_i P_i \ln(r/d_i) \frac{\partial W(.)}{\partial P_i} - \frac{\partial W(.)}{\partial \tau} + \sum_{i} \sigma_i \sigma_P \rho_i P_i \frac{\partial^2 W(.)}{\partial P_i \partial \tau} \quad (i = S, A, D) \quad [6.7]
\]

where \( P_i \) denotes the initial price of asset \( i \), subject to the boundary conditions:

\[
W_s(S,T,X,A(T),D(T),0) = \max(0, \phi S(T) - \phi X) \text{ if } A(t) > D(t) \text{ for all } t \in [t_o, T],
\]

\[
W_s(S,T,X,A(T),D(T),0) = 0 \text{ if } A(t) \leq D(t) \text{ for any } t \in [t_o, T],
\]

\[
W_s(0,X,A(t),D(t),T-t) = 0 \text{ if } S(t) = 0 \text{ for any } t \in [t_o, T],
\]

and \( 0 \leq W_s(S,T,X,A(t),D(t),T-t) \leq S(t) \text{ for all } t \in [t_o, T]. \)

For this specification of the model, consider a change of numeraire for the asset triggering default. That is, let \( B(t) = A(t)/D(t) \) (where \( B(t) = B \)), redefine the default boundary to be static at a level of unity, and rewrite the value of this option as \( W_s(S,X,B,1,\tau) \). Then observe from Itô's lemma that the relative asset price, \( B(t) \), also obeys the stochastic differential equation

\[
dB(t) = \mu_B B dt + \sigma_B B dZ_B(t)
\]

where

\[
\mu_B = \ln(d_P/d_A) - (\sigma_A \sigma_D P_{A,D}^2 - \sigma_D^2),
\]

\[
\sigma_B dZ_B(t) = \sigma_A dZ_A(t) - \sigma_D dZ_D(t),
\]

\[
\sigma_B^2 = \sigma_A^2 + \sigma_D^2 - 2 \sigma_A \sigma_D \rho_{A,D}
\]

and the continuously compounded return on this asset follows arithmetic Brownian motion:

\[
d\ln(B(t)/B) = (\ln(d_P/d_A) - \frac{1}{2}(\sigma_A^2 + \sigma_D^2)) dt + \sigma_B dZ_B(t).
\]

Substituting these changes into the derivation outlined in Appendix VIA, the solution for the intertemporal vulnerable Black-Scholes options with a stochastic default boundary is written as:

\[
W_s(S,X,A,D,\tau) = W_s(S,X,B,1,\tau)
\]

\[
= \phi \left[ S d_s^{-1} M(\phi x_1, \xi_2, \phi \rho_{S,B}) - X r^{-\tau} M(\phi(x_1 - \sigma_s \sqrt{\tau}), (\xi_2 - \sigma_s \rho_{S,B} \sqrt{\tau}), \phi \rho_{S,B}) \right]
\]

\[
- \phi \left[ \frac{D}{A} \right]^t \left[ S d_s^{-1} \left( \frac{D}{A} \right) 2 \sigma_x^2 \sigma_s^2 M(\phi x_2, \xi_4, \phi \rho_{S,B}) - X r^{-\tau} M(\phi(x_2 - \sigma_s \sqrt{\tau}), (\xi_4 - \sigma_s \rho_{S,B} \sqrt{\tau}), \phi \rho_{S,B}) \right]\]
\]

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where

\[
\hat{\gamma} = 2 \frac{\ln(d_D/d_A) - 5(\sigma_A^2 - \sigma_D^2)}{[\sigma_A^2]} \\
\hat{x}_2 = \frac{\ln(A/D) + (\ln(d_D/d_A) - 5(\sigma_A^2 - \sigma_D^2 - 2\sigma_{S,D}))\tau}{[\sigma_A\sqrt{\tau}]}; \\
\hat{x}_3 = x_1 + 2\ln(D/A)\rho_{S,D}/[\sigma_A\sqrt{\tau}] \\
\hat{x}_4 = \hat{x}_2 + 2\ln(D/A)/[\sigma_A\sqrt{\tau}] \\
\rho_{S,D} = \frac{\text{Cov}(\ln(S(T)), \ln(A(T)/D(T)))/[\sigma_A\sigma_{S,D}]}{[\sigma_A]} = [\sigma_A\rho_{A,S} - \sigma_D\rho_{S,D}]/[\sigma_A]
\]

and all other notation is as previously defined. Equation [6.8] satisfies [6.7]. Notice that equations [6.4] and [6.8] are identical when D is constant (i.e., when \(\mu_D = \sigma_D = \sigma_{A,D} = 0\)).

Just as in Section VI.B.1, the value of this vulnerable option is written as the sum of an otherwise default-free Black-Scholes option less the value of the writer's option to default. It is readily confirmed that the default option value is:

\[
\phi \left[ S_d S^{-1} \left( \frac{D}{A} \right)^{\gamma} \right] M(\hat{x}_1, -\hat{x}_2, -\phi \rho_{S,D}) - X r^{-\gamma} M(\hat{x}_1 - \sigma_A \sqrt{\tau}, -\hat{x}_2 - \sigma_D \rho_{S,D} \sqrt{\tau}, -\phi \rho_{S,D}) \\
+ \phi \left[ S_d S^{-1} \left( \frac{D}{A} \right)^{\gamma} \right] 2\sigma_A \rho_{S,D}^2 \left[ M(\hat{x}_3, \hat{x}_4, \phi \rho_{S,D}) - X r^{-\gamma} M(\hat{x}_3 - \sigma_A \sqrt{\tau}, \hat{x}_4 - \sigma_D \rho_{S,D} \sqrt{\tau}, \phi \rho_{S,D}) \right].
\]

The economic interpretation of the writer's option to default is also similar to before. The first (top) bracketed term is the current value of a path-independent option that pays off \(\max(\phi S(T) - \phi X, 0)\) if default occurs at \(T\) (i.e., \(B(T) = 1\)) and the second bracketed term is the current value of a path-dependent option that pays off \(\max(\phi S(T) - \phi X, 0)\) if default occurs prior to, but not at, \(T\) (i.e., \(B(t) \leq 1\) for some \(t \in [t_n, T]\)). Thus, the value of the default option, again, is the discounted expected value of the option finishing in-the-money and default occurring.

More importantly however, by decomposing this model into its respective parts, the relationship between the stochastic default boundary model and the Johnson and Stulz (1987, p. 277) model is established. It is shown that in one special case the two models are identical. This relationship is established for the vulnerable call option.

Recall in the Johnson and Stulz framework default can occur only at expiration when the vulnerable call option pays off the minimum of the value of the writer's assets and the intrinsic value. Their analytical solution is written as the sum of two double integral expressions, "the first double integral corresponds to the expected payoff if the option writer is bankrupt at
maturity, while the second integral corresponds to the expected payoff in the other case" (p. 278). To make their framework correspond with ours, consider only their second double integral expression, the current value of an option that pays off \( \max(S(T)-X,0) \) if \( V(T) > [S(T)-X] \). That is, ignore partial recoveries. An analysis of their payoff function then shows that when \( X=0 \) they have a closed-form solution for the value of an option that pays off the asset \( S(T) \) if \( V(T) > S(T) \) and zero otherwise. Or restated, default occurs if \( V(T) \leq S(T) \). In our model, default is defined to occur if \( V(t) \leq D(t) \) for any \( t \in [t_0,T] \). However, if the second (lower) bracketed term in [6.9] is ignored, default is allowed to occur only at expiration if \( V(T) \leq D(T) \). Define \( D(T)=S(T) \) to make the frameworks comparable. As \( X \rightarrow 0 \) (and consequently, \( x_1 \) and \( x_2 \rightarrow \infty \)), it is seen that the two models converge in value. Only in this highly restrictive context, however, are the two models identical.

Prior to closing this section, a caveat is offered on the use of the vulnerable Black-Scholes formulation with a stochastic default boundary. The reader is cautioned against defining \( D(t) \) as the value of the writer's liability portfolio at time \( t \) (i.e., defining the default boundary as zero net worth). This structure of the model presents a theoretical problem. Since the option is, itself, an outstanding liability for the writer, it is inconsistent to define the value of an option as a function of the value of the writer's liability portfolio. For practical purposes, however, if the contingent claim value accounts for only a negligible percentage of the overall value of the writer's liability portfolio\(^*\), equation [6.8] provides a reasonable approximation for the value of the option under these conditions.

VI.D Partial Recovery Options

The general framework outlined in Sections VI.B and VI.C can easily be extended to include partial recoveries. When partial recoveries are allowed, the total value of the vulnerable option is then determined as equation [6.4] (or equation [6.8]) plus the current value of the partial recovery option. The reader will recognize that the partial recovery options discussed in this section are simply the "out" rebate option pricing formulas derived in Chapters III and IV.

\(^*\) This structure of default is recognized as being similar to the "zero-correlation case" considered by Hull and White (1994, p. 13).
First, consider the case in which the vulnerable option with a static default boundary is secured by a prespecified margin deposit of $R(t)$. Define $\tau^*$ as the first passage time of $A(t)$ to $D$ and let $I_1(R(\tau^*), \tau)$ represent the current value of this partial recovery option. $I_1(R(\tau^*), \tau)$ has a deterministic payoff at an uncertain time. That is, $I_1(R(\tau^*), \tau)$ has a payoff of $R(t)$ at the first passage time (i.e., the option pays off $R(t)$ the instant default occurs) if default occurs in the interval $[t_0, T]$. To make the discussion as general as possible, allow the margin deposit to change exponentially with time to expiry. Specifically, let $R(t) = R e^{t\theta - \frac{1}{2} \sigma^2 t^2}$ denote a time-dependent deterministic margin deposit that changes exponentially with calendar time at rate $\theta$, where $R$ and $\theta$ are prespecified non-negative constants and $(\tau^* - \tau)$ represents the time remaining until maturity of the option. Define $\xi$ to be a binary variable that takes the value of one if the margin deposit is an increasing function of time, and negative one if the deposit is decreasing over time. If the margin deposit is constant over time, $\theta$ would be defined as zero and the specification of $\xi$ would be irrelevant.

The solution to this partial recovery option must satisfy the one asset fundamental partial differential equation

$$
\ln(r) I_1(\cdot) = \ln(r/d_A) A \frac{\partial I_1(\cdot)}{\partial A} - \frac{\partial I_1(\cdot)}{\partial \tau} + 0.5 \sigma^2 A^2 \frac{\partial^2 I_1(\cdot)}{\partial A^2}, \quad [6.10]
$$

which is defined over all $\infty > A(t) > D$ for $t \in [t_0, T]$, subject to $I_1(R(\tau^*), \tau - \tau^*) = R e^{t\theta - \frac{1}{2} \sigma^2 t^2}$ for $\tau^* \leq \tau$ and $I_1(R(\tau^*), \tau - \tau^*) = 0$ for $\tau^* > \tau$. By evaluating the discounted expected payoff, the solution to [6.10] can be obtained as:

---

$^9$ $I_1(\cdot)$ is chosen to represent the value of this (first) function because this option acts as an indicator function in terms of monitoring whether the writer is in a state of default or not.

$^{100}$ It is highly unlikely that a margin deposit would ever be decreasing over time; however, it is possible. In a related valuation problem, Merton (1973, p. 175, footnote 69) discusses a down-and-out option in which the rebate decreases over time. Presumably this is because the option is an increasing function of time to expiration, and therefore, the value of the option and the size of the rebate decrease as expiration approaches.
\[ I_1(R(\tau^*),\tau) = Re^{-\tau^*} \int_0^{\tau^*} e^{\tau^*} h(\tau^*) d\tau^* \]

\[ = Re^{-\tau^*} \left[ \left( \frac{D}{A} \right)^{5r^{*}m} N(x_2) + \left( \frac{D}{A} \right)^{5r^{*}m} N(x_2-2m\sigma_A/\sqrt{\tau}) \right] \]  

[6.11]

where

\[ h(\tau^*) = \frac{\ln(A/D)}{\sigma_A\tau^*\sqrt{2\pi\tau^*}} \exp \left\{ -\frac{1}{2} \left( \frac{\ln(A/D)+[\ln(\tau/d_A)-.5\sigma_A^2] \tau^*}{\sigma_A/\sqrt{\tau}} \right)^2 \right\} \]

\[ x_2 = \frac{[\ln(D/A)+m\sigma_A^2\tau^*]/[\sigma_A(\sqrt{\tau})]}{m = \sqrt{(.5\gamma\sigma_A)^2+2(\ln(\tau)-\xi\theta)/[\sigma_A]}} \]

\( h(\tau^*) \) is the first passage probability density function of \( A(t) \) to \( D \) and all other notation is as previously defined.\(^{101}\) Equation [6.11] can easily be verified to satisfy [6.10] (and the boundary conditions) by substitution; see Appendix VID.

As a second possible scenario, suppose that it is prespecified that the holder of the vulnerable option (with a static default boundary) receives \( \alpha \) (a constant) percent of the value of the writer’s assets at the time of default.\(^{102}\) Define \( I_2(\alpha A, \tau) \) as the current value of this partial recovery option. \( I_2(\alpha A, \tau) \), like \( I_1(R(\tau^*), \tau) \), has a deterministic payoff at an uncertain time. The payoff is deterministic because the value of the writer’s assets at the time of default must be equal to \( D \). \( I_2(\alpha A, \tau) \) must satisfy [6.10] for \( D < A(t) < \infty \) subject to \( I_2(\alpha A, \tau^* \tau) = A(\tau^* \tau) = D \) for \( \tau^* \leq \tau \) and \( I_2(\alpha A, \tau^* \tau) = 0 \) for \( \tau^* > \tau \). Accordingly,

\[ I_2(\alpha A, \tau) = \alpha D \int_0^{\tau^*} h(\tau^*) d\tau^* = \alpha D \left[ \left( \frac{D}{A} \right)^{5r^{*}m} N(x_2) + \left( \frac{D}{A} \right)^{5r^{*}m} N(x_2-2m^*\sigma_A/d_{\sqrt{\tau}}) \right] \]  

[6.12]

where \( m^* = \left( .5\gamma\sigma_A \right)^2 - 2 \ln(\tau)/[\sigma_A] \) and all other notation is as previously defined.

Thus far, only partial recovery options for the intertemporal vulnerable option pricing

\(^{101}\) This first passage density is derived in Rich (1994).

\(^{102}\) This recovery specification is similar to that specified by Longstaff and Schwartz (1992) in their valuation of risky debt.
model with a static default boundary have been considered. When the size of partial recovery is specified to be \( \alpha \) percent of the value of the writer's at the time of default (the second scenario), the valuation problem is shown to be trivial (after considering the first scenario). However, when the partial recovery is specified to be \( \alpha \) percent of the value of the writer's at the time of default, and the default boundary is allowed to be stochastic, the valuation problem becomes conceptually different. This is because the payment of such a partial recovery option is uncertain and occurs at an uncertain time. This option, which has a random payoff at a random time, is referred to as a first passage option.

Define \( I_3(\alpha A, D, \tau) \) as the current value of a first passage option that has a payoff of \( A(\tau^*) \) at the first passage time of \( A(t) \) to \( D(t) \) (\( t \in [t_0, T] \)). \( I_3(.) \) must satisfy the two asset partial differential equation listed in [6.3], with \( I_3(.) = H(.) \), \( P_1 = D \), and \( P_2 = A \), subject to \( I_3(\alpha A(\tau^-), A(\tau^-), \tau^-) = \alpha A(\tau^-) \) for \( \tau \geq \tau^- \) and \( I_3(\alpha A(\tau^-), A(\tau^-), \tau^-) = 0 \) for \( \tau < \tau^- \). Because of linear homogeneity, one can write \( I_3(\alpha A, D, \tau) = \alpha AI_4(Q, \tau) \), where \( Q = (D/\alpha A) \), and transform this threedimensional problem into a two-dimensional one. If it is noted that

\[
\frac{\partial I_4(\cdot)}{\partial D} = \frac{\partial I_4(\cdot)}{\partial Q}, \quad \frac{\partial^2 I_4(\cdot)}{\partial D^2} = \frac{\partial I_4(\cdot)}{\partial Q^2} \left( \frac{1}{\alpha A} \right), \quad \frac{\partial I_4(\cdot)}{\partial D} = -\frac{\partial I_4(\cdot)}{\partial Q} \left( \frac{Q}{A} \right), \quad \frac{\partial^2 I_4(\cdot)}{\partial A^2} = \frac{\partial I_4(\cdot)}{\partial Q^2} \left( \frac{Q^2 \alpha}{A} \right),
\]

equation [6.3] can be rewritten as

\[
\alpha A \left[ \ln(d_A) I_4(\cdot) \right] = \alpha A \left[ \ln(d_A/d_D) Q \frac{\partial I_4(\cdot)}{\partial Q} - \frac{\partial I_4(\cdot)}{\partial \tau} + 0.5 \left[ \sigma_d^2 + \sigma_A^2 - 2 \sigma_d \sigma_A \rho_{A,D} \right] Q^2 \frac{\partial^2 I_4(\cdot)}{\partial Q^2} \right] \tag{6.13}
\]

which is defined over all \( \infty > Q(t) > 1 \) for \( t \in [t_0, T] \), subject to \( I_4(Q(\tau^-), A(\tau^-)) = 1 \) for \( \tau^- \leq \tau \) and \( I_4(Q(\tau^-), A(\tau^-)) = 0 \) for \( \tau^- > \tau \). Equation [6.13] is recognized as \( \alpha A \) times the single asset partial differential equation listed in [6.10] subject to identical boundary conditions when: (1) the default boundary is unity, (2) the underlying asset is \( Q \), whose proportional variance is \( \sigma_Q = \sigma_A^2 + \sigma_d^2 - 2 \sigma_A \sigma_d \rho_{A,D} \) and payout rate is \( (d_D - 1) \), and (3) \( \tau = d_A, \theta = 0, R = 1 \). Accordingly, the closed-form analytical valuation formula for this first passage option is:
\[ I_2(A,D, \tau) = \alpha A \left[ \left( \frac{A}{D} \right)^{5\eta + \delta} N(x) + \left( \frac{A}{D} \right)^{5\eta - \delta} N(x - 2\bar{m}c\sqrt{\tau}) \right] \]  

where

\[
\begin{align*}
\dot{\gamma} &= 2[\ln(d_d/d_D) - .5\sigma_Q^2]/[\sigma_Q^2] \\
x_\delta &= [\ln(A/D) + \bar{m}\sigma_Q^2]/[\sigma_Q^2] \\
\bar{m} &= \sqrt{(.5\dot{\gamma}^2 + 2\ln(d_d/d_D)/[\sigma_Q^2]}
\end{align*}
\]

and all other notation is as previously defined. See Appendix VIE for verification.

As the popularity and volume of customized derivative securities continues to grow, it becomes increasingly more important for institutions to accurately assess credit risk. The exposure of financial loss resulting from default is decreased by possibility of a partial recovery. In this section three partial recovery scenarios have been considered. Little is known, however, about how customized claims are handled in the event of default. Therefore, whether the model developed in this chapter (combined with a partial recovery option) can accurately depict market prices remains a question for future research.

VI.E Chapter Summary

Twenty years ago option pricing theory was revolutionized with the development of the Black-Scholes option pricing model. Several of the assumptions underlying the Black-Scholes model have since been successfully relaxed. However, Black and Scholes assumed there was no possibility of writer default and heretofore, no closed-form solution exists for Black-Scholes options subject to intertemporal default risk. This chapter presents a systematic theory for valuing Black-Scholes options under such conditions. Closed-form expressions are derived for calls and puts restricted by static and stochastic default boundaries. In addition, the models are extended to include partial recoveries. One version of the partial recovery option leads to the development of an explicit valuation formula for an option with a random payoff at a random time.
VI.F Appendix VIA: Derivation of the Vulnerable Black-Scholes Model

In this appendix the required transition probability density function is derived, followed by the derivation of equation [6.4].

VI.F.1 The Generalized Reflection Principle and Derivation of the Density

The reflection principle is a powerful graphic depiction of the strong Markov property that can be used to simplify complex probabilities without explicit reference to measure theory. Unfortunately however, the reflection principle as commonly presented (see Harrison (1985), Chapter I) can be applied only in 2-space. Next, it is shown how the reflection principle can be generalized to k-space. A rigorous proof of the generalized reflection principle is beyond the purpose of this thesis. However, the bivariate normal density that is obtained from using the generalized reflection principle in three-space can be seen to satisfy the appropriate two-dimensional forward equation (which is presented below).

For a sketch of what follows, note that at time zero a well-behaved solution to the forward equation is characterized by a superposition of sources of strength. In the absence of a drift for each random process, it is well-known that each source of strength is unity. Therefore, we begin by assuming that the drift coefficients, \( \mu_s \) and \( \mu_a \), in [6.1] and [6.2] are zero so that our focus may be centered solely on determining the image sources.

To evaluate the following complex probability begin by employing the law of total probability.

\[
\begin{align*}
Pr\left(S(T) \geq X, \sup_{t \in [t_0, T]} A(t) > D\right) &= Pr(S(T) \geq X, A(T) > D, \sup_{t \in [t_0, T]} A(t) > D) \\
&= Pr(S(T) \geq X, A(T) > D) - Pr(S(T) \geq X, A(T) > D, \sup_{t \in [t_0, T]} A(t) \leq D)
\end{align*}
\]

[6.16]

where \( \sup \), the infimum of the process, extends over all \( \{A(t), t \in [t_0, T]\} \) of \( \Omega \). Note that the region in which the infimum is extended over has been suppressed for notational convenience. Converting into continuously compounded returns by letting

\[
Z_s(T) = \ln(S(T)/S), \; z_a = \ln(X/S), \; Z_a(T) = \ln(A(T)/A), \text{ and } z_a = b = \ln(D/A),
\]
equation [6.16] can be written as\(^{103}\)

\[
Pr\left(Z_s(T) > \frac{Z_A}{\sigma_A^{\sqrt{\tau}}}, Z_A(T) > \frac{Z_A}{\sigma_A^{\sqrt{\tau}}}\right) - Pr\left(Z_s(T) < \frac{Z_s}{\sigma_s^{\sqrt{\tau}}}, Z_A(T) > \frac{Z_A}{\sigma_A^{\sqrt{\tau}}} , \inf Z_A(t) < \frac{b}{\sigma_A^{\sqrt{\tau}}}\right). \tag{6.17}
\]

Since \(S(T)\) and \(A(T)\) are lognormally distributed (by construction), \(Z_s(T)\) is normally distributed.

If \(Z_s\) and \(Z_A\) are independent, the 2-space reflection principle tells us that a well behaved time zero solution to the (one-dimensional) Kolmogorov equations must have a superposition of a source of unit strength distributed somewhere along the \(Z_A\) axis (and a source of unit strength at the origin). To locate this source, imagine placing a mirror on the default boundary (facing the time axis) and locating the image of the origin of the time axis along the \(Z_A\) axis.\(^{104}\) The image source can be seen to be \([2b]/[\sigma_s^{\sqrt{\tau}}]\).

In the more general case, when \(Z_s\) and \(Z_A\) are not independent, there is an image source (from the mirror plane) along the \(Z_s\) axis, as well as an image source along the \(Z_A\) axis. The image source along the \(Z_s\) axis results from the correlation between \(Z_s\) and \(Z_A\) (i.e., in the uncorrelated case there is only one image source). Thus, one concludes that there is an image point along the \(Z_A\) axis at \([2b]/[\sigma_s^{\sqrt{\tau}}]\) (because this is the one-to-one effect of the reflection principle) and there is an additional image point along the \(Z_s\) axis at \([\rho_s\sigma_{2b}]/[\sigma_s^{\sqrt{\tau}}]\) (because this is the "correlation effect" of the reflection principle).

Accordingly, equation [6.17] reduces to

\[
Pr\left(Z_s(T) > \frac{Z_s}{\sigma_s^{\sqrt{\tau}}}, Z_A(T) > \frac{Z_A}{\sigma_A^{\sqrt{\tau}}}\right) - Pr\left(Z_s(T) < \frac{\rho_s\sigma_{2b}}{\sigma_s^{\sqrt{\tau}}}, Z_A(T) < \frac{2b}{\sigma_A^{\sqrt{\tau}}} - \frac{Z_A}{\sigma_A^{\sqrt{\tau}}}\right). \tag{6.18}
\]

For \(\mu_s \neq 0\) and \(\mu_A \neq 0\) (in equations [6.1] and [6.2]), Girsanov's Theorem is applied to restate [6.18] as

\(^{103}\) To make the derivation as general as possible, a different place holder is used for \(Z_A\) and \(b\). Of course, in this case they are the same.

\(^{104}\) See Cox and Miller (1965, p. 221) and Papoulis (1991, p. 608).
\[ M_1 \left( \frac{z_s - (\mu_S - 5\sigma_S)\tau}{\sigma_S\sqrt{\tau}}, \frac{z_A - (\mu_A - 5\sigma_A)\tau}{\sigma_A\sqrt{\tau}} ; \rho_{SA} \right) \]

\[ - \left( \frac{D}{A} \right)^\gamma M_2 \left( \frac{\rho_{sa} z_s - (\mu_S - 5\sigma_S)\tau}{\sigma_S\sqrt{\tau}}, \frac{z_A - (\mu_A - 5\sigma_A)\tau}{\sigma_A\sqrt{\tau}} ; \rho_{SA} \right) \]  

[6.19]

\[ M_i(q_1, q_2, \rho) (j = 1,2) \] is the bivariate cumulative normal distribution with upper limits of integration \( q_1 \) and \( q_2 \) and a correlation coefficient of \( \rho \). It is assumed that each distribution function in [6.19] is continuously differentiable, in which case, differentiation yields:

\[ \frac{\partial^2 M_i(\cdot)}{\partial Z_2 \partial Z_A} = f_{Z_A}(n, q_A, \tau)(z_S z_A) \]

\[ = \frac{1}{2\pi \sigma_S \sigma_A \tau \sqrt{1 - \rho_{SA}^2}} e^{-\frac{1}{2(1-\rho_{SA}^2)} \left( \frac{(z_s - 2b)^2}{\sigma_S^2} + \frac{(z_A - \mu_A - 5\mu_A)^2}{\sigma_A^2} + \frac{(\mu_S - 5\mu_S)^2}{\sigma_A^2} \right)} \]  

[6.20]

and

\[ \frac{\partial^2 M_2(\cdot)}{\partial Z_2 \partial Z_A} = g_{Z_A}(n, q_A, \tau)(z_S z_A) = (D/A)^\gamma \frac{1}{\sigma_S \sigma_A \tau \sqrt{1 - \rho_{SA}^2}} n(z_S z_A) \]  

[6.21]

where

\[ n(z_S z_A) = \frac{1}{2\pi e} \frac{1}{2(1-\rho_{SA}^2)} \left[ \frac{(z_s - 2b)^2}{\sigma_S^2} + \frac{2\rho_{sa}(z_s - 2b - \mu_s)\sigma_s(z_s - 2b - \mu_s)}{\sigma_s^2\sigma_a^2} + \frac{(z_A - \mu_A - 5\mu_A)^2}{\sigma_A^2} \right] \]  

[6.22]

\[ \mu_i = \mu_i - 5\sigma_i^2 \] (i=S,A) and \( f(z_s, z_A) \) and \( n(z_s, z_A) \) are bivariate normal probability density functions.

Following Ingersoll (1987, p. 369), define the “defective density”, \( \omega(z_s, z_A) \), as \( f(z_s, z_A) \) less \( g(z_s, z_A) \). The constant term, \( (D/A)^\gamma \), premultiplying the \( n(z_s, z_A) \) density is calculated by setting the "defective density" equal to zero when \( Z_A(T) \) is evaluated at \( z_A = b \) and \( Z_S(T) \) is concurrently evaluated at \( z_s = b\rho_{sa}\sigma_s\sigma_a^{-1} \). Proper specification of the constant term, \( (D/A)^\gamma \), ensures that the
"defective (joint) density" is the appropriate solution to the forward equation\(^{105}\)

\[
\frac{5\sigma_A^2}{\partial Z_A^2} \frac{\partial^2 \omega(\cdot)}{\partial Z_A^2} + 5\sigma_S^2 \frac{\partial^2 \omega(\cdot)}{\partial Z_s^2} + \sigma_A \sigma_S \frac{\partial^2 \omega(\cdot)}{\partial Z_s \partial Z_A} - \mu_s \frac{\partial \omega(\cdot)}{\partial Z_s} - \mu_A \frac{\partial \omega(\cdot)}{\partial Z_A} = \frac{\partial \omega(\cdot)}{\partial t} \quad (Z_A > b) \quad [6.23]
\]

subject to the initial condition

\[
\omega(z_A, z_s; t=0) = \delta(z_s, z_A) \quad [6.24]
\]

where \(\delta(z_A, z_s)\) is the Dirac delta function\(^{106}\), and the boundary condition

\[
\omega(b, z_s; t) = 0 \quad (0 < t \leq T). \quad [6.25]
\]

This solution can be regarded as a time zero superposition of sources distributed along the \(Z_s\) and \(Z_A\) axes. There is a source of unit strength at the origin of the time axis and a source of \((D/A)^n\) strength which is seen along the \(Z_A\) axis at \([2b]/[\sigma_A \sqrt{\tau}]\) and is seen along the \(Z_s\) axis at \([\rho_{s,A}2b]/[\sigma_s \sqrt{\tau}]\).

The "defective density" has a clear economic interpretation. Since \(f(z_s, z_A)\) represents the joint probability density of \(Z_s(T)\) being at \(z_s\) and \(Z_A(T)\) being at \(z_A\) at time \(T\) (path independent), and \(g(z_s, z_A)\) depicts the probability density of crossing the boundary \(b\) and \(Z_s(T)\) being at \(z_s\) and \(Z_A(T)\) being at \(z_A\) at \(T\) (path dependent), the "defective density" represents the joint probability density of not crossing \(b\) and \(Z_s(T)\) being at \(z_s\) and \(Z_A(T)\) being at \(z_A\) at \(T\) (path dependent).

The above methodology can quickly be extended to \(k\)-space. If there are \(k\) Weiner processes and \((k-1)\) are not restricted by an absorbing barrier (e.g., the default boundary), the generalized reflection principle says that each "correlation effect" must be considered. That is, every dependency with the writer's asset is seen in the mirror hyperplane as an image source.

\(^{105}\) For a derivation of the forward equation, see Cox and Miller (1965, p. 247).

\(^{106}\) A delta function has the character of a discrete probability distribution where all the weight is concentrated at the initial point \(\{Z_s(t_0), Z_A(t_0)\}\). Formally, the Dirac delta function is characterized by the properties:

\[
\delta(z_s, z_A) = \begin{cases} 
0 & \text{if } z_s = 0, z_A = 0 \\
\infty & \text{if } z_s = 0, z_A = 0
\end{cases}
\]

and

\[
\int \int \delta(z_s, z_A) dz_A dz_s = 1.
\]
VI.F.2 Derivation of the Intertemporal Vulnerable Black-Scholes Model

Define \( \bar{z}_S = \frac{z_S - \tilde{\mu}_S \tau}{\sigma_S \sqrt{\tau}} \) and \( \bar{z}_A = \frac{z_A - \tilde{\mu}_A \tau}{\sigma_A \sqrt{\tau}} \) and notice under risk neutrality, that \( z_i = \ln(r/d_i) \) (i=S,A). Also recall that the solution for the stochastic differential equation listed in [6.1] implies,

\[
S(T) = S e^{(\ln(d_f) - 5\sigma_X^2)\tau} + \sigma_X \sqrt{\tau} = r^T d_S e^{5\sigma_X^2 \tau} + \sigma_X \sqrt{\tau}.
\]

[6.26]

Risk neutral valuation is used to write the discounted expected terminal payoff as

\[
W_v(S,X,A,D,\tau) = r^{-\tau} E(\Phi S(T) - \Phi X | \Phi S(T) \geq \Phi X, \inf A(t) > D)
\]

\[
= r^{-\tau} E(\Phi S(T) | \Phi S(T) \geq \Phi X, A(T) > D) - \Phi r^{-\tau} X Pr(\Phi S(T) \geq \Phi X, A(T) > D)
\]

\[
- r^{-\tau} E(\Phi S(T) \Phi S(T) \geq \Phi X, A(T) > D, \inf A(t) \leq D)
\]

\[
+ \Phi r^{-\tau} X Pr(\Phi S(T) \geq \Phi X, A(T) > D, \inf A(t) \geq D).
\]

[6.27]

\( E \) denotes the expectation operator which is taken under an equivalent martingale measure over all first passage paths (i.e. over all paths of \( A(t) > D \)) to the non-negative payoff space (i.e., \( S(T) \geq X \)). Combining equations [6.20], [6.21], [6.22], and [6.26] with [6.27] gives

\[
W_v(S,X,A,D,\tau) = \Phi S d_S^{-\tau} e^{-5\sigma_X^2 \tau} \int_{-\Phi \sigma_X \sqrt{\tau}}^{\Phi \sigma_X \sqrt{\tau}} \int_{-\Phi \sigma_X \sqrt{\tau}}^{\Phi \sigma_X \sqrt{\tau}} e^{\sigma_X^2 \tau} (\Phi S - \Phi X) d\sigma d\phi
\]

\[
- \Phi r^{-\tau} X \int_{-\Phi \sigma_X \sqrt{\tau}}^{\Phi \sigma_X \sqrt{\tau}} \int_{-\Phi \sigma_X \sqrt{\tau}}^{\Phi \sigma_X \sqrt{\tau}} (\Phi S - \Phi X) d\sigma d\phi
\]

[6.28]

which when evaluated yields equation [6.4]; see Appendix VA.
VI.G Appendix VIB: Comparative Statics for the Vulnerable Model

In this appendix the comparative statics for the vulnerable option pricing model, equation [6.4], are presented. Note that the comparative statics are defined only over the non-default states (i.e., $A(t) > D$ for all $t \in [t_0, T]$). Also note that [6.4] can be verified to linearly homogeneous with respect to $S, X, D, A$, and $\alpha$, by observing that Euler’s theorem holds. To save space let $a = [\gamma + 2 \sigma_\alpha \sigma_\alpha^{-2}], \rho = \rho_{sa}, x_1^* = (x_1 - \sigma_3'\sqrt{\tau}), x_2^* = (x_2 - \sigma_3'\rho_s a \sqrt{\tau}), x_3^* = (x_3 - \sigma_3'\sqrt{\tau}),$ and $x_4^* = (x_4 - \sigma_3'\rho_s a \sqrt{\tau})$.

\[
\frac{\partial[6.4]}{\partial S} = \phi_d^{-1} [M(\phi x_1, x_2, \phi \rho) - (D/A)^\gamma M(\phi x_3, x_4, \phi \rho)] 
\]

\[
\frac{\partial[6.4]}{\partial A} = \phi \left( \frac{D}{A} \right)^\gamma \left[ \left( \frac{D}{A} \right)^2 \sigma^2 \Sigma S d^{-1} M(\phi x_1, x_2, \phi \rho) - \gamma X r^{-\gamma} M(\phi x_3, x_4, \phi \rho) \right] 
\] 

\[ + \phi \left( \frac{D}{A} \right)^\gamma \frac{1}{A \sigma \sqrt{\tau}} \left[ \int \left( \frac{\phi x_1 - \phi \rho x_2}{\sqrt{1 - \rho^2}} \right) n(x_2) \Sigma S d^{-1} - N \left( \frac{\phi x_1 - \phi \rho x_2}{\sqrt{1 - \rho^2}} \right) n(x_3) X r^{-\gamma} \right] \]

\[ + \phi \left( \frac{D}{A} \right)^\gamma \frac{1}{A \sigma \sqrt{\tau}} \left[ \int \left( \frac{\phi x_1 - \phi \rho x_2}{\sqrt{1 - \rho^2}} \right) n(x_2) \Sigma S d^{-1} - N \left( \frac{\phi x_1 - \phi \rho x_2}{\sqrt{1 - \rho^2}} \right) n(x_3) X r^{-\gamma} \right] \]

\[
\frac{\partial^2[6.4]}{\partial S^2} = \frac{d S^{-1}}{S \sigma \sqrt{\tau}} \left[ \int \left( \frac{x_2 - \rho x_3}{\sqrt{1 - \rho^2}} \right) n(x_1) - \left( \frac{D}{A} \right)^\gamma \int \left( \frac{x_4 - \rho x_3}{\sqrt{1 - \rho^2}} \right) n(x_2) \right] 
\]

\[6.29\]

\[6.30\]

\[6.31\]
\[ \frac{\partial^2 \rho(6.4)}{\partial A^2} = - \Phi \left( \frac{D}{A} \right)^n \left[ a(a+1) \left( \frac{D}{A} \right)^{2\alpha A \sigma^2} S_{d}^{-\tau} M(\Phi x_1^* x_4^\phi, \Phi \rho) \right] \\
- \frac{1}{A^2 \sigma_A^{\tau}} \left[ \left( \frac{D}{A} \right)^{\sigma_A^{2\tau}} \left( \frac{x_4 - \rho x_2}{\sqrt{1-\rho^2}} \right) n(x_2) \right] \sigma_\alpha \sqrt{\tau} + \Phi \left( \frac{\Phi x_1^* - \Phi \rho x_2}{\sqrt{1-\rho^2}} \right) n(x_2) \sigma_\alpha \sqrt{\tau} (x_2 + \sigma_A \sqrt{\tau}) \\
- \Phi \left( \frac{\Phi x_1^* - \Phi \rho x_2}{\sqrt{1-\rho^2}} \right) n(x_2) \sigma_\alpha \sqrt{\tau} (x_2 + \sigma_A \sqrt{\tau}) \] \[ [6.32] \]

\[ + \frac{\Phi}{A^2 \sigma_A^{2\tau}} \left( \frac{D}{A} \right)^n \left[ \left( \frac{\Phi x_3^* - \Phi \rho x_4}{\sqrt{1-\rho^2}} \right) n(x_4) \sigma_\alpha \sqrt{\tau} \right] \left( 2 + 2\gamma + 1\sigma_A \sqrt{\tau} - (x_4 - \sigma_\alpha \rho A \sqrt{\tau}) \right) \\
- \left( \frac{D}{A} \right)^{2\alpha A \sigma^2} \left\{ \left( \frac{\Phi x_3^* - \Phi \rho x_4}{\sqrt{1-\rho^2}} \right) n(x_4) \right\} \sigma_\alpha \sqrt{\tau} (2 + 1) \sigma_A \sqrt{\tau} - x_4 \]

\[ \frac{\partial^2 \rho(6.4)}{\partial A \partial S} = \frac{d_{\alpha} \sqrt{\tau}}{A \sigma_A^{\sqrt{\tau}}} \Phi \left( \frac{\Phi x_1^* - \Phi \rho x_2}{\sqrt{1-\rho^2}} \right) n(x_2) \\
+ \left( \frac{D}{A} \right)^n \left[ \left( \frac{x_4 - \rho x_2}{\sqrt{1-\rho^2}} \right) n(x_2) \right] \Phi \left( \frac{\Phi x_1^* - \Phi \rho x_2}{\sqrt{1-\rho^2}} \right) n(x_2) \Phi a \sigma_A \sqrt{\tau} M(\Phi x_3 x_4, \Phi \rho) \] \[ [6.33] \]

\[ \frac{\partial \rho(6.4)}{\partial x} = - \Phi \left( \frac{\Phi x_1^* x_4^\phi}{\Phi x_3^* x_4^\phi} \Phi \rho \right) - \left( \frac{D}{A} \right)^n M(\Phi x_3 x_4, \Phi \rho) \] \[ [6.34] \]

\[ \frac{\partial \rho(6.4)}{\partial D} = - \Phi \left( \frac{D}{A} \right)^n \left\{ \left( \frac{D}{A} \right)^{2\alpha A \sigma^2} S_{d}^{-\tau} M(\Phi x_3 x_4, \Phi \rho) \right\} \\
- \frac{1}{D \sigma_A^{\sqrt{\tau}}} \left[ \left( \frac{\Phi x_1^* - \Phi \rho x_2}{\sqrt{1-\rho^2}} \right) n(x_2) \right] \sigma_\alpha \sqrt{\tau} - \left( \frac{\Phi x_1^* - \Phi \rho x_2}{\sqrt{1-\rho^2}} \right) n(x_2) \sigma_\alpha \sqrt{\tau} \] \[ [6.35] \]
\[ \frac{\partial [6.4]}{\partial \tau} = -\phi S_d^\tau \ln(d_3[M(\phi x_1, x_2, \phi \rho) - (D/A)^a M(\phi x_3, x_4, \phi \rho)] \\
+ \phi Xr^{-\tau} \ln(r[M(\phi x_1^*, x_2^*, \phi \rho) - (D/A)^a M(\phi x_3^*, x_4^*, \phi \rho)] \\
+ S_d^\tau \left[ N \left( x_2^- - \rho x_1 \right) \frac{n(x_1)}{\sqrt{1 - \rho^2}} \right] - \left( \frac{D}{A} \right)^a \left[ N \left( x_4^- - \rho x_3 \right) \frac{n(x_3)}{\sqrt{1 - \rho^2}} \right] \\
+ \phi \frac{S_d^\tau}{2\tau} \left[ \left( \frac{D}{A} \right)^a N \left( x_3^* - \rho \phi x_4^* \right) \frac{n(x_4^*)}{\sqrt{1 - \rho^2}} \right] \\
+ \phi \frac{D}{A} \left( \frac{\phi x_1^* - \rho \phi x_2^*}{\sqrt{1 - \rho^2}} \right) \frac{n(x_2^*)}{Xr^{-\tau} (\gamma \sigma_A \sqrt{\tau} - x_4^*)} \\
- \phi N \left( \frac{\phi x_1^* - \rho \phi x_2^*}{\sqrt{1 - \rho^2}} \right) \frac{n(x_2^*)}{Xr^{-\tau} \left[ \ln(\tau/d_4) - 5(\sigma_A^2 - \sigma_{A_D}^2) \frac{x_2}{\sigma_A \sqrt{\tau}} \right]}. \tag{6.36} \]

\[ \frac{d [6.4]}{d \rho} = \phi \left[ S_d^\tau n(\phi x_1, x_2) - Xr^{-\tau} n(\phi x_1^*, x_2^*) - (D/A)^a S_d^\tau \phi \ln(D/A) \phi(\phi x_3, x_4, \phi \rho) \right] \\
- \phi \left( \frac{D}{A} \right)^a \left[ \left( \frac{D}{A} \right)^a S_d^\tau n(\phi x_3, x_4) - Xr^{-\tau} n(\phi x_3^*, x_4^*) \right]. \tag{6.37} \]

\[ \frac{\partial [6.4]}{\partial \sigma_A} = \phi \sqrt{\tau} S_d^\tau \nabla \left[ N \left( \frac{\phi x_1 - \rho \phi x_2}{\sqrt{1 - \rho^2}} \right) \frac{n(x_2)}{\sqrt{1 - \rho^2}} \right] - \left( \frac{D}{A} \right)^a \nabla \left[ N \left( \frac{\phi x_3 - \rho \phi x_4}{\sqrt{1 - \rho^2}} \right) \frac{n(x_4)}{\sqrt{1 - \rho^2}} \right]. \tag{6.38} \]
\[
\frac{\partial [6.4]}{\partial \sigma_A} = \phi \ln(D/A) \frac{2(D/A)}{\sigma_A} \left( \frac{D}{A} \right)^{2\alpha_2 \sigma_A^2} Sd^*_M(\phi x_2 x_4 \phi \rho) (y + 1 + \alpha_2 \rho \sigma_A^2) \\
+ \frac{\phi (x_2^2 - \alpha_2 \sqrt{2})}{\sigma_A} \left[ N \left( \frac{\phi x_1^* - \phi \rho x_2^*}{\sqrt{1 - \rho^2}} \right) n(x_2) Xr^{-\tau} - N \left( \frac{\phi x_1^* - \phi \rho x_2^*}{\sqrt{1 - \rho^2}} \right) n(x_2) Sd^{-\tau} \right] \\
+ \frac{\phi (x_2^2 - \alpha_2 \sqrt{2})}{\sigma_A} \left( \frac{D}{A} \right)^{2\alpha_2 \sigma_A^2} N \left( \frac{\phi x_3^* - \phi \rho x_4^*}{\sqrt{1 - \rho^2}} \right) n(x_4) Sd^*_M - N \left( \frac{\phi x_3^* - \phi \rho x_4^*}{\sqrt{1 - \rho^2}} \right) n(x_4) Xr^{-\tau} 
\]

[6.39]
VI.H Appendix VIC: The Standard Down-and-Out Barrier Model as the Limiting Solution

This appendix verifies that the intertemporal vulnerable Black-Scholes model, equation [6.4], is equivalent to the European down-and-out pricing formula when the writer’s assets are identical to the spot asset.

Observe that when \( S = A \), \( \sigma_S = \sigma_A = \sigma_{S,A} \), \( \rho_{S,A} = 1 \), and \( d_S = d_A \),

\[
\gamma = 2 \left[ \ln \left( \frac{r}{d_2} \right) - 0.5 \sigma_A^2 \right] / [\sigma_A^2] \\
x_2 = \frac{\ln (S/D) + (\ln (r/d_2) + 0.5 \sigma_A^2 \tau)}{[\sigma_A^2 \tau]} \\
x_3 = \frac{\ln (D^2/\mathcal{X} S) + (\ln (r/d_2) + 0.5 \sigma_A^2 \tau)}{[\sigma_A^2 \tau]} \\
x_4 = \frac{\ln (D/S) + (\ln (r/d_2) + 0.5 \sigma_A^2 \tau)}{[\sigma_A^2 \tau]} 
\]

Further note that when \( X \geq D \), \( M(\phi x_1, x_2, 1) = N(\phi x_1) \), \( M(\phi (x_1 - \sigma_S^2 \sqrt{\tau}), (x_2 - \sigma_S^2 \sqrt{\tau}), 1) = N(\phi (x_1 - \sigma_S^2 \sqrt{\tau})) \), \( M(\phi x_3, x_4, 1) = N(\phi x_3) \), and \( M(\phi (x_3 - \sigma_S^2 \sqrt{\tau}), (x_4 - \sigma_S^2 \sqrt{\tau}), 1) = N(\phi (x_3 - \sigma_S^2 \sqrt{\tau})) \). Therefore, when \( X \geq D \) equation [6.4] becomes

\[
W_r(S, X S \leq D, \tau) = \phi S d_3^{-\frac{1}{2}} \left[ N(\phi x_1) \left( \frac{D}{S} \right)^{\gamma/2} N(\phi x_2) \right] - \phi X r^{-\gamma} \left[ N(\phi (x_1 - \sigma_S^2 \sqrt{\tau})) \left( \frac{D}{S} \right)^{\gamma} N(\phi (x_2 - \sigma_S^2 \sqrt{\tau})) \right]
\]

When \( X < D \) and \( \phi = 1 \), \( M(\phi x_1, x_2, 1) = N(x_2) \), \( M(\phi (x_1 - \sigma_S^2 \sqrt{\tau}), (x_2 - \sigma_S^2 \sqrt{\tau}), 1) = N(x_1 - \sigma_S^2 \sqrt{\tau}) \), \( M(\phi x_3, x_4, 1) = N(x_4) \), \( M(\phi (x_3 - \sigma_S^2 \sqrt{\tau}), (x_4 - \sigma_S^2 \sqrt{\tau}), 1) = N(x_3 - \sigma_S^2 \sqrt{\tau}) \), and equation [6.4] becomes

\[
C_r(S, X S \leq D, \tau) = \phi d_3^{-\frac{1}{2}} \left[ N(x_2) \left( \frac{D}{S} \right)^{\gamma/2} N(x_4) \right] - \phi X r^{-\gamma} \left[ N(x_2 - \sigma_S^2 \sqrt{\tau}) \left( \frac{D}{S} \right)^{\gamma} N(x_4 - \sigma_S^2 \sqrt{\tau}) \right]
\]

When \( X < D \) and \( \phi = -1 \), \( P_r(S, X S \leq D, \tau) = 0 \) because there is a zero probability of this put option finishing in-the-money.

These results coincide with the down-and-out valuation formulas derived in Rubinstein and Reiner (1991a).
VI.I Appendix VI D: Comparative Statics for the First Recovery Option

In this appendix the comparative statics for equation [6.11] are provided. The comparative statics are defined over $A(t) > D$ for all $t \in [t_0, T]$. Equation [6.11] is readily verified to satisfy the partial differential equation [6.10].

\[
\frac{\partial [6.11]}{\partial A} = -m(Re^{-t_{0r}/A})(D/A)^{5\gamma - m}N(x_2) - 2(Re^{-t_{0r}/A})(D/A)^{5\gamma - m}n(x_2) / [\sigma_A \sqrt{\tau}] \\
+ m(Re^{-t_{0r}/A})(D/A)^{5\gamma - m}N(x_2) - 2m \sigma_A \sqrt{\tau} - [6.11].5\gamma/A \tag{6.43}
\]

\[
\frac{\partial^2 [6.11]}{\partial A^2} = (5\gamma + m)(5\gamma + m + 1)(Re^{-t_{0r}/A^2})(D/A)^{5\gamma - m}N(x_2) \\
+ (5\gamma - m)(5\gamma - m + 1)(Re^{-t_{0r}/A^2})(D/A)^{5\gamma - m}N(x_2) - 2m \sigma_A \sqrt{\tau} \tag{6.44}
\]

\[
\frac{\partial [6.11]}{\partial \tau} = -Re^{-t_{0r}}(D/A)^{5\gamma - m}n(x_2) [x_2/\tau - m \sigma_A \sqrt{\tau}] - \xi \theta [6.11] \tag{6.45}
\]
VI.J Appendix VIE: Comparative Statics for the First Passage Option

In this appendix the comparative statics for equation [6.14] are provided. The comparative statics are defined over \( A(t) > D(t) \) for all \( t \in [t_0, T] \). Equation [6.14] can be readily verified to satisfy the partial differential equation [6.3].

\[
\begin{align*}
\frac{\partial [6.14]}{\partial D} &= -(5\dot{\gamma} + \dot{m})(\alpha A/D)(A/D)^{5\gamma + \alpha} N(x_0) \\
&- (A/D)^{5\gamma + \alpha} n(x_0) 2aA/[D\sigma_0/\tau] - (5\dot{\gamma} - \dot{m})(\alpha A/D)(A/D)^{5\gamma + \alpha} N(x_0 - 2\dot{m}\sigma_0/\tau) \\
\frac{\partial^2 [6.14]}{\partial D^2} &= (5\dot{\gamma} + m)(5\dot{\gamma} + m + 1)(\alpha A/D^2)(A/D)^{5\gamma + \alpha} N(x_0) \\
&+ (5\dot{\gamma} - \dot{m})(5\dot{\gamma} - \dot{m} + 1)(\alpha A/D^2)(A/D)^{5\gamma + \alpha} N(x_0 - 2\dot{m}\sigma_0/\tau) \\
&+ (\dot{\gamma} + m + 1)(A/D)^{5\gamma + \alpha} n(x_0) 2aA/[D^2\sigma_0/\tau^2] - (A/D)^{5\gamma + \alpha} n(x_0) 2a\sigma_0 A/[D^2\sigma_0/\tau] \\
\frac{\partial [6.14]}{\partial A} &= -(5\dot{\gamma} + m + 1)\alpha(A/D)^{5\gamma + \alpha} N(x_0) \\
&+ (A/D)^{5\gamma + \alpha} n(x_0) 2aA/[\sigma_0/\tau] + (5\dot{\gamma} - \dot{m} + 1)\alpha(A/D)^{5\gamma + \alpha} N(x_0 - 2\dot{m}\sigma_0/\tau) \\
\frac{\partial^2 [6.14]}{\partial A^2} &= (5\dot{\gamma} + m)(5\dot{\gamma} + m + 1)(\alpha A/D)(A/D)^{5\gamma + \alpha} N(x_0) \\
&+ (5\dot{\gamma} - \dot{m})(5\dot{\gamma} - \dot{m} + 1)(\alpha A/D)(A/D)^{5\gamma + \alpha} N(x_0 - 2\dot{m}\sigma_0/\tau) \\
&+ (\dot{\gamma} + m + 1)(A/D)^{5\gamma + \alpha} n(x_0) 2aA/[A\sigma_0/\tau] - (A/D)^{5\gamma + \alpha} n(x_0) 2a\sigma_0 A/[A^2\sigma_0/\tau^2] \\
\frac{\partial [6.14]}{\partial A \partial D} &= -(5\dot{\gamma} + m)(5\dot{\gamma} + m + 1)(\alpha D/D)(A/D)^{5\gamma + \alpha} N(x_0) \\
&- (5\dot{\gamma} - \dot{m})(5\dot{\gamma} - \dot{m} + 1)(A/D)^{5\gamma + \alpha} (\alpha D/D)N(x_0 - 2\dot{m}\sigma_0/\tau) \\
&- (\dot{\gamma} + m + 1)(A/D)^{5\gamma + \alpha} n(x_0) 2aA/[D\sigma_0/\tau] + (A/D)^{5\gamma + \alpha} n(x_0) 2a\sigma_0 A/[D^2\sigma_0/\tau^2] \\
\frac{\partial [6.14]}{\partial \tau} &= -\alpha A(A/D)^{5\gamma + \alpha} n(x_0) (x_0 - \dot{m}\sigma_0/\tau)/\tau \\
\end{align*}
\]

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CHAPTER VII
Applications of Stochastic Barrier Options
to Performance Incentive Contracts

VII.A Introduction

In Chapter VI, the detached barrier option pricing model is used to incorporate default risk into an otherwise standard Black-Scholes framework. In this chapter, a brief discussion is offered on how the barrier exchange option (developed in Chapter IV) and the attached barrier option pricing model (developed in Chapter V) is applied to the valuation of performance incentive fee contracts.

Performance incentive fees are becoming an increasingly popular method of rewarding portfolio managers.\textsuperscript{107} Performance based fees are designed to discriminate between performance generated by managerial decisions and performance generated by external forces (e.g., general market conditions). Asset-based fees, a previously popular form of compensation, remunerate a money manager according to the market value of the managed portfolio; hence, performance is rewarded regardless of its origin - be it chance or skill. "Asset-based fees are akin to compensating corporate officers according to the size of their company or division, instead of profits" (Davanzo and Nesbitt (1987, p. 14)). Incentive fees are structured to overcome this obstacle by dichotomizing performance according to its origin. In theory, labor can then be rewarded based on the marginal product of management. The move from theory to practice is, however, obstructed by several factors.

\textsuperscript{107} Davanzo and Nesbitt (1987) argue that performance based fees have recently increased in popularity simply because prior to 1985 "regulation prohibited application of performance fees to SEC-registered investment advisors except in special circumstances" (p. 14).
A typical incentive fee contract is characterized by a base fee, which is fixed, and a contingent fee. The contingent fee allows a manager to participate in the return of the managed portfolio in excess of the capital change experienced by a prespecified benchmark portfolio. Marginal performance is, thus, measured by the incremental (absolute) return generated relative to a benchmark.

VII.B Valuing the Option Feature of the Contract

Compensation, when structured in this manner, is analogous to a call option. Margrabe (1978) is the first to identify this isomorphic relationship between the contingent fee of performance-based contracts and exchange call options. He argues that an incentive fee is analogous to issuing the money manager an exchange option, if the manager is not obligated to losses in the event of underperformance. To illustrate, suppose at time zero, the current time, the managed portfolio has a value of $40 million ($S_1=40$) and the benchmark portfolio has a value of $400 million ($S_2=400$). If the portfolio manager is entitled to a payoff of $\alpha$ (a constant) percent of the incremental return earned relative to a benchmark portfolio, the current value of the contingent fee is written as $(\alpha/S_1)$ exchange call options written on $S_1$ and struck at $\gamma S_2$, where $\gamma=.1$ in this case (i.e., $(\alpha/S_1) C_x(S_1,1S_2,T))$. To see this, note that when $\gamma=.1$, $S_1=\gamma S_2$ so that the option only has a positive payoff when the holding period return for asset one exceed

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108 The economic justification for the base fee is not clear from existing literature. Presumably, this fixed fee serves as compensation for the idiosyncratic risk that a portfolio manager faces. Bailey (1990, p. 34) suggests the base fee should be thought of as a maintenance fee which "keeps the manager 'in the game' financially during periods of poor performance."

109 According to the new SEC requirements, performance can be measured over any length time greater than or equal to one year; see Record and Tynan (1987, p. 42). The effectiveness of performance-based contracting is critically dependent on the measurement period length. For example, if a money manager is seeking long-term capital growth, a short measurement period is clearly inappropriate. Nevertheless, the optimal contract length is not addressed in this chapter. Readers interested in this topic are referred to Davanzo and Nesbitt (1987).

110 For an excellent discussion on this point, see Kritzman (1987, p. 23). As a caveat, Bookstaber and Clarke (1985) point out that if the managed portfolio actually contains options, the return distribution for this portfolio will not be symmetric, as is typically assumed in option pricing theory.
the holding period return for asset two (i.e., \( \frac{S_1(T) - S_1}{S_1} > \frac{\gamma S_2(T) - S_1}{S_1} \)). To confirm that the payoff is correct, observe that

\[
(\alpha|S_1) C_x(S_1(T), \gamma S_2(T), 0) = \alpha \max \left( \frac{S_1(T) - \gamma S_2(T)}{S_1}, 0 \right) = \alpha \max \left( \frac{S_1(T) - \gamma S_2(T)}{S_1} - 1 + 1, 0 \right)
\]

\[
= \alpha \max \left( \frac{S_1(T) - S_1}{S_1} - \frac{\gamma S_2(T) - \gamma S_2}{\gamma S_2}, 0 \right) = \alpha \max \left( \frac{S_1(T) - S_1}{S_1} - \frac{S_2(T) - S_2}{S_2}, 0 \right)
\]

which is \( \alpha \) percent of the incremental excess return, as required.

VII.C The Benchmark Portfolio

A critical issue is overlooked in the above discussion; namely, that the proper evaluation of performance rests critically on the benchmark portfolio that is specified in the contract. A logical choice for the target portfolio may be the S&P 500 index. However, Grinold and Rudd (1987) caution against this specification in many circumstances.

"Growth" managers, for example, will typically hold portfolios with high market risk, low yield and a bias toward low-capitalization securities. "Value" managers will hold assets with low P/E ratios and relatively high book values. In addition, some managers focus on certain industries or on assets with within a certain capitalization range. Some managers hold about 50 or 60 securities, others 30 or 40. Some managers tend to equal-weight; others tend toward capitalization-weighting. (p. 29)

As Record and Tynan (1987) put it, "one must first determine whether there is a relative benchmark against which the performance of the account can be measured" (p. 42). This issue is addressed by Rennie and Cowhey (1990) and Ankrim (1992) and is not considered any further in this paper. However, Record and Tynan raise a second issue which has not been previously addressed. Specifically, "even in the absence of a directly comparable relative benchmark, it might be appropriate to consider a threshold of relative returns that must be achieved before the manager is entitled to receive a share of the net gains in the account" (p. 42). For instance, in the spirit of Record and Tynan, suppose that it is specified that the manager receives a contingent payoff at time T if, and only if, the account outperforms the return on the benchmark portfolio by \( \alpha \) percent at some point during the measurement period. More specifically, consider the case
in which the contingent fee is structured such that the manager is rewarded by a contingent payoff of \( \max \ (S_1(T) - X, 0) \) (where \( X \) is a constant and \( S_1 = \$40 \) is the account value) if the account ever outperforms the S&P 500 index (\( S_2 = \$400 \)) by six percent or more. The value of this contract is determined using the up-and-in attached barrier call option formula developed in Chapter V (see Table VA), with zero rebate. This can be written as \( C_{ui}(S_1, X, (1.06)\gamma S_2, T) \). To see this, observe again, that if \( \gamma = 0.1 \), \( S_1 = 0.5 S_2 \) and, therefore, the option comes into existence at time \( t' \) only if the capital change in the value of the account exceeds the percentage change in the value of the index by 6 percent or more (i.e., \( \frac{S_1(t') - S_1}{S_1} > (1.06) \frac{\gamma S_2(t') - S_1}{S_1} \)). If this condition is ever met, the option feature of the contract pays off \( \max(S_1(T) - X, 0) \) at expiration, as required. In addition, if \( X \) is chosen such that 

\[ X = S_1 + \text{(the base fee)} \]

the contingent fee in the contract described above will not be paid until the manager generates absolute benefits in excess of at least the base fee.

**VII.D The Moral Hazard Problem**

The option feature of the incentive fee contract serves to tie a money manager’s rewards more directly to his/her skill. However, the option feature also serves to encourage a manager to alter the risk composition of the account. This can lead to the classical principal-agent problem. Managers that seek to maximize their pecuniary benefits (and are immune from downside risk) can increase the value of their option by increasing the risk of the portfolio. This creates a perverse incentive, which partially reduces the advantage of having a manager’s reward tied to his/her skill. Rather than a manager making investment decisions based purely on his/her expectations, the effects of a volatility change also become important.

To discourage a manager from taking excessive risks, many performance-based fee agreements stipulate a maximum payoff (i.e., a cap).\(^{111}\) A cap can serve to mitigate excessive

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\(^{111}\) It is unclear as to why the threat of firing does not serve as a complete control mechanism. Bailey (1990, p. 34) offers a more general explanation for the cap which may explain this phenomenon. "Clients prefer not to reward a manager for extraordinary results, presumably in

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risk taking but it simultaneously introduces an additional moral hazard problem.\textsuperscript{112}

Consider the case where a portfolio is performing well ahead of the benchmark. The manager can exercise his option by reducing the portfolio's net risk relative to the benchmark. Risk reduction, within this context, is tantamount to making the portfolio as similar to the benchmark as possible. (Kritzman (1987, p. 22))

In sum, the cap, when reached, encourages a manager to follow a passive strategy. The composition of the portfolio is altered to lock in the gain.

The cap creates a valuation problem in which, even if a benchmark exists against which the performance of the account can accurately be measured, the Margrabe exchange model is no longer applicable. However, the up-and-out barrier exchange call model developed in Chapter IV can be applied to this problem. For example, suppose the contract is written such that the payoff is capped at 15 percent. If the holding period return for asset one (the managed portfolio) ever exceeds the holding period return for asset two (the benchmark portfolio) by more than (or equal to) 15 percent, the option matures immediately and the manager is paid at this time. The current value of this option can be determined as \( (e/S) \) up-and-out barrier exchange call options written on \( S(t) \), struck at \( gS_2(t) \) (with \( g \) selected such that \( S_1 = gS_2 \)), with a barrier at \( (1.15)gS_2(t) \), and rebate amount of \( g(1.15-1)S_2(t) \) (i.e., \( C_{X=0}(S_1, gS_2, 1.15gS_2, 15gS_2, T) \); see Table IVA when \( X=1 \), and \( B(r) = .15g \) (a constant)).

I propose that the option mature when the cap is reached for three reasons. First, such an approach removes the incentive for a manager to follow a passive strategy. Second, if the client believes the superior performance is achieved by luck, rather than skill, they can transfer their account to another money manager whose investment strategy is more aligned with their ideology. On the other hand, if the client is satisfied with the current manager the contract can presumably in the belief either that these results are more a function of luck than skill, or that an unlimited fee may encourage too much risk taking." Hence, a manager may feel that being fired after generating extraordinary results, will leave his/her reputation capital largely unscarred.

\textsuperscript{112} In this section, I discuss how the up-and-out barrier exchange model (developed in Chapter IV) can be used to mitigate excessive risk taking. At times, however, when the opposite problem exists, a compensation contract is desired that encourages a manager to increase his/her risk taking. For this problem, the up-and-in barrier exchange model can be used. That is, the contingent fee contract is valid only (or comes into existence) after a certain level of performance has been achieved.
be renegotiated. Third, if the contract is renegotiated, the manager is likely to receive more favorable terms because of the performance generated. If the contract is not renegotiated, the manager is paid off immediately and he/she avoids having to mask (to preserve their reputation capital) a passive investment strategy.

VII.E Chapter Summary

In this chapter a brief discussion is offered on how the barrier exchange option pricing model and the attached barrier model is applied to the valuation of performance incentive fee contracts. However, unlike Chapter VI, the applications discussed in this chapter are not fully developed. This is because, as stated in Chapter I, the primary reason for developing the barrier exchange model and the attached barrier model is not to apply these models to specific valuation problems per se, but to progressively establish the theoretical foundation for the vulnerable Black-Scholes model. Nevertheless, hopefully it has been established that there are results developed in this dissertation that potentially can provide us with a new approach to the valuation of compensation contracts. When management is compensated on a relative basis or on a combination relative-absolute basis, an underlying boundary potentially can serve as a continuous surveillance mechanism to minimize the moral hazard problem. Perhaps one day, this will provide us with a new approach for quantifying the economic impact of the agency problem. At the very least, performance plans appear to be rich areas for future research.
CHAPTER VIII

Summary

VIII.A Introduction

The previous chapters have presented an introduction to stochastic barrier option pricing theory which culminated in the first analytical model to incorporate intertemporal default risk into the Black-Scholes framework. The purpose of this chapter is to briefly review and summarize the theoretical findings and developments that have been made.

The theoretical foundation needed to value vulnerable Black-Scholes options is progressively established. This approach is arguably inefficient as it resulted in the development of a number of models and interim results that are never specifically used. Nevertheless, the main problem of interest is ultimately solved, valuable insight is gained and a solid foundation is established for further research. Therefore, suggestions for future research are also offered in the discussion that follows.

VIII.B Recapitulation and Suggestions for Future Research

This thesis opened in Chapter III with a generalization to the existing literature on standard barrier options (i.e., barrier options with time-dependent deterministic barrier levels). Our knowledge of the valuation of such options has become fairly complete and there appears to be little need for further theoretical work in this area. However, future numerical studies are needed in this area. When the barrier is discontinuous (i.e., when, for example, the option can only be "knocked-out" or "knocked-in" at the close of a trading day), it is unclear as to which numerical pricing method is most efficient. The current popularity of these options (in the OTC market), combined with the inability to observe prices in continuous time, merits further
investigation into this issue.

The first stochastic barrier option is then introduced in Chapter IV. From a valuation perspective, the barrier exchange model is shown to be a natural extension to the pricing of standard barrier options. It is safe to say that no substantial theoretical advancements are made in this chapter. However, I believe there are many future applications of this model that may improve our understanding of some aspects of finance. A brief discussion is offered on how this might may be applied to (1) the pricing of equity in the presence of deposit insurance (in Chapter IV), and (2) the valuation of performance incentive fee contracts (in Chapter VII). Neither application is fully developed. In addition, it has been suggested (by William Zame) that this model may be applicable to the valuation of the options embedded in callable convertible bonds. Furthermore, it has been suggested (by Raman Kumar) that barrier exchange options may be applicable to the analysis of impending security exchange offers.

The major theoretical advancements/extensions to barrier option pricing theory are presented in Chapter V. The barrier exchange model developed in Chapter IV is generalized to allow for a non-stochastic strike price; this model is referred to as the attached barrier model. An application of the attached barrier model to the valuation of compensation contracts is briefly discussed in Chapter VII. Moreover, it appears that the attached model will readily apply to the valuation of warrants in the presence of intertemporal default risk. Also, it may be possible to use this model to value options outstanding around a takeover threat. In the event of a takeover, the options expire prematurely, thus, resembling "out" options. Knowledge gained from the valuation of the attached model is then applied to the valuation of detached barrier options. Perhaps, from a theoretical perspective, the major contribution of this chapter (and the dissertation) lies in the valuation methodology that is proposed. The generalized reflection principle admits explicit analytic valuation formula for stochastic barrier options in a fairly simple and straightforward way. This technique generalizes the literature on pricing European barrier options and should serve to introduce the valuation of stochastic barrier options to a wider audience.

The principle objective of this dissertation is to incorporate intertemporal default risk into the Black-Scholes framework. This is accomplished by a direct application of the down-and-out detached barrier model. The first closed-form analytic valuation formula for this problem is
presented. Whether this model accurately depicts observed OTC market prices, however, remains a question for future research.

VIII.C Closing Thoughts

As is often the case, this dissertation has raised many new questions. Nevertheless, this research marks the initial use of stochastic barrier option theory and its application to credit risk. The results developed here provide additional insight into credit risk management as it pertains to customized options; yet this research is far from exhaustive. A great deal more needs to be done, but a solid foundation has now been established and should lead to many new findings in the future.
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VITA

Don Rich was born on Easter morning April 10, 1966 in Pontiac, IL to Donna and Kenneth Rich. He married Lisa Benter in 1987. In 1988, he graduated from the University of Illinois with a B.S. degree in Education. Don completed his M.S. degree in 1989 in Agricultural/Applied Economics at the University of Illinois. After spending one year (Fall, 1989 to Spring, 1990) in the Agricultural Economics Ph.D. department at Virginia Tech, he transferred into the Virginia Tech Finance Ph.D. program. Don received his Ph.D. in Finance from Virginia Tech in the Fall of 1993.