VIBRATION OF STRESSED SHELLS OF DOUBLE CURVATURE

by

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V. INTRODUCTION

In the design of shell structures for launch vehicles, planetary atmospheric entry probes, or similar structures, knowledge of the natural frequencies and mode shapes of the systems is of fundamental importance in determining their dynamic behavior. Shells of double curvature are common structural elements in aerospace and related industries, but due to the complexity of their configurations and governing equations, little has been done to classify their general dynamic behavior. Reference 1 gives a survey on the present state of the art in the area of analysis of free vibrations of shell structures. The subject of this dissertation is the determination of the effect of the meridional curvature on the natural vibrations of a class of axisymmetrically prestressed doubly curved shells of revolution. Two methods of approach are used in this analysis. In Chapter VII the exact equations of motion are solved using approximate techniques of solution, while in Chapter VIII approximations are made to the exact equations so that closed form techniques of solution are possible. Both of the above methods are used in this investigation to complement one another.

Since the shells are too complex to allow closed form solutions of the exact equations, numerical techniques are used in Chapter VII to analyze the dynamic behavior. Several numerical methods have recently been developed for static stress analysis and unstressed free vibration analysis of general shells of revolution. Static stress analysis is considered in references 2 to 5 and unstressed vibrations are considered in references 6 and 7. Membrane and flexural vibrations of toroidal shells are treated in references 8 and 9 on the basis of the numerical approach of reference 2. In the present investigation, differential and difference equations that govern the asymmetric linear vibration of a general class of pre-stressed shells of revolution are derived using the nonlinear shell theory of reference 10. A finite difference procedure similar to that of reference 2 (utilizing and extending the ideas of references 8 and 9) is then formulated to obtain solutions of the equations.

Numerical methods yield accurate results; however, it becomes impractical to attempt an extensive parametric study of the behavior of shell systems even with high speed computers. Rapid closed form techniques of solution are possible when approximations are made to the equations of motion; however, the range of application is restricted. These techniques are used in Chapter VIII. The solutions are limited to shells of revolution with shallow constant meridional curvature with no shallowness limitation in the circumferential direction.

By perturbing the geometry of the cylinder and referencing the zero strain state at this new configuration, a set of nonlinear homogeneous field equations governing the dynamic behavior of doubly curved shells with positive or negative constant shallow meridional curvature is developed. The procedure used in developing these equations is similar to that used in references 11 and 12 where Donnell-type equations are developed from flat plate equations. The nonlinear cylinder strain-displacement relations from reference 10 are modified slightly in the twisting curvature relation and small nonlinear terms are deleted from the middle surface strain relations. The linearization and geometric perturbation of these equations coupled with the assumption of constant prestress rotations result in a set of strain-displacement relations which have corresponding equilibrium equations with constant coefficients. These equations govern the natural vibrations of prestressed nearly cylindrical shells of revolution and are solved here for the natural vibrations of shells with freely supported edges similar to that which was done previously for cylindrical shells, see for example references 13, 14, and 15.

Results from the shallow shell analysis of Chapter VIII are compared with those obtained by the use of techniques and equations of Chapter VII to indicate the range of application of the present analysis. This simplified analysis is then used to investigate the effects of meridional curvature on the fundamental vibrations of freely supported shells as the length and thickness parameters are varied.

The equations described above are also solved for the natural vibration of membrane shells and inextensional shells with freely supported edges. The characteristic roots of the membrane field equations are also investigated and compared with the corresponding characteristic roots of the bending field equations to show the dependence of frequencies on shell membrane resistance. The solutions to the membrane theory afford a simple method of determining shell configurations of negative curvature for which bending action predominates during vibratory motion in certain modes.

The approximate analysis is then used to investigate the effects of meridional curvature on the fundamental vibrations of shells as the length and thickness parameters are varied, and to determine the degree of stabilization afforded by the application of constant directional lateral pressures to the shell systems. Finally, the approximate solution is used to determine the unstressed fundamental frequencies of clamped shells and to determine the stiffening action due to the additional edge restraint.

VI. SYMBOLS

a	reference length
A_j	displacement amplitude coefficients
B	extensional stiffness (eq. (16))
c	central rise of the shell meridian
D	bending stiffness (eq. (16))
$\bar{e}_x, \bar{e}_\theta$	nondimensional prestress parameters (see eqs. (19))
E	Young's modulus of elasticity
f_j	functions defined by equations (77)
g_j, h_j	in-plane displacement amplitude coefficients (eq. (75))
h	shell thickness
i	station number, $i = 0, 1, 2, \cdots, N$
k_x, k_θ	nondimensional curvatures (eqs. (12))
m	number of axial half-waves
m_ξ	moment variable (see eqs. (20))
$M_\xi, M_\theta, M_{\xi\theta}, M_{\theta\xi}$	modified moment resultants associated with perturbed state
n	number of circumferential waves
N	total number of intervals along the meridian
$N_\xi, N_\theta, N_{\xi\theta}, N_{\theta\xi}, Q_\xi, Q_\theta$	modified stress resultants associated with perturbed state
$\bar{N}_\xi, \bar{N}_\theta$	modified prestress stress resultants

P, P_ξ, P_θ	surface loading
r	nondimensional radius of cross section, ρ/a
R	radius of cylinder
R_ξ, R_θ	principal radii of curvature (see fig. 1)
s	total meridional arc length
S	total nondimensional meridional arc length, s/a; s/R
t	time
u,v,w	displacement variables in meridional (ξ), circumferential (θ), and normal directions, respectively, of undeformed middle surface defining perturbed state
x	nondimensional meridional coordinate
β	ratio of the circumferential to axial wavelength
$\gamma = \frac{1}{r}\frac{dr}{dx}$	
Δ	length of interval between stations, S/N
δ	modal normalizing factor
δ_{jk}	cofacters of 4×4 matrix given in eq. (51)
$\epsilon_\xi, \epsilon_\theta, \epsilon_{\xi\theta}$	middle-surface strains associated with the perturbed state
θ	circumferential coordinate in undeformed shell
$\kappa_\xi, \kappa_\theta, \kappa_{\xi\theta}$	middle-surface bending strains associated with the perturbed state
λ_j	characteristic roots

λ	thickness parameter, $\frac{h}{a}$; $\frac{h}{R}$
μ	Poisson's ratio
ν	mass density
ξ	meridional coordinate in undeformed shell
ρ	cross-sectional radius (fig. 1)
τ	percent ratio of meridional rise to length
$\varphi_\xi, \varphi_\theta, \varphi$	middle-surface rotations associated with perturbed state
$\overline{\varphi}_\xi$	prestress meridional rotation
ω	natural frequency
$\Omega^2 = \frac{a^2 \omega^2 \nu \left(1 - \mu^2\right)}{E}$	frequency parameter

Notations Used to Identify Load and Deformation Variables:

	Unmarked variables indicate variables associated with perturbed state only
$(\tilde{\ })$	indicates modified variables associated with the total deformation measured with respect to an unstressed shell
$(\hat{\ })$	indicates variables associated with total deformation measured with respect to an unstressed undeformed cylinder
$(\bar{\ })$	indicates modified variables associated with the prestress state only
$(\ )^{\circ}$	indicates physical stress resultant quantities

The comma before a subscript denotes differentiation with respect to the following subscripted variable. A dot over a symbol indicates differentiation of the quantities with respect to time.

1×4 column matrices:

Z_i	dependent variable

1×8 column matrices:

A_j	amplitude coefficients

4×4 matrices:

$A_i, B_i, C_i, D_O, D_N, E_O, E_N$	difference-equation coefficients
e_{jk}, f_{jk}	boundary-equation coefficients
F_{jk}, G_{jk}, H_{jk}	equilibrium-equation coefficients
P_i	recursion coefficients
α, β	boundary-condition-selection matrices

8×8 matrices:

N_{jk}	boundary stress coefficients
Y_{jk}	boundary displacement coefficients
Φ, ψ	boundary-condition-selection matrices

VII. NUMERICAL ANALYSIS OF VIBRATIONS OF GENERAL PRESTRESSED SHELLS OF REVOLUTION*

A. Development of General Equations of Motion

In this section, the linear equations of motion are derived for a symmetrically prestressed shell of revolution with an arbitrary meridional configuration. The nonlinear shell theory given in reference 10 forms the basis for all analytical work done in this and subsequent sections.

The shell geometry is illustrated in figure 1. The location of points on the middle surface of the shell is described by the principal coordinates (ξ,θ), where ξ is the meridional distance measured on the middle surface from one boundary, and θ is the circumferential angle. Since the shell is axisymmetric, it is completely described by the meridional shape parameter $\rho(\xi)$ which is the radial distance from the axis of revolution to the middle surface of the shell.

The principal radii of curvature of the middle surface, $R_\xi(\xi)$ and $R_\theta(\xi)$, are given by:

$$R_\xi = -\frac{\sqrt{1-\left(\frac{d\rho}{d\xi}\right)^2}}{\frac{d^2\rho}{d\xi^2}}$$

*Most of the material covered in Chapter VII has been pre-published in the report "Vibration and Buckling of Prestressed Shells of Revolution" NASA TN D-3831, March 1967.

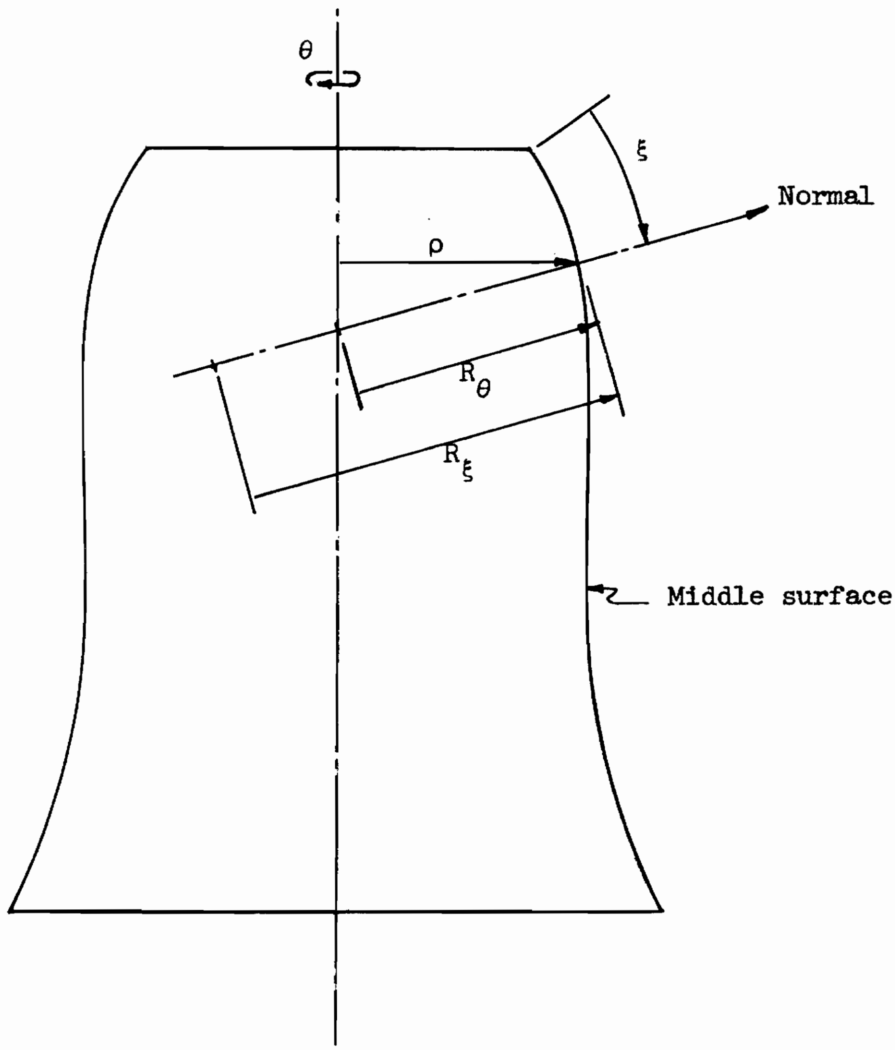


Figure 1.- Shell middle-surface geometry.

$$R_\theta = \frac{\rho}{\sqrt{1 - \left(\frac{d\rho}{d\xi}\right)^2}} \qquad (1)$$

The shell is assumed to have a constant thickness h measured along the normal to the middle surface and boundaries at $\xi = 0$ and $\xi = s$, where s is the total meridional arc length. The material is assumed homogeneous and isotropic with mass density ν, Young's modulus of elasticity E, and Poisson's ratio μ.

1. Governing nonlinear equations.- General nonlinear shell equations in which strains are assumed to be small and rotations, moderately small, are given in reference 10. For a shell of revolution, these equations become, when inertia terms are added,

$$\left(\rho\tilde{N}_\xi\right)_{,\xi} + \tilde{N}_{\xi\theta,\theta} - \frac{d\rho}{d\xi}\tilde{N}_\theta + \frac{\rho}{R_\xi}\tilde{Q}_\xi + \frac{1}{2}\left(\frac{1}{R_\xi} - \frac{1}{R_\theta}\right)\tilde{M}_{\xi\theta,\theta}$$

$$- \frac{\rho}{R_\xi}\left(\tilde{\varphi}_\xi\tilde{N}_\xi + \tilde{\varphi}_\theta\tilde{N}_{\xi\theta}\right) - \frac{1}{2}\left[\tilde{\varphi}\left(\tilde{N}_\xi + \tilde{N}_\theta\right)\right]_{,\theta} + \rho\tilde{P}_\xi = \rho\nu h\ddot{U} \qquad (2a)$$

$$\left(\rho\tilde{N}_{\xi\theta}\right)_{,\xi} + \tilde{N}_{\theta,\theta} + \frac{d\rho}{d\xi}\tilde{N}_{\xi\theta} + \frac{\rho}{R_\theta}\tilde{Q}_\theta + \frac{\rho}{2}\left[\left(\frac{1}{R_\theta} - \frac{1}{R_\xi}\right)\tilde{M}_{\xi\theta}\right]_{,\xi}$$

$$- \frac{\rho}{R_\theta}\left(\tilde{\varphi}_\xi\tilde{N}_{\xi\theta} + \tilde{\varphi}_\theta\tilde{N}_\theta\right) + \frac{\rho}{2}\left[\tilde{\varphi}\left(\tilde{N}_\xi + \tilde{N}_\theta\right)\right]_{,\xi} + \rho\tilde{P}_\theta = \rho\nu h\ddot{V} \qquad (2b)$$

$$\left(\rho\tilde{Q}_\xi\right)_{,\xi} + \tilde{Q}_{\theta,\theta} - \rho\left(\frac{\tilde{N}_\xi}{R_\xi} + \frac{\tilde{N}_\theta}{R_\theta}\right) - \left(\rho\tilde{\varphi}_\xi\tilde{N}_\xi\right)_{,\xi} - \left(\rho\tilde{\varphi}_\theta\tilde{N}_{\xi\theta}\right)_{,\xi}$$

$$- \left(\tilde{\varphi}_\xi\tilde{N}_{\xi\theta}\right)_{,\theta} - \left(\tilde{\varphi}_\theta\tilde{N}_\theta\right)_{,\theta} + \rho\tilde{P} = \rho h\nu\ddot{W} \tag{2c}$$

$$\left(\rho\tilde{M}_\xi\right)_{,\xi} + \tilde{M}_{\xi\theta,\theta} - \frac{d\rho}{d\xi}\tilde{M}_\theta - \rho\tilde{Q}_\xi = 0 \tag{2d}$$

$$\left(\rho\tilde{M}_{\xi\theta}\right)_{,\xi} + \tilde{M}_{\theta,\theta} + \frac{d\rho}{d\xi}\tilde{M}_{\xi\theta} - \rho\tilde{Q}_\theta = 0 \tag{2e}$$

where the comma before a subscript denotes partial differentiation with respect to the succeeding subscripted independent variables (ξ or θ) and dots over a quantity denote differentiation with respect to time. The equilibrium equations (2a), (2b), and (2c) represent the sum of forces along coordinates of the undeformed surface, and the perturbed displacements of the middle surface U, V, and W are measured in the direction of the coordinates of the undeformed surface with W measured positive along the outward normal.

The boundary conditions considered on the edges $\xi = 0$ and $\xi = s$ may be chosen from any combination of the following four pairs of quantities in which either quantity (but not both) of each pair is prescribed:

$$\left.\begin{array}{r} \tilde{N}_\xi \quad \text{or} \quad \tilde{U} \\[2ex] \tilde{N}_{\xi\theta} + \left(\dfrac{3}{2R_\theta} - \dfrac{1}{2R_\xi}\right)\tilde{M}_{\xi\theta} + \dfrac{1}{2}\left(\tilde{N}_\xi + \tilde{N}_\theta\right)\tilde{\varphi} \quad \text{or} \quad \tilde{V} \\[2ex] \tilde{Q}_\xi + \dfrac{1}{\rho}\tilde{M}_{\xi\theta,\theta} - \tilde{\varphi}_\xi\tilde{N}_\xi - \tilde{\varphi}_\theta\tilde{N}_{\xi\theta} \quad \text{or} \quad \tilde{W} \\[2ex] \tilde{\varphi}_\xi \quad \text{or} \quad \tilde{M}_\xi \end{array}\right\} \quad (3)$$

The equations have been derived by use of the Kirchhoff-Love assumptions; that is, normals to the undeformed middle surface remain normal to the deformed middle surface, normal strain is zero, and the normal stress is negligible. Rotary inertia terms have been neglected in the moment equations. Modified stress and moment resultants have been used in the development of equations (2) and (3) and are defined as follows:

$$\left.\begin{array}{l} \tilde{N}_\xi = N_\xi{}^{o} - \dfrac{M_\xi{}^{o}}{R_\xi} \\[2ex] \tilde{N}_\theta = N_\theta{}^{o} - \dfrac{M_\theta{}^{o}}{R_\theta} \\[2ex] \tilde{N}_{\xi\theta} = N_{\xi\theta}{}^{o} - \dfrac{M_{\theta\xi}{}^{o}}{R_\theta} \\[2ex] \tilde{N}_{\theta\xi} = N_{\theta\xi}{}^{o} - \dfrac{M_{\xi\theta}{}^{o}}{R_\xi} \\[2ex] \tilde{M}_\xi = M_\xi{}^{o} \end{array}\right\}$$

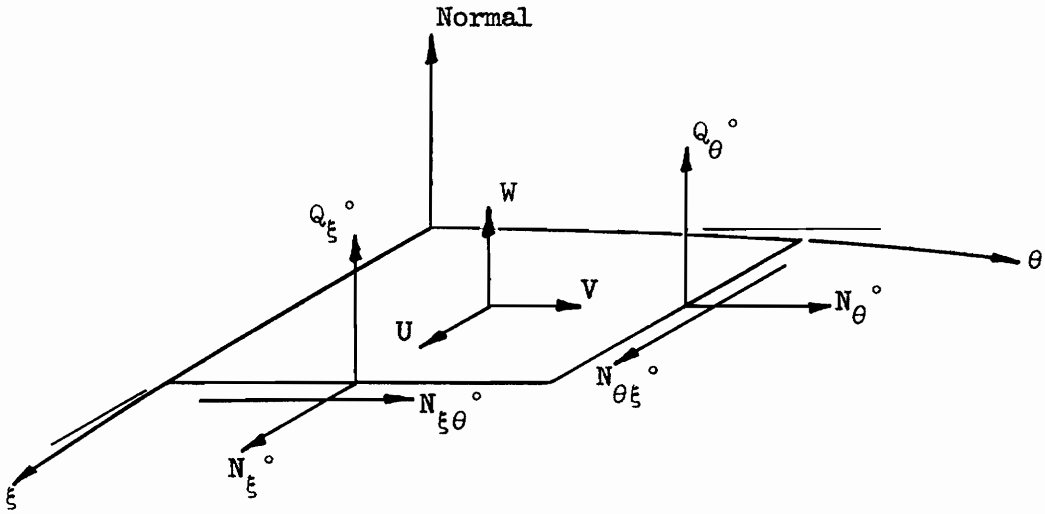
$$\left.\begin{aligned} \tilde{M}_\theta &= M_\theta{}^{o} \\ \tilde{M}_{\xi\theta} &= \tilde{M}_{\theta\xi} = \frac{1}{2}\left(M_{\xi\theta}{}^{o} + M_{\theta\xi}{}^{o}\right) \end{aligned}\right\} \tag{4}$$

The modified transverse shear stress resultants $\tilde{Q}_\xi$ and $\tilde{Q}_\theta$ may be found by applying the definitions in equations (4) to the moment equilibrium equations (eqs. (2d) and (2e)).

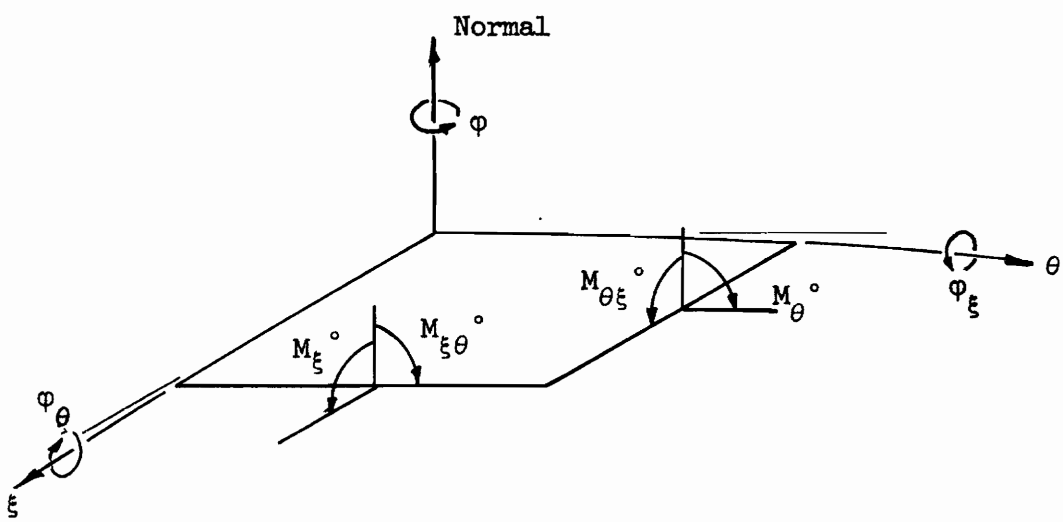
The quantities $N_\xi{}^{o}$, $N_\theta{}^{o}$, $N_{\xi\theta}{}^{o}$, $N_{\theta\xi}{}^{o}$, $Q_\xi{}^{o}$, and $Q_\theta{}^{o}$ represent the total middle-surface stress resultants (fig. 2(a)), and the quantities $M_\xi{}^{o}$, $M_\theta{}^{o}$, $M_{\xi\theta}{}^{o}$, and $M_{\theta\xi}{}^{o}$ represent the total middle-surface moment resultants (fig. (2(b)). No attempt is made to relate these stress and moment resultants to the distribution of stress through the thickness of the shell. The equations of reference 10 have been derived without dependence on such a relationship; thus, any formulation consistent with thin-shell theory is acceptable. According to reference 10, the addition of terms like M^{o}/R to the N^{o} quantities in the stress-strain relations does not introduce errors any greater than those already introduced by neglecting transverse shear flexibility in the Kirchhoff-Love hypothesis. Consequently, the $\tilde{N}$ quantities may be treated as stress resultants without introducing an inconsistency in the thin-shell analysis.

The sum of the moments about the normal direction is

$$N_{\xi\theta}{}^{o} - \frac{M_{\theta\xi}{}^{o}}{R_\theta} - N_{\theta\xi}{}^{o} + \frac{M_{\xi\theta}{}^{o}}{R_\xi} = 0$$



(a) Stress resultants and displacements.



(b) Moment resultants and rotations.

Figure 2.- Middle surface quantities.

Hence, from the definition of the modified stress resultants

$$\tilde{N}_{\theta\xi} = \tilde{N}_{\xi\theta} \tag{5}$$

Therefore, the sixth equilibrium equation, that of equilibrium of moments about the normal, is identically satisfied by symmetric modified stress resultants.

2. Vibration equations.- In the derivation of the vibration equations, the total rotations and total stress and moment resultants are separated into the parts associated with the initial axisymmetric prestress and the parts associated with the infinitesimal perturbed time dependent displacements about the prestressed state. Symbols with bars represent those quantities associated with the prestress conditions and symbols without bars represent those associated with the perturbed state. Thus, the total stress state may be completely described by these quantities from the relations:

$$\left.\begin{aligned}
\tilde{N}_{\xi} &= \overline{N}_{\xi}(\xi) + N_{\xi}(\xi,\theta,t) \\
\tilde{N}_{\theta} &= \overline{N}_{\theta}(\xi) + N_{\theta}(\xi,\theta,t) \\
\tilde{N}_{\xi\theta} &= N_{\xi\theta}(\xi,\theta,t) \\
\tilde{Q}_{\xi} &= \overline{Q}_{\xi}(\xi) + Q_{\xi}(\xi,\theta,t) \\
\tilde{Q}_{\theta} &= Q_{\theta}(\xi,\theta,t)
\end{aligned}\right\}$$

$$\left.\begin{aligned}\tilde{M}_{\xi} &= \overline{M}_{\xi}(\xi) + M_{\xi}(\xi,\theta,t)\\ \tilde{M}_{\theta} &= \overline{M}_{\theta}(\xi) + M_{\theta}(\xi,\theta,t)\\ \tilde{M}_{\xi\theta} &= M_{\xi\theta}(\xi,\theta,t)\end{aligned}\right\rbrace \quad (6)$$

The total rotations are

$$\left.\begin{aligned}\tilde{\varphi}_{\xi} &= \overline{\varphi}_{\xi}(\xi) + \varphi_{\xi}(\xi,\theta,t)\\ \tilde{\varphi}_{\theta} &= \varphi_{\theta}(\xi,\theta,t)\\ \tilde{\varphi} &= \varphi(\xi,\theta,t)\end{aligned}\right\rbrace \quad (7)$$

and the total surface loading is described by

$$\left.\begin{aligned}\tilde{P}_{\xi} &= \overline{P}_{\xi}(\xi)\\ \tilde{P} &= \overline{P}(\xi)\end{aligned}\right\rbrace \quad (8)$$

since no additional surface loading is assumed to be associated with the vibrating state.

Substitution of equations (6), (7), and (8) in equation (2a) yields

$$\left\{\frac{d\rho}{d\xi}\overline{N}_{\xi} + \rho\frac{d\overline{N}_{\xi}}{d\xi} - \frac{d\rho}{d\xi}\overline{N}_{\theta} + \frac{\rho}{R_{\xi}}\overline{Q}_{\xi} - \frac{\rho}{R_{\xi}}\overline{\varphi}_{\xi}\overline{N}_{\xi} + \rho\overline{P}_{\xi}\right\} + \left\{\frac{d\rho}{d\xi}N_{\xi} + \rho N_{\xi,\xi}\right.$$

$$\left. + N_{\xi\theta,\theta} - \frac{d\rho}{d\xi}N_{\theta} + \frac{\rho}{R_{\xi}}Q_{\xi} + \frac{1}{2}\left(\frac{1}{R_{\xi}} - \frac{1}{R_{\theta}}\right)M_{\xi\theta,\theta} - \frac{\rho}{R_{\xi}}\varphi_{\xi}N_{\xi}\right.$$

$$
\left.\left. - \frac{\rho}{R_\xi} \varphi_\theta N_{\xi\theta} - \frac{1}{2}\left[\varphi\left(N_\xi + N_\theta\right)\right]_{,\theta}\right\} + \left\{- \frac{\rho}{R_\xi} \overline{\varphi}_\xi N_\xi\right\}
+\left\{- \frac{\rho}{R_\xi} \varphi_\xi \overline{N}_\xi - \frac{1}{2} \varphi_{,\theta}\left(\overline{N}_\xi + \overline{N}_\theta\right)\right\} = \rho\nu h\ddot{U} \right\} \quad (9)
$$

where θ-derivatives of barred quantities vanish as a result of axisymmetry of the prestressed state. The prestressed shell is in equilibrium; thus, the sum of the terms enclosed by the first set of braces in equation (9) vanishes identically. Furthermore, the perturbation of the shell away from the prestressed configuration is governed by linear theory. Therefore, the nonlinear terms enclosed by the second pair of braces are neglected.

The term in equation (9) enclosed by the third pair of braces (i.e., the interaction between the prestress deformation and the perturbation stress resultants) is usually neglected in the procedure followed in the classical linearization process for a cylinder. If this term is neglected, the general assumption is made that the prestress rotation is uniformly zero throughout the shell. The error introduced is usually negligible, but for certain boundary conditions or for sharply varying surface loads, this term may be significant and is consequently retained in this analysis. If $\overline{\varphi}_\xi$ and $\overline{\varphi}_{\xi,\xi}$ are neglected, the problem may be reinterpreted as that of a prestressed but undeformed shell of revolution perturbed about the undeformed state. With this term retained, the equilibrium equation in the meridional direction (eq. (9)) reduces to

$$\left(\rho N_{\xi}\right)_{,\xi} + N_{\xi\theta,\theta} - \frac{d\rho}{d\xi} N_{\theta} + \frac{\rho}{R_{\xi}} Q_{\xi} + \frac{1}{2}\left[\left(\frac{1}{R_{\xi}} - \frac{1}{R_{\theta}}\right) M_{\xi\theta}\right]_{,\theta} - \frac{\rho}{R_{\xi}} \overline{N}_{\xi}\varphi_{\xi}$$

$$- \frac{1}{2}\left(\overline{N}_{\xi} + \overline{N}_{\theta}\right) \varphi_{,\theta} - \frac{\rho}{R_{\xi}} \overline{\varphi}_{\xi} N_{\xi} = \rho h\nu\ddot{U} \qquad (10a)$$

By the same procedure, the remaining equilibrium equations (eqs. (2b) to (2e)) are linearized, and reduce to:

$$\left(\rho N_{\xi\theta}\right)_{,\xi} + N_{\theta,\theta} + \frac{d\rho}{d\xi} N_{\xi\theta} + \frac{\rho}{R_{\theta}} Q_{\theta} + \frac{\rho}{2}\left[\left(\frac{1}{R_{\theta}} - \frac{1}{R_{\xi}}\right) M_{\xi\theta}\right]_{,\xi}$$

$$- \frac{\rho}{R_{\theta}} \overline{N}_{\theta}\varphi_{\theta} + \frac{\rho}{2}\left[\left(\overline{N}_{\xi} + \overline{N}_{\theta}\right)\varphi\right]_{,\xi} - \frac{\rho}{R_{\theta}} \overline{\theta}_{\xi} N_{\xi\theta} = \rho h\nu\ddot{V} \qquad (10b)$$

$$\left(\rho Q_{\xi}\right)_{,\xi} + Q_{\theta,\theta} - \rho\left(\frac{N_{\xi}}{R_{\xi}} + \frac{N_{\theta}}{R_{\theta}}\right) - \left(\rho\overline{N}_{\xi}\varphi_{\xi}\right)_{,\xi} - \overline{N}_{\theta}\varphi_{\theta,\theta} - \left(\rho\overline{\varphi}_{\xi}N_{\xi}\right)_{,\xi}$$

$$- \overline{\varphi}_{\xi} N_{\xi\theta,\theta} = \rho h\nu\ddot{W} \qquad (10c)$$

$$\left(\rho M_{\xi}\right)_{,\xi} + M_{\xi\theta,\theta} - \frac{d\rho}{d\xi} M_{\theta} - \rho Q_{\xi} = 0 \qquad (10d)$$

$$\left(\rho M_{\xi\theta}\right)_{,\xi} + M_{\theta,\theta} + \frac{d\rho}{d\xi} M_{\xi\theta} - \rho Q_{\theta} = 0 \qquad (10e)$$

Solving equations (10d) and (10e) for Q_{ξ} and Q_{θ} and substituting for them into equations (10a), (10b), and (10c), eliminates these quantities from the system. The parameters defining the geometry of the middle surface can be nondimensionalized by using a reference length a, as follows:

$$\left.\begin{aligned} x &= \frac{\xi}{a} \\ r &= \frac{\rho}{a} \end{aligned}\right\} \qquad (11)$$

and nondimensional curvatures can be defined as

$$\left.\begin{aligned} k_x &= \frac{a}{R_\xi} \\ k_\theta &= \frac{a}{R_\theta} \end{aligned}\right\} \qquad (12)$$

Upon completion of these manipulations, the following equilibrium equations result:

$$a\left(\frac{dr}{dx} N_\xi + rN_{\xi,x} + N_{\xi\theta,\theta} - \frac{dr}{dx} N_\theta\right) + k_x \frac{dr}{dx} M_\xi + rk_x M_{\xi,x} - k_x \frac{dr}{dx} M_\theta$$
$$+ \frac{1}{2}\left(3k_x - k_\theta\right) M_{\xi\theta,\theta} - a\left[rk_x \overline{N}_\xi \varphi_\xi + \frac{1}{2}\left(\overline{N}_\xi + \overline{N}_\theta\right) \varphi_{,\theta}\right.$$
$$\left. + rk_x \overline{\varphi}_\xi N_\xi\right] = ra^2 h\nu \ddot{U} \qquad (13a)$$

$$a\left(2\frac{dr}{dx} N_{\xi\theta} + rN_{\xi\theta,x} + N_{\theta,\theta}\right) + \frac{1}{2}\left[\left(k_x + 3k_\theta\right)\frac{dr}{dx} - \frac{dk_x}{dx} r\right] M_{\xi\theta}$$
$$+ \frac{r}{2}\left(3k_\theta - k_x\right) M_{\xi\theta,x} + k_\theta M_{\theta,\theta} + a\left[-rk_\theta \overline{N}_\theta \varphi_\theta\right.$$
$$\left. + \frac{r}{2}\frac{d}{dx}\left(\overline{N}_\xi + \overline{N}_\theta\right)\varphi + \frac{r}{2}\left(\overline{N}_\xi + \overline{N}_\theta\right)\varphi_{,x} - rk_\theta \overline{\varphi}_\xi N_{\xi\theta}\right] = ra^2 h\nu \ddot{V}$$
$$(13b)$$

$$-ar\left(k_x N_\xi + k_\theta N_\theta\right) + \frac{d^2r}{dx^2} M_\xi + 2\frac{dr}{dx} M_{\xi,x} + rM_{\xi,xx} - \frac{d^2r}{dx^2} M_\theta - \frac{dr}{dx} M_{\theta,x}$$

$$+ \frac{2}{r}\frac{dr}{dx} M_{\xi\theta,\theta} + 2M_{\xi\theta,x\theta} + \frac{1}{r} M_{\theta,\theta\theta} + a\left(- \frac{dr}{dx} \overline{N}_\xi \varphi_\xi - r\overline{N}_\xi \varphi_{\xi,x}\right.$$

$$- r\frac{d\overline{N}_\xi}{dx} \varphi_\xi - \overline{N}_\theta \varphi_{\theta,\theta} - \frac{dr}{dx} \overline{\varphi}_\xi N_\xi - r\frac{d\overline{\varphi}_\xi}{dx} N_\xi - r\overline{\varphi}_\xi N_{\xi,x}$$

$$\left.- \overline{\varphi}_\xi N_{\xi\theta,\theta}\right) = ra^2 h\nu \ddot{W} \tag{13c}$$

Similarly, the boundary conditions (eqs. (3)) are given as

$$\left.\begin{aligned}
N_\xi &= 0 \quad \text{or} \quad U = 0 \\
N_{\xi\theta} + \frac{1}{2a}\left(3k_\theta - k_x\right) M_{\xi\theta} + \frac{1}{2}\left(\overline{N}_\xi + \overline{N}_\theta\right) \varphi &= 0 \quad \text{or} \quad V = 0 \\
\frac{1}{a}\left[M_{\xi,x} + \frac{1}{r}\frac{dr}{dx}\left(M_\xi - M_\theta\right) + \frac{2M_{\xi\theta,\theta}}{r}\right] - \overline{N}_\xi \varphi_\xi - \overline{\varphi}_\xi N_\xi &= 0 \quad \text{or} \quad W = 0 \\
\varphi_\xi &= 0 \quad \text{or} \quad M_\xi = 0
\end{aligned}\right\} \tag{14}$$

The modified stress-resultant—strain relationships, if physical linearity is assumed, are

$$\left.\begin{aligned}
N_\xi &= B\left(\epsilon_\xi + \mu\epsilon_\theta\right) \\
N_\theta &= B\left(\epsilon_\theta + \mu\epsilon_\xi\right) \\
N_{\xi\theta} &= B(1 - \mu)\ \epsilon_{\xi\theta} \\
M_\xi &= D\left(\kappa_\xi + \mu\kappa_\theta\right)
\end{aligned}\right\}$$

$$\left.\begin{aligned} M_{\theta} &= D\left(\kappa_{\theta} + \mu\kappa_{\xi}\right) = \mu M_{\xi} + D\left(1 - \mu^{2}\right)\kappa_{\theta} \\ M_{\xi\theta} &= D(1 - \mu)\,\kappa_{\xi\theta} \end{aligned}\right\} \quad (15)$$

where

$$\left.\begin{aligned} B &= \frac{Eh}{1 - \mu^{2}} \\ D &= \frac{Eh^{3}}{12\left(1 - \mu^{2}\right)} \end{aligned}\right\} \quad (16)$$

The linearized strain-displacement relationships, from reference 10, reduce to

$$\left.\begin{aligned} \epsilon_{\xi} &= \frac{1}{a}\left(U_{,x} + k_{x}W + \overline{\varphi}_{\xi}k_{x}U - \overline{\varphi}_{\xi}W_{,x}\right) \\ \epsilon_{\theta} &= \frac{1}{a}\left(\frac{dr}{dx}\frac{U}{r} + \frac{V_{,\theta}}{r} + k_{\theta}W\right) \\ \epsilon_{\xi\theta} &= \frac{1}{a}\left(\frac{U_{,\theta}}{2r} + \frac{V_{,x}}{2} - \frac{dr}{dx}\frac{V}{2r} + \frac{\overline{\varphi}_{\xi}}{2}k_{\theta}V - \frac{\overline{\varphi}_{\xi}}{2r}W_{,\theta}\right) \\ \kappa_{\xi} &= \frac{1}{a^{2}}\left(k_{x}U_{,x} + \frac{dk_{x}}{dx}U - W_{,xx}\right) \\ \kappa_{\theta} &= \frac{1}{a^{2}}\left(\frac{dr}{dx}\frac{k_{x}}{r}U + \frac{k_{\theta}}{r}V_{,\theta} - \frac{W_{,\theta\theta}}{r^{2}} - \frac{dr}{dx}\frac{W_{,x}}{r}\right) \\ \kappa_{\xi\theta} &= \frac{1}{a^{2}}\left[\frac{\left(3k_{x} - k_{\theta}\right)}{4r}U_{,\theta} + \frac{\left(3k_{\theta} - k_{x}\right)}{4}V_{,x} + \frac{\left(k_{x} - 3k_{\theta}\right)}{4r}\frac{dr}{dx}V\right. \\ &\qquad \left. - \frac{W_{,x\theta}}{r} + \frac{dr}{dx}\frac{W_{,\theta}}{r^{2}}\right] \end{aligned}\right\} \quad (17)$$

The middle-surface rotations are given in terms of displacements as

$$\left.\begin{aligned}\varphi_\xi &= \frac{1}{a}\left(k_x U - W_{,x}\right)\\ \varphi_\theta &= \frac{1}{a}\left(k_\theta V - \frac{W_{,\theta}}{r}\right)\\ \varphi &= \frac{1}{2a}\left(V_{,x} + \frac{dr}{dx}\frac{V}{r} - \frac{U_{,\theta}}{r}\right)\end{aligned}\right\}\tag{18}$$

The prestress terms are given nondimensionally as

$$\bar{N}_\xi = B\bar{e}_x(x) \qquad \bar{N}_\theta = B\bar{e}_\theta(x)\tag{19}$$

3. Reduction to ordinary differential equations.- With equations (13) and (15) to (19), the equilibrium equations can be reduced to three partial differential equations with the displacements as the unknown dependent variables where the highest order derivative in x is a fourth-order derivative. However, since the solution, in general, can only be achieved by numerical techniques, the procedure of reference 2 is followed, where dependence on θ is removed by assuming a solution of the separable type and introducing M_ξ as an additional unknown. This procedure yields a set of four second-order ordinary differential equations with variable coefficients. The fourth equation is simply the equation for M_ξ in terms of the displacements. This reduction in order is essential for the numerical treatment that follows.

A solution is assumed of the form

$$\left.\begin{aligned} U &= u(x)\ (\cos n\theta)\ e^{i\omega t} \\ V &= v(x)\ (\sin n\theta)\ e^{i\omega t} \\ W &= w(x)\ (\cos n\theta)\ e^{i\omega t} \\ M_\xi &= \frac{Eh^3}{a^2}\, m_\xi(x)\ (\cos n\theta)\ e^{i\omega t} \end{aligned}\right\} \quad (20)$$

Defining the perturbation displacements in this manner assures compatibility in the θ-coordinate.

Performing the operations indicated and utilizing the following geometric relationships

$$\left.\begin{aligned} \frac{dk_\theta}{dx} &= \frac{1}{r}\frac{dr}{dx}\left(k_x - k_\theta\right) \\ \frac{d^2r}{dx^2} &= -rk_xk_\theta \end{aligned}\right\} \quad (21)$$

which are the Codazzi and Gauss equations, respectively, yields the governing equations, as follows:

$$F_{11}u'' + G_{11}u' + H_{11}u + G_{12}v' + H_{12}v + F_{13}w'' + G_{13}w' + H_{13}w$$

$$+ G_{14}m_\xi' + H_{14}m_\xi = 0 \qquad (22a)$$

$$G_{21}u' + H_{21}u + F_{22}v'' + G_{22}v' + H_{22}v + F_{23}w'' + G_{23}w' + H_{23}w$$

$$+ H_{24}m_\xi = 0 \tag{22b}$$

$$F_{31}u'' + G_{31}u' + H_{31}u + F_{32}v'' + G_{32}v' + H_{32}v + F_{33}w'' + G_{33}w'$$

$$+ H_{33}w + F_{34}m_\xi'' + G_{34}m_\xi' + H_{34}m_\xi = 0 \tag{22c}$$

$$G_{41}u' + H_{41}u + H_{42}v + F_{43}w'' + G_{43}w' + H_{43}w + H_{44}m_\xi = 0 \tag{22d}$$

The same procedure yields the boundary conditions, as follows:

$$e_{11}u' + f_{11}u + f_{12}v + e_{13}w' + f_{13}w = 0 \quad \text{or} \quad u = 0 \tag{23a}$$

$$f_{21}u + e_{22}v' + f_{22}v + e_{23}w' + f_{23}w = 0 \quad \text{or} \quad v = 0 \tag{23b}$$

$$e_{31}u' + f_{31}u + e_{32}v' + f_{32}v + e_{33}w' + f_{33}w + e_{34}m_\xi' + f_{34}m_\xi = 0$$

$$\text{or} \quad w = 0 \tag{23c}$$

$$f_{41}u + e_{43}w' = 0 \quad \text{or} \quad m_\xi = 0 \tag{23d}$$

Primes denote total differentiation with respect to the nondimensional variable x, and the coefficients are subscripted for convenience in subsequent matrix manipulations. The coefficients F_{jk}, G_{jk}, H_{jk} are given in appendix A in terms of the parameters γ and λ, where

$$\left.\begin{aligned} \gamma &= \frac{1}{r}\frac{dr}{dx} \\ \lambda &= \frac{h}{a} \end{aligned}\right\} \quad (24)$$

the frequency parameter Ω^2 occurs in H_{11}, H_{22}, and H_{33}, where

$$\Omega^2 = \frac{a^2\omega^2\nu\left(1 - \mu^2\right)}{E} \quad (25)$$

B. Closed-Form Solution for Cylinder Vibrations

The vibratory characteristics of a "freely supported" cylinder (simply supported but unrestrained in the axial direction, i.e. $N_\xi = v = w = M_\xi = 0$) with prestress deformations neglected are well known. (See refs. 16, 17, and 18.) Thus, these known results can be used as a check of the validity of the governing equations and of the accuracy of the numerical techniques to be suggested subsequently.

When prestress deformations are neglected and the in-plane stresses are constant, the equations (22) reduce to ordinary differential equations with constant coefficients which, for freely supported boundary conditions, have a solution of the form

$$\left.\begin{aligned} u(x) &= A_m \cos\frac{m\pi x}{S} \\ v(x) &= B_m \sin\frac{m\pi x}{S} \\ w(x) &= C_m \sin\frac{m\pi x}{S} \\ m_\xi(s) &= D_m \sin\frac{m\pi x}{S} \end{aligned}\right\} \quad m = 1, 2, \cdots \quad (26)$$

where S is the length-radius ratio of the cylinder (a = cylinder radius). The classical procedure of neglecting prestress deformations to ensure constant coefficients in the field equations implies that the cylinder is initially prestressed as a shell with free edges and then subsequently supported for vibration.

Equations (26) are substituted into equations (22) to yield a set of linear homogeneous algebraic equations. For a nontrivial solution to exist, the determinant of the coefficient matrix of the resultant set of equations must vanish. This procedure leads to the characteristic equation

$$\Lambda_3\left(\Omega^6\right) + \Lambda_2\left(\Omega^4\right) + \Lambda_1\left(\Omega^2\right) + \Lambda_0 = 0 \tag{27}$$

where the coefficients Λ_i are given in appendix A.

C. Development of Numerical Solution

1. Development of difference equations.- A numerical procedure is needed for those shells of revolution and loading conditions which do not admit a solution in closed form. The meridian of the shell is divided into increments and a three-point central difference method is used to reduce the differential equations to algebraic form. The distance measured along the meridian between adjacent stations is constant and is represented nondimensionally by Δ where

$$\Delta = x_i - x_{i-1} = \frac{S}{N} \tag{28}$$

and where the subscript i on symbols and matrices indicates the evaluation of the subscripted variable or matrix at the ith station, $i = 0, 1, 2, \cdots, N$ and where

S total meridional arc length of the nondimensional shell, s/a

N total number of intervals

The three-point difference formulas, when applied at the ith station for some function $z(x)$, are

$$\left.\begin{aligned} z_i'' &\approx \frac{1}{\Delta^2}\left(z_{i-1} - 2z_i + z_{i+1}\right) \\ z_i' &\approx \frac{1}{2\Delta}\left(-z_{i-1} + z_{i+1}\right) \end{aligned}\right\} \tag{29}$$

Reference 3 indicates that this simple approximation leads to sufficiently accurate results.

The governing equations (22) may be written in matrix form at station i as

$$F_i Z_i'' + G_i Z_i' + H_i Z_i = 0 \tag{30}$$

where

$$F_i = \begin{bmatrix} F_{11} & 0 & F_{13} & 0 \\ 0 & F_{22} & F_{23} & 0 \\ F_{31} & F_{32} & F_{33} & F_{34} \\ 0 & 0 & F_{43} & 0 \end{bmatrix}_i \tag{31a}$$

$$G_i = \begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & 0 \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & 0 & G_{43} & 0 \end{bmatrix}_i \qquad (31b)$$

$$H_i = \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix}_i \qquad (31c)$$

and

$$Z_i = \begin{Bmatrix} u \\ v \\ w \\ m_\xi \end{Bmatrix}_i \qquad (31d)$$

Similarly, all the boundary equations (23) may be written in matrix form at the boundaries, $i = 0$ and $i = N$, as

$$\left.\begin{aligned} \alpha_0 e_0 Z_0' + \left(\alpha_0 f_0 + \beta_0\right) Z_0 &= 0 \\ \alpha_N e_N Z_N' + \left(\alpha_N f_N + \beta_N\right) Z_N &= 0 \end{aligned}\right\} \qquad (32)$$

where

$$e_{0,N} = \begin{bmatrix} e_{11} & 0 & e_{13} & 0 \\ 0 & e_{22} & e_{23} & 0 \\ e_{31} & e_{32} & e_{33} & e_{34} \\ 0 & 0 & e_{43} & 0 \end{bmatrix}_{0,N} \tag{33a}$$

$$f_{0,N} = \begin{bmatrix} f_{11} & f_{12} & f_{13} & 0 \\ f_{21} & f_{22} & f_{23} & 0 \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & 0 & 0 & 0 \end{bmatrix}_{0,N} \tag{33b}$$

and where

$$\left.\begin{aligned} \alpha_{0,N} &= \begin{bmatrix} \alpha_{11} & 0 & 0 & 0 \\ 0 & \alpha_{22} & 0 & 0 \\ 0 & 0 & \alpha_{33} & 0 \\ 0 & 0 & 0 & \alpha_{44} \end{bmatrix}_{0,N} \\ \beta_{0,N} &= \begin{bmatrix} (1-\alpha_{11}) & 0 & 0 & 0 \\ 0 & (1-\alpha_{22}) & 0 & 0 \\ 0 & 0 & (1-\alpha_{33}) & 0 \\ 0 & 0 & 0 & (1-\alpha_{44}) \end{bmatrix}_{0,N} \end{aligned}\right\} \tag{34}$$

The elements α_{jj} take on the value 1 or 0 depending on the prescribed conditions.

The α- and β-matrices (eqs. (34)) are used to select the prescribed boundary conditions. If, for example, $u = 0$ is prescribed at $i = 0$ then $(\alpha_{11})_0 = 0$ and if u is not prescribed at $i = N$ (i.e., if the u displacement is unrestrained in the meridional direction of the undeformed shell), then $(\alpha_{11})_N = 1$. If desired, the present theory can be extended to allow for elastic and directional supports in the boundary conditions by appropriate redefinition of the α- and β-matrices.

When equations (29) are applied to equations (30) and (32), the governing equations become

$$\left(\frac{F_i}{\Delta^2} - \frac{G_i}{2\Delta}\right) Z_{i-1} + \left(H_i - \frac{2F_i}{\Delta^2}\right) Z_i + \left(\frac{F_i}{\Delta^2} + \frac{G_i}{2\Delta}\right) Z_{i+1} = 0 \tag{35}$$

$$(i = 0, 1, 2, \ldots, N - 1, N)$$

and the boundary equations become

$$\left.\begin{aligned} &-\frac{\alpha_0 e_0}{2\Delta} Z_{-1} + \left(\alpha_0 f_0 + \beta_0\right) Z_0 + \frac{\alpha_0 e_0}{2\Delta} Z_1 = 0 \\ &-\frac{\alpha_N e_N}{2\Delta} Z_{N-1} + \left(\alpha_N f_N + \beta_N\right) Z_N + \frac{\alpha_N e_N}{2\Delta} Z_{N+1} = 0 \end{aligned}\right\} \tag{36}$$

Equations (36) can be solved for Z_{-1} and Z_{N+1} and the results can be substituted into equation (35) to yield, at $i = 0$,

$$\left\{\alpha_0 \left[\frac{e_0}{2\Delta}\left(\frac{F_0}{\Delta^2} - \frac{G_0}{2\Delta}\right)^{-1}\left(H_0 - \frac{2F_0}{\Delta^2}\right) + f_0\right] + \beta_0\right\} Z_0$$

$$+ \frac{\alpha_0 e_0}{2\Delta}\left[\left(\frac{F_0}{\Delta^2} - \frac{G_0}{2\Delta}\right)^{-1}\left(\frac{F_0}{\Delta^2} + \frac{G_0}{2\Delta}\right) + I\right] Z_1 = 0 \tag{37}$$

and, at $i = N$,

$$\left\{ \alpha_N \left[-\frac{e_N}{2\Delta}\left(\frac{F_N}{\Delta^2} + \frac{G_N}{2\Delta}\right)^{-1}\left(H_N - \frac{2F_N}{\Delta^2}\right) + f_N \right] + \beta_N \right\} Z_N$$

$$- \frac{\alpha_N e_N}{2\Delta}\left[\left(\frac{F_N}{\Delta^2} + \frac{G_N}{2\Delta}\right)^{-1}\left(\frac{F_N}{\Delta^2} - \frac{G_N}{2\Delta}\right) + I\right] Z_{N-1} = 0 \qquad (38)$$

where

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The difference equations (35), (37), and (38) constitute a complete set of field equations governing the behavior of the perturbed state.

2. Numerical solution.- The problem is now one of solving a set of homogeneous equations (eqs. (35), (37), and (38)). This set constitutes an eigenvalue problem such that the mode shape Z_i is the eigenvector and the frequency parameter Ω^2 is the corresponding eigenvalue for the vibration problem.

The fourth equation, being simply the definition of m_x in terms of displacements, will not contain an eigenvalue. For a nontrivial solution to exist, the determinant of the coefficient matrix must vanish.

The coefficient matrix will be a 12-element-wide band matrix. A convenient technique for solution of this problem can be formulated by modifying the method of references 8 and 9. Such a modification is presented herein to handle the free vibrations of a prestressed shell governed by four second-order difference equations with two-point boundary conditions.

Define the following (4×4) matrices:

$$\left.\begin{aligned}
A_i &= \frac{F_i}{\Delta^2} - \frac{G_i}{2\Delta} \\
B_i &= H_i - \frac{2F_i}{\Delta^2} \\
C_i &= \frac{F_i}{\Delta^2} + \frac{G_i}{2\Delta} \\
D_0 &= \alpha_0 \left[\frac{e_0}{2\Delta} \left(\frac{F_0}{\Delta^2} - \frac{G_0}{2\Delta} \right)^{-1} \left(H_0 - \frac{2F_0}{\Delta^2} \right) + f_0 \right] + \beta_0 \\
D_N &= \alpha_N \left[- \frac{e_N}{2\Delta} \left(\frac{F_N}{\Delta^2} + \frac{G_N}{2\Delta} \right)^{-1} \left(H_N - \frac{2F_N}{\Delta^2} \right) + f_N \right] + \beta_N \\
E_0 &= \alpha_0 \frac{e_0}{2\Delta} \left[\left(\frac{F_0}{\Delta^2} - \frac{G_0}{2\Delta} \right)^{-1} \left(\frac{F_0}{\Delta^2} + \frac{G_0}{2\Delta} \right) + I \right] \\
E_N &= - \alpha_N \frac{e_N}{2\Delta} \left[\left(\frac{F_N}{\Delta^2} + \frac{G_N}{2\Delta} \right)^{-1} \left(\frac{F_N}{\Delta^2} - \frac{G_N}{2\Delta} \right) + I \right]
\end{aligned}\right\} \tag{39}$$

Equations (35), (37), and (38) may now be written as

$$A_i Z_{i-1} + B_i Z_i + C_i Z_{i+1} = 0 \qquad (i = 1, 2, \ldots, N - 1) \tag{40}$$

$$D_0 Z_0 + E_0 Z_1 = 0 \tag{41}$$

$$E_N Z_{N-1} + D_N Z_N = 0 \tag{42}$$

For such a set of homogeneous equations, a recursion formula for Z_i may be written as

$$Z_i + P_i Z_{i+1} = 0 \qquad \begin{cases} (i = 1, 2, \cdots, N - 2) \\ \text{and} \\ (i = 0 \text{ if } Z_0 \neq 0) \\ (i = N - 1 \text{ if } Z_N \neq 0) \end{cases} \tag{43}$$

where P_i is a (4×4) recursion matrix. To find P_i, combine equation (43) and equation (40) to obtain

$$Z_i + \left(B_i - A_i P_{i-1}\right)^{-1} C_i Z_{i+1} = 0 \qquad (i = 1, 2, \cdots, N - 1) \tag{44}$$

Comparison of equation (44) with equation (43) shows that

$$P_i = \left(B_i - A_i P_{i-1}\right)^{-1} C_i \qquad (i = 1, 2, \cdots, N - 1) \tag{45}$$

Comparison of equation (41) with equation (43), the latter written at $i = 0$, shows that

$$P_0 = D_0^{-1} E_0 \tag{46}$$

From equations (45) and (46), P_i may be found at all points with the exception of the point $i = N$. This process of determining all required values of P_i in terms of P_0 is in essence a Gaussian elimination process.

Equation (43) written at $i = N - 1$, in combination with equation (42), yields

$$\left(D_N - E_N P_{N-1}\right) Z_N = 0 \tag{47}$$

If $Z_N \neq 0$, then for a solution to exist,

$$\left| D_N - E_N P_{N-1} \right| = 0 \tag{48}$$

Therefore, any frequency parameter Ω^2 which satisfies equation (48) contains a natural frequency of the system. The natural frequencies can be found by trial and error by selecting successive values for Ω^2, calculating the matrices of equations (39), and using equations (45) and (46) to evaluate the determinant in equation (48). This procedure is continued until the desired zeroes of the determinant are found.

The method must be slightly modified for the case $Z_N = 0$. Substituting equation (43) written at $N - 2$ into equation (40) written at $N - 1$ yields

$$\left(B_{N-1} - A_{N-1} P_{N-2} \right) Z_{N-1} = 0 \tag{49}$$

If $Z_{N-1} = 0$, then $Z_i = 0$ from equation (43), and the solution is trivial. Therefore

$$\left| B_{N-1} - A_{N-1} P_{N-2} \right| = 0 \tag{50}$$

Consequently, for the case $Z_N = 0$, equation (50) is used in place of equation (48) in the elimination process.

After the natural frequencies have been found, the corresponding mode shapes are determined from equations (43) and (47). Equation (47) is used to find a normalized Z_N. For the case where $Z_N = 0$, equation (49) is used in place of (47) to solve for Z_{N-1}. The remaining Z_i's are then determined by using the recursion formula, equation (43). To find a normalized Z_N, it is noted that the components of Z_N are proportional to the cofactors of the elements of a dependent row of the coefficient matrix in equation (49). If δ_{jk} is defined as the cofactor of the jth element in the kth row, where row k contains the coefficients of a linearly dependent equation, then the normalizing factor δ can be defined as

$$\delta = \sqrt{\left(\delta_{1k}\right)^2 + \left(\delta_{2k}\right)^2 + \left(\delta_{3k}\right)^2 + \left(\delta_{4k}\right)^2} \tag{51}$$

so that Z_N may be given as

$$Z_N = \begin{Bmatrix} u \\ v \\ w \\ m_\xi \end{Bmatrix} = \frac{1}{\delta} \begin{Bmatrix} \delta_{1k} \\ \delta_{2k} \\ \delta_{3k} \\ \delta_{4k} \end{Bmatrix} \tag{52}$$

The index k indicates any row which is a linearly dependent row of the matrix $\left(D_N - E_N P_{N-1}\right)$.

This numerical procedure is particularly well suited for use with a large number of stations since only the band elements need be retained during the computation process. References 2, 3, and 5 give further advantages in using this general method of solution.

Although equation (48) (or eq. (50)) contains all the roots of the system of equations (40), (41), and (42), this method of elimination introduces spurious singularities in the determinants of equation (48) or (50). For some shell configurations and boundary conditions, it is found that the roots and singularities very nearly coincide, and the usual root searching methods fail to indicate a root if the increments given to the frequency parameter are too large. Moreover, some of these singularities are associated with a change in sign in the value of the determinant even though no zero exists at that value of the frequency parameter. The technique used in this investigation for avoiding this difficulty is presented in Appendix B.

A set of linear equations governing the infinitesimal vibrations of axisymmetrically prestressed shells has been developed and both the in-plane inertia and prestress deformation effects have been retained in the analysis. The equations derived are consistent with first-order thin-shell theory and can be used to describe the behavior of shells with arbitrary meridional configuration having moderately small prestress rotation.

A numerical procedure has been given for solving the governing equations for the natural frequencies and associated mode shapes for a general shell of revolution with homogeneous boundary conditions. The numerical procedure uses matrix methods in finite-difference form coupled with a Gaussian elimination to solve the governing eigenvalue problem.

Accurate results can be found for specific shell structures with this procedure. However, even with high-speed computers, it becomes impractical to attempt an extensive parameter study of the behavior of shell systems. To achieve a more rapid analysis of the effects of the different parameters governing the dynamic behavior of shell systems, an approximate set of equations of motion is developed in the next section. The equations are limited to cylindrical like shells having a shallow meridional curvature. The finite difference solution of the general shell equations developed in this section can then be used to determine the accuracy of the approximate solutions.

VIII. APPROXIMATE SHELL EQUATIONS

A. Derivation of Approximate Shell Equations

In this section, a set of linear homogeneous field equations is developed which governs the dynamic behavior of doubly curved shells with positive or negative constant shallow meridional curvature. The nonlinear strain-displacement relations given by reference (10) are specialized for a cylinder and modified slightly. The cylinder is given a small axisymmetric deformation, then, by removing the strains caused by this initial geometric perturbation, an unstrained state is established in the deformed configuration. The nonlinear equilibrium equations corresponding to this unstrained state are developed and linearized.

The first approximation nonlinear strain-displacement relations developed in reference (10) based on the assumption of small strains and moderately small rotations become, for a cylinder with radius R,

$$\left.\begin{aligned}
\hat{\epsilon}_{\xi} &= \hat{u},_{\xi} + \frac{1}{2}\left(\hat{w},_{\xi}\right)^2 + \frac{1}{8}\left(\hat{v},_{\xi} - \frac{\hat{u},_{\theta}}{R}\right)^2 \\
\hat{\epsilon}_{\theta} &= \frac{\hat{v},_{\theta}}{R} + \frac{\hat{w}}{R} + \frac{1}{2R^2}\left(\hat{w},_{\theta} - \hat{v}\right)^2 + \frac{1}{8}\left(\hat{v},_{\xi} - \frac{\hat{u},_{\theta}}{R}\right)^2 \\
\hat{\epsilon}_{\xi\theta} &= \frac{1}{2}\left[\hat{v},_{\xi} + \frac{\hat{u},_{\theta}}{R} + \frac{\hat{w},_{\xi}}{R}\left(\hat{w},_{\theta} - \hat{v}\right)\right] \\
\hat{\kappa}_{\xi} &= -\,\hat{w},_{\xi\xi}
\end{aligned}\right\}$$

$$\left.\begin{aligned}\hat{\kappa}_{\theta} &= -\frac{1}{R^2}\left(\hat{w},_{\theta} - \hat{v}\right),_{\theta} \\ \hat{\kappa}_{\xi\theta} &= -\frac{1}{R}\left[\left(\hat{w},_{\theta} - \frac{3\hat{v}}{4}\right),_{\xi} + \frac{\hat{u},_{\theta}}{4R}\right]\end{aligned}\right\} \tag{53}$$

where the symbol $\hat{}$ is introduced to indicate quantities measured with respect to an unstressed cylindrical surface.

To permit the method of solution which follows, the term $\hat{\kappa}_{\xi\theta}$ is modified so that the term $-\dfrac{\hat{u},_{\theta}}{4R^2}$ is eliminated. The quantity $\hat{\kappa}_{\xi\theta}$ may be written as

$$\hat{\kappa}_{\xi\theta} = -\frac{1}{R}\left(\hat{w},_{\theta} - \hat{v}\right),_{\xi} - \frac{\left(\hat{\epsilon}_{\xi\theta}\right)_L}{2R} \tag{54}$$

where $\left(\hat{\epsilon}_{\xi\theta}\right)_L$ contains only the linear terms of $\hat{\epsilon}_{\xi\theta}$. It is shown in reference (19) that linear terms of the type $\dfrac{\epsilon_L}{R}$ multiplied by a numerical factor of the order of unity may be added to the linear expressions for bending strains without introducing an error greater than that originally introduced through application of the Kirchhoff-Love hypothesis. Hence, $\hat{\kappa}_{\xi\theta}$ is sufficiently defined by

$$\hat{\kappa}_{\xi\theta} = -\frac{1}{R}\left(\hat{w},_{\theta} - \hat{v}\right),_{\xi} \tag{55}$$

To increase the manageability of the strain-displacement relations, only the Von Karman type nonlinear terms are retained (i.e., non-linear terms composed of products of derivatives of the normal displacement). The final form of the cylinder strain-displacement relations reduces to

$$\left.\begin{aligned}
\hat{\epsilon}_{\xi} &= \hat{u},_{\xi} + \frac{1}{2}\left(\hat{w},_{\xi}\right)^{2} \\
\hat{\epsilon}_{\theta} &= \frac{\hat{v},_{\theta}}{R} + \frac{\hat{w}}{R} + \frac{1}{2R^{2}}\left(\hat{w},_{\theta}\right)^{2} \\
\hat{\epsilon}_{\xi\theta} &= \frac{1}{2}\left(\hat{v},_{\xi} + \frac{\hat{u},_{\theta}}{R} + \frac{1}{R}\,\hat{w},_{\xi}\,\hat{w},_{\theta}\right) \\
\hat{\kappa}_{\xi\theta} &= -\,\hat{w},_{\xi\xi} \\
\hat{\kappa}_{\theta} &= -\,\frac{1}{R^{2}}\left(\hat{w},_{\theta} - \hat{v}\right),_{\theta} \\
\hat{\kappa}_{\xi\theta} &= -\,\frac{1}{R}\left(\hat{w},_{\theta} - \hat{v}\right),_{\xi}
\end{aligned}\right\} \qquad (56)$$

A set of strain-displacement relations governing the behavior of a shell of revolution with a shallow meridional curvature is developed in a fashion similar to that of references (11) and (12) by the introduction of an initial displacement $w_o(\xi)$ to the cylinder middle surface. Let

$$\hat{w}(\xi,\theta,t) = \tilde{w}(\xi,\theta,t) + w_o(\xi) \qquad (57)$$

where $\tilde{w}$ is the displacement measured from the initially deformed surface along the normal to the cylindrical surface. Since the meridional curve is assumed shallow, it is sufficient as a first approximation to represent the initially deformed curve by

$$w_o(\xi) = \frac{-\xi^2}{2R_\xi} \tag{58}$$

where R_ξ is the constant radius of curvature of the initially deformed meridional curve. This representation restricts the subsequent analysis to shells of the form shown in the sketches in figure 3. The axial displacement $\hat{u}$ may be represented by

$$\hat{u} = \tilde{u} + \frac{\xi\tilde{w}}{R_\xi} \tag{59}$$

where for a shallow meridional curve (i.e., R_ξ large), $\tilde{u}$ is the meridional displacement measured along a tangent to the initially deformed surface. With application of equations (57), (58), and (59) the strains caused by the total deformation can be written as

$$\left.\begin{aligned}
\hat{\epsilon}_\xi &= \tilde{u},_\xi + \frac{\tilde{w}}{R_\xi} + \frac{1}{2}\left(\tilde{w},_\xi\right)^2 + \frac{1}{2}\left(\frac{\xi}{R_\xi}\right)^2 \\
\hat{\epsilon}_\theta &= \frac{\hat{v},_\theta}{R} + \frac{\tilde{w}}{R} + \frac{1}{2}\left(\frac{\tilde{w},_\theta}{R}\right)^2 - \frac{\xi^2}{2R_\xi R} \\
\hat{\epsilon}_{\xi\theta} &= \frac{1}{2}\left(\hat{v},_\xi + \frac{\tilde{u},_\theta}{R} + \frac{\tilde{w},_\xi \tilde{w},_\theta}{R}\right) \\
\hat{\kappa}_\xi &= -\tilde{w},_{\xi\xi} - \frac{1}{R_\xi} \\
\hat{\kappa}_\theta &= -\frac{1}{R^2}\left(\tilde{w},_\theta - \hat{v}\right),_\theta \\
\hat{\kappa}_{\xi\theta} &= -\frac{1}{R}\left(\tilde{w},_\theta - \hat{v}\right),_\xi
\end{aligned}\right\} \tag{60}$$

The strains caused by the initial deformation w_o are

$$\left.\begin{aligned}
\epsilon_{\xi_o} &= \frac{1}{2}\left(\frac{\xi}{R_\xi}\right)^2 \\
\epsilon_{\theta_o} &= -\frac{\xi^2}{2R_\xi\ R} \\
\epsilon_{\xi\theta_o} &= 0 \\
\kappa_{\xi_o} &= -\frac{1}{R_\xi} \\
\kappa_{\theta_o} &= 0 \\
\kappa_{\xi\theta_o} &= 0
\end{aligned}\right\} \qquad (61)$$

The initially deformed system is now taken to be in an unstrained equilibrium state. Any deformation $\tilde{u}, \tilde{v}, \tilde{w}$ away from this new equilibrium state introduces strains given approximately by $\tilde{\epsilon} = \hat{\epsilon} - \epsilon_o$, thus, from equations (60) and (61)

$$\left.\begin{aligned}
\tilde{\epsilon} &= \tilde{u},_\xi + \frac{\tilde{w}}{R_\xi} + \frac{1}{2}\left(\tilde{w},_\xi\right)^2 \\
\tilde{\epsilon}_\theta &= \frac{\tilde{v},_\theta}{R} + \frac{\tilde{w}}{R} + \frac{1}{2}\left(\frac{\tilde{w},_\theta}{R}\right)^2 \\
\tilde{\epsilon}_{\xi\theta} &= \frac{1}{2}\left(\tilde{v},_\xi + \frac{\tilde{u},_\theta}{R} + \frac{\tilde{w},_\xi\ \tilde{w},_\theta}{R}\right)
\end{aligned}\right.$$

$$\left.\begin{aligned}
\tilde{\kappa}_\xi &= - \tilde{w},_{\xi\xi} \\
\tilde{\kappa}_\theta &= - \frac{1}{R^2}\left(\tilde{w},_\theta - \tilde{v}\right),_\theta \\
\tilde{\kappa}_{\xi\theta} &= - \frac{1}{R}\left(\tilde{w},_\theta - \tilde{v}\right),_\xi
\end{aligned}\right\} \qquad (62)$$

where ξ is now a length measure along the meridian and where the quantity $\tilde{v} = \hat{v}$ is introduced for convenience in notation. Equations (62) are the strain-displacement relations for a shell of revolution with a shallow meridional curvature. To obtain a set of equilibrium equations consistent with equations (62), the principle of minimum potential energy is applied.

The total potential energy for an isotropic, linearly elastic, homogeneous, vibrating shell acted on by a uniform lateral conservative loading p is

$$\begin{aligned}
\Pi = \frac{E}{2\left(1 - \mu^2\right)} \int_{\xi_1}^{\xi_2} \int_0^{2\pi} \int_{-\frac{h}{2}}^{\frac{h}{2}} \Big\{ & \left(\tilde{\epsilon}_\xi + z\,\tilde{\kappa}_\xi\right)^2 + \left(\tilde{\epsilon}_\theta + z\,\tilde{\kappa}_\theta\right)^2 \\
& + 2\mu\left(\tilde{\epsilon}_\xi + z\,\tilde{\kappa}_\xi\right)\left(\tilde{\epsilon}_\theta + z\,\tilde{\kappa}_\theta\right) + 2(1 - \mu)\left(\tilde{\epsilon}_{\xi\theta} + z\,\tilde{\kappa}_{\xi\theta}\right)^2 \Big\} R\,dz\,d\theta\,d\xi \\
& - p \int_{\xi_1}^{\xi_2} \int_0^{2\pi} \tilde{w}\, R\,d\theta\,d\xi - \frac{1}{2}\nu \int_{\xi_1}^{\xi_2} \int_0^{2\pi} \int_{-\frac{h}{2}}^{\frac{h}{2}} \Big[\left(\tilde{u},_t\right)^2 \\
& + \left(\tilde{v},_t\right)^2 + \left(\tilde{w},_t\right)^2\Big] R\,dz\,d\theta\,d\xi
\end{aligned} \qquad (63)$$

Performing the integration over z and allowing the variation of the energy to vanish yields

$$\delta\Pi = \int_{\xi_1}^{\xi_2}\int_0^{2\pi}\left[\tilde{N}_\xi\ \delta\tilde{\epsilon}_\xi + \tilde{N}_\theta\ \delta\tilde{\epsilon}_\theta + 2\tilde{N}_{\xi\theta}\ \delta\tilde{\epsilon}_{\xi\theta} + \tilde{M}_\xi\ \delta\tilde{\kappa}_\xi + \tilde{M}_\theta\ \delta\tilde{\kappa}_\theta\right.$$

$$\left. + 2\tilde{M}_{\xi\theta}\ \delta\tilde{\kappa}_{\xi\theta} - p\delta w - \nu h\left(\tilde{u},_t\ \delta\tilde{u},_t + \tilde{v},_t\ \delta\tilde{v},_t + \tilde{w},_t\ \delta\tilde{w},_t\right)\right] R d\theta d\xi = 0 \tag{64}$$

where

$$\left.\begin{aligned}
\tilde{N}_\xi &= B\left(\tilde{\epsilon}_\xi + \mu\ \tilde{\epsilon}_\theta\right)\\
\tilde{N}_\theta &= B\left(\tilde{\epsilon}_\theta + \mu\ \tilde{\epsilon}_\xi\right)\\
\tilde{N}_{\xi\theta} &= B\ (1 - \mu)\ \tilde{\epsilon}_{\xi\theta}\\
\tilde{M}_\xi &= D\left(\tilde{\kappa}_\xi + \mu\ \tilde{\kappa}_\theta\right)\\
\tilde{M}_\theta &= D\left(\tilde{\kappa}_\theta + \mu\ \tilde{\kappa}_\xi\right)\\
\tilde{M}_{\xi\theta} &= D\ (1 - \mu)\ \tilde{\kappa}_{\xi\theta}
\end{aligned}\right\} \tag{65}$$

and where B and D are defined in equation (16). Integration by parts in equation (64) leads to the following nonlinear equilibrium equations

$$\left.\begin{aligned}
&\tilde{N}_{\xi,\xi} + \frac{\tilde{N}_{\xi\theta,\theta}}{R} - \nu h\, \tilde{u},_{tt} = 0 \\
&\frac{\tilde{N}_{\theta,\theta}}{R} + \tilde{N}_{\xi\theta,\xi} + \frac{\tilde{M}_{\theta,\theta}}{R^2} + \frac{2\tilde{M}_{\xi\theta,\xi}}{R} - \nu h\, \tilde{v},_{tt} = 0 \\
&\tilde{M}_{\xi,\xi\xi} + \frac{2\tilde{M}_{\xi\theta,\xi\theta}}{R} + \frac{\tilde{M}_{\theta,\theta\theta}}{R^2} - \frac{\tilde{N}_{\xi}}{R_{\xi}} - \frac{\tilde{N}_{\theta}}{R} + \left(\tilde{N}_{\xi}\, \tilde{w},_{\xi} + \tilde{N}_{\xi\theta} \frac{\tilde{w},_{\theta}}{R}\right)_{,\xi} \\
&\quad + \frac{1}{R}\left(\tilde{N}_{\theta} \frac{\tilde{w},_{\theta}}{R} + \tilde{N}_{\xi\theta}\, \tilde{w},_{\xi}\right)_{,\theta} + \tilde{p} - \nu h\, \tilde{w},_{tt} = 0
\end{aligned}\right\} \quad (66)$$

with the following prescribed at the ξ = constant boundaries

$$\left.\begin{aligned}
&\tilde{N}_{\xi} \quad \text{or} \quad \tilde{u} \\
&\tilde{N}_{\xi\theta} + \frac{2\tilde{M}_{\xi\theta}}{R} \quad \text{or} \quad \tilde{v} \\
&\tilde{M}_{\xi,\xi} + \frac{2\tilde{M}_{\xi\theta,\theta}}{R} + \tilde{N}_{\xi}\, \tilde{w},_{\xi} + \tilde{N}_{\xi\theta} \frac{\tilde{w},_{\theta}}{R} \quad \text{or} \quad \tilde{w} \\
&\tilde{M}_{\xi} \quad \text{or} \quad \tilde{w},_{\xi}
\end{aligned}\right\} \quad (67)$$

These equations parallel equations (2) and (3) in Chapter VII. The equations are linearized for prestressed linear vibrations by the same procedure as that used in Chapter VII. Let

$$\left.\begin{aligned}
\tilde{N}_\xi &= \overline{N}_\xi(\xi) + N_\xi(\xi,\theta,t) \\
\tilde{N}_\theta &= \overline{N}_\theta(\xi) + N_\theta(\xi,\theta,t) \\
\tilde{N}_{\xi\theta} &= N_{\xi\theta}(\xi,\theta,t) \\
\tilde{M}_\xi &= \overline{M}_\xi(\xi) + M_\xi(\xi,\theta,t) \\
\tilde{M}_\theta &= \overline{M}_\theta(\xi) + M_\theta(\xi,\theta,t) \\
\tilde{M}_{\xi\theta} &= M_{\xi\theta}(\xi,\theta,t)
\end{aligned}\right\} \tag{68}$$

Also let

$$\left.\begin{aligned}
\tilde{u} &= \overline{u}(\xi) + u(\xi,\theta,t) \\
\tilde{v} &= v(\xi,\theta,t) \\
\tilde{w} &= \overline{w}(\xi) + w(\xi,\theta,t)
\end{aligned}\right\} \tag{69}$$

where the barred terms are associated with the stresses and deformations due to an axisymmetric loading and the unbarred terms are associated with stresses and deformations due to a subsequent linear vibration about the prestressed state. The equations governing the linear vibration about the prestressed state are found by applying equations (68) and (69) to equations (66), applying equilibrium of the prestressed state, and neglecting vibration terms. They are as follows:

$$\left.\begin{aligned}
&N_{\xi,\xi} + \frac{N_{\xi\theta,\theta}}{R} - \nu h\, u_{,tt} = 0 \\[2ex]
&\frac{N_{\theta,\theta}}{R} + N_{\xi\theta,\xi} + \underline{\frac{M_{\theta,\theta}}{R^2} + \frac{2M_{\xi\theta,\xi}}{R}} - \nu h\, v_{,tt} = 0 \\[2ex]
&M_{\xi,\xi\xi} + \frac{2M_{\xi\theta,\xi\theta}}{R} + \frac{M_{\theta,\theta\theta}}{R^2} - \frac{N_\xi}{R_\xi} - \frac{N_\theta}{R} + \overline{N}_\xi\, w_{,\xi\xi} + \frac{\overline{N}_\theta}{R^2}\, w_{,\theta\theta} \\
&\qquad - \nu h\, w_{,tt} = 0
\end{aligned}\right\} \tag{70}$$

where the classical assumption $\overline{w}_{,\xi} = 0$ has been made and where it is assumed that the prestress loading is such that $\overline{N}_{\xi\theta} = 0$.

The corresponding boundary conditions are

$$\left.\begin{aligned}
&N_\xi = 0 \quad \text{or} \quad u = 0 \\[1ex]
&N_{\xi\theta} + \underline{\frac{2M_{\xi\theta}}{R}} = 0 \quad \text{or} \quad v = 0 \\[1ex]
&M_{\xi,\xi} + \frac{2M_{\xi\theta,\theta}}{R} + \overline{N}_\xi\, w_{,\xi} = 0 \quad \text{or} \quad w = 0 \\[1ex]
&M_\xi = 0 \quad \text{or} \quad w_{,\xi} = 0
\end{aligned}\right\} \tag{71}$$

The underlined terms would be omitted for a Donnell type approximation. The corresponding terms which would be omitted as a result of this approximation are similarly underlined in all subsequent calculations. If these terms were omitted, the system of equations would reduce to those given in reference 11. Equations (70) and (71) parallel equations (13) and (14) in Chapter VII.

The vibratory stress and moment resultants are found from equations (62), (65), (68), and (69). With the prestress deformations neglected, the resultants become

$$\left.\begin{aligned}
N_\xi &= B\left[u,_\xi + \frac{w}{R_\xi} + \mu\left(\frac{v,_\theta}{R} + \frac{w}{R}\right)\right] \\
N_\theta &= B\left[\frac{v,_\theta}{R} + \frac{w}{R} + \mu\left(u,_\xi + \frac{w}{R_\xi}\right)\right] \\
N_{\xi\theta} &= \frac{B(1-\mu)}{2}\left(\frac{u,_\theta}{R} + v,_\xi\right) \\
M_\xi &= -D\left[w,_{\xi\xi} + \frac{\mu}{R^2}\left(w,_\theta - \underline{v}\right),_\theta\right] \\
M_\theta &= -D\left[\frac{1}{R^2}\left(w,_\theta - \underline{v}\right),_\theta + \mu\, w,_{\xi\xi}\right] \\
M_{\xi\theta} &= \frac{-D(1-\mu)}{R}\left(w,_\theta - \underline{v}\right),_\xi
\end{aligned}\right\} \quad (72)$$

With the aid of equations (72), equations (70) are written in terms of displacements as

$$\left.\begin{aligned}
&u,_{\xi\xi} + \left(\frac{1-\mu}{2}\right)\frac{u,_{\theta\theta}}{R^2} + \left(\frac{1+\mu}{2}\right)\frac{v,_{\theta\xi}}{R} + \left(\frac{1}{R_\xi} + \frac{\mu}{R}\right) w,_\xi - \frac{\nu h}{B} u,_{tt} = 0 \\
&\left(\frac{1+\mu}{2}\right)\frac{u,_{\xi\theta}}{R} + \frac{1}{R^2}\left(1 + \underline{\frac{\lambda^2}{12}}\right) v,_{\theta\theta} + \left(\frac{1-\mu}{2}\right)\left(1 + \underline{\frac{\lambda^2}{3}}\right) v,_{\xi\xi} \\
&\qquad + \left(\frac{1}{R} + \frac{\mu}{R_\xi}\right)\frac{w,_\theta}{R} \underline{- \frac{\lambda^2}{12}\frac{w,_{\theta\theta\theta}}{R^2} - \frac{(2-\mu)\lambda^2}{12} w,_{\xi\xi\theta}} \\
&\qquad - \frac{\nu h}{B} v,_{tt} = 0 \\
&\left(\frac{1}{R_\xi} + \frac{\mu}{R}\right) u,_\xi + \left(\frac{1}{R} + \frac{\mu}{R_\xi}\right)\frac{v,_\theta}{R} \underline{- \frac{\lambda^2}{12 R^2} v,_{\theta\theta\theta} - \frac{\lambda^2(2-\mu)}{12} v,_{\xi\xi\theta}} \\
&\quad + \frac{\lambda^2}{12 R^2}\left(R^4 w,_{\xi\xi\xi\xi} + 2R^2 w,_{\xi\xi\theta\theta} + w,_{\theta\theta\theta\theta}\right) + \left(\frac{1}{{R_\xi}^2} + \frac{2\mu}{R_\xi R}\right. \\
&\quad \left. + \frac{1}{R^2}\right) w - \frac{\overline{N}_\xi}{B} w,_{\xi\xi} - \frac{\overline{N}_\theta}{B}\frac{w,_{\theta\theta}}{R^2} + \frac{\nu h}{B} w,_{tt} = 0
\end{aligned}\right\} \quad (73a)$$

where now $\lambda = \frac{h}{R}$. Similarily, the corresponding boundary conditions at ξ = constant are

$$\left.\begin{aligned}
&u,_\xi + \frac{w}{R_\xi} + \mu\left(\frac{v,_\theta}{R} + \frac{w}{R}\right) = 0 \quad \text{or} \quad u = 0 \\
&\frac{u,_\theta}{R} + \left(1 + \underline{\frac{\lambda^2}{3}}\right) v,_\xi - \underline{\frac{\lambda^2}{3}} w,_{\theta\xi} = 0 \quad \text{or} \quad v = 0
\end{aligned}\right\}$$

$$\frac{\lambda^2}{12}\left[\underline{-(2-\mu)\, v,_{\xi\theta}} + R^2\, w,_{\xi\xi\xi} + (2-\mu)\, w,_{\theta\theta\xi}\right] - \frac{\overline{N}_\xi}{B}\, w,_\xi = 0 \quad \text{or} \quad w = 0$$

$$w,_{\xi\xi} + \frac{\mu}{R^2}\left(w,_\theta - \frac{v}{}\right),_\theta = 0 \quad \text{or} \quad w,_\xi = 0 \qquad (73b)$$

With prestress deformations neglected, the prestress stress resultants are defined by

$$\left.\begin{aligned} \overline{N}_\xi &= \overline{N} \\ \overline{N}_\theta &= R\left(\overline{p} - \frac{\overline{N}}{R_\xi}\right) \end{aligned}\right\} \qquad (74)$$

where $\overline{N}$ is an applied meridional stress resultant and $\overline{p}$ is an applied constant lateral internal pressure. Equations (73) are constant coefficient partial differential equations. If prestress deformations had been retained, additional terms with variable coefficients would occur and approximate methods of solution similar to those used in Chapter VII would be required. The solution to the prestress equations are given in Appendix C.

B. Solution of Approximate Vibration Equations

Equations (73) are satisfied by a solution of the form

$$\left.\begin{aligned} u &= A_j\, h_j\, e^{\lambda_j x} \sin n\theta\, e^{i\omega t} \\ v &= A_j\, g_j\, e^{\lambda_j x} \cos n\theta\, e^{i\omega t} \\ w &= A_j\, e^{\lambda_j x} \sin n\theta\, e^{i\omega t} \end{aligned}\right\} \qquad (75)$$

where λ_j are the characteristic roots of equations (73) and the nondimensional variable $x = \frac{\xi}{R}$ has been introduced.

Application of the assumed solution to equations (73) yields

$$\begin{bmatrix} \lambda_j^{\,2} + f_1 & f_2\,\lambda_j & f_3\,\lambda_j \\ f_2\,\lambda_j & f_4\,\lambda_j^{\,2} + f_5 & \underline{f_6\,\lambda_j^{\,2}} + f_7 \\ f_3\,\lambda_j & \underline{f_6\,\lambda_j^{\,2}} + f_7 & f_8\,\lambda_j^{\,4} + f_9\,\lambda_j^{\,2} + f_{10} \end{bmatrix} \begin{Bmatrix} h_j\,A_j \\ g_j\,A_j \\ A_j \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{76}$$

where

$$f_1 = -\left(\frac{1-\mu}{2}\right) n^2 + \Omega^2$$

$$f_2 = -\left(\frac{1+\mu}{2}\right) n$$

$$f_3 = k_x + \mu$$

$$f_4 = -\left(\frac{1-\mu}{2}\right)\left(1 + \underline{\frac{\lambda^2}{3}}\right)$$

$$f_5 = \left(1 + \underline{\frac{\lambda^2}{12}}\right) n^2 - \Omega^2$$

$$\underline{f_6 = \frac{\lambda^2}{12}(2-\mu)n}$$

$$f_7 = -\left(1 + \mu\,k_x\right) n - \underline{\frac{\lambda^2 n^3}{12}}$$

$$f_8 = \frac{\lambda^2}{12}$$

$$f_9 = - \frac{\lambda^2 n^2}{6} - \frac{\overline{N}}{B}$$

$$f_{10} = \frac{\lambda^2 n^4}{12} + \left(k_x^2 + 2\mu k_x + 1\right) + \frac{\overline{N}_\theta}{B} n^2 - \Omega^2 \qquad (77)$$

where

$$k_x = \frac{R}{R_\xi} \quad \text{and} \quad \Omega^2 = \frac{\omega^2 R^2 \nu(1 - \mu^2)}{E}$$

The characteristic equation is found by setting the determinant of the coefficient matrix of equation (76) to zero. This yields the biquartic

$$\lambda_j^8 + a_6 \lambda_j^6 + a_4 \lambda_j^4 + a_2 \lambda_j^2 + a_o = 0 \qquad (78)$$

where

$$a_o = \frac{- f_1 \left(f_7\right)^2 + f_1 f_5 f_{10}}{f_4 f_8}$$

$$a_2 = \frac{f_3\left(2f_2f_7 - f_3f_5\right) - 2 f_1f_6f_7 - \left(f_7\right)^2 + f_1f_5f_9 + f_{10}\left[f_5 + f_1f_4 - \left(f_2\right)^2\right]}{f_4 f_8}$$

$$(79)$$

$$a_4 = \frac{f_3 \left(2f_2f_6 - f_3f_4\right) - f_6\left(2f_7 + f_1f_6\right) + f_1f_5f_8 + f_9 \left[f_5 + f_1f_4 - \left(f_2\right)^2\right] + f_4f_{10}}{f_4 f_8}$$

$$a_6 = \frac{-\left(f_6\right)^2 + f_8 \left[f_5 + f_1 f_4 - \left(f_2\right)^2\right] + f_4 f_9}{f_4 f_8}$$

The amplitude coefficients h_j and g_j may be found from any two of the equations (26). From the last two equations of equations (26), they become

$$\left.\begin{aligned} h_j &= \frac{-\left(f_6 \lambda_j^2 + f_7\right)^2 + \left(f_8 \lambda_j^4 + f_9 \lambda_j^2 + f_{10}\right)\left(f_4 \lambda_j^2 + f_5\right)}{f_2 \lambda_j \left(f_6 \lambda_j^2 + f_7\right) - f_3 \lambda_j \left(f_4 \lambda_j^2 + f_5\right)} \\ g_j &= \frac{-f_2 \left(f_8 \lambda_j^4 + f_9 \lambda_j^2 + f_{10}\right) + f_3 \left(f_6 \lambda_j^2 + f_7\right)}{f_2 \left(f_6 \lambda_j^2 + f_7\right) - f_3 \left(f_4 \lambda_j^2 + f_5\right)} \end{aligned}\right\} \quad (80)$$

In general λ_j can be complex. Since the displacements u, v, and w are real then A_j, h_j, and g_j are complex whenever λ_j is complex. Since the complex roots λ_j occur in complex conjugate pairs when complex roots do occur, it can be shown that the corresponding quantities A_j, h_j, and g_j will also occur in complex conjugate pairs. For axisymmetric vibrations (n = o) the terms g_j vanish and the circumferential equilibrium equation uncouples from the remaining equations so that the procedure must be modified (see Appendix D).

The complete solution for u, v, and w is found by summing the contributions from each characteristic root thus

$$\left.\begin{aligned} u &= U(x) \sin n\theta \, e^{i\omega t} \\ v &= V(x) \cos n\theta \, e^{i\omega t} \\ w &= W(x) \sin n\theta \, e^{i\omega t} \end{aligned}\right\} \quad (81)$$

where

$$U(x) = \sum_{j=1}^{8} A_j \; h_j \; e^{\lambda_j x}$$

$$V(x) = \sum_{j=1}^{8} A_j \; g_j \; e^{\lambda_j x} \qquad (82)$$

$$W(x) = \sum_{j=1}^{8} A_j \; e^{\lambda_j x}$$

The stress resultants are found from equations (72) to be

$$N_\xi = \frac{B}{R} n_x(x) \sin n\theta \; e^{i\omega t}$$

$$N_\theta = \frac{B}{R} n_\theta(x) \sin n\theta \; e^{i\omega t}$$

$$N_{\xi\theta} = \frac{B}{R} n_{x\theta}(x) \cos n\theta \; e^{i\omega t}$$

$$M_\xi = \frac{D}{R^2} m_x(x) \sin n\theta \; e^{i\omega t} \qquad (83)$$

$$M_\theta = \frac{D}{R^2} m_\theta(x) \sin n\theta \; e^{i\omega t}$$

$$M_{\xi\theta} = \frac{D}{R^2} m_{x\theta}(x) \cos n\theta \; e^{i\omega t}$$

where

$$
\left.
\begin{aligned}
n_x &= \sum_{j=1}^{8} A_j \left[\mu\left(1 - ng_j\right) + k_x + h_j\ \lambda_j\right] e^{\lambda_j x} \\
n_\theta &= \sum_{j=1}^{8} A_j \left[1 - ng_j + \mu k_x + \mu h_j\ \lambda_j\right] e^{\lambda_j x} \\
n_{x\theta} &= \sum_{j=1}^{8} \left(\frac{1 - \mu}{2}\right) A_j \left[nh_j + g_j\ \lambda_j\right] e^{\lambda_j x} \\
m_x &= \sum_{j=1}^{8} A_j \left[n\mu\ \left(n - \underline{g_j}\right) - \lambda_j^{\,2}\right] e^{\lambda_j x} \\
m_\theta &= \sum_{j=1}^{8} A_j \left[n\ \left(n - \underline{g_j}\ \right) - \mu\ \lambda_j^{\,2}\right] e^{\lambda_j x} \\
m_{x\theta} &= \sum_{j=1}^{8} (1 - \mu)\ A_j \left[\left(\underline{g_j} - n\right) \lambda_i\right] e^{\lambda_j x}
\end{aligned}
\right\} \qquad (84)
$$

The boundary conditions may be found in terms of displacements from equations (71) with the use of equations (72) and with the solutions (81) and (82) may be written in general at two edges as

$$
\left[\Phi N + \Psi Y\right] \left\{A\right\} = \left\{0\right\} \qquad (85)
$$

where $\{A\}$ is an 8 element column matrix whose elements are A_j and where N, Y, Φ, and ψ are 8×8 matrices whose elements are given by

$$N_{1j} = \left\{\mu\left(1 - ng_j\right) + k_x + h_j \lambda_j\right\} e^{\frac{\lambda_j S}{2}}$$

$$N_{2j} = \left\{nh_j + \left[\underline{g_j + \frac{\lambda^2}{3}\left(g_j - n\right)}\right] \lambda_i\right\} e^{\frac{\lambda_j S}{2}}$$

$$N_{3j} = \left\{\left[\frac{\lambda^2}{12}(2-\mu)\, n\left(\underline{g_j} - n\right) - \frac{\overline{N}}{B}\right]\lambda_j + \frac{\lambda^2}{12}\lambda_j^3\right\} e^{\frac{\lambda_j S}{2}}$$

$$N_{4j} = \left\{\mu n\left(\underline{g_j} - n\right) + \lambda_j^2\right\} e^{\frac{\lambda_j S}{2}}$$

$$N_{5j} = \left\{\mu\left(1 - ng_j\right) + k_x + h_j \lambda_j\right\} e^{-\frac{\lambda_j S}{2}}$$

$$N_{6j} = \left\{nh_j + \left[\underline{g_j + \frac{\lambda^2}{3}\left(g_j - n\right)}\right] \lambda_j\right\} e^{-\frac{\lambda_j S}{2}}$$

$$N_{7j} = \left\{\left[\frac{\lambda^2}{12}(2 - \mu)\, n\left(\underline{g_j} - n\right) - \frac{\overline{N}}{B}\right]\lambda_j + \frac{\lambda^2}{12}\lambda_j^3\right\} e^{-\frac{\lambda_j S}{2}}$$

$$
\left.
\begin{aligned}
N_{8j} &= \left\{\mu n \left(\underline{g_j} - n\right) + \lambda_j^{2}\right\} e^{-\frac{\lambda_j S}{2}} \\
Y_{1j} &= h_j \, e^{\frac{\lambda_j S}{2}} \\
Y_{2j} &= g_j \, e^{\frac{\lambda_j S}{2}} \\
Y_{3j} &= e^{\frac{\lambda_j S}{2}} \\
Y_{4j} &= \lambda_j \, e^{\frac{\lambda_j S}{2}} \\
Y_{5j} &= h_j \, e^{-\frac{\lambda_j S}{2}} \\
Y_{6j} &= g_j \, e^{-\frac{\lambda_j S}{2}} \\
Y_{7j} &= e^{-\frac{\lambda_j S}{2}} \\
Y_{8j} &= \lambda_j \, e^{-\frac{\lambda_j S}{2}}
\end{aligned}
\right\} \tag{86}
$$

where $S = \frac{s}{R}$ is the total nondimensional axial length of the shell, and

$$
\left.
\begin{aligned}
\Phi_{lj} &= 0 \quad \text{for} \quad l \neq j \\
\Psi_{lj} &= 0 \quad \text{for} \quad l \neq j \\
\Psi_{jj} &= 1 - \Phi_{jj}
\end{aligned}
\right\} \tag{87}
$$

The elements Ψ_{jj} take on the value 1 or 0 according to the prescribed conditions. The Φ_{lj} and Ψ_{lj} matrices are used to select the prescribed boundary conditions in a similar fashion as in Chapter VII with the α and β matrices. If a stress-free condition is desired, the corresponding Φ_{jj} term is set equal to one. If a displacement variable or meridional slope is constrained such that it must vanish at the boundary, the corresponding Φ_{jj} term is set equal to zero. The Φ and Ψ matrices may be generalized to enforce linear elastic or directional constraints.

For a particular set of homogeneous boundary conditions and a particular circumferential harmonic mode number (n), the natural frequencies of the system are those contained in the frequency parameter Ω which cause the determinant of the coefficient matrix in equation (85) to vanish, that is,

$$\left| \Phi N + \Psi Y \right| = 0 \tag{88}$$

Due to the highly transcendental character of equation (88), a trial and error procedure must be used to find the natural frequencies. The procedure is exact in the sense that the frequency parameter can be found to any desired degree of accuracy.

A trial value for Ω is selected and the quantities in equations (77) and (79) are computed. The characteristic roots are determined from equation (78) and h_j and g_j are found from equations (80). The values for Φ_{jj} which give the desired boundary conditions are selected and the determinant of equation (88) is evaluated using equations (86). The value of the determinant is then compared with zero. The frequency parameter is then increased by fixed increments until the determinant of equation (88) changes sign, indicating the presence of a root. The root is located to within the accuracy desired by successive halving of increments.

The residual in equation (88) is in general a complex number, but since the characteristic roots occur in complex conjugate pairs, the determinant must have complex conjugate columns and thus its value must be either real or pure imaginary. This facilitates the automatic search procedures in the root searching operation since only one number must be corrected to zero.

The success of this method as a rapid means of obtaining solutions hinges on the investigator's ability to make good initial estimates of the frequency. To aid in the selection of an initial estimate, a closed form solution is obtained in the next section for the vibration frequencies of a shallow doubly curved shell with freely supported edges. The solutions to this system may be used to select initial trial frequencies for systems with other boundary conditions.

C. Solution of Approximate Equations for a Freely Supported Shell

A direct solution to equations (73) is available for the freely supported boundary conditions $\left(N_\xi = v = w = M_\xi = 0\right)$. Assume a solution of the form

$$\left.\begin{aligned} u &= U_{fs} \cos \frac{m\pi x}{S} \sin n\theta \; e^{i\omega t} \\ v &= V_{fs} \sin \frac{m\pi x}{S} \cos n\theta \; e^{i\omega t} \\ w &= W_{fs} \sin \frac{m\pi x}{S} \sin n\theta \; e^{i\omega t} \end{aligned}\right\} \quad (89)$$

where m is the number of meridional half waves. This solution satisfies the freely supported conditions at $x = 0$ and $x = S$. Equations (89) are substituted into equations (73) to yield a set of linear homogeneous equations. For a nontrivial solution to exist, the determinant of the coefficient matrix of the resultant set of equations must vanish. This procedure leads to the characteristic equation

$$-\Omega^6 + \overline{\Lambda}_2 \, \Omega^4 - \overline{\Lambda}_1 \, \Omega^2 + \overline{\Lambda}_0 = 0 \qquad (90)$$

where the coefficients $\overline{\Lambda}_j$ are given in Appendix A. Hence the natural frequencies of a freely supported doubly curved shell of revolution with shallow meridional curvature can be found for any specific mode by solving equation (90). This equation leads to three frequencies for each set of m,n considered since in-plane inertia terms have been retained.

D. Membrane Solution of Approximate Vibration Equations

It is of value to inspect the extreme case of zero bending stiffness ($D = 0$). The vibratory behavior in this case is associated with only the extensional properties of the shell. The membrane equations are found directly from equations (70) by deleting all moment terms, yielding

$$\left.\begin{aligned} N_{\xi,\xi} + \frac{N_{\xi\theta,\theta}}{R} &= 0 \\ \frac{N_{\theta,\theta}}{R} + N_{\xi\theta,\xi} &= 0 \\ \frac{N_\xi}{R_\xi} + \frac{N_\theta}{R} - \overline{N}_\xi\, w,_{\xi\xi} - \overline{N}_\theta \frac{w,_{\theta\theta}}{R^2} + \nu h\, w,_{tt} &= 0 \end{aligned}\right\} \quad (91)$$

where for convenience in this limiting case the in-plane inertias have been neglected in the first two equations. The first two equilibrium equations are identically satisfied by the introduction of the stress function ψ defined by

$$\left.\begin{aligned} \frac{N_\xi}{B} &= \frac{\psi,_{\theta\theta}}{R^2} = u,_\xi + \frac{w}{R_\xi} + \mu\left(\frac{v,_\theta}{R} + \frac{w}{R}\right) \\ \frac{N_\theta}{B} &= \psi,_{\xi\xi} = \frac{v,_\theta}{R} + \frac{w}{R} + \mu\left(u,_\xi + \frac{w}{R_\xi}\right) \\ \frac{N_{\xi\theta}}{B} &= -\frac{\psi,_{\xi\theta}}{R} = \left(\frac{1-\mu}{2}\right)\left(v,_\xi + \frac{u,_\theta}{R}\right) \end{aligned}\right\} \quad (92)$$

In terms of the nondimensional variable x, the third equilibrium equation becomes

$$\frac{k_x \psi,_{\theta\theta}}{R} + \frac{\psi,_{xx}}{R} - \frac{\overline{N}_x}{B} w,_{xx} - \frac{\overline{N}_\theta}{B} w,_{\theta\theta} + \frac{\nu h R^2}{B} w,_{tt} = 0 \tag{93}$$

It can be seen from equation (93) that for a negative Gaussian curvature $\left(k_x \text{ negative}\right)$ unstressed membrane shell, the governing equation has a hyperbolic character. The equation for the cylinder has a parabolic character and for the positive Gaussian curvature shell has an elliptic character.

1. Solution of membrane vibration equations for a freely supported shell.- Equation (93) is satisfied by

$$\left.\begin{aligned} \psi &= \psi_m \sin\frac{m\pi x}{S}\cos n\theta\; e^{i\omega t} \\ w &= W_m \sin\frac{m\pi x}{S}\cos n\theta\; e^{i\omega t} \end{aligned}\right\} \tag{94}$$

where it follows from (92) that

$$\left.\begin{aligned} u &= U_m \cos\frac{m\pi x}{S}\cos n\theta\; e^{i\omega t} \\ v &= V_m \sin\frac{m\pi x}{S}\sin n\theta\; e^{i\omega t} \end{aligned}\right\} \tag{95}$$

This solution satisfies the freely supported boundary conditions.

With the application of equations (94) and (95) to equations (92), equations (92) may be written matrix form as

$$\begin{bmatrix} -\frac{m\pi}{S} & \mu n & k_x + \mu \\ -\mu\frac{m\pi}{S} & n & 1 + \mu k_x \\ -n & \frac{m\pi}{S} & 0 \end{bmatrix} \begin{Bmatrix} U_m \\ V_m \\ W_m \end{Bmatrix}_{mem} = \frac{\psi_m}{R} \begin{Bmatrix} -n^2 \\ -\left(\frac{m\pi}{S}\right)^2 \\ \frac{2n\left(\frac{m\pi}{S}\right)}{1-\mu} \end{Bmatrix} \tag{96}$$

where from Cramer's rule

$$W_{mem} = -\left(\frac{\psi_{mem}}{R}\right)\frac{\left[\left(\frac{m\pi}{S}\right)^2 + n^2\right]^2}{\left(1-\mu^2\right)\left[\left(\frac{m\pi}{S}\right)^2 + n^2 k_x\right]} \tag{97}$$

Substitution of equations (94), (95), and (97) into equation (93) yields the following membrane frequency equation

$$\left(\Omega_{mem}\right)^2 = \frac{\left(1-\mu^2\right)\left(k_x + \beta^2\right)^2}{\left(1+\beta^2\right)^2} + \left(\frac{\overline{N}_\theta}{B} + \frac{\overline{N}_x}{B}\beta^2\right) n^2 \tag{98}$$

where $\beta = \frac{m\pi}{nS}$ is the ratio of the circumferential to the axial wavelength of the vibration mode. The frequency determined from equation (98) vanishes for an initially unstressed membrane for meridional curvatures given by

$$k_x = -\beta^2 \tag{99}$$

Thus, for certain mode shapes the vibration of a negative curvature membrane shell with freely supported edges is not sustained by the membrane stiffness of the shell.

2. <u>Characteristic roots from membrane theory</u>.- The membrane characteristic equation is found from the exponential solution form:

$$\left.\begin{aligned}
\psi &= \psi_j^{*}\ e^{\lambda_j x}\ \cos n\theta\ e^{i\omega t} \\
w &= W_j^{*}\ e^{\lambda_j x}\ \cos n\theta\ e^{i\omega t} \\
u &= U_j^{*}\ e^{\lambda_j x}\ \cos n\theta\ e^{i\omega t} \\
v &= V_j^{*}\ e^{\lambda_j x}\ \sin n\theta\ e^{i\omega t}
\end{aligned}\right\} \tag{100}$$

Substitution of this solution into equations (92) yields

$$W_j^{*} = \frac{\psi_j^{*}}{R}\ \frac{\left(\lambda_j^{2} - n^{2}\right)^{2}}{\left(1 - \mu^{2}\right)\left(k_x\, n^{2} - \lambda_j^{2}\right)} \tag{101}$$

Substitution of equations (100) and (101) into equations (93) with prestress terms deleted yields the biquadratic characteristic equation

$$\left[\left(1 - \mu^{2}\right) - \Omega^{2}\right] \lambda_j^{4} + 2n^{2}\left[\Omega^{2} - k_x\left(1 - \mu^{2}\right)\right] \lambda_j^{2} + n^{4}\left[\left(1 - \mu^{2}\right) k_x - \Omega^{2}\right] = 0 \tag{102}$$

This equation may also be found from equation (78) by deleting in-plane inertia terms and products of λ^2 in the functions given by equations (77). Solution of equation (102), yields the following membrane characteristic roots

$$\lambda_j{}^2{}_{\text{membrane}} = \frac{n^2}{\left(1 - \mu^2\right) - \Omega^2}\left[k_x\left(1 - \mu^2\right) - \Omega^2 \pm \Omega\left(1 - k_x\right)\sqrt{1 - \mu^2}\right] \tag{103}$$

E. Pure Bending Solution of Approximate Vibration Equations

The extreme case of pure bending is considered in this section. The vibratory behavior in this case is purely flexural and the middle surface extensional strains are assumed negligible. By assuming that the extensional stiffness B remains finite, the pure bending equations are found directly from equations (70) by deleting all vibratory stress resultants. The displacement formulation may be written down directly from equations (73a) by retaining all terms containing λ^2, v and w inertia terms, and membrane prestress terms. The following equations result:

$$\left.\begin{aligned}
&\underline{\frac{\lambda^2}{12} v,_{\theta\theta} + \left(\frac{1 - \mu}{6}\right)\lambda^2 v,_{xx} - \frac{\lambda^2}{12} w,_{\theta\theta\theta} - \frac{(2 - \mu)}{12}\lambda^2 w,_{xx\theta}} \\
&\qquad - \frac{\nu R^2\left(1 - \mu^2\right)}{E} v,_{tt} = 0 \\
&\underline{- \frac{\lambda^2}{12} v,_{\theta\theta\theta} - \frac{\lambda^2(2 - \mu)}{12} v,_{xx\theta}} + \frac{\lambda^2}{12}\nabla^4 w - \frac{\overline{N}_\xi}{B} w,_{xx} - \frac{\overline{N}_\theta}{B} w,_{\theta\theta} \\
&\qquad + \frac{\nu R^2\left(1 - \mu^2\right)}{E} w,_{tt} = 0
\end{aligned}\right\} \tag{104}$$

The meridional displacement u and the meridional curvature k_x do not appear in equations (104), thus in the approximate formulation the pure bending state is independent of u and k_x.

The freely supported boundary conditions are satisfied by

$$\left.\begin{aligned} w &= W_{pb} \sin \frac{m\pi x}{S} \sin n\theta \ e^{i\omega_{pb} t} \\ v &= V_{pb} \sin \frac{m\pi x}{S} \cos n\theta \ e^{i\omega_{pb} t} \end{aligned}\right\} \quad (105)$$

If the Donnell approximation was made, the v inertia term would vanish and the natural frequency of pure bending would be equivalent to the natural frequency of a simply supported rectangular plate and would be given by

$$\left(\Omega^2_{pb}\right)_{Donnell} = \frac{\lambda^2}{12}\left(\beta^2 + 1\right)^2 n^4 + \left(\frac{\overline{N}_x}{B}\beta^2 + \frac{\overline{N}_\theta}{B}\right) n^2 \qquad n \neq 0 \quad (106)$$

Substitution of equations (105) into equations (104) yields the following pure bending frequency equation

$$\Omega^2_{pb} = \frac{a_{11} + a_{22}}{2} \pm \sqrt{\left(\frac{a_{11} + a_{22}}{2}\right)^2 - \left[a_{11}\, a_{22} - \left(a_{12}\right)^2\right]} \quad (107)$$

where

$$\left.\begin{aligned} a_{11} &= \frac{\lambda^2}{12}\left[1 + 2(1 + \mu)\, \beta^2\right] n^2 \\ a_{12} &= -\frac{\lambda^2}{12}\left[1 + (2 - \mu)\, \beta^2\right] n^3 \\ a_{22} &= \left(\Omega^2_{pb}\right)_{Donnell} \end{aligned}\right\} \quad (108)$$

and where again

$$\beta = \frac{m\pi}{nS} \quad .$$

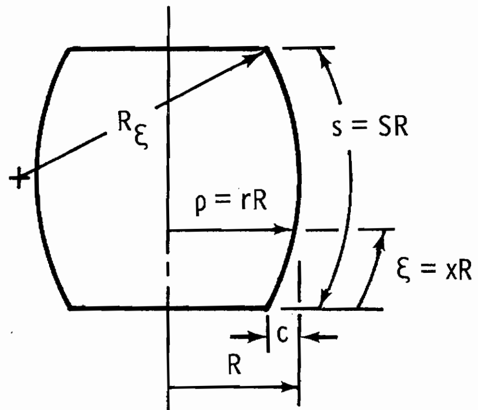
Equation (107) gives two pure bending frequencies one of which is close to zero. This frequency is associated with a predominant in-plane v displacement mode and is usually several orders of magnitude higher than the frequency associated with a w predominant mode. These frequencies being associated primarily with in-plane motion are governed almost entirely by membrane action; thus, the pure bending theory cannot give a reasonable estimate of these frequencies since its stiffness contribution is negligible, so that frequencies close to zero are expected and can be ignored. Furthermore for $n > 3$ the pure bending frequencies found from Donnell theory which are associated with the w inertia term only (equation (106)) will closely approximate those found from equation (107).

IX. RESULTS AND DISCUSSION

Vibration calculations have been made for some simple shell of revolution configurations to determine the accuracy of the approximate analysis developed in this paper and to determine the influence of meridional curvature on vibration modes and frequencies. The accuracy is assessed by comparing the results of the approximate analysis with results obtained from the more accurate numerical analysis presented in Chapter VII. The influence of meridional curvature is assessed by performing a parameter study for shells with differing meridional curvatures as the length-radius and thickness-radius ratios are held constant. Results of additional calculations based on membrane theory and pure bending theory clarify the relative roles of membrane action and bending action and their dependence on modal wavelength ratios during the vibration of these shells. The results obtained are for a specific class of shell configurations; however, the effects determined many provide insight into the general behavior of more complex doubly-curved shells of revolution.

A. Shell Configurations Investigated

The class of shells of revolution with constant shallow meridional curvature was elected for investigation since it has the simplest configurations for fulfilling the parameter study of interest. The equations defining these shells are given in Figure 3 and are used whenever the more accurate numerical method of Chapter VII is

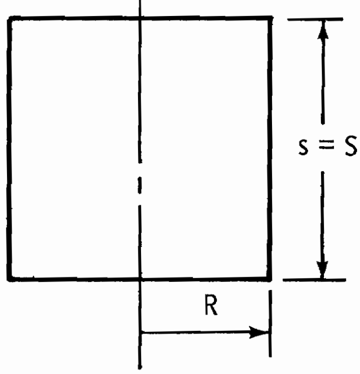


Positive Gaussian Curvature Shell $R_\xi > 0$

$$k_x = \frac{R}{R_\xi}$$

$$r = 1 + \frac{1}{k_x}\left\{\cos\left[\left(\frac{S}{2} - x\right)k_x\right] - 1\right\}$$

$$k_\theta = \frac{k_x \cos\left[\left(\frac{S}{2} - x\right)k_x\right]}{k_x + \cos\left[\left(\frac{S}{2} - x\right)k_x\right] - 1}$$

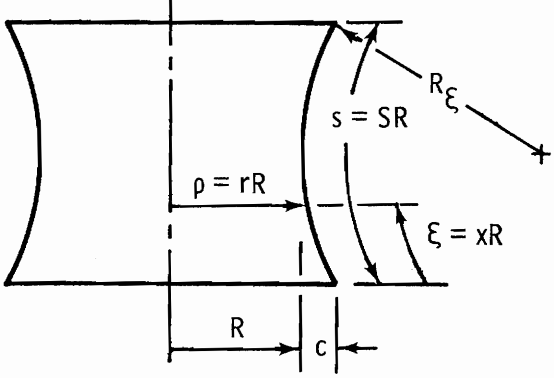


Zero Gaussian Curvature Shell (Cylinder)

$$k_x = 0$$

$$r = 1$$

$$k_\theta = 1$$



Negative Gaussian Curvature Shell $R_\xi < 0$

Equations same as positive shell with

R_ξ taken as a negative number

Figure 3.- Geometry of class of shells investigated.

employed. Since the approximate theory is limited by shallowness in the meridional direction, it is sufficient to represent the meridional curve of this class of shells by equation (58) when using the approximate theory (see reference 20). The effect of curvature is determined for both the approximate solutions presented here and the more accurate numerical method of Chapter VII by varying k_x while holding the length-radius and thickness-radius ratios constant. The mass distribution varies slightly as k_x is changed, but if meridional shallowness is maintained, the effect of change in mass distribution on the natural frequencies is negligible; and any change in natural frequencies can be attributed solely to the change in Gaussian curvature of the shell and the resultant shell stiffness change.

When the numerical procedure of Chapter VII is employed, 200 equally spaced intervals along the meridian are used. In all calculations, Poisson's ratio is taken to be equal to 0.3 (i.e. $\mu = 0.3$).

The central rise of the shell meridian (c in figure 3) is given by

$$c = R_\xi \left(1 - \cos \frac{s}{2R_\xi}\right) \tag{109}$$

where s is the total meridional length. The shell meridian can be approximated by the first term of a series expansion of the right-hand side of equation (109). The percent ratio of the central meridional rise to length can thus be represented by

$$\tau = \frac{Sk_x}{8} (100) \tag{110}$$

This quantity is introduced as an auxiliary parameter which is used to give an indication of the degree of shallowness of the meridian.

B. Accuracy of Solutions

Before a parametric study is performed using the approximate theory, a level of confidence in the accuracy of its solutions must be established. To make an assessment of the errors involved when using the approximate theory, the results obtained with this theory are compared with corresponding results obtained with the more accurate numerical method of Chapter VII for several specific doubly curved unstressed shells in figures 4, 5, and 6. In these figures, the lowest frequencies are plotted for successive circumferential mode numbers for both positive and negative Gaussian curvature shells. The cylinder results, found using the approximate method, are repeated on each figure for purposes of comparison. Since the only approximation made for the cylinder results was the neglect of small nonlinear terms before linearization, only terms related to prestress would be effected; thus, the unstressed cylinder results presented in these figures are exact.

For the slightly curved shells ($\tau = \pm 1.87$) of figure 4, the approximate theory is shown to be quite accurate with a larger percentage error evident for the negative Gaussian curvature shells. The accuracy diminishes as the curvature of the shell meridian increases as is shown in figures 5 ($\tau = \pm 3.75$) and 6 ($\tau = \pm 5.62$). The positive curvature shell results never differ from the more accurate results by more than 8 percent. The negative curvature shell results have larger errors but still indicate the general character of the actual solution.

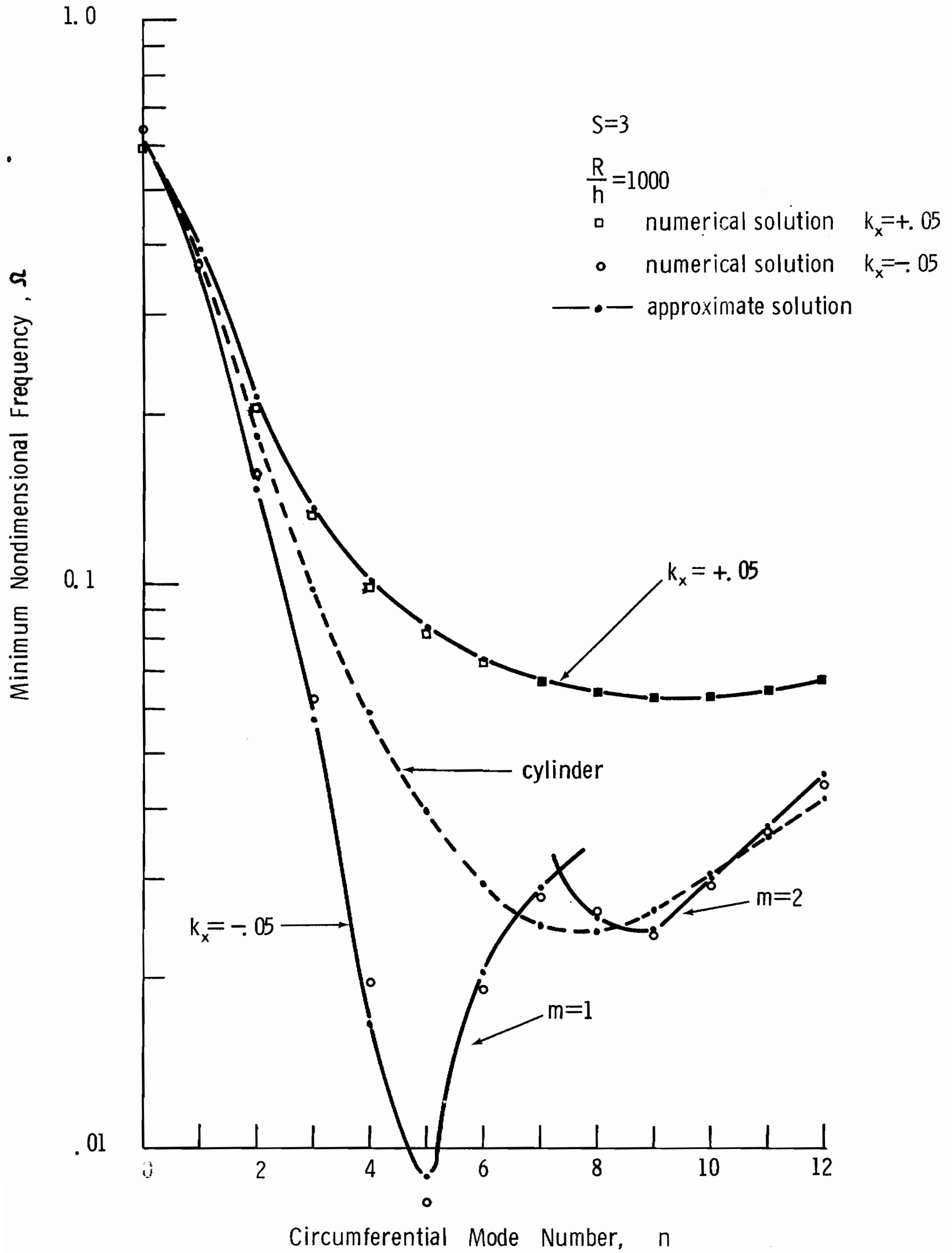


Figure 4.- Comparison of minimum frequencies from approximate analysis with those from numerical analysis for freely supported shells $\left(k_x = \pm\, 0.05;\ S = 3;\ \frac{R}{h} = 1000\right)$.

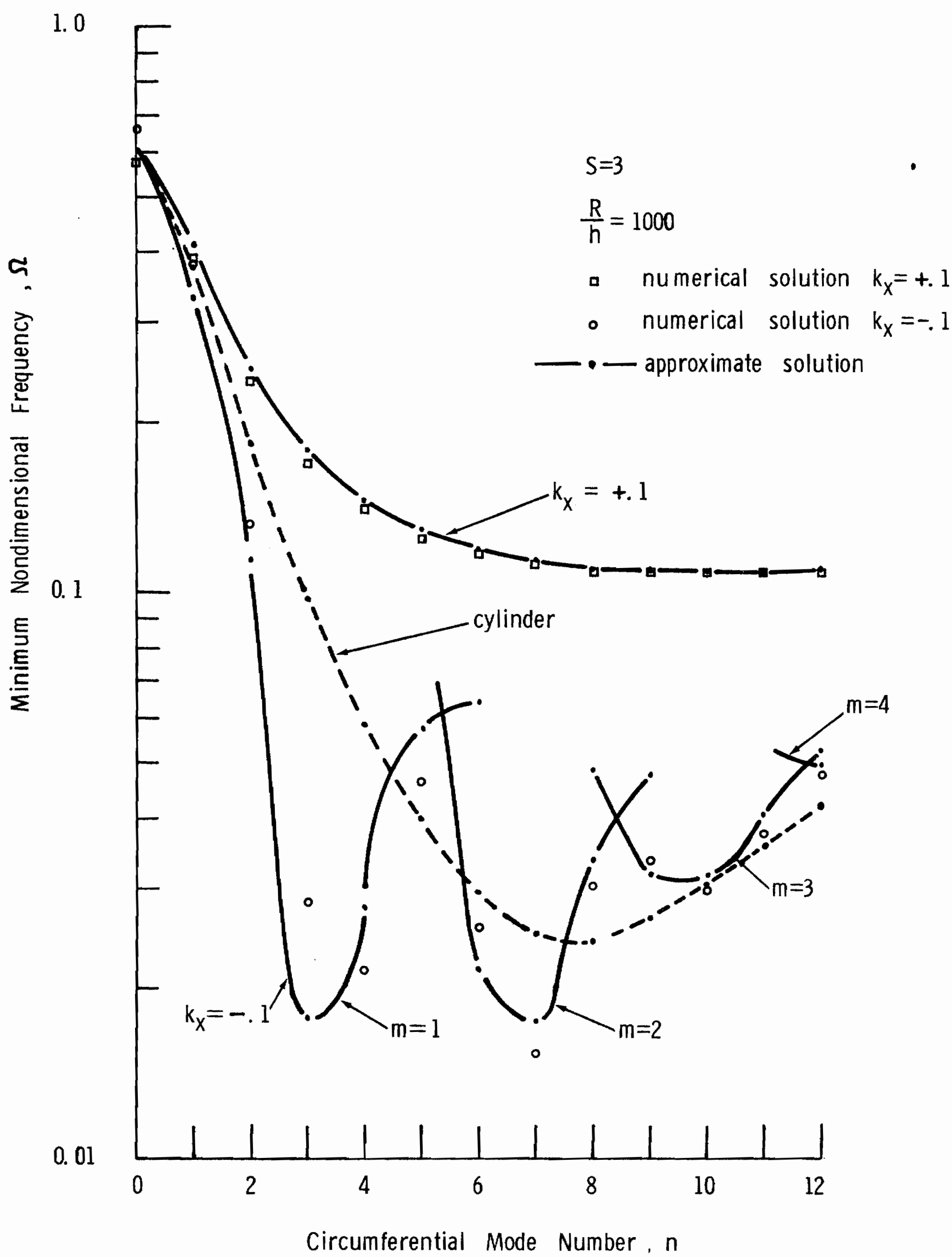


Figure 5.- Comparison of minimum frequencies from approximate analysis with those from numerical analysis for freely supported shells $\left(k_x = \pm 0.1;\ S = 3;\ \frac{R}{h} = 1000\right)$.

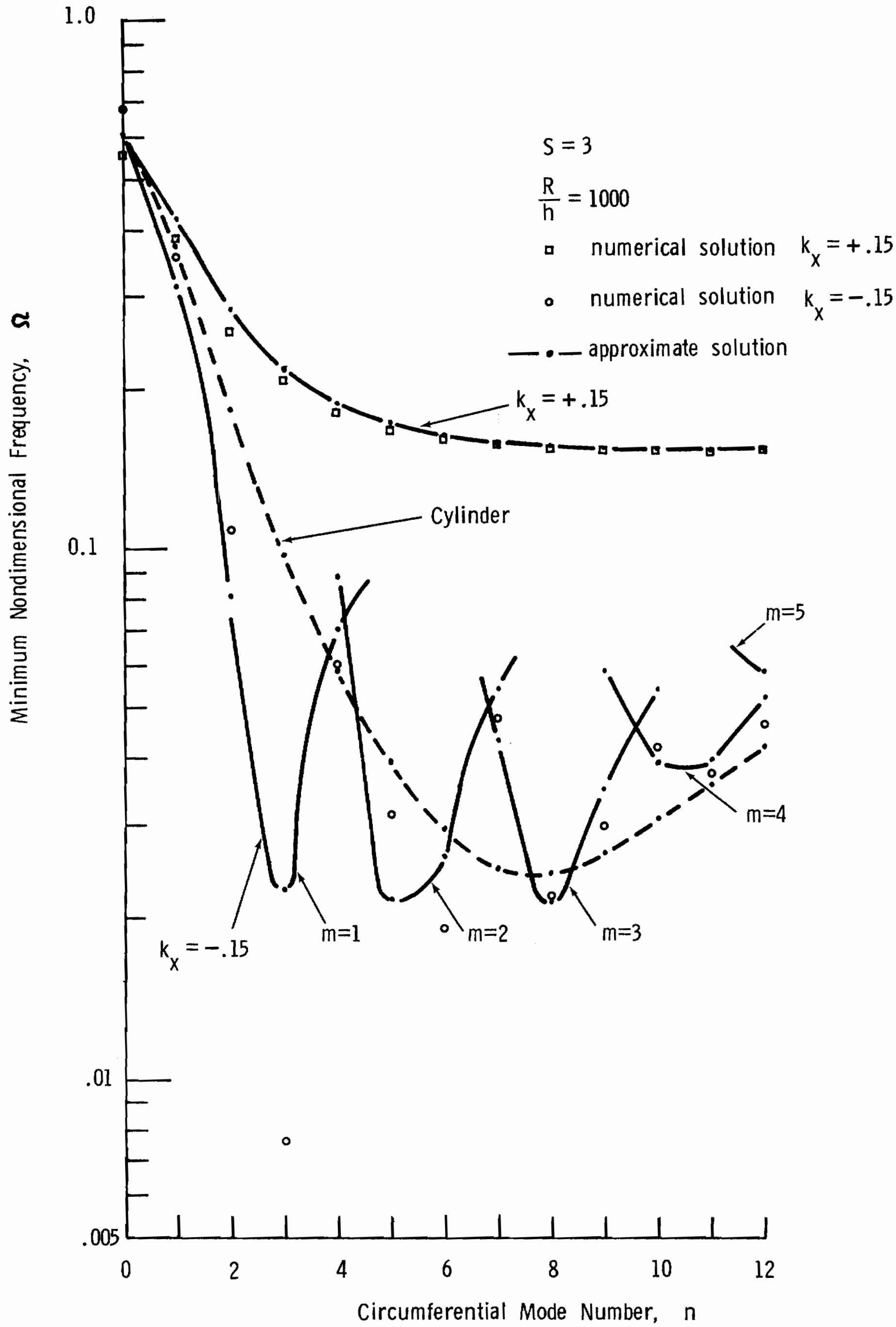


Figure 6.- Comparison of minimum frequencies from approximate analysis with those from numerical analysis for freely supported shells $\left(k_x = \pm\, 0.15;\ S = 3;\ \frac{R}{h} = 1000\right)$.

The lowest natural frequency for the axisymmetric vibration mode $n = 0$ is associated with a pure torsional mode and is independent of the meridional curvature in the approximate theory. The more accurate results, however, indicate that this frequency is dependent on curvature to a small degree as can be seen by inspection of figures 4, 5, and 6.

The approximate procedure has a modal solution of sinusoidal form along the meridian (see equation (89)) whereas the mode shape in the numerical procedure is calculated once the frequency is determined and need not necessarily be sinusoidal. Plots of the minimum frequency normal displacement mode (w) determined using the numerical procedure are given in figure 7 for particular values of n for the negative curvature shells of figures 4, 5, and 6. In each case, the number of axial half-waves (m) determined by the approximate theory agrees with the modes given in figure 7. As the curvature increases; the mode shape begins to deviate from sinusoidal form, thus the sinusoidal modal solution of the approximate theory becomes a less accurate representation of the true modal configuration.

A comparison of the results found by the approximate theory and the numerical procedure for a long negative Gaussian curvature shell with a rise-length ratio greater than 6 percent is given in figure 8. Large percentage errors are evident, with the approximate theory in general overestimating the lowest natural frequencies. The corresponding normal displacement modes given in figure 9 are found using the numerical procedure and exhibit noticeable deviations from the assumed sinusoidal form of the approximate solution especially as n increases. Nevertheless the approximate solution

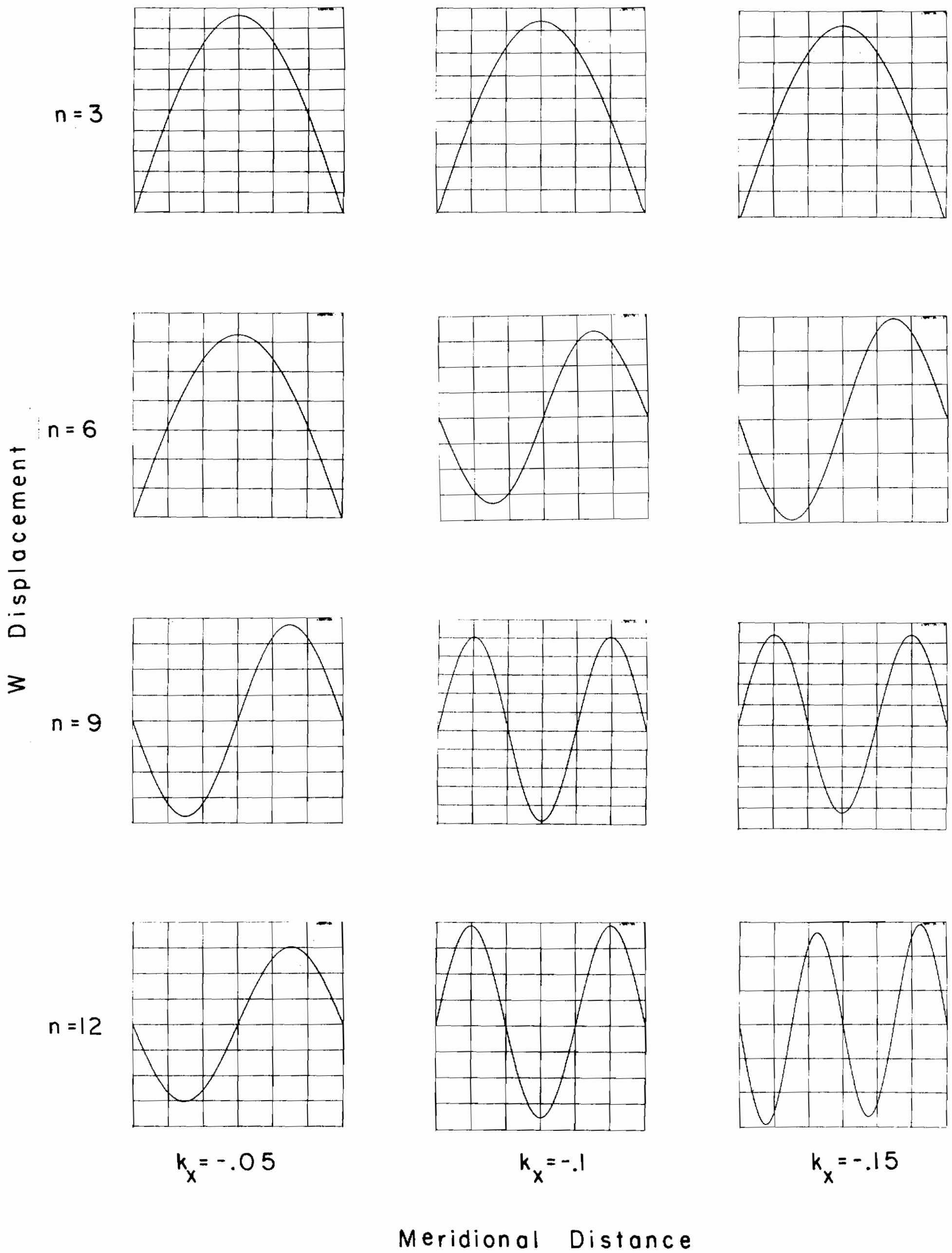


Figure 7.- Comparison of w meridional modes for several n for successive degrees of curvature, $\left(k_x = -0.01, -0.1, -0.15;\ S = 3;\ \frac{R}{h} = 1000\right)$.

(All calculations based on numerical analysis)

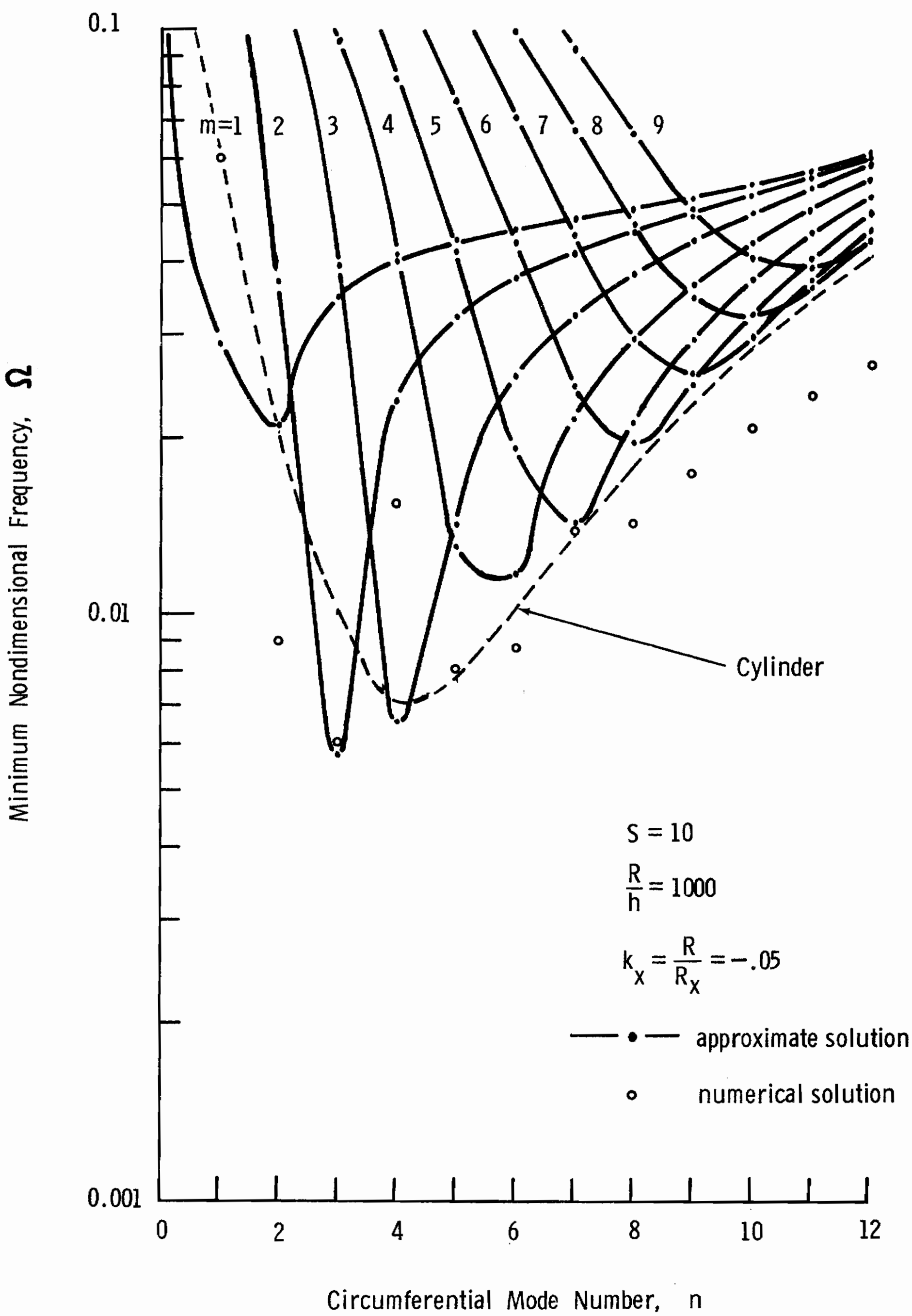


Figure 8.- Comparison of minimum frequencies from approximate analysis with those from numerical analysis for freely supported shells $\left(k_x = \pm 0.05,\ S = 10;\ \frac{R}{h} = 1000\right)$.

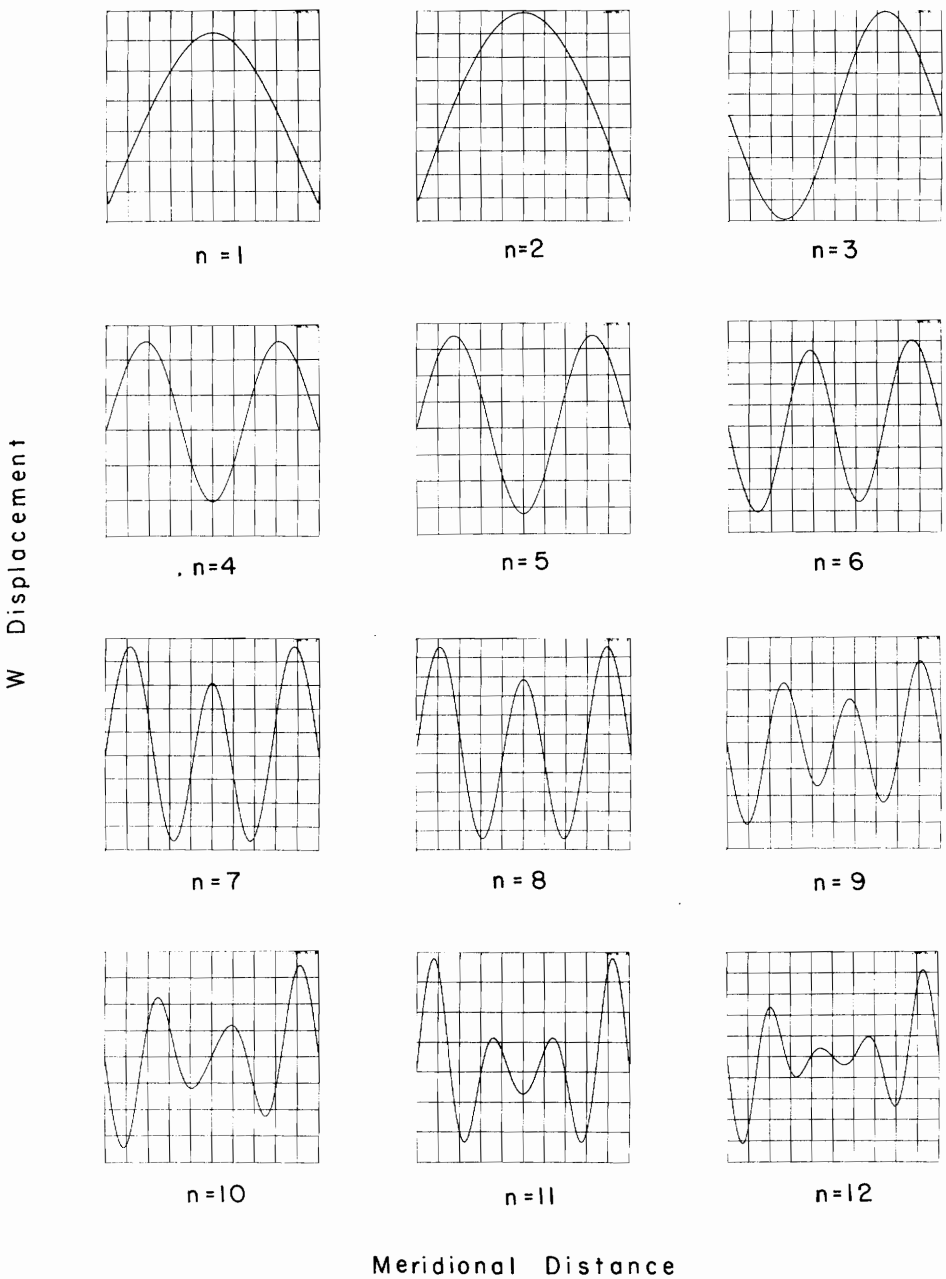


Figure 9.- Meridional modes of w for several successive n $\left(k_x = -0.05;\ S = 10;\ \frac{R}{h} = 1000\right)$.

(All calculations based on numerical analysis) still exhibits the general shell behavior with respect to both the modes and frequencies. The radial displacements are larger near the ends of the shell as these areas behave as regions of low stiffness due to the larger crossectional radius. Since the approximate theory can only admit a sinusiodal solution, it cannot reflect this property of the shell. In positive Gaussian curvature shells, the region of low stiffness is in the center of the shell meridian and the modes calculated by numerical procedure (not shown) would generally yield larger radial displacements in this area.

Based on the comparisons made in this section, it is believed that the approximate theory gives reasonable estimates of frequencies and predicts trends adequately for values of τ between ± 5 percent. All analyses in the remainder of this report will be limited to shells contained within this range.

C. Effects of Meridional Curvature

The natural frequencies of a shell structure are closely related to the effective stiffness of the structure in the sense that as the effective stiffness increases, the fundamental frequency increases. The positive curvature shells exhibit a strong stiffening character with the lowest natural frequencies increasing as the curvature of the meridian increases. For example, the fundamental frequency of the positive Gaussian curvature shell in figure 4 is 150 percent higher than that of the cylinder, while in figure 6, the lowest natural frequency is over 300 percent higher. On the other hand, negative curvature shells exhibit rapid losses in effective stiffness and the minimum frequencies seem to be highly dependent on the circumferential mode numbers.

The lowest frequencies for each circumferential mode number always occur at the simplest meridional mode $m = 1$ for positive and zero (cylinders) Gaussian curvature shells. However, the negative curvature shells exhibit lowest frequencies at higher meridional modes in the higher circumferential mode number range. This is well demonstrated in figure 8 where the fundamental frequency occurs for $m = 2$ and higher meridional modes are associated with lowest frequencies for $n > 3$. This figure shows an interesting phenomenon in the spacing of the frequencies. As n increases, the spacing between frequencies associated with successive meridional mode numbers decreases. This behavior suggests that experimental resolution of individual natural modes would be difficult to achieve at the higher n range.

The results of figures 4, 5, and 6 do not show a well defined trend in the behavior of negative Gaussian curvature shells, thus a more extensive parameter study is necessary. The fundamental frequencies and modes of a series of freely supported shells with a length-radius ratio of three ($S = 3$) and with the rise-length percent ratio varying from -5 percent to +5 percent are presented in figure 10. The fundamental frequencies for the shells of figures 4 and 5 are located by the vertical dashed lines. The horizontal dashed lines will be discussed later. As the positive curvature increases, the

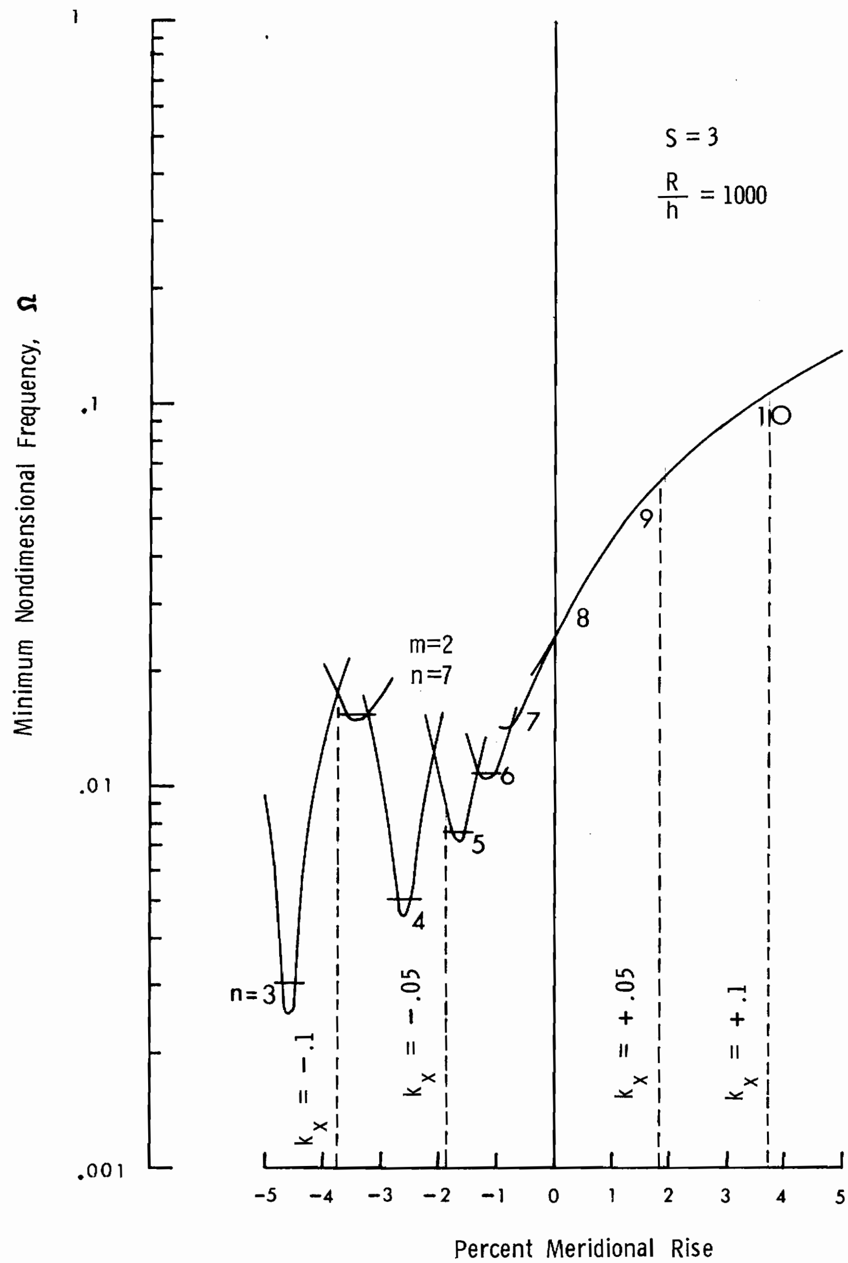


Figure 10.- Effect of meridional curvature on the fundamental frequencies of freely supported unstressed shells $\left(\tau = -5 \text{ percent to } +5 \text{ percent}; \; S = 3, \; \frac{R}{h} = 1000\right)$.

The meridional mode $m = 1$ unless otherwise noted, the circumferential mode n is given on curves. (All calculations based on approximate theory.)

fundamental frequency and circumferential mode number increase monotonically. As the negative curvature increases from zero, the circumferential mode numbers decrease and each branch associated with a particular modal configuration has a distinct minimum. That is, for each branch of the envelope of fundamental frequencies in the negative curvature range, there are specific values of curvature at which a large decrease in effective stiffness occurs.

The fundamental frequency of cylinders and positive curvature shells occur for $m = 1$. However, an $m = 2$ meridional mode is associated with the fundamental frequency for shells within a small range of negative curvature.

Figure 11 is a compilation of nine plots of the same type as figure 10. The length-radius ratio ranges from 1 to 10 and the radius-thickness ratio ranges from 100 to 1000. The minimum frequencies in general decrease as the length increases and as the thickness decreases. The reductions in stiffness in the negative Gaussian curvature range are more prominent for the thinner shells, but the effect is noticeable to some degree for all the shells. The minimum frequency for each branch of the envelope in the negative curvature range occurs at the same rise-length percent ratio for a given length regardless of the thickness of the shell. This suggests that the curvatures at which the minimums occur are related to membrane action. The degree of decrease in effective stiffness on the other hand is highly dependent on the thickness, thus bending action must be a prominent factor in the effective stiffness of the negative curvature shell in the region of these minimum frequencies.

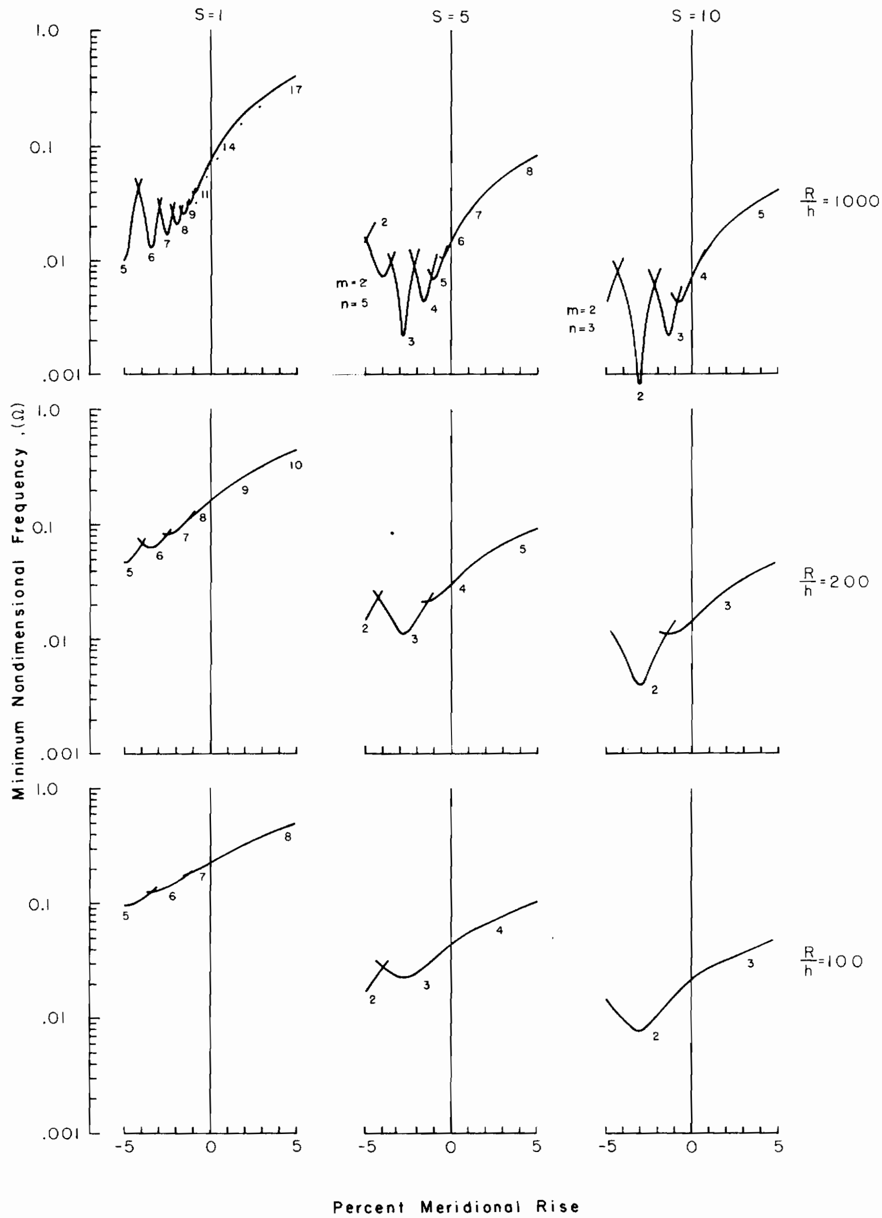


Figure 11.- Effect of meridional curvature on the fundamental frequencies of freely supported unstressed shells for various lengths and thicknesses $\left(\tau = -5 \text{ percent to } +5 \text{ percent; } S = 1, 5, 10; \frac{R}{h} = 100, 200, 1000\right)$.

The meridional mode $m = 1$ unless otherwise noted, the circumferential mode n is given on curves. (All calculations based on approximate theory.)

D. Membrane and Pure Bending Analysis

The results of the previous section imply that membrane behavior is closely related to the large reductions in effective stiffness observed for negative Gaussian curvature shells. The membrane equations which correspond to these shells have been solved in closed form for freely supported edge conditions. The solution for membrane natural frequencies is given in equation (98) in terms of the nondimensional meridional curvature $\left(k_x\right)$, modal wavelength ratio (β), circumferential mode number (n), and prestress quantities. This solution is plotted in figure 12 for particular wavelength ratios for unstressed shells. The membrane frequency is a continuous linear function of k_x for a given wavelength ratio and decreases to zero as k_x approaches the negative value given by equation (99). Since the total mass and mass distribution are essentially constant as k_x is varied, this decrease in frequency must correspond to a decrease in effective membrane stiffness. Therefore, for a given modal wavelength ratio, there exists a negative Gaussian curvature membrane shell with a nondimensional curvature k_x given by equation (99) which vibrates without developing any effective membrane stiffness (or for that matter any membrane stresses). As the meridional curvature increases negatively from this critical value, the membrane shell regains its stiffening characteristics.

To show how this membrane behavior is related to the previous results noted in figures 10 and 11, the modes ($m = 1$, $n = 3$) and

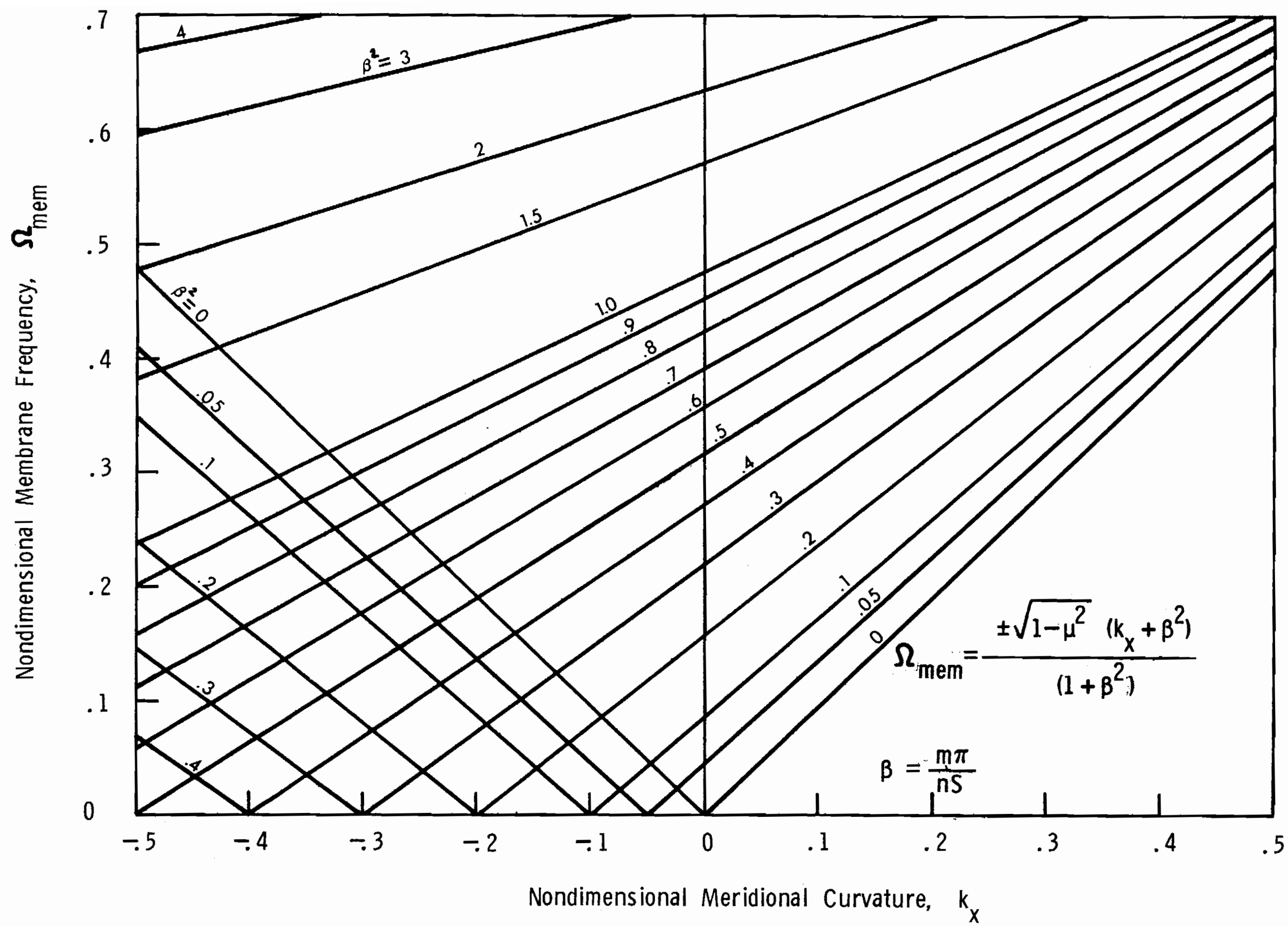


Figure 12.- Effect of meridional curvature on the natural frequencies of freely supported unstressed membrane shells $\left(k_x = -0.5 \text{ to } +0.5\right)$.

(All calculations based on approximate membrane theory)

($m = 1$, $n = 4$) are investigated for shells with a length-radius ratio of 3 and a radius-thickness ratio of 1000 (shells of figure 10). The wavelength ratios for these modes are $\beta = 0.349$ and $\beta = 0.209$, respectively. From equation (99), the meridional curvatures at which no membrane action is developed are $k_x = -0.1218$ and $k_x = -0.0685$, respectively, which for the length-radius ratio in question are equivalent to the percent meridional rise ratios $\tau = -4.57$ percent and $\tau = -2.57$ percent, respectively. These rise ratios very nearly coincide with the rise ratios locating the minimum frequencies in figure 10 at the modes in question. Hence, these reductions in fundamental frequency occur at curvatures for which the membrane theory predicts no membrane action. This can be shown to be true for the remaining minimums in figure 10 and all the minimums in figure 11. The higher fundamental frequencies on either side of these minimums must be due to the reinstatement of membrane stiffness for that wavelength ratio as the curvature changes as indicated in figure 12.

The preceding suggests that membrane action is not developed in negative Gaussian curvature shells for certain wavelength ratios and that this accounts for the radical reductions in frequency observed in figures 10 and 11. This would further suggest that at these critical combinations of wavelength ratio and meridional curvature only pure bending action exists. If this be true then the fundamental frequencies found with a pure bending theory should yield frequencies near the observed minimums. To verify this, the pure bending theory was solved. The pure bending solution given in equation (107) is independent of curvature and is in general a poor representation of shell frequencies since membrane stiffness is usually a quite prominent mode of resistance. The pure bending frequencies associated with predominant w displacement for a shell of length-radius ratio of three, radius-thickness ratio of 1000 and modes $m = 1, n = 3$; $m = 1, n = 4$; $n = 1, n = 5$; $m = 1, n = 6$; $m = 2, n = 7$ are plotted in figure 10 as horizontal lines at appropriate places for comparison with general shell results. In each case, even though the pure bending results are independent of meridional curvature, the pure bending frequency closely approximates the minimum frequencies obtained using the approximate theory for corresponding m, n modes. Hence, it may be concluded that the reductions in frequency observed for certain negative curvature shells are due to a loss in membrane action and that for these critical combinations of modal wavelength and meridional curvature the shell is very nearly vibrating in a pure bending mode. Furthermore these combinations may be predicted from a simple membrane equation (equation (99)).

Further insight into the relative role of membrane action and bending action in the vibration behavior of doubly curved shells may be gained by inspecting the characteristic roots of the governing equations. Table I contains a tabulation of a sample of the characteristic roots found from the characteristic equation (78) for a particular circumferential harmonic mode number as the meridional

TABLE I.- COMPARISON OF CHARACTERISTIC ROOTS OBTAINED FROM MEMBRANE AND BENDING THEORIES ($n = 4$, $\mu = 0.3$)

k_x \ Ω	0.005		0.01		0.025		0.05	
	λ_j^2 bending	λ_j^2 membrane	λ_j^2 bending	λ_j^2 membrane	λ_j^2 bending	λ_j^2 membrane	λ_j^2 bending	λ_j^2 membrane
-0.3	No natural frequency of 0.005 is possible for k_x = -0.3 for any S or boundary conditions		-4.61 -4.99 36.80±3304i	-4.58 -5.02	-4.26 -5.37 36.81±3303i	-4.27 -5.36	-3.73 -5.99 36.85±3300i	-3.76 -5.95
-0.15	-2.37 -2.42 34.40±3304i	-2.3 -2.5	-2.23 -2.58 34.40±3304i	-2.21 -2.60	-1.92 -2.90 34.41±3303i	-1.93 -2.90	-1.46 -3.45 34.45±3300i	-1.48 -3.42
0	0.0470 -0.0467 32.00±3304i	0.083 -0.084	0.156 -0.158 32.00±3304i	0.166 -0.170	0.415 -0.438 32.01±3303i	0.409 -0.431	0.815 0.912 32.04±3300i	0.797 -0.885
0.15	2.35 2.45 29.60±3304i	2.33 2.47	2.26 2.53 29.60±3304i	2.26 2.54	2.03 2.75 29.61±3303i	2.03 2.74	1.63 3.09 29.64±3300i	1.65 3.08
0.3	4.75 4.85 27.20±3304i	4.74 4.86	4.68 4.91 27.20±3304i	4.68 4.92	4.49 5.09 27.21±3303i	4.50 5.09	4.16 5.36 27.23±3300i	4.18 5.36

curvature is varied from positive to negative values with particular values of frequency maintained as constant. The roots are a function of the governing field equations only and are thus independent of shell length and edge boundary conditions. The character of the equations changes radically with the sign of the curvature. A comparison with the roots of the corresponding indicial equation from membrane theory given by equation (103) shows that this change in the characteristic behavior of the shell is associated with membrane (i.e. in-plane) rather than bending terms. Moreover, the membrane roots are very nearly unchanged by the presence of bending terms and in the shell theory roots associated with bending terms are very nearly insensitive to changes in curvature. Thus, in general, differences in the dynamic behavior between negative and positive curvature shells are associated with membrane behavior.

It should be noted that every negative Gaussian curvature shell will vibrate in a pure bending mode for some particular wavelength ratio, however; if the m or n number is large, there may be enough bending stiffness present to maintain a frequency well above the fundamental frequency of the shell.

E. Prestressing Effects of Lateral Pressure

Figure 13 indicates the vibration behavior of a shell subjected to a small internal and small external constant directional lateral pressure. The prestress deformation is assumed negligible so that the circumferential stress is constant and the governing equations

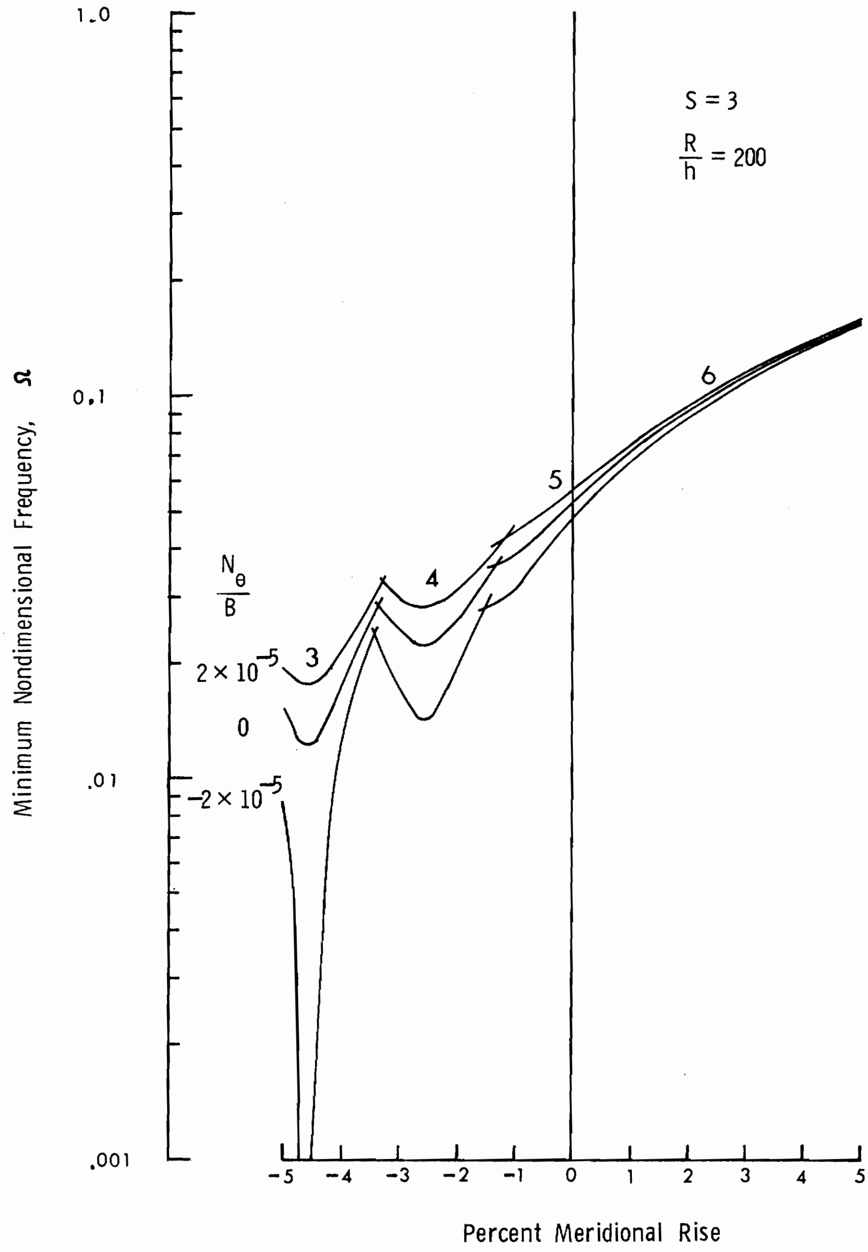


Figure 13.- Effect of meridional curvature on the fundamental frequencies of freely supported shells subjected to a constant circumferential tensile and compressive prestress $\left(\tau = -5 \text{ percent to } +5 \text{ percent}; S = 3, \frac{R}{h} = 200\right)$.

The meridional mode $m = 1$, the circumferential mode n is given on curves. (All calculations based on approximate theory.)

retain their constant coefficient character. The internal pressure develops a constant positive stress in the shell wall which introduces a stiffening effect. This effect is more prominent in the negative curvature range causing the fundamental frequencies to increase slightly. The same small level of negative circumferential stress produced by an applied external pressure causes a radical deviation from the unstressed behavior in the negative curvature range with complete loss in stiffness (buckling) occuring for shells with a percent meridional rise in the vicinity of -4.7 percent. Only a slight decrease in frequency is evident for the positive curvature shells. This demonstrates the highly unstable character of the negative curvature shell in the vicinity of curvatures at which membrane resistance is ineffective and bending stiffness is small.

The introduction of large internal lateral pressure is a possible method of stabilizing the negative curvature shells. Figure 14 shows the effects of the application of a large internal pressure on the minimum frequencies of doubly curved shells. The fundamental frequencies are raised on the order of 1000 percent in the negative curvature shell so that the large reductions in stiffness at critical curvatures are no longer evident. The envelope of fundamental frequencies has flattened considerably with the result that only a small difference exists between the stiffness characteristics of negative and positive curvature shells.

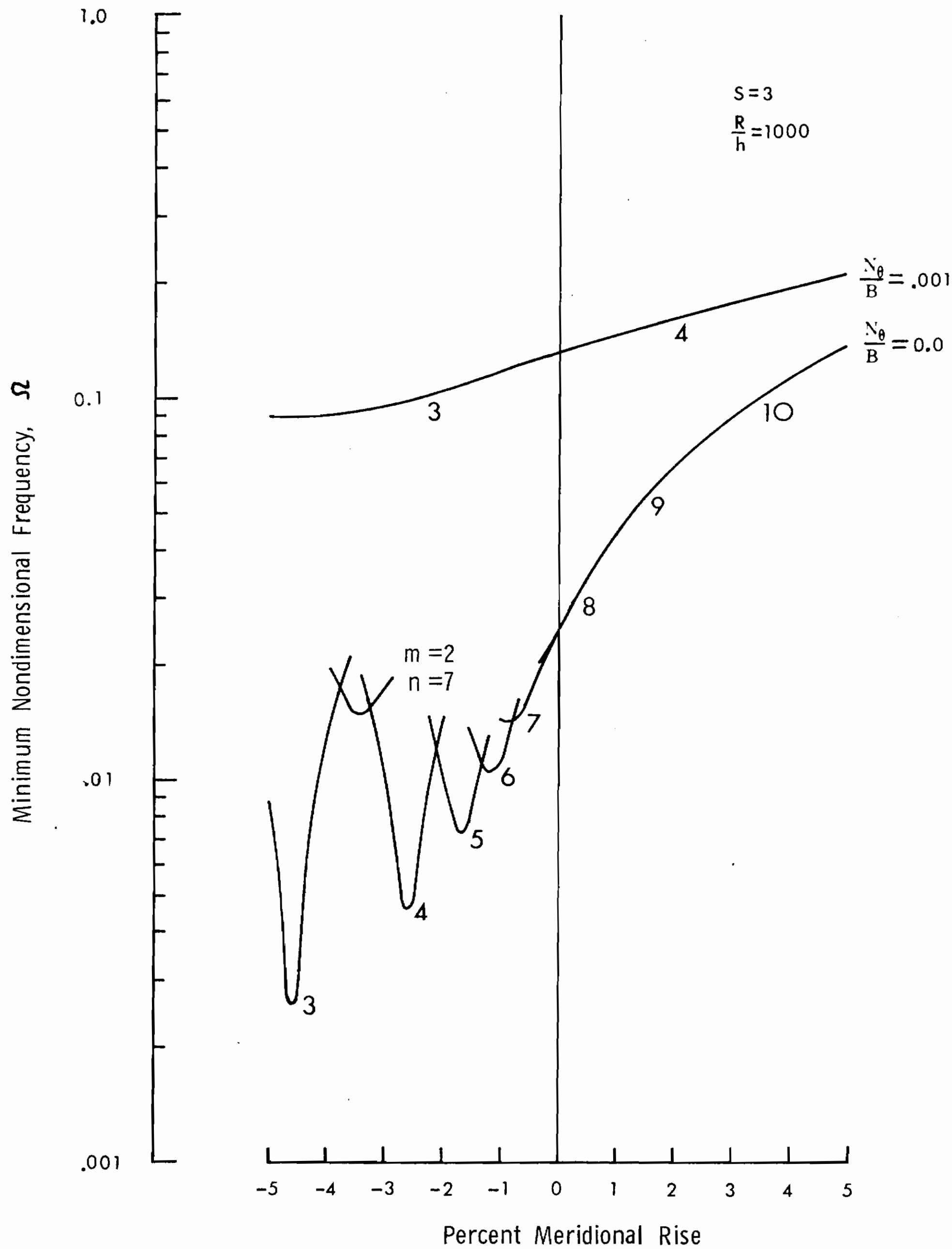


Figure 14.- Pressure stabilization of freely supported shells $\left(\tau = -5\right.$ percent to +5 percent; $S = 3$; $\left.\frac{R}{h} = 1000\right)$.

The meridional mode $m = 1$ unless otherwise noted, the circumferential mode n is given on curves. (All calculations based on approximate theory.)

The general result of varying the lateral pressure may be determined from figure 15. This figure is a plot of the fundamental frequencies for a series of shells as the circumferential stress varies. The fundamental frequency occured for $m = 1$ for all shells inspected in this investigation. The rate of reduction in stiffness as the compressive stress is increased is larger than the corresponding rate of increase in stiffness as the tensile stress is increased. The intersection of a curve with the abscissa yields the compressive buckling stress. The curves show that the negative curvature shells are considerably more susceptible to buckling at low compressive stresses than are the positive curvature shells.

F. Effects of Edge Restraint

The effect of edge restraint on the vibration frequencies of specific positive and negative curvature shells are shown in figure 16 and in Table II. The figure shows an increase in minimum frequencies of clamped shells $\left(u = v = w = w_{,x} = 0\right)$ over those of freely supported shells $\left(N_x = v = w = m_x = 0\right)$ for all curvatures examined. The boundary effects had a sizeable influence for moderate values of n but diminished in influence for high and low values of n. The clamped positive and zero curvature shells have an $m = 1$ type mode at the lowest frequency for each n whereas the negative curvature shells exhibit more complicated modes at higher n. Since in the approximate theory the lowest frequency for the axisymmetric mode $(n = 0)$ is the torsional frequency which has been shown to be

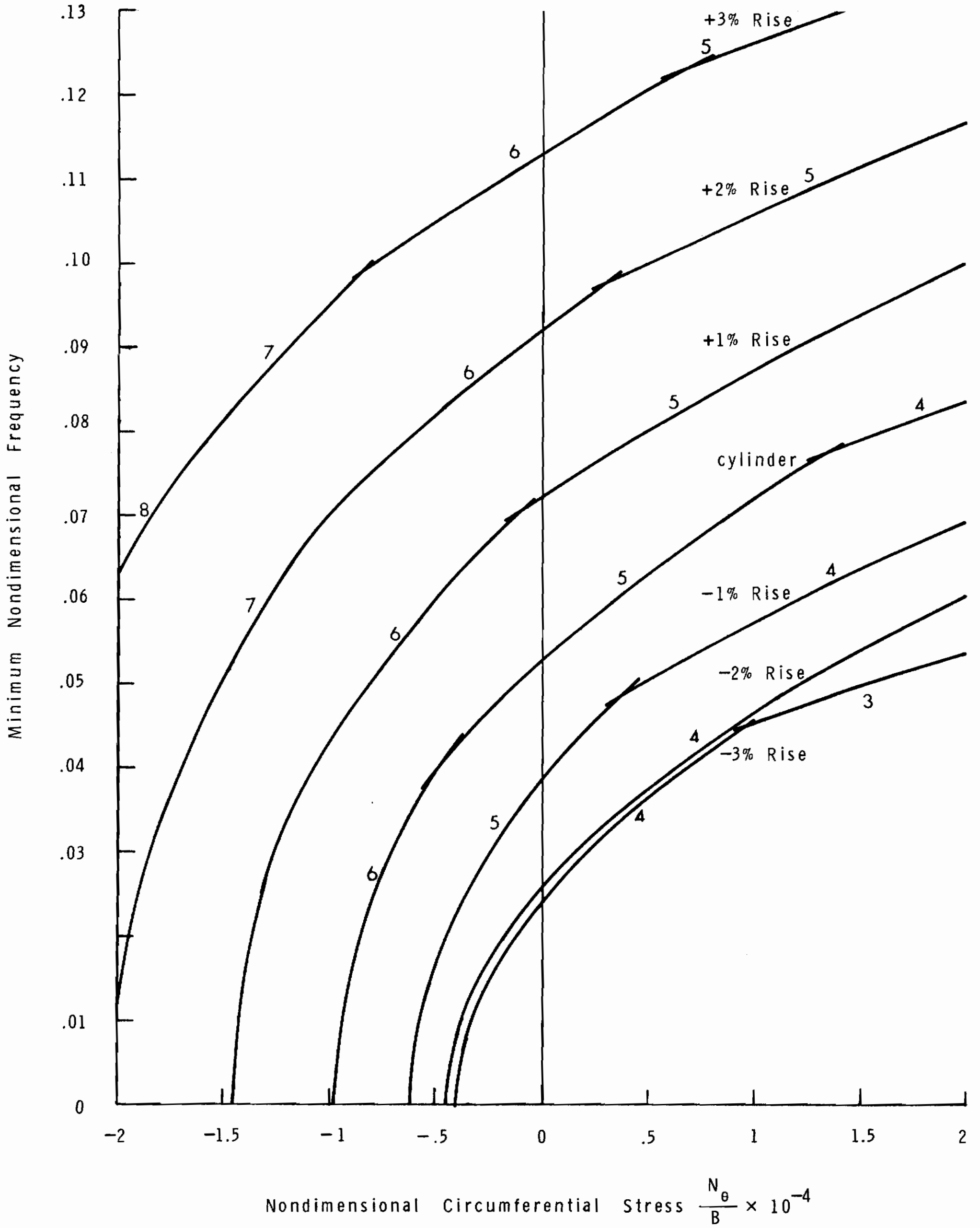


Figure 15. Effect of constant circumferential prestress on the fundamental frequencies of freely supported shells $\left(\tau = -3\right.$ percent to +3 percent; $S = 3$; $\frac{R}{h} = 200$; $\frac{N_\theta}{B} = -2 \times 10^{-4}$ to $\left.+2 \times 10^{-4}\right)$.

The meridional mode $m = 1$, the circumferential mode n is given on curves. (All calculations based on approximate theory.)

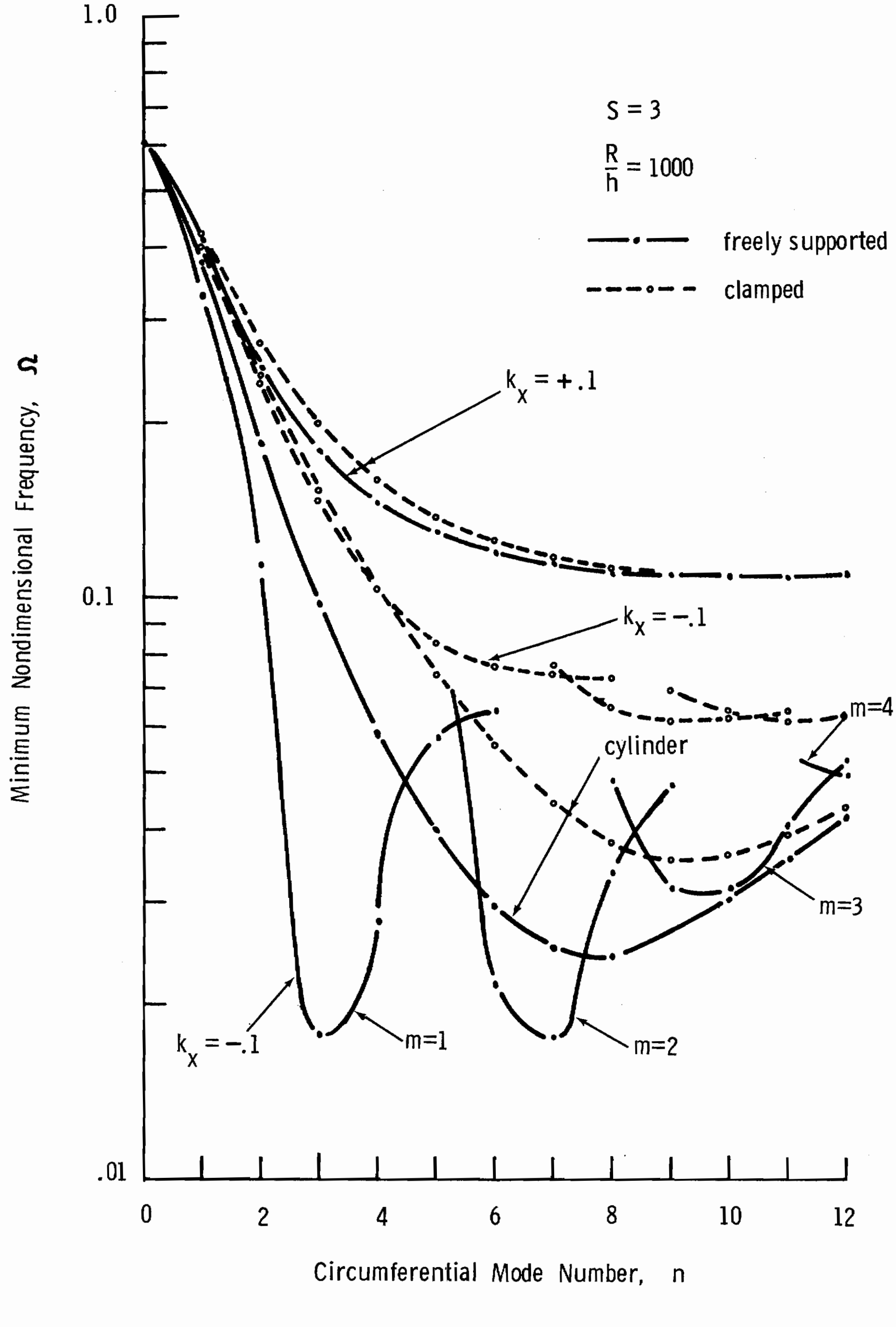


Figure 16.- Effect of edge restraint on the natural vibration of unstressed shells $\left(k_x = \pm 0.1;\ s = 3;\ \frac{R}{h} = 1000\right)$.

(All calculations based on approximate theory)

TABLE II.- COMPARISON OF THE EFFECT OF EDGE RESTRAINT ON THE NATURAL FREQUENCIES OF A NEGATIVE CURVATURE SHELL $\left(k_x = -0.05,\ S = 3,\ \frac{R}{h} = 1000\right)$

$$\Omega = R\omega\sqrt{\frac{\nu(1 - \mu^2)}{E}}$$

n	$N_\xi = v = w = M_\xi = 0$ (freely supported)	$N_\xi = v = w = w_{,x} = 0$ (clamped with no meridional restraint)	$u = v = w = M_\xi = 0$ (simply supported)	$u = v = w = w_{,x} = 0$ (clamped)
1	0.3534	0.3534	0.4029	0.4030
2	.1482	.1482	.2355	.2356
3	.0577	.0577	.1459	.1460
4	.0166	.0169	.0960	.0961
5	.00897	.00948	.0678	.0679
6	.0206	.0208	.0526	.0527
7	.0291	.0293	.0456	.0456
8	.0253	.0261	.0436	.0436
9	.0245	.0252	.0446	.0446
10	.0302	.0308	.0474	.0474

independent of u, w and the meridional curvature k_x, the minimum frequency curves have the same value at $n = 0$ regardless of the boundary conditions (as long as $v = 0$ is maintained on each edge) and curvature.

The clamped negative curvature shells do not have the large reductions in minimum frequency which occured in the freely supported shells. The results for different combinations of edge restraint of u and $w,_x$ given in Table II show that the condition $u = 0$ and not $w,_x = 0$ caused the increase in frequencies above those of the freely supported shells. Thus, the membrane constraint condition $u = 0$ has apparently prevented a loss in membrane stiffness for the negative curvature shells. The slope restraint $w,_x = 0$ was essentially ineffectual in its ability to raise the fundamental frequencies. Results similar to these have been given in reference 13 for circular cylindrical shells.

X. CONCLUDING REMARKS

A set of linear equations governing the infinitesimal vibrations of axisymmetrically prestressed shells is developed from Sander's nonlinear shell theory and both in-plane inertia and prestress deformation effects are retained in the development. The equations derived are consistent with first-order thin-shell theory and can be used to describe the behavior of shells with arbitrary meridional configuration having moderately small prestress rotations.

A numerical procedure is given for solving the governing equations for the natural frequencies and associated mode shapes for a general shell of revolution with homogeneous boundary conditions. The numerical procedure uses matrix methods in finite-difference form coupled with a Gaussian elimination to solve the governing eigenvalue problem. The solutions obtained by this method are used to determine the accuracy of the approximate solutions used in the vibration analysis.

An approximate set of governing equations of motion with constant coefficients which are based on shallowness of the meridian are developed as an alternate more rapid method of solution and are solved in an exact manner for all boundary conditions. The membrane and pure bending equations which correspond to this approximate set of equations are solved for a specific boundary condition. The character of the characteristic roots of these membrane equations are also inspected.

The effect of the meridional curvature on the fundamental frequencies of a class of cylindrical-like shells with shallow meridional curvature is investigated. The positive Gaussian curvature shells have fundamental frequencies well above those of corresponding cylindrical shells. The fundamental frequencies of the negative Gaussian curvature shells generally are below those of the corresponding cylinders and evidence wide variations in value with large reductions in magnitude occuring at certain critical curvatures. The corresponding membrane and pure bending equations are also solved for the same edge conditions. Comparison of the membrane, pure bending and complete shell analyses shows that these critical curvatures represent configurations at which the fundamental mode of vibration of the shell is in a state close to pure bending. The membrane theory affords a simple method of determining the modal wavelength ratio at which the pure bending state exists for a given negative Gaussian curvature shell, while the pure bending theory gives a good estimate of the magnitude of the frequency for this wavelength ratio. Meridional edge restraints and internal lateral pressure reduce the wide variation of the natural frequencies in the negative curvature shells and in general raise the natural frequencies. External lateral pressure accentuates the reduction in natural frequencies of the negative curvature shells and causes instability at low compressive stress levels.

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XII. VITA

Mr. Cooper was born on June 12, 1940 in Boston, Massachusetts. He graduated from Boston Technical High School in 1957. He attended Northeastern University receiving a B.S.M.E. degree in 1962. While an undergraduate he was a member of Pi Tau Sigma, ASME and ASM. He held a cooperative work assignment with Artisan Industries in Waltham, Massachusetts from 1959 to 1962. He received a teaching assistantship appointment in the Mechanical Engineering Department at Northeastern University in 1962 and taught undergraduate courses until August 1964. In June of 1964 he received an M.S.M.E. degree from Northeastern University. In August of 1964 he joined the National Aeronautics and Space Administration at Langley Research Center where he is currently employed. He is an associate member of ASME and is the author of two technical papers, one in the field of metallurgy and one in the field of shell dynamics and stability. He is married and has two children.

Paul A. Cooper

XIII. APPENDIX A

COEFFICIENTS OF EQUATIONS (22), (23), (27), AND (90)

A. Coefficients of Equations (22)

The coefficients of terms in the governing equations (22) are defined as follows:

$$F_{11} = 1$$

$$F_{13} = -\overline{\varphi}_{\xi}$$

$$F_{22} = \frac{(1-\mu)}{2} + \frac{\lambda^2 (1-\mu)}{96} \left(3k_\theta - k_x\right)^2 + \frac{1}{4}\left(\overline{e}_x + \overline{e}_\theta\right)$$

$$F_{23} = \frac{\lambda^2 (1-\mu)n}{24r} \left(3k_\theta - k_x\right)$$

$$F_{31} = F_{13}$$

$$F_{32} = F_{23}$$

$$F_{33} = \frac{\lambda^2 (1-\mu)}{12} \left[\frac{2n^2}{r^2} + (1+\mu)\gamma^2\right] + \overline{e}_x + \overline{\varphi}_{\xi}^{\,2}$$

$$F_{34} = \lambda^2 \left(1 - \mu^2\right)$$

$$F_{43} = F_{34}$$

$$G_{11} = \gamma$$

$$G_{12} = \frac{(1 + \mu)n}{2r} + \frac{\lambda^2 n\ (1 - \mu)}{96r} \left(3k_x - k_\theta\right) \left(3k_\theta - k_x\right) - \frac{n}{4r} \left(\bar{e}_x + \bar{e}_\theta\right)$$

$$G_{13} = k_x + \mu k_\theta + \frac{\lambda^2\ (1 - \mu)}{12} \left[(1 + \mu)\ \gamma^2\ k_x + \frac{n^2\left(3k_x - k_\theta\right)}{2r^2}\right] + k_x \bar{e}_x$$

$$- (1 - \mu)\ \gamma \bar{\varphi}_\xi + k_x\ \bar{\varphi}_\xi^{\ 2} - \frac{d\bar{\varphi}_\xi}{dx}$$

$$G_{14} = \lambda^2 \left(1 - \mu^2\right) k_x$$

$$G_{21} = - G_{12}$$

$$G_{22} = \frac{(1 - \mu)}{2}\ \gamma - \frac{\lambda^2\ (1 - \mu)}{96} \left(3k_\theta - k_x\right)\left[2\ \frac{dk_x}{dx} - \gamma\ \left(5k_x - 3k_\theta\right)\right]$$

$$+ \frac{1}{4}\ \frac{d}{dx} \left(\bar{e}_x + \bar{e}_\theta\right) + \frac{\gamma}{4} \left(\bar{e}_x + \bar{e}_\theta\right)$$

$$G_{23} = \frac{\lambda^2\ (1 - \mu)n}{24r} \left[2(1 + \mu)\ \gamma k_\theta - \frac{dk_x}{dx} + 3\gamma\ \left(k_x - k_\theta\right)\right] + \frac{(1 + \mu)n}{2r}\ \bar{\varphi}_\xi$$

$$G_{31} = - G_{13} - 2\gamma \bar{\varphi}_\xi - 2\ \frac{d\bar{\varphi}_\xi}{dx}$$

$$G_{32} = \frac{\lambda^2\ (1 - \mu)n}{24r} \left[3\gamma k_x - \gamma k_\theta\ (5 + 2\mu) - \frac{dk_x}{dx}\right] - \frac{n}{2r}\ (1 + \mu)\ \bar{\varphi}_\xi$$

$$G_{33} = - \frac{\lambda^2\ (1 - \mu)}{12} \left[(1 + \mu)\ \left(2\gamma k_x\ k_\theta + \gamma^3\right) + \frac{2n^2\gamma}{r^2}\right] + \gamma \bar{e}_x + \frac{d\bar{e}_x}{dx}$$

$$+ \gamma \bar{\varphi}_\xi^{\ 2} + 2\bar{\varphi}_\xi\ \frac{d\bar{\varphi}_\xi}{dx}$$

$$G_{34} = \lambda^2 (2 - \mu) \left(1 - \mu^2\right)\gamma$$

$$G_{41} = - G_{14}$$

$$G_{43} = \lambda^2 \left(1 - \mu^2\right)\mu\gamma$$

$$H_{11} = - \mu k_x k_\theta - \gamma^2 - \frac{(1 - \mu)n^2}{2r^2} - \frac{\lambda^2 (1 - \mu)}{12}\left[(1 + \mu) \gamma^2 k_x{}^2\right.$$

$$\left. + \frac{n^2 \left(3k_x - k_\theta\right)^2}{8r^2}\right] - k_x{}^2 \bar{e}_x - \frac{n^2}{4r^2}\left(\bar{e}_x + \bar{e}_\theta\right)$$

$$+ \left[(1 - 2\mu) \gamma k_x + \frac{dk_x}{dx}\right] \bar{\varphi}_\xi - k_x{}^2 \bar{\varphi}_\xi{}^2 + k_x \frac{d\bar{\varphi}_\xi}{dx} + \Omega^2$$

$$H_{12} = - \frac{(3 - \mu)n\gamma}{2r} - \frac{\lambda^2 (1 - \mu)n\gamma}{12r}\left[\frac{\left(3k_x - k_\theta\right)\left(3k_\theta - k_x\right)}{8}\right.$$

$$\left. + (1 + \mu) k_x k_\theta\right] - \frac{n\gamma}{4r}\left(\bar{e}_x + \bar{e}_\theta\right) - \frac{\mu n}{r} k_x \bar{\varphi}_\xi + \frac{(1 - \mu)n}{2r} k_\theta \bar{\varphi}_\xi$$

$$H_{13} = \frac{dk_x}{dx} + \gamma\left(k_x - k_\theta\right) - \frac{\lambda^2 (1 - \mu)n^2\gamma}{12r^2}\left[\frac{\left(3k_x - k_\theta\right)}{2} + (1 + \mu) k_x\right]$$

$$- k_x \left(k_x + \mu k_\theta\right) \bar{\varphi}_\xi + \frac{(1 - \mu)n^2}{2r^2} \bar{\varphi}_\xi$$

$$H_{14} = \lambda^2 \left(1 - \mu^2\right) (1 - \mu)\gamma k_x$$

$$H_{21} = -\frac{(3-\mu)n\gamma}{2r} + \frac{\lambda^2(1-\mu)n}{12r}\left[-(1+\mu)\gamma k_x k_\theta + \frac{\gamma}{8}\left(6k_x k_\theta - 7k_x^{\ 2} - 3k_\theta^{\ 2}\right) - \frac{1}{4}\frac{dk_x}{dx}\left(5k_\theta - 3k_x\right)\right] + \frac{n}{4r}\frac{d}{dx}\left(\overline{e}_x + \overline{e}_\theta\right) - \frac{n\gamma}{4r}\left(\overline{e}_x + \overline{e}_\theta\right)$$

$$- \frac{\mu n}{r} k_x \overline{\varphi}_\xi + \frac{(1-\mu)n}{2r} k_\theta \overline{\varphi}_\xi$$

$$H_{22} = -\gamma G_{22} + \frac{(1-\mu)}{2} k_x k_\theta - \frac{n^2}{r^2} - \frac{\lambda^2(1-\mu)}{12}\left[\frac{(1+\mu)n^2}{r^2} k_\theta^{\ 2} - \frac{k_x k_\theta}{8}\left(3k_\theta - k_x\right)^2\right] + \frac{\gamma}{2}\frac{d}{dx}\left(\overline{e}_x + \overline{e}_\theta\right) - k_\theta^{\ 2}\,\overline{e}_\theta - \frac{1}{4} k_x k_\theta\left(\overline{e}_x + \overline{e}_\theta\right)$$

$$+ \frac{(1-\mu)\gamma}{2}\left(k_x + 2k_\theta\right)\overline{\varphi}_\xi - \frac{(1-\mu)}{2} k_\theta^{\ 2}\,\overline{\varphi}_\xi^{\ 2}$$

$$+ \frac{(1-\mu)}{2} k_\theta \frac{d\overline{\varphi}_\xi}{dx} + \Omega^2$$

$$H_{23} = -\frac{n}{r}\left(k_\theta + \mu k_x\right) + \frac{\lambda^2(1-\mu)n}{24r}\left[\gamma\frac{dk_x}{dx} - 2\gamma^2 k_x - \frac{2(1+\mu)n^2}{r^2} k_\theta + \left(3k_\theta - k_x\right)\left(\gamma^2 + k_x k_\theta\right)\right] - \frac{n}{r} k_\theta\,\overline{e}_\theta + \frac{(1-\mu)n\gamma}{2r}\,\overline{\varphi}_\xi$$

$$- \frac{(1-\mu)n}{2r} k_\theta\,\overline{\varphi}_\xi^{\ 2} + \frac{(1-\mu)n}{2r}\frac{d\overline{\varphi}_\xi}{dx}$$

$$H_{24} = -\frac{\lambda^2\mu\left(1-\mu^2\right)n}{r} k_\theta$$

$$H_{31} = -\gamma\left(k_\theta + \mu k_x\right) + \frac{\lambda^2 (1-\mu)}{12}\left[(1+\mu)\gamma\left(\gamma^2 k_x - \gamma \frac{dk_x}{dx} - \frac{n^2}{r^2} k_x\right.\right.$$

$$\left.\left. + 2k_x{}^2 k_\theta\right) + \frac{n^2}{2r^2}\left(\gamma k_x - \gamma k_\theta - 3\frac{dk_x}{dx}\right)\right] - \left(\gamma k_x + \frac{dk_x}{dx}\right)\bar{e}_x$$

$$- k_x \frac{d\bar{e}_x}{dx} + \left[\frac{(1-\mu)n^2}{2r^2} - k_x{}^2\right]\bar{\varphi}_\xi - \left(\gamma k_x + \frac{dk_x}{dx}\right)\bar{\varphi}_\xi{}^2$$

$$- \left(2k_x \bar{\varphi}_\xi + \mu\gamma\right)\frac{d\bar{\varphi}_\xi}{dx}$$

$$H_{32} = -\frac{n}{r}\left(k_\theta + \mu k_x\right) + \frac{\lambda^2 (1-\mu)n}{24r}\left[2(1+\mu)\left(k_x k_\theta{}^2 - \gamma^2 k_x\right.\right.$$

$$\left.\left. + 2\gamma^2 k_\theta - \frac{n^2}{r^2} k_\theta\right) + \gamma\frac{dk_x}{dx} + 3\gamma^2\left(k_\theta - k_x\right) + k_x k_\theta\left(3k_\theta - k_x\right)\right]$$

$$- \frac{n}{r} k_\theta \bar{e}_\theta + \frac{(1-\mu)n\gamma}{2r}\bar{\varphi}_\xi - \frac{(1-\mu)n}{2r} k_\theta \bar{\varphi}_\xi{}^2 - \frac{\mu n}{r}\frac{d\bar{\varphi}_\xi}{dx}$$

$$H_{33} = -k_x{}^2 - 2\mu k_x k_\theta - k_\theta{}^2 + \frac{\lambda^2 (1-\mu)n^2}{12r^2}\left[(1+\mu)\left(k_x k_\theta - \frac{n^2}{r^2} + 2\gamma^2\right)\right.$$

$$\left. + 2\left(\gamma^2 + k_x k_\theta\right)\right] - \frac{n^2}{r^2}\bar{e}_\theta - \left[\gamma k_x (1+\mu) + \frac{dk_x}{dx}\right]\bar{\varphi}_\xi$$

$$- \frac{(1-\mu)n^2}{2r^2}\bar{\varphi}_\xi{}^2 - \left(k_x + \mu k_\theta\right)\frac{d\bar{\varphi}_\xi}{dx} + \Omega^2$$

$$H_{34} = -\lambda^2\left(1-\mu^2\right)\left[(1-\mu) k_x k_\theta + \frac{\mu n^2}{r^2}\right]$$

$$H_{41} = -\lambda^2\left(1-\mu^2\right)\left(\frac{dk_x}{dx} + \mu\gamma k_x\right)$$

$$H_{42} = H_{24}$$

$$H_{43} = -\frac{\lambda^2\left(1-\mu^2\right)\mu n^2}{r^2}$$

$$H_{44} = 12\lambda^2\left(1-\mu^2\right)^2$$

Coefficients of Equations (23)

The coefficients of the terms associated with the boundary conditions (eqs. (23)) are defined in the following equations:

$$e_{11} = 1$$

$$e_{13} = -\overline{\varphi}_\xi$$

$$e_{22} = \frac{(1-\mu)}{2} + \frac{\lambda^2(1-\mu)}{96}\left(3k_\theta - k_x\right)^2 + \frac{1}{4}\left(\overline{e}_x + \overline{e}_\theta\right)$$

$$e_{23} = \frac{\lambda^2(1-\mu)n}{24r}\left(3k_\theta - k_x\right)$$

$$e_{31} = e_{13}$$

$$e_{32} = e_{23}$$

$$e_{33} = \frac{\lambda^2(1-\mu)}{12}\left[\frac{2n^2}{r^2} + (1+\mu)\,\gamma^2\right] + \overline{e}_x + \overline{\varphi}_\xi{}^2$$

$$e_{34} = \lambda^2\left(1-\mu^2\right)$$

$$e_{43} = e_{34}$$

$$f_{11} = \mu\gamma + k_x\,\bar{\varphi}_\xi$$

$$f_{12} = \frac{\mu n}{r}$$

$$f_{13} = k_x + \mu k_\theta$$

$$f_{21} = -\frac{(1-\mu)n}{2r} - \frac{\lambda^2\,(1-\mu)n}{96r}\left(3k_x - k_\theta\right)\left(3k_\theta - k_x\right) + \frac{n}{4r}\left(\bar{e}_x + \bar{e}_\theta\right)$$

$$f_{22} = -\frac{(1-\mu)\gamma}{2} - \frac{\lambda^2\gamma\,(1-\mu)}{96}\left(3k_\theta - k_x\right)^2 + \frac{\gamma}{4}\left(\bar{e}_x + \bar{e}_\theta\right) + \frac{(1-\mu)}{2}\,k_\theta\bar{\varphi}_\xi$$

$$f_{23} = -\frac{\lambda^2\gamma\,(1-\mu)n}{24r}\left(3k_\theta - k_x\right) + \frac{(1-\mu)n}{2r}\,\bar{\varphi}_\xi$$

$$f_{31} = -\frac{\lambda^2\,(1-\mu)}{12}\left[(1+\mu)\,\gamma^2\,k_x + \frac{n^2}{2r^2}\left(3k_x - k_\theta\right)\right] - k_x\,\bar{e}_x$$

$$- \mu\gamma\,\bar{\varphi}_\xi - k_x\,\bar{\varphi}_\xi^{\;2}$$

$$f_{32} = -\frac{\lambda^2\,(1-\mu)n\gamma}{24r}\left[3k_\theta - k_x + 2(1+\mu)\,k_\theta\right] - \frac{\mu n}{r}\,\bar{\varphi}_\xi$$

$$f_{33} = -\frac{\lambda^2\,(1-\mu)}{12}\,(3+\mu)\,\frac{n^2\gamma}{r^2} - \left(k_x + \mu k_\theta\right)\bar{\varphi}_\xi$$

$$f_{34} = \lambda^2\,(1-\mu)\left(1-\mu^2\right)\gamma$$

$$f_{41} = - \lambda^2 \left(1 - \mu^2\right) k_x$$

C. Coefficients of Equation (27)

The coefficients of equation (27) are defined as follows:

$$\Lambda_0 = \left[a_{11}a_{22}a_{33} - a_{11}\left(a_{23}\right)^2 - a_{22}\left(a_{13}\right)^2 - a_{33}\left(a_{12}\right)^2 + 2a_{12}a_{23}a_{13}\right] a_{44}$$

$$+ \left[- a_{11}a_{22}a_{34} + 2a_{11}a_{23}a_{24} + \left(a_{12}\right)^2 a_{34} - 2a_{12}a_{24}a_{13}\right] a_{34}$$

$$+ \left[\left(a_{13}\right)^2 - a_{11}a_{33}\right] \left(a_{24}\right)^2$$

$$\Lambda_1 = \left[a_{22}a_{33} + a_{11}a_{22} + a_{11}a_{33} - \left(a_{12}\right)^2 - \left(a_{23}\right)^2 - \left(a_{13}\right)^2\right] a_{44}$$

$$- \left(a_{24}\right)^2 \left(a_{11} + a_{33}\right) - \left(a_{34}\right)^2 \left(a_{11} + a_{22}\right) + 2a_{23}a_{24}a_{34}$$

$$\Lambda_2 = \left(a_{11} + a_{22} + a_{33}\right) a_{44} - \left(a_{34}\right)^2 - \left(a_{24}\right)^2$$

$$\Lambda_3 = a_{44}$$

where

$$a_{11} = H_{11} - F_{11}\left(\frac{m\pi}{S}\right)^2 - \Omega^2$$

$$a_{12} = G_{12}\left(\frac{m\pi}{S}\right)$$

$$a_{13} = G_{13}\left(\frac{m\pi}{S}\right)$$

$$a_{22} = H_{22} - F_{22}\left(\frac{m\pi}{S}\right)^2 - \Omega^2$$

$$a_{23} = H_{23} - F_{23}\left(\frac{m\pi}{S}\right)^2$$

$$a_{24} = H_{24}$$

$$a_{33} = H_{33} - F_{33}\left(\frac{m\pi}{S}\right)^2 - \Omega^2$$

$$a_{34} = H_{34} - F_{34}\left(\frac{m\pi}{S}\right)^2$$

$$a_{44} = H_{44}$$

D. Coefficients of Equation (90)

The coefficients of equation (90) are defined as follows:

$$\bar{\Lambda}_0 = b_{11}\left[b_{22}b_{33} - \left(b_{23}\right)^2\right] - b_{12}\left[b_{12}b_{33} - b_{13}b_{23}\right]$$
$$+ b_{13}\left[b_{12}b_{23} - b_{13}b_{22}\right]$$

$$\bar{\Lambda}_1 = b_{22}b_{33} - \left(b_{23}\right)^2 + b_{33}b_{11} - \left(b_{13}\right)^2 + b_{11}b_{22} - \left(b_{12}\right)^2$$

$$\bar{\Lambda}_2 = b_{11} + b_{22} + b_{33}$$

where

$$b_{11} = \left(\frac{m\pi}{S}\right)^2 + \left(\frac{1 - \mu}{2}\right) n^2$$

$$b_{12} = \left(\frac{1 + \mu}{2}\right)\left(\frac{m\pi}{S}\right) n$$

$$b_{13} = -\left(\frac{m\pi}{S}\right)\left(k_x + \mu\right)$$

$$b_{22} = \left(1 + \frac{\lambda^2}{\underline{12}}\right) n^2 + \left(\frac{1 - \mu}{2}\right)\left(1 + \frac{\lambda^2}{\underline{3}}\right)\left(\frac{m\pi}{S}\right)^2$$

$$b_{23} = -n\left(1 + \mu k_x\right) - \underline{\frac{\lambda^2}{12}\, n \left[n^2 + (2 - \mu)\left(\frac{m\pi}{S}\right)^2\right]}$$

$$b_{33} = \frac{\lambda^2}{12}\left[\left(\frac{m\pi}{S}\right)^2 + n^2\right]^2 + k_x{}^2 + 2\mu k_x + 1 + \frac{\overline{N}}{B}\left(\frac{m\pi}{S}\right)^2 + \frac{\overline{N}_\theta}{B} n^2$$

APPENDIX B

REFINEMENT OF THE NUMERICAL PROCEDURE OF CHAPTER VII

The method of Gaussian elimination used in Chapter VII introduces spurious singularities into the determinantal equation (48). These singularities are associated with sign changes of the determinant even though no zero of the determinant exists for these values of Ω. This causes difficulties in the search procedure for finding the frequencies. The actual value of the determinant of the coefficient matrix of the set of equations (40), (41), and (42) may be written as

$$R = \left| D_0 \right| \left| B_1 - A_1 P_0 \right| \cdot \cdot \cdot \cdot \left| B_{N-1} - A_{N-1} P_{N-2} \right| \left| D_N - E_N P_{N-1} \right| \tag{B1}$$

where the zeros of R are contained in the last determinant if $Z_N \neq 0$ and in the next to last determinant if $Z_N = 0$ (simply supported). The last determinant is infinite at the frequencies of a simply supported system since P_{N-1} is found by inverting a null matrix in equation (45). Similarly, the determinant containing the zeros for the simply supported system tends towards infinity as the frequencies for a simply supported system of length $S - \Delta$ is approached.

To remove the singularities, equation (B1) is used in place of equation (48). Although individual terms in (B1) will increase without bound for certain values of the trial frequencies, R will remain bounded since there will always be corresponding terms approaching zero at the same rate. If $Z_N = 0$, the last term in equation (B1) is dropped and the modified R is used in place of equation (50). This procedure does not increase the time for calculation appreciably since the determinants required in equation (B1) have been found during the computational procedure followed in determining P_i with equation (45).

APPENDIX C

SOLUTION OF PRESTRESS EQUATION FOR CHAPTER VIII

The equations governing the axisymmetric prestress deformations are found by applying equations (68) and (69) to equations (66) and retaining the prestress terms

$$\left.\begin{aligned} \frac{d\overline{N}_\xi}{d\xi} &= 0 \\ \frac{d}{d\xi}\left[\overline{N}_{\xi\theta} + \frac{2\overline{M}_{\xi\theta}}{R}\right] &= 0 \\ \frac{d^2\,\overline{M}_\xi}{d\xi^2} - \frac{\overline{N}_\xi}{R_\xi} - \frac{\overline{N}_\theta}{R} + \overline{N}_\xi\,\frac{d^2\overline{w}}{d\xi^2} + p &= 0 \end{aligned}\right\} \qquad \text{(C1)}$$

The corresponding boundary conditions are

$$\left.\begin{aligned} &\overline{N}_\xi = \overline{N} \quad \text{or} \quad \overline{u} = \overline{U} \\ &\overline{N}_{\xi\theta} + \frac{2\overline{M}_{\xi\theta}}{R} = \overline{T} \quad \text{or} \quad \overline{v} = \overline{V} = 0 \quad \text{from symmetry} \\ &\frac{d\overline{M}_\xi}{d\xi} + \overline{N}_\xi\,\frac{d\overline{w}}{d\xi} = \overline{Q} \quad \text{or} \quad \overline{w} = \overline{W} \\ &\overline{M}_\xi = \overline{M} \quad \text{or} \quad \frac{d\overline{w}}{d\xi} = \frac{d\overline{W}}{d\xi} \end{aligned}\right\} \qquad \text{(C2)}$$

where $\overline{N}$, $\overline{T}$, $\overline{Q}$, and $\overline{M}$ are applied edge forces and $\overline{U}$, $\overline{V}$, $\overline{W}$, and $\frac{d\overline{W}}{d\xi}$ are applied displacement conditions.

The first of equations (C1) and (C2) and the second of equations (C1) and (C2) yield

$$\left.\begin{aligned} \overline{N}_\xi &= \overline{N} \\ \overline{N}_{\xi\theta} + \frac{2\overline{M}_{\xi\theta}}{R} &= \overline{T} \end{aligned}\right\} \quad (C3)$$

for all ξ

If no shear loading $\overline{T}$ is applied to the edges then

$$\overline{N}_{\xi\theta} + \frac{2\overline{M}_{\xi\theta}}{R} = 0 \quad \text{for all} \quad \xi \qquad (C4)$$

The stress resultants may be written as

$$\left.\begin{aligned} \overline{N}_\theta &= B\,(1 - \mu^2)\,\frac{\overline{w}}{R} + \mu\overline{N} \\ \overline{M}_\xi &= -\,D\,\frac{d^2\overline{w}}{d\xi^2} \end{aligned}\right\} \quad (C5)$$

and the third equation in (C1) becomes with the introduction of the nondimensional variable x

$$\frac{\lambda^2}{12}\,\frac{d^4\overline{w}}{dx^4} - \frac{\overline{N}}{B}\,\frac{d^2\overline{w}}{dx^2} + (1 - \mu^2)\,\overline{w} = \frac{pR^2\,(1 - \mu^2)}{Eh} - \left(k_x + \mu\right)\,R\,\frac{\overline{N}}{B} \qquad (C6)$$

Equation (C6) has the same form as the Föppl equation solved in reference 21 and in fact reduces to this equation when $k_x = 0$. If the boundary conditions are the same on both edges of the shell, the solution to equation (C6) is

$$\frac{\overline{w}}{R} = A_1 \sin a_1 \left(x - \frac{S}{2}\right) \sinh a_2 \left(x - \frac{S}{2}\right)$$

$$+ A_2 \cos a_1 \left(x - \frac{S}{2}\right) \cosh a_2 \left(x - \frac{S}{2}\right) + \frac{P}{1 - \mu^2} \tag{C7}$$

where

$$P = \frac{Rp\ (1 - \mu^2)}{Eh} - \left(k_x + \mu\right) \frac{\overline{N}}{B} \tag{C8}$$

and

$$\left.\begin{aligned} a_1 &= \sqrt{\frac{3}{\lambda} \sqrt{1 - \mu^2} + \frac{3\overline{N}}{\lambda^2 B}} \\ a_2 &= \sqrt{\frac{3}{\lambda} \sqrt{1 - \mu^2} - \frac{3\overline{N}}{\lambda^2 B}} \end{aligned}\right\} \tag{C9}$$

For a simply supported boundary condition ($\overline{W} = \overline{M} = 0$)

$$A_1 = \left(\frac{P}{1 - \mu^2}\right) \frac{\left(a_2{}^2 - a_1{}^2\right) \cos a_1 \frac{S}{2} \cosh a_2 \frac{S}{2} - 2a_1 a_2 \sin a_1 \frac{S}{2} \sinh \frac{a_2 S}{2}}{2a_1 a_2 \left(\sin^2 \frac{a_1 S}{2} \sinh^2 \frac{a_2 S}{2} + \cos^2 a_1 \frac{S}{2} \cosh^2 \frac{a_2 S}{2}\right)}$$

$$\tag{C10}$$

$$A_2 = \left(\frac{-P}{1 - \mu^2}\right) \frac{\left(a_2{}^2 - a_1{}^2\right) \sin \frac{a_1 S}{2} \sinh \frac{a_2 S}{2} + 2a_1 a_2 \cos \frac{a_1 S}{2} \cosh \frac{a_2 S}{2}}{2a_1 a_2 \left(\sin^2 \frac{a_1 S}{2} \sinh^2 \frac{a_2 S}{2} + \cos^2 \frac{a_1 S}{2} \cosh^2 \frac{a_2 S}{2}\right)}$$

For a clamped boundary condition $\left(\overline{W} = \frac{d\overline{w}}{dx} = 0\right)$

$$\left.\begin{aligned}
A_1 &= \left(\frac{P}{1-\mu^2}\right)\frac{a_2 \cos a_1 \frac{S}{2} \sinh \frac{a_2 S}{2} - a_1 \sin a_1 \frac{S}{2} \cosh a_2 \frac{S}{2}}{a_2 \sin \frac{a_1 S}{2} \cos \frac{a_1 S}{2} + a_1 \sinh \frac{a_2 S}{2} \cosh \frac{a_2 S}{2}} \\[2ex]
A_2 &= \left(\frac{-P}{1-\mu^2}\right)\frac{a_1 \cos a_1 \frac{S}{2} \sinh \frac{a_2 S}{2} + a_2 \sin a_1 \frac{S}{2} \cosh a_2 \frac{S}{2}}{a_2 \sin \frac{a_1 S}{2} \cos \frac{a_1 S}{2} + a_1 \sinh \frac{a_2 S}{2} \cosh \frac{a_2 S}{2}}
\end{aligned}\right\} \quad (C11)$$

This solution could be used to determine the prestress quantities in the numerical solution of Chapter VII, in order that the effect of prestress deformations on the natural frequencies of the doubly curved shallow shells may be investigated. If there is no axially applied load ($\overline{N} = 0$), then from equation (C6) the deformations and rotations are independent of k_x.

APPENDIX D

AXISYMMETRIC VIBRATIONS

For the particular case of axisymmetric vibrations ($n = 0$), the circumferential equilibrium equation, that is, the second of equations (73), uncouples from the remaining equations hence the torsional frequency is independent of u and w. This equation, written in terms of the nondimensional meridional length x, becomes

$$\left(\frac{1 - \mu}{2}\right)\left(1 + \frac{\lambda^2}{3}\right) v,_{xx} - \frac{\nu h R^2}{B} v,_{tt} = 0 \qquad \text{(D1)}$$

Since k_x does not occur in this equation, the torsional frequency will be independent of the meridional curvature.

The general solution of equation (D1) is

$$v = \left[v_1 \sin\left(\frac{\Omega x}{\sqrt{\left(\frac{1 - \mu}{2}\right)\left(1 + \frac{\lambda^2}{3}\right)}}\right) + v_2 \cos\left(\frac{\Omega x}{\sqrt{\left(\frac{1 - \mu}{2}\right)\left(1 + \frac{\lambda^2}{3}\right)}}\right)\right] e^{i\omega t} \qquad \text{(D2)}$$

so that for the circumferential boundary conditions the torsional vibration frequencies is given by

$$\frac{\Omega S}{\sqrt{\left(\frac{1 - \mu}{2}\right)\left(1 + \frac{\lambda^2}{3}\right)}} = l\pi \qquad l = 1, 3, 5, \cdots$$

The minimum axisymmetric vibration frequency is thus given by

$$\Omega_{torsion} = \frac{\pi \sqrt{\left(\frac{1 - \mu}{2}\right)\left(1 + \frac{\lambda^2}{3}\right)}}{S} \tag{D3}$$

The remaining two equilibrium equations of equations (73) reduce to

$$\left.\begin{aligned} & u,_{xx} + \left(k_x + \mu\right) w,_x - \frac{\nu h R^2}{B} u,_{tt} = 0 \\ & \left(k_x + \mu\right) u,_x + \frac{\lambda^2}{12} w,_{xxxx} + \left(k_x^2 + 2\mu\, k_x + 1\right) w - \frac{\overline{N}_\xi}{B} w,_{xx} \\ & \quad + \frac{\nu h R^2}{B} w,_{tt} = 0 \end{aligned}\right\} \tag{D4}$$

These equations are handled in the same manner as were equations (73). The characteristic roots of equations (D4) are determined from a sixth degree equation formed by equating the determinant of the matrix which results after deleting the second row and column of the coefficient matrix of equation (76) to zero. With these modifications, the terms g_j vanish as they should since there is no longer any interdependency between w and v. Since only six roots are present, the sum of linear solutions in equations (82) range over six rather than eight terms. With the dependence on v removed, the boundary conditions in equations (85) associated with v and $N_{\xi\theta}$ must be deleted, that is, N_{2j}, N_{6j}, Y_{2j}, and Y_{6j} leaving three boundary conditions on each edge. The solution of the axisymmetric frequencies is determined in the same manner as is indicated in Chapter VIII.

XVII APPENDIX E

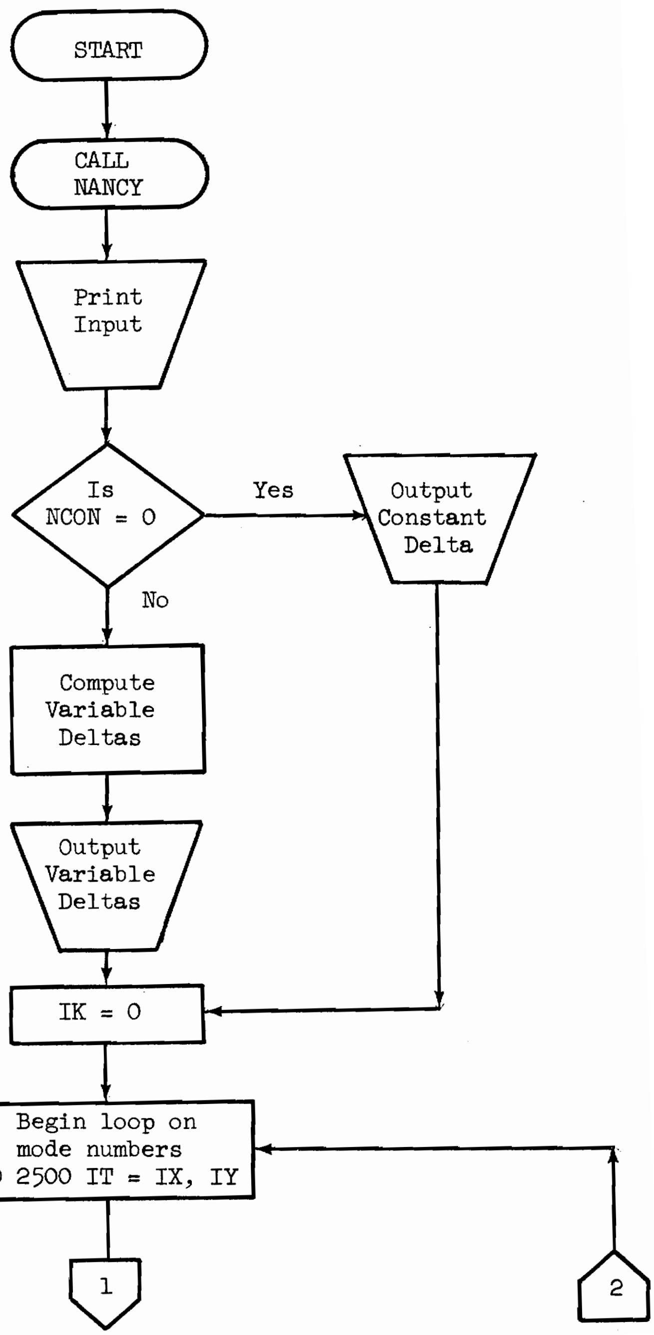
Printout of the three computer programs with sample output and flow diagrams used in the analysis.

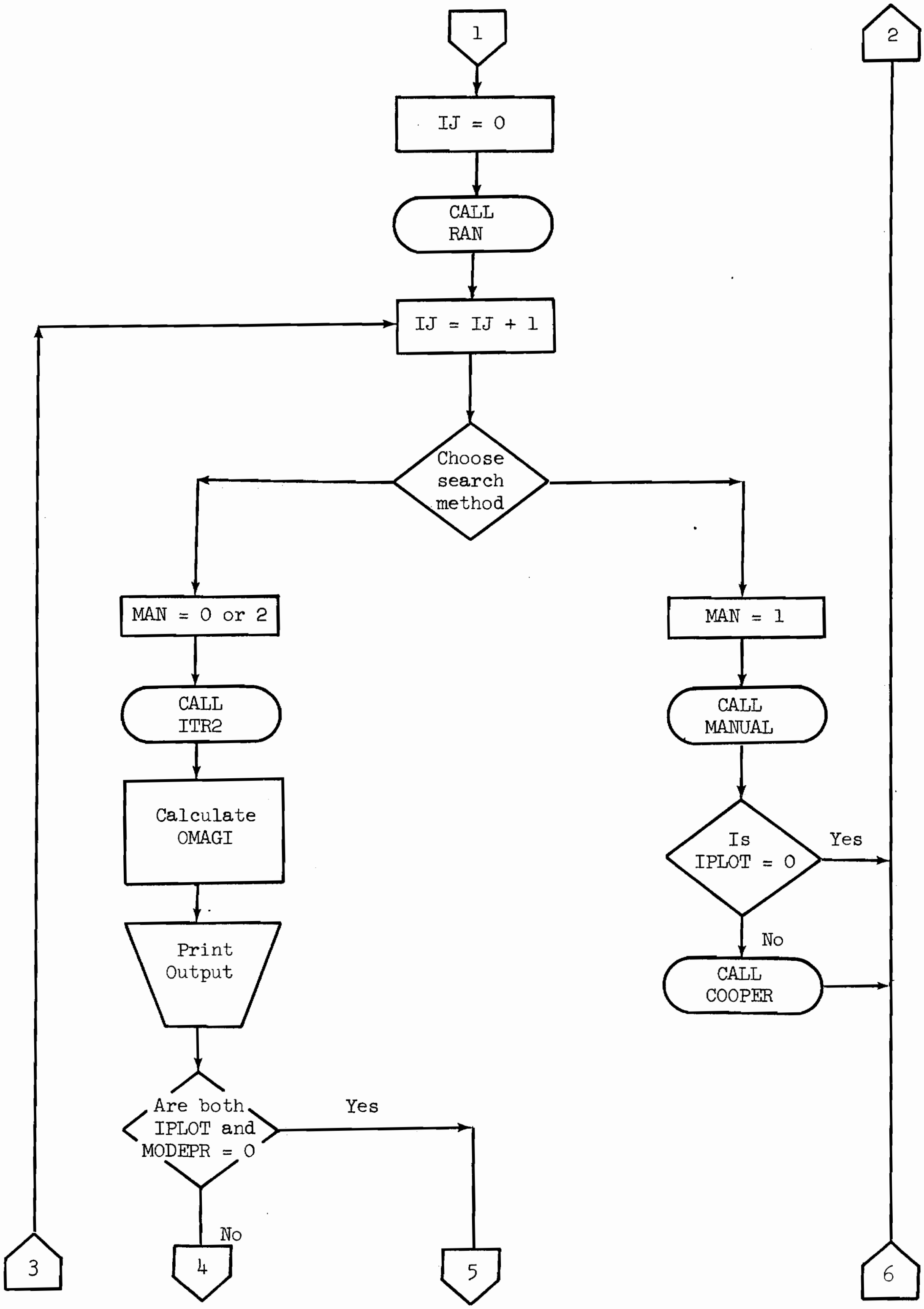
1. Computer program for numerical method of solution of the deep shell equations of Chapter VII.
2. Computer program for general method of solution of the approximate (shallow meridian) equations for Chapter VIII for several boundary conditions.
3. Computer program for method of solution of the approximate (shallow meridian) equations of Chapter VIII for a freely supported shell.

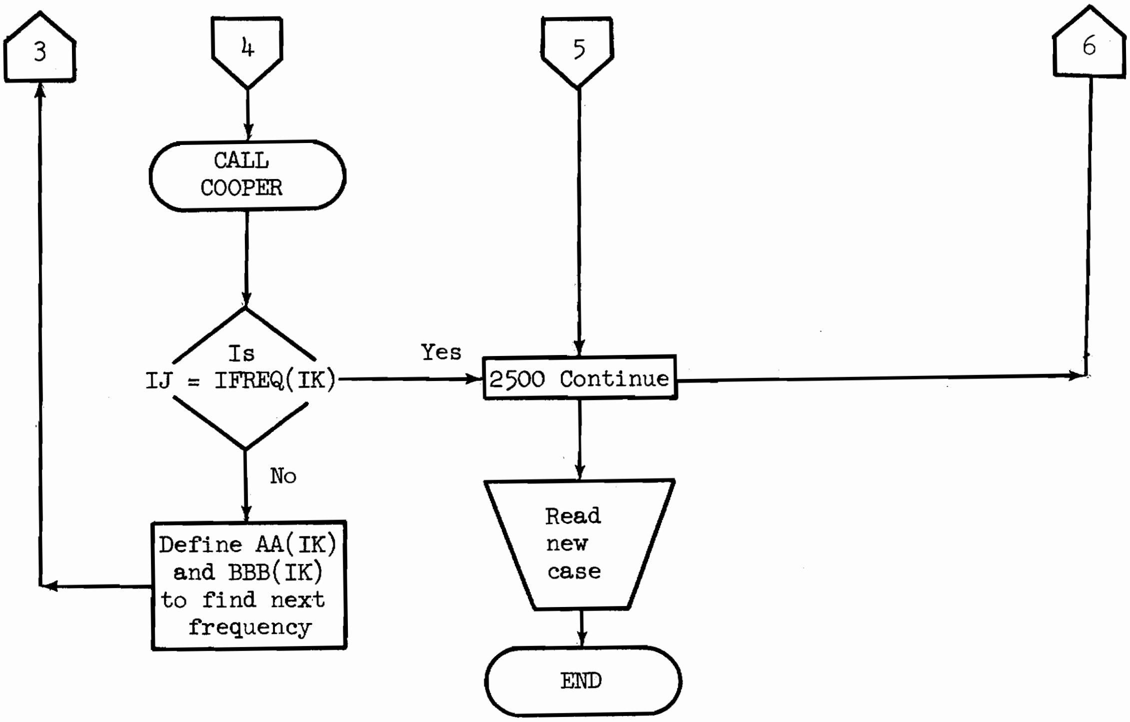
FLOW DIAGRAM OF MAIN PROGRAM FOR

NUMERICAL METHOD OF SOLUTION

OF THE DEEP SHELL EQUATIONS




1
2
IJ = 0
CALL
RAN
IJ = IJ + 1
Choose
search
method
MAN = 0 or 2
MAN = 1
CALL
ITR2
CALL
MANUAL
Calculate
OMAGI
Is
IPLOT = 0
Yes
No
Print
Output
CALL
COOPER
Are both
IPLOT and
MODEPR = 0
Yes
No
3
4
5
6


3
4
5
6
CALL
COOPER
Is
IJ = IFREQ(IK)
Yes
No
2500 Continue
Define AA(IK)
and BBB(IK)
to find next
frequency
Read
new
case
END

A. Main Program Variables

1. NCON - indicator for constant or variable delta values.
2. IK - subscript associated with each frequency interval.
3. IJ - counts the number of successive frequencies for a particular circumferential mode number.
4. MAN - indicator for search method used.
5. OMAGI - final interpolated value of the frequency.
6. IPLOT - indicator for use of plotting routine.
7. MODEPR - indicator for printing mode shapes.
8. IFREQ - array containing number of frequencies desired for each circumferential mode number.
9. AA, BBB - interval in which ITR2 subroutine searches for value of frequency.

B. Subroutines and Function Subprograms

1. NANCY - a user-written subroutine which supplies all geometry, prestress conditions, boundary conditions, and other input to the main program.
2. RAN - subroutine which calculates modified coefficient matrices of the differential equations and the difference equations. These matrices are independent of the frequency.
3. ITR2 - a NASA Langley Research Center library subroutine which, given $F(X) = 0$, searches for a sign change in a specified interval (A,B) and converges on X, using an interval halving procedure. ITR2 calls function subprogram FOFX, which in turn serves as a vehicle for calling COOPER.
4. MANUAL - subroutine which calculates the residuals when given a frequency interval and a constant frequency increment. MANUAL calls FOFX, which calls COOPER.
5. COOPER - subroutine which calculates the recursion matrices, the final characteristic determinant, and the mode shapes (if desired). Subroutines used by COOPER:
   a. MATINV - a NASA Langley Research Center library subroutine which finds the inverse of a matrix and calculates its determinant.
   b. DDIPLT - a NASA Langley Research Center library subroutine which generates plots on tape to be processed off-line on film or oscillograph paper.

```
      PROGRAM BAT (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT,TAPE21)

C   COMPUTER PROGRAM FOR NUMERICAL METHOD OF SOLUTION OF THE DEEP SHELL EQUATIONS
C        OF CHAPTER 7.   SAMPLE OUTPUT INCLUDED (SEE FIGURE 4 IN TEXT FOR
C        COMPARISON OF OUTPUT).         N. P. SYKES - P. A. COOPER

      DIMENSION  ALPHA1(4,4),ALPHA2(4,4),PERCNT(5),DELTA(201),NSTAT(5),
     1IFREQ(20),AA(20),BBB(20),DELTAX(20),XM(24),OMAG(2),DET(2),
     2ARCL(5),ISTAT(5),DEL(5)
     3,R(201),GAMMA(201),XK(201),TK(201),DXKDX(201),EPX(201),EPTH(201),
     4DEXDX(201),DETHDX(201),PHI(201),DPHIDX(201),
     3A(200,4,4),B(200,4,4),C(200,4,4),FF(4,4),FFF(4,4),HH(4,4),
     4HHH(4,4),SF(4,4),SFF(4,4),TT(4,4),TTT(4,4),EO(4,4),EN(4,4),
     5IPIVOT(4),INDEX(4,2),PAT(4,4),PAR(4,4),ACT(4,4),ACE(4,4),
     6BSAVE(201,3),H1(3),HM(3)
      COMMON /CDC/ VIBE,PPP                                             C
      COMMON /LAY20/ RESID,OMAGI                                        C
      COMMON /SUB/ ISUB                                                 C
      COMMON /BOUND/ALPHA1,ALPHA2
     1/GEOM/R,GAMMA,XK,TK,DXKDX
     1/PRESTR/EPX,EPTH,DEXDX,DETHDX,PHI,DPHIDX
     2/CONST/XLAM,XMU,S,NN,PERCNT,ALPHA,ELENG
     3/SPAC/DELTA,NSTAT,XN1,XN2,NCON,IFREQ,IPLOT,MAN,MODEPR
     4/CONITR/AA,BBB,DELTAX,E1,E2,MAXI,XM
     5/BATPCB/IM,OMAG,DET,NST,IJL,TERM
     6/BATFOF/XN
     7/BATRAN/XLAM2,XMU1
     5/RANPCB/A,B,C,FF,FFF,HH,HHH,SF,SFF,TT,TTT,EO,EN,IPIVOT,INDEX,
     6PAT,PAR,ACT,ACE
     8/SAVE/BSAVE,H1,HM
     9/FOPPL/IFOPPL,EBX,PEL
      EXTERNAL FOFX

C   CALL FOR GEOMETRY AND INPUT

    1 CALL OVERLAY (VIBE,3,0,PPP)                                       C
      XMU1=1.-XMU
      XLAM2=XLAM**2
      NST=ALPHA2(1,1)+ALPHA2(2,2)+ALPHA2(3,3)+ALPHA2(4,4)

C   WRITE TITLE, INPUT PARAMETERS, AND CODES
```

```
      WRITE (6,1177) (XM(I),I=1,24)                                      C
 1177 FORMAT (1H1,8A10/1X,8A10/1X,8A10)                                  C
      WRITE(6,1840)(ALPHA1(I,I),I=1,4),(ALPHA2(I,I),I=1,4)
 1840 FORMAT(20HOBOUNDARY CONDITIONS/
     1       4HOX=0/4X,12HALPHA1(1,1)=,F8.2/4X,12HALPHA1(2,2)=,F8.2/
     14X,12HALPHA1(3,3)=,F8.2/4X,12HALPHA1(4,4)=,F8.2/4HOX=S/4X,
     212HALPHA2(1,1)=,F8.2/4X,12HALPHA2(2,2)=,F8.2/4X,12HALPHA2(3,3)=,
     3F8.2/4X,12HALPHA2(4,4)=,F8.2/)
 1841 FORMAT(9HOGEOMETRY/4X7HLAMBDA=F9.5,3X13HAXIAL LENGTH=E16.8/5X6HALP
     1HA=F9.5,2X14HNO. INTERVALS=I5/9X2HS=F9.5,13X3HMU=F5.2)
 1842 FORMAT(//* CODES*//
     14X*NSTAT=*I4,4I3,3X*REGION BOUNDARIES AT THESE POINTS*//
     24X*NCON=*I2,3X*0 FOR CONSTANT DELTA, 1 FOR VARIABLE DELTAS*//
     34X*IPLOT=*I2,3X*0 FOR NO PLOTS, 1 FOR PLOTS OF MODE SHAPES*//
     44X*MAN=*I2,3X*0 FOR USING ITR2, 1 FOR MANUAL SEARCH, 2 FOR ITR2 WI
     5TH PRINTOUT*//
     64X*MAXI=*I3,3X*MAXIMUM NUMBER OF ITERATIONS FOR ITR2 SUBROUTINE*//
     74X*MODEPR=*I2,3X*0 FOR NO PRINTOUT OF MODE SHAPE VALUES, 1 FOR PRI
     8NTING THEM*//
     94X*E1=*E11.4,4X*E2=*E11.4,4X*CONVERGENCE TESTS USED IN ITR2*//
     14X*RANGE OF N IS*F4.0,* TO*F4.0//
     24X*IFOPPL=*I2,3X*0 FOR CONSTANT PRESTRESS (MEMBRANE),  1 FOR PREST
     3RESS DEFORMATIONS (FOPPL)*)
 1843 FORMAT(*0*4X*LOAD PARAMETER=*E16.8,*MERIDIONAL PRESTRESS=*E16.8)
      WRITE(6,1841) XLAM, ELENG, ALPHA, NN, S, XMU
      WRITE(6,1842) NSTAT, NCON, IPLOT, MAN, MAXI, MODEPR, E1,E2,XN1,XN2
     1,IFOPPL
      IF(IFOPPL.EQ.1) WRITE(6,1843)PEL,EBX
      IF(NCON.EQ.0)GO TO 2860

C  COMPUTE AND WRITE VARIABLE DELTAS IN REGIONS

      DO 2850 I=1,5
      ARCL(I)=PERCNT(I)*S
 2850 PERCNT(I)=PERCNT(I)*100.0
      ISTAT(1)=NSTAT(1)-1
      DO 2840 I=2,5
 2840 ISTAT(I)=NSTAT(I)-NSTAT(I-1)
      DO 2830 I=1,5
      K=NSTAT(I)-1
 2830 DEL(I)=DELTA(K)
      WRITE(6,2820)(ARCL(I),PERCNT(I),ISTAT(I),DEL(I),I=1,5)
 2820 FORMAT(10HOREGION(1)/6X,5HS(1)=,E18.8,4X,16HQ(1) PERCENTAGE=,
```

```
     1E18.8,4X,5HN(1)=,I6,4X,9HDELTA(1)=,E18.8/
     1        10HOREGION(2)/6X,5HS(2)=,E18.8,4X,16HQ(2) PERCENTAGE=,
     1E18.8,4X,5HN(2)=,I6,4X,9HDELTA(2)=,E18.8/
     1        10HOREGION(3)/6X,5HS(3)=,E18.8,4X,16HQ(3) PERCENTAGE=,
     1E18.8,4X,5HN(3)=,I6,4X,9HDELTA(3)=,E18.8/
     1        10HOREGION(4)/6X,5HS(4)=,E18.8,4X,16HQ(4) PERCENTAGE=,
     1E18.8,4X,5HN(4)=,I6,4X,9HDELTA(4)=,E18.8/
     1        10HOREGION(5)/6X,5HS(5)=,E18.8,4X,16HQ(5) PERCENTAGE=,
     1E18.8,4X,5HN(5)=,I6,4X,9HDELTA(5)=,E18.8)
      GO TO 2800

C  WRITE CONSTANT DELTA

 2860 WRITE(6,2810) DELTA(1)
 2810 FORMAT(///16H CONSTANT DELTA=F8.5)
 2800 IX=XN1+1.0
      IY=XN2+1.0
      IK=0

C  BEGIN LOOP ON VALUES OF N
C  IK ASSOCIATES A COUNTING INTEGER WITH VALUES OF N (MODE NUMBER)

      DO 2500 IT=IX,IY
      IM=0
      IK=IK+1
      NT=IT-1
      XN=FLOAT(NT)

C  IJ COUNTS FREQUENCIES CALCULATED FOR EACH N (TO COMPARE WITH IFREQ)

      IJ=0


C  CALL RAN SUBROUTINE TO CALCULATE MATRICES INDEPENDENT OF OMEGA

      CALL OVERLAY (VIBE,1,0,PPP)                                     C
 2100 IJ=IJ+1
      IJL=0

C  CHOOSE SEARCH METHOD
C   IF MAN=0  ITR2 IS USED WITH NO EXTRA PRINTOUT
C   IF MAN=1  ITR2 IS NOT USED, VALUES PRINTED OVER INTERVAL WITH CONSTANT DELTA
C   IF MAN=2  ITR2 IS USED WITH PRINTOUT AT EVERY VALUE
```

```
      WRITE(6,8887) AA(IK), BBB(IK), DELTAX(IK)
 8887 FORMAT(29H1INITIAL FREQUENCY INSPECTED=F10.7,11X12HUPPER LIMIT=F10
     1.7/2X27HINCREMENT USED IN INTERVAL=F10.7)
      IF(MAN.EQ.0) GO TO 4321
      WRITE(6,8888)NT
 8888 FORMAT(10HOMODE NO.=I6///6X9HITERATION,4X11HOMEGA VALUE,9X6HSCALED
     1/7X6HNUMBER,6X11H(FREQUENCY),8X8HRESIDUAL/)
      IF(MAN.EQ.2) GO TO 4321
      CALL MANUAL(X,AA(IK),BBB(IK),DELTAX(IK),FOFX)
      WRITE(6,8889) IJL
 8889 FORMAT(/////8X*NUMBER OF ITERATIONS=*I3)
      IF (IPLOT. EQ. 0) GO TO 2500
      OMAGI=AA(IK)
      ISUB=1

C  CALL COOPER TO CALCULATE MODE SHAPES FOR PARTICULAR FREQUENCY (WHICH
C                  WAS INPUT AS AA(1))

      CALL OVERLAY (VIBE,2,0,PPP)
      GO TO 2500
 4321 CALL ITR2(X,AA(IK),BBB(IK),DELTAX(IK),FOFX,E1,E2,MAXI,ICODE)

C  SUBROUTINE WHICH, GIVEN F(X)=0, CALCULATES X IN INTERVAL (A,B) USING AN
C        INTERVAL HALVING PROCEDURE.
C  SUBROUTINE IS ON TAPE AND AVAILABLE FOR LANGLEY PROGRAMMERS.

      IF(ICODE.EQ.1)WRITE(6,1191)
      IF(ICODE.EQ.2)WRITE(6,1192)
      IF(ICODE.EQ.3)WRITE(6,1193)
      IF(ICODE.EQ.4)WRITE(6,1194)
      IF(ICODE.NE.0) GO TO 2500
      OMAGI=OMAG(1)-DET(1)*((OMAG(2)-OMAG(1))/(DET(2)-DET(1)))
      WRITE(6,8889) IJL
      WRITE(6,8890) OMAG(1),OMAG(2),DET(1),DET(2),OMAGI
 8890 FORMAT(/2X27HBOUNDS FOR FINAL FREQUENCY=F10.7,2H ,,F10.7/
     15X24HCORRESPONDING RESIDUALS=E15.8,3H , ,E15.8//
     218X31HINTERPOLATED (FINAL) FREQUENCY=F11.8)
      IF(IPLOT.EQ.0.AND.MODEPR.EQ.0) GO TO 2300
      ISUB=1                                                            C

C  CALL COOPER TO CALCULATE MODE SHAPES AND PLOT OR PRINT THEM
```

```
      CALL OVERLAY (VIBE,2,0,PPP)                                        C
      WRITE(6,8891) TERM
 8891 FORMAT(19X23HCORRESPONDING RESIDUAL=E16.8)
      IF(IPLOT.NE.0) WRITE(6,8892)
 8892 FORMAT(*OMODE SHAPES HAVE BEEN PLOTTED*)
 1191 FORMAT(  31HOMAX NO. OF ITERATIONS EXCEEDED)
 1192 FORMAT( 26HOHDELTX IS LESS THAN OR = 0)
 1193 FORMAT(85HONO ROOT FOUND IN INTERVAL, MODE SHAPE CALCULATIONS AND
     1PLOTTING HAVE BEEN SUPPRESSED)
 1194 FORMAT(  20HOA IS GREATER THAN B)
      GO TO 2301
 2300 WRITE(6,9999)
 9999 FORMAT(21X36HMODE SHAPES NOT REQUESTED, THEREFORE/23X32HFINAL RESI
     1DUAL IS NOT CALCULATED)
 2301 IF (IFREQ(IK).EQ.IJ) GO TO 2500
      AA(IK)=X+DELTAX(IK)
      BBB(IK)=BBB(IK)+X
      GO TO 2100
 2500 CONTINUE
      GO TO 1
      END
```

```
      FUNCTION FOFX(R)                                                  C

C  FUNCTION CALLED FROM ITR2 ITERATION ROUTINE
C  HERE IT IS ESSENTIALLY A DUMMY FOR THE PURPOSE OF CALLING ANOTHER
C         LARGE SUBROUTINE

      COMMON /CDC/ VIBE,PPP                                             C
      COMMON /SUB/ ISUB                                                 C
      COMMON /OVER20/ DZ,T                                              C
      COMMON /BATFOF/XN
      T=R                                                               C
      ISUB=0                                                            C
      CALL OVERLAY (VIBE,2,0,PPP)                                       C
      FOFX=DZ
      RETURN
      END
```

```
      SUBROUTINE MANUAL(X,A,B,D,FOFX)
C  SUBROUTINE CALLED WHEN MANUAL SEARCH IS DESIRED
      X=A
  100 Y=FOFX(X)
      X=X+D
      IF(X.LE.B) GO TO 100
      RETURN
      END
```

```
      BLOCK DATA
C   NECESSARY IN OVERLAY PROCEDURE
      COMMON /CDC/ VIBE,PPP                                          C
      DATA VIBE/3LVIB/
      DATA PPP/6HRECALL/                                             C
      END
```

```
      SUBROUTINE DDIPLT (IEC,IN,N,X,Y,XMIN,XMAX,YMIN,YMAX,NXM,XM,NYM,YM,
     1                   ISYM,DDITPE)
C  LANGLEY SUBROUTINE FOR PLOTTING ON FILM OR OSCILLOGRAPH PAPER
C  SUBROUTINE IS ON TAPE AND AVAILABLE FOR LANGLEY PROGRAMMERS
C        IEC  DISPLAY END CODE
C             0    DATA INCOMPLETE
C             1    DISPLAY COMPLETE
C        IN   PROGRAMMER INITIALS AND JOB IDENTIFICATION (2 WORD ARRAY)
C        N    NUMBER OF DATA POINTS TO BE PLOTTED
C        X    ARRAY CONTAINING X COORDINATES
C        Y    ARRAY CONTAINING Y COORDINATES
C        XMIN ****
C        XMAX    *** DATA RANGE TO BE PLOTTED. IF VALUES ARE ZERO THE
C        YMIN    *** RANGE WILL BE DETERMINED ON FIRST CURVE OF DISPLAY
C        YMAX ****
C        NXM  NUMBER OF WORDS IN HORIZONTAL MESSAGE ARRAY
C        XM   HORIZONTAL MESSAGE
C        NYM  NUMBER OF WORDS IN VERTICAL MESSAGE ARRAY
C        YM   VERTICAL MESSAGE
C        ISYM SYMBOL CODE
C        DDITPE TAPE NAME IN PROGRAM CARD
C
      DIMENSION IN(2),ID(3),INIT(9),PLOT(100),IPLOT(100)
      DIMENSION X(N), Y(N), ISTAB(14), SM(3)
      DATA (ISTAB(I),I=1,14)/07620365533757575757575,
     1                       07620365534757575757575,
     2                       07620365520757575757575,
     3                       07620365553757575757575,
     4                       07620365554757575757575,
     5                       07620365540757575757575,
     6                       07620365561757575757575,
     7                       07620365515757575757575,
     8                       07620365516757575757575,
     9                       07620365513757575757575,
     1                       07620365514757575757575,
     2                       07620365532757575757575,
     3                       07620365552757575757575,
     4                       07660557575757575757575/
      DATA (SM(I),I=1,3)/07775757575757575757575,07575757575757575757575,
     10757575757575757575757575/
```

```
      DIMENSION XGRID(14), YGRID(14),IXGRID(14),IYGRID(14)
      DIMENSION XSC(14),IXPOS(14),YSC(14),IXSC(14),IYSC(14)
      EQUIVALENCE (PLOT,IPLOT)
      EQUIVALENCE (XGRID,IXGRID),(YGRID,IYGRID)
      EQUIVALENCE  (XSC,IXSC),(YSC,IYSC)
      INTEGER DDITPE
      DATA NPAGE/0/,IEND/1/,NCAM/07600227575757575757575/
      DATA (ID(I),I=1,3)/07600227654123717377 5,00,075757575757575760044/
      DATA (INIT(I),I=1,9)/07640341337377575757 5,00,076403413371775757 57
     15,00,075757575757575757540,00,07640341336177575757 5,00,07575757575
     27575760044/
C
C     TRANSFER ON FIRST ENTRY
C
      IF (INIT(2).EQ.0) GO TO 100
C
C     TRANSFER FOR START OF NEW DISPLAY (IEND=1) OR SET ICODE = 1
C
      IF (IEND.EQ.1) GO TO 110
      K=1
      GO TO 140
C
C     SEND INITIAL IDENTIFICATION FRAME
C
  100 CALL DISBCD (IN(1),INIT(2),1)
      ID(2)=INIT(2)
      WRITE (DDITPE) ID
C
C     INITIALIZE FOR A DISPLAY
C
  110 NPAGE=NPAGE+1
      LPCT=0
      IPLOT(1) = NCAM
      K=2
      CALL DISBCD (IN(2),INIT(4),1)
C
C     CHECK FOR X MAX-MIN VALUES
C
      IF (XMIN.EQ.0.0.AND.XMAX.EQ.0.0) GO TO 120
      XMN=XMIN
      XMX=XMAX
      GO TO 125
  120 XMN=X(1)
```

```
      XMX=X(1)
      DO 122 I=2,N
      IF (X(I).LT.XMN) XMN=X(I)
      IF (X(I).GT.XMX) XMX=X(I)
  122 CONTINUE
C
C     CHECK FOR Y MAX-MIN VALUES
C
  125 IF (YMIN.EQ.0.0.AND.YMAX.EQ.0.0) GO TO 127
      YMN=YMIN
      YMX=YMAX
      GO TO 130
  127 YMN=Y(1)
      YMX=Y(1)
      DO 129 I=2,N
      IF (Y(I).LT.YMN) YMN=Y(I)
      IF (Y(I).GT.YMX) YMX=Y(I)
  129 CONTINUE
C
C     SETUP HORIZONTAL(X) MESSAGE
C
  130 IPOS=520-NXM*40
      IHIGH=IPOS/40B
      ILOW=IPOS-IHIGH*40B
      IPLOT(K)=7575757576400000000B+(IHIGH*100B+ILOW)*10000B
      K=K+1
      CALL DISBCD (XM,PLOT(K),NXM)
      K=K+NXM
C
C     SETUP VERTICAL(Y) MESSAGE
C
      IPOS=520-NYM*40
      IHIGH=IPOS/40B
      ILOW=IPOS-IHIGH*40B
      IPLOT(K)=7575757576420000000B+IHIGH*100B+ILOW
      K=K+1
      CALL DISBCD (YM,PLOT(K),NYM)
      K=K+NYM
C
C     STORE MINIMUM VALUES FOR DISPLAY
C
      ENCODE (10,1,XMNT) XMN
    1 FORMAT (E10.3)
```

```
      ENCODE (10,1,YMNT) YMN
      GO TO 150
*
*     ADDITIONAL DATA FOR DISPLAY
*     CHECK X MAX-MIN VALUES
*
  140 IF (XMIN.NE.0.0.OR. XMAX.NE.0.0) GO TO 142
      ICODE2=0
      GO TO 145
  142 XMN=XMIN
      XMX=XMAX
      ICODE2=1
C
C     CHECK Y MAX-MIN VALUES
C
  145 IF (YMIN.NE.0.0.OR. YMAX.NE.0.0) GO TO 147
      IF (ICODE2) 150,1247,150
  147 YMN=YMIN
      YMX=YMAX
C
C     CHECK DATA RANGE, ADJUST IF ZERO
C
  150 XRANGE=XMX-XMN
      YRANGE=YMX-YMN
      IF (XRANGE.NE.0.0) GO TO 154
      IF (XMN.EQ.0.0)GO TO 152
      XMN=XMN-.1*XMN
      XMX=XMX+.1*XMX
      GO TO 154
  152 XMN=-1.0
      XMX=1.0
  154 IF (YRANGE.NE.0.0) GO TO 160
      IF (YMN.EQ.0.0) GO TO 156
      YMN=YMN-.1*YMN
      YMX=YMX+.1*YMX
      GO TO 160
  156 YMN=-1.0
      YMX=1.0
C
C     FIND INCREMENT AND  ADJUST MAX-MIN VALUES
C
  160 XINC=(XMX-XMN)/10.0
      I=0
```

```
  162 IF (XINC.LT.10.0) GO TO 164
      I=I+1
      XINC=XINC/10.0
      GO TO 162
  164 IF (XINC.GE.1.0) GO TO 166
      I=I-1
      XINC=XINC*10.0
      GO TO 164
  166 IF (XINC.EQ.1.0) GO TO 1167
      IF (XINC.GT.2.0) GO TO 167
      XINC=2.0
      GO TO 169
 1167 XINC=1.0
      GO TO 169
  167 IF (XINC .GT.5.0) GO TO 168
      XINC=5.0
      GO TO 169
  168 XINC=10.0
  169 IIX=I
  170 IF (I) 171,173,172
  171 I=I+1
      XMN=XMN*10.0
      XMX=XMX*10.0
      GO TO 170
  172 I=I-1
      XINC=XINC*10.0
      GO TO 170
  173 XR=AINT(XMN/XINC)*XINC
      IF (XR.GT.XMN) XR=XR-XINC
      XMN=XR
      XR=AINT(XMX/XINC)*XINC
      IF (XMX.GT.XR) XR=XR+XINC
      XMX=XR
C
      YINC=(YMX-YMN)/10.0
      I=0
  174 IF (YINC.LT.10.0) GO TO 175
      I=I+1
      YINC=YINC/10.0
      GO TO 174
  175 IF (YINC.GE.1.0) GO TO 176
      I=I-1
      YINC=YINC*10.0
```

```
      GO TO 175
  176 IF (YINC.EQ.1.0) GO TO 1177
      IF (YINC.GT.2.0) GO TO 177
      YINC=2.0
      GO TO 179
 1177 YINC=1.0
      GO TO 179
  177 IF (YINC.GT.5.0) GO TO 178
      YINC=5.0
      GO TO 179
  178 YINC=10.0
  179 IIY=I
  180 IF (I) 181,183,182
  181 I=I+1
      YMN=YMN*10.0
      YMX=YMX*10.0
      GO TO 180
  182 I=I-1
      YINC=YINC*10.0
      GO TO 180
  183 YR=AINT(YMN/YINC)*YINC
      IF (YR.GT.YMN ) YR=YR-YINC
      YMN=YR
      YR=AINT(YMX/YINC)*YINC
      IF (YMX.GT.YR) YR=YR+YINC
      YMX=YR
C
C     TABLE FOR X AND Y GRID
C
      XSCALE=(XMX-XMN)/983.0
      YSCALE=(YMX-YMN)/983.0
      IF (IEND.EQ.0) GO TO 246
      IX=1
      XGRID(1)=XMN
  184 IX=IX+1
      XGRID(IX)=XGRID(IX-1)+XINC
      IF (XGRID(IX).LT.XMX) GO TO 184
C
      IY=1
      YGRID(1)=YMN
  186 IY=IY+1
      YGRID(IY)=YGRID(IY-1)+YINC
      IF (YGRID(IY).LT.YMX) GO TO 186
```

```
C
C     TABLE FOR X AND Y LABELS
C
      DO 190 I=1,IX
  190 IXSC(I)=XGRID(I)
  192 IF (IABS(IXSC(1)).LT.1000.AND.IABS(IXSC(IX)).LT.1000) GO TO 196
      DO 193 I=1,IX
  193 IXSC(I)=IXSC(I)/10
      GO TO 192
  196 DO 205 I=1,IX
      IF (IABS(IXSC(I)).GT.9) GO TO 197
      IXPOS(I)=-24
      GO TO 200
  197 IF (IABS(IXSC (I)).GT. 99) GO  TO 198
      IXPOS(I)= -20
      GO  TO 200
  198 IXPOS(I)=-16
  200 ENCODE (10,2,IXSC(I))IXSC(I)
    2 FORMAT (I4)
  205 CONTINUE
      IXPOS(IX)=-24
      CALL DISBCD (IXSC(1),IXSC(1),IX)
      DO 210 I=1,IY
  210 IYSC(I)=YGRID(I)
  212 IF (IABS(IYSC(1)).LT.1000.AND.IABS(IYSC(IY)).LT.1000) GO TO 216
      DO 213 I=1,IY
  213 IYSC(I)=IYSC(I)/10
      GO TO 212
  216 DO 220 I=1,IY
      ENCODE (10,2,IYSC(I))IYSC(I)
  220 CONTINUE
      CALL DISBCD (IYSC(1),IYSC(1),IY)
      DO 230 I=1,IX
  230 IXGRID(I)=(XGRID(I)-XMN)/XSCALE+40.5
      DO 235 I=1,IY
  235 IYGRID(I)=(YGRID(I)-YMN)/YSCALE+32.5
C
C     DRAW GRID LINES
C
  238 IA1=IXGRID(1)/40B
      IA2=IXGRID(1)-IA1*40B
      IA3=IXGRID(IX)/40B
      IA4=IXGRID(IX)-IA3*40B
```

```
      IA5=7660000000000000000B+(IA1*100B+IA2)*1000000000000B+(IA3*100B+
     1IA4)*10000B
      IA6=76400014000075757575B
      DO 240 I=1,IY
      IA1=IYGRID(I)/40B
      IA2=IYGRID(I)-IA1*40B
      IA3=IA1*100B+IA2
      IPLOT(K)=IA5+IA3*100000000B+IA3
      IPLOT(K+1)=IA6+IA3*100000000B
      IPLOT(K+2)=IYSC(I)
  240 K=K+3
      IA1=IYGRID(1)/40B
      IA2=IYGRID(1)-IA1*40B
      IA3=IYGRID(IY)/40B
      IA4=IYGRID(IY)-IA3*40B
      IA5=7660000000000000000B+(IA1*100B+IA2)*100000000B+(IA3*100B+IA4)
      IA6=76400000001775757575B
      DO 245 I=1,IX
      IA1=IXGRID(I)/40B
      IA2=IXGRID(I)-IA1*40B
      IA3=IA1*100B+IA2
      IA7=IXGRID(I)+IXPOS(I)
      IA8=IA7/40B
      IA9=IA7-IA8*40B
      IA10=IA8*100B+IA9
      IPLOT(K)=IA5+IA3*1000000000000B+IA3*10000B
      IPLOT(K+1)=IA6+IA10*1000000000000B
      IPLOT(K+2)=IXSC(I)
  245 K=K+3
  246 IF (IIX.GE.0) GO TO 247
      IIX=IIX+1
      XMN=XMN/10.0
      XMX=XMX/10.0
      XSCALE=XSCALE/10.0
      XINC=XINC/10.0
      GO TO 246
  247 IF (IIY.GE.0) GO TO 1180
      IIY=IIY+1
      YMN=YMN/10.0
      YMX=YMX/10.0
      YSCALE=YSCALE/10.0
      YINC=YINC/10.0
      GO TO 247
```

```
C
C      DISPLAY MINIMUM VALUES AND INCREMENT
C
 1180 IF (IEND.NE.1) GO TO 1247
      IPLOT(K)=76400024373775756060B
      IPLOT(K+1)=60606760443145601360B
      CALL DISBCD (XMNT,IPLOT(K+2),1)
      IPLOT(K+3)=60314523512544254563B
 1185 ENCODE (10,1,IPLOT(K+4)) XINC
      CALL DISBCD (IPLOT(K+4),IPLOT(K+4),1)
      IPLOT(K+5)=60607060443145601360B
      CALL DISBCD (YMNT,IPLOT(K+6),1)
      IPLOT(K+7)=60314523512544254563B
 1195 ENCODE (10,1,IPLOT(K+8)) YINC
      CALL DISBCD (IPLOT(K+8),IPLOT(K+8),1)
      K=K+9
C
C      SET SYMBOL AND SCALE DATA TO PLOT
C
 1247 IPLOT(K)=ISTAB(ISYM)
      K=K+1
      DO 250 I=1,N
      IF (X(I).GT.XMX.OR.X(I).LT.XMN.OR.Y(I).GT.YMX.OR.Y(I).LT.YMN) GO
     1TO 248
      IA1=(X(I)-XMN)/XSCALE+40.5
      IA2=IA1/40B
      IA3=IA1-IA2*40B
      IA4=(Y(I)-YMN)/YSCALE+32.5
      IA5=IA4/40B
      IA6=IA4-IA5*40B
      IPLOT(K)=75757575757500000000B+(IA2*100B+IA3)*10000B+IA5*100B+IA6
      K=K+1
      GO TO 249
  248 LPCT=LPCT+1
  249 IF (K.LT.101) GO TO 250
      WRITE (DDITPE)(IPLOT(J),J=1,100)
      K=1
  250 CONTINUE
      IF (K.EQ.1) GO TO 252
      K=K-1
      WRITE (DDITPE) (IPLOT(I),I=1,K)
  252 IF (IEC.EQ.1) GO TO 255
      IEND=0
```

```
      GO TO 1000
  255 ENCODE (10,2,INIT(6))NPAGE
      CALL DISBCD (INIT(6),INIT(6),1)
      IEND=1
      IF (LPCT.EQ.0) GO TO 256
      ENCODE (10,2,INIT(8))LPCT
      CALL DISBCD (INIT(8),INIT(8),1)
      GO TO 260
  256 INIT(8)=60606060606060606060B
  260 WRITE (DDITPE)(INIT(I),I=1,9)
      WRITE (DDITPE)(SM(I),I=1,3)
      IEND=1
 1000 RETURN
      END
```

```
       PROGRAM RAN
       DIMENSION  ALPHA1(4,4),ALPHA2(4,4),R(201),GAMMA(201),XK(201),
      1TK(201),DXKDX(201),EPX(201),EPTH(201),DEXDX(201),DETHDX(201),
      2PHI(201),DPHIDX(201),PERCNT(5),DELTA(201),NSTAT(5),IFREQ(20),
      3A(200,4,4),B(200,4,4),C(200,4,4),FF(4,4),FFF(4,4),HH(4,4),
      4HHH(4,4),SF(4,4),SFF(4,4),TT(4,4),TTT(4,4),EO(4,4),EN(4,4),
      5IPIVOT(4),INDEX(4,2),PAT(4,4),PAR(4,4),ACT(4,4),ACE(4,4),
      6BSAVE(201,3),H1(3),HM(3),
      7F(4,4),G(4,4),H(4,4),EL(4,4),FL(4,4)
       COMMON /BATFOF/ XN                                                  C
       COMMON /BOUND/ALPHA1,ALPHA2  /GEOM/R,GAMMA,XK,TK,DXKDX
      1/PRESTR/EPX,EPTH,DEXDX,DETHDX,PHI,DPHIDX
      2/CONST/XLAM,XMU,S,NN,PERCNT,ALPHA,ELENG
      3/SPAC/DELTA,NSTAT,XN1,XN2,NCON,IFREQ,IPLOT,MAN,MODEPR
      5/RANPCB/A,B,C,FF,FFF,HH,HHH,SF,SFF,TT,TTT,EO,EN,IPIVOT,INDEX,
      6PAT,PAR,ACT,ACE
      7/BATRAN/XLAM2,XMU1
      8/SAVE/BSAVE,H1,HM
       MM=1
       M=NN+1
       DO 2000 I=1,M
C
C     COMPUTE BIG F MATRIX
C
       F(1,1)=1.0
       F(1,2)=0.0
       F(1,3)=-PHI(I)
       F(1,4)=0.0
       F(2,1)=0.0
       F(2,2)=XMU1/2.+(XLAM2*XMU1/96.)*(3.*TK(I)-XK(I))**2+.25*(EPX(I)+
      1EPTH(I))
       F(2,3)=(XLAM2*XMU1*XN/(24.*R(I)))*(3.*TK(I)-XK(I))
       F(2,4)=0.0
       F(3,1)=-PHI(I)
       F(3,2)=F(2,3)
       F(3,3)=(XLAM2*XMU1/12.)*(2.*XN**2/R(I)**2+(1.+XMU)*XLAM2)+EPX(I)+
      1PHI(I)**2
       F(3,4)=XLAM2*(1.-XMU**2)
       F(4,1)=0.0
       F(4,2)=0.0
       F(4,3)=F(3,4)
       F(4,4)=0.0
```

```
C     COMPUTE G MATRIX

      G(1,1)=GAMMA(I)
      G(1,2)=(1.+XMU)*XN/(2.*R(I))+(XLAM2*XN*XMU1/(96.*R(I)))*(3.*
     1XK(I)-TK(I))*(3.*TK(I)-XK(I))-XN/(4.*R(I))*(EPX(I)+EPTH(I))
      G(1,3)=XK(I)+XMU*TK(I)+(XLAM2*(1.-XMU)/12.)*((1.+XMU)*GAMMA(I)**2
     1*XK(I)+(XN**2/(2.*R(I)**2))*(3.*XK(I)-TK(I)))+XK(I)*EPX(I)-(1.-
     2XMU)*GAMMA(I)*PHI(I)+XK(I)*PHI(I)**2-DPHIDX(I)
      G(1,4)=XLAM2*(1.-XMU**2)*XK(I)
      G(2,1)=-G(1,2)
      G(2,2)=XMU1/2.*GAMMA(I)-(XLAM2*XMU1/96.)*(3.*TK(I)-XK(I))*(2.*
     1DXKDX(I)-GAMMA(I)*(5.*XK(I)-3.*TK(I)))+.25*(DEXDX(I)+DETHDX(I))
     2+GAMMA(I)/4.*(EPX(I)+EPTH(I))
      G(2,3)=(XLAM2*XMU1*XN/(24.*R(I)))*(2.*(1.+XMU)*GAMMA(I)*TK(I)
     1-DXKDX(I)+3.*GAMMA(I)*(XK(I)-TK(I)))+(1.+XMU)*XN/(2.*R(I))*PHI(I)
      G(2,4)=0.0
      G(3,1)=-G(1,3)-2.*GAMMA(I)*PHI(I)-2.*DPHIDX(I)
      G(3,2)=(XLAM2*XMU1*XN/(24.*R(I)))*(3.*GAMMA(I)*XK(I)-GAMMA(I)
     1*TK(I)*(5.+2.*XMU)-DXKDX(I))-XN/(2.*R(I))*(1.+XMU)*PHI(I)
      G(3,3)=-(XLAM2*XMU/12.)*((1.+XMU)*(2.*GAMMA(I)*XK(I)*TK(I)
     1+GAMMA(I)**3)+2.*XN**2*GAMMA(I)/R(I)**2)+GAMMA(I)*EPX(I)
     2+DEXDX(I)+GAMMA(I)*PHI(I)**2+2.*PHI(I)*DPHIDX(I)
      G(3,4)=XLAM2*(2.-XMU)*(1.-XMU**2)*GAMMA(I)
      G(4,1)=-G(1,4)
      G(4,2)=0.0
      G(4,3)=XLAM2*(1.-XMU**2)*XMU*GAMMA(I)
      G(4,4)=0.0

C     COMPUTE H BAR MATRIX

      H(1,1)=-XMU*XK(I)*TK(I)-GAMMA(I)**2-(1.-XMU)*XN**2/(2.*R(I)**2)
     1-(XLAM2*XMU1/12.)*((1.+XMU)*GAMMA(I)**2*XK(I)**2+XN**2*(3.
     2*XK(I)-TK(I))**2/(8.*R(I)**2))-XK(I)**2*EPX(I)-XN**2/(4.*R(I)**2)
     3*(EPX(I)+EPTH(I))+((1.-2.*XMU)*GAMMA(I)*XK(I)+DXKDX(I))*PHI(I)
     4-XK(I)**2*PHI(I)**2+XK(I)*DPHIDX(I)
      H(1,2)=-(3.-XMU)*XN*GAMMA(I)/(2.*R(I))-(XLAM2*XMU1*XN*GAMMA(I)
     1/(12.*R(I)))*((3.*XK(I)-TK(I))*(3.*TK(I)-XK(I))/8.+(1.+XMU)
     2*XK(I)*TK(I))-XN*GAMMA(I)/(4.*R(I))*(EPX(I)+EPTH(I))-XMU*XN/R(I)
     3*XK(I)*PHI(I)+XMU1*XN/(2.*R(I))*TK(I)*PHI(I)
      H(1,3)=DXKDX(I)+GAMMA(I)*(XK(I)-TK(I))-XLAM2*XMU1*XN**2*GAMMA(I)
     1/(12.*R(I)**2)*((3.*XK(I)-TK(I))/2.+(1.+XMU)*XK(I))-XK(I)
     2*(XK(I)+XMU*TK(I))*PHI(I)+XMU1*XN**2/(2.*R(I)**2)*PHI(I)
```

```
      H(1,4)=XLAM2*(1.-XMU**2)*XMU1*GAMMA(I)*XK(I)
      H(2,1)=-(3.-XMU)*XN*GAMMA(I)/(2.*R(I))+(XLAM2*XMU1*XN/(12.*R(I)))
     1*(-(1.+XMU)*GAMMA(I)*XK(I)*TK(I)+GAMMA(I)/8.*(6.*XK(I)*TK(I)-7.
     2*XK(I)**2-3.*TK(I)**2)-.25*DXKDX(I)*(5.*TK(I)-3.*XK(I)))+XN/(4.
     3*R(I))*(DEXDX(I)+DETHDX(I))-XN*GAMMA(I)*.25/R(I)*(EPX(I)+EPTH(I))
     4-XMU*XN/R(I)*XK(I)*PHI(I)+XMU1*XN*.5/R(I)*TK(I)*PHI(I)
      H(2,2)=-GAMMA(I)*G(2,2)+.5*XMU1*XK(I)*TK(I)-XN**2/R(I)**2-XLAM2
     1*XMU1/12.*((1.+XMU)*XN**2/R(I)**2*TK(I)**2-XK(I)*TK(I)/8.*(3.
     2*TK(I)-XK(I))**2)+.5*GAMMA(I)*(DEXDX(I)+DETHDX(I))-TK(I)**2
     3*EPTH(I)-.25*XK(I)*TK(I)*(EPX(I)+EPTH(I))+.5*XMU1*GAMMA(I)
     4*(XK(I)+2.*TK(I))*PHI(I)-.5*XMU1*TK(I)**2*PHI(I)**2+(XMU1*.5)
     5*TK(I)*DPHIDX(I)
      H(2,3)=-XN/R(I)*(TK(I)+XMU*XK(I))+XLAM2*XMU1*XN/(24.*R(I))*(GAMMA
     1(I)*DXKDX(I)-2.*GAMMA(I)**2*XK(I)-2.*(1.+XMU)*XN**2/R(I)**2*TK(I)
     2+(3.*TK(I)-XK(I))*(GAMMA(I)**2+XK(I)*TK(I)))-XN/R(I)*TK(I)*EPTH(I)
     3+XMU1*XN*GAMMA(I)*.5/R(I)*PHI(I)-XMU1*XN*.5/R(I)*TK(I)*PHI(I)**2
     4+XMU1*XN*.5/R(I)*DPHIDX(I)
      H(2,4)=-XLAM2*XMU*(1.-XMU**2)*XN/R(I)*TK(I)
      H(3,1)=-GAMMA(I)*(TK(I)+XMU*XK(I))+XLAM2*XMU1/12.*((1.+XMU)*GAMMA
     1(I)*(GAMMA(I)**2*XK(I)-GAMMA(I)*DXKDX(I)-XN**2/R(I)**2*XK(I)+2.
     2*XK(I)**2*TK(I))+XN**2*.5/R(I)**2*(GAMMA(I)*XK(I)-GAMMA(I)*TK(I)
     3-3.*DXKDX(I)))-(GAMMA(I)*XK(I)+DXKDX(I))*EPX(I)-XK(I)*DEXDX(I)
     4+(XMU1*XN**2*.5/R(I)**2-XK(I)**2)*PHI(I)-(GAMMA(I)*XK(I)+DXKDX(I))
     5*PHI(I)**2-(2.*XK(I)*PHI(I)+XMU*GAMMA(I))*DPHIDX(I)
      H(3,2)=-XN/R(I)*(TK(I)+XMU*XK(I))+XLAM2*XMU1*XN/(24.*R(I))*(2.
     1*(1.+XMU)*(XK(I)*TK(I)**2-GAMMA(I)**2*XK(I)+2.*GAMMA(I)**2*TK(I)
     2-XN**2/R(I)**2*TK(I))+GAMMA(I)*DXKDX(I)+3.*GAMMA(I)**2*(TK(I)
     3-XK(I))+XK(I)*TK(I)*(3.*TK(I)-XK(I)))-XN/R(I)*TK(I)*EPTH(I)
     4+XMU1*XN*GAMMA(I)*.5/R(I)*PHI(I)-XMU1*XN*.5/R(I)*TK(I)*PHI(I)**2
     5-XMU*XN/R(I)*DPHIDX(I)
      H(3,3)=-XK(I)**2-2.*XMU*XK(I)*TK(I)-TK(I)**2+(XLAM2*XMU1*XN**2
     1/(12.*R(I)**2))*((1.+XMU)*(XK(I)*TK(I)-XN**2/R(I)**2+2.*GAMMA(I)
     2**2)+2.*(GAMMA(I)**2+XK(I)*TK(I)))-XN**2/R(I)**2*EPTH(I)-(GAMMA
     3(I)*XK(I)*(1.+XMU)+DXKDX(I))*PHI(I)-XMU1*XN**2*.5/R(I)**2*PHI(I)
     4**2-(XK(I)+XMU*TK(I))*DPHIDX(I)
      H(3,4)=-XLAM2*(1.-XMU**2)*(XMU1*XK(I)*TK(I)+XMU*XN**2/R(I)**2)
      H(4,1)=-XLAM2*(1.-XMU**2)*(DXKDX(I)+XMU*GAMMA(I)*XK(I))
      H(4,2)=H(2,4)
      H(4,3)=-XLAM2*(1.-XMU**2)*XMU*XN**2/R(I)**2
      H(4,4)=12.*XLAM2*(1.-XMU**2)**2
      IF (I.EQ.1) GO TO 100
      IF (I.EQ.M) GO TO 100
      GO TO 200
```

```
COMPUTE LITTLE E MATRIX

  100 DO 150 J=1,4
      DO 150 K=1,4
  150 EL(J,K)=F(J,K)

COMPUTE LITTLE F MATRIX

      FL(1,1)=XMU*GAMMA(I)+XK(I)*PHI(I)
      FL(1,2)=XMU*XN/R(I)
      FL(1,3)=XK(I)+XMU*TK(I)
      FL(1,4)=0.0
      FL(2,1)=-XMU1*XN*.5/R(I)-XLAM2*XMU1*XN/(96.*R(I))*(3.*XK(I)-TK(I))
     1*(3.*TK(I)-XK(I))+XN*.25/R(I)*(EPX(I)+EPTH(I))
      FL(2,2)=-XMU1*GAMMA(I)*.5-XLAM2*GAMMA(I)*XMU1/96.*(3.*TK(I)-XK(I))
     1**2+GAMMA(I)*.25*(EPX(I)+EPTH(I))+XMU1*.5*TK(I)*PHI(I)
      FL(2,3)=-GAMMA(I)*EL(2,3)+XMU1*XN*.5/R(I)*PHI(I)
      FL(2,4)=0.0
      FL(3,1)=-XLAM2*XMU1/12.*((1.+XMU)*GAMMA(I)**2*XK(I)+XN**2*.5/R(I)
     1**2*(3.*XK(I)-TK(I)))-XK(I)*EPX(I)-XMU*GAMMA(I)*PHI(I)-XK(I)
     2*PHI(I)**2
      FL(3,2)=-XLAM2*XMU1*XN*GAMMA(I)/(24.*R(I))*(3.*TK(I)-XK(I)+2.
     1*(1.+XMU)*TK(I))-XMU*XN/R(I)*PHI(I)
      FL(3,3)=-XLAM2*XMU1/12.*(3.+XMU)*XN**2*GAMMA(I)/R(I)**2-(XK(I)
     1+XMU*TK(I))*PHI(I)
      FL(3,4)=XLAM2*XMU1*(1.-XMU**2)*GAMMA(I)
      FL(4,1)=-XLAM2*(1.-XMU**2)*XK(I)
      FL(4,2)=0.0
      FL(4,3)=0.0
      FL(4,4)=0.0
      IF (I.EQ.M) GO TO 170
      DO 160 J=1,4
      DO 160 K=1,4
      FF(J,K)=F(J,K)
      HH(J,K)=H(J,K)
  160 SF(J,K)=FL(J,K)
      DO 161 J=1,3
  161 H1(J)=HH(J,J)
      GO TO 200
  170 DO 180 J=1,4
      DO 180 K=1,4
      FFF(J,K)=F(J,K)
```

```
      HHH(J,K)=H(J,K)
  180 SFF(J,K)=FL(J,K)
      DO 181 J=1,3
  181 HM(J)=HHH(J,J)
  200 IF (I.NE.1) GO TO 500

COMPUTE EO MATRIX

      DO 250 J=1,4
      DO 250 K=1,4
  250 PAR(J,K)=F(J,K)/DELTA(I)**2-G(J,K)*.5/DELTA(I)
      CALL MATINV (PAR,4,PAT,0,DETERM,IPIVOT,INDEX,4,ISCALE)
      DO 260 J=1,4
      DO 260 K=1,4
      ACE(J,K)=EL(J,K)*.5/DELTA(I)
  260 TT(J,K)=PAR(J,K)
      DO 270 J=1,4
      DO 270 K=1,4
      ACT(J,K)=0.0
      DO 270 L=1,4
  270 ACT(J,K)=ACT(J,K)+ALPHA1(J,L)*ACE(L,K)
      DO 275 J=1,4
      DO 275 K=1,4
      ACE(J,K)=0.0
      DO 275 L=1,4
  275 ACE(J,K)=ACE(J,K)+ACT(J,L)*PAR(L,K)
      DO 280 J=1,4
      DO 280 K=1,4
  280 PAR(J,K)=F(J,K)/DELTA(I)**2+G(J,K)*.5/DELTA(I)
      DO 290 J=1,4
      DO 290 K=1,4
      EO(J,K)=0.0
      DO 290 L=1,4
  290 EO(J,K)=EO(J,K)+ACE(J,L)*PAR(L,K)
      DO 300 J=1,4
      DO 300 K=1,4
  300 EO(J,K)=EO(J,K)+ACT(J,K)
      GO TO 2000
  500 IF (I.NE.M) GO TO 800

COMPUTE EN MATRIX

      DO 550 J=1,4
```

```
      DO 550 K=1,4
  550 PAR(J,K)=F(J,K)/DELTA(I-1)**2+G(J,K)*.5/DELTA(I-1)
      CALL MATINV (PAR,4,PAT,0,DETERM,IPIVOT,INDEX,4,ISCALE)
      DO 560 J=1,4
      DO 560 K=1,4
      ACE(J,K)=EL(J,K)*.5/DELTA(NN)
  560 TTT(J,K)=PAR(J,K)
      DO 570 J=1,4
      DO 570 K=1,4
      ACT(J,K)=0.0
      DO 570 L=1,4
  570 ACT(J,K)=ACT(J,K)-ALPHA2(J,L)*ACE(L,K)
      DO 575 J=1,4
      DO 575 K=1,4
      ACE(J,K)=0.0
      DO 575 L=1,4
  575 ACE(J,K)=ACE(J,K)+ACT(J,L)*PAR(L,K)
      DO 580 J=1,4
      DO 580 K=1,4
  580 PAR(J,K)=F(J,K)/DELTA(NN)**2-G(J,K)*.5/DELTA(NN)
      DO 590 J=1,4
      DO 590 K=1,4
      EN(J,K)=0.0
      DO 590 L=1,4
  590 EN(J,K)=EN(J,K)+ACE(J,L)*PAR(L,K)
      DO 600 J=1,4
      DO 600 K=1,4
  600 EN(J,K)=EN(J,K)+ACT(J,K)
      GO TO 2000
  800 IF (NCON.EQ.0) GO TO 1000
      IF (NSTAT(MM).NE.I) GO TO 1000
      MM=MM+1

COMPUTE A, B, C MATRICES AT EVERY STATION

      DO 850 J=1,4
      DO 850 K=1,4
      A(I,J,K)=(2.*F(J,K)-G(J,K)*DELTA(I))/(DELTA(I-1)*(DELTA(I)+DELTA
     1(I-1)))
      B(I,J,K)=H(J,K)-(1./(DELTA(I)*DELTA(I-1)))*(2.*F(J,K)-G(J,K)
     1*(DELTA(I)-DELTA(I-1)))
  850 C(I,J,K)=(2.*F(J,K)+G(J,K)*DELTA(I-1))/(DELTA(I)*(DELTA(I)+DELTA
     1(I-1)))
```

```
      DO 851 J=1,3
  851 BSAVE(I,J)=B(I,J,J)
      GO TO 2000
 1000 DO 1100 J=1,4
      DO 1100 K=1,4
      A(I,J,K)=F(J,K)/DELTA(I)**2-G(J,K)*.5/DELTA(I)
      B(I,J,K)=H(J,K)-2.*F(J,K)/DELTA(I)**2
 1100 C(I,J,K)=F(J,K)/DELTA(I)**2+G(J,K)*.5/DELTA(I)
      DO 1101 J=1,3
 1101 BSAVE(I,J)=B(I,J,J)
 2000 CONTINUE
      RETURN
      END
```

```
      PROGRAM CONTROL

C  NECESSARY IN OVERLAY PROCEDURE

      COMMON /BATFOF/ XN                                                 C
      COMMON /SUB/ ISUB                                                  C
      COMMON /OVER20/ DZ,T                                               C
      COMMON /LAY20/ RESID,OMAGI                                         C
      IF (ISUB.EQ.0) CALL COOPER (DZ,T,0,XN)                             C
      IF (ISUB.EQ.1) CALL COOPER (RESID,OMAGI,1,XN)                      C
      RETURN                                                             C
      END                                                                C
```

```
      SUBROUTINE COOPER (DQ,DELTX,MAZE,XN)
      DIMENSION  ALPHA1(4,4),ALPHA2(4,4),PERCNT(5),DELTA(201),NSTAT(5),
     1IFREQ(20),A(200,4,4),B(200,4,4),C(200,4,4),FF(4,4),FFF(4,4),
     2HH(4,4),HHH(4,4),SF(4,4),SFF(4,4),TT(4,4),TTT(4,4),E0(4,4),
     3EN(4,4),IPIVOT(4),INDEX(4,2),PAT(4,4),PAR(4,4),ACT(4,4),ACE(4,4),
     4OMAG(2),DET(2),BSAVE(201,3),H1(3),HM(3),
     5XM1(5),XM2(5),XM3(5),XM4(5),YM(13),RW(2),NW(2),IN1(2),XX(201),      C
     6YY(201),VR(4,201),D0(4,4),DN(4,4),P(200,4,4),RIS(4,4)
      COMMON /BOUND/ALPHA1,ALPHA2
     2/CONST/XLAM,XMU,S,NN,PERCNT,ALPHA,ELENG
     3/SPAC/DELTA,NSTAT,XN1,XN2,NCON,IFREQ,IPLOT,MAN,MODEPR
     5/RANPCB/A,B,C,FF,FFF,HH,HHH,SF,SFF,TT,TTT,E0,EN,IPIVOT,INDEX,
     6PAT,PAR,ACT,ACE
     5/BATPCB/IM,OMAG,DET,NST,IJL,TERM
     8/SAVE/BSAVE,H1,HM
      IM=IM+1
      IJL=IJL+1
      DXXT=DELTX**2

CALCULATE D0 MATRIX AND ITS INVERSE

      DO 50 I=1,3
   50 HH(I,I)=H1(I)   +DXXT
      DO 100 I=1,4
      DO 100 J=1,4
  100 ACE(I,J)=FF(I,J)*.5/DELTA(1)
      DO 110 I=1,4
      DO 110 J=1,4
      ACT(I,J)=0.0
      DO 110 K=1,4
  110 ACT(I,J)=ACT(I,J)+ACE(I,K)*TT(K,J)
      DO 120 I=1,4
      DO 120 J=1,4
  120 ACE(I,J)=HH(I,J)-2.*FF(I,J)/DELTA(1)**2
      DO 130 I=1,4
      DO 130 J=1,4
      PAR(I,J)=0.0
      DO 130 K=1,4
  130 PAR(I,J)=PAR(I,J)+ACT(I,K)*ACE(K,J)
      DO 140 I=1,4
      DO 140 J=1,4
      ACT(I,J)=0.0
```

```
      DO 140 K=1,4
  140 ACT(I,J)=ACT(I,J)+ALPHA1(I,K)*PAR(K,J)
      DO 150 I=1,4
      DO 150 J=1,4
      ACE(I,J)=0.0
      DO 150 K=1,4
  150 ACE(I,J)=ACE(I,J)+ALPHA1(I,K)*SF(K,J)
      DO 160 I=1,4
      DO 160 J=1,4
      DO(I,J)=ACT(I,J)+ACE(I,J)
      IF (I.EQ.J) DO(I,J)=DO(I,J)+FLOAT(1-IFIX(ALPHA1(I,J)))
  160 CONTINUE
      CALL MATINV (DO,4,PAT,0,DETERM,IPIVOT,INDEX,4,ISCALE)
      Q=DETERM*1.E18**ISCALE

CALCULATE P MATRIX AT FIRST STATION

      DO 200 I=1,4
      DO 200 J=1,4
      P(1,I,J)=0.0
      DO 200 K=1,4
  200 P(1,I,J)=P(1,I,J)+DO(I,K)*EO(K,J)
      NNM=NN
      IF (NST.EQ.0) NNM=NN-1

CALCULATE P MATRICES AT REMAINDER OF STATIONS

      DO 1000 L=2,NNM
      DO 250 I=1,3
  250 B(L,I,I)=BSAVE(L,I)+DXXT
      DO 300 I=1,4
      DO 300 J=1,4
      PAR(I,J)=0.0
      DO 300 K=1,4
  300 PAR(I,J)=PAR(I,J)+A(L,I,K)*P(L-1,K,J)
      DO 400 I=1,4
      DO 400 J=1,4
  400 PAR(I,J)=-PAR(I,J)+B(L,I,J)
      CALL MATINV (PAR,4,PAT,0,DETERM,IPIVOT,INDEX,4,ISCALE)
      Q=Q*DETERM*1.E18**ISCALE
      SCALE1=1.E200
      SCALE2=1.E-200
      IF(ABS(Q).LE.SCALE1) GO TO 450
```

```
      Q=Q/SCALE1
      GO TO 475
  450 IF(ABS(Q).GE.SCALE2) GO TO 475
      Q=Q*SCALE1
  475 DO 500 I=1,4
      DO 500 J=1,4
      P(L,I,J)=0.0
      DO 500 K=1,4
  500 P(L,I,J)=P(L,I,J)+PAR(I,K)*C(L,K,J)
 1000 CONTINUE

CALCULATE DN MATRIX

      IF (NST.EQ.0) GO TO 1500
      DO 1050 I=1,3
 1050 HHH(I,I)=HM(I)    +DXXT
      DO 1060 I=1,4
      DO 1060 J=1,4
 1060 ACE(I,J)=-FFF(I,J)*.5/DELTA(NN)
      DO 1070 I=1,4
      DO 1070 J=1,4
      ACT(I,J)=0.0
      DO 1070 K=1,4
 1070 ACT(I,J)=ACT(I,J)+ACE(I,K)*TTT(K,J)
      DO 1080 I=1,4
      DO 1080 J=1,4
 1080 ACE(I,J)=HHH(I,J)-2.*FFF(I,J)/DELTA(NN)**2
      DO 1090 I=1,4
      DO 1090 J=1,4
      PAR(I,J)=0.0
      DO 1090 K=1,4
 1090 PAR(I,J)=PAR(I,J)+ACT(I,K)*ACE(K,J)
      DO 1100 I=1,4
      DO 1100 J=1,4
      ACT(I,J)=0.0
      DO 1100 K=1,4
 1100 ACT(I,J)=ACT(I,J)+ALPHA2(I,K)*PAR(K,J)
      DO 1110 I=1,4
      DO 1110 J=1,4
      ACE(I,J)=0.0
      DO 1110 K=1,4
 1110 ACE(I,J)=ACE(I,J)+ALPHA2(I,K)*SFF(K,J)
      DO 1120 I=1,4
```

```
      DO 1120 J=1,4
      DN(I,J)=ACT(I,J)+ACE(I,J)
      IF (I.EQ.J) DN(I,J)=DN(I,J)+FLOAT(1-IFIX(ALPHA2(I,J)))
 1120 CONTINUE
      DO 1200 I=1,4
      DO 1200 J=1,4
      PAR(I,J)=0.0
      DO 1200 K=1,4
 1200 PAR(I,J)=PAR(I,J)+EN(I,K)*P(NN,K,J)
      DO 1300 I=1,4
      DO 1300 J=1,4
 1300 PAR(I,J)=-PAR(I,J)+DN(I,J)
      DO 1400 I=1,4
      DO 1400 J=1,4
 1400 RIS(I,J)=PAR(I,J)
      CALL MATINV (PAR,4,PAT,0,DETERM,IPIVOT,INDEX,4,ISCALE)
      Q=Q*DETERM*1.E18**ISCALE
      GO TO 2000
 1500 DO 1600 I=1,4
      DO 1600 J=1,4
      PAR(I,J)=0.0
      DO 1600 K=1,4
 1600 PAR(I,J)=PAR(I,J)+A(NN,I,K)*P(NN-1,K,J)
      DO 1700 I=1,4
      DO 1700 J=1,4
 1700 PAR(I,J)=-PAR(I,J)+B(NN,I,J)
      DO 1800 I=1,4
      DO 1800 J=1,4
 1800 RIS(I,J)=PAR(I,J)
      CALL MATINV (PAR,4,PAT,0,DETERM,IPIVOT,INDEX,4,ISCALE)

* Q IS CUMULATIVE PRODUCT OF DETERMINANTS AND IS PRINTED OUT AS THE
*       RESIDUAL

      Q=Q*DETERM*1.E18**ISCALE
 2000 IF(IJL.GT.1) GO TO 2001
      XLOG=ALOG10(ABS(Q))
      ISCALE=IFIX(XLOG)
      SCALE=10.**ISCALE
 2001 DQ=Q/SCALE
      TERM=DQ
      NNM=NN
      IF (NST.EQ.0) NNM=NN-1
```

```
      IF (MAZE.EQ.1) GO TO 2600
      IF(MAN.EQ.0) GO TO 2400
      WRITE(6,8888)IJL,DELTX,DQ
 8888 FORMAT(8XI3,8XF10.7,4XE16.8)
 2400 IF (IM.EQ.1) OMAG(1)=DELTX
      IF (IM.EQ.1) DET(1)=DQ
      IF (IM.EQ.1) GO TO 2500
      IF (DET(1)*DQ.GT.0.) OMAG(1)=DELTX
      IF (DET(1)*DQ.GT.0.) DET(1)=DQ
      IF (DET(1)*DQ.LT.0.) OMAG(2)=DELTX
      IF (DET(1)*DQ.LT.0.) DET(2)=DQ
 2500 GO TO 5000

CALCULATE MODE SHAPES

 2600 IF (IFIX(ALPHA2(4,4)).EQ.1) GO TO 3000
      IF(IFIX(ALPHA2(3,3)).EQ.1)GO TO 3010
      IF(IFIX(ALPHA2(2,2)).EQ.1)GO TO 3020
      IF(IFIX(ALPHA2(1,1)).EQ.1)GO TO 3030
      IF (NST.EQ.0) GO TO 3000
      WRITE(6,3334)
 3334 FORMAT(7HOERRORO)
 3010 DO 3040 I=1,4
 3040 RIS(3,I)=RIS(4,I)
      GO TO 3000
 3020 DO    3050 I=1,4
      RIS(2,I)=RIS(3,I)
 3050 RIS(3,I)=RIS(4,I)
      GO TO 3000
 3030 DO 3060 I=1,4
      RIS(1,I)=RIS(2,I)
      RIS(2,I)=RIS(3,I)
 3060 RIS(3,I)=RIS(4,I)
 3000 UU7=RIS(1,2)*RIS(2,3)*RIS(3,4)+RIS(2,2)*RIS(3,3)*RIS(1,4)+
     1RIS(3,2)*RIS(2,4)*RIS(1,3)-RIS(1,4)*RIS(2,3)*RIS(3,2)
     1-RIS(2,4)*RIS(3,3)*RIS(1,2)-RIS(3,4)*RIS(2,2)*RIS(1,3)
      VV7=-(RIS(1,1)*RIS(2,3)*RIS(3,4)+RIS(2,1)*RIS(3,3)*RIS(1,4)
     1+RIS(3,1)*RIS(2,4)*RIS(1,3)-RIS(1,4)*RIS(2,3)*RIS(3,1)
     2-RIS(2,4)*RIS(3,3)*RIS(1,1)-RIS(3,4)*RIS(2,1)*RIS(1,3))
      WW7=RIS(1,1)*RIS(2,2)*RIS(3,4)+RIS(2,1)*RIS(3,2)*RIS(1,4)
     1+RIS(3,1)*RIS(2,4)*RIS(1,2)-RIS(1,4)*RIS(2,2)*RIS(3,1)
     2-RIS(2,4)*RIS(3,2)*RIS(1,1)-RIS(3,4)*RIS(2,1)*RIS(1,2)
      XXMX=-(RIS(1,1)*RIS(2,2)*RIS(3,3)+RIS(2,1)*RIS(3,2)*RIS(1,3)
```

```
     1+RIS(3,1)*RIS(2,3)*RIS(1,2)-RIS(1,3)*RIS(2,2)*RIS(3,1)
     2-RIS(2,3)*RIS(3,2)*RIS(1,1)-RIS(3,3)*RIS(2,1)*RIS(1,2))
      DC2=1./SQRT(UU7**2+VV7**2+WW7**2+XXMX**2)
      VR(1,1)=DC2*UU7
      VR(2,1)=DC2*VV7
      VR(3,1)=DC2*WW7
      VR(4,1)=DC2*XXMX
      NNM=NN+1
      IF (NST.EQ.0) NNM=NN
      DO 2900 L=2,NNM
      NET=NNM-L+1
      DO 2800 I=1,4
      VR(I,L)=0.
      DO 2800 J=1,4
 2800 VR(I,L)=VR(I,L)-1.*(P(NET,I,J)*VR(J,L-1))
 2900 CONTINUE
      IF(MODEPR.EQ.0) GO TO 2910
 1357 FORMAT(////13X1HU,19X1HV,19X1HW,18X2HMX/)
 2468 FORMAT(4E20.8)
      WRITE(6,1357)
      DO 3579 L=1,NNM
 3579 WRITE(6,2468) (VR(I,L),I=1,4)

C     SET UP PLOT TITLE INFORMATION

 2910 ITAPE=6LTAPE21
      DATA IN1/20H  SRD NPS VIBRATION /
      DATA YM(1)/10H        N=/
      DATA YM(3)/10H    OMEGA=/
      DATA YM(6)/10H STATIONS=/
      DATA YM(8)/10H   LAMBDA=/
      DATA YM(10)/10H    ALPHA=/
      DATA YM(12)/10H        S=/
      DATA XM1   /42H MERIDIONAL DISTANCE  VS  NORMALIZE  U    /        C
      DATA XM2   /42H MERIDIONAL DISTANCE  VS  NORMALIZE  V    /        C
      DATA XM3   /42H MERIDIONAL DISTANCE  VS  NORMALIZE  W    /        C
      DATA XM4   /42H MERIDIONAL DISTANCE  VS  NORMALIZE  M(X) /        C
      ENCODE(10,2,YM(2)) XN
    2 FORMAT(F8.2)
      ENCODE(20,3,YM(4)) DELTX
    3 FORMAT(E20.8)
      NNN=NN+1
      ENCODE(10,4,YM(7)) NNN
```

```
    4 FORMAT(I5)
      ENCODE(10,5,YM(9)) XLAM
    5 FORMAT(F10.6)
      ENCODE(10,5,YM(11)) ALPHA
      ENCODE(10,5,YM(13)) S
      K=NN+1
      IF (NST.EQ.0) K=NN
      DO 3100 I=1,K
      NET=K+1-I
 3100 YY(I)=VR(1,NET)
      NSA=NSTAT(1)
      XX(1)=0.0
      DO 3313 I=2,NSA
 3313 XX(I)=DELTA(1)*FLOAT(I-1)
      IF(NSTAT(2).EQ.0)GO TO 3133

C     SET UP X VALUES (FOR PLOTTING) FOR VARIABLE SPACING

      ISA=NSA+1
      JSA=NSTAT(2)
      DO 3323 I=ISA,JSA
 3323 XX(I)=XX(NSA)+DELTA(NSA)*FLOAT(I-NSA)
      IF(NSTAT(3).EQ.0)GO TO 3133
      ISA=JSA+1
      NSA=NSTAT(3)
      DO 3343 I=ISA,NSA
 3343 XX(I)=XX(JSA)+DELTA(JSA)*FLOAT(I-JSA)
      IF(NSTAT(4).EQ.0)GO TO 3133
      ISA=NSA+1
      JSA=NSTAT(4)
      DO 3353 I=ISA,JSA
 3353 XX(I)=XX(NSA)+DELTA(NSA)*FLOAT(I-NSA)
      IF(NSTAT(5).EQ.0)GO TO 3133
      ISA=JSA+1
      NSA=NSTAT(5)
      DO 3373 I=ISA,NSA
 3373 XX(I)=XX(JSA)+DELTA(JSA)*FLOAT(I-JSA)
 3133 NI=NN+1

CALL PLOTTING ROUTINE
C  LANGLEY SUBROUTINE FOR PLOTTING ON FILM OR OSCILLOGRAPH PAPER
C  SUBROUTINE IS ON TAPE AND AVAILABLE FOR LANGLEY PROGRAMMERS
```

```
      CALL DDIPLT (1,IN1,NI,XX,YY,0,0,0,0,13,YM,5,XM1,14,ITAPE)
      DO 3200 I=1,K
      IF (NST.EQ.0) NI=NN
      NET=K+1-I
3200  YY(I)=VR(2,NET)
      CALL DDIPLT (1,IN1,NI,XX,YY,0,0,0,0,13,YM,5,XM2,14,ITAPE)
      DO 3300 I=1,K
      NET=K+1-I
3300  YY(I)=VR(3,NET)
      CALL DDIPLT (1,IN1,NI,XX,YY,0,0,0,0,13,YM,5,XM3,14,ITAPE)
      DO 3400 I=1,K
      NET=K+1-I
3400  YY(I)=VR(4,NET)
      CALL DDIPLT (1,IN1,NI,XX,YY,0,0,0,0,13,YM,5,XM4,14,ITAPE)
5000  CONTINUE
      RETURN
      END
```

```
      PROGRAM NANCY
      DIMENSION  ALPHA1(4,4),ALPHA2(4,4),R(201),GAMMA(201),XK(201),
     1TK(201),DXKDX(201),EPX(201),EPTH(201),DEXDX(201),DETHDX(201),
     2PHI(201),DPHIDX(201),PERCNT(5),DELTA(201),NSTAT(5),IFREQ(20),
     3AA(20),BBB(20),DELTAX(20),XM(24)
      DIMENSION XX(201),WA(201),DER1(201),DER2(201)
      COMMON
     1/BOUND/ALPHA1,ALPHA2
     2/GEOM/R,GAMMA,XK,TK,DXKDX
     3/PRESTR/EPX,EPTH,DEXDX,DETHDX,PHI,DPHIDX
     4/CONST/XLAM,XMU,S,NN,PERCNT,ALPHA,ELENG
     5/SPAC/DELTA,NSTAT,XN1,XN2,NCON,IFREQ,IPLOT,MAN,MODEPR
     6/CONITR/AA,BBB,DELTAX,E1,E2,MAXI,XM
     7/FOPPL/IFOPPL,EBX,PEL
      NAMELIST/INPUT/ALPHA1,ALPHA2,XLAM,XMU,S,NN,NSTAT,XN1,XN2,NCON,
     1IFREQ,IPLOT,MAN,AA,BBB,DELTAX,E1,E2,MAXI,ALPHA,MODEPR
     3,IFOPPL,EBX,PEL
     2/EXTRA/PERCNT

C       READ ALL INPUT AND COMPUTE LENGTH

      READ(5,1)XM
    1 FORMAT (8A10)                                                     C
      IF (EOF,5) 999,888                                                C
  888  CONTINUE                                                         C
      READ(5,INPUT)
      IF(ALPHA.EQ.0.) GO TO 50
      ELENG=2.*SIN(S*ALPHA/2.)/ALPHA
      GO TO 51
   50 ELENG=S
   51 N1=NN+1
      IF(NCON.EQ.0) GO TO 100

COMPUTE VARIABLE DELTAS

      READ(5,EXTRA)
      DO 10 I=1,5
      KKI=6-I
      N2=NN+2-NSTAT(KKI)
      N3=NSTAT(I)-1
      DO 10 J=N2,N3
   10 DELTA(J)=PERCNT(I)*S/FLOAT(N3-N2+1)
```

```
      GO TO 200
C
COMPUTE CONSTANT DELTA
C
  100 DO 15 I=1,N1
   15 DELTA(I)=S/FLOAT(NN)
  200 IF(ALPHA.EQ.0.) GO TO 300
C
COMPUTE DOUBLY CURVED SHELL GEOMETRY
C
      X=0.
      DO 20 I=1,N1
      ARG=S*ALPHA/2.-X*ALPHA
      R(I)=1.0+(COS(ARG)-1.)/ALPHA
      GAMMA(I)=ALPHA*SIN(ARG)/(ALPHA+(COS(ARG)-1.))
      XK(I)=ALPHA
      TK(I)=ALPHA*COS(ARG)/(ALPHA+(COS(ARG)-1.))
      DXKDX(I)=0.
   20 X=X+DELTA(I)
      GO TO 310
C
COMPUTE CYLINDER GEOMETRY
C
  300 DO 25 I=1,N1
      R(I)=1.0
      GAMMA(I)=0.
      XK(I)=0.
      TK(I)=1.0
   25 DXKDX(I)=0.
  310 IF(IFOPPL.EQ.0) GO TO 400
C
COMPUTE PRESTRESS DEFORMATIONS
C
      A1=SQRT(SQRT(3.)*XLAM*SQRT(1.-XMU**2)-3.*EBX)/XLAM
      A2=SQRT(SQRT(3.)*XLAM*SQRT(1.-XMU**2)+3.*EBX)/XLAM
      A1A2=2.*A1*A2
      A1L=A1*S/2.
      A2L=A2*S/2.
      SA1L=SIN(A1L)
      CA1L=COS(A1L)
      SHA2L=.5*(EXP(A2L)-EXP(-A2L))
      CHA2L=.5*(EXP(A2L)+EXP(-A2L))
      PTILDE=PEL*(1.-XMU**2)-(ALPHA+XMU)*EBX
```

```
      PM=PTILDE/(1.-XMU**2)
      IF(ALPHA1(4,4).EQ.1.) GO TO 320

* FREELY (OR SIMPLY) SUPPORTED

      ADEN=A1A2*(SA1L**2*SHA2L**2+CA1L**2*CHA2L**2)
      CAPA1=       PM*((A2**2-A1**2)*CA1L*CHA2L-A1A2*SA1L*SHA2L)/ADEN
      CAPA2=      -PM*((A2**2-A1**2)*SA1L*SHA2L+A1A2*CA1L*CHA2L)/ADEN
      GO TO 330

* CLAMPED

  320 ADEN=A2*SA1L*CA1L+A1*SHA2L*CHA2L
      CAPA1=PM*(A2*CA1L*SHA2L-A1*SA1L*CHA2L)/ADEN
      CAPA2=-PM*(A1*CA1L*SHA2L+A2*SA1L*CHA2L)/ADEN
  330 DX=S/FLOAT(NN)
      X=0.
      DO 80 I=1,N1
      XX(I)=X
      SX=SIN(A1*(X-S/2.))
      CX=COS(A1*(X-S/2.))
      SHAX=SINH(A2*(X-S/2.))
      CHAX=COSH(A2*(X-S/2.))
      WA(I)=CAPA1*SX*SHAX+CAPA2*CX*CHAX+PM
      DER1(I)=(CAPA1*A2-CAPA2*A1)*SX*CHAX+(CAPA1*A1+CAPA2*A2)*CX*SHAX
      DER2(I)=(CAPA2*A2**2+2.*CAPA1*A1*A2-CAPA2*A1**2)*CX*CHAX+(CAPA1*A2
     1**2-2.*CAPA2*A1*A2-CAPA1*A1**2)*SX*SHAX
      EPX(I)=EBX
      EPTH(I)=WA(I)*(1.-XMU**2)+XMU*EBX
      DEXDX(I)=0.
      DETHDX(I)=(1.-XMU**2)*WA(I)
      PHI(I)=-DER1(I)
      DPHIDX(I)=-DER2(I)
   80 X=X+DX
      GO TO 998

COMPUTE CONSTANT PRESTRESS (MEMBRANE)

  400 DO 30 I=1,N1
      EPX(I)=0.
      EPTH(I)=0.
      DEXDX(I)=0.0
      DETHDX(I)=0.0
```

```
      PHI(I)=0.0
   30 DPHIDX(I)=0.0
  998 RETURN
  999 STOP
      END
```

C

```
SAMPLE OUTPUT FOR NUMERICAL METHOD OF SOLUTION OF THE DEEP SHELL EQUATIONS OF
CHAPTER 6.          NATURAL VIBRATIONS OF FREELY SUPPORTED UNSTRESSED NEGATIVE
CURVATURE SHELL WITH KX=-.05, S=3, RADIUS/THICKNESS=1000, N=4-8.     N.P.SYKES

BOUNDARY CONDITIONS

X=0
   ALPHA1(1,1)=     1.00
   ALPHA1(2,2)=     0.
   ALPHA1(3,3)=     0.
   ALPHA1(4,4)=     0.

X=S
   ALPHA2(1,1)=     1.00
   ALPHA2(2,2)=     0.
   ALPHA2(3,3)=     0.
   ALPHA2(4,4)=     0.


GEOMETRY
   LAMBDA=    .00100   AXIAL LENGTH=  2.99718829E+00
    ALPHA=   -.05000  NO. INTERVALS=  200
        S=   3.00000             MU=  .30


CODES

   NSTAT= 201   0   0   0   0    REGION BOUNDARIES AT THESE POINTS

   NCON= 0    0 FOR CONSTANT DELTA, 1 FOR VARIABLE DELTAS

   IPLOT= 0    0 FOR NO PLOTS, 1 FOR PLOTS OF MODE SHAPES

   MAN= 2    0 FOR USING ITR2, 1 FOR MANUAL SEARCH, 2 FOR ITR2 WITH PRINTOUT

   MAXI= 25    MAXIMUM NUMBER OF ITERATIONS FOR ITR2 SUBROUTINE

   MODEPR= 0    0 FOR NO PRINTOUT OF MODE SHAPE VALUES, 1 FOR PRINTING THEM

   E1= 1.0000E-05    E2= 1.0000E-06    CONVERGENCE TESTS USED IN ITR2

   RANGE OF N IS    4 TO    8

   IFOPPL= 0    0 FOR CONSTANT PRESTRESS (MEMBRANE),  1 FOR PRESTRESS DEFORMATIONS (FOPPL)
```

CONSTANT DELTA= .01500

INITIAL FREQUENCY INSPECTED= .0050000 UPPER LIMIT= 1.0000000
INCREMENT USED IN INTERVAL= .0050000

MODE NO.= 4

ITERATION NUMBER	OMEGA VALUE (FREQUENCY)	SCALED RESIDUAL
1	.0050000	5.26875619E+00
2	.0100000	4.14303296E+00
3	.0150000	2.31343991E+00
4	.0200000	-1.52209838E-01
5	.0175000	1.15494353E+00
6	.0187500	5.19225565E-01
7	.0193750	1.87877667E-01
8	.0196875	1.89142307E-02
9	.0198437	-6.63792502E-02
10	.0197656	-2.36651808E-02
11	.0197266	-2.35861576E-03
12	.0197070	8.28202932E-03
13	.0197168	2.96276091E-03
14	.0197217	3.02338131E-04
15	.0197241	-1.02807118E-03
16	.0197229	-3.62850809E-04
17	.0197223	-3.02566179E-05
18	.0197220	1.36050040E-04
19	.0197221	5.28941224E-05

NUMBER OF ITERATIONS= 19

BOUNDS FOR FINAL FREQUENCY= .0197221 , .0197223
CORRESPONDING RESIDUALS= 5.28941224E-05 , -3.02566179E-05

INTERPOLATED (FINAL) FREQUENCY= .01972223
MODE SHAPES NOT REQUESTED, THEREFORE
FINAL RESIDUAL IS NOT CALCULATED

INITIAL FREQUENCY INSPECTED= .0020000 UPPER LIMIT= 1.0000000
INCREMENT USED IN INTERVAL= .0020000

MODE NO.= 5

ITERATION NUMBER	OMEGA VALUE (FREQUENCY)	SCALED RESIDUAL
1	.0020000	3.29511671E+00
2	.0040000	2.60031920E+00
3	.0060000	1.44978214E+00
4	.0080000	-1.45379697E-01
5	.0070000	7.06915203E-01
6	.0075000	2.94323634E-01
7	.0077500	7.78447118E-02
8	.0078750	-3.29264017E-02
9	.0078125	2.26697123E-02
10	.0078437	-5.07574815E-03
11	.0078281	8.81014161E-03
12	.0078359	1.87047879E-03
13	.0078398	-1.60181101E-03
14	.0078379	1.34531923E-04
15	.0078389	-7.33591348E-04
16	.0078384	-2.99518737E-04
17	.0078381	-8.24891105E-05
18	.0078380	2.60403017E-05
19	.0078381	-2.82289783E-05

NUMBER OF ITERATIONS= 19

BOUNDS FOR FINAL FREQUENCY= .0078380 , .0078381
CORRESPONDING RESIDUALS= 2.60403017E-05 , -2.82289783E-05

INTERPOLATED (FINAL) FREQUENCY= .00783804
MODE SHAPES NOT REQUESTED, THEREFORE
FINAL RESIDUAL IS NOT CALCULATED

INITIAL FREQUENCY INSPECTED= .0050000 UPPER LIMIT= 1.0000000
INCREMENT USED IN INTERVAL= .0050000

MODE NO.= 6

ITERATION NUMBER	OMEGA VALUE (FREQUENCY)	SCALED RESIDUAL
1	.0050000	6.19911848E+00
2	.0100000	4.71329997E+00
3	.0150000	2.41159835E+00
4	.0200000	-4.58929073E-01
5	.0175000	1.02897549E+00
6	.0187500	2.95653957E-01
7	.0193750	-7.93078642E-02
8	.0190625	1.08796954E-01
9	.0192187	1.48953647E-02
10	.0192969	-3.21691916E-02
11	.0192578	-8.62756428E-03
12	.0192383	3.13624541E-03
13	.0192480	-2.74507121E-03
14	.0192432	1.95733394E-04
15	.0192456	-1.27463212E-03
16	.0192444	-5.39443882E-04
17	.0192438	-1.71851023E-04
18	.0192435	1.19420358E-05
19	.0192436	-7.99576182E-05

NUMBER OF ITERATIONS= 19

BOUNDS FOR FINAL FREQUENCY= .0192435 , .0192436
CORRESPONDING RESIDUALS= 1.19420358E-05 , -7.99576182E-05

INTERPOLATED (FINAL) FREQUENCY= .01924349
MODE SHAPES NOT REQUESTED, THEREFORE
FINAL RESIDUAL IS NOT CALCULATED

INITIAL FREQUENCY INSPECTED= .0050000 UPPER LIMIT= 1.0000000
INCREMENT USED IN INTERVAL= .0050000

MODE NO.= 7

ITERATION NUMBER	OMEGA VALUE (FREQUENCY)	SCALED RESIDUAL
1	.0050000	3.66026420E+00
2	.0100000	3.09142559E+00
3	.0150000	2.25381145E+00
4	.0200000	1.29974712E+00
5	.0250000	4.17219725E-01
6	.0300000	-1.92200553E-01
7	.0275000	6.59007747E-02
8	.0287500	-7.63033168E-02
9	.0281250	-8.30576111E-03
10	.0278125	2.80447349E-02
11	.0279687	9.67836010E-03
12	.0280469	6.38151896E-04
13	.0280859	-3.84588687E-03
14	.0280664	-1.60688246E-03
15	.0280566	-4.85118025E-04
16	.0280518	7.63289969E-05
17	.0280542	-2.04441454E-04
18	.0280530	-6.40683628E-05
19	.0280524	6.12709529E-06
20	.0280527	-2.89710945E-05
21	.0280525	-1.14222028E-05

NUMBER OF ITERATIONS= 21

BOUNDS FOR FINAL FREQUENCY= .0280524 , .0280525
CORRESPONDING RESIDUALS= 6.12709529E-06 , -1.14222028E-05

INTERPOLATED (FINAL) FREQUENCY= .02805242
MODE SHAPES NOT REQUESTED, THEREFORE
FINAL RESIDUAL IS NOT CALCULATED

INITIAL FREQUENCY INSPECTED= .0050000 UPPER LIMIT= 1.0000000
INCREMENT USED IN INTERVAL= .0050000

MODE NO.= 8

ITERATION NUMBER	OMEGA VALUE (FREQUENCY)	SCALED RESIDUAL
1	.0050000	1.64692911E+00
2	.0100000	1.33873122E+00
3	.0150000	9.01024881E-01
4	.0200000	4.35007614E-01
5	.0250000	5.86500663E-02
6	.0300000	-1.13111627E-01
7	.0275000	-5.93025833E-02
8	.0262500	-7.59745686E-03
9	.0256250	2.38085279E-02
10	.0259375	7.66332034E-03
11	.0260937	-7.91958486E-05
12	.0260156	3.76422604E-03
13	.0260547	1.83553153E-03
14	.0260742	8.76418759E-04
15	.0260840	3.98173913E-04
16	.0260889	1.59379428E-04
17	.0260913	4.00644744E-05
18	.0260925	-1.95724417E-05
19	.0260919	1.02442160E-05
20	.0260922	-4.66457699E-06
21	.0260921	2.78968537E-06

NUMBER OF ITERATIONS= 21

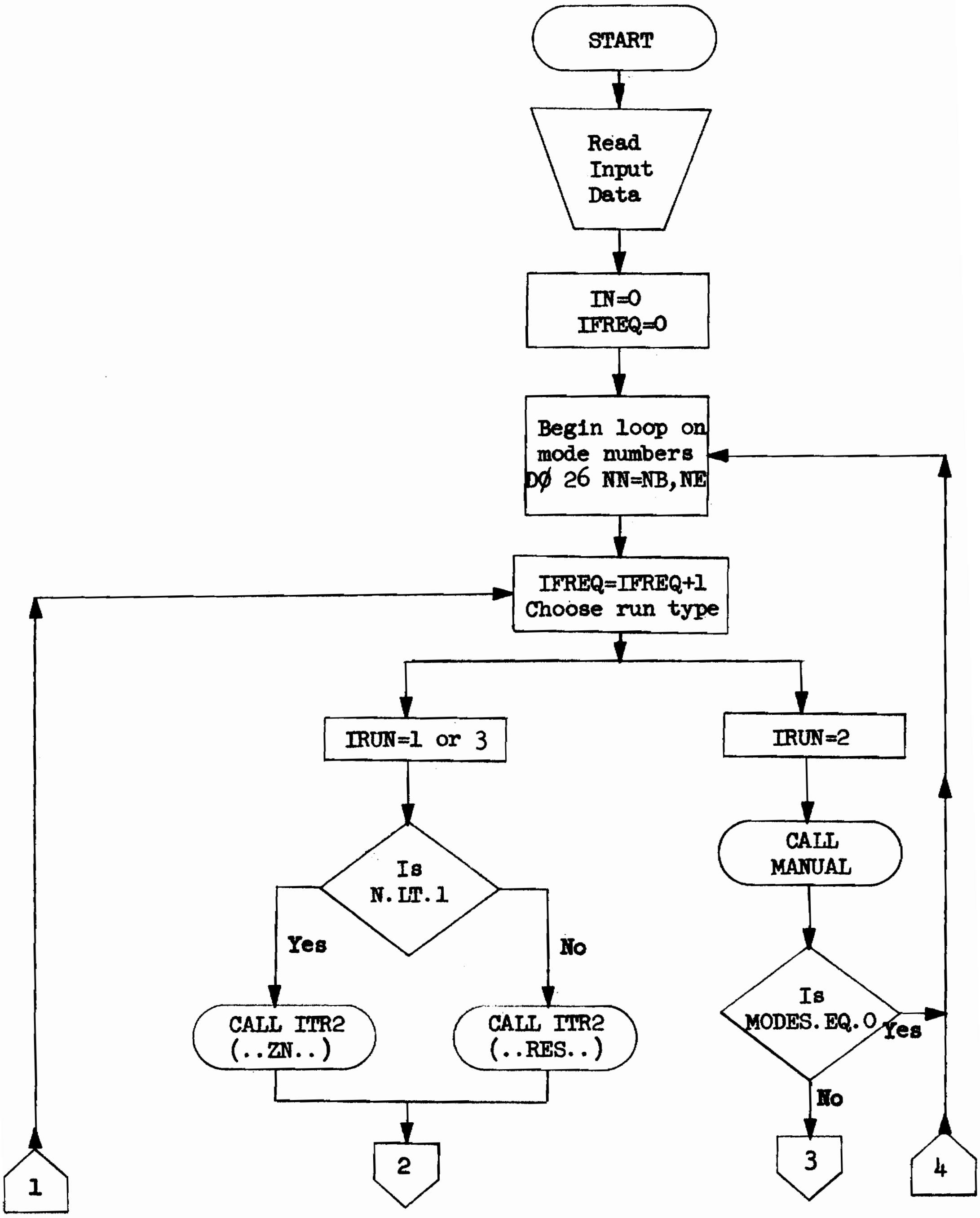
BOUNDS FOR FINAL FREQUENCY= .0260921 , .0260922
CORRESPONDING RESIDUALS= 2.78968537E-06 , -4.66457699E-06

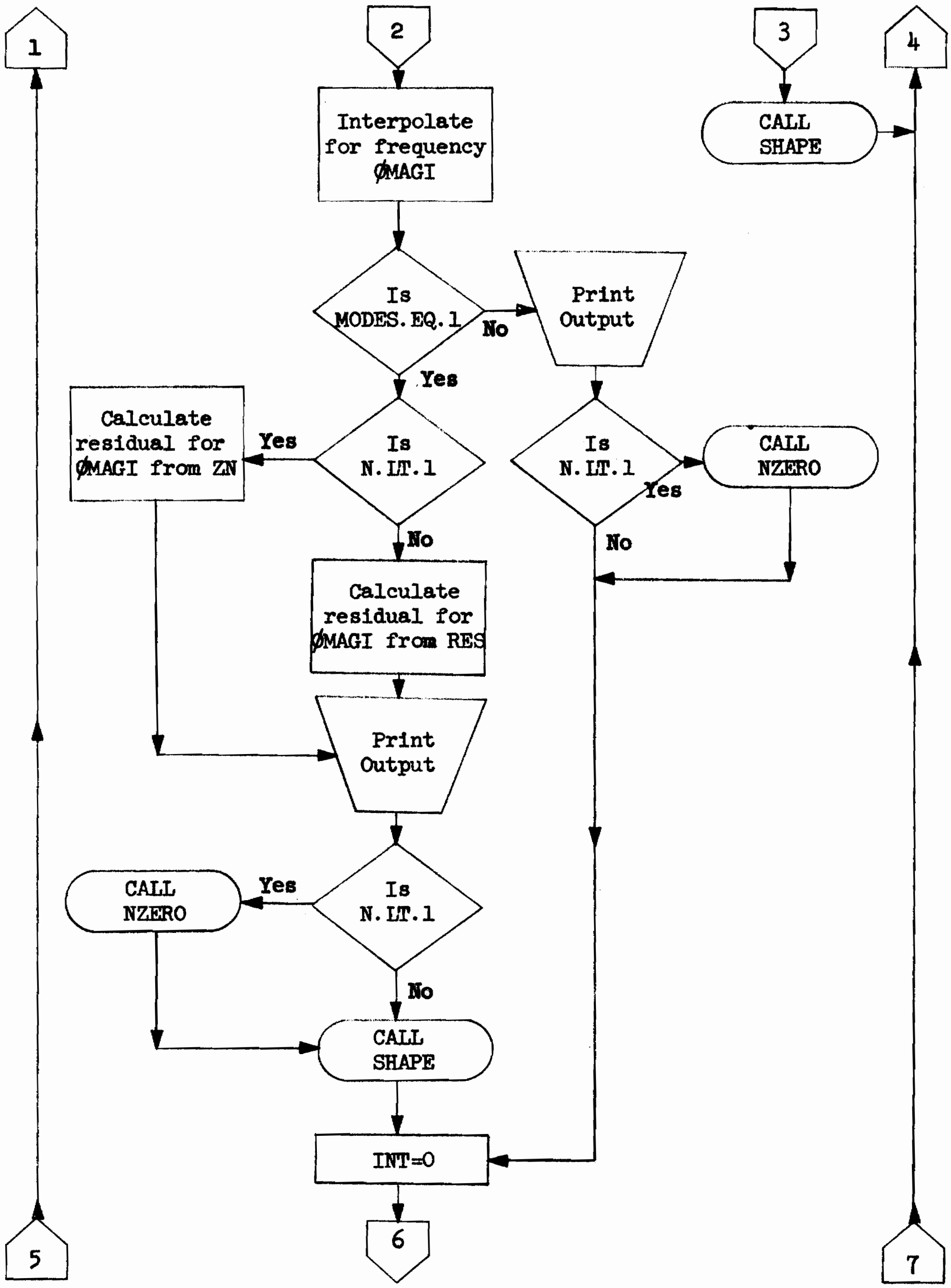
INTERPOLATED (FINAL) FREQUENCY= .02609213
MODE SHAPES NOT REQUESTED, THEREFORE
FINAL RESIDUAL IS NOT CALCULATED

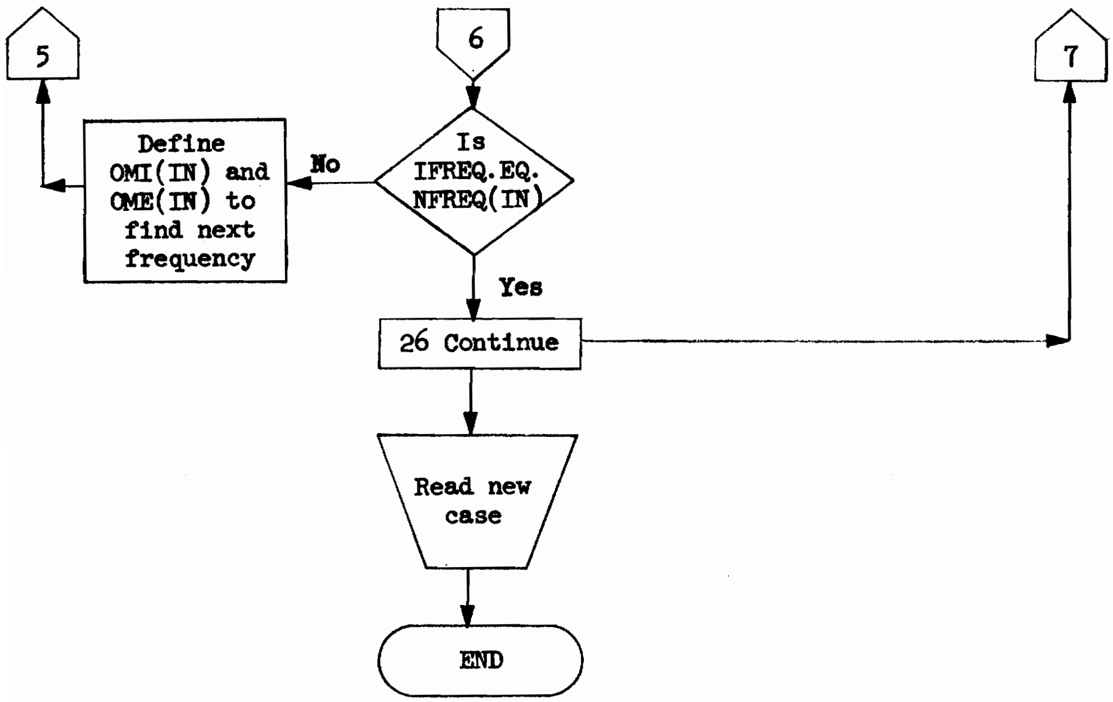
FLOW DIAGRAM OF MAIN PROGRAM FOR GENERAL METHOD

OF SOLUTION OF THE APPROXIMATE

(SHALLOW MERIDIAN) EQUATIONS




1
2
3
4
Interpolate for frequency ØMAGI
CALL SHAPE
Is MODES.EQ.1
No
Print Output
Yes
Calculate residual for ØMAGI from ZN
Yes
Is N.LT.1
Is N.LT.1
Yes
CALL NZERO
No
No
Calculate residual for ØMAGI from RES
Print Output
CALL NZERO
Yes
Is N.LT.1
No
CALL SHAPE
INT=0
5
6
7


5
6
7
Define
OMI(IN) and
OME(IN) to
find next
frequency
No
Is
IFREQ.EQ.
NFREQ(IN)
Yes
26 Continue
Read new
case
END

A. Main Program Variables

1. See comment cards in the main program for a description of of the input data.
2. IN - subscript associated with each frequency interval.
3. IFREQ - counts the number of successive frequencies for a particular circumferential mode number.
4. N - circumferential mode number.
5. ØMAGI - interpolated value of the frequency.
6. INT - counts the number of iterations in the ITR2 subroutine for each frequency.

B. Subroutines and Function Subprograms

1. MANUAL - subroutine which calculates the residuals when given a frequency interval and a constant frequency increment.
2. ITR2 - iterative halving subroutine which searches for a sign change and then proceeds to the frequency within a specified error limit.
3. SHAPE - subroutine which calculates the mode shapes and stresses; results were not used in the thesis.
4. NZERØ - subroutine which calculates the minimum torsional frequency when the circumferential mode number is zero.
5. RES - function subprogram for solving the characteristic equation, calculating the modal amplitude and boundary condition coefficients and evaluating the residuals; used for all circumferential mode numbers except zero. Subroutines used by RES:
    a. FALG - calculates the roots of the characteristic equation.
    b. ABCDI - examines and orders the roots of the characteristic equation.
    c. CDETERM - calculates the complex residual.
6. ZN - function subprogram similar to RES; used only for circumferential mode number zero.

```
C         COMPUTER PROGRAM FOR GENERAL METHOD OF SOLUTION OF THE APPROXIMATE
C         (SHALLOW MERIDION) EQUATIONS OF CHAPTER 8.  SAMPLE OUTPUT INCLUDED.
C         (SEE FIGURE 4 IN TEXT FOR COMPARISON OF OUTPUT.)
C         M.P. ROBINSON - P.A. COOPER

      PROGRAM CMPLXDT (INPUT,OUTPUT,TAPE5=INPUT,TAPE21)
      EXTERNAL RES,ZN
      DIMENSION TITLE(24),SU(201),SV(201),SW(201),XA(201),NX(201),
     1NTHETA(201),NXTHETA(201),MX(201),MTHETA(201),MXTHETA(201),
     2RESULT(2),NFREQ(20),OMI(20),OME(20),OMD(20)
      REAL MU,N,LAMBDA,NXB,NYB,KX,NX,NTHETA,NXTHETA,MX,MTHETA,MXTHETA
      COMMON/INPUTD/MU,N,KX,NXB,NYB,LAMBDA,S,IRUN
     1/MODSHA/MUVW,IPRINT,IPLOT
     2/BC/ABC(8,8),BBC(8,8)
     3/RESIDL/OMAG(2),DETR(2),DETI(2),IRTEST,ITEST,INCASE,INT
     4/CDET/DETERM,ISCALE
     5/CASE/LAMDAI(8),ROOT(8),ICASE
     6/FREQM/OMAGI
     7/CMR/CM(8,8)
     8/CMRZ/CMRZ(6,6)
     9/BCN/ABCN(8,8),BBCN(8,8)
      COMPLEX DETERM,CM,LAMDAI,ROOT,CMZ
      NAMELIST/INPUT/MU,KX,NXB,NYB,LAMBDA,S,IRUN,NROOTS,NB,NE,NFREQ,OMI,
     1OME,OMD,MAXI,E1,E2,MODES,NMODES,MUVW,IPRINT,IPLOT,ABC
 1000 PRINT 1
    1 FORMAT(1H1////1X*VIBRATIONS OF SHALLOW SHELLS WITH DEEP CIRCUMFERE
     1NTIAL CURVATURE*/1X*COMPLEX DETERMINANT METHOD      ROBINSON-COOPE
     2R  SRD-A1556  RDK-258*/)
      CALL DAYTIM (RESULT)
      PRINT 2, RESULT(1)
    2 FORMAT(*ODATE*3XA10/)

C         READ THREE TITLE CARDS OF USERS CHOICE TO DESCRIBE PROBLEM

      READ 3, (TITLE(I),I=1,24)
    3 FORMAT(8A10)
      PRINT 4, (TITLE(I),I=1,24)
    4 FORMAT(*0*5X8A10/6X8A10/6X8A10/)

C         SET BOUNDARY CONDITION MATRICES ABC AND BBC EQUAL TO ZERO
```

```
      DO 5 I=1,8
      DO 5 J=1,8
      ABC(I,J)=0.0
    5 BBC(I,J)=0.0

C        READ INPUT DATA

C          MU      POISSONS RATIO
C          KX      MERIDIONAL CURVATURE PARAMETER
C          NXB     MERIDIONAL PRESTRESS
C          NYB     CIRCUMFERENTIAL PRESTRESS
C          LAMBDA  THICKNESS-RADIUS RATIO
C          S       LENGTH-RADIUS RATIO

C          IRUN    INDICATOR FOR TYPE OF RUN
C                     1  USE ITR2  (ITERATIVE HALVING PROCESS FOR FINDING
C                           RESIDUAL)  MINIMUM PRINT-OUT -- FINAL FREQUENCY AND
C                           RESIDUAL ONLY
C                     2  USE MANUAL SEARCH  (SIMPLY GIVES CALCULATED RESIDUALS
C                           BY GOING THRU THE GIVEN FREQUENCY INTERVAL WITH A
C                           CONSTANT INCREMENT)  PRINT-OUT AT EACH INCREMENT
C                     3  USE ITR2  PRINT-OUT FROM EACH ITERATION

C          NROOTS  INDICATOR FOR PRINTING ROOTS OF CHARACTERISTIC EQUATION
C                     0  DO NOT PRINT ROOTS
C                     1  DO PRINT ROOTS

C          NB      INITIAL MODE NUMBER
C          NE      FINAL MODE NUMBER
C                     NB AND NE GIVE RANGE OF CIRCUMFERENTIAL MODE NUMBERS FOR
C                        WHICH FREQUENCIES ARE TO BE FOUND  (A FREQUENCY
C                        INTERVAL IS READ IN ASSOCIATED WITH EACH MODE NUMBER)

C          NFREQ   ARRAY OF THE NUMBERS OF SUCCESSIVE FREQUENCIES THAT ARE TO
C                     BE FOUND - EACH ELEMENT OF THE ARRAY IS ASSOCIATED WITH A
C                        PARTICULAR MODE NUMBER

C          OMI     ARRAY OF INITIAL VALUES FOR FREQUENCY INTERVALS
C          OME     ARRAY OF FINAL VALUES FOR FREQUENCY INTERVALS
C          OMD     ARRAY OF INCREMENTS FOR FREQUENCY INTERVALS
C          MAXI    MAXIMUM NUMBER OF ITERATIONS ALLOWED IN ITR2
C          E1      RELATIVE ERROR CRITERION FOR ITR2
C          E2      ABSOLUTE ERROR CRITERION FOR ITR2
```

```
C          MODES INDICATOR FOR CALLING MODE SHAPE SUBROUTINE
C                     0  DO NOT CALCULATE MODE SHAPES
C                     1  DO CALCULATE MODE SHAPES

C          NMODES  NUMBER OF X-AXIS STATIONS FOR MODE SHAPES

C          MUVW    INDICATOR USED IN MODE SHAPE SUBROUTINE
C                     0  CALCULATE U,V,W MODES, STRESSES AND MOMENTS
C                     1  CALCULATE U,V,W MODES ONLY

C          IPRINT  INDICATOR FOR HAVING MODES, STRESSES AND MOMENTS PRINTED
C                     0  DO NOT PRINT
C                     1  DO PRINT

C          IPLOT   INDICATOR FOR HAVING MODES, STRESSES AND MOMENTS PLOTTED
C                     0  DO NOT PLOT
C                     1  DO PLOT

C          ABC     ELEMENTS FOR BOUNDARY CONDITION MATRIX - ONLY DIAGONAL
C                     ELEMENTS NEED TO BE READ IN

      IF (EOF,5) 27,6
    6 READ(5,INPUT)

C         REDEFINE BOUNDARY CONDITIONS - PUT IN DIAGONAL ELEMENTS

      DO 7 I=1,8
    7 BBC(I,I)=1.-ABC(I,I)

C         PRINT INPUT DATA

      PRINT 8, ABC(1,1),ABC(5,5),ABC(2,2),ABC(6,6),ABC(3,3),ABC(7,7),
     1ABC(4,4),ABC(8,8),LAMBDA,NXB,S,NYB,KX,MU
    8 FORMAT(*0BOUNDARY CONDITIONS*//1X*X=-S/2*,17X*X=S/2*/
     15X*ABC(1,1)=*F5.2,8X*ABC(5,5)=*F5.2/
     25X*ABC(2,2)=*F5.2,8X*ABC(6,6)=*F5.2/
     35X*ABC(3,3)=*F5.2,8X*ABC(7,7)=*F5.2/
     45X*ABC(4,4)=*F5.2,8X*ABC(8,8)=*F5.2///
     51X*GEOMETRY*//5X*LAMBDA=*F7.3,5X*NXB=*F7.3/
     610X*S=*F7.3,5X*NYB=*F7.3/9X*KX=*F7.3,6X*MU=*F7.3)
      PRINT 9, NROOTS,IRUN,MAXI,E1,E2,MODES,NMODES,MUVW,IPRINT,IPLOT,NB,
     1NE
```

```
    9 FORMAT(*0CODES*//5X*NROOTS=*I2,3X*1 PRINT ROOTS OF CHARACTERISTIC
     1EQUATION,    0 DO NOT*/
     27X*IRUN=*I2,3X*1 USE ITR2,    2 USE MANUAL SEARCH,    3 USE ITR2 AND
     3 PRINT*/
     47X*MAXI=*I2,3X*MAXIMUM NUMBER OF ITERATIONS FOR ITR2 SUBROUTINE*/
     59X*E1=*E8.1,3X*E2=*E8.1,3X*CONVERGENCE CRITERIA FOR ITR2*/
     66X*MODES=*I2,3X*1 CALL MODE SHAPE SUBROUTINE,    0 DO NOT*/
     75X*NMODES=*I2,3X*NUMBER OF X-AXIS POINTS FOR MODE SHAPES*/
     87X*MUVW=*I2,3X*1 CALCULATE U,V,W MODES ONLY,    0 CALCULATE STRESSE
     9S ALSO*/
     A5X*IPRINT=*I2,3X*1 PRINT CALCULATED MODE SHAPE VALUES ONLY/AND STR
     BESSES,    0 DO NOT*/
     C6X*IPLOT=*I2,3X*1 PLOT MODE SHAPE VALUES ONLY/AND STRESSES,    0 DO
     D NOT*/
     E5X*MODE NUMBERS RANGE FROM*I3* TO*I3)
C
C        IN - SUBSCRIPT ASSOCIATED WITH EACH FREQUENCY INTERVAL
C
      IN=0
      NB=NB+1
      NE=NE+1
      DO 26 NN=NB,NE
      IN=IN+1
C
C        IRTEST - INDICATOR WHICH TELLS PROGRAM WHICH COLUMN OF THE COMPLEX
C        RESIDUAL TO FOLLOW, DECIDES BY EXAMINING ROOTS OF THE CHARACTERISTIC
C        EQUATION, SEE ABCDI SUBROUTINE
C
      ITEST=0
      N=NN-1
C
C        INT - COUNT ON NUMBER OF ITERATIONS OF ITR2
C
      INT=0
C
C
C        IFREQ - COUNT ON NUMBER OF SUCCESSIVE FREQUENCIES
C
      IFREQ=0
   99 IFREQ=IFREQ+1
C
C        PRINT INFORMATION ON INTERVAL LIMITS, INCREMENT AND TYPE OF RUN
```

```
      PRINT 10, OMI(IN),OME(IN),OMD(IN),N
   10 FORMAT(///*1INITIAL FREQUENCY INSPECTED=*F8.5,3X*FINAL FREQUENCY L
     1IMIT=*F8.5,3X*INSPECTION INTERVAL=*F8.5//1X*MODE NUMBER=*F4.0)
      IF (IRUN.EQ.1) GO TO 13
      PRINT 11
   11 FORMAT(*0 ITERATION*,3X*OMEGA VALUE*,20X*RESIDUAL*,24X*CASE*/
     13X*NUMBER*,5X*(FREQUENCY)*,19X*(COMPLEX)*,16X*ISCALE*,1X*NUMBER*/)
      IF (IRUN.EQ.3) GO TO 13
      CALL MANUAL (OMI(IN),OME(IN),OMD(IN))
      PRINT 12, INT
   12 FORMAT(*ONUMBER OF ITERATIONS=*I3)
      IF (MODES.EQ.0) GO TO 26
      OMAGI=OMI(IN)
      CALL SHAPE(NMODES,SU,SV,SW,XA,NX,NTHETA,NXTHETA,MX,MTHETA,
     1MXTHETA)
      GO TO 26
   13 IF (N.LT.1.) GO TO 135

C        IRTEST - INDICATOR WHICH ALLOWS PROGRAM TO ENTER RES (OR ZN) FUNCTION
C        SUBPROGRAM ONCE TO DETERMINE CASE NUMBER

      IRTEST=0
      RETEST=RES(OMI(IN))
      INCASE=ICASE
      IRTEST=1
      ITEST=1
      IF (ICASE.EQ.2) ITEST=2
      CALL ITR2 (OMEGA,OMI(IN),OME(IN),OMD(IN),RES,E1,E2,MAXI,ICODE)
      GO TO 134

C        CIRCUMFERENTIAL MODE NUMBER EQUAL TO ZERO IS A SPECIAL CASE, EIGHTH
C        DEGREE CHARACTERISTIC EQUATION BECOMES SIXTH DEGREE, A SEPARATE
C        FUNCTION SUBPROGRAM ZN HANDLES THIS CASE, ALL ADJUSTMENTS TO BOUNDARY
C        CONDITIONS, ETC. ARE INTERNAL TO THE PROGRAM

  135 DO 1360 I=1,8
      ABCN(1,I)=ABC(1,I)
 1360 BBCN(1,I)=BBC(1,I)
      DO 136 I=2,4
      DO 136 J=1,8
      ABCN(I,J)=ABC(I+1,J)
  136 BBCN(I,J)=BBC(I+1,J)
      DO 137 I=5,6
```

```
      DO 137 J=1,8
      ABCN(I,J)=ABC(I+2,J)
  137 BBCN(I,J)=BBC(I+2,J)
      DO 138 I=1,6
      DO 138 J=2,4
      ABCN(I,J)=ABCN(I,J+1)
  138 BBCN(I,J)=BBCN(I,J+1)
      DO 139 I=1,6
      DO 139 J=5,6
      ABCN(I,J)=ABCN(I,J+2)
  139 BBCN(I,J)=BBCN(I,J+2)
      IRTEST=0
      RETEST=ZN(OMI(IN))
      INCASE=ICASE
      IRTEST=1
      ITEST=1
      IF (ICASE.EQ.3) ITEST=2
      CALL ITR2 (OMEGA,OMI(IN),OME(IN),OMD(IN),ZN,E1,E2,MAXI,ICODE)
  134 IF (ICODE.EQ.1) PRINT 14, ICODE
   14 FORMAT(*0MAXIMUM NUMBER OF ITERATIONS EXCEEDED*,3X*ICODE=*I2)
      IF (ICODE.EQ.2) PRINT 15, ICODE
   15 FORMAT(*0DELX=0 OR IS NEGATIVE*,3X*ICODE=*I2)
      IF (ICODE.EQ.3) PRINT 16, ICODE
   16 FORMAT(*0NO ROOT*,3X*ICODE=*I2)
      IF (ICODE.EQ.4) PRINT 17, ICODE
   17 FORMAT(*0A.GT.B*,3X*ICODE=*I2)
      IF (ICODE.NE.0) GO TO 26
C
C        INTERPOLATION FOR FINAL FREQUENCY AND RESIDUAL
C
      Q1=OMAG(1)
      Q2=OMAG(2)
      Q3=DETR(1)
      Q4=DETR(2)
      Q5=DETI(1)
      Q6=DETI(2)
      IF (ITEST.NE.1) GO TO 18
      OMAGI=Q1-Q3*((Q2-Q1)/(Q4-Q3))
      GO TO 19
   18 OMAGI=Q1-Q5*((Q2-Q1)/(Q6-Q5))
   19 IF (MODES.EQ.1) GO TO 23
      PRINT 12, INT
      PRINT 20, Q1,Q2,Q3,Q5,Q4,Q6,OMAGI,ICASE
```

```
   20 FORMAT(*0BOUNDS FOR FINAL FREQUENCY*,3XF9.7,*,*3XF9.7/
     11X*CORRESPONDING RESIDUALS*,3X*(*E15.8,*,*1XE15.8*)*,3X*(*E15.8,*,
     2*1XE15.8*)*//5X*INTERPOLATED (FINAL) FREQUENCY=*F9.7,5X*CASE=*I2)
      PRINT 21
   21 FORMAT(*0MODE SHAPES NOT REQUESTED, THEREFORE FINAL RESIDUAL NOT C
     1ALCULATED*)
      IF (NROOTS.EQ.1) PRINT 22, (LAMDAI(I),I=1,8)
   22 FORMAT(*0ROOTS OF CHARACTERISTIC EQUATION*/(5X*(*E15.8*,*1XE15.8*)
     1*))
      IF (N.LT.1.) CALL NZERO
      GO TO 25
   23 IF (N.LT.1.) RESID=ZN(OMAGI)
      IF (N.LT.1.) GO TO 233
      RESID=RES(OMAGI)
  233 PRINT 12, INT

C         PRINT OUTPUT
C           Q1,Q2,Q3,Q4,Q5,Q6   INTERPOLATION DATA
C           OMAGI    INTERPOLATED FREQUENCY
C           DETERM   RESIDUAL ASSOCIATED WITH OMAGI
C           ICASE    CASE OF CHARACTERISTIC ROOTS
C           ISCALE   DETERMINANT (RESIDUAL) = DETERM*10.E18**ISCALE

      PRINT 20, Q1,Q2,Q3,Q5,Q4,Q6,OMAGI,ICASE
      PRINT 24, DETERM,ISCALE
   24 FORMAT(*0INTERPOLATED (FINAL) RESIDUAL=*,1X*(*E15.8*,*1XE15.8*)*,5
     1X*ISCALE=*I2)
      IF (NROOTS.EQ.1) PRINT 22, (LAMDAI(I),I=1,8)
      IF (N.LT.1.) CALL NZERO
      CALL SHAPE (NMODES,SU,SV,SW,XA,NX,NTHETA,NXTHETA,MX,MTHETA,
     1MXTHETA)
   25 INT=0
      IF (IFREQ.EQ.NFREQ(IN)) GO TO 26
      OMI(IN)=OMAGI+.005*OMD(IN)
      OME(IN)=OME(IN)+OMAGI
      GO TO 99
   26 CONTINUE
      GO TO 1000
   27 STOP
      END
```

```
      FUNCTION RES(OMEGA)
      REAL MU,N,LAMBDA,NXB,NYB,KX
      DIMENSION A4(5),IPIVOT(8),INDEX(8,2)
      COMPLEX TEMP(10)
      COMMON/INPUTD/MU,N,KX,NXB,NYB,LAMBDA,S,IRUN
     1/BC/ABC(8,8),BBC(8,8)
     2/RESIDL/OMAG(2),DETR(2),DETI(2),IRTEST,ITEST,INCASE,INT
     3/CASE/LAMDAI(8),ROOT(8),ICASE
     4/CDET/DETERM,ISCALE
     5/HANDG/H(8),G(8)
     6/CHANGE/ICHANGE
     7/CMR/CM(8,8)
C        FUNCTION SUBPROGRAM WHICH DETERMINES THE RESIDUAL (DETERM) FOR A
C        PARTICULAR FREQUENCY (OMEGA)
      COMPLEX H,G,RL,RL2,RL4,CN(8,8),U(8,8),CM1(8,8),CM2(8,8),CM,
     1LAMDA4(4),LAMDAI,DETERM,P1,P2,P3,P4,C1,C2,ROOT,CMS(8,8)
      INT=INT+1
      QM=(1.-MU)/2.
      QP=(1.+MU)/2.
      QL=1.+LAMBDA**2/12.
      F2=-QP*N
      F3=MU+KX
      F4=-QM*(1.+LAMBDA**2/3.)
      F6=LAMBDA**2*(2.-MU)*N/12.
      F7=-(1.+MU*KX)*N-LAMBDA**2*N**3/12.
      F8=LAMBDA**2/12.
      F9=-LAMBDA**2*N**2/6.-NXB
      F1=-QM*N**2+OMEGA    **2
      F5=QL*N**2-OMEGA    **2
      F10=LAMBDA**2*N**4/12.+KX**2+2.*MU*KX+1.+NYB*N**2-OMEGA**2
C        A4 - COEFFICIENTS OF CHARACTERISTIC EQUATION - SINCE EQUATION IS A
C        BI-QUARTIC, EQUATION IS HANDLED AS FOURTH DEGREE
      A4(1)=1.0
      A4(2)=(-F6**2+F8*(F5+F1*F4-F2**2)+F4*F9)/(F4*F8)
      A4(3)=(F3*(2.0*F2*F6-F3*F4)-F6*(2.  *F7+F1*F6)+F1*F5*F8+F9*(F5+
     1F1*F4-F2**2)+F4*F10)/(F4*F8)
      A4(4)=(F3*(2.  *F2*F7-F3*F5)-2.  *F1*F6*F7-F7**2+F1*F5*F9+F10*
     1(F5+F1*F4-F2**2))/(F4*F8)
```

```
      A4(5)=(-F1*F7**2+F1*F5*F10)/(F4*F8)
  301 CALL FALG(A4,4,0,LAMDA4,TEMP,IERR)
      IF (IERR.NE.0) PRINT 1, IERR
    1 FORMAT(*0IERR=*I2,3X*ABNORMAL RETURN FROM FALG SUBROUTINE*)
      IF (IERR.NE.0) PRINT 8, (A4(I),I=1,5)
    8 FORMAT(*0COEFFICIENTS//(5E25.8))

C        CALCULATE THE EIGHT ROOTS

      DO 2 J=1,4
      LAMDAI(2*J-1)=CSQRT(LAMDA4(J))
    2 LAMDAI(2*J)=-LAMDAI(2*J-1)
      CALL ABCDI
      IF (IRUN.EQ.2) GO TO 11
      IF (IRTEST.EQ.0) 10,20

C        ENTRY FOR DETERMING CASE TO START WITH TERMINATED HERE

   10 RES=0.0
      INT=0
      RETURN

C        STATEMENTS FOR HANDLING A CASE CHANGE - IN A PARTICULAR FREQUENCY
C        INTERVAL THE CASE MAY CHANGE SUCH THAT THE COLUMN OF THE COMPLEX
C        RESIDUAL TO BE FOLLOWED MAY SWITCH

   20 ICHANGE=0
      IF (ICASE.EQ.INCASE) GO TO 11
      IF (IRUN.EQ.3) PRINT 25, INT,OMEGA,ICASE
   25 FORMAT(5XI2,5XE15.8,9X*CASE CHANGE OBSERVED AND INSPECTED*8XI2)
      INCASE=ICASE
      ITEST=1
      IF (ICASE.EQ.2) ITEST=2
      ICHANGE=1
      RETURN
   11 DO 9 I=1,8
    9 LAMDAI(I)=ROOT(I)
   15 DO 3 I=1,8
      RL=LAMDAI(I)
      RL2=RL**2
      RL4=RL**4

C        H AND G  MODAL AMPLITUDE COEFFICIENTS
```

```
      H(I)=(-(F6*RL2+F7)**2+(F8*RL4+F9*RL2+F10)*(F4*RL2+F5))/(F2*RL*(F6*
     1RL2+F7)-F3*RL*(F4*RL2+F5))
      G(I)=(-(F2*RL)*(F8*RL4+F9*RL2+F10)+F3*RL*(F6*RL2+F7))/(F2*RL*(F6*
     1RL2+F7)-F3*RL*(F4*RL2+F5))
      P1=H(I)*RL-MU*N*G(I)+KX+MU
      P2=N*H(I)+(1.+LAMBDA**2/ 6.)*RL*G(I)-LAMBDA**2*N*RL/6.
      P3=LAMBDA**2*(2.-MU)*N*RL*G(I)/12.+LAMBDA**2*(RL**3-(2.-MU)*N**2*
     1RL)/12.-NXB*RL
      P4=MU*N*G(I)+RL2-MU*N**2
      C1=CEXP(-RL*S/2.)
      C2=CEXP(RL*S/2.)

C        CN AND U  BOUNDARY CONDITION COEFFICIENTS

      CN(1,I)=C1*P1
      CN(2,I)=C1*P2
      CN(3,I)=C1*P3
      CN(4,I)=C1*P4
      CN(5,I)=C2*P1
      CN(6,I)=C2*P2
      CN(7,I)=C2*P3
      CN(8,I)=C2*P4
      U(1,I)=C1*H(I)
      U(2,I)=C1*G(I)
      U(3,I)=C1
      U(4,I)=C1*RL
      U(5,I)=C2*H(I)
      U(6,I)=C2*G(I)
      U(7,I)=C2
    3 U(8,I)=C2*RL
      DO 4 I=1,8
      DO 4 J=1,8
      CM1(I,J)=(0.,0.)
      CM2(I,J)=(0.,0.)
      DO 4 K=1,8
      CM1(I,J)=CM1(I,J)+ABC(I,K)*CN(K,J)
    4 CM2(I,J)=CM2(I,J)+BBC(I,K)*U(K,J)
      DO 5 I=1,8
      DO 5 J=1,8
      CM(I,J)=CM1(I,J)+CM2(I,J)
    5 CMS(I,J)=CM(I,J)
```

```
C         EVALUATION OF COMPLEX RESIDUAL

      CALL CDETRM (CMS,8,DETERM,ISCALE,IPIVOT,INDEX)
      X=REAL(DETERM)
      Y=AIMAG(DETERM)
      IF (ITEST.EQ.0) RES=0.0
      IF (IRUN.EQ.2) RETURN
      IF (ITEST.EQ.1) RES=X
      IF (ITEST.EQ.2) RES=Y
   16 IF (INT.EQ.1) OMAG(1)=OMEGA
      IF (INT.EQ.1) DETR(1)=X
      IF (INT.EQ.1) DETI(1)=Y
      IF ((INT.EQ.1).AND.(IRUN.EQ.3)) PRINT 6, INT,OMEGA,DETERM,ISCALE,
     1ICASE
      IF (INT.EQ.1) RETURN
      IF (ITEST.EQ.2) 13,12

C         STORAGE OF INFORMATION FOR INTERPOLATION OF FINAL RESIDUAL AND
C         FREQUENCY

   12 IF (X.GT.0.) OMAG(1)=OMEGA
      IF (X.GT.0.) DETR(1)=X
      IF (X.GT.0.) DETI(1)=Y
      IF (X.LE.0.) OMAG(2)=OMEGA
      IF (X.LE.0.) DETR(2)=X
      IF (X.LE.0.) DETI(2)=Y
      GO TO 14
   13 IF (Y.GT.0.) OMAG(1)=OMEGA
      IF (Y.GT.0.) DETR(1)=X
      IF (Y.GT.0.) DETI(1)=Y
      IF (Y.LE.0.) OMAG(2)=OMEGA
      IF (Y.LE.0.) DETR(2)=X
      IF (Y.LE.0.) DETI(2)=Y
   14 IF (IRUN.EQ.3) PRINT 6, INT,OMEGA,DETERM,ISCALE,ICASE
    6 FORMAT(5XI2,5XE15.8,5X*(*E15.8*,*E16.8,*)*,2(5XI2))
      RETURN
      END
```

```
      FUNCTION ZN(OMEGA)
      DIMENSION A4(4),IPIVOT(6),INDEX(6,2)
      COMPLEX TEMP(8)
      COMMON/INPUTD/MU,N,KX,NXB,NYB,LAMBDA,S,IRUN
     9/BCN/ABCN(8,8),BBCN(8,8)
     2/RESIDL/OMAG(2),DETR(2),DETI(2),IRTEST,ITEST,INCASE,INT
     3/HANDG/H(8),G(8)
     4/CASE/LAMDAI(8),ROOT(8),ICASE
     5/CHANGE/ICHANGE
     6/CDET/DETERM,ISCALE
     7/CMZR/CMZ(6,6)
      COMPLEX LAMDA4(3),LAMDAI,ROOT,H,G,RL,RL2,RL4,CN(6,6),U(6,6),
     1CMZ,CMZ1(6,6),CMZ2(6,6),CMZS(6,6),P1,P3,P4,C1,C2,DETERM
      REAL MU,N,KX,NXB,NYB,LAMBDA

C        SPECIAL FUNCTION SUBPROGRAM FOR N=0 WHICH DETERMINES THE RESIDUAL
C        (DETERM) FOR A PARTICULAR FREQUENCY (OMEGA) - SEE FUNCTION SUBPROGRAM
C        RES

      INT=INT+1
      F3=MU+KX
      F8=LAMBDA**2/12.
      F9=-NXB
      F1=OMEGA**2
      F10=KX**2+2.*MU*KX+1.-OMEGA**2
      A4(1)=F8
      A4(2)=F1*F8+F9
      A4(3)=F1*F9+F10-F3**2
      A4(4)=F1*F10
      CALL FALG (A4,3,0,LAMDA4,TEMP,IERR)
      IF (IERR.NE.0) PRINT 1, IERR
    1 FORMAT(*0IERR=*I2,3X*ABNORMAL RETURN FROM FALG SUBROUTINE*)
      IF (IERR.NE.0) PRINT 2, (A4(I),I=1,4)
    2 FORMAT(*0COEFFICIENTS*//(4E20.8))
      DO 43 J=1,3
      LAMDAI(2*J-1)=CSQRT(LAMDA4(J))
   43 LAMDAI(2*J)=-LAMDAI(2*J-1)
      LAMDAI(7)=(0.,1.)
      LAMDAI(8)=(0.,-1.)
      CALL ABCDI
      IF (IRUN.EQ.2) GO TO 11
      IF (IRTEST.EQ.0) 10,20
```

```
10 ZN=0.0
   INT=0
   RETURN
20 ICHANGE=0
   IF (ICASE.EQ.INCASE) GO TO 11
   INCASE=ICASE
   ITEST=1
   IF (ICASE.EQ.3) ITEST=2
   ICHANGE=1
   RETURN
11 IF (ABS(AIMAG(ROOT(5))).EQ.1.) ROOT(5)=ROOT(7)
   IF (ABS(AIMAG(ROOT(6))).EQ.1.) ROOT(6)=ROOT(8)
   DO 9 I=1,6
 9 LAMDAI(I)=ROOT(I)
   DO 34 I=1,6
   RL=LAMDAI(I)
   RL2=RL**2
   RL4=RL**4
   H(I)=(F8*RL4+F9*RL2+F10)/(-F3*RL)
   P1=H(I)*RL+KX+MU
   P3=LAMBDA**2*RL**3/12.-NXB*RL
   P4=RL2
   C1=CEXP(-RL*S/2.)
   C2=CEXP(RL*S/2.)
   CN(1,I)=C1*P1
   CN(2,I)=C1*P3
   CN(3,I)=C1*P4
   CN(4,I)=C2*P1
   CN(5,I)=C2*P3
   CN(6,I)=C2*P4
   U(1,I)=C1*H(I)
   U(2,I)=C1
   U(3,I)=C1*RL
   U(4,I)=C2*H(I)
   U(5,I)=C2
34 U(6,I)=C2*RL
   DO 4 I=1,6
   DO 4 J=1,6
   CMZ1(I,J)=(0.,0.)
   CMZ2(I,J)=(0.,0.)
   DO 4 K=1,6
   CMZ1(I,J)=CMZ1(I,J)+ABCN(I,K)*CN(K,J)
 4 CMZ2(I,J)=CMZ2(I,J)+BBCN(I,K)*U(K,J)
```

```
      DO 5 I=1,6
      DO 5 J=1,6
      CMZ(I,J)=CMZ1(I,J)+CMZ2(I,J)
    5 CMZS(I,J)=CMZ(I,J)
      CALL CDETRM (CMZS,6,DETERM,ISCALE,IPIVOT,INDEX)
      X=REAL(DETERM)
      Y=AIMAG(DETERM)
      IF (ITEST.EQ.0) ZN=0.0
      IF (IRUN.EQ.2) RETURN
      IF (ITEST.EQ.1) ZN=X
      IF (ITEST.EQ.2) ZN=Y
      IF (INT.EQ.1) OMAG(1)=OMEGA
      IF (INT.EQ.1) DETR(1)=X
      IF (INT.EQ.1) DETI(1)=Y
      IF ((INT.EQ.1).AND.(IRUN.EQ.3)) PRINT 6, INT,OMEGA,DETERM,
     1ISCALE,ICASE
      IF (INT.EQ.1) RETURN
      IF (ITEST.EQ.2) 13,12
   12 IF (X.GT.0.) OMAG(1)=OMEGA
      IF (X.GT.0.) DETR(1)=X
      IF (X.GT.0.) DETI(1)=Y
      IF (X.LE.0.) OMAG(2)=OMEGA
      IF (X.LE.0.) DETR(2)=X
      IF (X.LE.0.) DETI(2)=Y
      GO TO 14
   13 IF (Y.GT.0.) OMAG(1)=OMEGA
      IF (Y.GT.0.) DETR(1)=X
      IF (Y.GT.0.) DETI(1)=Y
      IF (Y.LE.0.) OMAG(2)=OMEGA
      IF (Y.LE.0.) DETR(2)=X
      IF (Y.LE.0.) DETI(2)=Y
   14 IF (IRUN.EQ.3) PRINT 6, INT,OMEGA,DETERM,ISCALE,ICASE
    6 FORMAT(5XI2,5XE15.8,5X*(*E15.8*,*E16.8*)*,2(5XI2))
      RETURN
      END
```

```
      SUBROUTINE NZERO
      COMMON/INPUTD/MU,N,KX,NXB,NYB,LAMBDA,S,IRUN
      REAL MU,N,KX,NXB,NYB,LAMBDA

C        SPECIAL SUBROUTINE FOR N=0 CASE -- CALCULATES MINIMUM TORSIONAL
C        FREQUENCY
      PI=3.14159265358979
      TORMIN=PI*SQRT(((1.-MU)/2.)*(1.+LAMBDA**2/3.))/S
      PRINT 1, TORMIN
    1 FORMAT(*OMINIMUM TORSIONAL FREQUENCY FOR N=0     *E15.8)
      RETURN
      END
```

```
      SUBROUTINE ABCDI
      COMMON/CASE/LAMDAI(8),ROOT(8),ICASE
      COMPLEX LAMDAI,ROOT
      INTEGER SUM

C        BOOKKEEPING SUBROUTINE FOR ROOTS OF THE CHARACTERISTIC EQUATION --
C        DETERMINES CASE AND REORDERS ROOTS

C                  REAL COLUMN FOLLOWED FOR CASES 1,3,4
C                  IMAGINARY COLUMN FOLLOWED FOR CASE 2

C              CASE 1  4 FULL COMPLEX
C                      4 PURE REAL

C              CASE 2  4 FULL COMPLEX
C                      2 PURE REAL
C                      2 PURE IMAGINARY

C              CASE 3  4 FULL COMPLEX
C                      4 PURE IMAGINARY

C              CASE 4  8 FULL COMPLEX

      J1=0
      J2=0
      J3=0
      J4=0
      J5=0
      J6=0
      J7=0
      J8=0
      SUM=0
      DO 7 I=1,8
      X=REAL(LAMDAI(I))
      Y=AIMAG(LAMDAI(I))
      IF ((X.GT.0.).AND.(Y.GT.0.)) SUM=SUM+1
      IF ((X.GT.0.).AND.(Y.GT.0.)) J1=J1+1
      IF ((X.GT.0.).AND.(Y.GT.0.).AND.(J1.EQ.1)) A=X
      IF ((X.GT.0.).AND.(Y.GT.0.).AND.(J1.EQ.1)) B=Y
      IF ((X.GT.0.).AND.(Y.GT.0.).AND.(J1.EQ.1)) ROOT(1)=LAMDAI(I)
      IF ((X.GT.0.).AND.(Y.GT.0.).AND.(J1.EQ.2)) C=X
      IF ((X.GT.0.).AND.(Y.GT.0.).AND.(J1.EQ.2)) D=Y
```

```
      IF ((X.GT.0.).AND.(Y.GT.0.).AND.(J1.EQ.2)) ROOT(5)=LAMDAI(I)
      IF ((X.LT.0.).AND.(Y.LT.0.)) SUM=SUM+2
      IF ((X.LT.0.).AND.(Y.LT.0.)) J2=J2+1
      IF ((X.LT.0.).AND.(Y.LT.0.).AND.(J2.EQ.1)) ROOT(2)=LAMDAI(I)
      IF ((X.LT.0.).AND.(Y.LT.0.).AND.(J2.EQ.2)) ROOT(6)=LAMDAI(I)
      IF ((X.GT.0.).AND.(Y.LT.0.)) SUM=SUM+3
      IF ((X.GT.0.).AND.(Y.LT.0.)) J3=J3+1
      IF ((X.GT.0.).AND.(Y.LT.0.).AND.(J3.EQ.1)) ROOT(3)=LAMDAI(I)
      IF ((X.GT.0.).AND.(Y.LT.0.).AND.(J3.EQ.2)) ROOT(7)=LAMDAI(I)
      IF ((X.LT.0.).AND.(Y.GT.0.)) SUM=SUM+4
      IF ((X.LT.0.).AND.(Y.GT.0.)) J4=J4+1
      IF ((X.LT.0.).AND.(Y.GT.0.).AND.(J4.EQ.1)) ROOT(4)=LAMDAI(I)
      IF ((X.LT.0.).AND.(Y.GT.0.).AND.(J4.EQ.2)) ROOT(8)=LAMDAI(I)
      IF ((X.GT.0.).AND.(Y.EQ.0.)) SUM=SUM+5
      IF ((X.GT.0.).AND.(Y.EQ.0.)) J5=J5+1
      IF ((X.GT.0.).AND.(Y.EQ.0.).AND.(J5.EQ.1)) C=X
      IF ((X.GT.0.).AND.(Y.EQ.0.).AND.(J5.EQ.1)) ROOT(5)=LAMDAI(I)
      IF ((X.GT.0.).AND.(Y.EQ.0.).AND.(J5.EQ.2)) D=X
      IF ((X.GT.0.).AND.(Y.EQ.0.).AND.(J5.EQ.2)) ROOT(7)=LAMDAI(I)
      IF ((X.LT.0.).AND.(Y.EQ.0.)) SUM=SUM+6
      IF ((X.LT.0.).AND.(Y.EQ.0.)) J6=J6+1
      IF ((X.LT.0.).AND.(Y.EQ.0.).AND.(J6.EQ.1)) ROOT(6)=LAMDAI(I)
      IF ((X.LT.0.).AND.(Y.EQ.0.).AND.(J6.EQ.2)) ROOT(8)=LAMDAI(I)
      IF ((X.EQ.0.).AND.(Y.GT.0.)) SUM=SUM+7
      IF ((X.EQ.0.).AND.(Y.GT.0.)) J7=J7+1
      IF ((X.EQ.0.).AND.(Y.GT.0.).AND.(J7.EQ.1)) D=Y
      IF ((X.EQ.0.).AND.(Y.GT.0.).AND.(J7.EQ.1)) ROOT(7)=LAMDAI(I)
      IF ((X.EQ.0.).AND.(Y.GT.0.).AND.(J7.EQ.2)) C=Y
      IF ((X.EQ.0.).AND.(Y.GT.0.).AND.(J7.EQ.2)) ROOT(5)=LAMDAI(I)
      IF ((X.EQ.0.).AND.(Y.LT.0.)) SUM=SUM+8
      IF ((X.EQ.0.).AND.(Y.LT.0.)) J8=J8+1
      IF ((X.EQ.0.).AND.(Y.LT.0.).AND.(J8.EQ.1)) ROOT(8)=LAMDAI(I)
      IF ((X.EQ.0.).AND.(Y.LT.0.).AND.(J8.EQ.2)) ROOT(6)=LAMDAI(I)
    7 CONTINUE
      IF (SUM.EQ.32) ICASE=1
      IF (SUM.EQ.32) RETURN
      IF (SUM.EQ.36) ICASE=2
      IF (SUM.EQ.36) RETURN
      IF (SUM.EQ.40) ICASE=3
      IF (SUM.EQ.40) RETURN
      IF (SUM.EQ.20) ICASE=4
      IF (SUM.EQ.20) RETURN
      PRINT 100
```

```
100 FORMAT(*OERROR IN ABCDI SUBROUTINE*)
    RETURN
    END
```

```
      SUBROUTINE CDETRM (A,N,DETERM,ISCALE,IPIVOT,INDEX)
      COMPLEX A(N,N),AMAX,SWAP,T,DETERM,PIVOT,PIVOTI
      DIMENSION IPIVOT(N),INDEX(N,2)
      EQUIVALENCE(AMAX,SWAP,T)

C         SUBROUTINE FOR EVALUATING A COMPLEX DETERMINANT

C         INITIALIZATION

      ISCALE=0
      R1=10.E18
      R2=1./R1
      DETERM=(1.,0.)
      DO 20 J=1,N
   20 IPIVOT(J)=0
      DO 550 I=1,N
      RN=I

C         SEARCH FOR PIVOT ELEMENT

      AMAX=(0.0,0.0)
      DO 105 J=1,N
      IF (IPIVOT(J)-1) 60,105,60
   60 DO 100 K=1,N
      IF (IPIVOT(K)-1) 80,100,740
   80 IF (CABS(AMAX)-CABS(A(J,K))) 85,100,100
   85 IROW=J
      ICOLUM=K
      AMAX=A(J,K)
  100 CONTINUE
  105 CONTINUE
  110 IPIVOT(ICOLUM)=IPIVOT(ICOLUM)+1

C         INTERCHANGE ROWS TO PUT PIVOT ELEMENT ON DIAGONAL

      IF (IROW-ICOLUM) 140,260,140
  140 DETERM=-DETERM
      DO 200 L=1,N
      SWAP=A(IROW,L)
      A(IROW,L)=A(ICOLUM,L)
  200 A(ICOLUM,L)=SWAP
  260 INDEX(I,1)=IROW
```

```
      INDEX(I,2)=ICOLUM
      PIVOT=A(ICOLUM,ICOLUM)

C        SCALE DETERMINANT

      PIVOTI=PIVOT
      IF (CABS(DETERM)-R1) 1030,1010,1010
 1010 DETERM=DETERM/R1
      ISCALE=ISCALE+1
      IF (CABS(DETERM)-R1) 1060,1020,1020
 1020 DETERM=DETERM/R1
      ISCALE=ISCALE+1
      GO TO 1060
 1030 IF (CABS(DETERM)-R2) 1040,1040,1060
 1040 DETERM=DETERM*R1
      ISCALE=ISCALE-1
      IF (CABS(DETERM)-R2) 1050,1050,1060
 1050 DETERM=DETERM*R1
      ISCALE=ISCALE-1
 1060 IF (CABS(PIVOTI)-R1) 1090,1070,1070
 1070 PIVOTI=PIVOTI/R1
      ISCALE=ISCALE+1
      IF (CABS(PIVOTI)-R1) 320,1080,1080
 1080 PIVOTI=PIVOTI/R1
      ISCALE=ISCALE+1
      GO TO 320
 1090 IF (CABS(PIVOTI)-R2) 2000,2000,320
 2000 PIVOTI=PIVOTI*R1
      ISCALE=ISCALE-1
      IF (CABS(PIVOTI)-R2) 2010,2010,320
 2010 PIVOTI=PIVOTI*R1
      ISCALE=ISCALE-1
  320 DETERM=DETERM*PIVOTI

C        DIVIDE PIVOT ROW BY PIVOT ELEMENT

      A(ICOLUM,ICOLUM)=(1.0,0.0)
      DO 350 L=1,N
  350 A(ICOLUM,L)=A(ICOLUM,L)/PIVOT

C        REDUCE NON PIVOT ROWS

      DO 550 L1=1,N
```

```
      IF (L1-ICOLUM) 400,550,400
  400 T=A(L1,ICOLUM)
      A(L1,ICOLUM)=(0.0,0.0)
      DO 450 L=1,N
  450 A(L1,L)=A(L1,L)-A(ICOLUM,L)*T
  550 CONTINUE
  740 RETURN
      END
```

```
      SUBROUTINE MANUAL (OMI,OME,OMD)
C         MANUAL SEARCH SUBROUTINE
      COMMON/CASE/LAMDAI(8),ROOT(8),ICASE
     1/CDET/DETERM,ISCALE
     2/RESIDL/OMAG(2),DETR(2),DETI(2),IRTEST,ITEST,INCASE,INT
     3/CMR/CM(8,8)
     4/CMRZ/CMZ(6,6)
     5/INPUTD/MU,N,KX,NXB,NYB,LAMBDA,S,IRUN
      REAL MU,N,KX,NXB,NYB,LAMBDA
      COMPLEX DETERM,CM,LAMDAI,ROOT,CMZ
      K=ABS((OME-OMI)/OMD)+2.
      DO 10 I=1,K
      IF (I.EQ.1) OMEGA=OMI
      IF (I.NE.1) OMEGA=OMI+OMD
      INT=0
      IF (N.EQ.0.) Q=ZN(OMEGA)
      IF (N.EQ.0.) GO TO 10
      Q=RES(OMEGA)
   10 PRINT 11, INT,OMEGA,DETERM,ISCALE,ICASE
   11 FORMAT(5XI2,5XE15.8,5X*(*E15.8*,*E16.8,*)*,5XI2)
      RETURN
      END
```

```
      SUBROUTINE ITR2 (X,A,B,DELTX,FOFX,E1,E2,MAXI,ICODE)
C        SUBROUTINE FOR THE ITERATIVE HALVING PROCESS WHICH FINDS FREQUENCY FOR
C        WHICH RESIDUAL IS ZERO -- CONTAINS LOGIC TO HANDLE A CASE CHANGE IF ONE
C        OCCURS
      COMMON/CHANGE/ICHANGE
      ABEGIN=A
      DELBEG=DELTX
      X=A
      IF (DELTX.LE.0.0) GO TO 16
      IF (B.LE.A) GO TO 17
  100 ITEST=0
    1 I=0
      ITEST=ITEST+1
      TEST=FOFX(A)
      IF (TEST) 2,20,7
    2 XB1=X
      X=X+DELTX
      IF (X.GT.B) GO TO 18
      TEST=FOFX(X)
      IF (ICHANGE.EQ.0) GO TO 23
      SAVEX=X
      IF (ITEST.LT.3) GO TO 43
      A=SAVEX
      DELTX=DELBEG
      GO TO 100
   43 OLDA=X
      A=X-DELTX
      DELTX=DELTX/10.
      X=A
      GO TO 1
   23 IF (TEST) 2,20,3
    3 XB=X
      X=X-DELTX/(2.**(I+1))
    4 I=I+1
      IF (MAXI.LT.I) GO TO 19
      IF (FOFX(X)) 5,20,6
    5 L=1
      XX=XB
      GO TO 12
    6 L=2
```

```
      XX=XB1
      GO TO 12
    7 XB1=X
      X=X+DELTX
      IF (X.GT.B) GO TO 18
      TEST=FOFX(X)
      IF (ICHANGE.EQ.0) GO TO 25
      SAVEX=X
      IF (ITEST.LT.3) GO TO 53
      A=SAVEX
      DELTX=DELBEG
      GO TO 100
   53 OLDA=X
      A=X-DELTX
      DELTX=DELTX/10.
      X=A
      GO TO 1
   25 IF (TEST) 8,20,7
    8 XB=X
      X=X-DELTX/(2.**(I+1))
    9 I=I+1
      IF (MAXI.LT.I) GO TO 19
      IF (FOFX(X)) 11,20,10
   10 L=3
      XX=XB
      GO TO 12
   11 L=4
      XX=XB1
   12 IF ((ABS(X)-E1).LE.0.0) GO TO 13
      IF ((ABS((XX-X)/X)-E1).LE.0.0) 20,14
   13 IF ((ABS(XX-X)-E2).LE.0.0) GO TO 20
   14 GO TO (15,3,15,8), L
   15 XB1=X
      X=X+DELTX/(2.**(I+1))
      GO TO (4,3,9,8),L
   16 ICODE=2
      RETURN
   17 ICODE=4
      RETURN
   18 ICODE=3
      RETURN
   19 ICODE=1
      RETURN
```

```
20 ICODE=0
   A=ABEGIN
   DELTX=DELBEG
   RETURN
   END
```

```
      SUBROUTINE SHAPE (NMODES,SU,SV,SW,XA,NX,NTHETA,NXTHETA,MX,MTHETA,
     1MXTHETA)
C         MODE SHAPE SUBROUTINE
      COMMON/MODSHA/MUVW,IPRINT,IPLOT
     1/HANDG/H(8),G(8)
     2/CDET/DETERM,ISCALE
     3/INPUTD/MU,N,KX,NXB,NYB,LAMBDA,S,IRUN
     4/CASE/LAMDAI(8),ROOT(8),ICASE
     5/FREQM/OMAGI
     6/CMR/CM(8,8)
      COMPLEX H,G,DETERM,CM,LAMDAI,ROOT,CM7(7,7),CM7S(7,7),SUM,DELTA,
     1USUM,VSUM,WSUM,A(8),S1,S2,S3,S4,S5,S6,DET,CM7SS(7,7)
      DIMENSION SU(NMODES),SV(NMODES),SW(NMODES),XA(NMODES),NX(NMODES),
     1NTHETA(NMODES),NXTHETA(NMODES),MX(NMODES),MTHETA(NMODES),
     2MXTHETA(NMODES),XM(11),YM1(3),YM2(3),YM3(2),YM4(3),YM5(4),
     3YM6(3),YM7(3),YM8(3),YM9(2),IPIVOT(7),INDEX(7,2),IN(2)
      REAL NX,NTHETA,NXTHETA,MX,MTHETA,MXTHETA,MU,N,NXB,NYB,LAMBDA,KX
C         CRAMERS RULE APPLIED TO CALCULATE MODE SHAPES AND STRESSES
      A(8)=(1.,0.)
      DO 1 I=1,7
      DO 1 J=1,7
    1 CM7(I,J)=CM(I+1,J)
      DO 2 I=1,7
      DO 2 J=1,7
    2 CM7S(I,J)=CM7(I,J)
      CALL CDETRM (CM7S,7,DET,ISCALE,IPIVOT,INDEX)
      I8=ISCALE
      DET=10.E18**I8*DETERM
      DO 3 I=1,7
      CM7SS(I,1)=CM7(I,1)
    3 CM7(I,1)=-CM(I+1,8)
      DO 4 I=1,7
      DO 4 J=1,7
    4 CM7S(I,J)=CM7(I,J)
      CALL CDETRM (CM7S,7,A(1),ISCALE,IPIVOT,INDEX)
      I1=ISCALE
      A(1)=10.E18**I1*A(1)
   56 A(1)=A(1)/DET
```

```
57 DO 5 I=1,7
   CM7(I,1)=CM7SS(I,1)
   CM7SS(I,2)=CM7(I,2)
 5 CM7(I,2)=-CM(I+1,8)
   DO 6 I=1,7
   DO 6 J=1,7
 6 CM7S(I,J)=CM7(I,J)
   CALL CDETRM (CM7S,7,A(2),ISCALE,IPIVOT,INDEX)
   I2=ISCALE
   A(2)=10.E18**I2*A(2)
59 A(2)=A(2)/DET
60 DO 7 I=1,7
   CM7(I,2)=CM7SS(I,2)
   CM7SS(I,3)=CM7(I,3)
 7 CM7(I,3)=-CM(I+1,8)
   DO 8 I=1,7
   DO 8 J=1,7
 8 CM7S(I,J)=CM7(I,J)
   CALL CDETRM (CM7S,7,A(3),ISCALE,IPIVOT,INDEX)
   I3=ISCALE
   A(3)=10.E18**I3*A(3)
62 A(3)=A(3)/DET
63 DO 9 I=1,7
   CM7(I,3)=CM7SS(I,3)
   CM7SS(I,4)=CM7(I,4)
 9 CM7(I,4)=-CM(I+1,8)
   DO 10 I=1,7
   DO 10 J=1,7
10 CM7S(I,J)=CM7(I,J)
   CALL CDETRM (CM7S,7,A(4),ISCALE,IPIVOT,INDEX)
   I4=ISCALE
   A(4)=10.E18**I4*A(4)
65 A(4)=A(4)/DET
66 DO 11 I=1,7
   CM7(I,4)=CM7SS(I,4)
   CM7SS(I,5)=CM7(I,5)
11 CM7(I,5)=-CM(I+1,8)
   DO 30 I=1,7
   DO 30 J=1,7
30 CM7S(I,J)=CM7(I,J)
   CALL CDETRM (CM7S,7,A(5),ISCALE,IPIVOT,INDEX)
   I5=ISCALE
   A(5)=10.E18**I5*A(5)
```

```
 68 A(5)=A(5)/DET
 69 DO 31 I=1,7
    CM7(I,5)=CM7SS(I,5)
    CM7SS(I,6)=CM7(I,6)
 31 CM7(I,6)=-CM(I+1,8)
    DO 32 I=1,7
    DO 32 J=1,7
 32 CM7S(I,J)=CM7(I,J)
    CALL CDETRM (CM7S,7,A(6),ISCALE,IPIOVT,INDEX)
    I6=ISCALE
    A(6)=10.E18**I6*A(6)
 71 A(6)=A(6)/DET
 72 DO 33 I=1,7
    CM7(I,6)=CM7SS(I,6)
    CM7SS(I,7)=CM7(I,7)
 33 CM7(I,7)=-CM(I+1,8)
    DO 34 I=1,7
    DO 34 J=1,7
 34 CM7S(I,J)=CM7(I,J)
    CALL CDETRM (CM7S,7,A(7),ISCALE,IPIVOT,INDEX)
    I7=ISCALE
    A(7)=10.E18**I7*A(7)
 74 A(7)=A(7)/DET
    XINC=S/FLOAT(NMODES-1)
    SWSAVE=0.0

C        NORMALIZE MODE SHAPES AND STRESSES TO W MODE

C               SU        MERIDIONAL DISPLACEMENT
C               SV       CIRCUMFERENTIAL DISPLACEMENT
C               SW       NORMAL DISPLACEMENT
C               NX       MERIDIONAL STRESS RESULTANT
C               NTHETA   CIRCUMFERENTIAL STRESS RESULTANT
C               NXTHETA  SHEAR STRESS RESULTANT
C               MX       MERIDIONAL BENDING MOMENT
C               MTHETA   CIRCUMFERENTIAL BENDING MOMENT
C               MXTHETA  TWISTING MOMENT

    DO 12 K=1,NMODES
    IF (K.EQ.1) Z=-S/2.
    IF (K.NE.1) Z=Z+XINC
    XA(K)=Z
    USUM=(0.,0.)
```

```
      VSUM=(0.,0.)
      WSUM=(0.,0.)
      S1=(0.,0.)
      S2=(0.,0.)
      S3=(0.,0.)
      S4=(0.,0.)
      S5=(0.,0.)
      S6=(0.,0.)
      DO 13 I=1,8
      USUM=USUM+A(I)*H(I)*CEXP(LAMDAI(I)*Z)
      VSUM=VSUM+A(I)*G(I)*CEXP(LAMDAI(I)*Z)
      WSUM=WSUM+A(I)*CEXP(LAMDAI(I)*Z)
      IF (MUVW.EQ.1) GO TO 13
      S1=S1+(H(I)*LAMDAI(I)-N*MU*G(I)+KX    +MU)*A(I)*CEXP(LAMDAI(I)*Z)
      S2=S2+(MU*H(I)*LAMDAI(I)-N*G(I)+1.+MU*KX    )*A(I)*CEXP(LAMDAI(I)
     1*Z)
      S3=S3+(N*H(I)+LAMDAI(I)*G(I))*A(I)*CEXP(LAMDAI(I)*Z)
      S4=S4+(N*N*MU-N*MU*G(I)-LAMDAI(I)**2)*A(I)*CEXP(LAMDAI(I)*Z)
      S5=S5+(N*N-N*G(I)-MU*LAMDAI(I)**2)*A(I)*CEXP(LAMDAI(I)*Z)
      S6=S6+ (G(I)-N)*LAMDAI(I)*A(I)*CEXP(LAMDAI(I)*Z)
   13 CONTINUE
      SU(K)=REAL(USUM)
      SV(K)=REAL(VSUM)
      SW(K)=REAL(WSUM)
      ABSS=ABS(SW(K))
      IF (ABSS.GT.SWSAVE) SWSAVE=ABSS
      IF (MUVW.EQ.1) GO TO 12
      NX(K)=REAL(S1)
      NTHETA(K)=REAL(S2)
      NXTHETA(K)=((1.-MU)/2.)*REAL(S3)
      MX(K)=REAL(S4)
      MTHETA(K)=REAL(S5)
      MXTHETA(K)=(1.-MU)*REAL(S6)
   12 CONTINUE
      DO 80 K=1,NMODES
      SU(K)=SU(K)/SWSAVE
      SV(K)=SV(K)/SWSAVE
      SW(K)=SW(K)/SWSAVE
      IF (MUVW.EQ.1) GO TO 80
      NX(K)=NX(K)/SWSAVE
      NTHETA(K)=NTHETA(K)/SWSAVE
      NXTHETA(K)=NXTHETA(K)/SWSAVE
      MX(K)=MX(K)/SWSAVE
```

```
      MTHETA(K)=MTHETA(K)/SWSAVE
      MXTHETA(K)=MXTHETA(K)/SWSAVE
   80 CONTINUE
      IF (IPLOT.EQ.0) GO TO 14
      ITAPE=6LTAPE21
      DATA IN/20HSRDMPR188CVIBRATIONS/
      DATA XM(1)/10H        N=/
      DATA XM(3)/10H    OMEGA=/
      DATA XM(6)/10H   LAMBDA=/
      DATA XM(8)/10H    ALPHA=/
      DATA XM(10)/10H        S=/
      DATA YM1/30HMERIDIONAL DISPLACEMENT       /
      DATA YM2/30HCIRCUMFERENTIAL DISPLACEMENT  /
      DATA YM3/20HNORMAL DISPLACEMENT /
      DATA YM4/30HMERIDIONAL STRESS RESULTANT   /
      DATA YM5/40HCIRCUMFERENTIAL STRESS RESULTANT
      DATA YM6/30HSHEAR STRESS RESULTANT        /
      DATA YM7/30HMERIDIONAL BENDING MOMENT     /
      DATA YM8/30HCIRCUMFERENTIAL BENDING MOMENT/
      DATA YM9/20HTWISTING MOMENT     /
      ENCODE(10,90,XM(2)) N
   90 FORMAT(F4.0)
      ENCODE(20,91,XM(4)) OMAGI
   91 FORMAT(E16.8)
      ENCODE(10,92,XM(7)) LAMBDA
   92 FORMAT(F5.3)
      ENCODE(10,93,XM(9)) KX
      ENCODE(10,93,XM(11)) S
   93 FORMAT(F5.2)
      CALL DDIPLT (1,IN,NMODES,XA,SU,0.,0.,0.,0.,11,XM,3,YM1,14,ITAPE)
      CALL DDIPLT (1,IN,NMODES,XA,SV,0.,0.,0.,0.,11,XM,3,YM2,14,ITAPE)
      CALL DDIPLT (1,IN,NMODES,XA,SW,0.,0.,0.,0.,11,XM,2,YM3,14,ITAPE)
      IF (MUVW.EQ.1) GO TO 14
      CALL DDIPLT (1,IN,NMODES,XA,NX,0.,0.,0.,0.,11,XM,3,YM4,14,ITAPE)
      CALL DDIPLT (1,IN,NMODES,XA,NTHETA,0.,0.,0.,0.,11,XM,4,YM5,14,
     1ITAPE)
      CALL DDIPLT(1,IN,NMODES,XA,NXTHETA,0.,0.,0.,0.,11,XM,3,YM6,14,
     1ITAPE)
      CALL DDIPLT(1,IN,NMODES,XA,MX,0.,0.,0.,0.,11,XM,3,YM7,14,ITAPE)
      CALL DDIPLT(1,IN,NMODES,XA,MTHETA,0.,0.,0.,0.,11,XM,3,YM8,14,
     1ITAPE)
      CALL DDIPLT(1,IN,NMODES,XA,MXTHETA,0.,0.,0.,0.,11,XM,2,YM9,14,
     1ITAPE)
```

```
14 IF (IPRINT.EQ.0) GO TO 16
   PRINT 17
17 FORMAT(*1    MERIDIONAL*,5X*CIRCUMFERENTIAL*,8X*NORMAL*/
  13X*DISPLACEMENT*,6X*DISPLACEMENT*,6X*DISPLACEMENT*/
  27X*(U)*,15X*(V)*,15X*(W)*/)
   DO 18 I=1,NMODES
18 PRINT 19, SU(I),SV(I),SW(I)
19 FORMAT(1XE15.8,5(3XE15.8))
   IF (MUVW.EQ.1) GO TO 16
   PRINT 21
21 FORMAT(*1    MERIDIONAL*,5X*CIRCUMFERENTIAL*,8X*SHEAR*,11X*MERIDION
  1AL*,5X*CIRCUMFERENTIAL*,7X*TWISTING*/
  26X*STRESS*,12X*STRESS*,12X*STRESS*,11X*BENDING*,11X*BENDING*,12X*B
  3ENDING*,12X*MOMENT*/
  44X*RESULTANT*,9X*RESULTANT*,9X*RESULTANT*,11X*MOMENT*,12X*MOMENT*/
  5)
   DO 22 I=1,NMODES
22 PRINT 19, NX(I),NTHETA(I),NXTHETA(I),MX(I),MTHETA(I),MXTHETA(I)
16 RETURN
   END
```

```
VIBRATIONS OF SHALLOW SHELLS WITH DEEP CIRCUMFERENTIAL CURVATURE
COMPLEX DETERMINANT METHOD      ROBINSON-COOPER  SRD-A1556  RDK-258


DATE    08/18/67.


     SAMPLE OUTPUT FOR GENERAL METHOD OF SOLUTION OF THE APPROXIMATE (SHALLOW MERID-
     ION) EQUATIONS OF CHAPTER 7.  NATURAL VIBRATIONS OF FREELY SUPPORTED UNSTRESSED
     NEGATIVE CURVATURE SHELL WITH KX=-.05, S=3, RADIUS/THICKNESS=1000, N=4-8. MPR-PC


BOUNDARY CONDITIONS

X=-S/2                   X=S/2
    ABC(1,1)= 1.00          ABC(5,5)= 1.00
    ABC(2,2)= 0.            ABC(6,6)= 0.
    ABC(3,3)= 0.            ABC(7,7)= 0.
    ABC(4,4)= 1.00          ABC(8,8)= 1.00


GEOMETRY

    LAMBDA=   .001      NXB=  0.
         S=  3.000      NYB=  0.
        KX=  -.050       MU=   .300

CODES

    NROOTS= 1   1 PRINT ROOTS OF CHARACTERISTIC EQUATION,   0 DO NOT
      IRUN= 3   1 USE ITR2,   2 USE MANUAL SEARCH,   3 USE ITR2 AND PRINT
      MAXI=99   MAXIMUM NUMBER OF ITERATIONS FOR ITR2 SUBROUTINE
        E1= 1.0E-04   E2= 1.0E-05   CONVERGENCE CRITERIA FOR ITR2
     MODES= 0   1 CALL MODE SHAPE SUBROUTINE,   0 DO NOT
    NMODES= 0   NUMBER OF X-AXIS POINTS FOR MODE SHAPES
      MUVW= 0   1 CALCULATE U,V,W MODES ONLY,   0 CALCULATE STRESSES ALSO
    IPRINT= 0   1 PRINT CALCULATED MODE SHAPE VALUES ONLY/AND STRESSES,   0 DO NOT
     IPLOT= 0   1 PLOT MODE SHAPE VALUES ONLY/AND STRESSES,   0 DO NOT
    MODE NUMBERS RANGE FROM  4 TO  8
```

INITIAL FREQUENCY INSPECTED= .00500 FINAL FREQUENCY LIMIT= 1.00000 INSPECTION INTERVAL= .00500

MODE NUMBER= 4

ITERATION NUMBER	OMEGA VALUE (FREQUENCY)	RESIDUAL (COMPLEX)	ISCALE	CASE NUMBER
1	5.00000000E-03	(1.14618942E-06, -8.47032947E-21)	6	3
2	1.00000000E-02	(1.13464381E-05, 1.28749008E-19)	6	3
3	1.50000000E-02	(8.09811844E-06, -1.36202898E-18)	6	3
4	2.00000000E-02	(-3.32113607E-05, -1.74827600E-18)	6	3
5	1.75000000E-02	(-6.20517053E-06, 5.01867021E-19)	6	3
6	1.62500000E-02	(2.34159507E-06, 1.13835934E-18)	6	3
7	1.68750000E-02	(-1.55807513E-06, 1.69308651E-18)	6	3
8	1.65625000E-02	(4.82109507E-07, 7.46073899E-19)	6	3
9	1.67187500E-02	(-5.15007789E-07, 5.67776772E-19)	6	3
10	1.66406250E-02	(-1.07536772E-08, 1.26255869E-19)	6	3
11	1.66015625E-02	(2.37095729E-07, 7.02474036E-19)	6	3
12	1.66210937E-02	(1.13526236E-07, 2.82751116E-19)	6	3
13	1.66308594E-02	(5.14751758E-08, 7.00671325E-19)	6	3
14	1.66357422E-02	(2.03829853E-08, -9.82942935E-19)	6	3
15	1.66381836E-02	(4.82021483E-09, -1.29230696E-18)	6	3
16	1.66394043E-02	(-2.96534211E-09, 7.57720449E-19)	6	3

NUMBER OF ITERATIONS= 16

BOUNDS FOR FINAL FREQUENCY .0166382, .0166394
CORRESPONDING RESIDUALS (4.82021483E-09, -1.29230696E-18) (-2.96534211E-09, 7.57720449E-19)

INTERPOLATED (FINAL) FREQUENCY= .0166389 CASE= 3

MODE SHAPES NOT REQUESTED, THEREFORE FINAL RESIDUAL NOT CALCULATED

ROOTS OF CHARACTERISTIC EQUATION

```
( 4.08477227E+01,  4.04441851E+01)
(-4.08477227E+01, -4.04441851E+01)
( 4.08477227E+01, -4.04441851E+01)
(-4.08477227E+01,  4.04441851E+01)
( 0.            ,  1.04720189E+00)
(-0.            , -1.04720189E+00)
( 0.            ,  7.16340041E-01)
(-0.            , -7.16340041E-01)
```

INITIAL FREQUENCY INSPECTED= .00200 FINAL FREQUENCY LIMIT= 1.00000 INSPECTION INTERVAL= .00200

MODE NUMBER= 5

ITERATION NUMBER	OMEGA VALUE (FREQUENCY)	RESIDUAL (COMPLEX)	ISCALE	CASE NUMBER
1	2.00000000E-03	(7.21941430E-06, 1.21972744E-19)	6	4
2	4.00000000E-03	(4.50401528E-06, -1.35525272E-20)	6	4
3	6.00000000E-03	(1.31231804E-06, 3.38813179E-21)	6	4
4	8.00000000E-03	CASE CHANGE OBSERVED AND INSPECTED		3
5	6.00000000E-03	CASE CHANGE OBSERVED AND INSPECTED		4
6	6.20000000E-03	(1.03541385E-06, -3.38813179E-21)	6	4
7	6.40000000E-03	(7.74837836E-07, -3.38813179E-21)	6	4
8	6.60000000E-03	(5.32961614E-07, 2.54109884E-21)	6	4
9	6.80000000E-03	(3.12218358E-07, -5.92923063E-21)	6	4
10	7.00000000E-03	(1.15101516E-07, 4.23516474E-22)	6	4
11	7.20000000E-03	CASE CHANGE OBSERVED AND INSPECTED		3
12	7.00000000E-03	CASE CHANGE OBSERVED AND INSPECTED		4
13	7.02000000E-03	(9.67870489E-08, 1.90582413E-21)	6	4
14	7.04000000E-03	(7.87369072E-08, -2.11758237E-22)	6	4
15	7.06000000E-03	(6.09536801E-08, 5.29395592E-22)	6	4
16	7.08000000E-03	(4.34399650E-08, 0.)	6	4
17	7.10000000E-03	(2.61983630E-08, 0.)	6	4
18	7.12000000E-03	(9.23148090E-09, 1.98523347E-23)	6	4
19	7.14000000E-03	CASE CHANGE OBSERVED AND INSPECTED		3
20	7.14000000E-03	(7.45806743E-09, -2.58080351E-22)	6	3
21	9.14000000E-03	(-1.96637636E-07, -6.67303144E-20)	6	3
22	8.14000000E-03	(4.28963837E-07, -2.31345874E-20)	6	3
23	8.64000000E-03	(2.69826684E-07, -6.06819697E-20)	6	3
24	8.89000000E-03	(7.79583350E-08, -4.05153064E-20)	6	3
25	9.01500000E-03	(-4.86243007E-08, -3.16363497E-20)	6	3
26	8.95250000E-03	(1.72987994E-08, -4.75899754E-20)	6	3
27	8.98375000E-03	(-1.49989389E-08, 1.10909436E-19)	6	3
28	8.96812500E-03	(1.31514888E-09, -7.24608553E-20)	6	3
29	8.97593750E-03	(-6.80049837E-09, 6.87138224E-20)	6	3
30	8.97203125E-03	(-2.73233738E-09, -1.03645650E-20)	6	3
31	8.97007812E-03	(-7.06011148E-10, 9.07616789E-20)	6	3
32	8.96910156E-03	(3.05214468E-10, 1.26158235E-19)	6	3
33	8.96958984E-03	(-2.00236925E-10, 1.72655008E-19)	6	3

NUMBER OF ITERATIONS= 33

BOUNDS FOR FINAL FREQUENCY .0089691, .0089696
CORRESPONDING RESIDUALS (3.05214468E-10, 1.26158235E-19) (-2.00236925E-10, 1.72655008E-19)

```
    INTERPOLATED (FINAL) FREQUENCY= .0089694     CASE= 3

MODE SHAPES NOT REQUESTED, THEREFORE FINAL RESIDUAL NOT CALCULATED

ROOTS OF CHARACTERISTIC EQUATION
    ( 4.09647480E+01,  4.03343692E+01)
    (-4.09647480E+01, -4.03343692E+01)
    ( 4.09647480E+01, -4.03343692E+01)
    (-4.09647480E+01,  4.03343692E+01)
    ( 0.            ,  1.18415141E+00)
    (-0.            , -1.18415141E+00)
    ( 0.            ,  1.04719333E+00)
    (-0.            , -1.04719333E+00)
```

INITIAL FREQUENCY INSPECTED= .00500   FINAL FREQUENCY LIMIT= 1.00000   INSPECTION INTERVAL= .00500

MODE NUMBER= 6

ITERATION NUMBER	OMEGA VALUE (FREQUENCY)	RESIDUAL (COMPLEX)	ISCALE	CASE NUMBER
1	5.00000000E-03	(1.43016953E-04, -4.33680869E-19)	6	4
2	1.00000000E-02	(1.23853854E-05, 8.75493254E-18)	6	4
3	1.50000000E-02	CASE CHANGE OBSERVED AND INSPECTED		3
4	1.00000000E-02	CASE CHANGE OBSERVED AND INSPECTED		4
5	1.05000000E-02	CASE CHANGE OBSERVED AND INSPECTED		3
6	1.00000000E-02	CASE CHANGE OBSERVED AND INSPECTED		4
7	1.00500000E-02	(1.10292422E-05, 7.18283939E-18)	6	4
8	1.01000000E-02	(9.67626841E-06, -8.40256684E-19)	6	4
9	1.01500000E-02	(8.32659552E-06, -6.77626358E-20)	6	4
10	1.02000000E-02	(6.98035513E-06, 8.13151629E-20)	6	4
11	1.02500000E-02	(5.63767912E-06, 9.48676901E-20)	6	4
12	1.03000000E-02	(4.29869958E-06, -6.77626358E-21)	6	4
13	1.03500000E-02	(2.96354885E-06, -1.35525272E-20)	6	4
14	1.04000000E-02	(1.63235949E-06, -3.38813179E-21)	6	4
15	1.04500000E-02	(3.05264276E-07, 0.)	6	4
16	1.05000000E-02	CASE CHANGE OBSERVED AND INSPECTED		3
17	1.05000000E-02	(1.01760381E-06, -3.38813179E-21)	6	3
18	1.55000000E-02	(8.87406500E-05, -1.13841228E-18)	6	3
19	2.05000000E-02	(3.40576025E-06, -2.05082558E-18)	6	3
20	2.55000000E-02	(-3.39157482E-04, -1.51788304E-18)	6	3
21	2.30000000E-02	(-1.33911433E-04, 6.55942314E-18)	6	3
22	2.17500000E-02	(-5.67063675E-05, -1.06929439E-17)	6	3
23	2.11250000E-02	(-2.45340577E-05, -1.93496206E-17)	6	3
24	2.08125000E-02	(-1.00392108E-05, -5.64335701E-18)	6	3
25	2.06562500E-02	(-3.18610048E-06, 7.87168894E-18)	6	3
26	2.05781250E-02	(1.42404116E-07, 1.50046148E-18)	6	3
27	2.06171875E-02	(-1.51369412E-06, -2.58624305E-18)	6	3
28	2.05976562E-02	(-6.83607786E-07, -4.15692338E-18)	6	3
29	2.05878906E-02	(-2.70092695E-07, -1.00548767E-18)	6	3
30	2.05830078E-02	(-6.37170249E-08, -4.75316152E-18)	6	3
31	2.05805664E-02	(3.93753651E-08, -3.31509948E-18)	6	3
32	2.05817871E-02	(-1.21628785E-08, 1.72582644E-18)	6	3

NUMBER OF ITERATIONS= 32

BOUNDS FOR FINAL FREQUENCY .0205806, .0205818
CORRESPONDING RESIDUALS (3.93753651E-08, -3.31509948E-18) (-1.21628785E-08, 1.72582644E-18)

```
    INTERPOLATED (FINAL) FREQUENCY= .0205815     CASE= 3

MODE SHAPES NOT REQUESTED, THEREFORE FINAL RESIDUAL NOT CALCULATED

ROOTS OF CHARACTERISTIC EQUATION
    ( 4.11022796E+01,  4.01943923E+01)
    (-4.11022796E+01, -4.01943923E+01)
    ( 4.11022796E+01, -4.01943923E+01)
    (-4.11022796E+01,  4.01943923E+01)
    ( 0.            ,  1.58791841E+00)
    (-0.            , -1.58791841E+00)
    ( 0.            ,  1.04719137E+00)
    (-0.            , -1.04719137E+00)
```

INITIAL FREQUENCY INSPECTED= .00500   FINAL FREQUENCY LIMIT= 1.00000   INSPECTION INTERVAL= .00500

MODE NUMBER= 7

ITERATION NUMBER	OMEGA VALUE (FREQUENCY)	RESIDUAL (COMPLEX)	ISCALE	CASE NUMBER
1	5.00000000E-03	(1.23622679E-03, -2.01227923E-16)	6	4
2	1.00000000E-02	(6.06631178E-04, -1.90819582E-17)	6	4
3	1.50000000E-02	CASE CHANGE OBSERVED AND INSPECTED		3
4	1.00000000E-02	CASE CHANGE OBSERVED AND INSPECTED		4
5	1.05000000E-02	(5.34665477E-04, 5.20417043E-18)	6	4
6	1.10000000E-02	(4.62582400E-04, 1.96891115E-16)	6	4
7	1.15000000E-02	(3.90779147E-04, 8.67361738E-19)	6	4
8	1.20000000E-02	(3.19652150E-04, 1.73472348E-18)	6	4
9	1.25000000E-02	(2.49594795E-04, -8.67361738E-19)	6	4
10	1.30000000E-02	(1.80995120E-04, 0.)	6	4
11	1.35000000E-02	(1.14233492E-04, 4.33680869E-19)	6	4
12	1.40000000E-02	(4.96802825E-05, 0.)	6	4
13	1.45000000E-02	CASE CHANGE OBSERVED AND INSPECTED		3
14	1.40000000E-02	CASE CHANGE OBSERVED AND INSPECTED		4
15	1.40500000E-02	(4.33603286E-05, 1.08420217E-19)	6	4
16	1.41000000E-02	(3.70664020E-05, 0.)	6	4
17	1.41500000E-02	(3.07988518E-05, -2.25514052E-17)	6	4
18	1.42000000E-02	(2.45580255E-05, -2.02203705E-17)	6	4
19	1.42500000E-02	(1.83442691E-05, -1.08420217E-19)	6	4
20	1.43000000E-02	(1.21579273E-05, -2.71050543E-20)	6	4
21	1.43500000E-02	(5.99934278E-06, 2.71050543E-20)	6	4
22	1.44000000E-02	CASE CHANGE OBSERVED AND INSPECTED		3
23	1.44000000E-02	(1.31143099E-07, -4.23516474E-22)	6	3
24	1.94000000E-02	(4.21811934E-04, 1.73472348E-18)	6	3
25	2.44000000E-02	(3.68572635E-04, -7.37257477E-18)	6	3
26	2.94000000E-02	(-2.43115837E-05, -2.74269268E-18)	6	3
27	2.69000000E-02	(1.90279487E-04, -6.17995238E-18)	6	3
28	2.81500000E-02	(8.31383851E-05, -1.20617492E-18)	6	3
29	2.87750000E-02	(2.88316356E-05, 2.28190676E-18)	6	3
30	2.90875000E-02	(2.03311248E-06, -9.79009945E-18)	6	3
31	2.92437500E-02	(-1.12063700E-05, -3.77734343E-18)	6	3
32	2.91656250E-02	(-4.60210218E-06, 1.08778618E-17)	6	3
33	2.91265625E-02	(-1.28820025E-06, 1.49211471E-18)	6	3
34	2.91070312E-02	(3.71550083E-07, 1.95204533E-18)	6	3
35	2.91167969E-02	(-4.58554127E-07, 3.74824404E-18)	6	3
36	2.91119141E-02	(-4.35589639E-08, 2.70160974E-18)	6	3
37	2.91094727E-02	(1.63981361E-07, -2.25071445E-18)	6	3

```
NUMBER OF ITERATIONS= 37

BOUNDS FOR FINAL FREQUENCY     .0291095,     .0291119
CORRESPONDING RESIDUALS    ( 1.63981361E-07, -2.25071445E-18)   (-4.35589639E-08,  2.70160974E-18)

    INTERPOLATED (FINAL) FREQUENCY= .0291114     CASE= 3

MODE SHAPES NOT REQUESTED, THEREFORE FINAL RESIDUAL NOT CALCULATED

ROOTS OF CHARACTERISTIC EQUATION
    ( 4.12658934E+01,  4.00299895E+01)
    (-4.12658934E+01, -4.00299895E+01)
    ( 4.12658934E+01, -4.00299895E+01)
    (-4.12658934E+01,  4.00299895E+01)
    ( 0.            ,  1.96325830E+00)
    (-0.            , -1.96325830E+00)
    ( 0.            ,  1.04725225E+00)
    (-0.            , -1.04725225E+00)
```

INITIAL FREQUENCY INSPECTED= .00500 FINAL FREQUENCY LIMIT= 1.00000 INSPECTION INTERVAL= .00500

MODE NUMBER= 8

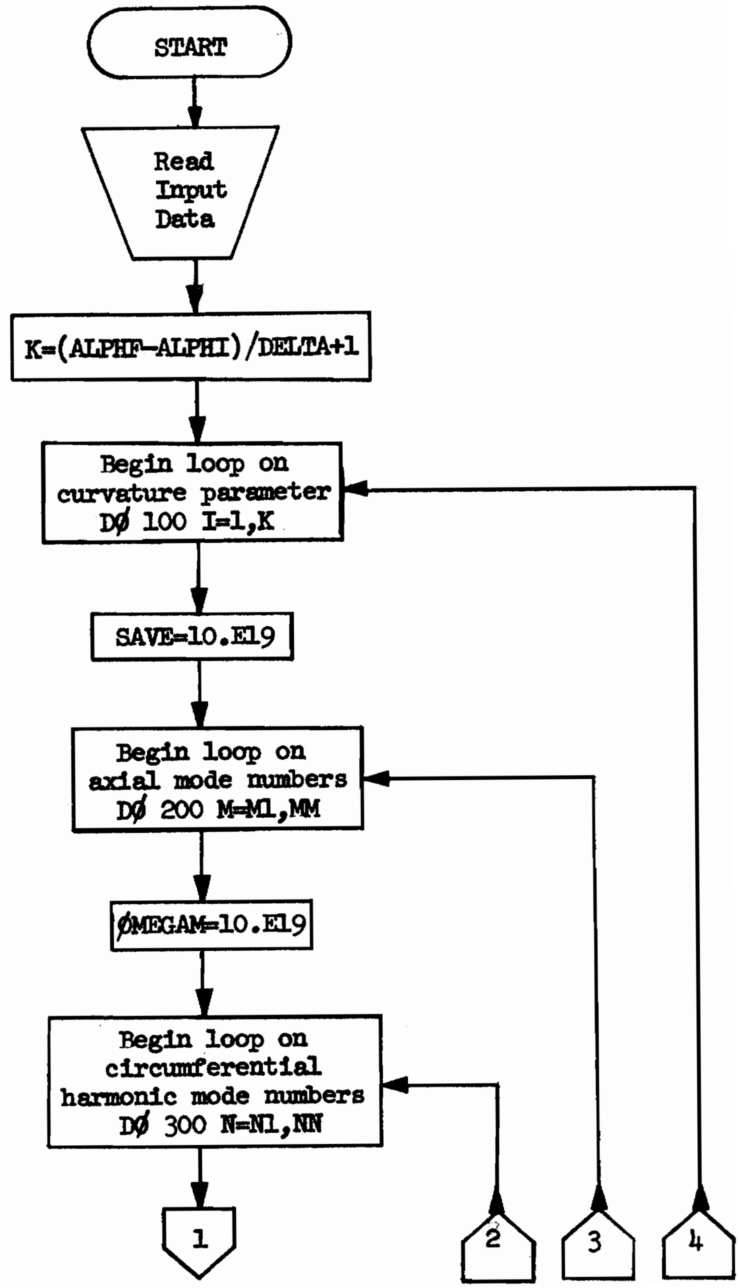
ITERATION NUMBER	OMEGA VALUE (FREQUENCY)	RESIDUAL (COMPLEX)	ISCALE	CASE NUMBER
1	5.00000000E-03	(7.92873887E-03, -4.16333634E-17)	6	4
2	1.00000000E-02	(4.89608282E-03, -6.10622664E-16)	6	4
3	1.50000000E-02	(1.63629461E-03, -2.63677968E-16)	6	4
4	2.00000000E-02	CASE CHANGE OBSERVED AND INSPECTED		3
5	1.50000000E-02	CASE CHANGE OBSERVED AND INSPECTED		4
6	1.55000000E-02	(1.36598828E-03, -3.46944695E-18)	6	4
7	1.60000000E-02	(1.11188266E-03, 3.46944695E-18)	6	4
8	1.65000000E-02	(8.75179457E-04, 3.46944695E-18)	6	4
9	1.70000000E-02	(6.56952935E-04, 3.46944695E-18)	6	4
10	1.75000000E-02	(4.58140315E-04, 2.60208521E-18)	6	4
11	1.80000000E-02	(2.79532722E-04, 4.33680869E-18)	6	4
12	1.85000000E-02	(1.21766705E-04, 1.06251813E-17)	6	4
13	1.90000000E-02	CASE CHANGE OBSERVED AND INSPECTED		3
14	1.85000000E-02	CASE CHANGE OBSERVED AND INSPECTED		4
15	1.85500000E-02	(1.07156090E-04, -2.16840434E-19)	6	4
16	1.86000000E-02	(9.27591556E-05, 7.80625564E-18)	6	4
17	1.86500000E-02	(7.85762661E-05, 6.50521303E-19)	6	4
18	1.87000000E-02	(6.46077701E-05, -6.50521303E-19)	6	4
19	1.87500000E-02	(5.08539990E-05, -2.16840434E-19)	6	4
20	1.88000000E-02	(3.73152669E-05, -1.08420217E-19)	6	4
21	1.88500000E-02	(2.39918703E-05, 5.42101086E-20)	6	4
22	1.89000000E-02	(1.08840886E-05, -1.35525272E-19)	6	4
23	1.89500000E-02	CASE CHANGE OBSERVED AND INSPECTED		3
24	1.89500000E-02	(2.00781614E-06, -1.35525272E-20)	6	3
25	2.39500000E-02	(2.37462878E-04, 5.80048162E-18)	6	3
26	2.89500000E-02	(-8.29689283E-04, -2.60208521E-18)	6	3
27	2.64500000E-02	(-2.50140305E-04, 7.69783542E-18)	6	3
28	2.52000000E-02	(2.40431806E-05, 1.05205727E-17)	6	3
29	2.58250000E-02	(-1.07575171E-04, 1.68051337E-18)	6	3
30	2.55125000E-02	(-4.01161766E-05, 1.60936260E-17)	6	3
31	2.53562500E-02	(-7.59024003E-06, -6.34528263E-18)	6	3
32	2.52781250E-02	(8.34217173E-06, -1.59113551E-17)	6	3
33	2.53171875E-02	(4.04378084E-07, 5.68194581E-18)	6	3
34	2.53367187E-02	(-3.58589244E-06, -8.31098140E-19)	6	3
35	2.53269531E-02	(-1.58898942E-06, 7.30359453E-18)	6	3
36	2.53220703E-02	(-5.91862764E-07, 1.30141852E-17)	6	3
37	2.53196289E-02	(-9.36314544E-08, 3.45492651E-18)	6	3

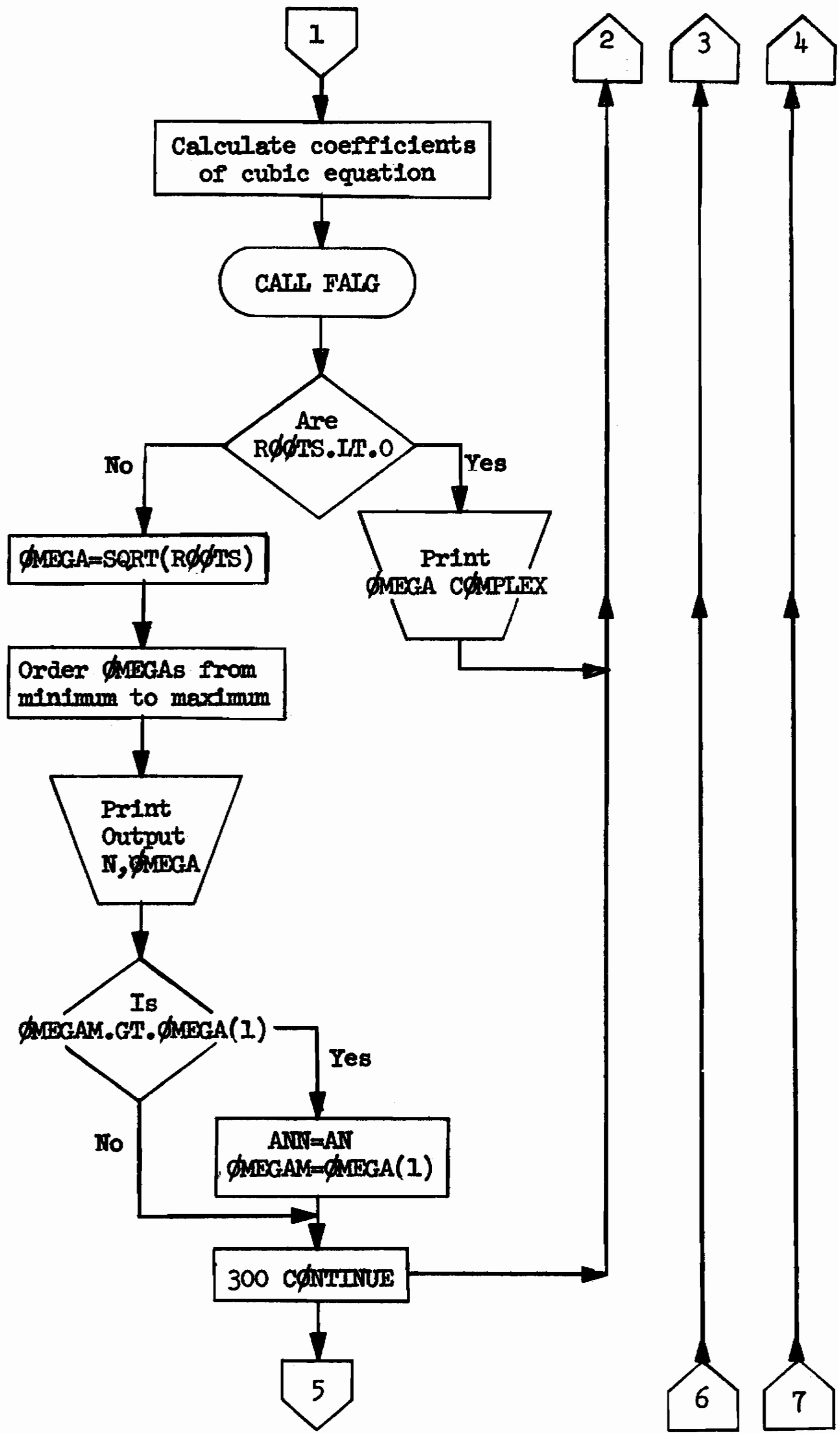
```
NUMBER OF ITERATIONS= 37

BOUNDS FOR FINAL FREQUENCY    .0253172,    .0253196
CORRESPONDING RESIDUALS   ( 4.04378084E-07,  5.68194581E-18)   (-9.36314544E-08,  3.45492651E-18)

    INTERPOLATED (FINAL) FREQUENCY= .0253192     CASE= 3

MODE SHAPES NOT REQUESTED, THEREFORE FINAL RESIDUAL NOT CALCULATED

ROOTS OF CHARACTERISTIC EQUATION
    ( 4.14636226E+01,  3.98502021E+01)
    (-4.14636226E+01, -3.98502021E+01)
    ( 4.14636226E+01, -3.98502021E+01)
    (-4.14636226E+01,  3.98502021E+01)
    ( 0.            ,  2.09440724E+00)
    (-0.            , -2.09440724E+00)
    ( 0.            ,  1.41519516E+00)
    (-0.            , -1.41519516E+00)
```

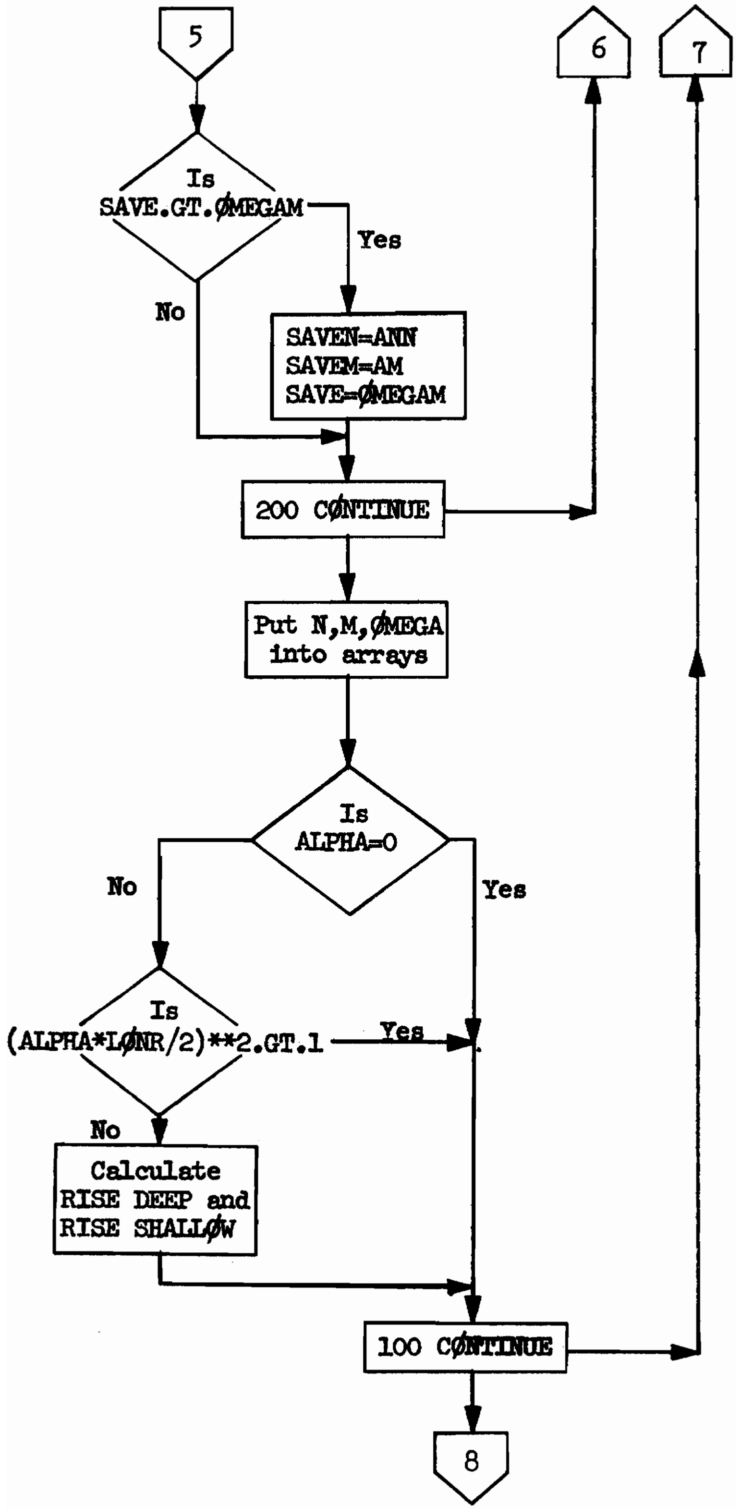
FLOW DIAGRAM OF MAIN PROGRAM FOR

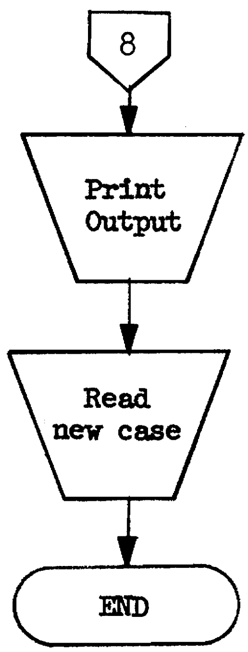
METHOD OF SOLUTION OF THE

APPROXIMATE (SHALLOW MERIDIAN) EQUATIONS FOR A FREELY SUPPORTED SHELL




1
2
3
4
Calculate coefficients of cubic equation
CALL FALG
Are RØØTS.LT.0
No
Yes
ØMEGA=SQRT(RØØTS)
Print ØMEGA CØMPLEX
Order ØMEGAs from minimum to maximum
Print Output N,ØMEGA
Is ØMEGAM.GT.ØMEGA(1)
Yes
No
ANN=AN
ØMEGAM=ØMEGA(1)
300 CØNTINUE
5
6
7


5
6
7
Is
SAVE.GT.ØMEGAM
Yes
No
SAVEN=ANN
SAVEM=AM
SAVE=ØMEGAM
200 CØNTINUE
Put N,M,ØMEGA into arrays
Is
ALPHA=0
No
Yes
Is
(ALPHA*LØNR/2)**2.GT.1
Yes
No
Calculate RISE DEEP and RISE SHALLØW
100 CØNTINUE
8


8
Print
Output
Read
new case
END

A. Main Program Variables

1. K - total number of curvature parameters considered.
2. ALPHF - final curvature parameter.
3. ALPHI - initial curvature parameter.
4. DELTA - difference between successive curvature parameters.
5. ØMEGA - frequency.
6. M - axial mode number.
7. N - circumferential harmonic mode number.
8. RØØTS - roots of cubic equation (frequency).
9. ØMEGAM - minimum frequency within a specific M-loop.
10. SAVE - minimum frequency.

B. Subroutine

1. FALG - NASA Langley Research Center library subroutine which solves for the roots of an equation.

```
C     COMPUTER PROGRAM FOR METHOD OF SOLUTION OF THE APPROXIMATION (SHALLOW
C     MERIDION) EQUATIONS OF CHAPTER 8 FOR A FREELY SUPPORTED SHELL.  SAMPLE
C     OUTPUT INCLUDED (SEE FIGURE 4 IN TEXT FOR COMPARISON OF OUTPUT).
C
C     C. DEXTER - F. COOPER
C
      PROGRAM NEWCOOP (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
      DIMENSION COEFFS(4),ROOTS(3),TEMP(8), ALPHA(1000)
      DIMENSION           OMEGA(3),XN(1000),XM(1000),XOMEGA(1000)
      DIMENSION RD(1000),RS(1000)
      REAL MU, LAMBDA,LONR
      COMPLEX ROOTS,TEMP
      NAMELIST/NAM1/M1,MM,N1,NN,LONR,ALPHI,DELTA,ALPHF,LAMBDA,EBARX,
     1EBARY,MU
      WRITE(6,1)
    1 FORMAT(1H1,///10X*SAMPLE OUTPUT FOR METHOD OF SOLUTION OF THE APPR
     1OXIMATE (SHALLOW MERIDION)*/
     110X*EQUATICNS OF CHAPTER 7 FOR A FREELY SUPPORTED SHELL.*/
     310X*NATURAL VIBRATIONS OF FREELY SUPPORTED UNSTRESSED NEGATIVE CUR
     4VATURE SHELL*/
     510X*WITH KX=-.05, S=3, RADIUS/THICKNESS=1000, N=4-8, M=1-5.*//)
      R=0
      PI=3.1415926
 1000 IF(EOF,5) 1001,1002
 1002 READ(5,NAM1)
      IF(R.NE.0.) WRITE(6,3)
    3 FORMAT(1H1///)
      NF=N1-1
      NT=NN-1
      MF=M1
      MT=MM
      WRITE(6,5) LONR,LAMBDA,NF,NT,MF,MT
    5 FORMAT(10X23HLENGTH TO RADIUS RATIO=F10.5,//10X26HTHICKNESS TO RAD
     1IUS RATIO=F10.5,//10X13HN TESTED FROMI4,2X2HTOI4,10X13HM TESTED FR
     2OMI4,2X2HTOI4////)
      K=(ALPHF-ALPHI)/DELTA+1.
      K=1
C
C     K INDICATES TOTAL NUMBER OF KX CONSIDERED
C     SET K=1 IF ONLY CONSIDERING ONE KX
C     IN THIS PROGRAM ALPHA=KX
C
```

```
      Q=0
      WRITE(6,7)
    7 FORMAT(10X,51HCURVATURE PARAMETER CALCULATIONS FOR SELECTED ALPHA)
C
C     DETERMINE KX (CURVATURE PARAMETER)
C
      DO 100 I=1,K
      ALPHA(I)=ALPHI+Q*DELTA
      WRITE(6,8) ALPHA(I)
    8 FORMAT(//10X6HALPHA=F8.4)
      Q=Q+1.
      SAVE=10.E19
C
C     DETERMINE M (AXIAL MODE NUMBER)
C
      DO 200 M=M1,MM
      AM=M
      GAM=AM*PI/LONR
      WRITE(6,9) AM
    9 FORMAT(/15X2HM=F4.0//21X1HN,10X5HOMEGA)
      OMEGAM=10.E19
C
C     DETERMINE N (CIRCUMFERENTIAL HARMONIC MODE NUMBER)
C     CALCULATE COEFFICIENTS OF CUBIC EQUATION
C
      DO 300 N=N1,NN
      AN=N-1
      A11=GAM**2+(1.-MU)/2.*AN**2
      A12=(1.+MU)/2.*GAM*AN
      A13=-GAM*(MU+ALPHA(I))
      A22=AN**2*(1.+LAMBDA**2/12.)+(1.-MU)/2.*GAM**2*(1.+LAMBDA**2/3.)
      A23=-AN*(1.+MU*ALPHA(I))-LAMBDA**2/12.*AN*(AN**2+(2.-MU)*GAM**2)
      A33=(LAMBDA**2/12.)*(GAM**2+AN**2)**2+ALPHA(I)**2+2.*MU*ALPHA(I)+1
     1.+(GAM**2)*EBARX+AN**2*EBARY
      THETA=A11+A22+A33
      PHI=A22*A33-A23**2+A33*A11-A13**2+A11*A22-A12**2
      PSI=A11*(A22*A33-A23**2)-A12*(A12*A33-A13*A23)+A13*(A12*A23-A13*A2
     12)
      COEFFS(1)=-1.
      COEFFS(2)=THETA
      COEFFS(3)=-PHI
      COEFFS(4)=PSI
      NDEG=3
```

```
      IREAL=0
C
C   CALL LIBRARY SUBROUTINE TO FIND ROOTS OF CUBIC EQUATION
C
      CALL FALG(CCEFFS,NDEG,IREAL,ROOTS,TEMP,IERR)
      IF(IERR.NE.0)WRITE(6,6)IERR
    6 FORMAT(/5X5HIERR=,I5//)
C
C   CALCULATE OMEGA (FREQUENCY) FROM POSITIVE ROOTS
C
      IF((REAL(ROOTS(1))).LT.0.) GO TO 77
      IF((REAL(ROOTS(2))).LT.0.) GO TO 77
      IF((REAL(ROOTS(3))).LT.0.) GO TO 77
      OMEGA(1)=SQRT(REAL(ROOTS(1)))
      OMEGA(2)=SQRT(REAL(ROOTS(2)))
      OMEGA(3)=SQRT(REAL(ROOTS(3)))
C
C   ORDER OMEGAS FROM MINIMUM TO MAXIMUM
C
      DO 500 IND=1,3
      L=3-IND
      DO 500 J=1,L
      IF(OMEGA(J)-OMEGA(J+1)) 500,500,50
   50 ALT=OMEGA(J)
      OMEGA(J)=OMEGA(J+1)
      OMEGA(J+1)=ALT
  500 CONTINUE
      GO TO 778
C
C   FOR NEGATIVE ROOTS OF CUBIC EQUATION WRITE N AND INDICATE OMEGA IS COMPLEX
C
   77 WRITE(6,777) AN
  777 FORMAT(17XF5.0,8X7HCOMPLEX)
      GO TO 300
C
C   WRITE N AND OMEGA
C
  778 WRITE(6,10) AN,OMEGA(1)
   10 FORMAT(17XF5.0,5XE15.8)
C
C   SAVE MINIMUM OMEGA AND CORRESPONDING N TO COMPARE WITH FOLLOWING VALUES OF
C   OMEGA IN ORDER TO FIND ABSOLUTE MINIMUM
C
```

```
   25 IF(OMEGAM.GT.OMEGA(1))ANN=AN
      IF(OMEGAM.GT.OMEGA(1))OMEGAM=OMEGA(1)
  300 CONTINUE
C
C    SAVE MINIMUM OMEGA AND CORRESPONDING N AND M
C
  110 IF(SAVE.GT.OMEGAM)SAVEN=ANN
      IF(SAVE.GT.OMEGAM)SAVEM=AM
      IF(SAVE.GT.OMEGAM)SAVE=OMEGAM
  200 CONTINUE
C
C    PUT N,M, AND OMEGA INTO ARRAY
C
      XN(I)=SAVEN
      XM(I)=SAVEM
      XOMEGA(I)=SAVE
      IF(ABS(ALPHA(I)).LT..0000000001) GO TO 100
      IF((ALPHA(I)*LONR/2.)**2.GT.1.) GO TO 100
C
C    CALCULATE RISE DEEP AND RISE SHALLOW
C
      RD(I)=1./ALPHA(I)*(1.-SQRT(1.-(ALPHA(I)*LONR/2.)**2))
      RS(I)=LONR**2*ALPHA(I)/8.
  100 CONTINUE
C
C    WRITE ABSOLUTE MINIMUM OMEGA WITH CORRESPONDING N,M,RISE DEEP, AND RISE
C    SHALLOW FOR EACH KX
C
      WRITE(6,11)
   11 FORMAT(///10X*MINIMUM FREQUENCY RESULTS*//12X*ALPHA*,11X*OMEGA*,12
     1X*M*,9X*N*,10X*RISE(DEEP)*,8X*RISE(SHALLOW)*)
      DO 400 I=1,K
      IF(ABS(ALPHA(I)).LT..0000000001) GO TO 401
      IF((ALPHA(I)*LONR/2.)**2.GT.1.) GO TO 401
      WRITE(6,12)ALPHA(I),XOMEGA(I),XM(I),XN(I),RD(I),RS(I)
   12 FORMAT( 9XF8.4,5XE15.8,5XF4.0,5XF5.0,5XF15.8,5XF15.8)
      GO TO 400
  401 WRITE(6,13) ALPHA(I),XOMEGA(I),XM(I),XN(I)
   13 FORMAT( 9XF8.4,5XE15.8,5XF4.0,5XF5.0)
  400 CONTINUE
   35 R=1
      GO TO 1000
 1001 STOP
```

```
SAMPLE OUTPUT FOR METHOD OF SOLUTION OF THE APPROXIMATE (SHALLOW MERIDION)
EQUATIONS OF CHAPTER 7 FOR A FREELY SUPPORTED SHELL.
NATURAL VIBRATIONS OF FREELY SUPPORTED UNSTRESSED NEGATIVE CURVATURE SHELL
WITH KX=-.05, S=3, RADIUS/THICKNESS=1000, N=4-8, M=1-5.


LENGTH TO RADIUS RATIO=    3.00000

THICKNESS TO RADIUS RATIO=     .00100

N TESTED FROM    4  TO    8          M TESTED FROM    1  TO    5



CURVATURE PARAMETER CALCULATIONS FOR SELECTED ALPHA


ALPHA=  -.0500

     M=    1

            N          OMEGA
            4      1.66389375E-02
            5      8.96939726E-03
            6      2.05814999E-02
            7      2.91114023E-02
            8      3.56720153E-02

     M=    2

            N          OMEGA
            4      1.62309464E-01
            5      9.99118837E-02
            6      6.12007153E-02
            7      3.73420602E-02
            8      2.53191686E-02

     M=    3

            N          OMEGA
            4      3.24916282E-01
```

5	2.31080696E-01
6	1.65753268E-01
7	1.20002342E-01
8	8.78861490E-02

M= 4

N	OMEGA
4	4.66095825E-01
5	3.59113457E-01
6	2.76996723E-01
7	2.14839732E-01
8	1.67953003E-01

M= 5

N	OMEGA
4	5.76221941E-01
5	4.69862926E-01
6	3.81157738E-01
7	3.09315696E-01
8	2.51957011E-01

MINIMUM FREQUENCY RESULTS

ALPHA	OMEGA	M	N	RISE(DEEP)	RISE(SHALLOW)
-.0500	8.96939726E-03	1	5	-.05632932	-.05625000

VIBRATION OF STRESSED SHELLS OF DOUBLE CURVATURE

By

Paul A. Cooper

ABSTRACT

Shells of double curvature are common structural elements in aerospace and related industries, but due to the complexity of their configurations and governing equations, little has been done to classify their general dynamic behavior. The subject of this dissertation is the determination of the effect of the meridional curvature on the natural vibrations of a class of axisymmetrically prestressed doubly curved shells of revolution.

A set of linear equations governing the infinitesimal vibrations of axisymmetrically prestressed shells is developed from Sander's nonlinear shell theory and both the in-plane inertia and prestress deformation effects are retained in the development. The equations derived are consistent with first-order thin-shell theory and can be used to describe the behavior of shells with arbitrary meridional configuration having moderately small prestress rotations.

A numerical procedure is given for solving the governing equations for the natural frequencies and associated mode shapes for a general shell of revolution with homogeneous boundary conditions. The numerical procedure uses matrix methods in finite-difference form coupled with a Gaussian elimination to solve the governing eigenvalue problem.

An approximate set of governing equations of motion with constant coefficients which are based on shallowness of the meridian are developed as an alternate more rapid method of solution and are solved in an exact manner for all boundary conditions. The solutions of the exact system of shell equations determined from the numerical procedure are used to determine the accuracy of the approximate solutions and with its accuracy established, the approximate equations are used exclusively to generate results. The membrane and pure bending equations which correspond to the approximate set of equations are solved for a specific boundary condition.

The effect of the meridional curvature on the fundamental frequencies of a class of cylindrical-like shells with shallow meridional curvature and freely supported edges are investigated. Results show that the positive Gaussian curvature shells have fundamental frequencies well above those of corresponding cylindrical shells. The fundamental frequencies of the negative Gaussian curvature shells generally are below those of the corresponding cylinders and evidence wide variations in value with large reductions in magnitude occuring at certain critical curvatures. Comparison of the membrane, pure bending and complete shell analyses shows that these critical curvatures represent configurations at which the fundamental mode of vibration of the shell is in a state close to pure bending. The membrane theory affords a simple method of determining the modal wavelength ratio at which the pure bending state exists for a given negative Gaussian curvature shell, while the pure bending theory gives a good estimate of the magnitude of the frequency for this wavelength ratio. Meridional edge restraints and internal lateral pressure reduce the wide variation of the natural frequencies in the negative curvature shells and in general raise the natural frequencies. External lateral pressure accentuates the reduction in natural frequencies of the negative curvature shells and causes instability at low compressive stress ratios.