

OPTIMIZATION OF MATERIAL FLOW IN THE NUCLEAR FUEL CYCLE USING
A CYCLIC MULTI-STAGE PRODUCTION-TO-INVENTORY MODEL

by

Elden Leo DePorter

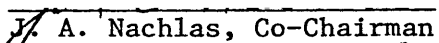
Dissertation submitted to the Graduate Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

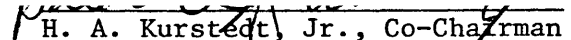
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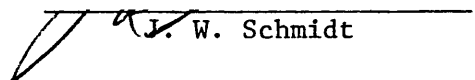
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Industrial Engineering and Operations Research

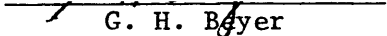
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ACKNOWLEDGEMENTS

The author is especially grateful to his two major advisors: Drs. Joel A. Nachlas and Harold A. Kurstedt, Jr. Their suggestions, guidance, and moral support were sincerely appreciated. Special recognition to committee members, Drs. Robert P. Davis, J. William Schmidt, and Gerhard H. Beyer is acknowledged for their patience, support, and academic contributions.

Special thanks are given to Dr. James M. Moore who originally offered employment at V.P.I. and S.U. and to Dr. Harry L. Snyder who continued it.

A note of appreciation is also due to _____ for extremely efficient typing under rather short deadlines.

Finally, the author is most deeply indebted to his patient, understanding, loving wife. Her contributions were immeasurable.

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CHAPTER I.

INTRODUCTION

In 1973 the world was jolted by the Arab oil embargo. Since that time the "Energy Crisis" has become a formidable problem that is being attacked scientifically and analytically by scientists and technologists; practically, and sometimes emotionally, by environmentalists; and cautiously by politicians. Nuclear reactor generated electricity provides a viable alternative to oil as a basic source of energy. However, energy is consumed in such vast quantities today that in the absence of some technological break-through, nuclear fuels could also be exhausted by the year 2000. Thus, nuclear fuel, in its infancy as an energy source, is already viewed as a depleting resource.

One of the obvious results of the oil embargo was that companies that controlled their own energy supply were in a commanding position. This result caused major energy companies, e.g., Exxon, and utilities, e.g., Tennessee Valley Authority, to attempt to gain control of basic nuclear fuels in order to assure their later availability when they are needed.

This series of developments has created an unique management problem. Assuming that an utility, in an effort to control the future availability of nuclear fuel, buys a basic uranium ore source, then, recognizing that uranium ore must pass through several stages before it is in useable form, at which stage and in what quantities should

it be stockpiled in order to realize a minimum cost over a specified time horizon? Aspects of this problem which make it unique are monotonically increasing demand for uranium fuel over the time horizon, monotonically increasing value of the uranium in relation to other commodities over the time horizon, a depleting resource, and the possibility of reprocessing spent fuels to re-capture (re-cycle) the unburned portion.

The national energy outlook is not optimistic. Nuclear power utilized to generate electrical energy can materially improve the national energy outlook. Nuclear energy is in its infancy, and numerous articles have called for the application of scientific management techniques to the nuclear fuel cycle^[23,26,37,38,41,46,48,60,63]. Although some progress has been made, concentrated efforts are still required. MASON^[47] approximates the initial core of a light water reactor to cost \$31 million and the annual reload to cost \$14 million. MASON summarizes, "Thus, very large sums of money are involved in the inventory requirements and in the annual fuel costs during the operation of large nuclear power plants. This in turn produces large incentives for the development of techniques that will lead to reductions in the cost . . ."^[47]. GALON and SALMON^[26] also emphasize the need for improved methods, "Uncertainties that are brought about by revision of long-term utility planning and by forced outages also contribute to a continuing need for dynamic optimization of nuclear unit operating and fuel cycle strategies"^[26].

The need for minimizing costs is obvious and imperative. Since there is no presently known production-to-inventory model to optimize costs for the nuclear fuel cycle, the motivation is obvious, yet a more dramatic motivator might well be KURSTEDT's evaluation, "In short, we will not freeze in the dark due to inadequate technology, rather due to the lack of good management decisions"^[37].

The objective of this research is to develop a production-to-inventory model that considers the aspects above and that can be utilized by the utilities, or governmental agencies, in deciding at what stage of the nuclear fuel cycle and in what quantities uranium fuel can be stockpiled in order to incur a minimum production and inventory cost over a defined time horizon.

The resultant model uses the calculus of variations to determine the production trajectories for each stage of the nuclear fuel cycle that meet the demand for energy while minimizing acquisition, production, and inventory holding costs. The solution demonstrates that it is optimal to stockpile larger quantities of ore in the explored resource (in uranium fields as is presently the case) and of fabricated fuel assemblies and lesser quantities of enriched uranium hexafluoride. All other stages essentially produce only sufficient quantities to meet demand. Analysis of the sensitivity of the model to variation in model parameters shows the greatest effect results from varying the assumed production capacities.

CHAPTER II.

THE PROBLEM DESCRIPTION

A. National Energy Outlook

Numerous reports and papers^[1,5,23,42,45,55,64] forecast the growth of energy requirements in the United States. In testimony before the U.S. Committee on Interior and Insular Affairs, Subcommittee on Energy and the Environment, R. W. A. LeGasse of the Energy Research and Development Administration (ERDA) gave a summary forecast of the U.S. energy growth through year 2000^[42]. This summary is shown in Table 1. LeGasse's estimates are particularly meaningful because they show three ranges of forecasts depending on different pricing and conservation practices. The importance of this summary is that, even for the low forecast, energy requirements are expected to increase by 178 percent by year 2000.

Confronted with this forecasted energy growth requirement, ERDA reports^[1] that the "available energy" picture is very bleak. In particular:

Domestic crude oil production peaked in 1970 and has declined by more than one million barrels per day since then. Production is now at a nine-year low.

Oil imports are about 37 percent of oil consumption and would rise to more than 50 percent of consumption of 12 million barrels per day by 1985 if no new actions are taken.

As a result of increasing import dependence, U.S. payments to foreign producers for imported oil increased from less than \$3 billion in 1970 to about \$27 billion in 1975 and will increase by another \$2 billion annually, largely because of the OPEC price increases.

Table 1

ERDA Summary Forecast of Total U.S. Energy Growth
(Quadrillion Btu)^[68]

	<u>Year</u>			
	1973	1980	1985	2000
Low	75.6	86.1	96.0	135.0
Moderate	75.6	89.7	105.0	174.0
High	75.6	95.3	117.0	195.0

Natural gas production peaked in 1973, declined by 6 percent in 1974 (the equivalent of over 230 million barrels of oil), and dropped another 8.5 percent during the first half of 1975.

Electric utility financial problems and regulatory delays have in part resulted in the cancellation or postponement of about three-fourths of all planned nuclear plants and about one-third of all coal plants previously scheduled to come into operation between now and 1985.

Some emerging technologies, such as synthetic fuels from coal, shale oil, solar, and methods to use energy more efficiently, have uncertain economics due to long lead times and technological uncertainties, and considerable risk^[1].

It is evident that present resources and technologies need to be utilized to the fullest^[32] and that a concentrated effort needs to be applied toward developing the energy of the future. The sources of energy today are oil, natural gas, coal, nuclear power, hydropower, geothermal power, solar energy, and energy produced from solid wastes^[64]. Of these options, the U.S. possesses the resources in large quantities for only two: indigenous coal and uranium supplies^[5]. To meet its interim demand and until the energy of the future has been developed, the United States must rely heavily upon its resources of coal and uranium.

B. Electrical Energy Outlook

Uranium and, to some extent, coal are primary sources of energy, i.e., they are not in a form readily useable by the consumer. Indeed an intermediate form of energy is generally necessary.

Electricity is a convenient form of energy for customers. It is normally available continuously and automatically, and the precise amount needed is instantly delivered so that the user needs no inventory. Discounting conversion and line

losses, it is efficient and it causes the consumer no pollution problems. For these reasons industry has turned increasingly to electricity with a resulting growth rate during the last decade of nearly 8 percent annually. Looking ahead, between 1970 and 1990, an annual growth rate of 6.4 percent is forecast[64].

BRADEN and BROWN stated that the advantages of electricity as an energy source are so great, so superior to other sources, that a mark of an economically advanced nation is a high degree of production and consumption of electricity^[10].

ERDA forecasts for electrical generating capacity growth follow the same pattern as total energy growth. Table 2 depicts this forecast.

The Atlantic Council of the United States estimates that the percent of primary energy converted to electricity in the United States will increase from the 1973 level of 23 percent to 50 to 60 percent by the year 2000^[5].

C. Nuclear Power Generated Electrical Energy Outlook

Concurrent with this growth in electrification is the growth in nuclear power generated electricity. ERDA forecasts for nuclear power generated electricity are shown in Table 3. Recall Table 2 shows the total electrical generating capacity growth.

Other projections are similar. GOLAN and SALMON^[26] project that in the period from 1980 to 2000, the average annual energy demand will increase from 3.2×10^3 to 10×10^3 gigawatts electric (313 percent), and that the nuclear electrical portion will increase from 150 to 1,400 gigawatts electric (933 percent). The Bureau of Mines of the U.S. Department of Interior, projects that

Table 2

ERDA 1975 Electrical Generating Capacity Projections
(Gigawatts Electric) [68]

	<u>Year</u>			
	1975	1980	1985	1990
Low	492	604	785	980
Moderate-Low	496	620	800	1,040
Moderate-High	500	630	820	1,075
High	505	654	875	1,180

Table 3

ERDA 1975 Nuclear Power Generated Capacity Projections
(Gigawatts Electric) [68]

	<u>Year</u>			
	1975	1980	1985	1990
Low	38	70	160	285
Moderate-Low	39	76	185	340
Moderate-High	41	70	205	345
High	43	92	245	470

electrical generating capacity will increase from 474,573 megawatts in 1974 to 1,887,000 megawatts in the year 2000 (398 percent), and that the nuclear contribution will increase from 31,662 megawatts in 1974 to 900,000 megawatts in 2000 (2,842 percent)^[26]. A further indication is given by the World List of Nuclear Power Plants which lists 58 nuclear power plants operating in the United States as of December 31, 1976, and projects 157 to be in operation by 1984^[60].

Figure 1 depicts the ERDA forecast as a monotonically increasing demand for electricity and the corresponding monotonically increasing demand for nuclear power generation capacity. This graphic portrayal strongly illustrates the increasing demands.

From these data, it is clear that nuclear powered electrical generation will play an increasingly important role in meeting the energy demands in the United States. Thus, the success of the nuclear power industry in meeting the needs for electrical energy, and in turn, the energy needs of the United States, will be determined in part by the progress attained in understanding, completing, and operating the nuclear fuel cycle^[5]. Before elaborating on the forecasted demand for nuclear fuels, a description of the nuclear fuel cycle is given so that fuel requirements can be viewed from a proper perspective.

D. Nuclear Fuel Cycle

Uranium fuels used in the nuclear reactors are the product of what is referred to as the "Nuclear Fuel Cycle". This "cycle" consists of the process operations performed on the nuclear fuel from the time

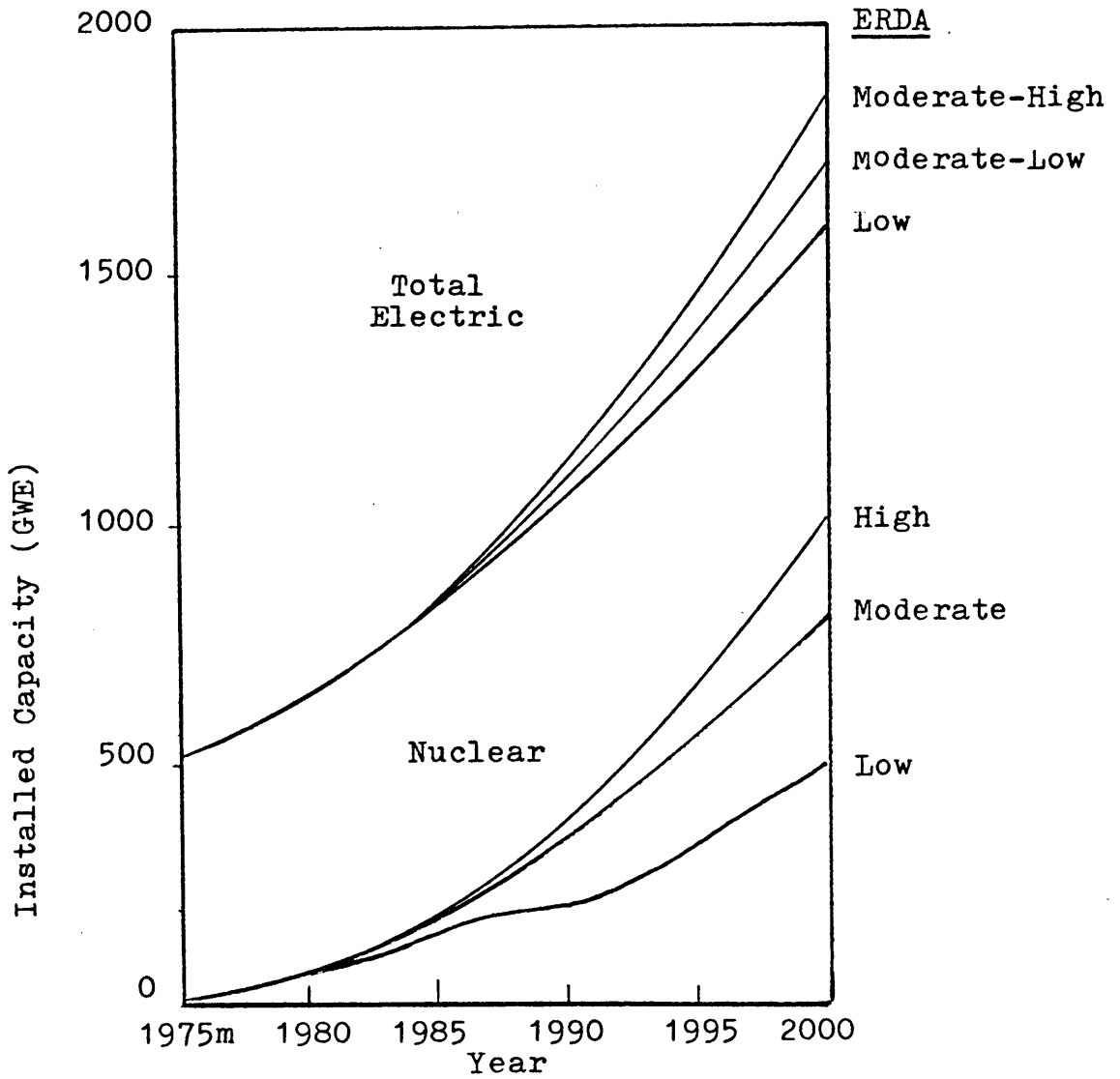


Figure 1. ERDA Forecasts of Demands for Electricity (Gigawatts Electric)⁽²³⁾

it is removed from the earth in the form of uranium ore until it is returned to the earth in some form that is safe for permanent storage. Generally, the cycle consists of mining ore from its location in the earth, milling it to remove uranium oxide (U_3O_8) (commonly referred to as yellowcake), converting this to uranium hexafluoride gas (UF_6), processing this gas through a mechanical separation process to generate a product that is enriched in the fissile U-235 isotope, converting this enriched gas to uranium dioxide (UO_2) pellets, fabricating these pellets into fuel assemblies, and inserting the assemblies into reactors. The assemblies are subsequently removed from the reactors and are stored as spent fuel. Finally, this spent fuel can be reprocessed, thereby re-capturing useable fuel and generating unuseable wastes which are stored in the earth in some acceptable form. This view of the nuclear fuel cycle is depicted in Figure 2.

Thus, the nuclear fuel cycle is seen to consist of approximately seven stages through which uranium ore passes. A more detailed description of these stages is given in Appendix D.

Another view of the nuclear fuel cycle is shown in Figure 3, which depicts the quantitative material-balance flows. This diagram begins at the mill and reflects the materials required at each stage to support a 1,000 megawatt electric reactor operating at 85 percent load factor, and chemically reprocessing spent fuel to re-capture the unburned fuel^[7]. Likewise from this figure the "demand" on the nuclear fuel cycle is deduced. The typical reactor experiences a burnup of 20,333 megawatt days per tonne of fuel. Assuming a standard

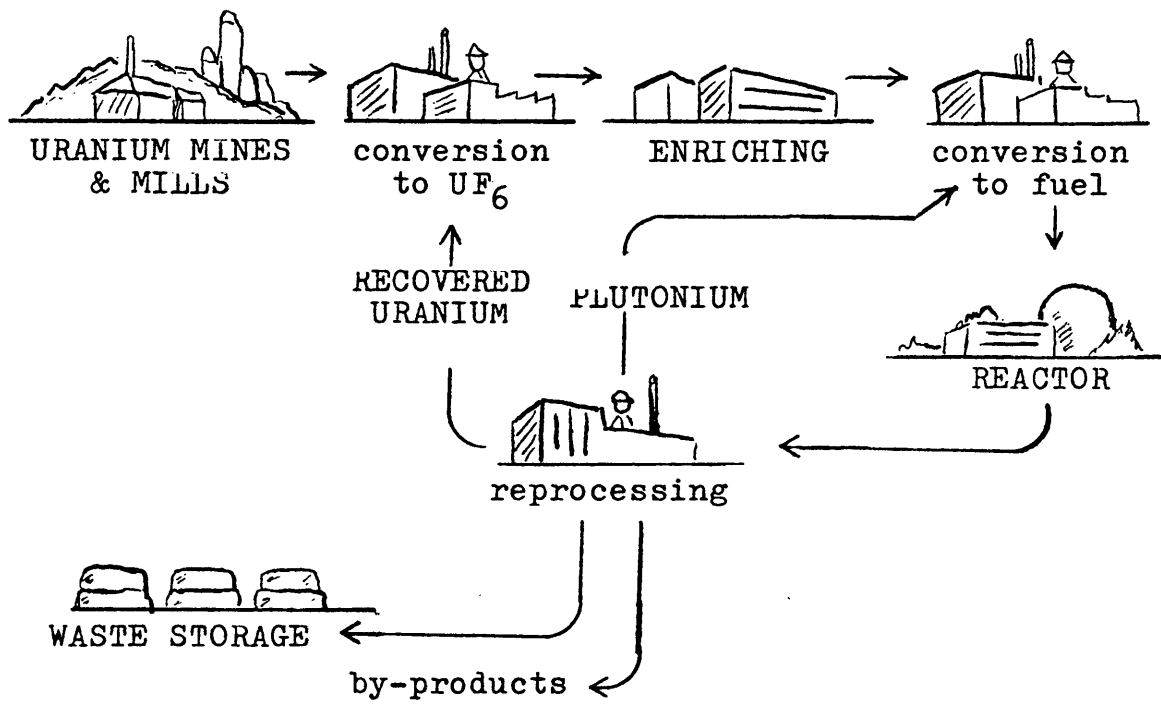
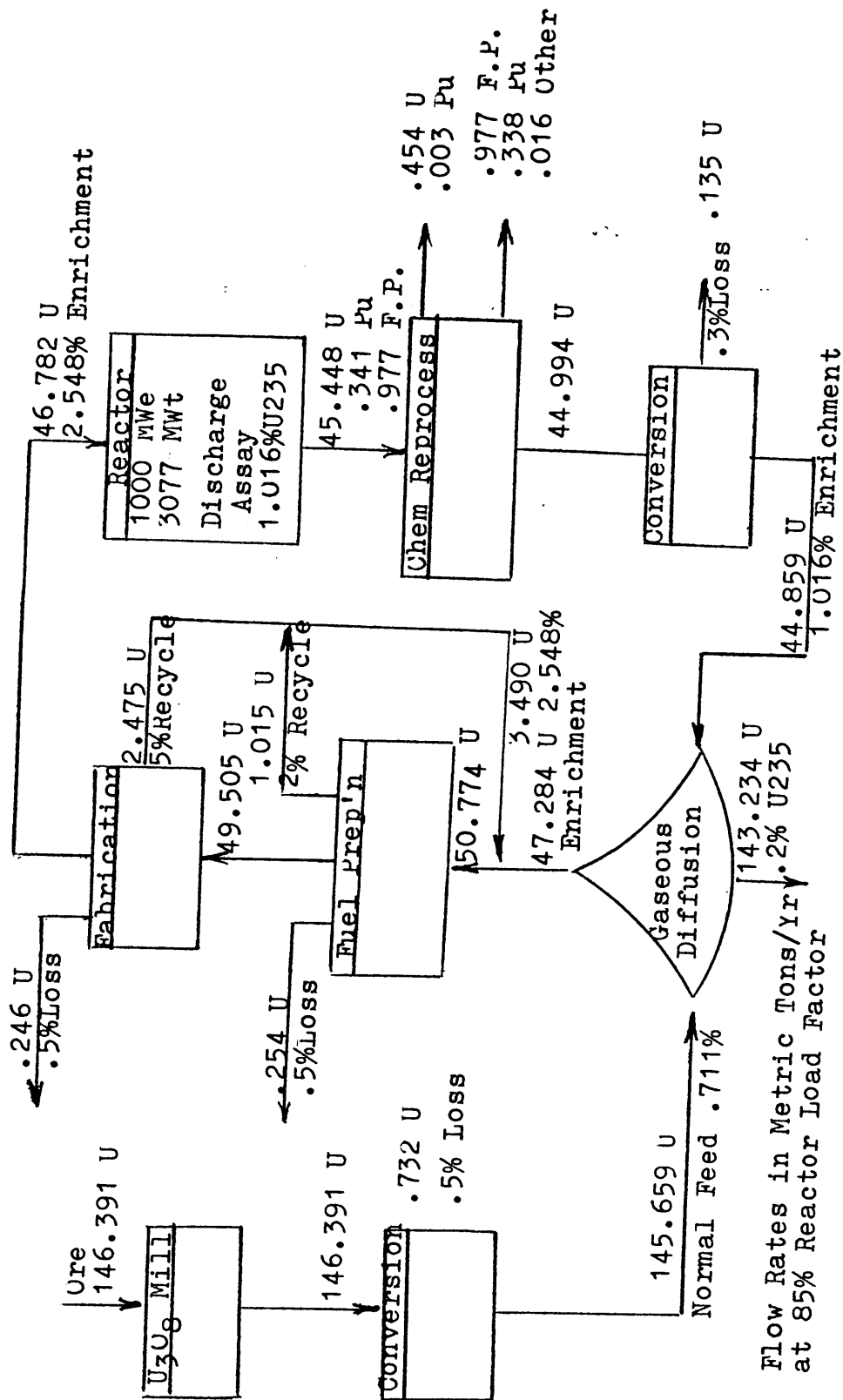


Figure 2. Overview of the Nuclear Fuel Cycle⁽⁴⁴⁾



Flow Rates in Metric Tons/Yr
at 85% Reactor Load Factor

Figure 3. Typical Material Balance Flowsheet (7)

fuel enrichment, the equivalent demand on each stage is defined per single reactor. Then, taking the schedule for activation and operation of nuclear reactors for generating electricity in the United States as shown in the "World List of Nuclear Reactors"^[60] and assuming each activation requires a full core of fuel, that each year one-third of the core is replaced with fresh fuel, and that the initial core is equivalent to three average reloads, the demand for reloads is computed and is shown in Table 4. It is readily discernible from the data in Table 4 that demand is an increasing function of time.

It is necessary at this point to digress and discuss "forward cost". The uranium industry traditionally calls the cost to produce U_3O_8 from uranium ore the "forward costs", even though the precise form of U_3O_8 may not even be present in processed fuel. Forward costs are the costs incurred in removing the overburden, mining and milling, e.g., if ore is excessively deep much overburden must be removed thus the forward costs increase, or if the assay is high, less ore is mined to yield a given amount of U_3O_8 and forward costs are lower. Forward costs do not include exploration costs, land acquisition costs, profits, taxes, or interest on capital investment. Understanding forward costs is imperative, for demands are usually represented in short tons or metric tons of U_3O_8 ; and U_3O_8 availability is normally projected on the basis of forward cost^[45]. The implication is that high-grade ore laying on top of the ground has a low forward cost, while low-grade ore deep in the ground has a high forward cost. Therefore, the low-forward-cost ore is exhausted first. The result

Table 4

Demand for Reloads for U.S. Nuclear Powered
Electrical Generators

<u>Year</u>	<u>Reloads</u>
1960 - - - - -	3
1961 - - - - -	4
1962 - - - - -	8
1963 - - - - -	7
1964 - - - - -	5
1965 - - - - -	5
1966 - - - - -	8
1967 - - - - -	6
1968 - - - - -	12
1969 - - - - -	17
1970 - - - - -	23
1971 - - - - -	27
1972 - - - - -	43
1973 - - - - -	48
1974 - - - - -	70
1975 - - - - -	67
1976 - - - - -	86

is that, as time passes, only the higher forward-cost ores remain. Uranium is thus a depleting resource whose value (price to obtain) increases with time.

LIEBERMAN^[45] shows a close relation between the amount of drilling and uranium discovery. Further, he concludes that a time lag of approximately one year between a surge in drilling and an increase in discoveries exists. LIEBERMAN then uses historical data to arrive at an equation defining the quantity of uranium at a given forward cost that will be discovered and produced as a function of time. Let $Q(t)$ be the cumulative amount of uranium of a given forward cost discovered and produced at time t , Q^∞ be the total cumulative amount of uranium available for discovery (must be determined from extrapolation of the discovery rate curve), "a" and "b" be constants determined by fitting a curve to the data, and t_0 be the arbitrary base point in time; then the equation for cumulative discoveries and production is:

$$Q(t) = \frac{Q^\infty}{1 + ae^{-b(t-t_0)}}$$

Using the cumulative discovery data for ore with a forward cost of \$8 per pound, LIEBERMAN finds for a base year of 1948, $a = 220$, $b = 0.41/\text{year}$ that $Q^\infty = 534,000$ short tons of U_3O_8 . Although the curve fit is not conclusive, LIEBERMAN concludes that the quantity of Q^∞ is between 500,000 and 800,000 short tons of U_3O_8 . Requiring

Q(1974) to be 270,000 short tons (a known amount) and knowing that the amount of presently held reserves are approximately 273,000 short tons of U_3O_8 , LIEBERMAN develops the overall picture of the history and future availability of uranium ore with the indicated forward cost of \$8 per pound as shown in Figure 4.

Further application of this equation for ores with \$8, \$15, and \$30 per pound forward costs enables LIEBERMAN to develop the more comprehensive picture shown in Figure 5. By interpolation of this curve, it is seen that ores with these forward costs will be exhausted by 1985, 1990, and 1993, respectively.

Although LIEBERMAN and others refer to an exhaustion date, actual exhaustion of resources will not occur at a specified time, but rather as HUBBERT states:

Resources won't be exhausted suddenly on a given day, but usage will be such that exhaustion will resemble a bell shaped curve with the tail of final exhaustion resulting in higher prices.

Indeed, this is already the case in the U.S. for natural gas and possibly for gasoline . . . [32].

Increasing demand and a depleting resource result in high prices for nuclear fuel. In fact, the 1976 prices for yellowcake are reported to be ten times the 1969 level^[25]. The history of the value of U_3O_8 to date is shown in Figure 6.

A logical question exists as to what part of the increased cost may be attributable to increased demand and the depleting resource concept and what part may be attributable to inflation. It is impossible to make a distinction; however, an attempt is made to

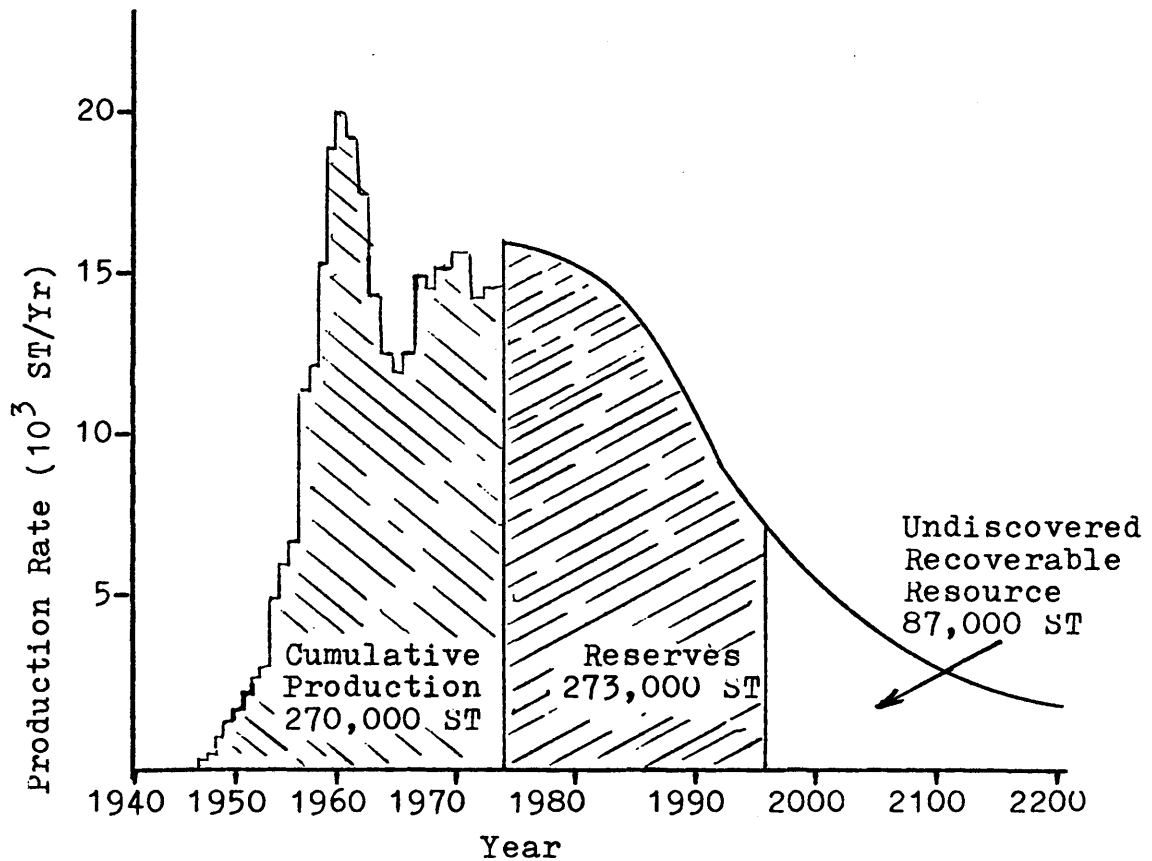


Figure 4. Actual Production and a Possible Cycle of Production for U_3O_8 at \$8 or Less per Pound(45)

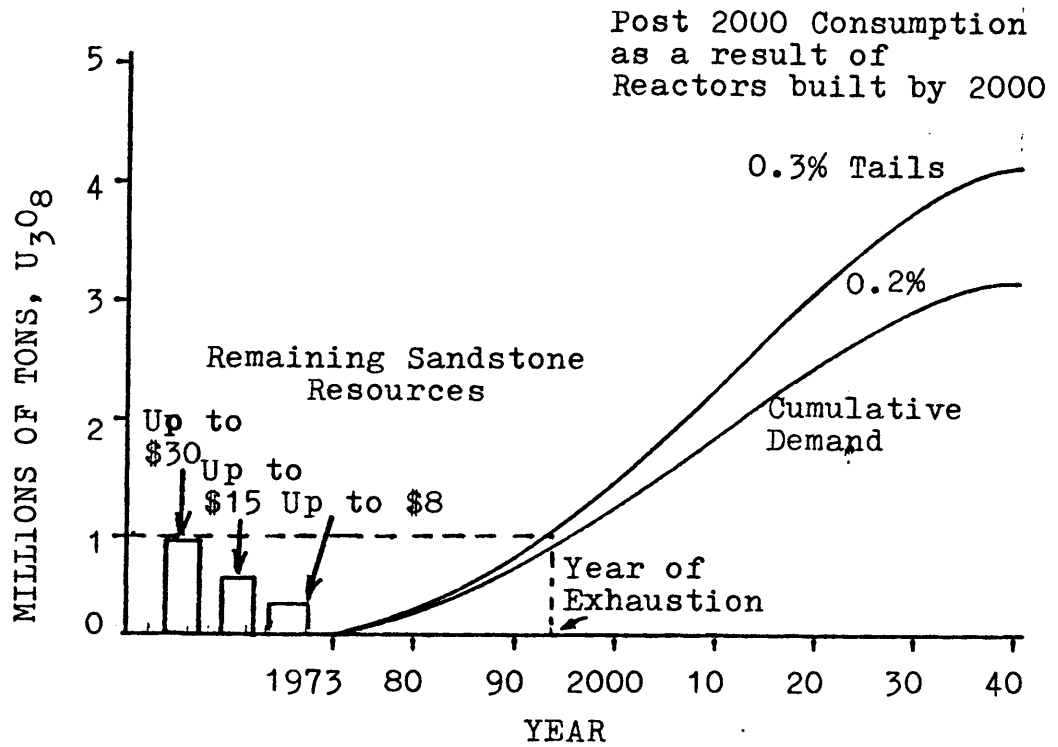


Figure 5. United States Cumulative Demand
for U_3O_8 (45)

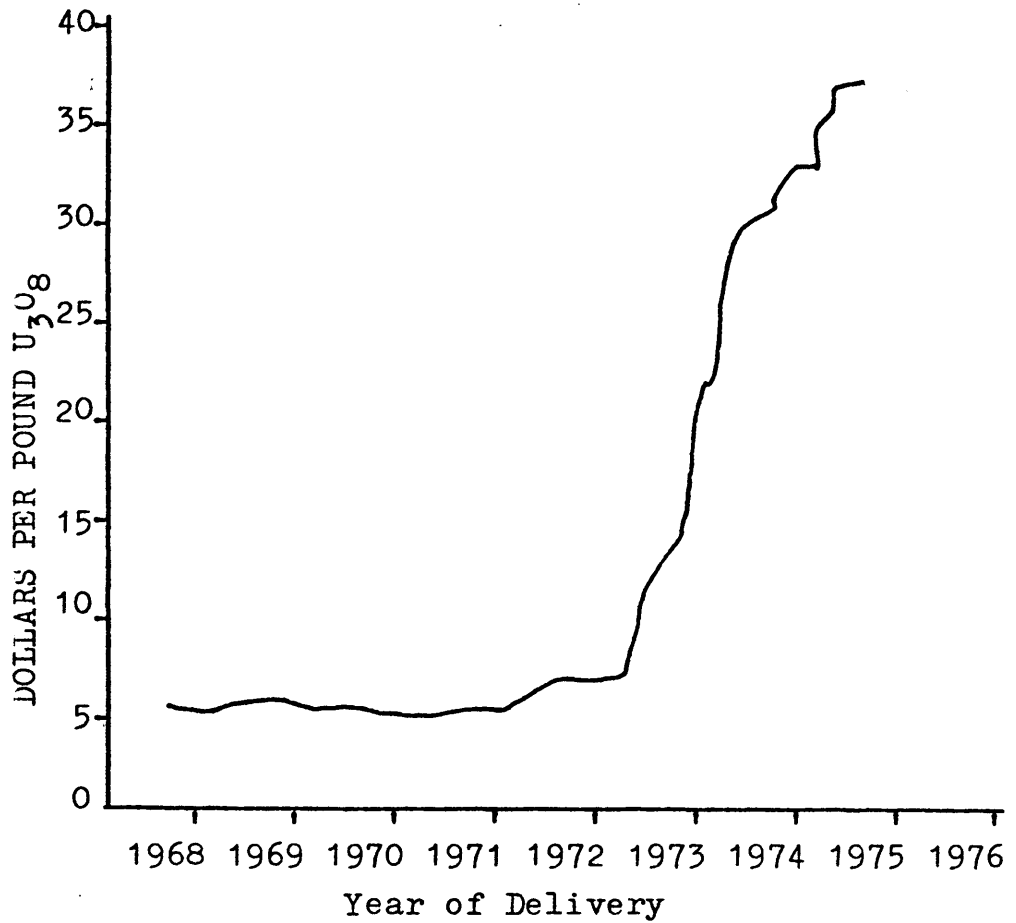


Figure 6. Historical Exchange Value for U_3O_8 for Immediate Delivery⁽²⁹⁾

reflect the cost of fuel as a function of time and in terms of constant 1972 U.S. dollars. As shown in Figure 7, the constant dollar curve clearly indicates a monotonically increasing cost of uranium fuel over time. It is assumed that this increase without inflation must reflect the increasing value of uranium as a result of the factors previously mentioned.

An alternate, perhaps more desirable, means for showing the increasing value of uranium fuel is to relate the cost to cumulative demand^[69]. From LIEBERMAN's equations predicting exhaustion of the reserves, this approach has more intuitive appeal. Figure 8 shows this increasing cost as a function of time.

Faced with this depleting resource, increasing demand, increasing value of the fuel in relation to other commodities, and the general outlook for energy in the world today, utilities are attempting to gain control of the basic fuel source^[74]. It is conceivable that utilities may become even more deeply involved in vertical integration of the entire nuclear fuel cycle in the future. Another alternative for utilities is to form a collective or corporation and to participate cooperatively in the fuel cycle. In fact, utilities are actively considering ". . . a corporation (that) would be set up and would buy fuel and lease it to utility companies. The entire fuel (cycle) would be financed through the issuance of commercial paper"^[48].

E. Statement of the Problem

In any case, a logical management decision must be made. If uranium ore is obtained; then, faced with increasing demand, increasing

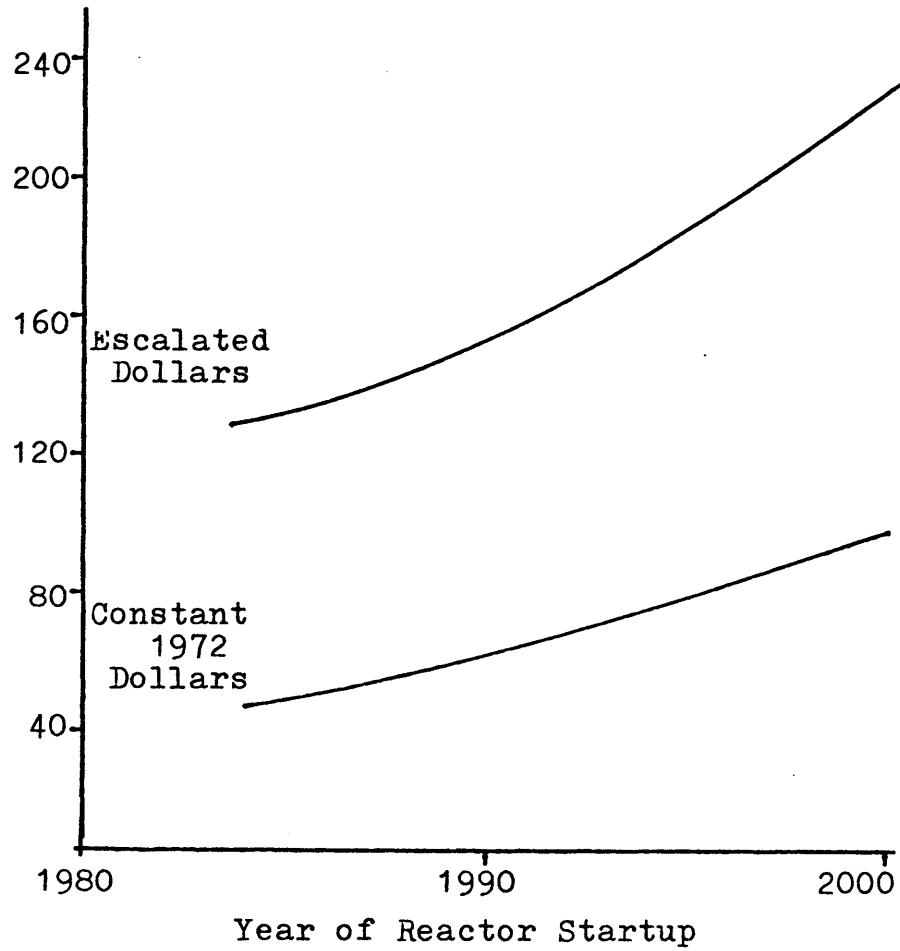


Figure 7. Increasing Costs of Uranium Fuels⁽⁶⁾

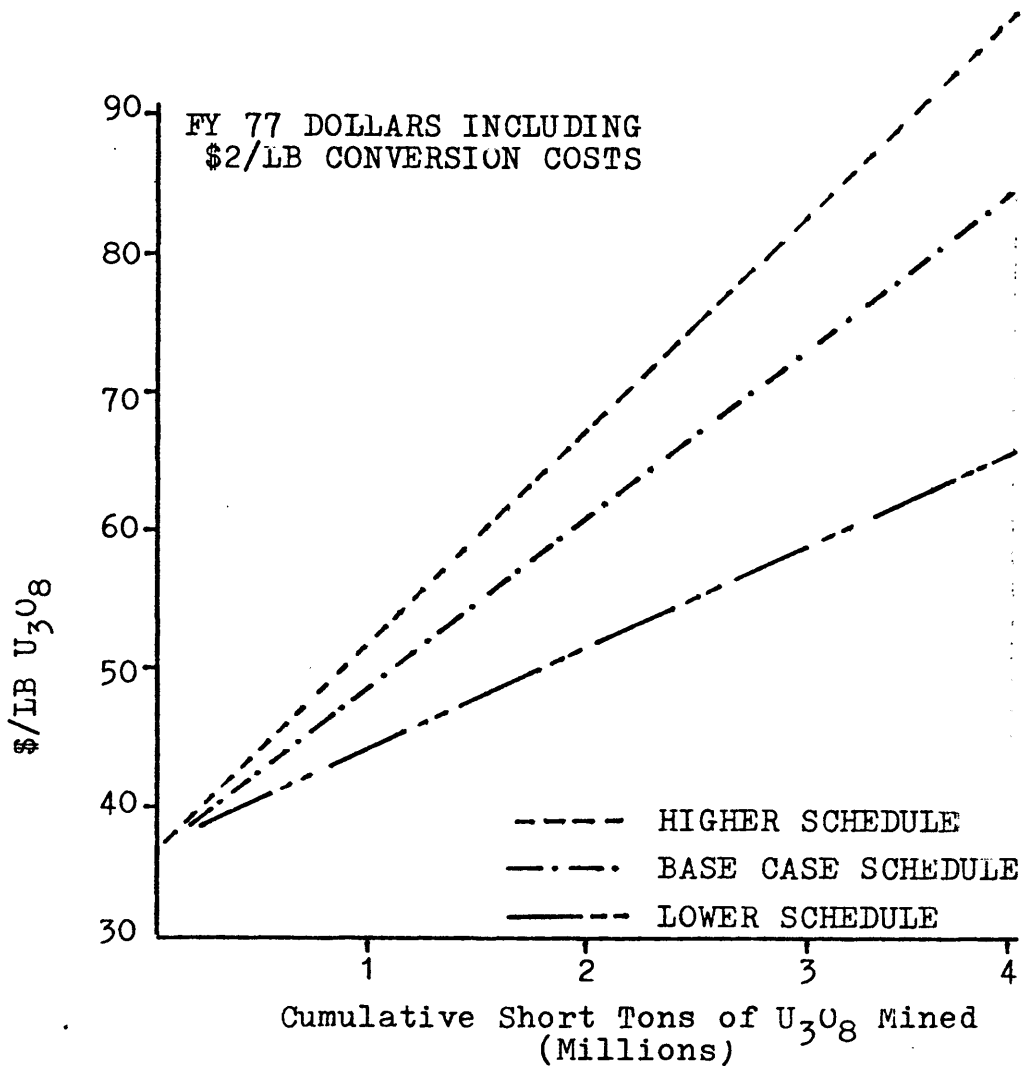


Figure 8. Uranium Price Schedules⁽⁶⁹⁾

procurement costs, and a multi-stage processing system, at what stage and in what quantities could uranium fuel be stockpiled in order to meet demand and incur a minimum cost over a defined horizon of time?

This is the question addressed here. What is required is a multi-stage production-to-inventory model that permits cyclic flow. The model should incorporate acquisition costs which increase as a function of time, production costs, and inventory holding costs. It should permit accounting of fuel-stage conversion factors and losses. It should allow for production lag time. It does not need to allow for shortages. Finally, the model should develop stage-wise production trajectories over a specified time horizon which minimize the total cost.

The final result should be a macroscopic management tool that can be used by utilities or government agencies in determining the production levels to be used (developed) and inventory levels to be maintained (stockpiled) for a valuable but depleting resource. ERDA predicts that this depleting resource could provide as much as 60 percent of our electrical energy needs by year 2000^[68], yet LIEBERMAN^[45] predicts that the \$30 forward cost ores will be exhausted by year 1994. Any tool that provides a better management capability is, then, of considerable value.

CHAPTER III.

LITERATURE SEARCH

A. Production and Inventory Theory Research

Probably the earliest work directly related to the production-to-inventory problems where a product is to be produced in given amounts over each of T periods such that production and holding costs are minimized, is that of MODIGLIANI and HOHN^[52]. Their single-stage model permits increasing marginal costs of production and constant holding cost per unit of product per unit of time and arrives at a "fundamental solution" that has far-reaching application. This article is a milestone in production-inventory theory but considers only the discrete case. MORIN^[53] restates the problem in continuous functions and utilizes the calculus of variations to arrive at a reasonably simple result for linear and quadratic production costs. WALVEKAR, SMITH, and DECICCO^[75] expand this result to include the concept of shortages. ARROW and KARLIN^[3,4] simplify the results of MORIN by analyzing problems where convexity properties can be assumed. Their work also involves the calculus of variations, but the assumption of convexity simplifies the form of the solution.

ZANGWILL in a series of articles^[77,78,79,80] makes a dramatic breakthrough, expanding single stage dynamic lot size models into multi-product, multi-stage production and inventory models. He ultimately includes non-decreasing demands, series flow, parallel flow, production lags and backlogging; however, he never permits

cyclic flow. Generally, ZANGWILL's model is based on a dynamic programming algorithm that optimizes through "dominant sets". Although his algorithm insures optimal solutions, it becomes unwieldy when the number of periods are large, i.e., twelve or more^[78].

Multi-stage process problems are considered in depth by MITTEN and NEMHAUSER^[50,51,56,57]. Schematically, the stage-wise chemical process appears similar to the nuclear fuel cycle. NEMHAUSER's development includes series flow, parallel flow, and forward and backward cycles; however, in the chemical processes there are no production lags nor are there accumulations (inventories) of materials, i.e., flow is direct. Also, the stage-wise decomposition is made with respect to production stages, and time is not explicitly modeled.

A further development of importance is BOWMAN's^[9] use of the transportation method of linear programming to solve the production scheduling problem. For linear costs and known deterministic demands, BOWMAN is able to optimize the production and inventory costs over T time periods for a single-stage single-product problem.

B. Production and Inventory Theory Applied to the Nuclear Fuel Cycle

The most intense efforts to apply production and inventory theory to the nuclear fuel cycle are the recent articles by KURSTEDT and NACHLAS^[13,17,18,19,38,41,46,55]. Their first article jointly authored with COCKRELL^[13] develops a static inventory model for

maintaining emergency reserves of fabricated nuclear fuel assemblies. Subsequently, KURSTEDT, NACHLAS, and LYONS expand this model to a non-stationary dynamic inventory model with delivery lags, but it is still applicable to emergency reserves of nuclear fuel assemblies^[46].

KURSTEDT, NACHLAS, and MACEK^[40] examine the inventory problem associated with spent nuclear fuel. DEPORTER, NACHLAS, and KURSTEDT^[17] present a production model to minimize costs associated with working inventories in the fabrication plant. While these efforts are recent and the only known work relating to production and inventory models within the nuclear fuel cycle, they are all directed toward single production stage inventories.

Recognizing that a total fuel cycle approach is necessary, DEPORTER, NACHLAS, and KURSTEDT^[18] model the material flows of the complete nuclear fuel cycle. While the model is not solved for an optimum, it is the first known attempt to link the entire cycle. Based on this model, the authors then simulate the entire fuel cycle as a production and inventory model^[19]. This simulation model incorporates the concepts of multi-stage single product and cycled production to inventories with production lags and lead times. However, this model remains descriptive, since the excessively large number of variables prevent optimization. There appear to be no other attempts to apply production and inventory theory to the nuclear fuel cycle.

Properties of the nuclear fuel cycle which cause all of the above techniques to be inadequate are:

1. Multi-stage System
2. Cyclic (feedback) Flow
3. Accumulation of Inventories Between Processes
4. Time Dependence

At least one of these properties is not represented or handled by each of the techniques; however, combinations or modifications of these techniques might permit examination of the nuclear fuel cycle.

CHAPTER IV.

DEVELOPMENT OF THE MODEL

The nuclear fuel cycle has been shown to be a multi-stage production-to-inventory system with possible cyclic (feedback) flow. Since this class of problems has not been solved before, a general model is developed here. After the mathematical foundations are laid, the general model is adapted to the nuclear fuel cycle.

In general the costs to be considered in a production-to-inventory system are:

1. Costs of purchasing raw material.
2. Costs of production.
3. Costs of holding material in inventory.

The objective of the model is to determine the production rate over time at each stage that minimizes the total of these costs. The underlying assumption is that each of these costs is related to the production rate. The model will be developed in the following three stages:

1. Single-stage production-to-inventory model.
2. Multi-stage production-to-inventory model.
3. Multi-stage production-to-inventory model with cyclic (feedback) flow.

A. Single-stage Production-to-Inventory Model

In the single-stage production-to-inventory systems, raw material is purchased from an external source, rendered into a final product

by one process, and is held in inventory awaiting an external demand. This flow is described in Figure 9.

Assume that external demand is described as a continuous function of time, $r(t)$. Further, assume that a cumulative demand function, $R(t)$, is given by:

$$R(t) = \int_0^t r(\tau) d\tau \quad \text{IV.1}$$

Now assume that the production rate for the process is described as a continuous function of time, $x(t)$. Further, assume that a cumulative production function, $X(t)$, is given by:

$$X(t) = \int_0^t x(\tau) d\tau \quad \text{IV.2}$$

The particular advantage of this formulation is that the rate of change in the cumulative production function is the production rate, or:

$$\frac{d}{dt} [X(t)] \equiv X'(t) \equiv x(t) \quad \text{IV.3}$$

The same is true for the demand functions:

$$\frac{d}{dt} [R(t)] \equiv R'(t) \equiv r(t) \quad \text{IV.4}$$

Now the inventory at any time t , $I(t)$, is equal to the initial inventory plus the cumulative production minus the cumulative demand.

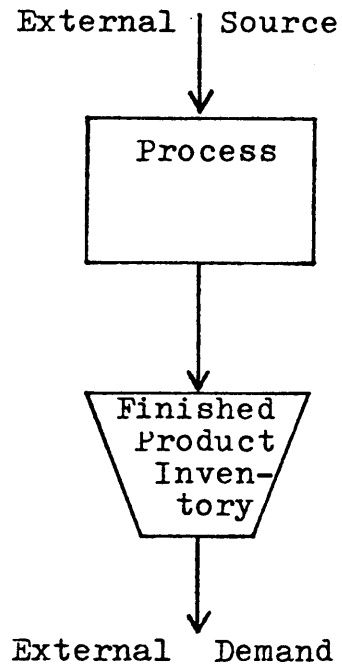


Figure 9. Schematic Diagram of a Single-Stage Production-to-Inventory System

$$I(t) = I(0) + X(t) - R(t)$$

IV.5

Shortages are not allowed; therefore, $I(0) + X(t) \geq R(t)$, $0 \leq t \leq T$, where T is the time horizon over which interest is expressed.

Clearly, the raw material purchasing costs can be written as a function of the production rate, $f[X'(t)]$. For example, let the cost per unit produced be k . Then the cost per unit of time at time t is:

$$f[X'(t)] = k \cdot X'(t),$$

IV.6

and the total purchasing cost over a time horizon, T , is:

$$\int_0^T f[X'(t)] dt$$

IV.7

Likewise, the production costs at any time t are written as a function of the production rate, $g[X'(t)]$. For example, a frequently used scheme in production models is to assume an optimal production rate for a process (plant)^[12]. Any variation in this optimal production rate results in costs per unit produced that are proportional to the square of the increase (decrease). An example is shown in Figure 10. Since this scheme is used here, it is appropriate to digress at this point and elaborate on it.

The production facilities in the nuclear fuel cycle operate generally as any manufacturing process. A given facility has some

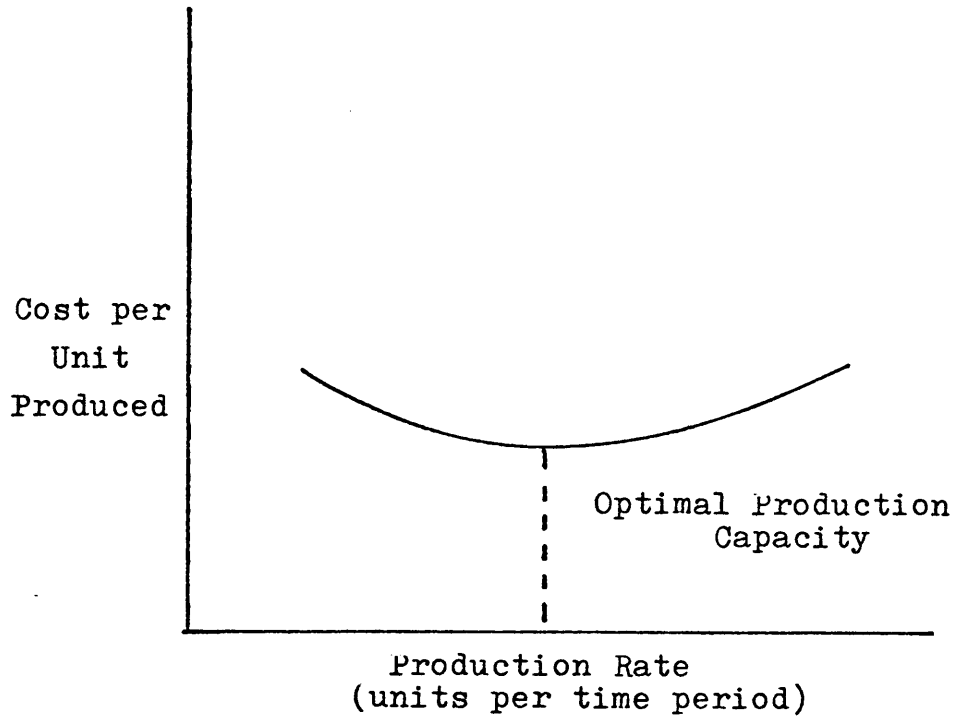


Figure 10. Graph of Production Cost versus Production Rate

optimal level of operating capacity where the production cost per unit produced is a minimum. Changes in the production rate to meet anticipated demand are accomplished by (1) hiring or firing personnel; (2) overtime or undertime; (3) adding a shift; or (4) adding plant capacity. Such actions incur costs that cause the cost per unit produced to increase. Recognizing this relationship, the usual approach^[12,52] is to approximate production costs as a quadratic function of the production rate. Assuming the optimal level of production for a minimum unit cost to be 100% of capacity, the cost function could have a relationship as shown in Figure 10.

This assumption is particularly appropriate to the nuclear fuel cycle even where the facility is a government owned enrichment process. In this case separative work is contracted at a specified cost per separative work unit. Variations in this schedule incur penalties. For example, decreasing the number of separative work units incur cancellation charges. Increases are generally not allowed but when they are made, they result in extra costs. In the general case, changes to the number of separative work units are not made, and any variations necessary are made by exercising the "tails options". The end effect is the same.

The total production cost over a specified time horizon, T, is given by:

$$\int_0^T g[X'(t)]dt \quad \text{IV.8}$$

where $g[X'(t)]$ will assume a quadratic form.

The costs of holding units in inventory is a function of the inventory on hand at time t and is defined as $h[I(t)]$. However, $I(t) = I(0) + X(t) - R(t)$; therefore, the total inventory holding cost over a specified time horizon, T , is given by:

$$\int_0^T h[I(t)] dt = \int_0^T h[I(0) + X(t) - R(t)] dt; \quad \text{IV.9}$$

$$I(0) + X(t) \geq R(t), \quad 0 \leq t \leq T$$

The model for the single stage is now complete. The total cost, C , is given as the sum of the purchasing cost, the production cost, and the holding cost.

$$C = \int_0^T f[X'(t)] dt + \int_0^T g[X'(t)] dt + \int_0^T h[I(0) + X(t) - R(t)] dt \quad \text{IV.10}$$

$$C = \int_0^T \{f[X'(t)] + g[X'(t)] + h[I(0) + X(t) - R(t)]\} dt \quad \text{IV.11}$$

Since shortages are not allowed, cumulative production must equal or exceed cumulative demand, or:

$$I(0) + X(t) \geq R(t) \quad 0 \leq t \leq T \quad \text{IV.12}$$

As a boundary condition, the initial inventory plus the cumulative production must equal the cumulative demand plus the

ending inventory, or:

$$I(0) + X(T) = R(T) + I(T) \quad \text{IV.13}$$

Since $X(t)$ and $R(t)$ are cumulative functions, their initial conditions can be specified. Let $R(0) = 0$ and $X(0) = 0$.

Finally, it is desired to minimize the total cost; therefore, the objective function for the single-stage model is defined as:

$$\min_{X(t)} C = \int_0^T \{f[X'(t)] + g[X'(t)] + h[I(0) + X(t) - R(t)]\} dt \quad \text{IV.14}$$

$$\text{subject to: } I(0) + X(t) \geq R(t) \quad 0 \leq t \leq T$$

$$I(0) + X(T) = R(T) + I(T)$$

$$R(0) = 0$$

$$X(0) = 0$$

B. Multi-stage Production-to-Inventory Model

Now consider a multi-stage production-to-inventory system. Again raw material is acquired from an external source and the finished product is subject to an external demand; however, there are several internal stages. Each internal stage acquires its "raw material" from the preceding inventory. Flow is represented in Figure 11.

Let each stage have the type of cost functions specified for the single-stage system. Further, for the moment, let the cumulative demand occurring on each i^{th} inventory be $R_i(t)$. Then the cost equation for each i^{th} stage is:

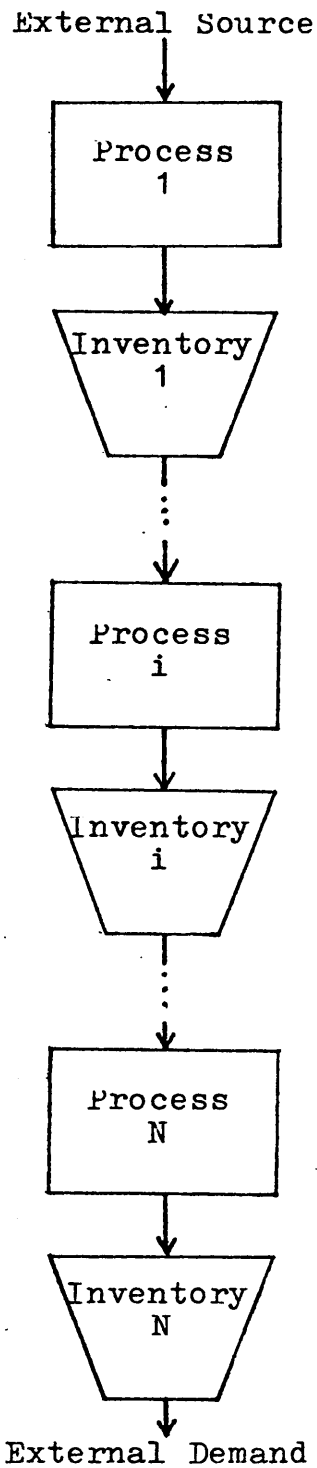


Figure 11. Schematic Diagram of a N-Stage Production-to-Inventory System

$$C_i = \int_0^T \{f_i[X_i'(t)] + g_i[X_i'(t)] + h_i[I_i(0) + X_i(t) - R_i(t)]\} dt \quad \text{IV.15}$$

subject to: $I_i(0) + X_i(t) \geq R_i(t); \quad 0 \leq t \leq T, \quad i=1, \dots, N$

$$I_i(0) + X_i(T) = R_i(T) + I_i(T) \quad i=1, \dots, N$$

$$R_i(0) = 0 \quad i=1, \dots, N$$

$$X_i(0) = 0 \quad i=1, \dots, N$$

The total cost, TC, for the system is:

$$TC = \sum_{i=1}^N C_i = \sum_{i=1}^N \int_0^T \{f_i[X_i'(t)] + g_i[X_i'(t)] + h_i[I_i(0) + X_i(t) - R_i(t)]\} dt \quad \text{IV.16}$$

subject to: $I_i(0) + X_i(t) \geq R_i(t); \quad 0 \leq t \leq T, \quad i=1, \dots, N$

$$I_i(0) + X_i(T) = R_i(T) + I_i(T) \quad i=1, \dots, N$$

$$R_i(0) = 0 \quad i=1, \dots, N$$

$$X_i(0) = 0 \quad i=1, \dots, N$$

In the multi-stage system, each process draws material from the previous inventory; therefore, the cumulative production rate in process $i+1$ is related to the cumulative demand on inventory i by:

$$X_{i+1}(t) = R_i(t) \quad i=1, \dots, N-1 \quad \text{IV.17}$$

Note that raw material for process one is acquired from an external source. Also, recall that the external demand, $R_N(t)$, is known.

Define $k_{i,i+1}$ as the amount of "raw material" in inventory i required to produce one unit of product entering inventory $i+1$. The relationship between production at stage $i+1$ and demand at stage i becomes:

$$(k_{i,i+1}) X_{i+1}(t) = R_i(t) \quad \text{IV.18}$$

Now, suppose that the production process requires some amount of time to convert raw materials to processed materials. In general, assume that ℓ_{i+1} time units elapse from the time raw material is withdrawn from inventory i until the processed product enters inventory $i+1$. The relationship that exists between production at stage $i+1$ and demand at stage i becomes:

$$(k_{i,i+1}) X_{i+1}(t+\ell_{i+1}) = R_i(t) \quad \text{IV.19}$$

Making the appropriate substitutions, the objective function for the multi-stage model is:

$$\min TC = \sum_{i=1}^N \int_0^T \{f_i[X_i'(t)] + g_i[X_i'(t)]\} dt \quad \text{IV.20}$$

$$+ h_i [I_i(0) + X_i(t) - (k_{i,i+1}) X_{i+1}(t+\ell_{i+1})] dt$$

$$\text{subject to: } I_i(0) + X_i(t) \geq (k_{i,i+1}) X_{i+1}(t+\ell_{i+1})$$

$$0 \leq t \leq T - \ell_{i+1}, \quad i=1, \dots, N-1$$

$$I_i(0) + X_i(T-\ell_{i+1}) = (k_{i,i+1}) X_{i+1}(T) + I_i(T)$$

$$i=1, \dots, N-1$$

$$X_i(0) = 0 \quad i=1, \dots, N$$

C. Multi-stage Production-to-Inventory Model with Cyclic Flow

Let the production-to-inventory system be expanded to include cyclic (feedback) flow. Assume that stage N , in producing good products, also produces defective products that can be recycled in process $N+1$. Subsequently, the recycled products are placed in inventory j . Numerous examples exist in addition to the nuclear fuel cycle. One example is steel forgings. The defective forgings are melted down and reenter the system as raw steel ingots. The flow for the general case is shown in Figure 12.

The cost functions remain as before except for stages j , $N-1$, N and $N+1$. For stage j the purchasing cost function and the production cost function remain as before; however, the holding cost function will now include the input production from stage $N+1$. The holding cost function becomes:

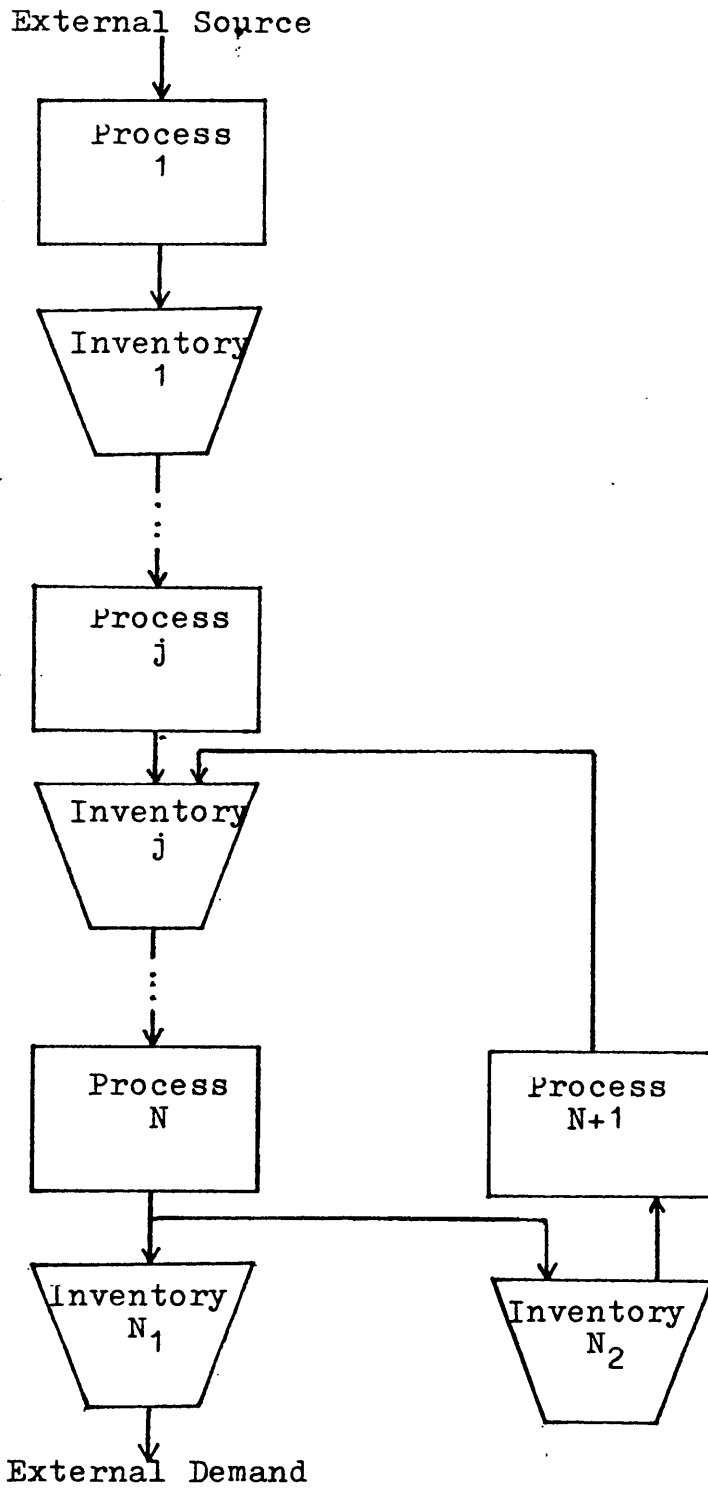


Figure 12. Schematic Diagram of a N-Stage Production-to-Inventory System with Cyclic Flow

$$h_j [I_j(0) + X_j(t) + X_{N+1}(t) - (k_{j,j+1}) X_{j+1}(t+l_{j+1})] \quad \text{IV.21}$$

Changes in stage N result from now having two inventories, e.g., finished products $[I_{N_1}(t)]$ and scrap $[I_{N_2}(t)]$. Let the fraction of production resulting in finished parts be α ($0 \leq \alpha \leq 1$) and the fraction scrap be $1-\alpha$. The amount of raw material now necessary to produce one unit of finished product in inventory N is $(k_{N-1,N})/\alpha$. For stage N the purchasing cost function and the production cost function remain as before. The holding cost functions become:

$$h_{N_1} [I_{N_1}(0) + X_N(t) \cdot \alpha - R_N(t)] + h_{N_2} [I_{N_2}(0) + (1-\alpha)X_N(t) - (k_{N,N+1}) X_{N+1}(t+l_{N+1})] \quad \text{IV.22}$$

The holding cost function for stage N-1 is revised to include the increased demand resulting from some of the produced parts in stage N being defective. The holding cost function for stage N-1 becomes:

$$h_{N-1} [I_{N-1}(0) + X_{N-1}(t) - (k_{N-1,N})X_N(t+l_N)/\alpha] \quad \text{IV.23}$$

Finally, stage N+1 has only a purchasing function and a production function, since it produces to replenish inventory j. The purchasing cost function and production cost function for stage N+1 are:

$$f_{N+1}[X_{N+1}'(t)] + g[X_{N+1}'(t)].$$

IV.24

Making the appropriate substitutions, the objective function for the multi-stage production to inventory system with cyclic (feedback) flow becomes:

$$\min TC = \int_0^T \left[\sum_{i=1}^{j-1} \{f_i[X_i'(t)] + g_i[X_i'(t)] + h_i[I_i(0) \right. \quad \text{IV.25}$$

$$\left. + X_i(t) - (k_{i,i+1})X_{i+1}(t+\ell_{i+1})\} + f_j[X_j'(t)]$$

$$+ g_j[X_j'(t)] + h_j[I_j(0) + X_j(t) + X_{N+1}(t)$$

$$- (k_{j,j+1})X_{j+1}(t+\ell_{j+1})] + \sum_{i=j+1}^{N-2} \{f_i[X_i'(t)]$$

$$+ g_i[X_i'(t)] + h_i[I_i(0) + X_i(t) - (k_{i,i+1})X_{i+1}(t+\ell_{i+1})]\}$$

$$+ f_{N-1}[X_{N-1}'(t)] + g_{N-1}[X_{N-1}'(t)] + h_{N-1}[I_{N-1}(0)$$

$$+ X_{N-1}(t) - (k_{N-1,N})X_N(t+\ell_N)/\alpha] + f_N[X_N'(t)$$

$$+ g_N[X_N'(t)] + h_{N-1}[I_{N-1}(0) + \alpha X_N(t) - R_N(t)]$$

$$+ h_{N-2}[I_{N-2}(0) + (1-\alpha)X_N - (k_{N,N+1})X_{N+1}(t+\ell_{N+1})]$$

$$+ f_{N+1}[X_{N+1}'(t)] + g_{N+1}[X_{N+1}'(t)] \Big] dt$$

subject to:

a. Boundary Conditions:

$$(1) X_i(0) = 0 \quad i=1, \dots, N+1$$

$$(2) I_i(0) + X_i(T-\ell_{i+1}) = (k_{i,i+1})X_{i+1}(T) + I_i(T)$$

$$i=1, \dots, N-1; \quad i \neq j-1$$

$$(3) I_{j-1}(0) + X_{j-1}(T-\ell_j) + X_{N+1}(T-\ell_j) = (k_{j-1,j})X_j(T)$$

b. Constraints:

$$(1) I_i(0) + X_i(t) \geq (k_{i,i+1})X_{i+1}(t+\ell_{i+1})$$

$$0 \leq t \leq T - \ell_{i+1}; \quad i=1, \dots, N-2; \quad i \neq j$$

$$(2) I_j(0) + X_j(t) + X_{N+1}(t) \geq (k_{j,j+1})X_{j+1}(t+\ell_{j+1});$$

$$0 \leq t \leq T - \ell_{j+1}$$

$$(3) I_{N-1}(0) + X_{N-1}(t) \geq (k_{N-1,N})X_N(t+\ell_N)/\alpha$$

$$0 \leq t \leq T - \ell_N$$

$$(4) I_{N_1}(0) + \alpha X_N(t) \geq R_N(t) \quad 0 \leq t \leq T$$

$$(5) I_{N_2}(0) + (1-\alpha)X_N(t) \geq (k_{N_2,N+1})X_{N+1}(t+\ell_{N+1})$$

The model is now complete. Its purpose is to find optimal stage-wise cumulative production trajectories over the horizon of interest which will minimize acquisition, production, and inventory holding

costs while insuring that demands are met. The objective function assumes the form of minimizing an integral containing several unknown functions all of which contain a single independent variable, time. This class of problems is called functionals. The following chapter develops the mathematical foundation for solving this class of problems.

CHAPTER V.

MATHEMATICAL FOUNDATION

In the previous chapter, an objective function is developed that assumes the form of finding the minimum of the integral of several unknown functions containing a single independent variable subject to given constraints. The solution of such an objective function falls properly into the calculus of variations.

The mathematical foundation for the solution of this class of problems is developed by considering first necessary and sufficient conditions for minimizing integrals with a single unknown function, then integrals with several unknown functions. Next, transversality conditions are developed for functionals with unknown boundary conditions. Finally, a method is demonstrated for solving functionals subject to inequality constraints.

A. Necessary Conditions for an Extremum

1. Single Unknown Function

Let $y(t)$ be an unknown function, $y'(t)$ be its first derivative with respect to the independent variable, and let t be the independent variable. Further, let $y(t)$ and $y'(t)$ be abbreviated by y and y' respectively. Assume the minimum is sought to the following functional:

$$\min I = \int_a^b F(y, y', t) dt \quad \text{V.1}$$

This is referred to as the "simplest problem" in the calculus of variations. The necessary condition for a relative minimum is given by the Euler-Lagrange equation:

$$\frac{\partial F}{\partial y} - \frac{d}{dt} \left(\frac{\partial F}{\partial y'} \right) = 0 \quad \text{V.2}$$

The proof of this necessary condition is found in several texts [27,28,62] and, therefore, is not repeated here.

2. Several Unknown Functions

The previously identified objective function contains several unknown functions; therefore, the Euler-Lagrange equation must be expanded to meet this requirement. The necessary conditions for extremizing a functional containing several unknown functions are rarely found [24]. Therefore, proof of these necessary conditions is developed in Appendix B. The resulting equations to extremize the functional:

$$\min J = \int_a^b F(y_1, y_1', y_2, y_2', \dots, y_n, y_n', t) dt \quad \text{V.3}$$

are:

$$\frac{\partial F}{\partial y_1} - \frac{d}{dt} \left(\frac{\partial F}{\partial y_1'} \right) = 0 \quad \text{V.4}$$

$$\frac{\partial F}{\partial y_2} - \frac{d}{dt} \left(\frac{\partial F}{\partial y_2'} \right) = 0$$

⋮

$$\frac{\partial F}{\partial y_n} - \frac{d}{dt} \left(\frac{\partial F}{\partial y_n'} \right) = 0.$$

B. Sufficient Conditions for an Extremum

Sufficient conditions for an extremum for unconstrained functionals are developed in Appendix B. These conditions are known as the Legendre conditions and are repeated here. Let

$$F_{y_i', y_j'} \equiv \frac{\partial^2 F}{\partial y_i' \partial y_j'} \quad \text{V.5}$$

Then, in order for the minimum of a functional to be obtained at the extremal, it is necessary to satisfy the following chain of inequalities:

$$F_{y_1', y_1'} \geq 0 \quad \text{V.6}$$

$$\begin{vmatrix} F_{y_1', y_1'} & F_{y_1', y_2'} \\ F_{y_2', y_1'} & F_{y_2', y_2'} \\ \vdots & \vdots \\ F_{y_1', y_1'} & \cdots & F_{y_1', y_n'} \\ \vdots & & \vdots \\ F_{y_n', y_1'} & \cdots & F_{y_n', y_n'} \end{vmatrix} \geq 0$$

Subsequent to the Legendre conditions, the strengthened Legendre conditions, the Jacobi condition, and the Weierstrass condition have been developed. PETROV^[62] states that it has been shown that

these conditions are also necessary. In this regard, the Euler-Lagrange equations establish necessary conditions locally while the Jacobi condition and the Weierstrass condition establish necessary conditions throughout a field. In any case, PETROV points out that investigation of the latter two sufficiency conditions is awkward and difficult and that wherever possible a more practical approach should be taken. Specifically, he states that if the minimum (maximum) of a functional in a given class of functions exists and the extremal is unique, it may be asserted without any analysis of the sufficient conditions, that the minimum (maximum) is reached on the extremal.

The recent works of KROTOV^[33,34,35,36] make this approach all the more appealing. KROTOV derived a test for determining the class of admissible functions in which the extremal will be found. Once it is known from the KROTOV test that the extremal falls in a "given class" there remains only the determination of uniqueness of the extremal, i.e., does the solution of the Euler-Lagrange differential equations yield an unique extremal.

KROTOV's works^[33,34,35,36] are summarized in Appendix E.

Recall that the extremals sought represent cumulative production functions. The production rates are assumed to be continuous; therefore the cumulative production function should be continuous and smooth. From Appendix E, the admissible class of functionals must be of the "first kind" and the KROTOV test must have the following solutions (see E.14 and E.15):

$$\lim_{y' \rightarrow +\infty} F(x, y, y') \frac{1}{y'} \rightarrow +\infty \quad \text{V.7}$$

$$\lim_{y' \rightarrow -\infty} F(x, y, y') \frac{1}{y'} \rightarrow -\infty \quad \text{V.8}$$

To summarize, the following steps will be taken to insure that the minimizing extremal is found:

1. The Legendre conditions will be applied to insure a local minimum.
2. The KROTOV test in V.7 and V.8 will be applied to assure that the extremal is in the given class of piecewise smooth functions.
3. The solution to the Euler-Lagrange differential equation will be examined to insure uniqueness.

C. Transversality Conditions

A most important result is now developed. This result is the key to solving the cyclic (feedback) production to inventory problems.

In the problem statement, all end points can be specified with the exception of the point where the feedback rejoins the mainstream flow. Demand here is unknown. Once this demand is known, all end points are known. What is required is a relaxation of the end point condition.

Consider equation B.14 (Appendix B) which is repeated here:

$$\begin{aligned}
\delta I = 0 = & \int_a^b \left[\frac{\partial F}{\partial y_1} \eta_1 + \frac{\partial F}{\partial y_2} \eta_2 + \dots + \frac{\partial F}{\partial y_n} \eta_n \right. \\
& - \eta_1 \frac{d}{dt} \left(\frac{\partial F}{\partial y_1'} \right) - \eta_2 \frac{d}{dt} \left(\frac{\partial F}{\partial y_2'} \right) - \dots - \eta_n \frac{d}{dt} \left(\frac{\partial F}{\partial y_n'} \right) \Big] dt \\
& + \left[\frac{\partial F}{\partial y_1'} \eta_1 + \frac{\partial F}{\partial y_2'} \eta_2 + \dots + \frac{\partial F}{\partial y_n'} \eta_n \right] \Big|_a^b
\end{aligned} \tag{V.9}$$

In the further development of this relationship, it is assumed that $\eta_i(a) = \eta_i(b) = 0$. The end points are known and fixed. Now assume the end points are unknown, i.e., $\eta_i(a), \eta_i(b) \neq 0$. The first variation must be equal to zero; therefore since $\eta_i(t)$ is arbitrary, the conclusion is that

$$\frac{\partial F}{\partial y_i'} = 0 \quad i=1, \dots, n \quad \text{for } t=a, b. \tag{V.10}$$

The remainder of the development of necessary and sufficient conditions is as shown in Appendix B. These results are as shown in PETROV^[62] and are known as transversality conditions.

D. Extremizing Inequality Constrained Functionals

The last development in the mathematical foundation is a little known proof for extremizing functionals subject to inequality constraints.

Assume the extremum is sought for the following functional:

$$I = \int_a^b F(y, y', t) dt \quad \text{V.11}$$

subject to

$$y(t) \geq g(t). \quad \text{V.12}$$

The equality $y = g(t)$ defines the boundary of the admissible domain within which the function achieving the extremum may be found. In other words, the extremum must lie in a closed region consisting of the domain and its boundary. The Euler-Lagrange equation developed in Appendix B has no such restriction. In fact, it assumes freedom of variation. Obviously, if the extremal lies on the boundary of the domain, it has no freedom of variation in one direction.

Introduce another function, $z(t)$, such that

$$z^2 = y - g \quad \text{V.13}$$

From equation V.13 it follows that $2zz' = y' - g'$, and that $y' = 2zz' + g'$. The functional in equation V.11 becomes:

$$I = \int_a^b F[z^2 + g, 2zz' + g', t] dt \quad \text{V.14}$$

No restrictions have been imposed on the new function $z(t)$, and the value $z=0$ merely corresponds to the domain boundary. Using the objective with the new variable, z , the extremal is sought in the usual manner.

The Euler-Lagrange equation for the functional in z must now be satisfied:

$$\frac{\partial F}{\partial z} - \frac{d}{dt} \left(\frac{\partial F}{\partial z'} \right) = 0 \quad \text{V.15}$$

Now,

$$\frac{\partial F}{\partial z} = \frac{\partial F}{\partial y} \frac{\partial y}{\partial z} + \frac{\partial F}{\partial y'} \frac{\partial y'}{\partial z} = \frac{\partial F}{\partial y} (2z) + \frac{\partial F}{\partial y'} (2z') \quad \text{V.16}$$

and

$$\frac{\partial F}{\partial z'} = \frac{\partial F}{\partial y'} \frac{\partial y'}{\partial z'} = \frac{\partial F}{\partial y'} (2z). \quad \text{V.17}$$

Therefore,

$$\frac{d}{dt} \left[\frac{\partial F}{\partial z'} \right] = \frac{d}{dt} \left[\frac{\partial F}{\partial y'} (2z) \right] = 2z' \frac{\partial F}{\partial y'} + 2z \frac{d}{dt} \left(\frac{\partial F}{\partial y'} \right). \quad \text{V.18}$$

Finally,

$$\frac{\partial F}{\partial z} - \frac{d}{dt} \left[\frac{\partial F}{\partial z'} \right] = 2z \frac{\partial F}{\partial y} - 2z \left[\frac{d}{dt} \left(\frac{\partial F}{\partial y'} \right) \right] \quad \text{V.19}$$

From equation V.19 above:

$$2z \left[\frac{\partial F}{\partial y} - \frac{d}{dt} \left(\frac{\partial F}{\partial y'} \right) \right] = 0 \quad \text{V.20}$$

The conclusion is either:

$$\frac{\partial F}{\partial y} - \frac{d}{dt} \left(\frac{\partial F}{\partial y'} \right) = 0 \quad \text{V.21}$$

which is the Euler-Lagrange equation for the objective functional,
or:

$$z=0$$

V.22

which means the solution is on the boundary of the closed region, i.e., on the constraint. In other words, the extremum of the functional is achieved on either the extremal or on the boundary of the admissible domain. Also, it may be a curve comprised of several extremal segments and the boundary segments.

To find the complete solution, the condition at the point of passage from the extremal to the boundary must be found.

Assume that the passage from the extremal to the boundary occurs at one point, t_1 . At this point, $y(t) = g(t)$. Now,

$$I = \int_a^{t_1} F(y, y', t) dt + \int_{t_1}^b F(g, g', t) dt \quad \text{V.23}$$

Consider first the portion of the composite curve which is the extremal:

$$I_1 = \int_a^{t_1} F(y, y', t) dt. \quad \text{V.24}$$

Since t_1 is not known, $y(t_1)$ is not known. Hence, this problem is similar to the problem with no fixed end point. (The difference is that here t_1 is not known).

Consider first the general case where both end points are not fixed. Let the variation in $y(t)$ be $y(t) + \epsilon h(t)$ and be as shown in Figure 13.

The increment in the functional as y passes to $y+\epsilon h$ is written as:

$$\Delta I = I(y+\epsilon h) - I(y) = \int_{t_0+\delta t_0}^{t_1+\delta t_1} F(y+\epsilon h, y'+\epsilon h', t) dt \quad \text{V.25}$$

$$- \int_{t_0}^{t_1} F(y, y', t) dt$$

$$= \int_{t_0}^{t_1} [F(y+\epsilon h, y'+\epsilon h', t) - F(y, y', t)] dt \quad \text{V.26}$$

$$+ \int_{t_1}^{t_1+\delta t_1} F(y+\epsilon h, y'+\epsilon h', t) dt$$

$$- \int_{t_0}^{t_0+\delta t_0} F(y+\epsilon h, y'+\epsilon h', t) dt$$

The first term in V.26 is reduced in identical steps as equation B.5 is reduced to B.10. The result is:

$$\int_{t_0}^{t_1} [F(y+\epsilon h, y'+\epsilon h', t) - F(y, y', t)] dt \quad \text{V.27}$$

$$= \int_{t_0}^{t_1} \left[\frac{\partial F}{\partial y} h + \frac{\partial F}{\partial y'} h' \right] dt$$

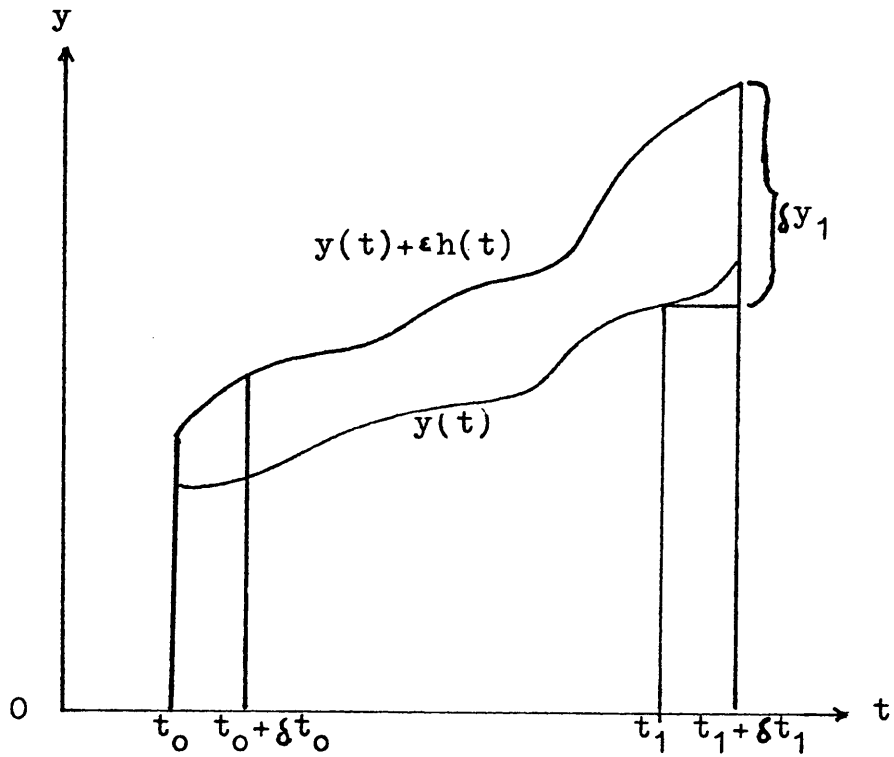


Figure 13. Functional with Variable Endpoints

Using a linear approximation for infinitesimal variations, the second and third terms of equation V.26 become:

$$\int_{t_1}^{t_1+\delta t_1} F(y+\epsilon h, y'+\epsilon h', t) dt \sim F \Big|_{t_1}^{\delta t_1} \quad \text{V.28}$$

$$\int_{t_0}^{t_0+\delta t_0} F(y+\epsilon h, y'+\epsilon h', t) dt \sim F \Big|_{t_0}^{\delta t_0} \quad \text{V.29}$$

The second member of equation V.27 is integrated by parts identically to equations B.10 to B.14, and the result becomes:

$$\begin{aligned} \delta I = & \int_{t_0}^{t_1} \left[\frac{\partial F}{\partial y} - \frac{d}{dt} \left(\frac{\partial F}{\partial y'} \right) \right] h dx + \frac{\partial F}{\partial y'} h \Big|_{t_0}^{t_1} + F \Big|_{t_1}^{\delta t_1} \\ & - F \Big|_{t_0}^{\delta t_0} \end{aligned} \quad \text{V.30}$$

Looking at Figure 13 it is seen that to the accuracy of higher order infinitesimals, $h(t_0)$ and $h(t_1)$ can be approximated by:

$$h(t_0) \sim \delta y_0 - y' \delta t_0; \quad h(t_1) \sim \delta y_1 - y' \delta t_1 \quad \text{V.31}$$

Substituting V.31 into V.30:

$$\delta I = \int_{t_0}^{t_1} \left[\frac{\partial F}{\partial y} - \frac{d}{dt} \left(\frac{\partial F}{\partial y'} \right) \right] h dt + \left. \frac{\partial F}{\partial y'} \right|_{t_1} (\delta y_1 - y' \delta t_1) \quad \text{V.32}$$

$$- \left. \frac{\partial F}{\partial y'} \right|_{t_0} (\delta y_0 - y' \delta t_0) + F \Big|_{t_1} \delta t_1 - F \Big|_{t_0} \delta t_0$$

$$\delta I = \int_{t_0}^{t_1} \left[\frac{\partial F}{\partial y} - \frac{d}{dt} \left(\frac{\partial F}{\partial y'} \right) \right] h dt + \left. \frac{\partial F}{\partial y'} \right|_{t_1} \delta y_1 \quad \text{V.33}$$

$$+ (F - y' \frac{\partial F}{\partial y'}) \Big|_{t_1} \delta t_1 - \left. \frac{\partial F}{\partial y'} \right|_{t_0} \delta y_0$$

$$- (F - y' \frac{\partial F}{\partial y'}) \Big|_{t_0} \delta t_0$$

In summary, equation V.33 expresses the variation of the functional as the variation of $y(t)$ within the original range of integration and variation outside the end points as a result of varying the end points. For the extremal, the integral part of V.33 must vanish leaving:

$$\delta I = \left. \frac{\partial F}{\partial y'} \right|_{t_1} \delta y_1 + (F - y' \frac{\partial F}{\partial y'}) \Big|_{t_1} \delta t_1 \quad \text{V.34}$$

$$- \left. \frac{\partial F}{\partial y'} \right|_{t_0} \delta y_0 - (F - y' \frac{\partial F}{\partial y'}) \Big|_{t_0} \delta t_0$$

Now, suppose that the two end points slide along two curves $y = m(t)$ and $y = n(t)$. To the accuracy of higher order infinitesimals,

$$\delta y_0 = m'(t)\delta t_0; \delta y_1 = n'(t)\delta t_1 \quad \text{V.35}$$

At the extremum $\delta I=0$ and the substitution of V.35 into V.33 becomes

$$\delta I = 0 = \left. \frac{\partial F}{\partial y'} \right|_{t_1} n' \delta t_1 + (F - y' \left. \frac{\partial F}{\partial y'} \right) \Big|_{t_1} \delta t_1 \quad \text{V.36}$$

$$- \left. \frac{\partial F}{\partial y'} \right|_{t_0} m' \delta t_0 + (F - y' \left. \frac{\partial F}{\partial y'} \right) \Big|_{t_0} \delta t_0$$

$$\delta I = 0 = \left[\left. \frac{\partial F}{\partial y'} n' + F - y' \frac{\partial F}{\partial y'} \right] \Big|_{t_1} \delta t_1 \quad \text{V.37}$$

$$- \left[\left. \frac{\partial F}{\partial y'} m' + F - y' \frac{\partial F}{\partial y'} \right] \Big|_{t_0} \delta t_0$$

Since δt_0 and δt_1 are arbitrary and independent increments, it follows that:

$$\left[(m' - y') \frac{\partial F}{\partial y'} + F \right] \Big|_{t_0} = 0 \quad \text{V.38}$$

$$\left[(n' - y') \frac{\partial F}{\partial y'} + F \right] \Big|_{t_1} = 0 \quad \text{V.39}$$

Returning now to equation V.23, it is seen that the left end point is fixed (equation V.38 does not apply) and that the right end point is not fixed (equation V.38 does apply). Combining these results:

$$\delta I_1 = \left[(g' - y') \frac{\partial F}{\partial y'} + F \right] \Big|_{t_1} \delta t_1 \quad \text{V.40}$$

Consider now the second integral in equation V.22. Since this functional lies on the boundary, the only variation possible at t_1 is δt_1 ; therefore,

$$\delta I_2 = \int_{t_1 - \delta t_1}^b F(g, g', t) dt - \int_{t_1}^b F(g, g', t) dt \quad \text{V.41}$$

or

$$\delta I_2 = [F(g, g', t)] \Big|_{t_1} (-\delta t_1) \quad \text{V.42}$$

Since the extremum is assumed on the composite curve, $\delta I = \delta I_1 + \delta I_2 = 0$, the result is:

$$\begin{aligned} \delta I = 0 = \delta I_1 + \delta I_2 &= [F(g, g', t)] \Big|_{t_1} (-\delta t_1) \quad \text{V.43} \\ &+ \left[(g' - y') \frac{\partial F}{\partial y'} + F \right] \Big|_{t_1} \delta t_1 \end{aligned}$$

Because δt_1 is arbitrary and because at t_1 , $y=g$, V.43 reduces to:

$$-F(y, g', t) + (g' - y') \left. \frac{\partial F}{\partial y'} \right|_{t_1} + F = 0 \quad \text{V.44}$$

$$F(y, y', t) - F(y, g', t) = (y' - g') \left. \frac{\partial F}{\partial y'} \right|_{t_1} = 0 \quad \text{V.45}$$

The difference of $F(y, y', t) - F(y, g', t)$ is transformed using the Lagrange theorem of the mean:

$$f(a) - f(b) = (a-b)f(c), \text{ where } a < c < b.$$

Adapting V.45 accordingly results in:

$$F(y, y', t) - F(y, g', t) = (y' - g') \left. \frac{\partial}{\partial y'} [F(y, q', t)] \right|_{t_1} \quad \text{V.46}$$

where $y' < q' < g'$.

Equating right hand sides of V.45 and V.46:

$$(y' - g') \left. \frac{\partial}{\partial y'} [F(y, y', t)] \right|_{t_1} = (y' - g') \left. \frac{\partial}{\partial y'} [F(y, q', t)] \right|_{t_1} \quad \text{V.47}$$

or

$$(y' - g') \left[\left. \frac{\partial}{\partial y'} \{F(y, q', t)\} - \left. \frac{\partial}{\partial y'} \{F(y, y', t)\} \right] = 0 \quad \text{V.48}$$

It follows immediately that $y'(t) = g'(t)$ because q is some intermediate value between y and g and as such has no restriction on q' . Therefore, in the general case the term in brackets in equation V.48 is not equal to zero. The conclusion is that at t_1 , the slope of tangent to y (the extremal) is equal to the slope of the tangent to g (the constraint). This important deduction enables all pieces of extremals and boundary curves of which the composite extremal exists to be found. Finally, since $g(t)$ is known, the intersections of all extremal pieces with the boundary can be found by solving a set of simultaneous equations numerically. Thus, all constants of integration can be found.

Recognition of the implications of this result, the transversality conditions, and KROTOV's works provides the key to solving the multi-stage production-to-inventory system with cyclic flow. This result demonstrates that the derivative of the extremal with respect to time at the point of passage to the constraint is equal to the derivative of the constraint with respect to time at the point of passage. Since the extremal intersects the constraint at the point of passage, the two curves share a common point. Two curves sharing a common point and having equal slopes at that point form a composite curve that is smooth about that point. KROTOV's test assures smoothness otherwise. Finally, the remaining obstacle is that the boundary condition at the point the recycle reenters the mainstream is not known. Application of the transversality conditions removes this obstacle. There remains only a discussion of the sufficiency conditions for functionals

constrained by inequalities.

E. Sufficient Conditions for Inequality Constrained Functionals

As previously stated the solution to inequality constrained functionals is generally a composite curve consisting of extremals and constraints. Sufficiency conditions for extremals are developed in section V.B. There remains only the question of what can be said about the portion(s) of the curve coincident with the domain boundary.

Because the curve is on the domain boundary, variation is only possible in one direction. Assume that a variation δy is appended to the curve where $y(t)$ is equal to $g(t)$. In order to avoid violation of the inequality, $\delta y > 0$. The variation in the functional at this point is:

$$\delta I = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial g} - \frac{d}{dt} \left(\frac{\partial F}{\partial g'} \right) \right] \delta y dx \quad V.49$$

If $y(t)$ is a minimum, then:

$$\delta I \geq 0 \quad V.50$$

However, no basis exists for deducing, as before, that:

$$\frac{\partial F}{\partial g} - \frac{d}{dt} \left(\frac{\partial F}{\partial g'} \right) = 0 \quad V.51$$

It is only asserted that:

$$\frac{\partial F}{\partial g} - \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{g}} \right) \geq 0.$$

V.52

The final conclusion is that in order for $y(x)$ to yield the minimum for the portions of the curve coincident with the domain boundary, it is necessary that the inequality in V.52 apply.

The mathematical development is now complete. Attention is now directed toward solving the specific problem.

CHAPTER VI.

SPECIFICS OF THE PROBLEM AND THE SOLUTION

To proceed with the solution to the specific problem, the nuclear fuel cycle is formulated as a multi-stage production-to-inventory system with cyclic (feedback) flow. The external demand for energy is translated into a continuous time-dependent equation representing the demand for fabricated fuel reloads. Cost equations are then developed for production, inventory holding, and acquisition costs. Representative model parameter values are then tabulated. At this point, the general form of the solution is found. From the general form, the specific solution is generated by applying boundary conditions, transversality conditions, and the technique for solving inequality constrained functionals. Sufficiency conditions are then applied to insure the solution is a minimum. Finally, the results are discussed.

A. Formulation of the Nuclear Fuel Cycle as a Multi-stage Cyclic Production-to-Inventory System

The basis of the model formulation is a conceptualization of the nuclear fuel cycle that is consistent with a production and inventory perspective. This view of the nuclear fuel cycle is constructed by treating each of the stages of the cycle as a production facility and by associating with each such facility an inventory of output. In addition, exploration for uranium ore reserves is treated as a production stage with an associated inventory corresponding to the reserves located. Thus, each stage of the fuel cycle is represented

as a basic building block of the type first used by ZANGWILL^[78] and shown in Figure 14.

Under this concept of the stages, the nuclear fuel cycle and its material flows can be represented in the form illustrated in Figure 15. It should be noted that each stage can be expanded into several parallel plants and that the stages of enrichment and irradiation in the reactor can be represented as several stages each of which corresponds to a separate product enrichment. In addition, aggregation of several parallel facilities or several enrichments into a single representative stage is also reasonable, and this approach is used here.

As is indicated in Figure 15, the output materials from a given stage become the input materials for another stage. However, a model using this representation of the cycle will be "driven" by the demand for energy from the reactor. Thus, this conceptual representation of the fuel cycle should permit the analysis of the implications of customer demand for energy upon the production operations throughout the fuel cycle. It will also permit the analysis of strategies for using vertical integration of the cycle to stockpile fuel materials.

One further adaptation is necessary for the nuclear fuel cycle. Reprocessed UF_6 , in general, has a different percent of U-235 enrichment than natural feed UF_6 . Different enrichment levels require different SWU, and, therefore the enrichment stage has different production and cost characteristics. Because of this, an additional stage, enrichment of reprocessed UF_6 , is introduced. The nuclear

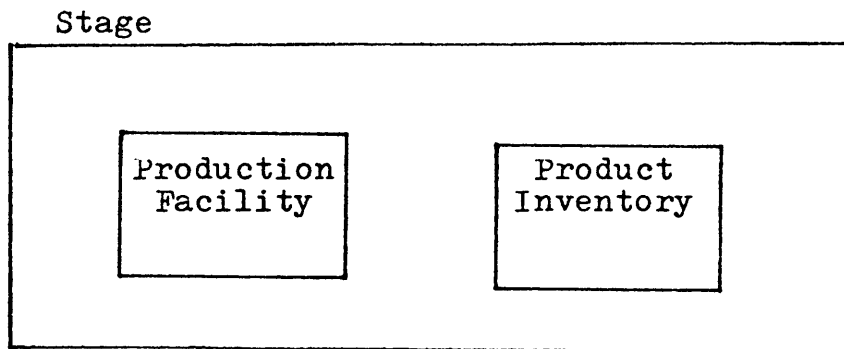


Figure 14. Basic Stage Composition⁽⁷⁸⁾

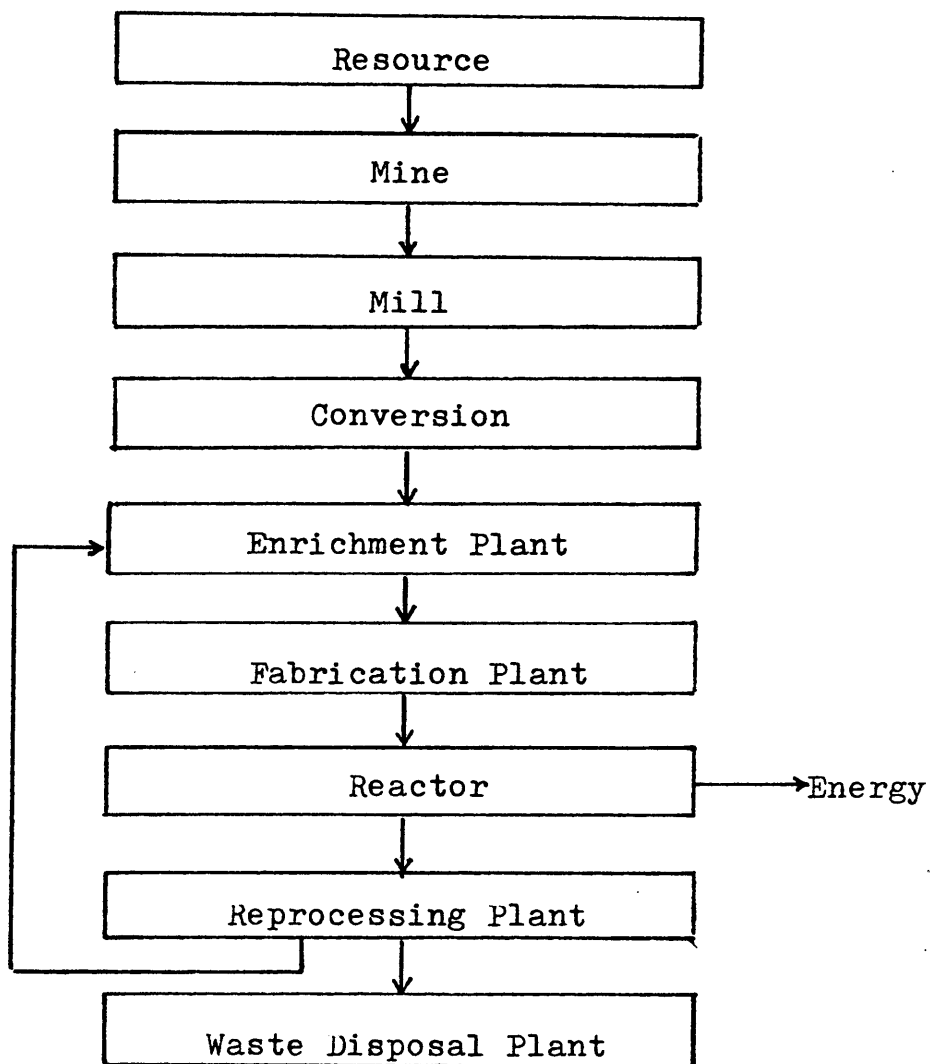


Figure 15. Nuclear Fuel Cycle as Set of Production Stages

fuel cycle production-to-inventory system, as adapted, is shown in Figure 16.

B. Development of the Nuclear Fuel Demand Equations

The purpose here is to develop a macroscopic management tool. Recent works^[39,40,41] attempting similar purposes tend to standardize assumptions as to reactor size, U-235 enrichment percentages, discharge enrichment assays, etc. These same assumptions are used here to permit comparison of results.

It is assumed that a typical reactor is in a one thousand megawatt electric generating plant. An average fuel enrichment of three percent U-235 is inserted into the reactor and the discharge assay of the spent fuel removed is .86 percent U-235. A full core load is inserted when the reactor comes on line, and each year one-third of the core is replaced with fresh fuel (a reload). Based on this scenario, the annual demand for fuel reloads can be approximated by assessing one reload for each reactor on line the previous year plus three reloads for each reactor that comes on line that year. The "World List of Nuclear Power Plants"^[60] gives the history of the operation of all nuclear power plants in the United States. It also projects all nuclear power plants scheduled to come on line through year 2000. Because approximately eight years are presently required to construct and license such a plant, the schedule for the next eight years is realistic. MORRISON's^[54] polynomial forecasting technique described in Appendix C uses the history and schedule to

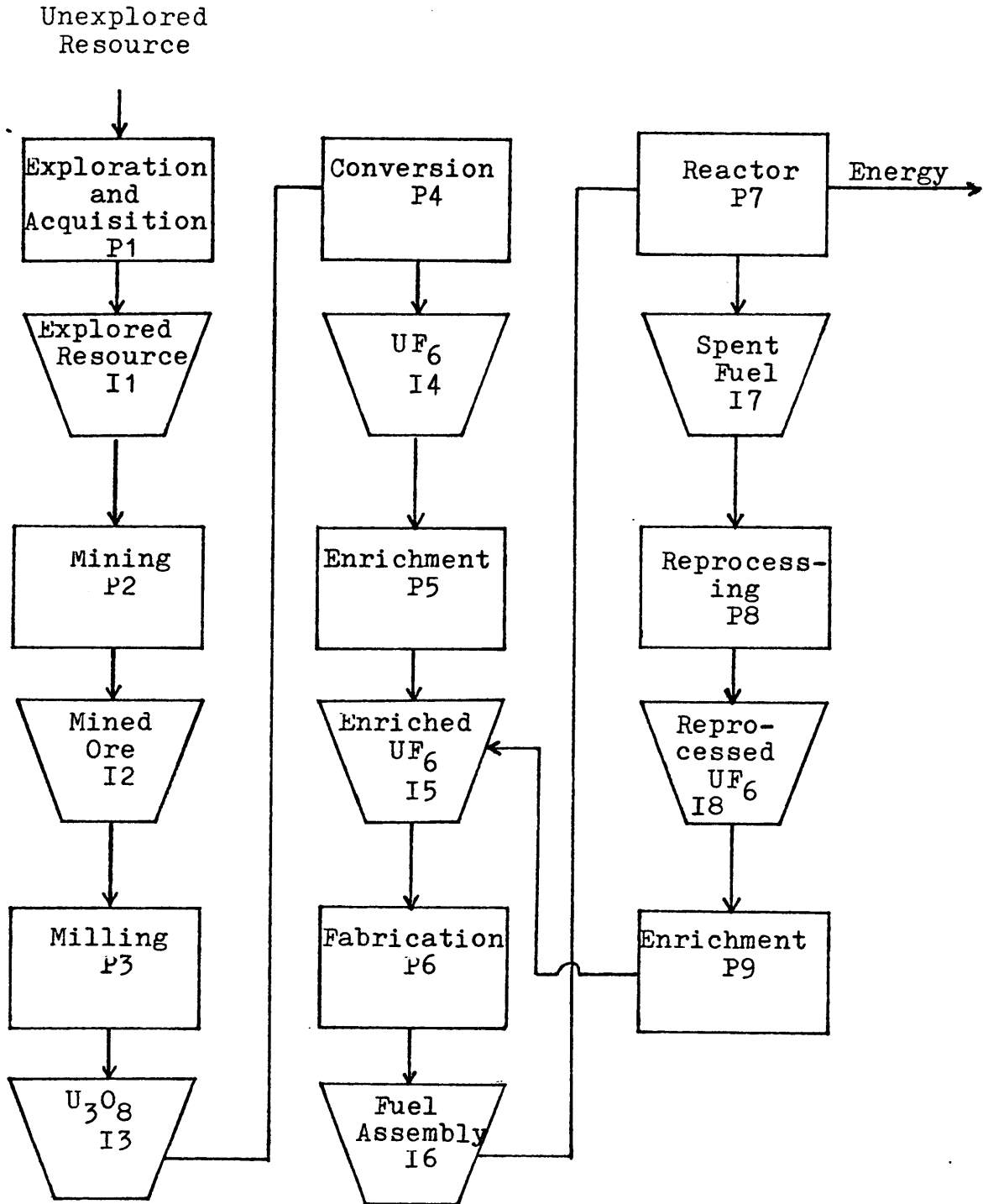


Figure 16. Nuclear Fuel Cycle as a Cyclic Production-to-Inventory System

predict nuclear reload demand through year 1994. The results of this analysis are depicted in Table 5. It should be noted that the forecast is based on cumulative reloads demanded. For ease of computation, this cumulative figure is adjusted to a base year of 1976.

A measure of the forecast error is obtained using the procedure in Appendix C. To estimate the variance of the error, the forecasted results are compared to ten years of known data. The comparison is shown in Table 6.

The estimate of the variance of the error is given by:

$$s^2 = \frac{\sum (\text{difference})^2}{n-1} = \frac{29,621}{9} = 3,291.2 \quad \text{VI.1}$$

The estimate of the variance of the error of the forecast for the ten-year-ahead forecast is determined from equation C.46:

$$\begin{aligned} \sigma^2 W(10)W(10)^T &\approx s^2 W(10)W(10)^T = 3,291.2 \left[\frac{40,883,512}{858^2} \right] \quad \text{VI.2} \\ &= 182,779 \end{aligned}$$

This corresponds to three standard deviations of 1,283 reloads.

A measure of the association of the forecasted demand to the forecast year is given by the index of determination, R^2 . For the data shown in Table 6, the total sum of the squares of the deviations is 105,081,273. The sum of the squares due to regression is 102,568,282. Thus, $R^2 = .976$.

For the model developed, a continuous demand function is assumed. To find the nominal trajectory for the forecast, OSTLE's^[61] quadratic

Table 5
Ten-Year Forecast of Reactor Reloads Based on
Presently Scheduled Reactors

<u>Year</u>	<u>Reactor Start-up</u>	<u>Cumulative Reactors</u>	<u>Equivalent Reloads Rqd.</u>	<u>Cumulative Reloads</u>	<u>Adjusted to Base Year (1976)</u>
1960	1	1	3	3	
61	1	2	4	7	
62	2	4	8	15	
63	1	5	7	22	
64	0	5	5	27	
65	0	5	5	32	
66	1	6	8	40	
67	0	6	6	46	
68	2	8	12	58	
69	3	11	17	75	
70	4	15	23	98	
71	4	19	27	125	
72	8	27	43	168	
73	7	34	48	216	
74	12	46	70	286	
75	8	54	70	356	
76	5	59	69	425	0
77	[10]*	[69]	[89]	[514]	[89]
78	[5]	[74]	[84]	[598]	[173]
79	[9]	[83]	[101]	[699]	[274]
80	[10]	[93]	[113]	[812]	[387]
81	[17]	[110]	[144]	[956]	[531]
82	[14]	[124]	[152]	[1108]	[683]
83	[17]	[141]	[175]	[1283]	[851]
84	[17]	[157]	[192]	[1475]	[1050]
85				(1661)**	(1235)
86				(1870)	(1445)
87				(2096)	(1671)
88				(2336)	(1911)
89				(2590)	(2165)
90				(2858)	(2433)
91				(3141)	(2715)
92				(3439)	(3014)
93				(3749)	(3324)
94				(4077)	(3652)

*Brackets represent expected reloads based on actual schedule of reactors to come on line from 1977-1984.

**Parenthesis represent expected reloads based on forecasting techniques.

Table 6

Error of the Forecast for Reloads

<u>Year</u>	<u>Actual</u>	<u>Forecast</u>	<u>Difference</u>	<u>(Difference)²</u>
1975	356	334	22	484
1976	425	400	25	625
1977	514	473	41	1681
1978	598	552	46	2116
1979	699	648	51	2601
1980	812	872	-60	3600
1981	956	972	-16	256
1982	1108	1069	56	3136
1983	1283	1164	119	14161
1984	1475	1444	31	961

$$\Sigma (\text{difference})^2 = 29621$$

regression technique is applied. The nominal trajectory for the cumulative reloads required by the reactors as a function of time is given by:

$$7.306 t^2 + 71.606 t \quad \text{VI.3}$$

Again $t=0$ corresponds to a base year of 1976.

Each year one-third of the core is replaced with a reload. The discharged spent fuel resides in "cooling off" storage for six months. It is as if the reactor were producing spent fuel reloads with a production lag time of $1 \frac{1}{2}$ years. It is necessary to express this "cumulative production rate" as a function of time. Table 7 shows the discharge of spent reloads developed in the same manner as the demand for reloads. For this case the estimate of the variance in the error of the forecast is 6,327. This corresponds to three standard deviations of 1,778 reloads. Here the index of determination, R^2 , is equal to .967.

Using the data in Table 7, the quadratic regression yields the following expression for the reloads discharged as a function of time:

$$4.292 t^2 + 73.86 t \quad \text{VI.4}$$

C. Development of the Cost Equations

1. Production Cost Equations

The form of the production cost equations is discussed in Chapter III. Specifically, an optimal level of production is assumed, and any variation of the actual production rate from this optimal

Table 7

Ten-Year Forecast of Reactor Discharge of Equivalent Reloads

<u>Year</u>	<u>Cumulative Reactors On Line</u>	<u>Reloads Discharged</u>	<u>Cumulative Reloads Discharged</u>	<u>Adjusted to Base Year (1976)</u>
1960	1	0	0	
61	2	1	1	
62	4	2	3	
63	5	4	7	
64	5	5	12	
65	5	5	17	
66	6	5	22	
67	6	6	28	
68	8	6	34	
69	11	8	40	
70	15	11	51	
71	19	15	66	
72	27	19	85	
73	34	27	112	
74	46	34	146	
75	54	46	192	
76	59	54	246	0
77	[69]*	[59]	[305]	[59]
78	[74]	[69]	[374]	[128]
79	[83]	[74]	[448]	[202]
80	[93]	[83]	[531]	[285]
81	[110]	[93]	[624]	[378]
82	[124]	[110]	[734]	[488]
83	[141]	[124]	[858]	[612]
84	[157]	[141]	[999]	[753]
85			(1127)**	(881)
86			(1276)	(1030)
87			(1435)	(1187)
88			(1603)	(1357)
89			(1782)	(1536)
90			(1971)	(1725)
91			(2169)	(1923)
92			(2376)	(2130)
93			(2612)	(2366)
94			(2966)	(2719)

*Brackets represent expected reloads discharged based on actual schedule of reactors to come on line.

**Parenthesis represent expected reloads discharged based on forecasting technique.

production rate results in costs that are proportional to the square of the increase (decrease).

Let L_i be the optimal production rate for stage i . Let P_i be the production cost per unit at level L_i . Let β_i be the constant of proportionality for the incremental increase in cost per unit for the square of the difference in the actual production and the optimal level of production. The production cost function for anytime t is:

$$g_i[X_i'(t)] = \beta_i P_i [X_i'(t) - L_i]^2 + P_i X_i'(t) \quad \text{VI.5}$$

Expanding on the right:

$$\beta_i P_i X_i'^2(t) - (2\beta_i P_i L_i + P_i) X_i'(t) + \beta_i P_i L_i^2 \quad \text{VI.6}$$

Make the following substitutions:

$$\text{a. } a_i = \beta_i P_i \quad \text{VI.7}$$

$$\text{b. } b_i = 2\beta_i P_i L_i + P_i$$

$$\text{c. } c_i = \beta_i P_i L_i^2$$

and the production cost function becomes:

$$g_i[X_i'(t)] = a_i X_i'^2(t) - b_i X_i'(t) + c_i \quad \text{VI.8}$$

To establish production costs, the β_i is first determined. For example, assume that an increase/decrease in production at stage i of twenty percent results in an increase of ten percent in production costs, or:

$$\text{a. } \beta_i P_i [1.2 L_i - L_i]^2 = .1 P_i L_i \quad \text{VI.9}$$

$$\text{b. } \beta_i = \frac{2.5}{L_i}$$

Table 8 shows the assumed optimal production rates, L_i , from which the β_i are determined. Of the production rates shown, all except for enrichment, are from references shown. Enrichment capacities must be computed from a total capacity of 27.6 million SWU annually^[71]. From the Fuel Management Module-3^[7], one kilogram of UF_6 enriched to 3% from natural feed (.711%) with tails of .25% requires 5.965 kilograms of feed and 3.811 SWU. For one kilogram of UF_6 enriched to 3% from reprocessed UF_6 (.86%) with tails of .25% requires 4.508 kilograms of feed and 2.857 SWU.

Because each stage converts the uranium fuel to a different form, a material conversion factor is necessary to balance the material flow. The factors are developed in Appendix D and are shown in Table 9. Applying these factors to a fuel reload that has been spent, reprocessed, and reenriched to its original level of enrichment yields the following results:

$$a. \frac{1. \text{ Reload}}{.0000566 \text{ Reloads/kg } UF_6} = 17,668 \text{ kg } UF_6 \quad \text{VI.10}$$

$$b. \frac{17,668 \text{ kg } UF_6}{4.508 \text{ kg } UF_6/\text{kg Enriched } UF_6} = 3,919 \text{ kg Enriched } UF_6$$

$$c. \frac{3,919 \text{ kg Enriched } UF_6}{41,354 \text{ kg Enriched } UF_6/\text{Reload}} = .095 \text{ Reloads}$$

From these computations it is concluded that not more than ten percent of the enrichment process capability should be applied to enriching reprocessed UF_6 . From the SWU conversion factors just

Table 8

Optimal Production Levels [40,71,72]

<u>Stage</u>	<u>Identifier</u>	<u>Optimal Production Level (Units/Year)</u>
Exploration	L ₁	29,542,863 tons ore/year
Mining	L ₂	11,920,804 tons ore/year
Milling	L ₃	20,909,090 kg U ₃ O ₈ /year
Conversion	L ₄	14,780,000 kg UF ₆ /year
Enrichment (Natural Feed)	L ₅	6,517,974 kg UF ₆ (3.0%)/year
Fabrication	L ₆	146.075 reloads/year
Reactor	L ₇	1.000 reloads/year
Reprocessing	L ₈	2,218,487 kg UF ₆ /year
Enrichment (Reprocessed)	L ₉	966,048 kg UF ₆ (3.0%)/year

Table 9

Material Flow Conversion Factors

<u>Stage</u>	<u>Identifier*</u>	<u>Conversion Factor</u>
Exploration	$k_{0,1}$	1
Mining	$k_{1,2}$	1
Milling	$k_{2,3}$.588
Conversion	$k_{3,4}$.8054
Enrichment (Natural Feed)	$k_{4,5}$	5.965
Fabrication	$k_{5,6}$	41,354
Reactor	$k_{6,7}$	1
Reprocessing	$k_{7,8}$.0000566
Enrichment (Reprocessed)	$k_{8,5}$	4.508

* $k_{i,j}$ is the number of units in inventory i required to produce one unit in inventory j . See Appendix D.

given, the optimal production level for enrichment of natural uranium feed is:

$$.9 \times 27,600,000 \text{ SWU}/3.811 = 6,517,974 \text{ kg UF}_6 \text{ (3\%)} \quad \text{VI.11}$$

The optimal production level for enrichment of reprocessed UF₆ is:

$$.1 \times 27,600,000 \text{ SWU}/2.857 = 966,048 \text{ kg UF}_6 \text{ (3\%)} \quad \text{VI.12}$$

Next, the individual unit production costs per unit produced, P_i , are required. Generally, production costs are proprietary and, therefore, not available^[69]. Table 10 lists production costs assumed by ERDA as of 1976. The only exception is the reprocessing cost, P_8 . ERDA assumes \$280 per kilogram UF₆. Other estimates are as low as \$90^[69]. This variance results from a lack of information (the last reprocessing occurred in 1972). The low figure will be used initially, and the sensitivity analysis will investigate the sensitivity of the model to this cost parameter.

From Tables 8 and 10 and from equations VI.7, the values of a_i , b_i , and c_i are now determined and are enumerated in Table 11.

To complete the production data, production lag times are assumed to be those shown in Table 12.

2. Inventory Holding Cost Equations

Inventory holding (carrying) costs are difficult to approximate^[30]. However, nuclear fuels, particularly after the milling stage, are

Table 10

Stage Production Costs [69,71]

<u>Stage</u>	<u>Identifier</u>	<u>Production Cost (\$/Unit)</u>
Exploration	P ₁	\$1.30/ton ore
Mining	P ₂	\$20/ton ore
Milling	P ₃	\$4.00/kg U ₃ O ₈
Conversion	P ₄	\$4.40/kg UF ₆
Enrichment (Natural Feed)	P ₅	\$381.10/kg Enr. UF ₆ (3.0%)
Fabrication	P ₆	\$2,826,000 reload
Reactor	P ₇	-
Reprocessing	P ₈	\$90/kg UF ₆
Enrichment (Reprocessed)	P ₉	\$285.70/kg Enr. UF ₆ (3.0%)

Table 11

Values of Production Cost Equation Coefficients

<u>Stage</u>	<u>Stage No.</u>	$\frac{a_i}{1}$	$\frac{b_i}{1}$	$\frac{c_i}{1}$
Exploration	1	1.1×10^{-6}	7.8	960×10^7
Mining	2	41.94×10^{-6}	120	59.6×10^7
Milling	3	4.783×10^{-6}	24	23×10^7
Conversion	4	7.442×10^{-6}	26.4	16.25×10^7
Enrichment (Natural Feed)	5	$1,461.7 \times 10^{-6}$	2,286.6	620×10^7
Fabrication	6	483,656	$16,956 \times 10^3$	103.2×10^7
Reprocessing	8	$1,014 \times 10^{-6}$	540	49.92×10^7
Enrichment (Reprocessed)	9	$7,393.5 \times 10^{-6}$	1,714.2	69×10^7

Table 12

Production Lag Times^[65]

<u>Stage</u>	<u>Identifier</u>	<u>Lag Time (Years)</u>
1-Exploration	l_1	1
2-Mining	l_2	1/6
3-Milling	l_3	1/4
4-Conversion	l_4	1/4
5-Enrichment (Natural Feed)	l_5	1/4
6-Fabrication	l_6	1/4
7-Reactor	l_7	1 1/2
8-Reprocessing	l_8	1/4
9-Enrichment (Reprocessed)	l_9	1/4

highly intensified energy sources. As such, they comprise a very expensive commodity costing at some points several hundred dollars per kilogram. As stated by HADLEY^[30], there are costs associated with insurance, tax, lights, heat, warehouse rent, security, etc.; however, opportunity costs, breakage, pilferage, interest, etc., are proportional to the investment. In the case of nuclear fuels the investment aspect dominates^[8,22].

Figure 7 shows an estimate of the increase in the value of uranium in terms of constant 1972 dollars. The slope of this curve appears nearly linear over the interval of interest. Let M_i be the value of one unit in inventory i at the base year (1976). Then the value of each unit of product in inventory i can be estimated over the interval of interest by:

$$M_i(1 + qt) \qquad \text{VI.13}$$

where $M_i q$ is the slope of the line.

The amount in inventory i at any time t is given by the initial inventory plus cumulative production minus cumulative demand, $I_i(0) + X_i(t) - R_i(t)$. Assume a time value of money factor for the holding cost to be d . Then the holding cost for inventory i at time t is given by:

$$h_i(t) = dM_i(1 + qt)[I_i(0) + X_i(t) - R_i(t)] \qquad \text{VI.14}$$

To quantify this relationship, values for d , q , $I_i(0)$, and M_i are required. Consider first the initial inventories, $I_i(0)$.

Production lags require either initial inventories or shortages. The purpose of this development is to demonstrate where fuels should be stockpiled; therefore, shortages are not allowed. Also, all processes are presently operating, except reprocessing. Therefore, working inventories are on hand. Except where actual initial inventories are estimated, they are computed to be the amount necessary for production in each production lag time. ERDA^[72] has estimated the tons of ore in the explored reserve. The reloads of spent fuel are estimated by taking the cumulative number of reloads discharged from Table 7 as of 1976 and subtracting the amount reprocessed. Assume a reload of spent fuel requires 30 tonnes. The reprocessing plant at West Valley reprocessed 244 tons of spent fuel in its operation^[71]. That is approximately eight reloads. From Table 7 the cumulative spent fuel reloads as of 1976 is 246. Consequently, the initial inventory of spent fuel reloads is assumed to be 238. Initial inventory values are shown in Table 13.

Final inventories are assumed to be zero at all stages where the model does not require otherwise. An example of where this might occur is in recycled material. The model may show that it is not economical to reprocess, in which case an ending inventory of spent fuel reloads can accumulate. Otherwise, assume that $I_i(T) = 0$ for all i .

Table 13

Initial Inventory on Hand^[60,72]

<u>Stage</u>	<u>Identifier</u>	<u>Units on Hand</u>
Exploration	I ₁ (0)	100 x 10 ⁶ tons ore
Mining	I ₂ (0)	5,444,900
Milling	I ₃ (0)	9,254,000
Conversion	I ₄ (0)	11,483,000
Enrichment (Natural Feed)	I ₅ (0)	2,051,000
Fabrication	I ₆ (0)	0
Reactor	I ₇ (0)	238
Reprocessing	I ₈ (0)	643,400
Enrichment (Reprocessed)	I ₉ (0)	0

The time value of money factor, d , is assumed to be $.14^{[47]}$.

The marginal increase in the value of uranium, q , is found by interpolation of the curve in Figure 7. Its value is $.05$.

The initial value of each unit in inventory, M_i , is computed from the base value of U_3O_8 . Conversion factors and unit production costs are used to compute the remaining values. Table 14 enumerates the values of M_i .

There is one exception to the holding cost function developed. Spent fuel reloads not reprocessed are assumed to be encapsulated and stored. This represents a penalty for not reprocessing. Assume that this penalty is a constant cost per reload, S . Then the holding cost function for the spent fuel (stage 7) is:

$$h_7(t) = S[I_7(0) + X_7(t) - k_{7,8}X_8(t+l_8)] \quad \text{VI.15}$$

The penalty for encapsulating and storing spent fuel is placed at \$90 per kilogram^[69]. The value for S is, then, assumed to be \$2,700,000.

3. Acquisition Cost Equations

The acquisition cost is assumed to be the purchase of rights to the ore located by exploration. In this respect it is a function of the production (exploration) rate and, implicitly, of time. Let M_0 represent the cost to purchase one ton of ore located in the base year. Applying the same cost function as VI.13, the acquisition cost function becomes:

Table 14

Present Value of Inventories

<u>Stage</u>	<u>Identifier</u>	<u>Value (\$/Unit)</u>
External Source	M ₀	\$7.00/ton ore
Exploration	M ₁	\$9.00/ton ore
Mining	M ₂	\$29.00/ton ore
Milling	M ₃	\$54.87/kg U ₃ O ₈
Conversion	M ₄	\$78.08/kg UF ₆
Enrichment (Natural Feed)	M ₅	846.85/kg UF ₆ (3.0%)
Fabrication	M ₆	\$37,846,519/reload
Reactor	M ₇	\$609,164/reload
Reprocessing	M ₈	124.47/kg UF ₆
Enrichment (Reprocessed)	M ₅	846.85/kg UF ₆ (3.0%)

$$f(t) = M_0(1 + qt)X_1'(t)$$

VI.16

The value for q has already been defined, and M_0 is shown in Table 14.

Finally, the time horizon of interest is taken to be 18 years (1977-1994). This corresponds to the forecast period previously developed.

D. Solution Equations in General Form

Making substitutions of the appropriate equation forms into the general objective function (IV.25) and collecting the equations for acquisition cost, production costs, and holding costs, the objective function becomes:

$$\begin{aligned} \min TC = & \int_0^T ([1+qt]M_0X_1'(t) + \sum_{i=1}^9 \{a_i X_i'^2(t) - b_i X_i'(t) \\ & + c_i\} + \sum_{i=1}^4 \{dM_i(1+qt)[I_i(0) + X_i(t) - k_{i,i+1} \\ & \cdot X_{i+1}(t+l_{i+1})\}] + dM_5(1+qt)[I_5(0) + X_5(t) + X_9(t) \\ & - k_{5,6}X_6(t+l_6)] + dM_6(1+qt)[I_6(0) + X_6(t) - R_6(t)] \\ & + S[I_7(0) + X_7(t) - k_{7,8}X_8(t+l_8)] + dM_8(1+qt)[I_8(0) \\ & + X_8(t) - k_{8,9} \cdot X_9(t+l_9)])dt \end{aligned} \quad \text{VI.17}$$

subject to:

a. Boundary conditions

VI.18

$$(1) X_i = 0 \quad i=1, \dots, 9$$

$$(2) I_i(0) + X_i(T-l_{i+1}) = k_{i,i+1} X_{i+1}(T) \\ i=1, \dots, 7; \quad i \neq 5$$

$$(3) I_5(0) + X_5(T-l_6) + X_9(T-l_6) = k_{5,6} X_6(T)$$

$$(4) I_6(0) + X_6(T) = R_6(T)$$

$$(5) I_8(0) + X_8(T-l_9) = k_{8,5} X_9(T)$$

b. Constraints

$$(1) I_i(0) + X_i(t) \geq k_{i,i+1} X_{i+1}(t+l_{i+1}) \\ 0 \leq t \leq T-l_{i+1}; \quad i=1, \dots, 7; \quad i \neq 5$$

$$(2) I_5(0) + X_5(t) + X_9(t) \geq k_{5,6} X_6(t+l_6) \\ 0 \leq t \leq T-l_6$$

$$(3) I_8(0) + X_8(t) \geq k_{8,5} X_9(t+l_9) \quad 0 \leq t \leq T-l_9$$

$$(4) I_6(0) + X_6(t) \geq R_6(t) \quad 0 \leq t \leq T$$

Applying the Euler-Lagrange equation for a functional dependent on many functions (V.4) results in the following system of differential equations:

$$a. \frac{\partial F}{\partial X_i(t)} - \frac{d}{dt} \left[\frac{\partial F}{\partial X_i'(t)} \right] = 0 \quad i=1, \dots, 9$$

VI.19

$$b. X_1''(t) = \frac{dM_1(1+qt) - qM_0}{2a_1}$$

$$c. \quad X_2''(t) = \frac{d(1+qt)(M_2 - k_{1,2} M_1)}{2a_2}$$

$$d. \quad X_3''(t) = \frac{d(1+qt)(M_3 - k_{2,3} M_2)}{2a_3}$$

$$e. \quad X_4''(t) = \frac{d(1+qt)(M_4 - k_{3,4} M_3)}{2a_4}$$

$$f. \quad X_5''(t) = \frac{d(1+qt)(M_5 - k_{4,5} M_4)}{2a_5}$$

$$g. \quad X_6''(t) = \frac{d(1+qt)(M_6 - k_{5,6} M_5)}{2a_6}$$

h. $X_7(t) \equiv$ a known function

$$i. \quad X_8''(t) = \frac{d(1+qt)(M_8 - k_{7,8} S)}{2a_8}$$

$$j. \quad X_9''(t) = \frac{d(1+qt)(M_5 - k_{8,5} M_8)}{2a_9}$$

The forms of VI.19 b-j are such that they may be immediately integrated. The twofold integration will result in two constants of integration. Recalling boundary condition VI.18 a(1); $X_i = 0$, $i=1, \dots, 9$; it is readily seen that the second constant of integration is zero. The general solutions are as follows:

$$a. \quad X_1(t) = \frac{dqM_1}{12a_1} t^3 + \frac{dM_1 - qM_0}{4a_1} t^2 + c_1 t \quad \text{VI.20}$$

$$b. \quad X_i(t) = \frac{dq(M_i^{-k_{i-1,i}} M_{i-1}^{M_{i-1}})}{12a_i} t^3 + \frac{d(M_i^{-k_{i-1,i}} M_{i-1}^{M_{i-1}})}{4a_i} t^2 + c_i t$$

$i=2,3,4,5,6.$

$$c. \quad X_8(t) = \frac{dqM_8}{12a_8} t^3 + \frac{dM_8^{-k_{7,8}} M_8^S}{4a_8} t^2 + c_8 t$$

$$d. \quad X_9(t) = \frac{dq(M_5^{-k_{8,5}} M_8^{M_8})}{12a_9} t^3 + \frac{d(M_5^{-k_{8,5}} M_8^{M_8})}{4a_9} t^2 + c_9 t$$

and, again, $X_7(t)$ is a known function.

E. Derivation of the Specific Solution

Note that the general solution equations in VI.20 are independent of each other except for their relation through the constraint inequalities. Each general equation becomes specific by the application of boundary conditions or transversality conditions. The result is an extremal for each stage representing the optimal cumulative production function for that stage if no constraints are given. This extremal is then compared with the constraints. If all constraints are inactive, the extremal is the extremizing cumulative production function. If the constraints are active then the extremizing cumulative production function is a composite curve consisting of the extremal and the constraint. The results in Chapter V.D are applied to determine the points of passage from the extremal to the constraint and vice versa.

Because each stage exacts a demand on its preceding stage, it is necessary to begin the solution at the point where external demand is known and work backwards through the stages. However, the cyclic (feedback) flow extremizing cumulative production functions must be determined before the extremizing cumulative production function at the mainstream reentry point is determined. For the nuclear fuel cycle, the stagewise extremizing cumulative production functions should be found in the order of fabrication, enrichment of reprocessed UF_6 , reprocessing, enrichment of natural feed, conversion, milling, mining and, finally, exploration and acquisition.

Following this technique, it is necessary to begin with the following given conditions:

1. The cumulative demand function for nuclear fuel reloads,

$R_6(t)$, is given by VI.3:

$$R_6(t) = 7.306 t^2 + 71.606 t \quad \text{VI.21}$$

2. The cumulative production function for the reactor, $X_7(t)$, is given by VI.4:

$$X_7(t) = 4.292 t^2 + 73.86 t \quad \text{VI.22}$$

1. Fabrication

Begin by finding the extremal for the fabrication stage because demand on this stage is known. The first step is to find the constant of integration.

Using VI.20b and substituting the values for the model parameters shown in Tables 11, 13, and 14:

$$a. \quad X_6(t) = \frac{dq(M_6 - k_{5,6} M_5)}{12a_6} t^3 + \frac{d(M_6 - k_{5,6} M_5)}{4a_6} t^2 + c_6 t \quad \text{VI.23}$$

$$b. \quad X_6(t) = \frac{.14 \times .05 (37,847,000 - 41,354 \times 846.85)}{12 \times 483,660} t^3 \\ + \frac{.14 (37,847,000 - 41,354 \times 846.85)}{4 \times 483,660} t^2 + c_6 t$$

$$c. \quad X_6(t) = .0034088 t^3 + .20453 t^2 + c_6 t$$

To solve for the constant of integration, c_6 , apply boundary condition VI.18 a(6):

$$a. \quad I_6(0) + X_6(18) = R_6(18) \quad \text{VI.24}$$

$$b. \quad .0034088(18)^3 + .20453(18)^2 + c_6(18) \\ = 7.306(18)^2 + 71.606(18)$$

$$c. \quad c_6 = 198.33$$

The extremal for the cumulative production function for fabrication is:

$$X_6(t) = .0034088 t^3 + .20453 t^2 + 198.33 t \quad \text{VI.25}$$

Next, it is determined if the extremizing function is the extremal or a composite function consisting of the extremal and the domain boundary. This is accomplished by comparing the extremal to the constraint VI.18 b(4):

$$a. \quad I_6(0) + X_6(t) \geq R_6(t) \quad 0 \leq t \leq 18 \quad \text{VI.26}$$

$$b. \quad .0034088 t^3 + .20453 t^2 + 198.33 t \\ \geq 7.306 t^2 + 71.606 t \quad 0 \leq t \leq 18$$

By inspection the extremal is equal to or greater than the constraint throughout the interval. The optimal cumulative production function for fabrication is given by VI.25.

The solution can proceed no further along the mainstream because the recycle flow reenters the inventory before fabrication. The feedback flow must now be found.

2. Enrichment of Reprocessed UF_6

At the point where the recycle reenters the mainstream, the boundary conditions are not known. The extremizing function for the cumulative production of enriching the reprocessed UF_6 must then meet two necessary conditions:

- a. The original Euler-Lagrange condition given by VI.19a and,
- b. Transversality conditions as stated in V.7.

The first necessary condition results in VI.20d, or:

$$\text{a. } X_9(t) = \frac{dq(M_5 - k_{8,5} M_8)}{12a_9} t^3 + \frac{d(M_5 - k_{8,5} M_8)}{4a_9} t^2 + c_9 t \quad \text{VI.27}$$

$$\text{b. } X_9(t) = \frac{.14 \times .05 (846.85 - 4.508 \times 124.47)}{12 \times 7,393.5 \times 10^{-6}} t^3$$

$$+ \frac{.14 (846.85 - 4.508 \times 124.47)}{4 \times 7,393.5 \times 10^{-6}} t^2 + c_9 t$$

$$\text{c. } X_9(t) = 22.544 t^3 + 1352.7 t^2 + c_9 t$$

The second necessary condition results in:

$$\text{a. } \left. \frac{\partial F}{\partial X_9'(t)} \right|_{t=T} = 0 \quad \text{VI.28}$$

$$\text{b. } 2a_9 X_9'(t) - b_9 \Big|_{t=18} = 0$$

$$\text{c. } X_9'(18) = \frac{b_9}{2a_9} = \frac{14,571}{7,393.5 \times 10^{-6}} = 1,970,800$$

From VI.27c and VI.28c

$$\text{a. } X_9(t) = 22.544 t^3 + 1,352.7 t^2 + c_9 t \quad \text{VI.29}$$

$$\text{b. } X_9'(t) = 67.632 t^2 + 2,705.4 t + c_9$$

$$\text{c. } X_9'(18) = 67.632(18)^2 + 2,705.4(18) + c_9 = 1,970,800$$

$$\text{d. } c_9 = 1,900,200$$

The conclusion is that the extremal for enrichment of reprocessed UF_6 is:

$$X_9(t) = 22.544 t^3 + 1,352.7 t^2 + 1,900,200 t \quad \text{VI.30}$$

3. Reprocessing

The enrichment of reprocessed UF_6 production function sets the demand for reprocessed UF_6 . To continue with the solution, proceed back down the cyclic flow to reprocessing of UF_6 .

From VI.20c and substitutions from Tables 11, 13, and 14:

$$\text{a. } X_8(t) = \frac{dqM_8}{12a_8} t^3 + \frac{dM_8 - k_{7,8}^S}{4a_8} t^2 + c_8 t \quad \text{VI.31}$$

$$\text{b. } X_8(t) = \frac{.14 \times .05 \times 124.47}{12 \times 1,014 \times 10^{-6}} t^3 \\ + \frac{.14 \times 124.47 - .0000566 \times 2,700,000}{4 \times 1,014 \times 10^{-6}} t^2 + c_8 t$$

$$\text{c. } X_8(t) = 71.605 t^3 - 33,381 t^2 + c_8 t$$

To find the constant of integration, apply boundary condition VI.18 a(5):

$$I_8(0) + X_8(T - \ell_9) = k_{8,5} X_9(T) \quad \text{VI.32}$$

First, $I_8(0)$ must be found. Recall that initial inventories, where not known, are equal to production lag time requirements.

$$a. \quad I_8(0) = k_{8,5} X_9(l_9) \quad \text{VI.33}$$

$$b. \quad I_8(0) = 4.508 \left[22.544 \left(\frac{1}{4}\right)^3 + 1,352.7 \left(\frac{1}{4}\right)^2 + 1,900,200 \left(\frac{1}{4}\right) \right]$$

$$c. \quad I_8(0) = 2,141,900$$

Now, from VI.32:

$$a. \quad 2,141,900 + 71.605 \left(18 - \frac{1}{4}\right)^3 - 33,381 \left(18 - \frac{1}{4}\right)^2 + c_8 \left(18 - \frac{1}{4}\right) = 4.508 \left[22.544 (18)^3 + 1,352.7 (18)^2 + 1,900,200 (18) \right]$$

$$b. \quad c_8 = 9,278,500$$

Thus, the extremal for reprocessing is:

$$X_8(t) = 71.605 t^3 - 33,381 t^2 + 9,278,500 t \quad \text{VI.35}$$

Next, the extremal must be checked against the constraint VI.18 b(3):

$$a. \quad I_8(0) + X_8(t) \geq k_{8,5} X_9(t+l_9) \quad 0 \leq t \leq T-l_9 \quad \text{VI.36}$$

$$b. \quad 2,141,900 + 71.605 t^3 - 33,381 t^2 + 9,278,500 t \\ \geq 4.508 \left[22.544 \left(t + \frac{1}{4} \right)^3 + 1,352.7 \left(t + \frac{1}{4} \right)^2 \right. \\ \left. + 1,900,200 \left(t + \frac{1}{4} \right) \right]$$

$$c. \quad 71.605 t^3 - 33,381 t^2 + 9,278,500 t + 2,141,900 \\ \geq 101.63 t^3 + 6,174.2 t^2 + 8,569,200 t + 2,141,200$$

The constraint is active throughout the interval. However, it is beneficial to note that the cumulative production extremal for reprocessing is also constrained by the cumulative production function for the reactor, $X_7(t)$. This function is known and is given by VI.4:

$$X_7(t) = 4.292 t^2 + 73.86 t \quad \text{VI.37}$$

Apply constraint VI.18 b(1) to VI.35:

$$a. \quad I_7(0) + X_7(t) \geq k_{7,8} X_8(t+l_8) \quad 0 \leq t \leq T-l_8 \quad \text{VI.38}$$

$$b. \quad 238 + 4.292 t^2 + 73.86 t \geq .0000566 \left[71.605 \left(t + \frac{1}{4} \right)^3 \right. \\ \left. - 33,381 \left(t + \frac{1}{4} \right)^2 + 9,278,500 \left(t + \frac{1}{4} \right) \right]$$

$$c. \quad 4.292 t^2 + 73.86 t + 238 \geq .004053 t^3 - 1.8863 t^2 \\ + 524.22 t + 131.17$$

The constraint is active in the interval. The extremizing function will, then, be a composite curve consisting of the extremal and the constraint. To find the point of passage from the extremal to the constraint the results of V.45 must be applied. At the point of passage the tangent to the extremal must equal the tangent to the constraint. Also, the two curves must be equal at the point. These two equations are necessary to solve for two unknowns: the point of passage and the constant of integration, c_8 .

The two equations in two unknowns (t, c_8) are:

$$a. \quad \frac{d}{dt} [I_7(0) + X_7(t)] = \frac{d}{dt} [k_{7,8} X_8(t+l_8)] \quad \text{VI.39}$$

$$b. \quad I_7(0) + X_7(t) = k_{7,8} X_8(t+l_8)$$

Since interest is in $X_8(t)$, translate above equations by $-l_8$ and solve:

$$a. \quad \frac{d}{dt} [238+4.292(t - \frac{1}{4})^2 + 73.86(t - \frac{1}{4})^2] = \frac{d}{dt} [.0000566 \text{ VI.40}$$

$$(71.605 t^3 - 33,381 t^2 + c_6 t)]$$

$$b. \quad 238+4.292(t - \frac{1}{4})^2 + 73.86(t - \frac{1}{4})^2 = .0000566$$

$$[71.605 t^3 - 33,381 t^2 + c_6 t]$$

Solving these two equations simultaneously results in:

$$\text{a. } t = 5.982 \quad \text{VI.41}$$

$$\text{b. } c_6 = 2,565,400$$

The extremizing function is the extremal over the interval $0 \leq t \leq 5.982$ and the constraint over the interval $5.982 \leq t \leq 18$.

Thus,

$$\text{a. } X_8(t) = 71.605 t^3 - 33,381 t^2 + 2,565,400 t \quad \text{VI.42}$$

$$0 \leq t \leq 5.982$$

$$\text{b. } X_8(t) = [238 + 4.292(t - \frac{1}{4})^2 + 73.86(t - \frac{1}{4})] / .0000566$$

$$= 75.830 t^2 + 1,267,000 t + 3,874,000$$

$$5.982 \leq t \leq 18$$

4. Enrichment of Reprocessed UF₆ Recomputed

Attention is now redirected to $X_9(t)$. Recall that $X_8(t)$ must be checked against the constraint indicated in VI.36a; however, $X_8(t)$ now has a different form. Reapply the constraint:

$$I_8(0) + X_8(t) \geq k_{8,5} X_9(t+l_9) \quad 0 \leq t \leq 18-l_9 \quad \text{VI.43}$$

Using VI.36c and checking the interval $0 \leq t \leq 5.982$ first:

$$2,141,900 + 71.605 t^3 - 33,381 t^2 + 2,565,400 t \quad \text{VI.44}$$

$$\frac{1}{4} 101.63 t^3 + 6,174.2 t^2 + 8,569,200 t + 2,141,900$$

The constraint is active throughout the interval $0 \leq t \leq 5.982$.

Now check the interval $5.982 \leq t \leq 18 - \frac{1}{4}$

$$75,830 t^2 + 1,267,000 t + 3,874,000 \quad \text{VI.45}$$

$$\geq 101.63 t^3 + 6,174.2 t^2 + 8,569,200 t + 2,141,900$$

$$5.982 \leq t \leq 18 - \frac{1}{4}$$

This constraint also is active everywhere in the interval. The conclusion is that throughout the interval $0 \leq t \leq 18$, $X_9(t)$ is constrained by $X_8(t)$. Thus:

$$X_9(t) = [X_8(t - \ell_9) + I_8(0)]/k_{8,5} \quad \text{VI.46}$$

Since the $X_8(t)$ solution is known over $0 \leq t \leq 5.982$, $X_9(t)$ is constrained over that interval plus lag time, or $0 \leq t \leq 6.232$.

$$\text{a. } X_9(t) = [71.605(t - \frac{1}{4})^3 - 33,381(t - \frac{1}{4})^2 \quad \text{VI.47}$$

$$+ 2,565,400(t - \frac{1}{4}) + 643,437]/4.508$$

$$0 \leq t \leq 5.982 + \ell_9$$

$$b. X_9(t) = 15.884 t^3 - 7,416.8 t^2 + 572,780 t$$

$$0 \leq t \leq 6.232$$

$$c. I_8(0) = 643,400$$

Also, $X_9(t)$ has the remainder of the interval adjusted by the lag time, or:

$$a. X_9(t) = [X_8(t-l_9)]/k_{8,5} \quad \text{VI.48}$$

$$b. X_9(t) = 75,830(t - \frac{1}{4})^2 + 1,267,000(t - \frac{1}{4})$$

$$+ 3,874,000]/4.508$$

$$c. X_9(t) = 16,821 t^2 + 272,690 t + 790,150$$

$$6.232 \leq t \leq 18$$

5. Enrichment of Natural Feed UF_6

Attention is now returned to the mainstream flow at the enrichment process. From VI.20b and substitutions from Tables 11, 13, and 14:

$$a. X_5(t) = \frac{dq(M_5-k_{4,5}M_4)}{12a_5} t^3 + \frac{d(M_5-k_{4,5}M_4)}{4a_5} t^2 + c_5 t \quad \text{VI.49}$$

$$\begin{aligned} \text{b. } X_5(t) &= \frac{.14 \times .05 (846.85 - 5.965 \times 78.08)}{12 \times 1,461.7 \times 10^{-6}} t^3 \\ &+ \frac{.14 (848.85 - 5.965 \times 78.08)}{4 \times 1,461.7 \times 10^{-6}} t^2 + c_5 t \end{aligned}$$

$$\text{c. } X_5(t) = 152.09 t^3 + 9,125.4 t^2 + c_5 t$$

To solve for the constant of integration apply boundary condition VI.18 a(3):

$$I_5(0) + X_5(T - \ell_6) + X_9(T - \ell_6) = k_{5,6} X_6(T) \quad \text{IV.50}$$

First, determine the initial inventory for the enrichment process, $I_5(0)$.

$$\text{a. } I_5(0) = k_{5,6} X_6(\ell_6) \quad \text{IV.51}$$

$$\begin{aligned} \text{b. } I_5(0) &= 41,354 \left[.0034088 \left(\frac{1}{4}\right)^3 + .20453 \left(\frac{1}{4}\right)^2 \right. \\ &\quad \left. + 198.33 \left(\frac{1}{4}\right) \right] \end{aligned}$$

$$\text{c. } I_5(0) = 2,051,000$$

Substituting into VI.50:

$$\begin{aligned}
 \text{a. } & 2,051,000 + 152.09\left(18 - \frac{1}{4}\right)^3 + 9,125.4\left(18 - \frac{1}{4}\right)^2 && \text{VI.52} \\
 & + c_5\left(18 - \frac{1}{4}\right) + 16,821\left(18 - \frac{1}{4}\right)^2 + 272,690\left(18 - \frac{1}{4}\right) \\
 & + 790,150 = 41,354[.0034088(18) + .20453(18)^2 \\
 & + 198.33(18)]
 \end{aligned}$$

$$\text{b. } c_5 = 7,621,300$$

The extremal for enrichment is:

$$X_5(t) = 152.09 t^3 + 9,125.4 t^2 + 7,621,300 t \quad \text{VI.53}$$

This extremal must now be compared to the constraint VI.18 b(2):

$$\text{a. } I_5(0) + X_5(t) + X_9(t) \geq k_{5,6} X_6(t+l_6) \quad \text{VI.54}$$

$$0 \leq t \leq 18-l_6$$

$$\begin{aligned}
 \text{b. } & 2,051,000 + 152.09 t^3 + 9,125.4 t^2 + 7,621,300 t \\
 & + 15.884 t^3 - 7,416.8 t^2 + 572,780 t \\
 & \geq 41,354[.0034088\left(t + \frac{1}{4}\right)^3 + .20453\left(t + \frac{1}{4}\right)^2 \\
 & + 198.33\left(t + \frac{1}{4}\right)] \quad 0 \leq t \leq 6.232
 \end{aligned}$$

$$c. \quad 2,051,000 + 167.974 t^3 + 1,708.6 t^2 + 8,215,100 t$$

$$\frac{1}{2} 140.97 t^3 + 8,563.8 t^2 + 8,206,000 t + 2,051,000$$

$$0 \leq t \leq 6.232$$

The constraint is active in the interval. Attention is turned to the second interval $6.232 \leq t \leq 18 - \frac{1}{4}$

$$a. \quad 2,051,000 + 152.09 t^3 + 9,125.4 t^2 + 7,621,300 t \quad \text{VI.55}$$

$$+ 16,821 t^2 + 272,690 t + 790,150 \geq 140.97 t^3$$

$$+ 8,563.8 t^2 + 8,206,000 t + 2,051,000$$

$$6.232 \leq t \leq 18 - \frac{1}{4}$$

$$b. \quad 2,051,000 + 152.09 t^3 + 25,946 t^2 + 7,894,000 t$$

$$+ 790,150 \frac{1}{2} 140.97 t^3 + 8,563.8 t^2$$

$$+ 8,206,000 t + 2,051,000$$

The constraint is active in this interval also. Thus, the extremizing function must be a composite curve consisting of the extremal and the constraint. Again the results of V.45 are applied. Using the forms of the equations in VI.55.b and following the steps taken in VI.39 and VI.40:

$$a. \quad 2,051,000 + 152.09 t^3 + 25,946 t^2 + c_5 t + 272,690 t \quad \text{VI.56}$$

$$+ 790,150 = 140.97 t^3 + 8,563.8 t^2 + 8,206,000$$

$$+ 2,051,000$$

$$b. \quad 456.28 t^2 + 51,893 t + c_5 + 272,690 = 422.91 t^2$$

$$+ 17,128 t + 8,206,000$$

Solving VI.56a and b simultaneously for t and c_5 results in:

$$a. \quad t = 6.713 \quad \text{VI.57}$$

$$b. \quad c_5 = 7,698,400$$

The result is:

$$X_5(t) = 152.09 t^3 + 9,125.4 t^2 + 7,698,400 t \quad \text{VI.58}$$

$$0 \leq t \leq 6.713$$

The new expression is reapplied against constraint VI.18 b(2) as in VI.54a, b, and c:

$$a. \quad 2,051,000 + 152.09 t^3 + 9,125.4 t^2 + 7,698,400 t \quad \text{VI.59}$$

$$+ 15.884 t^3 - 7,416.8 t^2 + 572,780 t \geq 140.97 t^3$$

$$+ 8,563.8 t^2 + 8,206,000 t + 2,051,000$$

$$0 \leq t \leq 6.713$$

This time the constraint is not active. Attention is turned to the interval $6.713 \leq t \leq 18 - \frac{1}{4}$. Over this interval $X_5(t)$ is the difference in the constraint equation and $X_9(t)$, or:

$$\begin{aligned} \text{a. } X_5(t) &= 140.97 t^3 + 8,563.8 t^2 + 8,206,000 t && \text{VI.60} \\ &+ 2,051,000 - 16,821 t^2 - 272,300 t - 790,150 \end{aligned}$$

$$\begin{aligned} \text{b. } X_5(t) &= 140.97 t^3 - 8,257.2 t^2 + 7,933,700 t \\ &+ 1,260,850 \quad 6.713 \leq t \leq 18 - \frac{1}{4} \end{aligned}$$

6. Conversion

Attention is now turned to the conversion process. From VI.20b and Tables 11, 13, and 14:

$$\text{a. } X_4(t) = \frac{dq(M_4^{-k_3,4} M_3)}{12a_4} t^3 + \frac{d(M_4^{-k_3,4} M_3)}{4a_4} t^2 + c_4 t \quad \text{VI.61}$$

$$\begin{aligned} \text{b. } X_4(t) &= \frac{.14 \times .05 (78.08 - .8054 \times 54.87)}{12 \times 7.442 \times 10^{-6}} t^3 \\ &+ \frac{.14 (78.08 - .8054 \times 54.87)}{4 \times 7.442 \times 10^{-6}} t^2 + c_4 t \end{aligned}$$

$$\text{c. } X_4(t) = 2,656.3 t^3 + 159,375 t^2 + c_4 t$$

To solve for the constant of integration apply boundary condition VI.18 a(2):

$$I_4(0) + X_4(T - \ell_5) = k_{4,5} X_5(T) \quad \text{VI.62}$$

First find the initial inventory, $I_4(0)$:

$$a. \quad I_4(0) = k_{4,5} X_5(\ell_5) \quad \text{VI.63}$$

$$b. \quad I_4(0) = 5.965[152.09\left(\frac{1}{4}\right)^3 + 9,125.4\left(\frac{1}{4}\right)^2 \\ + 7,698,000\left(\frac{1}{4}\right)]$$

$$c. \quad I_4(0) = 11,483,000$$

Now, from VI.62:

$$a. \quad 11,483,000 + 2,656.3\left(18 - \frac{1}{4}\right)^3 + 159,375\left(18 - \frac{1}{4}\right)^2 \quad \text{VI.64}$$

$$+ c_4\left(18 - \frac{1}{4}\right) = 5.965[140.97(18)^3 - 8,257.2(18)^2$$

$$+ 7,933,700(18) + 1,260,850]$$

$$b. \quad c_4 = 43,479,000$$

The extremal for conversion is:

$$X_4(t) = 2,656.3 t^3 + 159,375 t^2 + 43,479,000 t \quad \text{VI.65}$$

This extremal must be checked against the constraint in VI.20 b(1):

$$a. \quad I_4(0) + X_4(t) \geq k_{4,5} X_5(t+\ell_5) \quad 0 \leq t \leq 6.713-.25 \quad \text{VI.66}$$

$$\begin{aligned}
 \text{b. } & 11,483,000 + 2,656.3 t^3 + 159,375 t^2 + 43,479,000 t \\
 & \geq 5.965 \left[152.09 \left(t + \frac{1}{4} \right)^3 + 9,125.4 \left(t + \frac{1}{4} \right)^2 \right. \\
 & \quad \left. + 7,698,400 \left(t + \frac{1}{4} \right) \right] \quad 0 \leq t \leq 6.463
 \end{aligned}$$

$$\begin{aligned}
 \text{c. } & 11,483,000 + 2,656.3 t^3 + 159,375 t^2 + 43,479,000 t \\
 & \frac{1}{2} 907.22 t^3 + 55,114 t^2 + 45,948,000 t \\
 & + 11,483,000 \quad 0 \leq t \leq 6.463
 \end{aligned}$$

The constraint is active over the interval $0 \leq t \leq 6.463$. Now, check the constraint over the interval $6.463 \leq t \leq 18 - \frac{1}{4}$.

$$\begin{aligned}
 \text{a. } & 11,483,000 + 2,656.3 t^3 + 159,375 t^2 + 43,479,000 t \quad \text{VI.67} \\
 & \geq 5.965 \left[140.97 \left(t + \frac{1}{4} \right)^3 - 8,257.3 \left(t + \frac{1}{4} \right)^2 \right. \\
 & \quad \left. + 7,933,700 \left(t + \frac{1}{4} \right) + 1,260,850 \right] \quad 6.463 \leq t \leq 17.75
 \end{aligned}$$

$$\begin{aligned}
 \text{b. } & 11,483,000 + 2,656.3 t^3 + 159,375 t^2 + 43,479,000 t \\
 & \frac{1}{2} 840.80 t^3 - 48,624 t^2 + 47,300,000 t + 19,349,000 \\
 & 6.463 \leq t \leq 17.75
 \end{aligned}$$

The constraint is active throughout the interval $6.463 \leq t \leq 17.75$. Since the constraining equations are active throughout the total interval $0 \leq t \leq 18$, the cumulative production function for conversion is given by:

$$X_4(t) = 907.22 t^3 + 55,114 t^2 + 45,948,000 t \quad \text{VI.68}$$

$$0 \leq t \leq 17.75$$

The initial inventory is as shown before:

$$I_4(0) = 11,483,000 \quad \text{VI.69}$$

7. Milling

Attention is now turned to milling. From VI.20b and Tables 11, 13, and 14:

$$\text{a. } X_3(t) = \frac{dq(M_3 - k_{2,3} M_2)}{12a_3} t^3 + \frac{d(M_3 - k_{2,3} M_2)}{4a_3} t^2 + c_3 t \quad \text{VI.70}$$

$$\text{b. } X_3(t) = \frac{.14 \times .05 (54.87 - .588 \times 29.00)}{12 \times 4.783 \times 10^{-6}} t^3$$

$$+ \frac{.14 (54.87 - .588 \times 29.00)}{4 \times 4.783 \times 10^{-6}} t^2 + c_3 t$$

$$\text{c. } X_3(t) = 4,612.3 t^3 + 276,740 t^2 + c_3 t$$

To solve for the constant of integration, apply boundary condition VI.18a(2):

$$I_3(0) + X_3(18 - l_4) = k_{3,4} X_4(18) \quad \text{VI.71}$$

First, the initial inventory must be found.

$$\text{a. } I_3(0) = k_{3,4} X_4(l_4) \quad \text{VI.72}$$

$$\text{b. } I_3(0) = .8054 \left[907.22 \left(\frac{1}{4} \right)^3 + 55,114 \left(\frac{1}{4} \right)^2 + 45,948,000 \left(\frac{1}{4} \right) \right]$$

$$c. \quad I_3(0) = 9,254,400$$

From VI.71:

$$a. \quad 9,254,400 + 4,612.3 \left(18 - \frac{1}{4}\right)^3 + 276,740 \left(18 - \frac{1}{4}\right)^2 \quad \text{VI.73}$$

$$+ c_3 \left(18 - \frac{1}{4}\right) = .8054 [907.22(18)^3 + 55,114(18)^2$$

$$+ 45,948,000(18)]$$

$$b. \quad c_3 = 31,691,000$$

The extremal for milling is:

$$X_3(t) = 4,612.3 t^3 + 276,740 t^2 + 31,691,000 t \quad \text{VI.74}$$

The extremal must be checked against the constraint VI.18b(1):

$$a. \quad I_3(0) + X_3(t) \geq k_{2,3} X_4(t + \ell_4) \quad \text{VI.75}$$

$$b. \quad 9,254,400 + 4,612.3 t^3 + 276,740 t^2 + 31,691,000 t$$

$$\geq .8054 [907.22 \left(t + \frac{1}{4}\right)^3 + 55,114 \left(t + \frac{1}{4}\right)^2$$

$$+ 45,948,000 \left(t + \frac{1}{4}\right) \quad 0 \leq t \leq 18 - \frac{1}{4}$$

$$c. \quad 9,254,400 + 4,612.3 t^3 + 276,740 t^2 + 31,691,000 t$$

$$\geq \frac{1}{2} 730.7 t^3 + 44,390 t^2 + 37,029,000 t + 9,254,400$$

$$0 \leq t \leq 17.75$$

The constraint is active throughout the interval. The conclusion is that the extremizing cumulative production function for mining is the constraining equation, or:

$$X_3(t) = 730.7 t^3 + 44,390 t^2 + 37,029,000 t \quad \text{VI.76}$$

8. Mining

Attention is now turned to mining. From VI.20b and Tables 11, 13, and 14:

$$\text{a. } X_2(t) = \frac{dq(M_2 - k_{1,2} M_1)}{12a_2} t^3 + \frac{d(M_2 - k_{1,2} M_1)}{4a_2} t^2 + c_2 t \quad \text{VI.77}$$

$$\text{b. } X_2(t) = \frac{.14 \times .05 (29-1 \times 9)}{12 \times 41.94 \times 10^{-6}} t^3 + \frac{.14 (29-1 \times 9)}{4 \times 41.94 \times 10^{-6}} t^2 + c_2 t$$

$$\text{c. } X_2(t) = 278.18 t^3 + 16,691 t^2 + c_2 t$$

To find the constant of integration, apply boundary condition VI.18a(2):

$$I_2(0) + X_2(T-l_3) = k_{2,3} X_3(T) \quad \text{VI.78}$$

First, the initial inventory must be determined:

$$\text{a. } I_2(0) = k_{2,3} X_3(l_3) \quad \text{VI.79}$$

$$\text{b. } I_2(0) = .588 [730.7 \left(\frac{1}{4}\right)^3 + 44,390 \left(\frac{1}{4}\right)^2 + 37,029,000 \left(\frac{1}{4}\right)]$$

$$\text{c. } I_2(0) = 5,444,900$$

Substitute into VI.78:

$$\begin{aligned} \text{a. } 5,444,900 + 278.18\left(18 - \frac{1}{4}\right)^3 + 16,691\left(18 - \frac{1}{4}\right)^2 & \quad \text{VI.80} \\ + c_2\left(18 - \frac{1}{4}\right) & = .588[730.7(18)^3 + 44,390(18)^2 \\ + 37,029,000(18)] & \end{aligned}$$

$$\text{b. } c_2 = 22,007,000$$

The extremal for mining is:

$$X_2(t) = 278.18 t^3 + 16,691 t^2 + 22,007,000 t \quad \text{VI.81}$$

This extremal must now be checked against the constraint VI.20b(1):

$$\text{a. } I_2(0) + X_2(t) \geq k_{2,3} X_3(t + \ell_3) \quad 0 \leq t \leq 18 - \ell_3 \quad \text{VI.82}$$

$$\begin{aligned} \text{b. } 5,444,900 + 278.18 t^3 + 16,691 t^2 + 22,007,000 t \\ \geq .588[730.7\left(t + \frac{1}{4}\right)^3 + 44,390\left(t + \frac{1}{4}\right)^2 \\ + 37,029,000\left(t + \frac{1}{4}\right)] \end{aligned}$$

$$\begin{aligned} \text{c. } 5,444,900 + 278.18 t^3 + 16,691 t^2 + 22,007,000 t \\ \geq 429.65 t^3 + 26,105 t^2 + 21,786,000 t + 5,444,900 \end{aligned}$$

The constraint is active throughout the interval $0 \leq t \leq 17.75$.

The extremizing function for mining is the constraining equation, or:

$$X_2(t) = 429.65 t^3 + 26,105 t^2 + 21,786,000 t \quad \text{VI.83}$$

9. Acquisition and Exploration

Attention is now turned to acquisition and exploration. From VI.20a and Tables 11, 12, and 14:

$$\text{a. } X_1(t) = \frac{dqM_1}{12a_1} t^3 + \frac{dM_1 - qM_0}{4a_1} t^2 + c_1 t \quad \text{VI.84}$$

$$\text{b. } X_1(t) = \frac{.14 \times .05 \times 9}{12 \times 1.1 \times 10^{-6}} t^3 + \frac{.14 \times 9 - .05 \times 7}{4 \times 1.1 \times 10^{-6}} t^2 + c_1 t$$

$$\text{c. } X_1(t) = 4,772.9 t^3 + 206,820 t^2 + c_1 t$$

To find the constant of integration, apply boundary condition VI.18a(2):

$$\text{a. } I_1(0) + X_1(T - \ell_2) = k_{1,2} X_2(T) \quad \text{VI.85}$$

$$\begin{aligned} \text{b. } 100,000,000 + 4,772.7 \left(18 - \frac{1}{6}\right)^3 + 206,820 \left(18 - \frac{1}{6}\right)^2 \\ + c_1 \left(18 - \frac{1}{6}\right) &= 1(429.65(18)^3 + 26,105(18)^2 \\ &+ 21,786,000(18)^2 \end{aligned}$$

$$\text{c. } c_1 = 11,791,000$$

The extremal for exploration and acquisition is:

$$X_1(t) = 4,772.7 t^3 + 206,820 t^2 + 11,791,000 t \quad \text{VI.86}$$

This extremal must be checked against the constraint VI.20b(1):

$$\text{a. } I_1(0) + X_1(t) \geq k_{1,2} X_2(t+l_2) \quad 0 \leq t \leq 18-l_2 \quad \text{VI.87}$$

$$\begin{aligned} \text{b. } 100,000,000 + 4,772.7 t^3 + 206,820 t^2 + 11,791,000 t \\ \geq 1[429.65(t + \frac{1}{6})^3 + 26,105(t + \frac{1}{6})^2 \\ + 21,786,000(t + \frac{1}{6})] \end{aligned}$$

$$\begin{aligned} \text{c. } 100,000,000 + 4,772.7 t^3 + 206,820 t^2 + 11,791,000 t \\ \geq 429.65 t^3 + 26,427 t^2 + 21,788,000 t + 3,631,000 \end{aligned}$$

The constraint is inactive throughout the interval $0 \leq t \leq 17.833$.

The conclusion is that the extremizing function for exploration and acquisition is given by:

$$X_1(t) = 4,772.7 t^3 + 206,820 t^2 + 11,791,000 t \quad \text{VI.88}$$

$$0 \leq t \leq 17.833$$

F. Sufficient Conditions for an Extremum

To insure that the specific solution is an extremum and a minimum, the steps outlined in Chapter V.B and E are taken. The Legendre conditions are applied to insure a local minimum. The KROTOV test is applied to insure that the specific solution falls in a class of smooth curves only. The uniqueness of the solution is discussed.

1. To apply the Legendre conditions as given in V.5, the following partial derivatives must be found:

$$\frac{\partial^2 F}{\partial X_i \partial X_j} \quad \begin{array}{l} i,j=1, \dots, 9 \\ i,j \neq 7 \end{array} \quad \text{VI.89}$$

Examination of the objective function in VI.17 reveals that no cross-products of functions exist. The result is:

$$\frac{\partial^2 F}{\partial X_i \partial X_j} = 0 \quad \begin{array}{l} i \neq j \quad i,j=1, \dots, 9 \\ i,j \neq 7 \end{array} \quad \text{VI.90}$$

Further examination reveals that:

$$\frac{\partial^2 F}{\partial X_i \partial X_j} = 2a_i \quad \begin{array}{l} i=j; i=1, \dots, 9 \\ i \neq 7 \end{array} \quad \text{VI.91}$$

The values of a_i tabulated in Table 11 are all positive and greater than zero. The immediate conclusion is that the resulting matrices formed for the Legendre conditions are

diagonal matrices with all diagonal elements positive. This insures that the chain of inequalities in V.5 are satisfied.

2. For the KROTOV test, again refer to the objective function in VI.17. The objective function consists of individual terms consisting of X_i' and X_i . Forming the KROTOV test:

$$\lim_{X_i' \rightarrow \pm\infty} F(X_i', X_i, t) \frac{1}{X_i'} \quad \begin{array}{l} i=1, \dots, 9 \\ i \neq 7 \end{array} \quad \text{VI.92}$$

it is seen that the $a_i X_i'^2$ term causes the limit to $\rightarrow \pm\infty$, as $X_i' \rightarrow \pm\infty$. This result insures that the extremal portion of the specific solution falls in the class of smooth functions.

3. The form of the differential equations in VI.19 are such that the functions resulting from a twofold integration (with application of boundary conditions and transversality conditions) are unique.

Thus, the Legendre conditions insure that the specific solution is at least a local minimum. The KROTOV test shows that the specific solution occurs in a single class of smooth curves. Finally, the form of the specific differential equations insure uniqueness. The specific solution, then, is a global minimum.

G. Discussion of Solution

The cumulative production functions for all stages are now complete. To illustrate best the relationship, Figures 17 through

24 compare the initial inventory plus the cumulative production function to the cumulative demand function for each stage. Recall that the cumulative production function at stage $i+1$ establishes the cumulative demand function on stage i according to the relationships given in IV.19.

Examination of Figures 17-24 indicates that there are only two significant accumulations of material: explored reserves and fabricated reloads. Pertinent comments on each stage are:

1. From Figure 17, it is seen that an inventory of explored reserves is carried throughout the interval of interest; however, this inventory is gradually depleted. The most probable reason for this depletion is that all inventories are driven to zero by assumption in the problem statement.
2. Figures 18, 19, 20 and 24 show that production in mining, milling, conversion, and reprocessing of UF_6 is clearly the amount necessary to meet demand only.
3. Figure 21 shows that production in enrichment of natural uranium is essentially the amount necessary to meet demand; however, a relatively small inventory of 157,900 kilograms of enriched UF_6 builds up by 1982 and is then depleted.
4. Figure 22 shows that a significant inventory of 567 fabricated reloads builds up and is then depleted.
5. Figure 23 shows that the initial inventory of 238 spent fuel reloads is depleted by reprocessing by 1981, and all

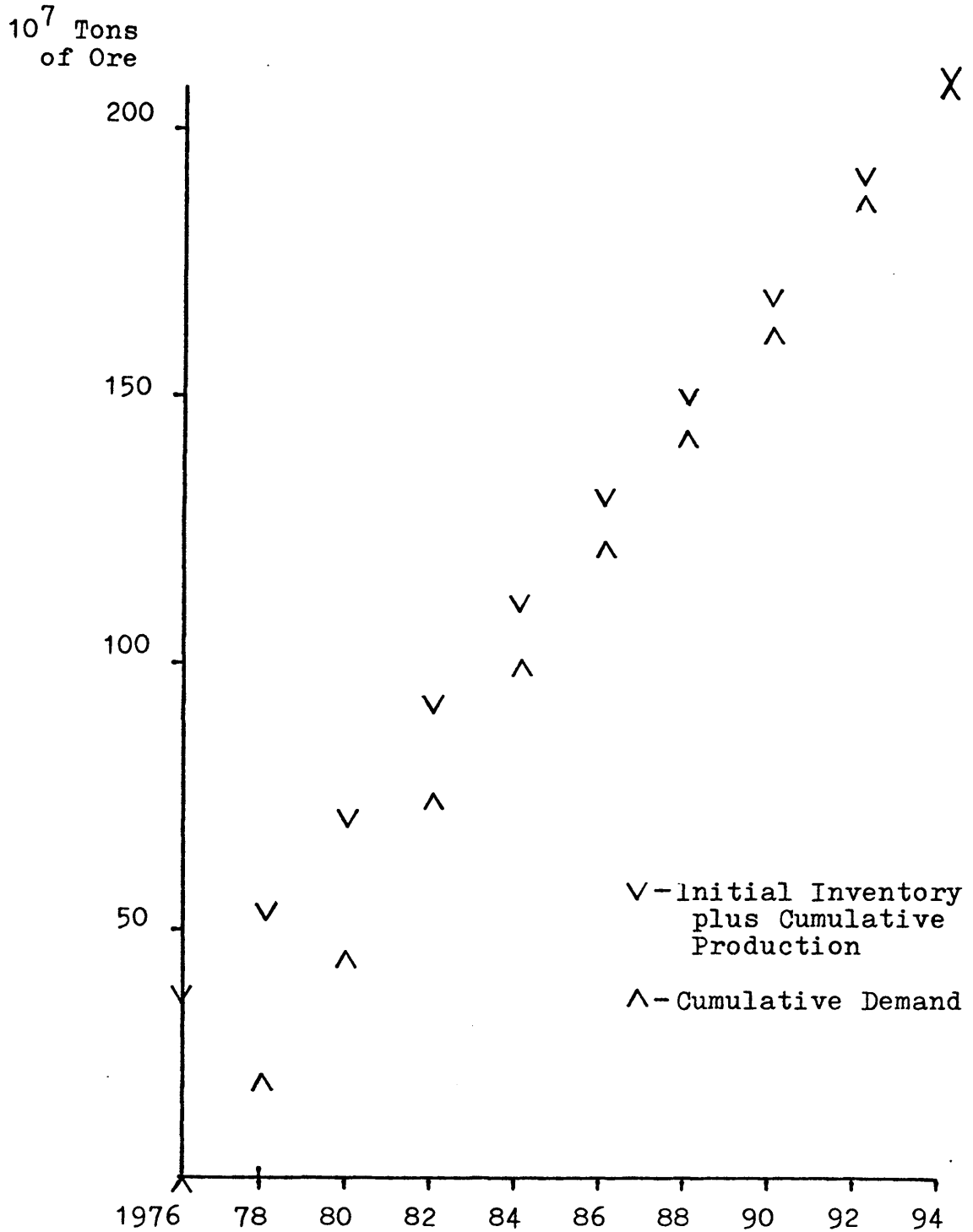


Figure 17. Initial Inventory Plus Cumulative Production versus Cumulative Demand for the Exploration and Acquisition Stage

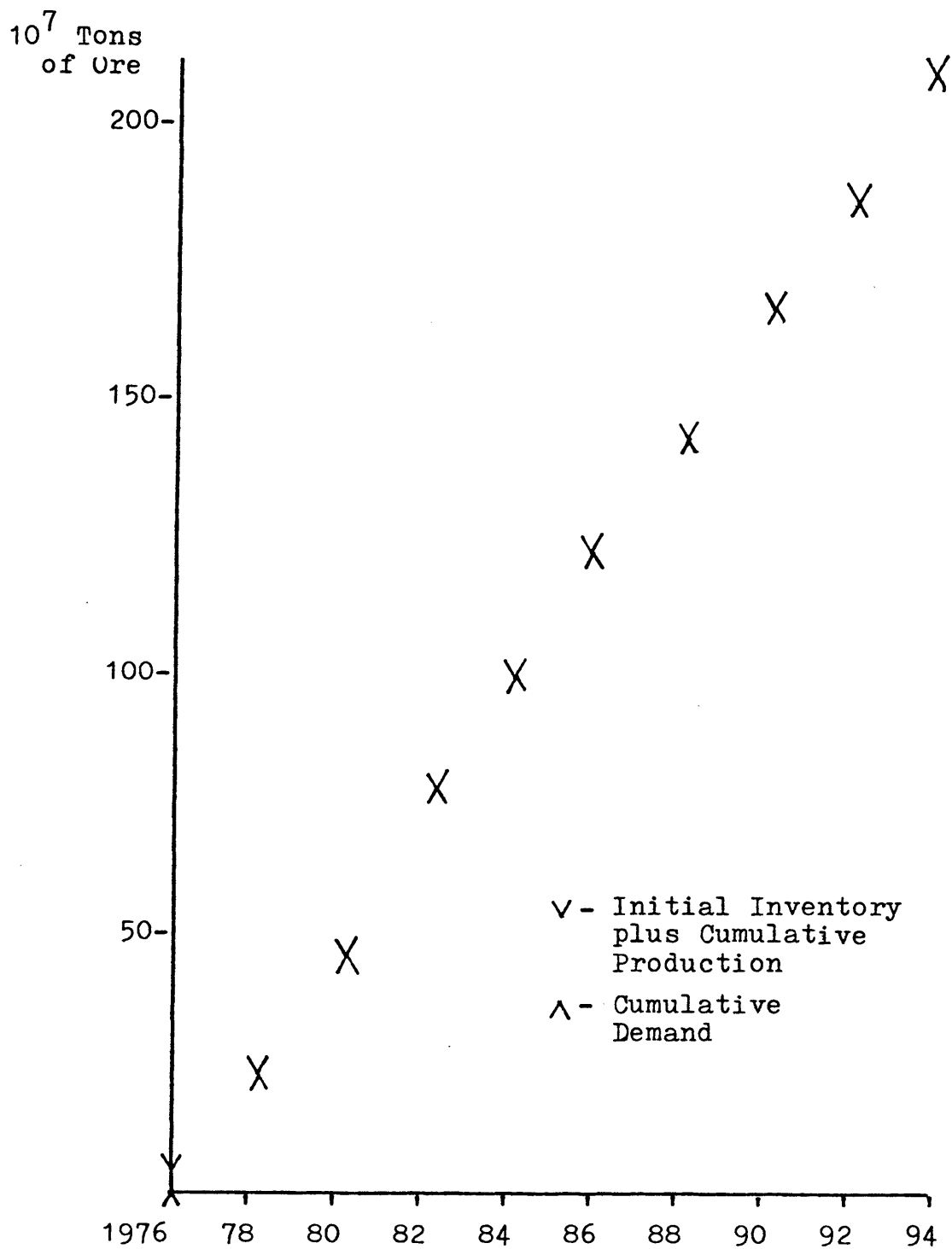


Figure 18. Initial Inventory Plus Cumulative Production versus Cumulative Demand for the Mining Stage

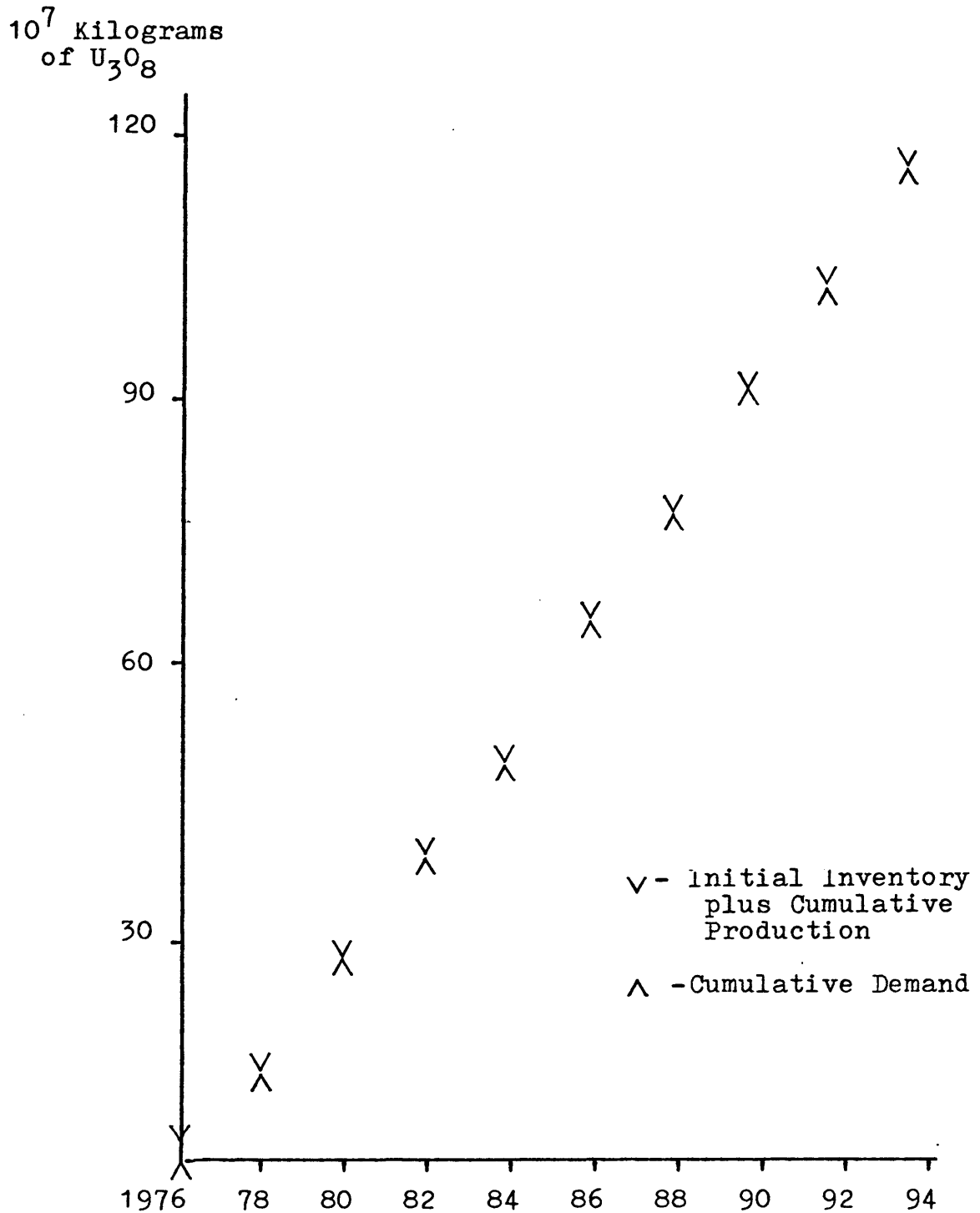


Figure 19. Initial Inventory Plus Cumulative Production versus Cumulative Demand for the Milling Stage

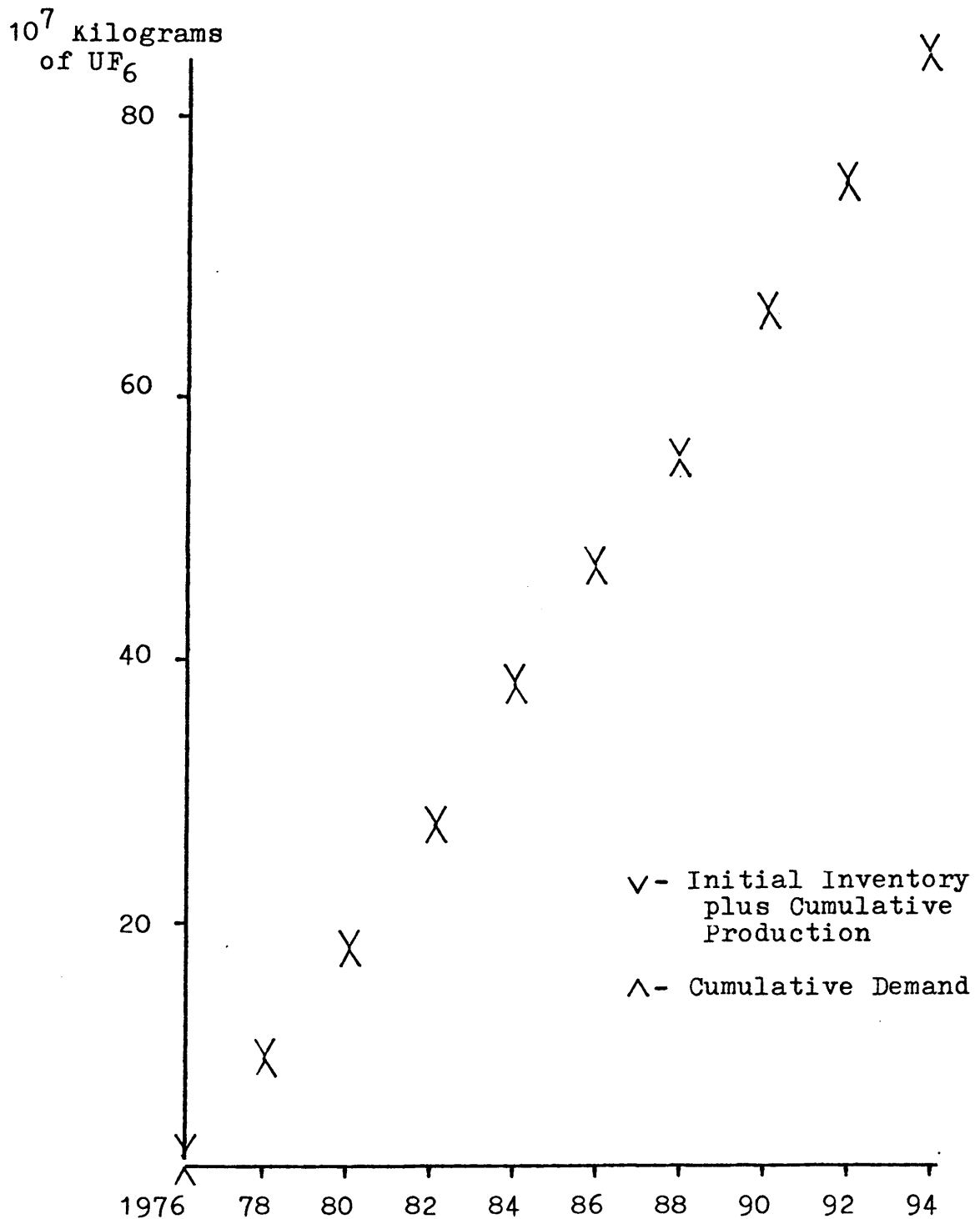


Figure 20. Initial Inventory Plus Cumulative Production versus Cumulative Demand for the Conversion Stage

10^6 Kilograms
of Enriched UF_6

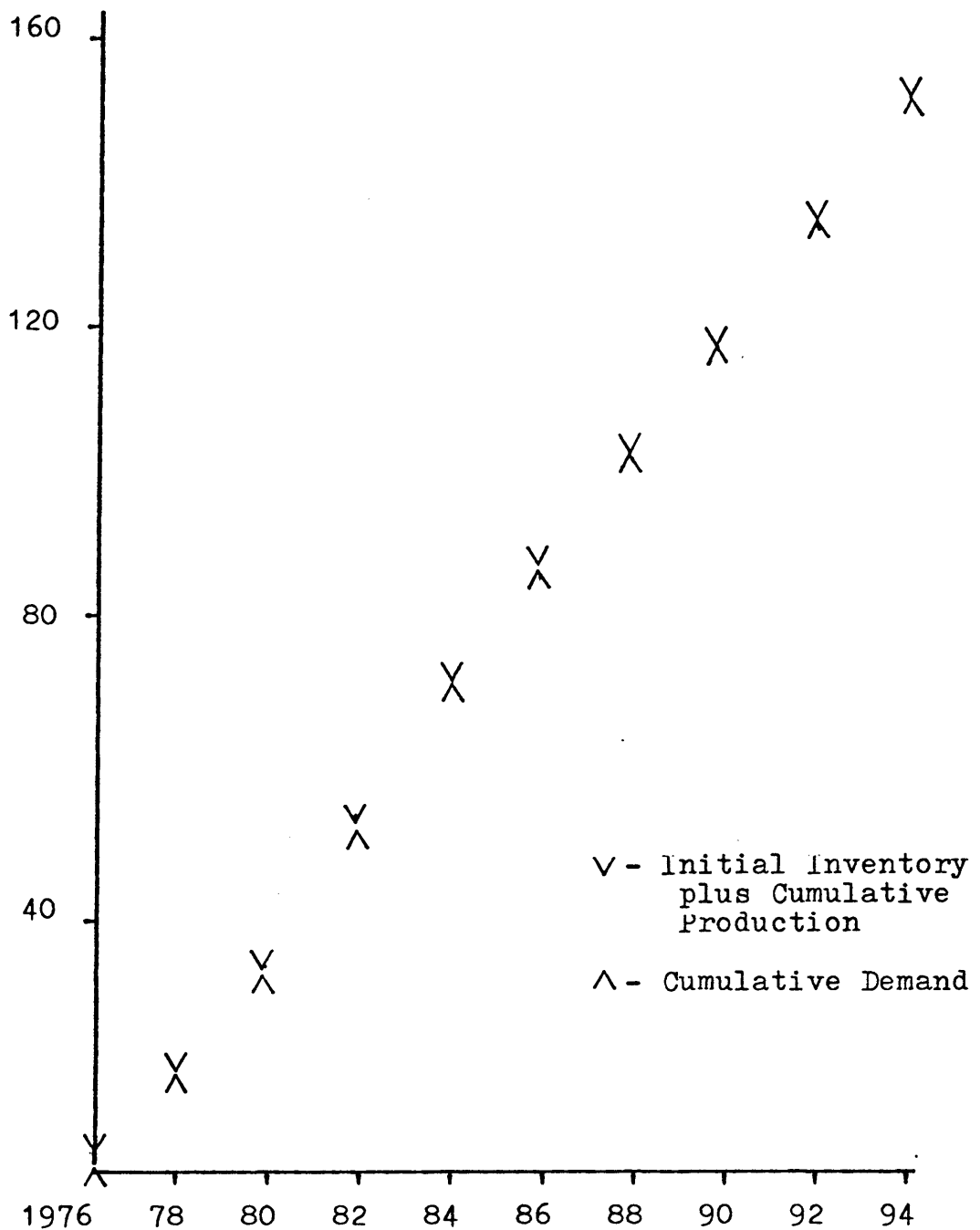


Figure 21. Initial Inventory Plus Cumulative Production versus Cumulative Demand for the Enrichment Stage

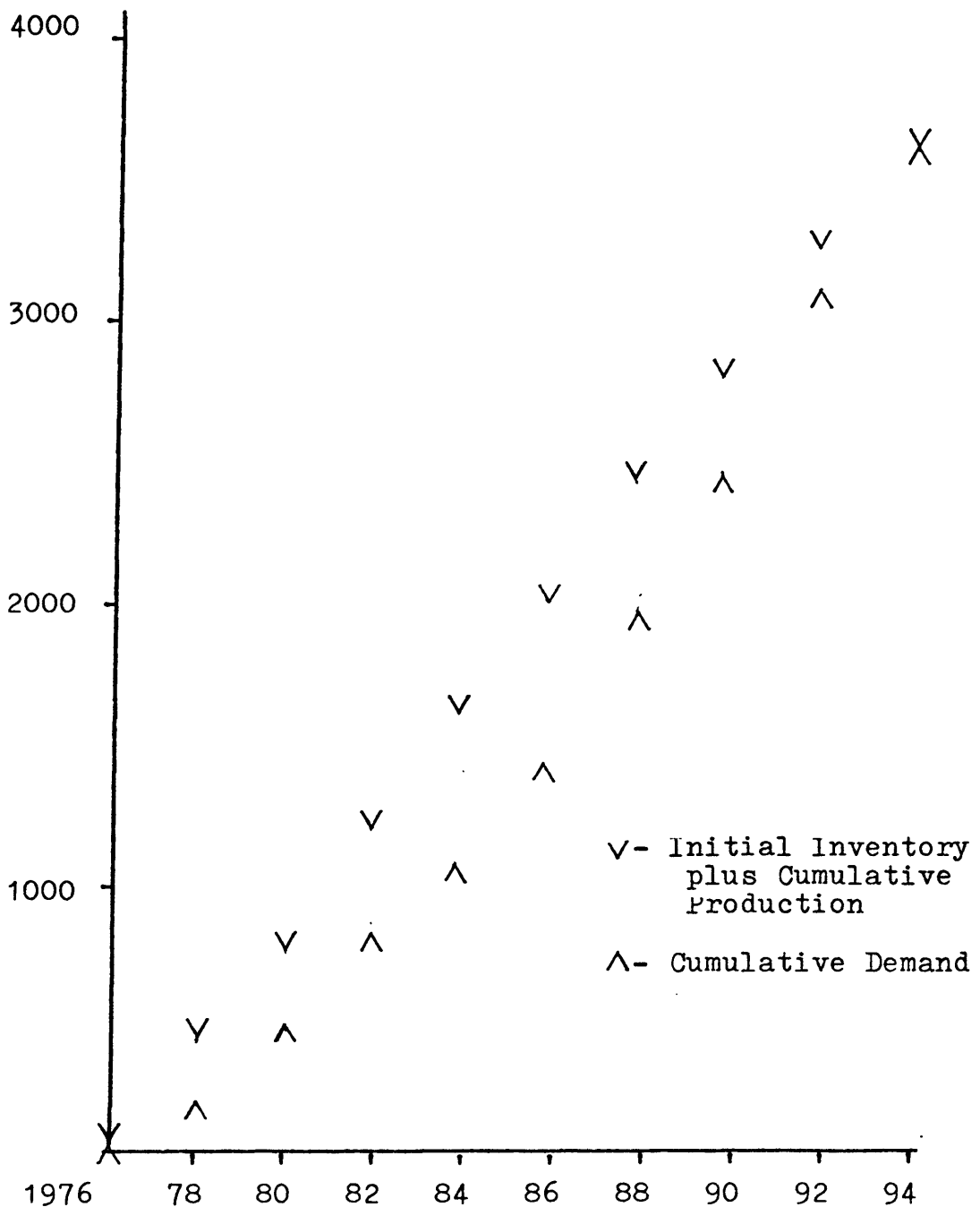
Fabricated
Reloads

Figure 22. Initial Inventory Plus Cumulative Production versus Cumulative Demand for the Fabrication Stage

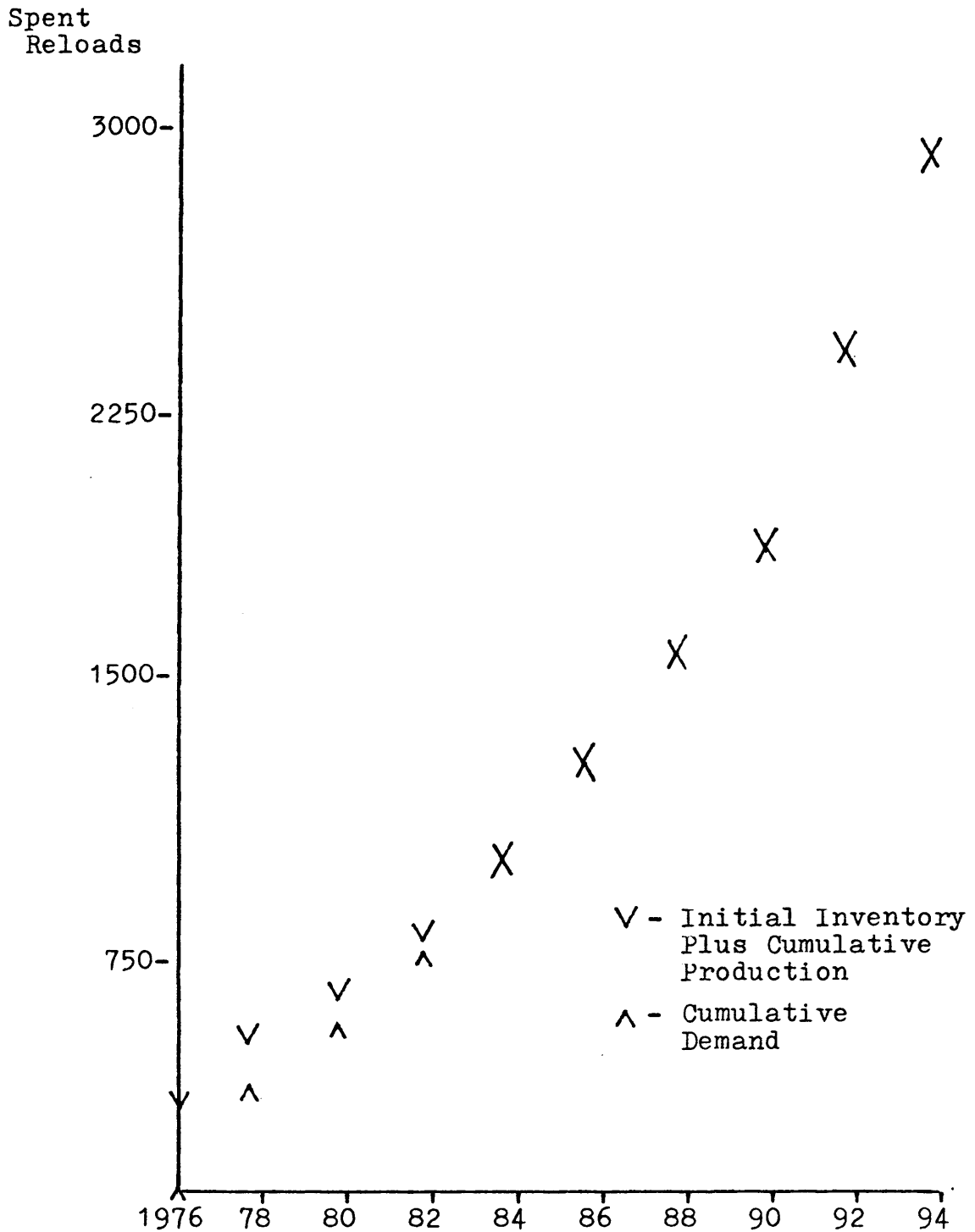


Figure 23. Initial Inventory Plus Cumulative Production versus Cumulative Demand for the Reprocessing Stage

10^5 Kilograms
of Reprocessed UF_6

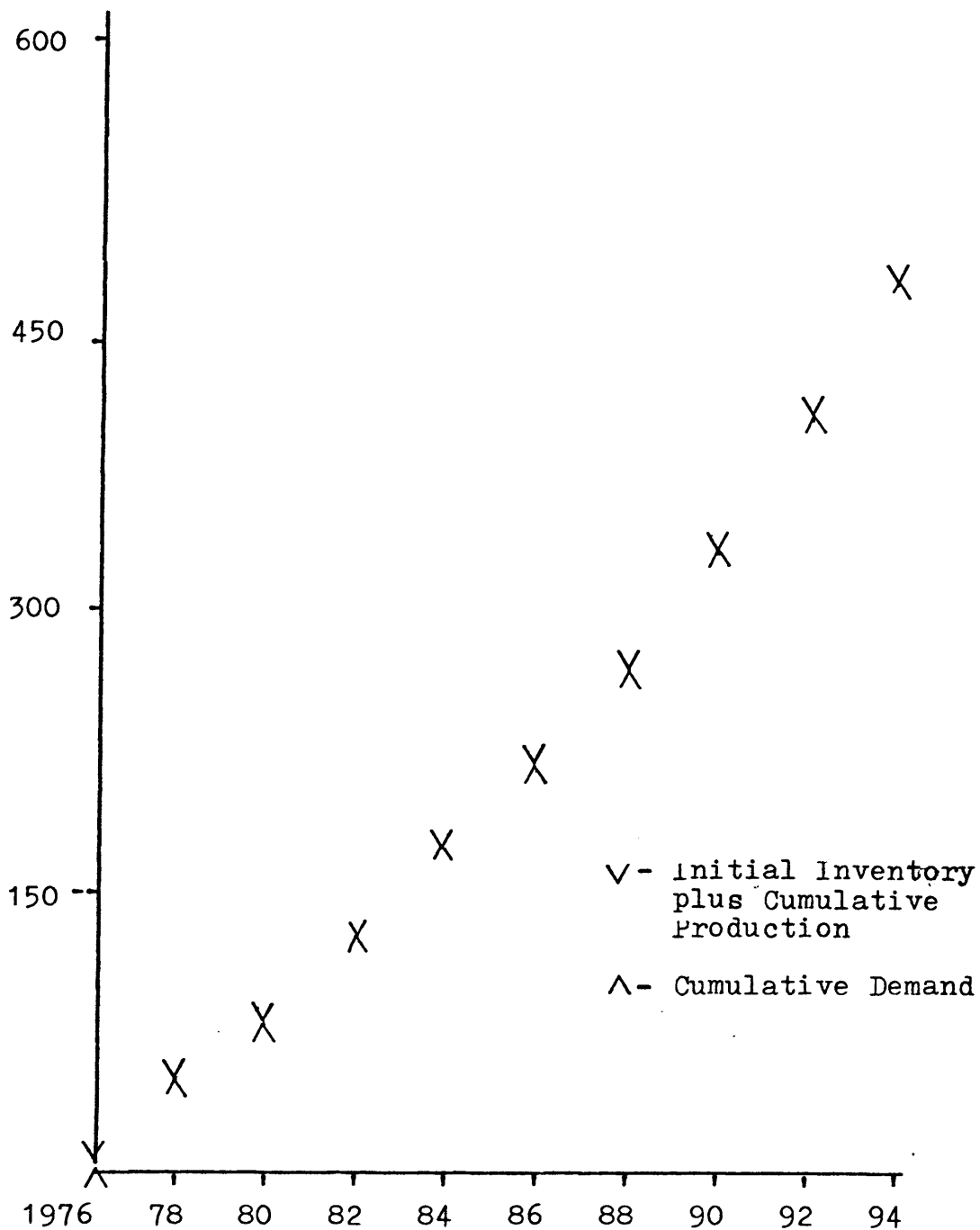


Figure 24. initial inventory Plus Cumulative Production versus Cumulative Demand for the Enrichment of Reprocessed UF_6 Stage

spent fuel reloads past that time are reprocessed as they are reproduced.

Recall that the objective was to minimize total costs where the costs considered were acquisition, production, and holding costs. Further, recall that production costs included costs associated with varying production rates from the rate considered optimal. Table 15 compares the derived production rate with the assumed production rate (see Table 8), and with available ERDA forecasts of what capacities should exist by 1990. Recall that the production rate is merely the first derivative of the cumulative production function with respect to time. From Table 15, it is readily concluded (for assumptions as given, specifically with reprocessing) that industry should concentrate on expanding production capability in mining, milling, and conversion, but not so in enrichment. Also, fabrication and reprocessing capacity will not be as critical as originally thought.

The final element of the solution is the total cost. Equation VI.17 is the objective function to be minimized. All extremizing functions have now been found and can be substituted into VI.17. Before substitution, some simplification can be made. Recall that in many cases, production was only sufficient to meet demand, i.e., there was no inventory. Where inventories are zero, obviously holding costs are zero and can be omitted.

To simplify the computations further, a fundamental law of calculus will be applied, i.e., the integral of the sum is equal to

Table 15

Comparison of Production Rates and Capacities

<u>Stage</u>	<u>Assumed Optimal Production Capacity (1977)</u>	<u>Derived Production Capacity (1990)</u>	<u>ERDA Forecast Production Capacity</u>
Exploration	29,542,863 tons/yr	20,388,000	108,429,000
Mining	11,920,804 tons/yr	22,770,000	95,673,000
Milling	20,909,090 kg U ₃ O ₈ /yr	38,702,000	54,545,000
Conversion	14,780,000 kg UF ₆ /yr	48,025,000	-
Enrichment (Total)	7,242,194 kg UF ₆ /yr	7,785,400	-
Fabrication	146.075 reloads/yr	206.060	-
Reprocessing	2,218,487 kg UF ₆ /yr	3,390,000	0

the sum of the integrals. This permits the total cost to be computed as the sum of the stagewise costs.

1. Exploration and Acquisition

Extracting exploration and acquisition costs from VI.17:

$$\begin{aligned}
 TC_1 = \int_0^T \{ & (1+qt)M_0 X_1'(t) + a_1 X_1'^2(t) - b_1 X_1'(t) \\
 & + c_1 + dM_1(1+qt)[I_1(0) + X_1(t) - k_{1,2} X_2(t+\ell_2)] \} dt
 \end{aligned} \quad \text{VI.93}$$

$X_1(t)$ is given in VI.88. $X_1'(t)$ is its first derivative with respect to time. Values for a_1 , b_1 , c_1 , q , d , M_0 , and M_1 have been given. The relationship inside the square brackets is developed in VI.87a-c. Finally, all computations are developed for the period of interest, 18 years. This requires production to be lag time ahead of demand. For this stage, the period of interest becomes $T-\ell_2$, or $18 - \frac{1}{6}$. Making these substitutions, VI.93 becomes:

$$\begin{aligned}
 TC_1 = \int_0^{17.833} \{ & (1+.05 t) \times 7 (14,318 t^2 + 413,640 t \\
 & + 11,791,000 + 1.1 \times 10^{-6} (14,318 t^2 + 413,640 t \\
 & + 11,791,000 - 7.8 (14,318 t^2 + 413,640 t + 11,791,000) \\
 & + 9.60 \times 10^7 + .14 \times 9 (1+.05 t)[(100,000,000+4,772.7 t^3 \\
 & + 206,820 t^2 + 11,791,000 t) - (429.7 t^3 + 26,105 t^2 \\
 & + 21,786,000 t] \} dt
 \end{aligned} \quad \text{VI.94}$$

Simplifying, integrating, and evaluating results in:

$$TC_1 = \$1.4161 \times 10^{10} \quad \text{VI.95}$$

2. Mining

Extracting mining costs from VI.17:

$$TC_2 = \int_0^T \{a_2 X_2'^2(t) - b_2 X_2'(t) + c_2 + dM_2 \cdot (1+qt)[I_2(0) + X_2(t) - k_{2,3} X_3(t+l_3)]\} dt \quad \text{VI.96}$$

$X_2(t)$ is given by VI.82. $X_2'(t)$ is the first derivative of $X_2(t)$ with respect to time. Values for a_2 , b_2 , c_2 , q , d , and M_2 have been given. The relationship inside the square brackets is developed in VI.81a-c, but for this stage is equal to zero. The period of interest is $18 - \frac{1}{4}$. Making these substitutions, VI.93 becomes:

$$TC_2 = \int_0^{17.75} \{41.94 \times 10^{-6} (1,289 t^2 + 52,210 t + 21,786,000)^2 - 120(1,289 t^2 + 52,210 t + 21,786,000) + 59.6 \times 10^7\} dt \quad \text{VI.97}$$

Simplifying, integrating, and evaluating results in:

$$TC_2 = \$3.2628 \times 10^{11} \quad \text{VI.98}$$

3. Milling

Extracting milling costs from VI.17:

$$TC_3 = \int_0^T \{a_3 X_3'^2(t) - b_3 X_3'(t) + c_3 + dM_3 \cdot (1+qt)[I_3(0) + X_3(t) - k_{3,4} X_4(t+l_4)]\} dt \quad \text{VI.99}$$

$X_3(t)$ is given by VI.75. $X_3'(t)$ is the first derivative with respect to time. Values for a_3 , b_3 , c_3 , q , d , and M_3 have been given. The relationship inside the square brackets is developed in VI.74a-c, but for this stage is equal to zero. The period of interest is $18 - \frac{1}{4}$. Making these substitutions, VI.96 becomes:

$$TC_3 = \int_0^{17.75} \{4.783 \times 10^{-6} (2,192 t^2 + 88,780 t + 37,029,000)^2 - 24 (2,192 t^2 + 88,780 t + 37,029,000) + 23 \times 10^7\} dt \quad \text{VI.100}$$

Simplifying, integrating, and evaluating results in:

$$TC_3 = \$1.090 \times 10^{11} \quad \text{VI.101}$$

4. Conversion

Extracting conversion costs from VI.17:

$$TC_4 = \int_0^T \{a_4 X_4'^2(t) - b_4 X_4'(t) + c_4 + dM_4 \cdot (1+qt)[I_4(0) + X_4(t) - k_{4,5} X_5(t+l_5)]\} dt \quad \text{VI.102}$$

$X_4(t)$ is given by VI.68. $X_4'(t)$ is the first derivative of $X_4(t)$ with respect to time. Values for a_4 , b_4 , c_4 , q , d , and M_4 have been given. The relationship in the square brackets is developed in VI.65a-c and VI.66a and b. Holding costs accrue over the interval $0 \leq t \leq 6.463$ but are zero over the interval $6.463 \leq t \leq 18$. The period of interest is $18 - \frac{1}{4}$. Making appropriate substitutions, VI.99 becomes:

$$\begin{aligned} TC_4 = \int_0^{17.75} \{ & 7.442 \times 10^{-6} (2,522 t^2 + 110,228 t \\ & + 45,948,000)^2 - 26.4(2,522 t^2 + 110,228 t \\ & + 45,948,000) + 16.25 \times 10^7 \} dt \end{aligned} \quad \text{VI.103}$$

Simplifying, integrating, and evaluating results in:

$$TC_4 = \$2.602 \times 10^{11} \quad \text{VI.104}$$

5. Enrichment and Natural Feed UF_6

Extracting enrichment costs from VI.17:

$$\begin{aligned} TC_5 = \int_0^T \{ & a_5 X_5'^2(t) - b_5 X_5'(t) + c_5 + dM_5 \\ & \cdot (1+qt)[I_5(0) + X_5(t) + X_9(t) - k_{5,6} X_6(t+l_6)] \} dt \end{aligned} \quad \text{VI.105}$$

$X_5(t)$ is given by VI.57 and VI.59b. $X_5'(t)$ is the first derivative of $X_5(t)$ with respect to time. Values for a_5 , b_5 , c_5 , q , d , and M_5 are given. The relationship inside the square brackets is developed in VI.58 for the interval $0 \leq t \leq 6.713$. Over the remainder of the interval the relationship equals zero. The period of interest is $18 - \frac{1}{4}$. Making the appropriate substitutions, VI.102 becomes:

$$\begin{aligned}
 TC_5 = & \int_0^{6.713} \{1,461.7 \times 10^{-6} (456.27 t^2 + 18,251 t \\
 & + 7,698,000)^2 - 2,286.6(456.27 t^2 + 18,251 t \\
 & + 7,698,000) + 620 \times 10^7 + .14 \times 846.85 \\
 & \cdot (1+.05 t)[(2,051,000+152.09 t^3 + 9,125.4 t^2 \\
 & + 7,698,000 t + 15.884 t^3 - 7,416.8 t^2 + 572,780 t) \\
 & - (140.97 t^3 + 8,563.8 t^2 + 8,206,000 t \\
 & + 2,051,000)]\}dt + \int_{6.713}^{17.75} \{1,461.7 \times 10^{-6} (422.91 t^2 \\
 & - 16,514 t + 7,933,700)^2 - 2,286.6(422.91 t^2 \\
 & - 16,514 t + 7,933,700) + 620 \times 10^7\}dt
 \end{aligned} \tag{VI.106}$$

Simplifying, integrating, and evaluating results in:

$$TC_5 = \$1.3487 \times 10^{12} \tag{VI.107}$$

6. Fabrication

Extracting fabrication costs from VI.17:

$$TC_6 = \int_0^T \{a_6 X_6'^2(t) - b_6 X_6'(t) + c_6 + dM_6(1+qt) \cdot [I_6(0) + X_6(t) - R_6(t)]\} dt \quad \text{VI.108}$$

$X_6(t)$ is given by VI.25. $X_6'(t)$ is the first derivative of $X_6(t)$ with respect to time. Values for a_6 , b_6 , c_6 , q , d , and M_6 have been given. The initial inventory, $I_6(0)$, is zero. $R_6(t)$ is known and given by VI.21. Making the appropriate substitutions, VI.105 becomes:

$$TC_6 = \int_0^{18} \{483,656(.01023 t^2 + .40906 t + 198.33)^2 - 16,956 \times 10^3 (.01023 t^2 + .40906 t + 198.33) + 103.2 \times 10^7 + .14 \times 37,846,519(1+.05 t)[(.0034088 t^3 + .20453 t^2 + 198.33 t) - (7.306 t^2 + 71.606 t)]\} dt \quad \text{VI.109}$$

Simplifying, integrating, and evaluating results in:

$$TC_6 = \$3.4172 \times 10^{11} \quad \text{VI.110}$$

7. Reactor

Extracting the costs relating to storage of spent fuel reloads from VI.17:

$$TC_7 = \int_0^T S[I_7(0) + X_7(t) - k_{7,8} X_8(t+l_8)] dt \quad \text{VI.111}$$

$X_7(t)$ is the reactor cumulative production function for spent fuel reloads and is given by VI.22. S and $I_7(0)$ are both known. $X_8(t)$ is developed in VIa and b over the intervals $0 \leq t \leq 5.982$ and $5.982 \leq t \leq 18 - \frac{1}{4}$; however, the relationship inside the brackets equals zero over the second interval. Making appropriate substitutions, VI.108 becomes:

$$\begin{aligned} TC_7 = \int_0^{5.982} \{2,700,000[(238+4.292 t^2 + 73.86 t) \\ - .0000566(71.605 t^3 - 33,381 t^2 + 2,565,400 t)]\}dt \end{aligned} \quad \text{VI.112}$$

Simplifying, integrating, and evaluating results in:

$$TC_7 = \$1.5970 \times 10^9 \quad \text{VI.113}$$

8. Reprocessing

Extracting reprocessing costs from VI.17:

$$\begin{aligned} TC_8 = \int_0^T \{a_8 X_8'^2(t) - b_8 X_8'(t) + c_8 + dM_8(1+qt) \\ \cdot [I_8(0) + X_8(t) - k_{8,9} X_9(t+l_9)]\}dt \end{aligned} \quad \text{VI.114}$$

$X_8(t)$ is given by VI.41 a and b. $X_8'(t)$ is the derivative of these functions with respect to time. Values of a_8 , b_8 , c_8 , d , q , and M_8 have been given. The relationship inside the square brackets is equal to zero throughout the interval of interest, which consists of $0 \leq t \leq 5.982$ and $5.982 \leq t \leq 18 - \frac{1}{4}$. Making appropriate

substitutions, VI.114 becomes:

$$\begin{aligned}
 TC_8 = & \int_0^{5.982} \{1,014 \times 10^{-6} (214.82 t^2 - 66,762 t \\
 & 2,565,400)^2 - 540(214.82 t^2 - 66,762 t + 2,565,400) \\
 & + 49.92 \times 10^7\} dt + \int_{5.982}^{17.75} \{1,014 \times 10^{-6} (151,660 t \\
 & + 1,267,000)^2 - 540(151,660 t + 1,267,000) + 49.92 \times 10^7\} dt
 \end{aligned} \tag{VI.115}$$

Simplifying, integrating, and evaluating results in:

$$TC_8 = \$1.3424 \times 10^{11} \tag{VI.116}$$

9. Enrichment of Reprocessed UF₆

Extracting the enrichment of reprocessed UF₆ costs from VI.17:

$$TC_9 = \int_0^T \{a_9 X_9'^2(t) - b_9 X_9'(t) + c_9\} dt \tag{VI.117}$$

$X_9(t)$ is given in VI.46b and VI.47c. $X_9'(t)$ is the first derivative of $X_9(t)$ with respect to time. Values of a_9 , b_9 , and c_9 have been given. The intervals of interest are $0 \leq t \leq 6.232$ and $6.232 \leq t \leq 18 - \frac{1}{4}$. Making appropriate substitutions, VI.114 becomes:

$$\begin{aligned}
TC_9 = & \int_0^{6.232} \{7,393 \times 10^{-6} (47.652 t^2 - 14,833.6 t \\
& + 572,820)^2 - 1,714.2(47.652 t^2 - 14,833.6 t \\
& + 5572,820) + 69 \times 10^7\} dt + \int_{6.232}^{17.75} \{7,393 \times 10^{-6} \\
& (33,642 t + 272,690)^2 - 1,714.2(33,642 t + 272,690) \\
& + 69 \times 10^7\} dt
\end{aligned} \tag{VI.118}$$

Simplifying, integrating, and evaluating results in:

$$TC_9 = \$4.6376 \times 10^{10} \tag{VI.119}$$

Finally, the total cost is the sum of the stage costs.

$$TC = \sum_{i=1}^9 TC_i = \$2.5823 \times 10^{12} \tag{VI.120}$$

This total cost figure is of little significance here but will be of more interest in the sensitivity analysis. Of more significance are the conclusions that can be drawn by reviewing the results of the model. The more important conclusions are:

- A. Accumulation of inventories occurs at the exploration stage and at the fabrication stage, but for apparently different reasons:

1. The inventory at exploration results from the increasing value of the uranium ore on the one hand, but also because there already exists a stockpile.
 2. At the fabrication stage the cumulative demand function is known and is a rapidly increasing function. As the model was designed to associate a cost with changes to the production rate, the results show that it is more cost effective to stockpile material in the form of reloads of fabricated fuel than to experience a rapidly increasing production rate. Recall that variation from the optimal production level results in costs proportional to the square of the difference.
- B. The derived production rates for mining, milling, and conversion show dramatic increases are necessary. Enrichment, fabrication and reprocessing should increase only moderately.

CHAPTER VII.

SENSITIVITY ANALYSIS AND RECOMMENDATIONS FOR FURTHER RESEARCH

Three parameters are selected to test the model for sensitivity to variation in model parameters. These parameters are:

1. The optimal production capacity for conversion, L_4 .
2. The cost to reprocess one kilogram of UF_6 , P_8 .
3. The constant of proportionality, β_i , to determine the increase in production cost per unit as the square of the difference in the assumed production capacity and the derived production rate.

Subsequent to this analysis, recommendations for further research are given.

A. Optimal Production Capacity for Conversion

The optimal production capacity for conversion is selected for analysis because the derived production rate is shown to be approximately three times the assumed production capacity. Recall that production costs increase proportionally to the square of the difference in the assumed production capacity and the derived production rate. Intuitively, the total cost should decrease if the assumed production capacity is increased. Assume that the conversion production capacity is doubled. L_2 is now equal to 30,000,000 kilograms of UF_6 per year. Solving in the same manner as in Chapter VI yields a stage cost of $\$4.7569 \times 10^{12}$ as compared to an originally computed cost of $\$1.7770 \times 10^{11}$, or a 31.7%

reduction. Note that no consideration is given to the cost of increasing mining production capacity. Further, the total fuel cycle cost is $\$2.4998 \times 10^{12}$ as compared to an originally computed cost of $\$2.5823 \times 10^{12}$, or a reduction of 9.4%.

B. Cost of Reprocessing

The cost to reprocess spent fuel into UF_6 is selected for analysis because little is known about the actual costs. As previously stated, estimated costs range from \$90 to \$280 per kilogram UF_6 . The original solution assumes the lower figure. Assume now that the cost to reprocess spent fuel is \$180 per kilogram of UF_6 . Recomputation of the solution yields a stage cost of $\$1.7611 \times 10^{11}$ as compared to an original solution cost of $\$1.3424 \times 10^{11}$, or an increase of 31.2% for that stage. The total fuel cycle costs increase from $\$2.5823 \times 10^{12}$ to $\$2.6242 \times 10^{12}$, or 1.6%. This result is as expected; however, a further increase might eliminate reprocessing because of economics.

C. The Constant of Proportionality in Production Costs

The constant of proportionality, β_1 , is selected for analysis because the originally assumed value is arbitrary. The actual cost to vary production rates is proprietary to the company concerned. The original assumption, a variation in production from the assumed optimal production capacity of 20% increases production costs by 10%, is the result of conversations with persons who perform similar studies [8,22,74].

Assume now that a variation in production from the assumed optimal production capacity of 20% increases production costs by 20%. Deriving a new solution as done in Chapter VI, yields the increasing costs shown in Table 16. Note that the nature of β_i is such that as β_i increases, the production rate tends to be "straight-lined", i.e., it costs more to vary the derived production rate from the assumed optimal capacity. This has the effect of dampening out the creation of early inventories (stockpiles) and forces the acquisition of materials to a later time when they are more expensive.

A further observation is that the stage costs increase where the derived production rate varies greatly from the assumed optimal production capacity. This relationship is best seen by comparing Tables 15 and 16. Note in Table 15 for stages where the derived production rate is significantly different from the assumed optimal production capacity (conversion) that in Table 16 the corresponding increases in production costs are higher (88.7%).

D. Recommendation for Further Research

In summary, the model developed here solves the cyclic, multi-stage production-to-inventory problem that, heretofore, had not been solved. It has a diverse range of application, specifically in any process where products are recycled. Examples include processes where defective products can be salvaged such as steel forgings. Further, the model permits analysis of the relationships among the

Table 16

Comparison of Fuel Cycle Cost Increases

<u>Stage</u>	<u>Original Solution Cost</u>	<u>Increased β_i Solution Cost</u>	<u>Percent Increase</u>
Exploration	1.416×10^{10}	1.562×10^{10}	10.3%
Mining	3.263×10^{11}	3.952×10^{11}	21.1%
Milling	1.090×10^{11}	1.311×10^{11}	20.2%
Conversion	2.602×10^{11}	4.911×10^{11}	88.7%
Enrichment	1.3487×10^{12}	1.976×10^{12}	46.5%
Fabrication	3.417×10^{11}	4.327×10^{11}	26.6%
Reprocessing	1.342×10^{11}	1.689×10^{11}	25.8%
Enrichment of Reprocessed UF_6	4.638×10^{10}	5.162×10^{10}	11.3%
Total Cycle	2.582×10^{12}	3.629×10^{12}	40.5%

model parameters. In the absence of such a solution technique, these relationships remain speculative.

In the nuclear fuel cycle the sensitivity analysis demonstrates that the model is highly sensitive to changes in the assumed optimal production capacities and to changes in the production cost constant of proportionality, β_1 . The model is less sensitive to the cost of reprocessing of spent fuel. These results suggest that a thorough investigation of assumed optimal production capacities is appropriate. One direction of this investigation should focus on the model developed here and should seek the actual production capacities that would yield a minimum total cost over the interval of interest. The other direction is improvement of the model. Recall that the optimal production capacity was assumed to be constant over the interval. Allowing this capacity to vary with time or to experience increases/decreases at intervals is a much desired improvement.

Further research is also recommended to increase the model's representation of the real-world system. Specifically, the model assumes that all nuclear fuel is enriched in the U-235 isotope to one standard percentage. The present model allows cyclic (feedback) flow. A more realistic model would allow a variety of fuel enrichments in a forward branching flow. The technique is similar to that developed here. A further investigation with more realism is an analysis of the original problem with a change in boundary conditions. The assumption of exhausting all existing inventories provides a better basis for economic comparison; however, a boundary condition

establishing specific ending inventories is more realistic.

Thus, the development here can contribute greatly to the cost-benefit analysis of the nuclear fuel cycle management. In doing so, it not only can improve the management decision process, but it can assist in improving the energy outlook of the United States.

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APPENDIX A.

DEFINITION OF NOTATION

a_i	coefficient in the quadratic production cost equation for stage i
α	percentage of non-defective products from a manufacturing process
b_i	coefficient in the quadratic production cost equation for stage i
β_i	proportionality constant for the increase in production cost versus the square of the difference in the optimal production level and the actual production level for stage i
c_i	coefficient in the quadratic production cost equation for stage i
C_i	total cost for stage i
d	time value of money factor
δ	small increment in a variable or a small variation in a functional
F	abbreviation for $F(y, y', t)$

- $F(y, y', t)$ functional dependent on the function y , the first derivative of y with respect to t , and the independent variable t
- $F_{y_i y_j}$ second partial derivative of F with respect to y_i and with respect to y_j
- ϵ an arbitrary parameter
- $f_i(t)$ acquisition cost function in time
- $g_i(t)$ production cost function in time
- $h_i(t)$ inventory holding cost function in time
- $I_i(t)$ number of units in inventory as a function of time
- J general functional
- $k_{i,j}$ number of input products i required to produce one product j
- l_i production lag time in stage i
- L_i assumed optimal production capacity for stage i
- M_i value of one unit of product in inventory i
- $n(t)$ an arbitrary function in t
- P_i cost to produce one unit of product in stage i

q	marginal increase per unit of uranium fuel
$r_i(t)$	demand rate as a function of time for stage i
$R_i(t)$	cumulative demand as a function of time for stage i
S	cost to encapsulate and store one reload of spent nuclear fuel
t	independent variable, time
T	horizon of time of interest
TC	total cost for fuel cycle
τ	substitute variable for t
$x_i(t)$	production rate as a function of time for stage i
$X_i(t)$	cumulative production as a function of time for stage i
$X_i'(t)$	first derivative of the cumulative production function for stage i ; also, identical to $x_i(t)$
$y(t)$	general unknown function in a functional
$y'(t)$	first derivative of $y(t)$ with respect to t

APPENDIX B.

NECESSARY AND SUFFICIENT CONDITIONS FOR EXTREMALS OF FUNCTIONALS DEPENDING ON SEVERAL UNKNOWN FUNCTIONS

Necessary and sufficient conditions are developed here for the type of problem presented only. Other necessary and sufficient conditions exist but will not be developed.

Assume that an unknown function, $y(t)$, is sought that will extremize the integral:

$$I = \max (\min) \int_a^b F[y(t), y'(t), t] dt \quad \text{B.1}$$

Clearly the value of I is dependent on the form of $y(t)$. Let $y(t)$ be specifically selected from a class of admissible functions having the following properties:

1. The functions are defined and have continuous first and second derivatives.
2. The functions pass through the points $[a, y(a)]$ and $[b, y(b)]$.

Any particular function having these properties and extremizing B.1 is called an extremal. Let $y(t)$ be an extremal. Consider some other function from the class of admissible functions that is in the near neighborhood of $y(t)$. Call this function:

$$\bar{y}(t) = y(t) + \epsilon \eta(t) \quad \text{B.2}$$

Assume that $\eta(t)$ is an arbitrary function such that $\eta(a) = \eta(b) = 0$ and that ϵ is an arbitrary parameter.

Now if $y(t)$ is the extremal and held in its form, and if $\eta(t)$ is arbitrary, known, and fixed, then the value of I becomes a function of

ϵ , $I(\epsilon)$. From this relationship it is seen that $I(0)$ is the extremal, that $I(\epsilon)$ is a near neighborhood functional, and that they have the following forms:

$$I(\epsilon) = \int_a^b F[y(t) + \epsilon\eta(t), y'(t) + \epsilon\eta'(t), t] dt \quad B.3$$

$$I(0) = \int_a^b F[y(t), y'(t), t] dt$$

Let the same line of reasoning apply to a functional dependent on n unknown functions, and for brevity, establish the following identities:

$$y_i(t) \equiv y_i \quad B.4$$

$$\frac{d}{dt} y_i(t) \equiv y_i'$$

$$\eta_i(t) \equiv \eta_i$$

$$\frac{d}{dt} \eta_i(t) \equiv \eta_i'$$

$I(\epsilon)$ becomes:

$$I(\epsilon) = \int_a^b F[y_1 + \epsilon\eta_1, y_2 + \epsilon\eta_2, \dots, y_n + \epsilon\eta_n, \quad B.5$$

$$y_1' + \epsilon\eta_1', y_2' + \epsilon\eta_2', \dots, y_n' + \epsilon\eta_n', t] dt$$

The first variation in $I(\epsilon)$, δI , is defined as:

$$\delta I \equiv \lim_{\epsilon \rightarrow 0} \frac{I(\epsilon) - I(0)}{\epsilon} \quad \text{B.6}$$

In order for a functional to be an extremal, it is necessary that $\delta I = 0$.

$$\lim_{\epsilon \rightarrow 0} \frac{I(\epsilon) - I(0)}{\epsilon} = 0 = \lim_{\epsilon \rightarrow 0} \left\{ \frac{1}{\epsilon} \int_a^b [F(y_1 + \epsilon \eta_1, y_2 + \epsilon \eta_2, \dots, y_n + \epsilon \eta_n, y_1' + \epsilon \eta_1', y_2' + \epsilon \eta_2', \dots, y_n' + \epsilon \eta_n', t) - F(y_1, y_2, \dots, y_n, y_1', y_2', \dots, y_n', t)] dt \right\} \quad \text{B.7}$$

The term inside the brackets expressed in a Taylor Series expansion with respect to ϵ and about $\epsilon = 0$ is:

$$F(y_1 + \epsilon \eta_1, y_2 + \epsilon \eta_2, \dots, y_n + \epsilon \eta_n, y_1' + \epsilon \eta_1', y_2' + \epsilon \eta_2', \dots, y_n' + \epsilon \eta_n', t) - F(y_1, y_2, \dots, y_n, y_1', y_2', \dots, y_n', t) = [F(y_1 + \epsilon \eta_1, y_2 + \epsilon \eta_2, \dots, y_n + \epsilon \eta_n, y_1' + \epsilon \eta_1', y_2' + \epsilon \eta_2', \dots, y_n' + \epsilon \eta_n', t) - F(y_1, y_2, \dots, y_n, y_1', y_2', \dots, y_n', t)] \quad \text{B.8}$$

$$+ \epsilon \eta_2', \dots, y_n' + \epsilon \eta_n', t) - F(y_1, y_2, \dots, y_n, y_1', y_2', \dots, y_n', t) = [F(y_1 + \epsilon \eta_1, y_2 + \epsilon \eta_2, \dots, y_n + \epsilon \eta_n, y_1' + \epsilon \eta_1', y_2' + \epsilon \eta_2', \dots, y_n' + \epsilon \eta_n', t) - F(y_1, y_2, \dots, y_n, y_1', y_2', \dots, y_n', t)]$$

$$+ \epsilon \eta_2', \dots, y_n' + \epsilon \eta_n', t) = [F(y_1 + \epsilon \eta_1, y_2 + \epsilon \eta_2, \dots, y_n + \epsilon \eta_n, y_1' + \epsilon \eta_1', y_2' + \epsilon \eta_2', \dots, y_n' + \epsilon \eta_n', t) - F(y_1, y_2, \dots, y_n, y_1', y_2', \dots, y_n', t)]$$

$$+ \epsilon \eta_2', \dots, y_n' + \epsilon \eta_n', t) = [F(y_1 + \epsilon \eta_1, y_2 + \epsilon \eta_2, \dots, y_n + \epsilon \eta_n, y_1' + \epsilon \eta_1', y_2' + \epsilon \eta_2', \dots, y_n' + \epsilon \eta_n', t) - F(y_1, y_2, \dots, y_n, y_1', y_2', \dots, y_n', t)]$$

$$y_n' + \varepsilon \eta_n', t) - F(y_1, y_2, \dots, y_n, y_1', y_2', \dots,$$

$$y_n', t)] \Big|_{\varepsilon=0} + \frac{\partial}{\partial (y_1 + \varepsilon \eta_1')} [F(y_1 + \varepsilon \eta_1, y_2 + \varepsilon \eta_2,$$

$$\dots, y_n + \varepsilon \eta_n, y_1' + \varepsilon \eta_1', y_2' + \varepsilon \eta_2', \dots, y_n'$$

$$+ \varepsilon \eta_n')] \Big|_{\varepsilon=0} \frac{\partial}{\partial \varepsilon} (y_1 + \varepsilon \eta_1) \Big|_{\varepsilon=0} \cdot (\varepsilon - 0)$$

$$+ \frac{\partial}{\partial (y_1' + \varepsilon \eta_1')} [F(y_1 + \varepsilon \eta_1, y_2 + \varepsilon \eta_2, \dots, y_n$$

$$+ \varepsilon \eta_n, y_1' + \varepsilon \eta_1', y_2' + \varepsilon \eta_2', y_n' + \varepsilon \eta_n')]$$

$$\Big|_{\varepsilon=0} \frac{\partial}{\partial \varepsilon} (y_1' + \varepsilon \eta_1') \Big|_{\varepsilon=0} \cdot (\varepsilon - 0) + \dots$$

$$+ \frac{\partial}{\partial (y_n + \varepsilon \eta_n)} [F(y_1 + \varepsilon \eta_1, y_2 + \varepsilon \eta_2, \dots, y_n$$

$$+ \varepsilon \eta_n, y_1' + \varepsilon \eta_1', y_2' + \varepsilon \eta_2', \dots, y_n' + \varepsilon \eta_n', t)]$$

$$\Big|_{\varepsilon=0} \frac{\partial}{\partial \varepsilon} (y_n + \varepsilon \eta_n) \Big|_{\varepsilon=0} \cdot (\varepsilon - 0) + \frac{\partial}{\partial (y_n' + \varepsilon \eta_n')}$$

$$[F(y_1 + \varepsilon \eta_1, y_2 + \varepsilon \eta_2, \dots, y_n + \varepsilon \eta_n, y_1' + \varepsilon \eta_1'$$

$$+ y_2' + \varepsilon \eta_2', \dots, y_n' + \varepsilon \eta_n', t)] \Big|_{\varepsilon=0} \frac{\partial}{\partial \varepsilon} (y_n' + \varepsilon \eta_n') \Big|_{\varepsilon=0} = 0$$

$$\cdot (\varepsilon - 0) + O(\varepsilon^2) \text{ and higher order terms.}$$

Taking the partials as shown, evaluating at $\epsilon = 0$, and ignoring higher order terms of ϵ results in:

$$F(y_1 + \epsilon \eta_1, y_2 + \epsilon \eta_2, \dots, y_n + \epsilon \eta_n, y_1' + \epsilon \eta_1' \quad \text{B.9}$$

$$y_2' + \epsilon \eta_2', \dots, y_n' + \epsilon \eta_n')$$

$$- F(y_1, y_2, \dots, y_n, y_1', y_2', \dots, y_n') = \frac{\partial}{\partial y_1} [F(y_1, y_2, \dots, y_n,$$

$$y_1', y_2', \dots, y_n', t)] \epsilon \eta_1 + \frac{\partial}{\partial y_1'} [F(y_1, y_2, \dots, y_n,$$

$$y_1', y_2', \dots, y_n', t)] \epsilon \eta_1' + \dots + \frac{\partial}{\partial y_n} [F(y_1, y_2, \dots, y_n,$$

$$y_1', y_2', \dots, y_n', t)] \epsilon \eta_n + \frac{\partial}{\partial y_n'} [F(y_1, y_2, \dots, y_n,$$

$$y_1', y_2', \dots, y_n')] \epsilon \eta_n' .$$

Using the identities in B.4 and substituting back into B.7:

$$0 = \lim_{\epsilon \rightarrow 0} \int_a^b \frac{1}{\epsilon} \left[\frac{\partial F}{\partial y_1} \epsilon \eta_1 + \frac{\partial F}{\partial y_1'} \epsilon \eta_1' + \dots + \frac{\partial F}{\partial y_n} \epsilon \eta_n + \frac{\partial F}{\partial y_n'} \epsilon \eta_n' \right] dt \quad \text{B.10}$$

Complete the division and take the limit. Recognizing the sum of the integrals is equal to the integral of the sum, each integral can be

analyzed separately. To integrate each i th term, use integration by parts on:

$$\frac{\partial F}{\partial y_i} \varepsilon \eta_i' . \quad \text{B.11}$$

Let:

$$u = \frac{\partial F}{\partial y_i} \quad \text{B.12}$$

$$du = \frac{d}{dt} \left(\frac{\partial F}{\partial y_i} \right) dt$$

$$dv = \eta_i' dt$$

$$v = \eta_i$$

The integrated result is:

$$\frac{\partial F}{\partial y_i} \eta_i \Big|_a^b - \int_a^b \eta_i \frac{d}{dt} \left(\frac{\partial F}{\partial y_i} \right) dt \quad i=1, \dots, n \quad \text{B.13}$$

Equation B.10 now becomes:

$$\begin{aligned} \delta I = 0 = & \int_a^b \left[\frac{\partial F}{\partial y_1} \eta_1 + \frac{\partial F}{\partial y_2} \eta_2 + \dots + \frac{\partial F}{\partial y_n} \eta_n \right. \\ & \left. - \eta_1 \frac{d}{dt} \left(\frac{\partial F}{\partial y_1} \right) - \eta_2 \frac{d}{dt} \left(\frac{\partial F}{\partial y_2} \right) - \dots - \eta_n \frac{d}{dt} \left(\frac{\partial F}{\partial y_n} \right) \right] dt \\ & + \left[\frac{\partial F}{\partial y_1} \eta_1 + \frac{\partial F}{\partial y_2} \eta_2 + \dots + \frac{\partial F}{\partial y_n} \eta_n \right] \Big|_a^b . \end{aligned} \quad \text{B.14}$$

Since $\eta_i(a) = \eta_i(b) = 0$, $i=1, \dots, n$, equation B.14 becomes:

$$\begin{aligned} \delta I = 0 = & \int_a^b \left[\frac{\partial F}{\partial y_1} - \frac{d}{dt} \left(\frac{\partial F}{\partial y_1'} \right) \right] \eta_1 dt + \int_a^b \left[\frac{\partial F}{\partial y_2} - \frac{d}{dt} \left(\frac{\partial F}{\partial y_2'} \right) \right] \eta_2 dt \\ & + \dots + \int_a^b \left[\frac{\partial F}{\partial y_n} - \frac{d}{dt} \left(\frac{\partial F}{\partial y_n'} \right) \right] \eta_n dt \end{aligned} \quad \text{B.15}$$

Since the function η is any arbitrary function the only way that each integral can equal zero is for each integrand to equal zero.

The result is the Euler-Lagrange equations:

$$\frac{\partial F}{\partial y_i} - \frac{d}{dt} \left(\frac{\partial F}{\partial y_i'} \right) = 0 \quad i=1, \dots, n \quad \text{B.16}$$

These equations are the necessary conditions for an extremal to exist for a functional dependent upon several unknown functions.

Now, assume that the necessary conditions are met and the extremal exists. This implies that the first variation of I with respect to ϵ is equal to zero ($\delta I=0$).

In the Taylor series expansion (equation B.8) of the variation in F , higher order terms of ϵ were ignored. Now, since the first variation is equal to zero, the magnitude of the second order terms of ϵ is important. Indeed examination of the second variation determines whether the functional increases or decreases (when the first variation is equal to zero) and, therefore, establishes whether the extremal is a maximizing or minimizing functional.

Ignoring the first variation terms and expanding the second variation terms results in:

$$F(y_1 + \epsilon\eta_1, y_2 + \epsilon\eta_2, \dots, y_n + \epsilon\eta_n, y_1' + \epsilon\eta_1', y_2' + \epsilon\eta_2', \dots, y_n' + \epsilon\eta_n', t) = \quad \text{B.17}$$

$$\frac{\partial}{\partial(y_1 + \epsilon\eta_1)} \left\{ \frac{\partial}{\partial(y_1 + \epsilon\eta_1)} [F(y_1 + \epsilon\eta_1, y_2 + \epsilon\eta_2, \dots, y_n + \epsilon\eta_n, y_1' + \epsilon\eta_1', y_2' + \epsilon\eta_2', \dots, y_n' + \epsilon\eta_n', t)] \left[\frac{\partial}{\partial \epsilon} (y_1 + \epsilon\eta_1) \right] \right\} \Big|_{\epsilon=0}$$

$$(\epsilon-0)^2 + \dots + \frac{\partial}{\partial(y_1 + \epsilon\eta_1)} \left\{ \frac{\partial}{\partial(y_n + \epsilon\eta_n)} [F(y_1 + \epsilon\eta_1, y_2 + \epsilon\eta_2, \dots, y_n + \epsilon\eta_n, y_1' + \epsilon\eta_1', y_2' + \epsilon\eta_2', \dots, y_n' + \epsilon\eta_n', t)] \left[\frac{\partial}{\partial \epsilon} (y_n + \epsilon\eta_n) \right] \right\} \Big|_{\epsilon=0}$$

$$(\epsilon-0)^2 + \dots + \dots + \frac{\partial}{\partial(y_n' + \epsilon\eta_n')}$$

$$[F(y_1 + \epsilon\eta_1, y_2 + \epsilon\eta_2, \dots, y_1' + \epsilon\eta_1', y_2' + \epsilon\eta_2', \dots, y_n' + \epsilon\eta_n', t)] \left[\frac{\partial}{\partial \epsilon} (y_n' + \epsilon\eta_n') \right] \Big|_{\epsilon=0} \cdot (\epsilon-0)^2$$

$$[F(y_1 + \epsilon\eta_1, y_2 + \epsilon\eta_2, \dots, y_1' + \epsilon\eta_1', y_2' + \epsilon\eta_2', \dots, y_n' + \epsilon\eta_n', t)] \left[\frac{\partial}{\partial \epsilon} (y_n' + \epsilon\eta_n') \right] \Big|_{\epsilon=0} \cdot (\epsilon-0)^2$$

$$+ 0(\epsilon^3) \text{ and higher order terms in } \epsilon.$$

Ignore the higher order terms in ϵ . Take the partial derivatives as shown and evaluate at $\epsilon=0$. Let the following abbreviated form represent the partial derivatives:

$$\frac{\partial}{\partial y_i} \left(\frac{\partial F}{\partial y_j} \right) \equiv F_{y_i y_j}. \quad \text{B.18}$$

Now, equation B.17 can be reduced to the matrix form below:

$$(\eta_1, \eta_2, \dots, \eta_n, \eta_1', \eta_2', \dots, \eta_n') \quad \text{B.19}$$

$$\begin{bmatrix} F_{y_1 y_1} & F_{y_1 y_2} & \dots & F_{y_1 y_n} & F_{y_1 y_1}' & F_{y_1 y_2}' & \dots & F_{y_1 y_n}' \\ F_{y_2 y_1} & F_{y_2 y_2} & \dots & F_{y_2 y_n} & F_{y_2 y_1}' & F_{y_2 y_2}' & \dots & F_{y_2 y_n}' \\ \vdots & & & & & & & \\ F_{y_n y_1} & F_{y_n y_2} & \dots & F_{y_n y_n} & F_{y_n y_1}' & F_{y_n y_2}' & \dots & F_{y_n y_n}' \\ F_{y_1' y_1} & F_{y_1' y_2} & \dots & F_{y_1' y_n} & F_{y_1' y_1}' & F_{y_1' y_2}' & \dots & F_{y_1' y_n}' \\ \vdots & & & & & & & \\ F_{y_n' y_1} & F_{y_n' y_2} & \dots & F_{y_n' y_n} & F_{y_n' y_1}' & F_{y_n' y_2}' & \dots & F_{y_n' y_n}' \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \\ \eta_1' \\ \eta_2' \\ \vdots \\ \eta_n' \end{bmatrix}$$

Investigation of this quadratic form reveals the following:

1. All products off the main diagonal have η as a multiplier.

Since η is an arbitrary function it can be selected as small as desired.

2. All products on the main diagonal for the first n rows and columns have η_i^2 , $i=1, \dots, n$. η_i^2 is also arbitrary and can be selected small.

3. The products on the main diagonal from the $n+1$ to $2n$ row and column contain $\eta_i'^2$, $i=1, \dots, n$. Now, η_i can be selected small, but not necessarily η_i' ; therefore, to a first order approximation, $\eta_i'^2$ dominates all other terms. Since η_i' is squared it will always be positive. The conclusion is that $Fy_i'y_i'$ determines whether a variation would increase or decrease the value of the function. The result as shown in PETROV^[62] is as follows:

(a) In order for the minimum of a functional to be obtained at the extremal it is necessary to satisfy the chain of inequalities:

$$Fy_1'y_1' \geq 0; \quad \begin{vmatrix} Fy_1'y_1' & Fy_1'y_2' \\ Fy_2'y_1' & Fy_2'y_2' \end{vmatrix} \geq 0; \dots; \quad \text{B.20}$$

$$\begin{vmatrix} Fy_1'y_1' & \dots & Fy_1'y_n' \\ \vdots & & \vdots \\ Fy_n'y_1' & \dots & Fy_n'y_n' \end{vmatrix} \geq 0$$

(b) In order for the maximum of a functional to be obtained at the extremal it is necessary for the above determinants to be negative definite.

These conditions are known as the Legendre conditions.

APPENDIX C.

POLYNOMIAL FORECASTING TECHNIQUE^[54]

Let there be $L+1$ observations taken at equally spaced intervals of time. Let the most recent observation be called y_n and each of all observations be identified as:

$$Y_n = (y_{n-L}, y_{n-L+1}, y_{n-L+2}, \dots, y_{n-1}, y_n) \quad C.1$$

Assume that these observations are as shown in Figure 25. The purpose is to find an appropriate polynomial that in the least squares sense estimates a best fitting curve through these points. Call the abscissa of this polynomial "r" and define $[P^*(r)]_n$ as the polynomial of selected degree that best estimates the points of the n-element observation vector in C.1. The "*" represents an estimate.

Consider the residual vector, $E(n)$, as the difference between the actual observation and the estimating curve. Then,

$$E(n) = \begin{bmatrix} y_n - [P^*(L)]_n \\ y_{n-1} - [P^*(L-1)]_n \\ \vdots \\ y_{n-L} - [P^*(0)]_n \end{bmatrix} \quad C.2$$

Then the sum of the residual squares is

$$e_n = \sum_{r=0}^L \{y_{n-L+r} - [P^*(r)]_n\}^2 \quad C.3$$

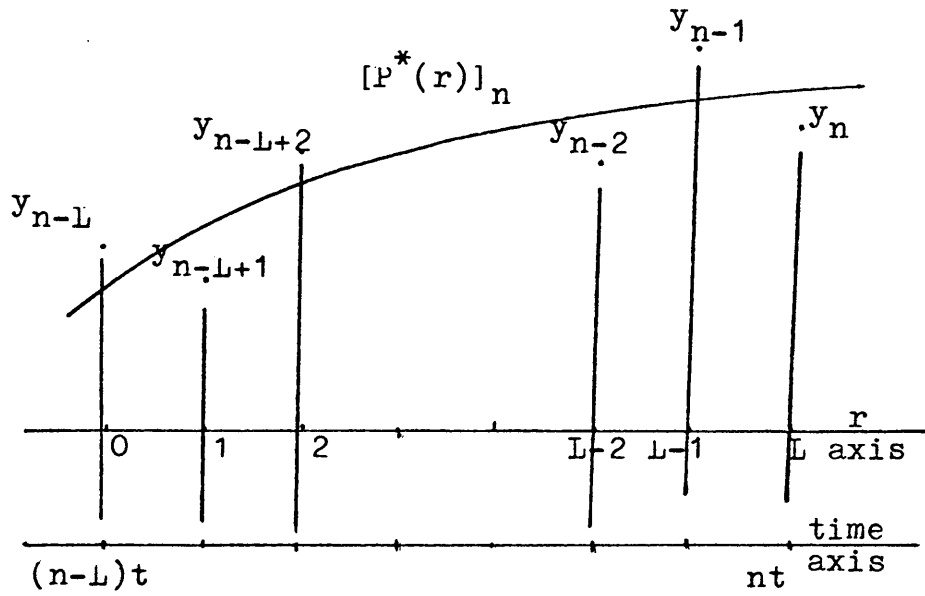


Figure 25. Relations of Observations to Polynomial in "r"

Consider now the discrete Legendre polynomial:

$$\sum_{x=0}^L p(x;i,L)p(x;j,L) = 0 \quad i \neq j \quad \text{C.4}$$

where $p(x;i,L)$ stands for a polynomial in x , of degree i , and orthogonal over the range $0 \leq x \leq L$. Now, let $f(x,k)$ be any polynomial in x of degree k . Clearly there exist constants β_i such that:

$$f(x,k) = \sum_{i=0}^k \beta_i p(x;i,L) \quad \text{C.5}$$

It follows then that:

$$\sum_{x=0}^L f(x,k)p(x;j,L) = 0 \quad 0 \leq k < j \quad \text{C.6}$$

and that:

$$p(x;0,L) = 1 \quad \text{C.7}$$

Now, define a function $g(x;j,L)$ such that

$$\nabla^j g(x;j,L) \equiv p(x;j,L) \quad \text{C.8}$$

where:

$$\nabla g(x;j,L) = g(x;j,L) - g(x-1;j,L) \quad \text{C.9}$$

Then, C.6 becomes:

$$\sum_{x=0}^L f(x,k) \nabla^j g(x;j,L) = 0 \quad j \geq 1 \quad \text{C.10}$$

Summing by parts where $u_{x-1} = f(x,k)$ and $v_x = \nabla^{j-1} g(x;j,L)$:

$$0 = f(x+1,k) \cdot \nabla^{j-1} g(x;j,L) \Big|_{-1}^L - \sum_{x=0}^L \nabla f(x+1,k) \quad \text{C.11}$$

$$\nabla^{j-1} g(x;j,L)$$

Repeating the summation by parts:

$$\begin{aligned} 0 = & f(x+1,k) \nabla^{j-1} g(x;j,L) \Big|_{-1}^L - \nabla f(x+2) \nabla^{j-2} g(x;j,L) \Big|_{-1}^L \quad \text{C.12} \\ & + \dots + (-1)^{j-1} [\nabla^{j-1} f(x+j,k) g(x;j,L) \Big|_{-1}^L \end{aligned}$$

The iteration terminates here since by assumption $\nabla^j f(x,k) \equiv 0$.

From C.6 let $j=1$, then $k=0$, and:

$$f(x+1,0) g(x;1,L) \Big|_{-1}^L = 0 \quad \text{C.13}$$

But since this must hold for any f , the conclusion is:

$$g(-1;1,L) = g(L;1,L) = 0 \quad \text{C.14}$$

Repeating for $j=2,3, \dots$, and recalling that C.12 must apply for any polynomial $f(x,k)$ of degree $j-1$ or less, the following set of boundary conditions can be obtained:

$$\begin{aligned} g(x;j,L) \Big|_{x=L} &= \nabla g(x;j,L) \Big|_{x=L} = \dots = \nabla^{j-1} g(x;j,L) \Big|_{x=L} & \text{C.15} \\ &= 0; g(x;j,L) \Big|_{x=-1} = \nabla g(x;j,L) \Big|_{x=-1} = \dots = \\ &\nabla^{j-1} g(x;j,L) \Big|_{x=-1} = 0 \end{aligned}$$

From C.8, $\nabla^j g(x;j,L)$ is specified as a j^{th} degree polynomial; therefore:

$$\nabla^{2j+1} g(x;j,L) = 0 \quad \text{C.16}$$

Using C.16 and the boundary conditions in C.15, $g(x;j,L)$ can be solved for:

$$g(x;j,L) = a(j,L) \sum_{v=0}^j (-1)^v \binom{j}{v} \frac{(x+y)^{(j+v)}}{(L+j)^{(j+v)}} \quad \text{C.17}$$

where $a(j,L)$ is arbitrary and will be set to:

$$a(j,L) \equiv \binom{L+j}{j} \quad \text{C.18}$$

Substituting back into C.8 results in:

$$p(x;j,L) = \sum_{v=0}^j (-1)^v \binom{j}{v} \binom{j+v}{v} \frac{x^{(v)}}{L^{(v)}} \quad \text{C.19}$$

This equation, C.19, is the discrete Legendre polynomial of degree j for $j \leq L$.

As a convenience, define a term $c(j,L)$ as:

$$[c(j,L)]^2 = \sum_{x=0}^L [p(x;j,L)]^2 \quad \text{C.20}$$

Substituting $p(x;j,L)$ from C.19 and solving yields:

$$[c(j,L)]^2 = \frac{(L+j+1) \binom{j+1}{j}}{(2j+1)L \binom{j}{j}} \quad \text{C.21}$$

Recall that the polynomial developed was the discrete Legendre polynomial. Now let $Q(x;j)$ be a continuous polynomial valid over the discrete points. Further, define the normalized discrete Legendre polynomial of degree j in variable x as:

$$Q_j(x) \equiv \frac{1}{c_j} p_j(x) \quad \text{C.22}$$

Now, suppose that $P^*(r)$ in C.3 can be written as a linear combination of this Legendre polynomial:

$$[P^*(r)]_n = \sum_{j=0}^m (\beta_j)_n Q_j(r) \quad \text{C.23}$$

where the β_i are not yet specified. Substituting C.23 into C.3:

$$e_n = \sum_{r=0}^L [y_{n-L+r} - \sum_{j=0}^m (\beta_j)_n Q_j(r)]^2 \quad \text{C.24}$$

To minimize e_n set $\frac{\partial e_n}{\partial \beta_j} = 0$, $j=0,1, \dots, m$. The resulting equations are:

$$\sum_{k=0}^L \sum_{j=0}^m (\beta_j) Q_j(k) Q_i(k) = \sum_{k=0}^L y_{n-L+k} Q_i(k) \quad \text{C.25}$$

$$i=0,1, \dots, m$$

Now, it can be shown that:

$$\sum_{r=0}^L Q_i(r) Q_j(r) = \delta_{ij} \quad \text{C.26}$$

where δ_{ij} is the kronecker delta.

Using C.26 and reversing the order of summation in C.25 results in:

$$(\beta_j)_n = \sum_{k=0}^L y_{n-L+k} Q_j(k) \quad j=0,1, \dots, m \quad \text{C.27}$$

Substituting C.27 into C.23 results in:

$$[P^*(r)]_n = \sum_{j=0}^m \left[\sum_{k=0}^L y_{n-L+k} Q_j(k) \right] Q_j(r) \quad \text{C.28}$$

Thus, C.28 is the required expression for the polynomial that best fits the data vector Y_n in the sense of least squares. By varying r the process can be estimated both in the past and in the future based on observations to the present. For example, if $r=L$ we have an estimate of the present observation. If $r=L+1$ an estimate of the one-step prediction is obtained.

Assume that a second order polynomial is appropriate. From this $m=2$. From C.19:

$$p(s;0,L) = 1 \quad \text{C.29}$$

$$p(s;1,L) = 1 - \frac{2s}{L}$$

$$p(s;2,L) = 1 - \frac{6s}{L} + \frac{6s(s-1)}{L(L-1)}$$

From C.21

$$[C(0,L)]^2 = L+1 \quad \text{C.30}$$

$$[C(1,L)]^2 = \frac{(L+1)(L+2)}{3L}$$

$$[C(2,L)]^2 = \frac{(L+1)(L+2)(L+3)}{5L(L-1)}$$

Rearranging C.28:

$$[P^*(r)]_n = \sum_{k=0}^L \left[\sum_{j=0}^m Q_j(k) Q_j(r) \right] y_{n-L+k} \quad \text{C.31}$$

By definition in C.22

$$Q_j(s) = \frac{1}{c_j} p_j(s) \quad \text{C.32}$$

Making the appropriate substitutions in C.31:

$$\begin{aligned} [P^*(r)]_n &= \sum_{k=0}^L \left[\sum_{j=0}^m Q_j(k) Q_j(r) \right] y_{n-L+k} \quad \text{C.33} \\ &= \sum_{k=0}^L [Q_0(k) Q_0(r) + Q_1(k) Q_1(r) + Q_2(k) Q_2(r)] y_{n-L+k} \\ &= \sum_{k=0}^L \left[\frac{1}{c_0} p_0(k) \frac{1}{c_0} p_0(r) + \frac{1}{c_1} p_1(k) \frac{1}{c_1} p_1(r) \right. \\ &\quad \left. + \frac{1}{c_2} p_2(k) \frac{1}{c_2} p_2(r) \right] y_{n-L+k} \end{aligned}$$

There are $L+1$ "weights" multiplied times $L+1$ observations. Call the k^{th} weight, w_k . Then:

$$w_k = \frac{1}{c_0} p_0(k) p_0(r) + \frac{1}{c_1} p_1(k) p_1(r) + \frac{1}{c_2} p_2(k) p_2(r) \quad \text{C.34}$$

Now assume that a ten-period ahead forecast is desired based on the eleven most recent observations. For this case, $L=10$ and $R=L+10=20$. Equation C.34 after substitution becomes:

$$w_k = \frac{1}{11} + \frac{3 \times 10}{11 \times 12} \left[1 - \frac{2k}{10}\right] \left[1 - \frac{2 \times 20}{10}\right] + \frac{5 \times 10 \times 9}{11 \times 12 \times 13} \quad \text{C.35}$$

$$\left[1 - \frac{2k}{10} + \frac{6 \times k(k-1)}{10.9}\right] \left[1 - \frac{2 \times 20}{10} + \frac{6 \times 20 \times 19}{10.9}\right]$$

C.35 reduces to:

$$w_k = \frac{1}{858} [2718 - 2033k + 215k^2] \quad \text{C.36}$$

Returning to C.33, substituting w_k , and solving yields:

$$[P^*(20)]_n = \sum_{k=0}^{10} w_k y_{n-10+k} \quad \text{C.37}$$

$$= \sum_{k=0}^n \left\{ \frac{1}{858} [2718 - 2033k + 215k^2] y_{n-10+k} \right\}$$

In similar fashion, predictions can be made for any successive periods based on observations up to the present.

Before leaving this topic, it is desirable to have some knowledge of the variances of the error of the forecast.

Call X_n the nominal trajectory at time t_n . Then the vector of observations can be written as:

$$Y_n = T_n X_n + E_n \quad \text{C.38}$$

Where T_n is the transformation matrix and E_n is the vector of errors in the differences in the transformed nominal trajectory and

the observed vector. Let the j -step prediction at time t_n be called:

$$Z_{n+j,n} \tag{C.39}$$

then equation C.36 can be written as:

$$Z_{n+j,n} = [P^*(n+j)]_n = W(j)Y_n \tag{C.40}$$

Then, C.37 becomes

$$W(j)Y_n = W(j)T_n X_n + W(j)E_n \tag{C.41}$$

Since the vector E_n originates a vector of random errors, the estimate of the errors is given by

$$E^*(n+j) = W(j)E_n \tag{C.42}$$

Assume that e_i for all i has:

1. mean of zero
2. finite variance
3. no autocorrelation.

Under these assumptions, C.42 also has zero mean. For this case, the covariance matrix would be:

$$S_{n+j}^* = W(k) R(n) W(k)^T \tag{C.43}$$

where $R(n)$ is the covariance matrix of E_n .

A further frequent assumption is that observational errors have zero mean and are uncorrelated with equal variances of residual errors, σ^2 . Under this assumption:

$$R(n) = \sigma^2 I \quad \text{C.44}$$

and finally:

$$S_{n+j}^* = \sigma^2 I W(j) W(j)^T . \quad \text{C.45}$$

Variance of the error of the forecast is the upper left of the resulting matrix, or:

$$\sigma^2 [W(j) \cdot W(j)^T] . \quad \text{C.46}$$

APPENDIX D

MATERIAL FLOW IN THE NUCLEAR FUEL CYCLE

In the following sections, each stage of the nuclear fuel cycle is described. Following each description, the material flow conversion factor for that stage is developed. These values are collected in Table 9.

Exploration

Although exploration is not considered a stage in the overview, it can rightfully be considered such because the process of exploration could be construed to be that process which converts an unexplored resource into a reserve ore field.

In the United States, uranium ore is found principally in sedimentary sandstone and mudstone deposits of the Colorado Plateau, the Wyoming Basin and the Gulf Coastal Plains of Texas^[45]. Exploration is generally done by aerial and ground radiation surveys, radon gas evolution measurements, and an extensive program of exploratory drilling. Once deposits of uranium have been located, they are held in reserve until they are mined.

Units of accountability are generally taken to be tons of ore. In the exploration process, a ton of ore in the unexplored resource is converted to a ton of ore in a reserve. The conversion factor, $k_{0,1}$, is then unity (1).

Mining

In the mining process uranium ore bearing stone is extracted from the earth and delivered to a mill in similar fashion to copper ore mining. Open pit mining accounts for about half of the ore produced and is utilized when ore is located at depths of 400 feet or less. Underground mining is employed for greater depths or when excessive blasting would be required. As of 1974, 29 open pit mines produced 4,549,336 tons annually while 193 underground mines produced 1,992,953 tons of ore annually. Of this ore, approximately 0.2 percent or four pounds per ton of ore is U_3O_8 [72].

For accountability, the mining process is assumed to take tons of ore from the inventory of reserve ore fields and add tons of ore to the inventory of mined ore. The conversion factor, $k_{1,2}$, is unity (1).

Milling

In the milling process, uranium ore is crushed and ground. It is then leached with either sulphuric acid or sodium carbonate to extract several uranium compounds. The most common is ammonium diuranate, commonly called "yellowcake". As of 1976, approximately 20 mills with a total annual production of 21,000 metric tonnes of U_3O_8 were operating in the Western United States and principally very near the mines [72].

It is assumed that the milling process takes tons of ore from the mined ore inventory and adds kilograms of U_3O_8 to the inventory of U_3O_8 .

The average mined ore contains approximately 0.2 percent or four pounds of U_3O_8 per ton of mined ore^[66]. Assuming an efficiency of .935 in separating the U_3O_8 from other materials yields approximately 3.74 pounds, 1.698 kilograms, of U_3O_8 per ton of ore. Then to produce one kilogram of U_3O_8 , .589 tons of ore are required. Therefore,

$$k_{2,3} = .589$$

Conversion

The U_3O_8 extracted from the ore is converted into uranium hexafluoride, UF_6 , by either a wet or dry chemical solvent process. Two commercial plants convert approximately 15,000 metric tonnes of U_3O_8 to UF_6 annually^[66].

The conversion process takes kilograms of U_3O_8 from the milled ore inventory and adds kilograms of UF_6 to the UF_6 inventory. The process is extremely efficient in separating the uranium. The efficiency is generally assumed to be .99^[66].

In one kilogram of UF_6 , there are approximately .6761 kilograms of uranium. Applying the efficiency factor of .99 results in .6830 kilograms of uranium being required. The molecular weight of U_3O_8 is approximately 842. Thus, .8054 kilograms of U_3O_8 produces 1.0 kilograms of UF_6 . The conversion factor, $k_{3,4}$, is then .8054.

Enrichment

Of the uranium present in UF_6 , approximately 0.7 percent is the isotope U-235 required for fission. This low percentage must be enhanced to 2-4 percent for most commercial reactors. Isotopic

enrichment is presently accomplished by the gaseous diffusion process. Three gaseous diffusion plants owned by the United States Government have a total capacity of 27.6 million separative work units [71]. "A separative work unit (SWU) is a measure of the effort expended to separate a quantity of uranium of a given assay into two components; one having a higher percentage of uranium 235 and one having a lower percentage" [61].

The enrichment process takes kilograms of UF_6 from the conversion plant inventory and adds kilograms of enriched UF_6 to its production inventory. It also takes UF_6 of a different enrichment from the reprocessing plant and adds it to its production inventory.

The ratio of feed to product is given by the ratio of the percent of U-235 in the product stream minus the percent U-235 in the waste stream to the percent U-235 in the feed stream minus the percent in the waste stream. For one kilogram of product of enrichment 3.0 percent, a natural feed of .711 percent, and a waste stream of .25 percent gives the conversion factor, $k_{4,5} = 5.965$.

For reprocessed UF_6 at an enrichment of .86 percent, the conversion factor, $k_{8,5} = 4.508$.

Fabrication

The fabrication stage is a multi-stage process in itself in which UF_6 gas is reduced to uranium dioxide (UO_2) powder, the powder is formed into pellets, the pellets are sintered to uniform density and inserted into rods, and finally, the rods are assembled into fixed arrays known

as fuel assemblies. These assemblies then comprise the reactor reloads. Ten commercial plants presently fabricate all assembly requirements for commercial reactors [66].

The fabrication stage draws kilograms of enriched UF_6 from the enriched inventory and adds reloads of UO_2 to its production inventory. One kilogram of enriched UF_6 has .6761 kilograms of uranium. Assume a loss of 1 percent. The resulting uranium is .6693 kilograms of uranium. UO_2 has a molecular weight of approximately 269.8. Then the amount of UO_2 formed per kilogram of UF_6 is .7593 kilograms.

One reload is assumed to have 31.4 metric tonnes of UO_2 [40]. Since one kilogram of UF_6 is required for .7593 kilograms of UO_2 , the amount of UF_6 required for one reload (conversion factor, $k_{5,6}$) is 41,354.

Reactor

In the reactor, the U-235 isotope is fissioned releasing thermal energy which in turn is used to generate electricity. The fissions reduce the amount of U-235 and increase the amounts of fission products. Fission products must ultimately be separated and safely stored. Generally, the reactor has an initial core of fresh fuel, and approximately one-third of this core is replaced each year. The discharged fuel, or spent fuel, is held for a period of time to cool down, and then can be reprocessed [66].

In the reactor burnup process, assemblies are withdrawn from the fabrication inventory and are added to the spent fuel inventory. The conversion factor, $k_{6,7}$, is unity (1).

Reprocessing

The discharged fuel assemblies generally contain about one-third of the U-235 isotope that was originally in the fuel plus some plutonium and other marketable isotopes. Through reprocessing the unused U-235, the plutonium saleable isotopes, and the U-238 are separated from the unuseable fission products.

The U-235 and U-238 can be re-cycled as reactor reload fuel, the plutonium is isolated and stored, and the fission products are reduced to a safe form and stored permanently. There are no commercial reprocessing plants in operation now; however, three are under construction or alteration, and one did operate from 1966-72^[71].

The reprocessing facilities take fuel assemblies from the spent fuel inventory and adds kilograms of UF_6 of enrichment .86 percent to the enrichment facility. Each reload contains approximately 257.6^[40] kilograms of U-235 upon discharge. At an average enrichment of .86 percent, this results in an uranium content of 26,403 kilograms of uranium. Assume a loss in conversion of one percent. This leaves 26,139 kilograms of uranium. Each kilogram of uranium results in 1.479 kilograms of UF_6 . The conversion factor then for reloads of UO_2 to kilograms of UF_6 is $k_{7,8} = .0000566$.

APPENDIX E

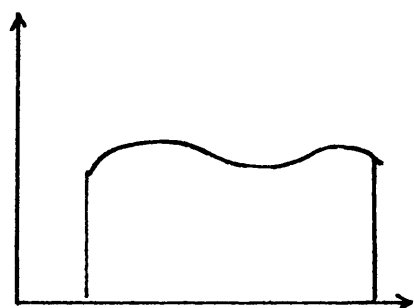
SUMMARY OF THE WORKS OF KROTOV [33,34,35,36]

A. Introduction

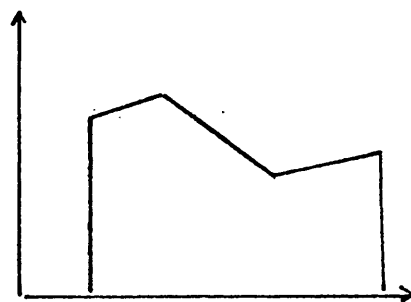
V. F. KROTOV is a Russian mathematician who published prolifically in the years 1960 to 1964 in the area of calculus of variations and optimal control theory. Of the twelve articles he published, only four are translated into English. These four articles are follow-on articles to his dissertation and other basic articles. Since they are intended for dissemination in Russia where the basic material resides, they merely refer to his theorems previously proven. This appendix briefly summarizes the more important aspects of KROTOV's work.

B. Classes of Functions

Five classes of functions must first be defined. Class one is called smooth functions and are characterized by having continuous first derivatives. An example of a smooth function is shown in Figure 26a. Class two is called a continuous function. An example of a continuous function is shown in Figure 26b. Class three is called discontinuous functions and examples are shown in Figures 26c and 26d. The remaining two classes are narrower definitions. Class four is called piecewise-smooth. For example, the function in Figure 26b is smooth except for certain individual points; therefore it is piecewise-smooth. Class five is piecewise-continuous. For example, the function in Figure 26d is continuous except at individual points; therefore



a. Smooth Curve



b. Continuous Curve

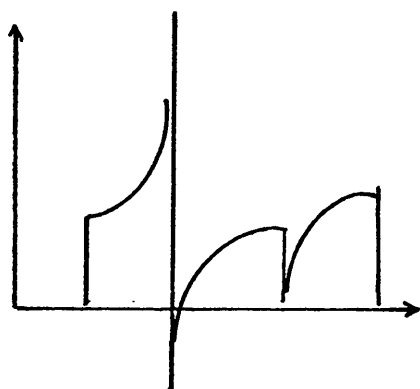
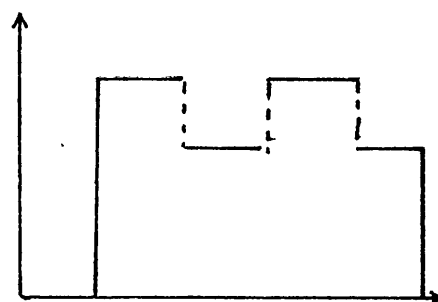
c. Discontinuous
Curved. Discontinuous
Curve

Figure 26. Examples of Curves

it is called piecewise-continuous.

KROTOV's work centers around determining into which class an extremal falls. WEIERSTRASS^[57] first identified this problem. The "WEIERSTRASS Problem" demonstrates a piecewise-continuous solution where it had been thought that no solution existed. This is because the solution had been sought in the class of piecewise-smooth functions.

C. KROTOV's Works

In the calculus of variations, an extremal is sought to the functional

$$I(u) = \int_a^b F(x, y, y') dx \quad E.1$$

where u is a line on the set U of lines whose properties will be described later. Assume that the line u is piecewise-continuous and has a vertical segment as shown in Figure 27.

This type of function is not Riemann integrable over the interval $[a, b]$ because $y(x)$ is not single-valued over the interval; therefore incline the vertical segment such that it makes an angle of $1/m$ with the vertical. The changed extremal now appears as shown in Figure 28.

$y(x)$ is now single-valued over the interval of interest. Define the extremal in Figure 28 as u^m . Further, if $m > 0$, the inclination is clockwise, and if $m < 0$, the inclination is counter-clockwise. From this convention, the sign of m agrees with the sign $y_1 - \bar{y}_1$ where $y_1 = y_1(x_0^+)$ and $\bar{y}_1 = y_1(x_0^-)$. Assume that the line u^m has a

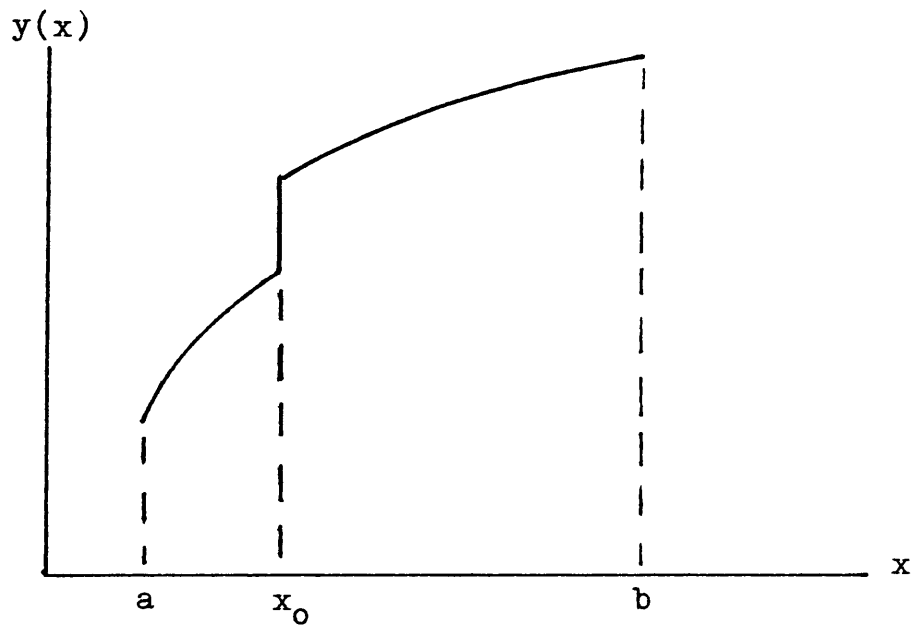


Figure 27. Piecewise-Smooth Function with a Vertical Segment

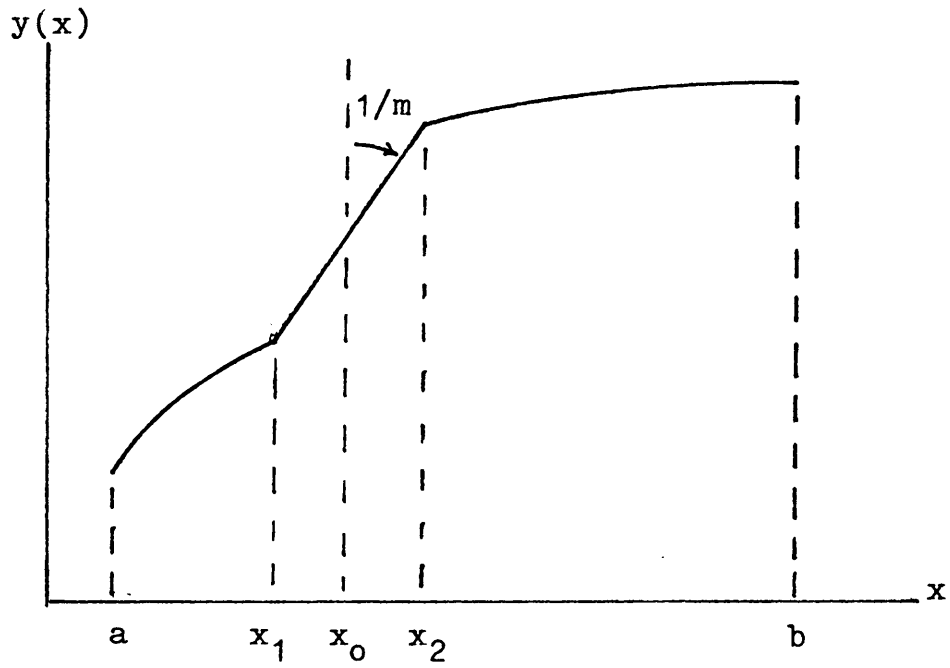


Figure 28. Piecewise-Smooth Function with the Vertical Segment Inclined

discontinuity at x_0 . With the vertical segment inclined, as stated, the functional J is now the sum of three functionals:

$$J = J_1 + J_2 + J_3 \quad \text{E.2}$$

or:

$$J = \int_a^b F(x,y,y')dx = \int_a^{x_1} Fdx + \int_{x_1}^{x_2} Fdx + \int_{x_2}^b Fdx. \quad \text{E.3}$$

The breakpoints in the above integrals correspond to the points on the abscissa in Figure 28.

Since the angle that the inclined segment makes with the vertical is $1/m$, it is clear that as $m \rightarrow \infty$, the angle $\rightarrow 0$; therefore:

$$\lim_{m \rightarrow \infty} J(u^m) = J(u) \quad \text{E.4}$$

but,

$$J(u) = \int_a^b F(x,y,y')dx. \quad \text{E.5}$$

Recall that y' is the first derivative of the function y , and therefore is the slope of the function throughout. If y is in the class of curves under consideration and $F(x,y,y')$ is a function containing y and y' , then if the quotient of F and y' is formed letting $y' \rightarrow \infty$, the anticipated results should be instructive. If, as $y' \rightarrow \infty$, the quotient also goes to k , then the function F must also contain a y' that goes to ∞ . If y' goes to ∞ at one of more points

in the interval, then the slope of the function goes to ∞ , or the function is vertical at those points. However, if, as $y' \rightarrow \infty$, F also goes to ∞ , then F must depend on higher order of magnitude relationships of y' . This is the essence of KROTOV's test.

Now return to the example. Look at J_2 .

$$J_2 = \int_{x_1}^{x_2} F(x,y,y') dx. \quad \text{E.6}$$

Multiply J_2 by the following:

$$\frac{\frac{dy}{dx}}{\frac{dy}{dx}} = 1 \quad \text{E.7}$$

$$J_2 = \int_{x_1}^{x_2} F(x,y,y') dx \left(\frac{\frac{dy}{dx}}{\frac{dy}{dx}} \right) = \int_{x_1}^{x_2} F(x,y,y') \frac{dy}{dx} \quad \text{E.8}$$

$$J_2 = \int_{x_1}^{x_2} \frac{F(x,y,y') dy}{y'} \quad \text{E.9}$$

With an appropriate change of variables on the integral limits and with the application of the limit as $y' \rightarrow \infty$, the result is a line integral from y_1 to y_2 at the point x_0 , or:

$$J_2 = \int_{x_1}^{x_2} F(x,y,y') dx = \lim_{y' \rightarrow \infty} \int_{y_1}^{y_2} \frac{F(x,y,y') dy}{y'} \quad \text{E.10}$$

The heart of KROTOV's work is the passage of this limit. Let u be a line in the set U with the following properties:

1. The x and y coordinates of the points on the line u may be given as continuous functions of some parameter t .
2. The function $y(x)$ is continuous and single-valued everywhere on $[a,b]$ except on a finite set of points x_i ($i=1,2, \dots, k$), where it may have discontinuities of the first kind (jumps).
3. The derivative of $y(x)$ is continuous and bounded on the intervals $(a,x_1), \dots, (x_i,x_{i+1}), \dots, (x_k,b)$.
4. $y(x)$ satisfies the conditions $y(a) = a_1$ and $y(b) = b_1$.
5. There exists a simply-connected closed domain B of the XY plane in which $F(x,y,y')$ together with its partial derivatives $F_x, F_y, F_{y'}$, is continuous from the right ($y' \equiv z$) with respect to all three arguments and all lines of the set U lie in this domain.
6. Everywhere in the domain B assumes the existence of the limits

$$W(x,y,\text{sign } m) = \lim_{m \rightarrow \infty} \frac{1}{m} F(x,y,m). \quad \text{E.11}$$

Define $I(u) = \lim_{m \rightarrow \infty} I(u^m)$. Extending the past development to include the possibility that vertical segments may also exist at the endpoints of the interval $[a,b]$ and using W to represent the above limit, the composite integral is written as:

$$I(u) = \sum_{i=1}^k \int_{y_{1i}}^{y_{2i}} W dy + \sum_{i=1}^{k-1} \int_{x_i^+}^{x_{i+1}^-} F dx + \int_{a^+}^{x_1^-} F dx \quad E.12$$

$$+ \int_{x_k^+}^b F dx + \int_{y_{a1}}^{y_{a2}} W dy + \int_{y_{b1}}^{y_{b2}} W dy.$$

In words, the above equation says that the functional is equal to the sum of all the vertical segments, plus the sum of the continuous lines, plus the vertical segment on the left side of the interval, plus the vertical segment on the right side of the interval. If the limit defined on W exists everywhere from the left and from the right, then the above integral is Riemann integrable.

From this analysis, KROTOV concludes that the behavior of the extremal is related to the function $W(x,y)$. Recognizing that m here is merely the slope of the inclined segment, a substitution of y' for m results in:

$$W(x,y) = \lim_{y' \rightarrow \infty} F(x,y,y') \frac{1}{y'} \quad E.13$$

Now, the specific results of KROTOV's works can be summarized. By investigation of the limit in E.13, KROTOV concludes:

1. If the right or left limit of E.13 do not exist, or

$$\lim_{y' \rightarrow \pm\infty} F(x,y,y') \frac{1}{y'} \rightarrow \pm\infty \quad E.14$$

and

$$\lim_{y' \rightarrow -\infty} F(x, y, y') \frac{1}{y'} \rightarrow -\infty \quad \text{E.15}$$

then the extremal will fall in the class of piecewise-smooth functions.

2. If the right and left limits exist and are equal at a finite number of points in the interval of interest, then the extremal will have vertical segments at these points. The extremal will fall in the class of piecewise-continuous functions.
3. If the right and left limits exist and are equal everywhere in the interval of interest, there may exist an infinite quantity of minimal curves each of which may have any quantity of points of discontinuity.
4. If the right and left limits exist, but they are not equal at individual points in the interval of interest, or they are not equal everywhere in the interval of interest, then the extremal may have vertical segments at the endpoints of the interval.

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OPTIMIZATION OF MATERIAL FLOW IN THE NUCLEAR FUEL CYCLE USING
A CYCLIC MULTI-STAGE PRODUCTION-TO-INVENTORY MODEL

by

Elden Leo DePorter

ABSTRACT

The nuclear fuel cycle is modelled as a cyclic, multi-stage production-to-inventory system. The objective is to meet a known deterministic demand for energy while minimizing acquisition, production, and inventory holding costs for all stages of the fuel cycle. The model allows for cyclic flow (feedback) of materials, material flow conversion factors at each stage, production lag times at each stage, and for escalating costs of uranium ore. It does not allow shortages to occur in inventories. The model is optimized by the application of the calculus of variations and specifically through recently developed theorems on the solution of functionals constrained by inequalities. The solution is a set of optimal cumulative production trajectories which define the stagewise production rates. Analysis of these production rates reveals the optimal nuclear fuel cycle costs and that inventories (stockpiles) occur in uranium fields, enriched uranium hexafluoride, and fabricated fuel assemblies. An analysis of the sensitivity of the model to variation in three important parameters is performed.