THE EFFECT OF RANDOM INTERNAL MOTIONS ON THE ANGULAR ORIENTATION OF A FREE BODY WITH LIMIT CONTROL

by

John Richard Davidson

Thesis submitted to the Graduate Faculty of the

Virginia Polytechnic Institute

in partial fulfillment for the degree of

DOCTOR OF PHILOSOPHY

in

Engineering Mechanics

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III. ACKNOWLEDGMENTS

Many people contributed to this report. The idea to study the effect of random internal motions on spacecraft rotation originated in a conversation with Dr. Leonard Meirovitch. A perceptive comment by Dr. T. S. Chang concerning the applicability of Markov processes saved many hours of random search for suitable analytical methods.
Dr. James Pickands, III, suggested the diffusion equation approach, which proved most fruitful, and the study of the transformed Kolmogorov equations which led to the development of the iteration method. Dr. T. A. Filippas has made many knowledgeable observations concerning computational and analytical methods; discussions with him about the implications of the results have been of great benefit. The thesis program has been directed by Dr. R. L. Armstrong, whose timely suggestions have greatly improved the results. The author wishes to thank Mrs. Pauline Pyle for her editing suggestions and her preparation of the manuscript, and Mrs. Edna Davidson for her patience and skill in typing the manuscript for reproduction.

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V. INTRODUCTION

Since space flight has become a practical accomplishment there has been an increased interest in the motion of bodies in a reduced-force or zero-force environment. In near-earth orbits, gravitational-gradient torques and small atmospheric drag forces have some influence on the attitude of orbiting bodies; as the radial distance from the earth increases, these forces diminish. At large distances from the earth (or planets), the major torques exerted upon the spacecraft are due to the internal motions of the crew or attached machinery. Although spin stabilization such as considered by Meirovitch [1] might be used in unmanned spacecraft, a manned mission would necessarily require a more complex mechanism such as control jets or reaction wheels. Either system would require some expenditure of power, and because energy is a valuable commodity in extended space flight, it is desirable to have some way to predict the amount needed.

The attitude control energy is related to the amount of internal disturbance and the stringency of the permissible disorientation limits. It is intuitive that frequent violent internal motions will cause greater deviations, or drift, from the nominal angular orientation than will small, infrequent motions. Also, if reaction jets are used for attitude control such that they fire when preset deviation limits are reached, the closer the limits are to the nominal orientation the more frequently these jets will fire.

Tewell and Murrich [2] consider some of the effects of crew motion. This reference contains an experimental study of the actual motions of a man and the forces and moments generated by the motion. The study is of a form for future application to computer analysis of the deterministic effect of the man's motion. The purpose of the study was to define an experiment for future Apollo missions.

The presence of a crew in a manned spacecraft lends a degree of unpredictability to the calculation of the angular drift rate due to crew motion. It would not be expected that a given crew member would use exactly the same motions each time he performed a specified task. However, one would expect the motions to be statistically stationary so that frequency versus magnitude histograms, measured for specific and/or varied tasks, could be used to predict the magnitude of the crew-induced disturbances.

This dissertation presents a method for determining the effect of internal disturbances on an otherwise force-free, nonrotating body. The disturbances are considered random but statistically stationary in time. The problem divides naturally into two parts: (1) the determination of the net angular change of the spacecraft due to one internal disturbing motion of given magnitude, and (2) the determination of the (stochastic) magnitude of the sum of these motions. In particular, the "waiting time" is determined; this is the period of time the spacecraft takes to move (or be moved) from its nominal orientation to the preset limits on the angular deviation. If the mean waiting time is known, the frequency of control jet firing can be determined, thus determining the rate of fuel consumption.

The response to internal motion is a problem of dynamics of rotation. The basic principle used herein is that the total angular momentum of the system must at all times be zero. The additive effect of the motions, considered in the second part, has the analogy of a random walk or Brownian motion problem with preferred drift. Three solutions are obtained in this second part: the first solution is based upon the extended diffusion equation and is applicable when a large number of single internal motions occur between the nominal state and a boundary; the second solution is based upon Markov chain theory and should be used when only a few motions are required to change the spacecraft orientation from the nominal position to a preset limit; the third solution (iteration solution) is again based upon Markov chain theory but is computationally more efficient. The iteration solution should be used whenever the standard deviation of the period between arrivals at close preset limits is not needed.

VI. SYMBOLS

A	represents an event; the event is defined when used
a	product of the variance of the one-step histogram and the average number of moves per unit time
B	number of subperiods of length h
b	product of the mean of the one-step histogram and the mean number of moves per unit time
$E(y)$	expectation (mean) of y; first moment of the distribution function for y
$E(y^2)$	second moment of the distribution function for y
e	base of the natural logarithm
$\exp(y)$	equals e^y
$F(s)$	integral of $f(s)$ over x from $x = 0$ to $x = x_b$
$\mathcal{F}(t)$	integral of $u(x,t)$ over x
$f(s)$	equals $f(x,s)$; Laplace transform of the probability density distribution $u(x,t)$
$G(s)$	Laplace transform of $\beta(t)$
$\mathcal{G}(y)$	generating function defined by equation (89)
$g(p,s)$	iterated Laplace transform of $u(x,t)$
H	angular momentum, kg-m/sec
$\mathcal{H}(y)$	generating function for the numbers on a die
h	length of subperiods, sec
I	inertia, kg/m^2
$\tilde{\mathcal{I}}$	identity matrix
i	index

$J(y)$	generating function for the Poisson distribution
j	index
K	reduced mass, $K = \frac{Mm}{M + m}$, kg
k	index
$\mathcal{L}\{f(t)\}$	Laplace transform of $f(t)$
$\mathcal{L}^{-1}\{f(s)\}$	inverse of the Laplace transform of $f(s)$
$L(y)$	generating function for the sum of the numbers on a pair of dice
l	function of s, defined by equation (71)
l	index
M	mass of spacecraft, kg
m	mass of internal moving point mass, kg
N	index
n	index or number of motions
$O[\ ]$	order of
$P[A]$	probability that event A occurs
P_{ij}	matrix of transition probabilities. This matrix is usually a function of time
p	Laplace transform parameter
p_{ij}	matrix of one-step transition probabilities. This matrix is independent of time
$Q(y)$	generating function defined by equation (91)
r	distance measured with respect to inertial coordinates, m
S	area enclosed by the path of the moving mass m, m^2
s	Laplace transform parameter

T	period, sec
t	time, sec
$U(x - x_0)$	unit step function defined by equation (73)
u	probability density function for x, $u = u(x,t)$. See equation (63)
$Var(y)$	variance of y, $Var(y) = E(y^2) - E^2(y)$
W	waiting time, sec
$\bar{W}$	nondimensional mean waiting time, $\bar{W} = \frac{bE(W)}{x_b}$
X	inertial coordinate axis, or random variable
x	angle of rotation or angular deviation measured with respect to shifted coordinates where x_0 represents the nominal position
Y	inertial coordinate axis
y	dummy variable
Z	inertial coordinate axis
z	distance to elemental spacecraft mass dM, m
α_i	a coefficient equal to the probability of being in state i
α_{ij}	a coefficient equal to the probability of moving from state i to state j
$\beta(t)$	probability density function for the waiting time, W
γ	represents fictitious boundary state at positive boundary
δ_{ij}	Kronecker delta
$\delta(x - x_0)$	Dirac delta
ϵ_{ijk}	permutation symbol

ζ	off period. See discussion preceding equation (111)
η	defined by equation (107)
θ	dimensional angle of rotation
λ	rate at which motions are made, 1/sec
$1/\lambda_0$	dwell time in (reflecting) boundary state, 1/sec
μ	mean number of moves, $\mu = \lambda T$
ν_j	value of P_{ij} as $t \to \infty$, defined by equation (40)
Π_{ij}	Laplace transform of P_{ij}
ρ	distance with respect to spacecraft (rotating) coordinates, m
$\sum$	summation
$\bar{\sigma}^2$	nondimensional variance for W
τ	time, sec
Φ	angle or rotation (random variable)
φ	angle of rotation (realized value)
Ω	angular velocity defined by equation (17), m^2/sec
ω	angular velocity, 1/sec

Superscripts

(n)	nth convolution
$\rightarrow$	vector
$\sim$	matrix
$-$	modified matrix, absorbing boundaries
$\cdot$	a dot over the quantity denotes differentiation with respect to time

Subscripts

b	positive boundary
c	center of mass
r	relative to moving coordinates. Time rates of change of vector quantities so subscripted are taken as if viewed from the rotating system of coordinates
i,j,k	indices
N	index
0	boundary state, negative boundary
γ	boundary state, positive boundary
$-$	modified matrix, reflecting boundaries
opt	value which makes the average waiting time a maximum

Special notation

$\{ \}$	column matrix
$\lfloor \rfloor$	row matrix
$[ ]$	square matrix
$\bar{p}_{ij}^{(n)}$	the term in the ith row, jth column of the modified (absorbing boundary) matrix after the matrix had been convoluted n times

Note on notation: Latin subscripts are used to denote tensor components in Chapter VII, "Relation Between Internal Movement and Spacecraft Net Reorientation." Latin subscripts do not represent tensor components after Chapter IX A.

VII. RELATION BETWEEN INTERNAL MOVEMENT AND SPACECRAFT NET REORIENTATION

In this chapter the relationship between a single internal movement and the angular reorientation of the spacecraft is obtained. The final equations are generally applicable to manned spacecraft, where the spacecraft principal moments of inertia are large and where some limit is placed upon the maximum angular deviation.

Some typical spacecraft data (three Apollo configurations) are listed in table I. A man moving his entire body would provide a moving mass of 100 kilograms and, inasmuch as he would usually be inside of the spacecraft, the moment of inertia associated with the man would be much less than the principal moments of inertia of the spacecraft. These points will be useful in the subsequent analysis to simplify the equations.

In addition, the analysis is greatly simplified by restricting the maximum angular deviation to $\pm 5^{\circ}$ from the nominal position, so that the direction cosines of the angles of deviation remain near unity. Doolin [3] lists some typical attitude requirements. A solar power array, which is a focused paraboloidal reflector used to heat a boiler for a heat engine, requires an alignment accuracy of 0.1 second of arc to be maintained. An orbital telescope would require 0.01 second accuracy. Most mission specifications are not so stringent and can be expected to require from 0.1° to a few degrees.

The angular deviation of the spacecraft is determined by measuring the angle between two sets of Cartesian coordinates. Figure 1 is a sketch of these coordinates. The axes X, Y, and Z are inertial axes in space. The axes (1), (2), and (3) are along the principal axes of inertia of the spacecraft, are fixed to the spacecraft, and rotate with it. In figure 1 the position of the center of mass of the system is shown for convenience in depicting the vectors; equation (6) will show that this point actually lies between the origin (at the spacecraft center of mass, M) and the moving mass, m.

If no net external torques act upon the spacecraft and the spacecraft and its contents are initially nonrotating, the angular momentum must be zero at all times. Let ρ_0 be the vector from the common origin of the coordinate systems to the center of mass of the m,M system. Let $\vec{z}$ be a vector to some elemental mass, dM, of the spacecraft, $\vec{\rho}$ be a vector to mass m. The vectors $\vec{r}_1$ and $\vec{r}_2$ are from the center of mass of the m,M system to the masses m and dM, respectively. From Greenwood [4], the angular momentum about the system center of mass is, in either coordinate system,

$$\vec{H}_c = 0 = \sum_i \vec{r}_i \times m_i\dot{\vec{r}}_i \tag{1}$$

But

$$\left.\begin{aligned} \vec{r}_1 &= \vec{\rho} - \vec{\rho}_c \\ \vec{r}_2 &= \vec{z} - \vec{\rho}_c \end{aligned}\right\} \tag{2}$$

and, therefore, equation (1) becomes

$$\vec{H}_c = 0 = (\vec{\rho} - \vec{\rho}_c) \times m(\dot{\vec{\rho}} - \dot{\vec{\rho}}_c) + \int_M (\vec{z} - \vec{\rho}_c) \times (\dot{\vec{z}} - \dot{\vec{\rho}}_c)dM \tag{3}$$

But (see Greenwood [4]) the (1), (2), (3) coordinate system rotates at some angular velocity, $\vec{\omega}$, and therefore,

$$\left.\begin{aligned} \dot{\vec{\rho}} &= (\dot{\vec{\rho}})_r + \vec{\omega} \times \vec{\rho} \\ \dot{\vec{z}} &= (\dot{\vec{z}})_r + \vec{\omega} \times \vec{z} \\ \dot{\vec{\rho}}_c &= (\dot{\vec{\rho}}_c)_r + \vec{\omega} \times \vec{\rho}_c \end{aligned}\right\} \tag{4}$$

where the subscript r denotes a velocity measured relative to the moving coordinates, and $\vec{\omega}$ is the absolute angular velocity of the moving coordinate system. Substituting equation (4) into equation (3) yields

$$\begin{aligned} 0 = {} & m(\vec{\rho} - \vec{\rho}_c) \times \left[(\dot{\vec{\rho}})_r + \vec{\omega} \times \vec{\rho} - (\dot{\vec{\rho}}_c)_r - \vec{\omega} \times \vec{\rho}_c\right] \\ & + \int_M \left\{(\vec{z} - \vec{\rho}_c) \times \left[(\dot{\vec{z}})_r + \vec{\omega} \times \vec{z} - (\dot{\vec{\rho}}_c) - \vec{\omega} \times \vec{\rho}_c\right] dM\right\} \end{aligned} \tag{5}$$

where the integral is taken over the spacecraft mass M. We note, however, that the equation for the center of mass is

$$(M + m)\vec{\rho}_c = M\vec{z}_M + m\vec{\rho} = m\vec{\rho}$$

Therefore,

$$\vec{\rho}_c = \frac{m}{M + m}\vec{\rho} \tag{6}$$

where

$$M\vec{z}_M = \int_M \vec{z}\, dM = 0$$

because the center of mass of the spacecraft was chosen as the origin of the coordinate system.

The coordinate system is fixed to, and moves with, the rigid-body spacecraft M, and therefore,

$$(\dot{\vec{z}})_r = 0$$

This relation and equation (6) are substituted into equation (5) to obtain

$$\begin{aligned} 0 = {} & m\left(1 - \frac{m}{M + m}\right)^2\left\{\vec{\rho} \times \left[(\dot{\vec{\rho}})_r + \vec{\omega} \times \vec{\rho}\right]\right\} \\ & + \int_M \vec{z} \times \left[\cancelto{0}{(\dot{\vec{z}})_r} + \vec{\omega} \times \vec{z}\right]dM \\ & - \left(\frac{m}{M + m}\right)\int_M \vec{z} \times \left[(\dot{\vec{\rho}})_r + \vec{\omega} \times \vec{\rho}\right]dM \\ & - \left(\frac{m}{M + m}\right)\int_M \vec{\rho} \times \left[\cancelto{0}{(\dot{\vec{z}})_r} + \vec{\omega} \times \vec{z}\right]dM \\ & + \left(\frac{m}{M + m}\right)^2\int_M \vec{\rho} \times \left[(\dot{\vec{\rho}})_r + \vec{\omega} \times \vec{\rho}\right]dM \end{aligned} \tag{7}$$

All elements of a rigid body have the same angular velocity, and therefore, the angular velocity may be taken outside of the integral sign in the second, third, and fourth integrals. Also, the vector $\vec{\rho}$ is not included in the integration and can be taken outside of the integral sign. After using standard vector identities, equation (7) becomes

$$
\begin{aligned}
0 = {} & m\left(1 - \frac{m}{M + m}\right)^2 \left\{\vec{\rho} \times \left[(\dot{\vec{\rho}})_r + \vec{\omega} \times \vec{\rho}\right]\right\} \\
& + \int_M \vec{z} \times (\vec{\omega} \times \vec{z}) dM \\
& + \left(\frac{m}{M + m}\right)\left[(\dot{\vec{\rho}})_r + \vec{\omega} \times \vec{\rho}\right] \times \cancelto{0}{\int_M \vec{z}\, dM} \\
& - \left(\frac{m}{M + m}\right)\vec{\rho} \times \left[\vec{\omega} \times \cancelto{0}{\int_M \vec{z}\, dM}\right] \\
& + \left(\frac{m}{M + m}\right)^2 M\left[\vec{\rho} \times (\dot{\vec{\rho}})_r + \vec{\rho} \times (\vec{\omega} \times \vec{\rho})\right] \qquad (8)
\end{aligned}
$$

where use has been made of the relation

$$\int_M dM = M$$

The first and last terms may be combined and the vector identity

$$\vec{z} \times (\vec{\omega} \times \vec{z}) = (\vec{z} \cdot \vec{z})\vec{\omega} - (\vec{z} \cdot \vec{\omega})\vec{z}$$

be used to obtain

$$0 = \left(\frac{Mm}{M + m}\right)\left[\vec{\rho} \times (\dot{\vec{\rho}})_r + \vec{\rho} \times (\vec{\omega} \times \vec{\rho})\right] + \int_M \left[(\vec{z} \cdot \vec{z})\vec{\omega} - (\vec{z} \cdot \vec{\omega})\vec{z}\right] dM \qquad (9)$$

In equation (9) the quantity $\left(\frac{Mm}{M + m}\right)$ is a "reduced mass" expression which accounts for shifts in the center of mass as m moves relative to M. It can be seen that

$$\lim_{m \ll M} \left(\frac{Mm}{M + m}\right) \to m$$

as would be expected. By expanding out the integral in equation (9), cross product terms can be canceled because the coordinate system was chosen along principal coordinates of the spacecraft. Equation (9) reduces to

$$0 = \frac{Mm}{M + m}\left[\vec{\rho} \times (\dot{\vec{\rho}})_r + \vec{\rho} \times (\vec{\omega} \times \vec{\rho})\right] + \vec{\vec{I}} \cdot \vec{\omega} \qquad (10)$$

where the dyadic $\vec{\vec{I}}$ can be represented by a second-order tensor I_{ij}, and

$$I_{ij} = 0 \qquad i \neq j \qquad i,j = 1,2,3$$

$$I_{JJ} = \text{principal moment of inertia about the jth axis}$$

By taking the dot product of equation (10) with $\vec{\rho}$ and interchanging the dots and crosses in the vector triple product, one finds that

$$\vec{\rho} \cdot \vec{\vec{I}} \cdot \vec{\omega} = 0$$

or, the angular momentum of the spacecraft is always perpendicular to the vector $\vec{\rho}$.

Because it is desired to obtain the angular difference between the inertial coordinates and the moving coordinates, equation (10) must be solved for the angular velocity $\vec{\omega}$. This is most easily done by expanding equation (10) into components and using matrix algebra (Cramer's rule) to solve for ω_1. Although somewhat lengthy, the algebra is straightforward. In the following result, use has been made of the fact that, because $m \ll M$ and $|\vec{\rho}|$ is, in general, less than or equal to $|\vec{z}|$

$$I_{NN} \gg \left(\frac{Mm}{M + m}\right)\rho_i\rho_j \tag{11}$$

Let

$$K = \frac{Mm}{M + m} \tag{12}$$

Then

$$\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = -K\begin{bmatrix} \frac{1}{I_{11}} & \frac{K}{I_{11}I_{22}}\rho_1\rho_2 & \frac{K}{I_{11}I_{33}}\rho_1\rho_3 \\ \frac{K}{I_{11}I_{22}}\rho_1\rho_2 & \frac{1}{I_{22}} & \frac{K}{I_{22}I_{33}}\rho_2\rho_3 \\ \frac{K}{I_{11}I_{33}}\rho_1\rho_3 & \frac{K}{I_{22}I_{33}}\rho_2\rho_3 & \frac{1}{I_{33}} \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} \tag{13}$$

where the terms in the column matrix on the right are

$$A_i = \epsilon_{ijk}\rho_j(\dot{\rho}_r)_k$$

The first equation of the set (eq.(13)) is

$$-\omega_1 \cong \frac{K}{I_{11}}\left\{\rho_2(\dot{\rho}_r)_3 - \rho_3(\dot{\rho}_r)_2 + \frac{K}{I_{22}}\left[\rho_1\rho_2\rho_3(\dot{\rho}_r)_1 - \rho_1{}^2\rho_2(\dot{\rho}_r)_3\right]\right.$$

$$\left. + \frac{K}{I_{33}}\left[\rho_1{}^2\rho_3(\dot{\rho}_r)_2 - \rho_1\rho_2\rho_3(\dot{\rho}_r)_1\right]\right\} \tag{14}$$

where the notation $(\dot{\rho}_r)_i = \left[(\dot{\rho})_r\right]_i$ has been used for convenience, and where use has again been made of the relation (11) to eliminate terms which have the same velocity component. After using the relation (11) again in equation (14), there is left

$$-\omega_1 \cong \frac{K}{I_{11}}\left[\rho_2(\dot{\rho}_r)_3 - \rho_3(\dot{\rho}_r)_2 + \rho_1\rho_2\rho_3\left(\frac{K}{I_{22}} - \frac{K}{I_{33}}\right)(\dot{\rho}_r)_1\right] \tag{15}$$

Because of the relation (11), the last term of equation (15) is of smaller order. To have a net change in the orientation over a complete motion (see the discussion following eq. (18)), all components of $(\dot{\vec{\rho}})_r$ will be of approximately the same order of magnitude, especially for a series of complex motions; this last term will be taken as negligible. It should be noted that the last term is identically zero if the spacecraft ellipsoid of inertia is spherical.

For practical purposes then, equation (15) reduces to

$$-\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = K \begin{bmatrix} \frac{1}{I_{11}} & 0 & 0 \\ 0 & \frac{1}{I_{22}} & 0 \\ 0 & 0 & \frac{1}{I_{33}} \end{bmatrix} \begin{Bmatrix} \rho_2(\dot{\rho}_r)_3 - \rho_3(\dot{\rho}_r)_2 \\ \rho_3(\dot{\rho}_r)_1 - \rho_1(\dot{\rho}_r)_3 \\ \rho_1(\dot{\rho}_r)_2 - \rho_2(\dot{\rho}_r)_1 \end{Bmatrix}$$

$$+ \, 0\left\{\frac{K\rho_K\rho_J}{I_{JJ}I_{KK}}\right\} + 0\left\{\frac{K^2\rho_J\rho_K}{I_{JJ}I_{KK}I_{NN}}\right\} \tag{16}$$

Let

$$\Omega_N = \frac{\omega_N I_{NN}}{K} \tag{17}$$

so that equation (16) may be written in vector form

$$-\vec{\Omega} = \vec{\rho} \times (\dot{\vec{\rho}})_r \tag{18}$$

In the next chapter we will be concerned with "complete" motions. As an example, consider a crew member who is restrained in a chair, and who must operate switches on a control console. He moves a hand from the armrest to the console, operates one or more switches, and returns his hand to the armrest. This is a "complete" motion wherein the moving mass (in this example the arm represents an integrated mass) moves over a closed path with respect to the spacecraft coordinates. As will be demonstrated later, all motions may be considered to be of this type.

We are thus led to an integral of equation (18) over a closed path.

$$-\int_{t_1}^{t_2} \vec{\Omega}\, dt = \int_{t_1}^{t_2} \vec{\rho} \times (\dot{\vec{\rho}})_r\, dt$$

leads to, with

$$\dot{\vec{\theta}} = \vec{\Omega} \tag{19}$$

$$-\int_{\vec{\theta}_a}^{\vec{\theta}_b} d\vec{\theta} = \oint \vec{\rho} \times d\vec{\rho} \tag{20}$$

With the restriction that $\vec{\theta}$ is a small angle, as was discussed earlier, the left-hand side of equation (20) may be integrated directly. The right-hand side may be integrated by applying the second extension of Stokes theorem

$$\oint \vec{\rho} \times d\vec{\rho} = -\iint_S (\vec{n} \times \vec{\nabla}) \times \vec{\rho}\, dS$$

$$= \iint_S \left[\vec{n}(\vec{\nabla} \cdot \vec{\rho}) - \vec{n} \cdot \vec{\nabla}\vec{\rho}\right] dS$$

where $\vec{n}$ is the unit normal to the surface element dS.

But

$$\vec{\nabla} \cdot \vec{\rho} = 3$$

and

$$\vec{n} \cdot \vec{\nabla}\vec{\rho} = \vec{n}$$

so that

$$\oint \vec{\rho} \times d\vec{\rho} = 2 \iint_S \vec{n}\, dS \tag{21}$$

Equation (20) thus becomes

$$-\Delta\vec{\theta} = 2 \iint_S \vec{n}\, dS \tag{22}$$

If one wishes to obtain the components of $\Delta\vec{\theta}$, it is easily seen that the components of the integral on the right are the projections on the coordinate planes of the area enclosed by the path of the mass m. This is illustrated in figure 2. Equation (22) then becomes, after using equations (17) and (12) in equation (22) and defining $\Delta\vec{\varphi}$ as the change in the angle between the inertial coordinates and the spacecraft coordinates,

$$-\Delta\varphi_N = 2\left(\frac{Mm}{M + m}\right)\frac{S_N}{I_{NN}} \qquad (N = 1,2,3) \tag{23}$$

Equation (23) shows that the angular changes among the three coordinates are not coupled; this is a result of the motion being limited to small angular deviations, and the relationship (11) which specifies that the respective moments of inertia of the spacecraft are large compared with the effective moment of inertia of the moving point mass m with respect to the center of mass of the spacecraft. Thus, the angle changes about each coordinate can be considered independently. The 5^o angle restriction applies to the total angular deviation, which would be the sum of the angular changes $\Delta\varphi_N$. The total angular deviation, φ_N, is caused by adding the effects of many motions.

VIII. EXPERIMENTAL METHODS TO OBTAIN HISTOGRAMS OF THE MOTION

Equation (22) shows that one parameter, the product $\left(\frac{Mm}{M + m} S_i\right)$ determines the angular change about the ith axis for a given completed motion. During the remainder of this chapter, the discussion concerns motion about one individual axis only. The question of independence of motion about the three axes, in the sense of no statistical correlation, is taken up later in the mathematical analysis of the statistical aspects. In the most usual situations, one complete motion will not result in the spacecraft turning a sufficient amount to reach the pre-set limits (boundaries) to its angular deviation. The accumulated effect of several such motions must be considered. The exact pattern of motion that a crew member might make is not completely predictable; there will be variations among repetitions of even a simple task such as the operation of a switch. An experiment can be devised to measure $\left(\frac{Mm}{M + m} S_i\right)$, and a histogram (fig. 3(a)) obtained for the frequency of occurrence of a given magnitude of $\left(\frac{Mm}{M + m} S_i\right)$. Similarly, histograms can be obtained for more complex tasks, or combinations of tasks. An advantage here is that the crew member is not restricted to a specified type of motion; it is desirable that he perform these tasks in a natural manner which seems suitable to him.

As the tasks increase in length and complexity, a question arises concerning the definition of a "complete" motion, for one movement may naturally flow into the next. Also to be considered is the number of complete motions made per unit time. These two considerations are interdependent, as will be noted in the following sections.

Thus, the desired end result of the experimental measurements is to obtain the histogram of the magnitude of the single motion, which will be called the one-step histogram (using Markov chain terminology), and the frequency at which motions occur. The inverse of the frequency is the "dwell time," or period between steps. The following sections describe briefly alternate experimental methods for obtaining these data. Tewell and Murrich [2] give an excellent description of a series of such experiments.

At the present time, insufficient data have been obtained and reduced to provide an actual histogram. The histogram shown in figure 3(a) has features which are typical of those expected from actual experiment. The maximum magnitudes are within the maximums that have been measured (full-arm rotation about the shoulder, 5 kg-m^2). The intermediate values were deliberately chosen so that the final distribution is not quite normal; the stochastic analysis is not restricted to normal distributions.

A. Limb Sensor, Photographic, and Computer Analysis

Tewell and Murrich [2] describe an ingenious skeletal device which can be mounted to the limbs and torso of a man. The device is instrumented so that the flexure at the joints can be measured. Photographs are also taken of the subject. The data are used as input to a digital computer program which computes the accelerations, forces, and moments about prescribed points. Additional input is the masses and inertias of the various limbs and torso. The program accounts for the fact that motion must be made commensurate with allowable joint flexibility.

To determine the "complete" motion and frequency of motions, a nominal position can be defined. A motion is started whenever departure is made from the nominal position and ended when the nominal position is again attained.

B. Reaction Force Analysis

Another method to obtain the data is to place the crew member on an instrumented platform (Tewell and Murrich [2]) on which the three reaction moments and forces can be continually measured and recorded. By specifying the application of the reaction force with respect to the actual spacecraft coordinate system, the torques caused by these forces may be obtained. Thus, all moments can be determined and integrated twice to obtain effective angular displacements.

The start of a new motion is easily detected. The motion (about a specific axis) can be considered completed when the integral of the moment with respect to time (about that axis) passes through its first zero. This is equivalent to saying motion ends when the velocity is zero. The number of motions about an axis made during the measuring period is the number of times the velocity about that axis is zero.

From these data (oscillograph or digital tape) alone there is no way to ascertain if the subject has returned to his nominal position. A zero velocity may mean that the man has momentarily paused. Therefore, if this procedure is followed, the resulting histogram could show, comparatively, a large number of motions of smaller magnitude. However, the rate at which motions are made would increase (i.e., the dwell time would decrease) in compensation.

The advantage of this approach is that the data recording and reduction can be completely automated. The disadvantage is the presence of gravity force, which continually acts upon the man. The undesirable moment caused by gravity will vary with the man's position on the platform. During some of the experiments described in reference [2], the man was suspended by a complex supporting device which negated his weight upon the platform. Although cumbersome, this method has the aforementioned advantage of having a tractable data handling capability.

C. Gimbals and Free-Body Suspension

The obvious method of suspending the subject by a gimbal system where the inertias of the various moving components are known has the previously mentioned disadvantage of having to operate in a gravity force field. The next obvious step is to perform the experiment in an almost (orbiting) gravity-free spacecraft.

The spacecraft experiment requires extreme accuracy in the measurement of the angle changes; these angle changes may be of the order of seconds of arc. This is a severe instrumentation requirement. The results would have to be computer analyzed to eliminate gravity gradient torque and atmospheric-drag-caused torques. The gravity gradient torque on an Apollo configuration is small, but the angular impulse can be large because of the time period over which the torque acts (one-half orbital period, approximately).

IX. STOCHASTIC ANALYSIS

A. Independence of Axes

The present chapter will demonstrate that separate histograms may be obtained for each axis of rotation. Subsequent solutions to this problem of random walk (angular reorientation) can thus be applied to one axis at a time, provided the conditions of equation (23) are met.

For simplicity, the following derivation is initially restricted to two dimensions; the extension to three dimensions is obvious. For additional discussions, see Parzen [5].

When a motion is made internal to the spacecraft, there is an angular change about each of the axes. Let the event A represent the simultaneous occurrence of the two events

$$\varphi_{1a} \leq \Phi_1 < \varphi_{1b}$$

and (24)

$$\varphi_{2a} \leq \Phi_2 < \varphi_{2b}$$

In symbols, the probability that the event A will be observed is written as $P[A]$. The probability $P[A]$ is determined by the joint probability density function of Φ_1 and Φ_2, $f(\varphi_1, \varphi_2)$, in the following fashion:

$$P[A] = \int_{\varphi_{1a}}^{\varphi_{1b}} \int_{\varphi_{2a}}^{\varphi_{2b}} f(\varphi_1, \varphi_2) d\varphi_1 \, d\varphi_2 \tag{25}$$

The functions which express the probabilities that $\Phi_1 \leq \varphi_1$ or $\Phi_2 \leq \varphi_2$ are the probability distribution functions for Φ_1 and Φ_2.

These functions can be obtained by experimental measurement. Mathematically, the probability distribution functions are

$$\left.\begin{aligned} F_1(\varphi_1) &= P\left[\Phi_1 \le \varphi_1\right] \\ F_2(\varphi_2) &= P\left[\Phi_2 \le \varphi_2\right] \end{aligned}\right\} \tag{26}$$

where φ_1 and φ_2 are some given numbers. However,

$$F_1(\varphi_1) = \int_{-\infty}^{\varphi_1} dy_1 \int_{-\infty}^{\infty} f(y_1,y_2)dy_2 \tag{27a}$$

$$F_2(\varphi_2) = \int_{-\infty}^{\varphi_2} dy_2 \int_{-\infty}^{\infty} f(y_1,y_2)dy_1 \tag{27b}$$

where equation (27a) may be interpreted as the probability that the angle observed about the (1) axis is less than φ_1, no matter what angle φ_2 is observed about the (2) axis, and y_i is a dummy variable of integration.

From the definition of $F_1(\varphi_1)$, we can obtain the probability density function for φ_1 (and similarly for φ_2)

$$\frac{dF_1(\varphi_1)}{d\varphi_1} = f_1(\varphi_1) \tag{28a}$$

$$\frac{dF_2(\varphi_2)}{d\varphi_2} = f_2(\varphi_2) \tag{28b}$$

If we perform the operations (28) on equations (27), respectively, we obtain

$$f(\varphi_1) = \int_{-\infty}^{\infty} f(\varphi_1,\varphi_2)d\varphi_2 \tag{29a}$$

$$f(\varphi_2) = \int_{-\infty}^{\infty} f(\varphi_1,\varphi_2)d\varphi_1 \tag{29b}$$

If the functions $F_1(\varphi_1)$ and $F_2(\varphi_2)$ can be determined by measurement, so can the functions $f_1(\varphi_1)$ and $f_2(\varphi_2)$. The problem is to determine $f(\varphi_1,\varphi_2)$, assuming $f_1(\varphi_1)$ and $f_2(\varphi_2)$ known.

A study of equations (29) shows that there is an infinity of possible functions $f(\varphi_1,\varphi_2)$ which will satisfy these equations. In order to determine $f(\varphi_1,\varphi_2)$ uniquely, an additional constraint must be imposed. Assume

$$f(\varphi_1,\varphi_2) = f_1(\varphi_1)f_2(\varphi_2) \tag{30}$$

Equation (30), together with equations (29) and the property of the distribution function

$$\int_{-\infty}^{\infty} f_1(\varphi_1)d\varphi_1 = \int_{-\infty}^{\infty} f_2(\varphi_2)d\varphi_2 = 1$$

uniquely determines $f(\varphi_1,\varphi_2)$. But equation (30) is a condition which, by definition, specifies that Φ_1 and Φ_2 are independent (see Parzen [5]). The additional constraint can be assumed of the form of equation (30) only if φ_1 and φ_2 are independent, and equation (22) must be examined to ascertain if the analysis implies a constraint which would inhibit such independence. The progression to three dimensions is obvious. From equation (22) and figure 2, it can be seen that, in general

$$S_1{}^2 + S_2{}^2 + S_3{}^2 \neq S^2$$

Suppose that two components only of the motion were known, say S_1 and S_2. The S_3 component is not determinable because of the inequality sign in the equation. Thus, S_3 is independent of S_1 and S_2. The argument applies similarly with a cyclical rearrangement of subscripts. Thus, functions $f_1(\varphi_1)$, $f_2(\varphi_2)$, and $f_3(\varphi_3)$ may be considered independent, histograms may be obtained by independent measurement for each of the three axes, and a general solution to the motion about any one axis is applicable to the other two (provided the proper respective histograms are used).

B. Derivation of the Basic Equations

The basic equations will be derived on a physical basis which is directly related to the present problem. The physical process is reviewed to prepare for these derivations.

First, a notational difficulty will be circumvented. It becomes necessary to use subscripts for purposes other than to indicate tensor components. The preceding section showed that motion about only one axis need be considered because the rotations are independent. The equations for motion about one axis apply equally well to motion about the other axes (with the appropriate constants and histograms, of course). To avoid confusion between a tensor component, φ_1, and an angle φ_1 at time t_1, let x represent the total angular displacement between one related pair of inertial and spacecraft axes. The value x_1 is the angular displacement at time t_1.

It is also obvious from equation (23) that $\Delta\varphi_i$ and KS_i differ only by a constant. The histogram of figure 3(a) can represent $\Delta\varphi_i$

as well as KS_i simply by changing the scale on the abscissa. In the following, figure 3(b) will be used and the variable considered will be the angular deviation, hereafter denoted by x. Figure 3(b) is the mass density distribution for Δx.

The total angular displacement at time t_n is

$$x_n = \Delta x_1 + \Delta x_2 + \ldots + \Delta x_n$$

where

$$\Delta x_i = x_i - x_{i-1}$$

The value of Δx_i, given that a motion occurs, is specified by the histogram of figure 3(b). With one exception, the value of Δx_i is independent of x_i, that is, the size of the step, given that a motion occurs, is not related to the present position; all single-step sizes have the distribution of figure 3(b). The exception occurs when the step is such that a preselected boundary is reached (and the control jets fire). Here, truncation occurs; the mathematical means for handling these boundary conditions will be developed in the appropriate sections.

The preceding discussion has carried the qualifying (conditional) statement "given that a motion occurs." The implication is that there exists a probability distribution for the number of motions made during the period. The total angular deviation depends upon both the size of the steps and the number of steps.

The distribution of the number of motions occurring during the period is derived in the next section. The associated dwell time between motions is also derived. Next, the Kolmogorov equations, which determine the probability of reaching x_j from x_i during period $t_j - t_i$ are obtained. The final derivation is that of the diffusion equation, which might be regarded as a limiting form of the Kolmogorov equations when the number of steps increases to infinity as the size of the steps decreases to zero.

C. The Kolmogorov Equations and the Diffusion Equation

1. Poisson distribution for the number of motions, n.- Suppose that the mass density function of figure 3 was obtained by observing the actions (or the effect of the actions) of a crew member over some period of length T. The number of motions made during this period was counted. Subdivide the period T into B subintervals of length $h = \frac{T}{B}$. Let the probability that one motion is made during this small interval be

$$P\left[n = 1\right] = \lambda h \tag{31}$$

where λ is not a function of t. The expression

$$\sum_{i=0}^{\infty} P\left[n = i\right] = 1 \tag{32}$$

states that the sum of the probabilities of all possible numbers of motions equals unity, that is, n must take on at least one of the values of the set $n = 0,1,2,3,\ldots$.

For the moment, regard a specific interval h_j. If the motions are independent, the probability that two motions occurred during h_j is of the order of $\lambda^2 h^2$. The probability that no motions were made during the interval h_j is

$$P\left[n = 0\right] = 1 - \lambda h - \sum_{i=2}^{\infty} P\left[n = i\right]$$

The argument applies to any arbitrary h_j, therefore,

$$P\left[n = 0\right] = 1 - \lambda h - O(h^2) \tag{33}$$

If the probability that a motion is made during period h_j is independent of the fact that a motion was made during period h_k $(k \neq j)$, the situation can be regarded as a series of B independent trials, with the outcome measured as success (a motion occurs) or failure (a motion does not occur). Under these circumstances the Bernoulli probability law applies, and the probability that the number of motions equals n is

$$P\left[n \middle| B \text{ trials during period } T\right] = \frac{B!}{(B - n)!n!}(\lambda h)^n(1 - \lambda h)^{B-n} \tag{34}$$

Note from the properties of the Bernoulli distribution that the expected number of movements is $B\lambda h$ during T, and that the expected number should remain the same regardless of the (artificial) manner in which T is subdivided. Therefore, let

$$B\lambda h = \mu \quad \text{as} \quad h \to 0 \quad \text{and} \quad B \to \infty \tag{35}$$

In other words, we will decrease the subinterval h_j by increasing the number of subintervals B to improve the approximation in equation (33), but do so in such a manner as to keep $B\lambda h$ constant. The algebra is straightforward; Stirling's approximation to the factorial function is used, together with the expansion

$$e^{-y} \cong 1 - y \qquad \text{for } y \text{ very small}$$

The final equation is

$$\lim_{\substack{B \to \infty \\ h \to 0 \\ B\lambda h = \mu}} P\left[n\right] = \frac{(\mu)^n e^{-\mu}}{n!} \tag{36}$$

Equation (36) gives the Poisson distribution. It measures the probability that the number of moves observed during the period T is n, provided that the expected number of moves is

$$\mu = B\lambda h = \lambda T \tag{37}$$

Directly associated with the Poisson distribution is the concept of the exponential dwell time. In equation (36) let $n = 0$

$$P\left[n = 0\right] = e^{-\mu} = e^{-\lambda T} \tag{38}$$

It is easily seen that $\frac{1}{\lambda}$ is the average dwell time. Equation (38) states that the probability that no motion will be observed during a period T decreases exponentially with T, or the probability that one will have to wait a time T for motion to occur decreases exponentially with T. Because of the assumptions of independence among the subintervals made during the derivation of equation (36), it has been assumed that past history has no effect on this dwell time; if no motion has occurred during a period T_1, the probability that no motion will occur during the immediately following period T_2 is

$$P\left[n = 0, T_2 \mid n = 0, T_1\right] = e^{-\lambda T_2}$$

A more detailed discussion of the implications will be found in Parzen [5].

Equation (36) with $n = 0$ is a probability function for the dwell time. Define $\epsilon(t)$ as the probability density function for the dwell time. The probability that the dwell time T is less than or equal to t is equal to the probability that a motion occurred during the interval t, or

$$P\left[T \leq t\right] = \int_0^t \epsilon(\tau)d\tau = 1 - e^{-\lambda t} \tag{39}$$

By Liebnitz rule,

$$\epsilon(t) = \lambda e^{-\lambda t} \tag{40}$$

2. <u>The Kolmogorov equations</u>.- The Kolmogorov equations determine the probability of moving from x_1 at time t_1 to x_2 at time t_2. These equations are derived in references [6], [7], [8], and [9]. The present derivation follows closely the (very brief) derivation found on page 458 of Feller [6]. It is common and convenient to use the term "state" such that when $x = i$, the system is said to be in state i. The variable x (for the present) can always be made a discrete integer by suitably choosing the units on the abscissa of figure 3(b). Although not essential to the derivation, it helps to visualize the process if such a transformation is assumed to have been made. The Kolmogorov equations pertain to such discrete (integer) steps when the dwell time between steps is exponentially distributed.

We seek to determine the probability P_{ij} of finding the system (angle) in state j at time $\tau + T$ if at time τ the system was in state i. The probability that the system goes from state i to state j in one step (one movement) is denoted by p_{ij}. Figure 3(b) determines p_{ij} for the problem under consideration. It can be seen that, except near the boundaries where truncation of the step size might occur, p_{ij} depends only upon the difference $j - i$ because of the assumptions of independence of steps. The mass density function for each step is given by figure 3(b).

If the system is initially in state i, the probability that one would find the system in state k at the end of n movements is

$$P_{ik}^{(n)} = \sum_j p_{ij} P_{jk}^{(n-1)} \tag{41}$$

However, we wish to have equation (41) expressed in terms of the time required to go from state i to state k. The waiting time in any state i has an exponential density function given by equation (40). We seek the probability that the system remains in state i until time τ, at which time it jumps to the intermediate state j, and then proceeds to state k through a series of intermediate steps in the remaining period $T - \tau$. Equation (41) has been summed over all possible states j; the following equation is also summed over all possible combinations of τ and $T - \tau$. Therefore, if $x(0) = i$

$$P_{ik}(T) = \sum_j \int_0^T \lambda_i e^{-\lambda_i \tau} p_{ij} P_{jk}(T - \tau) d\tau \qquad k \neq i \tag{42}$$

A term must be added to equation (42) to account for the possibility that the final state is i because no motion occurred at all, that is, the system started in state i and remained in state i throughout the entire period.

$$P_{ik}(T) = \delta_{ik}e^{-\lambda_i T} + \sum_j \int_0^T \lambda_i e^{-\lambda_i \tau} p_{ij}P_{jk}(T - \tau)d\tau \qquad (43)$$

where δ_{ik} is the Kronecker delta. Equation (43) is the backward Kolmogorov equation. The forward Kolmogorov equation is obtained similarly by starting with an equation in which the last jump is explicitly expressed

$$P_{ik}^{(n)} = \sum_j P_{ij}^{(n-1)} p_{jk}$$

to the result

$$P_{ik}(T) = \delta_{ik}e^{-\lambda_i T} + \int_0^T \sum_{j=1} P_{ij}(\tau)\lambda_j p_{jk} e^{-\lambda_k(T-\tau)}\, d\tau \qquad (44)$$

3. The Chapman-Kolmogorov equation.- In a sense, the Chapman-Kolmogorov equation has already been used in equation (41). The equation is

$$P_{ik}\left[\tau;T \mid x(\tau) = i\right] = \sum_j P_{ij}(\tau;t)P_{jk}(t;T) \qquad (45)$$

Equation (45) implies that the probability of finding the system in state k at time T depends only upon the fact that the system was in state i at time τ. Events before τ have no effect on the future probability of the system being in state k if it is known to be in state i at τ.

The following theorem will be needed. Proofs of the theorem may be found in Feller [6], [9].

4. Ergodic theorem.- Suppose that for the family of stochastic matrices $\tilde{P}(t)$

$$\tilde{P}(T + \tau) = \tilde{P}(T)\tilde{P}(\tau) \tag{46}$$

and

$$\tilde{P}(t) \to \tilde{\nu} \quad \text{as} \quad t \to 0$$

If no P_{ik} vanishes identically, then

$$\lim_{t \to \infty} P_{ik}(t) = \nu_k \tag{47}$$

where either $\nu_k = 0$ for all k, or else

$$\nu_k > 0, \ \sum_k \nu_k = 1 \tag{48}$$

and

$$\sum_j \nu_j P_{jk}(\tau) = \nu_k \tag{49}$$

The second alternative occurs whenever there exists a probability vector ν_k satisfying equation (49) for some $\tau > 0$. In this case equation (49) holds for all $\tau > 0$, and the probability vector ν_k is unique and is the reciprocal of the mean recurrence time of state k.

5. <u>The diffusion equation</u>.- The diffusion equation is used in place of the (discrete state) Kolmogorov equations when the angular displacement can be regarded as a continuous variable.

Let $f(x_1,t_1;x_2,t_2)$ be the conditional probability density function for finding the deviation at an angle x_2 at time t_2, given that the angular deviation was x_1 at time t_1. The analog to the Chapman-Kolmogorov equation (45) is, as derived on page 199 of Bailey [7]

$$f(x_1,t_1;x_3,t_3) = \int_{-\infty}^{\infty} f(x_1,t_1;x_2,t_2)f(x_2,t_2;x_3,t_3)dx_2 \tag{50}$$

There is a continuous analog to the backward Kolmogorov equation. Consider, for $t_1 < t_2$, the difference

$$f(x_1,t_1 - \Delta t;x_3,t_3) - f(x_1,t_1;x_3,t_3) \tag{51}$$

Using equation (50),

$$f(x_1,t - \Delta t;x_3,t_3) = \int_{-\infty}^{\infty} f(x_1,t_1 - \Delta t;x_2,t_2)f(x_2,t_2;x_3,t_3)dx_2 \tag{52}$$

But

$$\int_{-\infty}^{\infty} f(x_1,t_1 - \Delta t;x_2,t_2)dx_2 = 1 \tag{53}$$

so that

$$f(x_1,t_1;x_3,t_3) = f(x_1,t_1;x_3,t_3) \int_{-\infty}^{\infty} f(x_1,t_1 - \Delta t;x_2,t_2)dx_2 \tag{54}$$

Combine equations (52) and (54) to obtain the following equation for expression (51):

$$f(x_1,t_1 - \Delta t;x_3,t_3) - f(x_1,t_1;x_3,t_3)$$

$$= \int_{-\infty}^{\infty} f(x_1,t_1 - \Delta t;x_2,t_2) \left\{ f(x_2,t_2;x_3,t_3) - f(x_1,t_1;x_3,t_3) \right\} dx_2$$

$$= \int_{-\infty}^{\infty} f(x_1,t_1 - \Delta t;x_2,t_2) \left\{ (x_2 - x_1) \frac{\partial f(x_1,t_1;x_3,t_3)}{\partial x_1} \right.$$

$$\left. + \frac{1}{2} (x_2 - x_1)^2 \frac{\partial^2 f(x_1,t_1;x_3,t_3)}{\partial x_1{}^2} + \ldots \right\} dx_2$$

$$\cong \frac{\partial f(x_1,t_1;x_3,t_3)}{\partial x_1} \int_{-\infty}^{\infty} (x_2 - x_1) f(x_1,t_1 - \Delta t;x_2,t_2) dx_2$$

$$+ \frac{1}{2} \frac{\partial^2 f(x_1,t_1;x_3,t_3)}{\partial x_1{}^2} \int_{-\infty}^{\infty} (x_2 - x_1)^2 f(x_1,t_1 - \Delta t;x_2,t_2) dx_2 \qquad (55)$$

Let there exist infinitesimal means and variances for changes in x, defined by

$$b(x_1,t_1) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int (x_2 - x_1) f(x_1,t_1 - \Delta t;x_2,t_2) dx_2 \qquad (56)$$

and

$$a(x_1,t_1) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int (x_2 - x_1)^2 f(x_1,t_1 - \Delta t;x_2,t_2) dx_2 \qquad (57)$$

where the integration may have to be truncated in some cases to ensure convergence. Divide equation (55) by Δt and take the limit as $\Delta t \to 0$. The result is

$$- \frac{\partial f(x_1,t_1;x_3,t_3)}{\partial t} = b(x_1,t_1) \frac{\partial f(x_1,t_1;x_3,t_3)}{\partial x_1} + \frac{1}{2} a(x_1,t_1) \frac{\partial^2 f(x_1,t_1;x_3,t_3)}{\partial {x_1}^2} \tag{58}$$

Equation (58) is the continuous analog to the backward Kolmogorov equation. By starting with $t + \Delta t$ instead of $t - \Delta t$ and using a similar procedure, the continuous analog to the forward equation (sometimes called the Fokker-Planck equation) is obtained.

$$\frac{\partial f(x_1,t_1;x_3,t_3)}{\partial t} = - \frac{\partial}{\partial x_3} \left\{ b(x_3,t_3) f(x_1,t_1;x_3,t_3) \right\} + \frac{1}{2} \frac{\partial^2}{\partial {x_3}^2} \left\{ a(x_3,t_3) f(x_1,t_1;x_3,t_3) \right\} \tag{59}$$

In the present problem the transition probabilities for the changes in angle depend only upon the time elapsed $t_3 - t_1$ and on the difference in position $x_3 - x_1$. Therefore,

$$f(x_1,t_1;x_3,t_3) = u(x_3 - x_1, t_3 - t_1) \tag{60}$$

Equation (60) defines a process which is homogeneous in time and additive in the spatial coordinates. The average change in position during Δt is, from equation (56),

$$b(x,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int (x_2 - x_1) f(x_1, t_1 - \Delta t; x_2, t_2) dx_2$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\lambda \, \Delta t E(x_2 - x_1) \right] = \lambda E(\Delta x)$$

or

$$b = \lambda E(\Delta x) \tag{61}$$

where the integral has been evaluated by noting that the average change in position during time Δt is equal to the product of the average number of steps, $\lambda \, \Delta t$, and the average step size $E(\Delta x)$. Similarly, from equation (57)

$$a = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\lambda \, \Delta t \, \mathrm{Var}(\Delta x) + \lambda^2 (\Delta t)^2 E^2(\Delta x) \right]$$

$$= \lambda \, \mathrm{Var}(\Delta x) + \lim_{\Delta t \to 0} \lambda^2 \, \Delta t E^2(\Delta x)$$

$$= \lambda \, \mathrm{Var}(\Delta x) \tag{62}$$

Equations (61) and (62) show that, for the present problem, b and a are constants, independent of time and position. A more rigorous derivation of the values for the instantaneous mean and variance is given in the appendix.

If the forms of equations (60), (61), and (62) are substituted into either of equations (58) or (59), there results

$$\frac{\partial u}{\partial t} = \frac{a}{2} \frac{\partial^2 u}{\partial x^2} - b \frac{\partial u}{\partial x} \tag{63}$$

Equation (63) is the diffusion equation, so named because the equation describes the diffusion of heat through a solid or gas through a (porous) medium. Equation (63) is applicable to the present random walk problem when the number of steps (to reach a boundary) is large.

X. SOLUTION OF THE STOCHASTIC EQUATIONS

A. The Diffusion Equation

1. Applicability of the diffusion equation.- The applicability of the diffusion equation to a variety of situations can be illustrated by a simple calculation. In the discussion of experiments, a full arm rotation about the shoulder gave a mass-area of 5 kg-m^2. This is a relatively large movement, especially for a crew member restrained to a chair. From configuration 2 of table 1, the smallest principal value of inertia is 15,550 kg-m^2. By using these data in equation (23), the change in angle is found to be, for a 6.8 kg arm,

$$-\Delta\varphi_1 = \frac{2}{I_{11}} \frac{Mm}{M + m} S_1 \cong \frac{2(15)}{15,550} \frac{360}{2\pi} = 0.11 \text{ degrees}$$

For the intermediate principal axis of configuration 2,

$$-\Delta\varphi_2 \cong \frac{2(15)}{2,660,000} \frac{360}{2\pi} = 0.00064 \text{ degrees}$$

If the limits on the motion are set at $\pm 5^{\circ}$ from the nominal position, 45 complete arm rotations in the same direction will be required to reach the limit about the axis with the smallest principal moment of inertia. Almost 8,000 rotations will be required in the second case. Most random motions will be smaller, and some will be in an opposite direction. The average number of random motions to reach the boundary would be expected to be at least an order of magnitude larger than these calculations show. The diffusion equation should have less than 5 percent error (in time to reach a boundary) if the number of motions exceeds 500.

2. Basic equation and boundary conditions.- The basic equation is equation (63)

$$\frac{\partial u}{\partial t} = \frac{a}{2} \frac{\partial^2 u}{\partial x^2} - b \frac{\partial u}{\partial x} \tag{63}$$

where b and a are determined from equations (61) and (62). The variable x is the angular deviation, and u is the probability density function for the event that the angular deviation is x at time t, given that $x = x_0$ at time $t = 0$, from which the probability that

$$P\left[x_1 \leq X \leq x_2\right] = \int_{x_1}^{x_2} u(y)dy \tag{64}$$

can be calculated. Equation (64) is a basic definition for a probability density function. It will be convenient to make a shift in coordinates so that the nominal position, which is also the initial position, is denoted by $x_0 \neq 0$. The negative limit, or boundary, is at $x = 0$. The positive boundary is at $x = x_b$.

The desired result is an expression for the mean time to reach either boundary. This is directly related to the probability that $0 \leq X \leq x_b$, given that $x(0) = x_0$. Thus, a solution of equation (63) must be obtained first.

The initial and boundary conditions must be stated in probabilistic terms. The initial condition is obtained by stating that, at $t = 0$,

$x = x_0$ with probability one. That is, for $u(x,t)$,

$$u(x,0) = \delta(x - x_0) \tag{65}$$

where $\delta(x - x_0)$ is the Dirac delta function (see Bharucha-Reid [8]).

To obtain the proper boundary conditions, a bit of mathematical fiction may be introduced. Suppose that, when the angular limit is reached, the spacecraft immediately passes permanently beyond the physical boundary, rather than being returned to the nominal position. All positions outside a physical boundary will be regarded as points in a "boundary state." To determine if a physical boundary has been reached, one only observes whether or not a spacecraft is in a boundary state. A boundary through which the angular deviation may pass without the possibility of return is called an "absorbing boundary," and is mathematically represented as

$$u(x_b,t) = u(0,t) = 0 \tag{66}$$

There is a direct analogy to heat transfer problems where the boundary temperature is held constant (at zero); the thermal energy reaching the boundary is absorbed into some outside heat sink. Such a thermal problem has the boundary conditions on temperature given by equation (66).

3. Solution of diffusion equation for the waiting time.- The solution of equation (63), together with the conditions (65) and (66), can be obtained by using iterated Laplace transforms. Such a formulation will be convenient subsequently because difficult integrals can be avoided by using Laplace transform operations to determine the average waiting time to reach a boundary from the solution for $u(x,t)$.

Let $f(x,s)$ be the Laplace transform of $u(x,t)$. The transformed differential equation and boundary conditions are

$$sf = \frac{1}{2} a \frac{d^2f}{dx^2} - b \frac{df}{dx} + \delta(x - x_0) \tag{67}$$

and

$$f(0,s) = f(x_b,s) = 0 \tag{68}$$

Because the initial condition is a Dirac delta function, the solution to equation (67) is best obtained by transforming again with respect to x. Let $g(p,s)$ be the iterated transform of $f(x,s)$. Equations (67) and (68) become

$$- \frac{2s}{a} g + (p^2g - f'(0,s) - 0) - \frac{2b}{a} (pg - 0) = - \frac{2}{a} e^{-px_0} \tag{69}$$

where $f'(0,s)$ is the derivative of $f(x,s)$ with respect to x evaluated at $x = 0$. This will be evaluated later by using the boundary condition at $x = x_b$. Equation (69) becomes

$$g = \frac{f'(s,0) - \frac{2}{a} e^{-px_0}}{\left(p - \frac{b}{a}\right)^2 - l^2} \tag{70}$$

where

$$l = \frac{b}{a} \sqrt{1 + \frac{2as}{b^2}} \tag{71}$$

The inverse transform (with respect to p) of equation (70) is

$$f = \frac{1}{l} f'(s,0)e^{bx/a} \sinh lx - \frac{2}{la} e^{b(x-x_0)/a}\left[\sinh l(x - x_0)\right] U(x - x_0) \tag{72}$$

where

$$U(x - x_0) = \begin{cases} 0 & \text{for} \quad x < x_0 \\ 1 & \text{for} \quad x \geq x_0 \end{cases} \tag{73}$$

From the boundary condition at $x = x_b$, one obtains

$$f'(s,0) = \frac{2e^{-bx_0/a} \sinh l(x_b - x_0)}{a \sinh lx_b} \tag{74}$$

Equations (72) and (74) are combined, and after using identities for the hyperbolic functions, one obtains

$$f_2(s,x) = \frac{2e^{b(x-x_0)/a}}{la \sinh lx_b}\left[\sinh lx_0 \sinh l(x_b - x)\right] \quad \text{for} \quad x \geq x_0 \tag{75a}$$

and

$$f_1(s,x) = \frac{2e^{b(x-x_0)/a}}{la \sinh lx_b}\left[\sinh l(x_b - x_0)\sinh lx\right] \quad \text{for} \quad x \leq x_0 \tag{75b}$$

Equations (75) can be inverted to obtain $u(x,t)$. Because the next mathematical operations can be performed more easily on the transformed variable, the inverse will be taken later. However, these operations are more easily explained in terms of real variables. Therefore, in order to describe the mathematical procedure, assume that $u(x,t)$ has been found. The probability that a boundary has not been reached at time t is

$$P\left[0 < X < x_b, t\right] = \int_0^{x_b} u(y,t)dy \tag{76a}$$

Let W be the time required to reach a boundary starting from x_0 at $t = 0$. By definition, W is the waiting time. The probability that $W > t$ is equal to the probability that a boundary has not been reached by time t. Because the boundaries are absorbing, this is the same as not finding the spacecraft in an absorbing state at time t. Mathematically, these concepts lead to

$$P\left[W > t\right] = \int_0^{x_b} u(y,t)dy \tag{76b}$$

To find the mean waiting time, the probability density function for W must be obtained from equation (76b). Let $\beta(t)$ be this probability density function. Then,

$$\int_t^{\infty} \beta(\tau)d\tau = \int_0^{x_b} u(y,t)dy \tag{76c}$$

from which

$$\beta(t) = -\frac{\partial}{\partial t}\int_0^{x_b} u(y,t)dy \tag{76d}$$

Because $\beta(t)$ is a probability density function, $\beta(\infty) = 0$.

The mean value for the waiting time is

$$E(W) = \int_0^{\infty} t\beta(t)dt \tag{76e}$$

The definition of the variance of the waiting time is

$$\mathrm{Var}(W) = \int_0^{\infty} t^2\beta(t)dt - E^2(W) \tag{76f}$$

The order of related operations of equations (76) will be performed on the transformed variables, $f_1(s,x)$ and $f_2(s_1,x)$.

Equations (75) are changed to the exponential form for the hyperbolic functions. Equation (75a) is integrated from x_0 to x_b, and equation (75b) is integrated from 0 to x_0. The two results are added to obtain the transformed analog of equation (76a). This is

$$F(s) = \frac{2}{a\left[1 - \exp(-2lx_b)\right]\left[l^2 - \frac{b^2}{a^2}\right]}\left\{1 - \exp(-2lx_b) - \exp\left(-\frac{bx_0}{a}\right)\left[\exp(-lx_0) - \exp(-2lx_b + 2lx_0)\right] - \exp\left(\frac{bx_b}{a} - \frac{bx_0}{a}\right)\left[\exp(-lx_b + lx_0) - \exp(-lx_b - lx_0)\right]\right\} \tag{77}$$

The analog to the operation indicated in equation (76d) is, from reference [10],

$$sF(s) - \mathcal{F}(+0) \quad \text{represents} \quad \frac{d}{dt}\mathcal{L}^{-1}\left\{F(s)\right\} \tag{78}$$

From the initial condition of equation (65), the value of $\mathcal{F}(+0) = 1$ can be deduced. At $t = 0$, the angular deviation is between $x = 0$ and $x = x_b$ with probability one. The operation (78) is performed upon equation (77), and equation (71) used to get l in terms of s. The result is

$$G(s) = \frac{2}{a\left[1 - \exp\left(-2\,\frac{bx_b}{a}\right)\right]}\left\{\exp(-lx_0) - \exp(-2lx_b + 2lx_0)\right.$$

$$+ \exp\left(\frac{bx_b}{a} - \frac{bx_0}{a}\right)\left[\exp(-lx_b + lx_0)\right.$$

$$\left.\left. - \exp(-lx_b - lx_0)\right]\right\} \tag{79}$$

Equation (79) is the Laplace transform of the probability density function, $\beta(t)$, for the waiting time.

To perform the operation indicated in equation (76e), note that differentiation of the transform of a function with respect to s corresponds to multiplication of the function by t. Also, the final operation is greatly simplified by noting from the definition of the Laplace transform that

$$\lim_{s \to 0}\left[-\frac{d}{ds}\,G(s)\right] = \lim_{s \to 0}\int_0^\infty e^{-st}\,t\beta(t)dt$$

$$= \int_0^\infty t\beta(t)dt = E(W) \tag{80}$$

The expected waiting time is found from equation (79) by using equations (71) and (80) to be

$$E(W) = \frac{x_b}{b}\left\{\left[\frac{1 - \exp\left(-2\,\frac{bx_b}{a}\,\frac{x_0}{x_b}\right)}{1 - \exp\left(-2\,\frac{bx_b}{a}\right)}\right] - \frac{x_0}{x_b}\right\} \tag{81}$$

where b and a are determined by equations (61) and (62).

Equation (81) determines the mean waiting time to reach a boundary. No specification is made regarding which boundary.

The situation wherein $b \to 0$ is of interest. Physically, such a condition will exist when the magnitudes and directions of the crew member motions are such that the mean value of the histogram of figure 3 is zero ($E_1(\Delta x) = 0$).

$$\lim_{b \to 0} E(W) = \frac{2x_b x_0}{a}\left(1 - \frac{x_0}{x_b}\right) \tag{82}$$

Equation (82) shows that the expected waiting time to reach a boundary does not become infinite and that as the variance of the one-step histogram increases, the time between control jet operation decreases. Also, when $a \to 0$, equation (81) reduces to the deterministic solution for the time to reach the boundary in the preferred direction of movement.

For some applications, the variability in the waiting time may be of interest. A measure of the variability is the variance of the waiting time given by equation (76f). A procedure similar to equation (80) may be used.

$$\int_0^\infty t^2\beta(t)dt = \lim_{s \to 0} (-1)^2 \frac{d^2}{ds^2} g(s) \tag{83}$$

It is helpful to note that

$$\frac{d}{ds} = \frac{1}{al}\frac{d}{dl}$$

$$\frac{d^2}{ds^2}\,g(s) = \frac{1}{a^2l^2}\left(\frac{d^2g}{dl^2} - \frac{a}{al}\,\frac{dg}{dl}\right)$$

and that

$$E(W) = \lim_{l \to \frac{b}{a}} \frac{1}{al}\frac{dg}{dl}$$

because $l \to \frac{b}{a}$ as $s \to 0$. Therefore,

$$\mathrm{Var}(W) = \lim_{l \to \frac{b}{a}} \frac{1}{a^2l^2}\left\{\frac{d^2g}{dl^2} + aE(W) - a^2l^2E^2(W)\right\}$$

Equation (83) is used together with equation (77). The algebra is very lengthy, but after terms are collected and canceled, the result is

$$\mathrm{Var}(W) = \left(\frac{x_b}{b}\right)^2 \frac{1}{\left[1 - \exp\left(-2\,\frac{bx_b}{a}\right)\right]^2}\left\{\left[1 - \exp\left(-2\,\frac{bx_b}{a}\,\frac{x_0}{x_b}\right)\right]\exp\left(-2\,\frac{bx_b}{a}\,\frac{x_0}{x_b}\right)\right.$$

$$- 4\left(\frac{x_0}{x_b}\right)\left[1 - \exp\left(-2\,\frac{bx_b}{a}\right)\right]\exp\left(-2\,\frac{bx_b}{a}\,\frac{x_0}{x_b}\right)$$

$$\left. + 3\left[1 - \exp\left(-2\,\frac{bx_b}{a}\,\frac{x_0}{x_b}\right)\right]\exp\left(-2\,\frac{bx_b}{a}\right)\right\} + \frac{a}{b^2}\,E(W) \qquad (84)$$

Equations (81), (84), (61), and (62), together with the histogram of figure 3(b), constitute the solution to the problem.

To obtain a more useful range for the numerical evaluation of $E(W)$ and $Var(W)$, these quantities can be defined in nondimensional form as

$$\overline{W} = \frac{b}{x_b} E(W) = \left[\frac{1 - \exp\left(-2 \frac{bx_b}{a} \frac{x_0}{x_b}\right)}{1 - \exp\left(-2 \frac{bx_b}{a}\right)}\right] - \frac{x_0}{x_b} \tag{85}$$

and

$$\overline{\sigma}^2 = \left(\frac{b}{x_b}\right)^2 Var(W) = \frac{1}{\left[1 - \exp\left(-2 \frac{bx_b}{a}\right)\right]^2} \left\{\left[1 - \exp\left(-2 \frac{bx_b}{a} \frac{x_0}{x_b}\right)\right] \exp\left(-2 \frac{bx_b}{a} \frac{x_0}{x_b}\right)\right.$$

$$- 4\left(\frac{x_0}{x_b}\right)\left[1 - \exp\left(-2 \frac{bx_b}{a}\right)\right] \exp\left(-2 \frac{bx_b}{a} \frac{x_0}{x_b}\right)$$

$$\left. + 3\left[1 - \exp\left(-2 \frac{bx_b}{a} \frac{x_0}{x_b}\right)\right] \exp\left(-2 \frac{bx_b}{a}\right)\right\} + \frac{a}{bx_b} \overline{W} \tag{86}$$

It should be noted that

$$\lim_{\frac{bx_b}{a} \to \infty} \overline{W} = 1 - \frac{x_0}{x_b} \tag{87a}$$

and

$$\lim_{\frac{bx_b}{a} \to \infty} \overline{\sigma}^2 = 0 \tag{87b}$$

Numerical values for $\overline{W}$ and $\overline{\sigma}^2$ are plotted in figures 4 and 5.

4. Minimum fuel consumption.- If the mean rate of motion, b, is not zero, the rate of control fuel consumption may be reduced by

selecting the nominal position x_0 at some point other than halfway between zero and x_b so as to maximize the waiting time. Equation (81) is differentiated with respect to x_0 and the result set equal to zero.

$$\left(\frac{x_0}{x_b}\right)_{opt} = -\frac{1}{2}\frac{a}{bx_b}\,ln\left(\frac{1}{2}\frac{a}{bx_b}\right) \tag{88}$$

B. Markov Chain Analysis

When few motions are required to reach a preset boundary limit on the angular deviation, either because the moving mass is large or the spacecraft moment of inertia is small, the assumption of a continuum may lead to undesirable error. If the less restrictive assumptions made in the dynamic analysis are met so that the angles φ_1, φ_2, and φ_3 remain independent, an improvement in accuracy might be obtained by considering the discrete magnitude of each step. More information is used in the calculation of the waiting time if all values of the one-step histogram are used rather than just the mean and the variance; therefore, one would expect that a more accurate evaluation of the sum of the steps can be made. A disadvantage is that a digital computer must be used to evaluate the equations. Applications are limited by the capacity of the computer for matrix manipulation.

1. Generating functions.- The desired equations are most easily obtained by using generating functions (see Bailey [7] or Feller [9]). The concept is simple, and an intuitive derivation will be given here. During the derivation of the Kolmogorov equations, it was mentioned that the units on the abscissa of the histogram of figure 3(b) could be

chosen so that the magnitudes represented by the bars were integers. It is convenient to do so now. Let $\{\Delta x_i\}$ be some sequence of numbers with a common probability distribution (fig. 3(b)), and y be a dummy variable. Form a power series in y

$$\mathcal{G}(y) = \sum_n p_n y^n \tag{89}$$

such that p_n is the probability that $\Delta x = n$. $\mathcal{G}(y)$ is the generating function for Δx. Although n may sometimes be less than zero, $p_n \geq 0$ always. Generating functions have a useful convolution property; the product of two generating functions gives a generating function for the probability that the sum of the two random variables each represents will be equal to a given number. This is easily demonstrated by taking

$$\mathcal{H}(y) = \sum_{n=1} \frac{1}{6} y^n$$

By squaring $\mathcal{H}(y)$,

$$L(y) = \mathcal{H}(y) * \mathcal{H}(y) = \frac{1}{36}\Big[y^2 + 2y^3 + 3y^4 + 4y^5 + 5y^6 + 6y^7 + 5y^8 + 4y^9 + 3y^{10} + 2y^{11} + y^{12}\Big]$$

where $\mathcal{H}(y)$ is the generating function for the numbers on a die, and $L(y)$ can be recognized as the generating function for sum of the numbers on two dice.

Let $\mathcal{J}(y)$ be the generating function for Δx. The values of p_n are the heights of the bars of the histogram of figure 3(b). The convolution property will be used to obtain the probability mass density distribution for the sum of many Δx. The number of terms (Δx's) which constitute the sum is also a random variable, governed by the Poisson distribution previously derived.

2. <u>The compound Poisson distribution</u>.- The generating function for the Poisson distribution can be obtained from equation (36) and the definition of a generating function. This is

$$J(y) = \sum_{n=0}^{\infty} \frac{\mu^n e^{-\mu}}{n!} y^n \tag{90}$$

where the coefficient of y^n is the probability that n motions will occur. Equation (90) would give the sum of Δx's only if each and every $\Delta x = 1$, because then the sum of the Δx's would be numerically equal to the number of motions. But the Δx's have the distribution of figure 3(b) (or eq. (89)) and are not all equal to unity. The process where n is Poisson distributed and Δx has some probability mass distribution is called the compound Poisson process.

The generating function, $Q(y)$, for the compound Poisson distribution may be deduced from the definition of a generating function. Suppose that n motions occur and that the generating function for the size (Δx_i) of each of the n steps is given by equation (89). From the convolution property just demonstrated, the generating function for the sum of n Δx's is $[\mathcal{J}(y)]^n$.

The coefficient of y^k in $\left[\mathscr{G}(y)\right]^n$ can be regarded as the conditional probability that $\Delta x = k$ given that n motions occur. This coefficient must be multiplied by the probability that n motions will occur to obtain the unconditional probability that $\Delta x = k$. Consideration of all possible values of n leads to

$$
\begin{aligned}
Q(y) &= P\left[n = 0\right] \mathscr{G}^0(y) + P\left[n = 1\right] \mathscr{G}(y) + P\left[n = 2\right] \mathscr{G}^2(y) + \dots \\
&= \sum_{n=0}^{\infty} \frac{\mu^n e^{-\mu}}{n!} \mathscr{G}^n(y) \\
&= e^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \left[\sum_k p_k y^k\right]^n \\
&= \exp\left[-\mu\left(1 - \sum_{n=0}^{\infty} p_k y^k\right)\right] \qquad (91)
\end{aligned}
$$

where $\mathscr{G}^0(y) = 1$. See Bailey [7] for the derivation of the generating functions for general compound processes.

The coefficient of y^j in the power series expansion of equation (91), denoted by α_j

$$
Q(y) = \sum_{j=-\infty}^{\infty} \alpha_j y^j \qquad (92)
$$

is the probability that the sum of the motions has resulted in a total displacement $x = j$, given that the initial displacement was $x = 0$. Equation (92) can easily be generalized to other initial conditions by writing it as

$$Q(y) = \sum_{j=-\infty}^{\infty} \alpha_{ij} y^j \tag{93}$$

where α_{ij} is the probability of finding the system in state j, provided the system started at state i. The problem is to find α_{ij} from equation (91).

To be consistent with equation (93), p_k in equation (91) should be written as p_{ik}. The one-step probability p_{ik} is defined as the probability that the spacecraft will move from state $x = i$ to state $x = j$ in one step. It was mentioned in section IX.C.2 (p. 33) that, except near the boundaries on the motion, p_{ik} depends only upon the difference $(k - i)$ in the present problem. For the purpose of the present derivation, this restriction will be relaxed so that p_{ik} is a function of both i and $(k - i)$. In the next-to-last form of equation (91) an equivalent form, based upon p_{ik}, is needed. The more general form for p_{ik} can be represented as a matrix p. Then,

$$\left[\sum_k p_k y^k\right]^n \rightarrow \sum_k \tilde{p}_{ik}^{(n)} \, y^k \tag{94}$$

for the more general situation. Here, $\tilde{p}_{ik}^{(n)}$ is defined as the term in the ith row of the kth column of the matrix formed by the nth convolution of the matrix $\tilde{p}$. The term $\tilde{p}_{ik}^{(0)}$ is defined as such a similar term in the unitary matrix, so that if no motion is made the spacecraft remains in its present position (state) with probability one.

If equation (91) be modified by using the transformation (94), and set equal to equation (93), the result is

$$\alpha_{ik} = e^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \tilde{p}_{ik}{}^{(n)} = P_{ij}(t) \tag{95}$$

because y is a dummy variable which may take on any value (other than zero, to avoid a convergence problem when k is negative), and the coefficients of like powers of y^k therefore must be equal in the respective series. (See Feller 9 , p. 428, for reference to generalized Poisson processes.)

Because $\mu = \lambda T$, equation (95) gives the transition probabilities for the total displacement during the period T, provided no boundaries are encountered. In the physical situation, if a boundary is encountered the control jets fire and the spacecraft is returned to its nominal position. However, as in the solution to the diffusion equation, it will be convenient to employ the mathematical fiction of an absorbing boundary, where it is assumed that the spacecraft remains in the boundary state if a boundary is reached. If it were unnecessary to consider boundary limits, the doubly infinite square matrix $\tilde{p}_{ij}$ would be a band matrix whose elements depended only upon the difference $(j - i)$. The value for p_{ij} would be found by setting $\Delta x = (j - i)$ and using the histogram of figure 3(b). (Note that $P[\Delta x = j - i] = p_{ij} = 0$ for large values of $(j - i)$ which are beyond those measured and plotted on the histogram.) Figure 6 illustrates the matrix for the unbounded case.

For the situation where absorbing boundaries are used (fig. 7), the matrix $\tilde{p}$ must be changed to $\tilde{\tilde{p}}$ by two modifications: (1) One new absorbing state is added beyond each physical boundary such that $\bar{p}_{0j} = \delta_{0j}$ and $\bar{p}_{\gamma j} = \delta_{\gamma j}$, where the fictitious boundary states are located at $x = 0$ and $x = \gamma$, and (2) the probabilities are adjusted so that whenever a single step might take the spacecraft beyond a boundary, instead it takes the spacecraft into the (absorbing) boundary. (Because of space limitation, the matrix shown is smaller than the matrices which would actually be used; the principles of construction remain the same.)

In terms of the modified matrix, equation (95) becomes

$$\alpha_{ij} = e^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \tilde{\tilde{p}}_{ij}(n) \tag{96}$$

With $\mu = \lambda T$, the probability of being in one or the other boundary state, if the system started at state $x = i$ when $T = 0$, is

$$P\left[\text{a boundary has been reached}\right] = \alpha_{i0} + \alpha_{i\gamma}$$

$$= e^{-\lambda T} \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!} \left(\tilde{\tilde{p}}_{i0}(n) + \tilde{\tilde{p}}_{i\gamma}(n)\right) \tag{97}$$

Let $\beta(t)$ be the probability density function for the waiting time, W. The probability that the waiting time is less than T is equal to the probability that the spacecraft is in a boundary state at time T (has reached an absorbing boundary before T). Mathematically,

$$\int_0^T \beta(t)dt = e^{-\lambda T} \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!}\left(\tilde{\bar{p}}_{i0}(n) + \tilde{\bar{p}}_{i\gamma}(n)\right) \tag{98}$$

from which

$$\beta(T) = \lambda e^{-\lambda T} \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!}\left(\tilde{\bar{p}}_{i0}^{(n+1)} - \tilde{\bar{p}}_{i0}^{(n)} + \tilde{\bar{p}}_{i\gamma}^{(n+1)} - \tilde{\bar{p}}_{i\gamma}^{(n)}\right) \tag{99}$$

The mean value of the waiting time is

$$E(W) = \int_0^{\infty} t\beta(t)dt$$

$$= \frac{1}{\lambda} \sum_{n=0}^{\infty} (n + 1)\left(\tilde{\bar{p}}_{i0}(n+1) - \tilde{\bar{p}}_{i0}(n) + \tilde{\bar{p}}_{i\gamma}(n+1) - \tilde{\bar{p}}_{i\gamma}(n)\right) \tag{100}$$

given that the initial state was at $x = i$. The variance of the waiting time is

$$\mathrm{Var}(W) = \int_0^{\infty} t^2\beta(t)dt - E^2(W)$$

$$= \frac{1}{\lambda^2} \sum_{n=0}^{\infty} (n + 1)(n + 2)\left(\tilde{\bar{p}}_{i0}(n+1) - \tilde{\bar{p}}_{i0}(n)\right.$$

$$\left. + \tilde{\bar{p}}_{i\gamma}(n+1) - \tilde{\bar{p}}_{i\gamma}(n)\right) - E^2(W) \tag{101}$$

Equations (100) and (101), together with a modified matrix $\tilde{\tilde{p}}_{ij}$ similar to figure 7, constitute the solution for the waiting time and waiting time variation. The mean frequency of control jet operation is the inverse of the mean waiting time.

If a knowledge of the variance is not needed, computer storage and time may be saved by an alternate procedure for the mean waiting time calculation which is developed in the next section.

C. Iteration Solution

The iteration solution is based upon the Kolmogorov equations and the theorem cited in the section "Derivation of the Basic Equations." The iteration solution saves computer storage and time by reducing the number of matrices which must be manipulated. For a detailed derivation of the underlying equations, see Feller [6], Chapter XIV. The heuristic derivation which follows is related to the physical problem under consideration.

In the previous two solutions a fictitious absorbing boundary state was introduced into the transition matrix, $\tilde{p}$, to give a modified matrix $\tilde{\tilde{p}}$. The present analysis will consider a modified matrix $\underline{\tilde{p}}$ (fig. 8) which conforms more realistically to the physical situation. The added boundary states of $\underline{\tilde{p}}$ are again located immediately outside the physical limiting values of the angular deviation and are such that, once in this state, the spacecraft is reflected back to its nominal position with probability one when the next motion occurs. These boundary states are labeled "0" and "γ." Previously, the dwell time between motions $(1/\lambda)$ was considered independent of the position (state) of the spacecraft. The dwell time will again be considered constant for all states within the actual physical boundaries; for the fictitious added reflecting boundaries, the dwell time is $1/\lambda_0$. The purpose of taking $1/\lambda_0$ different from $1/\lambda$ will be to (eventually) let $\lambda_0 \to \infty$ to not introduce additional dwell times due to the added boundary states.

The Chapman-Kolmogorov equation (eq. (45)) stipulates that the probability of finding the spacecraft at an angle $x_k = k$ at time $\tau + T$ depends only upon the fact that the spacecraft is known to be at an angle $x_i = i$ at time T, and that the history of angular changes before time T has no effect upon these probabilities. Because the one-step transition matrix $\tilde{p}$ and the dwell times are independent of time, these conditions are met for the present physical problem.

Equation (46) is equation (45) in matrix form. As $t \to 0$, no motions will occur and the spacecraft will remain in its present state. This corresponds with the zeroth convolution of the transition matrix, similar to the discussion following equation (95); thus, the first condition of the theorem is met. There are no absorbing states in $\underline{\tilde{p}}$ (i.e., the spacecraft cannot be trapped in any fixed position. Therefore, there are no transient states; that is, there always remains a finite probability that each state will be reached again at some future time. Thus, equations (47), (48), and (49) hold, with $\nu_k > 0$.

Suppose the spacecraft started at an initial deviation angle $x_i = i$. After a very long period, during which the control jets have fired many times, the probability that $x_j = j$ becomes independent of the initial conditions. According to the ergodic theorem, this probability is ν_j. The problem of the present section is to find ν_j.

The following simple example will illustrate the theorem and its usefulness. Suppose the one-step transition matrix were

$$\underset{\sim}{\underline{p}} = \begin{array}{c} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \end{bmatrix} \end{array} \begin{array}{c} \underline{\text{States}} \\ 0 \\ 1 \\ 2 \end{array}$$

where the starting state was state No. 1 and one step was made each second. After n steps,

$$\underset{\sim}{\underline{p}}^{(n)} = \begin{bmatrix} 0 & 1 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \end{bmatrix}^{n}$$

For $n = 1$ (after one step), the probability of being in state 0 is 1/4, state 1 is 1/2, and state 2 is 1/4. For $n = 2$,

$$\underset{\sim}{\underline{p}}^{(2)} = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/8 & 5/8 & 1/4 \\ 1/8 & 1/2 & 3/8 \end{bmatrix}$$

which shows that after two steps the probability of being in state 0 is 1/8, state 1 is 5/8, and state 2 is 1/4. A few more convolutions will show that

$$\lim_{n \to \infty} \underset{\sim}{p}^{(n)} = \begin{bmatrix} 5/32 & 9/16 & 9/32 \\ 5/32 & 9/16 & 9/32 \\ 5/32 & 9/16 & 9/32 \end{bmatrix}$$

which shows that the probability of being in state 0 is 5/32, in state 1 is 9/16, and in state 2 is 9/32. The probability of being in these states is also seen to be independent of the initial state (starting position). This is a consequence of the matrix being ergodic, and is the stated result of the theorem.

The transition matrix $\tilde{\underline{P}}(t)$ will be determined from the Kolmogorov equation, and an iteration method used to find the limiting value of the matrix after a long period.

The Laplace transform of the forward Kolmogorov equation (eq. (44)) is, in terms of the new modified transition matrix,

$$\Pi_{ik}(s) = \frac{\delta_{ik}}{s + \lambda_i} + \sum_{j=0}^{b} \frac{\lambda_j}{s + \lambda_k} \Pi_{ij}(s)\underline{p}_{jk} \tag{102}$$

where Π_{ik} is the Laplace transform of P_{ik}. Equation (102) is more easily manipulated in matrix form. Define: a square diagonal matrix $\tilde{s}$ whose diagonal elements equal s; a square diagonal matrix $\tilde{\lambda}$ whose diagonal elements are equal to λ (except for elements $\lambda_{00} = \lambda_{\gamma\gamma} = \lambda_0$, for the added boundary states 0 and γ); and let $\tilde{\mathcal{J}}$ be the identity matrix. The summation runs from 0 to γ for the limited motion under consideration. In terms of the above defined matrices, equation (102) becomes, after switching i and k in the first term on the right,

$$\tilde{\Pi}(\tilde{s} + \tilde{\lambda}) = \tilde{\mathcal{J}} + \tilde{\Pi}\tilde{\lambda}\tilde{\underline{p}}(\tilde{s} + \tilde{\lambda})^{-1}(\tilde{s} + \tilde{\lambda})$$

or

$$\tilde{\Pi}(\tilde{s} + \tilde{\lambda}) = \tilde{\mathcal{J}} + \tilde{\Pi}\tilde{\lambda}\tilde{\underline{p}} \tag{103}$$

The result of the ergodic theorem (eq. (49)) may be expressed in matrix form as

$$\lim_{t \to \infty} \lfloor \nu \rfloor \left[\underline{P}(t) \right] = \lfloor \nu \rfloor \tag{104}$$

The Laplace transform of equation (104) is

$$s\lfloor \nu \rfloor \{\Pi\} = \lfloor \nu \rfloor$$

from which

$$s\lfloor \nu \rfloor \{\Pi\} = s\lfloor \nu \rfloor \tilde{\mathcal{I}} \{\Pi\} = \lfloor \nu \rfloor \tilde{s} \{\Pi\}$$

which yields finally

$$\lfloor \nu \rfloor \tilde{s}\tilde{\Pi} = \lfloor \nu \rfloor \tag{105}$$

Note that $\tilde{\Pi}\tilde{s} = \tilde{s}\tilde{\Pi} = s\tilde{\mathcal{I}}\tilde{\Pi}$, and substitute equation (105) into equation (103) to get

$$\tilde{s}\tilde{\Pi} + \tilde{\Pi}\tilde{\lambda} = \tilde{\mathcal{I}} + \tilde{\Pi}\tilde{\lambda}\tilde{\underline{p}}$$

whence

$$\lfloor \nu \rfloor \tilde{\Pi}\tilde{\lambda} = \lfloor \nu \rfloor \tilde{\Pi}\tilde{\lambda}\tilde{\underline{p}}$$

Use equation (105) again to get

$$\frac{1}{s}\lfloor \nu \rfloor \tilde{\lambda} = \frac{1}{s}\lfloor \nu \rfloor \tilde{\lambda}\tilde{\underline{p}}$$

Finally,

$$\lfloor \nu \rfloor \{\lambda\} = \lfloor \nu \rfloor \{\lambda\} \left[\underline{p}\right] \tag{106}$$

Define a new matrix $\lfloor \eta \rfloor$ such that

$$\lfloor \eta \rfloor = \lfloor \nu \rfloor \{\lambda\} \tag{107}$$

Equation (106) becomes

$$\lfloor \eta \rfloor = \lfloor \eta \rfloor \left(\underline{p} \right) \tag{108}$$

If the values of η_i in the row matrix in equation (108) are found, the values of ν_i will be determined by equation (107). The values of η_i can be determined within a constant; that is, the ratios of all η_i may be found with respect to the value of one component, say η_N. This is sufficient because the values of ν_i are the values of the terms in the respective rows of the $\underline{\tilde{P}}$ matrix which has been convoluted an infinite number of times. The $\underline{\tilde{P}}$ matrix, and thus the row matrix $\lfloor \nu \rfloor$, is stochastic. Because the matrices are stochastic, the sums of the terms in each row equal unity. That is,

$$\sum_j \underline{P}_{ij} = \sum_j \nu_j = 1 \tag{109}$$

which is the additional relationship needed to fix the values of ν_i.

Equation (108) can be solved numerically for the ratios of the η_i. The matrix $\underline{\tilde{p}}$ is known (from fig. 8, for instance). The process is started by choosing initial values for (η_i) and performing the operation indicated on the right. The new values, $(\eta_i)_2$, are better approximations and are used on the right for the second iteration. The process is continued until no change (within the desired accuracy) is noted in the ratios of the η_i.

Assume the η_i have been found by means of the numerical iteration process. The values of ν_i are the probabilities of observing a deviation angle, $x_i = i$, after the random process has been going on for a long time. From equation (107),

$$\lfloor \nu \rfloor = \lfloor \eta \rfloor \left(\lambda \right)^{-1} = \lfloor \eta \rfloor \left[\frac{1}{\lambda} \right]$$

which, upon expansion, yields

$$\lfloor \nu \rfloor = \left\lfloor \frac{\eta_0}{\eta_N \lambda_0}, \frac{\eta_1}{\eta_N \lambda}, \ldots, \frac{\eta_\gamma}{\eta_N \lambda_\gamma} \right\rfloor \tag{110a}$$

provided that

$$\frac{1}{\lambda_0}(\eta_0 + \eta_\gamma) + \frac{1}{\lambda} \sum_{i=1}^{\gamma-1} \eta_i = \eta_N \tag{110b}$$

where equation (110b) follows from equation (109). The final statement of the ergodic theorem is that the inverse of the probability of finding the system in state j at time $t \to \infty$ is proportional to the average period between arrivals at state j. For the purpose of discussion, define an "off period" as the period during which the spacecraft is not at the boundary state $x_\gamma = \gamma$. A cycle is defined as starting when the spacecraft enters the boundary state $x_\gamma = \gamma$, and ending when the spacecraft is at $x_\gamma = \gamma$ again at some later time. The average length of an "off period" is ζ_γ. The average length of time the system is in state $x_\gamma = \gamma$ is (from eq. (38)) equal to $1/\lambda_\gamma$. (It should be remembered that $1/\lambda_\gamma = 1/\lambda_0$.) Intuitively, it can be seen that the ratio of the average time spent in state γ to the average time spent in all states is equal to the probability of finding the system in state γ (at an arbitrary time when no information is given about the initial state, or after the motion has been going on for such a long time that the initial condition is unimportant). In equation form this is

$$\nu_\gamma = \frac{\frac{1}{\lambda_\gamma}}{\frac{1}{\lambda_\gamma} + \zeta_\gamma} = \frac{1}{1 + \zeta_\gamma \lambda_\gamma} \tag{111}$$

For a similar derivation of equation (111), see Feller [6], page 469.

The total time spent between entries to state $x_\gamma = \gamma$ is

$$T_\gamma = \left(\frac{1}{\lambda_\gamma} + \zeta_\gamma\right) = \frac{1}{\lambda_\gamma \nu_\gamma} \tag{112}$$

But, from equation (110a),

$$\nu_\gamma = \frac{\eta_\gamma}{\eta_N \lambda_\gamma}$$

so that

$$T_\gamma = \left(\frac{1}{\lambda_\gamma} + \zeta_\gamma\right) = \frac{\eta_N}{\eta_\gamma} \tag{113}$$

Substitute equation (110b) into equation (113)

$$T_\gamma = \left(\frac{1}{\lambda_\gamma} + \zeta_\gamma\right) = \frac{1}{\eta_\gamma}\left[\frac{1}{\lambda_0}\left(\eta_0 + \eta_\gamma\right) + \frac{1}{\lambda}\sum_{i=0}^{\gamma-1} \eta_i\right] \tag{114}$$

By definition, ζ_γ is the period between leaving and arriving at the boundary $x_\gamma = \gamma$. Because the boundary at $x_\gamma = \gamma$ is a reflecting boundary which sends the spacecraft to its nominal position (with probability one), ζ_γ is also the mean waiting time required for the spacecraft to reach the boundary $x_\gamma = \gamma$ starting from the nominal position. (It should be kept in mind that a possible path might include a reflection from the boundary $x_0 = 0$ before reaching $x_\gamma = \gamma$. This will be developed later.) As was mentioned in the beginning of this section, the cycle time should not be prolonged by the introduction of the fictitious reflecting boundary states at $x_0 = 0$ and $x_\gamma = \gamma$ with their associated dwell times $1/\lambda_0$. This problem can be circumvented by associating dwell periods of zero length with these boundary states. Therefore, let $\lambda_0 \to \infty$ in equation (114).

$$\lim_{\lambda_0 \to \infty} T_\gamma = \zeta_\gamma = E(W_\gamma) = \frac{1}{\lambda \eta_\gamma} \sum_{i=1}^{\gamma-1} \eta_i \tag{115}$$

where $E(W_\gamma)$ is the mean waiting time to reach the boundary at $x_\gamma = \gamma$.

Similarly, the mean waiting time to reach the boundary at $x_0 = 0$ is

$$E(W_0) = \frac{1}{\lambda \eta_\gamma} \sum_{i=1}^{\gamma-1} \eta_i \tag{116}$$

The mean waiting time to reach either boundary, $E(W)$, without specifying which, may be found from equations (115) and (116). In some given period of length τ, the upper boundary will be reached a mean number of times of

$$N_\gamma = \frac{\tau}{E(W_\gamma)}$$

times, and the lower boundary reached a mean of

$$N_0 = \frac{\tau}{E(W_0)}$$

times. The total mean number of times a boundary is reached during the period τ is

$$N_T = N_\gamma + N_0 = \frac{\tau}{E(W_\gamma)} + \frac{\tau}{E(W_0)}$$

from which

$$N_T = \lambda\tau(\eta_0 + \eta_\gamma)\left[\sum_{i=1}^{\gamma-1} \eta_i\right]^{-1}$$

But

$$N_T = \frac{\tau}{E(W)}$$

or

$$E(W) = \frac{\tau}{N_T}$$

Thus,

$$E(W) = \frac{1}{\lambda(\eta_0 + \eta_\gamma)} \sum_{i=1}^{\gamma-1} \eta_i \qquad (117)$$

Equations (108), (115), (116), and (117), together with a modified $\underline{\tilde{p}}$ matrix such as that shown in figure 8, constitute the solutions for the various waiting times.

XI. RESULTS AND DISCUSSION

A. Rotational Dynamics

The simple relationship between the internal movements and the angular displacement is given by equation (23) and is shown graphically in figure 2. In most practical cases the spacecraft mass and principal moments of inertia are so large that equation (23) should give excellent results. Typical values for the spacecraft parameters are given in table I. Equation (23) applies to small total angular deviations such that the direction cosines of the angular deviation remain near one. Large angular deviations would necessitate the use of Euler angles or matrix methods which, in turn, would lead to coupling between the motions about each of the axes. A strong correlation between the axes would considerably complicate the random walk analysis. Fortunately, a large number of situations of practical interest restrict the deviation to within $\pm 5^{\circ}$ from a nominal position; in such cases equation (23) is valid.

B. Stochastic Analysis

1. Comparison Between the Markov Chain and Iteration Methods.- Both the Markov chain and the iteration methods are Markov processes. The iteration method is called such herein only to distinguish between the two analyses.

The distinction between the two methods is the amount of computational effort required to determine the waiting time. Calculations were performed to obtain data for $b = 1.0625$, $a = 21.124$ with $\lambda = 2.0$.

The mean value, b/λ, and the variance, a/λ, were determined from the histogram of figure 3(b). The number of moves per second, λ, was arbitrarily chosen at two moves per second, which is a rather frantic practical rate. However, the equations and computed results are linearly dependent upon λ and can be read in terms of minutes as well as seconds. The choice of values was made for mathematical comparison of the results rather than to illustrate a practical example. The maximum square matrix size for the iteration method was limited by computer storage to 100×100 which, with the histogram of figure 3(b), corresponds to an actual case where the allowed maximum deviation is very small ("tight" boundaries). Because three matrices must be manipulated at a time in the Markov chain method, the maximum size was 60×60.

A numerical comparison between the Markov chain method and the iteration method showed exact agreement. Thus, from equations (100) and (117),

$$\frac{1}{\lambda} \sum_{n=0}^{\infty} (n + 1)\left(\tilde{\bar{p}}_{i0}^{(n+1)} - \tilde{\bar{p}}_{i0}^{(n)} + \tilde{\bar{p}}_{i\gamma}^{(n+1)} - \tilde{\bar{p}}_{i\gamma}^{(n)}\right)$$

$$= \frac{1}{\lambda} \frac{1}{(\eta_0 + \eta_\gamma)} \sum_{i=1}^{\gamma-1} \eta_i = E(W) \qquad (118)$$

The computer program took 826 seconds for the Markov chain method and less than 7 seconds for the iteration methods on a case where both matrices were 50×50. Less than 200 terms were necessary for the iteration method to converge to a relative error of 10^{-5} for all elements in a 50×50 matrix. About 300 terms are estimated for equivalent convergence in the Markov chain method; for the 50×50 matrix, the actual number of terms computed was 1,000. In both methods the number of iterations and series terms required for convergence increased with the matrix size. The computation time for the iteration method was always an order of magnitude less than the Markov chain method. A CDC 6600 series computer was used.

2. The diffusion equation.- The expected waiting time, as determined by the solution to the diffusion equation, is shown in nondimensional form in figure 4. The figure demonstrates the effect predicted by equation (87a); the waiting time essentially reduces to the deterministic solution for values of $bx_b/a \geq 10$. The figure also illustrates equation (88); the maximum mean waiting time occurs at smaller values of x_0/x_b as bx_b/a increases.

Figure 5 shows the variation of the nondimensional variance. The nondimensional variance reaches a maximum for $2.4 \leq bx_b/a \leq 5.3$ for $0.1 \leq x_0/x_b \leq 0.9$ or approximately at the value where the waiting time reduces to the deterministic form. Equation (87b) predicted that the variance would asymptotically approach zero as $bx_b/a \rightarrow \infty$.

3. Comparison between matrix analyses and the diffusion equation solution.- Either matrix method leads to the same result for the expected waiting time; the diffusion equation solution is compared to the iteration matrix method in figure 9. The matrix sizes used for the calculations ranged from 40×40 to 99×99, where the two fictitious boundary states were included in the matrix size. It should be remembered that

these boundary states do not contribute to the waiting time. The values of $b/\lambda = 0.53125$, $a/\lambda = 10.562$, and $\lambda = 2$ were used (see the preceding subsection for a discussion of these values). The value of x_b was obtained as follows: Let the square matrix size (including the boundary states) be $N \times N$. Denote the nominal state by MR. The following schematic was used for $x_0/x_b = 1/2$:

State	1	2	3	4	5	6	N
	0	x	x	$\otimes$	x	x	0

where 0 is a boundary state, x is a physical state, and $\otimes$ is the nominal position. From the schematic, it can be seen that

$$\begin{aligned} N &= (MR - 2) + MR + 1 \\ &= 2MR - 1 \end{aligned}$$

and, because x_b corresponds to the distance between the physical boundaries,

$$x_b = N - 2$$

Similarly, for $x_0/x_b = 1/3$,

State	1	2	3	4	5	N
	0	x	$\otimes$	x	x	0

which leads to

$$\begin{aligned} N &= 2(MR - 2) + MR + 1 \\ &= 3MR - 3 \end{aligned}$$

and

$$x_b = N - 2$$

The resulting values of $E(W)$ are shown in figure 9. The agreement between the iteration method and the diffusion equation is good for $x_0/x_b = 1/2$ and improving at the larger values of bx_b/a. The largest value of bx_b/a shown for x_0/x_b corresponds to a matrix 99×99, for which an average of 92 steps are required to reach a boundary. The difference is approximately 3 percent at this point and decreasing absolutely and relatively with increasing bx_b/a, as would be expected. As bx_b/a increases (with b and a fixed), the number of steps to reach a boundary increases, and the assumptions made in the diffusion equation derivation are more nearly approximated.

The case for $x_0/x_b = 1/3$ is also shown in figure 9, and the same general behavior between the curves is noted. However, the agreement is not quite as good as for $x_0/x_b = 1/2$, even though at the upper end the average number of steps required to reach a boundary is greater for $x_0/x_b = 1/3$. This may be due to the characteristics of the particular matrix used or to the method of determining x_b (see the previous schematics).

It is generally concluded that the diffusion equation will yield shorter waiting times than the matrix method, and that the matrix method solution will asymptotically approach the diffusion equation solution as the matrix size increases.

C. Choice of Method of Solution

The applicability of the diffusion equation was discussed in the first subsection of the chapter entitled "Solution of the Stochastic Equations." This discussion illustrated that the diffusion equation should be applicable in the majority of practical cases because the number of motions required to reach a boundary is large. The diffusion equation solution (eq. (85)) is also the simplest. Whenever the number of motions required to reach a boundary is small and the histogram of the motion is known, the iteration method would be expected to give more accurate results.

D. Experimental Determination of the Mean and Variance of the One-Step Motion

In some problems the complete histogram may not be known; only the mean value and variance may be available. For such problems the diffusion equation must be used. Other problems might arise in an attempt to infer the parameters of the histogram from measurement of the actual waiting times between control jet firings on a spacecraft. If the limits are known (x_b and x_0 are known) and the mean waiting time and the variance of the waiting time computed from the record of control jet firings, b and a may be determined from equations (85) and (86). In this manner useful information about internal motions may be obtained without the need for an extremely precise angular measurement apparatus.

XII. CONCLUDING REMARKS

The relationship between the internal motions of a spacecraft and the angular deviation of the spacecraft has been obtained. For manned spacecraft, where the mass is large, the principal moments of inertia large, and where the maximum allowable angular deviations are $\pm 5^{\circ}$ or less, the waiting time to reach the maximum allowable angular deviations is obtained by random walk methods. It has been shown that the rotations about each axis may be considered independently. Three random walk methods for obtaining the mean waiting time are derived.

The solution to the diffusion equation yields a particularly simple relationship when the number of movements is large. The two matrix methods are applicable when the number of movements required to reach a boundary is small.

XIII. APPENDIX

The values of the infinitesimal instantaneous mean and variance, b and a, are usually inferred from the physical aspects of the problem rather than by direct application of the definitions (eqs. (56) and (57)). Application of the definitions requires an a priori knowledge of the solution. Occasionally, only the nature of the variables a and b is needed; as will be seen shortly, a and b are constants in the present problem.

The values of a and b are most easily obtained by the use of generating functions, which are described in the Markov chain analysis. Feller [9] shows that, if $\mathcal{G}(y)$ is the generating function for Δx,

$$\lim_{y \to 1} \frac{d\mathcal{G}(y)}{dy} = E(\Delta x) \tag{A1}$$

and

$$\lim_{y \to 1} \left\{ \frac{d^2\mathcal{G}(y)}{dy^2} + \frac{d\mathcal{G}(y)}{dy} - \left[\frac{d\mathcal{G}(y)}{dy}\right]^2 \right\} = \text{Var } (\Delta x) \tag{A2}$$

These equations apply to general discrete random variables. The application of the process indicated by equation (A1) to equation (91) yields

$$\frac{dQ(1)}{dy} = e^{-\mu} \sum_{n=0}^{\infty} \frac{n\mu^n}{n!} \mathcal{G}^{(n-1)}(1)$$

$$= e^{-\mu} e^{\mu} \, \mu E(\Delta x)$$

$$= \mu E(\Delta x) = \lambda t E(\Delta x)$$

The quantity $\lambda t E(\Delta x)$ is the expected total change in position in time t. Therefore, the infinitesimal rate of change is

$$b = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\Delta t \lambda E(\Delta x)\right] = \lambda E(\Delta x)$$

which checks with equation (61).

Similarly, the application of the process indicated in equation (A2) to equation (91) will yield a value for a. The algebra is reduced by noting that

$$Q = e^{-\mu\left[1-\phi(y)\right]}$$

and that, because of the definition of $\phi(y)$ in equation (89) and the fact that the sum of the probabilities of all outcomes must be unity,

$$\phi(1) = 1$$

Then,

$$\frac{d^2Q}{dy^2} + \frac{dQ}{dy} - \left(\frac{dQ}{dy}\right)^2 = \left[\mu\frac{d^2\phi}{dy^2} + \mu^2\left(\frac{d\phi}{dy}\right)^2 + \mu\frac{d\phi}{dy} - \mu^2\left(\frac{d\phi}{dy}\right)^2\right]e^{-\mu\left[1-\phi(y)\right]}$$

or, substituting $\lambda\,\Delta t$ for μ, the variance in x in time Δt is

$$\mathrm{Var}(x) = \lim_{y \to 1}\left[\lambda\ \Delta t\frac{d^2\phi}{dy^2} + \lambda\ \Delta t\frac{d\phi}{dy}\right]e^{-\mu\left[1-\phi(y)\right]}$$

$$= \lim_{y \to 1} \lambda\,\Delta t\left[\frac{d^2\phi}{dy^2} + \frac{d\phi}{dy}\right] \qquad \text{(A3)}$$

Combine equations (A2) and (A3) to get

$$a = \lim_{\Delta t \to 0} \frac{1}{\Delta t} (\lambda \Delta t) \left[\mathrm{Var}(\Delta x) + \lambda \Delta t E^2(\Delta x) \right]$$

$$= \lambda \mathrm{Var}(\Delta x) \tag{A4}$$

Equation (A4) corresponds to equation (62).

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XV. VITA

The author was born in Derby, Connecticut, on June 17, 1929. He attended the Ansonia, Connecticut, public school through the seventh grade and was graduated cum laude from Hopkins Grammar School, New Haven, Connecticut, in June 1947. He entered Brown University in the fall of 1947 and was subsequently graduated with a degree of Bachelor of Science in Mechanical Engineering in June 1951. The author accepted a position as mechanical engineer with the National Advisory Committee for Aeronautics in 1951. He attended the Virginia Polytechnic Institute during the summers of 1955 and 1956 and was awarded the degree of Master of Science in Applied Mechanics in 1959. He is an associate member of Sigma XI. He is presently employed as an aero-space engineer with the National Aeronautics and Space Administration and is a registered professional engineer in the State of Virginia. He is married and has three children.

John R. Davidson

TABLE I.- TYPICAL VALUES FOR THE DYNAMIC PROPERTIES

OF APOLLO SPACECRAFT*

Configuration	Mass, kg	Inertia, $kg\text{-}m^2$		
		x_1	x_2	x_3
1	11,300	18,800	63,500	64,800
2	31,200	15,550	2,660,000	2,620,000
3	39,800	22,300	2,955,000	2,690,000

*From Tewell and Murrich [2].

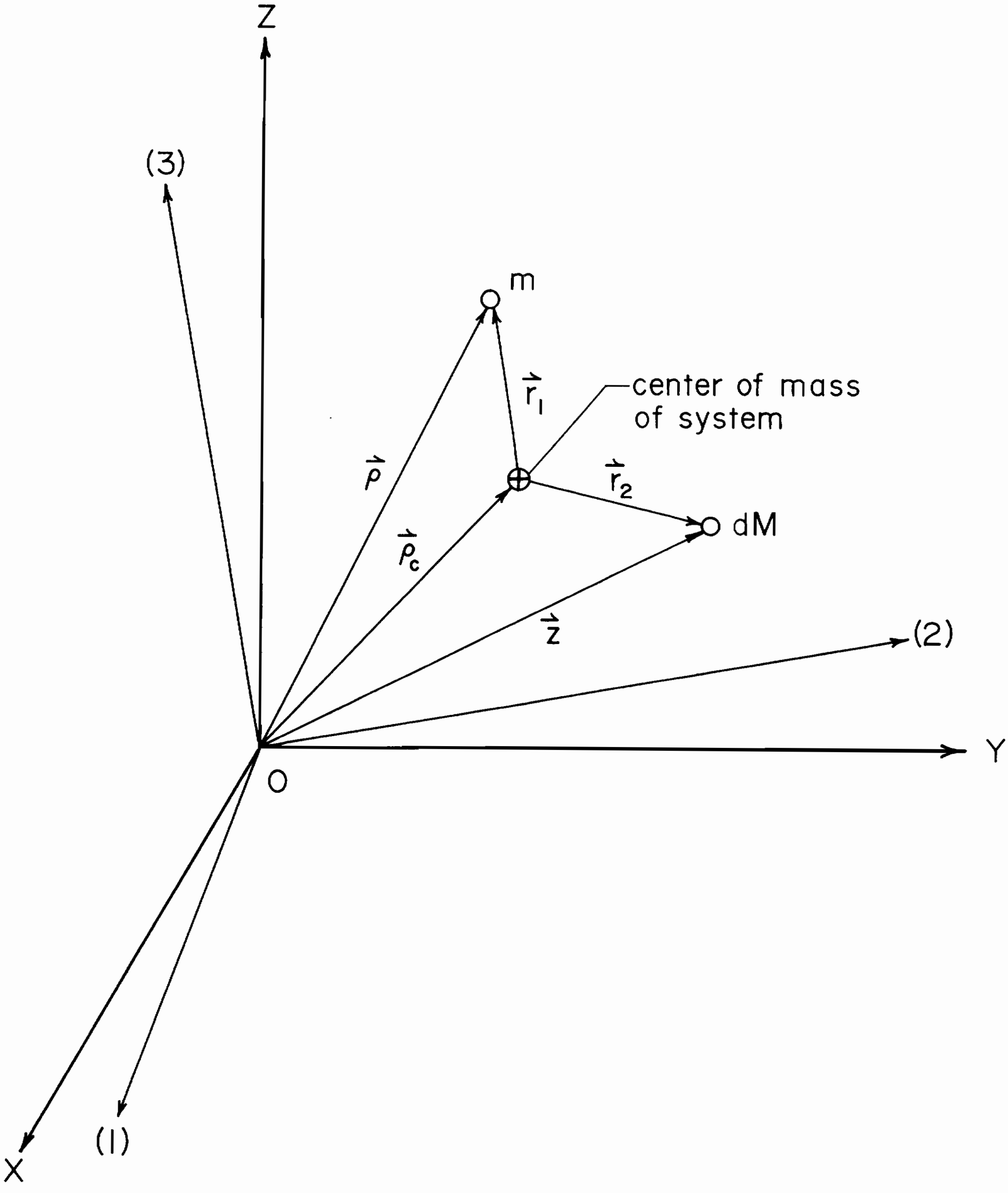


Figure 1.- Sketch of the coordinate systems. Axes X, Y, and Z are inertial. Axes (1), (2), and (3) are along the principal axes of inertia of the spacecraft, and the origins are located at the center of mass of the spacecraft.

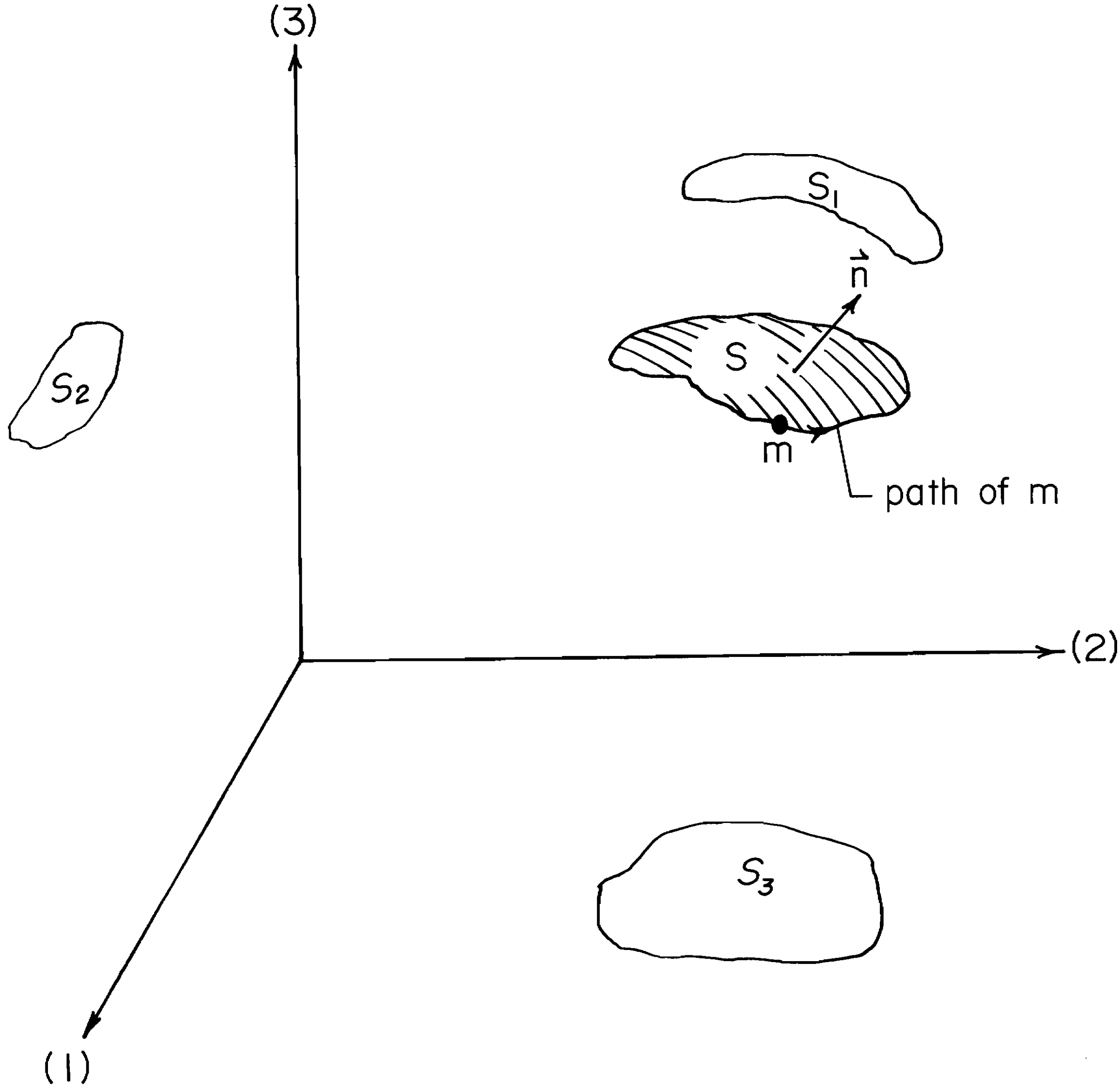
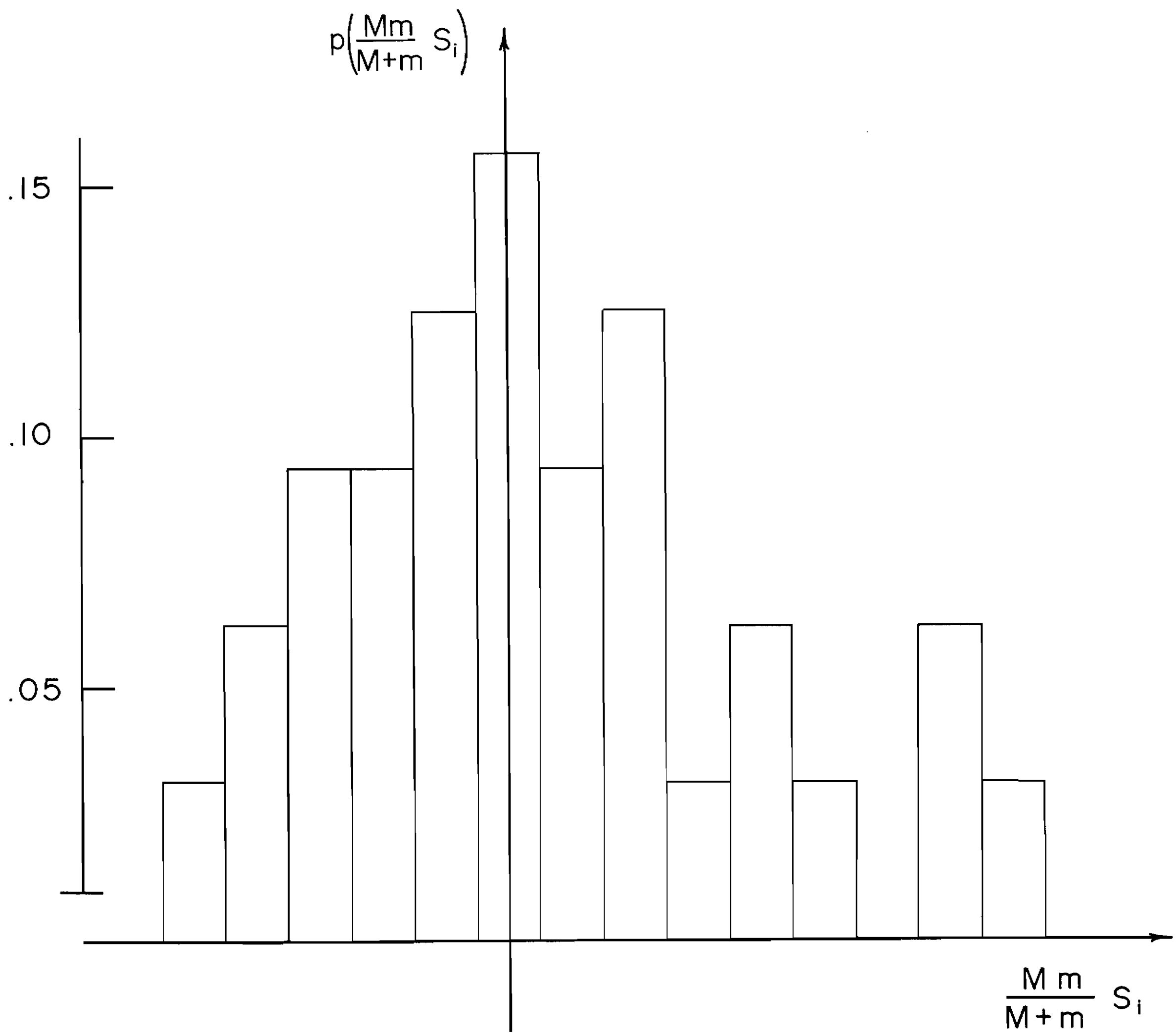
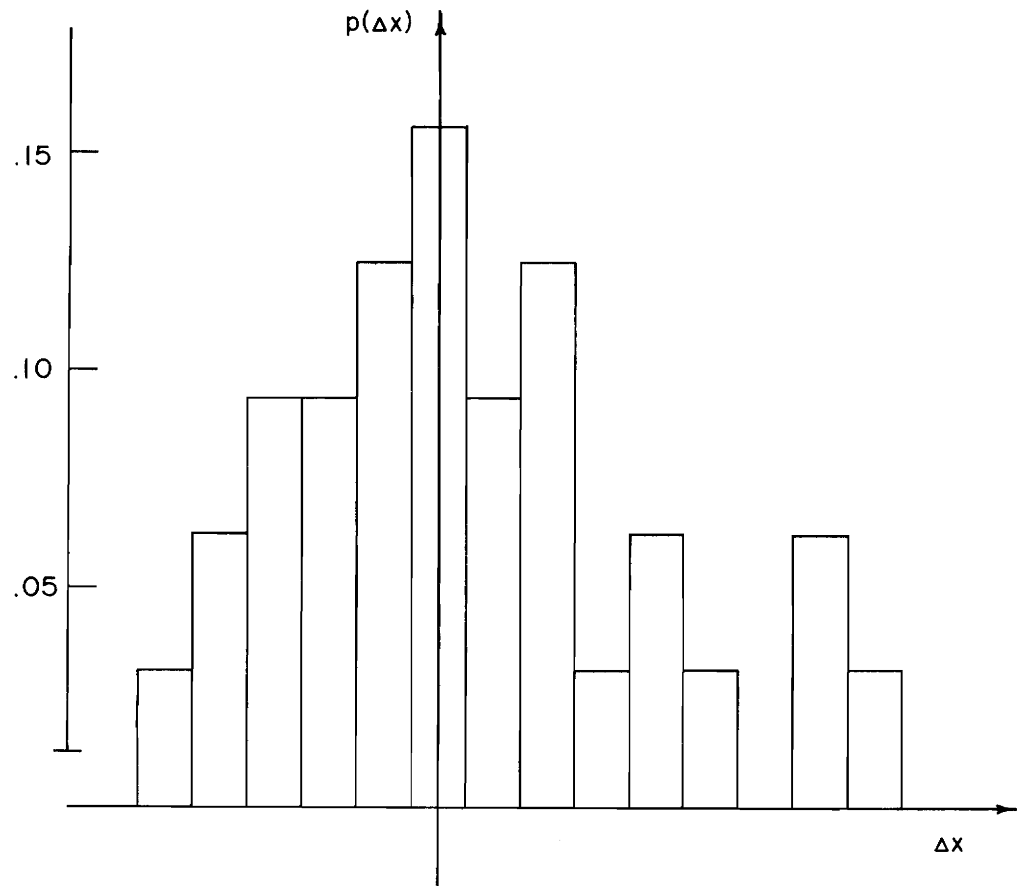


Figure 2.- The relation between internal mass movement and the areas on the coordinate planes. The curve described by the mass m encloses the area S. The projections of this area upon the coordinate planes are proportional to the angular change about correspondingly numbered axes.



(a) Histogram showing the probability that $\left(\frac{Mm}{M + m} S_i\right)$ will have a given magnitude. The heights of the bars sum to one. One such histogram would be determined for each of the three axes.

Figure 3.- Histogram of the motion.



(b) Histogram showing the probability that Δx will have a given magnitude. The abscissa for the histogram for Δx is linearly related to the abscissa for the histogram of $\left(\frac{Mm}{M + m} S_i\right)$. The ordinates are identical. There is one such histogram for Δx for each of the three axes.

Figure 3.- Concluded.

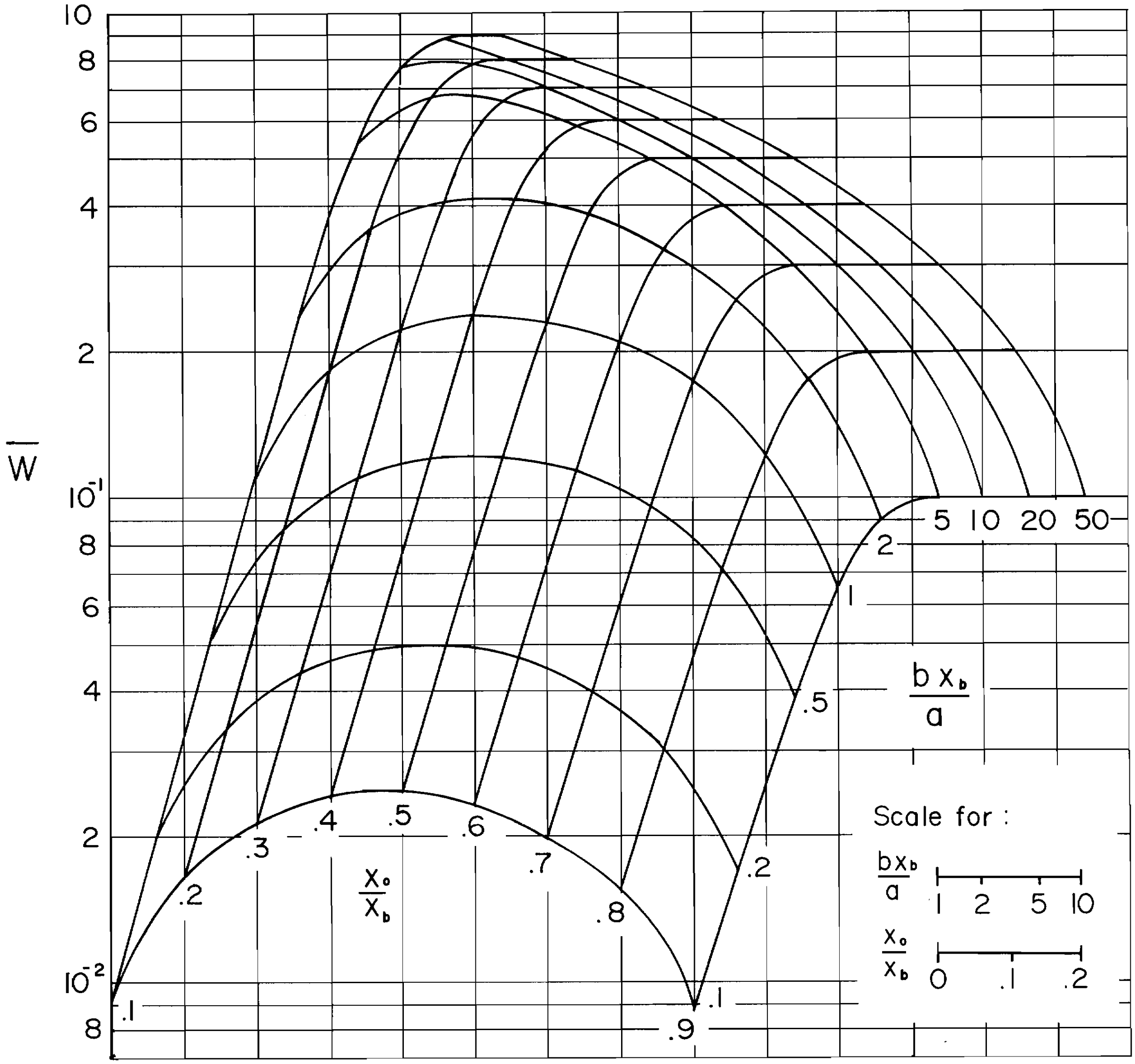


Figure 4.- Plot of the nondimensional mean waiting time as a function of the nondimensional starting position and the nondimensional distance between the boundaries. The data are from the solution to the diffusion equation, which is given in nondimensional form in equation (85) in the text. The lines are marked with the values of the parameter which is held constant.

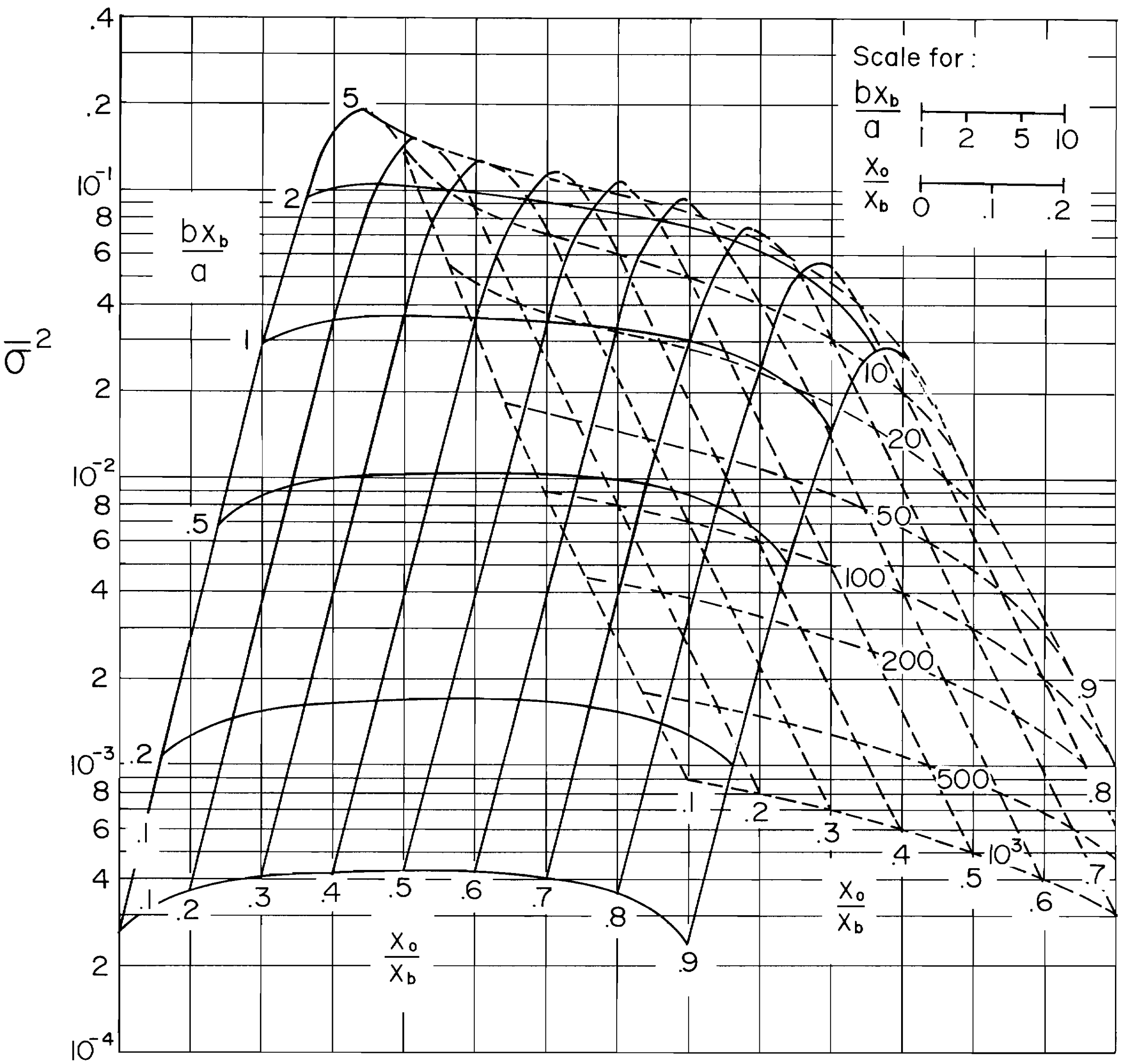


Figure 5.- Plot of the nondimensional variance for the waiting time as a function of the nondimensional starting position and distance between the boundaries. The data are from the solution to the diffusion equation, which is given in nondimensional form in equation (86) in the text. The lines are marked with the values of the parameter which is held constant.

$$
\frac{1}{32}\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 2 & 3 & 3 & 4 & 5 & 3 & 4 & 1 & 2 & 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 0 & 1 & 2 & 3 & 3 & 4 & 5 & 3 & 4 & 1 & 2 & 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 0 & 0 & 1 & 2 & 3 & 3 & 4 & 5 & 3 & 4 & 1 & 2 & 1 & 0 & 2 & 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 0 & 0 & 0 & 1 & 2 & 3 & 3 & 4 & 5 & 3 & 4 & 1 & 2 & 1 & 0 & 2 & 1 & 0 & 0 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 3 & 4 & 5 & 3 & 4 & 1 & 2 & 1 & 0 & 2 & 1 & 0 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 3 & 4 & 5 & 3 & 4 & 1 & 2 & 1 & 0 & 2 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}
$$

Figure 6.- Matrix $\tilde{p}$. The matrix is doubly infinite. The matrix was constructed from the histogram of figure 3(b) and should be interpreted such that $p_{NN} = \frac{5}{32}$.

$$
\frac{1}{32}\left(\begin{array}{ccccccccccccccccccccc}
32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hdashline
13 & 5 & 3 & 4 & 1 & 2 & 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
9 & 4 & 5 & 3 & 4 & 1 & 2 & 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 3 & 4 & 5 & 3 & 4 & 1 & 2 & 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & 4 & 5 & 3 & 4 & 1 & 2 & 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 3 & 3 & 4 & 5 & 3 & 4 & 1 & 2 & 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 3 & 3 & 4 & 5 & 3 & 4 & 1 & 2 & 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 3 & 3 & 4 & 5 & 3 & 4 & 1 & 2 & 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 2 & 3 & 3 & 4 & 5 & 3 & 4 & 1 & 2 & 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 3 & 3 & 4 & 5 & 3 & 4 & 1 & 2 & 1 & 0 & 2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 3 & 4 & 5 & 3 & 4 & 1 & 2 & 1 & 0 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 3 & 4 & 5 & 3 & 4 & 1 & 2 & 1 & 0 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 3 & 4 & 5 & 3 & 4 & 1 & 2 & 1 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 3 & 4 & 5 & 3 & 4 & 1 & 2 & 1 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 3 & 4 & 5 & 3 & 4 & 1 & 2 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 3 & 4 & 5 & 3 & 4 & 1 & 2 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 3 & 4 & 5 & 3 & 4 & 1 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 3 & 4 & 5 & 3 & 4 & 7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 3 & 4 & 5 & 3 & 11 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 3 & 4 & 5 & 14 \\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32
\end{array}\right)
\quad
\begin{array}{c}
\text{State} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \\ 18 \\ 19 \\ \gamma
\end{array}
$$

Figure 7.- Modified matrix $\tilde{\tilde{p}}$. The dashed lines correspond with the physical limiting deviations and represent the physical boundaries. States 0 and γ are the added fictitious absorbing boundary states. The one-step transition probabilities were obtained from the histogram of figure 3(b). Note that $\tilde{\tilde{p}}_{NN}$ is the probability that, although a motion occurs, the step size will be zero.

$$
\frac{1}{32}\left(\begin{array}{ccccccccccccccccccccc}
0&0&0&0&0&0&0&0&32&0&0&0&0&0&0&0&0&0&0&0&0\\ \hline
13&5&3&4&1&2&1&0&2&1&0&0&0&0&0&0&0&0&0&0&0\\
9&4&5&3&4&1&2&1&0&2&1&0&0&0&0&0&0&0&0&0&0\\
6&3&4&5&3&4&1&2&1&0&2&1&0&0&0&0&0&0&0&0&0\\
3&3&3&4&5&3&4&1&2&1&0&2&1&0&0&0&0&0&0&0&0\\
1&2&3&3&4&5&3&4&1&2&1&0&2&1&0&0&0&0&0&0&0\\
0&1&2&3&3&4&5&3&4&1&2&1&0&2&1&0&0&0&0&0&0\\
0&0&1&2&3&3&4&5&3&4&1&2&1&0&2&1&0&0&0&0&0\\
0&0&0&1&2&3&3&4&5&3&4&1&2&1&0&2&1&0&0&0&0\\
0&0&0&0&1&2&3&3&4&5&3&4&1&2&1&0&2&1&0&0&0\\
0&0&0&0&0&1&2&3&3&4&5&3&4&1&2&1&0&2&1&0&0\\
0&0&0&0&0&0&1&2&3&3&4&5&3&4&1&2&1&0&2&1&0\\
0&0&0&0&0&0&0&1&2&3&3&4&5&3&4&1&2&1&0&2&1\\
0&0&0&0&0&0&0&0&1&2&3&3&4&5&3&4&1&2&1&0&3\\
0&0&0&0&0&0&0&0&0&1&2&3&3&4&5&3&4&1&2&1&3\\
0&0&0&0&0&0&0&0&0&0&1&2&3&3&4&5&3&4&1&2&4\\
0&0&0&0&0&0&0&0&0&0&0&1&2&3&3&4&5&3&4&1&6\\
0&0&0&0&0&0&0&0&0&0&0&0&1&2&3&3&4&5&3&4&7\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&2&3&3&4&5&3&11\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&2&3&3&4&5&14\\ \hline
0&0&0&0&0&0&0&0&32&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right)
\quad
\begin{array}{ll}
\underline{\text{State}} & \\
0 & \\ 1 & \\ 2 & \\ 3 & \\ 4 & \\ 5 & \\ 6 & \\ 7 & \\ 8 & \text{Nominal} \\ 9 & \\ 10 & \\ 11 & \\ 12 & \\ 13 & \\ 14 & \\ 15 & \\ 16 & \\ 17 & \\ 18 & \\ 19 & \\ \gamma &
\end{array}
$$

Figure 8.- The modified matrix $\underline{\tilde{p}}$. The dashed lines correspond to the physical boundaries. The fictitious boundary states at 0 and γ are reflecting boundaries which reflect the spacecraft to the nominal position (chosen here as state 8) with probability one.

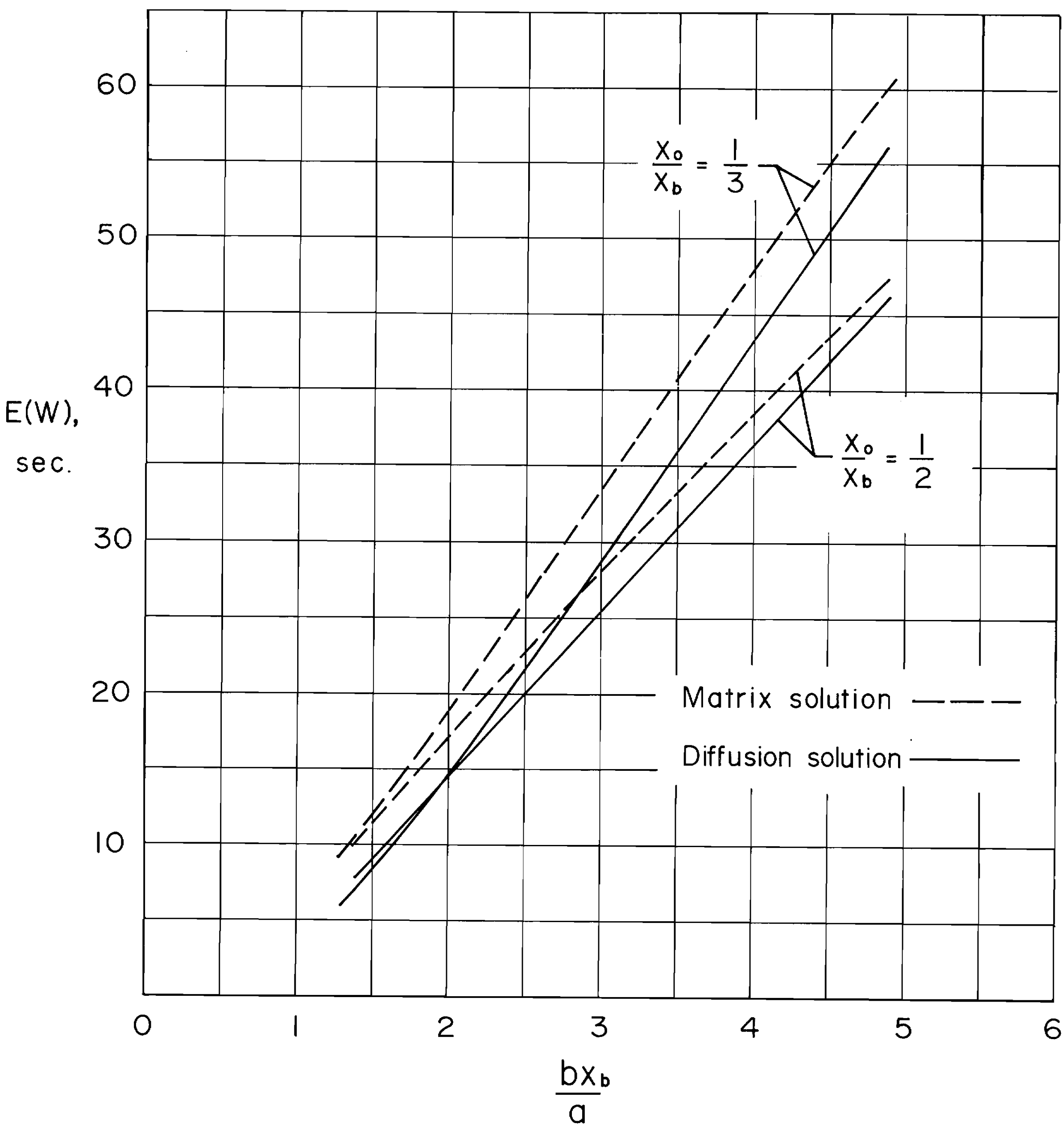


Figure 9.- Comparison of the expected waiting times for the matrix solutions and the diffusion equation solution. For this comparison $b = 1.0625$, $a = 21.124$, which were obtained from figure 3. Also, λ was assumed to be two movements per second for this calculation.

THE EFFECT OF RANDOM INTERNAL MOTIONS ON THE ANGULAR ORIENTATION OF A FREE BODY WITH LIMIT CONTROL

By

John R. Davidson

ABSTRACT

Crew motions can affect the angular orientation of a spacecraft. Such motions are of a somewhat random nature and must be treated stochastically. The equations of motion for the spacecraft are solved to obtain a simple relationship between individual crew motions and the angular change in spacecraft orientation. For the majority of manned spacecraft, the angular changes about the three principal axes of inertia can be considered independent. The additive effect of many motions, considered random in frequency and magnitude, is found by treating the motion as a form of random walk. Solutions are obtained for the time required to reach predetermined maximum allowable angular deviation limits. If reaction control jets are used for stabilization, this time determines the frequency of firing and the fuel consumption rate. For relatively heavy spacecraft, such as those designed to carry three or more crew members, the complete solution to the diffusion equation gives the time between control jet firing; this solution has a particularly simple form. For smaller spacecraft, where individual motions may cause a change in orientation which is several percent of the maximum allowable, additional solutions have been obtained using Markov chain matrix analysis; numerical values for the matrix equations can be obtained from a digital computer.