TWO PROBLEMS IN FUNCTION THEORY OF ONE COMPLEX VARIABLE: LOCAL PROPERTIES OF SOLUTIONS OF SECOND-ORDER DIFFERENTIAL EQUATIONS AND NUMBER OF DEFICIENT FUNCTIONS OF SOME ENTIRE FUNCTIONS

by

JIUYI CHENG

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APPROVED:

Dr. J. Rossi, Chairman

Dr. M. Murray

Dr. R. Olin

Dr. R. Snider

Dr. R. Wheeler

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JIUYI CHENG

Committee Chairman: John Rossi
Department of Mathematics

(ABSTRACT)

This dissertation investigates two problems in the function theory of one complex variable. In Chapter 1, we study the asymptotics and zero distribution of solutions of the differential equation

$$w'' + A(z)w = 0,$$

where $A(z)$ is a transcendental entire function of very slow growth. The result parallels the classical case when $A(z)$ is assumed to be a polynomial. An analogue concerning the case when $A(z)$ is a transcendental entire function whose series expansion satisfies the Hadamard gap condition is given.

In Chapter 2, we give upper bounds for the number of deficient functions of entire functions of completely regular growth and entire functions whose zeros have angular densities. In particular, the bound is $2\lambda + 1$ if the entire function is of completely regular growth with order λ, $0 < \lambda < \infty$.

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To

HUANAN WANG And SURONG ZHANG

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CONTENTS

CHAPTER 1 INTRODUCTION

§1.1 Basic Ideas and Notation of the Nevanlinna Theory

The value distribution theory of meromorphic functions studies the properties of meromorphic functions assuming values in the extended complex plane $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. The first major result in the theory was proved by Picard [25] in 1880. He showed that any transcendental meromorphic function assumes every value in $\overline{\mathbf{C}}$ infinitely often, with at most two exceptions. In 1897, Borel [4] proved another major result in the theory, which says that any meromorphic function $f(z)$ with finite order λ (see (1.4) and (1.5)) has at most two values $a \in \overline{\mathbf{C}}$ such that

$$\overline{\lim_{r\to\infty}} \frac{\log n(r, f = a)}{\log r} < \lambda, \tag{1.1}$$

where $n(r, f = a)$ denotes the number of zeros of $f(z) - a$ in $|z| \le r$ if $a \in \mathbf{C}$ and the number of poles of $f(z)$ in $|z| \le r$ if $a = \infty$. Because a exceptional value in Picard's result has to satisfy (1.1), Borel's result is an extention of Picard's.

In 1925, Nevanlinna [20] extended both results above and unified the theory. His theory contains two important theorems which are known as Nevanlinna's first and second fundamental theorems. Since we use much of the notation and techniques of the Nevanlinna theory in this dissertation, we will summarize the theory in this section. A more complete discussion may be found in [21] and [9].

The following formula plays an important role in the theory.

Poisson-Jensen Formula [21]. *Let $f(z)$ be a meromorphic function on $|z| \leq R$ $(0 < R < \infty)$, and a_j $(j = 1, 2, \cdots, m)$, b_k $(k = 1, 2, \cdots, n)$ are zeros and poles of $f(z)$ in $|z| < R$ respectively. If $z_0 = re^{i\theta}$ satisfies $r < R$, $z_0 \neq a_j$, $j = 1, 2, \cdots, m$ and $z_0 \neq b_k$, $k = 1, 2, \cdots, n$, then*

$$\log|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\varphi})| \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \varphi) + r^2}\, d\varphi + \sum_{j=1}^{m} \log\left|\frac{R(z_0 - a_j)}{R^2 - \bar{a}_j z_0}\right| - \sum_{k=1}^{n} \log\left|\frac{R(z_0 - b_k)}{R^2 - \bar{b}_k z_0}\right|. \tag{1.2}$$

Now let $f(z)$ be a meromorphic function in $\mathbf{C}$. We denote by $n(r, f)$ the number of poles of $f(z)$ in $|z| \leq r$ and write

$$N\left(r, \frac{1}{f-a}\right) = \int_0^r \frac{n\left(t, \frac{1}{f-a}\right) - n\left(0, \frac{1}{f-a}\right)}{t}\, dt + n\left(0, \frac{1}{f-a}\right)\log r$$

for any $a \in \mathbf{C}$ and

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t}\, dt + n(0, f)\log r.$$

Set $\log^+ x = \max\{0, \log x\}$ for $x \geq 0$,

$$m\left(r, \frac{1}{f-a}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|f(re^{i\theta}) - a\right|^{-1} d\theta$$

for $a \in \mathbf{C}$ and

$$m(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|f(re^{i\theta})\right| d\theta.$$

Then the characteristic function T of $f(z)$ is defined by

$$T(r,f) = m(r,f) + N(r,f).$$

Now we may state

First Fundamental Theorem. *Let $f(z)$ be a nonconstant meromorphic function in $\mathbf{C}$. Then for any $a \in \mathbf{C}$*

$$m\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-a}\right) = T(r,f) + O(1) \tag{1.3}$$

as $r \longrightarrow \infty$.

The theorem can be deduced from the Poisson-Jensen formula. From the definition, we notice that the functional $m\left(r, \frac{1}{f-a}\right)$ measures the average "closeness" of $f(z)$ to the value a on the circle $|z| = r$, and $N\left(r, \frac{1}{f-a}\right)$ is a weighted average of the number of times that $f(z)$ takes on the value a on the disk $|z| \leq r$. The functionals $m(r,f)$ and $N(r,f)$ measure the same averages with a replaced by ∞. Thus, interpreted in this way, Nevanlinna's first fundamental theorem says that the total affinity of the function $f(z)$ is essentially the same for all values in $\overline{\mathbf{C}}$.

If $f(z)$ is entire, $\log M(r,f)$, where $M(r,f) = \max_{|z|=r} |f(z)|$, is used to measure the growth of $f(z)$. If $f(z)$ is meromorphic, $M(r,f)$ becomes infinite on circles containing poles of $f(z)$ and so $\log M(r,f)$ is unsuitable for measuring the growth of $f(z)$. We use the functional $T(r,f)$ to measure the growth of meromorphic functions and remark that $T(r,f)$ and $\log M(r,f)$ share many of the same important properties. For instance, both are continuous, increasing, convex functions of $\log r$.

We may define the order $\lambda(f) = \lambda$ of a meromorphic function $f(z)$ by

$$\lambda = \overline{\lim_{r\to\infty}} \frac{\log T(r,f)}{\log r}. \tag{1.4}$$

If $f(z)$ is entire, λ is also defined by

$$\lambda = \overline{\lim_{r\to\infty}} \frac{\log\log M(r,f)}{\log r}. \tag{1.5}$$

An important relation between $T(r,f)$ and $\log M(r,f)$ when $f(z)$ is entire is that

$$T(r,f) \le \log^{+} M(r,f) \le \frac{R+r}{R-r} T(R,f)$$

for $0 < r < R$. So we easily see that definitions (1.4) and (1.5) are equivalent for entire functions.

To state Nevanlinna's second fundamental theorem, we denote by $S(r,f)$ any function which satisfies

$$S(r,f) = O\big(\log\big(rT(r,f)\big)\big) \tag{1.6}$$

as $r \to \infty$ without restriction if $\lambda(f) < \infty$, and as $r \to \infty$ outside an exceptional set of finite linear measure if $\lambda(f) = \infty$, and write

$$N_1(r,f) = 2N(r,f) - N(r,f') + N\left(r, \frac{1}{f'}\right).$$

It is not hard to see that $N_1(r,f) \ge 0$.

Then we state

Second Fundamental Theorem. *Let $f(z)$ be a nonconstant meromorphic function in* $\mathbf{C}$ *and let a_k, $k = 1, 2, \cdots, q$, be q distinct values in* $\mathbf{C}$. *Then*

$$m(r,f) + \sum_{k=1}^{q} m\left(r, \frac{1}{f - a_k}\right) \le 2T(r,f) - N_1(r,f) + S(r,f). \tag{1.7}$$

The proof of the second fundamental theorem depends essentially on the following estimate of the functional $m(r, \cdot)$ applied to the logarithmic derivative:

$$m\left(r, \frac{f'}{f}\right) = S(r,f)$$

or more generally

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r,f) \tag{1.8}$$

for all transcendental meromorphic functions $f(z)$ and $k \in \mathbf{N}$.

For any $a \in \mathbf{C}$, the Nevanlinna deficiency is defined by

$$\delta(a,f) = \varliminf_{r\to\infty} \frac{m\left(r, \dfrac{1}{f-a}\right)}{T(r,f)} = 1 - \varlimsup_{r\to\infty} \frac{N\left(r, \dfrac{1}{f-a}\right)}{T(r,f)},$$

and

$$\delta(\infty,f) = \varliminf_{r\to\infty} \frac{m(r,f)}{T(r,f)} = 1 - \varlimsup_{r\to\infty} \frac{N(r,f)}{T(r,f)}.$$

The value $a \in \overline{\mathbf{C}}$ is called a deficient value of $f(z)$ if $\delta(a,f) > 0$.

Since $N_1(r,f) > 0$, for transcendental $f(z)$ it follows from (1.7) that

$$\begin{aligned}\delta(\infty,f) + \sum_{k=1}^{q} \delta(a_k,f) &= \varliminf_{r\to\infty} \frac{m(r,f)}{T(r,f)} + \sum_{k=1}^{q} \varliminf_{r\to\infty} \frac{m\left(r, \dfrac{1}{f-a_k}\right)}{T(r,f)} \\ &\le 2 + \varliminf_{r\to\infty} \frac{S(r,f)}{T(r,f)} + o(1)\end{aligned}$$

as $r \to \infty$. So by (1.6), noticing that the transcendence of $f(z)$ implies $\log r = o\big(T(r,f)\big)$ as $r \to \infty$, we deduce that

$$\delta(\infty,f) + \sum_{k=1}^{q} \delta(a_k,f) \le 2.$$

This holds for any choice of q distinct values $a_1, a_2, \cdots, a_q$, so we deduce that the set of deficient values is a countable set and that

$$\sum_{a \in \overline{\mathbf{C}}} \delta(a, f) \leq 2. \tag{1.9}$$

Since

$$\delta(0, e^z) = \delta(\infty, e^z) = 1,$$

the function $f(z) = e^z$ shows that equality may hold in (1.9). In general, if $f(z)$ is a transcendental entire function, it is obvious that $\delta(\infty, f) = 1$. So we have

$$\sum_{a \neq \infty} \delta(a, f) \leq 1.$$

If $a \in \overline{\mathbf{C}}$ satisfies (1.1) for a function $f(z)$ with $\lambda(f) = \lambda < \infty$, it is not hard to see that

$$\overline{\lim_{r \to \infty}} \frac{\log N(r, f = a)}{\log r} < \lambda,$$

where $N(r, f = a)$ is equal to $N\left(r, \dfrac{1}{f - a}\right)$ if $a \in \mathbf{C}$ and $N(r, f)$ if $a = \infty$. From this, (1.7), (1.6) and Nevanlinna's first fundamental theorem, we can easily obtain Borel's result.

To obtain Picard's result, we just need to notice that if $f(z)$ assumes $a \in \overline{\mathbf{C}}$ finitely many times then $\delta(a, f) = 1$.

We can also define a deficient function. Let $f(z)$ be meromorphic. A meromorphic function $a(z)$ is said to be a deficient function for $f(z)$ if

$$T(r,a) = o\big(T(r,f)\big) \tag{1.10}$$

as $r \to \infty$ and

$$\delta\big(a(z),f\big) = \varliminf_{r\to\infty} \frac{m\left(r,\dfrac{1}{f-a}\right)}{T(r,f)} = 1 - \varlimsup_{r\to\infty} \frac{N\left(r,\dfrac{1}{f-a}\right)}{T(r,f)} > 0. \tag{1.11}$$

The conjecture that Nevanlinna's second fundamental theorem holds for deficient functions, was asked by Nevanlinna himself and had been open for a long time. It was independently proved by Osgood [22] in 1985 and Steinmetz [29] in 1986.

For later reference, we state the following theorem, which is also important in the value distribution theory of meromorphic functions.

Boutroux-Cartan Theorem [5]. *Let $P(z) = \prod_{k=1}^{n}(z-c_k)$, $c_k \in \mathbf{C}$, $k = 1,2,\cdots,n$, and $H>0$. Then there exist m open disks D_l with radii r_l, $l=1,2,\cdots,m$, such that*

(1) $m \le n$;

(2) $\sum_{l=1}^{m} r_l \le 2H$;

(3) *for any* $z \notin \bigcup_{l=1}^{m} D_l$, $|P(z)| > \left(\dfrac{H}{e}\right)^n$.

Finally, let z_1, z_2, $\cdots$ be the zeros of $f(z)$ other than $z=0$, and suppose that $|z_n| = r_n$ such that $0 < r_1 \le r_2 \le \cdots$. Then we define the exponent of convergence of the zeros of $f(z)$ to be the lower bound $\sigma(f)$ of numbers α such that $\sum_{n=1}^{\infty} r_n^{-\alpha}$ is convergent. If $f(z)$ has no zeros, then we define $\sigma(f)$ to be 0. Since the series $\sum_{n=1}^{\infty} r_n^{-\alpha}$

and the integral $\int\limits_0^\infty n\left(t, \frac{1}{f}\right) t^{-\alpha-1}\, dt$ converge simultaneously (see [9, Lemma 1.4]), we have

$$\sigma(f) = \inf\left\{\, \alpha > 0 : \int_0^\infty n\left(t, \frac{1}{f}\right) t^{-\alpha-1}\, dt < \infty \,\right\}.$$

Moreover (see [3, Theorem 2.5.8]),

$$\sigma(f) = \varlimsup_{r\to\infty} \frac{\log n\left(r, \frac{1}{f}\right)}{\log r}.$$

Finally it is not hard to see that $\sigma(f) \le \lambda(f)$ (see [6, Theorem 9]).

§1.2 Notation and Some Results in Wiman-Valiron Theory

In this section we give some notation and results in Wiman-Valiron theory which we will use in this dissertation. A complete survey may be found in [10].

Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1.12}$$

is a transcendental entire function. The Wiman-Valiron theory describes the local behaviour of $f(z)$, near a point where $|f(z)|$ is large, in terms of the power series of $f(z)$, and in particular gives some relations between $M(r, f)$ and the maximum term

$$\mu(r) = \sup_n |a_n| r^n = |a_N| r^N.$$

Here $N = N(r)$ is called the central index. If there are several maximum terms, we define $N(r)$ to be the largest of the corresponding indices. We easily see that $N(r)$ is an increasing function of r and $N(r) \to \infty$ with r. If write

$$N(r) = N \quad \text{for} \quad r_N \leq r < r_{N+1},$$

we see that $N(r)$ is continuous at the points r_N, and we can deduce that $\log \mu(r)$ is also a convex increasing function of $\log r$.

To state the results, we define the density and logarithmic density of a set $E \subset [0, \infty)$ respectively by

$$\operatorname{dens} E = \lim_{r \to \infty} \frac{\operatorname{meas}(E \cap [0, \infty))}{r} \tag{1.13}$$

and

$$\operatorname{logdens} E = \lim_{r \to \infty} \frac{\text{l. m.}(E \cap [1, \infty))}{\log r} \tag{1.14}$$

provided that the limits exist. Here $\mathrm{meas}(E) = \int\limits_{E\cap[0,\infty)} dr$ is the linear measure and $\mathrm{l.m.}(E) = \int\limits_{E\cap[1,\infty)} \frac{dr}{r}$ is the logarithmic measure of E.

The following theorem estimates the terms a_n in (1.12) in terms of the maximum term outside an exceptional set.

Theorem 1.A [10, Theorem 2]. *Suppose that $f(z)$ is an entire function of the form (1.12), $N(r)$ is the central index of $f(z)$ and K, $l > 0$, l an integer are given. Then denoting by $\mathcal{G}$ the range of $N(r)$ there exist sequences $\{r'_n\}_{n\in\mathcal{G}}$ and $\{s'_n\}_{n\in\mathcal{G}}$ satisfying $0 < r'_n < s'_n < r'_m < s'_m$ whenever $n < m$ and $n, m \in \mathcal{G}$ such that for $r = |z| \in \bigcup\limits_{n\in\mathcal{G}} (r'_n, s'_n)$ and $k \in \mathbf{N}$ we have*

$$|a_k|r^k \leq \exp\left\{-\frac{1}{2}b(|k-N|+N)(k-N)^2\right\} \cdot |a_N|r^N, \tag{1.15}$$

where $b(x)^{-1} = Kx\log x \cdots \log_l x\left(\log_{l+1} x\right)^{1+\delta}$, $\log_l x$ denotes the l times iterated logarithm, $\delta > 0$ and $N(r) = n$ whenever $r_n \in [r'_n, s'_n]$, $n \in \mathcal{G}$ and the logarithmic measure of $\bigcup\limits_{n\in\mathcal{G}} (s'_n, r'_{n+1})$ is finite.

The following theorem can be deduced from Theorem 1.A.

Theorem 1.B [10, Theorem 5]. *Suppose that $f(z)$ is an entire function of the form (1.12) and $N(r)$ is the central index of $f(z)$. Then there exists a set E of finite logarithmic measure such that for any $\epsilon > 0$, we have*

$$M(r,f) < (1+\epsilon)\mu(r)\left\{\frac{2\pi}{b(N)}\right\}^{\frac{1}{2}}$$

for all $r = |z| \notin E$ and $r \to \infty$, where $b(r)$ is defined exactly as in Theorem 1.A.

The following theorem estimates the maximum modulus of the derivatives of an entire function $f(z)$ in terms of the maximum modulus of $f(z)$ outside an exceptional set. It can be found in [10, Theorem 12].

Theorem 1.C. *Suppose that $f(z)$ is an entire function and $N(r)$ is the central index of $f(z)$. Let j be a fixed nonnegative integer. Then there exists a set E of finite logarithmic measure such that if $r = |z| \notin E$ and $r \to \infty$ we have*

$$M(r, f^{(j)}) = \big(1 + o(1)\big) \left(\frac{N(r)}{r}\right)^j M(r, f).$$

§1.3 Complex Differential Equations

In this section, we give some results concerning the properties, particularly the value distribution, of solutions of complex linear differential equations. A more complete discussion can be found in Laine's book [16].

Throughout the remainder of this section, $\lambda(f)$ always denotes the order of the function $f(z)$ (see (1.4) and (1.5)) and $\sigma(f)$ always denotes the exponent of convergence of the zeros of $f(z)$ (see the definition in section 1.1).

Consider the nth order homogeneous linear differential equation

$$f^{(n)} + a_{n-1}(z)f^{(n-1)} + \cdots + a_0(z)f = 0 \tag{1.16}$$

with entire coefficients $a_0(z)$, $a_1(z)$, $\cdots$, $a_{n-1}(z)$, where $a_0(z) \not\equiv 0$. As is well known, all solutions of (1.16) are entire functions. For the relation between the growth of the solutions and the coefficients, we have

Theorem 1.D (see [31, Satz 1] or [16, Theorem 4.1]). *The coefficients* $a_0(z)$, $a_1(z)$, $\cdots$, $a_{n-1}(z)$ *of* (1.16) *are polynomials if and only if all solutions of* (1.16) *are entire functions of finite order.*

Now let us consider the second order case, namely the equation

$$f'' + a_1(z)f' + a_0(z)f = 0, \tag{1.17}$$

where $a_0(z)$, $a_1(z)$ are entire. Let

$$g(z) = f(z)\exp\left(-\frac{1}{2}\int_0^z a_1(\zeta)\,d\zeta\right),$$

then by (1.17), we have

$$g'' + \left(a_0(z) - \frac{1}{4}a_1(z)^2 - \frac{1}{2}a_1'(z)\right) g = 0.$$

So we only need to consider the equation

$$f'' + A(z)f = 0, \tag{1.18}$$

where $A(z)$ is an entire function.

If $A(z)$ is a polynomial

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0,$$

where $a_n \neq 0$, many results concerning the zero distribution of the solutions have been proved. First we state

Theorem 1.E (see [16, prop.5.1]). *Any non-trivial solution $f(z)$ of*

$$f'' + P(z)f = 0 \tag{1.19}$$

has the order $\lambda(f) = \dfrac{n+2}{2}$.

Now let $f_1(z)$ and $f_2(z)$ be two linearly independent solutions of (1.19), and set $E(z) = f_1(z)f_2(z)$. Then noticing that the Wronskian determinant $W(f_1, f_2) = f_1(z)f_2'(z) - f_1'(z)f_2(z)$ is a constant C, we obtain an important equality

$$4A(z)E(z)^2 = E'(z)^2 - C^2 - 2E(z)E''(z). \tag{1.20}$$

Using (1.20), Theorem 1.E, (1.8) and some basic techniques in complex differential equations, we deduce

Theorem 1.F (see [16, Theorem 5.2]). *Let $f_1(z)$ and $f_2(z)$ be two linearly independent solutions of* (1.19). *Then*

$$\lambda(E) = \sigma(E) = \frac{n+2}{2}.$$

Moreover,

$$\max\big(\sigma(f_1),\, \sigma(f_2)\big) = \frac{n+2}{2}.$$

In 1984, Hellerstein, Shen and Williamson proved the following theorem, which inspired much work.

Theorem 1.G [12]. *Let $f_1(z)$, $f_2(z)$ and $E(z)$ be the same as those in Theorem* 1.F. *If $E(z)$ has only real zeros then $P(z)$ is constant.*

Gundersen extended this result by proving

Theorem 1.H [8]. *Let $f_1(z)$, $f_2(z)$ and $E(z)$ be the same as those in Theorem* 1.F. *If for $i = 1, 2$*

$$n_{\mathrm{NR}}(r, f_i) = o\big(r^{\frac{n+2}{2}}\big) \tag{1.21}$$

as $r \to \infty$, where $n_{\mathrm{NR}}(r, f)$ denotes the number of non-real zeros of $f(z)$ in $|z| \le r$, then $P(z)$ is constant.

In 1986, Hellerstein and Rossi considered the equation (1.18) in which the coefficient $A(z)$ is replaced by a rational function, and used Hille's method of asymptotic integration (see [14] or [15, Chap.7]) to prove

Theorem 1.I [11, Theorem 1]. *Let $f_1(z)$ and $f_2(z)$ be two linearly independent solutions of* (1.18) *in which*

$$A(z) = \frac{P(z)}{Q(z)} \tag{1.22}$$

where $P(z)$ and $Q(z)$ are polynomials. If (1.21) *holds and* $E(z) = f_1(z)f_2(z)$ *is transcendental then* $\text{degree}(P) - \text{degree}(Q) = 0$.

In that work, a result in the theory of asymptotic integration, which is related to our work in Chapter 2, played an important role. To state that result, let $A(z)$ be rational of the form (1.22) and let

$$n = \text{degree}(P) - \text{degree}(Q). \tag{1.23}$$

Then as $z \to \infty$,

$$A(z) = \big(1 + o(1)\big)a_n z^n; \quad a_n = |a_n|e^{i\alpha_n} \neq 0, \quad 0 \leq \alpha_n < 2\pi.$$

Define for $k = 0, 1, \cdots, n+1$

$$\theta_k = \frac{-\alpha_n + 2k\pi}{n+2},$$

$$V_k(\epsilon) = \{\, z : |\arg z - \theta_k| < \epsilon \,\},$$

$$S_k^+(\epsilon) = \{\, z : \theta_k + \epsilon \leq \arg z \leq \theta_{k+1} \,\},$$

$$S_k^-(\epsilon) = \{\, z : \theta_k \leq \arg z \leq \theta_{k+1} - \epsilon \,\},$$

$$S_k(\epsilon) = S_k^+(\epsilon) \cap S_k^-(\epsilon)$$

and

$$S_k = S_k^+(\epsilon) \cup S_k^-(\epsilon).$$

Now we can state

Theorem 1.J (see [14], [15, Chap.7] or [11, Theorem C]). *Let $A(z)$ be rational and of the form (1.22). Then there exist $n+2$ solutions $g_k(z)$ of (1.18), $k = 0, 1, \cdots, n+1$, meromorphic in $S_{k-1}^{+}(\epsilon) \cup S_k \cup S_{k+1}^{-}(\epsilon)$ with $g_{k-1}(z)$ and $g_k(z)$ linearly independent such that for all $z \in S_{k-1}^{+}(\epsilon) \cup S_k \cup S_{k+1}^{-}(\epsilon)$ and $|z| \to \infty$*

$$g_k(z) = \big(1+o(1)\big)(a_n z^n)^{-\frac{1}{4}} \exp\left\{\frac{2}{n+2}\sqrt{|a_n|}e^{i\alpha_n/2}(-1)^k i z^{\frac{n+2}{2}}\big(1+o(1)\big)\right\}. \quad (1.24)$$

Moreover, if $f(z)$ is a solution of (1.18), only finitely many of the zeros of $f(z)$ lie outside $\bigcup_{k=0}^{n+1} V_k(\epsilon)$. If $f(z)$ has infinitely many zeros in $V_k(\epsilon)$, then the number of zeros of $f(z)$ in $V_k(\epsilon) \cap \{z : |z| < r\}$, denoted by $n_k(r, f)$, satisfies

$$n_k(r, f) = \frac{2}{\pi(n+2)}\big(1+o(1)\big)\sqrt{|a_n|}r^{\frac{n+2}{2}} \quad (1.25)$$

as $r \longrightarrow \infty$.

Here we do not assume that the solution f is single value in $\mathbf{C}$. Instead, we assume, as did Hille, that it is defined on the Riemann surface of $\log z$. We notice that $S_{-1} = S_{n+1}$ only if all the g_k's are single valued meromorphic functions in $\mathbf{C}$, and otherwise, $S_0, S_1, \cdots, S_{n+1}$ are distinct regions on the Riemann surface of $\log z$. Moreover, if $A(z) = P(z)$ is a polynomial, then $n = \text{degree}(P)$.

It should be noted that Hille stated his results in [15] for the equation

$$(K(z)w')' + G(z)w = 0,$$

where $K(z)$ and $G(z)$ are polynomials. If we let $A(z) = \dfrac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials without common zeros, in (1.18), then the transformation $y(z) = Q(z)w(z)$ converts (1.18) to the form considered by Hille with $K(z) = Q(z)^2$.

From Theorem 1.J, we can deduce the following theorem concerning the zero deficiencies of meromorphic solutions of (1.18), where $A(z)$ is rational.

Theorem 1.K (see [11, Theorem 4]). *Suppose in* (1.18) *that* $A(z)$ *is rational and has the form* (1.22) *and* n, *defined by* (1.23), *is non-negative. Assume, in addition, that* (1.18) *has two linearly independent solutions meromorphic in* $\mathbf{C}$ *and that* $f(z) \not\equiv 0$ *is a solution of* (1.18). *Then* f *is meromorphic in* $\mathbf{C}$ *and* $s(f)$, *the number of* $V_k(\epsilon)$ *with only finitely many zeros of* $f(z)$, *is even and as* $r \to \infty$ *we have*

$$n\left(r, \frac{1}{f}\right) = 2(1 + o(1))\frac{(n+2) - s(f)}{\pi(n+2)}\sqrt{|a_n|}r^{\frac{n+2}{2}}$$

with $s(f) < n + 2$ *if and only if* $f(z)$ *has infinitely many zeros.*

$$N\left(r, \frac{1}{f}\right) = 4(1 + o(1))\frac{(n+2) - s(f)}{\pi(n+2)^2}\sqrt{|a_n|}r^{\frac{n+2}{2}},$$

$$T(r, f) = 2(1 + o(1))\frac{2(n+2) - s(f)}{\pi(n+2)^2}\sqrt{|a_n|}r^{\frac{n+2}{2}},$$

and consequently

$$\delta(0, f) = \frac{s(f)}{2(n+2) - s(f)}.$$

Moreover, $\delta(a, f) = 0$ *if* $a \neq 0$, $a \in \mathbf{C}$.

For a general entire function $A(z)$, a basic result concerning the zero distribution of the solutions of (1.18) is given by

Theorem 1.L [16, Theorem 5.6]. *Let* $f_1(z)$ *and* $f_2(z)$ *denote two linearly independent solutions of* (1.18) *and let* $f(z)$ *denote an arbitary non-trivial solution of* (1.18). *Let* $\bar{\sigma}(A)$ *denote the exponent of convergence for the distinct zeros of* $A(z)$. *Then*

(1) *If* $\lambda(A) \in (0, \infty) - \mathbf{N}$, *then* $\max(\sigma(f_1), \sigma(f_2)) \geq \lambda(A)$.

(2) *There exist* $f_1(z)$, $f(z)$ *having no zeros in* $\mathbf{C}$ *if and only if* $A(z)$ *can be represented as*

$$-4A(z) = h'(z)^2 + \varphi'(z)^2 - 2\varphi''(z),$$

where $\varphi(z)$ *is a non-constant entire function and* $h(z)$ *is a primitive of* $e^{\varphi(z)}$.

(3) *If* $\bar{\sigma}(A) < \lambda \in (0, \infty]$, *then* $\sigma(f) \geq \lambda$.

(4) *If* $\max\bigl(\sigma(f_1), \sigma(f_2)\bigr) < \infty$, *then* $\sigma(f) = \infty$ *for all* $f(z)$ *not being of the form* $\alpha f_1(z)$ *or* $\alpha f_2(z)$, $\alpha \in \mathbf{C}$.

We remark that actually some more precise result about the relation between $\lambda(A)$ and $\max\bigl(\sigma(f_1), \sigma(f_2)\bigr)$ were obtained by several people in the last decade. They can be stated as

$$\begin{aligned} \max\bigl(\sigma(f_1), \sigma(f_2)\bigr) &= \infty \quad \text{if} \quad \lambda(A) \leq \frac{1}{2}, \\ \max\bigl(\sigma(f_1), \sigma(f_2)\bigr) &> 1 \quad \text{if} \quad \frac{1}{2} < \lambda(A) < 1. \end{aligned}$$

The case $\lambda(A) < \frac{1}{2}$ was proved by Bank and Laine [1, Theorem 2(A)], and the case $\lambda(A) = \frac{1}{2}$ was proved by Rossi [26, Theorem 1] and [27, Theorem 1] independently. The assertion for the case $\frac{1}{2} < \lambda(A) < 1$ may be found in [26] and [28].

We conclude this section by mentioning that recently some analogues of results of Theorem 1.12 for third order equations were proved by Langley and for higher order equations (1.16) were proved by Bank and Langley. We refer the reader to [17] and [2].

CHAPTER 2

§2.1 Introduction

In section 1.3, we have seen many results concerning solutions of

$$w'' + A(z)w = 0, \tag{2.1}$$

where $A(z)$ is entire. We also gave the asymptotic representations (1.24) of solutions of the equation (2.1) in Theorem 1.J when $A(z)$ is rational. But for transcendental entire $A(z)$ not much is known about asymptotic representations of solutions of (2.1) even when the order of $A(z)$ is less than one.

In this chapter, we first consider the equation (2.1) with a transcendental entire function $A(z)$ which grows only slightly faster than a polynomial, namely it satisfies

$$\overline{\lim_{r\to\infty}} \frac{\log\log M(r,A)}{\log\log r} = p < 2. \tag{2.2}$$

We describe the solution of (2.1) in a union of a sequence of annuli which is quite large in the sense that the set of moduli of the points in its complement has logarithmic density (see (1.14) for the definition) zero in $(1, \infty)$. Specifically we obtain a local result similar to (1.24) in Theorem 1.J in the intersections of certain sectors and annuli. The idea can be stated as follows. Under the condition (2.2), a result (see

Lemma 2.1) in Wiman-Valiron theory implies that the entire function $A(z) = \sum_{n=0}^{\infty} a_n z^n$ equals $(1 + o(1))a_N z^N$ as $|z| \to \infty$ in the above union of annuli, where $N = N(r)$ is the central index (see section 1.2 for the definition) of $A(z)$. Then by use of the Liouville transform (2.32) in each annulus we can use the method of Hille locally on the corresponding equation

$$W'' + \big(1 - F(Z)\big)W = 0, \tag{2.3}$$

where

$$F(Z) = \frac{A''(z)}{4A(z)^2} - \frac{5A'(z)^2}{16A(z)^3} \tag{2.4}$$

and

$$W(Z) = A(z)^{\frac{1}{4}} w(z). \tag{2.5}$$

Finally, we transfer the result about the solution of (2.3) to that of (2.1), again via the Liouville transform.

To state our theorem, let

$$A(z) = \sum_{n=0}^{\infty} a_n z^n \tag{2.6}$$

and let $\alpha_n = \arg a_n$, $n = 0, 1, \cdots$. Now for any n define for $k = 0,\ 1,\ \cdots,\ n+1$

$$\mathbf{S}_{n,k}(\epsilon) = \left\{ z : \frac{-\alpha_n + 2(k-1)\pi}{n+2} + \epsilon < \arg z < \frac{-\alpha_n + 2(k+2)\pi}{n+2} - \epsilon \right\}$$

where $\epsilon > 0$.

Throughout the remainder of this chapter, we will use the letter C to denote a constant which depends on $A(z)$ (or on a function $F(z)$ which will ultimately depend on $A(z)$).

We now state our main result:

Theorem 2.1. *Let $A(z)$ be an entire function satisfying* (2.2) *and of the form* (2.6) *and let $0 < \eta < 2-p$. Then there exist an infinite sequence of positive integers $\mathcal{G}$ and two sequences $\{r_n\}_{n\in\mathcal{G}}$ and $\{s_n\}_{n\in\mathcal{G}}$ satisfying $0 < r_n < s_n < r_m < s_m$, whenever $n < m$; $n, m \in \mathcal{G}$, such that the logarithmic density of $\bigcup_{n\in\mathcal{G}} (s_n,\, r_{n+1})$ in $(1,\, \infty)$ is zero. Further for any $\epsilon > 0$ and $n \in \mathcal{G}$, the equation* (2.1) *has $n+2$ pairwise linearly independent solutions $u_{n,k}(z)$, $k = 0, 1, \cdots, n+1$, analytic in*

$$\mathbf{\Omega}_{n,k}(\epsilon) = \left\{\, z : r_n \le |z| \le s_n \,\right\} \cap \mathbf{S}_{n,k}(\epsilon)$$

such that for all $z \in \mathbf{\Omega}_{n,k}(\epsilon)$

$$u_{n,k}(z) = \big(1 + o(1)\big)(a_n z^n)^{-\frac{1}{4}} \exp\left\{ \frac{2}{n+2} \sqrt{|a_n|}\, e^{i\alpha_n/2} (-1)^k i z^{\frac{n+2}{2}} \big(1 + o(1)\big) \right\}. \quad (2.7)$$

Moreover, if $u(z)$ is a solution of (2.1), *then*

$$u(z) = \alpha_{n,k} u_{n,k}(z) + \beta_{n,k} u_{n+1,k}(z)$$

and either $\alpha_{n,k}\beta_{n,k} = 0$ and $u(z)$ has no zeros in $\mathbf{\Omega}_{n,k}(\epsilon) \cap \mathbf{\Omega}_{n,k+1}(\epsilon)$ or $\alpha_{n,k}\beta_{n,k} \ne 0$ and the set of all zeros of $u(z)$ in $\mathbf{\Omega}_{n,k}(\epsilon) \cap \mathbf{\Omega}_{n,k+1}(\epsilon)$ has the form

$$\left\{ \left[\frac{(-1)^k (n+2)}{4(1+\delta_{n,k,l})\sqrt{|a_n|}} e^{-i\alpha_n/2} \big((2l+1)\pi + i\gamma_{n,k} + \lambda_{n,k,l}\big) \right]^{\frac{2}{n+2}} : l \in \mathbf{Z} \right\}, \qquad (2.8)$$

where $\gamma_{n,k} = \log \dfrac{\alpha_{n,k}}{\beta_{n,k}}$ $\left(\arg \dfrac{\alpha_{n,k}}{\beta_{n,k}} \in [0, 2\pi]\right)$,

$$|\delta_{n,k,l}| < \exp\left\{ -\frac{1}{2} (\log r_n)^\eta \right\} \qquad (2.9)$$

and

$$|\lambda_{n,k,l}| < \frac{Ce^{n+2}(n+2)}{\sqrt{|a_n|}}(r_n)^{-\frac{n+2}{2}} \to 0 \tag{2.10}$$

as $n \to \infty$.

The theorem gives very good asymptotics for solutions of (2.1) in the spirit of Theorem 1.J. We emphasize however that our results are purely local. (So there is no result like (1.25) in this case.) This can be seen best in the result pertaining to the distribution of zeros. Indeed the theorem produces a family of Stoke's rays in each annulus similar to those of Theorem 1.J. The main difference is that the constant $\gamma_{n,k}$ may grow quite rapidly as $n \to \infty$. (This problem does not occur in Theorem 1.J, since there is but one annulus!) Thus closeness to the rays depends unfortunately but at least explicitly on the particular solution. It is fairly certain that any better result would depend on continuing the asymptotics of the $u_{n,k}(z)$ into neighboring annuli. With virtually no information on $A(z)$ in the exceptional set of logarithmic density zero, this strikes the author as a formidable task. It will become clear in the proof that the sequence $\mathcal{G}$ is merely the range of the central index $N(r)$ of $A(z)$ for r sufficiently large.

It is interesting to construct examples $A(z)$ satisfying the conditions of our Theorem for any p between 1 and 2. Specifically let $2 \leq k < \infty$ be an integer and define

$$A(z) = \sum_{n=0}^{\infty} \frac{z^n}{e^{n^k}}. \tag{2.11}$$

In section 2.4 we will show using the relation between the maximum modulus of an entire function and the maximum term of its series expansion that $A(z)$ actually satisfies (2.2) with $p = 1 + \dfrac{1}{k-1}$ (see Theorem 2.3).

We mention that a similar result even with $p = 2$ in (2.2) would be much more difficult since the power series in this case is dominated by a polynomial and not a monomial in each annulus and the degree and leading coefficient of this polynomial can not be precisely determined. However our method does indeed apply to an $A(z)$ of any order provided that its series expansion is sufficiently "gapped". Specifically if $A(z)$ is dominated by its maximum term in a sequence of annuli centered at the origin, the ratio of whose outer and inner radius approaches infinity, a similar version of our theorem still holds. A function with Hadamard gaps provides such an example. We can state the result as

Theorem 2.2. *Let*

$$A(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k} \tag{2.12}$$

be a transcendental entire function satisfying $\lambda_0 = 0$ and the Hadamard gap condition

$$\inf_{k \geq 1} \frac{\lambda_{k+1}}{\lambda_k} = \lambda > 1. \tag{2.13}$$

Then the conclusions of Theorem 2.1 are still true while "the logarithmic density of $\bigcup_{n \in \mathcal{G}} (s_n, r_{n+1})$ in $(1, \infty)$ is zero" in Theorem 2.1 is replaced by "the logarithmic measure of $\bigcup_{n \in \mathcal{G}} (s_n, r_{n+1})$ in $(0, \infty)$ is finite".

This theorem can be proved in the same way as Theorem 2.1 by Lemma 2.2 instead of Lemma 2.1 (see section 2.2). We will give some remarks on the proof in section 2.3, but we will omit the detail.

§2.2 Lemmas

An easy modification of the proof of Lemma 4 in [10] (see [10, P329]) gives the following lemma.

Lemma 2.1. *Suppose that $A(z)$ is an entire function satisfying (2.2) and of the form (2.6) with central index $N(r)$. Then denoting by $\mathcal{G}$ the range of $N(r)$ there exist two sequences $\{r'_n\}_{n\in\mathcal{G}}$ and $\{s'_n\}_{n\in\mathcal{G}}$ satisfying $0 < r'_n < s'_n < r'_m < s'_m$, whenever $n < m$; $n, m \in \mathcal{G}$ and $0 < \eta < 2-p$, such that the logarithmic density of $\bigcup_{n\in\mathcal{G}} (s'_n, r'_{n+1})$ is zero. Further if $r = |z| \in \bigcup_{n\in\mathcal{G}} [r'_n, s'_n]$ we have*

$$\sum_{j\neq N(r)} |a_j| r^j < |a_{N(r)}| r^{N(r)} \exp\{-(\log r)^{\eta}\}$$

where $N(r) \equiv n < (\log r)^{q-1}$ if $r \in [r'_n, s'_n]$ whenever $n \in \mathcal{G}$ and $p < q < 2 - \eta$.

The following lemma deals with transcendental entire functions satisfying the Hadamard gap condition and obtains a similar conclusion as that in Lemma 2.1.

Lemma 2.2. *Suppose that $A(z)$ is a transcendental entire function of the form (2.12) and (2.13). Let $N(r) = \lambda_v$ be the central index of $A(z)$. Then denoting by $\mathcal{G}$ the range of $N(r)$ there exist two sequences $\{r'_n\}_{n\in\mathcal{G}}$ and $\{s'_n\}_{n\in\mathcal{G}}$ satisfying $0 < r'_n < s'_n < r'_m < s'_m$ whenever $n < m$ and $n, m \in \mathcal{G}$ such that for sufficiently large $r = |z| \in \bigcup_{n\in\mathcal{G}} [r'_n, s'_n]$ we have*

$$\sum_{k\neq v} |a_k| r^{\lambda_k} \leq K N \exp\left\{-\left(1-\lambda^{-1}\right)^2 \sqrt{N}\right\} |a_v| r^N, \tag{2.14}$$

where K is a constant depending only on λ, $N(r) = n$ whenever $n \in \mathcal{G}$ and the logarithmic measure of $\bigcup_{n\in\mathcal{G}} (s'_n, r'_{n+1})$ is finite.

Proof. Since $\lim_{r\to\infty} N(r) = \infty$, we let $r \in \bigcup_{n\in\mathcal{G}} [r_n', s_n']$ be so large that

$$16\big(\log 2N(r)\big)^4 \le N(r). \tag{2.15}$$

Now we let $K = 1$, $l = 0$ and $\delta = 1$ in Theorem 1.A and use the theorem.

If $k < v$, by (2.13), we have $N \ge \lambda^{N-\lambda_k} \cdot \lambda_k \ge \lambda \cdot \lambda_k$. So by (1.15) and (2.15),

$$\begin{aligned} |a_k| r^{\lambda_k} &\le \exp\left\{ -\frac{(N-\lambda_k)^2}{2(2N-\lambda_k)\big(\log(2N-\lambda_k)\big)^2} \right\} \cdot |a_v| r^N \\ &\le \exp\left\{ -\frac{\left(N-\lambda^{-1}N\right)^2}{2\cdot 2N(\log 2N)^2} \right\} \cdot |a_v| r^N \\ &\le \exp\left\{ -\left(1-\lambda^{-1}\right)^2 \sqrt{N} \right\} \cdot |a_v| r^N. \end{aligned} \tag{2.16}$$

Therefore, by (2.16),

$$\sum_{k<v} |a_k| r^{\lambda_k} \le N \exp\left\{ -\left(1-\lambda^{-1}\right)^2 \sqrt{N} \right\} \cdot |a_v| r^N. \tag{2.17}$$

If $k > v$, by (2.13), we have $\lambda_k \ge \lambda^{\lambda_k - N} \cdot N \ge \lambda N$. So by (1.15) and (2.15) and noticing that $\lambda_k > \lambda_v = N(r)$, we have

$$\begin{aligned} |a_k| r^{\lambda_k} &\le \exp\left\{ -\frac{(\lambda_k - N)^2}{2\lambda_k(\log \lambda_k)^2} \right\} \cdot |a_v| r^N \\ &\le \exp\left\{ -\frac{\left(\lambda_k - \lambda^{-1}\lambda_k\right)^2}{2\lambda_k(\log \lambda_k)^2} \right\} \cdot |a_v| r^N \\ &\le \exp\left\{ -\left(1-\lambda^{-1}\right)^2 \sqrt{\lambda_k} \right\} \cdot |a_v| r^N. \end{aligned} \tag{2.18}$$

Now we claim that for any positive integer N and positive $T < 1$,

$$\sum_{k=N}^{\infty} T^{\sqrt{k}} \le \frac{2(N+1)T^{\sqrt{N}-1}}{(1-T)^2}. \tag{2.19}$$

In fact,

$$
\begin{aligned}
\sum_{k=N}^{\infty} T^{\sqrt{k}} &\leq \sum_{k=[\sqrt{N}]}^{\infty} \left((k+1)^2 - k^2\right) T^k \\
&= \sum_{k=[\sqrt{N}]}^{\infty} (2k+1) T^k \\
&\leq 2 \sum_{k=[\sqrt{N}]}^{\infty} (k+1) T^k \\
&= \frac{2([\sqrt{N}] + 1 - [\sqrt{N}]T) T^{[\sqrt{N}]}}{(1-T)^2} \\
&\leq \frac{2(N+1) T^{\sqrt{N}-1}}{(1-T)^2},
\end{aligned}
$$

where $[x]$ is the greatest integer in $[0, x]$. We obtain (2.19).

So from (2.18) and (2.19), we obtain

$$
\begin{aligned}
\sum_{k>v} |a_k| r^{\lambda_k} &\leq \sum_{k>v} \exp\left\{-\left(1-\lambda^{-1}\right)^2 \sqrt{\lambda_k}\right\} \cdot |a_v| r^N \\
&\leq \left(\sum_{m>N} \exp\left\{-\left(1-\lambda^{-1}\right)^2 \sqrt{m}\right\} \right) |a_v| r^N \\
&\leq \frac{2(N+2) \cdot \exp\left\{-\left(1-\lambda^{-1}\right)^2 \left((N+1)^{\frac{1}{2}} - 1\right)\right\}}{\left(1 - \exp\left\{-\left(1-\lambda^{-1}\right)^2\right\}\right)^2} \cdot |a_v| r^N \\
&\leq KN \exp\left\{-\left(1-\lambda^{-1}\right)^2 \sqrt{N}\right\} \cdot |a_v| r^N. \qquad (2.20)
\end{aligned}
$$

Combining (2.17) and (2.20), we obtain (2.14). This completes the proof.

The following lemma is a local version of Hille's method (see [15, Chap.7]). It is basically due to Langley [18, Lemma 1]. Since our regions differ somewhat from his, we offer a detailed proof.

Lemma 2.3. *Let* $0 < \epsilon < \dfrac{1}{4}$ *and let* $F(Z)$ *be an analytic function in* $\mathbf{\Omega}'_k$ *satisfying* $|F(Z)| \leq \dfrac{C}{|Z|^2}$ *for all* $Z \in \Omega'_k$, *where*

$$\mathbf{\Omega}'_k = \{\, Z : \max\{1, 10C\} \leq R \leq |Z| \leq S < \infty,\ (k-1+\epsilon)\pi \leq \arg Z \leq (k+2-\epsilon)\pi \,\},$$

is a closed region on the Riemann surface of $\log Z$ *with* $S > R(\sin \pi\epsilon)^{-1}$. *Then the equation* (2.3) *has a solution* $U_k(Z)$ *in*

$$\mathbf{\Omega}_k = \{\, Z : R(\sin \pi\epsilon)^{-1} \leq |Z| \leq S \,\} \cap \mathbf{\Omega}'_k$$

such that

$$U_k(Z) = \big(1 + \epsilon_k(Z)\big) \exp\{(-1)^k iZ\} \tag{2.21}$$

in $\mathbf{\Omega}_k$, *where* $\epsilon_k(Z)$ *satisfies* $|\epsilon_k(Z)| < \dfrac{C}{\epsilon|Z|}$ *in* $\mathbf{\Omega}_k$, $k \in \mathbf{Z}$.

Proof. It suffices to prove the lemma when $k = 0$ and $k = 1$. The general case follows similarly. We first assume that $k = 0$. Choose a solution $u(Z)$ of the equation

$$u'' + 2iu' - Fu = 0$$

such that $u(X) = 1$ and $u'(X) = 0$ where $X = Se^{\pi i/2}$. Now set

$$w(Z) = u(Z) - 1 + \frac{1}{2i} \int_X^Z \left(e^{2i(t-Z)} - 1\right) F(t)u(t)\, dt, \tag{2.22}$$

where the integral is independent of path in the closed region $\mathbf{\Omega}'_0$ on the Riemann surface of $\log Z$, since $F(Z)$ is defined and analytic there. Differentiation of (2.22) gives

$$w'(Z) = u'(Z) - \int_X^Z e^{2i(t-Z)} F(t)u(t)\, dt$$

and $w''(Z) = -2iw'(Z)$, so that since $w(X) = w'(X) = 0$, w vanishes identically on $\mathbf{\Omega}'_0$. Now let $Z \in \mathbf{\Omega}_0$. Let $(-1+\epsilon)\pi \leq \arg Z < -\dfrac{\pi}{2}$. Then noting that $S >$

$R(\sin \pi\epsilon)^{-1}$, we choose the path Γ of integration in $\mathbf{\Omega}_0'$ to be the vertical line-segment from X to $X' = \big(-|Z|\sin(\arg Z)\big)e^{\pi i/2}$, followed by the half circle $|t| = -|Z|\sin(\arg Z)$ from X' to $X'' = \big(-|Z|\sin(\arg Z)\big)e^{-\pi i/2}$ in the right half-plane, and then followed by the horizontal line-segment from X'' to Z. Since $\operatorname{Im}(t - Z) \geq 0$ for $t \in \Gamma$, if $d\tau$ denotes the arc-length on Γ, (2.22) and the fact that $w(Z)$ is identically 0, give

$$\big|u(Z) - 1\big| \leq \int_X^Z |F(t)u(t)|\, d\tau. \tag{2.23}$$

Set

$$W(\zeta) = \log\left(1 + \int_X^\zeta |F(t)u(t)|\, d\tau\right),$$

where $\zeta \in \Gamma$. Then by (2.23), $\dfrac{dW}{d\zeta} \leq |F(\zeta)|$. So

$$\begin{aligned}
W(Z) &\leq W(X) + \int_X^Z |F(t)|\, d\tau \\
&= \int_X^{X'} |F(t)|\, d\tau + \int_{X'}^{X''} |F(t)|\, d\tau + \int_{X''}^{Z} |F(t)|\, d\tau \\
&\leq \int\limits_{-|Z|\sin(\arg Z)}^{S} \frac{C}{y^2}\, dy + \frac{C}{|Z|^2\sin^2(\arg Z)} \cdot \pi\big(-|Z|\sin(\arg Z)\big) \\
&\quad + \int\limits_{0}^{-|Z|\cos(\arg Z)} \frac{C}{x^2 + |Z|^2\sin^2(\arg Z)}\, dx \\
&\leq C\left(-\frac{1}{|Z|\sin(\arg Z)} - \frac{1}{S}\right) - \frac{\pi C}{|Z|\sin(\arg Z)} - \frac{C\cos(\arg Z)}{|Z|\sin(\arg Z)} \\
&< \frac{C}{|Z|\sin \pi\epsilon}.
\end{aligned}$$

Since $0 < \epsilon < \dfrac{1}{4}$, we have

$$\begin{aligned}
\sin \pi\epsilon &= \pi\epsilon - \frac{(\pi\epsilon)^3}{3!} + \frac{(\pi\epsilon)^5}{5!} - \cdots \\
&> \pi\epsilon - \frac{(\pi\epsilon)^3}{3!} \\
&> \frac{\pi\epsilon}{2},
\end{aligned}$$

and hence

$$W(Z) < \frac{C}{\epsilon|Z|}.$$

Thus (2.23) implies that

$$\left|u(Z) - 1\right| \le \exp\left(\frac{C}{\epsilon|Z|}\right) - 1 \le \frac{C}{\epsilon|Z|}.$$

Now set $U_0(Z) = e^{iZ}u(Z)$, then $U_0(Z)$ solves (2.3), and (2.21) (when $k = 0$) follows at once. If $\frac{3\pi}{2} < \arg Z \le (2-\epsilon)\pi$, we choose a similar path of integration in $\mathbf{\Omega}_0'$, but now the half circle is in the left half-plane. If $-\frac{\pi}{2} \le \arg Z \le \frac{3\pi}{2}$, we just choose the path of integration in $\mathbf{\Omega}_0$ to be the vertical line-segment from X to $|Z|e^{\pi i/2}$, followed by the arc of the circle $|t| = |Z|$ from $|Z|e^{\pi i/2}$ to Z. Then

$$\begin{aligned} W(Z) &\le \int_X^{|Z|e^{\pi i/2}} |F(t)|\,d\tau + \int_{|Z|e^{\pi i/2}}^{Z} |F(t)|\,d\tau \\ &\le \int_{|Z|}^{S} \frac{C}{y^2}\,dy + \frac{C}{|Z|^2}\cdot\pi|Z| \\ &\le \frac{C}{|Z|}, \end{aligned}$$

and (2.21) (when $k = 0$) follows at once.

To prove the lemma in the case $k = 1$, choose a solution $v(Z)$ of

$$v'' - 2iv' - Fv = 0$$

such that $v(Y) = 1$ and $v'(Y) = 0$ where $Y = Se^{3\pi i/2}$. The integral equation for v is

$$v(Z) - 1 = \frac{1}{2i}\int_Y^Z \left(e^{-2i(t-Z)} - 1\right) F(t)v(t)\,dt$$

and we choose a path of integration on which $\mathrm{Im}(t - Z) \le 0$. Finally we set $U_1(Z) = e^{-iZ}v(Z)$.

Lemma 2.4. *Let C, ϵ, F, $\boldsymbol{\Omega}'_k$, $\boldsymbol{\Omega}_k$ and U_k satisfy all the conditions in Lemma 2.3 and let $R \geq 4\sqrt{2}\pi C$. Suppose that $U(Z) = \alpha_k U_k(Z) + \beta_k U_{k+1}(Z)$ is a solution of (2.3) in*

$$\boldsymbol{\Omega} = \left\{ Z : R(\sin \pi\epsilon)^{-1} \leq |Z| \leq S < \infty, \ -\infty < \arg Z < \infty \right\},$$

a closed region on the Riemann surface of $\log Z$. Then either $\alpha_k\beta_k = 0$ and $U(Z)$ has no zeros in $\boldsymbol{\Omega}_k \cap \boldsymbol{\Omega}_{k+1}$ or $\alpha_k\beta_k \neq 0$ and the set of all zeros of $U(Z)$ in $\boldsymbol{\Omega}_k \cap \boldsymbol{\Omega}_{k+1}$ is of the form

$$\left\{ \frac{(-1)^k}{2}\big[(2l+1)\pi + i\gamma_k + \lambda_{k,l}\big] : l \in \mathbf{Z} \right\}, \tag{2.24}$$

where $\gamma_k = \log \dfrac{\alpha_k}{\beta_k}$ $\quad (\arg \dfrac{\alpha_k}{\beta_k} \in [0, 2\pi])$ and $\lambda_{k,l} \in \mathbf{C}$ satisfies

$$|\lambda_{k,l}| < \frac{C}{R}. \tag{2.25}$$

Proof. By Lemma 2.3, the equation (2.3) has two linearly independent solutions $U_k(Z)$ and $U_{k+1}(Z)$ in $\boldsymbol{\Omega}_k \cap \boldsymbol{\Omega}_{k+1}$ such that

$$U_j(Z) = \big(1 + \epsilon_j(Z)\big) \exp\{(-1)^j iZ\}$$

and $|\epsilon_j(Z)| < \dfrac{C}{\epsilon|Z|}$, $j = k,\ k+1$ and by virtue of the range of $|Z|$, this is not greater than $\dfrac{1}{4}$. Now there exist α_k, $\beta_k \in \mathbf{C}$ such that for $Z \in \boldsymbol{\Omega}_k \cap \boldsymbol{\Omega}_{k+1}$

$$U(Z) = \alpha_k U_k(Z) + \beta_k U_{k+1}(Z).$$

If $\alpha_k\beta_k = 0$, it is easily seen that $U(Z)$ has no zeros in $\boldsymbol{\Omega}_k \cap \boldsymbol{\Omega}_{k+1}$. If $\alpha_k\beta_k \neq 0$ and $U(Z)$ has a zero Z_0 in $\boldsymbol{\Omega}_k \cap \boldsymbol{\Omega}_{k+1}$, then since

$$\alpha_k\big(1 + \epsilon_k(Z_0)\big) \exp\{2(-1)^k iZ_0\} + \beta_k\big(1 + \epsilon_{k+1}(Z_0)\big) = 0,$$

we have

$$Z_0 = \frac{(-1)^k}{2}\left[(2l+1)\pi + i\log\frac{\alpha_k}{\beta_k} + i\log\left(\frac{1+\epsilon_k(Z_0)}{1+\epsilon_{k+1}(Z_0)}\right)\right],$$

where $l \in \mathbf{Z}$ and the arguments of $\dfrac{\alpha_k}{\beta_k}$ and $\left(\dfrac{1+\epsilon_k(Z_0)}{1+\epsilon_{k+1}(Z_0)}\right)$ are in $[0, 2\pi]$. Letting $\gamma_k = \log \dfrac{\alpha_k}{\beta_k}$ and $\lambda_{k,l} = i \log \left(\dfrac{1+\epsilon_k(Z_0)}{1+\epsilon_{k+1}(Z_0)}\right)$, we obtain (2.24). Moreover, by Lemma 2.3, we have

$$\left|\log \left|\frac{1+\epsilon_k(Z_0)}{1+\epsilon_{k+1}(Z_0)}\right|\right| \leq \log \left(\frac{1+\left|\epsilon_k(Z_0)\right|}{1-\left|\epsilon_{k+1}(Z_0)\right|}\right) \leq \log \left(\frac{1+\dfrac{C}{\epsilon|Z_0|}}{1-\dfrac{C}{\epsilon|Z_0|}}\right) = \log \left(1+\frac{\dfrac{2C}{\epsilon|Z_0|}}{1-\dfrac{C}{\epsilon|Z_0|}}\right). \tag{2.26}$$

Let $x_0 = \dfrac{\dfrac{2C}{\epsilon|Z_0|}}{1-\dfrac{C}{\epsilon|Z_0|}}$, then by $\dfrac{C}{\epsilon|Z_0|} \leq \dfrac{1}{4}$,

$$0 < x_0 \leq \frac{\dfrac{2}{4}}{1-\dfrac{1}{4}} = \frac{2}{3}.$$

So

$$\begin{aligned}
\log(1+x_0) &= x_0 - \frac{1}{2!}x_0^2 + \frac{1}{3!}x_0^3 - \cdots \\
&< x_0 + \frac{1}{2}x_0^2 + \frac{1}{2^2}x_0^3 + \cdots \\
&= \frac{2x_0}{2-x_0} = \frac{2C}{\epsilon|Z_0| - 2C} \\
&\leq \frac{4C}{\epsilon|Z_0|}.
\end{aligned}$$

Thus by (2.26) and the fact that $|Z_0| \geq R(\sin \pi\epsilon)^{-1}$, we have

$$\left|\operatorname{Im} \lambda_{k,l}\right| < \frac{4C}{\epsilon|Z_0|} \leq \frac{4C \sin \pi\epsilon}{\epsilon R} < \frac{4\pi C}{R}. \tag{2.27}$$

Now by $R \geq 4\sqrt{2}\pi C$, we have

$$\begin{aligned}
\left|\frac{1+\epsilon_k(Z_0)}{1+\epsilon_{k+1}(Z_0)} - 1\right| &< \left|\frac{\epsilon_{k+1}(Z_0)-\epsilon_k(Z_0)}{1+\epsilon_{k+1}(Z_0)}\right| \\
&\leq \frac{\dfrac{2C}{\epsilon|Z_0|}}{1-\dfrac{C}{\epsilon|Z_0|}} \leq \frac{4C}{\epsilon|Z_0|} < \frac{4\pi C}{R} \\
&\leq \frac{1}{\sqrt{2}}.
\end{aligned}$$

Denoting $\arg\left(\dfrac{1+\epsilon_k(Z_0)}{1+\epsilon_{k+1}(Z_0)}\right)$ by θ, we consequently have that $|\sin\theta| < \dfrac{4\pi C}{R}$, $|\theta| \leq \dfrac{\pi}{4}$, and

$$\begin{aligned}
|\sin\theta| &= \left|\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right| \\
&\geq |\theta| - \left|\frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \cdots\right| \\
&> |\theta| - \left(\left(\frac{\theta}{2}\right)^2 + \left(\frac{\theta}{2}\right)^4 + \cdots\right) \\
&= |\theta| - \frac{\theta^2}{4-\theta^2} \\
&> \frac{1}{2}|\theta|.
\end{aligned}$$

Hence

$$\left|\operatorname{Re}\lambda_{k,l}\right| = \left|\arg\left(\frac{1+\epsilon_k(Z_0)}{1+\epsilon_{k+1}(Z_0)}\right)\right| = |\theta| < 2|\sin\theta| < \frac{8\pi C}{R}.$$

This and (2.27) imply (2.25).

§2.3 Proof of Theorem 2.1

Now we prove Theorem 2.1.

Since the exceptional set in Theorem 1.C has finite logarithmic measure, applying Theorem 1.C and Lemma 2.1 to the entire function $A(z)$, we see easily that there exists a sequence $\{ [r_n', s_n'] \}_{n\in\mathcal{G}}$ of closed intervals satisfying all the conclusions of Lemma 2.1 such that whenever $r = |z| \in \bigcup_{n\in\mathcal{G}} [r_n', s_n']$ and $r \to \infty$ we have

$$A(z) = a_{N(r)}\big(1 + \delta_1(z)\big) z^{N(r)}, \tag{2.28}$$

$$M(r, A^{(j)}) = \big(1 + o(1)\big) \left(\frac{N(r)}{r}\right)^j M(r, A), \quad j = 1,\ 2 \tag{2.29}$$

and

$$N(r) \equiv n, \quad r \in [r_n', s_n'], \quad n \in \mathcal{G}, \tag{2.30}$$

where $\delta_1(z)$ satisfies

$$|\delta_1(z)| < \exp\{-(\log|z|)^\eta\}, \quad 0 < \eta < 2 - p.$$

Moreover, without loss of generality, we can assume that

$$\lim_{n\to\infty} \frac{s_n'}{r_n'} = \infty. \tag{2.31}$$

Now for sufficiently large $n \in \mathcal{G}$, we consider the following transform in each

$$\mathcal{A}_n = \{ z : r_n' \le |z| \le s_n', -\infty < \arg z < \infty \},$$

a closed region on the Riemann surface of $\log z$. For convenience, we write

$$\mathcal{A} = \{ z : s \le |z| \le s', -\infty < \arg z < \infty \}$$

for any $\mathcal{A}_n$ above. Noting (2.30), consider the transform

$$Z = \int_s^z A(t)^{\frac{1}{2}}\, dt + \frac{2}{n+2} a_n^{\frac{1}{2}} s^{\frac{n+2}{2}}, \quad z \in \mathcal{A}, \tag{2.32}$$

where the integral is evidently independent of the path, which remains in $\mathcal{A}$. Like in [15, Chap.7], we define $W(Z)$ as in (2.5). Then if $w(z)$ is a solution of (2.1), $W(Z)$ solves (2.3), and $F(Z)$ is given by (2.4). Moreover, (2.29) and (2.30) imply that

$$Z = \frac{2}{n+2} a_n^{\frac{1}{2}} z^{\frac{n+2}{2}} + a_n^{\frac{1}{2}} \int_s^z t^{\frac{n}{2}} \delta_2(t)\, dt, \quad z \in \mathcal{A}, \tag{2.33}$$

where $\delta_2(z)$ satisfies $|\delta_2(z)| < \exp\{-(\log|z|)^\eta\}$ in $\mathcal{A}$. So, if $-3\pi \le \arg z \le 3\pi$, choose the path of integration in (2.33) to be the straight line segment from s to $|z|$, followed by the arc of $|t| = |z|$ from $|z|$ to z in $\mathcal{A}$. Letting $d\tau$ denote arc-length on the path, we have

$$\begin{aligned}
\left| Z - \frac{2}{n+2} a_n^{\frac{1}{2}} z^{\frac{n+2}{2}} \right| &\le |a_n|^{\frac{1}{2}} \int_s^z |t|^{\frac{n}{2}} \exp\{-(\log|t|)^\eta\}\, d\tau \\
&= |a_n|^{\frac{1}{2}} \int_s^{|z|} x^{\frac{n}{2}} \exp\{-(\log x)^\eta\}\, dx \\
&\qquad + |a_n|^{\frac{1}{2}} \int_{|z|}^z |z|^{\frac{n}{2}} \exp\{-(\log|z|)^\eta\}\, d\tau \\
&\le |a_n|^{\frac{1}{2}} \left\{ \exp\{-(\log s)^\eta\} \cdot |z|^{\frac{n+2}{2}} + 3\pi \exp\{-(\log|z|)^\eta\} \cdot |z|^{\frac{n+2}{2}} \right\} \\
&\le (1+3\pi)|a_n|^{\frac{1}{2}} |z|^{\frac{n+2}{2}} \exp\{-(\log s)^\eta\}.
\end{aligned} \tag{2.34}$$

According to Lemma 2.1,

$$N(r) \equiv n < (\log r)^{q-1} \tag{2.35}$$

if $s \le r \le s'$, where $p < q < 2 - \eta$. Thus (2.34) implies that

$$Z = \big(1 + \delta_3(z)\big) \frac{2}{n+2} a_n^{\frac{1}{2}} z^{\frac{n+2}{2}}, \tag{2.36}$$

where $\delta_3(z)$ satisfies

$$|\delta_3(z)| < \exp\left\{-\frac{1}{2}(\log s)^\eta\right\} \tag{2.37}$$

for all $z \in \mathcal{A}$ with $-3\pi \leq \arg z \leq 3\pi$. Therefore the transform (2.32) maps

$$\{\, z : s \leq |z| \leq s', \; -3\pi \leq \arg z \leq 3\pi \,\}$$

one-to-one onto a set containing

$$\mathcal{B} = \left\{ Z : S \leq |Z| \leq S', \; \frac{2\alpha_n - 5(n+2)\pi}{4} \leq \arg Z \leq \frac{2\alpha_n + 5(n+2)\pi}{4} \right\},$$

where $S = \dfrac{3}{n+2}\sqrt{|a_n|}\, s^{\frac{n+2}{2}}$, $S' = \dfrac{1}{n+2}\sqrt{|a_n|}\, s'^{\frac{n+2}{2}}$ and $\alpha_n = \arg a_n$. A routine calculation involving (2.28), (2.29), (2.30), (2.4) and (2.36), gives for $Z \in \mathcal{B}$ that

$$\big|F(Z)\big| < \frac{C}{|Z|^2}.$$

Certainly Lemma 2.1 implies that $A(z)$ has no zeros in $\mathcal{A}$ so that $F(Z)$ is analytic in $\mathcal{B}$. By (2.31) we can assume that $S' > S(\sin \pi\epsilon)^{-1}$. Then $F(Z)$ satisfies the conditions of Lemma 2.3. Hence (2.3) has a solution $U_k(Z)$ in

$$\mathcal{C}_k = \mathcal{B} \cap \{\, Z : S(\sin \pi\epsilon)^{-1} \leq |Z| \leq S', \; (k-1+\epsilon)\pi \leq \arg Z \leq (k+2-\epsilon)\pi \,\}$$

such that

$$U_k(Z) = \big(1 + \epsilon_k(Z)\big) \exp\{(-1)^k iZ\} \tag{2.38}$$

and

$$|\epsilon_k(Z)| < \frac{C}{\epsilon |Z|}$$

in $\mathcal{C}_k$, where $k \in \mathbf{Z}$. We can also assume that $n \in \mathcal{G}$ is sufficiently large so that $S > C^2\epsilon^{-2}$. Then we have

$$|\epsilon_k(Z)| < \frac{1}{|Z|^{\frac{1}{2}}}. \tag{2.39}$$

Further let $n \in \mathcal{G}$ be large enough so that $e^{n+2} > (\sin \pi\epsilon)^{-1}$ and let

$$\mathcal{D} = \left\{ z : 3e^2 s \le |z| \le \frac{s'}{3e^2},\ -2\pi \le \arg z \le 2\pi \right\}.$$

Then from (2.36) we know that the transform (2.32) maps $\mathcal{D}$ one-to-one onto a subset of $\mathcal{B} \cap \{ Z : S(\sin\pi\epsilon)^{-1} \le |Z| \le S' \}$. Hence the equation (2.1) has a solution $u_k(z)$ in the intersection of $\mathcal{D}$ and the preimage of $\mathcal{C}_k$ which contains

$$\mathcal{E}_k(\epsilon) = \mathcal{D} \cap \left\{ z : \frac{-\alpha_n + 2(k-1)\pi}{n+2} + \epsilon \le \arg z \le \frac{-\alpha_n + 2(k+2)\pi}{n+2} - \epsilon \right\}$$

where $\alpha_n = \arg a_n$ and $\epsilon > 0$. By (2.28), (2.30), (2.5), (2.36), (2.37), (2.38) and (2.39), for $z \in \mathcal{E}_k(\epsilon)$,

$$u_k(z) = \big(1 + o(1)\big)(a_n z^n)^{-\frac{1}{4}} \exp\left\{ \frac{2}{n+2} \sqrt{|a_n|}\, e^{i\alpha_n/2} (-1)^k i z^{\frac{n+2}{2}} \big(1 + o(1)\big) \right\}.$$

Since $\mathcal{A}$ denotes any $\mathcal{A}_n$ with sufficiently large $n \in \mathcal{G}$, if we let

$$r_n = 3e^2 r_n', \qquad s_n = \frac{s_n'}{3e^2} \tag{2.40}$$

for sufficiently large $n \in \mathcal{G}$, we have actually proved (2.7) in $\boldsymbol{\Omega}_{n,k}(\epsilon)$ (provided we redefine $\mathcal{G}$ to contain only sufficiently large n). Moreover, noting (2.40) and (2.35) and using the definition of the logarithmic density, we can easily see that the logarithmic density of $\bigcup_{n \in \mathcal{G}} \big[(r_n', r_n) \cup (s_n, s_n')\big]$ is zero.

Finally, if $u(z)$ is a solution of (2.1), then by Lemma 2.4 and (2.36) for $n \in \mathcal{G}$ either $u(z)$ has no zeros in $\boldsymbol{\Omega}_{n,k}(\epsilon) \cap \boldsymbol{\Omega}_{n,k+1}(\epsilon)$ or the zeros of $u(z)$ satisfy (2.8). We obtain (2.9) from (2.37) and (2.10) from (2.25), Lemma 2.1 and the fact that $a_N r_N^N$ increases to infinity with n (see section 1.2).

To prove Theorem 2.2, we use Theorem 1.C and Lemma 2.2. The proof is the same as that of Theorem 2.1 except that the estimates for $\delta_1(z)$, $\delta_2(z)$ and $\delta_3(z)$ are changed according to (2.14). Moreover, noticing that the exceptional sets in both Theorem 1.C and Lemma 2.2 have finite logarithmic measures, the exceptional set in Theorem 2.2 has also finite logarithmic measure.

§2.4 Examples

In this section, we show that an entire function $A(z)$ of the form (2.11) satisfies (2.2) with $p = 1 + \dfrac{1}{k-1}$ by proving Theorem 2.3. To do that, we need the following lemma.

Lemma 2.5. *Suppose that $f(z)$ is an entire function of the form (2.11) and $N(r)$ is the central index of $f(z)$. Then $N(r) = n$ if $r \in [r_n, s_n)$, $n = 1, 2, \cdots$, where*

$$r_n = \exp\left\{\sum_{j=0}^{k-1}(n-1)^j n^{k-j-1}\right\}, \tag{2.41}$$

$$s_n = \exp\left\{\sum_{j=0}^{k-1} n^j (n+1)^{k-j-1}\right\}, \tag{2.42}$$

$k \in \mathbf{N}$ and $k \geq 2$.

Proof. Let $r \in [r_n, s_n)$.

If $m \leq n-1$, then by (2.41),

$$\begin{aligned} r^{n-m} \geq r_n^{n-m} &= \exp\left\{(n-m)\sum_{j=0}^{k-1}(n-1)^j n^{k-j-1}\right\} \\ &\geq \exp\left\{(n-m)\sum_{j=0}^{k-1} m^j n^{k-j-1}\right\} \\ &= \exp\{n^k - m^k\}. \end{aligned}$$

So

$$\frac{r^m}{e^{m^k}} \leq \frac{r^n}{e^{n^k}}. \tag{2.43}$$

If $m > n$, then by (2.42),

$$\begin{aligned} r^{m-n} < s_n^{m-n} &= \exp\left\{(m-n)\sum_{j=0}^{k-1} n^j(n+1)^{k-j-1}\right\} \\ &\leq \exp\left\{(m-n)\sum_{j=0}^{k-j-1} n^j m^{k-j-1}\right\} \\ &= \exp\{m^k - n^k\}. \end{aligned}$$

So we obtain (2.43) again, and the lemma is proved.

Now we can prove

Theorem 2.3. *Suppose that $k \in \mathbf{N}$, $k \geq 2$ and $f(z)$ is an entire function with the form (2.11). Then we have*

$$\lim_{r\to\infty} \frac{\log\log M(r,f)}{\log\log r} = 1 + \frac{1}{k-1}. \tag{2.44}$$

Proof. Let $K = 1$, $l = 0$, $\epsilon = 1$ and $\delta = 1$ in Theorem 1.B. We have

$$M(r,f) < 2\mu(r)\left\{2\pi N(\log N)^2\right\}^{\frac{1}{2}}, \tag{2.45}$$

for $r \notin E$ and $\mathrm{l.m.}(E) < \infty$.

Now by Lemma 2.5, if $r \in [r_n, s_n)$, then

$$k(N-1)^{k-1} \leq \log r < k(N+1)^{k-1}.$$

So

$$\left(k^{-1}\log r\right)^{\frac{1}{k-1}} - 1 < N(r) \leq \left(k^{-1}\log r\right)^{\frac{1}{k-1}} + 1.$$

Thus by (2.45), we obtain

$$\begin{aligned} \log M(r,f) &< \log 2 + \log|a_N| + N\log r + \frac{1}{2}\left(\log 2\pi + \log N + 2\log\log N\right) \\ &= \left(1 + o(1)\right)N\log r - N^k \\ &= \left(1 + o(1)\right)(k-1)k^{-\frac{k}{k-1}}\left(\log r\right)^{\frac{k}{k-1}} \end{aligned}$$

for $r \notin E$ and $r \to \infty$. Thus

$$\varlimsup_{\substack{r \notin E \\ r \to \infty}} \frac{\log\log M(r,f)}{\log\log r} \leq 1 + \frac{1}{k-1}.$$

Noticing the fact that $\mathrm{l.\,m.}(E) < \infty$, we can easily see that

$$\varlimsup_{r \to \infty} \frac{\log\log M(r,f)}{\log\log r} \leq 1 + \frac{1}{k-1}. \tag{2.46}$$

On the other hand,

$$M(r,f) \geq \mu(r) = |a_N| r^N.$$

So we have

$$\log M(r,f) \geq \big(1 + o(1)\big)(k-1)k^{-\frac{k}{k-1}}\big(\log r\big)^{\frac{k}{k-1}}$$

as $r \to \infty$. Thus

$$\varliminf_{r \to \infty} \frac{\log\log M(r,f)}{\log\log r} \geq 1 + \frac{1}{k-1}. \tag{2.47}$$

Combining (2.46) and (2.47), we obtain (2.45).

CHAPTER 3

§3.1 Introduction

Let $f(z)$ be an entire function of finite order λ. A function $\lambda(r)$ is called a Lindelöf proximate order of $f(z)$ if

(1) there exists a $K \geq 0$ such that $\lambda(r)$ is real, continuous, and piecewise differentiable on $[K, \infty)$;

(2) $\lim_{r \to \infty} \lambda(r) = \lambda$;

(3) $\lim_{r \to \infty} \lambda'(r) r \log r = 0$, where $\lambda'(r)$ is either the right- or left-hand derivative at points where they are different;

(4) $\overline{\lim_{r \to \infty}} \dfrac{\log M(r, f)}{r^{\lambda(r)}} = 1$.

Now if $\lambda(r)$ is any one of the Lindelöf proximate orders of $f(z)$, then the corresponding Valiron growth function of $f(z)$ is defined as

$$V(r) = r^{\lambda(r)}.$$

A useful property of $V(r)$ (see [6, p.41]) is that for any $\sigma > 1$

$$\lim_{r \to \infty} \frac{V(\sigma r)}{V(r)} = \sigma^{\lambda}. \tag{3.1}$$

The indicator function $h(\theta)$ of Phragmén-Lindelöf is defined by

$$h(\theta) = \overline{\lim_{r \to \infty}} \frac{\log\left|f(re^{i\theta})\right|}{V(r)}, \tag{3.2}$$

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which is a continuous function, with $h(\theta) \neq \infty$ and $h(\theta) \not\equiv 0$ (see [6, p.41-46 and p.54]).

An entire function $f(z)$ of finite order λ is said to be of completely regular growth, with respect to a $V(r)$, if

$$\frac{\log\left|f(re^{i\theta})\right|}{V(r)} \to h(\theta) \tag{3.3}$$

uniformly in θ as $r \to \infty$ and avoids some possible exceptional set E, of density zero (see (1.13) for the definition).

Now we denote by C the set of all arguments θ satisfying $0 \leq \theta \leq 2\pi$, and let the points $\theta = 0$ and $\theta = 2\pi$ be "identified". Moreover, let

$$C_0 = \left\{ \theta \in C : h(\theta) = 0 \right\},$$

where $h(\theta)$ is the indicator function of an entire function of finite order λ, and of completely regular growth. From [23, Lemma 1], C_0 is the union of finitely many isolated points and of q $(0 \leq q \leq 2\lambda)$ disjoint closed intervals which do not reduce to points.

K.-C. Oum proved the following two theorems in [23].

Theorem 3.A [23, Theorem 2]. *Let $f(z)$ be an entire function of order λ satisfying $0 < \lambda < \infty$. Suppose that $f(z)$ is of completely regular growth with respect to one of its Valiron growth functions and let q be the number of component intervals, which do not reduce to points, of the set C_0. Then the number of deficient values of $f(z)$, other than 0 and ∞, cannot exceed q and, if $\nu(f)$ $(\leq \infty)$ denotes the* *number of all deficient values of $f(z)$,*

$$\nu(f) \le 2\lambda + 1. \tag{3.4}$$

Moreover, if 2λ is a positive integer, (3.4) is to be replaced by the sharper inequality

$$\nu(f) \le 2\lambda. \tag{3.5}$$

Theorem 3.B [23, Theorem 1]. *Let $f(z)$ be an entire function of finite non-integer order λ, and let the zeros of $f(z)$ have an angular density in the sense of Pfluger and Levin. Then (3.4) in Theorem 3.A is true. Moreover, (3.5) in Theorem B is true under the same condition.*

Here, following Pfluger [24, p.204], we say that the zeros of $f(z)$ have an angular density, with respect to $V(r)$, if, with any $\epsilon > 0$, it is possible to associate a finite number $k = k(\epsilon)$ of arguments:

$$\theta_1, \theta_2, \cdots, \theta_k \qquad (0 \le \theta_1 < \theta_2 < \cdots < \theta_k < 2\pi)$$

such that

$$\theta_{j+1} - \theta_j < \epsilon, \quad j = 1, 2, \cdots, k, \quad \theta_{k+1} = \theta_1 + 2\pi$$

and the limits

$$\lim_{r\to\infty} \frac{n(r, \theta_j, \theta_{j+1})}{V(r)}$$

exist and are finite, where $n(r, \theta_j, \theta_{j+1})$ denotes the number of zeros of $f(z)$ in the sector $\{\, z : |z| \le r,\, \theta_j < \arg z < \theta_{j+1} \,\}$, for $j = 1, 2, \cdots, k$.

Now an interesting question is whether Theorem 3.A and Theorem 3.B are still true when deficient value in the theorems is replaced by deficient function (see (1.9) and (1.10)). In this chapter we answer this question.

Now we state our results as

Theorem 3.1. *Theorem 3.A is true with deficient value replaced by deficient function.*

Theorem 3.2. *Theorem 3.B is true with deficient value replaced by deficient function.*

Because the conditions in Theorem 3.B imply the conditions in Theorem 3.A (see [24, Theorem 3] or [19, p.90]), we only need to prove Theorem 3.1.

We know from [30] that if $f(z)$ is a transcendental solution of (1.16) where $a_0(z)$, $a_1(z)$, $\cdots$, $a_{n-2}(z)$ are polynomials and $a_{n-1}(z) = 0$ then $f(z)$ is of completely regular growth with order λ, $0 < \lambda < \infty$. So we directly have

Theorem 3.3. *The conclusions of Theorem* 3.1 *hold if* $f(z)$ *is a transcendental solution of* (1.16) *where* $a_0(z)$, $a_1(z)$, $\cdots$, a_{n-2} *are polynomials and* $a_{n-1}(z) = 0$.

§3.2 Lemmas

Let $g(z)$ be a meromorphic function. Let

$$m(r,g;J)=\frac{1}{2\pi}\int_J \log^+\left|g(re^{i\theta})\right|\,d\theta,$$

where $J \subset C$ is measurable. Then we have

Lemma 3.1 [7, Lemma III]. *Let $g(z)$ be meromorphic. With each $r>0$ we associate a set $\Lambda(r)$ of values of θ such that*

$$\operatorname{meas}\Lambda(r)=\mu(r).$$

Then for $1<r<r_0$ we have

$$m\big(r,g;\Lambda(r)\big)\le \frac{11r_0}{r_0-r}\,T(r_0,g)\,\mu(r)\left(1+\log^+\frac{1}{\mu(r)}\right).$$

Lemma 3.2 (see [23, Lemma 3]). *Let $f(z)$ be an entire function of finite order and of completely regular growth with respect to one of its Valiron growth functions $V(r)$. Let $h(\theta)$ be given by (3.2), let*

$$h^+(\theta)=\begin{cases} h(\theta) & \text{if } h(\theta)\ge 0,\\ 0 & \text{if } h(\theta)<0,\end{cases}$$

and let

$$H_+=\frac{1}{2\pi}\int_0^{2\pi} h^+(\theta)\,d\theta.$$

Then $0<H_+<\infty$ and

$$T(r,f)=\big(1+o(1)\big)H_+V(r),\ \ r\to\infty. \tag{3.7}$$

From (3.7) and (3.1), we have

Lemma 3.3. *Let $\sigma > 1$ and let $f(z)$ satisfy the conditions in Lemma 3.2. Then*

$$\lim_{r\to\infty} \frac{T(\sigma r, f)}{T(r,f)} = \sigma^{\lambda}. \tag{3.8}$$

Lemma 3.4. *Let $\sigma_0 > 1$ and $R' > 1$. Suppose that the subset $E \neq \emptyset$ of the closed interval $[\sigma_0^{-1}R', \sigma_0 R']$ satisfies*

(1) $E \neq [\sigma_0^{-1}R', \sigma_0 R']$;

(2) *E is a union of finitely many disjoint closed intervals;*

(3) $\operatorname{meas} E \leq \dfrac{\sigma_0 - 1}{4\sigma_0} R'$.

Then there exists a σ satisfying

$$\frac{2\sigma_0}{\sigma_0 + 1} \leq \sigma \leq \sigma_0 \tag{3.9}$$

such that $\sigma^{-1}R', \sigma R' \notin E$.

Proof. From the conditions (2) and (3), without loss of generality, we assume that

$$\left[\frac{1}{\sigma_0}R', \frac{\sigma_0+1}{2\sigma_0}R'\right] - E = \bigcup_{i=1}^{s} \left(\frac{1}{\sigma_i}R', \frac{1}{\sigma_i'}R'\right), \tag{3.10}$$

where σ_i and σ_i', $i = 1, 2, \cdots, s\,(s \geq 1)$, satisfy

$$\sigma_0 \geq \sigma_1 > \sigma_1' > \sigma_2 > \sigma_2' > \cdots > \sigma_s > \sigma_s' \geq \frac{2\sigma_0}{\sigma_0 + 1}$$

(It is possible that at most two intervals on the right side of (3.10) are not open. But the assumption (3.10) does not interfere with the proof). Hence by condition (3)

$$\begin{aligned}
\frac{(\sigma_0+1)^2}{4\sigma_0{}^2}\cdot\sum_{i=1}^{s}(\sigma_i-\sigma_i')R' &= \sum_{i=1}^{s}\frac{1}{\left(\dfrac{2\sigma_0}{\sigma_0+1}\right)^2}(\sigma_i-\sigma_i')R' \\
&\geq \sum_{i=1}^{s}\frac{1}{\sigma_i\sigma_i'}(\sigma_i-\sigma_i')R' \\
&= \text{meas}\left\{\left[\frac{1}{\sigma_0}R', \frac{\sigma_0+1}{2\sigma_0}R'\right]-E\right\} \\
&\geq \left(\frac{\sigma_0+1}{2\sigma_0}-\frac{1}{\sigma_0}\right)R'-\frac{\sigma_0-1}{4\sigma_0}R' \\
&= \frac{\sigma_0-1}{4\sigma_0}R'. \qquad (3.11)
\end{aligned}$$

If the lemma were not true, then for any σ satisfying (3.9), if $\sigma^{-1}R' \in \left[\frac{1}{\sigma_0}R', \frac{\sigma_0+1}{2\sigma_0}R'\right] - E$, we would have

$$\sigma R' \in \left[\frac{2\sigma_0}{\sigma_0+1}R', \sigma_0 R'\right]\cap E.$$

Thus from (3.10),

$$\bigcup_{i=1}^{s}(\sigma_i'R', \sigma_i R') \subset \left[\frac{2\sigma_0}{\sigma_0+1}R', \sigma_0 R'\right]\cap E.$$

Therefore from (3.11),

$$\begin{aligned}
\text{meas}\, E &\geq \text{meas}\left\{\bigcup_{i=1}^{s}(\sigma_i'R', \sigma_i R')\right\} \\
&= \sum_{i=1}^{s}(\sigma_i-\sigma_i')R' \\
&\geq \frac{4\sigma_0{}^2}{(\sigma_0+1)^2}\cdot\frac{\sigma_0-1}{4\sigma_0}R' = \frac{\sigma_0(\sigma_0-1)}{(\sigma_0+1)^2}R' \\
&> \frac{\sigma_0-1}{4\sigma_0}R'.
\end{aligned}$$

This contradicts condition (3), and the lemma is proved.

Lemma 3.5 [7, Lemma I]**.** *Let $W(z)$ be analytic, except for poles $c_1, c_2, \cdots, c_t$ in the sector*

$$S = \left\{ z : \frac{r}{\sigma} \leq |z| \leq \sigma r, \, |\arg z - \xi| \leq \gamma \right\},$$

where $\xi \in \mathbf{R}$ and $r > 0$, $\sigma > 1$, $0 < \gamma \leq \pi$, and $2 \log \sigma \leq \pi\gamma$. Let $|W(z)| \leq 1$ on the boundary of S. Write J for the interval of θ's

$$J = \{ \theta : |\theta - \xi| \leq (1 - 2\delta)\gamma \}$$

where $0 < \delta < \dfrac{1}{4}$. Then for $\theta \in J$,

$$\log\left|W(re^{i\theta})\right| \leq -\frac{64\delta^6}{\log \sigma} \exp\left(-\frac{16\pi\gamma}{\log \sigma}\right) m\left(r, \frac{1}{W}; J\right) + \log\left|\frac{(2\gamma\sigma)^t}{\prod_{l=1}^{t} \left(re^{i\theta} - c_l\right)}\right|.$$

Lemma 3.6. *Let $f(z)$, $a_1(z)$ and $a_2(z)$ be meromorphic functions. Let the function $S(r)$ defined in $(0, \infty)$ be positive and nondecreasing and satisfy*

$$T(r, a_k) = o\{S(r)\}, \; r \to \infty, \quad k = 1, 2. \tag{3.12}$$

Then

$$\lim_{r \to \infty} \operatorname{meas}\left\{ E_S\big(r, a_1(z), f\big) \cap E_S\big(r, a_2(z), f\big) \right\} = 0, \tag{3.13}$$

where

$$E_S\big(r, a_k(z), f\big) = \left\{ \theta \in C : \left|f(re^{i\theta}) - a_k(re^{i\theta})\right| < e^{-S(r)} \right\}, \quad k = 1, 2.$$

Proof. Denote $E_S\big(r, a_1(z), f\big) \cap E_S\big(r, a_2(z), f\big)$ by $J(r)$. If $\theta \in J(r)$, we have

$$\left|a_1(re^{i\theta}) - a_2(re^{i\theta})\right| \leq \left|f(re^{i\theta}) - a_1(re^{i\theta})\right| + \left|f(re^{i\theta}) - a_2(re^{i\theta})\right| < 2e^{-S(r)}.$$

So by Nevanlinna's first fundamental theorem (see (1.3)),

$$
\begin{aligned}
S(r)\cdot \operatorname{meas} J(r) &\le \int_0^{2\pi}\left(\log\frac{1}{|a_1(re^{i\theta})-a_2(re^{i\theta})|}+\log 2\right)d\theta\\
&\le 2\pi\, m\left(r,\frac{1}{a_1-a_2}\right)+2\pi\log 2\\
&\le 2\pi\left(T(r,a_1-a_2)+\log^+\frac{1}{|C_0|}+\log 2\right)\\
&\le 2\pi\left(T(r,a_1)+T(r,a_2)+\log^+\frac{1}{|C_0|}+2\log 2\right),
\end{aligned}
$$

where C_0 is the first nonzero coeficient of the Laurent expansion of $a_1(z)-a_2(z)$ at $z=0$. Then by (3.12), (3.13) follows at once.

§3.3 Proof of Theorem 3.1

Now we prove Theorem 3.1. Basically, we follow Oum's idea, but we have to surmount some difficulties in a few key steps. We just need to prove the first conclusion. The proof of any other one is analogous to that of the corresponding one in Theorem 3.A (see [23]).

Let $[\alpha_j, \beta_j]$, $j = 1, 2, \cdots, q$, be all intervals of positive length on which $h(\theta) \equiv 0$. As in [23], consider the finite set X formed by all the following points:

(1) the $2q$ points $\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_q, \beta_q$;

(2) the points, if they exist, where $h(\theta) = 0$ and $\theta \notin \bigcup_{j=1}^{q} [\alpha_j, \beta_j]$.

Assume that X has p distinct elements. We prove the theorem in two cases.

Case 1: $p > 0$

Let $\theta_1, \theta_2, \cdots, \theta_p$ ($\theta_1 < \theta_2 < \cdots < \theta_p < \theta_1 + 2\pi = \theta_{p+1}$) be some circular arrangement of all elements in X and choose η satisfying

$$0 < \eta < \min\left\{\frac{2}{\pi}\log\frac{3}{2},\ \frac{1}{4}\min_{1\le j\le p}\{\theta_{j+1} - \theta_j\}\right\}. \tag{3.14}$$

Put

$$\Lambda = \bigcup_{j=1}^{p} [\theta_j - \eta, \theta_j + \eta]. \tag{3.15}$$

Then the set $C - \Lambda$ is the union of three disjoint subsets:

(1) O_+ on which $h(\theta) > 0$;

(2) O_- on which $h(\theta) < 0$;

(3)

$$\mathcal{R} = \bigcup_{j=1}^{q} [\alpha_j + \eta, \beta_j - \eta]. \tag{3.16}$$

Assume that there exist $q+1$ distinct meromorphic functions $a_1(z)$, $a_2(z)$,$\cdots$,$a_{q+1}(z)$, none of which is 0 or ∞, satisfying

$$\lim_{r\to\infty} \frac{T(r, a_k)}{T(r, f)} = 0 \tag{3.17}$$

such that $\delta\big(a_k(z), f\big) > 0,\ k = 1, 2, \cdots, q+1$. Put

$$4\kappa = \min_{1\le k\le q+1} \big\{\delta\big(a_k(z), f\big)\big\}, \tag{3.18}$$

and choose η in (3.15) so small that

$$m\left(r, \frac{1}{f - a_k}; \Lambda\right) < \kappa T(r, f),\ k = 1, 2, \cdots, q+1 \tag{3.19}$$

whenever r is sufficiently large. This is certainly possible because by Lemma 1, Nevanlinna's first fundamental theorem (see (1.3)), (3.8), (3.15), and (3.17), we have

$$\begin{aligned} m\left(r, \frac{1}{f - a_k}; \Lambda\right) &\le 23\, T(2r, f)\, \mu \left(1 + \log^+ \frac{1}{\mu}\right) \\ &\le 23(2^\lambda + 1)\, T(r, f)\, \mu \left(1 + \log^+ \frac{1}{\mu}\right), \end{aligned}$$

where $\mu = \operatorname{meas} \Lambda = 2p\eta$ and r is sufficiently large.

Now for any $\epsilon > 0$, from (3.3),

$$\big|\log|f(re^{i\theta})| - h(\theta)V(r)\big| < \epsilon V(r) \tag{3.20}$$

uniformly in $\theta \in C$ if $r \notin E$ is sufficiently large.

Choose R to be large enough. If $\{c_l\}_k$ is the set of the poles of $a_k(z)$ in $|z| < 3R$, then by the Poisson-Jensen formula (see (1.2)), for any $r < 3R$,

$$\log\left|a_k(re^{i\theta})\right| \leq \frac{3R+r}{3R-r}m(3R,a_k) + \log\frac{(6R)^{n(3R,a_k)}}{\prod_{l=1}^{n(3R,a_k)}\left|re^{i\theta}-c_l\right|}, \quad k = 1,2,\cdots,q+1.$$

Moreover, by the Boutroux-Cartan theorem (see the section 1.1),

$$\prod_{l=1}^{n(3R,a_k)} |z-c_l| > \left(\frac{\eta m R}{32e(q+1)\exp(\pi\eta/2)}\right)^{n(3R,a_k)}, \tag{3.21}$$

for all z at most outside the union of a sequence $\{D_l\}_k$ of finitely many disks whose number does not exceed $n(3R,a_k)$ and in which the sum of the diameters does not exceed $\dfrac{\eta m R}{8(q+1)\exp(\pi\eta/2)}$. Here $m = \min\{\exp(\pi\eta/2)-1,\ 1\}$. Hence if we write

$$\Gamma_k = \left\{ r \in (0,\infty) : \text{there exists a } \theta \in C \text{ such that } re^{i\theta} \in \{D_l\}_k \right\},$$

then we have

$$\operatorname{meas}\Gamma_k \leq \frac{\eta m R}{8(q+1)\exp(\pi\eta/2)}, \quad k = 1,2,\cdots,q+1.$$

Further, if we denote the constant $\log\dfrac{192e(q+1)\exp\pi\eta/2}{\eta m}$ by $C(q,\eta)$, then, for $r < 3R$, $r \notin \Gamma_k$ and any θ, we have

$$\log\left|a_k(re^{i\theta})\right| \leq \frac{3R+r}{3R-r}m(3R,a_k) + n(3R,a_k)C(q,\eta), \quad k = 1,2,\cdots,q+1. \tag{3.22}$$

Similarly, for any θ,

$$\log\frac{1}{\left|a_k(re^{i\theta})\right|} \leq \frac{3R+r}{3R-r}m\left(3R,\frac{1}{a_k}\right) + n\left(3R,\frac{1}{a_k}\right)C(q,\eta) \tag{3.23}$$

for all $r < 3R$ outside a set Γ'_k whose measure does not exceed $\dfrac{\eta m R}{8(q+1)\exp(\pi\eta/2)}$, $k = 1,2,\cdots,q+1$. Therefore, since the density of E is zero and

$$\operatorname{meas}\left\{\bigcup_{k=1}^{q+1}(\Gamma_k \cup \Gamma'_k)\right\} \leq \frac{\eta m R}{4\exp(\pi\eta/2)},$$

we find that

$$(R,2R)-\left[E\bigcup\left(\bigcup_{k=1}^{q+1}(\Gamma_k\cup\Gamma_k')\right)\right]\neq\emptyset.$$

Choose $R'\in(R,2R)-\left[E\bigcup\left(\bigcup_{k=1}^{q+1}(\Gamma_k\cup\Gamma_k')\right)\right]$. By the continuity of $h(\theta)$, $\delta_+=\inf_{\theta\in O_+}h(\theta)>0$. So by (3.20) and $R'\notin E$, for $\theta\in O_+$,

$$\begin{aligned}\left|f(R'e^{i\theta})\right| &> \exp\left\{\left(h(\theta)-\epsilon\right)V(R')\right\}\\ &\geq \exp\left\{(\delta_+-\epsilon)V(R')\right\}\\ &\geq \exp\left(\frac{\delta_+}{2}V(R')\right).\end{aligned}$$

Thus from (3.22), (3.7), (3.8), and (3.17), for $\theta\in O_+$,

$$\begin{aligned}\left|f(R'e^{i\theta})-a_k(R'e^{i\theta})\right| &\geq \left|f(R'e^{i\theta})\right|-\left|a_k(R'e^{i\theta})\right|\\ &> \exp\left(\frac{\delta_+}{2}V(R')\right)-\exp\left\{5\,m(3R,a_k)+n(3R,a_k)C(q,\eta)\right\}\\ &> \exp\left(\frac{\delta_+}{2}V(R')\right)-\exp\left(\frac{\delta_+}{4}V(R)\right)\\ &> \exp\left(\frac{\delta_+}{4}V(R')\right).\end{aligned}$$

Therefore

$$\log\frac{1}{\left|f(R'e^{i\theta})-a_k(R'e^{i\theta})\right|}<-\frac{\delta_+}{4}V(R').$$

So we obtain

$$m\left(R',\frac{1}{f-a_k};O_+\right)=0,\quad k=1,2,\cdots,q+1. \tag{3.24}$$

Similarly, by $\delta_-=-\sup_{\theta\in O_-}h(\theta)>0$, (3.20) and $R'\notin E$, for $\theta\in O_-$,

$$\begin{aligned}\left|f(R'e^{i\theta})\right| &< \exp\left\{\left(h(\theta)+\epsilon\right)V(R')\right\}\\ &\leq \exp\left\{(-\delta_-+\epsilon)V(R')\right\}\\ &\leq \exp\left(-\frac{\delta_-}{2}V(R')\right).\end{aligned}$$

Then from (3.23), (3.7), (3.8), and (3.17), for $\theta \in O_-$,

$$\begin{aligned}\left|f(R'e^{i\theta}) - a_k(R'e^{i\theta})\right| &\geq \left|a_k(R'e^{i\theta})\right| - \left|f(R'e^{i\theta})\right| \\ &\geq \exp\left\{-5\,m\left(3R', \frac{1}{a_k}\right) - n\left(3R', \frac{1}{a_k}\right) C(q,\eta)\right\} \\ &\quad - \exp\left(-\frac{\delta_-}{2} V(R')\right) \\ &> \exp\left\{-10\,m\left(3R', \frac{1}{a_k}\right) - 2n\left(3R', \frac{1}{a_k}\right) C(q,\eta)\right\}.\end{aligned}$$

Therefore

$$\log^+ \frac{1}{\left|f(R'e^{i\theta}) - a_k(R'e^{i\theta})\right|} < 10\,m\left(3R', \frac{1}{a_k}\right) + 2\,n\left(3R', \frac{1}{a_k}\right) C(q,\eta).$$

We obtain

$$m\left(R', \frac{1}{f - a_k}; O_-\right) < \kappa\, T(R', f), \quad k = 1, 2, \cdots, q+1. \tag{3.25}$$

Combining (3.24) and (3.25), we deduce that

$$m\left(R', \frac{1}{f - a_k}; O_+ \cup O_-\right) < \kappa\, T(R', f), \quad k = 1, 2, \cdots, q+1. \tag{3.26}$$

So, by the definition of deficiency, (3.16), (3.18), (3.19) and (3.26),

$$m\left(R', \frac{1}{f - a_k}; \mathcal{R}\right) > \kappa\, T(R', f), \quad k = 1, 2, \cdots, q+1.$$

Hence we know that $q > 0$. So by (3.16), for any $a_k(z)$, $1 \leq k \leq q+1$, there exists an interval $[\alpha_{j_k} + \eta, \beta_{j_k} - \eta]$, $1 \leq j_k \leq q$, such that

$$m\left(R', \frac{1}{f - a_k}; [\alpha_{j_k} + \eta, \beta_{j_k} - \eta]\right) > \frac{\kappa}{q} T(R', f).$$

Since there are $q+1$ deficient functions and q intervals, there exist $a_{k_1}(z)$, $a_{k_2}(z)$ and an interval $[\alpha_{j_0} + \eta, \beta_{j_0} - \eta]$ $(1 \leq k_1, k_2 \leq q+1,\ 1 \leq j_0 \leq q)$, denoted simply by $a_1(z)$, $a_2(z)$ and $[\alpha_1 + \eta, \beta_1 - \eta]$, respectively, such that

$$m\left(R', \frac{1}{f - a_k}; [\alpha_1 + \eta, \beta_1 - \eta]\right) > \frac{\kappa}{q} T(R', f), \quad k = 1, 2. \tag{3.27}$$

Now noting that $\Gamma_1 \cup \Gamma_2$ is a union of finitely many closed intervals and

$$\text{meas}\{\Gamma_1 \cup \Gamma_2\} \leq \frac{\eta m R}{8\exp(\pi\eta/2)} < \frac{\exp(\pi\eta/2)-1}{4\exp(\pi\eta/2)} R',$$

we use Lemma 3.4 and find that there exists a σ satisfying

$$\frac{2\exp(\pi\eta/2)}{\exp(\pi\eta/2)+1} \leq \sigma \leq \exp(\pi\eta/2) \tag{3.28}$$

such that $\sigma^{-1}R'$, $\sigma R' \notin \Gamma_1 \cup \Gamma_2$. Moreover, since the sum of diameters of the disks in $\{D_l\}_1 \cup \{D_l\}_2$ does not exceed

$$\frac{\eta m R}{8\exp(\pi\eta/2)} < \frac{R'}{\sigma} \cdot \frac{\eta}{4},$$

there exist $\theta_0 \in \left(\alpha_1 + \frac{\eta}{4}, \alpha_1 + \frac{\eta}{2}\right)$ and $\theta_0' \in \left(\beta_1 - \frac{\eta}{2}, \beta_1 - \frac{\eta}{4}\right)$ such that $re^{i\theta_0}$, $re^{i\theta_0'} \notin \{D_l\}_1 \cup \{D_l\}_2$ for any $r \in [\sigma^{-1}R', \sigma R']$.

Now consider the sector

$$\mathbf{S} = \left\{ z : \frac{R'}{\sigma} \leq |z| \leq \sigma R', \left|\arg z - \frac{\theta_0 + \theta_0'}{2}\right| \leq \frac{\theta_0' - \theta_0}{2} \right\}.$$

By the discussion above, $\partial\mathbf{S} \cap \left(\{D_l\}_1 \cup \{D_l\}_2\right) = \emptyset$, where $\partial\mathbf{S}$ denotes the boundary of $\mathbf{S}$. And by (3.14) and (3.28),

$$\mathbf{S} \subset \{ z : |z| < 3R \}. \tag{3.29}$$

So by (3.22), (3.7), (3.8) and (3.17), if $z = re^{i\theta} \in \partial\mathbf{S}$, then

$$\begin{aligned} \log\left|a_k(re^{i\theta})\right| &\leq 5\, m(3R, a_k) + n(3R, a_k) C(q, \eta) \\ &< \epsilon V(R) < \epsilon V(R'), \quad k = 1, 2. \end{aligned} \tag{3.30}$$

Moreover

$$\log\left|f(re^{i\theta})\right| < \epsilon V(r)$$

uniformly in $\theta \in \left[\alpha_1 + \frac{\eta}{4}, \beta_1 - \frac{\eta}{4}\right]$ if r is sufficiently large. So by (3.30) and (3.1), for $z = re^{i\theta} \in \partial\mathbf{S}$,

$$\begin{aligned}\log\left|f(re^{i\theta}) - a_k(re^{i\theta})\right| &\le \log^+\left|f(re^{i\theta})\right| + \log^+\left|a_k(re^{i\theta})\right| + \log 2\\ &< \epsilon V(r) + \epsilon V(R') + \log 2\\ &< 3\epsilon V(\sigma R')\\ &< 3\epsilon(\sigma^\lambda + 1)V(R'), \quad k = 1, 2.\end{aligned}$$

Therefore

$$W_k(z) = \big(f(z) - a_k(z)\big)\exp\left\{-3\epsilon(\sigma^\lambda + 1)V(R')\right\} \tag{3.31}$$

is analytic in $\mathbf{S}$ except for finitely many poles and $|W_k(z)| < 1$ for $z \in \partial\mathbf{S},\ k = 1, 2$. Letting

$$\xi = \frac{\theta_0 + \theta_0'}{2}, \ \gamma = \frac{\theta_0' - \theta_0}{2}, \ \delta = \frac{\eta}{2(\theta_0' - \theta_0)}, \tag{3.32}$$

we have from (3.14) and (3.28) that

$$\delta < \frac{1}{4}, \quad 2\log\sigma \le \pi\gamma.$$

By Lemma 3.5 it follows that for any $\theta \in \left[\theta_0 + \frac{\eta}{2}, \theta_0' - \frac{\eta}{2}\right]$

$$\begin{aligned}\log\left|W_k(R'e^{i\theta})\right| \le -\kappa_0\, m&\left(R', \frac{1}{W_k}; \left[\theta_0 + \frac{\eta}{2}, \theta_0' - \frac{\eta}{2}\right]\right)\\ &+ \log\left|\frac{(2\sigma R')^{t_k}}{\prod_{l=1}^{t_k}(R'e^{i\theta} - c_{k,l})}\right|,\end{aligned} \tag{3.33}$$

where

$$\kappa_0 = \frac{64\delta^6}{\log\sigma}\exp\left(-\frac{16\pi\gamma}{\log\sigma}\right) \tag{3.34}$$

and $\{c_{k,l}\}_{l=1}^{t_k}$ is the set of poles of $a_k(z)$ in $\mathbf{S},\ k = 1, 2$. Since $R'e^{i\theta} \notin \{D_l\}_1 \cup \{D_l\}_2$, and (3.29) and (3.21) hold,

$$(6R)^{n(3R,a_k)-t_k} \cdot \prod_{l=1}^{t_k}\left|R'e^{i\theta} - c_{k,l}\right| \ge \prod_{l=1}^{n(3R,a_k)}\left|R'e^{i\theta} - c_{k,l}\right|$$

$$> \left(\frac{\eta m R}{32e(q+1)\exp(\pi\eta/2)} \right)^{n(3R,a_k)}, \quad k = 1, 2.$$

Hence

$$\prod_{l=1}^{t_k} \left| R' e^{i\theta} - c_{k,l} \right| > \left(\frac{\eta m}{192e(q+1)\exp(\pi\eta/2)} \right)^{n(3R,a_k)} \cdot (6R)^{t_k}, \quad k = 1, 2.$$

By (3.33) and $\sigma R' < 3R$,

$$\log \left| W_k(R' e^{i\theta}) \right| < -\kappa_0 \, m \left(R', \frac{1}{W_k}; \left[\theta_0 + \frac{\eta}{2}, \theta_0' - \frac{\eta}{2} \right] \right) + n(3R, a_k) C(q, \eta). \qquad (3.35)$$

Clearly

$$\log^+ \frac{1}{|W_k(z)|} \geq \log^+ \frac{1}{|f(z) - a_k(z)|}$$

for any z. So we have

$$m \left(R', \frac{1}{W_k}; \left[\theta_0 + \frac{\eta}{2}, \theta_0' - \frac{\eta}{2} \right] \right) \geq m \left(R', \frac{1}{f - a_k}; \left[\theta_0 + \frac{\eta}{2}, \theta_0' - \frac{\eta}{2} \right] \right). \qquad (3.36)$$

From (3.34), (3.32) and (3.28), we see that

$$\kappa_0 > \kappa_1 = \frac{2}{\pi(\beta_1 - \alpha_1)} \cdot \exp \left\{ -8\pi(\beta_1 - \alpha_1) \left(\log \frac{2\exp(\pi\eta/2)}{\exp(\pi\eta/2) + 1} \right)^{-1} \right\}.$$

Noting that $[\alpha_1 + \eta, \beta_1 - \eta] \subset \left[\theta_0 + \frac{\eta}{2}, \theta_0' - \frac{\eta}{2} \right]$, we have by (3.31), (3.35), (3.36), (3.7), (3.8) and (3.17), that for $\theta \in [\alpha_1 + \eta, \beta_1 - \eta]$

$$\begin{aligned} \log \left| f(R' e^{i\theta}) - a_k(R' e^{i\theta}) \right| &< -\kappa_1 \, m \left(R', \frac{1}{f - a_k}; [\alpha_1 + \eta, \beta_1 - \eta] \right) \\ &\quad + 3\epsilon \left((\exp(\pi\eta/2))^{\lambda} + 1 \right) V(R') + n(3R', a_k) C(q, \eta) \\ &< -\kappa_1 \, m \left(R', \frac{1}{f - a_k}; [\alpha_1 + \eta, \beta_1 - \eta] \right) \\ &\quad + 4\epsilon \left((\exp(\pi\eta/2))^{\lambda} + 1 \right) V(R'). \end{aligned}$$

Hence by (3.27) and (3.7), for any $\theta \in [\alpha_1 + \eta, \beta_1 - \eta]$,

$$\log\left|f(R'e^{i\theta}) - a_k(R'e^{i\theta})\right| < \left\{-\frac{\kappa\kappa_1}{q} + 4\epsilon\left(\left(\exp(\pi\eta/2)\right)^{\lambda} + 1\right)\cdot\frac{2}{H_+}\right\}T(R',f)$$
$$< -\frac{\kappa\kappa_1}{2q}T(R',f), \quad k = 1,2.$$

Equivalently

$$\left|f(R'e^{i\theta}) - a_k(R'e^{i\theta})\right| < \exp\left\{-\frac{\kappa\kappa_1}{2q}T(R',f)\right\}, \quad k = 1,2.$$

Setting $S(r) = \dfrac{\kappa\kappa_1}{2q}T(r,f)$,

$$E_S\big(R', a_1(z), f\big) \cap E_S\big(R', a_2(z), f\big) \supset [\alpha_1 + \eta, \beta_1 - \eta].$$

Therefore,

$$\operatorname{meas}\left\{E_S\big(R', a_1(z), f\big) \cap E_S\big(R', a_2(z), f\big)\right\} \geq 2\eta. \tag{3.37}$$

Here $2\eta > 0$ is a constant, and $R' > R$ can be made arbitrarily large. So (3.37) contradicts Lemma 3.6. This completes the proof in the case 1.

Case 2: $p = 0$

In this case $h(\theta) > 0$ for all $\theta \in C$. That means that $C = O_+$. As in (3.24), for any meromorphic function $a(z) \not\equiv 0, \infty$ satisfying (3.6), we have

$$m\left(R', \frac{1}{f-a}\right) = 0.$$

Then, since R' can be made arbitrarily large, we have $\delta\big(a(z), f\big) = 0$. The theorem is proved in this case.

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VITA

Jiuyi Cheng was born in Nanchang, Jiangxi province, P. R. China on February 7, 1962. He received the Bachelor of Science degree in mathematics from East China Normal University in July, 1984, and his Master of Science degree in mathematics from the same university in July, 1987. During 1987-1989, he served as an instructor in the department of mathematics, Shanghai College of Education.

He entered the Ph.D graduate program in mathematics at Virginia Polytechnic Institute and State University in August of 1989, and obtained the Ph.D degree in January, 1994.

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