Polynomial Approximation And Carleson Measures On A General Domain And Equivalence Classes Of Subnormal Operators

by

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Abstract

This thesis consists of eight chapters. Chapter 1 contains the preliminaries: the background, notation and results needed for this work.

In Chapter 2 we study the problem of when $\mathcal{P}$, the set of analytic polynomials, is dense in the Hardy space $H^t(G)$ or the Bergman space $L^t_a(G)$, where G is a bounded domain and $t \in [1, \infty)$. Characterizations of special domains are also given.

In Chapter 3 we generalize the definition of a Carleson measure to an arbitrary simply connected domain. Let G be a bounded simply connected domain with harmonic measure ω. We say a positive measure τ on G is a Carleson measure if there exists a positive constant c such that for each $t \in [1, \infty)$ and each polynomial p we have

$$\|p\|_{L^t(\tau)} \leq c \, \|p\|_{L^t(\omega)}.$$

We characterize all Carleson measures on a normal domain-definition: a domain G where $\mathcal{P}$ is dense in $H^1(G)$. It turns out that $\mathcal{P}$ is dense in $H^t(G)$ for all t when G is normal.

In Chapter 4 we describe some special simply connected domains and describe how they are related to each other via various types of polynomial approximation.

In Chapter 5 we study the various equivalence classes of subnormal operators under the relations of unitary equivalence, similarity and quasisimilarity under the assumption that G is a normal domain.

In Chapter 6 we characterize the Carleson measures on a finitely connected domain. We are able to push our techniques in the latter setting to characterize those subnormal operators similar to the shift on the closure of $R(K)$ in $L^2(\sigma)$ when $R(K)$ is a hypodirichlet algebra.

In Chapter 7 we illustrate our results by looking at their implications when G is a crescent. Several intersting function theory problems are studied.

In Chapter 8 we study arclength and harmonic measures. Let G be a Dirichlet domain with a countable number of boundary components. Let ω be the harmonic measure of G. We show that if J is a rectifiable curve and $E \subset \partial G \cap J$ is a subset with $\omega(E) > 0$, then E has positive length.

To My Parents

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Contents

Chapter 1

Preliminaries

Throughout this thesis we let μ be a finite positive measure with compact support in the complex plane $\mathbf{C}$; we let $\mathcal{P}$ denote the set of polynomials and if $1 \leq t < \infty$ we let $P^t(\mu)$ be the closure of $\mathcal{P}$ in $L^t(\mu)$. If K is a compact subset, we let $R(K)$ be the uniform closure of rational functions with poles off K. Let $R^t(K,\mu)$ be the closure of $R(K)$ in $L^t(\mu)$.

Analytic Bounded Point Evaluations

A point w in $\mathbf{C}$ is a bounded point evaluation (**bpe**) for $P^t(\mu)$ if there exists a constant $c > 0$ such that

$$|p(w)| \leq c\|p\|, \quad p \in \mathcal{P}.$$

In this case it is clear that $p \to p(w)$ extends to a bounded linear functional on $P^t(\mu)$. By the *Riesz Representation Theorem* there exists a k_w in $L^q(\mu)$ such that

$$p(w) = \int p k_w d\mu.$$

Let $\hat{f}(w) = \int f k_w d\mu$. A point w is called an **analytic bounded point evaluation** (**abpe**) if there exists a neighborhood G of w such that each point in G is a *bpe* and $\hat{f}$ is analytic in G for each $f \in P^t(\omega)$.

For a bounded domain G let $H^\infty(G)$ be the Banach space of bounded analytic functions on G endowed with the supremum norm.

Recently James E. Thomson proved the following remarkable theorem:

Theorem 1.1 (James E. Thomson) *Let μ be a finite measure with compact support and let $t \in [1, \infty)$. There exists a Borel partition $\{\Delta_i\}_0^\infty$ of the support of μ such that*

$$P^t(\mu) = L^t(\mu|\Delta_0) \oplus \bigoplus_1^\infty P^t(\mu|\Delta_i)$$

and for each $i \geq 1$, the space $P^t(\mu|\Delta_i)$ contains no nontrivial characteristic functions. If $i \geq 1$ and W_i is the set of abpe's for $P^t(\mu|\Delta_i)$, then W_i is a simply connected domain and $\Delta_i \subseteq \overline{W_i}$. Moreover, for $i \geq 1$, the evaluation map E: $f \to \hat{f}$ is one-to-one on $P^t(\mu_i)$. Finally, for $i \geq 1$, the Banach algebras $P^t(\mu_i) \cap L^\infty(\mu_i)$ and $H^\infty(W_i)$ are algebraically and isometrically isomorphic and weak-star homeomorphic via E.

The space $P^t(\mu)$ is called pure if $\Delta_0 = \emptyset$, where Δ_0 is the set referred to in the last theorem.

Lemma 1.1 *Let $G = abpe P^t(\mu)$. If $P^t(\mu)$ is pure, then $A(G) \subset P^t(\mu)$.*

Remark: R. Olin and L. Yang first showed $A(G) = C(supp(\mu)) \cap P^2(\mu)$, see [47]. For $t \in [1, \infty)$, Thomson had a proof for the last lemma which is essentially the same as Lemma 5.5 of [64]. The author extended this result to the rational case later [54].

Harmonic Measures

A domain G is called a Dirichlet domain if the Dirichlet problem is solvable for every real-valued continuous function on its boundary, ∂G. Let G be a Dirichlet domain. Let $C(\partial G)$ be the *Banach* space of continuous functions on ∂G. If G is a Dirichlet domain and if $u \in C(\partial G)$, let $\hat{u}$ be the unique function that is continuous on $\overline{G}$ and harmonic in G. Fix a point $a \in G$ and observe the mapping $u \rightarrow \hat{u}(a)$ defines a positive functional on $C(\partial G)$ with norm one. An application of *the Riesz representation theorem* shows there is a unique positive measure ω_a on ∂G so that

$$\hat{u}(a) = \int_{\partial G} \hat{u} d\omega_a, \quad u \in C(\partial G).$$

The measure ω_a is called the harmonic measure of G evaluated at a (consult [17, 24] or [67]). A simple application of Harnack's inequality shows that the harmonic measures of two points of G are boundedly equivalent. If G has smooth boundary (when G is simply connected, ∂G being smooth means that the derivative of a conformal map of D to G has a continuous extension to ∂D), then it is well-known that its harmonic measure and arclength measure on ∂G are boundedly equivalent.

When G is a simply connected domain, then $\omega_a = m \circ \varphi^{-1}$, where φ is the unique conformal map of D onto G (here, we use φ to denote its boundary value on ∂D too).

If K is a compact subset with connected interior, then the harmonic measure of K is defined to be the harmonic measure of its interior.

Nicely Connected Domains

Following Glicksburg ([29]), we call a simply connected domain nicely connected if the conformal map φ of the unit disc D onto G is univalent almost everywhere with respect to the normalized Lebesgue measure, m, on the unit circle. In this case, there is a Borel subset E of ∂D with $m(E) = 1$ such that the conformal map performs a one-to-one correspondence between the *Borel* subsets of E and $\varphi(E)$.

Let K be a compact subset of $\mathbf{C}$. A uniform algebra $A \subset C(K)$ is called a Dirichlet algebra if $\{\mathrm{Re}(f) : f \in A\}$ is uniformly dense in $C(K)$. The algebra A is a hypodirichlet algebra if the uniform closure of the space $\{\mathrm{Re}(r) : r \in A\}$ has finite codimension in $C_R(\partial K)$ and the linear span of $\{log|f| : f, f^{-1} \in A\}$ is dense in $C_R(K)$. In Chapter 4 we will describe the similarity class of the shift on the $R^2(\sigma)$, where σ is harmonic measure of K.

Let $A(G) = \{f : f \in C(\overline{G})$ and f is analytic on $G\}$. The following theorem, which can be found in [20], characterizes nicely connected domains and will be used repeatedly in this thesis.

Theorem 1.2 (Davie; Gamelin-Garnett) *Let G be a simply connected domain in the complex plane. The following are equivalent:*

1) Every bounded analytic function on G is the pointwise limit on G of a bounded sequence in $A(G)$.

2) G is a nicely connected domain.

3) $A(G)$ is a Dirichlet algebra on ∂G.

The following lemma is also used many times in this thesis.

Lemma 1.2 *If G is a nicely connected domain, then for each $f \in L^1(\omega)$*

$$\int f d\omega_a = \int f \circ \varphi dm$$

and for each $g \in L^1(m)$

$$\int g \circ \psi d\omega_a = \int g dm.$$

Here, m *is normalized Lebesgue measure on the unit circle,* φ *is the unique conformal map of* D *onto* G *with* $\varphi(0) = a$ *such that* $\omega_a = m \circ \varphi^{-1}$ *and* $\psi = \varphi^{-1}$.

Hardy Spaces $H^t(G)$

Let G be a bounded domain and let $t \in [1, \infty)$. The Hardy space $H^t(G)$ is defined to be the set of all analytic functions f such that $|f|^t$ has a harmonic majorant on G. The norm of a function f in the Hardy space $H^t(G)$ can be defined as follows: Fix a point z_0 in G. Then the norm $\|f\| = u(z_0)^{1/t}$, for $f \in H^t(G)$, where u is the least harmonic majorant of $|f|^t$ (see [22]). Under this norm, $H^t(G)$ is a Banach space. Harnack's inequality guarantees that the norms of $H^t(G)$ induced by two points of G are equivalent. When $t = \infty$, the space $H^\infty(G)$ is the Banach space of bounded analytic functions on G equipped with the supremum norm.

In the case that G is a nicely connected domain, we can embed $H^t(G)$ in $L^t(\omega)$. In fact, from the discussing earlier, there is a conformal map φ of D onto G that effects a point isomorphism between the measure spaces $(\partial D, m)$ and $(\partial G, \omega)$. Let $\tilde{f}$ be the boundary value function of $f \in H^t(D)$. Now fix $h \in H^t(G)$. It also has boundary value, $\tilde{h}$, where $h = (\widetilde{h \circ \varphi}) \circ \tilde{\psi}$. This mapping, $h \to \tilde{h}$, defines an isometric isomorphism of $H^t(G)$ onto its image in $L^t(\omega)$. In this thesis we shall not distinguish between $h \in H^t(G)$ and $\tilde{h} \in L^t(\omega)$.

The Sweep of a Measure

Let U be a bounded Dirichlet domain and μ be a finite positive measure on $\overline{G}$. Let $C(\partial G)$ denote the set of continuous functions on ∂G. For $u \in C(\partial G)$, let $\hat{u}$ be the unique function that is continuous on $\overline{G}$ and harmonic in G ([17, 24] or [67]). Then $u \to \int_{\overline{G}} \hat{u}\, d\mu$ defines a positive functional on $C(\partial G)$ with norm one. So using *the Riesz representation theorem*, there is a unique positive measure $\hat{\mu}$ so that

$$\int_{\overline{G}} \hat{u}\, d\mu = \int_{\partial G} u\, d\hat{\mu}, \quad u \in C(\partial G).$$

The measure $\hat{\mu}$ is called the **sweep** of μ.

Theorem 1.3 *If μ is a measure on $\overline{G}$, then*

$$\hat{\mu} = \mu|\partial G + \widehat{\mu|G}.$$

Subnormal Operators

Let H be a Hilbert space. An operator N is normal if $NN^* = N^*N$. An operator S on H is said to be a subnormal operators it there exists a Hilbert space K that contains H and there exists a normal operator N such that $NH \subset H$ and S is the restriction of N to H. A vector $v \in H$ is called a cyclic vector for S if the set $\{p(S)v : p \in \mathcal{P}\}$ is dense in H; the vector v is called a rationally cyclic vector, if $\{r(S)v : r \in R(\sigma(S))\}$ is dense in H. The operator S is called cyclic if S has a cyclic vector; and the operator S is called rationally cyclic if it has a rationally cyclic vector.

The symbol S_μ always stand for the operator induced by the multiplication by z on $P^2(\mu)$; that is, $S_\mu f = zf$ for all $f \in P^2(\mu)$; and R_μ stands for the operator induced by the multiplication by z on $R^2(K, \mu)$. Here K is a compact set containing the support of the measure μ.

Theorem 1.4 (Bram and S. Singer) *1) A subnormal operator is cyclic if and* *only if there exists a positive finite measure μ with compact support such that S is unitarily equivalent to S_μ.*

2) A subnormal operator is rationally cyclic if and only if there exists a positive finite measure μ with compact support such that S is unitarily equivalent to R_μ.

Chapter 2

Density of Polynomials

Let X be a Banach space of analytic functions on some bounded domain $G \subseteq \mathbf{C}$ such that $\mathcal{P} \subseteq X$. A natural question is to inquire if

$$\mathcal{P} \text{ is dense in } X? \tag{1.1}$$

We could also ask if

$$\mathcal{P} \quad \text{is weak-star dense in } X \tag{1.2}$$

when X is the dual of another Banach space.

We first address question (1.1) for the case that X is equal to $H^t(G)$. Theorem 2.1 gives several equivalent conditions for (1.1) to hold in this case. A consequence of this result is a characterization of those analytic Toeplitz operators, T_f, on $H^2(D)$ that have 1 as a cyclic vector. In the study of question (1.2) where $X = H^\infty(G)$, we obtain a characterization of a perfectly connected domain in terms of if its Riemann map can be approximated by polynomials (a domain is perfectly connected if its the image of a weak-star generator of $H^\infty(D)$).

During this course of investigation, we also obtain a new characterization of a

Carathéodory domain - Definition: a domain that is a component of the interior of the polynomially convex hull of its closure. We then turn our study to question (1.1) where $X = L_a^t(G)$, the Bergman space consisting of all analytic funstions f such that $\int_G |f| dA < \infty$here A denotes area measure on G. A generalization of Farrell's theorem [23] is given: We show $\mathcal{P}$ is dense in $L_a^t(G)$ for all $t \geq 1$ when G is perfectly connected. (The case $t = 2$ was done by P. Bourdon [9] using a different method.) Another consequence arising from our study is a characterization of those analytic Toeplitz oeprators, T_f, on $H^2(D)$ that have 1 as a cyclic vector.

2.1 Density of Polynomials in Hardy Spaces

When G is a simply connected domain, then the harmonic measure on G evaluated a A, denoted ω_a, is equal to $m \circ \tilde{\varphi}^{-1}$, where φ is the conformal map of D onto G mapping 0 to a, consult [17].

For a given domain G, a point $a \in \partial G$ is called removable for $H^t(G)$ if each $f \in H^t(G)$ can extended analyticly to a neighborhood of a. S. Axler showed that each isolated boundary point is removable for $H^t(G)$.

The next theorem is our main result concerning the density of polynomials in Hardy spaces in this chapter.

Theorem 2.1 *Let $t \in [1, \infty)$ and let G be a bounded domain with harmonic measure ω such that no point of ∂G is removable for $H^t(G)$. The following are equivalent:*

1) $\mathcal{P}$ is dense in $H^t(G)$.

2) $abpe P^t(\omega) = G$.

3) G is a nicely connected domain and if ψ is a Riemann map of G onto D, then its boundary value function on ∂G belongs to $P^t(\omega)$.

4) G is simply connected and if ψ is a Riemann map of G onto D, then ψ can be approximated in $H^t(G)$ by polynomials.

Proof. 1) $\Longrightarrow$ 2): Suppose that $\mathcal{P}$ is dense in $H^t(G)$. For each $a \in G$, it follows that by the definition of harmonic measure

$$p(a) = \int p \ d\omega_a, \quad p \in \mathcal{P}.$$

By Hölder's inequality, we see $a \in bpeP^t(\omega)$ and so $G \subseteq bpeP^t(\omega)$. Now using Harnack's inequality we can easily show that every *bpe* is an *abpe*, hence, $G \subseteq abpeP^t(\omega)$.

On the other hand, if $h \in H^t(G)$, by our hypothesis there exists a sequence of polynomials $\{q_n\}$ such that $q_n \to h$ in $H^t(G)$. So $\{q_n\}$ is a Cauchy sequence in $H^t(G)$. But, if we let $\widehat{|q_n|}^t$ denote the harmonic extension of $|q_n|^t$ on G and we choose the evaluated point of the harmonic measure of G properly, we have

$$\begin{aligned} \{\int |q_n|^t \ d\omega\}^{\frac{1}{t}} &= \{\int \widehat{|q_n|}^t \ d\omega\}^{\frac{1}{t}} \\ &= \|q_n\|_{H^t(G)}. \end{aligned}$$

This implies that $\{q_n\}$ is a Cauchy sequence in $P^t(\omega)$; and hence $\{q_n\}$ uniformly converges to an analytic function g on compact subsets of $abpeP^t(\omega)$ (see [17, p.171]). But g clearly is an extension of h. So it follows by our hypothesis that $abpeP^t(\omega) \subset G$. Hence $abpeP^t(\omega) = G$.

2) $\Longrightarrow$ 3): Assume that $abpeP^t(\omega) = G$. Let ψ be a Riemann map of G onto D and let φ be the inverse of ψ. Without loss of generality we may assume that $\omega = m \circ \varphi^{-1}$. By Thomson's theorem [64], there exists a unique function

$f \in P^t(\omega) \cap L^\infty(\omega)$ such that

$$\psi(z) = \hat{f}(z), \quad z \in G.$$

Choose a sequence $\{p_n\} \subset \mathcal{P}$ such that

$$\int |p_n - f|^t d\omega \to 0.$$

Then

$$\int |p_n \circ \varphi - f \circ \varphi|^t dm \to 0.$$

Thus, $f \circ \varphi \in P^t(m)$. Since $p_n \to \psi$ pointwise on G, it follows that $p_n \circ \hat{\varphi} \to \hat{z}$ (here, we regard φ and z as functions in $P^t(m)$. Also note that $abpeP^t(m) = D$) pointwise on D. This implies that $p_n \circ \varphi \to z$ in $P^t(m)$. Hence $f \circ \varphi = z$ in $P^t(m)$. This clearly implies that φ is one-to-one a.e. $[m]$. Hence G is nicely connected. Apparently,

$$f = \varphi^{-1} = \psi, \text{ a.e. } [\omega] \text{ on } \partial G.$$

Hence $\psi \in P^t(\omega)$.

3) $\Longrightarrow$ 1): Assume that G is a nicely connected domain and assume that $\psi \in P^t(\omega)$. Clearly

$$\mathcal{P} \circ \psi = \{p \circ \psi : p \in \mathcal{P}\} \subseteq P^t(\omega).$$

Let φ be the inverse function of ψ. Since $h \circ \varphi \in H^t(D)$ for each $h \in H^t(G)$, there is a sequence of polynomials $\{p_n\}$ such that

$$\int |h \circ \varphi - p_n|^t \ dm \rightarrow 0.$$

So for such a function h we see

$$\int |h \circ \varphi \circ \upsilon - p_n \circ \psi|^t \ d\omega \rightarrow 0.$$

This means that, $\tilde{h}$, the boundary value function of h, is in the closure of $\mathcal{P} \circ \psi$. Thus,

$$\widetilde{H}^t(G) \subset P^t(\omega).$$

On the other hand, clearly $P^t(\omega) \subseteq \widetilde{H}^t(G)$; and so $P^t(\omega) = \widetilde{H}^t(G)$. Now if $h \in H^t(G)$, then there is a sequence of polynomials $\{q_n\}$ such that

$$\int |\tilde{h} - q_n|^t \ d\omega \rightarrow 0.$$

If we choose the evaluated point of the harmonic measure of G properly, then

$$\{\int |\tilde{h} - q_n|^t \ d\omega\}^{\frac{1}{t}} = \|h - q_n\|_{H^t(G)}.$$

It follows that h can be approximated by polynomials in $H^t(G)$. Since h is arbitrary, we conclude that $\mathcal{P}$ is dense in $H^t(G)$.

1) $\Longrightarrow$ 4): This is obvious.

4) $\Longrightarrow$ 3): Suppose that a Riemann map ψ can be approximated by a sequence

of polynomials $\{q_n\}$ in $H^t(G)$. That is, suppose

$$\int |\psi - q_n|^t \ d\omega \to 0.$$

Therefore,

$$\int |z - q_n \circ \varphi|^t \ dm \to 0.$$

By passing to a subsequence if necessary, this implies that $q_n \circ \varphi(z) \to z$ almost everywhere on ∂D with respect to m, from which it follows clearly that φ is univalent almost everywhere on ∂D. Hence G is nicely connected.

Now we can define $\tilde{\psi}$, the boundary function of ψ on ∂G, in $L^t(\omega)$; and clearly $\tilde{\psi}$ is approximable by polynomials in $P^t(\omega)$. The proof of the theorem is completed.

$\square$

2.2 Analytic Toeplitz Operators

Let $f \in L^\infty(m)$ and let Q be the orthogonal projection of $L^2(m)$ onto $P^2(m)$. Recall that a Toeplitz operator T_f with symbol f is defined by $T_f(h) = Q(fh)$ for each $h \in P^2(m)$. If $f \in P^\infty(m)$, then T_f is called an analytic Toeplitz operator. Ronald Douglas's book [21] is an excellent reference for the Toeplitz operators.

The problem of when an analytic Toeplitz operator is cyclic or has the function 1 as a cyclic vector has been studied by a number of authors. Recently it has been shown [4] that there is a cyclic analytic Toeplitz operator T_f such that 1 is not a cyclic vector for T_f.

The following result is a consequence of Theorem 2.1.

Theorem 2.2 *Let $g \in P^\infty(m)$. An analytic Toeplitz operator T_g has 1 as a cyclic* *vector if and only if $\hat{g}$ is conformal on D and the inverse of $\hat{g}$ can be approximated by polynomials in $H^2(\hat{g}(D))$, where $\hat{g}(D)$ is the image of D under $\hat{g}$.*

Proof. First, it is well-known that $\hat{f} \in H^2(D)$ for every $f \in P^2(m)$. Suppose that $\hat{g}$ is conformal and $\hat{g}^{-1}$ can be approximated by polynomials in $H^2(\hat{g}(D))$. We define an operator U from $H^2(\hat{g}(D))$ to $H^2(D)$ by

$$U(f) = f \circ \hat{g}, \quad f \in H^2(\hat{g}(D)).$$

We can choose a proper norm on $H^2(\hat{g}(D))$ so that for each $f \in H^2(\hat{g}(D))$

$$\begin{aligned} \|f\|_{H^2(\hat{g}(D))} &= \|U(f)\|_{H^2(D)} \\ &= \|f \circ \hat{g}\|_{H^2(D)} \\ &= \{\int |\widetilde{f \circ \hat{g}}|^2 dm\}^{\frac{1}{2}}. \end{aligned}$$

where $\widetilde{f \circ \hat{g}}$ is the boundary value of $f \circ \hat{g}$ on ∂D. This says that U is an isometry. Clearly $H^\infty(D)$ is contained in the range of U. Hence, U is an isometrical isomorphism. Moreover, for each $f \in H^2(\hat{g}(D))$

$$\begin{aligned} UM_z(f) &= U(zf) \\ &= (z \circ \hat{g})(f \circ \hat{g}) \\ &= \hat{g}(f \circ \hat{g}) \\ &= M_{\hat{g}}U(f) \end{aligned}$$

where $M_{\hat{g}}$ is the operator induced by the multiplication by $\hat{g}$ on $H^2(D)$. Hence, M_z and $M_{\hat{g}}$ are unitarily equivalent. It is easy to see that $M_{\hat{g}}$ can be identified with T_g; it follows that M_z and T_g are unitarily equivalent. Using our hypothesis and

Theorem 2.1, we see that $\mathcal{P}$ is dense in $H^2(\hat{g}(D))$. It follows that 1 is a cyclic vector of M_z. Consequently, 1 is a cyclic vector of T_g.

Conversely, assume that 1 is a cyclic vector of T_g. Choose a sequence $\{p_n\} \subset \mathcal{P}$ such that $p_n(T_g)1 \to z$ in $P^2(m)$. That implies that $p_n(\hat{g}) \to z$ in $H^2(D)$. In particular, we see that $p_n(\hat{g}) \to z$ pointwise on D. Hence, $\hat{g}$ must be univalent on D. As before, we have that M_z and T_g are unitarily equivalent. The rest of the proof can be done by reversing the steps in the previous argument.

□

2.3 On perfectly connected domains and Carathéodory domains

The weak-star topology of $H^\infty(D)$ can be defined as follows: The map $f \to \tilde{f}$ from $H^\infty(D)$ to $P^\infty(m)$ is an isometrical isomorphism. Since $P^\infty(m)$ is the dual space of $L^1(m)/P^\infty(m)^\perp$, we have a weak-star topology on $H^\infty(D)$ that is induced from $P^\infty(m)$ via the isometrical isomorphism. A sequence $\{f_n\}$ in $H^\infty(D)$ converges to a function f in $H^\infty(D)$ if and only it is uniformly bounded and converges to f pointwisely on D [58, Lemma 1]. For a simply connected domain G, a conformal map of D onto G induces an isometrical isomorphism from $H^\infty(G)$ onto $H^\infty(D)$ in the obvious way. Thus the weak-star topology on $H^\infty(G)$ is defined to be the one induced from $H^\infty(D)$ by that map. Therefore, a sequence in $H^\infty(G)$ is a weak-star Cauchy sequence if and only if it is uniformly bounded on G and a Cauchy sequence in the topology of pointwise convergence[1].

We begin our study of problem 1.2, if the polynomials are weak-star dense, with

[1] Actually, it is well-known that the predual of $P^\infty(m)$ is unique [6, Ando]. Using Lemma 5.7 of [64] we see that the predual of $H^\infty(G)$ is unique too. Therefore, the weak-star topology on $H^\infty(G)$ is uniquely defined.

the following lemma.

Lemma 2.1 *For a bounded simply connected domain G, $\mathcal{P}$ is weak-star dense in $H^\infty(G)$ if and only if G is a perfectly connected domain.*

Proof. The proof of necessity. Suppose that $\mathcal{P}$ is weak-star dense in $H^\infty(G)$. Then $\mathcal{P} \circ \varphi$ is weak-star dense in $H^\infty(D)$, where φ is a conformal map of D onto G. It follows by definition that φ is a weak-star generator; hence, G is perfectly connected.

The proof of sufficiency is also straightforward. In fact, if $G = g(D)$ for some weak-star generator, g, then $\{p \circ g : p \in \mathcal{P}\}$ is weak-star dense in $H^\infty(D)$ and therefore, $\mathcal{P}$ is weak-star dense in $\{f \circ g^{-1} : f \in H^\infty(D)\} = H^\infty(G)$. (D. Sarason shows [57] that g is 1-1 a.e. [m] if g is a weak-star generator. So g^{-1} does exists.)

$\square$

Theorem 2.3 *A bounded simply connected domain G is perfectly connected if and only if the Riemann map of G onto D can be weak-star approximated by polynomials in $H^\infty(G)$.*

Proof. Suppose a Riemann map, ψ, of G can be weak-star approximated by polynomials in $H^\infty(D)$. Let φ be the inverse of a Riemann map of G. Without loos of generality, we may assume that $\omega = m \circ \varphi^{-1}$. Choose a net of polynomials $\{p_\alpha\}$ in $H^\infty(G)$ such that $\{p_\alpha \circ \varphi\}$ weak-star converges to z. We show z belongs to the closure of $\{p \circ \varphi : p \in \mathcal{P}\}$ in $P^2(m)$. In fact, since a convex set is norm closed if and only if it is weakly closed. Since p_α belongs to the weak closure, it belongs the weak-star closure. Hence the conclusion follows.

Now we want to show that G is a nicely connected domain. Let $\{q_n\}$ be a sequence of polynomials such that $\{q_n \circ \varphi\}$ converges to z in $P^2(m)$. By passing to a subsequence if necessary, we have that $q_n \circ \varphi \to z$ almost everywhere on ∂D; this clearly implies that φ is univalent almost everywhere on the unit circle. Hence, G is nicely connected. Now every $f \in H^\infty(G)$ has a well-defined boundary value function $\tilde{f}$. It follows by our hypothesis that

$$\tilde{\psi} \in P^\infty(\omega).$$

Let $f \in H^\infty(G)$ (we also use f to denote its boundary value function). Since $f \circ \varphi \in H^\infty(D)$, there exists a sequence of polynomials $\{q_n\}$ such that $\{q_n\}$ converges weak-star to $f \circ \varphi$ in $L^\infty(m)$. This, in turn, is equivalent to saying that $\{q_n \circ \psi\}$ converges weak-star to f in $L^\infty(\omega)$. Since $q_n \circ \psi$ is in $P^\infty(\omega)$ for all n, it follows that $f \in P^\infty(\omega)$. Hence $H^\infty(G) \subseteq P^\infty(\omega)$. Obviously $P^\infty(\omega) \subseteq H^\infty(G)$, so $P^\infty(\omega) = H^\infty(G)$. Hence $\mathcal{P}$ is weak-star dense in $H^\infty(G)$. By Lemma 2.1, we see that G is perfectly connected.

□

For a compact subset K of the complex plane, the *outer boundary* of the set K is defined to be the boundary of the unbounded component of its complement.

A domain G in the plane is called a Carathéodory domain if its boundary is equal to the outer boundary of its closure. The interior of the polynomially convex hull of $\overline{G}$ is known as the Carathéodory hull of G; and we denote it by G^*. A domain G is a Carathéodory domain if and only if it is equal to a component of G^*.

In 1934, O. J. Farrell showed [23] that a domain G is a Carathéodory domain if and only if each bounded analytic function on G is the pointwise limit on G of a bounded sequence of polynomials in $H^\infty(G)$. Nearly three decades later, Rubel and Shield extended this result to an arbitrary bounded open set in the plane [56].

The following theorem gives a new characterization of a Carathéodory domain.

Theorem 2.4 *A simply connected domain G is a Carathéodory domain if and only if a Riemann map of G onto D is the pointwise limit on G of a bounded sequence of polynomials in $H^\infty(G)$.*

Proof. Let ψ be a Riemann map of G onto D and assume that ω is the harmonic measure evaluated at $\psi^{-1}(\{0\})$. Suppose that ψ is the pointwise limit of a bounded sequence of polynomials $\{q_n\}$. By Lemma 2.1 and Theorem 2.2, we have

$$P^\infty(\omega) = H^\infty(G).$$

So for $f \in H^\infty(G)$, there exists a sequence of polynomials $\{p_n\}$ such that $\{p_n \circ \psi\}$ converges weak-star to f (since, every function in $H^\infty(D)$ is the pointwise limit on D of a bounded sequence of polynomials). Set

$$r_n = p_n \circ q_n.$$

We want to show:

$$\{r_n\} \quad \text{is bounded and} \quad \{r_n\} \quad \text{converges to } f \quad \text{pointwise.}$$

The former is clear. For the proof of the latter, let $z_0 \in G$ and let $\delta > 0$. Since $\{q_n\}$ is uniformly convergent to ψ on every compact subset of G, it follows that there is an integer N_0 and $c > 0$ so that when $n > N_0$,

$$|q_n(z) - \psi(z)| < \delta/3 \qquad \text{for} \quad |z - z_0| < c.$$

Note, $\{p_n\}$ is a normal family; thus, by the Arzela-Ascoli theorem, $\{p_n\}$ is equicontinuous. We can find $0 < d < \delta/2$ such that for each n

$$|p_n(u) - p_n(w)| < \delta/3 \qquad \text{whenever} \quad |u - w| < d.$$

Since $\{p_n\}$ uniformly converges to $f \circ \varphi(w)$ on compact subset of D, there is an integer N_1 so that when $n > N_1$

$$|p_n(w) - f \circ \varphi(w)| < \delta/3 \qquad \text{on} \quad \{w : |w - \psi(z_0)| < d\},$$

where φ is the inverse function of ψ. Set $M = max\{N_0, N_1\}$. Then

$$\begin{aligned} |p_n \circ q_n(z_0) - f(z_0)| &\leq |p_n \circ q_n(z_0) - p_n \circ \psi(z_0)| + |p_n \circ \psi(z_0) - f(z_0)| \\ &\leq |p_n \circ q_n(z_0) - f \circ \psi(z_0)| + |f \circ \psi(z_0) - p_n(\psi(z_0))| \\ &+ |p_n \circ \psi(z_0) - f(z_0)| \\ &\leq \delta/3 + \delta/3 + \delta/3 \\ &= \delta \qquad\qquad \text{when} \quad n > M. \end{aligned}$$

Thus, $\{r_n\}$ converges to f at every point of G. Since $\{r_n\} \subset \mathcal{P}$ and since it is uniformly bounded on $\overline{G}$, by passing a subsequence if necessary, we see that $\{r_n\}$ pointwise converges to a bounded function that is analytic on G^*, the Carathéodory hull of G. This function clearly is the analytic extension of f on G^*. Since f is an arbitrary element in $H^\infty(G)$, it follows that G must be equal to a component of G^*. Hence, G is a Carathéodory domain.

The converse can be proved by using an argument of Farrell in [23] and a result of Carathéodory. In fact, let K be the polynomially convex hull of $\overline{G}$. Then there is a sequence of Jordon domains $\{G_n\}$ such that for each natural number n

$$G \subset \overline{G_{n+1}} \subset G_n, \quad \text{and} \quad \bigcap G_n = K.$$

Fix a point $a \in G$. For each n, let ψ_n be the Riemann map of G onto D such $\psi_n(a) = 0$ and $\psi_n(a) > 0$. Since $\partial G = \partial K$, the sequence $\{G_n\}$ converges to G in the Carathéodory sense and $\{\psi_n\}$ uniformly converges to a Riemann map of G on compact subsets of G[2]. Since each ψ_n can be uniformly approximated by polynomials on $\overline{G}_n$ (by Runge's theorem), the conclusion clearly follows.

□

2.4 Density of Polynomials in Bergman Spaces

For a bounded domain G and a number$t$ with $1 \leq t < \infty$, the Bergman space $L^t_a(G)$ is the Banach space of all analytic functions f on G such that

$$\int |f|^t \; dA < \infty,$$

where A is area measure on G. Over many years, the problem of deciding when $\mathcal{P}$ is dense in $L^t_a(G)$ has attracted considerable interest (see [23], [40], [41], [42], [10], [62], [63] and [9]). In 1934 Farrell and Markusevic proved that $\mathcal{P}$ is dense in $L^t_a(G)$ if G is a Carathéodory domain. Is this result also valid for a perfectly connected domain? (Note, a Carathéodory domain is clearly perfectly connected.) Using Hilbert space method, P. S. Bourdon showed in [9, 1987] that the answer is 'yes' when $t = 2$. We

[2]See [12, p.76]. In Farrell's original paper [23, p.708], it is not clear whether his proof works or not for a domain G with G^* having has more than one component, a fact pointed out in both [56] and [25]. When G^* has just one component, the proof of this fact is an application of Runge's theorem and an elementary fact about conformal maps [56, p.148] and [25, p.152]. If G^* has more than one component, the proof is not obvious. Fortunately, the argument can be carried over in general. The interested reader should consult [18, p.71].

now show that $\mathcal{P}$ is dense in the Bergman space $L_a^t(G)$ for each $t \in [1, \infty)$ if G is perfectly connected.

Theorem 2.5 *If G be perfectly connected, then $P^t(A) = L_a^t(G)$ for all $t \geq 1$.*

For the proof of this theorem, we need some lemmas.

The next lemma is elementary.

Lemma 2.2 *Let μ be a finite positive measure. then $P^\infty(\mu) \subseteq P^t(\mu)$ for all $t \in [1, \infty)$.*

Lemma 2.3 *If G is perfectly connected, then $\mathcal{P}$ is dense in $H^t(G)$ for each $t \geq 1$*

Proof. Fix a number $t \in [1, \infty)$. Suppose that G is perfectly connected and let ω be the harmonic measure of G. Applying Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned} H^\infty(G) &= P^\infty(\omega) \\ &\subseteq P^t(\omega) \\ &\subseteq H^t(G). \end{aligned}$$

This clearly implies that $\mathcal{P}$ is dense in $H^t(G)$ (since, $H^\infty(G)$ is dense in $H^t(G)$).

$\square$

The proof of Theorem 2.5: Suppose that G is perfectly connected. First we want to show

$$H^\infty(G) = P^\infty(A) \qquad (*)$$

Fix a number t with $1 \leq t < \infty$. We can find a positive measure μ with the property that $[\mu] \equiv [A]$ such that (see [64], or [37])

$$P^\infty(A) = P^t(\mu) \cap L^\infty(\mu).$$

Let $W = abpeP^t(\mu)$. It follows from Thomson's theorem that

$$P^\infty(A) = P^t(\mu) \cap L^\infty(\mu) = H^\infty(W).$$

Let $\hat{\mu}$ be the sweep of μ (see [18, p.201]). Since $\mu|\partial G = 0$, it follows that $\hat{\mu} \ll \omega$. This implies that

$$P^\infty(\omega) \subset P^\infty(\hat{\mu}).$$

Also, using the maximum principle for harmonic functions, we have

$$\int |p|^t d\mu \leq \int |p|^t d\hat{\mu}, \quad p \in \mathcal{P}.$$

This implies that $P^t(\hat{\mu})$ can be regarded as a subspace of $P^t(\mu)$. Now it follows that

$$\begin{aligned} H^\infty(G) &= P^\infty(\omega) \\ &\subset P^\infty(\hat{\mu}) \\ &\subset P^t(\hat{\mu}) \cap L^\infty(\hat{\mu}) \\ &\subset P^t(\mu) \cap L^\infty(\mu) \\ &= H^\infty(W) \\ &= P^\infty(A). \end{aligned}$$

Obviously, $H^\infty(G) \supseteq P^\infty(A)$. Hence $H^\infty(G) = P^\infty(A)$. Using Lemma 2.4, we have

$$P^\infty(A) \subseteq P^t(A) \subseteq L^t_a(G).$$

So

$$H^\infty(G) \subseteq P^t(A) \subseteq L^t_a(G).$$

Now the density of the polynomials in $L^t_a(G)$ follows from Hedberg's theorem, which says that $H^\infty(G)$ is dense in $L^t_a(G)$ for all $t \geq 1$ (see [7] for a proof).

The case that $t = \infty$ is just $(*)$. So we are done.

$\square$

Chapter 3

Carleson Measure and Polynomial Approximation

Let G be a simply connected domain with harmonic measure ω. We call a positive measure τ on G a Carleson measure if there is a positive constant c such that for each $t \in [1, \infty)$ and all $p \in \mathcal{P}$ we have

$$\|p\|_{L^t(\tau)} \leq c \, \|p\|_{L^t(\omega)}$$

A well-known theorem of L. Carleson [28, p.238] says that a measure τ on the unit disc D is a Carleson measure if and only if there exists a constant $A > 0$ such that

$$\tau(C_h) \leq Ah,$$

for each Carleson square

$$C_h = \{z = re^{it} : 1 - h \leq r < 1; t_0 \leq t \leq t_0 + h\}.$$

Now the question is:

What are the Carleson measures on G?

The purpose of this chapter is to answer this question. Using Theorem 2.1, we prove our main result in this section; it describes all Carleson measures on a normal domain and offers a characterization of such a domain via the Carleson measures on it. In particular, we show the density property of $\mathcal{P}$ in $H^t(G)$ is independent of t. As a corollary, we obtain that $abpeP^t(\omega)$ is the same for each $t \in [1, \infty)$ if G is a crescent (A simply connected domain is called a crescent if its enclosed by two Jordon curves.) This result was proved by J. Akeroyd recently for a large class of crescents [3]. The author has been informed that J. Thomson has shown this is not the case for a general measure [65]; i.e., the set $abpeP^t(\omega)$ does change with the index t.

We conclude this chapter by describing the inclusion relationships among the types of simply connected domains we have discussed in this thesis.

3.1 The Main Result

Lemma 3.1 *Let G be a simply connected domain with harmonic measure ω and let φ be a conformal map of D onto G. Suppose τ is a positive measure on G. If $\tau \circ \varphi$ is a Carleson measure on D, then τ is a Carleson measure on G.*

Proof. Suppose that $\tau \circ \varphi$ is a *Carleson* measure on D. There is a constant $c > 0$ such that for all $t \in [1, \infty)$

$$\|q\|_{L^t(\tau\circ\varphi)} \leq c\, \|q\|_{L^t(m)}, \quad q \in \mathcal{P}.$$

Fix a polynomial q. Since $q \circ \varphi \in H^\infty(D)$, it follows by a classical result that $q \circ \varphi$ is a pointwise limit on D of a bounded sequence in $\mathcal{P}$. By the Lebesgue dominated convergence theorem we see that $q \circ \varphi$ can be approximated by polynomials in $L^t(\tau \circ \varphi)$. Using the inequality above, we have for each $t \in [1, \infty)$

$$\begin{aligned} \|q\|_{L^t(\tau)} &= \|q \circ \varphi\|_{L^t(\tau\circ\varphi)} \\ &\leq c\,\|q \circ \tilde{\varphi}\|_{L^t(m)}. \end{aligned}$$

By definition, $\omega = m \circ \tilde{\alpha}^{-1}$ for some conformal map α of D onto G. Since any two harmonic measures of a domain are boundedly equivalent, it follows that if we let $m_1 = m \circ (\tilde{\varphi}^{-1} \circ \alpha)$ (for a subset $B \subset \partial G$, $m_1(B) = m(\tilde{\varphi}(\alpha(B)))$), then there exists a constant c_1 such that

$$\begin{aligned} \|q \circ \tilde{\varphi}\|_{L^t(m)} &= \|q \circ \tilde{\alpha} \circ (\tilde{\alpha}^{-1} \circ \tilde{\varphi})\|_{L^t(m)} \\ &= \|q \circ \tilde{\alpha}\|_{L^t(m_1)} \\ &\leq c_1\,\|q \circ \tilde{\alpha}\|_{L^t(m)} \\ &= c_1\,\|q\|_{L^t(\omega)}. \end{aligned}$$

Thus, we conclude

$$\|q\|_{L^t(\tau)} \leq cc_1\,\|q\|_{L^t(\omega)}.$$

Since q is arbitrary in $\mathcal{P}$, it follows by Carleson's theorem that τ is a *Carleson* measure.

$\square$

Theorem 3.1 *Let G be a simply connected domain with harmonic measure ω. The* *following are equivalent:*

1) Every Carleson measure τ on G has the form $\tau = \eta \circ \alpha^{-1}$ for a Carleson measure η on D and a conformal map α of D onto G.

2) $\mathcal{P}$ is dense in $H^t(G)$ for all $t \in [1, \infty)$.

3) G is a normal domain.

Remark: Another characterization of Carleson measures on a normal domain G can be found in Chapter 4: there, we characterize all subnormal operators that are similar to S_ω in terms of Carleson measures on G.

Proof. 1) $\Longrightarrow$ 2). Let ω denote the harmonic measure of G. For a fixed t, we know, by Theorem 2.1, that $\mathcal{P}$ is dense in $H^t(G)$ if and only if $abpeP^t(\omega) = G$. Assume that $\mathcal{P}$ is not dense in $H^t(G)$ for some $1 \le t < \infty$, then

$$G \neq abpeP^t(\omega).$$

Set

$$W = abpeP^t(\omega).$$

Using the fact that each ω_a is a representing measure at a for the function algebra $A(G)$ and using Harnack's inequality, one can easily show that

$$G \subset W \quad \text{ and } \quad \partial W \subset \partial G.$$

It follows that

$$\partial G \cap W \neq \emptyset.$$

Therefore, there exists a point $x \in \partial G \cap W$ and an open disc $V \subset W$ so that $x \in V$.

Let γ be a diameter of V that is not contained in ∂G. Then γ separates V into two disjoint open semi-discs. Obviously, at least one of such semi-discs has non-empty intersection with ∂G because of the simply connectivity of G. Let us denote this open semi-disc by V^+ and set

$$E = \partial V^+ \cup \partial G.$$

Since ∂G is connected, it follows clearly that E is a connected compact subset; and hence $V^+ - E$ is open. Let

$$V^+ - E = V_1 \cup V_2 \cup \ldots \cup V_n \ldots \ldots,$$

where each V_n is a component of $V^+ - E$. Fix a number n. Now we claim:

$$\omega(\partial V_n \cap \partial V^+) \neq 0.$$

To prove the claim, it is sufficient to show

$$\sigma_a(\partial V_n \cap \partial V^+) \neq 0 \quad \text{for some} \quad a \in V_n$$

by the maximum principle of harmonic functions, where σ_a is the harmonic measure of V_n evaluated at a. Set

$$F = \partial V_n \cap \partial V^+.$$

Remembering that G is simply connected, we see that

$$\text{neither } F = \emptyset \text{ nor } F = \partial V_+.$$

Moreover, not every component of F contains just a single point; otherwise, since

$$\partial V_n = F \cup (\partial G \cap V^+),$$

we get that $\partial V_n \subset \partial G$; this is impossible. Hence, F must contain an open arc of ∂V^+. Since V^+ is a Dirichlet domain, there there a non-zero function that is harmonic in V^+ and has zero boundary value on $\partial V^+ - F$. If we denote this function by h, then for every $a \in V_n$

$$h(a) = \int_F h \ \ d\sigma_a.$$

But, by Harnack's inequality the restriction of h to V_n is not identically zero; so we conclude

$$\sigma_a(\partial V_n \cap \partial V^+) \neq 0 \quad \text{for some} \ \ a \in V_n.$$

This proves the claim.

Let φ be a conformal map of D onto G. For the sake of convenience, we also use φ to denote its boundary on ∂D. Fix a natural number n and set

$$B = \varphi^{-1}(\partial V_n \cap \partial V^+).$$

Clearly

$$\lambda(B) \neq 0,$$

where λ is the harmonic measure of $\varphi^{-1}(V_n)$. So there exists least a point $b \in B$ such that b is an accessible point of the domain $\varphi^{-1}(V_n)$. That is, there exists a Jordon arc whose endpoints are b and another point in $\varphi^{-1}(V_n)$. For a natural number k, let I_k be the the closed arc on ∂D that is centered at b and has the length $|I_k| = \frac{1}{k}$, and let C_k be the Carleson square such that

$$\overline{C_k} \cap \partial D = I_k.$$

Since b is an accessible boundary point, for each k

$$C_k \cap \varphi^{-1}(V_n) \neq \emptyset.$$

Now let $z_k \in C_k \cap \varphi^{-1}(V_n)$ and set

$$\eta = \sum_{1}^{\infty} x_k \delta_{z_k},$$

where

$$x_k = \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}$$

and

$$\delta_{z_k}(\Delta) = \begin{cases} 1 & \text{if } z_k \in \Delta \\ 0 & \text{otherwise.} \end{cases}$$

Clearly η is a positive finite measure on D. Moreover,

$$\begin{aligned} \eta(C_k) &= \sum_{m=1}^{\infty} x_m \delta_{z_m}(C_k) \\ &\geq \sum_{m=k}^{\infty} x_m \delta_{z_m}(C_k) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=k}^{\infty}\left(\frac{1}{\sqrt{m}} - \frac{1}{\sqrt{m+1}}\right) \\
&= \frac{1}{\sqrt{k}} \\
&= \sqrt{|I_k|}.
\end{aligned}
$$

This shows that η is not a Carleson measure on D.

On the other hand, let us consider the measure $\tau = \eta \circ \varphi^{-1}$. By our construction of τ, we see that the support of τ is contained in the closed disc $\overline{V}$ (this is a compact subset of $abpeP^t(\omega)$). So it follows that there is a constant $c > 0$ such that for all $z \in \overline{V}$

$$|p(z)| \le c \, \{\int |p|^t d\omega\}^{\frac{1}{t}}, \quad p \in \mathcal{P}.$$

Consequently, we obtain

$$\{\int |p|^t d\tau\}^{\frac{1}{t}} \le (\tau(\overline{V}))^{\frac{1}{t}} \{\int |p|^t d\omega\}^{\frac{1}{t}}, \quad p \in \mathcal{P}.$$

The last inequality implies that τ is a Carleson measure on G. It is clear from our construction that τ is not of the form $\eta' \circ \alpha^{-1}$ where η' is a Carleson measure on D and α is a conformal map of D onto G. [Otherwise, $\eta = \tau \circ \varphi = \eta' \circ (\alpha^{-1} \circ \varphi)$ would be a Carleson measure on D, a contradiction.]

2) $\Longrightarrow$ 3) is obvious.

3) $\Longrightarrow$ 1) Suppose that G is a normal domain. Let τ be a Carleson measure on G. So there is $c > 0$ such that

$$\|p\|_{L^1(\tau)} \le c \, \|p\|_{L^1(\omega)}, \quad p \in \mathcal{P}.$$

Let φ be the conformal map of D onto G such that $\omega = m \circ \varphi$, where we still use φ to denote the boundary value of φ on ∂D. Let ψ be the inverse function of φ. By hypothesis, $\mathcal{P}$ is dense in $H^1(G)$, it follows by Theorem 2.1 that ψ has a boundary value on ∂G that is 1-1 and equal to φ^{-1} almost everywhere with respect to ω and $\psi \in P^1(\omega)$. Fix a polynomial $p \in \mathcal{P}$. So $p \circ \psi$ can be approximated by polynomials in $P^1(\omega)$. It follows from the last inequality that

$$\begin{aligned}
\|p\|_{L^1(\tau\circ\varphi)} &= \|p \circ \psi\|_{L^1(\tau)} \\
&\leq c\,\|p \circ \psi\|_{L^1(\omega)} \\
&= c\,\|p\|_{L^1(m)}.
\end{aligned}$$

Using Carleson's theorem [28, p.63, p238], we conclude that $\tau \circ \varphi$ is a Carleson measure on D.

□

Corollary 3.1 *Let G be a normal domain. A positive measure on G is a Carleson measure if and only if there is $c > 0$ such that for some $1 \leq t < \infty$*

$$\|p\|_{L^t(\tau)} \leq c\,\|p\|_{L^t(\omega)}, \quad p \in \mathcal{P}.$$

Proof. Let τ be a positive measure on G. Suppose that there exist a constant $c > 0$ such that for some $t \in [1, \infty)$

$$\|p\|_{L^t(\tau)} \leq c\,\|p\|_{L^t(\omega)}, \quad p \in \mathcal{P}.$$

By Theorem 3.1 and our hypothesis that G is normal, we see that $\mathcal{P}$ is dense in $H^t(G)$. As in the last paragraph above, if we let φ be the conformal map such that

$\omega = m \circ \varphi$, then there exists $c > 0$ so that for each $p \in \mathcal{P}$

$$\|p\|_{L^t(\tau\circ\varphi)} \quad \leq \quad c\ \|p\|_{L^t(m)}.$$

Hence, $\tau \circ \varphi$ is a Carleson measure on D. Now the conclusion follows from Theorem 3.1.

$\square$

Corollary 3.2 *If G is a crescent, then $abpeP^t(\omega)$ is the same for each $t \in [1, \infty)$.*

Proof. Let W be the Jordon domain enclosed by the outer boundary of $\overline{G}$. For a fixed $t \in [1, \infty)$, it is easy to see that

$$\text{either} \quad abpeP^t(\omega) = G, \qquad\qquad \text{or } abpeP^t(\omega) = W.$$

If $abpeP^t(\omega) = W$ for all $t \in [1, \infty)$, then we are done. Suppose that

$$abpeP^q(\omega) = G \quad \text{for some } q \in [1, \infty).$$

It follows by Theorem 2.1 that $\mathcal{P}$ is dense in $H^q(G)$. By Theorem 3.1 we get that $\mathcal{P}$ is dense in $H^t(G)$ for all $t \in [1, \infty)$. Using Theorem 2.1 again, we conclude that for all $t \in [1, \infty)$

$$abpeP^t(\omega) = G.$$

This proves the corollary.

Chapter 4

A Classification Of Some Simply Connected Domains

This chapter is devoted to describing the inclusions among the special simply connected domains we have discussed. Recall that a Jordon domain is a domain whose boundary is a Jordon curve. A famous theorem of Carathéodory says the Riemann maps of a Jordon domains G can be extended to be homeomorphisms $\overline{G}$ onto D. Using a classical result (for example, Theorem 9.1 in [25, p.48]), one can show that G is a Jordon domain if and only if a Riemann map of G onto D belongs to uniform closure (on G) of the polynomials.

A domain that is that more general than a Jordon domain is a Carathéodory domain: - Definition: a domain that is a component of the interior of the polynomial convex hull of its closure. Theorem 2.4 shows that G is a Carathéodory domain if and only if a Riemann map of G onto D is a pointwise limit on G of a bounded sequences of polynomials.

A domain G is called perfectly connected if G is the image of a weak-star generator of $H^\infty(D)$. D. Sarason characterizes those domains in [58]; his method of proof produces an algorithm for deciding if a Riemann map is weak-star generator. Theorem 2.3 shows that G is perfectly connected if and only if a Riemann map of G

onto D can be weak-star approximated by polynomials in $H^\infty(G)$. Some other characterizations of perfectly connected domains can be found in Chapter 4. Perfectly connected domains are domains that share many properities with the unit disc D.

Theorem 3.1 shows that the property $\mathcal{P}$ is dense in $H^t(G)$ for some t is a geometric one of the domain, and is independent of the norm of $H^t(G)$. Normal domains have many properties that the disc D possesses too. However, they are far away from the disc D with regards to the function theory structure on them. Reader may consult Chapter 4 for more information about function theory and operator theory in Hardy spaces on a normal domain. The simplest non-perfectly connected, normal domain is a simply connected domain obtained by deleting finitely many disjoint closed sub-discs from the unit disc D; for example, a crescent [3].

Finally, following Glicksburg [29] and Davie [20] we call G nicely connected if a conformal map of D onto G is almost one-to-one with respect to Lebesgue measure on the unit circle ∂D. A theorem of M. Davie-Gamelin-Garnett (see [20]) gives characterizations of a nicely connected domain.

All the domains we have discussed so far are nicely connected. In summary we have:

Theorem 4.1 *Let G be a simply connected domain. Then*

$$\begin{aligned} G \text{ is Jordon} &\Longrightarrow G \text{ is Carathéodory} \\ &\Longrightarrow G \text{ is perfectly connected} \\ &\Longrightarrow G \text{ is normal} \\ &\Longrightarrow G \text{ is nicely connected;} \end{aligned}$$

where the symbol $\Longrightarrow$ means 'imply'.

Chapter 5

The Equivalence Classes of Subnormal Operators

Throughout this chapter we tacitly assume that

$$G \quad \textbf{is a normal domain.}$$

Recall a simply connected domain is normal if $\mathcal{P}$ is dense in the Hardy space $H^1(G)$. Normal domains are characterized in Theorem 3.1.

Let H_1 and H_2 be Hilbert spaces and let A and B be bounded operators on H_1 and H_2, respectively. The operators A and B are unitarily equivalent if there is an isometric isomorphism U of H_1 onto H_2 such that $UA = BU$. The operators are similar if there exists an invertible operator X from H_1 to H_2 such that $XA = BX$. A weaker equivalence relation among operators is quasisimilarity. We say A and B are quasisimilar if there are one-to-one and dense-range bounded operators X: $H_1 \to H_2$ and Y: $H_2 \to H_1$ such that $XA = BX$ and $YB = AY$. We denote these equivalence relations by $\cong$, $\simeq$ and $\sim$, respectively.

In this chapter we study the following problem: When is a subnormal operator unitarily equivalent, similar, or quasisimilar to S_ω? In [14, 1973] Clary answered the latter two questions for the unilateral shift (consult [14], or [18]). W. Hastings [31, 1979] extended Clary's quasisimilar theorem to isometries of finite cyclic multiplicity. J. McCarthy [38, 1990] also extended Clary's quasisimilar result to the rationally cyclic shift operator M_z on $R^2(\sigma)$, where σ is harmonic measure on a compact subset K and $R(K)$ is a hypodirichlet algebra.

In this chapter we extend Clary's results in another direction. We generalize both his similarity result and his quasisimilarity result to the cyclic shift M_z on $H^2(G)$ in the case that G is a normal domain (when G is a normal domain, the shift M_z on $H^2(G)$ is unitarily equivalent to S_ω). We also characterize all operators that are unitarily equivalent to this shift.

We say that $x \in P^2(\omega)$ is an **outer** function if $A(\overline{G})x$ is dense in $P^2(\omega)$. Now our similarity theorem can be stated as follows:

Theorem 5.1 Let G be a normal domain with harmonic measure ω and let S be a subnormal operator. The operators S and S_ω are similar if and only if there exists a measure μ on $\overline{G}$ such that $S_\mu \cong S$ and has the following properties:

1) $abpeP^2(\mu) \subseteq G$.

2) $\mu|\partial G \ll \omega$ and $\log \frac{d\mu|\partial G}{d\omega} \in L^1(\omega)$.

3) If $x \in P^2(\omega)$ is outer and $|x|^2 = \frac{d\mu|\partial G}{d\omega}$, then $|\hat{x}|^{-2}\mu|G$ is a Carleson measure on G. (Here, $\hat{x}$ is the analytic extension of x to $abpeP^2(\omega) = G$.)

If we remove the Carleson measure property in 3) and keep the other two properties, then they are the necessary and sufficient conditions that $S_\mu \sim S_\omega$ (Theorem 5.5).

Note, the last two results were first proved by Clary in the case that G is the unit disc. Clary's theorem says that Conditions 2 and 3 are necessary and sufficient. Condition 1) is necessary in general. However, the only domain where we can keep the validity of Theorem 5.1 and remover Condition 1 is characterized in Theorem 5.3.

A classical result of Hardy space theory says that a function f in $P^2(m)$ is outer if and only if $\mathcal{P}f$ is dense in $P^2(m)$. This is no longer true, in general. At the end of this chapter we characterize all those domains on which our definition of $f \in P^2(\omega)$ being an outer function is equivalent to the condition that $\mathcal{P}f$ is dense in $P^2(\omega)$.

5.1 Outer and Inner Functions

Lemma 5.1 *Let f be a positive function in $L^1(\omega)$. In order for there to exist an outer function g in $P^2(\omega)$ such that $|g|^2 = f$ a.e. $[\omega]$, it is necessary and sufficient that* $\log f$ *is in $L^1(\omega)$.*

Proof. Assume that $\log f \in L^1(\omega)$. We may assume $\omega = \omega_a$ for a fixed point a in G. Let φ be the conformal map of D onto G with $\varphi(0) = a$ such that $\omega = m \circ \varphi^{-1}$ and let $\psi = \varphi^{-1}$. Then both $f \circ \varphi$ and $\log(f \circ \varphi)$ are in $L^1(m)$. By the classical Hardy space theory on ∂D, there exists an outer function x in $P^2(m)$ such that $|x|^2 = f \circ \varphi$. Let $g = x \circ \psi$. Then $|g|^2 = |x \circ \psi|^2 = f$ a.e. $[\omega]$ on ∂G. Choose $\{p_n\} \subset \mathcal{P}$ such that $p_n \to x$. It follows that $p_n \circ \psi \to g$ in $L^2(\omega)$. Since $\psi \in P^2(\omega)$ by Theorem 2.1, we see that $g \in P^2(\omega)$.

Since G is normal, G is nicely connected and thus $A(G)$ is a Dirichlet algebra on ∂G (Theorem 1.2); moreover,

$$\widetilde{H}^\infty(G) = P^2(\omega) \cap L^2(\omega),$$

where $\widetilde{H}^\infty(G)$ is the image of the Hardy space $H^2(G)$ under the map $f \to \tilde{f}$, the boundary value of f (note, since G is nicely connected, this map is well-defined and is one-to-one). Using Theorem 2.1 again. we have that $A(G)$ is boundedly pointwise dense in $H^\infty(G)$, and this implies that $A(G)$ is weak-star dense in $\widetilde{H}^\infty(G)$. It follows that $A(G)$ is dense $P^2(\omega)$. Using an abstract version of Szegö's theorem [32, p.103], we conclude that $A(G)g$ is dense in $P^2(\omega)$.

Conversely, assume that $f = |g|^2$ for some (outer) function $g \in P^2(\omega)$. Using the argument before, we get that $g \circ \varphi \in P^2(m)$. It follows by a result of the classical Hardy space theory that $\log |g \circ \varphi| \in L^1(m)$. Hence $\log f \in L^1(\omega)$. Now we are done.

□

We call a function f in $P^2(\omega)$ an **inner function** if $|f| = 1$ a.e. $[\omega]$. A consequence of the proof of Lemma 4.1 is the following inner and outer factorization result.

Corollary 5.1 *If $g \in P^2(\omega)$, then there is an outer function h and there is an inner function f such that $g = fh$.*

Proof. Suppose that $g \in P^2(\omega)$. Let φ be a conformal map of D onto G. As in the proof of Lemma 4.1, we have that $g \circ \varphi \in P^2(m)$. It follows by Szegö's theorem that

$$\int \log |g| d\omega = \int \log |g \circ \varphi| dm > \infty.$$

By a classical Hardy space result, there exists an outer function f_1 in $P^2(m)$ and an inner function h_1 in $P^2(m)$ so that $g \circ \varphi = f_1 h_1$. Let $f = f_1 \circ \varphi^{-1}$ and let $h = h_1 \circ \varphi^{-1}$. From the proof of Lemma 4.1, we see that f is outer and h is inner.

□

It is well-known that if G is the unit disc D, then $\mathcal{P}f$ is dense for each outer function f. (This is clear from our definition since $A(D)$ is the uniform closure of $\mathcal{P}$.) But this is not true in general. So it is natural to ask the following question.

Question 5.1 *For what domain G do we have $\mathcal{P}\{$ is dense in $P^2(\omega)$ for every outer function $f \in P^2(\omega)$.*

The answer to the question is given in Theorem 5.6.

5.2 Similarity and Unitary Equivalence

We begin this section with several lemmas. The first two lemmas are well-known and we include them here for the reader's convenience.

Lemma 5.2 *Let μ and ν be two compactly supported positive measures. If there exists a bounded operator A from $P^2(\mu)$ to $P^2(\nu)$ such that A has dense range and $S_\nu A = AS_\mu$, then $bpeP^2(\nu) \subseteq bpeP^2(\mu)$.*

Proof. Suppose that $w \in bpeP^2(\nu)$. Proposition 9.2 in [17] implies a point $w \in bpeP^2(\nu)$ if and only if $Ran[(S_\nu - w)]$ is not dense in $P^2(\mu)$. Thus, $Ran[(A)(S_\nu - w)]$ is not dense in $P^2(\nu)$. Since A has dense range, it follows that $Ran(S_\mu - w)$ is not dense in $P^2(\mu)$. Consequently, $w \in bpe(P^2(\nu))$. So the proof is complete.

□

Lemma 5.3 *Let μ and ν be two compactly supported positive measures. If S_μ is quasisimilar to S_ν, then $bpeP^2(\nu) = bpeP^2(\mu)$.*

Proof. This follows directly from the definition of the quasisimilarity and the previous lemma.

□

Note: L. Yang shows in his thesis that quasisimilar subnormal operators have the same essential spectrum [68].

Let μ and ν be two measures. We use the symbol $\mu \ll \nu$ to state that μ is absolutely continuous with respect to ν. The next lemma is crucial to our main results.

Lemma 5.4 *Let μ be a finite compactly supported measure. If S_μ is quasisimilar to S_ω, then the following hold:*

1) supp$(\mu) \subset \overline{G}$.

2) abpe$P^2(\mu) =$ abpe$P^2(\omega)$

3) $\mu|\partial G \ll \omega$ and $\log(\frac{d\mu|\partial G}{d\omega})$ is in $L^1(\omega)$.

□

To prove Lemma 5.4 we need the following lemma which was obtained by Olin and Yang for an arbitrary simply connected domain in [47]. The author generalized their result to the rational space case [54].

Lemma 5.5 *Let U be a nicely connected domain with harmonic measure ω. If μ is a finite positive measure such that abpe$P^t(\mu) = U$ and $P^t(\mu)$ is pure, then $\mu|\partial U \ll \omega$.*

First, We need the following version of *the Abstract F and M Riesz Theorem* (consult [26, p.158] or [18]).

The Abstract F & M Riesz Theorem for the Algebra $A(U)$:

Let $\eta \perp A(U)$. Then η can be expressed as a series $\eta = \sum_{j\geq 0} \eta_j$, where each $\eta_j \perp A(U)$, the η_j's are pairwise mutually singular, η_0 is singular to all representing measures for all points of $\overline{U}$, and for $j \geq 1$, η_j is absolutely continuous with respect to a representing measure for some point of $\overline{U}$.

Proof of Lemma 5.5. First, since U is nicely connected, it follows by Theorem 1.2 that $A(U)$ is a Dirichlet algebra on ∂U, and hence every point in ∂U is a peak point for $A(U)$. Second, the maximal ideal space of $A(U)$ is $\overline{U}$ (see [26]), and the nontrivial Gleason part contains U [26, Section 15], and thus every trivial Gleason part consists of a single point because $A(U)$ is a Dirichlet algebra (consult Section 15 [26] for an explanation of our terminology and results). Consequently, U is the only nontrivial Gleason part of $A(U)$.

Now let $\eta \perp A(U)$ and we claim: $\eta \ll \nu$ for a representing measure ν at some point of U. By the *Abstract F & M Riesz theorem*, we have $\eta = \sum_{j \geq 0} \eta_j$, where each η_j is as in the theorem above. Using Wilkin's Theorem (see [26, p.162]), $\eta_0 = 0$. Let a be a peak point and let $f \in A(U)$ be a peak function for a. Clearly the sequence $f^n(z)$ boundedly pointwise, $\chi_{\{a\}}$, the characteristic function of $\{a\}$ at a. Thus, $0 = \int \lim_{n \to \infty} f^n d\eta = \eta(\{a\})$. This implies that $\eta \ll \nu$ for a representing measure of $A(U)$ at some $a \in U$. We denote it by ν.

Since $A(U)$ is a Dirichlet algebra on ∂U, each point $z \in U$ has a unique representing measure whose support is contained in ∂U for $A(U)$ [26, Lemma 31.1]. Now if we denote $\widehat{\nu}$ by the sweep of ν on ∂U, we then have for all $g \in A(U)$

$$\int g(z) d\widehat{\nu} = \int \widehat{g(z)} d\nu = g(a) = \int g d\omega_a.$$

By uniqueness, we conclude $\widehat{\nu} = \omega_a$. Hence

$$\eta|\partial U \ll \nu|\partial U \ll \widehat{\nu}|\partial U = \omega_a.$$

Now suppose that $g \in L^q(\mu)$ such that

$$\int fgd\mu = 0 \quad \text{for each} \quad f \in P^t(\mu).$$

where $1/q + 1/t = 1$. Since $A(U) \subseteq P^t(\mu)$, it follows that

$$\int fgd\mu = 0, \quad f \in A(U).$$

That is, $g\mu \perp A(U)$ and so $g\mu|\partial U \ll \omega_a \ll \omega$. This implies that $(g\mu)_s = 0$, where $(g\mu)_s$ is the singular part of the Lebesgue decomposition of $g\mu$ with respect to ω. Consequently, $g \perp \chi_\Delta$, where Δ is the carrier of μ_s and μ_s is the singular part of the Lebesgue decomposition of μ with respect to ω. Now an application of the Hahn-Banach theorem yields $\chi_\Delta \in P^t(\mu)$. Since $P^t(\mu)$ is pure, it follows that $\chi_\Delta = 0$ $a.e.$ $[\mu]$ and hence $\mu_s = 0$.

□

The proof of Lemma 5.4 : Assume that $S_\mu \sim S_\omega$. The space $P^2(\mu)$ is pure [17, p.223] since $P^2(\omega)$ is pure. Now Theorem 4.11 in [64] together with Lemma 4.3 imply that

$$abpeP^2(\mu) = bpeP^2(\mu) = bpeP^2(\omega) = abpeP^2(\omega) = G.$$

It follows Lemma 4.5 that $\mu|\partial G \ll \omega$. The fact that $supp(\mu) \subseteq \overline{G}$ follows from Theorem 4.10 of [64].

Let $A : P^2(\mu) \to P^2(\omega)$ be a quasi-infinity such that $AS_\mu = S_\omega A$. For simplicity we may assume $\|A\| = 1$. Let φ be a conformal map of D onto G with $\varphi(0) = a$ such that $\omega = m \circ \varphi$ and let ψ be its inverse function. We may assume, without loss of generality, that $\omega = \omega_a$. Set $u = A(1)$. It is easy to check that

$$A(f) = uf \quad \text{for each} \quad f \in A(G).$$

Since $|\psi| = 1$ on ∂G a.e. $[\omega]$, for all $n \geq 1$ and for each $f \in A(G)$

$$\begin{aligned} \int_{\partial G} |f|^2 \, |u|^2 \, d\omega &= \int_{\partial G} |f|^2 \, |u|^2 \, |\psi|^{2n} \, d\omega \\ &= \|A((\psi)^n f)\|^2 \\ &\leq \|(\psi)^n f\|^2 \\ &= \int_{\overline{G}} |\psi|^{2n} \, |f|^2 d\mu. \end{aligned}$$

If we let $n \to \infty$, then for each $f \in A(G)$

$$\int_{\partial G} |f|^2 \, |u|^2 \, d\omega \leq \int_{\partial G} |f|^2 \, d\mu. \tag{5.1}$$

Now we claim that $\{t \circ \varphi : t \in A(G)\}$ is dense in $P^2(|u|^2 m)$ for all $u \in P^2(m)$. In fact, since $A(G)$ is bounded pointwise dense in $H^\infty(G)$ (Theorem 1.2), we see that $\{t \circ \varphi : t \in A(G)\}$ is bounded pointwise dense in $H^\infty(D)$; that is, each $f \in H^\infty(D)$ is the pointwise limit of a bounded sequence of functions in $\{t \circ \varphi : t \in A(G)\}$ on D. Now the Lebesgue dominated convergence theorem together with the density of $H^\infty(D)$ in $P^2(|u|^2 m)$ implies that $\{t \circ \varphi : t \in A(G)\}$ is dense $P^2(|u|^2 m)$ and so the claim is proved.

Using (5.1), we have for each $f \in A(G)$

$$\int |f \circ \varphi|^2 \, |u \circ \varphi|^2 \, dm \leq \int |f \circ \varphi|^2 \, (\frac{d\mu|\partial G}{d\omega}) \circ \varphi \, dm.$$

Consequently, for each g in $P_0^2(m)$

$$\int |1-g|^2 \, |u \circ \varphi|^2 \, dm \leq \int |1-g|^2 \, (\frac{d\mu|\partial G}{d\omega}) \circ \varphi \, dm.$$

Notice that the function $u \circ \varphi$ is in $P^2(m)$. So by Szegö's theorem (see [32], p.49)

$$\inf_{g \in P_0^2(m)} \int |1-g|^2 \, (\frac{d\mu|\partial G}{d\omega} \circ \varphi) dm > -\infty.$$

Again, by Szegö's theorem, we see

$$\int \log(\frac{d\mu|\partial G}{d\omega} \circ \varphi) dm > -\infty;$$

i.e.,

$$\int \log \frac{d\mu|\partial G}{d\omega} d\omega > -\infty,$$

and hence $\log \frac{d\mu|\partial G}{d\omega}$ is in $L^1(\omega)$. The proof is complete.

$\square$

Lemma 5.6 *Suppose that A is an invertible operator from $P^2(\mu)$ to $P^2(\omega)$ such that $AS_\mu = S_\omega A$. Let $u = A(1)$ and let $\alpha = \mu|\partial G$. Then there exists an outer function x in $P^2(\omega)$ such that $|x|^2 = \frac{d\alpha}{d\omega}$. Moreover, there exists an invertible function $h \in P^2(\omega) \cap L^\infty(\omega)$ such that that $hx = u$.*

Proof. Since $AS_\mu = S_\omega A$, it is easy to verify that for every $f \in A(G)$ (as in the proof of Lemma 5.4)

$$\|A^{-1}\|^{-1} \|f\|_{L^2(\mu)} \leq \|uf\|_{L^2(\omega)} \leq \|A\| \|f\|_{L^2(\mu)},$$

Replacing f by $\psi^n f$ (here, ψ is a conformal map of G onto D. Since G is normal, ψ is in $P^2(\omega)$) and letting $n \to \infty$, we obtain

$$\|A^{-1}\|^{-1} \|f\|_{L^2(\alpha)} \leq \|u f\|_{L^2(\omega)} \leq \|A\| \, \|f\|_{L^2(\omega)} \quad \text{for every} \quad f \in A(G) \tag{5.2}$$

By Lemma 5.4 $\log(\frac{d\mu|\partial G}{d\omega}) \in L^1(\omega)$. It follows by Lemma 5.1 that there exists an outer function x such that $|x|^2 = \frac{d\alpha}{d\omega}$. Define an operator B on the manifold $\{fx : f \in A(G)\}$ via $B(xf) = uf$ for each $f \in A(G)$. Then for each $f \in A(G)$

$$\begin{aligned}
\|uf\|_\omega^2 &= \int |uf|^2 \, d\omega \\
&= \int |A(f)|^2 \, d\omega \\
&\leq \|A\|^2 \int |f|^2 d\alpha \\
&= \|A\|^2 \int |f|^2 \frac{d\alpha}{d\omega} \, d\omega \\
&= \|A\|^2 \|fx\|_\omega^2.
\end{aligned}$$

Hence B can be extended boundedly on $P^2(\omega)$. We use B to denote this extension too. Notice that the operator B commutes with S_ω. By Yoshino's theorem ([17] p.147), there is a function h in $P^2(\omega) \cap L^\infty(\omega)$ such that $B = M_h$, the multiplication operator induced by h on $P^2(\omega)$. So $hx = u$. Finally (5.2) clearly indicates that M_h is bounded below, and hence h is invertible in $L^\infty(\omega)$. The proof is complete.

□

Recall that a positive measure τ on G is a Carleson measure if there is a constant $c > 0$ such that for every t with $1 \leq t < \infty$

$$\|p\|_{L^t(\tau)} \leq c\|p\|_{L^t(\omega)}, \quad p \in \mathcal{P}$$

Theorem 3.1 shows if G is normal, then τ is a Carleson measure on G if and only if $\tau \circ \varphi$ is a Carleson measure on D, where φ is a conformal map of D onto G.

Lemma 5.7 *Let A be an invertible operator from $P^2(\mu)$ to $P^2(\omega)$. Let $u = A(1)$ and let $v = A^{-1}(1)$. Then*

$$v = A^{-1}(1) = \frac{1}{A(1)} = \frac{1}{u} \quad a.e. \quad [\mu].$$

Proof. Let $\{p_n\} \subset \mathcal{P}$ such that p_n converges to $A^{-1}(1)$ in $P^2(\mu)$. By passing a subsequence if necessary, we see that $p_n \to A^{-1}(1)$ a.e. $[\mu]$. But the continuity of A implies that $up_n \to u$ in $P^2(\omega)$. So there exists a subsequence $\{p_{n_i}\}$ such that up_{n_i} converges to 1 a.e. $[\omega]$. By Lemma 5.4 we see that $abpeP^2(\mu) = G$; it follows then that p_{n_i} converges uniformly on compact subsets of G to $\hat{v}$, the analytic extension of v on G. But $v = \hat{v}$ on G, so $uv = 1$ a.e. $[\mu]$. Hence, $v = \frac{1}{u}$ a.e. $[\mu]$. The proof is completed.

Lemma 5.8 *If τ is a Carleson measure on G, then $S_{\omega+\tau} \simeq S_\omega$.*

Proof. By hypothesis, there exists $c > 0$ such that

$$\|p\|_\tau \le c\|p\|_\omega, \quad p \in \mathcal{P}.$$

So if we define operator $I : P^2(\omega) \to P^2(\omega + \tau)$ densely via $I(p) = p$ for each $p \in \mathcal{P}$, then it is easy to verify that I is an invertible operator and $IS_\omega = S_{\omega+\tau}I$. Hence $S_{\omega+\tau} \simeq S_\omega$.

□

Recall that if $x \in P^2(\omega)$, then $\hat{x}$ denotes the analytic extension of x to $abpeP^2(\omega) = G$. Now we can state our first main result in this chapter, which generalizes a well-known result of W. S. Clary ([14] or [18, p.370-373]).

Theorem 5.1 *Let S be a subnormal operator. Then the following are equivalent:*

1) $S \simeq S_\omega$.

2) There is a Carleson measure τ on G such that $S \cong S_{\omega+\tau}$.

3) There exists a finite measure μ on $\overline{G}$ such that $S_\mu \cong S$ and μ has the following properties:

a) $abpeP^2(\mu) \subseteq G$.

b) $\alpha = \mu|\partial G \ll \omega$ and $\log \frac{d\alpha}{d\omega} \in L^1(\omega)$.

c) If x is an outer function in $P^2(\omega)$ such that $|x|^2 = \frac{d\alpha}{d\omega}$, then $|\hat{x}|^{-2}\mu|G$ is a Carleson measure.

Remarks:

I) Condition $a)$ in 3) can be replaced by $abpeP^2(\mu) = G$ since $S_\mu \simeq S_\omega$ implies $abpeP^2(\mu) = abpeP^2(\omega)$ (see Lemma 5.4).

II) If G is the unit disc D, then $a)$ is satisfied for all μ with $supp(\mu) \subseteq \overline{D}$ and therefore it can be removed. However, $a)$ can not be dropped in general (See Example 1 below). A natural question now arises:

Question 5.2 *What normal domains have the property that Theorem 4.1 is valid with condition 3a omitted?*

The answer is that G is perfectly connected, see Theorem 5.3 and ther remark afterwards.

The proof of Theorem 5.1. 1) $\Longrightarrow$ 2). Assume $S \simeq S_\omega$. By Bram and Singer's theorem there is a measure μ such that $S_\mu \cong S$. So $S_\mu \simeq S_\omega$ also. It follows by

Lemma 5.4 that

$$abpe P^2(\mu) \subseteq G, \quad \mu|\partial G \ll \omega, \quad \text{and} \quad \log \frac{d\mu|\partial G}{d\omega} \in L^1(\omega).$$

Let T be an invertible operator from $P^2(\mu)$ to $P^2(\omega)$ so that $TS_\mu = S_\omega T$. Let x, h, and u be the functions as in Lemma 5.6 and extend each of the three functions to G in the obvious way. Using Lemma 5.7, one has that

$$\frac{1}{x} = \frac{h}{u} = hT^{-1}(1) \text{ on } \partial G \text{ a.e. } [\mu]$$

and

$$\frac{1}{\hat{x}} = \frac{\hat{h}}{\hat{u}} = \hat{h}\widehat{T^{-1}}(1) \text{ on } G.$$

Therefore, there exist x_1 in $P^2(\mu)$ and h_1 in $P^2(\mu) \cap P^\infty(\mu)$ such that $x = x_1$ and $h = h_1$ a.e. $[\mu]$ and

$$\frac{1}{x_1} = h_1 v = h_1 T^{-1}(1) \text{ a.e. } [\mu]$$

Set $\eta = |x|^{-2}\mu$. Then for each $p \in \mathcal{P}$

$$\begin{aligned} \|p\|_\eta &= \left(\int |\frac{p}{x}|^2 d\mu\right)^{\frac{1}{2}} \\ &= \left(\int |\frac{ph}{u}|^2 d\mu\right)^{\frac{1}{2}} \\ &= \|hT^{-1}(p)\|_\mu \\ &\leq \|h\|_\infty \|T^{-1}\| \|p\|_\omega. \end{aligned}$$

But this implies that $abpe P^2(\eta) \subseteq G$. It follows by Lemma 1.1 that $A(G) \subseteq P^2(\eta)$.

Now define an operator $F : P^2(\eta) \to P^2(\mu)$ densely[1] via

$$F(f) = (\frac{1}{x}) f \quad \text{for every} \quad f \in A(G).$$

Clearly F is an isometry and $S_\eta F = F S_\mu$. We claim that F is onto. In fact, since $F[A(G)] = hT^{-1}[A(G)]$ and since h is invertible in $P^2(\mu) \cap P^\infty(\mu)$, it follows that F has dense range; and hence F is an isometric isomorphism. Therefore, if we let $\tau = |x|^{-2}\mu|G$, then $\eta = \omega + \tau$, and

$$S \cong S_\mu \cong S_\eta = S_{\omega+\tau}.$$

Clearly

$$\|p\|_\tau \leq \|p\|_\eta \leq \|h\|_\infty \|T^{-1}\| \|p\|_\omega,$$

we see that τ is a Carleson measure.

2) $\Longrightarrow$ 3). If there exists a Carleson measure τ such that $S \cong S_{\omega+\tau}$, then $S \simeq S_\omega$ (Lemma 5.8). Now the rest proof is the same as the one above.

3) $\Longrightarrow$ 1). Let μ be as in 3). Set $\beta = \omega + |\hat{x}|^{-2}\mu|G$. Since $|\hat{x}|^{-2}\mu|G$ is a Carleson measure, it follows by Lemma 5.8 that $S_\beta \simeq S_\omega$. If we repeat the process of the proof of 1) $\Longrightarrow$ 2), we see that $S_\beta \equiv S_\omega$. So the proof is complete.

□

Example 1 Let G be a crescent domain formed by two internally tangent circles such that $\mathcal{P}$ is not dense in $L^2_a(G)$ (For the existence of such domain, consult [10]). This implies that $abpeP^2(A) \neq G$, where A is area measure on G. If let $\mu = A + \omega$,

[1] Here, densely means the operator is defined on a manifold that is dense in the space. We will use this terminology repeatly throughout this paper.

then $abpeP^2(\mu) \neq G$ since

$$G \subseteq abpeP^2(A) \subseteq abpeP^2(\mu).$$

Let $\widehat{\mu}$ be the sweep of μ on ∂G, then $\widehat{\mu} \ll \omega$ by Theorem 1.3. Clearly, $\log(\frac{d\widehat{\mu}}{d\omega})$ is in $L^1(d\omega)$ and $|\widehat{x}|^{-2}\widehat{\mu}|G = 0$ is a Carleson measure (here, $|x|^2 = \log(\frac{d\widehat{\mu}}{d\omega})$ as in Theorem 5.1. On the other hand, by J. AKeroyd's theorem [3] that $\mathcal{P}$ is dense in $H^2(G)$, it follows by Theorem 2.1 that $G = abpeP^2(\omega)$. Hence $abpeP^2(\widehat{\mu}) \neq G$. So we conclude that $S_{\widehat{\mu}}$ is not even quasisimilar to S_ω.

□

Note: In the previous example area measure is not a Carleson measure on the crescent G (since, $abpeP^2(A)$ strictly contains G).

The next result is our unitarily equivalence theorem.

Theorem 5.2 *Let H be a Hilbert space and let B be a bounded operator on H. Then $B \cong S_\omega$ if and only if B is a cyclic subnormal operator and there exists a measure μ on $\overline{G}$ such that $B \cong S_\mu$ and μ has the properties:*
1) $abpeP^2(\mu) \subseteq G$.
2) $[\mu] \equiv [\omega]$.
3) $\int \log(\frac{d\mu|\partial G}{d\omega})d\omega > -\infty$.

Remark: Example 1 also shows that condition 1) in Theorem 5.2 is necessary. We will characterize all those domains G for which property 1) can be removed, see Theorem 5.3.

Proof. Suppose that $B \cong S_\omega$. Clearly B is a cyclic operator since S_ω. Since S_ω is subnormal, it is easy to show B is subnormal too. (for example, one may apply (f) of Theorem 1.9 [17, p.118]). Using Bram and Singer's theorem, we have a measure μ on $\overline{G}$ such that $S_\mu \cong B$. So $S_\mu \cong S_\omega$. It follows by Proposition 5 [17, p.128] that their minimal normal extensions of the operators are unitarily equivalent. But this is equivalent to saying that μ and ω are mutually absolutely continuous ([17, p.77]). By Theorem 5.1, we have

$$abpe P^2(\mu) \subset G \quad \text{and} \quad \frac{d\mu|\partial G}{d\omega} \in L^1(\omega).$$

Conversely, assume that $\int \log \frac{d\mu|\partial G}{d\omega} d\omega > -\infty$. By Lemma 5.1, we can find an outer function x in $P^2(\omega)$ so that $|x|^2 = \frac{d\mu|\partial G}{d\omega}$. Since

$$abpe P^2(\mu) \subseteq abpe P(\omega) = G,$$

it follows that $A(G) \subseteq P^2(\mu)$. Now if we define an operator A from $P^2(\mu)$ to $P^2(\omega)$ densely via $A(p) = xp$ for each $p \in A(G)$, then A clearly is isometric isomorphism and $AS_\mu = S_\omega A$. Thus $S_\mu \cong S_\omega$. This proves the theorem.

□

Using Lemma 5.1, we obtain another version of Theorem 5.2 as follows:

Theorem Let μ be a finite positive measure. Then $S_\mu \cong S_\omega$ if and only if μ has the following properties:

1) $abpe P^2(\mu) \subseteq G$.

2) $[\mu] \equiv [\omega]$.

3) There is a function $x \in P^2(\omega)$ such that $|x|^2 = \frac{d\mu|\partial G}{d\omega}$.

Recall that a function g in $H^\infty(D)$ is a *weak-star generator* provided polynomials in g are weak-star dense in $H^\infty(D)$ (consult [57, 58, 59, D. Sarason]). We say a simply connected domain U is **perfectly connected** if U is the image of a weak-star generator of $H^\infty(D)$.

The following theorem not only answers Question 5.2 but also gives a characterization of a perfectly connected domain.

Theorem 5.3 *Let U be a bounded domain with harmonic measure ω such that no point of ∂U is removable for $H^2(U)$. In order that S_μ be unitarily equivalent to S_ω for each positive measure μ with the properties that $[\mu] \equiv [\omega]$ and $\int \log(\frac{d\mu|\partial G}{d\omega})d\omega > -\infty$, it is necessary and sufficient that U is a perfectly connected domain.*

Remark: This theorem also says: Theorem 5.1 with hypothesis 3a omitted is valid for a domain U if and only if U is perfectly connected. In fact, suppose that $S_\mu \simeq S_\omega$ for each measure μ that satisfies conditions b) and c) in Theorem 5.1. We then have, in particular, that $S_\mu \simeq S_\omega$ for all those μ that satisfies b) and c) in Theorem 5.1 and $\mu|G = 0$. Now, using Theorem 5.2 gives that $S_\mu \cong S_\omega$ for all those μ with the properties that $[\mu] = [\omega]$ and $\log \frac{d\mu|\partial G}{d\omega} \in L^1(\omega)$. Hence U is perfectly connected.

For the proof of Theorem 5.3, we need to recall Sarason's weak-star density theorem for polynomials. For a compact set K, let $R(K)$ denote the uniform closure in $C(K)$ of the set of the rational functions with poles off K. Recall that $R(K)$ is called a Dirichlet algebra if $\{Re(f) : f \in R(K)\}$ is dense in $C_R(\partial K)$, the real continuous function algebra on ∂K. Sarason's theorem for weak-star density for polynomials now can be stated as [18, p.301]:

Sarason's theorem: For any compactly supported positive measure μ on the complex plane, there is a compact set K and measures μ_a and μ_s having the following properties:

1) $\mu = \mu_a + \mu_s$, $\mu_a \perp \mu_s$ and $P^\infty(\mu) = L^\infty(\mu_s) \oplus P^\infty(\mu_a)$.

2) K contains the support of μ_a, the algebra $R(K) \subseteq P^\infty(\mu_a)$ and $R(K)$ is a Dirichlet algebra.

3) There is an isometric isomorphism α from $H^\infty(int(K))$ to $P^\infty(\mu_a)$ such that α is also a weak-star homeomorphism and $\alpha(f) = f$ for every $f \in R(K)$.

Remark: The set K is known as the Sarason hull of μ.

For a compactly supported measure μ, let $bpeP^\infty(\mu)$ denote the set of points λ such that the linear map $p \to p(\lambda) \quad p \in \mathcal{P}$ can be extended to a weak-star continuous linear functional on $P^\infty(\mu)$.

The following lemma can be found in [18, p.306].

Lemma 5.9 *If μ is a finite measure with compact support, then $bpeP^\infty(\mu_a) = int(K)$.*

The proof of Theorem 5.3. Suppose that U is a perfectly connected domain. Looking at Lemma 2.2 , we see that $P^\infty(\omega) = H^\infty(U)$. Let μ be measure such that $[\mu] \equiv [\omega]$ and $\int \log \frac{d\mu|\partial G}{d\omega} d\omega > -\infty$. It follows from Lemma 2.2 and Lemma 2.3 that

$$\begin{aligned} A(U) &\subseteq H^\infty(U) \\ &= P^\infty(\omega) \\ &= P^\infty(\mu) \\ &\subseteq P^2(\mu) \end{aligned}$$

By Lemma 5.1, there is an outer function x in $P^2(\omega)$ such that $|x^2| = \frac{d\mu|\partial G}{d\omega}$ and so there is an isometric isomorphism $A : P^2(\mu) \to P^2(\omega)$, defined by extending the isometrical map $A(a) = ax$ for each $a \in A(U)$. Clearly the operator A has the property that $AS_\mu = S_\omega A$. Hence $S_\mu \cong S_\omega$.

For the proof of the other conclusion, let us assume that $S_\mu \cong S_\omega$ for each positive μ that has the properties:

$$[\mu] \equiv [\omega] \quad \text{and} \quad \int \log \frac{d\mu|\partial G}{d\omega} d\omega > -\infty.$$

Let $f \in L^1(\omega)$. Clearly

$$(|f| + 1)\omega \equiv \omega \quad \text{and} \quad \log(|f| + 1) \in L^1(\omega).$$

It follows by our assumption that $S_{(|f|+1)\omega} \cong S_\omega$. Thus,

$$abpeP^2(|f|\omega) \subseteq abpeP^2((|f| + 1)\omega) \subseteq abpeP^2(\omega).$$

Now if η is a measure such that $[\eta] \equiv [\omega]$, then by the Randon-Nikodym theorem there exists $h \in L^1(\omega)$ such that $h = \frac{d\eta}{d\omega}$. Using the previous argument, we get that

$$abpeP^2(|\eta|) \subseteq abpeP^2(\omega) \quad \text{for each } \eta \text{ with } [\eta] \equiv [\omega].$$

Since ∂U has no removable point for $H^2(U)$, it follows that from Theorem 2.1 that $abpeP^2(\omega) \subset U$; and hence we have

$$abpeP^2(|\eta|) \subset U \quad \text{for each } \eta \text{ with } [\eta] \equiv [\omega].$$

We now claim:

$$bpeP^\infty(\omega) = U.$$

Actually, Applying Theorem 5.9 in [64]), we can find a finite measure ν such that $[\nu] \equiv [\omega]$ and

$$P^{\infty}(\omega) = P^2(\nu) \cap L^{\infty}(\nu).$$

Therefore, if we apply Sarason's theorem to $P^{\infty}(\omega)$ and Thomson's theorem to $P^2(\nu)$, respectively, then it follows that

$$H^{\infty}(bpe P^{\infty}(\omega)) = H^{\infty}(abpe P^2(\nu)).$$

Consequently,

$$bpe P^{\infty}(\omega) = abpe P^2(\nu).$$

Clearly $U \subseteq bpe P^{\infty}(\omega)$ and so the last equality implies that $U \subseteq abpe P^2(\nu)$. Thus,

$$U = abpe P^2(\nu) = bpe P^{\infty}(\omega).$$

Finally, since $bpe P^{\infty}(\omega) = U$, it follows from Sarason's theorem that $\mathcal{P}$ is weak-star dense in $H^{\infty}(U)$. Using Lemma 2.1, we see that G is perfectly connected.

$\square$

It is natural to ask whether we can extend Theorem 5.1 and Theorem 5.2 to a domain U that does not have the property that $\mathcal{P}$ is dense in $H^2(U)$. The question should be phrased as follows (Keep in mind that U is strictly contained in $abpe P^2(\omega)$ when U is not a normal domain).

Question 5.3 *Let U be a simply connected domain with harmonic measure ω. Suppose that $abpe P^2(\omega) \neq U$. Let μ be a positive measure with the following properties:*

1) $abpeP^2(\mu) = abpeP^2(\omega)$.

2) $[\mu] \equiv [\omega]$ and $\int \log \frac{d\mu|\partial G}{d\omega} d\omega > -\infty$.

In addition, assume that $S_{\mu_0} \cong S_{\omega_0}$, where μ_0 and ω_0 are the restrictions of μ and ω on the boundary of $abpeP^2(\omega)$.

Is $S_\mu \simeq S_\omega$?

Note: The measure μ stated in Question 5.3 satisfies all the conditions in both Theorem 5.1 and Theorem 5.2 (except U does not have the property that $abpeP^2(\omega) = U$).

The following example shows that the above question has a negative answer.

Example 2 Let V be an open disc whose boundary contains the origin such that one of its diameters lies on the nonnegative real axis. Let J be a closed segment which joins 0 and a point inside the disc such that J forms an angle $\pi/3$ at 0 with the nonnegative imaginary axis. Let E be the closed domain enclosed by the triangle that is symmetric to the real axis and has J as one of its sides. Now set $U = V - E$. Let ω be harmonic measure of U and let ω_0 and ω_1 be the restrictions of ω on ∂V and ∂E, respectively. By Lemma 4.8 in [4], ω is boundedly equivalent to measure $|z|^2 s$ near 0, where s is the arclength measure on ∂U. Set $s_0 = s|\partial V$ and $s_1 = s|\partial E$. Then $|z|^{-2}\omega_1$ is boundedly equivalent to s_1. It is obvious that s_1 is a Carleson measure on V (see [22]); and so it follows that there is $c > 0$ such that

$$\|p\|_{L^2(|z|^{-2}\omega_1)} \leq c\|p\|_{L^2(s_0)}, \quad p \in \mathcal{P}.$$

This implies that there is some $c_0 > 0$ such that

$$\|p\|_{L^2(\omega_1)} \leq c_0\|p\|_{L^2(\omega_0)}, \quad p \in \mathcal{P}.$$

Now we define $A : P^2(\omega) \to P^2(\omega_0)$ densely via $A(p) = p$ for each $p \in \mathcal{P}$. The last inequality implies that A is invertible. Apparently, $AS_\omega = S_{\omega_0}A$; and, hence, S_ω is similar to S_{ω_0}. Since $\log \frac{d\omega_0}{ds_0} \in L^1(s_0)$, it follows by Theorem 5.3 that $S_{s_0} \cong S_{\omega_0}$. (Note, s_0 is boundedly equivalent to harmonic measure of V.) Thus S_{s_0} is similar to S_ω.

Now we define a measure μ on ∂U by setting $\mu = s_0 + |z|^{-\frac{1}{2}}s_1$. The measure $|z|^{-\frac{1}{2}}s_1$ is not a Carleson measure since it does not satisfy the 'window' condition of the original definition of Carleson measures [22, p.156]; therefore, S_μ is not similar to S_{s_0}. Combining this with the argument above, we conclude that S_μ is not similar to S_ω. However, one can easily verify that we do have

1) $abpeP^2(\mu) = abpeP^2(\omega) = V$.

2) $[\mu] = [\omega]$ and $\log \frac{d\mu|\partial G}{d\omega} \in L^1(\omega)$.

3) $S_{\omega_0} \cong S_{\mu_0}$, where $\mu_0 = \mu|\partial V$.

Note: The above domain V is one of the simplest domain that does not have the property that $abpeP^2(\omega) = U$. (The domain U is an A-type crescents, see the introduction of Chapter 6 for definition and for more information about A-type crescent.) However, if we only work on an operator that has the form $S_{\omega+\tau}$, where τ is a positive measure on U, then we show in Theorem 7.3 that $S_{\omega+\tau} \simeq S_\omega$ if and only if τ is a Carleson measure on U.

To study operators that are similar to the shifts associated with more general regions, more examples are needed. Very little is know in this regards. We close our study of similarity and unitary equivalence relations by presenting the following theorem which deals with arbitrary domains.

Theorem 5.4 *Let U be a bounded domain with harmonic measure ω such that $abpe P^2(\omega) \neq U$. Let μ be a finite positive measure. Let μ_0 and ω_0 be the restrictions of the measures μ and ω to the boundary of $abpe P^2(\omega)$, respectively. Then*

1) In order that $S_\mu \cong S_\omega$, it is necessary that $S_{\mu_0} \cong S_{\omega_0}$.

2) In order that $S_\mu \simeq S_\omega$, it is necessary that $S_{\mu_0} \simeq S_{\omega_0}$.

Proof. The proofs of 1) and 2) are almost identical, and so we are only going to prove 1). Suppose $S_\mu \cong S_\omega$. Let X be an isometric isomorphism from $P^2(\mu)$ to $P^2(\omega)$ such that $XS_\mu = S_\omega X$. If we let $v = X(1)$ and $u = X^{-1}(1)$, then for every $p \in \mathcal{P}$

$$\|p\|_\omega = \|up\|_\mu$$

and

$$\|p\|_\mu = \|vp\|_\omega.$$

Let g be the Riemann map from $abpe P^2(\omega)$ to D and extend g to the boundary by defining its values there to be those obtained in the nontangential limit sense. Then g can be regarded as a function in both $P^2(\omega)$ and $P^2(\mu)$. Replacing p by pg^n in the above equalities, respectively, and letting $n \to \infty$, we see that for each $p \in \mathcal{P}$

$$\|p\|_{\omega_0} = \|up\|_{\mu_0} \tag{5.3}$$

and

$$\|p\|_{\mu_0} = \|vp\|_{\omega_0}.$$

Now define A: $P^2(\mu_0) \to P^2(\omega_0)$ densely by $A(p) = vp$ for each $p \in \mathcal{P}$. The operator A obviously is an isometry and $AS_{\mu_0} = S_{\omega_0}A$. So if we can show that A is onto, then we are done. To do this let us pick a function $f \in P^2(\omega_0)$. Then (5.3) implies that $uf \in P^2(\mu_0)$ and so uvf is in the range of (A). But $uv = 1$ almost every where (Lemma 5.7), and so f is the range of A. Therefore, A is onto. Hence it is an isometric isomorphism.

□

5.3 Quasisimilarity Theorem

Quasisimilar subnormal operators has been studied by a number of authors. W. S. Clary [14, 1973] first characterized all subnormal operators quasisimilar to the unilateral shift. W. Hastings [31, 1979] extended his result to the finite direct sum of unilateral shifts. A decade later J. McCarthy [38, 1990] extended Clary's result to rationally cyclic subnormal operators (where a hypodirichlet algebra condition was imposed). The next theorem is our primary result concerning quasisimilar subnormal operators. *Again we remind our reader of the standing hypothesis in this chapter, namely, G is a normal domain.*

Theorem 5.5 *Let S be a subnormal operator. Then $S \sim S_\omega$ if and only if there exists a finite measure μ on $\overline{G}$ such that $S_\mu \cong S$ and μ has each of the following two properties:*

1) $abpeP^2(\mu) \subseteq G$.

2) $\mu|\partial G \ll \omega$ and $\log \frac{d\mu|\partial G}{d\omega} \in L^1(\omega)$.

Remark: We'd like to point out the differences between McCarthy's similarity theorem for rationally cyclic subnormal operators in [38] and Theorem 5.5. In [38]

McCarthy shows that if $R(K)$ is a hypodirichlet algebra, then $R_\mu \sim R_\omega$ if and only if $\mu|\partial K \ll \omega$ and $\log \frac{d\mu|\partial K}{d\omega} \in L^1(\omega)$, where μ is a finite measure on K and ω is the harmonic measure of $int(K)$. McCarthy's result fails to apply to the case when $\overline{G}^\circ = G$ and $R(\overline{G})$ is a Dirichlet algebra. An obvious reason why it doesn't is $P^2(\omega)$ is not the same as the space $R^2(\overline{G}, \omega)$ in general even if $R(\overline{G})$ is a Dirichlet algebra. An even when the two spaces are equal, the spaces $P^2(\mu)$ and $R^2(\mu)$ are different; thus, the operators S_μ and R_μ are different.

Proof. Suppose $S \sim S_\omega$. It is easy to verify that S is a cyclic operator because S_ω is cyclic. By Bram and Singer's theorem, there is a measure μ such that $S_\mu \cong S$ [17, p.146] and so it follows that $S_\omega \sim S_\mu$. By Lemma 4.4 we have that $supp(\mu) \subseteq \overline{G}$, $\mu|\partial G \ll \omega$, $abpeP^2(\mu) = abpeP^2(\omega) = G$, and $\log \frac{d\mu|\partial G}{d\omega} \in L^1(\omega)$.

For the proof of sufficiency. Assume that μ is a measure such that $supp(\mu) \subseteq \overline{G}$ and satisfies properties 1) and 2) listed in the theorem. We first want to show that the inclusion operator $I : P^2(\mu) \to P^2(\mu|\partial G)$, defined by via $I(a) = a$ for all $a \in P^2(\mu)$, is one-to-one. In fact, suppose $f \in P^2(\mu)$ and $I(f) = 0$; choose a sequence of functions $\{a_n\} \subseteq A(G)$ such that a_n converges to f in $P^2(\mu)$ norm. We see then that

$$I(a_n) = a_n \to 0 \quad \text{in} \quad P^2(\mu|\partial G).$$

Observing $S_{\mu|\partial G} \cong S_\omega$ by Theorem 5.2, we see $abpeP^2(\mu|\partial G) = G$. This implies that $a_n \to 0$ uniformly on compact subsets of G. Now it follows that $a_n \to 0$ in $P^2(\mu)$ since $a_n \to 0$ in $P^2(\mu|\partial G)$. Thus, $f = 0$.

Since $S_{\mu|\partial G} \cong S_\omega$, there is an isometric isomorphism $J : P^2(\mu|\partial G) \to P^2(\omega)$ such that $JS_{\mu|\partial G} = S_\omega J$. Now let $Y = JI$. Then Y is one-to-one and has dense range, and moreover, $YS_\mu = S_\omega Y$.

To finish our proof we need to find another operator X from $P^2(\omega)$ to $P^2(\mu)$ that is one-to-one, that has dense range, and that shares the property that $XS_\omega = S_\mu X$.

Let φ be a conformal map of D onto G which sends 0 to b. Without loss of generality, we may assume that ω is the harmonic measure evaluated at $b \in G$. Extend φ to ∂D almost everywhere with respect to m by defining its boundary values to be the nontangential limits $\tilde{\varphi}$. Define a measure ν on $\overline{D}$ via $\nu(B) = \mu(\varphi(B))$ for every measurable subset B. We first want to find an operator C from $P^2(m)$ to $P^2(\nu)$ such that C is one-to-one and has dense range and shares the obvious intertwining property. Since $\mu|\partial G \ll \omega$ and $\log(\frac{d\mu|\partial G}{d\omega}) \in L^1(\omega)$, we can easily verify that $\nu|\partial D \ll m$ and $\log(\frac{d\nu|\partial D}{dm}) \in L^1(m)$. Recall that if $\widehat{\nu}$ is the sweep of ν, then $\widehat{\nu} = \nu|\partial D + \widehat{\nu|D}$. So we have that $\log(\frac{d\widehat{\nu}}{dm}) \in L^1(m)$. It follows by Theorem 5.2 that $S_{\widehat{\nu}} \cong S_m$ and hence there is an isometric isomorphism R from $P^2(m)$ to $P^2(\widehat{\nu})$ such that $RS_m = S_{\widehat{\nu}}R$. Now if we show that the linear map A from the dense subspace $A(D)$ of $P^2(\widehat{\nu})$ to the dense subspace $A(D)$ of $P^2(\nu)$, defined by $A(a) = a$ for each $a \in A(D)$, can be extended to be a bounded operator from $P^2(\widehat{\nu})$ to $P^2(\nu)$, then the operator $C = AR$ is the desired one. The proof of the boundedness of A is as follows. If we denote that the harmonic extension of $|a|^2|\partial D$ by $\widehat{|a|^2}$ for $a \in A(D)$, then it follows by the the maximum principle of subharmonic functions that for each a in A(D)

$$\begin{aligned} \|a\|_\nu &= \{\int |a|^2 d\nu\}^{\frac{1}{2}} \\ &\leq \{\int \widehat{|a|^2} d\nu\}^{\frac{1}{2}} \\ &= \int |a|^2 d\widehat{\nu} \\ &= \|a\|_{\widehat{\nu}}. \end{aligned}$$

So A is bounded.

Finally, let $u = C(1)$. The set $A(D)u$ is dense in $P^2(\nu)$ since C has dense range. Let ψ be the inverse of φ and let $v = u \circ \psi$. It is easy to verify that $v \in P^2(\mu)$. Now if we can show that $A(G)v$ is dense in $P^2(\mu)$, then the operator X from $P^2(\omega)$ to $P^2(\mu)$, defined densely by $X(a) = va$ for each $a \in A(G)$, is the desired operator, which has dense range, is one-to-one and satisfies $S_\mu X = XS_\omega$. Hence $S_\mu \sim S_\omega$.

To show that $A(G)v$ is dense in $P^2(\mu)$, we pick a function $f \in P^2(\mu)$. Then $f \circ \varphi \in P^2(\nu)$, and hence it can be approximated by the functions in $A(D)u$ in the $P^2(\nu)$-norm. This is equivalent to saying that f is approximable by functions in $\{v(a \circ \psi) : a \in A(D)\}$ in the $P^2(\mu)$-norm. Now the Lebesgue dominated convergence theorem together with the fact that $A(G)$ is pointwise bounded dense in $H^\infty(G)$ implies that $A(G)v$ is dense in $P^2(\mu)$.

$\square$

Note: As in Theorem 5.1 and Theorem 5.2, condition 1) ($abpeP^2(\mu) \subseteq G$) is necessary. The following corollary shows that condition 1) may be removed if and only if G is perfectly connected.

Corollary 5.2 *In order that S_μ be quasisimilar to S_ω for every finite measure μ with $\mu \ll \omega$ and $\log(\frac{d\mu|\partial G}{d\omega}) \in L^1(\omega)$, it is necessary and sufficient that G is perfectly connected.*

Proof. The sufficiency follows from previous theorem . For the proof of necessity, let us assume that $S_\mu \sim S_\omega$ for every finite positive measure μ with $\log \frac{d\mu|\partial G}{d\omega} \in L^1(\omega)$. Then in particular, $S_\mu \sim S_\omega$ for each μ with $[\mu] = [\omega]$ and $\log \frac{d\mu|\partial G}{d\omega} \in L^1(\omega)$. Now it follows by Theorem 5.2 that $S_\mu \cong S_\omega$ for every finite positive measure μ with $[\mu] = [\omega]$ and $\log \frac{d\mu|\partial G}{d\omega} \in L^1(\omega)$ (since, that $S_\mu \sim S_\omega$ implies that $abpeP^2(\mu) \subseteq$

$abpeP^2(\omega)$). Hence G is perfectly connected by Theorem 5.3.

□

It is natural to ask whether Theorem 5.5 can be generalized to those domains U where $\mathcal{P}$ is not dense in $H^2(U)$. One might ask the following question.

Question 5.4 *Let U be a simply connected domain with harmonic measure ω such that $abpeP^2(\omega) \neq U$. Assume μ is a finite positive measure such that:*

1) $P^2(\mu)$ contains no non-trivial characteristic function

2) $abpeP^2(\omega) = abpeP^2(\mu)$.

3) $\log \frac{d\mu|\partial G_0}{d\omega_0} \in L^1(\omega)$, where μ_0 and ω_0 are the restrictions of μ and ω on the boundary of $abpeP^2(\omega)$ respectively.

Is S_μ quasisimilar to S_ω?

The answer is negative. Here is an example:

Example 3 Let V be an open crescent enclosed by two circles C_0 and C_1 such that C_0 is its outer boundary. Let D_0 and D_1 be the open discs enclosed by C_0 and C_1, respectively. Let l be a proper closed arc of C_1 which intersects C_0 and has its endpoints in D_0. Set $U = D_0 - l$. Let ω be harmonic measure of U.

Claim: $abpeP^2(\omega) = D_0$.

For the proof of the claim, we first need to show that is a nicely connected domain. First, since ω is the harmonic measure of U, we have observe that $U \subseteq abpeP^2(\omega)$. Second, using Thomson's theorem and Theorem 94 in [44, Miller-Olin-Thomson], we see that $abpeP^2(\omega)$ must be nicely connected. Third, D_0 is the smallest nicely connected domain containing U. Since it is obvious that $abpeP^2(\omega)$ must be contained in $\overline{D}_0$, so we conclude that $abpeP^2(\omega) = D_0$.

On the other hand, $\mathcal{P}$ is dense in $H^2(V)$ [3, Akeroyd]. It follows by Theorem 2.1 that $abpeP^2(\lambda) = V$, where λ is harmonic measure of V. If we denote the restriction of λ on C_0 by λ_0 and the arclength measure of C_0 by s, respectively, then Szegö's theorem implies that $\log(\frac{d\lambda_0}{ds})$ is not in $L^1(s)$. Now a well-known argument about harmonic measure shows that ω_0 (harmonic measure restricted to C_0) is boundedly equivalent to λ_0 on ∂C_0. So we conclude that $\log(\frac{d\omega_0}{ds})$ does not belong to $L^1(s)$. By Theorem 5.4, S_ω is not similar to S_s. (Note, s is boundedly equivalent to harmonic measure of D_0, therefore, Theorem 5.4 can be applied here.) But, obviously we have

1) $P^2(s)$ contains no non-trivial characteristic function.

2) $abpeP^2(s) = D_0 = abpeP^2(\omega)$.

3) $\log(\frac{ds}{d\lambda_0}) \in L^1(\lambda_0)$.

□

Now we present the theorem that answers Question 5.1.

Theorem 5.6 *In order that $\mathcal{P}\}$ be dense in $P^2(\omega)$ for every outer function g in $P^2(\omega)$ it is necessary and sufficient that G is a perfectly connected domain.*

Proof. Suppose that G is a perfectly connected domain. Let $g \in P^2(\omega)$ be an outer function. The hypothesis implies that $abpeP^2(|g|^2(\omega)) \subsetneq G$; (since, $G = bpeP^\infty(|g|^2\omega))$, and therefore $A(G) \subsetneq P^2(|g|^2\omega)$ and so it is dense in $P^2(|g|^2\omega)$. This implies that $g\mathcal{P}$ is dense in $gA(G)$. It follows that $g\mathcal{P}$ is dense in $P^2(\omega)$.

Conversely, suppose that $g\mathcal{P}$ dense in $P^2(\omega)$ for every outer function $g \in P^2(\omega)$. Let μ be any measure such that $[\mu] = [\omega]$ and $\log \frac{d\mu|\partial G}{d\omega} \in L^1(\omega)$. Using Lemma 4.1, there exists an outer function x such that $\frac{d\mu|\partial G}{d\omega} = |x^2|$. So if we define an operator A from $P^2(\mu)$ to $P^2(\omega)$ via $A(p) = xp$ for each polynomial p, then A is an isometric isomorphism and $AS_\mu = S_\omega A$. Therefore $S_\mu \cong S_\omega$. Now it follows from Theorem

4.3 that U is perfectly connected.

□

The following theorem is slightly different from Theorem 5.6 and it deals with a general domain U.

Theorem 5.7 *Let U be a bounded domain. In order for there to exist a function g in $P^2(\omega)$ such that $|g|^2 = f$ and $\mathcal{P}\}$ is dense in $P^2(\omega)$ for every positive integrable function f with $\log f \in L^1(\omega)$, it is necessary and sufficient that U is perfectly connected.*

Proof. The proof of necessity. We want to show that the hypothesis of Theorem 5.3 is satisfied. Let μ be such that $[\mu] = [\omega]$ and $\log(\frac{d\mu|\partial G}{d\omega}) \in L^1(\omega)$. By our hypothesis there exists a function g such that $\frac{d\mu|\partial G}{d\omega} = |g^2|$ and so it follows that the operator A from $P^2(\mu)$ to $P^2(\omega)$, defined via $A(p) = gp$ for each $p \in \mathcal{P}$, is an isometric isomorphism and $AS_\mu = S_\omega A$. Thus $S_\mu \cong S_\omega$. Now it follows by Theorem 5.3 that U is perfectly connected.

The proof of sufficiency follows from Lemma 5.1 and the above theorem.

□

Remark: All the results in this chapter can be extended to the space $P^t(\omega)$ for any $t \in [1, \infty)$, if we restrict the equivalence relation to those subnormal operators S_μ, where μ is a positive finite measure with compact support in the plane.

Chapter 6

Carleson Measures On A General Domain And Rationally Cyclic Subnormal Operators

In a private communication, J. McCarthy asked: What are the Carleson measures on a circular domain? More general, if G is a finite connected domain that is not simply connected, one could ask:

What are the Carleson measures on G?

Suppose G is a finitely connected Dirichlet domain with harmonic measure σ and suppose G is not a simply connected domain. We say a positive measure τ on G is a Carleson measure if there exists a positive constant c such that for all $t \in [1, \infty)$

$$\{\int_G |r|^t d\mu\}^{\frac{1}{t}} \leq C\{\int_{\partial G} |r|^t d\sigma\}^{\frac{1}{t}}, \quad r \in Rat(\overline{G}),$$

where $Rat(\overline{G})$ the set of rational functions with poles off $\overline{G}$[1]

[1]The reader should notice that if G is a simply connected domain, then the above definition is more restrictive than the one we give for a simply connected domain in Chapter 3. If we use the definition as given here, then the conclusion 1) of Theorem 3.1 holds for every simply connected domain G with the property that $\overline{G}^{\circ} = G$. Note though, the last definition is a natural one on a multiply connected domain G that is equal to the interior of its closure, as it is equivalent to

We generalize a well-known theorem of Carleson to those bounded domains G such that $G = int(\overline{G})$ and $\overline{G}$ is finitely connected. In particular, we extend the celebrated Carleson theorem for the disc that gives an equivalence between a Carleson measure and the Carleson measure inequality. As an application, let K be a compact subset with harmonic measure σ, and let $R(K)$ is a hypodirichlet algebra. With these assumptions we generalize W. S. Clary's theorem regarding the unilateral shift and the unit disc to the shift operator on the closure of $R(K)$ in $L^2(\sigma)$. (Note, if $R(K)$ is a hypodirichlet algebra, then K° must be finitely connected even though K itself may not be finitely connected). For the sake of simplicity, we make the assumption that K° is connected in this study. One can extend our results to the general case without to much additional effort.

We begin. Let K be a compact subset in the complex plane plane $\mathbf{C}$. We denote the uniform closure of rational functions with poles off K by $R(K)$. If μ is a finite positive measure with compact support on K, let $R^2(K, \mu)$ denote the closure of $R(K)$ in $L^2(\mu)$. The function algebra $R(K)$ is called a hypodirichlet algebra if the uniform closure of the space $\{\mathrm{Re}(r)|\partial K : r \in R(K)\}$ has finite codimension in $C_{\mathbf{R}}(\partial K)$ and the linear span of $\{log|r| : r, r^{-1} \in R(K)\}$ is dense in $C_{\mathbf{R}}(K)$.

In section 2 we generalize the notion of Carleson squares in the unit disc to a circular domain and characterize the *Carleson* measures on such a domain. We then generalize the result to finitely connected domains via a conformal map.

In [14] W.S. Clary characterized all subnormal operators similar or quasisimilar to the unilateral shift. These results were later extended in different directions see

the original Carleson definition in the unit disk. Let us also note, the results concerning Carleson measures given in this chapter are not generalizations of those given in Chapter 3. They do, though, generalize Carleson's result in another direction. The reader will see that our results concerning similarity of subnormal operators in this chapter are not generalizations of the corresponding ones given in Chapter 4 either.

[31], [38] and Chapter 5 of this thesis. In section 3 we extend Clary's similarity theorem to the case where the fundamental operator (the operator analogous to the unilateral shift) is M_z on $R^2(K, \sigma)$.

6.1 Carleson Measures On A Circular Domain

The next theorem is the essential one in this chapter; it characterizes Carleson measures on circular domains.

Theorem 6.1 *Let G be a circular domain with boundary components C_0, C_1, ... C_n where C_0 denotes the outer boundary. Let a_i be the center of C_i and let r_i be the radius of the circle C_i for each $i = 0, 1, 2, ...n$. Let $f_0 = \frac{z-a_0}{r_0}$ and $f_i = \frac{r_i}{z-a_i}$ for each $i > 0$. Let s be Lebesgue measure on ∂G. A positive measure μ on G is a Carleson measure if and only if $\mu \circ f_i^{-1}$ is a Carleson measure on the unit disc D for each i.*

Proof. Suppose that $\mu \circ f_i^{-1}$ is a Carleson measure on D for each i. Observe that there is a constant $\delta > 0$ such that $G_i = \{z : \mathrm{dis}(z, C_i) < \delta\} \cap G$ is a ring domain for each i. Let $\nu = \mu|(\cup_i G_i)$ and $\nu_i = \mu|G_i$. We first want to show ν is a Carleson measure on G.

Fix i and let $\mathcal{S}$ be the collection of all subsets S in the complex plane $\mathbf{C}$ such that S can be mapped onto a Carleson square by the map f_i. It is clear that there are finite many elements $S_1, S_2, ...S_k$ in $\mathcal{S}$ such that G is covered by the union of $S_1, S_2, ...S_k$. We want to show that there is a constant c_i such that for $t \in [1, \infty)$

$$\{\int_G |r|^t d\nu_i\}^{1/t} \leq c_i\{\int_{\partial G} |r|^t ds\}^{1/t}, \quad r \in Rat(\overline{G}),$$

where s is arclength measure on ∂G. Let $B_j = f_i(S_j)$. Then B_j is a Carleson square in D for each $j = 1, 2, ...k$. We can chose δ (in the definition of G_i's) so small that

$$f_i(G_i) \subset \{z : 1 > |z| > 1 - \frac{\delta}{2}\}.$$

Let

$$E_j = B_j \cap \{z : |z| > 1 - \frac{\delta}{2}\}.$$

It is obvious that we can find a Jordon domain $F_j \subset D$ with smooth boundary so that

$$\overline{E}_j - \partial D \subset F_j.$$

Let η_j be the measure that is defined to be the restriction of $\mu \circ f_i^{-1}$ to E_j and to be zero on $(F_j - E_j)$. We claim that η_j is a Carleson measure on F_j; i.e., there is a constant x_j such that for $t \in [1, \infty)$

$$\{\int_{F_j} |p|^t d\eta_j\}^{1/t} \leq x_j \{\int_{\partial F_j} |p|^t ds_j\}^{1/t}, \quad p \in \mathcal{P},$$

where s_j is arclength measure on ∂F_j and $\mathcal{P}$ is the set of the polynomials. For the proof of the claim it is sufficient to show (Theorem 3.1) that $\eta_j \circ \alpha$ is a Carleson measure on D , where α is a conformal map of D onto F_j. Let

$$C_h = \{z = re^{it} : 1 - h \leq r < 1; t_0 \leq t \leq t_0 + h\}.$$

be a Carleson square. Since

$$\alpha(C_h) \subset F_j \subset D,$$

it is not difficult to see that the smoothness of F_j implies that $\alpha(C_h)$ must be contained in some Carleson square C_{h_1} with $h_1 \leq 2s_j[\partial(\alpha(C_h)) \cap \partial D]$ (note, C_{h_1} may be not contained in F_j). From the definition of η_j and our hypothesis that $\mu \circ f_i^{-1}$ is a Carleson measure on D, we can view η_j as a Carleson measure on D (which contains F_j). Thus, there exists a poistive constant λ such that

$$\eta_j(C_{h_1}) \leq \lambda h_1 \text{ for each Carleson square } C_{h_1}.$$

It follows that

$$\begin{aligned}
\eta_j \circ \alpha(C_h) &= \eta_j(\alpha(C_h)) \\
&\leq \eta_j(C_{h_1}) \\
&= \eta_j[C_{h_1} \cap E_j] \\
&\leq min\{s_j(\partial E_j \cap \partial D),\ 2\lambda s_j[\partial(\alpha(C_h)) \cap \partial D]\} \\
&\leq s_j[\partial(\alpha(C_h)) \cap \partial D] \\
&\leq 2\lambda \|\alpha'\|_\infty h.
\end{aligned}$$

where α' is the derivative of α and $\|\alpha'\| = \sup_{z\in D} |\alpha'(z)|$. This implies that $\eta_j \circ \alpha$ is a Carleson measure D and the claim is proved.

Next we want to show that there exists $y_j > 0$ such that

$$(*) \qquad \{\int_{\partial F_j} |p|^t ds_j\}^{1/t} \leq y_j\{\int_{\partial D} |p|^t ds \circ f_i^{-1}\}^{1/t}, \qquad p \in \mathcal{P}.$$

Let $a \in F_j$. Let ω_j and β be the harmonic measures evaluated at a for F_j and D respectively. For $p \in \mathcal{P}$, we let $\widehat{|p|^t_{F_j}}$ be the harmonic extension of the restriction of $|p|^t$ to ∂F_j and we let $\widehat{|p|^t_D}$ be the harmonic extension of $|p|^t$ to ∂D. Using the maximum principle, we have for each $p \in \mathcal{P}$

$$\begin{aligned}
\{\int_{\partial F_j} |p|^t d\omega_j\}^{1/t} &= \{\int_{\partial F_j} \widehat{|p|}^t_{F_j} d\omega_j\}^{1/t} \\
&= \widehat{|p|}^t_{F_j}(a) \\
&\leq \widehat{|p|}^t_D(a) \\
&= \{\int_{\partial D} \widehat{|p|}^t_D d\beta\}^{1/t} \\
&= \{\int_{\partial D} |p|^t d\beta\}^{1/t}.
\end{aligned}$$

Since harmonic measure on a domain with smooth boundary is boundedly equivalent to arclength measure on the boundary, $(*)$ follows.

Since $\mathcal{P}$ is dense in $R(\overline{F}_j)$ for each j, it follows that for each $r \in Rat(\overline{G})$

$$\begin{aligned}
\{\int_G |r|^t d\nu_i\}^{1/t} &= \{\int_{f_i(G)} |r \circ f_i^{-1}|^t d\nu_i \circ f_i^{-1}\}^{1/t} \\
&\leq \Sigma_j\{\int_{F_j} |r \circ f_i^{-1}|^t d\eta_j\}^{1/t} \\
&\leq \Sigma_j x_j\{\int_{\partial F_j} |r \circ f_i^{-1}|^t ds_j\}^{1/t} \\
&\leq \Sigma_j x_j y_j\{\int_{\partial D} |r \circ f_i^{-1}|^t ds \circ f_i^{-1}\}^{1/t} \\
&= c_i\{\int_{\partial G} |r|^t ds\}^{1/t},
\end{aligned}$$

where x_j is a positive constant for each j and $c_i = \Sigma_j x_j y_j$ is a constant. Consequently, we have

$$\{\int_G |r|^t d\nu\}^{1/t} \leq c\{\int_{\partial G} |r|^t ds\}^{1/t},$$

for all $r \in Rat(\overline{G})$ and some $c > 0$. Hence ν is a Carleson measure.

Now let us go back to μ. First note that if $a \in G$, then $r(a) = \int r d\sigma_a$ for each $r \in Rat(\overline{G})$, where σ_a is the harmonic measure evaluated at a. It follows by Hölder inequality that for $t \in [1, \infty)$

$$|r(a)| \leq \|r\|_{L^t(\sigma_a)}, \quad r \in Rat(\overline{G}).$$

Now we note, since each $|r|$ assume its maximum value on the closed set $G - \cup_i G_i$, that there is λ_r such that $|r(\lambda_r)| = max\{|r(z)| : z \in G - \cup_i G_i\}$. It follows by a classical result that

$$\begin{aligned} \{\int_{G-\cup_i G_i} |r|^t d\mu\}^{1/t} &\leq |r(\lambda_r)|\mu(G - \cup_i G_i) \\ &\leq A\{\int |r|^t d\sigma\}^{1/t}. \end{aligned}$$

where A is some positive constant. Finally we note, since

$$\{\int_G |r|^t d\mu\}^{1/t} = \{\int_{\cup_i G_i} |r|^t d\mu\}^{1/t} + \{\int_{G-\cup_i G_i} |r|^t d\mu\}^{1/t},$$

that the conclusion follows immediately.

To see the proof of the other direction, assume that there exists $c > 0$ such that for $t \in [1, \infty)$

$$\{\int_G |r|^t d\mu\}^{1/t} \leq c\{\int_{\partial G} |r|^t ds\}^{1/t}, \quad r \in Rat(\overline{G}).$$

We want to show that $\mu \circ f_i^{-1}$ is Carleson measure for each i. To do this, let us fix i and let $p \in \mathcal{P}$. It follows that

$$\begin{aligned} \int_D |p|^2 d\mu \circ f_i^{-1} &= \int_G |p \circ f_i|^2 d\mu \\ &\leq c\int_{\partial G} |p \circ f_i|^2 ds \\ &= c\int_{\partial f_i(G)} |p|^2 ds \circ f_i^{-1}. \end{aligned}$$

Let m be the normalized Lebesgue measure on ∂D. If we define operator I: $P^2(m) \to P^2(ds \circ f_i^{-1})$ by $I(p) = p$ for each $p \in \mathcal{P}$, then we deduce from the last inequality that I is a bounded operator. Thus,

$$\int_D |p|^2 d\mu \circ f_i^{-1} \le c\|I\| \int_{\partial D} |p|^2 dm.$$

Now applying Carleson's theorem [22, p. 157], we conclude that $\mu \circ f_i^{-1}$ is a Carleson measure on D. Since i is arbitrary, it follows that μ is a Carleson measure on G.

□

6.2 Carleson Measures On A Finitely Connected Domain

Let G be a finitely connected domain such that every component of ∂G has more than one point. A classical result says that there is a circular domain W such that W and G are conformally equivalent ([67, p. 424]). Let $R(K)$ be a hypodirichlet algebra and let G be the interior of K. It is well-know that there is a circular domain W such that if φ maps W conformally onto G , then the boundary value function of φ is one-to-one almost everywhere with respect to the harmonic measure of W. For the sake of convenience, we still use φ to denote this boundary map. We also have

$$\int f d\sigma_z = \int f \circ \varphi d\omega_{\varphi(z)} \qquad\qquad f \in L^1(\sigma)$$

and

$$\int g \circ \varphi^{-1} d\sigma_z = \int g d\omega_{\varphi(z)} \qquad\qquad g \in L^1(s)$$

where σ_z and $\omega_{\varphi(z)}$ are the harmonic measures evaluated at z and $\varphi(z)$ respectively.

The following theorem is the principal result of the section; it generalizes a theorem of L. Carleson (see [13] or [22]).

Theorem 6.2 *Let G be a connected domain with harmonic measure σ. Suppose that $G = int(\overline{G})$ and $\overline{G}$ is finitely connected. Let W be a circular domain and φ be a conformal map of W onto G. Then a finite measure μ on G is a Carleson measure if and only if $\mu \circ \varphi$ is a Carleson measure on W.*

Proof. Since $\overline{G}$ is finitely connected, it is well-known that $R(\overline{G})$ is a hypodirichlet algebra. Assume that $\mu \circ \varphi$ is a Carleson measure on W. Since W is a circular domain, it follows that $Rat(\overline{W})$ is pointwise boundedly dense in $H^\infty(W)$ (see [2]). It now follows by Lebesgue dominated convergence theorem that $r \circ \varphi$ can be approximated by functions in $R^t(\sigma \circ \varphi)$ for each $r \in Rat(\overline{W})$. Using Theorem 5.1, we have for each $r \in R(\overline{G})$

$$\begin{aligned} \|r\|_{L^t(\mu)} &= \|r \circ \varphi\|_{L^t(\mu \circ \varphi)} \\ &\leq c\|r \circ \varphi\|_{L^t(\sigma \circ \varphi)} \\ &= c\|r\|_{L^t(\sigma)}. \end{aligned}$$

Hence μ is a Carleson measure on G.

For the proof in the other direction, suppose μ is a Carleson measure on G. There is a constant $c > 0$ such that for $t \in [1, \infty)$

$$\{\int_G |r|^t d\mu\}^{1/t} \leq c\{\int_{\partial G} |r|^t\}^{1/t} d\sigma, \quad r \in Rat(\overline{G}).$$

Fix a function q in $Rat(\overline{G})$. Again, using the fact that $R(\overline{G})$ is a hypodirichlet algebra implies it is boundedly pointwise dense in $H^\infty(G)$. We have by the Lebesgue dominated convergence theorem, that

$$\begin{aligned}\{\int_W |q|^t d\mu \circ \varphi\}^{1/t} &= \{\int_G |q \circ \varphi^{-1}|^t d\mu\}^{1/t} \\ &\leq c\{\int_{\partial G} |q \circ \varphi^{-1}|^t\}^{1/t} d\sigma \\ &= c\{\int_{\partial W} |q|^t \sigma \circ \varphi\}^{1/t}.\end{aligned}$$

Since $\sigma \circ \varphi$ is boundedly equivalent to arclength on ∂W, therefore we conclude $\mu \circ \varphi$ is a Carleson measure on W, this proves the theorem.

6.3 Similarity of Rationally Cyclic Subnormal Operators

Throughout this section K will be a compact subset with the property that $R(K)$ is a hypodirichlet algebra. Let G be the interior of K and let σ denote the harmonic measure of K (which is defined to be the harmonic measure of G, consult [17, p. 322]). In addition we assume that G is connected. It follows, as we have noted earlier, that G is a finitely connected domain (see[1]). Let $R^2(K, \sigma)$ be the closure of $Rat(K)$ in $L^2(\sigma)$. For a finite measure with compact support μ, we use the symbol R_μ to represent the operator defined by $R_\mu(f) = zf$ for each $f \in R^2(K, \mu)$.

Recall that a function $f \in R^2(K, \sigma)$ is said to be an outer function if the set $\{rf : r \in Rat(K)\}$ is dense in $R^2(K, \sigma)$. A result in [1] says that if $f \in L^1(\sigma)$ and $\log|f| \in L^1(\sigma)$, then there exists an outer function $x \in R^2(K, \sigma)$ such that $|x|^2 = |f|$ a.e. $[\sigma]$.

Lemma 6.1 *Let μ be a finite measure such that $R_\mu \simeq R_\sigma$. Let T be an invertible* *operator from* $R^2(K,\mu)$ *to* $R^2(K,\sigma)$ *such that* $TR_\mu = R_\sigma T$. *Let* $u = T(1)$ *and* $\alpha = \mu|\partial G$. *There exists an outer function* x *in* $R^2(K,\sigma)$ *with* $|x|^2 = \frac{d\alpha}{d\sigma}$ *and there exists an invertible function* h *in the Banach algebra* $R^2(K,\sigma) \cap L^\infty(\sigma)$ *such that* $hx = u$.

Proof. Since $TR_\mu = T_\sigma A$, it is easy to see that

$$\|T^{-1}\|^{-1}\|r\|_\mu \le \|ur\|_\sigma \le \|T\|\|r\|_\mu \quad \text{for all} \quad r \in Rat(\overline{G}). \tag{6.1}$$

Using Theorem 2.1 in [38, McCarthy], we see that $\mu| \ll \sigma$ and $log\frac{d\mu}{d\sigma} \in L^1(\sigma)$; therefore, there exists an outer function x such that $|x|^2 = \frac{d\alpha}{d\sigma}$. Define a densely defined operator B on $R^2(K,\sigma)$ via

$$B(xr) = ur$$

for every $r \in R(\overline{G})$. Noting that for each $r \in R(\overline{G})$

$$\begin{aligned} \|ur\|_\sigma^2 &= \int |T(r)|^2\, d\sigma \\ &= \|T\|^2 \int |r|^2 \frac{d\alpha}{d\sigma}\, d\sigma \\ &= \|T\|^2 \|rx\|_\alpha^2, \end{aligned}$$

we conclude B is bounded on $R^2(K,\sigma)$. Also observe that B commutes with R_σ. It now follows by a theorem of Yoshino ([17] p.147) that there is $h \in R^2(K,\sigma) \cap L^\infty(\sigma)$ such that $B = M_h$, where M_h is the multiplication operator induced by multiplication by the function h on $R^2(K,\sigma)$. So $hx = u$. Clearly (6.1) implies that M_h is bounded below; it follows then that h is invertible in $L^\infty(\sigma)$. $\square$

Now we present the main result of this section.

Theorem 6.3 *Suppose S is a subnormal operator. The following are equivalent:*

1) $S \simeq R_\sigma$.

2) S is rationally cyclic and if μ is any measure such that $S \cong R_\mu$, then the measure μ is supported on $\overline{G}$, the measure $\alpha = \mu|\partial G \ll \sigma$ and the function $\log \frac{d\alpha}{d\sigma} \in L^1(\sigma)$. Furthermore, if x is an outer function in $R^2(K, \sigma)$ so that $|x|^2 = \frac{d\alpha}{d\sigma}$, then $|\widehat{x}|^{-2}\mu|G$ is a Carleson measure.

3) There is a Carleson measure τ on G such that $S \cong R_{\sigma+\tau}$.

Remark: This theorem is not a generalization of Theorem 5.1 for the following reasons (also, see the remark for Theorem 5.5):

1) The definition of a Carleson measure on a multiply connected domain is not equivalent to that one on a simply connected domain, as we point out in the begining of this chapter.

2) Even when the two definitions of a Carleson measure coincides on K°, the spaces $P^2(\mu)$ and $R^2(K, \mu)$ are different and thus, the two R_μ operators are different, in general.

Therefore, Theorem 5.1 and Theorem 6.2 are the generalizations of Clary's result in two different directions.

Proof. 1) $\Longrightarrow$ 2). Suppose that $S \simeq R_\sigma$. Then S is rationally cyclic and there is a finite measure μ such that $R_\mu \cong S$. So $R_\mu \simeq R_\sigma$ also. Let $\alpha = \mu|\partial G$ and $\beta = \mu|G$. Then $\mu = \alpha + \beta$ and $\log(d\alpha/d\sigma) \in L^1(\sigma)$.

Let T be an invertible operator from $R^2(K, \mu)$ to $R^2(K, \sigma)$ so that $TR_\mu = R_\sigma T$. As in Lemma 5.7 we have that $T^{-1}(r) = 1/T(1)r = r\frac{1}{u}$ *a.e.* $[\mu]$.

Now let x, h, and u be the functions as in Lemma 6.1. We now have $T^{-1}(r) = r\frac{1}{u}$; hence,

$$\frac{r}{x} = h\frac{r}{u} = hT^{-1}(r) \qquad \text{a.e.} \quad [\mu], \qquad \text{for all} \quad r \in Rat(\overline{G}).$$

Using Lemma 6.1, we have

$$\begin{aligned} \|r\|_{|\hat{x}|^{-2}\mu|G} &= \|r\|_{|\frac{h}{u}|^{-2}\mu|G} \\ &\leq \|h\|_\infty \|r\frac{1}{u}\|_\mu \\ &= \|h\|_\infty \|T^{-1}(r)\|_\mu \\ &\leq \|h\|_\infty \|T^{-1}\| \, \|r\|_\sigma \end{aligned}$$

for every $r \in Rat(\overline{G})$. From Theorem 2 in this section we see that μ is a Carleson measure on G.

2) $\Longrightarrow$ 3). Let $\alpha = \sigma + |\hat{x}|^{-2}\mu|G$. Apparently, the operator T defined from $R^2(K,\mu)$ to $R^2(\alpha)$ by $T(a) = xa$ for each $a \in Rat(\overline{G})$ is an isometry. By hypothesis, $|\hat{x}|^{-2}\mu|G$ is a Carleson measure. So this implies that that $\| \, \|_\sigma$ and $\| \, \|_\alpha$ are equivalent norms. Since $Rat(\overline{G})x$ is dense in $R^2(K,\sigma)$, it follows that $Rat(\overline{G})x$ is dense in $R^2(\alpha)$. Thus, T is an isomorphism. Clearly $TR_\mu = R_\alpha T$, and therefore $R_\mu \cong R_\alpha$.

3) $\Longrightarrow$ 1). Suppose that there is a Carleson measure τ such that $S \cong R_{\sigma+\tau}$. It sufficient to prove that $R_{\sigma+\tau} \simeq R_\sigma$. By Theorem 6.2 there is $c > 0$ such that for each $r \in Rat(\overline{G})$

$$\|r\|_\tau \leq c\|r\|_\sigma \quad \text{for all} \quad r \in$$

Define operator T from $R^2(K,\sigma)$ to $R^2(K,\sigma+\tau)$ via $T(r) = r$ for each $r \in Rat(\overline{G})$.

Then T is bounded and invertible. Moreover $T R_\sigma = R_{\sigma+\tau} T$, and hence, R_σ is similar to $R_{\sigma+\tau}$.

Chapter 7

Boundary Values Of Analytic Functions In The Banach Space $P^t(\sigma)$ On Crescents

A simply connected domain Ω is called a crescent if it is enclosed by two Jordon curves, which intersect at a single point. We call this point the multiple boundary point. The theory of Banach spaces of analytic functions on crescents has been studied by a number of authors, but there still are many unanswered questions. Though a crescent Ω has a nice and simple boundary topologically, it does not have many of the nice properties that a Jordon domain possesses. For example, $\mathcal{P}$, the set of polynomials, is not always dense in the Hardy space $H^t(\Omega)$ (where $t \in [1, \infty)$) and this density property depends on the geometrical properties near the multiple boundary point (see, [4]). J. Akeroyd shows [4] that if Ω is bounded by two tangent circles, then $\mathcal{P}$ is always dense in $H^t(\Omega)$; however, this is not always the case for the corresponding Bergman space $L^t_a(\Omega)$ (see [10]). We say a crescent is A-type, if it is contained in D, it is enclosed by ∂D, and another Jordon curve whose part near the multiple boundary point coincides with the sides of a triangle. It is not difficult to show the polynomials are not dense in $H^t(\Omega)$ if Ω is an A-type crescent.

Fix a crescent Ω with harmonic measure σ. For the sake of simplicity, we assume that $\Omega \subset D$ and $\partial D \subset \partial\Omega$. An elementary fact is:

$$\text{either } abpe P^t(\sigma) = \Omega, \text{ or } abpe P^t(\sigma) = D;$$

the former equality is equivalent $\mathcal{P}$ being dense in $H^t(\Omega)$ (see Theorem 2.1). In this chapter we tacitly assume that (for a fixed t in $[1, \infty)$[1])

$$abpe P^t(\sigma) = D.$$

We show (Theorem 7.2) that every function $f \in P^t(\sigma)$ has nontangential limits, at almost everywhere with respect to m on ∂D. Moreover,

$$f(\alpha) = \lim_{z \to \alpha} \hat{f}(z) \quad \text{a.e. } [m] \text{ on } \partial D,$$

where the limits are taken in nontangential sense. In other words, every $\hat{f}$ in $P^t(\sigma)$ has a boundary value (function) on the circle. Now a natural question is raised:

$$\text{If } f \in P^t(\sigma) \text{ and } f|\partial D \text{ is bounded, is } \hat{f}(z) \text{ bounded?}$$

In the classical Hardy space case it is well-known that if $f \in P^t(m)$ and $\hat{f}(z)$ has a bounded boundary value, then $\hat{f}(z)$ itself is a bounded analytic function. The measure m actually is a harmonic measure for D; it would be very reasonable for us to expect the same is true for the functions in the space $P^t(\sigma)$. Unfortunately, this is no longer the case in general.

In Section 7.1 we present a counter-example. In fact, we construct a domain Ω

[1]By Corollary 3.2, the assumption implies that $abpe P^t(\sigma) = D$ for all $t \in [1, \infty)$. We do not need this fact here.

(with harmonic measure σ) and an unbounded function $h \in P^2(\sigma)$ such that $\widehat{h}$ has a continuous boundary value on ∂D.

Can we have a positive answer to the question for some of these crescents Ω ?

In Section 7.2, we give an affirmative answer if Ω is an A-type region.

Lastly, let μ be a finite positive measure with compact support in the plane and let S_μ be the operator defined by $S_\mu(f) = zf$ for each $f \in P^2(\mu)$. As an application of the last result, we prove that if τ is a positive measure carried by D and if Ω is an A-type crescent, then $S_{\sigma+\tau}$ and S_σ are similar if and only if τ is a Carleson measure on D.

7.1 A Counter-Example

The proof of the following lemma is elementary.

Lemma 7.1 *Let a and b be two positive numbers with $b > a$. Let $G_{a,b}$ denote the crescent enclosed by the circles $\{z : |z - (1 - \frac{1}{a})| = \frac{1}{a}\}$ and $\{z : |z - (1 - \frac{1}{b})| = \frac{1}{b}\}$. Then*

$$f_{a,b} = \frac{-i(e^{-\frac{a\pi i}{b-a}})e^{-(\frac{2\pi i}{b-a})\frac{1}{z-1}} - 1}{-i(e^{-\frac{a\pi i}{b-a}})e^{-(\frac{2\pi i}{b-a})\frac{1}{z-1}} + 1}$$

is a conformal map of $G_{a,b}$ onto D.

The next result is well-known and it can be proved by applying a famous theorem of F. and M. Riesz ([34, Koosis's book, p.70]).

Lemma 7.2 *With above notions, if let $x = f_{ab}^{-1}(0)$ and let s be arclength measure on ∂G_{ab}, then*

$$d\omega_x = \frac{1}{2\pi}|f'_{ab}|\ ds.$$

Let W_1 be the crescent enclosed by circles $C_0 = \partial D$ and $C_1 = \{z : |z - \frac{2}{3}| = \frac{1}{3}\}$; let W_2 be the crescent enclosed by C_1 and $C_2 = \{z : |z - \frac{4}{5}| = \frac{1}{5}\}$; let W_3 be the crescent enclosed by C_2 and $C_3 = \{z : |z - \frac{6}{7}| = \frac{1}{7}\}$; and let W_0 be the crescent enclosed by ∂D and C_3.

Now if we connect $z = 1$ and $z = \frac{i}{100}$ by a segment l, then l separates W_2 into two parts. We use V to denote the part completely contained in the upper plane. Set

$$U = W_0 \cap \{z : \text{Im z} > 0\}$$

and set

$$G = U - \overline{V}.$$

G is a crescent and we use ω to denote its harmonic measure. Besides the point $z = 1$, ∂V has two other singular points (which are the intersection points of l with C_1 and C_2), and ∂G has four more singular points (which are the intersection points of l with C_2 and C_2 and the intersection points of the real axis with C_0 and C_3). For some technical reasons we, in addition, modify ∂G and ∂V slightly at a small neighborhood of each of those 'bad' points so that G and V have smooth boundaries except at $z = 1$. With these notations, now we have:

Lemma 7.3 *Let h_j be the restriction of $\frac{e^{\frac{-\pi i}{z-1}}+1}{e^{\frac{-\pi i}{z-1}}-1}$ to W_j, $j = 1, 3$, and let h_2 be the restriction of $\frac{e^{\frac{-\pi i}{z-1}}-1}{e^{\frac{-\pi i}{z-1}}+1}$ to W_2. The function h_j maps W_j conformally onto D for each $j = 1, 2, 3$. Moreover, if s is arclength measure on ∂G and τ is harmonic measure of V, then ω is boundedly equivalent to the measure $(\frac{1}{|z-1|^2}e^{-\frac{|Imz|}{|z-1|^2}\pi})s$ and τ is boundedly equivalent to the restriction of $(\frac{1}{|z-1|^2}e^{-\frac{|Imz|}{|z-1|^2}\pi})s$ to ∂V.*

Proof. Using Lemma 7.1 with $a = 1$ and $b = 3$, we see $h_1 = f_{1,3}$ maps W_1 conformally onto D. Similarly, with $a = 3$ and $b = 5$ for W_2, we see $f_{3,5} = h_2$. Setting $a = 5$ and $b = 7$ for W_3 in Lemma 7.3 yields the desired result for $f_{5,7} = h_3$. This proves the first part of the lemma.

Now an easy computation gives that

$$h_j' = \left(\frac{e^{\frac{-\pi i}{z-1}}+1}{e^{\frac{-\pi i}{z-1}}-1}\right)' = -\frac{\frac{2\pi i}{(z-1)^2}e^{-\frac{\pi i}{z-1}}}{(e^{\frac{-\pi i}{z-1}}-1)^2} \quad \text{for } j = 1, 3$$

and

$$h_2' = \left(\frac{e^{\frac{-\pi i}{z-1}}-1}{e^{\frac{-\pi i}{z-1}}+1}\right)' = \frac{\frac{2\pi i}{(z-1)^2}e^{-\frac{\pi i}{z-1}}}{(e^{\frac{-\pi i}{z-1}}+1)^2}.$$

So we see that

$$|h_j'|s \quad \text{and} \quad \left(\frac{1}{|z-1|^2}e^{-\frac{|Imz|}{|z-1|^2}\pi}\right)s \quad \text{are boundedly equivalent.}$$

But ω and σ_i are boundedly equivalent on W_j, $j = 1, 3$, so using Lemma 7.2, we conclude that

$$\omega \quad \text{and} \quad \left(\frac{1}{|z-1|^2}e^{-\frac{|Imz|}{|z-1|^2}\pi}\right)s \quad \text{are boundedly equivalent.}$$

Similarly, τ and the restriction of $(\frac{1}{|z-1|^2}e^{-\frac{|Imz|}{|z-1|^2}\pi})s$ to ∂V are boundedly equivalent.

□

Lemma 7.4 *With above notations, $abpeP^t(\omega) = U$ for all $t \in [1, \infty)$.*

Proof. First, we have (see the proof of Theorem 2.1)

$$G \subset bpeP^t(\omega).$$

From Lemma 7.3 we see that τ and $\omega|\partial V$ are boundedly equivalent; it follows that

$$V \subset bpeP^t(\tau) \subset bpeP^t(\omega|\partial V) \subset bpeP^t(\omega).$$

Consequently,

$$G \cup V \subset bpeP^t(\omega).$$

An appeal to Harnack's inequality (see [5]), one can easily show that

$$bpeP^t(\omega) = abpeP^t(\omega).$$

Thus,

$$G \cup V \subset abpeP^t(\omega).$$

Next we show that $abpeP^t(\omega)$ is connected. This is equivalent to show that $P^t(\omega)$ contains no non-trivial characteristic function (see [64]). Let $\Delta \subset \partial G$ such that $\chi_\Delta \in P^t(\omega)$. So there exists $\{p_n\} \subset \mathcal{P}$ such that $\int |\chi_\Delta - p_n|^t d\omega \to 0$. Recall the Hardy space $H^t(G)$ consists of all analytic functions f such that $|f|^t$ has a harmonic majorant on G. For $f \in H^t(G)$, the norm can be defined as $\|f\| = u(a)^{\frac{1}{t}}$, where u is the least harmonic majorant of f. As a sequence in $H^t(G)$, our given sequence $\{p_n\}$ converges to a function, say x, in $H^t(G)$. Since $\chi_\Delta^2 = \chi_\Delta$, it follows that $x^2 = x$. Since G is connected, this implies that either $x = 1$ or $x = 0$. Hence, $\chi_\Delta = 0$ or $\chi_\Delta = 1$.

Let $W = abpeP^t(\omega)$. If $W \neq U$, then W is a slit simply connected domain whose boundary is contained in the union of two Jordon curves. This implies that no conformal map from D onto W is almost one-to-one on ∂W with respect to m. On the other hand, according to Thomson's theorem ([64], we have an isometrical isomorphism map from $H^\infty(W)$ to $P^t(\omega) \cap L^\infty(\omega)$. Using Theorem 94 of [44, Miller-Olin-Thomson], we conclude that W must be nicely connected. So the proof is complete.

□

Lemma 7.5 *If $g(z) = (z-1)^{\frac{2}{t}} e^{\frac{\pi}{t}\frac{z+1}{1-z}}$, then $g \in P^t(\omega)$.*

Proof. We first show that $g \in L^t(\omega)$. Let $z = x + iy$. By Lemma 7.3 there is a constant $c > 0$ such that

$$\begin{aligned}
\int |g_(z)|^t d\omega &\leq c \int |g_(z)|^t \frac{1}{|z-1|^2} e^{-\frac{|Imz|}{|z-1|^2}\pi} ds \\
&\leq c \int \{|(z-1)|^{\frac{2}{t}} e^{\frac{\pi}{t}\frac{1-x^2-y^2}{|1-z|^2}}\}^t \frac{1}{|z-1|^2} e^{-\frac{|Imz|}{|z-1|^2}\pi} ds \\
&\leq c \int e^{\pi\frac{1-x^2-y^2}{|1-z|^2}} e^{\frac{-|y|}{|1-z|^2}\pi} ds \\
&\leq c \int e^{\pi\frac{1-x^2-y^2}{|1-z|^2}} ds.
\end{aligned}$$

It is easy to verify that $e^{\pi\frac{1-x^2-y^2}{|1-z|^2}}$ is constant on each circle C_i, $i = 1, 2, 3$. So we conclude that $g \in L^t(\omega)$. Now set

$$g_n(z) = (z-1)^{\frac{2}{t}} e^{\frac{\pi}{t}\frac{z+1}{1-z-\frac{i}{n}}}.$$

We see g_n belongs to the disc algebra

$$A(U) = \{f : \text{f is analytic on } U \quad \text{and is continuous on } \overline{U}\}$$

and, hence,

$$g_n \in P^t(\omega) \quad \text{for each} \quad n.$$

Moreover, since

$$|z-1| < |z-(1-\frac{i}{n})| = |1-z-\frac{i}{n}| \quad \text{for all} \quad z \in G,$$

we have

$$\begin{aligned} |g_n(z)| &= |(z-1)^{\frac{2}{t}}||e^{\frac{\pi}{t}\frac{z+1}{1-z-\frac{i}{n}}}| \\ &= |(z-1)|^{\frac{2}{t}} e^{\frac{\pi}{t}\frac{1-x^2-y^2-\frac{y}{n}}{|1-z-\frac{i}{n}|^2}} \\ &\leq |(z-1)|^{\frac{2}{t}} e^{\frac{\pi}{t}\frac{1-x^2-y^2}{|1-z-\frac{i}{n}|^2}} \\ &\leq |(z-1)|^{\frac{2}{t}} e^{\frac{\pi}{t}\frac{1-x^2-y^2}{|1-z|^2}} \\ &= |g(z)|. \end{aligned}$$

Apparently, $g_n \to g$ pointwise. It follows by Lebesgue dominated convergence theorem that $g \in P^t(\omega)$.

$\square$

Theorem 7.1 *There is a crescent* Ω *with harmonic measure* σ *and there is an unbounded function* $h \in P^t(\sigma)$ *such that*

$$abpe P^t(\sigma) = D \quad \textit{and } h|\partial D \quad \textit{is continuous.}$$

Proof. Choose the region G and the function g as in Lemma 7.5. Let φ be a conformal map of D onto U, where U is the region as in Lemma 7.4. Since U is a Jordon domain, it follows from a theorem of Carathéodory that φ can be extended to be a homeomorphism from $\overline{D}$ onto $\overline{U}$ ([67, p. 353]). For the sake of simplicity, we still denote this homeomorphism by φ. Let $\Omega = \varphi^{-1}(G)$ and let σ be its harmonic measure. Now we claim:

$$\varphi|\partial\Omega \in P^t(\sigma) \quad \text{and} \quad \varphi^{-1}|\partial G \in P^t(\omega).$$

The proofs for them are very similar and we only prove the first one. To show that $\varphi|\partial\Omega \in P^t(\sigma)$, it suffices to prove that $\varphi|\partial\Omega \in P^\infty(\sigma)$. Since $\varphi \in P^\infty(m)$, there is a sequence of polynomials $\{p_n\}$ such that it weak-star converges to φ. That is,

$$\int_{\partial D} (\varphi - p_n) f \, dm \to 0 \quad \text{for each} \quad f \in L^1(m).$$

Since $\Omega \subset D$, it is well-known fact that the measure $\sigma|\partial D$ is absolutely continuous with respect to m. It now follows that

$$\int_{\partial D} (\varphi - p_n) f \, d\sigma \to 0 \quad \text{for each} \quad f \in L^1(\sigma).$$

Also, the weak-star convergence implies that $\{p_n\}$ is uniformly bounded on D and it pointwise converges to φ on D. Using the Bounded Convergence Theorem, we conclude

$$\int_{\partial\Omega\cap D}(\varphi - p_n)f\ d\sigma \to 0 \text{ for each } f \in L^1(\sigma).$$

Therefore, we have

$$\int_{\partial\Omega}(\varphi - p_n)f\ d\sigma \to 0 \text{ for each } f \in L^1(\sigma).$$

That is, $\{p_n\}$ weak-star converges to φ; and hence $\varphi \in P^\infty(\sigma)$. The claim is proved.

The restriction of φ to Ω is a conformal map of Ω onto G. So $\sigma \circ \varphi^{-1}$ is a harmonic measure of G. Thus, $\sigma \circ \varphi^{-1}$ is boundedly equivalent to ω. Let g be the function as in Lemma 7.5. There is a sequence of polynomials $\{q_n\}$ such that

$$q_n \to g \quad \text{in} \quad P^t(\omega).$$

Let $h = g \circ \varphi$. We have

$$\int_{\partial\Omega} |h - q_n \circ \varphi|^t\ d\sigma = \int_{\partial G} |g - q_n|^t\ d(\sigma \circ \varphi^{-1}) \to 0.$$

Since $\varphi \in P^t(\sigma) \cap L^\infty(\sigma)$, and since $P^t(\sigma) \cap L^\infty(\sigma)$ is a Banach algebra, it follows that

$$q_n \circ \varphi \in P^t(\sigma) \cap L^\infty(\sigma).$$

Consequently,

$$h \in P^t(\sigma).$$

Now we want to show that

$$abpeP^t(\sigma) = D.$$

Let α be a conformal map of Ω onto D. In light of Theorem 2.1, the fact $abpeP^t(\sigma) = D$ is equivalent to the fact that α is not in $P^t(\sigma)$. To prove the latter, we argue by contradicting; suppose there is a sequence of polynomials $\{p_n\}$ such that

$$\int |p_n - \alpha|^t d\sigma \to 0.$$

Note that

$$\int_{\partial G} |\alpha \circ \varphi^{-1} - p_n \circ \varphi^{-1}|^t \, d(\sigma \circ \varphi^{-1}) = \int_{\partial\Omega} |\alpha - p_n|^t \, d\sigma \to 0.$$

Also note that $p_n \circ \varphi^{-1} \in P^t(\omega)$ for each n. It follows that

$$\alpha \circ \varphi^{-1} \in P^t(\omega).$$

On the other hand, the restriction of $\alpha \circ \varphi^{-1}$ to G is also a conformal map of G onto D. Using Theorem 2.1 again, we conclude

$$abpeP^t(\omega) = G \neq U,$$

a contradiction to Lemma 7.4. Hence $abpeP^t(\sigma) = D$.

Finally, one verifies that $g(z) = (z-1)^{\frac{2}{t}} e^{\frac{\pi}{t}\frac{z+1}{1-z}}$ is continuous on ∂D and g is unbounded on G. Since φ is a homeomorphism from $\overline{D}$ onto $\overline{U}$, we see that $h = g \circ \varphi$ is the desired function. The proof is complete.

□

7.2 On A-type Crescents

The following theorem shows that $\hat{f}$ has boundary values on ∂D for each $f \in P^t(\sigma)$.

Theorem 7.2 *Let Ω be a crescent and let σ be its harmonic measure. If $abpeP^t(\sigma) = D$, then $\hat{f}$ has nontangential limits almost everywhere with respect to m on ∂D for every $f \in P^t(\tau)$. Moreover,*

$$f(\sigma) = \lim_{z \to \sigma} \hat{f}(z) \quad \textit{nontangentially a.e. } [m] \textit{ on } \partial D.$$

Remark The hypothesis $abpeP^t(\sigma) = D$ implies that Ω is a crescent which has ∂D as its outer boundary. Also note that we do not require Ω to be an A-type crescent (see the definition of A-type crescent at the beginning of this chapter). The theorem holds for all crescent Ω with $abpeP^t(\sigma) = D$.

Proof. Suppose that $f \in P^t(\sigma)$. Choose $p_n \in \mathcal{P}$ such that

$$\int |p_n - f|^t d\sigma \to 0.$$

Let φ be a conformal map of D onto Ω and let $\tilde{\varphi}$ denote the boundary value function on ∂D. Note

$$\int |p_n \circ \tilde{\varphi} - f \circ \tilde{\varphi}|^t dm \to 0.$$

Since $p_n \circ \tilde{\varphi} \in P^t(m)$ for each n, we get

$$f \circ \tilde{\varphi} \in P^t(m).$$

Now we claim:

$$[f \circ \varphi](z) = \widehat{f \circ \tilde{\varphi}}(z) \text{ for all } z \in D.$$

Let $L = \tilde{\varphi}^{-1}(\partial\Omega - \partial D)$ (note, ∂D is a part of $\partial\Omega$). It follows that L is an open arc on the unit circle ∂D (the proof uses Carathëodory's theorem). Since $\hat{f}(z)$ is continuous at every point of $\partial\Omega - \partial D$, it follows that for every $e^{ix} \in L$

$$\begin{aligned} \lim_{z \to e^{ix}} \hat{f} \circ \varphi(z) &= \hat{f}(\lim_{z \to e^{ix}} \varphi(z)) \\ &= \hat{f}(\tilde{\varphi}(e^{ix})) \\ &= f(\tilde{\varphi}(e^{ix})), \end{aligned}$$

where all limits are taken nontangentially. On the other hand, if we take the nontangential limit

$$\lim_{z \to e^{ix}} \widehat{f \circ \tilde{\varphi}}(z) = (f \circ \tilde{\varphi})(e^{ix}) = f(\tilde{\varphi}(e^{ix}))$$

for almost every $x \in [-\pi, \pi]$. Consequently, the nontangential limit

$$\lim_{z \to e^{ix}} [\hat{f} \circ \varphi(z) - \widehat{f \circ \tilde{\varphi}}(z)] = 0$$

for almost every point e^{ix} on the arc L. Applying Lusin-Privaloff's theorem [67, p.320], we conclude

$$\hat{f} \circ \varphi(z) = \widehat{f \circ \tilde{\varphi}}(z) \text{ for } z \in D.$$

The claim is proved.

Now let $J = \widetilde{\varphi}^{-1}(\partial D)$ (again, ∂D is a part of the boundary of Ω). Again applying Carathëodory's theorem (locally) we see J is an arc on the unit circle. Since both ∂D and J are smooth, φ^{-1} preserves angles at almost every point of ∂D. Therefore, for almost every $\beta \in \partial D$ we have

$$\begin{aligned}
\lim_{w \to \beta} \hat{f}(w) &= \lim_{z \to \widetilde{\varphi}^{-1}(\beta)} \hat{f}(\varphi(z)) \\
&= \lim_{z \to \widetilde{\varphi}^{-1}(\beta)} \hat{f} \circ \varphi(z) \\
&= \lim_{z \to \widetilde{\varphi}^{-1}(\beta)} \widehat{f \circ \widetilde{\varphi}}(z) \\
&= f \circ \widetilde{\varphi}(z)(\widetilde{\varphi}^{-1}(\beta)) \\
&= f(\beta)
\end{aligned}$$

where all limits are nontangential. The proof is complete.

□

The next lemma can be found in [4], a proof can also be given using Lemma 7.2 (one may also consult page 236 in [5]). We state it here for reader's convenience.

Lemma 7.6 *Let G be a Jordon domain such that ∂G is smooth except at one point a. Suppose that ∂G forms an angle α at a with $\alpha < \pi$. Then harmonic measure for G and the measure $(|z-a|^{\frac{\pi}{\alpha}-1})s$ are boundedly equivalent, where s is arclength on ∂G.*

Lemma 7.7 *Let Ω be an A-type crescent and let σ be its harmonic measure. Then*

$$abpe P^t(\sigma) = D.$$

Proof. Let γ be the Jordan curve such that $\gamma \cup \partial D = \partial\Omega$. Without loss of generality we may assume $z = 1$ is the multiple boundary point of $\partial\Omega$. Since Ω is an A-type domain, there exist two segments $l_1 \subset \gamma$ and $l_2 \subset \gamma$ such that l_1 and l_2 together form an angle at $z = 1$ (note, by definition a crescent is enclosed by **two** Jordon curves, so γ has only a intersection point with ∂D). Clearly l_i also forms an angle α_i with the vertical line Re $z = 1$, for each $i = 1, 2$. We may assume that $\alpha_1 < \alpha_2 < \pi$ ($\alpha_i > 0$ by the definition of an A-type crescent). Using the above lemma we can find constants $c > 0$ and $r > 0$ such that

$$d\sigma|\partial D \geq c|z-1|^r \ dm.$$

This implies that $\log \frac{d\sigma}{dm} \in L^1(m)$. Now a simple application of Szegö's Theorem [25, p. 136] shows that $abpeP^t(\sigma) = D$. So the conclusion follows by the above measure inequality.

□

The next theorem is a maximum principle type result for functions in $P^t(\sigma)$ on A-type crescents.

Theorem 7.3 *Let Ω be an A-type crescent with harmonic measure σ. If $f \in P^t(\sigma)$ and $f|\partial D \in L^\infty(\sigma)$, then $f \in P^t(\sigma) \cap L^\infty(\sigma)$.*

Proof. Suppose that $f \in P^t(\sigma)$ and $f|\partial D \in L^\infty(\sigma)$. So there exists $\{p_n\} \subset \mathcal{P}$ such that

$$\int_{\partial\Omega} |p_n - f|^t d\sigma \to 0 \quad \text{as} \quad n \to 0.$$

We may assume that $z = 1$ is the multiple boundary point. As in the proof of the previous lemma, there exist two constants $c > 0$ and $r > 0$ such that

$$d\sigma|\partial D \geq c|z-1|^r \, dm.$$

Hence,

$$\int_{\partial\omega} |p_n - f|^t |z-1|^r dm \to 0.$$

We express the last limit as

$$\int_{\partial D} |p_n(z-1)^{\frac{r}{t}} - f_0(z-1)^{\frac{r}{t}}|^t dm \to 0,$$

where $f_0 = f|\partial D$. Set $u = f_0(z-1)^{\frac{r}{t}}$, and note that

$$u = f_0(z-1)^{\frac{r}{t}} \in P^t(m).$$

Since $(z-1)^{\frac{r}{t}}$ is bounded, it follows from the hypothesis that

$$u \in L^\infty(m) \cap P^t(m) = P^\infty(m).$$

Now if we let $v = f_0^{\frac{t}{2r}}$, then

$$v = u^{\frac{t}{2r}}(z-1)^{-\frac{1}{2}}.$$

One can directly check that

$$\sup_{r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{1}{1-re^{i\theta}}\right|^{\frac{1}{2}} d\theta\right\} < \infty,$$

so $(\frac{1}{1-z})^{\frac{1}{2}} \in P^1(m)$ (see [22], or [28]). Obviously $u^{\frac{t}{2r}} \in P^{\infty}(m)$, it now follows that

$$v = u^{\frac{t}{2r}}(z-1)^{-\frac{1}{2}} \in P^1(m);$$

i.e.,

$$f_0 = v^{\frac{2r}{t}} \in P^{\frac{t}{2r}}(m).$$

But $f_0 \in L^\infty(m)$, so

$$f_0 \in L^\infty(m) \cap P^{\frac{t}{2r}}(m) = P^\infty(m).$$

Now let $\widehat{f_0}$ the analytic extension of f_0 on D. We note $\widehat{f_0} \in H^\infty(D)$ and $\lim_{z\to\lambda} \widehat{f_0}(\lambda) = f_0(\lambda)$ nontangentially a.e. $[m]$ on ∂D. Combining this fact with the proceeding theorem, we obtain

$$\lim_{z\to\lambda}[\widehat{f}(z) - \widehat{f_0}(z)] = 0 \text{ nontangentially a. e. } [m] \text{ on } \partial D.$$

Using Lusin-Privaloff's theorem [67, p. 320], we conclude

$$\widehat{f}(z) = \widehat{f_0}(z) \text{ for each } z \in D.$$

Hence $\widehat{f}$ is bounded on D. But $\widehat{f}(z) = f(z)$ a.e. $[\sigma]$ for $z \in \partial\Omega \cap D$, thus, we obtain

$$f \in L^\infty(\sigma) \cap P^t(\sigma).$$

The proof is complete.

□

Application to Operator Theory

A positive measure τ on D is a *Carleson measure* if there is a constant $c > 0$ such that for all $t \in [1, \infty)$

$$\|p\|_{L^t(\tau)} \leq c \ \|p\|_{L^t(m)} \ \text{ for } \ p \in \mathcal{P}.$$

A theorem of *Carleson* [28, p. 238] shows that a measure τ on the unit disc is a *Carleson* measure if and only if there exists a constant $A > 0$ such that

$$\mu(C_h) \leq Ah$$

for each *Carleson* square

$$C_h = \{z = re^{it} : 1 - h \leq r < 1; t_0 \leq t \leq t_0 + h\}.$$

Theorem 7.4 *Let Ω be an A-type crescent and let σ be harmonic measure of Ω. If τ is a finite positive measure on D, then $S_{\sigma+\tau}$ and S_σ are similar if and only if τ is a Carleson measure on D.*

Proof. Suppose that $S_{\sigma+\tau}$ and S_σ are similar. Let $A : P^2(\sigma+\tau) \to P^2(\sigma)$ be an invertible operator such that

$$AS_{\tau+\sigma} = S_\sigma A.$$

For every $p \in \mathcal{P}$, one verifies that

$$A(p) = (A(1))p, \ \text{ and } \ A^{-1}(p) = (A^{-1}(1))p.$$

Moreover, if we let $u = A(1)$ and $v = A^{-1}(1)$, then

$$\|A^{-1}\|^{-1}\|p\|_{\sigma+\tau} \leq \|up\|_\sigma \leq \|A\|\|p\|_{\sigma+\tau} \tag{7.1}$$

and

$$\|A\|^{-1}\|p\|_\sigma \leq \|vp\|_{\sigma+\tau} \leq \|A^{-1}\|\|p\|_\sigma. \tag{7.2}$$

Replacing p by $z^n p$ in (7.1) and letting $n \to \infty$, we obtain (note, $|z| < 1$ on D)

$$\|up\|_{\sigma_0} \leq \|A\|\|p\|_{\sigma_0} \quad \text{for each} \quad p \in \mathcal{P}, \tag{7.3}$$

where $\sigma_0 = \sigma|\partial D$. Now we claim:

$$u \in L^\infty(\sigma_0).$$

In fact, (7.3) implies that the operator M_u, defined by

$$M_u(p) = up \quad \text{for each} \quad p \in \mathcal{P},$$

is a bounded linear operator on $P^2(\sigma_0)$. So we have

$$\begin{aligned}
\int |u^n|^2 d\sigma_0 &= \int |M_u^n(1)|^2 d\sigma_0 \\
&\leq \|M_u\|^{2n} \int d\sigma_0 \\
&= \|M_u\|^{2n} \sigma_0(\partial D) \quad \text{for each} \quad n \geq 1.
\end{aligned}$$

Thus,

$$\int \left|\frac{u}{\|M_u\|}\right|^{2n} d\sigma_0 \leq \sigma_0(\partial D) \qquad \text{for all } n.$$

Hence,

$$|u(z)| \leq \|M_u\| \quad \text{a.e. } [\sigma_0] \quad \text{on } \partial D,$$

which proves the claim. By Theorem 7.3 we conclude

$$|u| \leq \|M_u\| \quad \text{a.e.} \quad [\sigma].$$

Now we claim:

$$v = A^{-1}(1) = \frac{1}{A(1)} = \frac{1}{u} \text{ a.e. } \quad [\sigma + \tau].$$

Let $\{p_n\} \subset \mathcal{P}$ such that p_n converges to $A^{-1}(1)$ in $P^2(\sigma + \tau)$. By passing to a subsequence if necessary, we see that $p_n \to A^{-1}(1)$ a.e. $[\sigma + \tau]$. Now the continuity of A implies that

$$up_n \to 1 \text{ in } P^2(\sigma).$$

So there exists a subsequence $\{p_{ni}\}$ such that up_{ni} converges to 1 a.e. $[\sigma]$. Since

$$D \supset abpe P^2(\sigma + \tau) \supset abpe P^2(\sigma) = D,$$

it follows that p_{ni} converges to $\hat{v}$, the analytic extension of v on D, uniformly on compact subsets of D. But $v = \hat{v}$ a.e. on D, so $uv = 1$ a.e. $[\sigma + \tau]$. Thus $v = \frac{1}{u}$

a.e. $[\sigma+\tau]$. The claim is proved. Now for $p \in \mathcal{P}$, we have

$$\begin{aligned} \|p\|_\tau &\leq \|u(vp)\|_{\sigma+\tau} \\ &\leq \|M_u\|\|vp\|_{\sigma+\tau} \\ &= \|M_u\|\|A^{-1}(p)\|_{\sigma+\tau} \\ &\leq \|M_u\|\|A^{-1}\|\|p\|_\sigma \end{aligned}$$

It follows that τ is a *Carleson* measure on D.

Conversely, assume that τ is a *Carleson* measure on D. There exists a constant $c > 0$ such that

$$\|p\|_\tau \leq c\ \|p\|_\sigma \text{ for each } p \in \mathcal{P}. \tag{7.4}$$

Define an operator $A\colon P^2(\sigma) \to P^2(\sigma+\tau)$ via $A(p) = p$ for each $p \in \mathcal{P}$. Then (7.4) implies that A is bounded. A is one-to-one and onto, so it follows by Open Mapping Theorem that A is invertible. Clearly, $A\,S_\sigma = S_{\sigma+\tau}A$. So S_σ and $S_{\sigma+\tau}$ are similar.

Chapter 8

On Arclength And Harmonic Measure

Let G be a Dirichlet domain with a countable number boundary components. Let ω be harmonic measure of G. We show in this chapter that if J is a rectifiable curve and $E \subset \partial G \cap J$ is a subset with $\omega(E) > 0$, then E has positive length. C. Bishop and P. Jones first prove this result for simply connected domains. We use their result to obtain ours.

If G is a Jordon domain with rectifiable boundary, F. and M. Riesz showed in 1916 (see Sec. D, Chapter II in [34]) that harmonic measure and arc length measure are equivalent (i.e., mutually absolutely continuous). This is not true in general for domains with nonrectifiable boundary. There is a Jordon domain for which neither harmonic measure is absolutely continuous with respect to arclength nor arclength is absolutely continuous with respect to harmonic measure; see [35, 36, 39].

C. Bishop and P. Jones in [8] proved the following notable result. If we intersect a rectifiable curve with the boundary of a simply connected domain, then any subset of this intersection having positive harmonic measure must also have positive arc length measure.

Prior to the work of Bishop and Jones, Øksendal [48] established the last result for the case where the rectifiable curve is a straight line and Kaufman and Wu [33] extended the result to chord-arc curves.

Bishop and Jones remark in the beginning of [8] that their result is also valid for finitely connected domains and to some infinitely connected ones they dubbed fully accessible. We began by carrying out their homework assignment. In so doing, we have extended their theorem to all domains that have a countable number of boundary components.

Throughout this chapter we use ω to represent a harmonic measure of G without specifying its evaluating point and we refer it as the harmonic measure of G.

Let G be a Dirichlet domain with harmonic measure ω. An observation is that if $\Delta \in \partial G$ is ω-measurable, then the function, $\hat{\chi}_\Delta(z) = \int_\Delta \omega_z$, defined for each $z \in G$, is a harmonic function in G and

$$\limsup_{z \to w} \hat{\chi}_\Delta(z) = \chi_\Delta \quad \text{a.e} \quad [\omega],$$

where z are taken from the inside of G; moreover,

$$\omega_z(\Delta) = \hat{\chi}_\Delta(z) \qquad \text{for each} \quad z \in G.$$

Lemma 8.1 *If G is a simply connected domain whose boundary contains more than one point in the closed Riemann sphere $\mathbf{C}_\infty = \mathbf{C} \cup \{\infty\}$ and if h is a conformal map of the unit disc D onto G, then h has nontangential limits almost everywhere with respect to Lebesgue measure on the unit circle.*

Proof. If G is a bounded domain, the result is well-known from classical Hardy space theory. Suppose G is not bounded. We first consider the case that ∞ is not in ∂G; hence, $\infty \in G$. Let a be the pre-image of ∞ under h and D_a be a domain obtained by deleting a small closed disc centered at a from D. Note D_a is a circular domain, therefore $h|D_a$ has nontangential limits everywhere with respect to its harmonic measure. But harmonic measure for D_a is equivalent to arclength on each component of the boundary of D_a. Hence we conclude that h has nontangential limits almost everywhere.

Now we cover the case that G is unbounded and $\infty \in \partial G$. Pick $a \in \mathbf{C} \setminus \partial G$; the function $1/(z-a)$ maps G conformally onto a simply connected domain with bounded boundary. The preceding case shows that $1/(h-a)$ has nontangential limits almost everywhere on ∂D. The result is now transparent.

A point $a \in \partial G$ is said to have a barrier if there is a positive harmonic function α defined on G which satisfies the following three conditions:

1) α can be continuously extended to V, a neighborhood (in $\overline{G}$), of a.

2) $\alpha(a) = 0$ and $\alpha(z) > 0$ if $z \neq a$ for $z \in V$.

3) There is a positive constant M such that

$$\limsup_{z \to b} \alpha(z) \geq M$$

for all $b \in \partial G \setminus V$.

If f is a bounded function on ∂G, let $\overline{f}(\lambda) = \limsup_{z \to \lambda} f(z)$ and $\underline{f}(\lambda) = \liminf_{z \to \lambda} f(z)$.

The prove of the following lemma is almost identical to the proof of Lemma 2 in [5, p.250].

Lemma 8.2 *Suppose $a \in \partial G$ has a barrier. If f is a bounded function on ∂G, the* *corresponding harmonic function u determined by Perron's method satisfies*

$$\underline{f}(a) \leq \liminf_{z \to a} u(z) \leq \limsup_{z \to a} u(z) \leq \overline{f}(a).$$

Corollary 8.1 *If a connected domain has a barrier at each boundary point, then it is a Dirichlet domain.*

Lemma 8.3 *Let G be a Dirichlet domain contained in a simply connected domain W. Suppose that g is a conformal map of W onto the unit disc D, then the image of G under g in D is also a Dirichlet region.*

Proof. It is sufficient to show that $g(G)$ has a barrier at each point on its boundary. Fix $a \in \partial g(G)$. If a is on the unit circle, we can draw a line segment so that one of its end points is a but all other points in the segment are exterior to $g(G)$. It is well-known that a has a barrier in this case.

Now assume that $a \in D$. For each $z \in \partial G$, let $\alpha(z) = dist(z, g^{-1}(a))$. Clearly α is continuous on ∂G. Since G is a Dirichlet region, α has a harmonic extension to G.

It is easy to verify that the harmonic extension of α is clearly a barrier at $g^{-1}(a)$. It follows immediately from Lemma 8.1 and the definition of barrier that $h \circ g^{-1}$ is a barrier for a. Hence $g(G)$ is a Dirichlet region.

We need one more well-known classical result. For the sake of completeness, we also give its proof.

Lemma 8.4 *Let G be a Dirichlet domain. The harmonic measure of G, ω, has no atoms.*

Proof. Let $a \in \partial G$ and $h(z) = \frac{1}{dist(z,a)+1}$ for each $z \in \partial G$. Obviously h is a continuous function on ∂G. Recall our notation; if $t \in C(\partial G)$, then $\hat{t}$ denotes the harmonic extension of t to G that is continuous on $\overline{G}$. Usage of the maximum principle for harmonic functions and Harnack's theorem yield the following:

i) $0 \leq \widehat{h^n}(z) < 1$ if $z \neq a$.

ii) $\{\widehat{h^n}\}$ is a monotone decreasing sequence of harmonic functions on $\overline{G}$.

iii) $\widehat{h^n}(z) \to \chi_{\{a\}}(z)$ for each $z \in \partial G$.

iv) $\{\widehat{h^n}\}$ converges uniformly on compact subset of G.

Let $u(z) = \lim \widehat{h^n}(z)$ for each $z \in G$. Clearly $u(z) \leq \widehat{h^n}(z)$ on G for each n. Thus, for $\lambda \in \partial G$ and $\lambda \neq a$, we have $\overline{\lim}_{z\to\lambda} u(z) \leq \overline{\lim}_{z\to\lambda} \widehat{h^n}(z) = h^n(\lambda)$ for all n. Letting $n \to \infty$, we infer from Lemma 7.2 that u is identically equal zero. Now, using the Lebesgue dominated convergence theorem we conclude

$$\begin{aligned} \omega(\{a\}) &= \int h^n d\omega \\ &= \hat{h}^n(x) \to 0 \end{aligned}$$

where x is the point at which ω is evaluated. Since a is arbitrary, the lemma is proved.

If E is a subset of a rectifiable curve, we use $|E|$ to denote the length of E. Now we are ready to state our main theorem.

Theorem 8.1 *Let G be a Dirichlet domain with a countable number of boundary components. Let ω be harmonic measure of G. If J is a rectifiable curve and $E \subset \partial G \cap J$ is a subset with positive harmonic measure in G, then $|E| > 0$.*

Proof. Since ω is a probability measure, it follows that ∂G has at most a countable number of components that have positive measure. This together with the hypothesis implies that there exists a component F such that

$$\omega(F \cap E) > 0.$$

Let $\mathbf{C}_\infty = \mathbf{C} \cup \{\infty\}$ denote the closed Riemann sphere. The component, W, of $\mathbf{C}_\infty \setminus F$ that contains G is simply connected since F is a component of ∂G; moreover, $\partial W = F$. If W is contained in the plane C and ω_W is the harmonic measure for W, we can easily show that $\omega(F \cap E) > 0$.. Using Bishop and Jone's theorem, we are done. Thus, we assume that W is not a planer simply connected domain in the rest of the proof. We are going to find a planar simply connected domain such that the set $E \cap F$ still has positive harmonic measure with respect to this new domain, then apply Bishop and Jones's theorem to finish the proof.

Lemma 7.4 implies that W has more than one boundary point. Appealing to the Riemann mapping theorem. there exists a conformal function g that maps W onto the unit open disc D. Let h denote the inverse function of g on D. By Lemma 7.1, h has nontangential limits almost everywhere on the unit circle with respect to the Lebesgue measure m. Let $\tilde{h}$ be the boundary value function of h on ∂D. The function $\tilde{h}$ is well-defined except on a subset of zero Lebesgue measure.

The domain $g(G)$ is a Dirichlet domain, a fact that follows from Lemma 7.3. Without loss of generality, we may assume that $\omega = \omega_b$, the harmonic measure of G evaluated at b, for some $b \in G$. Let σ be harmonic of $g(G)$ evaluated at $g(b)$ and let $B = \tilde{h}^{-1}(F \cap E)$. It is well-known, σ is absolutely continuous with respect to Lebesgue measure on $\partial D \cap \partial g(G)$. (This can easily be proved by using the definition of harmonic measure and the maximum principle for harmonic function.) Hence $\tilde{h}$

is well-defined on $\partial D \cap \partial g(G)$, except perhaps on a set with σ-measure 0. We claim that

$$\omega(\tilde{h}(L)) = \sigma(L) \qquad \text{for all measurable subsets} \quad L \subset \partial g(G) \cap \partial D.$$

In fact, using our observation and the associated notations in the beginning, we have $\hat{\chi}_{\tilde{L}}(b) = \omega(\tilde{h}(L))$. Now the function $\hat{\chi}_{\tilde{L}} \circ \tilde{h}$ is harmonic in $g(G)$; furthermore, we clearly have

$$\limsup_{z \to \lambda} \hat{\chi}_{\tilde{L}} \circ \tilde{h}(z) = \chi_L \quad \text{a.e. } [\sigma]$$

Applying the maximum principle for harmonic function, we conclude that $\hat{\chi}_{\tilde{L}} \circ \tilde{h} = \hat{\chi}_L$ in $g(G)$. Hence,

$$\begin{aligned} \omega(\tilde{h}(L)) &= \hat{\chi}_{\tilde{L}}(b) \\ &= \hat{\chi}_L(\tilde{h}(b)) \\ &= \sigma(L). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \sigma(B) &= \omega(\tilde{h}(B)) \\ &= \omega(F \cap E) \\ &> 0. \end{aligned}$$

Consequently, we conclude that $m(B) > 0$.

It is obvious that there is a subarc of the unit circle so that its intersection with B has positive Lebesgue. Therefore, we may assume that B is contained in an subarc I of the unit circle. Clearly we can construct a smooth Jordon arc α which lies in D but its end points coincide with those of I. Now the union of α and the arc I form a smooth Jordon curve.

Let $\mathcal{O}$ be the domain enclosed by that Jordon curve. Obviously $\mathcal{O}$ is contained in D. Now the domain $\mathcal{O}$ is a simply connected domain with smooth Jordon boundary, so its harmonic measures and arclength measure are equivalent on its boundary.

Note that the domain W may contains ∞. By our construction of $\mathcal{O}$, we can clearly require, in addition, that $g(\infty) \overline{\in} \mathcal{O}$ and $g(b) \in \mathcal{O}$. Thus, if we denote λ by the harmonic measure of $\mathcal{O}$ evaluated at a and τ by the harmonic measure of $h(\mathcal{O})$ evaluated at $h(a)$, we have

$$0 < \int_B d\lambda = \int_{\tilde{f}(B)} d\tau,$$

where $\tilde{f}$ is the boundary value of the function $f = h|\mathcal{O}$.

Finally, since $h(\mathcal{O})$ is a simply connected domain, it follows form Bishop and Jones's theorem that $|\tilde{f}(B)| > 0$. Now the set $f|\mathcal{O}$ is contained in E, hence $|E| > 0$. The proof is complete.

Bibliography

[1] P. Ahern, D. Sarason, *The H^p Spaces of a Class of Function Algebras*, Acta Math., 117(1967), 123-163

[2] P. Ahern, D. Sarason. *On Some Hypo-Dirichlet Algebras of Analytic Functions,* Amer. J. Math., 89(1967), 932-941.

[3] J. Akeroyd, *Polynomial Approximation in the Mean With Respect to Harmonic Measure on Crescents II*, Trans. Amer. Math. Soc. 303(1987), 193-199.

[4] J. Akeroyd, D. Khavinson, and H. S. Shapiro, *Remarks Concerning Cyclic Vectors in Hardy Spaces and Bergman Spaces*, Michigan Math. J. 38(1991), No. 2, 191-205.

[5] L. V. Ahlfors, *Complex Analysis*, Third Ed., McGraw-Hill, New York, 1979.

[6] T. Ando, *On the Pre-dual of H^∞*, Commen. Math. Spec. Issue 1(1978), 33-40.

[7] S. Axler and P. Bourdon, *Finite Codimentional Invariant Subspaces of Bergman Spaces*, Trans. Amer. Math. Soc., 306(1988), 805-817.

[8] C. Bishop, P. Jones, *Harmonic Measure and Arclength*, Annals of Math, 132(1990) 511-541.

[9] P. S. Bourdon, *Density of the Polynomials in* Bergman *Spaces*, Pacific J. Math., 130(1987), 215-221.

[10] J. E. Brennan, *Approximation in the Mean by Polynomials on Non-*Carathéodory *Domains* Ark. Mat., 15(1977), 117-168.

[11] S. W. Brown, *Some Invariant Subspaces for Subnormal Operators*, Integral Eqs. Op. Theory 1 (1978), 310-333.

[12] C. Carathéodory, *Conformal Representation.* Combridge University Press, Combridge, 1952.

[13] L. Carleson, *Interpolations by Bounded Analytic Functions and the Corona problem*, Ann. of Math. 76(1962), 547-559.

[14] W. S. Clary, *Quasi-Similarity and Subnormal Operator*, Ph.D. thesis, University of Michigan, 1973.

[15] W. S. Clary, *Equality of Spectra of Quasi-Similar Hyponormal Operator*, Proc. Amer. Math. Soc. 53(1975), 88-90.

[16] J. B. Conway, *Functions of One Complex Variable*. Second Ed., Springer-Verlag, New York, 1978.

[17] J. B. Conway, *Subnormal Operators*, Pitman Publishing Inc. , Research Notes in Math. 51, Marshfield, Mass., 1982.

[18] J. B. Conway, *The Theory of Subnormal Operators*. Math. Surveys and Monographs Vol. 36, Amer. Math. Soc. 1991.

[19] J. B. Conway, N. Elias, *Analytic Bounded Point Evaluations for Spaces of Rational Functions*, Preprint, to appear.

[20] A. M. Davie, *Dirichlet Algebras of Analytic Functions*, J. Funct. Analysis 6(1970), 348-356.

[21] R. G. Douglas, *Banach Algebra Techniques in Operator Theory*, Academic Press, New York, 1972.

[22] P. L. Duren, *Theory of H^p-spaces*, Academic Press, New York, 1970.

[23] O. J. Farrell, *On Approximation to an Analytic Function by Polynomials*, Bull. Amer. Math. Soc., 40(1934), 908-914.

[24] W. H. J. Fuchs, *Topic in the Theory of One Complex Variable*, D. Von Nostrand Co., Princeton, N. J., 1967.

[25] T. W. Gamelin, *Uniform Algebras*, Prentice Hall, Englewood Cliffs, N. J., 1969.

[26] T. W. Gamelin, *Rational Approximation Theory*, unpublished UCLA lecture notes, 1975.

[27] T. W. Gamelin, J. Garnett, *Constructive techniques in Rational Approximation,* Trans. A.M.S. 143(1969), 187-200.

[28] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, 1981.

[29] I. Glicksburg, *Dominant Representing Measure and Rational Approximation*, Trans. Amer. Math. Soc. 130(1968), 425-462.

[30] G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, Translations of Mathematical Monographs. Vol. 26. Amer. Math. Soc., Providence, 1969.

[31] W. W. Hastings, *Subnormal Operators Quasisimilar to an Isometry*, Trans. Amer. Math. Soc., 256(1979), 145-161.

[32] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, NJ., 1962.

[33] R. Kaufman, J.-M. Wu, *Distortion of the Boundary under Conformal Mapping*, Mich. Math. J. 29(1982), 267-286.

[34] P. Koosis, *Introduction to H_p Spaces*, London Math. Soc. Lecture Note Ser., Vol. 40, Cambridge Univ. Press, Cambridge, 1980.

[35] M.A. Lavrentiev, *Boundary Problems in the Theory of Univalent Functions*, Mat. Sb. 43(1936), 815-846(Russian), Amer. Math. Soc Transl. 32(1963), 1-15.

[36] A. J. Lohwater, W. Seidel, *An Example in Conformal Mapping*, Duke Math. J. 15(1948), 137-143.

[37] J. E. McCarthy, *Analytic Structures for Subnormal Operator*, Int. Eq. and Op. Theory Vol. 13 (1990), 251-270.

[38] J. E. McCarthy, *Quasisimilarity of Rationally Cyclic Subnormal Operators*, J. Operator Theory 24(1990), 105-116.

[39] J. Mcmillan, G. Piranian, *Compression and Expansion of Boundary Sets*, Duke Math. J. 40(1973), 599-605.

[40] A. I. Markusevic, *Conformal Mapping of Regions with Variable Boundary and Application to the Approximation of Analytic Functions by Polynomials*, Dissertation, Moscow, 1934.

[41] S. N. Mergeljan, *On the Completeness of Systems of Analytic Functions*, Amer. Math. Soc. Translations 19(1962), 109-166, Uspehi Mat. Nauk 8(1953), 3-63.

[42] S. N. Mergeljan, A. P. Talmadjan, *On the Completeness in a Class of Non-Jordon Regions*, Amer. Math. Soc. Translation, 35(1964), 79-94; Izv. Akad. Nauk Armjan. SSR, 7(1954), 1-17.

[43] T. L. Miller, R. C. Smith, *Nontangential Limits of Functions in Some $P^2(\mu)$ Spaces*, Ind. Univ. Math. J., Vol. 39, No. 1, (1990), 19-26.

[44] T. Miller, R. Olin, J. Thomson, *Subnormal Operators and Representations of Algebra of Bounded Analytic Functions and Other Uniform Algebras*, Memo. of AMS, No. 354, Sept 1986.

[45] R. F. Olin, J. Z. Qiu, *On Arclength and Harmonic Measure*, to appear.

[46] R. F. Olin, J. Z. Qiu, J. E. Thomson, *Carleson Measures on a Multiply Connected Domain and Rationally Cyclic Subnormal operators*, to appear.

[47] R. Olin, L. Yang, *The Commutant of Multiplication by* z *on the Closure of Polynomials in* $L^t(\mu)$, to appear.

[48] B. Øksendal, *Brownian Motion and Sets of Harmonic Zero*, Pacific J. Math. 95(1981), 179-172.

[49] G. Pommerenke, *On the Green's Function of Fuchsian Groups*, Ann. Acad. Sci. Fenn., Series A.I. Math. 2(1976), 409-427.

[50] G. Pommerenke, *On Fuchsian Groups of Accessible Type*, Ann. Acad. Sci. Fenn., Series A.I. Math. 7(1982), 249-258.

[51] J. Z. Qiu, *Boundary Values of Analytic Functions in the Banach Spaces $P^t(\sigma)$ on Crescents*, to appear in Illinois Math. Journal.

[52] J. Z. Qiu, *Density of polynomials*, to appear.

[53] J. Z. Qiu, *Carleson Measure and Polynomial Approximation*, 1993. to appear.

[54] J. Z. Qiu, *The Structure of $R^t(K, \tau) \cap L^\infty(\tau)$ and Rational Approximation*, to appear.

[55] J. Z. Qiu. *Equivalence Classes of Subnormal Operators*, to appear in J. Op. Theory.

[56] L. Rubel, A. L. Shields, *Bounded Approximation by Polynomials*, Acta Math., 112 (1964), 145-162.

[57] D. Sarason, Invariant Subspaces and Unstarred Operator Algebras, Pacific J. of Math. Vol. 17, No.3, 1966.

[58] D. Sarason, Weak-Star Generators of H^∞, pacific J. Math., 17(1966), 519-528.

[59] D. Sarason, *On the Order of a Simply Connected Domain*, Michigan Math. J.. 15(1968), 129 - 133.

[60] D. Sarason, *Weak-star Density of Polynomials*. J. Reine Angew. Math., 252(1972), 1 - 15.

[61] A. L. Shields, *Weighted Shift Operator and Analytic Function Theory*, Math. Surveys, 13: Topics in Operator Theory, ed. C.Pearcy, Amer. Math. Soc., Providence, R. I., 1974, 49-128.

[62] S. O. Sinanjan, *Approximation by Polynomials and Analytic Functions in the areal Mean*, Amer. Math. Soc. Translation, 74(1968), 91-124; Mat. Sb., 69(1966), 546-578.

[63] S. O. Sinanjan, *Approximation by Polynomials in the Mean with Respect to Area*, Math. USSR Sbornik 11(1970), 411-421, Mat. Sb. 82(1970), 444-455.

[64] J. E. Thomson, *Approximation in the Mean by Polynomials*, Ann. Math. 133 (1991), 477-507.

[65] J. E. Thomson, *Bounded Point Evaluations and Polynomial Approximation*, Preprint, to appear.

[66] T. T. Trent, *$H^2(\mu)$ Spaces and Bounded Point Evaluations*, Pacific J. Math. 80, 279-292.

[67] M. Tsuji, *Potential Theory in Modern Function Theory*, Chelsa, New York, 1975.

[68] L. Yang, *Equality of essentil spectra of quasisimilar subnormal operators*, Integral Equations and Operator Theory, 13 (1990), 433-441.

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