

ESSAYS ON BARGAINING THEORY AND VOTING BEHAVIOR

by

Wen Mao

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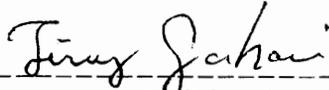
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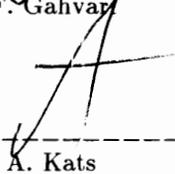
M. Kaneko, Chairman



F. Gahvari



Y. Ioannides



A. Kats



B. Lebrun

August 1994

Blacksburg, Virginia

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Wen Mao

Committee Chairman: Mamoru Kaneko

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(ABSTRACT)

This dissertation consists of four chapters, and the first chapter gives a brief summary of the dissertation. Chapters 2 and 3 analyze the Nash bargaining problem with endogenously determined threats in the n -person case. Chapter 4 discusses seemingly inconsistent behavior in voting for incumbents and for term limitation.

In Chapter 2, we provide two bargaining models with variable threats in the n -person case: commitments to threat strategies are available and required in the first model, but they are not available in the second one. We formulate these two models as extensive games, and give two sets of axioms for the solution concepts in these models. The two sets of axioms uniquely determine the Nash bargaining solution relative to an equilibrium threat and relative to a Nash equilibrium of the underlying noncooperative game, respectively, in the first and second models. Furthermore, we show that when the number of players is large, in a game with sidepayments, the difference between the two models becomes insignificant.

In Chapter 3, we discuss the voluntary decisions of commitments to threat strategies in Nash bargaining. Players are allowed to choose whether or not to commit to threat strategies. We consider three possible extensions where the voluntary decisions of commitments are included. In the two-person case, both players choose to commit to threat strategies voluntarily in all three extensions. However, an example is given to show that this is not the case for more than two players; the result depends upon an extension.

In Chapter 4, we try to explain seemingly inconsistent behavior in voting for incumbents and for term limitation. Voters in several states voted recently for the introduction of a term limit, while at the same time, re-electing a large percentage of their incumbents. We formulate the candidate election within a district as a two-person game and integrate the results of the election games over all districts in a state. By doing so, we explain the voting behavior of voters in the candidate election and in the term limitation election, based on the assumption that voters are self-interested in both elections.

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Chapter 1. Summary of the Dissertation

The dissertation consists of four chapters, and this chapter gives a brief summary of the dissertation. Chapters 2 and 3 analyze the Nash bargaining problem with endogenously determined threats in the n person case. Chapter 4 discusses seemingly inconsistent behavior in voting for incumbents and term limitation. In the following sections, we summarize the contents of those chapters.

1.1 Background of the Nash Bargaining Problem and Brief Summary of Chapters 2 and 3

The *Nash bargaining problem* was first introduced in Nash (1950). Two players bargain over a set of feasible outcomes. They can communicate and obtain mutual benefits. If they agree on a particular outcome, it becomes the final outcome of the bargaining game; otherwise, they end up with an exogenously given outcome, called a (fixed) *disagreement point*. A solution concept, called the *Nash bargaining solution*, is derived from certain axioms. The bargaining process is treated as a two-person cooperative game, and Nash's method is known as the axiomatic or cooperative approach. The existence and uniqueness of the Nash bargaining solution are obtained in the two-person case, but can be directly extended to the n -person case.

Nash (1953) described the above bargaining problem as a noncooperative game by specifying players' moves in negotiation. The major contribution of Nash (1953) is not only the noncooperative view of the bargaining problem but also the endogenous determination of the disagreement point. Bargaining is formulated as a two-stage game. In the first stage, each player chooses a threat strategy which would be used if the two players do not reach an agreement in the second stage, and they inform each other of their threats simultaneously. Thus the disagreement point is endogenously determined as a (variable) *threat point*. In the second stage, each player, without communication, demands a utility. If the demands by both players are feasible, each player gets what he demanded. Otherwise, they end up with the threat point chosen in the first

stage. Note that commitments to threat strategies are required in this model: players have to play the threat strategies if they disagree. This is known as the strategic or noncooperative approach. The two approaches are complementary.

In many economic situations, it is difficult to find a natural definition of an exogenously given disagreement point. The endogenous determination of a threat point allows much wider applications of the bargaining theory than the bargaining theory with exogenously given disagreement points. As was mentioned, in Nash (1953), however, players are presumed to commit to threat strategies in the first stage of the bargaining process. This presumption has not yet been fully explored. A crucial requirement of the presumption is the enforcement of the threats when the disagreement actually occurs. The availability of a device for such enforcement should be considered. We also ask the question of whether or not all the players are willing to commit to threat strategies in this game. Furthermore, the Nash bargaining model with variable threats is well defined and understood only for two-person games, which certainly limits its application. In the next two chapters, we consider the problem in the n -person case, emphasizing the presumption of players' commitments to threat strategies.

In Chapter 2, we provide two bargaining models with variable threats in the n -person case. In the first model, the commitments to threat strategies are available and required. In the second one, the commitments are not available. A basic game environment is described as a noncooperative strategic game G , on which the two bargaining models are formulated. The first model is a direct extension of Nash's original 2-person bargaining model with variable threats to the n -person case. In the second model, the players bargain for a cooperative outcome without a prechosen disagreement point; if they do not reach an agreement, they go to the original game G and play it noncooperatively. The main difference between those two models is the availability of the commitments.

We formulate these two bargaining models in extensive games. Unlike Nash's original

axioms which are often regarded as belonging to cooperative game theory, our axioms do not involve the cooperative aspect of the axiomatic approach to the bargaining problem. The axiom of Pareto optimality is not required directly. It can be obtained from subgame perfect equilibrium together with other axioms. We show that in the first model, our axioms determine equilibrium threat strategies for the threat stage and the Nash bargaining outcome for the demand stage. In the second model, our axioms determine the Nash bargaining outcome relative to a Nash equilibrium in the case of disagreement.

It is also argued that the structure of equilibrium threats for a game with $n+1$ players may be as complex as the Nash equilibria for a noncooperative game with n players. For a constant-sum game G , we show that the set of Nash equilibrium for the underlying game G is identical to the set of final equilibrium outcomes in both models. Furthermore, in the last part of Chapter 2, we show that when the number of players is large, in a game with sidepayments, any equilibrium threat point becomes approximately a Nash equilibrium in the underlying noncooperative game, and *vice versa*. This result suggests that, when the number of players is large, commitments do not play an important role.

In Chapter 3, we discuss the voluntary decisions of commitments to threat strategies, instead of the availability of the commitments. Players are allowed to choose whether or not to commit to threat strategies. There is more than one way of extending Nash's model so that the voluntary decision of commitments is included: three possible extensions are considered in Chapter 3. In the two person case, all three extensions give essentially the same result as that given by Nash (1953). That is, both players choose to commit to threat strategies voluntarily. However, this is not the case for more than two players; the result depends upon an extension. In one extension, Nash's result always holds. In the other two extensions, however, we find a three person example where not all players voluntarily choose commitments in equilibrium. This counterexample shows that, in some game situations, some players would be better off not to

commit to threat strategies. Therefore the presumption of commitments may be violated when the commitment decision is voluntary.

Chapters 2 and 3 investigate the presumption of commitments to threat strategies in Nash bargaining problems from different perspectives. The results from each chapter support each other. The key to the Nash (1953) bargaining model is the threat. The results of Chapter 2, however, indicate that the threats become insignificant in games with a large number of players. Thus a question to be asked here is whether or not players in large games are willing to commit to threat strategies. According to the result of Chapter 3, the answer, in general, is no for games with more than two players.

1.2 Background of the Term Limitation Election and Brief Summary of Chapter 4

Chapter 4 tries to explain a seemingly inconsistent behavior in voting for incumbents and term limitation as a consistent behavior. This seemingly inconsistent voting behavior has been observed in recent years: for example, in 1990, voters in California, Colorado and Oklahoma voted for the introduction of a term limit, while at the same time, re-elected a large percentage of their incumbents, who possibly would have been prevented from running if the term limit would have been enacted. If the incumbent can bring the voters more benefits than a challenger and if the voters vote only based on the benefits they obtain, they would vote for an incumbent and vote against the introduction of a term limit. In Chapter 4, we show that such seemingly inconsistent behavior can occur even if voters pursue their self-interests in both the candidate and the term limitation elections.

First, the candidate election at the district level is formulated as a two-person game, where each candidate maximizes votes by proposing a distribution of benefits into two groups of voters. A voter votes for a candidate based on the economic benefits proposed and his political preferences. Since the game might not have a Nash equilibrium, we use maximin strategies of the

candidates as the outcomes of the game. Then we describe an incumbent's maximin strategies of dividing the benefits between two groups of voters. One result states that an incumbent wins the election since he can generate more total benefits. We show that the benefits that majority voters receive from their incumbent increase as the incumbent becomes more experienced, up to some critical level and after this critical level the benefits drop to a low level and stay low thereafter.

The introduction of a term limit in a state leads to an early occurrence of elections with two new candidates in all districts of the state. We assume that for a term limitation election, each voter compares the average benefits from the incumbent of his district over the terms he serves in that office in cases with, and without, a term limit. We show, in the last part of Chapter 4, the possibility that for a majority of voters in the state, the average benefits with a term limit are larger than those without a term limit. This is caused by the fact that the pursuit of votes by an incumbent may dilute benefits for a majority of voters. Thus those voters would vote for term limitation, while they still vote for the incumbent in the candidate elections. By formulating the candidate election within a district as a two-person game and integrating the results of the campaign games over all districts in a state, we reconcile the inconsistent behavior in voting for incumbents and term limitation.

Chapter 2.

N-Person Nash Bargaining Problems with Variable Threats

2.1 Introduction

An important constituent of bargaining is the specification of the payoff vector when players fail in achieving an agreement. This payoff vector is called a *disagreement point*. With the major exception of Nash (1953), a disagreement point is frequently assumed to be exogenously given.¹ In economic examples, disagreement points such as an endowment point in an exchange economy are often exogenously but naturally determined. In general game situations and in economic examples, e.g., ones with externalities, however, we may not find a natural definition of an exogenously given disagreement point. This fact requires us to consider a bargaining model with an endogenous determination of a disagreement point. Nash (1953) gave a model of bargaining with two players and the endogenous determination of a disagreement point. The purpose of this chapter is to investigate the behavior of this model but with n-players. We compare, from both axiomatic and noncooperative game theoretic viewpoints, the n-player extension of Nash's model with an alternative model where a disagreement point is also endogenously determined.

Nash's (1953) model -- the model described in Section 2.2.2 -- is as follows. A basic game environment is described by a noncooperative game G . The players are allowed to cooperate for obtaining higher payoffs, and they bargain over possible cooperative payoffs. Bargaining may result in disagreement. If disagreement occurs, the players return to the original noncooperative game environment. Each player has to choose a strategy for the event of disagreement, prior to bargaining for cooperative payoffs. The model has two stages: each player chooses a strategy and

¹The role of a disagreement point has extensively been studied in axiomatic bargaining theory within the assumption of exogenously given disagreement point, cf., Chun-Thomson (1990), Peters-Van Damme (1991) and their references.

announces it to the other players -- *threat stage*; and then the players bargain for cooperative payoffs -- *demand stage*. In the threat stage, a player threatens his opponents with a strategy to be played in the case of disagreement so as to gain an advantage in the demand stage. Here it is presumed that each player plays his threat strategy in the case of disagreement; a commitment to play the threat strategy is required. A commitment is, however, not always possible. When a commitment is not available, a player may not use his threat strategy if disagreement actually occurs. The second model we consider in Section 2.2.3 describes such a situation.

In the second model the players bargain for cooperative payoffs without a prechosen disagreement outcome; if they do not agree on a cooperative outcome, they go to the original game G and play it noncooperatively. This means that in the first stage, the players bargain for cooperative payoffs; and if they do not achieve an agreement, then they go to the second stage of the noncooperative game G . The main difference between these two models is that commitments to play threat strategies are available in the first but not in the second. The primary purpose of the introduction of the second model is to investigate the behavior of the first model by making comparisons between them.

We formulate these two models as extensive games. Solutions are strategy combinations in these extensive games that satisfy the subgame perfect equilibrium property (Selten (1975)) and Nash's axioms except Pareto optimality -- the Subgame Perfectness Axiom together with the other axioms implies Pareto optimality. Since our axiomatizations can be regarded as ones in noncooperative game theory, they give a unified view to the axiomatic and noncooperative approaches.

We show that in the first model, our axioms determine equilibrium threat strategies for the threat stage and the Nash bargaining outcome for the demand stage. In the second model, the corresponding axioms determine the Nash bargaining outcome relative to a Nash equilibrium in the case of disagreement. These axiomatizations and results are discussed in Section 2.2. In

Section 2.3, we compare these axiomatizations with Nash's (1950) axiomatization for bargaining problems with fixed disagreement points. Using these comparisons, we prove the results of Section 2.2.

In the two person case, the first model gives a unique equilibrium payoff outcome, and moreover, it becomes a strictly competitive game when we restrict our attention to the threat stage. In Section 2.4, we show that these properties are not preserved for games with more than two players. We show that for a constant-sum game G , the set of Nash equilibria for G is identical to the set of final equilibrium outcomes in both models. This implies that the structure of equilibrium threats for a game with $n+1$ players may be as complex as the Nash equilibria for a noncooperative game with n players.

In Section 2.5, we consider the behavior of the solutions for both models with a large number of players. We show that for a large game with sidepayments, any equilibrium threat point becomes approximately a Nash equilibrium in the underlying noncooperative game, and *vice versa*. In the first model, a change in a player's threat strategy affects his own final payoff in two ways: directly, by guaranteeing a higher payoff to himself and indirectly by threatening the other players. Our theorem implies that when the number of players becomes larger, the indirect effect becomes negligible but the direct effect remains of a similar magnitude. This result suggests that the difference between the two models becomes insignificant when the number of players is large.

2.2 Two Models of Bargaining with Endogenous Determination of Disagreement Points

In this section, we present two models of n -person bargaining where disagreement points are endogenously chosen. The first model, called the *variable threat model*, is the direct generalization of Nash's (1953) original one with two players. The second model is one where in the case of disagreement, the players play the underlying noncooperative game G without prechosen threat strategies. These two models differ in that players commit to play threat

strategies in the first but not in the second. We formulate these models as extensive games. In these extensive games, we provide axioms, corresponding to Nash's (1950, '53), for solution concepts.

2.2.1 Underlying Noncooperative Games and Bargaining Problems

We denote a finite n -person game by $G = (N, \{X_i\}, \{h_i\})$, where $N = \{1, \dots, n\}$ ($n \geq 2$) is the player set and for each $i \in N$, X_i is a finite set of pure strategies and $h_i: \prod_{i \in N} X_i \rightarrow R$ is a payoff function. The class of all finite games with a fixed player set N is denoted by \mathcal{G} . We denote the set of all mixed strategies of player i in game G by $M_i(G)$, and denote the product $\prod_{i \in N} M_i(G)$ by $M(G)$. The set of all jointly feasible payoff vectors is denoted by $F(G)$, i.e., $F(G)$ is the convex hull of the set $\{h(x): x \in \prod_{i \in N} X_i\}$. We call $F(G)$ the *feasible region* generated by game G . This means that the players are allowed to coordinate their mixed strategies without any transfer of goods. The finiteness assumption of the pure strategy space X_i is made for simplicity.

A *bargaining problem* is given as a pair (G, T) of an n -person game $G = (N, \{X_i\}, \{h_i\})$ and a compact convex subset T of R^n with $F(G) \subseteq T$. The set T is called the *bargaining region*. We denote the set of all bargaining games (G, T) by \mathcal{B} . The bargaining region T coincides with the feasible region $F(G)$ when the cooperative payoff vectors in T are obtained purely from the coordination of mixed strategies. When some transfer of a commodity is allowed in addition to the coordination of their mixed strategies, the bargaining region T is larger than the feasible region $F(G)$ in general. One example is a *bargaining problem with sidepayments*: the bargaining region T is defined by

$$T = \{u \in R^n: \sum_i u_i \leq \max_{p \in M(G)} \sum_i h_i(p) \text{ and } u_i \geq \min_{p \in M(G)} h_i(p) \text{ for all } i \in N\}. \quad (2.1)$$

In this case, the players maximize the total utility measured by money (a composite good) and

may transfer some amount of money among them. The boundary condition $u_i \geq \min_{\rho \in M(G)} h_i(\rho)$ for all $i \in N$ is a kind of arbitrary choice. Bargaining problems with sidepayments and large numbers of players will be considered in Section 2.5.

2.2.2 Bargaining Model with Variable Threats

We associate an extensive game $\Gamma(G,T)$ with each bargaining problem (G,T) in \mathbb{B} , which we call the *associated bargaining game* (with commitments). The associated bargaining game $\Gamma(G,T)$ has two stages - Stage 1 is the *threat stage* and Stage 2 is the *demand stage*. The game tree of $\Gamma(G,T)$ is described in Figure 2.1.

Stage 1: Each player $i \in N$ chooses a (mixed) strategy ρ_i from $M_i(G)$ simultaneously, and announces it to the other players.

Stage 2: Each player $i \in N$ simultaneously chooses a utility demand u_i from R . If the vector $u = (u_1, \dots, u_n)$ belongs to the bargaining region T , then the final outcome is $u = (u_1, \dots, u_n)$, and otherwise, it is $h(\rho) = (h_1(\rho), \dots, h_n(\rho))$. The final outcome gives a payoff to each player.

The game $\Gamma(G,T)$ has two types of subgames. For each n-tuple $\rho = (\rho_1, \dots, \rho_n)$ of strategies, the demand stage forms a proper subgame of $\Gamma(G,T)$, which we denote by $\Gamma(G,T;\rho)$, and the other is $\Gamma(G,T)$ itself.

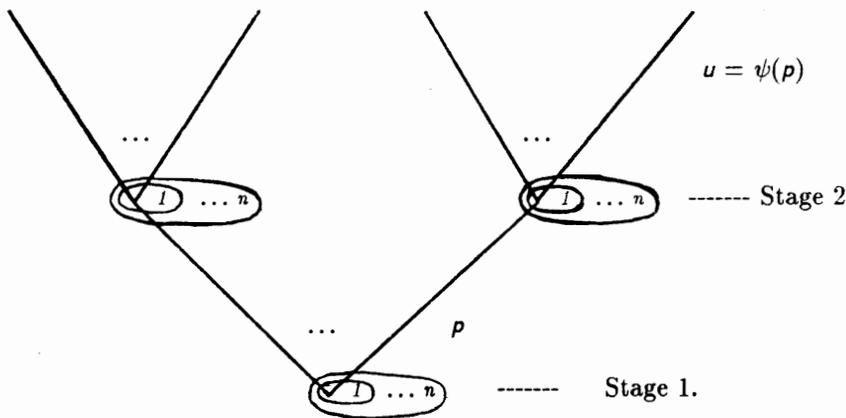


Figure 2.1 $\Gamma(G,T)$ (Simultaneous Move Form)

In $\Gamma(G,T)$, a *strategy* of player i is a pair of a point $q_i \in M_i(G)$ and a function ψ_i from $M(G)$ to R . That is, player i chooses q_i in Stage 1, and announces it to the other players. In Stage 2, he chooses a demand $u_i = \psi_i(q)$ depending upon $q = (q_1, \dots, q_n)$. The function ψ_i gives an action to the subgame $\Gamma(G,T;\rho)$ determined by each possible $\rho = (\rho_1, \dots, \rho_n) \in M(G)$, i.e., ψ_i is a complete plan of contingent actions. An n -tuple of strategies $(q, \psi) = ((q_1, \dots, q_n), (\psi_1, \dots, \psi_n))$ in $\Gamma(G,T)$ is called a *strategy combination*. The *payoff function* $H_i(q, \psi)$ of player $i \in N$ in $\Gamma(G,T)$ is defined by

$$\begin{aligned} H_i(q, \psi) &= \psi_i(q) \quad \text{if } \psi(q) \in T \\ &= h_i(q) \quad \text{otherwise.} \end{aligned} \tag{2.2}$$

The *induced payoff function* $H_i^P(u)$ of player i on the subgame $\Gamma(G,T;\rho)$ is defined by $H_i^P(u) = u_i$ if $u \in T$ and $H_i^P(u) = h_i(\rho)$ otherwise.

A *solution function* Ψ is a function which assigns to each bargaining problem (G,T) in \mathbb{B} a strategy combination (q, ψ) in $\Gamma(G,T)$. Now we consider the following axioms.

Axiom V.0 (Feasibility). For any (G,T) in \mathbb{B} with $\Psi(G,T) = (q, \psi)$, $\psi(\rho) \in T$ for all $\rho \in M(G)$.

A strategy combination (q, ψ) is called a *subgame perfect equilibrium* iff for every subgame of $\Gamma(G,T)$, the restriction of (q, ψ) is a Nash equilibrium in the subgame. In the present context, this means that 1) (q, ψ) is a Nash equilibrium in $\Gamma(G,T)$ and 2) $\psi(\rho)$ is a Nash equilibrium in the subgame $\Gamma(G,T;\rho)$ for all $\rho \in M(G)$.

Axiom V.1 (Subgame Perfect Equilibrium). For any (G,T) in \mathbb{B} , $\Psi(G,T) = (q, \psi)$ is a subgame perfect equilibrium in $\Gamma(G,T)$.

Let $a = (a_1, \dots, a_n) > 0$ and $b = (b_1, \dots, b_n)$ be vectors in R^n , and let $(G,T) = ((N, \{X_i\}), \{h_i\}, T)$. By $aG+b$, we denote the game $G' = (N, \{X_i\}, \{h_i'\})$ obtained from G by affine

transformations of payoff functions, i.e., $h_i'(x) = a_i h_i(x) + b_i$ for all $x \in \prod_{i \in N} X_i$ and all $i \in N$. Also, $a\psi + b$ means $(a_1 \psi_1(\cdot) + b_1, \dots, a_n \psi_n(\cdot) + b_n)$. Finally let $aT + b = \{(a_1 v_1 + b_1, \dots, a_n v_n + b_n) : v \in T\}$.

Axiom V.2 (Invariance under Affine Transformations). Let (G, T) be a bargaining problem in \mathbb{B} with $\Psi(G, T) = (q, \psi)$. Then $\Psi(aG + b, aT + b)$ coincides with $(q', a\psi + b)$ for any $a > 0$ and b in R^n (where q' is allowed to be different from q).

Let π be a permutation of the player set $N = \{1, \dots, n\}$. Then $\pi G = (N, \{X_i'\}, \{h_i'\})$ is obtained from game G by giving player $\pi(i)$ in G a new name i in the new game πG . Formally, $X_i' = X_{\pi(i)}$, $h_i'(x_{\pi(1)}, \dots, x_{\pi(n)}) = h_{\pi(i)}(x_1, \dots, x_n)$ for all $x \in \prod_{i \in N} X_i$. For example, when $N = \{1, 2, 3\}$ and $\pi = (3, 1, 2)$, the domain of h_1' is $X_1' \times X_2' \times X_3' = X_3 \times X_1 \times X_2$, and $h_1'(x_1', x_2', x_3') = h_1'(x_3, x_1, x_2) = h_{\pi(1)}(x_1, x_2, x_3) = h_3(x_1, x_2, x_3)$. Also, $\pi\psi = (\psi_1', \dots, \psi_n')$ is defined by $\psi_i'(\rho_{\pi(1)}, \dots, \rho_{\pi(n)}) = \psi_{\pi(i)}(\rho_1, \dots, \rho_n)$ for all $i \in N$ and $(\rho_1, \dots, \rho_n) \in M(G)$. Finally, $\pi T = \{(v_{\pi(1)}, \dots, v_{\pi(n)}) : v \in T\}$.

Axiom V.3 (Anonymity). Let (G, T) be a bargaining problem in \mathbb{B} with $\Psi(G, T) = (q, \psi)$ and let π be any permutation of $N = \{1, \dots, n\}$. Then $\Psi(\pi G, \pi T) = (q', \pi\psi)$ (where q' is allowed to be different from $\pi q = (q_{\pi(1)}, \dots, q_{\pi(n)})$).

Axiom V.4 (Independence of Irrelevant Alternatives). Let $(G, T), (G', T')$ be bargaining problems in \mathbb{B} with $\Psi(G, T) = (q, \psi)$, $\Psi(G', T') = (q', \psi')$, and let $\rho \in M(G)$, $\rho' \in M(G')$. If $h(\rho) = h(\rho')$, $T \subseteq T'$ and $\psi'(\rho') \in T$, then $\psi(\rho) = \psi'(\rho')$.

Axiom V.0 requires that the demand vector $\psi(\rho)$ be feasible in the second stage for any threat strategies ρ chosen in the first stage. Axiom V.1 states that the strategy combination (q, ψ) forms a subgame perfect equilibrium in $\Gamma(G, T)$. Axiom V.1 eliminates some infeasible payoff vector for $\psi(\rho)$, but, in general, Axiom V.1 does not imply Axiom V.0, since if every player i demands a very high utility level u_i , this infeasible n -tuple $u = (u_1, \dots, u_n)$ is an equilibrium in

$\Gamma(G,T;p)$ and the final outcome is the threat payoff vector $h(p)$.

The subgame perfect equilibrium requirement for proper subgames in Axiom V.1 corresponds to Pareto optimality in Nash's (1950,'53) original axiomatization. In Figure 2.2, the point $u = (u_1, u_2)$ is not an equilibrium point in the subgame, since player 1 (and 2) can improve his final payoff by demanding slightly higher u_1 (and u_2). In this example, any point in the Pareto frontier which locates in the northeast of the threat payoffs $h(p)$ is an equilibrium point in the subgame. Axiom V.1 does not, however, imply Pareto optimality for some games; the argument above is not applied to the game described by Figure 2.3. Nevertheless, Axiom V.1 together with Axioms V.0 and V.4 implies that $\psi(p)$ is Pareto optimal.

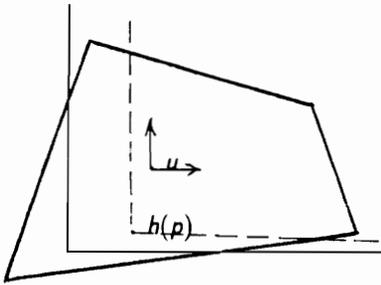


Figure 2.2

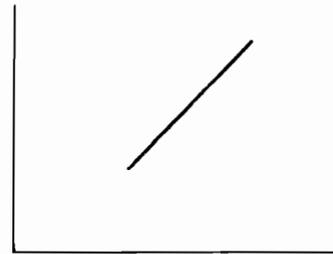


Figure 2.3

Axiom V.2 requires the function ψ to be invariant under affine transformations of payoff functions. This axiom comes from the basic presumption that payoff functions are representations of preference relations satisfying the von Neumann-Morgenstern expected utility axioms (cf., Herstein-Milnor (1953)). Such representations are uniquely determined up to affine transformations. Bargaining problems (G,T) and $(aG+b, aT+b)$ are identical except the payoff representations; consequently, the solutions must also be identical except their payoff representations. We allow q' to be different from q . See Remark 2.1.

Axiom V.3 requires that the solution function Ψ , actually ψ , depend upon the game structure but not the names of players, i.e., although the names of players change, the solution

does not change. See also Remark 2.1.

Axiom V.4 plays a key role in our axiomatization. Observe that Axioms V.0 and V.4 imply

$$T = T' \text{ and } h(\rho) = h'(\rho') \Rightarrow \psi(\rho) = \psi'(\rho'). \quad (2.3)$$

That is, for (G, T) , (G', T') with $T = T'$, if the threat payoffs are identical in two subgames, then the final outcomes are also identical. In this case, the final outcomes are independent of the strategic structures of underlying games; once threat strategies are chosen, only the bargaining regions and threat strategies are relevant, since the players are committed to play those threat strategies.

Axiom V.4 makes a further requirement on the final outcomes in two subgames, one of which is obtained from the other by keeping the threat strategies and eliminating some payoff vectors. If the outcome of the original game remains feasible in the new bargaining problem, it is also the final outcome in the new bargaining problem. This is essentially the same as Nash's (1950,'53) original axiom of the same name (see Section 2.3).

To state the main theorem of this subsection, we define Nash's fixed disagreement bargaining problem. A *fixed disagreement bargaining problem* is a pair (S, d) of a convex compact subset S of R^n and a point d in S . We denote the set of all fixed disagreement bargaining problems by \mathbb{F} . The *Nash solution function* is the function $f^N : \mathbb{F} \rightarrow R^n$ defined by

$$f^N(S, d) \in S \text{ and } f^N(S, d) \text{ maximizes } \prod_{i \in N(d)} (x_i - d_i) \text{ over } x \in S \text{ with } x \geq d, \quad (2.4)$$

where $N(d) = \{i \in N : x \geq d \text{ and } x_i > d_i \text{ for some } x \in S\}$. The value $f^N(S, d)$ is called the *Nash outcome* for (S, d) . Since we need to allow degenerated bargaining regions S , the Nash product is taken over the relevant players $N(d)$. However, since there is an $x \in S$ with $x \geq d$ such that $x_i > d_i$ for all $i \in N(d)$, we can apply the standard argument for the Nash outcome with the

nondegeneracy assumption by restricting N to the relevant player set $N(d)$.

The following is the main theorem of this subsection, which will be proved in Section 2.3.

Theorem 2.2.A. A solution function Ψ satisfies Axioms V.0-V.4 if and only if for any bargaining problem $(G,T) \in \mathbb{B}$, $\Psi(G,T) = (q,\psi)$ satisfies

- i) $\psi(\rho)$ coincides with the Nash outcome $f^N(T,h(\rho))$ for all $\rho \in M(G)$;
- ii) q is a Nash equilibrium in ψ , i.e., $\psi_i(q) \geq \psi_i(q_{-i},p_i)$ for all $p_i \in M_i(G)$ and all $i \in N$.

This theorem states that if Ψ satisfies Axioms V.0 - V.4, then Ψ gives the Nash outcome to each subgame $\Gamma(G,T;\rho)$ of the second stage --i); and the threat strategies q are chosen to be a Nash equilibrium in the first stage on condition that the Nash outcome is played in the second stage --ii). The solution described by i) and ii) is a generalization of what Nash (1953) described for the variable threat bargaining game with two players. We call q an *equilibrium threat point*.

In Section 2.3, we make a comparison between the above axiomatization and that of Nash's (1950) fixed disagreement bargaining model. As already mentioned, Axiom V.1 (Subgame Perfect Equilibrium) virtually implies Pareto optimality. The other axioms correspond to those of Nash's original axiomatization. Therefore the Nash outcome is derived from those axioms. The second assertion of Theorem 2.2.A follows from Axiom V.1. For details, see Section 2.3.

Nash (1950,'53) provided two complementary approaches to the Nash bargaining solution: axiomatic and noncooperative approaches. They are often regarded as quite different, but some authors noticed that they are "quite similar in spirit" (Luce-Raiffa (1957, p.143)). Our variable threat bargaining model described above is indeed a noncooperative game. An entire outcome for the associated bargaining game $\Gamma(G,T)$ is a strategy combination, and Axioms V.0 through V.4 are requirements for a strategy combination. Axiom V.1 is a requirement in noncooperative game theory, while Axioms V.0, V.2, V.3 and V.4 are reformulations of the axiomatic approach to the Nash bargaining solution. The axiomatic approach is often regarded

as belonging to cooperative game theory. Axioms V.0,V.2,V.3,V.4, however, do not involve cooperative aspects; we do not assume Pareto optimality, which expresses cooperative aspects. In our approach, cooperation is described in the rules of the bargaining game, and is derived virtually from Axiom V.1. Thus the approach in this paper shows that these two approaches are not only similar in spirit but also have an explicit relationship.

Our entire axiomatization can also be viewed as a refinement of a subgame perfect equilibrium in our specific context, i.e., Axioms V.0, V.2 - V.4 are requirements for refining a subgame perfect equilibrium in the associated bargaining game $\Gamma(G,T)$.

Remark 2.1. In Axioms V.2 and V.3, we require nothing for the threat strategy combination q for the purpose of direct comparison with Nash's (1950) axioms for fixed disagreement bargaining. If we require invariance in Axiom V.2 and V.3 for the threat point q , then, accordingly, we need to add the following two statements to Theorem 2.2.A:

- iii) if $\Psi(G,T) = (q,\psi)$ and $\Psi(aG+b,aT+b) = (q',\psi')$ for any $a > 0$ and b , then $q' = q$;
- iv) if $\Psi(G,T) = (q,\psi)$ and $\Psi(\pi G,\pi T) = (q',\psi')$ for any permutation π of N , then $q' = \pi q$.

2.2.3 Bargaining Model without Commitments to Threat Strategies

In the previous model, each player makes a commitment to play a threat strategy; he needs to ask somebody neutral or to set some device to enforce himself to play the threat strategy. Unless a commitment is available for a player, in the case of disagreement, he might be expected to change his mind if the threat strategy hurts him. In some situations, commitments are simply not possible. In such a case, players may make another strategy choice after disagreement occurs. In this subsection, we consider a bargaining model where no player can make a commitment to play a threat strategy. The primary purpose of the introduction of this model is to investigate the behavior of the previous model by making comparisons with the new

model. Specifically, in Section 2.5, such a comparison is made to show that these two models yield approximately the same results for bargaining problems with large numbers of players.

The bargaining situation is described by an extensive game $\Delta(G,T)$ for each (G,T) in \mathbb{B} , which we also call the *associated bargaining game* (without commitments). The game $\Delta(G,T)$ has two stages: the first stage is the *demand stage* and the second is the *noncooperative game stage*. The game tree of $\Delta(G,T)$ is described in Figure 2.4.

Stage 1: Every player $i \in N$ simultaneously chooses a utility demand u_i from R and announces it to the other players. If the vector $u = (u_1, \dots, u_n)$ does not belong to T , then the game goes to Stage 2. If u belongs to T , the game is over with probability 1 and the final payoff vector is $u = (u_1, \dots, u_n)$, but with probability 0, the game goes to Stage 2. (See Remark 2.2.)

Stage 2: The players simply play the noncooperative game G . (The strategy combination chosen here might depend upon $u = (u_1, \dots, u_n)$.)

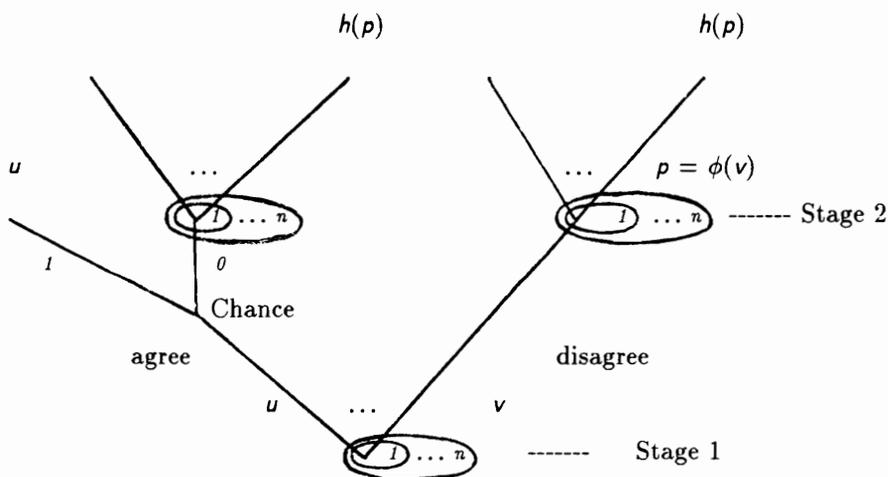


Figure 2.4: $\Delta(G,T)$

Remark 2.2. The game $\Delta(G,T)$ has a subgame G in Stage 2 even in the case of agreement, though the probability to go the Stage 2 is zero. This is necessary since we regard the case of agreement as the limit of a case where bargaining results in disagreement with a small probability even if u is feasible. This is essentially the same as the idea of trembling-hand perfection due to Selten (1975).² Nevertheless, we do not consider this problem formally here.

In $\Delta(G,T)$, a *strategy* of a player i is a pair of a point $u_i \in R$ and a function ϕ_i from R^n to $M_i(G)$. That is, player i chooses demand u_i and announces u_i to the other players in Stage 1. If the game goes to Stage 2, player i chooses a strategy $p_i = \phi_i(v)$ from $M_i(G)$. A *strategy combination* is an n -tuple of strategies $(u, \phi) = ((u_1, \dots, u_n), (\phi_1, \dots, \phi_n))$. The *payoff function* $H_i(u, \phi)$ of player i in $\Delta(G,T)$ is given as $H_i(u, \phi) = u_i$ if $u \in T$ and $H_i(u, \phi) = h_i(\phi(u))$ otherwise. The induced payoff function $H_i^V(p)$ of player i for the subgame determined by $v = (v_1, \dots, v_n) \in R^n$ is given as $H_i^V(p) = h_i(p)$.

A *solution function* Φ is a function which assigns to each (G,T) in \mathbb{B} a strategy combination (u, ϕ) in the associated bargaining game $\Delta(G,T)$. We consider the following axioms.

Axiom U.0 (Feasibility). For any (G,T) in \mathbb{B} with $\Phi(G,T) = (u, \phi)$, u belongs to T .

Axiom U.1 (Subgame Perfect Equilibrium). For any (G,T) in \mathbb{B} , $\Phi(G,T) = (u, \phi)$ is a subgame perfect equilibrium in $\Delta(G,T)$.

Axiom U.2 (Invariance under Affine Transformations). Let (G,T) be in \mathbb{B} with $\Phi(G,T) = (u, \phi)$, let $a > 0$ and b be points in R^n , and let $\Phi(aG+b, aT+b) = (u', \phi')$. If $\phi'(u') = \phi(u)$, then $u' = au+b$.

Axiom U.3 (Anonymity). Let (G,T) be in \mathbb{B} with $\Phi(G,T) = (u, \phi)$, let π be any permutation of $N = \{1, \dots, n\}$ and let $\Phi(\pi G) = (u', \phi')$. If $\phi'(u') = \pi\phi(u)$, then $u' = \pi u$.

Axiom U.4 (Independence of Irrelevant Alternatives). Let (G,T) and (G',T') be in \mathbb{B} , and let $\Phi(G,T) = (u, \phi)$ and $\Phi(G',T') = (u', \phi')$. If $h(\phi(u)) = h'(\phi'(u'))$, $T \subseteq T'$ and $u' \in T$, then $u = u'$.

Axioms U.0 - U.4 are parallel to Axioms V.0 - V.4 in the sense that they are the same types of requirements for the games of different structures. The main difference is that the

²Nash (1953) used a similar argument which he called the "smoothing procedure" to derive the Nash outcome.

See Kaneko (1981) and Binmore (1987) for detailed argument.

second stage of the previous model is the first stage of the present model. Since Axioms V.0, V.2 - V.4 are about the outcome of the second stage in the previous model, Axioms U.0, U.2 - U.4 apply to the behavior of the outcome in the first stage of the present model.

Theorem 2.2.B. A solution function Φ satisfies Axioms U.0 -U.4 if and only if for any (G,T) in \mathbb{B} with $\Phi(G,T) = (u,\phi)$,

- i) u coincides with the Nash outcome $f^N(T,d)$ where $d = h(\phi(u))$;
- ii) $\phi(v)$ is a Nash equilibrium in game G for all $v \in R^n$;
- iii) for all $i \in N$, $u_i \geq h_i(\phi(u_{-i},v_i))$ for all $v_i \in R$ with $(u_{-i},v_i) \notin T$.

The first assertion states that the Nash outcome u relative to disagreement point $h(\phi(u))$ results in Stage 1. The second states that $\phi(v)$ is a Nash equilibrium in noncooperative game G for all v , including infeasible ones. The third states that any player cannot improve the bargaining outcome u by breaking cooperation and playing a disagreement (Nash equilibrium) point in the second stage. The second and third assertions look to imply the subgame perfect equilibrium property, but, in fact, need Pareto optimality included in i) to form it. The first assertion needs the other axioms. Essentially, these axioms, together with Pareto optimality induced by Axioms U.0, U.1 and U.4, correspond to the axioms of the Nash fixed disagreement model. The first assertion can be regarded as Nash's (1950) theorem on the fixed disagreement model. We will discuss the explicit relationship between the present model and Nash's model in Section 2.3.3.

To illustrate the difference between the results of Sections 2.2.2 and 2.2.3, we consider the following game.

Example 2.1. (Battle of the Sexes): Let $N = \{1,2\}$, $X_1 = \{\alpha_1, \alpha_2\}$, $X_2 = \{\beta_1, \beta_2\}$ and let payoffs be given as follows:

	β_1	β_2
α_1	2,1	0,0
α_2	0,0	1,2

We assume that the bargaining region T is given as the feasible region $F(G)$. Since this game has two pure strategy equilibria $((1,0),(1,0)) = (\alpha_1,\beta_1)$, $((0,1),(0,1)) = (\alpha_2,\beta_2)$, and one mixed strategy equilibrium $((2/3,1/3),(1/3,2/3))$, the second model gives the corresponding three types of solutions;

$$1) \ u = (2,1), \phi(u) = ((1,0),(1,0)); \text{ and } 2) \ u = (1,2), \phi(u) = ((0,1),(0,1));$$

$$3) \ u = (1.5,1.5), \phi(u) = ((2/3,1/3),(1/3,2/3)).$$

The first model gives the unique solution to this game;

$$q = ((1,0),(0,1)) \text{ -- equilibrium threat; and } \psi(q) = (1.5,1.5) \text{ -- the final outcome.}$$

This gives the same final outcome as Solution 3) in the second model, though the disagreement points are different. Solutions 1) and 2) are degenerated; the disagreement points and the bargaining outcome coincide. This degeneracy occurs sometimes not only in the second model but also in the variable threat model. The class of games discussed in Section 2.4 has this property. The necessity of Assertion iii) of Theorem 2.2.B can be observed in this example; if ϕ in 3) has $\phi(u_1,v_2) = ((0,1),(0,1))$ for some $v_2 > u_2$ -- ϕ jumps from $\phi(u) = ((2/3,1/3),(1/3,2/3))$ to another equilibrium $((0,1),(0,1))$, then (u,ϕ) is not a subgame perfect equilibrium.

Observe that in the second model, each Nash equilibrium of the underlying game G gives a Nash outcome of the associated bargaining game $\Delta(G,T)$. The underlying game G may have multiple Nash equilibria, which give possibly different Nash outcomes. Hence the uniqueness of the final outcome does not hold for $\Delta(G,T)$ even with two players, while the game $\Gamma(G,T)$ with two players has a unique final outcome. In this sense, these models give quite different results, but when the number of players becomes large, the behavior of the first model becomes similar to

that of the second model. These will be discussed in Sections 2.4 and 2.5.

Remark 2.3. For the sake of simplicity, we formulated the present model as a two stage model instead of an extension of the model of Section 2.2.2. We can add the threat strategy choice to the model of this section before the first stage. The addition of the threat strategy choice to the present model is, however, not substantive if no player can make a commitment. In the last stage of the new model, every player pursues his payoff instead of playing threat strategy he announced; threat is not credible. In this formulation, we obtain essentially the same result as Theorem 2.2.B with appropriate modifications of strategies and of Axioms U.0 - U.4.³

2.3 Relationship to the Bargaining Model with Fixed Disagreement Points

In this section, we compare the bargaining models given in Section 2.2 with Nash's (1950) model with fixed disagreement points. Indeed, we prove the equivalence between the axiomatizations of Section 2.2.2 and of Nash (1950), when we restrict our attention in the second stage of the model of Section 2.2.2. A similar equivalence result is obtained between the axiomatizations of Section 2.2.3 and of Nash (1950). Theorems 2.2.A and 2.2.B will be proved using these equivalences.

2.3.1 The Nash Bargaining Model with Fixed Disagreement Points

Recall that a bargaining problem with a fixed disagreement point is given as a pair (S, d) of a bargaining region S and a disagreement point d in S , and that \mathbb{F} is the set of all bargaining problems with fixed disagreement points. Here a *solution function* f assigns a point $f(S, d)$ in R^n to

³We can modify this game further so that the game includes the choice of a commitment to play a threat strategy or not. Mao (1992) considers such a model and shows that in the case of two-players, each player chooses a commitment to play a threat strategy, i.e., the model of Section 2.2.2 results endogenously. It is, however, also shown that this result holds no longer for $n \geq 3$.

each bargaining problem (S, d) in \mathbb{F} .

The following axioms for the fixed disagreement bargaining model are given by Nash (1950). Let $f: \mathbb{F} \rightarrow \mathbb{R}^n$ be a fixed threat solution function, and let $(S, d), (S', d')$ be bargaining problems in \mathbb{F} .

Axiom N.0 (Feasibility). $f(S, d) \in S$.

Axiom N.1 (Pareto Optimality). $a \in S$ and $a \geq f(S, d)$ imply $a = f(S, d)$.

Axiom N.2 (Invariance under Affine Transformations). For any vector $a > 0$ and b , $f(aS+b, ad+ab) = af(S, d) + b$.

Axiom N.3 (Anonymity). $f(\pi S, \pi d) = \pi f(S, d)$ for any permutation π of $N = \{1, \dots, n\}$.

Axiom N.4 (Independence of Irrelevant Alternatives). $S \subseteq S', d = d'$ and $f(S', d') \in S$ imply $f(S, d) = f(S', d')$.

The following theorem is a direct extension of Nash's (1950) theorem with $n = 2$ into $n \geq 2$.

Theorem 2.3.A. A solution function $f: \mathbb{F} \rightarrow \mathbb{R}^n$ satisfies Axioms N.0 - N.4 if and only if f coincides with the Nash solution function f^N .

The proof of the above theorem is obtained from the standard proof of the Nash (1950) theorem (cf., Roth (1979)) with small modifications. Since we do not make the nondegeneracy assumption that $d < x$ for some $x \in S$, the Nash product is taken over the relevant player set $N(d)$ in definition (2.4), instead of the whole player set N . For the Only-If part of the theorem, we can apply the standard argument to the relevant bargaining region. For the If part, the verification of Axiom N.4 needs a small modification, but those for the other axioms are straightforward.

2.3.2 Comparison with the Variable Threat Model

A fixed disagreement bargaining problem (S, d) can be represented as a pair $(T, h(\rho))$ of the bargaining region T and the threat payoff vector $h(\rho)$ with $S = T$ and $d = h(\rho)$ for bargaining problem (G, T) in \mathbb{B} . This implies that (S, d) can be regarded as representing the proper subgame $\Gamma(G, T; \rho)$, determined by strategies ρ , of the associated bargaining game $\Gamma(G, T)$. We find that Axioms V.0 - V.4 in Section 2.2.2 has almost a perfect correspondence with Axioms N.0 - N.4, except the endogenous determination of threat strategies. While Axiom V.1 looks very different from N.1, V.1 may be viewed as a natural extension of N.1 to the two-stage variable threat model. If we restrict our attention to the second stage, these axiomatizations are virtually equivalent. Here the following is a modification of Axiom V.1, which makes the closeness of the relationship more explicit.

Axiom V.1^o (Equilibrium in Proper Subgames). For any (G, T) in \mathbb{B} with $\Psi(G, T) = (q, \psi)$, $\psi(\rho)$ is a Nash equilibrium in the proper subgame $\Gamma(G, T; \rho)$ of $\Gamma(G, T)$ for any $\rho \in M(G)$.

Of course, Axiom V.1 implies Axiom V.1^o. The following theorem shows the exact relationship between the variable threat model and Nash's fixed disagreement model.

Theorem 2.3.B. Let solution functions Ψ and f satisfy

$$\begin{aligned} &\text{for any } (G, T) \in \mathbb{B}, \rho \in M(G) \text{ and } (S, d) \in \mathbb{F}, \\ &\text{if } \Psi(G, T) = (q, \psi), T = S \text{ and } h(\rho) = d, \text{ then } \psi(\rho) = f(S, d). \end{aligned} \quad (2.5)$$

Then Ψ satisfies Axioms V.0, V.1^o, V.2, V.3, V.4 if and only if f satisfies Axioms N.0 - N.4.

Condition (2.5) connects two solution functions Ψ and f so that the function ψ coincides with solution function f for the same bargaining region $T = S$ and the same disagreement points $d = h(\rho)$. Then Theorem 2.3.B states that the restriction of the axiomatization of the variable

threat model to the second stage is equivalent to that of the fixed disagreement model.

With the help of Theorem 2.3.B we prove Theorem 2.2.A by deriving a fixed disagreement solution function f from Ψ . If Ψ satisfies Axiom V.4, it satisfies condition (2.3), which enables us to define a function $f: \mathbb{F} \rightarrow \mathbb{R}^n$ by (2.5). Then we apply Theorem 2.3.A to this f to show Assertion i) of Theorem 2.2.A. Conversely, if Ψ satisfies Assertions i) and ii) of Theorem 2.2.A, then we can again define $f: \mathbb{F} \rightarrow \mathbb{R}^n$ by (2.5). Then we apply Theorems 2.3.A and 2.3.B to f and Ψ . The following is an exact proof.

Proof of Theorem 2.2.A. (Only-If): Suppose that Ψ satisfies Axioms V.0 - V.4. By (2.3), we can define $f: \mathbb{F} \rightarrow \mathbb{R}^n$ by $f(S, d) = \psi(\rho)$ if $\Psi(G, S) = (q, \psi)$ with $h(\rho) = d$. Then Ψ and f satisfy (2.5). Thus it follows from Theorem 2.3.B that f satisfies Axioms N.0 through N.4. Therefore, by Theorem 2.3.A, $\psi(\rho) = f(S, d)$ coincides with the Nash outcome $f^N(S, d)$, which is Assertion i) of Theorem 2.2.A. Assertion ii) follows directly from Axiom V.1.

(If): Suppose Assertions i) and ii) of Theorem 2.2.A. Since $\psi(\rho)$ depends upon the bargaining region T and the threat payoffs $h(\rho)$ by Assertion i), we can define $f: \mathbb{F} \rightarrow \mathbb{R}^n$ by $f(S, d) = \psi(\rho)$ if $\Psi(G, S) = (q, \psi)$ with $h(\rho) = d$. Then f satisfies N.0 - N.4, and also, Ψ, f satisfy (2.5). Hence it follows from Theorem 2.3.B that Ψ satisfies V.0, V.1^o, V.2, V.3, V.4. Axiom V.1 follows from Assertion ii) of Theorem 2.2.A together with Pareto optimality implied by Assertion i). \square

Proof of Theorem 2.3.B: We prove only the Only-If part, i.e., Axioms V.0, V.1^o, V.2, V.3, V.4 imply Axioms N.0 - N.4. A proof of the If-part is given in the Appendix. Let $(S, d), (S', d') \in \mathbb{F}$.

N.0) Feasibility: Choose $(G, T) \in \mathbb{B}$ and $\rho \in M(G)$ so that $T = S$ and $h(\rho) = d$. Then $f(S, d) = \psi(\rho) \in T$ by (2.5) and Axiom V.0.

N.1) Pareto Optimality: We show that $f(S, d)$ is Pareto optimal in S . First, we choose a vector $b = (b_1, \dots, b_n)$ with $b \leq v$ for all $v \in S$. We extend the bargaining region S to the set $S^* = \{v \in \mathbb{R}^n: b \leq v \leq u \text{ for some } u \in S\}$. Now we have two problems (S, d) and (S^*, d) . The set S^* has the same

Pareto optimal surface as that of S . There are (G,S) and (G^*,S^*) in \mathbb{B} such that $d = h(\rho) = h^*(\rho^*)$ for some $\rho \in M(G)$ and $\rho^* \in M(G^*)$. By (2.5) $\psi(\rho) = f(S,d)$ and $\psi^*(\rho^*) = f(S^*,d)$.

Now we show that $\psi^*(\rho^*) = f(S^*,d)$ is Pareto optimal in S^* . On the contrary, suppose that for some $u \in S^*$, $u \geq \psi^*(\rho^*) = f(S^*,d)$ and $u_i > \psi_i^*(\rho^*)$ for some i . Then the vector v defined by

$$\begin{aligned} v_j &= \psi_j^*(\rho^*) & \text{if } j \neq i \\ &= u_i & \text{if } j = i \end{aligned}$$

belongs to S^* because $b \leq \psi^*(\rho^*) \leq v \leq u$. Hence, in the subgame $\Gamma(G^*,S^*; \rho^*)$, player i can improve his payoff $\psi_i^*(\rho^*)$ by demanding u_i , a contradiction to Axiom V.1^o.

Since the Pareto surface of S^* coincides with that of S , $\psi^*(\rho^*) = f(S^*,d)$ is on the Pareto surface of S , which implies $\psi^*(\rho^*) = f(S^*,d) \in S$. Since $S \subseteq S^*$, $\psi^*(\rho^*) \in S$ and $h(\rho) = h^*(\rho^*) = d$, Axiom V.4 implies $\psi(\rho) = \psi^*(\rho^*)$. This implies that $f(S,d) = \psi(\rho)$ is Pareto optimal in S .

N.2) Invariance under Affine Transformations: We show that $f(aS+b, ad+b) = af(S,d)+b$ for any $a > 0$ and b . Choose (G,T) in \mathbb{B} and $\rho \in M(G)$ so that $T = S$ and $h(\rho) = d$. By (2.5), $f(S,d) = \psi(\rho)$. Let $\Psi(aG+b, aS+b) = (q', \psi')$. Since $ah(\rho)+b = ad+b$, $f(aS+b, ad+b) = \psi'(\rho)$ by (2.5), which coincides with $a\psi(\rho)+b$ by Axiom V.2. Thus $f(aS+b, ad+b) = a\psi(\rho)+b = af(S,d)+b$.

N.3) Anonymity: We show $f(\pi S, \pi d) = \pi f(S,d)$. Choose (G,T) and $\rho \in M(G)$ with $T = S$ and $d = h(\rho)$. By (2.5) $f(S,d) = \psi(\rho)$. Consider the new game $\pi G = (N, \{h_i'\}, \{X_i'\})$. Then $h'(\pi\rho) = \pi d$. Let $\Psi(\pi G, \pi T) = (q', \psi')$. By (2.5), $f(\pi S, \pi d) = \psi'(\pi\rho)$. By Axiom V.3, $\psi_i'(\pi\rho) = \psi_{\pi(i)}(\rho)$ for all $i \in N$. Thus $f(\pi S, \pi d) = \psi'(\pi\rho) = \pi\psi(\rho) = \pi f(S,d)$.

N.4) Independence of Irrelevant Alternatives: Let $S \subseteq S'$ and $f(S',d) \in S$. We show $f(S,d) = f(S',d)$. Choose $(G,T), (G',T')$ in \mathbb{B} and $\rho \in M(G), \rho' \in M(G')$ so that $T = S, T' = S'$ and $h(\rho) = h(\rho') = d$. Let $\Psi(G,T) = (q, \psi)$ and $\Psi(G',T') = (q', \psi')$. By (2.5), $f(S,d) = \psi(\rho)$ and $f(S',d) = \psi'(\rho')$. Here $T = S \subseteq S' = T'$. Since $\psi'(\rho') = f(S',d) \in S = T$ and $h(\rho) = h'(\rho')$, we have $\psi(\rho) = \psi'(\rho')$ by Axiom V.4. Thus $f(S,d) = \psi(\rho) = \psi'(\rho') = f(S',d)$. \square

2.3.3 Comparison with the Second Bargaining Model

The bargaining model of Section 2.2.3 is even closer to the fixed disagreement model than the variable threat model. The second stage of this model results in a Nash equilibrium of the underlying game G , which gives a disagreement point. If G has a unique Nash equilibrium, then this Nash equilibrium can be regarded as a fixed disagreement point in the sense that it is independent of choices in the first stage. In the comparison between the variable threat model and the fixed disagreement model, Axiom V.1 was restricted to the second stage. In this subsection, the comparison is made between the first stage of the second model and the fixed disagreement model. Hence the corresponding axiom, U.1, needs a different modification than that of V.1. Unfortunately, we do not have a clear-cut noncooperative weakening of U.1; the Pareto Optimality Axiom is simply used here. Axiom U.1 together with Axioms U.0 and U.4 implies Pareto optimality:

Axiom U.1^o(Pareto Optimality). For any $(G,T) \in \mathbb{B}$ with $\Phi(G,T) = (u,\phi)$, $v \geq u$ and $v \in T$ imply $v = u$.

With this modification of Axiom U.1, we obtain the following equivalence theorem. A proof of Theorem 2.3.C is given in the Appendix.

Theorem 2.3.C. Let solution functions Φ and f satisfy:

$$\text{if } \Phi(G,T) = (u,\phi), T = S \text{ and } h(\phi(u)) = d, \text{ then } u = f(S,d). \quad (2.6)$$

Then Φ satisfies Axioms U.0, U.1^o, U.2, U.3, U.4 if and only if f satisfies Axioms N.0 - N.4.

With connection (2.6), Theorem 2.3.C states that the restriction of the model of Section 2.2.3 to the first stage is equivalent to the Nash fixed disagreement model. This theorem is parallel to Theorem 2.3.A with substitution of (2.6) for (2.5). The main purpose of this theorem

is to facilitate the proof of Theorem 2.2.B rather than the equivalence itself.

Proof of Theorem 2.2.B. (Only If): Suppose that Φ satisfies Axioms U.0 - U.4. Then it follows from Axioms U.0 and U.4 that for any $(G,T), (G',T) \in \mathbb{B}$ with $\Phi(G,T) = (u,\phi), \Phi(G',T) = (u',\phi')$,

$$\text{if } T = T' \text{ and } h(\phi(u)) = h'(\phi'(u')), \text{ then } u = u', \quad (2.7)$$

where $G = (N, \{X_i\}, \{h_i\})$ and $G' = (N, \{X'_i\}, \{h'_i\})$. From this, for any $(S,d) \in \mathbb{F}$, we can define $f(S,d)$ by $f(S,d) = u$ if $\Phi(G,T) = (u,\phi), S = T$ and $h(\phi(u)) = d$ for some game G . (The game G where each player i has only one pure strategy α_i with $h(\alpha) = h(\alpha_1, \dots, \alpha_n) = d$ can be used for this definition.) Then Φ and f satisfy (2.6). It follows from Theorems 2.3.A and 2.3.C that u is the Nash outcome of the bargaining problem $(T, h(\phi(u)))$. This is Assertion i) of Theorem 2.2.B. Assertions ii) and iii) follow from Axiom U.1.

(If): By Assertion i) of Theorem 2.2.B, we can define $f(S,d)$ for any $(S,d) \in \mathbb{F}$ by $f(S,d) = u$ if $\Phi(G,T) = (u,\phi), S = T$ and $h(\phi(u)) = d$. Then $f(S,d)$ is the Nash outcome $f^N(S,d)$ by Assertion i) of Theorem 2.2.B, and satisfies (2.7). It follows from Theorem 2.3.A that f satisfies Axioms N.0 - N.4. Hence by Theorem 2.3.C, Φ satisfies Axioms U.0, U.1^o, U.2, U.3, U.4. Axiom U.1 follows from Assertions i), ii) and iii) of Theorem 2.2.B. \square

2.4 The Structure of the Set of Equilibrium Threat Points

In the two-person case, Nash (1953) proved that the equilibrium payoff vector is uniquely determined in the variable threat model of Section 2.2.2, though the equilibrium threat points themselves may not be unique. The variable threat model is regarded as the game of threat choice, assuming the Nash outcome for each subgame $\Gamma(G,T;p)$. It is a strictly competitive game, which implies that equilibrium threat strategies are maximin strategies and also satisfy interchangeability in Nash's (1951) sense. The model of Section 2.2.3 does not share these

properties, as was discussed in Example 2.1. In this section we argue that when the number of players is more than two, the variable threat model loses such properties, and show that the structure of equilibrium threat points in the case of $n+1$ players may be as complex as that of Nash equilibria of a noncooperative game with n players.

The following theorem states that for a constant-sum game G with the bargaining region $T = F(G)$, there is no difference between the bargaining models of Sections 2.2.2 and 2.2.3, and that the set of final bargaining outcomes in both models coincides with the set of Nash equilibrium payoffs in G .

Theorem 2.4.A. Let $G = (N, \{X_i\}, \{h_i\})$ be a constant-sum game. Then the set of Nash equilibria in G coincides with the set of equilibrium threat points in the associated bargaining game $\Gamma(G, F(G))$. Furthermore, the set of final bargaining outcomes in $\Gamma(G, F(G))$ (in $\Delta(G, F(G))$) coincides with the set of Nash equilibrium payoff vectors in G .

Proof. Since G is constant-sum, the Pareto optimal surface of the feasible region $F(G)$ is $F(G)$ itself. Since any threat point is on the Pareto surface, the Nash outcome for a given threat point is the threat point itself. Thus $f^N(F(G), h(\rho)) = h(\rho)$ for any $\rho \in M(G)$. This implies that a Nash equilibrium in game G is also an equilibrium threat point in $\Gamma(G, F(G))$, and *vice versa*. Also, the set of final bargaining outcomes in $\Gamma(G, F(G))$ coincides with the set of Nash equilibrium payoff vectors in G . (The same arguments apply to $\Delta(G, F(G))$.) \square

Theorem 2.4.A states that bargaining is redundant for a constant-sum game G , i.e., constant-sum games are regarded as special from the viewpoint of bargaining. Nevertheless, this theorem together with the next theorem helps us consider the structure of the set of equilibrium threat points.

Theorem 2.4.B. For every n -person game G , there exists an $(n+1)$ -person constant-sum game G^*

such that there is a bijection between the Nash equilibrium sets of G and G^* .

The reasoning for Theorem 2.4.B, due to von Neumann-Morgenstern (1944), is as follows: for any game $G = (N, \{X_i\}, \{h_i\})$, we add one fictitious player $n+1$ having one strategy to the player set N so that the payoff functions of the original players are unchanged and the new player's payoff is the negative sum of the others' payoffs. The new extended game is an $(n+1)$ -person zero-sum game. Since the structure of the new game is identical to the original game except the nominal existence of a fictitious player, the set of Nash equilibria remains unchanged with this additional player. This argument is illustrated in Example 2.2.

Theorems 2.4.A and 2.4.B imply that for any n -person game $G = (N, \{X_i\}, \{h_i\})$, there is an $(n+1)$ -person game G^* such that the Nash equilibrium set of game G is isomorphic to the set of equilibrium threat points of the associated bargaining game $\Gamma(G^*, F(G^*))$. Thus when we go to games with many players, the set of equilibrium threats may become as complex as the set of Nash equilibria of a normal form game. It is known that the Nash equilibrium set for a general n -person game is possibly quite complex, in particular, uniqueness holds only for a very special class of games. Thus the uniqueness of the equilibrium outcome in the variable threat model is no longer valid when $n \geq 3$.

Constant-sum games are "degenerated" from the viewpoint of Nash bargaining. We can, however, modify a game slightly so that they are not constant-sum but the above two theorems remain true. In this sense, the assumption of constant-sum is not very crucial for showing that the set of equilibrium threats for $n+1$ players is possibly as complex as that of Nash equilibria of a noncooperative game with n players.

Example 2.2. We modify the game of "Battle of the Sexes", using the argument for Theorem 2.4.B. Formally, let $N = \{1, 2, 3\}$, $X_1 = \{\alpha_1, \alpha_2\}$, $X_2 = \{\beta_1, \beta_2\}$, $X_3 = \{\gamma\}$ and let payoffs be given as follows:

Table 2.1

	β_1	β_2
α_1	2, 1, -3	0, 0, 0
α_2	0, 0, 0	1, 2, -3
	γ	

This game has two pure strategy equilibria $((1,0),(1,0),1) = (\alpha_1,\beta_1,\gamma)$, $((0,1),(0,1),1) = (\alpha_2,\beta_2,\gamma)$, and one mixed strategy equilibrium $((2/3,1/3),(1/3,2/3),1)$. These correspond to the Nash equilibria of the game in Example 2.1. The variable threat model of Section 2.2.2 gives the corresponding three types of solutions;

- 1) $q = ((1,0),(1,0),1)$, $\psi(q) = (2,1,-3)$; and 2) $q = ((0,1),(0,1),1)$, $\psi(q) = (1,2,-3)$;
- 3) $q = ((2/3,1/3),(1/3,2/3),1)$, $\psi(q) = (2/3,2/3, -4/3)$.

The model of Section 2.2.3 gives the same outcomes to this game. The addition of a fictitious player changes the structure of the equilibrium threat points.

We give a nonconstant-sum game example to illustrate the nonuniqueness of the equilibrium outcomes in the variable threat model with $n \geq 3$. For such an example and also for the purpose of Section 2.5, we give the following (known) lemma.

Lemma 2.4.1. Let a, b be in \mathbb{R}^n with $a > 0$ and $\sum_j b_j/a_j \leq 1$. Consider a bargaining problem (S, d) where $S = \{x \in \mathbb{R}^n: \sum_j x_j/a_j \leq 1 \text{ and } x \geq b\}$ and $d \in S$. Then the Nash outcome $f^N(S, d)$ is given by

$$f_i^N(S, d) = a_i(1 - \sum_j d_j/a_j)/n + d_i \text{ for all } i \in N. \quad (2.8)$$

Proof.⁴ If $\sum_j d_j/a_j = 1$, then $f^N(S, d) = d$, which is (2.8). Let $\sum_j d_j/a_j < 1$. The Nash outcome

⁴This lemma is usually proved using the Lagrangian multiplier method. But we prove it in an algebraic way, since the algebraic method may lead to a finite algorithm to calculate the Nash outcome in general. See Kaneko (1992) for a related topic.

$f^N(S, d)$ maximizes $\prod_j(x_j - d_j)$ over x 's in S with $x \geq d$. Let x be a point on the Pareto surface of S , i.e., $\sum_j x_j/a_j = 1$. Recall that the arithmetic average $\sum_j(x_j - d_j)/a_j n$ is at least as large as the geometric averages $(\prod_j(x_j - d_j)/a_j)^{1/n}$ with equation for identical components. The arithmetic average is constant over the Pareto surface since $\sum_j(x_j - d_j)/a_j = 1 - \sum_j d_j/a_j$. Thus if equation holds at x° , then x° maximizes the Nash product $\prod_j(x_j - d_j)$ over x in S with $x \geq d$. Since $(x_j^\circ - d_j)/a_j$ is identical for all j , we have $n(x_i^\circ - d_i)/a_i = \sum_j(x_j^\circ - d_j)/a_j = 1 - \sum_j d_j/a_j$. Thus $x_i^\circ = a_i(1 - \sum_j d_j/a_j)/n + d_i$. \square

Example 2.3. Let $N = \{1, 2, 3\}$, $X_1 = \{\alpha_1, \alpha_2\}$, $X_2 = \{\beta_1, \beta_2\}$, $X_3 = \{\gamma_1, \gamma_2\}$, and let payoffs be given by the following matrices:

Table 2.2.

		β_1	β_2			β_1	β_2	
α_1	1, 2, 7	1, 1, 5	α_1	4, 0, 6	0, 10, 0	α_2	3, 5, 2	10, 0, 0
α_2	0, 0, 0	0, 0, 10	α_2	3, 5, 2	10, 0, 0			
		γ_1			γ_2			

Consider the bargaining region $T = F(G)$, which is given as the convex hull of four vectors $(0,0,0)$, $(10,0,0)$, $(0,10,0)$ and $(0,0,10)$. We can apply Lemma 2.4.1 to calculate the Nash solution for each triple of pure strategies. For example, strategy triple $(\alpha_1, \beta_2, \gamma_1)$ determines the threat payoff vector $(1,1,5)$. Lemma 2.4.1 implies that the Nash outcome for $(F(G), (1,1,5))$ is $(2,2,6)$. In the case of $(\alpha_1, \beta_1, \gamma_2)$, since $(4,0,6)$ is on the Pareto surface of $F(G)$, $(4,0,6)$ itself is the Nash outcome of $(F(G), (4,0,6))$. Similarly, we have the Nash outcomes relative to the other pure strategy triples described in Table 2.3.

Equilibrium threat points in the associated bargaining game $\Gamma(G, F(G))$ are obtained by calculating the Nash equilibria in the matrix game in Table 2.3. This game has Nash equilibria

$(\alpha_1, \beta_2, \gamma_1)$, $(\alpha_2, \beta_1, \gamma_1)$ (in pure strategies) and the corresponding Nash outcomes are $(2, 2, 6)$, $(10/3, 10/3, 10/3)$. Thus this example shows that the variable threat model has at least two distinct final outcomes.⁵ Since the underlying game of Table 2.2 has a Nash equilibrium $(\alpha_1, \beta_1, \gamma_1)$, the model of Section 2.2.3 gives a final bargaining outcome $(1, 2, 7)$, but neither $(2, 2, 6)$ nor $(10/3, 10/3, 10/3)$.

Table 2.3.

	β_1	β_2		β_1	β_2
α_1	1, 2, 7	2, 2, 6	α_1	4, 0, 6	0, 10, 0
α_2	10/3, 10/3, 10/3	0, 0, 10	α_2	3, 5, 2	10, 0, 0
	γ_1			γ_2	

2.5 Equilibrium Threats in Large Bargaining Games with Sidepayments

When the number of players is small, e.g., $n = 2, 3$, the first and second models of Section 2.2 give quite different disagreement points (and *a fortiori*, quite different final outcomes), i.e., the equilibrium threat points in the first model are, in general, very different from the Nash equilibria in the underlying noncooperative game. In one player's choice of a threat strategy in the first model, there is a tradeoff between threatening the other players and guaranteeing his own higher threat payoff. In this section we show that when the number of players is large, this tradeoff may disappear, i.e., guaranteeing his own higher threat payoff may become more significant. Formally, we show under a certain condition that when the number of players is large enough,

⁵In this paper, we do not consider coalitional behavior. If we introduce coalitional behavior, we meet similar variable threat bargaining inside a coalition; if a coalition considers a deviation from a final outcome, the players in the coalition bargain for a new cooperative outcome based on a new threat point among themselves. An investigation of such coalitional behavior is possibly made in term of the "coalition-proof" Nash equilibrium by Bernheim-Peleg-Whinston (1987). In the above example, $(\alpha_1, \beta_2, \gamma_1)$ is not coalition-proof, and the other equilibrium threat point is coalition-proof. It is, however, easy to find another example which does not have a coalition-proof equilibrium.

any equilibrium threat point in the associated bargaining game $\Gamma(G,T)$ for a bargaining problem (G,T) with sidepayments becomes approximately a Nash equilibrium in the underlying game G and *vice versa*.

We consider a sequence $\{(G^\nu, T^\nu)\} = \{((N^\nu, \{X_i^\nu\}, \{h_i^\nu\}), T^\nu)\}$ of bargaining problems with sidepayments defined by (2.1) with $|N^\nu| \rightarrow +\infty$ as $\nu \rightarrow +\infty$, and assume the following condition:

$$\text{Max}_{i \in N^\nu} \text{Max}_{s_{-i}} \text{Max}_{s_i, t_i} \sum_{j \neq i} (h_j^\nu(s_{-i}, s_i) - h_j^\nu(s_{-i}, t_i)) / |N^\nu| \rightarrow 0 \text{ as } \nu \rightarrow +\infty. \quad (2.9)$$

For this sequence of bargaining problems, the following theorem holds.

Theorem 2.5.A. For any $\epsilon > 0$, there is a positive integer n_ϵ such that for any (G^ν, T^ν) with $|N^\nu| \geq n_\epsilon$,

i) any Nash equilibrium ρ in G^ν is an ϵ -equilibrium threat point in $\Gamma(G^\nu, T^\nu)$, i.e., for all $i \in N^\nu$,

$$f_i^N(T^\nu, h^\nu(\rho)) + \epsilon \geq f_i^N(T^\nu, h^\nu(\rho_{-i}, r_i)) \text{ for all } r_i \in M_i(G^\nu);$$

ii) any equilibrium threat point q in $\Gamma(G^\nu, T^\nu)$ is an ϵ -Nash equilibrium in game G^ν , i.e., for all $i \in N^\nu$,

$$h_i(q) + \epsilon \geq h_i(q_{-i}, r_i) \text{ for all } r_i \in M_i(G^\nu).$$

Theorem 2.5.A asserts, under condition (2.9), that for a large bargaining problem (G^ν, T^ν) with sidepayments, the equilibrium threat points in $\Gamma(G^\nu, T^\nu)$ and the Nash equilibria in the underlying noncooperative game G^ν are approximately equivalent. This means that threatening the other players becomes less significant than guaranteeing his own higher payoff when the number of players is getting larger. Thus the bargaining models of Section 2.2 yield virtually the same outcomes -- the advantage of a commitment to play a threat strategy almost disappears -- when the number of players is large.

Condition (2.9) states that the total effect of a strategy change by a single player on the other players' payoffs becomes small relative to the number of players. For example, if one player affects the other players equally, the per capita effect becomes negligible for a large game. Alternatively, a single player affects a small number of players with the same magnitude even when n becomes large. Condition (2.9) excludes the possibility that the total effect remains comparable with the total number of players. Condition (2.9) is necessary for the above theorem, the reason for which is as follows. Observe that the formula of the Nash outcome, given in Lemma 2.4.1, has two parts -- threatening the other players and guaranteeing his own higher threat payoff. In the formula, the first effect appears as the average of total effects on the other players' payoffs but the second effect appears directly. Hence if a sequence of bargaining problems satisfies (2.9), then the effect of a threat strategy on the other players becomes negligible when the number of players is large.

Proof of Theorem 2.5.A. Since T^ν is defined by (2.1), it can be described as

$$T^\nu = \{u \in \mathbb{R}^{|N^\nu|} : \sum_{i \in N^\nu} u_i/A^\nu \leq 1 \text{ and } u_i \geq B_i^\nu \text{ for all } i \in N^\nu\},$$

where $A^\nu = \max_{\rho \in M(G^\nu)} \sum_i h_i^\nu(\rho)$ and $B_i^\nu = \min_{\rho \in M(G^\nu)} h_i^\nu(\rho)$ for $i \in N$. Hence, by Lemma 2.4.1,

the Nash outcome $f^N(T^\nu, d)$ is given by

$$f_i^N(T^\nu, d) = (A^\nu - \sum_j d_j)/|N^\nu| + d_i \text{ for all } i \in N^\nu. \quad (2.10)$$

Let $\epsilon > 0$ be given. Denote the numerator, $\text{Max}_{i \in N^\nu} \text{Max}_{s_{-i}} \text{Max}_{s_i, t_i} \sum_{j \neq i} (h_j^\nu(s_{-i}, s_i) - h_j^\nu(s_{-i}, t_i))$, of (2.9) by K^ν . By (2.9), we can take an integer n_o so that $\epsilon > K^\nu/(|N^\nu|-1)$ for all $\nu \geq n_o$. Consider $G^\nu = (N^\nu, \{X_i^\nu\}, \{h_i^\nu\})$ with $\nu \geq n_o$. Let ρ be any Nash equilibrium in G^ν , and let q be an equilibrium threat point in $\Gamma(G^\nu, T^\nu)$. It suffices to show that for all $i \in N^\nu$, i) $f_i^N(T^\nu, h^\nu(\rho)) + \epsilon \geq f_i^N(T^\nu, h^\nu(\rho_{-i}, r_i))$ for all $r_i \in M_i(G^\nu)$; and ii) $h_i^\nu(q) + \epsilon \geq h_i^\nu(q_{-i}, r_i)$ for

all $r_i \in M_i(G^\nu)$.

Consider inequality i). Let r_i be an arbitrary strategy in $M_i(G^\nu)$. Since ρ is a Nash equilibrium, $h_i^\nu(\rho_{-i}, r_i) \leq h_i^\nu(\rho)$ for an arbitrary $r_i \in M_i(G^\nu)$. Using (2.10), the difference between $f_i^N(T^\nu, h^\nu(\rho_{-i}, r_i))$ and $f_i^N(T^\nu, h^\nu(\rho))$ is evaluated as follows:

$$\begin{aligned}
& f_i^N(T^\nu, h^\nu(\rho_{-i}, r_i)) - f_i^N(T^\nu, h^\nu(\rho)) \\
&= \left((A^\nu - \sum_j h_j^\nu(\rho_{-i}, r_i)) / |N^\nu| + h_i^\nu(\rho_{-i}, r_i) \right) - \left((A^\nu - \sum_j h_j^\nu(\rho)) / |N^\nu| + h_i^\nu(\rho) \right) \\
&= \left(\sum_{j \neq i} h_j^\nu(\rho) - \sum_{j \neq i} h_j^\nu(\rho_{-i}, r_i) \right) / |N^\nu| + \left(h_i^\nu(\rho_{-i}, r_i) - h_i^\nu(\rho) \right) (|N^\nu| - 1) / |N^\nu| \\
&\leq \left(\sum_{j \neq i} h_j^\nu(\rho) - \sum_{j \neq i} h_j^\nu(\rho_{-i}, r_i) \right) / |N^\nu| \quad \left(\text{by } h_i^\nu(\rho_{-i}, r_i) \leq h_i^\nu(\rho) \right) \\
&\leq K^\nu / |N^\nu| \leq K^\nu / (|N^\nu| - 1) < \epsilon.
\end{aligned}$$

This implies inequality i).

Consider ii). Let $r_i \in M_i(G^\nu)$. Since q is an equilibrium threat point, we have, by Lemma 2.4.1,

$$\begin{aligned}
0 &\leq f_i^N(T^\nu, h^\nu(q)) - f_i^N(T^\nu, h^\nu(q_{-i}, r_i)) \\
&= \left((A^\nu - \sum_j h_j^\nu(q)) / |N^\nu| + h_i^\nu(q) \right) - \left((A^\nu - \sum_j h_j^\nu(q_{-i}, r_i)) / |N^\nu| + h_i^\nu(q_{-i}, r_i) \right) \\
&= \left(\sum_{j \neq i} h_j^\nu(q_{-i}, r_i) - \sum_{j \neq i} h_j^\nu(q) \right) / |N^\nu| + \left(h_i^\nu(q) - h_i^\nu(q_{-i}, r_i) \right) (|N^\nu| - 1) / |N^\nu|.
\end{aligned}$$

This is equivalent to

$$-\left(\sum_{j \neq i} h_j^\nu(q_{-i}, r_i) - \sum_{j \neq i} h_j^\nu(q) \right) / |N^\nu| \leq \left(h_i^\nu(q) - h_i^\nu(q_{-i}, r_i) \right) (|N^\nu| - 1) / |N^\nu|.$$

The left hand side is greater than $-K^\nu / |N^\nu|$, which implies

$$-K^\nu / |N^\nu| \leq \left(h_i^\nu(q) - h_i^\nu(q_{-i}, r_i) \right) (|N^\nu| - 1) / |N^\nu|. \quad (2.11)$$

Since $\epsilon > K^\nu / (|N^\nu| - 1)$, we have $\epsilon \times (|N^\nu| - 1) > K^\nu$. Hence we have, by (2.11),

$$-\epsilon \times (|N^\nu| - 1) / |N^\nu| < \left(h_i^\nu(\mathbf{q}) - h_i^\nu(\mathbf{q}_{-i}, r_i) \right) (|N^\nu| - 1) / |N^\nu|,$$

$$\text{i.e., } h_i^\nu(\mathbf{q}) + \epsilon > h_i^\nu(\mathbf{q}_{-i}, r_i). \quad \square$$

Chapter 3

Commitments to Threat Strategies in Nash Bargaining

3.1 Introduction

Nash (1953) introduced the two-person bargaining model with variable threats. The model consists of two stages, where players choose threat strategies in the first stage and then bargain for a cooperative outcome in the second stage. The threat strategies chosen in the first stage will be played if the players fail in achieving an agreement in the second stage. It is assumed, in this model, that once a player chooses a threat strategy, he has to play it in the case of disagreement; the model presumes commitments to play the threat strategies. In principle, however, a commitment to play a threat strategy is voluntary (if it is possible); each player can decide whether or not he makes a commitment. This suggests some extensions of Nash's model so that they include players' commitment choices. In this chapter, we consider three extensions. Then the question arises as to whether the same outcome results in the extensions as in Nash's original model. In the two person case, these three extensions give essentially the same result as that given by Nash. This is not the case for more than two players; the result depends upon an extension. In one extension, Nash's result always holds. In other two extensions, however, we give a three person example where not all players choose commitments in equilibrium. Since one extension is clear-cut and the other two are regarded as variants, we confine ourselves mainly to that extension, and consider the other two in the last subsection.

The main extension model consists of four stages. In the first stage, each player decides whether or not to choose a commitment and announces it to the other players. In the second stage, the players who chose commitments make threat strategy choices, and those who did not commit do not move. In the third stage, players bargain for a cooperative outcome. If they agree on a cooperative outcome, this outcome gives final payoffs, and otherwise, the outcome of

the fourth stage gives final payoffs. In the fourth stage, those who committed in the first stage simply play their threat strategies chosen in the second stage, and those who did not commit choose strategies freely. A disagreement outcome, on which bargaining in the third stage is based, consists of the threat strategies of the committed players and the strategies chosen in the last stage by the uncommitted players. We need the fourth stage because of the introduction of the commitment choice in the first stage -- the fourth stage is hidden in Nash's original model.

In our four stage model, we consider a modification of the solution concept given by Nash (1953). The solution, called a solution equilibrium, induces the Nash (1950) bargaining solution, to the bargaining stage, relative to the threat strategies chosen in the threat stage; and in the threat stage, each player chooses a threat strategy based on the assumption that the Nash bargaining solution will be played in the bargaining stage. Although this is a combination of cooperative and noncooperative solution concepts, it can be regarded as a refinement of a subgame perfect equilibrium in the two stage model (see Kaneko-Mao (1992)).

We show, in the two-person case, that each player chooses to commit to play a threat strategy in a solution equilibrium. This means that Nash's (1953) model results endogenously. In the case of more than two players, however, this does not hold. We give a three-person example in which, in any solution equilibrium, not all the players commit to play threat strategies.

The first variant, which we will discuss in Section 3.5, is obtained by eliminating the first stage from the main model. In this variant, commitment and threat strategy choices are simultaneously made. Contrary to the main model, commitment and threat strategy are always chosen in a solution equilibrium, regardless of the number of players. However, when we generalize the main model to include the variant as a subgame, the same conclusion holds as in the main model, in the two-person and more than two person cases.

3.2 Bargaining Game $\Gamma(G)$ with Commitment Choices

In this section, we define a bargaining game $\Gamma(G)$ with commitment choices. Let G denote a finite n -person noncooperative game with $G = (N, \{X_i\}, \{h_i\})$, where $N = \{1, \dots, n\}$ ($n \geq 2$) is the *player set*, and for player i , X_i is the finite set of *pure strategies*, and $h_i: \prod_{i \in N} X_i \rightarrow R$ is the *payoff function*. We denote the set of all mixed strategies of player i in game G by $M_i(G)$. The set of all jointly feasible payoff vectors is denoted by $F(G)$, i.e., $F(G)$ is the convex hull of the set $\{h(x) = (h_1(x), \dots, h_n(x)): x \in \prod_{i \in N} X_i\}$. We call $F(G)$ the *bargaining region* generated by G . The *bargaining game* $\Gamma(G)$ is defined based on this noncooperative game.

The bargaining game $\Gamma(G)$ is an extensive game with four stages. Stage 1 is the commitment choice stage, Stage 2 is the threat strategy choice stage, Stage 3 is the utility demand stage and Stage 4 is the noncooperative game stage.

Stage 1. Each player $i \in N$ chooses $a_i = 1$ (commitment) or $a_i = 0$ (noncommitment), and announces it to the other players.

Stage 2. Each player $i \in N$ chooses a strategy $p_i \in M_i(G) \cup \{\phi\}$ and announces it to the other players. It is required that if player i chose $a_i = 1$, then $p_i \in M_i(G)$; and if $a_i = 0$, then $p_i = \phi$.

Stage 3. Each player $i \in N$ chooses a utility demand u_i and announces it to the other players.

Stage 4. Each player $i \in N$ chooses a strategy $q_i \in M_i(G)$. It is required that if player i chose $p_i \in M_i(G)$ in Stage 2, then $q_i = p_i$; and otherwise, no restriction is required.

By the symbol ϕ we mean that player does not choose a threat strategy in Stage 2.

If the utility demand vector $u = (u_1, \dots, u_n)$ chosen in the third stage belongs to the bargaining region $F(G)$, the *final payoff* for player i is given by u_i , and otherwise, it is given by $h_i(q)$.

We assume that the above extensive game has the fourth stage even though the players agree on a cooperative outcome in the third stage, i.e., $u = (u_1, \dots, u_n) \in F(G)$. In the case of $u \in$

$F(G)$, however, the probability to go to the fourth stage is zero. This zero probability should be viewed as representing arbitrarily small but positive probabilities. That is, our model is the limit case of ones where with small positive probabilities, the game goes to the fourth stage even though the utility demand vector u is feasible.⁶

The bargaining game $\Gamma(G)$ has three types of proper subgames. We denote by $\Gamma(G; \mathbf{a})$ the maximal proper subgame determined by commitment decisions $\mathbf{a} = (a_1, \dots, a_n)$. $\Gamma(G; \mathbf{a}, \rho)$ is the subgame determined by commitment decisions \mathbf{a} and threat strategies ρ chosen in the second stage. $\Gamma(G; \mathbf{a}, \rho, u)$ is the minimal subgame determined by \mathbf{a} , ρ and u .

The bargaining game $\Gamma(G)$ is an extension of Nash's (1953) bargaining model with variable threats in the sense that $\Gamma(G)$ has his model as a subgame. The second and third stages of our model correspond to the first and second stages of Nash's model respectively. In the case of two players, the subgame $\Gamma(G; \mathbf{a})$ with $\mathbf{a} = (1, 1)$ can be regarded as exactly the same as Nash's model: if each player chooses a commitment, neither player makes a strategy decision in the fourth stage, i.e., the fourth stage does not exist virtually. The fourth stage of our model is hidden in Nash's original model. In other subgames, some player chooses a noncommitment, and makes a strategy choice in the fourth stage in the case of disagreement. Those subgames are different from Nash's model.

The question raised in Section 3.1 is restated in the present framework as whether or not the subgame determined by $\mathbf{a} = (1, \dots, 1)$ is chosen in equilibrium. To discuss this question, we have to extend also the solution concept given by Nash (1953) to our model.

First we describe a strategy concept for the above extensive game $\Gamma(G)$. Since each player observes the outcomes of the previous stages in $\Gamma(G)$, a strategy is a function of information of the previous stages. More precisely, a *strategy* of player i is defined as a four-tuple $s_i = (s_i^1, s_i^2, s_i^3, s_i^4)$, where

⁶This is the same idea as that of "trembling-hand" perfection of Selten (1975).

$$\begin{aligned}
s_i^1 \in \{0,1\}; \quad s_i^2: \{0,1\}^n \rightarrow M_i(G) \cup \{\phi\} \quad \text{with } s_i^2(a) = \phi \text{ if } a_i = 0, \text{ and } s_i^2(a) \in M_i(G) \text{ if } a_i = 1; \\
s_i^3: \prod_j \left(\{(0,\phi)\} \cup (\{1\} \times M_j(G)) \right) \rightarrow R; \quad \text{and } s_i^4: \prod_j \left(\{(0,\phi)\} \cup (\{1\} \times M_j(G)) \right) \times R^n \rightarrow M_i(G) \\
\text{with } s_i^4(a,p,u) = p_i \text{ if } p_i \in M_i(G). \quad (3.1)
\end{aligned}$$

In Stage 1, player i chooses simply either to commit or not to commit. In Stage 2, he acts depending upon $a = (a_1, \dots, a_n)$ announced in Stage 1, i.e., s_i^2 is a function of a . The requirement is that only committed players choose threat strategies, and the uncommitted players do not move. In Stage 3, his behavior depends upon a announced in Stage 1 and threat strategies p_j chosen in Stage 2 by the players j with $a_j = 1$. In Stage 4, his choice depends upon the demands u in Stage 3 as well as decisions in Stages 1 and 2. The additional requirement “ $s_i^4(a,p,u) = p_i$ if $p_i \in M_i(G)$ ” for Stage 4 expresses the commitment made in Stages 1 and 2, i.e., if player i chooses to commit to play a threat strategy, then in Stage 4 he plays the threat strategy chosen in Stage 2, and otherwise he can choose any strategy.

We consider the following solution concept for our bargaining game $\Gamma(G)$, which is an extension of that given in Nash (1953). A strategy combination $s = (s_1, \dots, s_n)$ is called a *solution equilibrium* of $\Gamma(G)$ iff

$$s \text{ is a subgame perfect equilibrium in } \Gamma(G); \text{ and} \quad (3.2)$$

for any pair (a,p) , $s^3(a,p) = u$ is the Nash bargaining solution

$$\text{relative to the disagreement point } q = s^4(a,p,u). \quad (3.3)$$

Here the *Nash bargaining solution* u relative to the disagreement point $q = s^4(a,p,u)$ is defined by

$$\prod_{i \in N(q)} (u_i - h_i(q)) \geq \prod_{i \in N(q)} (v_i - h_i(q)) \text{ for all } v \in F(G) \text{ with } v \geq h(q), \quad (3.4)$$

where $N(q) = \{i \in N: v_i > h_i(q) \text{ for some } v \in F(G) \text{ with } v \geq h(q)\}$. The Nash bargaining solution u relative to the disagreement point q is denoted by $f^N(q) = (f_1^N(q), \dots, f_n^N(q))$.

Since the disagreement point q is endogenously chosen, we do not make the nondegeneracy assumption that for some $v \in F(G)$, $v_i > h_i(q)$ for all $i \in N$. Therefore the Nash product is taken over the relevant players $N(q)$.

For two players, the solution concept of Nash (1953) is a part of our solution equilibrium: the restriction of our solution equilibrium to the subgame $\Gamma(G;(1,1))$ is the same as Nash's. If both players commit to play threat strategies in the first stage, then they choose threat strategies in the second stage, and do not make another strategy choice in the fourth stage. In this case, it follows from condition (3.3) that the outcome of the third stage is the Nash outcome relative to the threat point chosen in the second stage. Furthermore, (3.2) implies that each player maximizes the payoff given by the Nash outcome determined in the third stage. Thus our solution equilibrium induces Nash's (1953) solution concept to the subgame $\Gamma(G;(1,1))$. Of course, in the n -person case, the subgame $\Gamma(G;(1, \dots, 1))$ is regarded as a direct generalization of Nash's model. That is, a solution equilibrium induces *an equilibrium threat* (ρ_1, \dots, ρ_n) to the subgame $\Gamma(G;(1, \dots, 1))$.

The question raised in Section 3.1 is now formulated as whether the subgame $\Gamma(G;(1, \dots, 1))$ is realized in a solution equilibrium, more precisely, whether or not s^1 is equal to $(1, \dots, 1)$, for a solution equilibrium $s = (s^1, s^2, s^3, s^4)$. This question will be answered in the following sections. In the next section we show that for two players, this question is answered affirmatively. In Section 3.4, however, we show that this question is negatively answered by giving a three-person example.

3.3 Two-Person Case of Bargaining Game $\Gamma(G)$

The first theorem states that $a = (1,1)$ -- both players choose commitments -- is in a path

of a solution equilibrium.

Theorem 3.3.A. There is a solution equilibrium $s = (s^1, \dots, s^4)$ of $\Gamma(G)$ with $s^1 = (1, 1)$.

Our next step is to show that only the commitment choice (1,1) is possible in the solution equilibrium. To obtain this result, however, we exclude some nondegenerated cases; for example, when the payoff function $h_i(x)$ is constant for all $i \in N$, every outcome can be a solution equilibrium.

For the next result, we introduce the following notations: Let $s = (s^1, \dots, s^4)$ be an arbitrary solution equilibrium in the bargaining game $\Gamma(G)$. For each $a \in \{0, 1\}^2$, we denote by u^a the payoff vector induced by s to subgame $\Gamma(G; a)$, i.e., $u^a = s^3(a, s^2(a))$. For example, $u^{00} = (u_1^{00}, u_2^{00})$ is the payoff vector induced by s to the subgame $\Gamma(G; (0, 0))$, i.e., $u^{00} = f^N(q)$ and q is a Nash equilibrium in the underlying game G . The payoff vector depends on the strategies chosen in the fourth stage in this case. Note that the vector u^a is Pareto optimal in $F(G)$, since it is the Nash outcome relative to the threat strategy $q = s^4(a, s^2(a), s^3(a, s^2(a)))$.

Theorem 3.3.B. Let $s = (s^1, \dots, s^4)$ be an arbitrary solution equilibrium in the bargaining game $\Gamma(G)$ with two players. Then i) if $u^{00} \neq u^{11}$, $u^{10} \neq u^{11}$ or $u^{01} \neq u^{11}$, then $s^1 \neq (0, 0)$; ii) if $u^{10} \neq u^{11}$, then $s^1 \neq (1, 0)$; and iii) if $u^{01} \neq u^{11}$, then $s^1 \neq (0, 1)$.

From Theorems 3.3.A and 3.3.B, we have the following corollary.

Corollary. Let $s = (s^1, \dots, s^4)$ be an arbitrary solution equilibrium in the bargaining game $\Gamma(G)$ with two players. If $u^{10} \neq u^{11}$ and $u^{01} \neq u^{11}$, then $s^1 = (1, 1)$.

The corollary states, under the assumption $u^{10} \neq u^{11}$ and $u^{01} \neq u^{11}$, that both players choose commitments in any solution equilibrium in game $\Gamma(G)$. The assumption $u^{10} \neq u^{11}$ and

$u^{01} \neq u^{11}$ can be regarded as a nondegeneracy condition. In fact, the following lemma holds.

Lemma 3.3.1. We always have $u_1^{10} \geq u_1^{11}$ and $u_2^{01} \geq u_2^{11}$. By Pareto optimality, they are equivalent to $u_2^{10} \leq u_2^{11}$ and $u_1^{01} \leq u_1^{11}$.

These inequalities hold in equations for some examples, but they are rare cases. The restriction of a solution equilibrium to subgame $\Gamma(G;(1,0))$ is regarded as a Stackelberg equilibrium in the sense that player 2 (follower) maximizes his payoff, given 1's strategy and player 1 (leader) maximizes his payoff expecting 2's reactions to his move. Payoff vector u^{10} is determined by a Stackelberg equilibrium in this sense. On the other hand, u^{11} is the outcome of a "Nash equilibrium" in the game $\Gamma(G;(1,1))$. In general, the "leader" in Stackelberg equilibrium obtains a higher payoff than the payoff in Nash equilibrium. They coincide only when the underlying game is "degenerated".

The following example shows the properties stated in the theorems.

Example 3.1. (Battle of the Sexes). A game $G = (N, \{X_i\}, \{h_i\})$ is given as follows: $N = \{1,2\}$, $X_1 = \{\alpha_1, \alpha_2\}$, $X_2 = \{\beta_1, \beta_2\}$, and the payoffs are given in Table 3.1.

Table 3.1

	β_1	β_2
α_1	2, 1	-1, -1
α_2	-1, -1	1, 2

Table 3.2

	β_1	β_2
α_1	2, 1	1.5, 1.5
α_2	1.5, 1.5	1, 2

The bargaining region $F(G)$ is given as the convex hull of (2,1), (1,2) and (-1, -1). We calculate the Nash bargaining outcome for each threat point in pure strategy. For example, the threat payoff vector (-1,-1) determined by (α_1, β_2) leads to the Nash bargaining outcome (1.5,1.5). Also

the threat payoff vector (2,1) determined by (α_1, β_1) leads to the Nash outcome (2,1) itself. Similarly, all Nash outcomes from threat payoff vectors are given in Table 3.2. The equilibrium threat in the subgame $\Gamma(G, (1,1))$ is obtained by calculating the Nash equilibrium in the matrix game in Table 3.2, which is given as (α_1, β_2) . The Nash bargaining outcome is (1.5,1.5), which gives $u^{11} = (1.5, 1.5)$.⁷

In this example, $u^{10} = (2,1)$ and $u^{01} = (1,2)$, but u^{00} is not uniquely determined. Observe that the assumption $u^{10} \neq u^{11}$ and $u^{01} \neq u^{11}$ for the corollary hold here. Thus it follows from the corollary that $s^1 = (1,1)$ for any solution equilibrium $s = (s^1, \dots, s^4)$. Since the underlying game G has three Nash equilibria in mixed strategies, there are three corresponding solution equilibria where they differ only in the subgames $\Gamma(G; a, \rho, u)$.

Before giving the proofs of the above results, we introduce some terminologies and mention certain necessary conditions of a solution equilibrium $s = (s^1, \dots, s^4)$:

1) For subgame $\Gamma(G; (1,1))$: $s^2(1,1) = \hat{\rho}$ and $s^3((1,1), \hat{\rho})$ satisfy

$$f_i^N(\hat{\rho}) \geq f_i^N(\rho_i, \hat{\rho}_{-i}) \text{ for all } \rho_i \in M_i(G) \text{ and } i = 1, 2; \quad (3.5)$$

$$s^3((1,1), \hat{\rho}) = f^N(\hat{\rho}) = u^{11}. \quad (3.6)$$

2) For subgame $\Gamma(G; (1,0))$: $s^2(1,0) = (\bar{\rho}_1, \phi)$ satisfies

$$f_1^N(\bar{\rho}_1, \xi_2(\bar{\rho}_1)) \geq f_1^N(\rho_1, \xi_2(\rho_1)) \text{ for all } \rho_1 \in M_1(G), \quad (3.7)$$

where $\xi_2: M_1(G) \rightarrow M_2(G)$ is a best response function defined by

$$\text{for any given } \rho_1 \in M_1(G), h_2(\rho_1, \xi_2(\rho_1)) \geq h_2(\rho_1, \rho_2) \text{ for all } \rho_2 \in M_2(G) \quad (3.8);$$

and $s^3((1,0), (\bar{\rho}_1, \phi)) = u^{10}$ is defined to be $f^N(\bar{\rho}_1, \xi_2(\bar{\rho}_1))$.

⁷For the calculation of the threat strategies and the bargaining outcome in two person problems, see Owen (1982), p.151-152.

- 3) For subgame $\Gamma(G;(0,1))$: $s^2(0,1) = (\phi, \bar{p}_2)$, ξ_1 and $s^3((0,1),(\phi, \bar{p}_2)) = u^{01}$ are defined in the symmetric way to 2).
- 4) For subgame $\Gamma(G;(0,0))$: $s^3((0,0),(\phi, \phi)) = f^N(q) = u^{00}$, where $q = (q_1, q_2)$ is a Nash equilibrium of the underlying game G .

The above four conditions are necessary for $s = (s^1, s^2, s^3, s^4)$ to be a solution equilibrium. Conversely, for these subgames, there is a combination (s^2, s^3, s^4) such that the restriction of it to each subgame $\Gamma(G;a)$ ($a \in \{0,1\}^2$) satisfies conditions 1) - 4). The existence of (s^2, s^3, s^4) satisfying condition 1) is proved by Nash (1953). The existence of (s^2, s^3, s^4) satisfying condition 2), 3) and 4) is straightforward. Since the four cases are mutually exclusive, we can require a common (s^2, s^3, s^4) to satisfy conditions 1) - 4). The existence of a solution equilibrium $s = (s^1, \dots, s^4)$ is ensured by finding an appropriate s^1 . First we prove Lemma 3.3.1.

Proof of Lemma 3.3.1: Recall $u^{10} = f^N(\bar{p}_1, \xi_2(\bar{p}_1))$, $u^{01} = f^N(\xi_1(\bar{p}_2), \bar{p}_2)$ and $u^{11} = f^N(\hat{p}_1, \hat{p}_2)$. By (3.7), $u_1^{10} = f_1^N(\bar{p}_1, \xi_2(\bar{p}_1)) \geq f_1^N(\hat{p}_1, \xi_2(\hat{p}_1))$. By (3.5), $f_2^N(\hat{p}_1, \hat{p}_2) \geq f_2^N(\hat{p}_1, \xi_2(\hat{p}_1))$, which is equivalent to $f_1^N(\hat{p}_1, \xi_2(\hat{p}_1)) \geq f_1^N(\hat{p}_1, \hat{p}_2)$, since $f^N(p)$ is Pareto optimal for any $p \in M(G)$. Therefore $u_1^{10} \geq f_1^N(\hat{p}_1, \xi_2(\hat{p}_1)) \geq f_1^N(\hat{p}_1, \hat{p}_2) = u_1^{11}$. Similarly we have $u_2^{11} \leq u_2^{01}$. \square

Proof of Theorem 3.3.A: We show that there is a solution equilibrium s with $s^1 = (1,1)$. Let $s^1 = (1,1)$ and we define s^2, s^3, s^4 in a consistent manner with 1) - 4). As discussed above, it suffices to show that no unilateral deviation from $a = (1,1)$ in the first stage makes a player better off. By Lemma 3.3.1, $u_2^{11} \geq u_2^{10}$ and $u_1^{11} \geq u_1^{01}$. This implies that neither player is better off by a deviation from $a = (1,1)$. \square

Proof of Theorem 3.3.B: i) We show that $s^1 \neq (0,0)$ if $u^{00} \neq u^{11}$, $u^{10} \neq u^{11}$ or $u^{01} \neq u^{11}$. It suffices to show that player 1 or 2 can improve his payoff by deviating from $(0,0)$. Note that it follows from the Pareto optimality of u^a that if $u^a \neq u^{a'}$, then $u_i^a > u_i^{a'}$ is equivalent to $u_j^a < u_j^{a'}$.

Let $u^{00} \neq u^{11}$. Then $u_1^{00} < u_1^{11}$ or $u_1^{00} > u_1^{11}$. The second inequality is equivalent to u_2^{00}

$< u_2^{11}$. Since $u_1^{11} \leq u_1^{10}$ and $u_2^{11} \leq u_2^{01}$ by Lemma 3.3.1, we have $u_1^{00} < u_1^{10}$ or $u_2^{00} < u_2^{01}$. Therefore player 1 or 2 can obtain a higher payoff by deviating from (0,0).

Let $u^{10} \neq u^{11}$. Then $u_1^{11} < u_1^{10}$ by Lemma 3.3.1. If $u_1^{00} \leq u_1^{11}$, then $u_1^{00} < u_1^{10}$, which implies that player 1 is better off by deviating from (0,0). If $u_1^{00} > u_1^{11}$, equivalently, $u_2^{00} < u_2^{11}$, then $u_2^{00} < u_2^{01}$ since $u_2^{11} \leq u_2^{01}$ by Lemma 3.3.1. In this case, player 2 is better off by deviating from (0,0).

In the case of $u^{01} \neq u^{11}$, it is shown in the symmetric way that player 1 or 2 can improve u^{00} .

ii) Let $u^{10} \neq u^{11}$. Then $u_2^{10} < u_2^{11}$ by lemma 3.3.1, which implies that player 2 is better off by his deviation from (1,0). Assertion iii) is Symmetric to ii). \square

3.4 Three-Person Example with Noncommitment Solution Equilibria

In Section 3.3, we showed that for a two-person game G with the nondegeneracy assumption, Nash's (1953) model results in a solution equilibrium of game $\Gamma(G)$. In this section, we show, giving an example with three players, that this result is no longer true for more than two players. In the following example, for any solution equilibrium $s = (s^1, \dots, s^4)$, $s^1 \neq (1,1,1)$, i.e., not all the players choose commitments. We conclude from this example that no extension of the result of Section 3.3 is expected in the case of more than two players.

Table 3.3

	β_1	β_2
α_1	0, 16, 2	0, 0, 18
α_2	7, 4, 7	1, 1, 16

γ_1

Table 3.4

	β_1	β_2
α_1	1, 2, 0	0, 0, 0
α_2	2, 2, 5	18, 0, 0

γ_2

Example 3.2. Let $G = (N, \{X_i\}, \{h_i\})$, where $N = \{1,2,3\}$, $X_1 = \{\alpha_1, \alpha_2\}$, $X_2 = \{\beta_1, \beta_2\}$, $X_3 =$

$\{\gamma_1, \gamma_2\}$ and the payoffs are given in Tables 3.3 and 3.4. In this example, $s^1 = (1,1,1)$ is not sustained by a solution equilibrium, indeed only $s^1 = (1,1,0)$ and $s^1 = (1,0,0)$ are sustained by solution equilibria of $\Gamma(G)$.

First we show that $s^1 \neq (1,1,1)$ for any solution equilibrium s . Consider subgame $\Gamma(G;(1,1,1))$. To determine a solution equilibrium in subgame $\Gamma(G;(1,1,1))$, we calculate the Nash bargaining solution relative to the disagreement point determined by each pure strategy triple, which gives new payoff matrices. A solution equilibrium in subgame $\Gamma(G;(1,1,1))$ will be calculated based on these payoff matrices. The Pareto surface of the bargaining region $F(G)$ is the convex hull of $\{(18,0,0), (0,16,2), (0,0,18)\}$. Every payoff vector determined by pure strategies is either on or under the Pareto surface. If the disagreement point $d = (d_1, d_2, d_3)$ is on the Pareto surface, then d itself is the Nash bargaining solution $f^N(F(G), d)$. For example, $(\alpha_1, \beta_1, \gamma_1)$ gives $f^N(F(G), d) = d = (0,16,2)$. If d is under the Pareto surface, then the Nash bargaining solution $f^N(F(G), d)$ is given as

$$f_i^N(F(G), d) = \left(18 - (d_1 + d_2 + d_3)\right)/3 + d_i \quad \text{for all } i = 1, 2, 3. \quad (3.9)$$

For example, $(\alpha_1, \beta_1, \gamma_2)$ determines threat payoff vector $d = (1,2,0)$, which gives $f^N(F(G), d) = (6,7,5)$. Similarly each payoff vector in Tables 3.3 and 3.4 gives a Nash bargaining outcome. Thus we obtain Tables 3.5 and 3.6.

Table 3.5

	β_1	β_2
α_1	0, 16, 2	0, 0, 18
α_2	7, 4, 7	1, 1, 16
	γ_1	γ_2

Table 3.6

	β_1	β_2
α_1	6, 7, 5	6, 6, 6
α_2	5, 5, 8	18, 0, 0
	γ_1	γ_2

It follows from the linearity of (3.9) with respect to $d = h(p)$ that an equilibrium of the game in Tables 3.5 and 3.6 represents an equilibrium threat of subgame $\Gamma(G;(1,1,1))$. The game in Table 3.5 and 3.6 has a unique Nash equilibrium $(\alpha_1, \beta_1, \gamma_2)$ in mixed strategies. Thus the equilibrium threat strategy point of $\Gamma(G;(1,1,1))$ is $(\alpha_1, \beta_1, \gamma_2)$, which leads to the final bargaining outcome $u^{111} = (6,7,5)$.

To show that $(1,1,1)$ is not sustained by a solution equilibrium, it suffices to find out a successful deviation. Consider the subgame $\Gamma(G;(1,1,0))$, where player 3 does not commit to play a threat strategy. In the fourth stage of this subgame, only player 3 will move and maximize his payoff given strategies (p_1, p_2) chosen by players 1 and 2 in the second stage -- player 3 is the follower and the other players are the leaders for 3. Player 3 chooses a best response to the threat strategies chosen in Stage 2. Being independent of threat strategies (p_1, p_2) , however, player 3 always chooses γ_1 as the best response, since γ_1 dominates γ_2 in Table 3.3 and 3.4. Hence players 1 and 2 play the game of Table 3.5 to make their threat strategy decisions in Stage 2. In the game of Table 3.4.3, (α_2, β_1) is the unique Nash equilibrium in mixed strategies. Thus, (α_2, β_1) is chosen in Stage 2 and γ_1 is chosen in Stage 4, which leads to the bargaining payoff $u^{110} = (7,4,7)$. By deviating from $s^1 = (1,1,1)$ to $(1,1,0)$, player 3 improves his final bargaining payoff from 5 in $(6,7,5)$ to 7 in $(7,4,7)$. Therefore $s^1 \neq (1,1,1)$.

In this example, $s^1 = (1,1,0)$ and $s^1 = (1,0,0)$ are sustained by solution equilibria. To see this, first we list the bargaining payoffs of subgame $\Gamma(G;a)$ for all a :

$$\begin{aligned} u^{000} &= (7,4,7); & u^{100} &= (7,4,7); & u^{010} &= (7,4,7); & u^{001} &= (5,5,8); \\ u^{110} &= (7,4,7); & u^{101} &= (6,7,5); & u^{011} &= (5,5,8); & u^{111} &= (6,7,5). \end{aligned}$$

We have already shown that $s^1 \neq (1,1,1)$. Comparing u^{010} with u^{011} , player 3 has the incentive to deviate from $s^1 = (0,1,0)$ to $(0,1,1)$, since he can improve his bargaining outcome from 7 to 8. Thus $s^1 \neq (0,1,0)$. Similarly, we have $s^1 \neq (0,0,0)$, $s^1 \neq (0,0,1)$, $s^1 \neq (1,0,1)$ and $s^1 \neq (0,1,1)$. We can see that only $(1,1,0)$ and $(1,0,0)$ are sustained by solution equilibria, since no

successful deviation is possible from $(1,1,0)$ and $(1,0,0)$.

In the above example, some players commit to play threat strategies, but some others don't in the solution equilibria. This shows that Nash's original variable threat model does not result in the bargaining game $\Gamma(G)$ with more than two players.

3.5 Two Variants of the Bargaining Game $\Gamma(G)$

In the model of Section 3.2, the threat strategy choice is made after the announcement of commitment decisions. That is, when each player chooses his threat strategy, he knows the other players' commitment decisions. This implies that a change in the commitment decision of one player may affect other players' threat strategy choices. If we change the rules of the game so that commitment decision and threat strategy choice are made simultaneously, then this effect disappears, and the counterexample of Section 3.4 is no longer valid. Indeed, Nash's equilibrium threat results, regardless of the number of players. This is the first variant of the bargaining game $\Gamma(G)$ we consider in this section. If, however, we extend the bargaining game $\Gamma(G)$ keeping the original structure so that at the first stage he can postpone the commitment to the second stage, the model includes the first variant as a subgame, then the same conclusion holds as in the bargaining game $\Gamma(G)$.

1) The first variant is obtained from the bargaining game $\Gamma(G)$ by eliminating Stage 1 and allowing each player to choose freely a strategy from $M_i(G) \cup \{\phi\}$ in Stage 2. In this variant, if player i chooses a strategy p_i from $M_i(G)$, then he automatically commits to play p_i in Stage 4, and if he chooses ϕ , he does not commit and chooses a strategy q_i from $M_i(G)$ in Stage 4. In this sense, commitment choice and threat strategy choice are simultaneously made.

In this variant, the solution equilibrium is defined in the same way as in Section 3.2, while the definition of a strategy is modified in accordance with the rules of the variant model. Then the following theorem holds.

Theorem 3.5.A For any noncooperative game $G = (N, \{X_i\}, \{h_i\})$, the variant model based on G has a solution equilibrium $s = (s^2, s^3, s^4)$ such that $s^2 = \hat{p} = (\hat{p}_1, \dots, \hat{p}_n)$ is an equilibrium threat point in Nash's (1953) sense, i.e., $f_i^N(\hat{p}) \geq f_i^N(p_i, \hat{p}_{-i})$ for all $p_i \in M_i(G)$ and $i \in N$.

The reasoning for this theorem is as follows. Suppose that $s^2 = (p_1, \dots, p_n)$ is an equilibrium threat point. Each player has two possible deviations: 1) a different threat strategy and 2) noncommitment ϕ . Since $(\hat{p}_1, \dots, \hat{p}_n)$ is an equilibrium threat point, a different threat strategy does not give a higher payoff. In the case of noncommitment ϕ , he chooses a mixed strategy in Stage 4. Again, since $(\hat{p}_1, \dots, \hat{p}_n)$ is an equilibrium threat point, this does not give a higher payoff. Thus $s^2 = (\hat{p}_1, \dots, \hat{p}_n)$ together with appropriate choices of s^3 and s^4 forms a solution equilibrium.

In fact, except in some degenerated cases, only the equilibrium threat point is sustained by a solution equilibrium. Since a player who chose ϕ in Stage 2 behaves as a follower of the players who chose threat strategies, it would be better for him to choose a threat strategy in Stage 2 -- using the same strategy in Stage 2 ensures that he will obtain at least the same payoff as by being a follower.

The conclusion of the consideration of the above variant is that if the commitment choices and threat strategy choices are simultaneously made, no counterexamples such as in Section 3.4 appear. It is possible, however, to think that each player can make simultaneous choices of a commitment with a threat strategy or separate choices of a commitment and a threat strategy. To consider this possibility, we provide a unified model of the bargaining game $\Gamma(G)$ and the above variant. In the unified model, the counterexample of Section 3.4 becomes valid again.

2) The unified model is obtained from the bargaining game $\Gamma(G)$ by changing Stage 1 to the following.

Stage 1*. Each player $i \in N$ chooses either to announce his commitment decision, $a_i = 1$ (commitment), $a_i = 0$ (noncommitment), or not to announce, $a_i = \phi$.

In Stage 1*, we allow each player to be silent, i.e., to choose ϕ . If player i chooses ϕ in Stage 1*, he can choose a threat strategy from $M_i(G)$ or ϕ again in Stage 2. In this unified model, each player can still announce his commitment or noncommitment in the first stage. And if all players choose ϕ in the first stage, the subgame after this stage becomes the above variant. In this sense, this model unifies the model of Section 3.2 and the above variant.

In the unified model, Nash's (1953) model can be regarded as corresponding to a subgame determined by (a_1, \dots, a_n) with $a_i = 1$ or ϕ for all i . If $(a_1, \dots, a_n) = (1, \dots, 1)$, then the subgame is exactly Nash's model, and if $(a_1, \dots, a_n) = (\phi, \dots, \phi)$, the subgame becomes the above variant. We can prove that an equilibrium threat $(\hat{p}_1, \dots, \hat{p}_n)$ is always sustained by a solution equilibrium $s = (s^1, \dots, s^4)$ to the subgame $\Gamma(G; a)$ determined by $a = (a_1, \dots, a_n)$ with $a_i = 1$ or ϕ for all i . Therefore our question is now whether or not (a_1, \dots, a_n) with $a_i = 1$ or ϕ for all i is sustained by a solution equilibrium.

In this unified model, we have the same results as those of Sections 3.3 and 3.4. For two player games, Nash's model is always sustained by a solution equilibrium. In the unified model, $s^1 = (\phi, \phi)$, $(1, \phi)$ or $(\phi, 1)$ are regarded as equivalent to $(1, 1)$. We can prove that these are sustained by solution equilibria. For more than two players, it is not true: the example of Section 3.4 again becomes credible.

In the example of Section 3.4, only $s^1 = (1, 0, 0)$, $(\phi, 0, 0)$, $(1, 1, 0)$, $(1, \phi, 0)$, $(\phi, 1, 0)$ and $(\phi, \phi, 0)$ are sustained by solution equilibria in the unified model. Let us see that (ϕ, ϕ, ϕ) is not sustained by a solution equilibrium. An equilibrium threat results in the subgame determined by $a = (\phi, \phi, \phi)$. From the result of Section 3.4, $u^{\phi\phi\phi} = u^{111} = (6, 7, 5)$. Consider the deviation of player 3 to $a_3 = 0$ in Stage 1*. In Stage 2, player 3 does not move, and players 1, 2 choose their threat strategies, based on the prediction of player 3's reaction in Stage 4. This is essentially the

same as in the case of the deviation $a_3 = 0$ from $(1,1,1)$ in Section 3.4. By the argument of Section 3.4, $a_3 = 0$ gives a better payoff. Thus (ϕ, ϕ, ϕ) is not on a solution equilibrium path. Similarly, we can show that $a = (a_1, a_2, a_3)$ with $a_i = 1$ or ϕ for all i is not on a solution equilibrium path either. Following almost the same argument as in Section 3.4, we can verify that only $s^1 = (1,0,0), (\phi,0,0), (1,1,0), (1,\phi,0), (\phi,1,0)$ and $(\phi,\phi,0)$ are sustained by solution equilibria.

Nash's bargaining model is always on an equilibrium path in our extensions for two players. For more than two players, however, this result depends upon an extension. Thus Nash's (1953) result is not affected by the introduction of commitment choices of threat strategies in the case of two players, but relies on an extension for more than two players.

Chapter 4

On the Inconsistent Behavior in Voting for Incumbents and Term Limitation

4.1. Introduction

This chapter considers seemingly inconsistent behavior in voting for candidates and for limitations on the terms of legislators. Some people have voted for the introduction of a term limit, while in the candidate elections, the same people have voted for incumbents who possibly would have been prevented from running if the term limit would have been enacted. Voters would vote for an incumbent if they choose a candidate only on the basis of the benefits he generates for them and if the incumbent can generate greater benefits. In that case, the voter might lose benefits by the introduction of term limitation and is expected to vote against it. This expectation is inconsistent with the recent observations of voting on term limitation. The purpose of this chapter is to show that such seemingly inconsistent behavior can happen even if voters pursue their self-interests in both the candidate and the term limitation elections. Indeed, an incumbent does generate more benefits, but his pursuit of votes may dilute such benefits for some voters.

Historically, in political campaigns in the USA at federal level and at most state levels, there was no restriction on the length of tenure; a candidate was officially allowed to run in elections regardless of the number of terms he had served in that office.⁸ Recently, voters in several states, e.g., California, Colorado and Oklahoma in 1990, voted for the introduction of a term limit, while at the same time, re-elected a large percentage of their incumbents of legislatures.⁹ In 1992, twelve states adopted term limitation for state legislators, and fourteen

⁸Except for the term limit on the president of the United States, which was imposed only after the second world war.

⁹For example, 96% of the incumbents were re-elected in California in 1992.

states enacted it for members of Congress.¹⁰ More states are expected to impose term limit in the near future.

Some work has been done which is relevant to this issue. Using a life cycle model, Adams and Kenny (1986) tried to calculate the optimal tenure of officials. One of their results is that if the cost of government were to fall throughout tenure, the unlimited tenure could be uniformly preferred. This does not seem to support the phenomenon mentioned above. Dick and Lott (1993) explained the seemingly inconsistent behavior, assuming that voters care about social welfare for term limitation but care about their own benefits in candidate elections. An incumbent being in the office for more terms can bring more benefits to voters and obtain advantage in campaign, which may lead him to shirking behavior and decrease social welfare. Thus voters vote for term limitation to avoid that loss of social welfare.¹¹ In our approach, contrary to theirs, such an inconsistency may occur even if voters pursue their self-interests in both term limitation and candidate elections. If a voter is concerned only about the benefits he will receive from a candidate, it would be more natural for him also to be concerned about the benefits he will get as a result of the imposition of term limitation, though the latter is more related to social welfare than the former.

In Section 4.2, we formulate a candidate competition *within a district* as a two-person game where the players are candidates from parties 1 and 2. The voters in the district are

¹⁰In Ohio state, voters have voted to limit the terms of their house representatives and also U.S. senators.

¹¹One might explain the seemingly inconsistent behavior, using the Prisoner's Dilemma: (1) if the legislators of other districts are fixed, it is better to have a more senior incumbent for a district, but (2) it is better to have younger legislators as a whole. Thus each voter votes for an incumbent in a district, and votes for term limitation on a state level. This argument may be applied to some cases, but has some weaknesses. It would be more appropriate to assume that having more experience gives a higher efficiency to an incumbent. Excluding the shirking issue, it is difficult to have Assumption (2), but taking shirking into account, Assumption (1) is problematic. Furthermore, the Prisoner's Dilemma argument cannot be applied to the inconsistency in voting for congressmen (and U.S. senators), while voting also for their term limits.

divided into two groups of supporters respectively; they differ only in their political preferences. All voters are assumed to be interested in the economic benefits they would get from the candidates. Each candidate offers a platform which specifies a distribution of the economic values to those two supporter groups. A voter votes for a candidate based on the economic benefits he will get from the platform as well as his political preferences. The objective of each candidate is to maximize the votes he receives.¹² Since the total votes are divided between two candidates, the game is constant-sum. This two-person game may not have a Nash equilibrium in pure strategies. For that reason, we use the maximin strategies of the candidates as the outcome of the game.

In Section 4.3, we derive the incumbent's maximin strategies of dividing the benefits between the two political groups. One result states that an incumbent would win the election since he can always generate more total benefits. We show that the benefits that a majority of voters receive from their incumbent increases as the incumbent becomes more experienced, up to some critical level and after it the benefits drop to a low level and stay low thereafter. This result plays a central role in the analysis of the term limitation in Section 4.4.

The introduction of a term limit in a state leads to an early occurrence of elections with two new candidates in all the districts. We assume that each voter compares the average benefits from the incumbent of his district over the terms he serves in office in cases with, and without, a term limit. In Section 4.4, we show the possibility that for a majority of voters in a state, the average benefits with a term limit is larger than those without term limit. This is caused by the fact that the pursuit of votes by an incumbent may dilute benefits for a majority of voters. Thus, those voters would vote for term limitation, while they still vote for the incumbent in the candidate elections.

¹²Here candidates are assumed to be self-interested; we do not consider logrolling and ideology as controllable variables for the candidates.

4.2. A Campaign Game within a District

We consider a state with a finite number of districts, each of which has candidate elections. The term limitation issue, however, is raised at the state level, that is, all voters in the state vote for or against the introduction of a term limit, and once it is passed, all districts in the state follow the restricted length of tenure. Although our ultimate purpose is to consider whether or not voters vote for the introduction of a term limit, we first consider candidate elections in one district. Then, in Section 4.4, we consider the possibility of the introduction of a term limit in a state by integrating the results for each district.

Consider a candidate election game within a district. There are two candidates, 1 and 2, nominated by two parties (e.g., the Republican and the Democratic parties). The set of voters is $I = [0,1]$. The voters in the district are divided into two groups, denoted by M_1 and M_2 , respectively. The meaning of these groups is as follows: a voter is in M_i , $i = 1,2$, if whenever the economic benefits offered him by the two candidates are equal he will vote for candidate i . It is assumed that voters vote for candidates by evaluating their policies but they are not strategic players.

We assume that candidate i , $i = 1,2$, can generate *total benefit* T_i . The total benefit T_i depends on the seniority of candidate i . In Sections 4.2 and 4.3, T_i is fixed since we consider the one-district problem, and in Section 4.4, T_i is treated as a variable since we consider the term limitation problem over all districts in a state.

In the competition, each candidate $i = 1,2$ chooses an economic package (or platform). Voters receive economic benefits from the package. A package is a distribution of the total benefits T_i between the groups of supporters. That is, it is a vector (b_{i1}, b_{i2}) , where b_{ip} ($p = 1,2$) represents the total economic benefits to all the voters in M_p . The set S_i of *strategies* for candidate i is given as

$$S_i = \{(b_{i1}, b_{i2}) : b_{i1} + b_{i2} = T_i \text{ and } b_{i1}, b_{i2} \geq 0\}. \quad (4.1)$$

Each supporter group p has a distribution rule applied to the benefits from either candidate. For example, the equal division rule states that when b_{ip} is given, every voter of M_p receives the same benefits, b_{ip} divided by the cardinality (or measure) of M_p .

Although each supporter group has the same distribution rule applied to the benefits proposed by either candidate, voters have different political preferences towards the two parties. Such preferences are expressed by political discount factors. Voter k in M_p ($p = 1, 2$), who is a supporter for party p , has the discount factor α_k , with which he discounts the benefits proposed by candidate $j \neq p$. We assume that when candidates $i = p$ and $j \neq p$ propose benefits b_{ip} and b_{jp} to the voters in M_p ,

$$\text{voter } k \text{ votes for candidate } i \text{ if and only if } \psi_k(b_{ip}) \geq \psi_k(\alpha_k b_{jp}), \quad (4.2^0)$$

where ψ_k is the distribution rule of group p . Assuming that ψ_k is monotone, (4.2⁰) is written as

$$\text{voter } k \text{ votes for candidate } i \text{ if and only if } b_{ip} \geq \alpha_k b_{jp}. \quad (4.2)$$

Since it is also assumed that every voter does vote, (4.2) implies that voter k votes for candidate j if $b_{ip} < \alpha_k b_{jp}$.

We assume that $M_1 = [0, \theta)$ and $M_2 = [\theta, 1)$. Here θ is the proportion of the supporters for party 1. We assume $1/2 < \theta < 1$, that is, the first group forms a majority.

When 1 and 2 propose packages $b_1 = (b_{11}, b_{12})$ and $b_2 = (b_{21}, b_{22})$, respectively, voter k in M_p chooses one candidate by the criterion stated in (4.2). Thus the total votes for candidate i is:

$$V_i(b_1, b_2) = \mu(\{k \in [0, 1): k \text{ votes for candidate } i\}), \quad (4.3)$$

where μ is the Lebesgue measure on $[0, 1)$. The *payoff* of candidate i from strategy pair $(b_1, b_2) \in S_1 \times S_2$ is $V_i(b_1, b_2)$.

The candidate who wins more than one half of the total votes is the winner (it is a winner-takes-all election). Following Downs (1957), however, we assume that each player maximizes his total votes, $V_i(b_1, b_2)$.

Now we have completed the description of the election game, which is summarized as $G = (\{1, 2\}, \{S_i, V_i\}_{i=1,2})$. Since a voter always votes, the total votes cast is $V_1(b_1, b_2) + V_2(b_1, b_2) = 1$ for any $(b_1, b_2) \in S_1 \times S_2$. Thus, the election game G is constant-sum.

In the following sections, we consider the case where the total benefit T_2 is a constant T and $T_1 \geq T_2 = T$. Since the total benefit T_i is generated by candidate i if he is elected, the case $T_1 = T_2 = T$ is interpreted as the election game of new comers, and the case $T_1 > T = T_2$ as the game of an incumbent 1 and a challenger 2. The ratio $\rho = T_1/T_2 = T_1/T$ represents the seniority level of candidate 1. The assumptions $T_1 > T_2$ and $\theta > 1/2$ mean, respectively, that candidate 1 has greater seniority and more supporters than candidate 2. The case $T_1 < T_2$ and $\theta < 1/2$ is symmetric. We do not consider the case of $T_1 > T_2$ and $\theta < 1/2$.

According to the description of the election game, each district is characterized by the following three parameters:

- (1) the seniority level of incumbent 1: $\rho = T_1/T = T_1/T_2$;
- (2) the population structure: θ , where $M_1 = [0, \theta)$ and $M_2 = [\theta, 1)$ ($1/2 < \theta < 1$);
- (3) the distribution of the political discount factors.

The above three parameters may differ across districts in the same state. Recall that the purpose of the chapter is not to fully describe the results of the election game but to show the possibility of the introduction of a term limit by self-interested voting behavior. For this purpose, we further restrict our attention to the following.

For (3), we assume that in each district, discount factors for both supporter groups are uniformly distributed over $[\delta_0, \delta_1)$, where $1/2 < \delta_0 < \delta_1 \leq 1$. This assumption consists of two parts: i) the uniform distribution and ii) the same support $[\delta_0, \delta_1)$ for both groups of supporters.

The same distribution with the same support means that the discount factors in group 2 for candidate 1 are as diverse as the ones in group 1 for candidate 2, even though the population size of the latter is larger than that of the former. The uniform distribution is assumed for simplicity. These δ_0 and δ_1 are also assumed to be the same over districts in the theorems stated in this and next sections.

For (1), we assume that the seniority level ρ is in the interval $[1, \delta_1 + \frac{1}{\delta_1}]$. We regard $\rho \in [1, \delta_1 + \frac{1}{\delta_1}]$ as reasonable; the upper bound $\delta_1 + \frac{1}{\delta_1} \geq 2$ is sufficiently large for the ratio of the earning ability of the senior candidate to that of the challenger.

Finally we give the definition of the solution concept with which we investigate the election game within a district. Although the concept of Nash equilibrium is commonly used, we sometimes face the nonexistence of a Nash equilibrium in pure strategies in voting models.¹³ Also, in our election game, a Nash equilibrium does not necessarily exist. In this game, the payoff functions are not quasi-concave, which is known to be a sufficient condition for the existence of a Nash equilibrium.¹⁴ In particular, the Nash equilibrium in our game fails to exist for a large range of parameter values. Appendix B shows the nonexistence of a Nash equilibrium with certain parameter values. As a substitute for Nash equilibrium, we will use the maximin strategies.

A strategy b_1 is called a *maximin strategy* iff b_1 maximizes the guaranteed payoff $\text{Min}_{b_2} V_1(b_1, b_2)$. Since the election game G is a two-person constant-sum game, a Nash equilibrium exists if and only if the minimax theorem holds, i.e., $\text{Max}_{b_1} \text{Min}_{b_2} V_1(b_1, b_2) = \text{Min}_{b_2} \text{Max}_{b_1} V_1(b_1, b_2)$, in which case every maximin strategy becomes a Nash strategy. When a Nash equilibrium does not exist, $\text{Max}_{b_1} \text{Min}_{b_2} V_1(b_1, b_2) < \text{Min}_{b_2} \text{Max}_{b_1} V_1(b_1, b_2)$. In the election game,

¹³For example, in variations of the Hotelling spatial model, it is known that a Nash equilibrium does not exist (cf., Denzau, Kats and Slutsky (1985)).

¹⁴See Friedman (1990, p.72, Theorem 3.1).

a maximin strategy can be regarded as a reasonable outcome in the sense that 1) it maximizes the guaranteed payoff; and 2) it is a Nash equilibrium strategy when a Nash equilibrium exists. For a maximin strategy b_1^* , a strategy b_2^* which minimizes $V_1(b_1^*, b_2)$ is called an *associated minimizing strategy* for candidate 2.

We make an additional requirement on the maximin strategies. A maximin strategy b_1^* may satisfy

$$\text{almost all voters in } M_1 \text{ vote for candidate 1 in } (b_1^*, b_2) \text{ for any } b_2. \quad (*)$$

We restrict our attention on only the maximin strategies b_1^* satisfying

$$b_1^* \text{ satisfies Condition } (*) \text{ if } G \text{ has such a maximin strategy.} \quad (4.4)$$

Condition (4.4) does not impose any restriction on a game without a maximin strategy satisfying (*). In the sequel, when we refer to a maximin strategy, it means a maximin strategy satisfying (4.4). By (4.4), we eliminated some of the maximin strategies for candidate 1¹⁵, but the payoffs stay the same.

4.3. Maximin Strategies for the Incumbent in the Campaign Game

In this section, we describe the incumbent's maximin strategies in the election game within a district with different parameter values ρ , θ , δ_0 and δ_1 . The results obtained in this and in the next sections are proved in Appendix A.

The first theorem states that the incumbent has two types of maximin strategies. Either one guarantees his winning in the election game. Recall that we consider the seniority level ρ up to $\delta_1 + \frac{1}{\delta_1}$, since $\rho > \delta_1 + \frac{1}{\delta_1} \geq 2$ is regarded as too large.

¹⁵So far we cannot find an example G in which a maximin strategy satisfies (*) and another does not. However, we do not prove either that the existence of a maximin strategy satisfying (*) implies (*) for all maximin strategies.

Theorem 4.3.1. Let $\hat{\rho}(\theta, \delta_0, \delta_1) = \frac{2\theta-1}{\theta} \delta_1 + \frac{1-\theta}{\theta} \delta_0 + \frac{1}{\delta_1}$.

(a): If $1 \leq \rho \leq \hat{\rho}(\theta, \delta_0, \delta_1)$, the set of maximin strategies for candidate 1 is $\{b_1 = (b_{11}, b_{12})$:

$l_0 \leq b_{11} \leq \rho T$ and $b_{12} = \rho T - b_{11}\}$, where the lower bound l_0 satisfies

$$\left(\frac{2\theta-1}{\theta} \delta_1 + \frac{1-\theta}{\theta} \delta_0\right)T \leq l_0 \leq \delta_1 T. \quad (4.5)$$

The payoffs are $V_1(b_1, b_2) = \theta$ and $V_2(b_1, b_2) = 1 - \theta$ for any maximin strategy b_1 and associated minimizing strategy b_2 .

(b): If $\hat{\rho}(\theta, \delta_0, \delta_1) < \rho \leq \delta_1 + \frac{1}{\delta_1}$, the unique maximin strategy (b_{11}^*, b_{12}^*) for candidate 1 is

$$b_{11}^* = \frac{\theta(\rho + \delta_1 - \delta_0) + \delta_0 - \sqrt{A}}{2\theta} T; \quad \text{and} \quad b_{12}^* = \rho T - b_{11}^*, \quad (4.6)$$

where $A = \left((\rho + \delta_0 - \delta_1)^2 - 4\right)\theta^2 + \left(4 - 2\delta_0^2 + 2\delta_0\delta_1 - 2\delta_0\rho\right)\theta + \delta_0^2$. Furthermore,

$V_1(b_1^*, b_2^*) > \theta$ and $V_2(b_1^*, b_2^*) < 1 - \theta$ for any associated minimizing strategy b_2^* .

First, since $\frac{2\theta-1}{\theta} + \frac{1-\theta}{\theta} = 1$ and $\frac{(2\theta-1)}{\theta}, \frac{(1-\theta)}{\theta} \geq 0$, we have $\frac{2\theta-1}{\theta}\delta_1 + \frac{1-\theta}{\theta}\delta_0 \leq \delta_1$ and $\hat{\rho}(\theta, \delta_0, \delta_1) \leq \delta_1 + \frac{1}{\delta_1}$, i.e., (4.5) and the above classification of cases (a) and (b) are meaningful. Note that (b) includes the assertion that b_{11}^* given by (4.6) is a strategy, i.e., $A \geq 0$ and $0 \leq b_{11}^* \leq \rho T$.

It follows from Theorem 4.3.1 that in the candidate election, incumbent 1 is always elected, since $V_1(b_1, b_2) = \theta$ in case (a) and $V_1(b_1, b_2) > \theta$ in case (b). In case (a), he wins all the votes from his own group; in case (b), he obtains some votes from the opposite group as well. In case (b), he gives some of the benefits to voters in the other group in order to get some of their votes. The restriction (4.4) on maximin strategies is used in case (a).

Since δ_0 and δ_1 are fixed in the sequel, except for one example in Section 4, we write $\hat{\rho}(\theta, \delta_0, \delta_1)$ as $\hat{\rho}(\theta)$, which we call *the critical seniority level*. It is an increasing function of θ and $\hat{\rho}(1) = \delta_1 + \frac{1}{\delta_1}$.

Although all maximin strategies in case (a) give the same number of votes to each

candidate, voters receive different benefits from those strategies. A maximin strategy $b_1 = (b_{11}, b_{12})$ gives b_{11} to voters in $[0, \theta)$ and b_{12} to the ones in $[\theta, 1)$. When studying the voting for term limitation, we assume that each voter considers candidate 1 to have a uniform probability distribution for the selection of a maximin strategy, and so considers his payoff to be the average payoff of all these strategies. Since the maximin strategy is uniquely determined in case (b), the average benefit to a voter in group 1 is

$$\begin{aligned} B_1(\rho; \theta) &= (l_0 + \rho T)/2 && \text{if } 1 \leq \rho \leq \hat{\rho}(\theta) \text{ -- case (a)} \\ &= b_{11}^*(\rho) && \text{if } \hat{\rho}(\theta) < \rho \leq \delta_1 + \frac{1}{\delta_1} \text{ -- case (b)}. \end{aligned} \quad (4.7)$$

The average benefit $B_2(\rho; \theta)$ to a voter in Group 2 is $\rho T - B_1(\rho; \theta)$.

The behavior of $B_1(\rho; \theta)$ as a function of ρ is important for the consideration of the term limitation election. For this purpose, we consider a lower bound of $B_1(\rho; \theta)$ in case (a). Since (4.7) gives a lower bound for l_0 in case (a),

$$\hat{B}_1(\rho; \theta) = \left(\left(\frac{2\theta - 1}{\theta} \delta_1 + \frac{1 - \theta}{\theta} \delta_0 \right) T + \rho T \right) / 2 \quad (4.8)$$

is a lower bound of $B_1(\rho; \theta)$, which is a linear function of ρ in the domain of case (a). In fact, the function $B_1(\rho; \theta)$ jumps down at $\rho = \hat{\rho}(\theta)$, and the following theorem states that the ranges of the function $B_1(\rho; \theta)$ over the domains $[1, \hat{\rho}(\theta)]$ and $(\hat{\rho}(\theta), \delta_1 + \frac{1}{\delta_1}]$ are separated. This separation, together with the linearity of $\hat{B}_1(\rho; \theta)$, becomes essential in Section 4.4.

Theorem 4.3.2. There exists a constant c such that

$$\hat{B}_1(\rho'; \theta) \geq c \geq B_1(\rho; \theta) \text{ for all } \rho' \in [1, \hat{\rho}(\theta)] \text{ and } \rho \in (\hat{\rho}(\theta), \delta_1 + \frac{1}{\delta_1}]. \quad (4.9)$$

The benefit function $B_1(\rho; \theta)$ is illustrated in Figure 4.1: the dotted line segment represents the lower bound $\hat{B}_1(\rho'; \theta)$ and the solid curves represent the function $B_1(\rho; \theta)$. Thus $B_1(\rho; \theta)$ is discontinuous at $\rho = \hat{\rho}(\theta)$; after $\hat{\rho}(\theta)$, it is lower than any value of $B_1(\rho; \theta)$ on $[1, \hat{\rho}(\theta)]$.

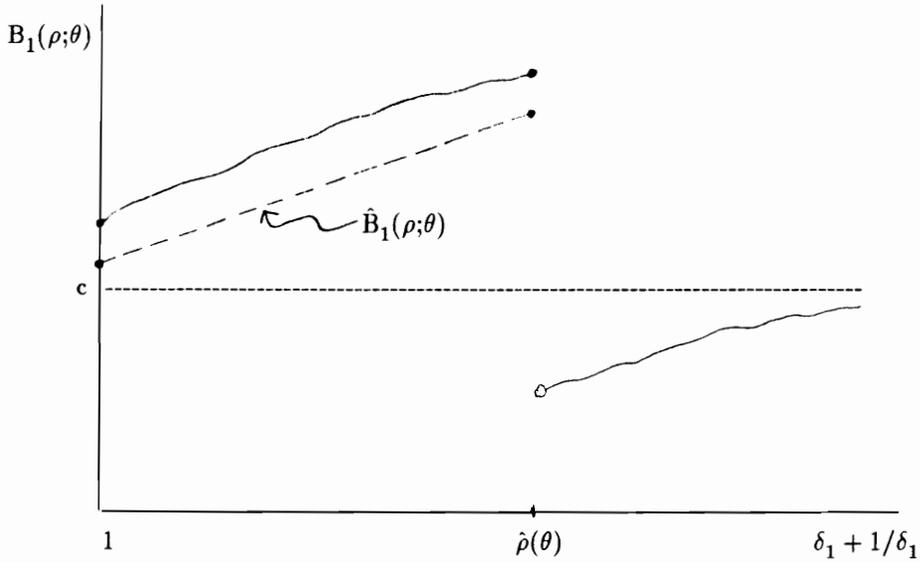


Figure 4.1: $B_1(\rho; \theta)$ (θ is fixed).

The discontinuity of $B_1(\rho; \theta)$ is caused by the incumbent's desire to maximize the number of votes. Up to the critical seniority level $\hat{\rho}(\theta)$, the incumbent can obtain votes only from his own supporters, i.e., $V_1(b_1, b_2) = \theta$; beyond $\hat{\rho}(\theta)$, he is able to gain votes from the opposite group as well, i.e., $V_1(b_1, b_2) > \theta$. This behavior of obtaining votes from the opposite group dilutes the benefits for his own supporters.

As the incumbent serves more terms, he generates more total benefits. But his supporters will get a smaller fraction of these from him after his seniority exceeds the critical level. Therefore his supporters may vote for the introduction of a term limit to avoid such behavior. This will be discussed in more detail in the next section.

4.4. Term Limitation for Districts in a State

The vote on term limitation is held at the state level. Most states have several districts, whose incumbents may have various seniority levels. Once a term limit passes in a state, it has an impact on all districts of the state eventually, though the timing of the effect depends upon a

district. That is, the incumbents might or might not face the immediate consequences of term limitation: those who do not face the immediate effect will also be forced out of office sometime in the future.

Let $\rho_\tau T$ be the total benefit generated by an incumbent in term τ . We assume that $1 = \rho_1 < \rho_2 < \dots < \rho_N \leq \delta_1 + \frac{1}{\delta_1}$, where N is the natural limit without a term limit. In the term limitation election, voters vote for or against the introduction of a proposed term limit $L (< N)$. If the term limit L is introduced, candidates being already in the office for at least L terms will not be able to seek election in the future.

Each voter evaluates a term limit L by the present value of his future benefits. Consider the stream of future benefits with cycle $C (= N \text{ or } L)$ starting with $B_1(\rho_t; \theta)$;

$$B_1(\rho(1); \theta), B_1(\rho(2); \theta), \dots, B_1(\rho(C); \theta), B_1(\rho(C+1); \theta), \dots, B_1(\rho(2C); \theta), \dots \quad (4.10)$$

where $\rho(1) = \rho_t, \rho(2) = \rho_{t+1}, \dots, \rho(C-t+2) = \rho_1, \dots, \rho(C) = \rho_{t-1}, \rho(C+1) = \rho_t, \dots$

Recall that $B_1(\rho_\tau; \theta)$, the average benefits that a voter in the majority group receives, is a function of the incumbent's seniority level ρ_τ as well as the district's population structure θ . The present value of this stream is

$$DS_1(\theta; C) = \sum_{\tau=1}^{\infty} \beta^{\tau-1} B_1(\rho(\tau); \theta), \quad (4.11)$$

where $\beta \in (0, 1)$ is the time discount factor, which is assumed to be constant over all voters. A voter of group 1 votes for term limitation if $DS_1(\theta; L) > DS_1(\theta; N)$.

Before looking at the general case, it would be helpful to consider the case where every district in the state has an identical θ . In this case, the critical seniority level $\hat{\rho}(\theta)$ is the same for all districts. Note that the majority supporter group in each district is called group 1, but this does not require the majority groups of districts to have the same political identity (e.g., group 1 may be Democratic in one district and Republican in another). Suppose $\hat{\rho}(\theta) < \rho_N$, i.e., without a term limit, the incumbent stays in office after the critical seniority level $\hat{\rho}(\theta)$. Now choose the

term limit L so that $\rho_L \leq \hat{\rho}(\theta) < \rho_{L+1}$. The introduction of the term limit L eliminates all the low benefit terms after $\hat{\rho}(\theta)$ for the supporters in M_1 . That is, the benefit stream created by the term limit L consists of cycles of $B_1(\rho_1; \theta), \dots, B_1(\rho_L; \theta)$, which are in the left high region of Figure 4.2. Therefore $DS_1(\theta; N) < DS_1(\theta; L)$ for any $\beta \in (0, 1)$, regardless of the present seniority level of the incumbent in the district. This implies that all voters in group 1 vote for the introduction of the term limit L . All voters in group 2 vote against L since they receive higher benefits after $\hat{\rho}(\theta)$. However, since group 1 is a majority ($\theta > 1/2$), the term limit L will pass.

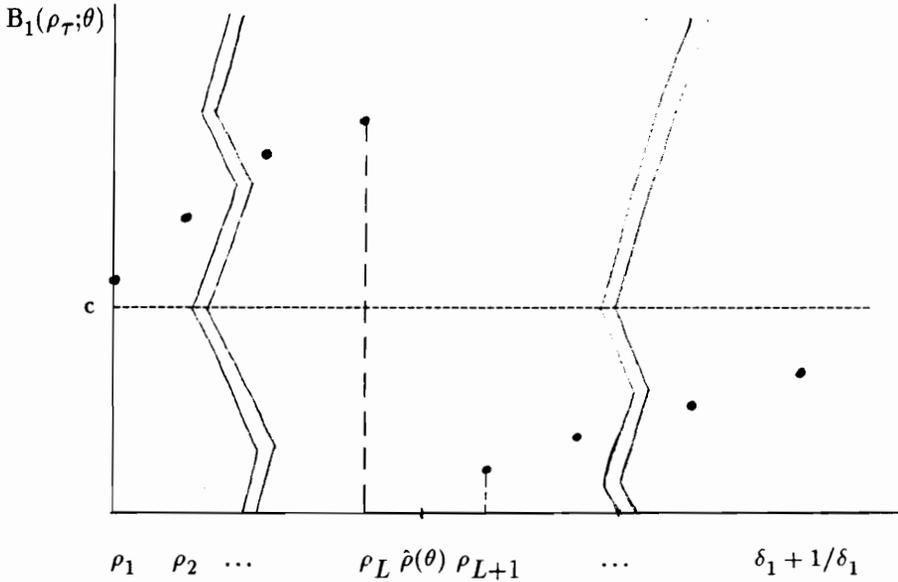


Figure 4.2

Consider now the case of nonuniform θ . In this case, the choice of a term limit is not so straightforward as in the uniform case, since the critical seniority level $\hat{\rho}(\theta)$ varies across districts. Therefore we need to further specify the structure of the state. Let $\Theta = \{\theta_1, \theta_2, \dots, \theta_d\}$ be the set of districts in the state and D the population size of each district. Then, the total population of the state is dD . We assume the existence of $\hat{\theta} < 1$ satisfying

$$\sum_{\theta_l \leq \hat{\theta}} \theta_l D > \frac{dD}{2}; \quad \text{and} \quad \hat{\rho}(\hat{\theta}) < \rho_N. \quad (4.12)$$

The first inequality states that the supporters of group 1 in the districts with $\theta_l \leq \hat{\theta}$ form a majority of the total population in the state. Again, note that the political identities of the majority groups in the districts may differ. The value $\hat{\theta}$, however, is required to have the property that its corresponding critical seniority level $\hat{\rho}(\hat{\theta})$ is less than the natural tenure length ρ_N .

Let $\hat{\theta}$ be a value satisfying (4.12). Consider the term limit L with $\rho_L \leq \hat{\rho}(\hat{\theta}) < \rho_{L+1}$. Then $B_1(\rho_\tau; \hat{\theta}) < c$ for all $\tau = L + 1, \dots, N$, where c is the number given in Theorem 4.3.2. For any district l with $\theta_l \leq \hat{\theta}$, we have $\hat{\rho}(\theta_l) \leq \hat{\rho}(\hat{\theta})$, since $\hat{\rho}(\theta)$ is an increasing function of θ . This implies, again by Theorem 4.3.2, that for $\theta_l \leq \hat{\theta}$, $B_1(\rho_\tau; \theta_l) < c$ for all $\tau = L + 1, \dots, N$. Thus, the term limit L eliminates some terms with low benefits for voters in group 1. Nevertheless, the elimination does not necessarily increase those voters' present value of their future benefits, since the term limit L may not eliminate all the low benefit terms because of the nonuniformness of θ_l . For example, suppose the stream of benefits (4.10) with $t = 1$ is given as:

$$10, 0, 6, 8, 10, 0, 6, 8, \dots, \text{ where } c = 9.$$

If $L = 2$, then the newly created stream is $10, 0, 10, 0, \dots$. For a discounted factor close to 1, the first stream gives a larger present value than the second. In this district, the optimal term limit is 1, but other districts may have longer optimal term limits because of different θ_l . Therefore, we should take different θ_l into account for the introduction of a term limit.

In the following, we give two sufficient conditions for the introduction of a term limit. The first theorem states that for a sufficiently small discount factor β , the introduction of the term limit L is supported without any additional condition. For a discount factor close to 1, we need a different condition, which is given in the second theorem.

Theorem 4.4.1. There is a $\beta_o \in (0, 1)$ such that for any $\beta \in (0, \beta_o]$, $DS_1(\theta_l; L) > DS_1(\theta_l; N)$ for all l with $\theta_l \leq \hat{\theta}$. Consequently, all voters in M_1 in the districts with $\theta_l \leq \hat{\theta}$ will vote for the

introduction of the term limit L .

The above theorem implies that if voters discount heavily their future, then each supporter of the majority group of each district votes for the introduction of a properly chosen term limit L . By condition (4.12), those voters form a majority in the state and the term limit L will pass at the state level.

The next theorem gives a sufficient condition for the inequality $DS_1(\theta_l; L) > DS_1(\theta_l; N)$ to hold when the discount factor β is close to 1.

Theorem 4.4.2. Assume that for any $\theta_l \leq \hat{\theta}$,

$$\sum_{\tau=1}^L B_1(\rho_\tau; \theta_l) / L > B_1(\rho_N; \theta_l). \quad (4.13)$$

Then there is a $\beta_o \in (0, 1)$ such that for any $\beta \in [\beta_o, 1)$, $DS_1(\theta_l; L) > DS_1(\theta_l; N)$ for all l with $\theta_l \leq \hat{\theta}$. Consequently, all voters in M_1 in the districts with $\theta_l \leq \hat{\theta}$ will vote for the introduction of the term limit L .

Condition (4.13) states that the average benefits voters of group 1 receive with the term limit L be larger than the benefit they receive in their incumbent's last term in the absence of a term limit. This condition implies that the inequality $DS_1(\theta_l; L) > DS_1(\theta_l; N)$ holds in the district l with $\theta_l \leq \hat{\theta}$, which together with Condition (4.12) guarantees a majority vote for the term limit L .

The following example illustrates the above results. In example 4.4.1, there are nine districts, i.e., $d = 9$, and voters' political preferences are uniformly distributed between $\delta_0 = 0.6$ and $\delta_1 = 0.9$. We give a specific formula of ρ_τ :

$$\rho_\tau = 2 - \left(\frac{1}{2}\right)^{\tau-1} \quad \text{for } \tau = 1, 2, \dots \quad (4.14)$$

In Table 4.1, θ_l describes the population structure in each district l . The critical seniority level $\hat{\rho}(\theta_l)$ can be calculated from the formula $\hat{\rho}(\theta_l) = \hat{\rho}(\theta_l, \delta_0, \delta_1) = \frac{2\theta-1}{\theta} \delta_1 + \frac{1-\theta}{\theta} \delta_0 + \frac{1}{\delta_1}$. By

formula (4.14), we can find $\hat{\tau}$ satisfying $\rho_{\hat{\tau}} \leq \hat{\rho}(\theta_l) < \rho_{\hat{\tau}+1}$ for each l .

Example 4.4.1.

Table 4.1

District	$\hat{\rho}(\theta_l)$	$\hat{\tau}$
$\theta_1=0.55$	1.7657	3
$\theta_2=0.60$	1.8111	3
$\theta_3=0.65$	1.8496	3
$\theta_4=0.70$	1.8825	4
$\theta_5=0.75$	1.9111	4
$\theta_6=0.80$	1.9361	4
$\theta_7=0.85$	1.9582	5
$\theta_8=0.90$	1.9778	6
$\theta_9=0.95$	1.9953	8

Now, let the natural limit be $N = 10$. Choose $\hat{\theta} = \theta_7$. Then Condition (4.12) is satisfied:

$$\sum_1^7 \theta_l D = 4.9D > \frac{9}{2} D = \frac{dD}{2}; \quad \text{and} \quad \hat{\rho}(\hat{\theta}) < \rho_N.$$

Moreover, L is determined to be 5 by $\rho_L \leq \hat{\rho}(\hat{\theta}) < \rho_{L+1}$. By Theorem 4.4.1, the term limit $L = 5$ will be supported by the majority of voters in districts 1-7, if the time discount factor β is close to 0.

Now consider this example from the viewpoint of Theorem 4.4.2. Suppose the time discount factor β is close to 1. Let $N = 10$ and $L = 5$. It can be shown that Condition (4.13) is satisfied:

$$\sum_{\tau=1}^5 B_1(\rho_{\tau}; \theta_l) / 5 > B_1(\rho_{10}; \theta_l) \text{ for districts } l = 1, \dots, 7.$$

Thus, by Theorem 4.4.2, voters of group 1 in districts 1-7 will vote for the introduction of the term limit L . Since they form a majority in the state, L will pass when β is close to 1.

Even when conditions (4.12) and (4.13) are not satisfied, a term limit might still pass in

the state. Example 4.4.2 illustrates a state where for some districts condition (4.13) is not satisfied, and where voters' time discount factor β is close to 1. In this example, we allow the political preference parameters δ_0 and δ_1 to vary across districts. Suppose that the state consists of 12 districts A, B, C, ..., L, listed in Table 4.2, and the natural length for tenure is $N = 10$.

Example 4.4.2:

Table 4.2

District	(1): θ	(2): δ_0	(3): δ_1	(4): $\hat{\rho}$	(5): $\hat{\tau}$
A	0.6	0.6	0.9	1.811	3
B	0.6	0.7	0.9	1.877	3
C	0.6	0.7	0.8	1.983	6
D	0.6	0.8	0.9	1.944	5
E	0.7	0.6	0.9	1.883	4
F	0.7	0.7	0.9	1.925	4
G	0.7	0.7	0.8	2.007	/
H	0.7	0.8	0.9	1.789	3
I	0.8	0.6	0.9	1.936	4
J	0.8	0.7	0.9	1.961	5
K	0.8	0.7	0.8	2.025	/
L	0.8	0.8	0.9	1.986	7

In Table 4.2, (1)-(3) characterize a district, (4) gives the critical seniority level $\hat{\rho}(\theta, \delta_0, \delta_1)$ and (5) gives the corresponding term $\hat{\tau}$ with $\rho_{\hat{\tau}} \leq \hat{\rho}(\theta, \delta_0, \delta_1) < \rho_{\hat{\tau}+1}$. Since the critical seniority level $\hat{\rho}(\theta, \delta_0, \delta_1)$ is greater than 2 in districts G and K, given formula (4.14), then ρ_{τ} never reaches the critical seniority level in these districts. Thus $\hat{\tau}$ does not exist in those two districts. In this example, we consider the case where the time discount factor β is sufficiently close to 1.

Now let the term limit be $L = 6$. In district C, $\hat{\tau}$ coincides with L . Thus, by Theorem

4.3.2, the supporters of group 1 will vote for the term limit L . For districts A, B, D, E, F, H, I and J, the term limit $L = 6$ eliminates some high benefit terms as well as all the low benefit terms. For district L, it does not eliminate all the low benefit terms. However, it can be shown that for any β sufficiently close to 1, $DS_1(\theta, \delta_0, \delta_1; L) > DS_1(\theta, \delta_0, \delta_1; N)$ for all districts except G, K. Thus, the voters of group 1 in those districts will vote for $L = 6$. For districts G and K, any term limit would eliminate the high benefits terms. Thus, all voters in G and K will vote against $L = 6$.

Now we calculate the proportion of votes in favor of $L = 6$. The supporters of group 1 in districts A, B, C, D, E, F, H, I, J and L vote for the term limit $L = 6$, and the supporters of group 2 in every district vote against the term limit. For districts G and K, all voters in both groups vote against the term limit L . The state consists of 12 districts; each district has 1/12 of the whole population. The portion of group 1 in each districts is given by θ . Thus the proportion of votes for $L = 6$ is

$$(1/12) \times (0.6 \times 4 + 0.7 \times 3 + 0.8 \times 3) = 0.575 > 0.5.$$

Therefore, the proposed term limit $L = 6$ will pass in this state.

The proposed length of tenure affects the result of the term limitation election significantly. Let $L = 3$. Then all voters in district L vote against term limit $L = 3$. It can be shown that a majority in districts A, B, C, D, E, F, H, I, J will vote for $L = 3$. The proportion of voters in favor of a term limit in this case is

$$(1/12) \times (0.6 \times 4 + 0.7 \times 3 + 0.8 \times 2) = 0.508.$$

In this case, the term limit $L = 3$ will still pass. However, $L < 3$ would not pass in this example.

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Appendices:

Appendix to Chapter 2

Proof of the If-part of Theorem 2.3.B. We prove that Axioms N.0-N.4 imply Axioms V.0,V.1^o, V.2-V.4.

V.0): Let $(G,T) \in \mathbf{B}$ with $\Psi(G,T) = (q,\psi)$, and let $\rho \in M(G)$. Then $(T,h(\rho)) \in \mathbf{F}$. By (2.5) $\psi(\rho) = f(T,h(\rho))$. Since $f(T,h(\rho)) \in T$ by Axiom N.0, $\psi(\rho) \in T$.

V.1^o): We show that for any $(G,T) \in \mathbf{B}$ with $\Psi(G,T) = (q,\psi)$, $\psi(\rho)$ is the Nash equilibrium in the subgame $\Gamma(G,T;\rho)$ of $\Gamma(G,T)$ for any $\rho \in M(G)$. Suppose not. Denote $u = \psi(\rho)$ and $(S,d) = (T,h(\rho))$. By Theorem 2.3.A, $u = f^N(S,d)$, which implies $u \geq d = h(\rho)$. Then there is a $v_i \in R$ with $(u_{-i}, v_i) \in T$ and $v_i > u_i$. By (2.5) $u = \psi(\rho) = f(S,d)$. Thus $v_i > \psi_i(\rho) = f_i(S,d)$ and $u_{-i} = \psi_{-i}(\rho) = f_{-i}(S,d)$. Since $(u_{-i}, v) \in T = S$, $f(S,d)$ is not Pareto optimal in S , a contradiction to Axiom N.1.

V.2): Let $\Psi(G,T) = (q,\psi)$, $\Psi(aG+b, aT+b) = (q',\psi')$ for $a > 0$ and b . We show $\psi'(\rho) = a\psi(\rho)+b$ for any $\rho \in M(G)$. Let $(S,d) = (T,h(\rho))$. Then $(aS+b, ad+b) = (aT+b, ah(\rho)+b)$. By (2.5), $\psi(\rho) = f(S,d)$ and $\psi'(\rho) = f(aS+b, ad+b)$. Since $f(aS+b, ad+b) = af(S,d) + b$ by Axiom N.2, $\psi'(\rho) = a\psi(\rho) + b$.

V.3): Let $\Psi(G,T) = (q,\psi)$ and $\Psi(\pi G, \pi T) = (q',\psi')$. We show that $\psi'(\pi\rho) = \pi\psi(\rho)$ for any $\rho \in M(G)$. Let $(S,d) = (T,h(\rho))$. Then $(\pi S, \pi d) = (\pi T, \pi h(\rho)) = (\pi T, h'(\pi\rho))$. Since $f(S,d) = \psi(\rho)$ and $f(\pi S, \pi d) = \psi'(\pi\rho)$ by (2.5), and since $f(\pi S, \pi d) = \pi f(S,d)$ by Axiom N.3, we have $\psi'(\pi\rho) = \pi\psi(\rho)$.

V.4): Let $(G,T), (G',T') \in \mathbf{G}$ with $\Psi(G,T) = (q,\psi)$ and $\Psi(G',T') = (q',\psi')$. We show that for any $\rho \in M(G)$ and $\rho' \in M(G')$, if $h(\rho) = h(\rho')$, $T \subseteq T'$ and $\psi'(\rho') \in T$, then $\psi(\rho) = \psi'(\rho')$. Let $(S,d) = (T,h(\rho))$ and $(S',d') = (T',h(\rho'))$. By (2.5), $f(S,d) = \psi(\rho)$, $f(S',d') = \psi'(\rho')$. Since $S = T \subseteq T' = S'$ and also $f(S',d') = \psi'(q') \in T = S$, Axiom N.4 implies $f(S',d') = f(S,d)$. Thus $\psi(\rho) = \psi'(\rho')$.

□

Proof of Theorem 2.3.C. First we prove the If part. Suppose that f satisfies Axioms N.1 - N.4.

Then we show that Φ with (2.6) satisfies Axioms U.0, U.1^o, U.2, U.3, U.4. Axioms U.0 and U.1^o follow from Axioms N.0 and N.1 respectively with (2.6).

U.2): Let $\Phi(G, T) = (u, \phi)$ and $\Phi(aG+b, aT+b) = (u', \phi')$ with $\phi'(u') = \phi(u)$. We show $u' = au+b$. By (2.6), $u = f(T, h(\phi(u)))$ and $u' = f(aT+b, ah(\phi'(u'))+b)$. Since $\phi'(u') = \phi(u)$, we have $f(aT+b, ah(\phi'(u'))+b) = f(aT+b, ah(\phi(u))+b)$. By Axiom N.2, $f(aT+b, ah(\phi(u))+b) = af(T, h(\phi(u))) + b$, i.e., $u' = au+b$.

U.3): Let $\Phi(G, T) = (u, \phi)$ and $\Phi(\pi G, \pi T) = (u', \phi')$ with $\phi'(u') = \pi\phi(u)$. We show $u' = \pi u$. By (2.6), $u = f(T, h(\phi(u)))$ and $u' = f(\pi T, h'(\phi'(u')))$. Since $h'(\phi'(u')) = \pi h(\phi(u))$, we have $f(\pi T, h'(\phi'(u')))$ = $f(\pi T, \pi h(\phi(u)))$. By Axiom N.2, $f(\pi T, \pi h(\phi(u))) = \pi f(T, h(\phi(u)))$, i.e., $u' = \pi u$.

U.4): Let $\Phi(G, T) = (u, \phi)$ and $\Phi(G', T') = (u', \phi')$. We show that $h(\phi(u)) = h'(\phi'(u'))$, $T \subseteq T'$ and $u' \in T$ imply $u = u'$. By (2.6), $u = f(T, h(\phi(u)))$ and $u' = f(T', h'(\phi'(u')))$. Since $h(\phi(u)) = h'(\phi'(u'))$, $T \subseteq T'$ and $u' \in T$, we have $f(T, h(\phi(u))) = f(T', h'(\phi'(u')))$ by Axiom N.4, i.e., $u = u'$.

Now we prove the Only-If part. Axioms N.0 and N.1 follow from Axioms U.0 and U.1 respectively with (2.6).

N.2): We show $f(aS+b, ad+b) = af(S, d)+b$. Consider (G, T) with $T = S$ and $X_i = \{\alpha_i\}$ (i has only one strategy α_i) for all i , and $h(\alpha) = h(\alpha_1, \dots, \alpha_n) = d$. Let $\Phi(G, T) = (u, \phi)$ and $\Phi(aG+b, aT+b) = (u', \phi')$. Then $\phi'(u') = \phi(u) = \alpha$. By Axiom U.2, $u' = au+b$. By (2.6), $u = f(T, h(\phi(u)))$ and $u' = f(aT+b, ah(\phi'(u'))+b)$. Since $\phi'(u') = \phi(u) = \alpha$, we have $f(aS+b, ad+b) = f(aT+b, ah(\phi'(u'))+b) = af(T, h(\phi(u))) + b = af(S, d)+b$.

N.3): We show $f(\pi S, \pi d) = \pi f(S, d)$. Consider (G, T) with $T = S$, $X_i = \{\alpha_i\}$ for all $i \in N$ and $h(\alpha) = d$. Let $\Phi(G, T) = (u, \phi)$ and $\Phi(\pi G, \pi T) = (u', \phi')$. Then $\phi'(u') = \pi\phi(u) = (\alpha_{\pi(1)}, \dots, \alpha_{\pi(n)})$. By Axiom U.3, $u' = \pi u$. By (2.6), $u = f(T, h(\phi(u)))$, and also since $h'(\phi'(u')) = \pi h(\phi(u))$, $u' = f(\pi T, h'(\phi'(u')))$ = $f(\pi T, \pi h(\phi(u)))$. Then $f(\pi S, \pi d) = f(\pi T, \pi h(\phi(u))) = u' = \pi u = \pi f(S, d)$.

N.4): We show that $S \subseteq S'$, $d = d'$ and $f(S', d') \in S$ imply $f(S, d) = f(S', d')$. Consider (G, T) with T

$= S$, $X_i = \{\alpha_i\}$ for all $i \in N$ and $h(\alpha) = d$. Let $\Phi(G, T) = (u, \phi)$. Consider (G, T') with $T' = S'$. Let $\Phi(G, T') = (u', \phi')$. Then $T \subseteq T'$ and $h(\phi(u)) = h(\phi'(u')) = d = d'$. By (2.6), $u = f(T, d)$ and $u' = f(T', d')$. Thus $u' \in S = T$. By Axiom U.4, $u' = u$, i.e., $f(T, d) = f(T', d')$. \square

Appendices to Chapter 4.

Appendix A. In this appendix, we prove the theorems given in Sections 4.3 and 4.4.

First we calculate the payoffs of the candidates given a strategy pair (b_1, b_2) . The payoff of candidate 1 is given as $V_1(b_1, b_2) = V_1^1(b_1, b_2) + V_1^2(b_1, b_2)$, i.e., the total votes from groups 1 and 2. Each is described as

$$\begin{aligned} V_1^1(b_1, b_2) &= \theta && \text{if } \frac{b_{11}}{b_{21}} \geq \delta_1, \\ &= \frac{b_{11}/b_{21} - \delta_0}{\delta_1 - \delta_0} \theta && \text{if } \delta_0 \leq \frac{b_{11}}{b_{21}} < \delta_1, \\ &= 0 && \text{if } \frac{b_{11}}{b_{21}} < \delta_0; \end{aligned} \tag{A.1}$$

$$\begin{aligned} V_1^2(b_1, b_2) &= 0 && \text{if } \frac{b_{22}}{b_{12}} \geq \delta_1, \\ &= \frac{\delta_1 - b_{22}/b_{12}}{\delta_1 - \delta_0} (1 - \theta) && \text{if } \delta_0 \leq \frac{b_{22}}{b_{12}} < \delta_1, \\ &= 1 - \theta && \text{if } \frac{b_{22}}{b_{12}} < \delta_0. \end{aligned} \tag{A.2}$$

In the first case of (A.1), even the voter of M_1 with political discount factor arbitrarily close to δ_1 votes for candidate 1; thus candidate 1 gets all votes from his supporters. In the last case, the voter in M_1 with δ_0 votes for candidate 2; thus 1 gets no votes from his supporters. In the second case, $\frac{b_{11}}{b_{21}}$ is the last voter in group 1 to vote for candidate 1; every voter of M_1 with a smaller political discount factor also votes for candidate 1. The argument for (A.2) is similar.

Recall that $\hat{\rho}(\theta, \delta_0, \delta_1) = \frac{2\theta - 1}{\theta} \delta_1 + \frac{1 - \theta}{\theta} \delta_0 + \frac{1}{\delta_1}$. The following lemma will be used in the proof of Theorem 4.3.1.(a).

Lemma 1. Let $1 \leq \rho \leq \hat{\rho}(\theta, \delta_0, \delta_1)$.

(I): Let $b_1 = (\rho T, 0)$. Then all voters of M_1 vote for candidate 1 in (b_1, b_2) for any strategy

$b_2 \in S_2$.

(II): For any $b_1 \in S_1$, there is a strategy $b_2 \in S_2$ such that $V_1(b_1, b_2) \leq \theta$.

(III): \tilde{b}_1 is a maximin strategy if and only if all voters in M_1 vote for candidate 1 in (\tilde{b}_1, b_2) for any b_2 .

Proof: (I) is straightforward: since $\rho \geq 1$ and voters of M_1 discount the benefits proposed by candidate 2, $b_1 = (\rho T, 0)$ guarantees candidate 1 to win the entire M_1 , even if $b_2 = (T, 0)$.

For the proof of (II), consider the following two possibilities of $b_1 = (b_{11}, b_{12})$.

$$(a) \ b_{11} < \frac{2\theta-1}{\theta}\delta_1 + \frac{1-\theta}{\theta}\delta_0; \text{ and } (b) \ b_{11} \geq \frac{2\theta-1}{\theta}\delta_1 + \frac{1-\theta}{\theta}\delta_0.$$

In case (a), let $b_2 = (T, 0)$. By (A.1), $V_1^1(b_1, b_2) < 2\theta - 1$. Also, $V_1^2(b_1, b_2) \leq 1 - \theta$, since the size of group 2 is $1 - \theta$. Thus $V_1(b_1, b_2) = V_1^1(b_1, b_2) + V_1^2(b_1, b_2) < \theta$. In case (b), let $b_2 = (0, T)$. Since $1 \leq \rho \leq \hat{\rho}(\theta, \delta_0, \delta_1)$, $b_{12} \leq \frac{1}{\delta_1}$. By (A.1), $V_1^2(b_1, b_2) = 0$. Thus $V_1(b_1, b_2) \leq \theta$.

Consider (III). Suppose that all voters in M_1 vote for candidate 1 in (\tilde{b}_1, b_2) for any b_2 . Then $V_1(\tilde{b}_1, b_2) \geq \theta$, which, together with (II), implies that \tilde{b}_1 is a maximin strategy. Conversely, suppose that \tilde{b}_1 is a maximin strategy. Assertion (I) states that for $b_1 = (\rho T, 0)$ and for any b_2 , all voters in M_1 vote for 1. By (II), $b_1 = (\rho T, 0)$ is a maximin strategy. By (4.4), this implies that all voters in M_1 vote for candidate 1 in (\tilde{b}_1, b_2) for any b_2 . \square

Proof of Theorem 4.3.1.(a): We prove the following three claims.

(1): Any maximin strategy $b_1 = (b_{11}, b_{12})$ satisfies

$$\left(\frac{2\theta-1}{\theta}\delta_1 + \frac{1-\theta}{\theta}\delta_0\right)T \leq b_{11} \leq \rho T. \quad (\text{A.3})$$

(2): Any strategy $b_1 = (b_{11}, b_{12})$ for candidate 1, with $\delta_1 T \leq b_{11} \leq \rho T$, is a maximin strategy.

(3): The set of maximin strategies for candidate 1 is convex.

Since the set of maximin strategies is closed, the above three claims imply the assertion in the

theorem.

(1): Suppose that b_1 does not satisfy (A.3). This implies $b_{11} < (\frac{2\theta-1}{\theta} \delta_1 + \frac{1-\theta}{\theta} \delta_0)T$. Let

$b_2 = (T, 0)$. For this b_2 , we have, by (A.1) and (A.2), that

$$\begin{aligned} V_1(b_1, b_2) &= \theta \frac{b_{11}/T - \delta_0}{\delta_1 - \delta_0} + (1 - \theta) \\ &< \theta \frac{(\frac{2\theta-1}{\theta} \delta_1 + \frac{1-\theta}{\theta} \delta_0)T/T - \delta_0}{\delta_1 - \delta_0} + (1 - \theta) = \theta. \quad (\text{since } b_{11} < (\frac{2\theta-1}{\theta} \delta_1 + \frac{1-\theta}{\theta} \delta_0)T) \end{aligned}$$

However, by Lemma 1 (I), candidate 1 can obtain at least θ by $(\rho T, 0)$. Hence b_1 is not a maximin strategy.

(2): Let $b_{11} \geq \delta_1 T$. In this case, all voters of group 1 vote for candidate 1 even if $b_2 = (T, 0)$.

Thus, by Lemma 1 (III), b_1 is a maximin strategy.

(3): We show that if b_1 with $b_{11} < \delta_1 T$ is a maximin strategy, then $\bar{b}_{11} \in [b_{11}, \delta_1 T]$ is also a maximin strategy for candidate 1. This, together with (2), implies the convexity of the set of maximin strategies. Since b_1 is a maximin strategy, then, by Lemma 1 (III), all voters in M_1 vote for candidate 1. This, together with $\bar{b}_{11} \geq b_{11}$, implies that \bar{b}_1 makes all voters in M_1 vote for 1. Thus, by Lemma 1 (III), \bar{b}_1 is a maximin strategy for candidate 1. \square

Theorem 4.3.1.(b) follows from Lemmas 2 and 3. Since the existence of a maximin strategy follows from the continuities of V_1 and V_2 and the compactness of the strategy spaces B_1 and B_2 , Lemmas 2 and 3 imply that the formula described in (4.6) gives a strategy.

Lemma 2. Let $\hat{\rho}(\theta, \delta_0, \delta_1) < \rho \leq \delta_1 + \frac{1}{\delta_1}$. If $b_1 \in S_1$ satisfies (i) $\theta(\delta_1 - b_{11}/T) = (1 - \theta)(T/b_{12} - \delta_0)$ and (ii) $b_{11}/T \in [\delta_0, \delta_1]$, then b_1 is given by (4.6).

Proof. Denoting b_{11}/T and T/b_{12} by α and β in (i), we get, $\theta(\delta_1 - \alpha) = (1 - \theta)(\beta - \delta_0)$. Since $b_{11} + b_{12} = \rho T$, we have $\alpha + 1/\beta = \rho$. Eliminating β from these equations, we obtain:

$$\theta(\delta_1 - \alpha) = (1 - \theta)\left(\frac{1}{\rho - \alpha} - \delta_0\right)$$

$$\begin{aligned} &\Rightarrow \theta\alpha^2 - \theta(\rho + \delta_1 - \delta_0)\alpha - \delta_0\alpha + \theta\rho(\delta_1 - \delta_0) + \delta_0\rho + \theta - 1 = 0 \\ &\Rightarrow \alpha = \frac{\theta(\rho + \delta_1 - \delta_0) + \delta_0 \pm \sqrt{A}}{2\theta}, \end{aligned}$$

where A is given in (b) of Theorem 4.3.1. (Since $\alpha = b_{11}/T$, this α must be given by the formula above whenever b_{11} satisfies (i) and (ii)). Since $\rho \geq 1 \geq \delta_1 > \delta_0 > 1/2$ and $1 > \theta > 1/2$, we have

$$\theta(\rho - \delta_1) + \delta_0(1 - \theta) > 0, \quad \text{which is equivalent to } \frac{\theta(\rho + \delta_1 - \delta_0) + \delta_0}{2\theta} > \delta_1. \quad \text{Thus}$$

$\frac{\theta(\rho + \delta_1 - \delta_0) + \delta_0 + \sqrt{A}}{2\theta} > \delta_1$. Since $\alpha = b_{11}/T \in [\delta_0, \delta_1]$ by (ii), the solution for α should be

$$\alpha = \frac{\theta(\rho + \delta_1 - \delta_0) + \delta_0 - \sqrt{A}}{2\theta}. \quad \text{Thus } b_{11} = \frac{\theta(\rho + \delta_1 - \delta_0) + \delta_0 - \sqrt{A}}{2\theta} T \text{ and } b_{12} = \rho T - b_{11}, \text{ which}$$

are given by (4.6). \square

Lemma 3. Let $\hat{\rho}(\theta, \delta_0, \delta_1) < \rho \leq \delta_1 + \frac{1}{\delta_1}$. Then $b_1 \in S_1$ is a maximin strategy if and only if (i)

$\theta(\delta_1 - b_{11}/T) = (1 - \theta)(T/b_{12} - \delta_0)$ and (ii) $b_{11}/T \in [\delta_0, \delta_1]$. Note that (ii) implies $b_{12} > 0$.

Proof. If-Part: Suppose that $b_1 \in S_1$ satisfies (i) and (ii). First we prove

$$V_1(b_1, (T, 0)) = V_1(b_1, (0, T)) = \min_{b_2} V_1(b_1, b_2). \quad (\text{A.4})$$

To prove (A.4), we show that (1) $V_2(b_1, (T, 0)) = V_2(b_1, (0, T)) < 1 - \theta$; and (2) candidate 2 cannot get (even partial) votes from both parties simultaneously for any b_2 . These imply that either $(T, 0)$ and $(0, T)$ maximizes $V_2(b_1, b_2)$, and equivalently, minimizes $V_1(b_1, b_2)$.

(1): For this, it suffices to show that $b_2 = (T, 0)$ gives $V_2(b_1, b_2) < 1 - \theta$, since (i) implies that $(T, 0)$ and $(0, T)$ give candidate 2 the same payoff. By Lemma 2, $V_2(b_1, (T, 0)) = \frac{(\delta_1 - b_{11}/T)}{\delta_1 - \delta_0} \theta <$

$1 - \theta$ can be rewritten as

$$(\delta_1\rho - 2\delta_1^2 + \delta_0\delta_1 - 1)\theta^2 + (1 - 2\delta_0\delta_1 - \rho\delta_1 + 3\delta_1^2)\theta + (\delta_0\delta_1 - \delta_1^2) < 0$$

$$\Leftrightarrow \rho > \frac{2\theta - 1}{\theta} \delta_1 + \frac{1 - \theta}{\theta} \delta_0 + \frac{1}{\delta_1} = \hat{\rho}(\theta, \delta_0, \delta_1).$$

Since $\rho > \hat{\rho}(\theta, \delta_0, \delta_1)$, we have $V_2(b_1, (T, 0)) < 1 - \theta$.

(2): This can be formally written as either $b_{21} < b_{11}/\delta_1$ or $b_{22} < \delta_0 b_{12}$ for any b_2 . Suppose,

$b_{21} \geq b_{11}/\delta_1$ and $b_{22} \geq \delta_0 b_{12}$ for some b_2 . Then $T = b_{21} + b_{22} \geq b_{11}/\delta_1 + \delta_0 b_{12}$. Since, by (1), b_1 guarantees that candidate 2 does not win either party completely by $b_2 = (T, 0)$ or $(0, T)$, we have $b_{11} > \delta_0 T$ and $b_{12} > T/\delta_1$. Therefore $T \geq b_{11}/\delta_1 + \delta_0 b_{12} > \delta_0 T/\delta_1 + \delta_0 T/\delta_1 = 2\frac{\delta_0}{\delta_1}T$. Since $1/2 < \delta_0 < \delta_1 \leq 1$, we have $\delta_0/\delta_1 > 1/2$, so $T > 2\frac{\delta_0}{\delta_1}T > T$, a contradiction. Thus candidate 2 cannot win votes from both groups simultaneously by any b_2 .

We now prove that b_1 is a maximin strategy, i.e., $\min_{b_2} V_1(b_1, b_2) > \min_{b_2} V_1(b'_1, b_2)$ for any $b'_1 \neq b_1$. Let $b'_1 \neq b_1$. Since (i) and (ii) uniquely determine b_1 by Lemma 2, b'_1 violates (i) or (ii). Suppose that b'_1 violates (i). Since $b'_1 \neq b_1$, either $b_{11} > b'_{11}$ or $b_{12} > b'_{12}$. Then we have either

$$V_2(b_1, (T, 0)) = \frac{(\delta_1 - b_{11}/T)\theta}{\delta_1 - \delta_0} < \frac{(\delta_1 - b'_{11}/T)\theta}{\delta_1 - \delta_0} = V_2(b'_1, (T, 0)) \quad \text{or}$$

$$V_2(b_1, (0, T)) = \frac{(T/b_{12} - \delta_0)(1 - \theta)}{\delta_1 - \delta_0} < \frac{(T/b'_{12} - \delta_0)(1 - \theta)}{\delta_1 - \delta_0} = V_2(b'_1, (0, T)).$$

The above inequalities are equivalent to

$$V_1(b_1, (T, 0)) > V_1(b'_1, (T, 0)) \quad \text{or} \quad V_1(b_1, (0, T)) > V_1(b'_1, (0, T)). \quad (\text{A.5})$$

By (A.4) and (A.5), we have $\min_{b_2} V_1(b_1, b_2) > \min\{V_1(b'_1, (T, 0)), V_1(b'_1, (0, T))\} \geq \min_{b_2} V_1(b'_1, b_2)$. Suppose that b'_1 violates (ii), i.e., $b'_{11} < \delta_0 T$ or $b'_{11} > \delta_1 T$. In either case candidate 2 gets $V_2(b'_1, b_2) \geq 1 - \theta$ by $b_2 = (T, 0)$ or $(0, T)$. Equivalently, we have either $V_1(b'_1, (T, 0)) \leq \theta$ or $V_1(b'_1, (0, T)) \leq \theta$. Thus $\min\{V_1(b'_1, (T, 0)), V_1(b'_1, (0, T))\} \leq \theta$. Recall that $V_2(b_1, (T, 0)) < 1 - \theta$ by (i), i.e., $V_1(b_1, (T, 0)) > \theta$; and by (A.4), $\min_{b_2} V_1(b_1, b_2) = V_1(b_1, (T, 0))$. Thus we have $\min_{b_2} V_1(b_1, b_2) = V_1(b_1, (T, 0)) > \theta \geq \min\{V_1(b'_1, (T, 0)), V_1(b'_1, (0, T))\} \geq \min_{b_2} V_1(b'_1, b_2)$.

Only-If-Part: Let b'_1 be the strategy with $b'_{11} = (\frac{2\theta - 1}{\theta}\delta_1 + \frac{1 - \theta}{\theta}\delta_0)T$. Since $\rho > \hat{\rho}(\theta, \delta_0, \delta_1)$, $V_1(b'_1, b_2) > \theta$ for any b_2 by (A.1) and (A.2). Thus for a maximin strategy b_1 , $V_1(b_1, b_2) > \theta$ for any b_2 , which implies that for b_1 , candidate 2 cannot win one group completely by any b_2 . This,

together with $\rho > \hat{\rho}(\theta, \delta_0, \delta_1)$, implies that candidate 2 cannot win votes from both groups simultaneously either.¹⁶

Suppose that b_1 is a maximin strategy and that it violates (i). Since b_1 violates (i), we have either (1) $V_2(b_1, (T, 0)) > V_2(b_1, (0, T))$ or (2) $V_2(b_1, (T, 0)) < V_2(b_1, (0, T))$. First, we consider case (1). In this case, candidate 2's associated minimizing strategy is $b_2 = (T, 0)$, since he cannot win votes from both groups simultaneously. That is, $\min_{b_2} V_1(b_1, b_2) = V_1(b_1, (T, 0))$. Now consider a strategy \tilde{b}_1 such that $\tilde{b}_{11} = b_{11} + \epsilon$, where $\epsilon > 0$ is sufficiently small. Then we still have $V_2(\tilde{b}_1, (T, 0)) > V_2(\tilde{b}_1, (0, T))$ and $V_1(\tilde{b}_1, (T, 0)) = \min_{b_2} V_1(\tilde{b}_1, b_2)$. However, since $\tilde{b}_{11} > b_{11}$, we have $V_1(\tilde{b}_1, (T, 0)) > V_1(b_1, (T, 0))$, which is equivalent to $\min_{b_2} V_1(\tilde{b}_1, b_2) > \min_{b_2} V_1(b_1, b_2)$. Thus, b_1 is not a maximin strategy, a contradiction. The proof of case (2) is similar.

Suppose that b_1 is a maximin strategy and that it violates (ii). Then $b_{11} < \delta_0 T$ or $b_{11} > \delta_1 T$. In either case, candidate 2 gets $V_2(b_1, b_2) \geq 1 - \theta$ by $(T, 0)$ or $(0, T)$. However, since b_1 is a maximin strategy, we have that for any b_2 , $V_1(b_1, b_2) > \theta$, or, equivalently, $V_2(b_1, b_2) < 1 - \theta$, a contradiction. \square

The following lemma will be used in the proof of Theorems 4.3.2 and 4.4.2.

Lemma 4. For any ρ with $\hat{\rho}(\theta) < \rho \leq \delta_1 + \frac{1}{\delta_1}$, $B_1(\rho; \theta)$ is an increasing function of ρ .

Proof: Take the derivative

$$\partial B_1(\rho; \theta) / \partial \rho = \theta - \frac{(\rho + \delta_0 - \delta_1)\theta^2 - \delta_0\theta}{\sqrt{A}}.$$

$\partial B_1(\rho; \theta) / \partial \rho > 0$ is equivalent to $\sqrt{A} > (\rho - \delta_0 + \delta_1)\theta + \delta_0$, which is also equivalent to $1 - \theta > 0$.

Since $1/2 < \theta < 1$, this inequality holds. \square

Proof of Theorem 4.3.2: Let $c = \hat{B}_1(1; \theta)$. Since $\hat{B}_1(\rho'; \theta)$ is an increasing function of ρ' with $1 \leq \rho' \leq \hat{\rho}(\theta)$, we have $c \leq \hat{B}_1(\rho'; \theta)$ for $1 \leq \rho' \leq \hat{\rho}(\theta)$. Also, by Lemma 4, $B_1(\rho; \theta)$ is an

¹⁶The argument is similar to the one in the If-Part. See (1) in the proof of (A.4).

increasing function of ρ with $\hat{\rho}(\theta) < \rho \leq \delta_1 + \frac{1}{\delta_1}$. Thus it suffices to show that $c \geq B_1(\delta_1 + \frac{1}{\delta_1}; \theta)$.

This is equivalent to the following:

$$\hat{B}_1(1; \theta) \geq B_1(\delta_1 + \frac{1}{\delta_1}; \theta) \quad \Leftrightarrow$$

$$\left(\frac{2\theta-1}{\theta}\delta_1 + \frac{1-\theta}{\theta}\delta_0 + 1\right)T/2 \geq \frac{\theta(\delta_1 + \frac{1}{\delta_1} + \delta_1 - \delta_0) + \delta_0 - \sqrt{A(1)}}{2\theta} T = \left(\frac{2\delta_1 - \delta_0 + \frac{1}{\delta_1}}{2} + \frac{\delta_0 - \sqrt{A(1)}}{2\theta}\right) T,$$

$$\text{where } A(1) = \left(\delta_1 + \frac{1}{\delta_1} + \delta_0 - \delta_1\right)^2 - 4 + \left(4 - 2\delta_0^2 + 2\delta_0\delta_1 - 2\delta_0\left(\delta_1 + \frac{1}{\delta_1}\right)\right)\theta + \delta_0^2$$

$$= \left(\frac{1}{\delta_1^2} + \delta_0^2 + 2\frac{\delta_0}{\delta_1} - 4\right)\theta^2 + \left(4 - 2\delta_0^2 - 2\frac{\delta_0}{\delta_1}\right)\theta + \delta_0^2.$$

This inequality is further transformed into:

$$\sqrt{A(1)} \geq \delta_1 + \left(\frac{1}{\delta_1} - 1\right)\theta$$

$$\Leftrightarrow \left(\frac{1}{\delta_1^2} + \delta_0^2 + 2\frac{\delta_0}{\delta_1} - 4\right)\theta^2 + \left(4 - 2\delta_0^2 - 2\frac{\delta_0}{\delta_1}\right)\theta + \delta_0^2 \geq \left(\delta_1 + \theta\left(\frac{1}{\delta_1} - 1\right)\right)^2$$

$$\Leftrightarrow \left(\frac{2}{\delta_1} + 2\frac{\delta_0}{\delta_1} + \delta_0^2 - 5\right)\theta^2 - 2\left(\frac{\delta_0}{\delta_1} + \delta_0^2 - \delta_1 - 1\right)\theta - (\delta_1^2 - \delta_0^2) \geq 0.$$

Let $F(\theta) = \left(\frac{2}{\delta_1} + 2\frac{\delta_0}{\delta_1} + \delta_0^2 - 5\right)\theta^2 - 2\left(\frac{\delta_0}{\delta_1} + \delta_0^2 - \delta_1 - 1\right)\theta - (\delta_1^2 - \delta_0^2)$. We now prove that $F(1/2) > 0$ and $F(1) \geq 0$, which implies that if the coefficient $\left(\frac{2}{\delta_1} + 2\frac{\delta_0}{\delta_1} + \delta_0^2 - 5\right)$ of θ^2 is nonpositive, then $F(\theta) \geq 0$ for all $\theta \in (1/2, 1)$. When the coefficient of θ^2 is positive, we additionally prove that the minimum point of $F(\theta)$ is taken at some point smaller than $1/2$, so $F(\theta) \geq 0$ for all $\theta \in (1/2, 1)$.

Let us show $F(1/2) > 0$ and $F(1) \geq 0$. We can calculate

$$4 \times F(1/2) = \frac{2}{\delta_1} - 2\frac{\delta_0}{\delta_1} + \delta_0^2 - 1 + 4\delta_1 - 4\delta_1^2 = (1 - \delta_0)\left(\frac{2}{\delta_1} - (\delta_0 + 1)\right) + 4\delta_1(1 - \delta_1);$$

$$F(1) = \frac{2}{\delta_1} - 3 + 2\delta_1 - \delta_1^2 = 2\left(\frac{1}{\delta_1} + \delta_1\right) - 3 - \delta_1^2.$$

Since $1/2 < \delta_0 < \delta_1 \leq 1$ and $2 \leq \frac{1}{\delta_1} + \delta_1$, we have $F(1/2) > 0$ and $F(1) \geq 0$.

Suppose that the coefficient $\left(\frac{2}{\delta_1} + 2\frac{\delta_0}{\delta_1} + \delta_0^2 - 5\right)$ of θ^2 is positive. Since $F'(\theta) = 2\left(\frac{2}{\delta_1} + 2\frac{\delta_0}{\delta_1} + \delta_0^2 - 5\right)\theta - 2\left(\frac{\delta_0}{\delta_1} + \delta_0^2 - \delta_1 - 1\right)$, the minimum is at $\theta_o = \left(\frac{\delta_0}{\delta_1} + \delta_0^2 - \delta_1 - 1\right) / \left(\frac{2}{\delta_1} + 2\frac{\delta_0}{\delta_1} + \delta_0^2 - 5\right)$.

We prove that $\theta_o < 1/2$. This inequality is equivalent to

$$2\delta_1^2 + 2 > \delta_1\delta_0^2 + 3\delta_1. \quad (\text{A.6})$$

The positivity of the coefficient $\left(\frac{2}{\delta_1} + 2\frac{\delta_0}{\delta_1} + \delta_0^2 - 5\right)$ can be rewritten as

$$2 + 2\delta_0 + 2\delta_1\delta_0^2 - 2\delta_1 > \delta_0^2\delta_1 + 3\delta_1. \quad (\text{A.7})$$

Since $1 > \delta_1 > \delta_0$, we have $\delta_1 + \delta_1^2 > \delta_0 + \delta_1\delta_0^2$, which is equivalent to $2\delta_1^2 + 2 > 2 + 2\delta_0 + 2\delta_1\delta_0^2 - 2\delta_1$. This, together with (A.7), implies (A.6). \square

Proof of Theorem 4.4.1: We show that for a sufficiently small β , $DS_1(\theta_l; L) > DS_1(\theta_l; N)$ for all l with $\theta_l \leq \hat{\theta}$. Recall that the term limit L is determined by $\rho_L \leq \hat{\rho}(\hat{\theta}) < \rho_{L+1}$. Let $\rho(1) = \rho_\tau$, i.e., voters are facing an incumbent with seniority level ρ_τ . We have the following:

$$\begin{aligned} DS_1(\theta_l; L) &= B_1(\rho_\tau; \theta_l) + \beta B_1(\rho_{\tau+1}; \theta_l) + \dots + \beta^{L-\tau} B_1(\rho_L; \theta_l) + \beta^{L-\tau+1} B_1(\rho_1; \theta_l) + \dots \\ DS_1(\theta_l; N) &= B_1(\rho_\tau; \theta_l) + \beta B_1(\rho_{\tau+1}; \theta_l) + \dots + \beta^{L-\tau} B_1(\rho_L; \theta_l) + \beta^{L-\tau+1} B_1(\rho_{L+1}; \theta_l) + \dots \\ &\quad + \beta^{N-\tau} B_1(\rho_N; \theta_l) + \beta^{N-\tau+1} B_1(\rho_1; \theta_l) + \dots \end{aligned}$$

Thus $DS_1(\theta_l; L) > DS_1(\theta_l; N) \Leftrightarrow$

$$\begin{aligned} &\beta^{L-\tau+1} (B_1(\rho_1; \theta_l) - B_1(\rho_{L+1}; \theta_l)) + \beta^{L-\tau+2} (B_1(\rho_2; \theta_l) - B_1(\rho_{L+1}; \theta_l)) + \\ &\quad \beta^{L-\tau+3} (B_1(\rho_3; \theta_l) - B_1(\rho_{L+2}; \theta_l)) \dots > 0 \quad \Leftrightarrow \\ &(B_1(\rho_1; \theta_l) - B_1(\rho_{L+1}; \theta_l)) + \beta (B_1(\rho_2; \theta_l) - B_1(\rho_{L+1}; \theta_l)) + \\ &\quad \beta^2 (B_1(\rho_3; \theta_l) - B_1(\rho_{L+2}; \theta_l)) + \dots > 0. \quad (\text{A.8}) \end{aligned}$$

For $\theta_l \leq \hat{\theta}$, we have $\hat{\rho}(\theta_l) \leq \hat{\rho}(\hat{\theta})$, since $\hat{\rho}(\theta)$ is an increasing function of θ . Also, the term limit L is chosen such that $\hat{\rho}(\hat{\theta}) < \rho_{L+1}$. Thus, for $\theta_l \leq \hat{\theta}$, $\hat{\rho}(\theta_l) < \rho_{L+1}$, i.e., for district l , the benefits $B_1(\rho_\tau; \theta_l)$ drops before term $L+1$. This implies $B_1(\rho_1; \theta_l) - B_1(\rho_{L+1}; \theta_l) > 0$ by Theorem 4.3.2. Note that some of the terms in (A.8) might be negative. Let $-M$ be a lower bound for all the

terms with coefficient β^t , $t = 1, 2, \dots$. Then the following implies (A.8):

$$\left(B_1(\rho_1; \theta_l) - B_1(\rho_{L+1}; \theta_l) \right) - \frac{\beta}{1-\beta} M > 0 \quad (\text{A.8}')$$

This is true for any sufficiently small β . \square

Proof of Theorem 4.4.2: For any $\theta_l \leq \hat{\theta}$, given $\sum_{\tau=1}^L B_1(\rho_\tau; \theta_l)/L > B_1(\rho_N; \theta_l)$, we show that for β close to 1, $DS_1(\theta_l; L) > DS_1(\theta_l; N)$. Let $\rho(1) = \rho_\tau$. We have

$$DS_1(\theta_l; L) = S(L) \sum_{m=0}^{\infty} \beta^{m(LN)} \quad \text{and} \quad DS_1(\theta_l; N) = S(N) \sum_{m=0}^{\infty} \beta^{m(LN)},$$

where $S(L) = N \sum_{t=1}^L \beta^{t-1} B_1(\rho(t); \theta_l)$ and $S(N) = L \sum_{t=1}^N \beta^{t-1} B_1(\rho(t); \theta_l)$. Thus $DS_1(\theta_l; L) > DS_1(\theta_l; N) \Leftrightarrow S(L) > S(N)$. As $\beta \rightarrow 1$, we have $S(L) \rightarrow N \sum_{\tau=1}^L B_1(\rho_\tau; \theta_l)$; and $S(N) \rightarrow L \sum_{\tau=1}^N B_1(\rho_\tau; \theta_l)$. We have to show that

$$\begin{aligned} N \sum_{\tau=1}^L B_1(\rho_\tau; \theta_l) &> L \sum_{\tau=1}^N B_1(\rho_\tau; \theta_l) \quad \Leftrightarrow \\ \sum_{\tau=1}^L B_1(\rho_\tau; \theta_l)/L &> \sum_{\tau=L+1}^N B_1(\rho_\tau; \theta_l)/(N-L). \end{aligned}$$

By assumption, $\sum_{\tau=1}^L B_1(\rho_\tau; \theta_l)/L > B_1(\rho_N; \theta_l)$. By Lemma 4, $B_1(\rho_N; \theta_l) > \sum_{\tau=L+1}^N B_1(\rho_\tau; \theta_l)/(N-L)$.

Hence $\sum_{\tau=1}^L B_1(\rho_\tau; \theta_l)/L > \sum_{\tau=L+1}^N B_1(\rho_\tau; \theta_l)/(N-L)$. Therefore, for β sufficiently close to 1, we have $DS_1(\theta_l; L) > DS_1(\theta_l; N)$. \square

Appendix B.

An Example for the Nonexistence of a Nash equilibrium in G.

Let $\rho = 1.5$, $\theta = \frac{2}{3}$ and $\delta_0 = 0.8$, $\delta_1 = 0.9$. We establish the following two properties.

- (1) For any strategy b_1 , there exists a strategy b_2 such that $V_2(b_1, b_2) > 0$.
- (2) For any strategy b_2 , there exists a strategy b_1 such that $V_2(b_1, b_2) = 0$.

Property (1) implies that any strategy combination (b_1, b_2) with $V_2(b_1, b_2) = 0$ cannot be a Nash equilibrium, and property (2) implies that any strategy combination (b_1, b_2) with $V_2(b_1, b_2) \neq 0$ cannot be a Nash equilibrium either. Therefore (1) and (2) imply the nonexistence of a Nash equilibrium.

To show (1), consider two cases: (i) $b_{11} \geq 0.9T$; and (ii) $b_{11} < 0.9T$. For case (i), $b_{12} = \rho T - b_{11} \leq 0.6T$. Thus, candidate 2 can choose $b_2 = (0, T)$ to win his own group, where $V_2(b_1, b_2) = \frac{1}{3}$. For case (ii), candidate 2 can choose $b_2 = (T, 0)$ to win some part of group 1, since $\frac{b_{11}}{b_{21}} < 0.9 = \delta_1$. From cases (i) and (ii), we obtain the assertion of (1).

To show (2), let $b_1 = (b_{11}, b_{12})$ with $b_{11} = 0.9b_{21}$. Thus, candidate 1 wins all votes from group 1. Now we show that $\frac{b_{22}}{b_{12}} \leq 0.8$, i.e., he wins the entire group 2 as well. Since $b_{12} = \rho T - b_{11} = 1.5T - 0.9b_{21}$, this is equivalent to show that $\frac{b_{22}}{1.5T - 0.9b_{21}} \leq 0.8 \Leftrightarrow 0.2T \geq -0.28b_{21}$. Therefore we have $V_1(b_1, b_2) = 1$, i.e., $V_2(b_1, b_2) = 0$. \square

Vitae

Wen Mao was born in Suzhou, China, on 30 March 1968. She received her B.A. in Shipping Management from Shanghai Maritime Institute in June 1990, entered the Virginia Polytechnic Institute and State University in August 1990, and received her Ph.D. in Economics in August 1994.

Wen Mao