The Cohomology Rings of Classical Brauer Tree Algebras

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Abstract

In this dissertation a simple algorithm is given for calculating minimal projective resolutions of nonprojective indecomposable modules over Brauer tree algebras. Those calculated resolutions lead to an algorithm for calculating a minimal set of generators for the cohomology ring of a Brauer tree algebra.
With All My Love
This Thesis is Dedicated to

My Wife
My Family
And My Friends
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Chapter 1

Introduction

Let $G$ be a finite group, $K$ a splitting field of characteristic $p \neq 0$, and $B$ a $p$-block of $KG$ having a cyclic defect group where $KG$ is the group ring. We note that a block is an indecomposable direct factor of $KG$. In 1941 in [2] Richard Brauer began using trees, or planar nondirected graphs with $n$ edges and $n + 1$ vertices, as a notational tool for describing the various characters, modules, and other objects associated with $B$.

In [12] D. Higman showed that if $K$ is a field of characteristic $p$ which divides the order of a group $G$, then $KG$ is of finite representation type if and only if the $p$-Sylow subgroups of $G$ are cyclic. Here finite representation type means that there are only a finite number of nonisomorphic indecomposable modules over $KG$. Later Kasch, Kneser, and Kupisch [14] and G. Janusz [13] provide detailed information about the number of nonisomorphic indecomposable modules in this case.

In [5] E. Dade describes the character theory developed by Brauer for $B$ when $B$ has a cyclic defect group. This includes the case where $G$ contains a cyclic $p$-Sylow subgroup. It is in that case that G. Janusz in [13] used Dade's results to construct the complete set of nonisomorphic indecomposable modules over $B$, a two sided ideal direct summand of $KG$. He also showed how to build the multiplication table
for $B$ from the Brauer tree and the character theory for $G$. Furthermore, he proved that once the multiplication table is known, the indecomposable modules can be determined without knowing the character theory.

A few years later, in [1] J. Alperin and G. Janusz began considering resolutions and periodicity. In particular, they proved that if $G$ is a finite group with a cyclic $p$-Sylow subgroup for a prime $p$ and $F$ is a splitting field of characteristic $p$ for $G$, then each term of the minimal projective resolution of $F$ over $FG$ is indecomposable. Actually they constructed the resolution and showed that it is periodic of period $2n$ where $n$ is the number of edges in the Brauer tree corresponding to $B$ the principal block of $FG$. From the construction of that resolution, they also obtained minimal projective resolutions for a few other indecomposable modules. In particular they obtained minimal projective resolutions for the socle of the projective indecomposables $v\Lambda$ where $v$ is a looped vertex of a Brauer quiver and $\Lambda$ is the associated Brauer tree algebra both of which we define in Chapter 2. For a definition of a looped vertex see section 5.1. We note that at about the same time J.A. Green [11] obtained similar results.

About a year later P. Gabriel [6] summarized many of the preceding results using, for the first time, a Brauer quiver, $\Gamma$, and the category $\text{rep}(\Gamma, \rho)$ which I also define in chapter 2. He credits Dade and Janusz with proving that if $G$ is a finite group and $B$ is a block of the group algebra $KG$, where $K$ is an algebraically closed field of characteristic $p > 0$, then if $B$ has only finitely many indecomposable $B$-modules, then $B$ is either semi-simple, or the category of $B$-modules is equivalent to the category $\text{rep}(\Gamma, \rho)$ for some Brauer quiver $\Gamma$ and some exceptional cycle. Thus in this paper we will focus on the category $\text{rep}(\Gamma, \rho)$ which is Morita equivalent the category of modules over the Brauer tree algebra corresponding to $\Gamma$. These ring
became known as Brauer tree algebras and are our focus in this thesis. We refer to them as Classical Brauer trees in the title since in recent years the term Brauer tree algebra has occasionally taken on different meanings.

Gabriel goes on to credit Janusz and Kupisch with showing that all of the indecomposable representations for $B$ in that case correspond to the set of string modules available for that Brauer quiver. We define those modules in Chapter 3.

Continuing to use the quiver approach M.C.R. Butler and C.M. Ringel [4] generalize the quiver approach. They defines a large class of algebras called string algebras. This class of algebras contains the Brauer tree algebras. They also gives a complete description of the indecomposable modules over string algebras. As an immediate consequence of their results we have that the set of string modules for the Brauer tree algebras is a complete set of nonprojective indecomposable modules for the Brauer tree algebra. So in studying the string modules over a Brauer tree algebra we will be studying the nonprojective indecomposable modules over the Brauer tree algebra.

In this thesis we will extend the resolution results of Alperin and Janusz. In particular we will give a simple constructive algorithm for computing the minimal projective resolutions for all indecomposable modules over a Brauer tree algebra. We will show that the periodicity is either $n$ or $2n$ where $n$ is the number of vertices in the Brauer quiver. We will use those results to compute a minimal set of generators for the cohomology ring of the Brauer tree algebra with respect to the Jacobson radical. We define the Jacobson radical and the cohomology ring in chapters 5 and 6 respectively. We note that in [3] P. Brown, using different techniques, determined these minimal generating sets less algorithmically for a larger class of algebras, namely representation-finite biserial algebras.
The basic layout for this thesis is as follows: In Chapter 5 we give an algorithm for constructing minimal projective resolutions of string modules over a Brauer tree algebra with exceptional number $N \geq 1$. In Chapter 6 we use those resolutions to compute a minimal set of generators for the cohomology ring of a Brauer tree algebra with exceptional number $N$. On some occasions we compute minimal projective resolutions for an indecomposable module over a Brauer tree algebra with exceptional number $N \geq 1$ by computing a minimal projective resolution for an indecomposable module over a Brauer tree algebra with exceptional number 1. We justify this technique using covering theory which was developed in [7] and [8] which we present in section 3.2. In Chapter 2 we define both path algebras and Brauer tree algebras which are quotients of certain path algebras.

In [15] H.Meltzer and A.Skowroński gave a short time line describing some of the events above and it was instrumental in the organization of this introduction. Although not explicitly referred to, we list the related works [10], [17], and [16].
Chapter 2

Preliminaries

2.1 Path Algebras

We recall the definition of a path algebra. By a quiver path or directed path P we will mean a finite sequence of arrows $a_1a_2\cdots a_r$ such that $t(a_i) = o(a_{i+1})$, where $r$ is a positive integer and $1 \leq i \leq r$. We will frequently leave off the word 'quiver' when referring to a quiver path. We also define $t(P)$ = the terminus of the path $P$, and $o(P)$ = the origin of the path $P$. The $K$-basis for $K\Gamma$ is the set of all vertices (i.e. paths of length zero) together with the set of all finite paths. We define the multiplication on basis elements and extend linearly. Recall that we are thinking of vertices as paths of length zero. Let $P$ and $Q$ be paths. Then

$$P \cdot Q = \begin{cases} PQ & \text{if } t(P) = o(Q) \\ 0 & \text{if } t(P) \neq o(Q). \end{cases}$$

2.2 Brauer Tree Algebras

We define a Brauer tree algebra, $\Lambda$, as a quotient of a path algebra, $K\Gamma/I$. In this section we describe the quiver $\Gamma$ that is associated with the Brauer tree and give a set of relations (or generators) which generate the ideal $I$. We will illustrate the construction with an example.
A *Brauer tree* is a finite connected undirected tree that has been placed in a plane with a clockwise orientation. The tree also has one exceptional node, i.e., one node that has a positive integer (the exceptional number) associated with it. In the diagrams we will place the exceptional number close to the exceptional node. Using the (underlying) Brauer tree we construct a quiver. For our example we use the underlying Brauer tree in figure 2.1.

![Diagram of Brauer tree](image)

**Figure 2.1:**

As in figure 2.2 we label each edge (approximately at the midpoint), $v_1, v_2, \ldots, v_n$. These will be the vertices of the directed quiver $\Gamma$ that we will build.
We enclose each node within a cycle of directed segments (arrows) which connect the midpoints (vertices) of the edges that have that node as an endpoint. These directed segments (which are the arrows of $\Gamma$) must follow the orientation of the plane. Label the arrows that have just been created. See figure 2.3.

From now on when we use the word cycle we will mean one of these cycles of arrows surrounding a node. We will call the cycle surrounding the exceptional node the exceptional cycle. And in diagrams we will place the exceptional number in the center of the exceptional cycle.
Thus, given a Brauer tree, we have constructed the quiver $\Gamma$ that we will associate to that tree. To present the relations for the ideal $I$, the set of which we will call $\rho$, we need some terminology. A path, $P$, will be called a maximal path if a) $P$ lies entirely within a cycle and b) if that cycle is not the exceptional cycle, the length of $P$ is the number of arrows in the cycle and c) if that cycle is the exceptional cycle, length of $P$ is $N \cdot$ (the number of arrows that are in the exceptional cycle) where $N$ is the exceptional number.

There are three types of relations that generate $I$.

**Type A:** Any path of length two that doesn't lie entirely within one cycle.

**Type B** Any path lying entirely within a cycle that contains a maximal path and is one arrow longer than that maximal path.

**Type C** The difference of any two maximal paths that have the same origin (and hence the same terminus).
For the example that we have been working with

\[ I = \langle ad, db, be, ec, cf, fh, hk, kn, nl, lo, om, mi, ip, pj, jg, aa, bb, cc, gdef, gdefg, defgd, efgde, fgdef, hijhijhij, hijhijhij, hijhijhij, hijhijhij, un, oo, pp, klmk, lmk, mklm, a - defg, b - efgd, c - fgde, hijhijhij - gdef, p - hijhijhij, ijhijhij - klm, n - lmk, o - mkl \rangle \]

### 2.3 \( \text{rep}(QUIV, \rho) \)

For the definitions we follow [7]. We let \( QUIV \) be a directed graph and we let \( \rho \) be a set of generators of an ideal \( I \) in \( K \text{QUIV} \). Also let \( \text{QUIV}_0 \) be the set of vertices in \( \text{QUIV} \), and \( \text{QUIV}^1 \) be the set of arrows in \( \text{QUIV} \). A \( K \)-representation of \( \text{QUIV} \) is a tuple \( X = (X_v, \alpha(a))_{v \in \text{QUIV}_0, a \in \text{QUIV}^1} \), where each \( X_v \) is a finite-dimensional \( K \) vector space, all but a finite number zero, and if \( a \) is an arrow from vertex \( v \) to vertex \( w \), then \( \alpha(a) \) is a \( K \)-linear map from \( X_v \) to \( X_w \). We will write the maps \( \alpha(a) \) on the right.

The category of \( K \)-representations of \( \text{QUIV} \), denoted \( \text{rep}(\text{QUIV}) \), has as objects the \( K \)-representations of \( \text{QUIV} \) and as morphisms \( B : (X_v, \alpha(a)) \rightarrow (Y_v, \beta(a)) \) tuples \( B = (B_v)_{v \in \text{QUIV}_0} \) such that each \( B_v \) is a \( K \)-linear map from \( X_v \) to \( Y_v \) such that if \( a \) is an arrow from vertex \( v \) to vertex \( w \) then \( B_w \circ \alpha(a) = \beta(a) \circ B_v \).

We will say that the \( K \)-representation \( X \) satisfies the relations \( \rho \) if whenever \( \prod_{i=1}^k a_{i_1}a_{i_2} \cdots a_{i_m} \in \rho \), we have \( \prod_{i=1}^k \alpha(a_{i_1})\alpha(a_{i_2}) \cdots \alpha(a_{i_m}) = 0 \). We let \( \text{rep}(\text{QUIV}, \rho) \) denote the full subcategory of \( K \)-representations of \( \text{QUIV} \) satisfying the relations \( \rho \).
Chapter 3

String Modules and Covering Theory

3.1 Strings and String Modules

Given a quiver, $QUIV$, which is either $\Gamma$ or $\Gamma^*$ and a set of relations $rel$, we define the formal inverse of an arrow $a$ to be $a^{-1}$ and define $o(a^{-1}) = t(a)$ and $t(a^{-1}) = o(a)$. We also define $(a^{-1})^{-1} = a$. A formal path, $a_1 \cdots a_n$, is a sequence where for $1 \leq i < n$, $a_i$ is of the form $a$ or $a^{-1}$ and $t(a_i) = o(a_{i+1})$. We define $(a_1 \cdots a_n)^{-1} = a_n^{-1} \cdots a_1^{-1}$, $t(a_1 \cdots a_n) = t(a_n)$, and $o(a_1 \cdots a_n) = o(a_1)$.

A string is a formal path $a_1 \cdots a_n$ where $a_{i+1} \neq a_i^{-1}$ for all $1 \leq i < n$, and no subpath nor its inverse is a maximal path, no subpath nor its inverse is in $rel$. We also define strings of length zero. For any vertex, $v$, of the quiver define $1_v$ to be a string of length zero. We define and always reduce

$$1_v a = \begin{cases} a & \text{if } o(a) = v \\ \text{undefined} & \text{otherwise} \end{cases} \quad \text{and} \quad a 1_v = \begin{cases} a & \text{if } t(a) = v \\ \text{undefined} & \text{otherwise} \end{cases}.$$ 

Following [4], given a string of length greater than zero $S = a_1 \cdots a_n$ we define a string module $M(S)$. We define it by giving its representation in $\text{rep}(QUIV, \rho)$. Let $u(i) = o(a_{i+1})$ for $0 \leq i < n$ and $u(n) = t(a_n)$. Then if $v$ is any vertex in $QUIV$, 

\[ u(i) = o(a_{i+1}) \] 

\[ u(n) = t(a_n) \]
we let $I_v = \{ i | u(i) = v \}$. $M(S)_v$ will be a vector space of dimension the cardinality of $I_v$ with basis vectors $z_i, i \in I_v$. If $a_i = a$, define $a(z_{i-1}) = z_i$, and if $a_i = a^{-1}$, then define $a(z_i) = z_{i-1}$. If $\gamma : w \rightarrow w'$ is an arrow and $z_j$ is one of the basis vectors of $M(S)$ and $\gamma(z_j)$ has not yet been defined, let $\gamma(z_j) = 0$. Note that $M(S)$ and $M(S^{-1})$ are isomorphic.

If $S = 1_v$ for some vertex in $v$ then we define $M(1_v)$ to be the simple representation corresponding to the vertex $v$ in QUIV. In other words $M(S)_v$ is a one dimensional vector space, $M(S)_u$ is a zero dimensional vector space for $u \neq v$, and all maps are the zero map.

In figure 3.1 we give a quiver $\Gamma$. Letting the relations be the Brauer tree relations, we give an example of a string $S$ and the representation for $M(S)$. In our illustration of the representation we will write $v$ when we are referring to a one dimensional subspace of $M(S)_v$. To distinguish two different one dimensional subspaces of $M(S)_v$ we would write $v$ once and $vt$ another time. We have drawn the nonzero maps which we note always map one dimensional subspaces to one dimensional subspaces.
For $S = k^{-1}(ij)(efgdef)^{-1}$
   = $k^{-1}ijf^{-1}c^{-1}d^{-1}g^{-1}f^{-1}e^{-1}$,

$M(S) =$

Figure 3.1:
These modules are nice to work with since up to isomorphism they are determined by only a few pieces of information. We will show that this is the case where the exceptional number is 1 by showing that string functions can be determined using only a few pieces of information. We introduce string functions in Chapter 4 and show that every string module is isomorphic to the image of some string function. But more than just another pretty module, Butler and Ringel [4] prove

**Theorem 3.1** \( \{M(S) : S \text{ is a string}\} \) is a complete set of the nonisomorphic nonprojective indecomposable modules over \( \Lambda \).

Therefore if we find the minimal projective resolutions for all string modules over \( \Lambda \) we will have found all of the minimal projective resolutions for the nonprojective indecomposable modules over \( \Lambda \).

### 3.2 Coverings

The covering theory that follows was developed by E.L. Green and can be found in [7] and [8]. Let \( \Gamma \) be the quiver for a Brauer tree algebra with exceptional number \( N \). We will construct a second Brauer tree, \( \Gamma^* \), and its associated Brauer tree algebra, \( \Lambda^* \), with exceptional number 1. The theory works more generally but we only concentrate on Brauer tree algebras. We define \( \Gamma_0 \) to be the set of vertices of \( \Gamma \), and \( \Gamma_1 \) be the set of arrows of \( \Gamma \). Let \( W \) be an arbitrary set function from \( \Gamma_1 \) to a group \( G \). For us \( G \) will be the group of integers modulo \( N \) which we denote \( \mathbb{Z}_N \).

A regular covering of \( \Gamma, \Gamma^* \), is constructed in [7]. We give review that construction below. The quiver is given as follows: \( \Gamma_0^* = \{v^g|v \in \Gamma_0, \text{ and } g \in G\} \), \( \Gamma_1^* = \{a^g|a \in \Gamma_1, \text{ and } g \in G\} \) where if \( v \xrightarrow{a} w \) in \( \Gamma \) then \( v^g \xrightarrow{a^g} w^{gW(a)} \). Since \( \Gamma \) and \( \Gamma^* \) are locally homeomorphic it is clear that \( \Gamma^* \) is a quiver associated to some Brauer
Theorem 3.2 $Λ^*$ is the Brauer tree algebra with exceptional number 1 associated to the quiver $Γ^*$.

Proof. Since $Γ$ and $Γ^*$ are locally homeomorphic it is clear that $Γ^*$ is a quiver
associated to some Brauer tree. Therefore we only need to show that $\rho^*$ is precisely the set of relations of types A, B, and C that define the relations for the Brauer tree algebra.

We first show that $\rho^*$ contains all of the relations of type A, B, and C. Since the quivers are locally homeomorphic it is clear that $\rho^*$ contains the relations of type A, products of arrows not lying in the same cycle.

To see that all of the relations of types B and C are contained in $\rho^*$ we first show that each maximal path in $\Gamma^*$ lifts from a maximal path in $\Gamma$. Note that each nonexceptional cycle in $\Gamma^*$ lifts from a homeomorphic nonexceptional cycle in $\Gamma$. Thus it is clear that each maximal path that lies in a nonexceptional cycle in $\Gamma^*$ lifts from a maximal path in $\Gamma$.

The exceptional cycle $C$ with exceptional number $N$ in $\Gamma$ lifts to an exceptional cycle $C^*$ with exceptional number 1 in $\Gamma^*$. These cycles are not homeomorphic. In fact $C^*$ has $N$ times the number of arrows that appear in $C$. We illustrate this in figure 3.4. Where we are letting $C$ be the cycle with arrows $a_1, \ldots, a_n$, and $W(a_1) = 1$. We construct $C^*$. 
Figure 3.4:
Note that the maximal path in $C^*$, 

$$(a_1 \bar{a}_2 \ldots a_n)(a_1 \bar{a}_2 \ldots a_n) \cdots (a_{N-2} \bar{a}_{N-1} \ldots \bar{a}_n)(a_{N-1} \bar{a}_n \ldots \bar{a}_1),$$

projects to the maximal path in $C$

$$\underbrace{(a_1 a_2 \ldots a_n)(a_1 a_2 \ldots a_n) \cdots (a_1 a_2 \ldots a_n)(a_1 a_2 \ldots a_n)}_{N \text{times}},$$

Similarly we see that each maximal path in $C^*$ lifts from a maximal path in $C$.

Now that we have every maximal path in $C^*$ lifts from a maximal path in $C$, we can easily see that all of the relations of types B and C are contained in $\rho^*$. This is because they are all either an arrow times a maximal path, or the difference of two maximal paths. It is also now clear that $\rho^*$ only contains relations of type A, B, and C. ■

We will let $\text{Mod}(\Lambda)$ and $\text{Mod}(\Lambda^*)$ be the categories of modules over $\Lambda$ and $\Lambda^*$ respectively. We let $\text{rep}(\Gamma, \rho)$ be the category of $K$-representations of $\Gamma$ satisfying the relations $\rho$ that generate $I$, which we defined in Section 2.3. In a similar fashion we define $\text{rep}(\Gamma^*, \rho^*)$ where $\rho^*$ are the relations that generate $I^*$. We let $F$ and $J$ be the standard functors: $F : \text{rep}(\Gamma, \rho) \rightarrow \text{Mod}(\Lambda)$ and $J : \text{Mod}(\Lambda^*) \rightarrow \text{rep}(\Gamma^*, \rho^*)$. See [7]

Let $X$ be a tuple in $\text{rep}(\Gamma^*, \rho^*)$ given by $(X_v, \alpha(a^r))_{v \in \Gamma_0, r, \tau, \epsilon \in G} \in \Gamma_1$ where $\alpha(a)$ is a linear map from the vector space $X_{\delta(a)}$ to $X_{\tau(a)}$. Then we define $H : \text{rep}(\Gamma^*, \rho^*) \rightarrow \text{rep}(\Gamma, \rho)$ to be the functor given by $H((X)) = (Y_v, \beta(a))$ where $Y_v = \bigcup_{t \in G} X_v$ and $\beta(a) = \sum \epsilon \alpha(a^\epsilon)$. It was shown in [9] that since $\Lambda$ and $\Lambda^*$ are finite dimensional over $K$, $H$ is exact and additive. We define the functor $E$ to be the composition of functors, $J \circ H \circ F$. Gordon and Green also showed that $E$ maps simple $\Lambda^*$-modules
(respectively $\Lambda^*$-projective covers) to simple $\Lambda^*$-modules (respectively $\Lambda$-projective covers). In addition they showed that if $\mathcal{L}$ and $\mathcal{L}^*$ are the Jacobson radicals for $\Lambda$ and $\Lambda^*$ respectively, then $E(\mathcal{L}^*M) = \mathcal{L}E(M)$ for any $\Lambda^*$-module. They proved these results in the case where $G = \mathbb{Z}$, but the proofs remain the same in the case where $G = \mathbb{Z}_N$. Thus $E$ maps $\Lambda^*$-minimal projective resolutions to $\Lambda$-minimal projective resolutions.

We add to the results above by by showing that $E$ maps indecomposable $\Lambda^*$-modules to indecomposable $\Lambda$-modules. In other words we will show that $E$ maps $\Lambda^*$-string modules to $\Lambda$-string modules. More than that we will show that given a $\Lambda$-string module, $M$, there is a $\Lambda^*$-string module $M^*$ such that $E(M^*) = M$. In Proposition 3.3 we will construct $M^*$. However, we begin with an illustration (see section 3.1 for the notation) of the action of $E$ by drawing part of the representation for a $\Lambda^*$-string module $R$ and the corresponding part of the representation for the $\Lambda$-string module $E(R)$. That $R^*$ is a string is an immediate consequence of the fact that maximal paths and relations in $\Lambda$ lift to maximal paths and relations in $\Lambda^*$. Later when we discuss the homomorphisms that appear in the minimal projective resolutions we will also describe the image of the homomorphism under $E$. In this way any constructive algorithms that we develop for computing the minimal projective resolutions of $\Lambda^*$-string modules will automatically become a constructive algorithms for computing minimal projective resolutions of $\Lambda$-string modules.

**Proposition 3.3** Given a $\Gamma$-string $S = s_1 \cdots s_n$ where the $s_i$ are quiver and formal arrows, there exists a $\Gamma^*$-string $S^*$ such that $E(M(S^*)) = M(S)$.

**Proof.** We make the following definition: if $s = p^{-1}$, where $p$ is a quiver arrow, and $\bar{t} \in Z_N$, we define $s^\bar{t} = (p^{\bar{t}})^{-1}$. We let $\tilde{a}$ be the arrow in $\Gamma$ such that $\tilde{w}(a) = \bar{1}$. 

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Then, if for each \( i \), \( w(s_i) = \overline{0} \), then we define \( S^* := s_1^{\overline{0}} \cdots s_n^{\overline{0}} \). If \( \{s_{t_1}, \ldots, s_{t_k}\} \) is the set of all arrows in \( S^* \) such that either \( s_{t_i} = a \) or \( a^{-1} \), then we define \( S^* = s_1^{\overline{0}} \cdots s_{t_1}^{\overline{0}} s_{t_1+1}^{\overline{1}} \cdots s_{t_2}^{\overline{0}} s_{t_2+1}^{\overline{1}} \cdots s_{t_k}^{\overline{0}} s_{t_k+1}^{\overline{1}} \cdots s_n^{\overline{0}} \). In either case \( E(M(S^*)) = M(S) \).

Remark: We could have begun the construction of \( S^* \) with \( s_1^1 \) instead of \( s_1^{\overline{0}} \) and changed the proof accordingly. We also could have used the inverses of any of the strings that we construct in this fashion. However, those are all of the strings \( S^* \) such that \( E(M(S^*)) = M(S) \). ■
Figure 3.5:
Chapter 4

String Functions and their Images

In this chapter we introduce string functions and show that for every string module there is a string function with an isomorphic image and vice-versa. We will also introduce $h$-strings and underlying strings two convenient notations for describing a string function. In Chapter 5 we will show that string functions are the homomorphisms that appear in a minimal projective resolution of a string module.

For the remainder of this paper we define a loop to be an arrow with the same origin and terminus. Suppose the number of vertices in the quiver for a classical Brauer tree algebra is $n$, then we are going to form a set of $n$ paths. We will call this set the set of fixed maximal paths. Recall that a maximal path, $p$, with origin $v$ is a path which is not a relation, yet every path properly containing $p$ is a relation. These paths always have the same origin and terminus, and there is always exactly two of them with the same origin. The set of fixed maximal paths will contain exactly one maximal path, which is not a loop, with origin $v$ for each vertex $v$ in the quiver.

Given a string $S$ we are going to define a new formal path, an $h$-string $hS$. While the notation seems to suggest that an $h$-string is a string it is not always the case. There are formal paths that are strings but not $h$-strings and vice-versa. Using an $h$-string, $hS$, we will construct a homomorphism $hS$ whose image is isomorphic
to $M(S)$. (The abuse of notation will cause no confusion.) We will call these homomorphisms string functions.

We define an $h$-string in the same way that we define a string, except that we do not allow formal paths of length less than 1, and we do allow formal paths that are in the set of fixed maximal paths. However, we do not allow any formal paths that contain maximal paths as proper subpaths.

Before we can create $hS$, we must first partition $S$. We say that a formal path, $S$, of length greater than 1, is partitioned if it is written in the form $S = s_1 \cdots s_n$ where $s_i$ is a formal path lying entirely within one cycle and for $1 \leq i < n$, $s_i$ and $s_{i+1}$ lie in different cycles. If $S = 1_v$ for some vertex $v$, then $S$ is also partitioned.

Note that, since any quiver path lying in two cycles contains a relation, if $s_i$ is a quiver path, then $s_{i+1}$ is a formal inverse of a quiver path. Also if $s_i$ is an inverse of a quiver path, then $s_{i+1}$ is a quiver path. This means that all of the formal paths $s_k$ where $k$ is odd are paths in $\Gamma$, and all of the formal paths $s_j$ where $j$ is even are formal inverses of paths in $\Gamma$, or vice-versa. But there is a one to one correspondence between strings and $h$-strings.

Given a partitioned string $S = s_1 \cdots s_n$, we define the partitioned $h$-string, $hS$ to be $s'_n \cdots s'_1$. Where $s'_i$ is defined as follows. For $1 \leq i \leq n$, if $s_i$ is a quiver path, then we define $s'_i$ to be the quiver path such that $s_i s'_i$ is a maximal quiver path. If $s_i$ is a formal inverse of a quiver path then define $s'_i$ to be the formal path such that $s_i s'_i$ is the formal inverse of a maximal quiver path. If $s_i = 1_v$, then we define $s'_i$ to be a fixed maximal path such that $o(s_i) = t(s_i) = v$. It is easy to see that $hS$ is indeed an $h$-string.

We show that there is a one to one correspondence between strings and $h$-strings. Given a partitioned $h$-string, $hS$, we show how to construct a string function. A
partitioned $h$-string $hS$ can be written in one of four different ways which we list below:

1) $hS = P_1Q_1^{-1}P_2Q_2^{-1} \cdots P_mQ_m^{-1}$ or
2) $hS = P_1Q_i^{-1}P_2Q_2^{-1} \cdots P_m$ or
3) $hS = Q_1^{-1}P_2Q_2^{-1} \cdots P_mQ_m^{-1}$ or
4) $hS = Q_1^{-1}P_2Q_2^{-1} \cdots P_m$ where $P_i$ and $Q_i$ are quiver paths.

For each of these four $h$-strings we define a string function $hS$. We do this in figures 4.1 - 4.4. Along with each definition we provide a diagram called an underlying string which assists in visualizing the string function. Also, whenever they are used, $c_i$ and $d_i$ are in the field $K$, $c_i, d_i \neq 0$, $v_i$ and $w_i$ are vertices in $\Gamma$ (or $\Gamma^*$), $\overline{P_i}$ and $\overline{Q_i}$ are the images of $P_i$ and $Q_i$ in $\Lambda$, and $p_i = c_i\overline{P_i}$ and $q_i = d_i\overline{Q_i}$. Also, for brevity, we identify $(0, \ldots, 0, \overline{w}, 0, \ldots, 0)$ of $v_1\Lambda \oplus \cdots \oplus v_n\Lambda$ with $\overline{w}$. After understanding the definition of a string function the reader should note that the functor $E$ introduced in Section 3.2 maps string functions to string functions, and that the corresponding $h$-strings are identical if the coefficients in $\text{Mod}(\Lambda^*)$ are ignored.
To the $h$-string $P_1 Q_1^{-1} P_2 Q_2^{-1} \cdots P_m Q_m^{-1}$, and to the constants $c_i$ and $d_i$, we associate a homomorphism $hS$: $v_1 \Lambda \oplus \cdots \oplus v_m \Lambda \to w_1 \Lambda \oplus \cdots \oplus w_m \Lambda$ where $hS(v_1) = (c_1 \overline{P}_1, d_1 \overline{Q}_1, 0, \ldots, 0)$, for $i = 2 \cdots (m - 1) hS(v_i) = (0, \ldots, 0, c_i \overline{P}_i, d_i \overline{Q}_i, 0, \ldots, 0)$ where $c_i \overline{P}_i$ is in the $(i)^{th}$ coordinate, and $hS(\overline{v}_m) = (0, \ldots, 0, c_m \overline{P}_m, d_m \overline{Q}_m)$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.1.png}
\caption{Figure 4.1:}
\end{figure}

To the $h$-string $P_1 Q_1^{-1} P_2 Q_2^{-1} \cdots P_m$, and to the constants $c_i$ and $d_i$, we associate a homomorphism $hS$: $v_1 \Lambda \oplus \cdots \oplus v_m \Lambda \to w_1 \Lambda \oplus \cdots \oplus w_m \Lambda$ where $hS(\overline{v}_1) = (c_1 \overline{P}_1, d_1 \overline{Q}_1, 0, \ldots, 0)$, for $i = 2 \cdots (m - 1) hS(\overline{v}_i) = (0, \ldots, 0, c_i \overline{P}_i, d_i \overline{Q}_i, 0, \ldots, 0)$ where $c_i \overline{P}_i$ is in the $(i)^{th}$ coordinate, and $hS(\overline{v}_m) = (0, \ldots, 0, c_m \overline{P}_m)$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.2.png}
\caption{Figure 4.2:}
\end{figure}

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To the $h$-string $P_1Q_1^{-1}P_2Q_2^{-1}\cdots P_m$, and to the constants $c_i$ and $d_i$, we associate a homomorphism $hS: v_1\Lambda\oplus\cdots\oplus v_m\Lambda \rightarrow hS: v_1\Lambda\oplus\cdots\oplus v_m\Lambda \rightarrow w_1\Lambda\oplus\cdots\oplus w_m\Lambda$ where $hS(\overline{v}_i) = (d_i\overline{Q}_i, 0, \ldots, 0)$, for $i = 2 \cdots (m-1)$ $hS(\overline{v}_i) = (0, \ldots, 0, c_i\overline{P}_i, d_i\overline{Q}_i, 0, \ldots, 0)$ where $c_i\overline{P}_i$ is in the $(i-1)^{st}$ coordinate, and $hS(\overline{v}_m) = (0, \ldots, 0, c_m\overline{P}_m, d_m\overline{Q}_m)$.

![Figure 4.3](image)

To the $h$-string $Q_1^{-1}P_2Q_2^{-1}\cdots P_m$, and to the constants $c_i$ and $d_i$, we associate a homomorphism $hS: v_1\Lambda\oplus\cdots\oplus v_m\Lambda \rightarrow hS: v_1\Lambda\oplus\cdots\oplus v_m\Lambda \rightarrow w_1\Lambda\oplus\cdots\oplus w_{m-1}\Lambda$ where $hS(\overline{v}_i) = (d_i\overline{Q}_i, q_i, \ldots, 0)$, for $i = 2 \cdots (m-1)$ $hS(\overline{v}_i) = (0, \ldots, 0, c_i\overline{P}_i, d_i\overline{Q}_i, 0, \ldots, 0)$ where $c_i\overline{P}_i$ is in the $(i-1)^{st}$ coordinate, and $hS(\overline{v}_m) = (0, \ldots, 0, c_m\overline{P}_m)$.

![Figure 4.4](image)

We next prove that all of the string functions associated with a single $h$-string are isomorphic.
Lemma 4.1 If $f$ and $g$ are string functions that are both associated with the same $h$-string, $hS$, then $\text{Im} f \simeq \text{Im} g$.

Proof. Without loss of generality we can assume that the coefficients in the definition of $f$ are all 1. We will only prove the lemma for string functions of the type described in figure 4.3. The proof is similar for string functions of the other three types. In figure 4.5 we give an underlying string for $f$ and $g$.

$$
\begin{align*}
  f: & \quad v_1 \rightarrow Q_1 \rightarrow P_2 \rightarrow Q_2 \rightarrow \cdots \rightarrow P_m \rightarrow Q_m \\
  g: & \quad v_1 \rightarrow d_1Q_1 \rightarrow c_2P_2 \rightarrow d_2Q_2 \rightarrow \cdots \rightarrow c_mP_m \rightarrow d_mQ_m
\end{align*}
$$

Figure 4.5:

To prove the lemma it suffices to show that there are isomorphisms $X$ and $Y$ such that the following diagram commutes.
We define the isomorphisms \( X \) and \( Y \) by defining them on the idempotents \( v_1, \ldots, v_m \). For \( 1 \leq i \leq m \) we will define \( X(v_i) = x_i v_i \) and \( Y(v_i) = y_i v_i \) where \( x_i, y_i \in K^* \) need to be determined. To ensure that the diagram commutes it suffices to choose the \( x_i \)'s and \( y_i \)'s so that the diagrams commute on the \( v_i \)'s.

For that to happen we need \( x_1 d_1 = y_1 \) and for \( 2 \leq i \leq m \), \( x_i c_i = y_{i-1} \) and \( x_i d_i = y_i \). Simplifying we get that for \( 2 \leq i \leq m \), \( x_i d_i = x_{i+1} c_{i+1} \). Thus by choosing \( x_1 = 1 \) and solving recursively we can find the necessary homomorphisms \( X \) and \( Y \).

Unfortunately using our definition two different \( h \)-strings can have associated string functions that have isomorphic images. So we need to identify when this can happen. We do this without proof in Lemma 4.2.

**Lemma 4.2** The string function \( hS \) is associated with only the \( h \)-strings \( hS \) and \( (hS)^{-1} \).

We conclude the chapter by proving

**Proposition 4.3** The image of the string function \( hS \) is \( M(S) \).

**Proof.** This is a sketch of the proof for a particular case. The proof clearly generalizes. However, the general case requires an even more burdensome notation.
We consider a graph \( QUIV \) and a particular string, \( S \). In figure 4.7 we have drawn a part of the graph \( QUIV \). We want to show that the string function \( hS \) which corresponds to the \( h \)-string \( hS = ((f_1 \cdots f_{n_f})(c_1 \cdots c_{n_c})^{-1}(a_1 \cdots a_{n_a}) \) has image isomorphic to \( M(S) \).

We let \( S = (b_1 \cdots b_{n_b})(d_1 \cdots d_{n_d})^{-1}(e_1 \cdots e_{n_e}) \) which is a sufficiently complicated to demonstrate all of the necessary proof techniques for the general case. We give the underlying string for \( hS \) in figure 4.8, and note that \( hS \) is a map from 

\[
t(f_{n_f})A \oplus t(a_{n_a})A \to t(e_{n_e})A \oplus t(d_{n_d})A.
\]

It is defined as follows: \( hS((t(f_{n_f}), 0)) = (f_1 \cdots f_{n_f}, e_1 \cdots e_{n_e}) \) and \( hS((0, t(a_{n_a})) = (0, a_1 \cdots a_{n_a}) \). After unwinding the definitions one can see that the image of this map is isomorphic to \( M(S) \). We give a representation of \( M(S) \) in figure 4.9. There we have drawn all of the nonzero maps between basis elements. Note that each time a vertex \( v \) appears in the diagram it represents a different basis element in \( M(S)_v \).

\[
\begin{figure}
\begin{center}
\includegraphics{figure}
\end{center}
\end{figure}

\textbf{Figure 4.7:}
Figure 4.8:

Figure 4.9:
Chapter 5

The Minimal Projective Resolutions of String Modules over Classical Brauer Tree Algebras

In this chapter we will prove that the periodicity of a string module is either $|\Gamma|$ or $2|\Gamma|$. To accomplish this we will develop algorithms for computing minimal projective resolutions for string modules over $\Lambda$. We remind the reader that $|\Gamma|$ is the number of vertices in $\Gamma$.

Let $\overline{r}$ be the Jacobson radical for $\Lambda$. Then we will be using the following definition for a minimal projective resolution of a module $M_0$: the exact sequence

$$\cdots P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_0} M_0$$

is a minimal projective resolution of the module $M_0$ if $\text{Ker}(f_i) \subseteq \overline{r}P_{i+1}$ and $P_i$ is projective for $i \geq 1$. 

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5.1 The Picture Algorithm

Our goal is to give a simple ‘walking’ (or follow the arrows) algorithm for calculating the minimal projective resolution of an indecomposable module over \( \Lambda \). But, before we are able to give the ‘walking’ algorithm we must first develop an intermediate algorithm which we will call the ‘picture algorithm’. This algorithm can be applied to string modules over \( \Lambda \). Hence it can also be applied to \( \Lambda^* \) since \( \Lambda^* \) is a Brauer tree algebra with exceptional number 1. The algorithm can in fact be applied to any path algebra whose quiver consists of cycles of arrows and employs the same type of relations that we have been using. In other words, we can relax the restriction that we look at a graph that is a tree of cycles.

**Lemma 5.1** Given a string function \( hS \). The string function \( hT \) that is found using figures 5.1 - 5.10 is such that \( \text{Im}(hT) = \text{Ker}(hS) \).

**Proof.** We define two types of vertices **looped** and **nonlooped**. We say that a vertex, \( v \), is a **looped** vertex if there is arrow, \( a \), with \( o(a) = t(a) = v \). We say that a vertex is a **nonlooped vertex** otherwise. The proof is broken up into ten cases. These ten cases correspond to ten different forms that an underlying string might assume. In each case we have given the underlying string for \( hS \) and directly above it the underlying string for \( hT \). Immediately following each diagram we prove that \( \text{Im}(hT) = \text{Ker}(hS) \). We do not concern ourselves with using only fixed maximal paths in the underlying strings. They are only used for ease of notation in other parts of the paper. Here it would be inconvenient. This will cause no problem when we return to examining \( h \)-strings. For if a maximal path does appear in an \( h \)-string our algorithms will only be concerned with the origin of the path.
Since our main concern will be in resolving simple modules, we can restrict our attention to the string functions where the field coefficients in the underlying string are all ones or alternate between +1 and -1. By that we mean $c_1, d_1, c_2, d_2, \ldots$ alternate between +1 and -1. The proof then becomes somewhat less burdensome with respect to notation. However, the approach remains the same for the more general case. After we have proven the lemma we will note the changes that occur when the sequence $c_1, d_1, c_2, d_2, \ldots$ alternates between +1 and -1.

We keep the same notation that we have been using in previous chapters. If $P$ is a path which lies entirely in one cycle in $\Gamma$, then $P'$ is the path such that $PP'$ is a maximal path.

**Case 1:**

\[ t(\alpha) \quad w_1 \quad w_{n-1} \quad t(\beta) \]

\[ hT : \]

\[ -\alpha \quad Q_1 \quad P'_2 \quad \ldots \quad Q'_{n-1} \quad P'_n \]

\[ -\beta \quad \]

\[ v_1 \quad v_2 \quad \ldots \quad v_{n-1} \quad v_n \]

\[ hS : \]

\[ Q_1 \quad P_2 \quad \ldots \quad Q_{n-1} \quad P_n \]

\[ w_1 \quad w_{n-1} \]

**Figure 5.1:** We let $\alpha$ be the arrow with origin $v_1$ which does not lie in the same cycle as $Q_1$ (or $w_1$). We let $\beta$ be the arrow with origin $v_n$ which does not lie in the same cycle as $P_n$ (or $w_{n-1}$). Assume $n > 1$. The left (resp. right) dotted in portion of $T$ will be present if $v_1$ (resp. $v_n$) is a nonlooped vertex.
For the remainder of the proof, when defined, we let \( q_i \) (resp. \( p_{i+1} \)) be a linear combination of distinct paths leaving \( v_i \) (resp. \( v_{i+1} \)) in the same cycle as \( Q_i \) (resp. \( P_{i+1} \)). We first assume that \( v_1 \) and \( v_n \) are looped. Then to prove \( hT \) maps onto \( \text{Ker} \, hS \) we will show that 
\[
\{(0, \ldots, 0, \overline{Q}_i, -P'_{i+1}, 0, \ldots, 0) : 0 \leq i < n \text{ and } \overline{Q}_i \text{ is in the } i^{th} \text{ component}\}
\]
is a generating set for \( \text{Ker} \, hS \).

Let \( x \in \text{Ker} \, hS \). Then
\[
x = (\overline{q}_1, \overline{p}_2, \ldots, \overline{q}_{n-1}, \overline{p}_n) = \sum_{i=1}^{n-1} (0, \ldots, 0, \overline{q}_i, \overline{p}_{i+1}, 0, \ldots, 0).
\]
Since the product of two paths lying in different cycles is in \( I \), we have that

\[
hS(x) = (\overline{Q}_1 \overline{q}_1 + \overline{P}_2 (\overline{p}_2 + \overline{q}_2), \overline{Q}_2 (\overline{p}_2 + \overline{q}_2) + \overline{P}_3 (\overline{p}_3 + \overline{q}_3), \ldots, \overline{Q}_{n-1} (\overline{p}_{n-1} + \overline{q}_{n-1}) + \overline{P}_n \overline{p}_n)
\]
\[
= (\overline{Q}_1 \overline{q}_1 + \overline{P}_2 \overline{p}_2, \overline{Q}_2 \overline{q}_2 + \overline{P}_3 \overline{p}_3, \ldots, \overline{Q}_{n-1} \overline{q}_{n-1} + \overline{P}_n \overline{p}_n)
\]
\[
= (0, \ldots, 0).
\]

So, in particular, for \( 0 \leq i < n \) we have that

\[
hS((0, \ldots, 0, \overline{q}_i, \overline{p}_{i+1}, 0, \ldots, 0)) = (0, \ldots, 0, \overline{Q}_i \overline{q}_i + \overline{P}_{i+1} \overline{p}_i, 0, \ldots, 0)
\]
\[
= (0, \ldots, 0).
\]

Therefore, if we define \( hS_i = \{y = (0, \ldots, 0, \overline{q}_i, \overline{p}_{i+1}, 0, \ldots, 0) : y \in \text{Ker} \, hS\} \), then \( \cup_{i=1}^{n-1} hS_i \) is a generating set for \( \text{Ker} \, hS \). So we only need to show that for a fixed \( i \) \( \{y = (0, \ldots, 0, \overline{q}_i, \overline{p}_{i+1}, 0, \ldots, 0) : y \in \text{Ker} \, hS\} \) is generated by 
\[
(0, \ldots, 0, \overline{Q}_i, -P'_{i+1}, 0, \ldots, 0).
\]

For the remainder of the proof for \( 0 \leq i < n \), we let \( b_i \) be the first arrow of \( Q_i \), and we let \( a_{i+1} \) be the first arrow of \( P_{i+1} \). If one of the terms of \( \overline{Q}_i \overline{q}_i \) (resp. \( \overline{P}_{i+1} \overline{p}_{i+1} \)) is zero in \( \Lambda^* \), then it must be a \( \Lambda^* \)-multiple of \( \overline{Q}_i \overline{q}_i b_i \) (resp. \( \overline{P}_{i+1} P'_{i+1} a_{i+1} \)) and hence

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generated by $\mathcal{Q}_i \mathcal{Q}'_i b_i$ (resp. $\mathcal{P}_i \mathcal{P}'_i a_{i+1}$).

So if we subtract from $\mathcal{Q}_i \bar{q}_i + \mathcal{P}_i \bar{p}_{i+1}$ all terms that are zero in $\Lambda^*$, then we either have nothing left or we have a linear combination of terms that is zero in $w_i \Lambda^*$. Up to a scalar multiple there is only one such combination. It is $\mathcal{Q}_i \mathcal{Q}'_i - \mathcal{P}_i \mathcal{P}'_i a_{i+1}$.

Since we also have for $0 < i < n$ $(\mathcal{Q}_i \mathcal{Q}'_i - \mathcal{P}_i \mathcal{P}'_i a_i) b_i = \mathcal{Q}_i \mathcal{Q}'_i b_i$ and $(\mathcal{Q}_i \mathcal{Q}'_i - \mathcal{P}_i \mathcal{P}'_i a_i) (-\mathcal{Q}'_i) = \mathcal{P}_i \mathcal{P}'_i a_i$, we have $\mathcal{Q}_i \mathcal{Q}'_i - \mathcal{P}_i \mathcal{P}'_i$ generates $\{\mathcal{Q}_i \mathcal{Q}'_i + \mathcal{P}_i \mathcal{P}'_i \bar{p}_{i+1} : (0, \ldots, 0, \bar{q}_i, \bar{p}_{i+1}, 0, \ldots, 0) \text{ Ker } hS\}$. Therefore we have that the element $(0, \ldots, 0, \bar{q}_i, -\mathcal{P}'_i, 0, \ldots, 0)$ generating $hS_i$ for each $i$, $0 < i < n$. We conclude that $hT$ maps onto $\text{Ker } hS$.

We now deal with the case where $v_1$ is nonlooped. We leave the case where $v_n$ is nonlooped to the reader. We will deal with this case by briefly going back through the the preceding proof, noting the places where the proof changes.

First we must define $p_1$ and $\alpha$. $p_1$ will be any linear combination of distinct paths starting at $v_1$ but not in the same cycle as $Q_1$. $\alpha$ will be the arrow leaving $v_1$ not in the same cycle as $Q_1$.

If $x \in \text{Ker } hS$, then

$$x = (\bar{p}_1 + \bar{q}_1, \ldots, \bar{p}_{n-1} + \bar{q}_{n-1}, \bar{p}_n),$$

and

$$hS(x) = (\bar{Q}_1(p_1 + q_1) + \bar{P}_2(p_2 + q_2), \bar{Q}_2(p_2 + q_2) + \bar{P}_3(p_3 + q_3), \ldots, \bar{Q}_{n-1}(p_{n-1} + q_{n-1}) + \bar{P}_n$$

$$= (\bar{Q}_1 q_1 + \bar{P}_2 \bar{p}_2, \bar{Q}_2 \bar{q}_2 + \bar{P}_3 \bar{p}_3, \ldots, \bar{Q}_{n-1} \bar{q}_{n-1} + \bar{P}_n \bar{p}_n)$$

$$= (0, \ldots, 0).$$

This implies that $(\bar{p}_1, 0, \ldots, 0) \in \text{Ker } hS$. So if we define $hS_0 = \{y = (\bar{p}_1, 0, \ldots, ) : y \in \text{Ker } hS\}$. Then $\bigcup_{i=0}^{n-1} hS_i$ is a generating set for $\text{Ker } hS$. 35
Finally if \( p \) is a path beginning with \( \alpha \) then \( \overline{Q}_1 \overline{p} = 0 \) in \( \Lambda \). So \( (-\overline{\alpha}, 0, \ldots, 0) \) generates \( hS_0 \). This completes the proof for case one.

Note that we used \( (-\overline{\alpha}, 0, \ldots, 0) \) when we could just have easily used \( (\overline{\alpha}, 0, \ldots, 0) \). We do this for the sake of consistency and ease of notation.

**Case 2:**

\[
\begin{aligned}
\text{hT:} & \quad -P'_{n-1} Q'_1 P'_2 \ldots \quad Q'_n \quad P'_{n+1} \\
\text{hS:} & \quad P_1 Q_1 P_2 \ldots \quad Q_{n-1} P_n Q_n
\end{aligned}
\]

Figure 5.2: The left (resp. right) dotted in portion of \( hT \) will be present if \( P_i \) (resp. \( Q_n \)) is not an arrow. Assume \( n > 1 \).

Case 2 uses the same format as Case 1 so we will move quickly through the proof noting only the differences.

If \( x \in \text{Ker} hS \), then

\[
x = (\overline{p}_1 + \overline{q}_1, \overline{p}_2 + \overline{q}_2, \ldots, \overline{p}_n + \overline{q}_n)
\]

\[
= (\overline{p}_1, 0, \ldots, 0) + \sum_{i=1}^{n-1} (0, \ldots, 0, \overline{q}_i, \overline{p}_{i+1}) + (0, \ldots, 0, \overline{q}_n)
\]
Hence we have

\[
hS(x) = (\overline{P}_1(p_1 + q_1), \overline{Q}_1(p_1 + q_1) + \overline{P}_2(p_2 + q_2), \ldots, \overline{Q}_{m-1}(p_{n-1} + q_{n-1}) + \overline{P}_n(p_n + q_n), \overline{Q}_n) \\
= (\overline{P}_1p_1, \overline{Q}_1q_1 + \overline{P}_2p_2, \ldots, \overline{Q}_{n-1}q_{n-1} + \overline{P}_n p_n, \overline{Q}_n q_n)
\]

So once again for \(1 \leq i < n\) we have \((0, \ldots, 0, \overline{q}_i, p_{i+1}, 0, \ldots, 0) \in \text{Ker} hS\). But in addition we have that \((\overline{p}_i, 0, \ldots, 0) \in \text{Ker} hS\) and \((0, \ldots, 0, \overline{q}_n) \in \text{Ker} hS\). Therefore, if we let 
\(S_0 = \{(\overline{p}_1, 0, \ldots, 0) \in \text{Ker} hS\}, S_n = \{(0, \ldots, 0, \overline{q}_n) \in \text{Ker} hS\}\), and for \(1 \leq i < n\) we let 
\(S_i = \{(0, \ldots, 0, \overline{q}_i, p_{i+1}, 0, \ldots, 0) \in \text{Ker} hS\}\), then \(\bigcup_{i=0}^{n} S_i\) is a generating set for \(\text{Ker} hS\). As before for \(1 \leq i < n\) \((0, \ldots, 0, \overline{q}_i, \overline{P}_i p'_{i+1}, 0, \ldots, 0)\) generates \(S_i\). All that remains is for us find generators for \(S_0\) and \(S_n\).

If \(\overline{P}_1p_1\) (resp. \(\overline{Q}_n q_n\)) is zero in \(\Lambda\), then since \(P_1 p_1\) (resp. \(Q_n q_n\)) contains only paths lying in a single cycle it must be that each of the paths in \(P_1 p_1\) (resp. \(Q_n q_n\)) contains \(P_1 P'_1 a_1\) (resp. \(Q_n Q'_n b_n\)). Therefore \(S_0\) (resp. \(S_n\)) is generated by \((-P'_1 a_1, 0, \ldots, 0)\) (resp. \((0, \ldots, 0, Q'_n b_n)\)). We have that \(hT\) maps onto \(\text{Ker} hS\) in either the case where \(P_1\) (resp. \(Q_n\)) is an arrow or not.

However, if \(P_1\) (resp. \(Q_n\)) were the arrow \(a_1\) (resp. \(b_n\)), then \(P_1\) (resp. \(Q_n\)) would be a maximal path. Thus the map \(hT\) including the dotted portions would not be a string function. Luckily, the dotted portions become unnecessary in that case. For then \(P'_1 a_i = \overline{Q}_1 Q'_{1}\) (resp. \(Q'_n b_n = P'_n P_n\)). Hence,

\[
(-P'_1 a_1, 0, \ldots, 0) = (\overline{Q}_1, -P'_2, 0, \ldots, 0)(-\overline{Q}_1) \\
\text{(resp.} (0, \ldots, 0, \overline{Q}_n b_n) = (0, \ldots, 0, \overline{Q}_{n-1}, P_n)(P_n)\text{)}
\]

and so \((\overline{Q}_1, -P'_2, 0, \ldots, 0)\) (resp. \((0, \ldots, \overline{Q}_n b_n)\)) generates \(S_0 \cup S_1\) (resp. \(S_{n-1} \cup S_n\)).

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Note that we used \((-\overline{P}_1 a_1, 0, \ldots, 0)\) instead of the more obvious \((\overline{P}_1 a_1, 0, \ldots, 0)\).

Again we do this for the sake of consistency and to ease later notation.

**Case 3:**

\[
\begin{align*}
\text{hT :} & \quad t(\alpha) \quad w_i \quad 0' \quad P' \quad 0' \quad \ldots \quad Q'_{n-1} \quad P' \quad Q'_{n} \quad t(b_n) \\
\text{hS :} & \quad \overline{Q}_1 \quad P_2 \quad \ldots \quad Q_{n-1} \quad P_n \quad Q_n
\end{align*}
\]

Figure 5.3: Once again \(\alpha\) is the arrow leaving \(v_1\) not in the same cycle as \(\overline{Q}_1\). The left dotted in part of hT is included if \(v_1\) is a nonlooped vertex. The right dotted in part of hT is included if \(Q_n\) is not an arrow. Assume \((n > 1)\).

This case as well as the remaining cases use proof techniques similar to the ones presented in Cases 1 and 2. Hence we list the Cases without proof.
Case 4:

Figure 5.4: We assume that $v_1$ nonlooped. We also assume that if exactly one of $P_1$ or $Q_1$ is an arrow, then it is $Q_1$. The dotted in part will appear only when $Q_1$ is not an arrow.
Case 5:

\[ t(\alpha) \quad t(b_1) \]
\[ hT : \quad -\alpha \quad Q'_1 \tilde{b}_1 \]
\[ v_1 \]
\[ hS : \quad Q_1 \]
\[ w_2 \]

Figure 5.5: We assume that \( v_1 \) nonlooped. The dotted in part is included only if \( Q_1 \) is not an arrow. If \( Q_1 \) is a maximal path, we define \( Q'_1 \) to be the vertex \( v_1 \).
Case 6:

\[ t(b_1) \]
\[ Q_1 b_1 \]
\[ v_1 \]
\[ Q_1 \]
\[ w_2 \]

Figure 5.6: We assume that \( v_1 \) looped. If \( Q_1 \) is a maximal path, we define \( Q_1' \) to be the vertex \( v_1 \).
Case 7:

Figure 5.7: We assume that \( n > 1 \). The left dotted in part of \( hT \) is included if \( P_1 \) is not an arrow, the right dotted in part is included if \( v_n \) is a nonlooped vertex.
Case 8:

\[ hT : \quad t(a_1) \quad \quad t(h_1) \]

\[ hS : \quad P_1 \quad Q_1 \]

\[ v_1 \quad P_1 \quad Q_1 \]

\[ w_1 \quad w_2 \]

Figure 5.8: We assume that \( v_1 \) nonlooped. We assume that \( Q_1 \) is never an arrow. The dotted in part of \( hT \) will be included if \( P_1 \) is not an arrow.
Case 9:

Figure 5.9: We assume that $v_1$ nonlooped. The dotted in part will be included if $P_1$ is not an arrow. If $P_1$ is a maximal path, we define $P'_1$ to be the vertex $v_1$.

Case 10:

Figure 5.10: We assume that $v_1$ looped. If $P_1$ is a maximal path, we define $P'_1$ to be the vertex $v_1$.

It can easily be shown that in each of the 10 cases above whenever the underlying string for $hS$ has coefficients that are all $+1$, the coefficients in $hT$ alternate between
+1 and -1. If instead the coefficients of $hS$ alternate between +1 and -1, then we would have the coefficients of $hT$ being all +1. ■

**Lemma 5.2** If $f: M \rightarrow N$ is a string function, then $\ker(f) \subseteq \mathcal{F}M$.

**Proof.** Without much work it can be shown that $\mathcal{F}$ is generated by \{\overline{a}: a is an arrow in the quiver\}. By Lemma 5.1 $\ker(f) = \text{Im}(g)$ for some string function $g$. By the definition of a string function it is clear that $\text{Im}(g) \subseteq \mathcal{F}M$. ■

**The Picture Algorithm**

Given $hS_0$ we construct the minimal projective resolution of $M(S_0)$ recursively. $hS_{i+1}$ is found by letting $hS_i$ play the role of $hS$ in Lemma 5.1. Use the appropriate case to find $hT$ which is $hS_{i+1}$. Lemmas 5.1 and 5.2 prove that the algorithm constructs the desired resolution.

### 5.2 A Simpler Notation

In this section we will restrict our attention to classical Brauer tree algebras with exceptional number 1. In other words, we will be considering $\Lambda^\ast$. For in this case we can simplify the notation. Since we are able to construct the minimal projective resolutions for $\Lambda$-string modules from $\Lambda^\ast$-string modules this is a worthy endeavor. It will primarily be beneficial when we are constructing the minimal set of generators for the cohomology ring of $\Lambda$.

We define a **string-tuple** to be a 4-tuple $(x, l, y, r)$ where $x, y \in \{\text{up, down}\}$ and $l$ and $r$ represent vertices of $\Gamma^\ast$, or the edges in the underlying Brauer tree on which the vertices lie. We also insist on a certain relationship between $x$ and $y$. Recall that the quiver for a classical Brauer tree has an underlying Brauer tree, where each vertex lies on an edge of the underlying Brauer tree. Recall also that we use the
same name for the edge and the vertex lying on it. Let \( w_1 \cdots w_j \) be the edge walk from the edge \( l \) to the edge \( r \) (\( w_1 = l \) and \( w_j = r \)). Then \( y \) = down if and only if \( x = up \) and \( j \) is even or \( x = down \) and \( j \) is odd.

**Proposition 5.3** There is a 1-1 correspondence between string-tuples and \( h \)-strings.

**Proof.** Let \( hS = s_1 \cdots s_n \) be a partitioned \( h \)-string. We describe the string-tuple \( (x, l, y, r) \) that corresponds to \( hS \) and show how we will use the information in the string-tuple to uniquely determine \( hS \). Let \( l = o(s_1) \), \( r = t(s_n) \), \( x = up \) if \( s_1 \) is the formal inverse of a quiver path and \( down \) otherwise, and \( y = up \) if \( s_n \) is a quiver path and \( down \) otherwise.

Recall that each vertex in \( \Gamma^* \) sits on an edge of the underlying Brauer tree for \( \Gamma^* \). These vertices and the edges that they sit on share the same name. The edges \( o(s_1)t(s_1)t(s_2)\cdots t(s_n) \) form a connected edge path. Since the exceptional number is 1 and since no \( s_i \) is a maximal path, the walk never doubles back on itself. Since this walk takes place on a tree, it is the unique walk from the edge \( o(s_1) \) to the edge \( t(s_n) \) with no repeating edges. So, given \( l = o(s_1) \) and \( r = t(s_n) \), we can build the sequence \( o(s_1)t(s_1)t(s_2)\cdots t(s_n) \). It this sequence together with the variable \( x \) that will allow us to build \( hS \).

In our construction whenever we are choosing a maximal path we will always choose from our collection of fixed maximal paths. We build the \( h \)-string \( s'_1 \cdots s'_n \). If \( x = down \), we choose \( s'_1 \) to be the quiver path lying in one cycle from \( o(s_1) \) to \( t(s_1) \). If \( x = up \), we choose \( s'_1 \) to be the formal inverse of that quiver path. Clearly \( s'_1 = s_1 \). For \( 2 \leq i < n \), we choose \( s'_i \) to be either the quiver path lying in one cycle from \( o(s_1) \) to \( t(s_1) \) or its formal inverse. Beginning with \( s'_1 \) we alternate between choosing a formal inverse and a quiver path. By construction the \( h \)-string, \( s'_1 \cdots s'_n \), is the \( h \)-string \( s_1 \cdots s_n \).
Note that we did not use $y$. The information contained in $y$ could be used in a similar fashion to the way $x$ was used to complete the description of $AS$. The two different approaches do not yield contradictory conclusions. In fact the variable $y$ is unnecessary. However, its presence simplifies later proofs. So we leave it in.

Lemma 4.2 now takes the form of Corollary 5.4.

**Corollary 5.4** A string function associated with $(x, l, y, r)$ can only be associated with the string-tuples $(x, l, y, r)$ or $(y, r, x, l)$. Hence up to an isomorphism of the image of a string function we have that each string function has only these two string-tuples associated with it.

Using the string-tuple notation we are going to restate some of the results of the picture algorithm in Corollary 5.5. But first we need some new notation. We will say that the vertex $v$ is one arrow away from the vertex $w$ if there is an arrow from $w$ to $v$. If $QUIV$ is a quiver for a classical Brauer tree algebra, $v$ is a nonlooped vertex in $QUIV$, and $w$ is any other vertex in $QUIV$, then we define a new smaller quiver $QUIV_{vw}$. We do this by considering the underlying Brauer tree. There are two subtrees that contain each edge. The quiver constructed using the subtree that contains $v$ and $w$ we will call $QUIV_{vw}$. We illustrate the idea in figure 5.11.

**Definition:** We will let $|QUIV_{vw}|$ be the number of vertices in $QUIV_{vw}$.
Corollary 5.5 Given that \( \{(x_i, l_i, y_i, r_i)\}_{i=0}^{\infty} \) is a minimal projective resolution of the image of \((x_0, l_0, y_0, r_0)\), we give selective information about \((x_i, l_i, y_i, r_i)\) in terms of \((x_{i-1}, l_{i-1}, y_{i-1}, r_{i-1})\). We do this by allowing \((x_{i-1}, l_{i-1}, y_{i-1}, r_{i-1})\) to play the role of \(hS\) in Lemma 5.1 and carefully considering the ten different cases. For each case we will list the cases that would need to be considered. We will be assuming the notation of Lemma 5.1.

A If \(l_{i-1} = r_{i-1}\) when \(l_i\) is a nonlooped vertex, then the next string-tuple will be \((u, l_i, u, w)\) where \(l_i\) and \(r_i\) will be the two vertices, say \(u\) and \(w\), one arrow away from \(l_{i-1}\). We are free to choose \(l_i\) to be \(u\) or \(w\). We make a particular choice when we present the ‘walking algorithm’. To verify this consider cases 5 and 9 of the picture algorithm. For B – F below we will assume that we do not have \(l_i = r_i\) when \(l_i\) is a nonlooped vertex.

B If \(x_{i-1} = \text{down and } l_{i-1}\) is looped, then \(l_i\) is one arrow away from \(l_{i-1}\) where
the arrow is not a loop. Consider cases 2, 4, 7, 8, 9, and 10. We have that \( i = v_1 \) when \( p_1 \) is an arrow, and otherwise \( l_i = t(a_1) \) where \( a_1 \) is the first arrow in \( P_1 \). In either case we have the result.

**C** If \( x_{i-1} = \text{down} \) and \( l_{i-1} \) is nonlooped, then \( l_i \) is one arrow away from \( i_{i-1} \) in \( \Gamma^*_{i_{i-1} r_{i-1}} \). The same argument used in B works here if we also note that \( P_1 \) lies in \( \Gamma^*_{i_{i-1} r_{i-1}} \).

**D** If \( x_{i-1} = \text{up} \) and \( l_{i-1} \) is nonlooped, then \( x_i = \text{up} \) and \( l_i \) is one arrow away from \( l_{i-1} \) not in \( \Gamma^*_{i_{i-1} r_{i-1}} \). Consider cases 1, 3, and 5. Note that \( \alpha \) is an arrow leaving \( v_1 \) not in the cycle containing \( Q_1 \) and hence not in \( \Gamma^*_{i_{i-1} r_{i-1}} \).

**E** If \( x_{i-1} = \text{up} \) and \( l_{i-1} \) is looped, then \( x_i = \text{down} \) and \( l_i = l_{i-1} \). See cases 1, 3, and 6.

**F** If \( l_i \) is looped and \( l_i \neq l_{i-1} \), then \( x_i = \text{up} \). We only get \( x_i = \text{down} \) if \( l_i = l_{i-1} \) where \( l_i \) is looped or if \( l_i = v_1 \) is nonlooped. Consider all cases.

**G** All of the (obvious) analogs for B – F concerning \( x_{i-1} \), \( l_{i-1} \), \( x_i \), and \( l_i \) hold for \( y_{i-1} \), \( r_{i-1} \), \( y_i \), and \( r_i \). The proofs look at different cases but the symmetry makes this easy.

Let \( v \leftrightarrow w \) represent the connected edge path in the underlying Brauer tree for \( \Gamma^* \) from \( v \) to \( w \) that contains no edge twice. Then we have the following corollary.

**Corollary 5.6** Let \( \{(x_i, l_i, y_i, r_i)\}_{i=0}^{\infty} \) be the minimal projective resolution of the image of \( (x_0, l_0, y_0, r_0) \). Then provided \( l_i \neq r_i \), \( x_i = \text{up} \) if and only if \( l_i \) does not appear in \( l_{i-1} \leftrightarrow r_{i-1} \). Also \( y_i = \text{up} \) if and only if \( r_i \) does not appear in \( l_{i-1} \leftrightarrow r_{i-1} \). If \( l_i = r_i \), then we may choose \( x_i = \text{up} \) and \( y_i = \text{down} \) or vice-versa.
The reader should examine the picture algorithm to see that this proposition is valid. Do this by noting that the edges in $l_i \leftrightarrow r_i$ are the vertices that appear from left to right in the underlying string corresponding to $(x_i, l_i, y_i, r_i)$.

5.3 The Walking Algorithm

Let $QUIV$ be the quiver associated with a classical Brauer tree algebra. If there are $m$ vertices in $QUIV$ then we are going to label $2m$ evenly spaced points on a unit circle using the $m$ vertices of $QUIV$. We will call this the $QUIV$-unit circle. Each vertex will be used exactly twice in the labeling process. Definition: We define any sequence that is formed by recording the labeled points in a clockwise direction as a $QUIV$-walking sequence. We will give two different algorithms for labeling the $QUIV$-unit circle the closed curve algorithm and the arrow algorithm. We will need to refer to each from time to time.

The Closed Curve Algorithm

We fix an embedding of $QUIV$ in a plane with a clockwise orientation. In other words we insist that every cycle has the arrows moving in a clockwise direction. A point will be considered to be in the exterior of $QUIV$ if it is not on the graph or bound by the graph. If $d$ is the smallest distance between any two vertices in $QUIV$, then we place in the plane a simple closed curve, $C$, with a clockwise orientation. We insist that $C$ follows the boundary of the exterior of $QUIV$ at a distance of $d/8$. See figure 5.12.

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Place the center of a circle, $D$, of radius $d/4$ anywhere on the curve $C$ such that no vertex of $QUIV$ lies within $D$. Then, keeping the center of the circle on $C$, begin to slide the circle $D$ in a clockwise direction. Choose any point on the $QUIV$-unit circle and label it the name of the first vertex to enter $D$. Continue to label the points on the $QUIV$-unit circle in a clockwise fashion using each successive vertex that enters $D$. Stop recording when $D$ returns to its starting location.

Consider figure 5.13 and note that every vertex enters $D$ exactly twice as it traverses $C$. Thus as we claimed we have labeled exactly $2m$ points on the $QUIV$-unit circle.

Figure 5.13:
The Arrow Algorithm

1. • Choose a looped vertex \( v \).
   
   • Label any two successive points on the \( QUIV \)-unit circle \( v \).
   
   • Choose the vertex one arrow away from \( v \). We insist that the arrow not be a loop.
   
   • Go to step 2.

2. • If the chosen vertex is \( v \), then stop. We have recorded every vertex twice.
   
   Otherwise
   
   • If the chosen vertex is a looped vertex, go to step 3.
   
   • If the chosen vertex is a nonlooped vertex, go to step 4.

3. • Continuing in a clockwise direction we label the next two points on the \( QUIV \)-unit circle as the chosen looped vertex.
   
   • Choose the vertex one arrow away from that looped vertex. We insist that the arrow is not a loop.
   
   • Go to step 2.

4. • Continuing in a clockwise direction we label the next point on the \( QUIV \)-unit circle as the chosen nonlooped vertex.
   
   • Choose the vertex one arrow away the nonlooped vertex not in the same cycle as the previously recorded vertex.
   
   • Go to step 2.
The Walking Algorithm

Recall that a \( \Gamma \)-walking sequence is any sequence formed by recording the points on the \( \Gamma \)-unit circle in a clockwise direction. A minimal projective resolution, \( \{(x_i, l_i, y_i, r_i)\}_{i=0}^{\infty} \), for the image of some string function \( hS_0 \in \Lambda^* \) can be computed by using the following algorithm.

1. Choose a string-tuple \((x_0, l_0, y_0, r_0)\) that corresponds to \( hS_0 \).

2. Use Corollaries 5.5 and 5.6 to construct (or choose) \((x_1, l_1, y_1, r_1)\).

3. Construct the \( \Gamma^* \)-walking sequence \( \{l_i\}_{i=0}^{\infty} \) which begins with the \( \Gamma^* \)-walking sequence \( \{l_0, l_1\} \).

4. Construct the \( \Gamma^* \)-walking sequence \( \{r_i\}_{i=0}^{\infty} \) which begins with the \( \Gamma^* \)-walking sequence \( \{r_0, r_1\} \).

5. Use Corollaries 5.6 to construct the sequences \( \{x_i\}_{i=0}^{\infty} \) and \( \{y_i\}_{i=0}^{\infty} \).

To validate this algorithm we only need to prove Proposition 5.7.

**Proposition 5.7** If Corollary 5.5 is used to construct the sequences \( \{l_i\}_{i=0}^{\infty} \) and \( \{r_i\}_{i=0}^{\infty} \), then those sequences can be chosen to be \( \Gamma^* \)-walking sequences.

**Proof.** We will show that for \( 0 \leq i \), that \( \{l_0, l_1, \ldots, l_{i+1}\} \) and \( \{r_0, r_1, \ldots, r_{i+1}\} \) are \( \Gamma^* \)-walking sequences. The proof will proceed by induction on \( i \). Using Corollary 5.5 to construct \( \{l_i\}_{i=0}^{\infty} \) (resp. \( \{r_i\}_{i=0}^{\infty} \)), it is clear that for \( i \geq 0 \), \( l_i \) (resp. \( r_i \)) is always one arrow (possibly a loop) away from \( l_{i-1} \) (resp. \( r_{i-1} \)). Since any two vertices one arrow (possibly a loop) away from each other appear successively in a clockwise direction somewhere on the \( \Gamma^* \)-unit circle, we have that \( \{l_0, l_1\} \) and \( \{r_0, r_1\} \) are \( \Gamma^* \)-walking sequences. We now assume that \( \{l_0, l_1, \ldots, l_i\} \) \( \{r_0, r_1, \ldots, r_i\} \) are \( \Gamma^* \)-walking
sequences and show that \( \{l_0, l_1, \ldots, l_i, l_{i+1}\} \) and \( \{r_0, r_1, \ldots, r_i, r_{i+1}\} \) are \( \Gamma^* \)-walking sequences. We do this by showing that the choice of \( l_{i+1} \) (resp. \( r_{i+1} \)) using Corollary 5.5 is consistent with the choice of \( l_{i+1} \) (resp. \( r_{i+1} \)) using the arrow algorithm.

We will first consider the proof when \( l_i \neq r_i \). We note that when \( l_i \neq r_i \), Corollary 5.5 allows no flexibility in the choice of \( l_{i+1} \) and \( r_{i+1} \). In that case by symmetry we can simply show that the choice of \( l_{i+1} \) using Corollary 5.5 is consistent with the choice of \( l_{i+1} \) using the arrow algorithm. We consider case where \( l_i \) is a looped vertex separately from the case where \( l_i \) is a nonlooped vertex.

1. **\( l_i \) is looped.** If \( l_{i-1} = l_i \), \( l_{i+1} \) is one arrow away from \( l_i \). On the other hand, if \( l_{i-1} \neq l_i \) then \( l_{i+1} = l_i \). In either case the looped vertices appear in \( \{l_i\}_{i=0}^{\infty} \) in a way that is consistent with the arrow algorithm.

2. **\( l_i \) is nonlooped.**

![Figure 5.14:](image)

Using figure 5.14 as a reference for our notation, our goal here is to show that Corollary 5.5 chooses \( l_{i+1} \) to be the vertex \( v \). We consider two cases.

(a) We assume \( r_{i-1} \) is in \( \Gamma_{\bar{l}_{i-1}}^* \) but that \( r_{i-1} \neq l_i \). Therefore \( l_i \) is not in \( l_{i-1} \leftrightarrow r_{i-1} \). So by Corollary 5.6, \( x_i = \text{up} \). Corollary 5.5 chooses \( l_{i+1} \)
to be one arrow away from \( l_i \) not in \( \Gamma_{i,r_i}^* \). Since \( r_i \) must be at most one arrow away from \( r_{i-1} \), \( r_i \) must be in \( \Gamma_{i,i-1}^* \). Therefore since \( r_i \neq l_i \), we have \( \Gamma_{i,r_i}^* = \Gamma_{i,l_i}^* \). Hence \( l_{i+1} = v \).

(b) We assume \( r_{i-1} \) is in \( \Gamma_{i,v}^* \). Then \( l_i \) is in \( r_{i-1} \leftrightarrow l_{i-1} \) and so \( x_i \) = down. Therefore \( l_{i+1} \) is one arrow from \( l_{i-1} \) in \( \Gamma_{i,r_i}^* \). Since \( r_i \) is at most one arrow away from \( r_{i-1} \) which is in \( \Gamma_{i,v}^* \) and \( r_i \neq l_i \), we have that \( r_i \) is in \( \Gamma_{i,v}^* \) and therefore \( \Gamma_{i,r_i}^* = l_{i}v \). So since \( x_i \) = down, \( l_{i+1} \) is one arrow away from \( l_i \) in \( \Gamma_{i,r_i}^* = \Gamma_{i,v}^* \). Hence \( l_{i+1} = v \).

We now consider the proof when \( l_i = r_i \). It is clear from Corollary 5.5 that the \( \Gamma^* \)-walking sequences \( \{l_{i-1}, l_i\} \) and \( \{r_{i-1}, r_i\} \) are distinct with \( l_i = r_i \) a vertex one arrow (not a loop) away from each of \( l_{i-1} \) and \( r_{i-1} \). Since \( l_i = r_i \), we may apply part A of Corollary 5.5. It says that we may choose \( l_{i+1} \) to be either of the two vertices that are one arrow away from \( l_i \), and choose \( r_{i+1} \) to be the other vertex. Therefore using Corollary 5.5 we can choose \( l_{i+1} \) and \( r_{i+1} \) to be consistent with the choice that would be made by the arrow algorithm. This completes the inductive step of the argument. We have the result. \( \blacksquare \)

Let \( \{(x_m, l_m, y_m, r_m)\}_{m=0}^\infty \) be the sequence of string-tuples corresponding to the string functions in the minimal projective resolution of the module \( M(S) \in \Gamma^* \) which is constructed using the walking algorithm. \textbf{Definition:} We will call such a sequence a \textit{walking sequence} for the module \( M(S) \).

### 5.4 The Periodicity of a String Module

Let \( \{h_i S_i\}_{i=0}^\infty \) be a minimal projective resolution of the string module \( M(S_0) \) in \( \Lambda \). If \( k \) is the least positive integer such that \( M(S_k) \simeq M(S_0) \), then we say that the \textit{periodicity} of \( M(S_0) \) is \( k \). Our main goal in this section is to give a method for

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determining the periodicity for any string module. We will prove that the periodicity is either \(|\Gamma|\) or \(2|\Gamma|\) for all string modules. In Theorem 5.13 we will give a test to determine the periodicity of a string module \(M(S_0)\).

For the remainder of the chapter we will let \(N\) be the exceptional number for \(\Gamma\). We let \(t\) be the number of times the arrow of weight \(\Gamma\) appears in \(S_0\). Then using the construction of Proposition 3.3 we find \(S_0^*\) has associated to it a string-tuple of the form \((x_0, \overrightarrow{y_0}, y_0, r_0)\). We let \(\{hS_i^*\}_{i=1}^\infty\) be the sequence of string functions in the minimal projective resolution of \(M(S_0^*)\). Also let \(\{(x_i, \overrightarrow{m_i}, y_i, r_i)\}_{i=0}^\infty\) be a \(\Gamma^*\)-walking sequence for \(M(S_0^*)\).

**Lemma 5.8**

1. If \(i\) is the least integer greater than 0 such that \(E(M(S_i^*)) = M(S_0)\), then the periodicity of \(M(S_0) = i\).

2. \(i\) satisfies (1) above if and only if \(S_i^*\) corresponds to a string-tuple of the form \((x_0, \overrightarrow{y_0+j}, y_0, r_0)\) or \((y_0, \overrightarrow{r_0+j}, x_0, \overrightarrow{y_0+j})\) where \(j\) is an integer.

**Proof.** (1) follows from the fact that the functor \(E\) preserves minimal projective resolutions. (2) follows from the proof of Proposition 3.3 □

**Lemma 5.9** If the superscripts of the vertices in a \(\Gamma^*\)-walking sequence are removed, then the \(\Gamma^*\)-walking sequence becomes a \(\Gamma\)-walking sequence. Also given the superscript of the first vertex of the \(\Gamma^*\)-walking sequence we can construct the \(\Gamma^*\)-walking sequence from the \(\Gamma\)-walking sequence.

**Proof.** The first statement is an immediate consequence of the fact that \(\Gamma^*\) and \(\Gamma\) are locally homeomorphic. Letting \(\overrightarrow{m_0}\) be the superscript of the first vertex of a
\( \Gamma^* \)-walking sequence. Then if \( \{w_i\}_{i=0}^\infty \) is the \( \Gamma \)-walking sequence that we construct by dropping the superscripts from a \( \Gamma^* \)-walking sequence, we construct the original \( \Gamma^* \)-walking sequence recursively.

For \( i > 0 \), if \( \phi(a) = w_{i-1} \) and \( t(a) = w_i \), then \( \mathfrak{m}_i = \mathfrak{m}_{i-1} + 1 \). Otherwise \( \mathfrak{m}_i = \mathfrak{m}_{i-1} \). \( \blacksquare \)

**Lemma 5.10** If \( \Gamma \) consists of a single vertex and two loops, then the periodicity for all string modules over \( \Lambda \) is 1 when the exceptional number is 1, and 2 otherwise.

**Proof.** If the exceptional number is 1, then there is only one string module. If the exceptional number is 2, then there are two string modules with the image of one being the kernel of the other. \( \blacksquare \)

For Lemmas 5.11 through 5.13 we assume that \( \Gamma \) is not the degenerate case that we just handled in Lemma 5.10.

**Lemma 5.11** Let \( k \) be the periodicity of the string module \( M(S_0) \), then \( k \) divides \( 2|\Gamma| \).

**Proof.** It suffices to show that \( E(M(S_{2|\Gamma|})) \simeq M(S_0) \). Since the number of labeled points on the \( \Gamma \)-unit circle is \( 2|\Gamma| \), by Lemma 5.9, we have that \( \mathfrak{l}_0^{2|\Gamma|} = \mathfrak{l}_0^\uparrow \) (resp. \( \mathfrak{r}_0^{2|\Gamma|} = \mathfrak{r}_0^{\uparrow\downarrow} \)). Suppose \( \mathfrak{l}_0^\uparrow = \mathfrak{r}_0^\uparrow \). Since \( \mathfrak{l}_0^\uparrow = \mathfrak{r}_0^\uparrow \) by Lemma 5.8 we only need to show that \( x_{2|\Gamma|} = x_0 \) and \( y_{2|\Gamma|} = y_0 \) or \( x_{2|\Gamma|} = y_0 \) and \( y_{2|\Gamma|} = x_0 \). But when we are looking at strings-tuples in \( \Gamma^* \). If we have \((x, v, y, v)\), either \( x \) or \( y \) is up and the other is down. So we have \( E(M(S_{2|\Gamma|})) = M(S_0) \).

Suppose \( \mathfrak{l}_0^\uparrow \neq \mathfrak{r}_0^\uparrow \). At the beginning of this proof we showed that \( \mathfrak{l}_0^{2|\Gamma|} = \mathfrak{l}_0^\uparrow \) and that \( \mathfrak{r}_0^{2|\Gamma|} = \mathfrak{r}_0^{\uparrow\downarrow} \). Therefore, if we prove that \( x_{2|\Gamma|} = x_0 \) and \( y_{2|\Gamma|} = y_0 \), we will have proven that \( E(M(S_{2|\Gamma|})) = M(S_0) \). However, since the number of edges in \( \langle \mathfrak{l}_0 \rangle \leftarrow \mathfrak{r}_0^\uparrow \) is
the same as the number of edges in $I_0 \leftrightarrow r_0^{T+1}$, and since that number together with $x_0$ (resp. $x_{2|\Gamma|}$) determines $y_0$ (resp. $y_{2|\Gamma|}$), we only need to show that $x_{2|\Gamma|} = x_0$.

Since, $\vec{d}_0 \neq r_0^T$, $l_1^{m_1}$ can be determined by $x_0$ and $\vec{t}_0 \leftrightarrow r_0^T$. Similarly, since $l_{2|\Gamma|} \neq r_{2|\Gamma|}^{T+1}$, $l_{2|\Gamma|+1}^{m_{2|\Gamma|+1}}$ can be determined by $x_0$ and $\vec{t}_{2|\Gamma|} \leftrightarrow r_{2|\Gamma|}^{T+1}$. Since $l_{2|\Gamma|} \leftrightarrow r_{2|\Gamma|}^{T+1}$ can be found by adding $T$ to each of the superscripts in $\vec{t}_0 \leftrightarrow r_0^T$, and since $l_{2|\Gamma|+1}^{m_{2|\Gamma|+1}} = l_{1}^{m_1}$ we must have $x_{2|\Gamma|} = x_0$. In other words, since those portions of $\Gamma^*$ are homeomorphic, we must have $x_{2|\Gamma|} = x_0$. In either case we have the result.

Lemma 5.12 If $0 < k < 2|\Gamma|$, then for $k \neq |\Gamma|$, $E(M(S^*_k))$ is not isomorphic to $M(S_0)$. If $E(M(S^*_k))$ is not isomorphic to $M(S_0)$, then the periodicity of $M(S_0)$ is $2|\Gamma|$.

Proof. The second statement follows from the first statement and Lemma 5.11. We prove the first statement. We first consider the case where $l_0 = r_0 = v$ where $v$ is some vertex. We assume that $i$ is such that $E(M(S^*_i))$ is isomorphic to $M(S_0)$. Hence, we must have $l_i = r_i = v$. Also using Corollary 5.5 we can see that sequences $\{l_i\}_{0}^{\infty}$ and $\{r_i\}_{i=0}^{\infty}$ have different starting places on the $\Gamma$-unit circle. Hence, the walking sequences $l_0, l_1, \ldots, l_{i-1}, r_1, r_2, \ldots, r_i$ and $l_0, l_1, \ldots, l_{2|\Gamma|}$ are identical. Therefore $i = \frac{2|\Gamma|}{2} = |\Gamma|$. (Note that this could only happen if $v$ is a nonlooped vertex, or $\Gamma$ is the degenerate case we are not considering here).

Suppose $l_0 \neq r_0$. Then we will assume that $i$ is such that $E(M(S^*_i))$ is isomorphic to $M(S)$ and proceed to identify the $i$'s for which this can be true. The sequence $l_0, l_1, \ldots, l_{2|\Gamma|}$ appears on the $\Gamma$-unit circle if we record the points in a clockwise direction. Recall that although there are $2|\Gamma|$ points on the unit circle, we only use the $|\Gamma|$ vertex labels to label them. Thus the label $l_0$ appears twice in the sequence $l_0, l_1, \ldots, l_{2|\Gamma|}$. We locate the point labeled $l_0$ which is not the first point in the
sequence. For the remainder of this lemma we refer to the point that we have just
located as $L_0$. So that now we will refer to the vertex $l_0$ as $l_0$ or $L_0$. We define $R_0$
in a similar fashion. We have then that the second time the vertex $v = l_0$ appears
in the $\Gamma$-walking sequence it will be labeled $L_0$. The third time it appears it will be
labeled $l_0$ once again. With this new notation we can now describe the four possible
scenarios that would lead to having the set of vertices $\{l_0, r_0\}$ being the same as the
set of vertices $\{l_i, r_i\}$, a necessary condition for $E(M(S_i^*)) = M(S_0)$.

**Case 1:** $l_i = L_0$, $r_i = R_0$

Since $l_i = L_0$ appears in a different location on the $\Gamma$-unit circle than $l_0$, $l_{i+1} \neq l_i$.
Similarly we have $r_{i+1} \neq r_1$. Yet if the periodicity of $M(S_0)$ is $i$, then we must have
$E(M(S_{i+1}^*)) = M(S_1)$. So it must be that $r_{i+1} = l_1$ and $l_{i+1} = r_1$. We have four
cases to consider. They each address what happens when only some of the equalities
of equations 5.1 - 5.4 hold.

\begin{align*}
l_0 & = l_1 \\
l_i & = l_{i+1} \\
r_0 & = r_1 \\
r_i & = r_{i+1}
\end{align*}

**Case 1a:** Suppose none of the equalities of equations 5.1 - 5.4 hold. In this case
it must be that part of $\Gamma$ must have the form given in figure 5.15. Since no tree of
cycles can have such an arrangement, we see that this case can not occur.

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Figure 5.15:

**Case 1b:** Suppose exactly one of the equalities of equations 5.1 - 5.4 holds. Then part of $\Gamma$ will have the form given in figure 5.16. No tree of cycle can have this form, so this case can not occur.

Figure 5.16:

**Case 1c:** Suppose exactly one of the following four pairs of equations hold: 5.1 and 5.2, 5.3 and 5.4, equations 5.3 and 5.2, or 5.1 and 5.4. Then part of $\Gamma$ will have the form given in figure 5.17. No tree of cycles can have this form, so this case can not occur.

Figure 5.17:

**Case 1d:** Suppose exactly one of the following two pairs of equations hold: 5.1 and 5.3, 5.2 and 5.4.
Then it must be that $\Gamma$ is the graph given in figure 5.18. Since $|\Gamma| = 2$ and $i$ must divide $2|\Gamma|$, we know that $i = 2$ or $i = 1$. We need to show that $i$ can not be 1. Suppose it is 1. Then we have $l_0 = l_i = l_1$ and $r_0 = r_i = r_1$. Since $l_0$ and $r_0$ are looped vertices, by Corollary 5.5 we have that $x_0 = y_0 = \text{up}$ and $x_1 = y_1 = \text{down}$. So in that case $E(M(S_i^*))$ is not isomorphic to $M(S_0)$. We are finished with Case 1.

![Figure 5.18:](image)

**Case 2:** $l_i = r_0$ and $r_i = l_1$

Then $l_{i+1} \neq r_1$ while $r_{i+1} = l_i$. So that the set of vertices \{l_1, r_1\} is not the same as the set of vertices \{l_{i+1}, r_{i+1}\}. Hence $E(M(S_i^*))$ is not isomorphic to $M(S)$. Which gives us that $E(M(S_i^*))$ is not isomorphic to $M(S)$.

**Case 3:** $l_i = R_0$ and $r_i = L_0$

This forces $l_{i+1} \neq r_0$ and $r_{i+1} \neq l_1$. So to have $E(M(S_{i+1}^*)) = M(S_1)$. This in turn forces $l_{i+1} = l_1$ and $r_{i+1} = r_1$. Just as in Case 1 there are four cases to consider. But the cases and the arguments are the same as they were in Case 1.

**Case 4:** $l_i = r_0$ and $r_i = l_0$

This implies that the walking sequences $l_0, l_1, \ldots, l_{i-1}, r_1, r_2, \ldots, r_i$ and $l_0, l_1, \ldots, l_2 |\Gamma|$ are identical. Therefore $i = \frac{2|\Gamma|}{2} = |\Gamma|$. (We note again that this could only happen if $v$ is a nonlooped vertex, or $\Gamma$ is the degenerate case we are not considering here). ■

For Lemma 5.13 we continue to use the notation that we developed in Lemma 5.12: the second time the label $l_0$ appears on the $\Gamma$-unit circle as we read the sequence.
\(\{l_i\}_{i=0}^{\infty}\) clockwise we relabel the point \(L_0\). Similarly we define \(R_0\). Thus if we say \(l_{|\Gamma|} = r_0\) we mean more than the two vertices are equal we mean that those two labels correspond to the same point on the \(\Gamma\)-unit circle.

**Theorem 5.13** The periodicity of \(M(S_0)\) is \(|\Gamma|\) precisely when \(l_{|\Gamma|} = r_0\) and \(r_{|\Gamma|} = l_0\) and the exceptional number is 1. The periodicity of \(M(S_0)\) is \(2|\Gamma|\) in all other cases.

**Proof.** The second statement follows from the first and Lemma 5.12. We prove the first statement. We have already shown that this is true when \(\Gamma\) contains only one looped vertex and when \(\Gamma\) contains exactly two vertices both of which are looped vertices. So we assume that we do not have either of these cases. We first let the exceptional number be 1. Then there is no need to form the graph \(\Gamma^*\). Our walking algorithm works for string modules over \(\Lambda\), and we can denote the string-tuples in the minimal projective resolution of \(M(S_0)\) without using superscripts.

If the vertex \(l_0\) is the vertex \(r_0\), then by arguments that we made in Lemma 5.11 we have \(E(M(S_{|\Gamma|}^*)) \simeq M(S_0)\). So assume that the vertex \(l_0\) is not the vertex \(r_0\). Then for the result to hold we need \(x_{|\Gamma|} = y_0\) and \(y_{|\Gamma|} = x_0\). Since \(l_{|\Gamma|} = r_0\) and \(r_{|\Gamma|} = l_0\), we know that the point \(l_{|\Gamma|+1}\) is the point \(r_1\) and that the point \(r_{|\Gamma|+1}\) is the point \(l_1\). Again using similar arguments to those used in the proof of Lemma 5.11 we know that this can only happen if \(x_{|\Gamma|} = y_0\) and \(y_{|\Gamma|} = x_0\). So we have that if the points \(l_{|\Gamma|} = r_0\) and \(r_{|\Gamma|} = l_0\) and the exceptional number is 1, the periodicity is \(|\Gamma|\).

We now show that if the exceptional number is not 1, the periodicity can not be n. By hypothesis and using Lemma 5.9 we have that the two \(\Gamma\)-walking sequences \(l_0, \ldots, l_{|\Gamma|}\) and \(l_0, \ldots, r_1, \ldots, r_{|\Gamma|}\) are identical. Let \(a\) be the arrow in \(\Gamma\) of weight \(1\). Since there is a unique \(i\) such that \(o(a) = l_i\) and \(t(a) = l_{i+1}\), it must be that there is a \(k \leq n\) such that either \(o(a) = l_k\) and \(t(a) = l_{k+1}\) or \(o(a) = r_k\) and
\( t(a) = l_{k+1} \). Thus the string-tuple for \( hS_1^* \) has either the form \((x_{[Γ]l}, r_{[Γ]}, y_{[Γ]}, i_{[Γ]})\) or \((x_{[Γ]}, r_{[Γ]}, y_{[Γ]}, i_{[Γ]})\). By Lemma 5.8 we know that in both of these cases \( E(M(S_1^*)) \) is not isomorphic to \( M(S_0) \). We have the result. ■

We define a short \( Γ \)-walking sequence, \( \{v, v_1, \ldots, v_n, w\} \), to be a \( Γ \)-walking sequence in which \( v, w \notin \{v_1, \ldots, v_n\} \).

**Corollary 5.14** Let \( v \in Γ \). Let 1\(_v\) be the string \( S_0 \) that we have used in the lemmas for this section. Keeping the same notation that we have been using throughout, the periodicity of \( M(1_v) \) is \( |Γ| \) if and only if \( |Γ_{l_0l_1}| = |Γ_{r_0r_1}| \) and the exceptional number is 1.

**Proof.** By Theorem 5.13 we only need to show that when the exceptional number is 1, \( l_0 = r_{[Γ]} \) and \( r_0 = l_{[Γ]} \) if and only if \( |Γ_{l_0l_1}| = |Γ_{r_0r_1}| \). Since in this case \( l_0 = r_0 \), this is equivalent to proving that \( l_0 = l_{[Γ]} \) and \( r_0 = r_{[Γ]} \) if and only if \( |Γ_{l_0l_1}| = |Γ_{r_0r_1}| \).

Suppose \( |Γ_{l_0l_1}| = |Γ_{r_0r_1}| \). Then since \( l_0 = r_0 \) and \( l_1 \neq r_1 \), by a simple counting argument it is easy to see that \( 2|Γ_{l_0l_1}| = |Γ| + 1 \). It is also not hard to see that the short \( Γ \)-walking sequence \( \{l_0, l_1, \ldots, l_0\} \) has length \( 2|Γ_{l_0l_1}| - 1 = |Γ| \). Therefore \( l_0 = l_{[Γ]} \). Similarly we have \( r_0 = r_{[Γ]} \).

Suppose \( l_0 = l_{[Γ]} \) and \( r_0 = r_{[Γ]} \). Since in every \( Γ \)-walking sequence of length \( 2|Γ| \) every vertex appears exactly twice, and since we know by previous experience that \( l_{[Γ]} = l_0 \) experience, we have that \( \{l_0, l_1, \ldots, l_{[Γ]}\} \) is a short \( Γ \)-walking sequence. Therefore it must be that the vertices of that walk are completely contained in \( Γ_{l_0l_1} \). In fact the sequence must have each of the vertices of \( Γ_{l_0l_1} \) appearing exactly twice. Thus \( |Γ| = 2|Γ_{l_0l_1}| - 1 \). Similarly \( |Γ| = 2|Γ_{r_0r_1}| - 1 \). Thus we have that \( |Γ_{l_0l_1}| = |Γ_{r_0r_1}| \). ■

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Chapter 6

A minimal set of generators for Ext(Λ)

In this chapter we construct a minimal set of generators for the cohomology ring or Ext-algebra of a classical Brauer tree algebra with respect to the Jacobson radical of Λ. By minimal we mean that there is no subset of the generating set that is also a generating set.

6.1 Definitions and Notations

For each \( v \in \Gamma_0 \), \( M(1_v) \) is a string module. So we can apply the algorithms of Chapter 5 to compute \( D^v \), a minimal projective resolution of \( M(1_v) \). We present the notation that we will use to describe \( D^v \) below.

\[
D^v : \cdots \rightarrow D_2^v \xrightarrow{hS_2^v} D_1^v \xrightarrow{hS_1^v} D_0^v \xrightarrow{hS_0^v} M(1_v) \rightarrow 0
\]

In this chapter unless otherwise stated a bar over a symbol will mean the equivalence class of that element. Though there will be more than one equivalence class appearing in the chapter no confusion should arise. Let \( \mathcal{R} \) be the Jacobson radical of \( Λ \). Then \( \mathcal{R} \) is generated by \( \{ \overline{a} : \text{where } a \text{ is an arrow in } \Gamma. \} \). It is not hard to show

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that $\Lambda/\mathfrak{r} \simeq \prod_{v \in \Gamma_0} v\Lambda/\mathfrak{r}(v\Lambda)$, and that for each $v \in \Gamma_0$, $v\Lambda/\mathfrak{r}(v\Lambda) \simeq M(1_v)$. Let $\overline{\Lambda} = \prod_{v \in \Gamma_0} M(1_v)$. Since the (natural) sum of two minimal projective resolutions is a minimal projective resolution, a minimal projective resolution of $\overline{\Lambda} \simeq \Lambda/\mathfrak{r}$ is $D = \prod_{v \in \Gamma_0} D^v$. In other words, if for $k \geq 0$, we define $d_k = \sum_{v \in \Gamma_0} kS_k^v$ and for $i \geq 0$, we define $D_i = \prod_{v \in \Gamma_0} D_i^v$, then we have

$$D : \cdots \rightarrow D_2 \xrightarrow{d_2} D_1 \xrightarrow{d_1} D_0 \xrightarrow{d_0} \overline{\Lambda} \rightarrow 0$$

is a minimal projective resolution of $\overline{\Lambda}$.

We are now in a position to define the Ext-algebra of $\Lambda$ with respect to $\mathfrak{r}$. As an abelian group we define $\text{Ext}(\Lambda) = \prod_{i=0}^{\infty} \text{Ext}^i(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$. To construct $\text{Ext}^i(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ we begin with a projective resolution of $\Lambda/\mathfrak{r}$. Since in our case $\Lambda$ is artinian, we begin with a minimal resolution $D$. We replace the module $\Lambda/\mathfrak{r}$ in the resolution with the zero module. We then form the complex

$$\text{Hom}(0, \Lambda/\mathfrak{r}) \xrightarrow{d_0^*} \text{Hom}(D_0, \Lambda/\mathfrak{r}) \xrightarrow{d_1^*} \text{Hom}(D_1, \Lambda/\mathfrak{r}) \xrightarrow{d_2^*} \cdots$$

where $d_0^*$ is the zero map and $d_i^*(\alpha) = \alpha \circ d_i$ for $i \geq 1$ and $\alpha \in \text{Hom}(D_i, \Lambda/\mathfrak{r})$. For $i \geq 1$ we define the abelian group $\text{Ext}^i(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}) = \text{Ker} d_{i+1}^*/\text{Im} d_i^*$. If $e \in \text{Ext}^i(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$, then we will say that $e$ is of degree $i$.

Since $D$ is minimal, and since $\Lambda$ is artinian, general theory gives us that $\text{Ext}^i(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}) \simeq \text{Hom}_\Lambda(D_i, \Lambda/\mathfrak{r})$. Hence $\text{Ext}(\Lambda) \simeq \prod_{i=0}^{\infty} \text{Hom}(D_i, \Lambda/\mathfrak{r})$. Next we use the Yoneda product to place a multiplicative structure on $\text{Ext}(\Lambda)$. Let $f \in \text{Hom}_\Lambda(D_i, \Lambda/\mathfrak{r})$ and $g \in \text{Hom}_\Lambda(D_j, \Lambda/\mathfrak{r})$. Since the all $D_k$'s are projective, we can find a collection of homomorphisms $\{u_k\}_{i=0}^{\infty}$ that make the diagram of figure 6.1 commute. We call these homomorphisms liftings. We define $f \ast g = g \circ u_j$. Since
\{ f \in \text{Hom}_\Lambda(D_i, \Lambda/\rho) : i \geq 0 \} \text{ is a basis for } \text{Ext}(\Lambda), \text{ we can extend this definition linearly to complete the definition of multiplication of any two elements of } \text{Ext}(\Lambda).

\[
\begin{array}{ccccccc}
D_{i+j} & \overset{d_{i+j}}{\longrightarrow} & D_{i+j-1} & \longrightarrow & \cdots & \overset{d_{i+2}}{\longrightarrow} & D_{i+1} & \overset{d_{i+1}}{\longrightarrow} & D_i \\
& & u_j & \downarrow & & u_{j-1} & \downarrow & \cdots & u_1 & \downarrow & u_0 & \downarrow & f \\
D_j & \overset{d_j}{\longrightarrow} & D_{j-1} & \longrightarrow & \cdots & \overset{d_1}{\longrightarrow} & D_1 & \overset{d_1}{\longrightarrow} & D_0 & \overset{d_0}{\longrightarrow} & \Lambda
\end{array}
\]

Figure 6.1:

6.2 A minimal set of generators

Lemma 6.1 Let \( hS : v_1 \text{QUIV} \oplus \cdots \oplus v_n \text{QUIV} \longrightarrow w_1 \text{QUIV} \oplus \cdots \oplus w_m \text{QUIV} \) be a string function.

1. If \( \text{QUIV} = \Gamma \), then the vertices \( v_1, \ldots, v_n \) are distinct.

2. If \( \text{QUIV} = \Gamma^* \) and we do not have both \( n = 1 \) and \( m = 1 \), then the vertices \( v_1, \ldots, v_n, w_1, \ldots, w_m \) are distinct.

Proof. Recall that for each of the vertices \( v_1, \ldots, v_n, w_1, \ldots, w_m \) there is a corresponding edge in the underlying Brauer tree on which that vertex lies. The edges \( v_1, \ldots, v_n, w_1, \ldots, w_m \) are the edges in some connected edge walk from \( o(v_1) \) to \( t(w_m) \). The edge walk can be found by reading the vertices that appear in the underlying string for \( hS \) from left to right. We let \( e_1 \iff e_{n+m} \) represent that edge walk. Since the walk is occurring on a tree, if that edge walk never doubles back on itself, the edges \( v_1, \ldots, v_n, w_1, \ldots, w_m \) are clearly unique and we have the result. If the walk doubles back on itself we will have some edge \( E \) appearing in the walk twice in a row. This implies that \( hS \) contains a formal path, \( P \), lying in one cycle that had the same terminus and origin.

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If $P$ lies in a nonexceptional cycle, then $P$ must be a maximal path or an inverse of one. So if $P$ lies in a nonexceptional cycle or is a maximal path in an exceptional cycle, (or an inverse of one), $hS = P$. Therefore we have (1) in the case where $P$ lies in a nonexceptional cycle. Since every cycle in $\Gamma^*$ is a nonexceptional cycle this constitutes a proof of (2). We now prove (1) in the case where $P$ lies in an exceptional cycle with exceptional number not equal to 1, but is not a maximal path, nor an inverse of one. In this case the vertex $E$ must also lie in the exceptional cycle. By the definition of a string function $EE$ can not be an edge walk contained in $e_1 \iff e_{n+m}$. Therefore, since this is a connected edge walk occurring on a tree of edges, if $e_1 \iff e_{n+m} \subset e_1 \iff vEEw \iff e_{n+m}$, then the edge walks $e_1 \iff v$ and $w \iff e_{n+m}$ must not contain any edges (or vertices) that lie in the exceptional cycle. Hence the walks $e_1 \iff E$ and $E \iff e_{n+m}$ are both connected edge walks that do not double back on themselves. If an edge $x$ appears in both of those walks, then the unique connected edge walk that doesn’t double back on itself from $x$ to $E$ has the same number of edges as the one from $E$ to $x$. Thus if $x = e_i = e_j$, and if $i$ is odd, $j$ must be even and vice-versa. Therefore, since the vertices $v_1, \ldots, v_n$ appear in $e_1 \iff e_{n+m}$ as every other edge, they must be distinct.

For each $k \geq 0$ and each $v \in \Gamma_0$, we will define $D^v_k = \bigcap_{w \in G^v_k} w\Lambda$ where $G^v_k$ is the appropriate set of vertices. If $w \in G^v_k$, then we will say that $w$ appears in $D^v_k$. We also define an index set $G_k = \{(v, w) : (v, w) \text{ is a formally defined symbol, } v \in \Gamma_0, \text{ and } w \in G^v_k\}$. Using this notation we have that $D_k \simeq \bigcap_{(v, w) \in G_k} w\Lambda$. To identify the term sitting in the $(v, w)$ component of $D_k$ we will append the subscript $(v, w)$ to that term.

We are now ready to create a very nice basis for $\text{Ext}(\Lambda)$. Let $v, y \in \Gamma_0$, $w \in G^v_k$, and $x \in G^y_k$, then we define the function $f_k^{(v, w)} : D_k \to \Lambda / \mathbb{L}$ to be the function such
that
\[
f_k^{(v,w)}((0, \ldots, 0, w_{(u,w)}, 0, \ldots, 0)) = (0, \ldots, 0, w, 0, \ldots, 0)
\] where and on the right \( w \) is in the \( M(w) \) component of \( \Lambda/\Sigma \), and
\[
f_k^{(v,w)}((0, \ldots, 0, x_{(y,x)}, 0, \ldots, 0)) = 0
\] where \((y, x) \neq (v, w)\).

By Lemma 6.1 the component \( w \Lambda \) appears only once in \( D_k^v \). Therefore, this function is well defined. We say that \( f_k^{(v,w)} \) only nontrivially projects the vertex \( w \) that appears in \( D_k^v \) to \( \Lambda/\Sigma \).

**Lemma 6.2** \( \{f_k^{(v,w)} : v \in \Gamma_0, k \geq 0, \text{ and } w \in G_k^v\} \) is a \( K \)-basis for \( \text{Ext}(\Lambda) \).

**Proof.** Consider the paragraph above and note that all homomorphisms, \( g \), from \( v \Lambda \) to \( \Lambda/\Sigma \) are defined as follows \( g(\overline{v}) = c(0, \ldots, 0, v, 0, \ldots, 0) \) where \( v \) is a vertex in the \( M(1_v) \) component of \( \Lambda/\Sigma \) and \( c \) is a constant.

We call the basis of Lemma 6.2 the standard basis and we refer to the vectors in this basis as standard basis vectors.

**Lemma 6.3** Let \( x, v, w \in \Gamma_0 \) with \( w \in G_k^x \). Then
\[
f_0^{(v,v)} \ast f_0^{(w,w)} = \begin{cases} f_0^{(v,v)} & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}
\] and
\[
f_0^{(v,v)} \ast f_i^{(x,w)} = \begin{cases} f_i^{(x,v)} & \text{if } x = v \\ 0 & \text{otherwise} \end{cases}
\] and
\[
f_0^{(x,w)} \ast f_0^{(v,v)} = \begin{cases} f_i^{(x,w)} & \text{if } w = v \\ 0 & \text{otherwise} \end{cases}.
\]

**Proof.** This is a good exercise to get the reader acclimated to the notation for subsequent arguments.

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Lemma 6.4 Let \( g \in \text{Hom}_{\Lambda}(D_j, \Lambda/\mathfrak{I}) \). Then a collection of liftings \( \{ u_r \}_{r=0}^{\infty} \) for the product \( f_i^{(v,w)} \ast g \) may be chosen so that for \( 0 \leq k \leq j \), \( u_k(D_{i+k}^v) \) sits in the \( D_k^v \) component of \( D_k \), and \( u_k(D_{i+k}^w) = \{ 0 \} \) for \( x \neq v \).

Proof. The proof is by induction on \( k \). We define \( D_{-1} = \overline{\Lambda} \), and for \( x \in \Gamma_0 \), \( D_{-1}^x = M(x) \). We first prove the result for \( k = 0 \). By definition \( f_i^{(v,w)}(D_i^x) = \{ 0 \} \) for \( x \neq v \), and \( f_i^{(v,w)}(D_i^v) \) sits in the \( D_{-1}^w \) component of \( D_{-1} \). Hence \( u_0 \) must be chosen so that \( d_0(u_0(D_i^v)) = \{ 0 \} \) for \( x \neq v \) and \( d_0(u_0(D_i^v)) \) sits in the \( D_{-1}^w \) component of \( D_{-1} \). Therefore, since \( d_0 = \sum_{x \in \Gamma_0} hS_0^w, hS_0^w(D_0^x) = \{ 0 \} \) for \( x \neq w \), and \( hS_0^w(D_0^w) \) sits in the \( D_{-1}^w \) component of \( D_{-1} \), \( u_0(D_i^v) \) is forced to lie in the \( D_0^w \) component of \( D_0 \), and \( u_0 \) may be chosen so that \( u_0(D_i^v) = \{ 0 \} \) for \( x \neq v \).

We now induct. Assume that we have the result for \( k = s \). Then the proof for \( k = s + 1 \) is nearly identical to the one just given for \( k = 0 \). Simply replace \( f_i^{(v,w)} \) in the argument with \( d_{i+s+1} \circ u_s \) and adjust the indices accordingly.

We call the lifting of Lemma 6.4 a standard lifting for the product.

Corollary 6.5 \( f_i^{(v,x)} \ast f_j^{(y,w)} = 0 \) when \( x \neq y \).

Proof. Let \( \{ u_k \}_{k=0}^{\infty} \) be a standard lifting for the product. By Lemma 6.4 \( u_j(D_{i+j}^z) = \{ 0 \} \) for \( z \neq v \), and \( u_j(D_{i+j}^v) \) sits in the \( D_j^v \) component of \( D_j \). Therefore, since \( f_j^{(y,w)} \) only nontrivially projects the vertex \( w \) that appears in \( D_j^y \) to \( \Lambda/\mathfrak{I} \), when \( y \neq w \), we must have \( u_j \circ f_j^{(y,w)} = 0 \). This gives us the result.

Corollary 6.6

Let \( \{ u_k \}_{k=0}^{\infty} \) be a standard lifting for the product \( f_i^{(v,x)} \ast f_j^{(x,w)} \in \text{Ext}(\Lambda) \). If \( u_j(0, \ldots, 0, w_{(v,w)}, 0, \ldots, 0) = (\text{other components}, c \overline{w}_{(x,w)}, \text{other components}) \) where \( c \) is a nonzero constant, then \( f_i^{(v,x)} \ast f_j^{(x,w)} = c f_i^{(v,w)} \). Otherwise \( f_i^{(v,x)} \ast f_j^{(x,w)} = 0 \).
Proof. This follows from Lemma 6.4 and because the only elements not in the kernel of $f_j^{(x,w)}$ must have a scalar multiple of $w$ in the $w\Lambda$ component of $D_J^x$. ■

It is well known that $\Lambda$ is self-injective and of finite representation type and hence $\text{Ext}(\Lambda)$ is finitely generated as an algebra. Therefore we can define $H$ to be a minimal set of generators for $\text{Ext}(\Lambda)$ which is also a subset of the standard basis. We will show that $H$ is unique.

Lemma 6.7 $f_k^{(v,w)} \notin H$ if and only if $cf_k^{(v,w)} = f_i^{(v,x)} * f_j^{(x,w)}$ for some $i, j$ with $i + j = k$, and $c \neq 0$.

Proof. This follows from Corollaries 6.5 and 6.6 and because $H$ is a subset of a basis. ■

Corollary 6.8 All standard basis vectors in $\text{Hom}_\Lambda(D_0, \Lambda_L)$ and $\text{Hom}_\Lambda(D_1, \Lambda_L)$ are in $H$.

Proof. Apply Lemmas 6.3 and 6.7. ■

Lemma 6.9 Let $\{u_k\}_{k=0}^\infty$ be a standard lifting for $f_i^{(\alpha,\beta)} * f_j^{(\beta,\gamma)} \in \text{Ext}(\Lambda)$. For some fixed $k$, $0 \leq k < j$, suppose $v \in G_{i+k}^\alpha$, $w \in G_{i+k+1}^\alpha$, $x \in G_k^\beta$, $y \in G_{k+1}^\beta$, $u_k((0, \ldots, 0, v(\alpha,v), 0, \ldots, 0)) = (\text{other components}, (c_1 \overline{a})_{(\beta, x)}, \text{other components})$, $d_{i+k+1}((0, \ldots, 0, w(\alpha,w), 0, \ldots, 0)) = (\text{other components}, (c_2 \overline{b})_{(\alpha, v)}, \text{other components})$, and $d_{k+1}((0, \ldots, 0, y(\beta, y), 0, \ldots, 0)) = (\text{other components}, (c_3 \overline{c})_{(\beta, x)}, \text{other components})$ where $a$ is a path of any length (including 0) that does not contain a relation, and $b, g$ are paths of length greater than 0. Also assume that $v, w, x, y, a, b, g$ all lie in the same cycle, and $c_1, c_2, c_3$ are constants not equal to 0.
1. If \( ab \) is a path that does not contain a maximal path, then \( u_{k+1}((0, \ldots, 0, w_{(\alpha, \omega)}, 0, \ldots, 0)) = (\text{other components}, (c_4 \overline{h})_{(\beta, y)}, \text{other components}) \) where \( h \) is the path such that \( gh = ab \), and \( c_4 \) is the constant such that \( c_3 c_4 = c_1 c_2 \).

**Proof.** We have \( u_k(d_{i+k+1}((0, \ldots, 0, v_{(\alpha, \omega)}, 0, \ldots, 0))) = (\text{other components}, (c_1 c_2 ab + P)_{(\beta, x)}, \text{other components}) \) where \( P \) is a path not in the same cycle as \( ab \). Also, since \( h S_{k+1}^\theta \) is a string function \( d_{k+1} \) maps at most two components of \( D_{k+1}^\theta \) to the \( x \Lambda \) component of \( D_k^\theta \). However, only one of those (possibly) two components, \( z \Lambda \), is such that \( z \) is in the same cycle as \( ab \). That component is the \( y \Lambda \) component. Therefore, since \( ab \) does not contain a maximal path, and since \( d_{k+1}((0, \ldots, 0, y_{\gamma}, 0, \ldots, 0)) = (\text{other components}, (c_3 \overline{y})_{x^\beta}, \text{other components}), \) we must have \( u_{k+1}((0, \ldots, 0, w_{\omega}, 0, \ldots, 0)) = (\text{other components}, (c_4 \overline{h})_{y^\beta}, \text{other components}) \), where \( h \) is a path such \( gh = ab \), and \( c_4 \) is a constant such that \( c_3 c_4 = c_1 c_2 \). 

Let \( h S \) be a string function. Then we will say that the vertices \( o(S) \) and \( t(S) \) are the end vertices of \( h S \). We will refer to all other vertices, \( w \), appearing in the underlying string for \( h S \), and with \( v \Lambda \) appearing in the domain for \( h S \) as middle vertices of \( h S \). We note that we will allow a vertex that appears twice in the underlying string to be both a middle vertex and an end vertex. However, we will say that an end vertex, \( v \), is an upper end vertex if \( v \Lambda \) appears in the domain for \( h S \). You will note that by Lemma 6.1 upper end vertices are never middle vertices. Note also that both of the cycles that a middle vertex lies within must have length at least 2. By the **length of a cycle** we mean the number of arrows in the cycle if the cycle is not exceptional and \( N \cdot (\text{the number of arrows in the cycle}) \) otherwise.

Figure 6.2 we supply as a reference for Theorem 6.10 and Lemma 6.11. We never
refer to it and simply intend for the reader to use it as a thinking device.

\[
\begin{align*}
D_k & \xrightarrow{d_k} D_{k-1} \xrightarrow{d_{k-1}} D_{k-2} \\
\downarrow u_2 & \quad \downarrow u_1 \quad \downarrow u_0 \quad \nabla f_{k-2}^{(v,w)} \\
D_2 & \xrightarrow{d_2} D_1 \xrightarrow{d_1} D_0 \xrightarrow{d_0} \overline{\lambda}
\end{align*}
\]

Figure 6.2:

For the following lemma we let the \textit{length of a cycle} be the number of arrows in the cycle when the cycle is not the exceptional cycle, and \(N\) (the number of arrows in the cycle) when the cycle is the exceptional cycle with exceptional number \(N\).

\textbf{Theorem 6.10} Let \(v \in G_2^w\) with \(v\) a middle vertex of \(hS_2^w\). Then \(f_2^{(v,w)} \notin H\) if and only if one of the two cycles that \(v\) sits in has length two.

\textbf{Proof.} Since \(v\) is a middle vertex of \(hS_2^w\), each cycle that \(v\) sits in must have length greater than 1. Let \(a\) and \(b\) be the two arrows with origin \(v\). Let \(a'\) and \(b'\) be the paths (not vertices) such that \(aa'\) and \(bb'\) are maximal paths. Using this notation we draw part of \(hS_1^w\) and \(hS_2^w\) in figure 6.3.
To argue both directions of the 'if and only if' statement we will use the following fact. By Lemmas 6.3 and 6.7 $f_2^{(v,w)} \notin H$ if and only if $f_2^{(v,w)} = cf_1^{(v,w)} * f_1^{(w,v)}$ for some $w \in \Gamma_0$ and some constant $c \neq 0$. Without loss of generality we first assume that the cycle containing $aa'$ has length two. Then $a'$ is an arrow and we have figure 6.4.

Let $\{u_k\}_{k=0}^{\infty}$ be a standard lifting for $f_1^{(v,x)} * f_1^{(x,v)}$. Since $u_0((0, \ldots, 0, x_{(v,v)}, 0, \ldots, 0)) = (0, \ldots, 0, x_{(x,x)}, 0, \ldots, 0)$, by Lemma 6.9 $u_1((0, \ldots, 0, v_{(v,v)}, 0, \ldots, 0)) = (0, \ldots, 0, v_{(x,v)}, 0, \ldots, 0)$. Hence by Corollary 6.4 $f_1^{(v,x)} * f_1^{(x,v)} = f_2^{(v,w)}$. This gives us that $f_2^{(v,w)} \notin H$. 

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We now assume that both cycles which \( v \) sits in have length greater than two. Let \( \alpha \) and \( \beta \) be the first arrows of the paths \( a' \) and \( b' \), and let \( A \) and \( B \) be the paths such that \( \alpha A = a' \) and \( \beta B = b' \). Using this notation we give \( hS^v_1 \) and \( hS^v_1 \) in figure 6.5.

\[
\begin{align*}
\text{Figure 6.5:} \\
 & hS^v_1 : \\
& \downarrow \alpha \\
x & \downarrow t(\alpha) \\
\text{\hspace{2cm}} & \downarrow t(\beta) \\
y & hS^v_1 : \\
& \downarrow \beta
\end{align*}
\]

Therefore, \( f_2^{(v,v)} \) will be in \( H \) unless either \( f_2^{(v,v)} = c f_1^{(v,x)} \ast f_1^{(x,v)} \) or \( f_2^{(v,v)} = c f_1^{(v,y)} \ast f_1^{(y,v)} \) where \( c \neq 0 \). We show that the first equality is not possible, and symmetry gives us that the second equality is not possible. Let \( \{ u_k \}_{k=0}^\infty \) be a standard lifting for \( f_1^{(x,z)} \ast f_1^{(z,v)} \). Since \( u_0((0, \ldots, 0, x_{(x,z)}, 0, \ldots, 0) = (0, \ldots, 0, x_{(z,v)}, 0, \ldots, 0) \), by Lemma 6.9 we have \( u_1((0, \ldots, 0, v_{(v,v)}, 0, \ldots, 0) = (\text{other components}, A_{(x,v)}, \text{other components}) \). Thus by Corollary 6.4 we have \( f_1^{(x,v)} \ast f_1^{(x,v)} = 0 \).

**Lemma 6.11** For \( k > 2 \), if \( w \in G^v_k \), where \( w \) is a middle vertex of \( hS^v_k \), then \( f_k^{(v,w)} = f_{k-2}^{(v,w)} \ast f_2^{(w,w)} \). Hence \( f_k^{(v,w)} \notin H \).

**Proof.** We first address the existence of \( f_2(w, w) \) and \( f_{k-2}^{(v,w)} \). Since \( w \) is a middle vertex, a quick study of its resolution using Corollary 5.1 establishes the existence of \( f_2^{(w,w)} \). Similarly, Corollary 5.1 can be used to see that if the middle vertex \( w \) appears in \( D^v_k \), it must also have appeared in \( D^v_{k-2} \). Thus \( f_{k-2}^{(v,w)} \) must exist.
In figure 6.6 we illustrate a portion of the string functions $hS_{k-1}^v, hS_k^v, hS_1^w, hS_2^w$. There we are letting $P, Q$ be paths of length greater than zero in the same cycles as the arrows $a$ and $b$ respectively. $P', Q', a'$, and $b'$ will be the paths such that $PP', QQ', aa'$, and $bb'$ are maximal paths. Also we let $c = \pm 1$. As you consider the figure recall that if we choose all of the coefficients in the underlying string for the first homomorphism of a resolution to be $+1$, then the coefficients of all subsequent underlying strings are either all $+1$'s or alternate between $+1$ and $-1$. 
Figure 6.6:
From figure 6.6 we see that
\[ d_1((0, \ldots, 0, \overline{x}_{(w,z)}, 0, \ldots, 0)) = (0, \ldots, 0, \overline{b}_{(w,w)}, 0, \ldots, 0), \]
\[ d_2((0, \ldots, 0, \overline{w}_{(w,w)}, 0, \ldots, 0)) = (0, \ldots, 0, \overline{b}_{(w,z)}' - \overline{a}_{(w,y)}, 0, \ldots, 0), \]
\[ d_{k-1}((0, \ldots, 0, \overline{m}_{(v,m)}, 0, \ldots, 0)) = \text{(other components, } \overline{Q}_{(w,v)}', \text{other components}) \]
and
\[ d_k((0, \ldots, 0, \overline{w}_{(v,w)}, 0, \ldots, 0)) = (0, \ldots, 0, \overline{Q}_{(v,m)}', -c\overline{P}_{(v,v)}, 0, \ldots, 0). \]

Note that \( u_0((0, \ldots, 0, \overline{w}_{(v,w)}, 0, \ldots, 0)) = (0, \ldots, 0, \overline{w}_{(v,w)}, 0, \ldots, 0) \), and apply Lemma 6.9 to see that
\[ u_1((0, \ldots, 0, \overline{m}_{(v,m)}, 0, \ldots, 0)) = \]
\( \text{(other components, } \overline{B}_{(w,z)}, \text{other components)} \) where \( B \) is the path such that \( bB = wQ = Q \). Since \( BQ' = B' \), we can reapply Lemma 6.9 to see that
\[ u_1((0, \ldots, 0, \overline{w}_{(v,w)}, 0, \ldots, 0)) = \text{(other components, } \overline{w}_{(w,w)}, \text{other components)} \). By Corollary 6.4 we have the result.

Following Lemma 6.11, the only standard basis vectors whose presence in \( H \) have not be determined algorithmically are of the form \( f_{k}^{(v,w)} \) where \( w \) is an upper end vertex of \( hS_k \) for \( k \geq 2 \). Those algorithmic conditions are given in Corollary 6.19. Actually the conditions given in that Corollary are for determining when a standard basis vector of the form \( f_{k}^{(l_1,k)} \) (or \( f_{k}^{(r_0,r_k)} \)) is in \( H \) where \( (x_m, l_m, y_m, r_m)_{m=0}^{\infty} \) is a walking sequence for \( M(1_{l_0}) \) and \( x_m = \text{up} \) (or \( y_m = \text{up} \)). However those are exactly the vectors of the form \( f_{k}^{(v,w)} \) where \( w \) is an upper end vertex of \( hS_k \). To obtain Corollary 6.19 we are going to construct algorithmic conditions for determining when the standard basis vectors of the form \( f_{k}^{(l_1,k)} \) (or \( f_{k}^{(r_0,r_k)} \)) are elements of the minimal generating set \( H^* \) for \( \text{Ext}(\Lambda^*) \) where \( (x_m, l_m, y_m, r_m)_{m=0}^{\infty} \) is a walking sequence for \( M(1_{l_0}) \in \text{Mod}(\Lambda^*) \) and \( x_k = \text{up} \) (or \( y_k = \text{up} \)). Lemma 6.12 justifies this approach. From Lemma 6.13 to Theorem 6.18 we are going to leave off the superscripts for the vertices of \( \Gamma^* \) when no confusion can arise. For that portion of the paper all
standard basis vectors will be assumed to be in $\text{Ext}(\Lambda^*)$.

**Lemma 6.12** Let $v \in G^v_k \subseteq \Gamma$, $x \in G^x_i \subseteq \Gamma$, and $w \in G^w_j \subseteq \Gamma$. Then $c_{f_k}^{(v,w)} = f_i^{(v,x)} \ast f_j^{(x,w)}$ if and only if $c_{f_k}^{(v,w)} = f_i^{(v,x_1)} \ast f_j^{(x_2,w_1)}$, where $\bar{1}_1$ and $\bar{1}_2$ are the unique elements of $Z_N$ such that $w^{\bar{1}_1} \in G^v_k \subseteq \Gamma^*$ and $x^{\bar{1}_2} \in G^x_i \subseteq \Gamma^*$.

**Proof.** Note that by Lemma 6.1 $\bar{1}_1$ and $\bar{1}_2$ are unique. Let $\{hS^w_m\}_{m=0}^{\infty}$ and $\{hS^v_m\}_{m=0}^{\infty}$ be the string functions in a minimal projective resolution of $M(1_v) \in \text{Mod}(\Lambda)$ and $M(1_v) \in \text{Mod}(\Lambda^*)$ respectively. Let $\{u_m\}_{m=0}^{\infty}$ and $\{u^*_m\}_{m=0}^{\infty}$ be standard liftings for $f_i^{(v,x)} \ast f_j^{(x,w)}$ and $f_i^{(v,x_1)} \ast f_j^{(x_2,w_1)}$ respectively.

By the proof of Lemma 3.3 if we disregard the superscripts in a string $S^v_m$, then the strings $S^v_m$ and $S^w_m$ are the same. Similarly if we disregard the superscripts in the $h$-string $hS^v_m$, then the $h$-strings $hS^v_m$ and $hS^w_m$ are the same. Since by Section 3 $E(M(S^v_m)) = M(S^w_m)$ for all $v$ and $m$, $u^*_m$ and $u_m$ will be the same if we disregard superscripts, and that is the essence of the proof. □

**Lemma 6.13** Let $\{(x_m, l_m, y_m, r_m)\}_{m=0}^{\infty}$ be a walking sequence for $M(1_v) \in \text{Mod}(\Lambda^*)$.

1. $l_s$ is not in $l_{s-1} \leftrightarrow r_{s-1}$ if and only if $r_{s-1}$ is not in $\Gamma_{l_s, l_{s+1}}^*$. 

2. $r_s$ is not in $l_{s-1} \leftrightarrow r_{s-1}$ if and only if $l_{s-1}$ is not in $\Gamma_{r_s, r_{s+1}}^*$.

**Proof.** Since $l_{s-1} \leftrightarrow r_{s-1}$ is an edge walk on the underlying Brauer tree, and since it never doubles back on itself, it is clear from the diagram in figure 6.7 that the vertex $r_{s-1} \in \Gamma_{l_{s-1}, l_{s+1}}^*$ if and only if $l_s \in l_{s-1} \leftrightarrow r_{s-1}$. The proof of the second statement follows in a similar fashion. □

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Recall that a short $\Gamma^*$-walking sequence $\{v, v_1, \ldots, v_n, w\}$ is a $\Gamma^*$-walking sequence in which $v, w \notin \{v_1, \ldots, v_n\}$.

**Lemma 6.14** In the construction of the walking sequence, $\{(x_m, l_m, y_m, r_m)\}_{m=0}^{\infty}$, for $M(l_0) \in \text{Mod}(\Lambda^*)$ we may choose $x_s = \text{up}$ if and only if $|\Gamma^*_{l_0l_1}| \geq |\Gamma^*_{l_sl_{s+1}}|$. Similarly we may choose $y_s = \text{up}$ if and only if $|\Gamma^*_{r_0r_1}| \geq |\Gamma^*_{r_sr_{s+1}}|$.

**Proof.** We prove the first statement. The second statement can be proven similarly. By Corollary 5.6, whenever $l_s = r_s$, we may choose $x_s = \text{up}$. Using the definition of a short $\Gamma^*$-walking sequence, one can see that $l_s = r_s$ if and only if the short $\Gamma^*$-walking sequence $\{l_s, l_{s+1}, \ldots, l_s\}$ and $\{l_s, l_{s+1}, \ldots, r_s\}$ have the same length. However, the length of $\{l_s, l_{s+1}, \ldots, r_s\}$ is the same as the length of $\{l_0, l_1, \ldots, r_0\}$ which is $|\Gamma^*_{l_0l_1}|$, and the length of $\{l_s, l_{s+1}, \ldots, l_s\}$ is $|\Gamma^*_{l_sl_{s+1}}|$. Therefore we have that $l_s = r_s$ if and only if $|\Gamma^*_{l_0l_1}| = |\Gamma^*_{l_sl_{s+1}}|$.

Suppose $l_s \neq r_s$. Then by Corollary 5.6 $x_s = \text{up}$ if and only if $l_s$ does not appear in $l_{s-1} \leftrightarrow r_{s-1}$. Using Lemma 6.13 we then have $x_s = \text{up}$ if and only if $r_{s-1}$ is not in $\Gamma^*_{l_sl_{s+1}}$. In addition, $r_{s-1}$ not appearing in $\Gamma^*_{l_sl_{s+1}}$ is equivalent to $r_{s-1}$ not appearing in the short $\Gamma^*$-walking sequence $\{l_s, l_{s+1}, \ldots, l_s\}$. However, $r_{s-1}$ is not in the sequence $\{l_s, l_{s+1}, \ldots, l_s\}$ if and only if the short $\Gamma^*$-walking sequence...
\{l_s, l_{s+1}, \ldots, r_{s-1}\} \text{ has length greater than the length of } \{l_s, l_{s+1}, \ldots, l_s\} \text{ which is } 2|\Gamma_{l_{s+1},l_s}^*|. \text{ Since } \{l_s, l_{s+1}, \ldots, r_{s-1}\} \text{ has the same length as } \{l_2, l_3, \ldots, r_0, r_1\} \text{ which is } 2|\Gamma_{l_0,l_1}^*| - 1, \text{ we have } x_s = \text{ up if and only if } 2|\Gamma_{l_0,l_1}^*| - 1 > 2|\Gamma_{l_{s+1},l_s}^*|. \text{ Hence, } x_s = \text{ up if and only if } |\Gamma_{l_0,l_1}^*| \geq |\Gamma_{l_{s+1},l_s}^*|.

Combining our results when \(l_s = r_s\) and those we just obtained when \(l_s \neq r_s\), we have that we may choose \(x_s = \text{ up if and only if } |\Gamma_{l_0,l_1}^*| \geq |\Gamma_{l_{s+1},l_s}^*|. \)

**Corollary 6.15** If we keep the same notation as in Lemma 6.14, and assume that 

\(|\Gamma_{l_0,l_1}^*| < |\Gamma_{l_{s+1},l_s}^*|\), then \(f_s^{(l_0,l_s)} \notin H^\ast\). In the same way we have that if \(|\Gamma_{r_0,r_1}^*| < |\Gamma_{r_{s+1},r_s}^*|\)

then \(f_s^{(r_0,r_s)} \notin H^\ast\).

**Proof.** The result follows immediately from Lemma 6.14. \(\blacksquare\)

**Lemma 6.16** Let \(v \in \Gamma^\ast\). Let \(\{(x_m,l_m,y_m,r_m)\}_{m=0}^\infty\) be a walking sequence for 

\(M(1_v) = M(1_{l_0}) \in \text{Mod}(\Lambda^\ast)\). Let \(k > 1, i + j = k, \text{ and let } c \text{ be a nonzero scalar. If } l_k \neq r_k \text{ and } c f_k^{(v,l_k)} = f_i^{(v,x)} \ast f_j^{(x,l_k)} \text{ and if } \{(x_m, i, y_m, j_m)\}_{m=0}^\infty \text{ is a walking sequence} \text{ for } M(x), \text{ then} \)

1. We may choose \(i_j = i_k \text{ with } x_j = \text{ up,} \)

2. \(|\Gamma_{l_0,l_1}^*| \geq |\Gamma_{i,i+k}^*|\).

3. \(|\Gamma_{i,i+k}^*| \geq |\Gamma_{i,i+k+1}^*|\).

If \(l_k \neq r_k \text{ and } c f_k^{(v,r_k)} = f_i^{(v,x)} \ast f_j^{(x,r_k)} \text{ and if } \{(x_m, i, y_m, j_m)\}_{m=0}^\infty \text{ is a walking sequence for } M(x), \text{ then} \)

1. We may choose \(i_j = i_k \text{ with } x_j = \text{ up,} \)

2. \(|\Gamma_{r_0,r_1}^*| \geq |\Gamma_{i,i+k+1}^*|\).
3. $|\Gamma^*_{r_k r_{k+1}}| \geq |\Gamma^*_{r_k r_{k+1}}|$.

**Proof.** We prove the first set of statements. The other set can be proven using symmetry. Let $\{u_m\}_{m=0}^\infty$ be a standard lifting for the product $f_i^{(v, x)} \ast f_j^{(x, l_k)}$.

We first prove (1). For ease of notation we let $w = l_k$. Since the product $f_i^{(v, x)} \ast f_j^{(x, l_k)} = cf_k^{(v, w)} \neq 0$, we know by Corollary 6.6 that $u_j((0, \ldots, 0, \overline{w(v, w)}, 0, \ldots, 0)) = (\text{other components}, c\overline{w}(x, w), \text{other components})$. Since $l_k$ is an upper end vertex of $hS^v_k$, we know that $d_k((0, \ldots, 0, \overline{w(v, w)}, 0, \ldots, 0)) = (0, \ldots, 0, c_1\overline{a}(v, e(a)), 0, \ldots, 0)$ for some nonzero scalar $c_1$ and path $a$. Hence $u_{j-1}(d_k((0, \ldots, 0, \overline{w(v, w)}, 0, \ldots, 0))) = (0, \ldots, 0, c_2\overline{b}(v, e(a)), 0, \ldots, 0)$ for some nonzero scalar $c_2$ and some path $b$. If $w$ is a middle vertex of $hS^v_j$, then we know $d_j(u_j((0, \ldots, 0, \overline{w(v, w)}, 0, \ldots, 0)) = (0, \ldots, 0, c_3\overline{A}(x, e(A), c_4\overline{B}(x, e(B)), 0, \ldots, 0)$ where $c_3, c_4$ are nonzero scalars, and $A$ and $B$ are paths that do not contain relations, i.e., $\overline{A}, \overline{B} \neq 0$. This leads us to the contradiction $d_j(u_j((0, \ldots, 0, \overline{w(v, w)}, 0, \ldots, 0))) \neq u_{j-1}(d_k((0, \ldots, 0, \overline{w(v, w)}, 0, \ldots, 0)))$.

Thus we can choose $\hat{t}_j = w$ with $\hat{x}_j = \text{up}$.

We now prove (2) and (3). Using (1) we now choose the walking sequence, $\{(\hat{x}_m, \hat{t}_m, \hat{y}_m, \hat{r}_m)\}_{m=0}^\infty$, for $M(x)$ so that $\hat{t}_j = w = t_{i+j} = l_k$. And $\hat{x}_j = \text{up}$. We will show that the walking sequence $\{(\hat{x}_m, \hat{t}_m, \hat{y}_m, \hat{r}_m)\}_{m=0}^\infty$ can in fact be chosen so that the sequences $\{\hat{t}_m\}_{m=0}^\infty$ and $\{l_{i+j}\}_{m=0}^\infty$ are identical with $x_j = \text{up}$. Since they are walking sequences, it suffices to show that the sequences $\{\hat{l}_{j-1}, \hat{t}_j\}$ and $\{l_{j-1}, l_j\}$ can be chosen to be identical.

In figure 6.8 we draw part of the underlying strings for $hS^v_{i+j}$ and $hS^r_j$ we do this by appealing to the following facts: $w = l_k$ and $x_k = \text{up}$, we may choose $\hat{l}_j = w$ with $\hat{x}_j = \text{up}$, while $u_j((0, \ldots, 0, \overline{w(v, w)}, 0, \ldots, 0)) = (\text{other components}, c\overline{w}(x, w), \text{other components})$ by Corollary 6.6. In the figure $c_5$ and $c_6$ are nonzero scalars. Since $l_k \neq r + k$, $P$ is not a maximal path. If $Q$ is not a
maximal path, then by Lemma 6.9 \( P \) and \( Q \) must lie in the same cycle. Hence, by Corollary 5.1 \( l_{i+j-1} \) (resp. \( \hat{l}_{j-1} \)) must be the unique vertex such that \( l_{i+j} \) (resp. \( \hat{l}_j \)) is one arrow away from \( l_{i+j-1} \) (resp. \( \hat{l}_{j-1} \)) in the same cycle as \( P \) (resp. \( Q \)). Since \( P \) and \( Q \) are in the same cycle and since \( \hat{l}_j = l_{i+j} \), we have \( \hat{l}_{j-1} = l_{i+j-1} \). Therefore the walking sequence \( \{(\hat{x}_m, \hat{l}_m, \hat{y}_m, \hat{r}_m)\}_{m=0}^{\infty} \) can be chosen so that \( \{\hat{l}_m\}_{m=0}^{j} = \{l_{i+m}\}_{m=0}^{j} \) are identical.

As an immediate consequence we have that \( l_i = \hat{l}_0 = x \). By Lemma 6.1 the vertices appearing in an underlying string for a module in \( \text{Mod}(\Gamma^*) \) are distinct. Therefore, since \( f^{(n,x)}_i \) is defined, we know that \( z_i = \text{up} \). By Lemma 6.14 we conclude that \( |\Gamma_{0i}| \geq |\Gamma_{li+1}| \). We have proven (2).

We had shown earlier that \( \hat{x}_j = \text{up} \). So again using that \( \{\hat{l}_m\}_{m=0}^{i} = \{l_{i+m}\}_{m=0}^{i} \) are identical and Lemma 6.14 we have that \( |\Gamma_{li+1}||\Gamma_{0i}| \geq |\Gamma_{j,i+1}||\Gamma_{k,i+1}| \). We have now proven (3).

\[ \begin{array}{c}
\text{Figure 6.8:}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
hS^w_{i+j} : \\
w
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
hS^w_j : \\
w
\end{array}
\end{array} \]

\[ \begin{array}{c}
\text{Lemma 6.17 Let } v \in \Gamma^* \text{ and let } \{(x_m, l_m, y_m, r_m)\}_{m=0}^{\infty} \text{ be a } \Gamma^* \text{-walking sequence for } M(1_v) \in \text{Mod}(\Lambda^*) \text{.}
\end{array} \]

1. If in accordance with Corollary 5.6 we can choose \( x_k = \text{up} \) and for some \( i < k \), we can choose \( x_i = \text{up} \). Let \( j = k - i \). Choose \( \{(\hat{x}_m, \hat{l}_m, \hat{y}_m, \hat{r}_m)\}_{m=0}^{\infty} \) be the
walking sequence for $M(1_{i_0})$ such that $\{\hat{i}_m\}_m=0^\infty = \{i_{i+m}\}_m=0^\infty$. If in accordance with Corollary 5.6 we can choose the sequences $\{\hat{x}_m\}_m=0^\infty$ and $\{i_{i+m}\}_m=0^\infty$ to be identical with $x_i = x_k = u_p$, then for some nonzero scalar $c$ we have $cf_k^{(x_0, i_k)} = f_i^{(x_0, i_i)} * f_j^{(x_0, i_j)} = f_i^{(x_0, i_i)} * f_j^{(x_0, i_j)}$.

2. If in accordance with Corollary 5.6 we can choose $y_k = u_p$ and for some $i < k$, we can choose $y_i = u_p$. Let $j = k - i$. Choose $\{(\hat{x}_m, \hat{i}_m, \hat{y}_m, \hat{r}_m)\}_m=0^\infty$ be the walking sequence for $M(r_i)$ such that $\{\hat{r}_m\}_m=0^\infty = \{r_{i+m}\}_m=0^\infty$. If in accordance with Corollary 5.6 we can choose the sequences $\{\hat{y}_m\}_m=0^\infty$ and $\{y_{i+m}\}_m=0^\infty$ to be identical with $y_i = y_k = u_p$, then for some nonzero scalar $c$ we have $cf_k^{(y_0, r_k)} = f_i^{(y_0, r_i)} * f_j^{(y_0, r_j)} = f_i^{(y_0, r_i)} * f_j^{(y_0, r_j)}$.

**Proof.** We prove (1). (2) follows using symmetry. For notations sake we let $u = \hat{i}_0 = \hat{i}_i$. By Corollary 6.6 we only need to show that $u_j((0, \ldots, 0, i_{k(u_i)}, 0, \ldots, 0)) = (\text{other components, } \hat{c}_{k(u_i)}, \text{other components})$. Using induction on $m$ we will prove the following more general statement. For $0 \leq m \leq j$ if $x_{i+m} = u_p$, then $u_m((0, \ldots, 0, i_{i+m(v_i, i+m)}, 0, \ldots, 0)) = (\text{other components, } c_m i_{i+m(v_i, i+m)}, \text{other components})$, and if $x_{i+m} = d_p$, then $u_{m-1}((0, \ldots, 0, i_{i+m(v_i, i+m)}, 0, \ldots, 0)) = (\text{other components, } c_{m-1} i_{i+m(v_i, i+m)}, \text{other components})$. We know from experience that $u_0((0, \ldots, 0, \hat{i}((v_i, i), 0, \ldots, 0)) = (0, \ldots, 0, \hat{i}(v_i, i), 0, \ldots, 0)$. Since $x_i = u_p$, we have that the general statement when $m = 0$. We now assume that it is true for $m = t$, $0 \leq t < j$.

We consider four cases, and, in each case, we appeal to figures 5.1 – 5.10 of Corollary 5.1 to create parts of the underlying strings $hS_{i+t}^w, hS_{i+t+1}^w, hS_t^w$, and $hS_{t+1}^w$ that occur in that case. We draw these in figures 6.9 – 6.12. Figure 6.9 represents
the case where \( x_{i+t} = \text{up} \) and \( x_{i+t+1} = \text{down} \). Figure 6.10 represents the case where \( x_{i+t} = \text{up} \) and \( x_{i+t+1} = \text{up} \). Figure 6.9 represents the case where \( x_{i+t} = \text{down} \) and \( x_{i+t+1} = \text{down} \). Figure 6.9 represents the case where \( x_{i+t} = \text{down} \) and \( x_{i+t+1} = \text{up} \). In the first case the inductive assumption gives us the result immediately. In the next two cases only one application of Lemma 6.9 gives us the result. In the last case we need to apply Lemma 6.9 twice in order to obtain the result. We will only write down the proof for the last case.

In this case since \( x_{t+i} = \text{down} \), we know that

\[
u_{t-1}((0, \ldots, 0, \overline{1}_{t+i}(v, l_{t+i}), 0, \ldots, 0)) =
\]

(other components, \( c_t \overline{l}_{t+i}(w, l_{t+i}) \), other components). By Lemma 6.9 we have that

\[
u_t((0, \ldots, 0, \overline{y}_{v,y}, 0, \ldots, 0)) =
\]

(other components, \( k_3 \overline{B}_{(w,z)} \), other components). where \( k_3 \) is some nonzero scalar and \( B \) is the path such that \( P = QB \). Note that \( (BP')a = (Q'a) \). Hence, we can reapply Lemma 6.9 to see that

\[
u_{t+i+1}((0, \ldots, 0, \overline{1}_{t+i+1}(v, l_{t+i+1}), 0, \ldots, 0)) =
\]

(other components, \( c_{t+i+1} \overline{l}_{t+i+1}(x, l_{t+i+1}) \), other components). Since \( x_{t+i+1} = \text{up} \), we have proven the inductive step of the argument.

\[
\begin{align*}
\text{Figure 6.9:} \\
hS_{t+i+1} : & \quad hS^w_{t+i+1} : \\
l_{t+i+1} = l_{t+i+1} & \quad l_t = l_{t+1} \\
hS^w_{t+i} : & \quad hS^w_t : \\

\end{align*}
\]
Figure 6.10: We let $k_1, k_2$ be nonzero scalars, and let $\alpha$ be an arrow.

Figure 6.11: We let $k_1, k_2$ be nonzero scalars, and let $\alpha$ be an arrow.
Figure 6.12: $k_1, k_2$ are nonzero scalars, and $P$ and $Q$ are paths lying in the same cycle which do not contain any relations. $a$ is the first arrow of $P$, and $P'$ and $Q'$ are the paths (or vertices) such that $PP'$ and $QQ'$ are maximal paths.


Theorem 6.18 Let $\{(x_m, l_m, y_m, r_m)\}_{m=0}^{\infty}$ be a walking sequence for $M(1_{l_0}) \in \text{Mod}(\Gamma^*)$.

1. $f^{(l_0,d_k)}_k \notin H^*$ if and only if $|\Gamma^*_{l_0,l_1}| < |\Gamma^*_{l_0,k+l_1}|$ or there is an $i$, $0 < i < k$ such that $|\Gamma^*_{l_0,i}| \geq |\Gamma^*_{l_0,i+1}| \geq |\Gamma^*_{l_0,k+i+1}|$.

2. $f^{(r_0,r_k)}_k \notin H^*$ if and only if $|\Gamma^*_{r_0,r_1}| < |\Gamma^*_{r_0,r_{k+1}}|$ or there is an $i$, $0 < i < k$ such that $|\Gamma^*_{r_0,i}| \geq |\Gamma^*_{r_0,i+1}| \geq |\Gamma^*_{r_0,r_{k+1}}|$.

Proof. We will prove (1). (2) follows from symmetry. Suppose $|\Gamma^*_{l_0,l_1}| < |\Gamma^*_{l_0,k+l_1}|$. Then by Corollary 6.15 $f^{(l_0,d_k)}_k \notin H$. Suppose, on the other hand, that there is an $i$, $0 < i < k$ such that $|\Gamma^*_{l_0,l_1}| \geq |\Gamma^*_{l_0,i+1}| \geq |\Gamma^*_{l_0,k+i+1}|$. Then we can find a $t$, $0 < t < k$ such that $|\Gamma^*_{l_0,t}| \geq |\Gamma^*_{l_0,t+1}| \geq |\Gamma^*_{l_0,k+t+1}|$ for all $j$ where $0 < j < k$, and $|\Gamma^*_{l_0,l_1}| \geq |\Gamma^*_{l_0,k+1}|$. Let $\{(x_m, l_m, y_m, r_m)\}_{m=0}^{\infty}$ be the walking sequence for $M(1_{l_1})$ such that $\{l_m\}_{m=0}^{\infty}$ and

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\{\hat{l}_{m+t}\}_{m=0}^{\infty} are the same sequence. By Lemma 6.17 we can show that \(f_k^{(l_0,l_k)} \notin H^*\) by showing that we can choose the sequences \(\{\hat{x}_a\}_{a=0}^{k-t} \) and \(\{x_{t+a}\}_{a=0}^{k-t} \) to be the same. Let \(0 < a \leq k - t\). By our choice of \(t\) we have \(|\Gamma_{l_{i+1}}^*| \geq |\Gamma_{l_{i}+a,l_{i+1}+1}^*|\) if and only if \(|\Gamma_{l_{i+1}}^*| \geq |\Gamma_{l_{i}+a,l_{i+1}+1}^*|\). Therefore we have \(|\Gamma_{l_{i+1}}^*| \geq |\Gamma_{l_{i}+a,l_{i+1}+1}^*|\) if and only if \(|\Gamma_{l_{i+1}}^*| \geq |\Gamma_{l_{i}+a,l_{i+1}+1}^*|\). Hence by Lemma 6.14, for \(0 < a \leq k - t\), we can choose \(x_{t+a} = \hat{x}_a\) to be up if and only if we can choose \(\hat{x}_a = \) up. In addition since \(|\Gamma_{l_{i+1}}^*| \geq |\Gamma_{l_{i}+a,l_{i+1}+1}^*|\), we can choose \(\hat{x}_{k-t} = x_k = \) up. Since \(|\Gamma_{l_{i+1}}^*| \geq |\Gamma_{l_{i}+a,l_{i+1}+1}^*|\), we can also choose \(x_t = \hat{x}_0 = \) up. Finally, since we can always choose \(\hat{x}_0 = \) up, we have \(f_k^{(l_0,l_k)} \notin H^*\). We have one direction of (1).

Suppose \(f_k^{(l_0,l_k)} \notin H^*\) and \(|\Gamma_{l_{i+1}}^*| \geq |\Gamma_{l_{i}+a,l_{i+1}+1}^*|\). If \(l_k \neq r_k\), then by Lemma 6.16 we have for some \(i, 0 < i < k, |\Gamma_{l_{i+1}}| \geq |\Gamma_{l_{i}+a,l_{i+1}+1}| \geq |\Gamma_{l_{i+k+1}}|\). So suppose \(l_k = r_k\), and that \(cf_k^{(l_0,l_k)} = f_i^{(l_0,x)} \ast f_j(x_{l_k})\). Then by (1) of Lemma 6.16 we may choose \(\hat{l}_j = l_k\) with \(\hat{x}_j = \) up. Therefore partial underlying strings for \(hS_j^\pi\) and \(hS_j^\pi_k\) and \(hS_j^\pi\) are as given in figure 6.8 with the dotted portions omitted. By Lemma 6.9, since \(u_j(0,\ldots,0,\overline{c_{l_{i+1}}(x_{l_k})},0,\ldots,0) = (0,\ldots,0,\overline{c_{l_{i+1}}(x_{l_k})},0,\ldots,0)\), and since \(P\) is maximal, we have \(Q\) is maximal. Thus \(hS_j^\pi_k = hS_j^\pi\) and hence \(hS_j^\pi_k = hS_j^\pi\) for \(0 \leq t \leq j\). Thus the walking sequence for \(M(x)\), \((\hat{x}_{m}, \hat{l}_{m}, \hat{y}_{m}, \hat{r}_{m})\), can be chosen so that the sequences \(\{(\hat{x}_m, \hat{l}_m, \hat{y}_m, \hat{r}_m)\}_{m=0}^{\infty}\) and \(\{(x_{m+i}, l_{m+i}, y_{m+i}, r_{m+i})\}_{m=0}^{\infty}\) are the same with \(l_i = \hat{l}_0 = \hat{r}_0 = r_i\). By the proof of Lemma 6.14, since \(l_i = r_i\) and \(l_k = r_k\), we have \(|\Gamma_{l_{i+1}}| = |\Gamma_{l_{i+1}}^*|\) and \(|\Gamma_{l_{i+1}}^*| = |\Gamma_{l_{i+k+1}}^*|\). Thus we also have the result when \(l_k = r_k\).

Let \(N\) be the exceptional number for \(\Gamma\). We define

\[
||\Gamma_{vw}|| = \begin{cases} 
|\Gamma_{vw}| & \text{if } \Gamma_{vw} \text{ does not contain the exceptional cycle} \\
|\Gamma_{vw}| + (N - 1)|\Gamma| & \text{otherwise.}
\end{cases}
\]

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Corollary 6.19 Let \( \{x_m, \bar{r}_m, y_m, r_m \} \) be a walking sequence for \( M(1, r_m) \in Mod(\Lambda^*) \). Then

1. \( f_k(l_0, t_k) \notin H \subseteq Ext(\Lambda) \) if and only if \( |\Gamma_{l_0l_1}| \geq |\Gamma_{k kl_{k+1}}| \) or there is an \( i, 0 < i < k \) such that

\[
|\Gamma_{l_0l_1}| \geq |\Gamma_{i_1i_{i+1}}| \geq |\Gamma_{i_{k+1}}|.
\]

2. \( f_k^{(r_0, r_k)} \notin H \subseteq Ext(\Lambda) \) if and only if \( |\Gamma_{l_0l_1}| \geq |\Gamma_{k kl_{k+1}}| \) or there is an \( i, 0 < i < k \) such that

\[
|\Gamma_{r_0 r_1}| \geq |\Gamma_{i_1 i_{i+1}}| \geq |\Gamma_{r_k r_{k+1}}|.
\]

Proof. It follows from the definition of \( |\Gamma_{l_0l_{m+1}}| \) that \( |\Gamma_{l_0l_{m+1}}| = |\bar{r}_{l_0} \bar{r}_{l_{m+1}}| \).

Thus by Theorem 6.18 and Lemma 6.12, we have (1). (2) follows similarly.

Corollary 6.19 is stronger than it appears at first. It seems to imply that we need to calculate a \( \Gamma^* \)-walking sequence for \( M(1, r_m) \). However, that is unnecessary. We can derive the sequences \( \{l_m\} \) and \( \{r_m\} \) without calculating the sequences \( \{l_m\} \) and \( \{r_m\} \).

Let \( d_0 \) be a \( \Lambda \)-string function corresponding to the \( h \)-string \( S_0 \) where \( S_0 \) is a maximal path with origin and terminus \( l_0 \). Let \( S_1 \) be the \( h \)-string corresponding to \( d_1 \) the string function acquired by applying the picture algorithm to \( d_0 \). Then let \( \bar{x}_1 = o(S_1) \) (the origin of \( S_1 \)). Similarly let \( \bar{y}_1 = t(S_1) \). Our choice for these two vertices may be the reverse from the choices that we would have made had we gone to \( \Lambda^* \) for our calculations. However, the symmetry makes this irrelevant. The reader should review Lemma 5.3.

Since \( \Lambda^* \)-walking sequences and \( \Lambda \)-walking sequences are identical when we ignore
the superscripts, the $\Lambda$-walking sequences $\{\hat{x}_m\}_{m=0}^{\infty}$ and $\{\hat{y}_m\}_{m=0}^{\infty}$ are the walking sequences $\{x_m\}_{m=0}^{\infty}$ and $\{y_m\}_{m=0}^{\infty}$.

6.3 Summary and Examples

We can view $\text{Ext}(\Lambda)$ as a quotient of a path algebra, $K\mathcal{T}$. Below we describe how to use Theorem 6.10 and Corollary 6.19 to construct $\mathcal{T}$. It is clear from Corollary 6.6 that the generators of the form $f_k^{(v,w)}$, for $k \geq 1$, behave like arrows from a vertex $v$ to a vertex $w$, and that the generators $f_0^{(v,v)}$ behave like a vertex $v$. Even though we have used the label $v$ in the graph $\Gamma$ we will use it again without confusion in the graph $\mathcal{T}$. We will use the label $f_k^{(v,w)}$ for the arrow from the vertex $v$ to the vertex $w$ corresponding to $f_k^{(v,w)}$.

It is clear from Corollary 6.8 that, for each vertex $v \in \Gamma$, $f_0^{(v,v)} \in H$. Hence, each of the vertices appearing in $\Gamma$ will also appear in $\mathcal{T}$. Using Theorem 6.10 we now decide which of the vertices $v$ will be an origin for an arrow of the form $f_2^{(v,v)}$. We do this by considering the lengths of the two cycles in which $v$ lies in $\Gamma$. According to the theorem the arrow $f_2^{(v,v)}$ must be included in $\mathcal{T}$ whenever both of the cycles have length greater than two. Recall that for the theorem we defined the length of a cycle to be the number of arrows in the cycle unless the cycle is the exceptional cycle. In that case, the length of the cycle is $N \cdot (\text{the number of arrows in the cycle})$ where $N$ is the exceptional number.

Let $v (= l_0 = r_0)$ be a vertex in $\Gamma$. Let $\{l_i\}_{i=0}^{\infty}$ and $\{r_i\}_{i=0}^{\infty}$ be the two $\Gamma$-walking sequences that begin with $v$. Then we now need to use Corollary 6.19 to determine when an arrow $f_k^{(l_0,l_k)}$ or an arrow $f_k^{(r_0,r_k)}$ is to be included in $\mathcal{T}$. We know by Lemma 6.11 that after doing this for each vertex $v \in \Gamma$ we will have completed the construction of the graph of $\mathcal{T}$. We warn the reader that the algorithm may yield
the following: \( f_k^{(v,l_k)} \in H \) and \( f_k^{(v,r_k)} \in H \) where \( l_k = r_k \). This does not mean that there will be two arrows of degree \( k \) from \( v \) to \( l_k \). The notation for the generators was well defined. It simply means that \( f_k^{(v,l_k)} = f_k^{(v,r_k)} \).

The reader will be pleased to note that for any \( \Gamma \)-walking sequences \( \{l_i\}_{i=0}^{\infty} \), \( l_2||\Gamma||_k = l_k \). Thus \( |\Gamma_{i_k,l_{k+1}}| = |\Gamma_{l_2||\Gamma||_k}l_{i+k+1}||_k \). So by Corollary 6.19 \( f_2^{(l_2,l_2||\Gamma||_k)} \notin H \) for \( k \geq 1 \). Thus the algorithm terminates rather quickly.

**Example:** In figure 6.13 we give a graph \( \Gamma \) and in figure 6.14 we give the constructed graph \( \Upsilon \). We give the reader that one of the \( \Gamma \)-walking sequences for \( v \) is \( uvwxuyzx \). We also list the sizes of the partial graphs in the order in which they should be considered for determining the arrows leaving the vertex \( v \) using that walk.

\[
\begin{align*}
|\Gamma_{uv}| &= 1 + 2||\Gamma|| = 1 + 2 \cdot 4 = 9 \\
|\Gamma_{uvw}| &= 4 \\
|\Gamma_{uw}| &= 1 \\
|\Gamma_{ux}| &= 4 + 2||\Gamma|| = 12 \\
|\Gamma_{xy}| &= 2 \\
|\Gamma_{yy}| &= 1 \\
|\Gamma_{yx}| &= 12 \\
|\Gamma_{xyv}| &= 11
\end{align*}
\]

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Figure 6.13: $\Gamma$

Figure 6.14: $\Theta$
Bibliography


Vita

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