Analytic Versions of The Zero Divisor Conjecture

by

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(Abstract)

One of the most famous and frustrating problems in algebra is the zero divisor conjecture. In this work we study some analytical versions of this conjecture. We will give sufficient conditions to determine when elements of $\mathbb{C}G$ are uniform nonzero divisors, we also give sufficient conditions to determine when elements of $\mathbb{C}G$ are p-zero divisors. Examples of p-zero divisors will also be given. Also a measure theoretic approach to the zero divisor conjecture will be given.

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Chapter 1

Introduction

In this dissertation we study some analytical versions of the zero divisor conjecture. To begin this story let f be a complex valued function on a discrete group G. We may represent f as a formal sum $\sum_{g\in G} a_g g$ where $a_g \in \mathbb{C}$ and $f(g) = a_g$. With respect to the counting measure we can form the following Lebesgue spaces, $L^\infty(G)$, $C_0(G)$ and $L^p(G), 1 \le p < \infty$. These spaces may be thought of in the following ways. $L^\infty(G)$ will consist of all formal sums $\sum_{g\in G} a_g g$ for which $\sup_{g\in G} |a_g| < \infty$, $C_0(G)$ will consist of those formal sums for which the set $\{g \mid |a_g| > \epsilon\}$ is finite for all $\epsilon > 0$, and $L^p(G)$ will consist of those formal sums for which $\sum_{g\in G} |a_g|^p < \infty$. Let $\mathbb{C}G$ be the group ring of G over $\mathbb{C}$, so $\mathbb{C}G$ consists of all formal sums $\sum_{g\in G} a_g g$ where $a_g = 0$ for all but finitely many g. Then $\mathbb{C}G$ can also be thought of as the complex valued functions on G with compact support. The following inclusions are clear:

$$\mathbb{C}G \subseteq L^p(G) \subseteq C_0(G) \subseteq L^\infty(G).$$

For $\alpha = \sum_{g\in G} a_g g \in L^1(G)$ and $\beta = \sum_{g\in G} b_g g \in L^p(G), 1 \le p \le \infty$, we define a multiplication $* : L^1(G) \times L^p(G) \to L^p(G)$ by

$$\alpha * \beta = \sum_{g,h} a_g b_h gh = \sum_{g\in G} (\sum_{h\in G} a_{gh^{-1}} b_h) g.$$

If $\beta \in C_0(G)$ the above product is in $C_0(G)$.

While working on a problem concerning L^2-cohomology of Groups, Joel Cohen made the following conjecture [2]. If G is a torsion free group, α is a nonzero element in $\mathbb{C}G$ and β is a nonzero element in $L^2(G)$, then $\alpha * \beta$ is nonzero. Cohen proved his conjecture for the case G torsion free abelian[1], and Peter Linnell has proved Cohen's conjecture for the cases G elementary amenable, G orderable [7],[8]. The classical zero divisor conjecture from algebra states that if G is torsion free, and α, β are nonzero elements in $\mathbb{C}G$, then $\alpha * \beta \neq 0$. Clearly this is a special case of Cohen's conjecture, which explains the title of this work.

A natural question to ask is what can we say about Cohen's conjecture if we replace 2 by any finite p. In this dissertation we will show that the conjecture is false for any $p > 2$, even in the case G torsion free abelian. We also give sufficient conditions on $\alpha \in \mathbb{C}\mathbb{Z}^n$ to ensure that $\alpha * \beta \neq 0$, whenever β is a nonzero element of $C_0(\mathbb{Z}^n)$. Similar results will also be proven for free groups. We will apply some of our results to a problem from the theory of trigonometric series. Also, we will give a measure theoretic approach to Cohen's conjecture.

Chapter 2

Some background from analysis

We begin with some algebra like definitions. Let $\alpha \in L^1(G)$. If there exists a nonzero β in $L^p(G)$, p finite, such that $\alpha * \beta = 0$, then we shall say that α is a *p-zero divisor.* If $\alpha * \beta \neq 0$ for all $\beta \in C_0(G) \setminus 0$, then we shall say that α is a *uniform nonzero divisor.* Now let's do some analysis. Given a real number $p \geq 1$, we will always let q denote the conjugate index of p, i.e., if $p > 1$ then $\frac{1}{p} + \frac{1}{q} = 1$, and $q = \infty$ if $p = 1$. Let $f = \sum_{g \in G} a_g g \in L^p(G), h = \sum_{g \in G} b_g g \in L^q(G)$ and define a map $\langle \cdot, \cdot \rangle : L^p(G) \times L^q(G) \to \mathbb{C}$ by

$$\langle f, h \rangle = \sum_{g \in G} a_g \overline{b_g}.$$

Fix $h \in L^q(G)$. Then $\langle \cdot, h \rangle$ is a continuous linear functional on $L^p(G)$. By the Riesz representation theorem every continuous linear functional on $L^p(G)$ is of this form. Let $y \in G$ and let $f \in L^p(G)$. The right translation of f by y will be denoted by f_y, where $f_y(x) = f(xy^{-1})$. Set $T^p[f]$ equal to the closure in $L^p(G)$ of the set of linear combinations of translates of f. Using the Hahn-Banach theorem we see that $T^p[f] = L^p(G)$ if and only if no nonzero continuous linear functional on $L^p(G)$ vanishes on all translates of f. For $\beta = \sum_{g \in G} a_g g \in L^\infty(G)$, set $\beta^* = \sum_{g \in G} \overline{a_g} g^{-1}$ and $\overline{\beta} = \sum_{g \in G} \overline{a_g} g$. The following proposition will give the link between multiplication (algebra) and the span of translates in a function space (analysis).

Proposition 1 *Let α be an element of $L^1(G)$. Then α is a p-zero divisor if and only if $T^q[\overline{\alpha^*}]$ is not equal to $L^q(G)$.*

Proof: Let $y \in G$ and write $\alpha = \sum_{g \in G} a_g g$. Then $\overline{\alpha^*} = \sum_{g \in G} a_g g^{-1}$ and $\overline{\alpha^*}_y = \sum_{g \in G} a_{yg^{-1}} g$. If $\beta = \sum_{g \in G} b_g g \in L^p(G)$, then $(\alpha * \beta)(y) = \sum_{g \in G} a_{yg^{-1}} b_g = \left\langle \overline{\alpha^*}_y, \overline{\beta} \right\rangle$, hence $\alpha * \beta = 0$ if and only if $\left\langle \overline{\alpha^*}_y, \overline{\beta} \right\rangle = 0$ for all $y \in G$. $\square$

Let's do some Fourier analysis. For the rest of this chapter we will assume that G is abelian. Let Γ be the group of homomorphisms from G to the unit circle in $\mathbb{C}$. Γ is known as the dual group of G. Also a topology can be defined on Γ that makes it a compact topological group. If $f = \sum_{g \in G} a_g g \in L^1(G)$, the Fourier transform of f is defined by

$$\hat{f}(\gamma) = \sum_{g \in G} a_g (g^{-1}, \gamma)$$

where $\gamma \in \Gamma$ and $(g^{-1}, \gamma) = \gamma(g^{-1})$. Let $Z(f)$ denote the elements in Γ such that $\hat{f}(\gamma) = 0$. The following lemma will be crucial for later work.

Lemma 1 *Let $f \in L^1(G)$. If $y \in G$, then $Z(f_y) = Z(f)$.*

Proof: Let $y \in G$

$$\begin{aligned} \hat{f}_y(\gamma) &= \sum_{g \in G} a_{gy^{-1}} (g^{-1}, \gamma) \\ &= \sum_{g \in G} a_{gy^{-1}} (y^{-1} y g^{-1}, \gamma) \\ &= \sum_{g \in G} a_{gy^{-1}} (y^{-1}, \gamma)(y g^{-1}, \gamma) \\ &= (y^{-1}, \gamma) \sum_{g \in G} a_{gy^{-1}} (y g^{-1}, \gamma) \\ &= (y^{-1}, \gamma) \hat{f}(\gamma). \end{aligned}$$

The lemma now follows since $|(y^{-1}, \gamma)| = 1$. $\square$

Denote by $M(\Gamma)$ the set of bounded regular measures on Γ, and let $M(E)$ be the elements in $M(\Gamma)$ that are concentrated on the closed set E. Let $\mu \in M(\Gamma)$. The Fourier-Stieltjes transform of μ is defined by

$$\hat{\mu}(x) = \int_{\Gamma} (x, \gamma) d\mu(\gamma)$$

where $x \in G$. Since $|(x, \gamma)| = 1$ for all $x \in G$ we see that $\hat{\mu} \in L^\infty(G)$, thus $\hat{\mu}$ is a continuous linear functional on $L^1(G)$. Set $\hat{\mu}^*(g) = \overline{\hat{\mu}(g^{-1})}$. The following lemma will be useful in constructing p-zero divisors.

Lemma 2 *If* $f = \sum_{g \in G} a_g g \in L^1(G)$ *and* $\mu \in M(Z(f))$, *then* $\langle f_y, \hat{\mu}^* \rangle = 0$ *for all* y *in* G. *Furthermore, if* $\hat{\mu}^* \in L^p(G)$ *and* p *is finite, then* $\overline{f}^*$ *is a p-zero divisor.*

Proof:

$$\begin{aligned} \langle f, \hat{\mu}^* \rangle &= \sum_{g \in G} a_g \overline{\hat{\mu}^*(g)} \\ &= \sum_{g \in G} a_g \int_\Gamma (g^{-1}, \gamma) d\mu(\gamma) \\ &= \int_\Gamma \sum_{g \in G} a_g (g^{-1}, \gamma) d\mu(\gamma) \\ &= 0 \end{aligned}$$

since $\sum_{g \in G} a_g (g^{-1}, \gamma) = \hat{f}(\gamma)$. The first part of the lemma now follows from lemma 1. The second part of the lemma follows immediately from proposition 1. □

We now develop the concept of spectral synthesis, which is important in proving that certain elements of $L^1(\mathbb{Z}^n)$ are uniform nonzero divisors. Let E be a closed subset of $\Gamma, I(E)$ the set of all $f \in L^1(G)$ such that $E \subseteq Z(f)$, and let $j(E)$ be the set of all $f \in L^1(G)$ such that $E \subseteq O \subseteq Z(f)$ where O is an open set in Γ. Denote by $J(E)$ the closure of $j(E)$ in the $L^1(G)$-norm, clearly $J(E) \subseteq I(E)$. Let $y \in G$, $f \in J(E)(I(E))$. It follows from Lemma 1 that $f_y \in J(E)(I(E))$, so both $I(E)$ and $J(E)$ are translation-invariant subspaces of $L^1(G)$. Let $\Phi(E) = \{h \in L^\infty(G) \mid \langle f, h \rangle = 0, f \in I(E)\}$ and $\Psi(E) = \{h \in L^\infty(G) \mid \langle f, h \rangle = 0, f \in J(E)\}$. E is said to be a set of *spectral synthesis* (S-set) if $I(E) = J(E)$. E is a set of *uniqueness* if $\Psi(E) \cap C_0(G) = 0$, if E is not a set of uniqueness then E is a set of *multiplicity*.

By taking the Fourier-Stieltjes transform of the point measures contained in $M(E)$, we can identify E with a subset of $L^\infty(G)$. The following are easily deduced from chapter 7 of [13], and will be used in the sequel.

1. $\Phi(E)$ is the weak*-closed subspace of $L^\infty(G)$ generated by E.

2. $\Phi(E)$ is the weak*-closure of $\{\hat{\mu} \mid \mu \in M(E)\}$.

3. If $\alpha * h = 0$, where $\alpha \in L^1(G)$ and $h \in L^\infty(G)$, then $h \in \Psi(Z(\alpha))$.

Chapter 3

Preliminaries for $G = \mathbb{Z}^n$

We begin the chapter by discussing Fourier analysis for the case $G = \mathbb{Z}^n$. It is well known that $\mathbb{T}^n$, where $\mathbb{T} = [-\pi, \pi]/\{-\pi \sim \pi\}$, is the dual group of $\mathbb{Z}^n$; see chapter 1 of [13] for the details. If $f \in L^1(\mathbb{Z}^n)$, the Fourier transform of f is

$$\hat{f}(t) = \sum_{m \in \mathbb{Z}^n} f(m) e^{-i(m \cdot t)}$$

where $t \in \mathbb{T}^n$ and $m \cdot t$ is the usual Euclidean inner product. The Fourier transform induces an isometry between $L^2(\mathbb{Z}^n)$ and $L^2(\mathbb{T}^n)$. For $g \in L^2(\mathbb{T}^n)$, the inverse map to the Fourier transform is given by

$$\check{g}(m) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} g(t_1, \ldots, t_n) e^{i(\sum_{k=1}^n m_k t_k)} dt_1 \ldots dt_n$$

where $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$. Recall that $C^\infty(\mathbb{T}^n)$ denotes the infinitely differentiable functions on $\mathbb{T}^n$. The next lemma is a generalization of exercise 4 from page 30 of [6].

Lemma 3 *If $g \in C^\infty(\mathbb{T}^n)$, then* $\sum_{m \in \mathbb{Z}^n} |\check{g}(m)| \leq \| g \|_{L^1(\mathbb{T}^n)}$

$$+\sqrt{\left(\sum_{k=1}^{\infty} \frac{2}{k^2}\right)^n} \left\| \frac{\partial^n g}{\partial x_1 \ldots \partial x_n} \right\|_{L^2(\mathbb{T}^n)}$$

$$+\sqrt{\left(\sum_{k=1}^{\infty} \frac{2}{k^2}\right)^{n-1}} \sum_{1 \leq i_1 < \cdots < i_{n-1} \leq n} \left\| \frac{\partial^{n-1} g}{\partial x_{i_1} \ldots \partial x_{i_{n-1}}} \right\|_{L^2(\mathbb{T}^n)}$$

$$+\sqrt{\left(\sum_{k=1}^{\infty} \frac{2}{k^2}\right)^{n-2}} \sum_{1 \leq i_1 < \cdots < i_{n-2} \leq n} \left\| \frac{\partial^{n-2} g}{\partial x_{i_1} \ldots \partial x_{i_{n-2}}} \right\|_{L^2(\mathbb{T}^n)}$$

$$+ \cdots + \sqrt{\sum_{k=1}^{\infty} \frac{2}{k^2}} \sum_{1 \leq i \leq n} \left\| \frac{\partial g}{\partial x_i} \right\|_{L^2(\mathbb{T}^n)}.$$

Proof: Since our proof is easy to generalize, we will prove the lemma for the case $n = 2$. To begin with

$$\sum_{(n_1,n_2)\in\mathbb{Z}^2} |\check{g}(n_1,n_2)| = |\check{g}(0,0)| + \sum_{n_1\neq 0} |\check{g}(n_1,0)|$$
$$+ \sum_{n_2\neq 0} |\check{g}(0,n_2)| + \sum_{n_1\neq 0, n_2\neq 0} |\check{g}(n_1,n_2)|.$$

Using integration by parts we obtain

$$\begin{aligned}
\frac{\partial^2 g}{\partial x_1 \partial x_2}\check{}(n_1,n_2) &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{\partial^2 g}{\partial x_1 \partial x_2}(x_1,x_2)e^{i(n_1x_1+n_2x_2)}dx_1dx_2 \\
&= \frac{-n_1n_2}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} g(x_1,x_2)e^{i(n_1x_1+n_2x_2)}dx_1dx_2 \\
&= -n_1n_2\check{g}(n_1,n_2).
\end{aligned}$$

With the help of the Cauchy-Schwarz inequality and the Parseval's relation we obtain

$$\sum_{n_1\neq 0, n_2\neq 0} |\check{g}(n_1,n_2)| = \sum_{n_1\neq 0, n_2\neq 0} \left|\frac{1}{n_1n_2}\frac{\partial^2 g}{\partial x_1 \partial x_2}\check{}(n_1,n_2)\right|$$
$$\leq \sqrt{\left(\sum_{k=1}^{\infty}\frac{2}{k^2}\right)^2}\left\|\frac{\partial^2 g}{\partial x_1 \partial x_2}\right\|_{L^2(\mathbb{T}^2)}.$$

It can be shown by similar calculations that

$$\sum_{n_1\neq 0} |\check{g}(n_1,0)| \leq \sqrt{\sum_{k=1}^{\infty}\left(\frac{2}{k^2}\right)^2}\left\|\frac{\partial g}{\partial x_1}\right\|_{L^2(\mathbb{T}^2)} \text{ and}$$

$$\sum_{n_2\neq 0} |\check{g}(0,n_2)| \leq \sqrt{\sum_{k=1}^{\infty}\left(\frac{2}{k^2}\right)^2}\left\|\frac{\partial g}{\partial x_2}\right\|_{L^2(\mathbb{T}^2)}.$$

The lemma follows after we make the observation that $|\check{g}(0,0)| \leq \|g\|_{L^1(\mathbb{T}^2)}$. □

Corollary 1 *If $g \in C^\infty(\mathbb{T}^n)$, then there exists an $f \in L^1(\mathbb{Z}^n)$ such that $\hat{f} = g$.*

Let E be a closed subset of $V = (-\pi,\pi)^n$ and set d equal to the Euclidean distance from E to $\mathbb{R}^n \setminus V$. Let U_r be the set of points of $\mathbb{R}^n$ at a distance less than $\frac{d}{r}$ from E, where r is a natural number. Edwards constructs a function F on $\mathbb{R}^n$ [4,pp229-230] such that $Z(F) = E$ and $F \in C^\infty(\mathbb{R}^n)$, the infinitely differentiable functions on $\mathbb{R}^n$. By using the U_r's defined above in place of the U_r's used in Edward's argument, we construct a function g such that $Z(g) = E$ and g is constant on the frontier of V, so $g \in C^\infty(\mathbb{T}^n)$. By the above corollary we see that given a closed set $E \subseteq V$ there exists a $f \in L^1(\mathbb{Z}^n)$ such that $Z(f) = E$. Now suppose that E is any closed set on $\mathbb{T}^n$. Write E as the union of two nonempty closed sets E_1, E_2 and let $f_1, f_2 \in L^1(\mathbb{Z}^n)$ such that $Z(f_i) = E_i$. Since $L^1(\mathbb{Z}^n)$ is a ring and $Z(f_1 * f_2) = Z(f_1) \cup Z(f_2) = E$, we can conclude that given a closed set E on $\mathbb{T}^n$ there exists an $f \in L^1(\mathbb{Z}^n)$ such that $Z(f) = E$.

We conclude this chapter by giving some geometric definitions that will be needed in chapter 5. Recall that $V = (-\pi, \pi)^n$. Identify V with an open subspace of $\mathbb{T}^n$. Let $f \in L^1(\mathbb{Z}^n)$ and $x_0 \in E = Z(f) \cap V$. We shall say that x_0 is a *regular point* if there exists an open neighborhood U of x_0 such that $F_{x_0} = U \cap E$ is a smooth m-dimensional submanifold of V, where m is a natural number. Consider F_{x_0} in a sufficiently small neighborhood of x_0 and write F_{x_0} as the image of a smooth mapping $\phi: W \to V$, where W is a neighborhood of the origin in $\mathbb{R}^m$. Also assume that the vectors $\frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_m}$ are linearly independent for each $x \in W$. Now suppose for each $y_0 \in W$ and each unit vector $\eta \in V$, there is a multi-index j, with $|j| \geq 1$, so that

$$\partial_x^j [\phi(x) \cdot \eta] \,|_{x=y_0} \neq 0$$

(where of course $\phi(x) \cdot \eta$ is the usual Euclidean inner product). The smallest k so that, for each unit vector η, there exists a j with $|j| \leq k$ for which

$$\partial_x^j [\phi(x) \cdot \eta] \,|_{x=y_0} \neq 0$$

is called the ***type*** of ϕ (and of F_{x_0}) at y_0. Also, if $W_1 \subset W$ is a compact set, the type of ϕ in W_1 is defined to be the maximum of the types of the $y_0 \in W_1$. If F_{x_0} is a $n-1$ dimensional submanifold of V, we can compute the $(n-1) \times (n-1)$ matrix

$$\left(\frac{\partial^2 \phi}{\partial x_j \partial x_k} \right)(x_0).$$

The eigenvalues of the above matrix are called the *principal curvatures* of F_{x_0} at x_0, and their product is the *Gaussian curvature* of F_{x_0} at x_0.

Chapter 4

Uniform nonzero divisors

Throughout this chapter we will assume that G is abelian. We begin with

Theorem 1 *Let G be a torsion free group and let $\alpha \in L^1(G)$. If $Z(\alpha)$ is a proper subgroup of Γ, then α is a uniform nonzero divisor.*

Proof: Let $A = \{a \in G \mid \gamma(a) = 1 \text{ for all } \gamma \in Z(\alpha)\}$. Then A is known as the annihilator subgroup of $Z(\alpha)$, and $A \neq 0$ since $Z(\alpha) \neq \Gamma$. Observe that A is infinite since G is torsion free. Let $h \in \Phi(Z(\alpha))$ and fix $g \in G$ such that $h(g) \neq 0$. Now $|h(ga)| = |h(g)|$ for all $a \in A$ since $\gamma(ga) = \gamma(g)\gamma(a) = \gamma(g)$, hence h is not in $C_0(G)$ because A is infinite. Using theorem 7.5.2(d),[13], we see that $Z(\alpha)$ is an S-set , so $\Psi(Z(\alpha)) \cap C_0(G) = 0$ and the theorem follows. □

Note that the theorem is still true if the subgroup generated by $Z(\alpha)$ is not all of Γ.

Corollary 2 *If $Z(\alpha)$ is a proper subgroup of Γ, then $Z(\alpha)$ is a set of uniqueness.*

We will now discuss uniform nonzero divisors for the case $G = \mathbb{Z}^n$. We begin with the following interesting

Proposition 2 *Suppose that E_1, E_2 are closed sets of uniqueness on $\mathbb{T}^n$. If $E_1 \cup E_2$ is a S-set, then $E_1 \cup E_2$ is a set of uniqueness.*

Proof: We know from the discussion in chapter 3 that there exist functions f_1, f_2 in $L^1(\mathbb{Z}^n)$ such that $E_i = Z(f_i)$. Let $h \in C_0(\mathbb{Z}^n)$. Since the E_i's are sets of uniqueness and $f_1 * f_2 \in L^1(\mathbb{Z}^n)$, it follows that $(f_1 * f_2) * h = 0$ if and only if $h = 0$. Since $E_1 \cup E_2 = Z(f_1 * f_2)$ and $E_1 \cup E_2$ is a S-set we see that $E_1 \cup E_2$ is a set of uniqueness. □

We consider the case $G = \mathbb{Z}$ first. Write G multiplicatively with generator g. By this we mean that g^n corresponds to the integer n, and $g^n g^m = g^{n+m}$ corresponds to $n+m$. Let $0 \neq \alpha \in \mathbb{C}\mathbb{Z}$. Thinking as above it is easy to see that α is a Laurent polynomial in the variable g. Hence, the set of zeros for $\hat{\alpha}$ is finite. Invoking theorem 4.2 of [4] we see that $T^q[\alpha] = L^q(\mathbb{Z})$ for all real $q > 1$, so by proposition 1 α is a uniform nonzero divisor. In the next chapter we will see that this is not true for the case $n \geq 2$. What is true for this case is

Theorem 2 *Let $\alpha \in L^1(\mathbb{Z}^n)$. If $Z(\alpha)$ is contained in a finite union of proper closed cosets, then α is a uniform nonzero divisor.*

Proof: For our purposes the case $n = 1$ is done above, so we will assume that $n \geq 2$. Let $\gamma(x_1, \ldots, x_n) = x_1 - x_n$, so $\hat{\gamma}(\theta_1, \ldots, \theta_n) = e^{-i\theta_1} - e^{-i\theta_n}$ and $Z(\gamma) = \{(\theta_1, \theta_2, \ldots, \theta_{n-1}, \theta_1) \mid -\pi \leq \theta_k \leq \pi\}$. $Z(\gamma)$ is a proper subgroup of $\mathbb{T}^n$, so it is a set of uniqueness. Assume for now that $Z(\alpha)$ is a coset on $\mathbb{T}^n$. By translation we may assume that $Z(\alpha) \subseteq Z(\gamma)$, so $\Psi(Z(\alpha)) \subseteq \Psi(Z(\gamma))$, thus $Z(\gamma)$ is a set of uniqueness. The theorem now follows from theorem 7.5.2[13] and proposition 2. $\square$

Chapter 5

p-zero divisors

We start by giving sufficient conditions on α for it to be a p-zero divisor, when $\alpha \in L^1(\mathbb{Z}^n)$. The notation and definitions from chapter 3 will be used freely in this chapter. Let's begin with

Theorem 3 *Let $\alpha \in L^1(\mathbb{Z}^n)$, $n \geq 2$, and suppose that x_0 is a regular point in $Z(\alpha)$. If F_{x_0} is of finite type k, then $\overline{\alpha}^*$ is a p-zero divisor for $p > nk$.*

Proof: Let x_0 be a regular point of $Z(\alpha)$ such that F_{x_0} is a submanifold of finite type k. Let μ be a smooth nonzero mass density on the closure of F_{x_0} in $\mathbb{R}^n$. The theorem will follow from lemma 2 once we show that $\hat{\mu}^* \in L^p(\mathbb{Z}^n)$ for $p > nk$. Let $0 \neq \eta \in \mathbb{Z}^n$, it is shown in chapter 8 of [14] that

$$|\hat{\mu}(\eta)| \leq C|\eta|^{-\frac{1}{k}}$$

where $|\cdot|$ is the usual Euclidean norm and C is some constant. Set $\eta = (m_1, \ldots, m_n)$, $\eta' = (m_1+1, \ldots, m_n+1)$ and let j be the least integer greater than $n^{\frac{1}{2}}$. The triangle inequality implies that $|\eta'| - n^{\frac{1}{2}} \leq |\eta|$, so for $|\eta| > n^{\frac{1}{2}} - 1$ we obtain

$$\begin{aligned} |\hat{\mu}(\eta)|^p &\leq \left(C(|\eta|)^{-\frac{1}{k}}\right)^p \\ &\leq \left(C(|\eta'| - n^{\frac{1}{2}})^{-\frac{1}{k}}\right)^p \\ &\leq \int_{m_n}^{m_n+1} \cdots \int_{m_1}^{m_1+1} (Cf(x))^p \, dx_1 \ldots dx_n \end{aligned}$$

where $f(x) = \left((x_1^2 + \cdots + x_n^2)^{\frac{1}{2}} - n^{\frac{1}{2}}\right)^{-\frac{1}{k}}$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. For $N \in \mathbb{N}$ we have

$$\sum_{\eta \in \mathbb{Z}^n, |\eta| \leq N} |\hat{\mu}(\eta)|^p \leq \sum_{|\eta| < n^{\frac{1}{2}}+1} |\hat{\mu}(\eta)|^p +$$

$$\sum_{1+n^{\frac{1}{2}}\leq|\eta|\leq N+j} \int_{m_n}^{m_n+1} \cdots \int_{m_1}^{m_1+1} (Cf(x))^p dx_1 \ldots dx_n. \tag{5.1}$$

Let p be a real number strictly greater than nk, so

$$\int_{|x|\geq 1+n^{\frac{1}{2}}} (f(x))^p dx_1 \ldots dx_n$$

is finite, hence

$$\sum_{\eta\in\mathbb{Z}^n, |\eta|\geq 1+n^{\frac{1}{2}}} \int_{m_n}^{m_n+1} \cdots \int_{m_1}^{m_1+1} (f(x))^p dx_1 \ldots dx_n$$

converges. Letting $N \to \infty$ in (5.1) we obtain $\sum_{\eta\in\mathbb{Z}^n} |\hat{\mu}(\eta)|^p < \infty$; therefore, $\hat{\mu}^* \in L^p(\mathbb{Z}^n)$ for $p > nk$. □

Corollary 3 *If α is as in theorem 3, then $T^q[\overline{\alpha^*}] \neq L^q(\mathbb{Z}^n)$ for q such that $1 \leq q < \frac{nk}{nk-1}$.*

Proof: Use proposition 1. □

The next corollary gives better estimates for certain α in $L^1(\mathbb{Z}^n)$.

Corollary 4 *Let $\alpha \in L^1(\mathbb{Z}^n), n \geq 2$. Suppose there exists an $x_0 \in Z(\alpha)$ such that F_{x_0} is a $n-1$ dimensional submanifold of V. If F_{x_0} has strictly positive Gaussian curvature, then $\overline{\alpha}^*$ is a p-zero divisor for $p > \frac{2n}{n-1}$.*

Proof: Let μ be a smooth nonzero mass density on the closure of F_{x_0} in $\mathbb{R}^n$, $0 \neq \eta \in \mathbb{Z}^n$. It is shown in [9] that

$$|\hat{\mu}(\eta)| \leq C|\eta|^{-\frac{n-1}{2}}.$$

Now proceed as in the theorem. □

Now we will show that Cohen's conjecture cannot be extended to any $p > 2$, in the case G torsion free abelian. Let $p > 2$ be given and pick an integer n such that $2 < \frac{2n}{n-1} < p$. Let

$$\alpha(x_1, \ldots, x_n) = \frac{2n-1}{2} - \frac{1}{2}(\sum_{k=1}^{n}(x_k + x_k^{-1})),$$

so $\alpha(x_1,\ldots,x_n) \in \mathbb{C}\mathbb{Z}^n, \hat{\alpha}(t_1,\ldots,t_n) = \frac{2n-1}{2} - \sum_{k=1}^{n} \cos t_k$. Near $(0,\ldots,0,\frac{\pi}{3})$, $Z(\alpha)$ is of the form $\{(t,g(t)) \mid t \in U\}$, where U is a bounded open set containing 0 in $\mathbb{R}^{n-1}, t = (t_1,\ldots,t_{n-1})$ and $g(t) = \cos^{-1}(\frac{2n-1}{2} - \sum_{k=1}^{n-1} \cos t_k)$. A computation shows that the rank of the matrix $\left(\frac{\partial^2 g(t)}{\partial t_i \partial t_k}\right)$ is $n-1$ at $t = 0$, thus $Z(\alpha)$ has strictly positive curvature. Therefore, $\alpha(x_1,\ldots,x_n)$ is a r-zero divisor for $r > \frac{2n}{n-1}$.

We conclude this chapter by giving an application to the theory of trigonometric series. It is well known that closed sets with strictly positive Haar measure on $\mathbb{T}^n$ are sets of multiplicity; however, not much is known about the uniqueness or multiplicity of a closed set if it has measure zero.

Proposition 3 *If $0 \neq \alpha \in \mathbb{C}\mathbb{Z}^n$, then $Z(\alpha)$ has Haar measure zero.*

Proof: It follows from [1] and proposition 1 that $T^2[\alpha] = L^2(\mathbb{Z}^n)$. The proposition now follows from theorem 7.2.9 [13]. □

For $n \geq 2$, we saw above that $\{(t_1,\ldots,t_n) \mid \frac{2n-1}{2} = \sum_{k=1}^{n} \cos t_k\}$ is a set of measure zero that is also a set of multiplicity. In chapter 4 we saw that $Z(x_1 - x_n)$ is a set of uniqueness. Hence, the multiplicity of a set of measure zero may depend on the geometry of the set.

Chapter 6

A measure theoretic approach

Why a measure theoretic approach to Cohen's conjecture? Suppose $f \in L^1(G)$ and we want to determine whether f is a 2-zero divisor. If $X = T^2[\overline{f}^*]$, then proposition 1 says that f is a 2-zero divisor if and only if $X \neq L^2(G)$. The question now becomes when is $X = L^2(G)$? The abelian case is taken care of by theorem 7.2.9[13], which says that $X = L^2(G)$ if and only if the Haar measure of $Z(f)$ on the dual group of G equals zero. We now proceed to generalize this theorem to include the noncommutative case.

Let $y \in G$ and $f \in L^2(G)$. Right translation of f by y is a unitary representation of G on $L^2(G)$. This representation is known as the right regular represtation of G and is denoted by ρ. A commutative set $\mathcal{P}$ of projections on a Hilbert space is a *Boolean algebra* of projections if it contains the zero projection, the identity projection 1, and contains along with any P and S, the projections $1 - P$ and PS. It is *complete* if, along with any subfamily $\mathcal{P}'$ of $\mathcal{P}$, $\mathcal{P}$ contains the projection which is the least upperbound of $\mathcal{P}'$. Let P be a projection on $L^2(G)$ and denote by $\mathcal{B}(P)$ a maximal complete Boolean algebra that contains P. Given a direct integral $\int_A H_\tau d\mu(\tau)$, where $(A, \mathcal{M}, \mu)$ is a standard measure space and $\{H_\tau\}$ is a measurable field of Hilbert spaces on A, define a function Q from the measurable subsets of A to $\int_A H_\tau d\mu(\tau)$ by

$$Q(E) = \{f \mid \text{supp } f \subseteq E\}.$$

Q is known as the *projection valued measure associated* to $\int_A H_\tau d\mu(\tau)$. By theorems (2.8 [10], 1.2 [11]) there exists a standard Borel space (A, μ), a measure field $\{H_\tau\}$ of Hilbert spaces on A, a measurable field $\{\pi_\tau\}$ of unitary representations of G, and an isometry $U: L^2(G) \mapsto \int_A H_\tau d\mu(\tau)$, such that

1. $U\rho(x)U^{-1} = \int_A \pi_\tau(x) d\mu(\tau)$ for $x \in G$;

2. almost all of the $\{\pi_\tau\}$ are irreducible;

3. to each projection S in $\mathcal{B}(P)$, there exists a Borel subset E of A such that $S(L^2(G))$ is unitarily equivalent to $Q(E)$, where Q is the projection valued measure associated to $\int_A H_\tau d\mu(\tau)$.

Let $f \in L^2(G)$ and set $Z(f) = \{\tau \in A \mid (Uf)(\tau) = 0\}$. We are now ready to prove the following generalization of theorem 7.2.9[13].

Theorem 4 *Let $f \in L^2(G)$. The right translates of f span $L^2(G)$ if and only if for each direct integral decomposition $\int_A H_\tau d\mu(\tau)$ of the right regular representation of G into irreducibles, $\mu(Z(f)) = 0$.*

Proof: Let $X = T^2[f]$ and suppose that $X \neq L^2(G)$. Let P be the projection associated with X, and let $\int_A H_\tau d\mu(\tau)$ be the direct integral decomposition of ρ such that the projection valued measure Q associated with $\int_A H_\tau d\mu(\tau)$ corresponds to $\mathcal{B}(P)$. Let E be the Borel subset of A for which $Q(E)$ is unitarily equivalent to $P(L^2(G))$. Since $(Uf) \in Q(E)$, we see that $(Uf)(\tau) = 0$ for $\tau \in A \setminus E$, hence $\mu(Z(f)) > 0$, which proves necessity.

To prove sufficiency, assume that there exists a direct integral $\int_A H_\tau d\mu(\tau)$ decomposition of ρ, such that $\mu(Z(f)) > 0$. Let χ be the characteristic function on $Z(f)$, thus

$$\left\langle f, U^{-1}\chi \right\rangle = \int_A \langle Uf, \chi \rangle_\tau \, d\mu(\tau) = 0.$$

Let P be the projection for which $P(L^2(G))$ is unitarily equivalent to $Q(A \setminus Z(f))$. Then $P \neq 0$, hence $T^2[f] \neq L^2(G)$. □

Let $f \in L^2(G)$ and let P be the projection associated with $T^2[f]$. Since either $T^2[f] = L^2(G)$ or $T^2[f] \neq L^2(G)$, it follows that for each direct integral decomposition of ρ that corresponds to some $\mathcal{B}(P)$, either $\mu(Z(f)) = 0$ or $\mu(Z(f)) > 0$. So in practice if we want to determine whether the right translates of f span $L^2(G)$, we only need to find one $\mathcal{B}(P)$ and corresponding decomposition of ρ.

Corollary 5 *Let $f \in L^1(G)$ and P the projection associated with $T^2[f]$. Let $\int_A H_\tau d\mu(\tau)$ be the decomposition of ρ that corresponds to some $\mathcal{B}(P)$. Then f is a 2-zero divisor if and only if $\mu(Z(\overline{f}^{\,*})) > 0$.*

Proof: Use proposition 1. $\square$

Chapter 7

Free groups and zero divisors

Throughout this chapter, F_k will denote the free group on k generators, where we will always assume that $k \geq 2$. What can we say about p-zero divisors for $\mathbb{C}F_k$? Peter Linnell[8] has shown that $\mathbb{C}F_k \setminus 0$ contains no 2-zero divisors. We will show in this chapter that this cannot be improved to any $p > 2$. We will prove this by first proving an analogue of lemma 2. Using this we will construct an example of an element in $\mathbb{C}F_k \setminus 0$ that is a p-zero divisor for $p > 2$.

Any element x of F_k has a unique expression as a finite product of generators and their inverses, which does not contain any two adjacent factors aa^{-1} or $a^{-1}a$. The number of factors in x is called the *length* of x and is denoted by $|x|$.

A complex valued function f on F_k will be called radial if the value $f(x), x \in F_k$, depends only on $|x|$. Let $E_n = \{x \in F_k \mid |x| = n\}$. The number of elements in E_n is $|E_n| = 2k(2k-1)^{n-1}$, to be denoted by e_n from now on. Let χ_n be the characteristic function on E_n (in group ring notation $\chi_n = \sum_{|x|=n} x$). Then every radial function has the form $\sum_{n=0}^{\infty} a_n \chi_n$, $a_n \in \mathbb{C}$. Let $L^p_r(F_k)$ denote the radial functions contained in $L^p(F_k)$, and let $(\mathbb{C}F_k)_r$ denote the radial functions contained in $\mathbb{C}F_k$, so $L^p_r(F_k)$ is the closure of $(\mathbb{C}F_k)_r$ in $L^p(F_k)$. Let $\omega = \sqrt{2k-1}$. It is shown in chapter 3.1 of [5] that

$$\chi_1 * \chi_1 = \chi_2 + 2k\chi_0 ,$$

$$\chi_1 * \chi_n = \chi_{n+1} + \omega^2 \chi_{n-1} , \; n \geq 2,$$

hence $(\mathbb{C}F_k)_r$ is a commutative algebra which is generated by χ_0 and χ_1.

For $f \in L^p_r(F_k)$, we shall say that $f * \chi_n$ is a radial translate of f. Set $TR^p[f]$ equal to the closure in $L^p_r(F_k)$ of the set of linear combinations of radial translates of f. The

Hahn-Banach theorem says that $TR^p[f] = L_r^p(F_k)$ if and only if no nonzero continuous linear functional on $L_r^p(F_k)$ vanishes on all radial translates of f.

Let $P_n, n \in \mathbb{N}$, denote the polynomials on $\mathbb{C}$ defined by

$$P_0(z) = 1,\ P_1(z) = z,\ P_2(z) = z^2 - 2k \text{ and}$$

$$P_{n+1}(z) = zP_n(z) - \omega^2 P_{n-1}(z) \,\text{for}\, n \geq 2.$$

For $f = \sum_{n=0}^{\infty} a_n \chi_n \in L_r^1(F_k)$, in [12] Pytlik shows the following

1. $E = \{x + iy \in \mathbb{C} \mid \left(\frac{x}{2k}\right)^2 + \left(\frac{y}{2k-2}\right)^2 \leq 1\}$ is the spectrum of $L_r^1(F_k)$.

2. The Gelfand transform of f, denoted by $\hat{f}$, is given by

$$\hat{f}(z) = \sum_{n=0}^{\infty} a_n P_n(z),\ z \in E.$$

Let $Z(f)$ denote the set of elements in E such that $\hat{f}(z) = 0$, where $f \in L_r^1(F_k)$.

Proposition 4 *Let $f \in L_r^1(F_k)$. If $Z(f) = \emptyset$, then $TR^1[f] = L_r^1(F_k)$.*

Proof: Suppose that $TR^1[f] \neq L_r^1(F_k)$. A slight modification of the proof of theorem 7.1.2 [13] shows that $TR^1[f]$ is an ideal in $L_r^1(F_k)$. Let J be a maximal ideal in $L_r^1(F_k)$ that contains $TR^1[f]$. By basic Gelfand theory there exists a z_0 in E, such that $J = \{g \mid \hat{g}(z_0) = 0\}$, a contradiction. □

The converse of the proposition is also true. Before we prove this we will need some more definitions.

Let f be a complex valued function on F_k, set $a_n(f) = \frac{1}{e_n}\sum_{x \in E_n} f(x)$, and denote by $P(f)$ the radial function $\sum_{n=0}^{\infty} a_n(f)\chi_n$. Recall that E denotes the spectrum of $L_r^1(F_k)$. For $z \in E$, we define an element in $L_r^{\infty}(F_k)$, the space of continuous linear functionals on $L_r^1(F_k)$ [3], by

$$\phi_z = \sum_{n=0}^{\infty} \frac{P_n(z)}{e_n}\chi_n.$$

Observe that the ϕ_z's are similar, in spirit, to the Fourier-Stieltjes transform of point measures on $\mathbb{T}^n$.

It follows from the linearity of the Gelfand transform that $(\chi_1 * \chi_n)\widehat{\ }(z) = zP_n(z) = P_1(z)P_n(z)$. Since χ_1 generates $(\mathbb{C}F_k)_r$ we conclude that $(\chi_n * \chi_m)\widehat{\ }(z) = P_n(z)P_m(z)$, for all n, m. Hence, multiplication of the characteristic functions on the E_n's, corresponds to pointwise multiplication of the $P_n(z)$'s on E.

Proposition 5 *Let $f \in L^1_r(F_k)$. If $Z(f) \neq \emptyset$, then $TR^1[f] \neq L^1_r(F_k)$.*

Proof: Let $z \in Z(f)$, let $y \in F_k$ and set $m = |y|$. Note that

$$\langle f_y, \phi_z \rangle = (f_y)\widehat{\ }(z) = (P(f * y))\widehat{\ }(z).$$

It follows from the first part of the proof of lemma 6.1 in [12] that $P(f * y) = f * P(y)$, so we obtain

$$\begin{aligned}\langle f_y, \phi_z \rangle &= (\left(\sum_{j=0}^{\infty} a_j \chi_j\right) * \tfrac{\chi_m}{e_m})\widehat{\ }(z) \\ &= \sum_{j=0}^{\infty} \frac{a_j}{e_m} P_j(z) P_m(z) \\ &= \frac{P_m(z)}{e_m} \hat{f}(z) \\ &= 0.\end{aligned}$$

Since ϕ_z vanishes on every translate of f, it vanishes on every radial translate of f. Hence, $TR^1[f] \neq L^1_r(F_k)$. □

Corollary 6 *If $z \in Z(f)$, then $f * \overline{\phi_z} = 0$. Thus if $\phi_z \in L^p_r(F_k)$, f is a p-zero divisor.*

Proof: Use proposition 1. □

The above corollary can be considered an analogue of lemma 2 and will be used to show that χ_1 is a p-zero divisor for $p > 2$. Since $\widehat{\chi_1}(z) = z$, we have that $Z(\chi_1) = 0$. Hence, $\chi_1 * \phi_0 = 0$, where

$$\phi_0 = \sum_{n=0}^{\infty} \frac{P_n(0)}{e_n} \chi_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2k-1)^n} \chi_{2n}.$$

Let p be a real number. Since the number of elements in F_k that have length $2n$ is e_{2n}, we obtain

$$\begin{aligned}\sum_{g\in F_k} |\phi_0(g)|^p &= \sum_{n=0}^{\infty} \frac{e_{2n}}{(2k-1)^{pn}} \\ &= \sum_{n=0}^{\infty} \frac{2k(2k-1)^{2n-1}}{(2k-1)^{pn}} \\ &= \frac{2k}{2k-1} \sum_{n=0}^{\infty} \frac{1}{(2k-1)^{n(p-2)}}.\end{aligned}$$

Thus $\sum_{g\in F_k} |\phi_0(g)|^p < \infty$ if $p > 2$. Therefore, χ_1 is a p-zero divisor for $p > 2$.

Remark 1 *The fact that $\phi_0 \in L^p(F_k)$ for $p > 2$ is a bit surprising, since the Fourier-Stieltjes transform of point measures on $\mathbb{T}^n$ are not even in $C_0(\mathbb{Z}^n)$.*

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