LOCAL PROPERTIES OF TRANSITIVE QUASI-UNIFORM SPACES

by

Massood Seyedin

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APPROVED:

Peter Fletcher

Dr. Peter Fletcher, Chairman

C. W. Patty

Dr. C. W. Patty

B. E. Reed

Dr. B. E. Reed

C. E. Aull

Dr. C. E. Aull

James Washenberger

Dr. J. K. Washenberger

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Blacksburg, Virginia

To those everywhere committed to the enhancement of human dignity.

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INTRODUCTION AND NOTATIONS

The concept of a quasi-uniformity on a set X was first introduced in 1950 by L. Nachbin in [27] and fifteen years later W. J. Pervin showed that each topological space has a compatible quasi-uniformity, called the Pervin quasi-uniformity [30]. The collection of all quasi-uniformities compatible with a given topological space may be divided into quasi-proximity classes. Each quasi-proximity class has exactly one totally bounded member and this totally bounded member is contained in every other member of the given quasi-proximity class. Thus with each totally bounded quasi-uniformity that is compatible with a given topological space there is a uniquely determined quasi-proximity class; in particular the Pervin quasi-uniformity is totally bounded and the corresponding Pervin quasi-proximity class contains many of the most studied quasi-uniformities.

In Chapter II we give three characterizations of those covering quasi-uniformities that are members of the Pervin quasi-proximity class. These characterizations are of use throughout, but one of them is of particular importance in establishing the main result of Chapter II, that if (X, τ) is a locally compact Hausdorff space and $\mathcal{U}$ is a compatible covering quasi-uniformity that is a member of the Pervin quasi-proximity class, then $(X, \mathcal{U})$ has a Hausdorff completion. In [33], R. Stoltenberg proved that every quasi-uniform space has a completion; however J. Carlson and T. Hicks [4] have given an example of a compatible quasi-uniformity $\mathcal{U}$ for a locally compact metric space (X, τ) such that $(X, \mathcal{U})$ does not have a Hausdorff completion. In that example, it is not difficult to show that $\mathcal{U}$ is a transitive quasi-uniformity. It follows from our results that $\mathcal{U}$ cannot be a member of the Pervin quasi-proximity class and that fact may also be verified directly.

In Chapter III, we investigate some of the local properties of quasi-uniform spaces. Local completeness, local precompactness, and local total boundedness are introduced. We show that a topological space (X, τ) is locally (countably) compact if and only if the fine transitive (upper semi-continuous) quasi-uniformity for (X, τ) is locally precompact. We note that if (X, τ) is a topological space and $\mathcal{U}$ is a member of the Pervin quasi-proximity class, then $\mathcal{U}$ is (weakly) locally symmetric if and only if (X, τ) is (R_0) regular. We also give an example of a completely normal quasi-uniform space that is not weakly locally symmetric. Product theorems for locally complete and locally precompact quasi-uniform spaces are obtained.

Chapter IV is concerned with function spaces of quasi-uniform spaces, and local symmetry plays an important role throughout this study. We extend the concept of equicontinuity and uniform convergence to function spaces of quasi-uniform spaces and obtain several generalizations of classical results in uniform spaces. For example, we extend the classical theorem that uniform convergence of a sequence of continuous functions implies the continuity of the limit function. In this chapter we also discuss an analogue of the Banach fixed-point theorem for the quasi-uniform spaces.

A well-known sufficient condition that a group of homeomorphisms, G, from a topological space X onto itself be a topological group relative to the topology of pointwise convergence is that X be uniformizable and G be equicontinuous. In Chapter V we show that if $\mathcal{U}$ is a weakly locally symmetric quasi-uniformity compatible with an R_0 topological space X and G is a subgroup of H(X) that is quasi-equicontinuous with respect to $\mathcal{U}$, then G is a topological group under the topology of pointwise convergence. In [16] R. Fuller defines a semi-uniformity and shows that every group of homeomorphisms which satisfies a condition of equicontinuity relative to a semi-uniformity (called semi-equicontinuity) is a topological group under the topology of pointwise convergence. We show that a semi-uniformity can be generated from a locally symmetric quasi-uniformity in such a way that if F is a quasi-uniformly equicontinuous family of functions, then F is also semi-equicontinuous relative to the semi-uniformity generated from $\mathcal{U}$. From this result it follows that if G is a group of homeomorphisms of a space (X, τ) and G is quasi-equicontinuous relative to a locally symmetric quasi-uniformity $\mathcal{U}$ such that $\tau \subset \tau_{\mathcal{U}}$, then G is a topological group under the topology of pointwise convergence with respect to τ. The main result of this chapter, Theorem 5.4, generalizes the classical result that an equicontinuous group of homeomorphisms is a topological group under the point-open topology in that Theorem 5.4 is applicable to R_0 spaces whereas the classical result only obtains in uniformizable (i.e. completely regular) spaces.

In the appendix we give an example of an R_0 space that is not regular whose homeomorphism group under the point-open topology contains, for each positive prime p, an isomorphic copy of $\Sigma_{i=1}^{\infty} Z_{p_i}$. Indeed these subgroups may be considered as topological vector spaces over Z_p. This example shows that the homeomorphism group of an R_0 space that is not regular may possess interesting quasi-equicontinuous subgroups. Our last example shows that the homeomorphism group of a compact metrizable space may have a subgroup that is equicontinuous relative to a locally symmetric quasi-uniformity that is not a uniformity.

NOTATIONS

Δ	Let X be a set. Then $\Delta = \{(x,x): x \in X\}$.
A_x^C	Let C be an open cover of a topological space (X,τ). Then $A_x^C = \cap\{C \in C: x \in C\}$.
U_C	Let C be an open cover of a topological space (X,τ). Then $U_C = \cup\{\{x\} \times A_x^C \mid x \in X\}$.
$A \circ B$	Let X be a set, and A,B subsets of $X \times X$. Then $A \circ B = \{(x,y)$: there exists $z \in X$ such that $(x,z) \in B$ and $(z,y) \in A$.
V^n	Let X be a set, and $V \subset X \times X$. Then $V^n = V^{n-1} \circ V$.
$\mathcal{P}(X)$	Let X be a set. Then $\mathcal{P}(X)$ in the collection of all subsets of X.
$\mathcal{P}$	Let (X,τ) be a topological space. Then $\mathcal{P}$ is the Pervin quasi-uniformity on X.
$H(X)$	The homeomorphism group of a topological space (X,τ).
$\mathcal{U} \vee \mathcal{V}$	Let $\mathcal{U}$ and $\mathcal{V}$ be quasi-uniformities or a set X. Then $\mathcal{U} \vee \mathcal{V}$ is the smallest quasi-uniformity that contains both $\mathcal{U}$ and $\mathcal{V}$.

CHAPTER I

BACKGROUND AND PRELIMINARIES

Definition 1.1: A quasi-uniformity on a set X is a filter $\mathcal{U}$ on $X \times X$ such that

i) each member of $\mathcal{U}$ contains the diagonal Δ of $X \times X$,

ii) for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V \circ V \subset U$.

Theorem 1.2 [26, p. 10, Theorem 1.10]: If $\mathcal{S} \subset 2^{X \times X}$ satisfies

i) $\Delta \subset S$, for each $S \in \mathcal{S}$

ii) for each $S \in \mathcal{S}$, there exists a $T \in \mathcal{S}$ with $T \circ T \subset S$,

then $\mathcal{S}$ is a subbase for a quasi-uniformity.

Theorem 1.3 [26, p. 7, Theorem 1.3]: Let β be a family of subsets of $X \times X$ such that

i) $\Delta \subset B$, for each $B \in \beta$

ii) for $B_1, B_2 \in \beta$, there exists $B_3 \in \beta$ such that $B_3 \subset B_1 \cap B_2$;

iii) for each $B \in \beta$, there exists an $A \in \beta$ such that $A \circ A \subset B$.

Then there exists a unique quasi-uniformity $\mathcal{U}$ on X for which β is a basis. $\mathcal{U}$ is said to be generated by β, and may be defined as the family

$$\{U \mid U \supset B \text{ for some } B \in \beta\}.$$

Definition 1.4: If $\mathcal{U}$ is a quasi-uniformity on X, then $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$ is also a quasi-uniformity on X and is called the conjugate quasi-uniformity on X.

Definition 1.5: Let X be a set; let $\mathcal{U}$ be a quasi-uniformity on X. Then $\tau_{\mathcal{U}} = \{A \subset X$: for each $a \in A$, there exists $U \in \mathcal{U}$ such that $U(a) \subset A\}$ is called quasi-uniform topology for X induced by $\mathcal{U}$. In fact $\tau_{\mathcal{U}}$ is a topology for X.

Definition 1.6: Let (X, τ) be a topological space; let $\mathcal{U}$ be a quasi-uniformity on X. Then $\mathcal{U}$ is a compatible quasi-uniformity provided that $\tau = \tau_{\mathcal{U}}$.

Theorem 1.7 [13, Lemma 1]: Let $\mathcal{U}$ be a quasi-uniformity for a set X. If $x \in X$ and $U \in \mathcal{U}$, then x is a member of $\tau_{\mathcal{U}}$- interior of U(x).

Theorem 1.8 [30, Theorem 1]: Let (X, τ) be a topological space. For each $A \in \tau$, let $S_A = (A \times A) \cup (X - A) \times X$ and $\mathcal{S} = \{S_A : A \in \tau\}$. Then $\mathcal{S}$ is a subbase for a compatible quasi-uniformity on X.

The quasi-uniformity defined in Theorem 1.8 is called Pervin quasi-uniformity and is denoted by $\mathcal{P}$.

The following discussion of construction of quasi-uniformities appears in [14].

Covering quasi-uniformities. A Q-cover of a topological space (X, τ) is an open cover $\mathcal{C}$ of X such that if $x \in X$, then $A_x^{\mathcal{C}} \in \tau$ [32].

Let α be a collection of Q-covers of a topological space (X, τ) such that if $x \in A \in \tau$, then there is $C \in \alpha$ such that $A_x^C \subset A$. For each $C \in \alpha$, let $U_C = \cup \{\{x\} \times A_x^C : x \in X\}$ and let $S = \{U_C : C \in \alpha\}$. Then S is a transitive subbase for a compatible quasi-uniformity $\mathcal{U}_\alpha$ for (X, τ) [11, Theorem 1]. The quasi-uniformity $\mathcal{U}_\alpha$ is called the <u>covering quasi-uniformity for</u> (X, τ) <u>with respect to</u> α. If α is, respectively, the collection of all (Q-covers, point finite open covers, locally finite open covers, finite open covers) then $\mathcal{U}_\alpha$ is the <u>fine transitive</u>, <u>point finite covering</u>, <u>locally finite covering</u>, <u>Pervin quasi-uniformity for</u> (X, τ). Since every transitive quasi-uniformity is a covering quasi-uniformity [11, Theorem 2], the fine transitive quasi-uniformity is, as the name suggests, the finest compatible transitive quasi-uniformity for a given topological space (X, τ). It is shown in [37] that Pervin quasi-uniformity is the same as the quasi-uniformity first defined by A. Császár in [5]. Moreover the quasi-uniformity defined by R. Nielsen and C. Sloyer in [29] is also the Pervin quasi-uniformity.

<u>Upper semi-continuous quasi-uniformities</u>. Let (X, τ) be a topological space and let F be a collection of upper semi-continuous functions which contains the collection of all characteristic functions on closed sets. For each $f \in F$ and each $\epsilon > 0$, let $U_{(f,\epsilon)} = \{(x,y) \in X \times X : f(y)-f(x) < \epsilon\}$.

Then $\mathfrak{p} = \{U_{(f,\varepsilon)} : f \in F,\ \varepsilon > 0\}$ is a subbase for a compatible quasi-uniformity $\mathcal{U}_F$ for (X,τ) [2, Theorem 1]. If U is the collection of all upper semi-continuous functions, then $\mathcal{U}_U$ is called the <u>upper semi-continuous quasi-uniformity</u> and is denoted by $\mathcal{U}SC$. It is clear that in general the subbase $\mathfrak{p}$ for $\mathcal{U}SC$ need not be a transitive subbase. Nevertheless it is known that $\mathcal{U}SC$ is a covering quasi-uniformity [14, Theorem 2.1].

<u>Definition 1.9</u> [31]: By a <u>quasi-proximity on a set</u> E we will mean a relation δ between the family of subsets of E satisfying the following axioms:

[P. 1] $(A,\phi) \notin \delta$ and $(\phi,A) \notin \delta$ for each $A \subset E$.

[P. 2] $(\{x\},\{x\}) \in \delta$ for every $x \in E$.

[P. 3] $(C, A \cup B) \in \delta$ if and only if $(C,A) \in \delta$ or $(C,B) \in \delta$ and $(A \cup B, C) \in \delta$ if and only if $(A,C) \in \delta$ or $(B,C) \in \delta$.

[P. 4] if $(A,B) \notin \delta$, then there exist two disjoint sets U and V such that $(A, E - U) \notin \delta$ and $(E - V, B) \notin \delta$.

In the following, $U_{(A,B)}$ will be used to denote the set $X \times X - (A \times B)$.

A process is now introduced which produces a quasi-proximity on X for a given quasi-uniformity on X. Let $\mathcal{U}$ be a quasi-uniformity on X and define a relation on $\mathcal{P}(X)$ by: $(A,B) \in \delta$ if and only if $U_{(A,B)} \notin \mathcal{U}$. The relation δ is denoted by $\delta_{\mathcal{U}}$.

Theorem 1.10 [25, p. 22, Theorem 9]: If $\mathcal{U}$ is a quasi-uniformity, then $\delta_{\mathcal{U}}$ is a quasi-proximity.

The quasi-proximity topology induced by $\delta_{\mathcal{U}}$ will be denoted by $\tau_{\delta_{\mathcal{U}}}$.

Theorem 1.11 [25, p. 22, Theorem 10]: Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $\tau_{\mathcal{U}} = \tau_{\delta_{\mathcal{U}}}$.

As in the theory of proximity spaces, in general for a given quasi-proximity δ, there may be many quasi-uniformities $\mathcal{U}$ such that $\delta_{\mathcal{U}} = \delta$. The collection of all such $\mathcal{U}$'s is called the quasi-proximity class of δ and is denoted by $\Sigma(\delta)$. Let (X,τ) be a topological space. Then $\delta_{\mathcal{P}}$ will denote the quasi-proximity induced by the Pervin quasi-uniformity and $\Sigma(\delta_{\mathcal{P}})$ will be called the Pervin quasi-proximity class.

Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $Q_B(\mathcal{U})$ will denote the set of all bounded quasi-uniformly upper semi-continuous functions from $(X,\mathcal{U})$.

Theorem 1.12 [25, p.27, Corollary 3]: Each quasi-proximity class contains a unique totally bounded member, and it is the coarsest element in the given quasi-proximity class.

Let $\mathcal{U}^*$ denote the totally bounded quasi-uniformity of $\Sigma(\delta_{\mathcal{U}})$.

Theorem 1.13 [20, Corollary 2]: Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $\{U_{(\epsilon,f)} : \epsilon > 0 \text{ and } f \in Q_B(\mathcal{U})\}$ is a subbase for $\mathcal{U}^*$.

Theorem 1.14 [20, Corollary 1]: If $\mathcal{U}_1$ and $\mathcal{U}_2$ are quasi-uniformities on X, then $\mathcal{U}_1$ and $\mathcal{U}_2$ are in the same quasi-proximity class if and only if $Q_B(\mathcal{U}_1) = Q_B(\mathcal{U}_2)$.

Definition 1.15 [7]: A topological space X is R_o (also called essentially T_1) provided that for $x,y \in X$ either $\{\overline{x}\} = \{\overline{y}\}$ or $\{\overline{x}\} \cap \{\overline{y}\} = \phi$.

Definition 1.16: A base for a quasi-uniformity $\mathcal{U}$ on a set X is transitive provided that for each $B \in \mathcal{B}$, $B \circ B = B$. A quasi-uniformity with a transitive base is called a transitive quasi-uniformity.

Definition 1.17: Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be quasi-uniform spaces, and suppose that $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$. Then f is $(U - \mathcal{V})$ quasi-uniformly continuous if for each $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that if $(x,y) \in U$, then $(f(x),f(y)) \in V$.

CHAPTER II

COMPLETIONS OF MEMBERS OF THE PERVIN QUASI-PROXIMITY CLASS

Although R. Stoltenberg [33] has shown that every quasi-uniform space has a completion, J. Carlson and T. Hicks [3] have given an example of a compatible quasi-uniformity $\mathcal{U}$ for a locally compact Hausdorff space (X,τ) such that $(X,\mathcal{U})$ does not have a Hausdorff completion. In that example, it is not difficult to show that $\mathcal{U}$ is a transitive quasi-uniformity and hence a covering quasi-uniformity.

In this chapter we characterize the quasi-uniformities which are members of $\Sigma(\delta_{\mathcal{P}})$ and use this characterization to show that if (X,τ) is a locally compact Hausdorff space and $\mathcal{U} \in \Sigma(\delta_{\mathcal{P}})$, then $(X,\mathcal{U})$ has a Hausdorff completion. In particular if $\mathcal{U}$ is the Pervin (point finite, locally finite, fine transitive) quasi-uniformity of a locally compact Hausforff space (X,τ) and $\mathcal{U}^*$ is the Pervin (point finite, locally finite, fine transitive) quasi-uniformity of a Hausdorff compactification (X^*,τ^*) of (X,τ), then $(X^*,\mathcal{U}^*)$ is a completion of $(X,\mathcal{U})$.

<u>Definition 2.1</u> [12]: Let (X,τ) be a topological space and let $A \in \tau$. A <u>fundamental cover</u> of X about A is an open cover C of X such that $A \in C$ and if $B \in C$ such that $A \cap B \neq \emptyset$ then $A \subset B$.

<u>Definition 2.2</u> [12]: Let (X, τ) be a topological space and let α be a collection of Q-covers of X such that for each $A \in \tau$, α contains a fundamental cover of X about A. Then $\mathcal{U}_\alpha$ is a <u>fundamental covering quasi-uniformity</u> for (X, τ).

<u>Definition 2.3</u>: A collection β of Q-covers of a topological space (X, τ) is said <u>to have property</u> S if for any $B \in \beta$ and any $G \in \tau$, the cover $B \cup \{G\}$ is a member of β.

<u>Definition 2.4</u>: A covering quasi-uniformity which is generated by a collection of Q-covers with property S will be called an S-<u>covering quasi-uniformity</u>.

Let M(X) denote the collection of all bounded real valued lower semi-continuous functions on a topological space (X, τ).

<u>Theorem 2.5</u> [29, Proposition 1.1]: Let (X, τ) be a topological space. The collection $\mathcal{U}$ of sets $U_{(\epsilon, f)} = \{(x,y): f(x) - \epsilon < f(y)\}$ for $\epsilon > 0$ and $f \in M(X)$ is a subbase for a compatible quasi-uniformity for (X, τ).

<u>Remark</u>: W. Hunsaker and W. Lindgren [19, Theorem 2] have shown that the above collection $\mathcal{U}$ is a subbase for the Pervin quasi-uniformity.

<u>Theorem 2.6</u> [12, Theorem 4]: Let (X, τ) be a topological space and let $\mathcal{U}$ be a fundamental covering quasi-uniformity for (X, τ). Then $\mathcal{P} \subset \mathcal{U}$.

<u>Theorem 2.7</u>: Let (X, τ) be a topological space and let $\mathcal{U}$ be a covering quasi-uniformity on X. Then the following statements are equivalent.

i) $\mathcal{P} \subset \mathcal{U}$.

ii) $\mathcal{U}$ is an S-covering quasi-uniformity.

iii) $\mathcal{U}$ is a member of $\Sigma(\delta_{\mathcal{P}})$.

iv) $\mathcal{U}$ is a fundamental covering quasi-uniformity.

<u>Proof</u>: i $\Rightarrow$ ii: Let $\mathcal{U}$ be a covering quasi-uniformity such that $\mathcal{P} \subset \mathcal{U}$. Let α be a collection of Q-covers such that $\mathcal{U} = \mathcal{U}_{\alpha}$ [11, Theorems 1 and 2]. Let γ be the collection of all finite open covers of X, and let $\beta = \gamma \cup \alpha$. We note that $\mathcal{U}_{\gamma \cup \alpha} = \mathcal{U}_{\gamma} \vee \mathcal{U}_{\alpha}$. Since $\mathcal{P} \subset \mathcal{U}$ by [9, Lemma 1] $\mathcal{U}_{\beta} = \mathcal{U}_{\alpha}$. Let $\beta' = \{C \cup \{A\} : C \in \beta \text{ and } A \in \tau\}$. We will show that $\mathcal{U}_{\alpha} = \mathcal{U}_{\beta'}$. Let $C \in \beta$, let $A \in \tau$ and let $C' = C \cup \{A\}$. By hypothesis U_C and $U_{\{A,X\}} \in \mathcal{U}_{\alpha}$ which implies that $U_C \cap U_{\{A,X\}} \in \mathcal{U}_{\alpha}$. For each $x \in X$, $(U_C \cap U_{\{A,X\}})(x) = U_C(x) \cap U(x)_{\{A,X\}} \subseteq U_{C'}(x)$. Thus $U_C \cap U_{\{A,X\}} \subset U_{C'}$ and consequently $U_{C'} \in \mathcal{U}_{\alpha}$. Finally since $\mathcal{U}_{\alpha} \subset \mathcal{U}_{\beta'}$ we conclude that $\mathcal{U} = \mathcal{U}_{\beta'}$.

ii $\Rightarrow$ i: Let β be a collection of Q-covers of X with property S and let α be the collection of all finite open covers of X. Let $C \in \beta$ and let

$C' \in \alpha$. Let $C'' = C \cup C'$. Then $C'' \in \beta$. Since, for each $x \in X$, $U_{C''}(x) \subset U_{C'}(x)$, $U_{C''} \subset U_{C'}$ so that $U_{C'} \in \mathcal{U}_\beta$. Finally by [9, Lemma 1] $\mathcal{P} \subset \mathcal{U}_\beta = \mathcal{U}$.

ii $\Rightarrow$ iii: It is sufficient to show that $\mathcal{P} \in \Sigma(\delta_{\mathcal{U}})$. Let $\mathcal{U}^*$ be the totally bounded member of $\Sigma(\delta_{\mathcal{U}})$. Since $\mathcal{U}^*$ is generated by $Q_B(\mathcal{U})$ and $\mathcal{P}$ has a subbase generated by the collection of all bounded upper semi-continuous functions, $\mathcal{U}^* \subset P$. Thus $Q_B(\mathcal{U}^*) \subset Q_B(\mathcal{P})$. By hypothesis $\mathcal{P} \subset \mathcal{U}$ so that $Q_B(\mathcal{P}) \subset Q_B(\mathcal{U}) = Q_B(\mathcal{U}^*)$. Therefore $Q_B(\mathcal{P}) = Q_B(\mathcal{U}^*)$ and consequently $\mathcal{P} \in \Sigma(\delta_{\mathcal{U}})$ [Theorem 1.14].

iii $\Rightarrow$ iv: Let $\mathcal{U} \in \Sigma(\delta_{\mathcal{P}})$. Let α be a collection of Q-covers of X such that $\mathcal{U}_\alpha = \mathcal{U}$. Let β be the collection of all finite open covers of X. Let $\alpha' = \alpha \cup \beta$. Since $\mathcal{P} \in \Sigma(\delta_{\mathcal{P}})$ and $\mathcal{P}$ is totally bounded, $\mathcal{P} \subset \mathcal{U}$ [Theorem 1.12]. Therefore $\mathcal{U}_{\alpha'} = \mathcal{U}$ and consequently $\mathcal{U}$ is a fundamental covering quasi-uniformity on X.

iv $\Rightarrow$ i: Theorem 2.6.

<u>Corollary 2.8</u>: Let (X, τ) be a topological space and let $\mathcal{U}$ be the fine transitive (point finite covering, locally finite covering, upper semi-continuous) quasi-uniformity on X. Then $\mathcal{U} \in \Sigma(\delta_{\mathcal{P}})$.

Definition 2.9: Let $(X,\mathcal{U})$ be a quasi-uniform space and $\mathcal{F}$ a filter on X. Then $\mathcal{F}$ is $\mathcal{U}$-Cauchy if for every $U \in \mathcal{U}$ there exists $x \in X$ such that $U(x) \in \mathcal{F}$.

Definition 2.10: A quasi-uniform space $(X,\mathcal{U})$ is complete if every $\mathcal{U}$-Cauchy filter converges.

Definition 2.11: Two quasi-uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are said to be quasi-uniformly isomorphic relative to $\mathcal{U}$ and $\mathcal{V}$ if there exists a one-to-one mapping f of X onto Y such that both f and f^{-1} are quasi-uniformly continuous.

Definition 2.12: A completion of a quasi-uniform space $(X,\mathcal{U})$ is a complete quasi-uniform space $(Y,\mathcal{V})$ such that X is quasi-uniformly isomorphic (relative to $\mathcal{U}$ and $\mathcal{V}$) to a dense subset of Y.

Remark: Let (X,τ) be a compact topological space and let $\mathcal{U}$ be a compatible quasi-uniformity for (X,τ). Then $\mathcal{U}$ is complete [26, Theorem 4.16].

Theorem 2.13: Let (X,τ) be a locally compact Hausdorff topological space and let $\mathcal{U}$ be a covering quasi-uniformity on X such that $\mathcal{U} \in \Sigma(\delta_\rho)$. Let (X^*,τ^*) be a Hausdorff compactification of (X,τ). There is a covering quasi-uniformity $\mathcal{U}^*$ on X^* such that $\mathcal{U}^* \in \Sigma(\delta_{\rho^*})$ and such that $(X^*, \mathcal{U}^*)$ is a T_2 completion of $(X, \mathcal{U})$.

Proof: Let (X^*, τ^*) be a Hausdorff compactification of (X, τ) and $f : X \to X^*$ be a homeomorphism of X onto a dense subspace of X^*. Let $\alpha = \{\mathcal{F} : \mathcal{F} \subset \tau^*, \mathcal{F}$ is finite and $X^* - f(X) \subset \cup \mathcal{F}\}$. Since $\mathcal{U} \in \Sigma(\delta_\rho)$, $\mathcal{U}$ is generated by a collection of Q-covers $\mathfrak{p}$ such that $\mathfrak{p}$ has property S. For each $\mathcal{C} \in \mathfrak{p}$ let $f(\mathcal{C}) = \{f(C) : C \in \mathcal{C}\}$. For each $\mathcal{C} \in \mathfrak{p}$ and each $\mathcal{F} \in \alpha$ let $\mathcal{C}^* = f(\mathcal{C}) \cup \mathcal{F}$ and let $\mathfrak{p}^* = \{\mathcal{C}^* : \mathcal{C} \in \mathfrak{p}$ and $\mathcal{F} \in \alpha\}$. Let $\mathcal{U}^* = \mathcal{U}_{\mathfrak{p}^*}$. Let $\mathcal{F} \in \alpha$, let $\mathcal{C} \in \mathfrak{p}$ and let $\mathcal{U}_{\mathcal{C}^*}$ be the corresponding entourage in $\mathcal{U}^*$. The open cover $\mathcal{C}' = \{X \cap f^{-1}(C^*) : C^* \in \mathcal{C}^*\}$ is a member of $\mathfrak{p}$. Let $x, y \in X$ such that $(x,y) \in \mathcal{U}_{\mathcal{C}'}$. Then $y \in A_x^{\mathcal{C}'} = \cap \{f^{-1}(C^*) : C^* \in \mathcal{C}^*$ and $f(x) \in C^*\}$. Hence $f(y) \in A_{f(x)}^{\mathcal{C}^*}$ and $(f(x), f(y)) \in U_{\mathcal{C}^*}$. Similarly we can show that f^{-1} is quasi-uniformly continuous. Hence we conclude that X is quasi-uniformly isomorphic (relative to $\mathcal{U}$ and $\mathcal{U}^*|f(x)$) to a dense subspace of X^*, namely $f(X)$.

Corollary 2.14: Let (X, τ) be a locally compact Hausdorff space. Let $\mathcal{U}$ be the fine transitive (point finite covering, locally finite covering, Pervin, upper semi-continuous) quasi-uniformity on X. Let

(X^*, τ^*) be a Hausdorff compactification of (X, τ). Then there is a T_2 completion $(X^*, \mathcal{U}^*)$ of $(X, \mathcal{U})$ such that $\mathcal{U}^* \in \Sigma(\delta_{\rho^*})$.

Theorem 2.15: Let (X, τ) be a locally compact Hausdorff space. Let $\mathcal{U}$ be the fine transitive (point finite covering, locally finite covering, Pervin) quasi-uniformity on X. Let (X^*, τ^*) be a Hausdorff compactification of (X, τ) and let $\mathcal{U}^*$ be the fine transitive (point finite covering, locally finite covering, Pervin) quasi-uniformity on X^*. Then $(X^*, \mathcal{U}^*)$ is a T_2 completion of $(X, \mathcal{U})$.

Proof: Let (X, τ) be a locally compact Hausdorff topological space and let $\mathcal{U}$ be the fine transitive (point finite covering, locally finite covering, Pervin) quasi-uniformity on X. Let (X^*, τ^*) be a Hausdorff compactification of (X, τ) and $\mathcal{U}^*$ be the fine transitive (point finite covering, locally finite covering, Pervin) quasi-uniformity on X^*. Let $f : X \to X^*$ be a homeomorphism of X onto a dense subspace of X^*. Let α and α^* be the collection of all Q-covers (point finite covers, locally finite covers, finite covers) of X and X^* respectively. Let $\mathcal{C}^*$ be any member of α^* and $U_{\mathcal{C}^*}$ be the corresponding entourage in $\mathcal{U}^*$. Then $\mathcal{C} = \{f^{-1}(C^*) : C^* \in \mathcal{C}^*\}$ is a member of α. Let $x,y \in X$ such that $(x,y) \in U_{\mathcal{C}}$. Then $y \in A_x^{\mathcal{C}}$ and $f(y) \in A_{f(x)}^{\mathcal{C}^*}$. Hence $(f(x), f(y)) \in U_{\mathcal{C}^*}$.

Now let $C \in \alpha$ and U_C be the subbasic entourage in $\mathcal{U}$. Then the collection $C^* = \{f(C) : C \in \mathcal{C}\}$ is a Q-cover (point finite cover, locally finite cover, finite cover) of $f(X)$. Since (X^*, τ^*) is compact and Hausdorff, we can find a finite collection $\mathcal{F}$ of open sets in τ^* such that $\mathcal{F}$ is a cover of $X^* - f(X)$. Thus $C^{**} = C^* \cup \mathcal{F}$ is a member of α^*. Let $U_{C^{**}}$ be the corresponding entourage in $\mathcal{U}^*$. Let $z, w \in f(X)$ such that $(z,w) \in U_{C^{**}}$. Clearly $(f^{-1}(z), f^{-1}(w)) \in U_C$. Thus we conclude that X is quasi-uniformly isomorphic (relative to $\mathcal{U}$ and $\mathcal{U}^* | f(X)$) to a dense subspace of X^*.

<u>Proposition 2.16</u>: Let (X, τ) be a locally compact Hausdorff space and let $\mathcal{U}$ be the fine transitive quasi-uniformity on X. Let (X^*, τ^*) be an n point compactification of (X, τ) and let $(X^*, \mathcal{U}^*)$ be the completion of $(X, \mathcal{U})$ given in Theorem 2.13. Then $\mathcal{U}^*$ is the fine transitive quasi-uniformity of X^*.

We have shown that if (X, τ) is locally compact Hausdorff topological space and $\mathcal{U}$ is a covering quasi-uniformity on X such that $\mathcal{U}$ does not have a T_2 completion, then $\mathcal{U}$ cannot be a member of $\Sigma(\delta_\rho)$.

<u>Example 2.17</u> [4, Example 2]: Let N denote the set of natural numbers. Let $U_n = \{(x,y) : x = y \text{ or } x \geq n\}$, $\beta = \{U_n : n \in N\}$, and let

$\mathcal{U}$ denote the quasi-uniformity on N generated by the base $\mathcal{B}$. Then $(N,\mathcal{U})$ is a locally compact Hausdorff quasi-uniform space that has no Hausdorff completion [4, Example 2]. It is not difficult to see that $\mathcal{U}$ is a transitive quasi-uniformity and hence a covering quasi-uniformity [11, Theorem 2] and it may be verified directly that $\mathcal{U} \notin \Sigma(\delta_{\wp})$.

CHAPTER III

LOCAL PROPERTIES OF QUASI-UNIFORM SPACES

In this chapter we study some local properties of quasi-uniformities. Local completeness, local precompactness, and local total boundedness are introduced. We show that a topological space (X, τ) is locally (countably) compact if and only if the fine transitive (upper semi-continuous) quasi-uniformity for (X, τ) is locally precompact. These results are natural analogues of [14, Corollary 4.9] and [2 , Theorem 3.3] respectively. Local symmetry and weak local symmetry were first studied in [26]. We note here that if (X, τ) is a topological space and $\mathcal{U} \in \Sigma(\delta_\rho)$, then $\mathcal{U}$ is (weakly) locally symmetric if and only if (X, τ) is (R_o) regular. We give an example of a weakly locally symmetric quasi-uniform space $(X, \mathcal{U})$ such that $\tau_{\mathcal{U}}$ is compact and metrizable, although $\mathcal{U}$ is neither a uniformity nor a member of $\Sigma(\delta_\rho)$. We also give an example of quasi-uniform space which is completely normal and T_1, (and hence R_o), but which is not weakly locally symmetric.

<u>Definition 3.1</u>: A quasi-uniform space $(X, \mathcal{U})$ is <u>precompact</u> if and only if for each $U \in \mathcal{U}$ there is a finite subset A of X such that $U(A) = X$.

<u>Definition 3.2</u>: A quasi-uniformity $\mathcal{U}$ on a set X is <u>totally</u> <u>bounded</u> if and only if for each $U \in \mathcal{U}$ there exists a finite collection $\{A_i\}_{i=1}^n$

of subsets of X such that,

i) $A_i \times A_i \subset U$, $1 \leq i \leq n$

ii) $\bigcup_{i=1}^{n} A_i = X$.

Definition 3.3: Let $(X, \mathcal{U})$ be a quasi-uniform space and let $S \subset X$. Then $\mathcal{U}|S = \{U \cap S \times S : U \in \mathcal{U}\}$ is called the trace of $\mathcal{U}$ on S. If $U \in \mathcal{U}$ then $U|S = U \cap S \times S$.

Theorem 3.4 [2.6]: Let $(X, \mathcal{U})$ be a quasi-uniform space and let $S \subset X$. Then the trace $\mathcal{U}|S$ of $\mathcal{U}$ on S is a quasi-uniformity on S and the topology for S induced by $\mathcal{U}|S$ is equivalent to the subspace topology for S.

Definition 3.5: A quasi-uniform space $(X, \mathcal{U})$ is locally complete, (locally totally bounded, locally precompact) if for each $x \in X$, there exists a closed neighborhood C of x such that $(C, \mathcal{U}|C)$ is complete, (totally bounded, precompact).

Lemma 3.6: Let $(X, \mathcal{U})$ be a precompact uniform space and let C be any subset of X. Then $(C, \mathcal{U}|C)$ is precompact.

Proposition 3.7: A uniform space $(X, \mathcal{U})$ is locally compact if and only if it is locally complete and locally precompact.

Proposition 3.8: A topological space (X, τ) is locally compact if and only if every compatible quasi-uniformity $\mathcal{U}$ on X is locally complete.

Proof: Let (X, τ) be a topological space such that every compatible quasi-uniformity $\mathcal{U}$ on X is locally complete. Let $x \in X$ and let $\mathcal{U}$ be the Pervin quasi-uniformity on X. Let $C \subset X$ and $x \in C$ such that $(C, \mathcal{U}|C)$ is complete. We know that $(C, \mathcal{U}|C)$ is totally bounded and hence precompact [26]. Thus $(C, \mathcal{U}|C)$ is compact. It is clear that if (X, τ) is locally compact then every compatible quasi-uniformity on X is locally complete.

Theorem 3.9 [26]: Let Λ be a nonempty set, and for each $\alpha \in \Lambda$, let $(X_\alpha, \mathcal{U}_\alpha)$ be a quasi-uniform space. Let $X = \Pi\{X_\alpha : \alpha \in \Lambda\}$. For each $\alpha \in \Lambda$, let $\Pi_\alpha : X \to X_\alpha$ be the projection mapping. For each $U_\alpha \in \mathcal{U}_\alpha$; let $U^*_\alpha = \{(x,y) \in X \times X : (\Pi_\alpha(x), \Pi_\alpha(y)) \in U_\alpha\}$ and $S = \{U_\alpha : \alpha \in \Lambda \text{ and } U_\alpha \in \mathcal{U}_\alpha\}$. Then S is a subbase for a quasi-uniformity $\mathcal{U}$ on X, which is compatible with the product topology for X. Furthermore, for each $\alpha \in \Lambda$, Π_α is a quasi-uniformly continuous function.

Theorem 3.10: Let $\{(X_\alpha, \mathcal{U}_\alpha) : \alpha \in \Lambda\}$ be a collection of quasi-uniform spaces. Then the product quasi-uniformity $\mathcal{U}$ on $X = \prod_{\alpha \in \Lambda} X_\alpha$

is locally precompact if all X_α's are locally precompact and at most finitely many are not precompact.

Proof: Let B be a finite subset of Λ and let $U \in \mathcal{U}$ such that $U = \Pi\{U_\alpha : \alpha \in \Lambda\}$, where $U_\alpha \in \mathcal{U}_\alpha$ and $U_\alpha = X_\alpha \times X_\alpha$ for each $\alpha \in \Lambda - B$. Let $x \in X$ and let B' be a finite subset of Λ such that for each $\alpha \in \Lambda$, C_α is a closed neighborhood of $x(\alpha)$, $(C_\alpha, \mathcal{U}_\alpha|C_\alpha)$ is precompact and $C_\alpha = X_\alpha$ for each $\alpha \in \Lambda - B'$. Then $C = \prod_{\alpha \in \Lambda} C_\alpha$ is a closed neighborhood of x. It remains to show that $(C, \mathcal{U}|C)$ is precompact. For each $\beta \in B \cap B'$ let F_β be a finite subset of C_β such that $U_\beta|C_\beta(F_\beta) = C_\beta$. Let $p \in C$ and let $F = \{x \in C : x(\beta) = p(\beta)$ for $\beta \in \Lambda - B \cap B'$ and $x(\beta) \in F_\beta$ for each $\beta \in B \cap B'\}$. Now let $z \in C$ and for each $\beta \in B \cap B'$ let $w_\beta \in F_\beta$ such that $z(\beta) \in U_\beta|C_\beta(w_\beta)$. Define $y : \Lambda \to X$ by $y(\beta) = w_\beta$ for each $\beta \in B \cap B'$ and $y(\beta) = p(\beta)$ for each $\beta \in \Lambda - B \cap B'$. Clearly $y \in F$ and $z \in U|C(y)$.

Lemma 3.11 [26, Theorem 4.3]: Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be quasi-uniform spaces, and let f be a quasi-uniformly continuous function from X into Y. Let $\mathcal{F}$ be a $\mathcal{U}$-Cauchy filter on X. Then $f(\mathcal{F})$ is a $\mathcal{V}$-Cauchy filter on Y.

Theorem 3.12: Let $\{(X_\alpha, \mathcal{U}_\alpha) : \alpha \in \Lambda\}$ be a collection of quasi-uniform spaces. Then the product quasi-uniformity $\mathcal{U}$ on $X = \prod_{\alpha \in \Lambda} X_\alpha$ is locally complete if all X_α's are locally complete and at most finitely many are not complete.

Proof: Suppose that $\{(X_\alpha, \mathcal{U}_\alpha) : \alpha \in \Lambda\}$ is a collection of quasi-uniform spaces with the above property. Let $x \in X$ and for each $\alpha \in \Lambda$ let C_α be a closed neighborhood of $x(\alpha)$ such that $(C_\alpha, \mathcal{U}_\alpha | C_\alpha)$ is complete where $C_\alpha = X_\alpha$ for all but finitely many α's. Let $C = \prod_{\alpha \in \Lambda} C_\alpha$; then C is a closed neighborhood of x. We now show that $(C, \mathcal{U}|C)$ is complete. Let $\mathcal{F}$ be a $\mathcal{U}|C$-Cauchy filter on C. By the preceding lemma, for each $\alpha \in \Lambda$, $\Pi_\alpha(\mathcal{F})$ is a $\mathcal{U}_\alpha|C_\alpha$-Cauchy filter on C_α. Thus for each $\alpha \in \Lambda$, there is $y_\alpha \in C_\alpha$ such that $\Pi_\alpha(\mathcal{F})$ converges to y_α. Let $y \in C$ such that $y(\alpha) = y_\alpha$. Clearly $\mathcal{F}$ converges to y and hence $(C, \mathcal{U}|C)$ is complete.

Corollary 3.13: Let $\{(X_\alpha, \mathcal{U}_\alpha) : \alpha \in \Lambda\}$ be a collection of quasi-uniform spaces and let $(X, \mathcal{U})$ be the product quasi-uniform space. Then $(X, \mathcal{U})$ is locally compact if all the $(X_\alpha, \mathcal{U}_\alpha)$ are locally compact and at most finitely many are not compact.

Definition 3.14: A topological space (X, τ) is locally countably compact if for each $x \in X$ there exists a closed neighborhood C of x such that C is countably compact with respect to the relative topology.

Theorem 3.15: Let (X, τ) be a topological space and let $\mathcal{U}$ be the upper semi-continuous quasi-uniformity on X. Then (X, τ) is locally countably compact if and only if $\mathcal{U}$ is locally precompact.

Proof: Let $x \in X$ and let $C \subset X$ be the closed neighborhood of x such that $(C, \mathcal{U}|C)$ is precompact. Let $\{A_i\}_{i=1}^{\infty}$ be a countable open cover of C and for each i let $G_i \in \tau$ such that $A_i = G_i \cap C$. Let $G_o = \{X - C\}$ and $\mathcal{C} = \{G_i\}_{i=o}^{\infty}$. Then $\mathcal{C}$ is a countable open cover of X. Define $f : X \to R$ as follows: for each $x \in X$, let $f(x)$ be the least nonnegative integer n such that $x \in G_n \in \mathcal{C}$. Since $f^{-1}(-\infty, n+1) = \bigcup_{i=o}^{n} G_i$, f is upper semi-continuous [2, Theorem 3.3]. Thus for each $\epsilon > 0$, $U_{(f,\epsilon)} \in \mathcal{U}$. By the hypothesis there exists a finite set $F = \{x_1, \ldots, x_n\} \subset C$, such that $U_{(f,\epsilon)}|C(F) = C$. Let $w \in F$ with the property that $f(w) = \sup\{f(x) : x \in F\}$. Then for all $x \in C$, $f(x) < f(w) + \epsilon$. Let k be the least positive integer such that

$f(w) + \epsilon < k$. Clearly the finite subcollection $\{A_i\}_{i=1}^{k}$ is a cover of C. Thus C is countably compact.

Now suppose that $x \in X$ and that $C \subset X$ is a countably compact closed neighborhood of x. Let τ_C denote the relative topology on C. Let $\mathcal{V}$ be the upper semi-continuous quasi-uniformity on (C, τ_C). Then $\mathcal{V}$ is precompact [2, Theorem 3.3]. But for each upper semi-continuous map $f : X \to R$, $f|C$ is upper semi-continuous on C. Thus $\mathcal{U}|C \subset \mathcal{V}$; consequently $(C, \mathcal{U}|C)$ is locally precompact.

Theorem 3.16 [1, Theorem 7.1]: Let (X, τ) be a topological space. Then (X, τ) is compact if and only if every open cover that is well ordered by set inclusion has a finite subcover.

Theorem 3.17: Let (X, τ) be a topological space and let $\mathcal{U}$ be the fine transitive quasi-uniformity on X. Then (X, τ) is locally compact if and only if $\mathcal{U}$ is locally precompact.

Proof: Suppose that $\mathcal{U}$ is locally precompact. Let $x \in X$ and let $C \subset X$ be a closed neighborhood of x such that $(C, \mathcal{U}|C)$ is precompact. Let $\{B_\alpha\}_{\alpha \in \Lambda}$ be a well ordered open cover of C. For each $\alpha \in \Lambda$, let $A_\alpha \in \tau$ with the property that $B_\alpha = A_\alpha \cap C$. Furthermore, for each

$\alpha \in \Lambda$ let $G_\alpha = A_\alpha \cup (X - C)$. Then $\mathcal{C} = \{G_\alpha\}_{\alpha \in \Lambda}$ is a cover of X, and clearly it is well ordered by set inclusion. Consequently the subbasic entourage $U_{\mathcal{C}} \in \mathcal{U}$. Now let $F = \{x_1, x_2, \ldots, x_n\}$ be a finite set such that $U_{\mathcal{C}}|C(F) = C$. For each $i = 1, \ldots, n$, let $B_{\alpha_i} = \cap \{B_\alpha : x_i \in B_\alpha\}$. Then for each $i = 1, \ldots, n$, $U_{\mathcal{C}}|C(x_i) = B_{\alpha_i}$ and $U_{\mathcal{C}}|C(F) = \bigcup_{i=1}^{n} B_{\alpha_i}$. Thus by Theorem 3.17 $(C, \mathcal{U}|C)$ is compact.

<u>Definition 3.18</u>: A Hausdorff topological space (X, τ) is <u>almost compact</u> (<u>H closed in the terminology</u> of [21]) provided that if $\mathcal{C}$ is an open cover of X, then there exists a subcollection $\{C_i : 1 \le i \le n\}$ of $\mathcal{C}$ such that $X = \bigcup_{i=1}^{n} \overline{C}_i$.

<u>Definition 3.19</u>: A Hausdorff topological space (X, τ) is <u>almost locally compact</u> provided that for each $x \in X$ there exists a closed neighborhood C of x such that C is almost compact with respect to the relative topology.

<u>Definition 3.20</u> [25]: A quasi-uniform space $(X, \mathcal{U})$ is <u>almost complete</u> provided that every open Cauchy filter has a cluster point.

Proposition 3.21: A Hausdorff topological space (X, τ) is almost locally compact if and only if it is almost locally complete with respect to every compatible quasi-uniformity on X.

We now discuss (weak) local symmetry. These concepts play an important role in the study of function spaces in Chapters IV and V.

Definition 3.22 [26]: A quasi-uniform space $(X, \mathcal{U})$ is locally symmetric provided for each $x \in X$ and $U \in \mathcal{U}$ there exists a symmetric entourage $V \in \mathcal{U}$ such that $V \circ V(x) \subset U(x)$.

Definition 3.23: Let (X, τ) be a topological space, and let F be the collection of all functions from (X, τ) into a quasi-uniform space $(Y, \mathcal{U})$. For each $x \in X$ and $U \in \mathcal{U}$ let $W(x,U) = \{(f,g) \in F \times F : (f(x),g(x)) \in U\}$. Then the collection $\mathcal{B} = \{W(x,U) : x \in X \text{ and } U \in \mathcal{U}\}$ is a subbase for the quasi-uniformity $\overline{\mathcal{U}}$ of pointwise convergence on F.

Definition 3.24 [26]: A quasi-uniform space $(X, \mathcal{U})$ is weakly locally symmetric provided that for each $x \in X$ and $U \in \mathcal{U}$ there exists a symmetric entourage $V \in \mathcal{U}$ such that $V(x) \subset U(x)$.

Proposition 3.25: Let (X, τ) be a topological space and let F be the collection of all functions from (X, τ) into a (weakly) locally symmetric quasi-uniform space $(Y,\mathcal{U})$. Then $(F,\overline{\mathcal{U}})$ is (weakly) locally symmetric.

Proof: We prove only the result in the case that $(Y,\mathcal{U})$ is locally symmetric. Let $x \in X$, $U \in \mathcal{U}$, $g \in F$ and let $W(x,U)$ be the subbasic entourage in $\overline{\mathcal{U}}$. By hypothesis there exists a symmetric entourage $V \in \mathcal{U}$ such that $V \circ V(g(x)) \subset U(g(x))$. We claim that $W(x,V)$ is symmetric and that $W(x,V) \circ W(x,V)(g) \subset W(x,U)(g)$. Let $(f,h) \in W(x,V)$. Then $(f(x),h(x)) \in V$. Since V is symmetric, $(h(x),f(x)) \in V$; thus $(h,f) \in W(x,V)$. Let $h \in W(x,V) \circ W(x,V)(g)$. Then there is $k \in F$ such that (g,k), $(k,h) \in W(x,V)$. Then $(g(x),h(x)) \in V \circ V$, which implies that $h(x) \in U(g(x))$ and consequently that $h \in W(x,U)(g)$.

Definition 3.26 [34]: A quasi-metric space (X,d) is called a strong quasi-metric space if $\tau_{d^{-1}} \supset \tau_d$.

Proposition 3.27: Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $\mathcal{U}$ is weakly locally symmetric if and only if $\tau_{\mathcal{U}} \subset \tau_{\mathcal{U}^{-1}}$.

Proof: Suppose that $(X,\mathcal{U})$ is weakly locally symmetric. Let $x \in X$ and let $U \in \mathcal{U}$. By hypothesis there exists $V \in \mathcal{U}$ such that $V^{-1}(x) \subset U(x)$. Thus $\tau_{\mathcal{U}} \subset \tau_{\mathcal{U}^{-1}}$. Now suppose that $\tau_{\mathcal{U}} \subset \tau_{\mathcal{U}^{-1}}$. Let $x \in X$ and let $U \in \mathcal{U}$. Let $V \in \mathcal{U}$ such that $V^{-1}(x) \subset U(x)$ and let $W = U \cap V$. Then $W \cup W^{-1}$ is a symmetric entourage in $\mathcal{U}$ and $(W \cup W^{-1})(x) = W(x) \cup W^{-1}(x) \subset U(x) \cup V^{-1}(x) \subset U(x)$. Thus $(X, \mathcal{U})$ is weakly locally symmetric.

Corollary 3.28: A quasi-metric d on a set X is a strong quasi-metric if and only if the associated quasi-uniformity is weakly locally symmetric.

Let (X, τ) be a topological space. It is known that if (X, τ) is regular (R_o) then the Pervin quasi-uniformity $\mathcal{P}$ on X is locally symmetric (weakly locally symmetric) [26, Theorems 3.6 and 3.17].

Proposition 3.29: Let (X, τ) be a regular (R_o) topological space and let $\mathcal{U} \in \Sigma(\delta_{\mathcal{P}})$. Then $\mathcal{U}$ is locally symmetric (weakly locally symmetric).

Example 3.30 [12, Example 1]: A compact metrizable space (X, τ) with a compatible weakly locally symmetric quasi-uniformity $\mathcal{U}$ on X which is not a uniformity and which is not a member of $\Sigma(\delta_{\mathcal{P}})$

Construction: Let I be the closed unit interval. For each α such that $0 < \alpha < 1$, let

$$V_\alpha = \Delta \cup \{(x,y) : x,y \neq 0,1 \text{ and } |x-y| < \alpha\}$$

$$\cup \{0\} \times \{y : 1-y < \alpha \text{ and } y \neq 1\} \cup \{y : y < \alpha \text{ and } y \neq 0\}.$$

Let $\mathcal{B} = \{V_\alpha : 0 < \alpha < 1\}$. Then $\mathcal{B}$ is a basis for the desired quasi-uniformity $\mathcal{U}$ on X.

Remark: If X is a set and $\mathcal{U}$ is a quasi-uniformity on X with a countable base such that $\tau_{\mathcal{U}}$ is T_1 but not metrizable, then $\mathcal{U}$ is not locally symmetric [10, Corollary 1]. Therefore if (X,τ) is a quasi-metrizable space that is not metrizable, there is a compatible quasi-uniformity that is not locally symmetric. It follows from Corollary 3.28 and [34, Theorem 3.1] that if $(X,\mathcal{U})$ is a weakly locally symmetric quasi-uniform space with a countable basis, and $(X,\tau_{\mathcal{U}})$ is T_1 then $(X,\tau_{\mathcal{U}})$ developable. Consequently the Sorgenfrey line is an example of a completely normal T_1 space that possesses a compatible quasi-uniformity that is not even weakly locally symmetric.

Definition 3.31: A quasi-uniform space $(X,\mathcal{U})$ is locally transitive provided that for each $x \in X$ and $U \in \mathcal{U}$ there exists a $V \in \mathcal{U}$ such that $V \circ V = V$ and $V(x) \subset U(x)$.

Having studied the local symmetry, it is natural to try to investigate local transitivity in quasi-uniform spaces. Many of the known quasi-uniformities, however, are transitive and hence locally transitive.

Proposition 3.32: Let (X, τ) be a topological space and let $\mathcal{U}$ be a compatible uniformity on X such that $\mathcal{U}$ is locally transitive. Then (X, τ) is zerodimensional.

Proof: Let $x \in X$ and let $U \in \mathcal{U}$. Let $V \in \mathcal{U}$ such that $V \circ V = V$ and $V(x) = U(x)$. Since $\mathcal{U}$ is a uniformity, $W = V \cap V^{-1} \in \mathcal{U}$. Since $W \circ W = W$, $W(x) \in \tau$. We now show that $W(x)$ is closed. Let $(x,y) \in (W \circ W)$. Then there exists $z \in X$ such that (x,z), $(z,y) \in V \cap V^{-1}$. Since (x,z), $(z,y) \in V$, $(x,y) \in V \circ V = V$. Also since (y,z), $(z,y) \in V$, $(y,x) \in V \circ V = V$ and $(x,y) \in V^{-1}$. Thus $(x,y) \in V \cap V^{-1}$. Consequently $W \circ W = W$. Now let $t \in X$ such that $t \notin W(x)$. Suppose that $q \in W(t) \cap W(x)$. Then (t,q), $(x,q) \in W$ and $(x,t) \notin W \circ W = W$ -- contradiction. Thus $W(t) \cap W(x) = \phi$, so that $W(x)$ is both open and closed. If follows that (X, τ) is zerodimensional.

It is known that every zerodimensional topological space possesses a compatible transitive uniformity. Thus we get the following corollary.

Corollary 3.33: Let (X, τ) be a topological space and let $\mathcal{U}$ be a compatible uniformity on X such that $\mathcal{U}$ is locally transitive. Then there exists a compatible uniformity $\mathcal{V}$ on X such that $\mathcal{V}$ is transitive.

CHAPTER IV

LOCAL SYMMETRY AND FUNCTION SPACES

In this chapter we extend the concepts of equicontinuity and uniform convergence to the function spaces of quasi-uniform spaces. Almost all of the developments concern the spaces of functions which are continuous relative to the topology of the domain space. Local symmetry plays an important role in some of our developments. In particular we obtain an extension of the classical theorem that uniform convergence of a sequence of continuous functions implies the continuity of the limit function.

Proposition 4.1: Let (X, τ) be a topological space and let F be the family of all functions from (X, τ) into a quasi-uniform space $(Y, \mathfrak{U})$. For each $x \in X$ and $U \in \mathfrak{U}$ let $W(x,U) = \{(f,g) \in F \times F : (f(x),g(x)) \in U\}$. Let $\mathfrak{U}' = \{W(x,U) : x \in X \text{ and } U \in \mathfrak{U}\}$. Then $\mathfrak{U}'$ is a subbase for a quasi-uniformity on F.

The quasi-uniformity for which $\mathfrak{U}'$ is a subbase will be denoted by $\overline{\mathfrak{U}}$.

Definition 4.2: Let (X, τ) be a topological space and let F be the family of all functions from (X, τ) into a topological space (Y, τ'). Let $x \in X$, $U \in \tau'$ and let $W(x,U) = \{f \in F : f(x) \in U\}$. Then $\mathfrak{p} = \{W(x,U): U \in \tau', x \in X\}$ is a subbase for a topology on F, called point open topology and is the relativized product topology.

It is commented on page 68 [26] that the topology of quasi-uniform convergence is finer than the point open topology. Actually it is not difficult to show that these topologies coincide. For the sake of completeness we include a proof.

Proposition 4.3: Let (X, τ) be a topological space and let F be the collection of all functions from (X, τ) into a quasi-uniform space $(Y, \mathcal{U})$. Then $\overline{\mathcal{U}}$ is compatible with the point open topology on F.

Proof: Let $x \in X$, $U \in \mathcal{U}$ and $f \in F$. Then $W(x,U)(f) = \{g \in F$ and $(f(x),g(x)) \in U\}$ is a subbasic open set in $\tau_{\overline{\mathcal{U}}}$ which contains f. Clearly, the open set $W(x, \text{int } U(f(x)))$ in the point open topology contains f and is a subset of $W(x,U)(f)$. Now let $f \in F$, $x \in X$ and $A \in \tau_{\mathcal{U}}$ such that $W(x,A)$ contains f. By hypothesis there exists a $U \in \mathcal{U}$ such that $U(f(x)) \subset A$. Clearly $W(x,U)(f)$ is a subset of $W(x,A)$.

Definition 4.4: Let F be a collection of maps from a topological space (X, τ) into a quasi-uniform space $(Y, \mathcal{U})$ and let $x \in X$. Then F is quasi-equicontinuous at x provided that for each $V \in \mathcal{U}$ there exists a neighborhood N of x such that for each $f \in F$, $f(N) \subset V(f(x))$. The collection F is quasi-equicontinuous provided F is quasi-equicontinuous at each $x \in X$.

Remark: It is an obvious consequence of the above definition that if F is an quasi-equicontinuous family of functions from a space (X,τ) into a quasi-uniform space $(Y,\mathcal{U})$, then for all $f \in F$, f is continuous on X.

Definition 4.5: Let F be a collection of functions from the space X into the space Y and let $\alpha : X \times F \to Y$ be defined by $\alpha(f,x) = f(x)$. A topology τ for F is said to be jointly continuous if the map α is continuous with respect to τ.

Proposition 4.6: Let (X,τ) be a topological space and let F be a collection of quasi-equicontinuous functions from (X,τ) into a quasi-uniform space $(Y,\mathcal{U})$. Then the point open topology on F is jointly continuous.

Proof: Let $f \in F$ and let $x \in X$. For any $U \in \mathcal{U}$, $U(f(x))$ is a neighborhood of $f(x)$ [Theorem 1.7]. Let $V \in \mathcal{U}$ such that $V \circ V \subset U$. By hypothesis there exists a neighborhood N of x such that for all $f \in F$, $f(N) \subset V(f(x))$. Consider the neighborhoods $W(x,V)(f)$ and N of f and x respectively. Let $z \in N$ and let $g \in W(x,V)(f)$. Then $(f(x),g(x))$, $(g(x),g(z)) \in V$ and $g(z) \in V \circ V(f(x)) \subset U(f(x))$.

Definition 4.7 [24]: A saturated topological space is a space in which any intersection of open sets is open; or, equivalently, a space is saturated provided that each point of the space possesses a minimum neighborhood.

Proposition 4.8: Let G be a family of functions from a saturated topological space (X, τ) into a quasi-uniform space $(Y, \mathcal{U})$ with a jointly continuous topology on G. Then G is quasi-equicontinuous on X.

Proof: Let $x \in X$, let $U \in \mathcal{U}$ and let N be the minimal neighborhood of x. Since the topology on G is jointly continuous, for each $f \in G$ there exists a neighborhood W of f such that for all $g \in W$ and all $y \in N$, $g(y) \in U(g(x))$.

Definition 4.9: [28]: Let (X, τ) be a topological space, let $(Y, \mathcal{U})$ be a quasi-uniform space and for each $V \in \mathcal{U}$ let $W(U) = \{(f,g) : Y^X \times Y^X : (f(x), g(x)) \in V$ for each $x \in X\}$. Then $\{W(U) : U \in \mathcal{U}\}$ is a subbase for the quasi-uniformity of quasi-uniform convergence on Y^X.

Remark: Let (X, τ) be a topological space and $(Y, \mathcal{U})$ be a quasi-uniform space. Let F be a collection of functions from X into Y. Then the quasi-uniformity $\overline{\mathcal{U}}$ on F is contained in the quasi-uniformity of quasi-uniform convergence on F.

Lemma 4.10: Let $(X,\mathcal{U})$ be a locally symmetric quasi-uniform space. Let $x \in X$ and $U \in \mathcal{U}$. Then for each positive integer n there exists a symmetric entourage $V \in \mathcal{U}$ such that $V^n(x) \subset U(x)$.

Proof: Let $x \in X$, $U \in \mathcal{U}$ and n be a positive integer such that $n > 2$. By hypothesis there exist symmetric entourages $W, W' \in \mathcal{U}$ such that $W \circ W(x) \subset U(x)$ and $W' \circ W'(x) \subset W(x)$. Let $V = W \cap W'$. Then $V \circ V(x) \subset W(x)$ and $W \circ (V \circ V)(x) \subset W \circ W(x)$, so that $V \circ (V \circ V)(x) \subset W \circ W(x) \subset U(x)$. By induction it follows that there exists a symmetric $Z \in \mathcal{U}$ such that $Z^n(x) \subset U(x)$.

Theorem 4.11: Let F be a family of functions from a topological space (X,τ) into a locally symmetric quasi-uniform space $(Y,\mathcal{U})$ such that F is $\mathcal{U}$-equicontinuous on X. Let g be a map of X into Y such that $g \in \mathrm{cl}\ F$ with respect to the topology of quasi-uniform convergence on $F \cup \{g\}$. Then $F \cup \{g\}$ is a $\mathcal{U}$-equicontinuous family.

Proof: Let g be a map from X into Y such that $g \in \mathrm{cl}\ F$. Let $x \in X$ and let $U \in \mathcal{U}$. Then there exists a neighborhood N of x such that for all $f \in F$, $f(N) \subset U(f(x))$. By Lemma 4.10 there exists a symmetric entourage $V \in \mathcal{U}$ such that $V \circ V \circ V(g(x)) \subset U(g(x))$. Let M be a neighborhood of x such that for each $f \in F$, $f(M) \subset V(f(x))$. Consider the neighborhoods $M \cap N$ and $W(V)(g)$ of x and g respectively. Let $y \in M \cap N$ and let $h \in W(V)(g) \cap F$. Then $(g(x),h(x))$, $(g(y),h(y))$, $(h(x),h(y)) \in V$. Thus $(g(x),g(y)) \in V \circ V \circ V$. Hence $g(y) \in V \circ V \circ V(g(x)) \subset U(g(x))$.

Remark: As a special case the above result obtains if the range is regular and $\mathcal{U} \in \Sigma(\delta_{\rho})$.

Definition 4.12: Let $(X,\mathcal{U})$ be a quasi-uniform space. A sequence $(x_i)_{i=1}^{\infty}$ in X is said to be $\mathcal{U}$-Cauchy if for each $V \in \mathcal{U}$ there exists a positive integer n such that for all $i \geq n$, $x_i \in V(x_n)$.

Definition 4.13: A quasi-uniform space $(X,\mathcal{U})$ is said to be sequentially complete if every $\mathcal{U}$-Cauchy sequence converges to a point in X.

Proposition 4.14: Let $(X,\mathcal{U})$ be a complete quasi-uniform space. Then $(X,\mathcal{U})$ is sequentially complete.

Proof: Let $\{x_i\}_{i=1}^{\infty}$ be a $\mathcal{U}$-Cauchy sequence in X. For each positive integer n let $F_n = \{x_i\}_{i=n}^{\infty}$. Let $\mathcal{F}$ be the filter generated by $\{F_n$: n is a positive integer$\}$. Clearly $\mathcal{F}$ is a $\mathcal{U}$-Cauchy filter. By hypothesis there exists a $y \in X$ such that $\mathcal{F}$ converges to y. Consequently for each $U \in \mathcal{U}$, there exists an $F_n \in \mathcal{F}$ such that $F_n \subset U(y)$.

Therefore for all positive integers $i \geq n$, $x_i \in U(y)$. Thus $\{x_i\}_{i=1}^{\infty}$ converges to y.

It is natural to investigate whether the converse of the Theorem 4.14 holds.

Next we give an example of a countably compact, first countable Hausdorff quasi-uniform space which is sequentially complete but not complete.

<u>Example 4.15</u>: Let (θ,W) be the space of all ordinals less than the first uncountable ordinal. It is known that (θ,W) is a countably compact, first countable Hausdorff topological space. Let $\mathcal{P}$ be the Pervin quasi-uniformity compatible with W. Since W is not compact, $(\theta,\mathcal{P})$ is not a complete quasi-uniform space [26, Theorems 4.11 and 4.14]. It is known that if A is a countably infinite subset of θ, then A has a limit point which is the least element of θ having infinitely many predecessors in A. Now let $\{x_i\}_{i=1}^{\infty}$ be a $\mathcal{P}$-Cauchy sequence in θ and let b be the least limit point of the set $\{x_i\}_{i=1}^{\infty}$ that has infinitely many predecessors in $\{x_i\}_{i=1}^{\infty}$. Let $<$ denote the well ordering on θ. Let $S = \{[0,\alpha) : \alpha < \Omega\} \cup \{(\alpha,\Omega) : \alpha < \Omega\}$. Then S is a subbase for W. We shall show that $\{x_i\}_{i=1}^{\infty}$ converges to b. Assume that $\{x_i\}_{i=1}^{\infty}$ does not converge to b. Then there exists a subbasic

open set $B \in S$ containing b such that if n is a positive integer then there exists $i \geq n$ for which $x_i \notin B$. Let $A_1 = \{x : x < b\}$ and let $A_2 = \{x : b < x\}$. Clearly $A_1 \cap A_2 = \emptyset$. Since (θ, W) is Hausdorff, B must contain infinitely many elements of $\{x_i\}_{i=1}^{\infty}$. We show next that A_1 and A_2 both must contain infinitely many elements of the sequence $\{x_i\}_{i=1}^{\infty}$. Suppose that A_2 contains only finitely many elements of $\{x_i\}_{i=1}^{\infty}$ and that there is $\beta > b$ such that $B = \{x : x < \beta\}$. Then it is clear that there exists a positive integer n such that for all $i \geq n$, $x_i \in B$ -- a contradiction. Now suppose that there is $\alpha \in \theta$ such that $\alpha < b$ and that $B = \{x : \alpha < x\}$. By hypothesis b is the least element of θ that has infinitely many predecessors in $\{x_i\}_{i=1}^{\infty}$. Thus the set $C = \{x : x < \alpha\}$ contains only a finite number of elements of the sequence $\{x_i\}_{i=1}^{\infty}$. Consequently there exists a positive integer n such that for all $i \geq n$, $x_i \in B$ -- a contradiction. Thus it is clear that A_1 and A_2 both contain infinitely many elements of $\{x_i\}_{i=1}^{\infty}$. Let $C = \{A_1, A_2, B\}$. Then C is a finite open cover of θ. Let U_C be the corresponding entourage in $\mathcal{P}$. Now it is not difficult to see that if n is a positive integer then $[\theta - U_C(x_n)] \cap \{x_i\}_{i=1}^{\infty}$ is infinite. Hence $\{x_i\}_{i=1}^{\infty}$ is not a $\mathcal{P}$-Cauchy sequence -- a contradiction. Consequently

$\{x_i\}_{i=1}^{\infty}$ must converge to b. It follows that $(\theta,\mathcal{P})$ is sequentially complete.

Definition 4.16: A sequence of functions $(f_i)_{i=1}^{\infty}$ from a topological space X into a quasi-uniform space $(Y,\mathcal{U})$ is a $\mathcal{U}$-Cauchy sequence if for each $U \in \mathcal{U}$ there exists a positive integer n (depending on U) such that for each $x \in X$ and for each $m \geq n$ $(f_n(x),f_m(x)) \in U$.

Definition 4.17: Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions from a topological space (X,τ) into a quasi-uniform space $(Y,\mathcal{U})$. Then $(f_n)_{n=1}^{\infty}$ is said to converge quasi-uniformly if there exists a function $g : X \to Y$, such that for each $U \in \mathcal{U}$ there exists a positive integer N (depending on U) such that for each $n \geq N$ and each $x \in X$, $(g(x),f_n(x)) \in U$.

Theorem 4.18: Let $(f_i)_{i=1}^{\infty}$ be a sequence of functions from a topological space X into a Hausdorff sequentially complete quasi-uniform space $(Y,\mathcal{U})$ such that $(f_i)_{i=1}^{\infty}$ is $\mathcal{U}$-Cauchy. Then $(f_i)_{i=1}^{\infty}$ converges quasi-uniformly.

Proof: By hypothesis for each $x \in X$, $(f_i(x))_{i=1}^{\infty}$ is a $\mathcal{U}$-Cauchy sequence. Let $x \in X$ and $y \in Y$ such that limit $(f_i(x))_{i=1}^{\infty} = y$. Define

$f : X \to Y$ by $f(x) = y = \text{limit } (f_i(x))_{i=1}^{\infty}$. Clearly $(f_i)_{i=1}^{\infty}$ converges quasi-uniformly to f.

<u>Theorem 4.19</u>: Let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous functions from a topological space (X, τ) into a locally symmetric quasi-uniform space $(Y, \mathcal{U})$ such that $(f_n)_{n=1}^{\infty}$ converges quasi-uniformly to a function $g : X \to Y$. Then g is continuous.

<u>Proof</u>: Let $x \in X$ and $U \in \mathcal{U}$. By Lemma 4.10 there exists a symmetric entourage W in $\mathcal{U}$ such that $W \circ W \circ W(g(x)) \subset U(g(x))$. Let N be a positive integer such that for all $n \geq N$ and for each $z \in X$, $(g(z), f_n(z)) \in W$. Let $n \geq N$ and let $y \in f^{-1}(W(f_n(x)))$; then $(f_n(x), f_n(y)) \in W$. We also have that $(g(y), f_n(y))$, $(g(x), f_n(x)) \in W$ so that $(g(x), g(y)) \in W \circ W \circ W$. Thus $g(y) \in W \circ W \circ W(g(x)) \subset U(g(x))$. Therefore g is a continuous function.

A well-known result for metric spaces is Banach's fixed-point theorem. We conclude this chapter with a discussion of a fixed-point theorem for quasi-uniform spaces. It is interesting to note that the fixed point theorem of A. Davis [7] can be obtained in terms of quasi-uniformities. Since the proof is a translation of A. Davis's argument to the setting of quasi-uniform spaces, we state the result without proof.

Definition 4.20: Let $(X,\mathcal{U})$ be a quasi-uniform space and let $x,y \in X$. We say x is linked to y if for any $V \in \mathcal{U}$ there exists a positive integer K such that $x \in V^K(y)$. $(X,\mathcal{U})$ is totally linked if for all $x,y \in X$, x is linked to y.

Definition 4.21: Let $(X,\mathcal{U})$ be a quasi-uniform space and let f be a function from $(X,\mathcal{U})$ into itself. Let m,n be positive integers such that $m \geq n$. Then f is said to be (m,n) contractive if there exists a basis $\mathcal{B}$ of $\mathcal{U}$ such that for all $B \in \mathcal{B}$, $f \circ B^m \subset B^n \circ f$. A function f is an (m,n) eventual contraction of $(X,\mathcal{U})$ if there is a positive integer K such that f^K is (m,n) contractive.

Theorem 4.22: Let $(X,\mathcal{U})$ be a sequentially complete, totally linked and T_o quasi-uniform space. Let f be any (m,n) eventual contraction of $(X,\mathcal{U})$. The f has a unique fixed point.

Proof: By [6, Theorem 2].

CHAPTER V

QUASI-UNIFORM SPACES AND TOPOLOGICAL HOMEOMORPHISM GROUP

It is well-known that if G is a subgroup of a homeomorphism group H(X) of a topological space X and G is equicontinuous with respect to a compatible uniformity for (X, τ), then G is a topological group under the topology of pointwise convergence. In this chapter we show that if $\mathcal{U}$ is a weakly locally symmetric quasi-uniformity compatible with an R_0 topological space X and G is a subgroup of H(X) which is equicontinuous with respect to $\mathcal{U}$, then G is a topological group under the topology of pointwise convergence. In [16] R. Fuller defines a semi-uniformity and shows that every group of homeomorphisms which satisfies a condition of equicontinuity relative to a semi-uniformity (called semi-equicontinuity) is a topological group under the topology of pointwise convergence. We show that a semi-uniformity can be generated from a locally symmetric quasi-uniformity in such a way that if F is a quasi-uniformly equicontinuous family of functions, then F is also semi-equicontinuous relative to the semi-uniformity generated from $\mathcal{U}$. From this result it follows that if G is a group of homeomorphisms of a space (X, τ) and G is quasi-equicontinuous relative to a locally symmetric quasi-uniformity $\mathcal{U}$ such that $\tau \subset \tau_{\mathcal{U}}$, then G is a topological group under the topology of pointwise convergence with respect to τ.

Remark: It is known that a topological space is T_1 if and only if it is both T_0 and R_0 and that a topological space admits a compatible weakly locally symmetry quasi-uniformity if and only if it is an R_o space [26, Corollary 3.9 and Theorem 3.6].

Note that the point open topology in Definition 4.2 is also called the topology of pointwise convergence.

Theorem 5.1: Let (X, τ) be an R_o topological space, let $\mathcal{U}$ be a compatible weakly locally symmetric quasi-uniformity on X, and let G be a quasi-equicontinuous group of homeomorphisms of X onto X. Let $I^{-1}: G \to G$ be defined by $I^{-1}(f) = f^{-1}$. Then $I^{-1}: G \to G$ is a continuous function with respect to the topology of pointwise convergence.

Proof: For each $x \in X$ define $\ell_x : G \to X$ by $\ell_x(g) = g(x)$ and define $\phi_x : G \to X$ by $\phi_x(g) = g^{-1}(x)$. It is clear that for each $x \in X$, ℓ_x is continuous and $\phi_x = \ell_x \circ I^{-1}$. Thus in order to show that I^{-1} is continuous, it suffices to show that for each $x \in X$, ϕ_x is continuous [8, p. 101, Theorem 2.2]. Let $f \in G$, let $x \in X$ and let $V \in \mathcal{U}$. There exists a symmetric $U \in \mathcal{U}$ such that $U(f^{-1}(x)) \subset V(f^{-1}(x))$. Since G is an quasi-equicontinuous collection, there exists a neighborhood N of x such that for $g \in G$, $g(N) \subset U(g(x))$. Let $W \in \mathcal{U}$ such that $W(x) \subset N$. There exists a symmetric entourage $A \in \mathcal{U}$ such that $A(x) \subset W(x)$. For each $g \in G$, $g(A(x)) \subset U(g(x))$. Thus for each $g \in G$, $(g^{-1}(y), g^{-1}(x)) \in U$ whenever $(x,y) \in A$. Suppose that $g \in W(f^{-1}(x), A)(f)$. Then

$(f(f^{-1}(x)),g(f^{-1}(x))) \in A$ so that $(g^{-1}(x),f^{-1}(x)) \in U$. Since U is symmetric $g^{-1}(x) \in U(f^{-1}(x)) \subset V(f^{-1}(x))$. Hence ϕ_x is continuous.

Theorem 5.2: Let (X,τ) be a topological space, let $\mathcal{U}$ be a compatible quasi-uniformity on X, and let G be a group of homeomorphisms of X onto X which is quasi-equicontinuous with respect to $\mathcal{U}$. Then G is a topological semigroup under the topology of pointwise convergence.

Proof: Throughout the proof, if $p \in X$ and $U \in \tau$, then $W(p,U)$ denotes $\{f \in G : f(p) \in U\}$. Let $g_1,g_2 \in G$ and let $x_o \in X$ and $B \in \tau$ such that $W(x_o,B)$ is a neighborhood of $g_1 \circ g_2$. Then $g_1 \circ g_2(x_o) \in B$. Let $y_o = g_1 \circ g_2(x_o)$ and let $U \in \mathcal{U}$ such that $U(y_o) \subset B$. Then $W(x_o, U(y_o))$ is a neighborhood of $g_1 \circ g_2$ which is contained in $W(x_o,B)$. Let $V \in \mathcal{U}$ such that $V \circ V \subset U$. Since G is quasi-equicontinuous with respect to $\mathcal{U}$, there exists a neighborhood N of $g_2(x_o)$ such that for each $g \in G$, $g(N) \subset V(g(g_2(x_o)))$. Let $Z \in \mathcal{U}$ such that $Z(g_2(x_o)) \subset N$. If $g \in G$ and $x \in Z(g_2(x_o))$, then $(g(g_2(x_o)),g(x)) \in V$. Let $C = W(g_2(x_o),V(y_o))$ and let $D = W(x_o,Z(g_2(x_o)))$. Then C and D are neighborhoods of g_1 and g_2 respectively. Let $g \in C$ and let $h \in D$. Then $(y_o,g(g_2(x_o))) \in V$ and $(g_2(x_o),h(x_o)) \in Z$. Since $(g_2(x_o),h(x_o)) \in Z$, $(g(g_2(x_o)),g(h(x_o))) \in V$.

Since $(y_o, g(g_2(x_o)))$ and $(g(g_2(x_o)), g(h(x_o))) \in V$, $(y_o, g(h(x_o))) \in$ $V \circ V \subset U$. It follows that $g \circ h(x_o) \in U(y_o) \subset B$.

Theorem 5.3: Let (X, τ) be an R_o space, let $\mathcal{U}$ be a compatible weakly locally symmetric quasi-uniformity, and let G be a group of homeomorphisms of X onto X such that G is quasi-equicontinuous with respect to $\mathcal{U}$. Then G is a topological group under the topology of pointwise convergence.

Corollary 5.4: Let (X, τ) be an R_o space, let $\mathcal{U}$ be a covering quasi-uniformity such that $\mathcal{U}$ is a member of the Pervin quasi-proximity class. Let G be a group of homeomorphisms of X onto X such that G is quasi-equicontinuous with respect to $\mathcal{U}$. Then G is a topological group under the topology of pointwise convergence.

We now discuss the full homeomorphism groups of saturated topological spaces. We note that there has been considerable recent interest in such spaces. See for example [36], [35], [17] and [18].

Lemma 5.5: Let (X, τ) be a saturated topological space and let F be the collection of all continuous functions from (X, τ) into a quasi-uniform space $(Y, \mathcal{U})$. Then F is quasi-equicontinuous.

Theorem 5.6: Let (X, τ) be an R_0 saturated topological space. Then the homeomorphism group $H(X)$ is a topological group under the topology of pointwise convergence.

In [36] it is shown that if G is a finitely generated group, then there is a T_0 saturated topological space X such that $H(X)$ is isomorphic to G. In light of Theorem 5.6 it would be interesting to know if the above result of [36] also obtains for R_0 saturated spaces. Note that it follows from the remarks immediately preceding Theorem 5.1 that every R_0, T_0 saturated space is discrete.

Let I denote the identity element of $H(X)$.

Definition 5.7 [23, Definition 2.3]: A topological space is a weak Galois if for each nonempty set $U \in \tau$, there is an $h \in H(X)$ and a point $p \in U$ such that $h|X - U = I|X - U$ and $h(p) \neq p$.

Theorem 5.8 [23, Corollary 2.18]: Every weak Galois space is strongly locally finite stable.

Theorem 5.9 [23, Theorem 4.14]: Let (X, τ) be a strongly locally finite stable topological space. Then $H(X)$ is nondiscrete under the topology of uniform convergence with respect to the coarsest uniformity containing both the locally finite quasi-uniformity and its conjugate.

Since the topology of pointwise convergence is contained in the topology of uniform convergence, it follows from Theorems 5.8 and 5.9 that if (X, τ) is a weak Galois space, then $H(X)$ is nondiscrete under the topology of pointwise convergence.

In Hausdorff spaces it is a simple matter to establish this result directly.

Theorem 5.10: Let (X, τ) be a Hausdorff weak Galois space. Then $H(X)$ is nondiscrete with respect to the topology of pointwise convergence.

Proof: We need only to show that any basic open set containing I is nondegenerate. For each $j = 1, \ldots, n$, let $x_j \in X$, let $N_j \in \tau$ and let $\bigcap_{j=1}^{n} W(x_j, N_j)$ be a neighborhood of I. By hypothesis there exists an open set N of x_1 such that $x_j \notin N$ for all $j > 1$. Let $M = N \cap N_1$. Since X is a weak Galois space, there exists an $h \in H(X)$ such that $h \neq I$ and $h|X - M = I|X - M$. Clearly for each $j > 1$, $h(x_j) = x_j$ and $h(x_1) \in N_1$. Thus $h \in \bigcap_{j=1}^{n} W(x_j, N_j)$. Consequently $\bigcap_{j=1}^{n} W(x_j, N_j)$ is nondegenerate.

Definition 5.11 [16]: Let Y be a topological space. A collection $\mathcal{V}^*$ of two-element open covers of Y is said to be a semiuniformity for

for Y if for each point q in Y and each neighborhood V of q there is a $\{V_1, V_2\}$ in $\mathcal{V}^*$ such that $q \in V_1 \subset V$ and $X - V_2$ is a neighborhood of q.

Remark [16]: One may easily show that a topological space has a semi-uniformity if and only if it is regular.

Definition 5.12 [16]: Let F be a family of functions from a topological space X to semi-uniform space $(Y, \mathcal{V}^*)$. Then F is semi-equicontinuous if for each $\mathcal{V} \in \mathcal{V}^*$ there is an open covering A of X such that A refines $f^{-1}(\mathcal{V})$ for each $f \in F$.

In the statement of the following theorem, which is the main result of [16], we have taken the liberty of correcting several misprints.

Theorem 5.13 [16, Theorem 6]: Let (X, τ) be a regular space, let $\mathcal{V}^*$ be a semi-uniformity for X and let G be a group of homeomorphisms of X onto X such that G is semi-equicontinuous with respect to $\mathcal{V}^*$. Then G is a topological group under the topology of pointwise convergence.

Theorem 5.14 [15, Theorem 2.2]: Let (X, τ) be a topological space and let $\mathcal{U}$ be a compatible quasi-uniformity for (X, τ). There is a base $\mathfrak{b}$ for $\mathcal{U}$ such that if $V \in \mathfrak{b}$ and $x \in X$, then $V(x) \in \tau$.

<u>Definition 5.15</u>: Let $(X,\mathcal{U})$ be a quasi-uniform space. If $x \in X$ and $U_1 \in \mathcal{U}$ such that $U_1(x) \in \tau$ and $U_2 \in \mathcal{U}$ such that $U_2 \circ U_2 \circ U_2 \circ U_2(x) \subset U_1(x)$ and $U_2 = U_2^{-1}$, then $\mathcal{L} = \{U_1(x), \cup\{\text{int } U_2(y) : y \notin U_2 \circ U_2(x)\}\}$ is a <u>two element quasi-uniform cover of</u> X.

<u>Theorem 5.16</u>: Let $(Y,\mathcal{U})$ be a locally symmetric quasi-uniform space. Then the collection of all two element quasi-uniform covers of Y is a semi-uniformity for Y.

<u>Proof</u>: Let $q \in Y$ and let V be a neighborhood of q. By Theorem 5.14, there exists an open entourage U_1 in $\mathcal{U}$ such that $U_1(q) \subset V$. By Lemma 4.10 there is a symmetric entourage U_2 in $\mathcal{U}$ such $U_2 \circ U_2 \circ U_2 \circ U_2(q) \subset U_1(q)$. We will show that $C = \{U_1(q), \cup\{\text{int } U_2(y): y \notin U_2 \circ U_2(q)\}\}$ is a quasi-uniform open cover which satisfies Definition 5.11. Suppose that $x \in Y$ and that $x \notin U_1(q)$. Note that if $z \in Y$ and $z \in U_2 \circ U_2(q)$, then $U_2(z) \subset U_1(q)$. Thus $x \notin U_2 \circ U_2(q)$ and $x \in U_2(x)$. Therefore C is an open cover of Y. Furthermore, let $p \in \text{int } U_2(q)$ and suppose that $p \in V_2 = \cup\{\text{int } U_2(y) : y \notin U_2 \circ U_2(q)\}$. Then there exists a $y \in Y$ such that $p \in \text{int } U_2(y)$ and $y \notin U_2 \circ U_2(q)$. But $y \in U_2(p) \subset U_2 \circ U_2(q)$--a contradiction. Then $\mathcal{V}^* = \{C : q \in Y \text{ and } V \text{ is a neighborhood } q\}$ is a semi-uniformity for Y.

Definition 5.17: The semi-uniformity $\mathcal{V}^*$ of the preceding theorem will be called a quasi-uniform semi-uniformity.

Theorem 5.18: Let $(Y, \mathcal{V})$ be a locally symmetric quasi-uniform space and let F be a family of quasi-equicontinuous functions from a topological space (X, τ) into $(Y, \mathcal{V})$. Then F is semi-equicontinuous with respect to the quasi-uniform semi-uniformity of $\mathcal{V}$.

Proof: Let $\mathcal{V}^*$ be the quasi-uniform semi-uniformity of $\mathcal{V}$, let $y, q \in Y$ and let U_1 and $U_2 \in \mathcal{V}$. Let $\mathcal{L} \in \mathcal{V}^*$ such that $\mathcal{L} = \{U_1(q), \cup\{\text{int } U_2(y) : y \notin U_2 \circ U_2(q)\}\}$. By hypothesis, for each $x \in X$ there exists a neighborhood N_x of x such that all $f \in F$, $f(N_x) \subset U_2(f(x))$. It may be seen that $U_2(f(x))$ is contained in either $U_1(q)$ or $A = \cup\{\text{int } U_2(y) : y \notin U_2 \circ U_2(q)\}$. Let $z_1, z_2 \in Y \cap U_2(f(x))$, so that $z_1 \notin U_1(q)$ and $z_2 \notin A$. Now if $z_2 \notin A$, then $z_2 \in U_2 \circ U_2(q)$ and $(q, z_2) \in U_2 \circ U_2$. Since $(z_2, f(x)) \in U_2$ and $(f(x), z_1) \in U_2$, $z_1 \in U_2 \circ U_2 \circ U_2 \circ U_2(q) \subset U_1(q)$ -- a contradication. Thus $\{N_x : x \in X\}$ is the desired open cover of X.

Theorem 5.19: Let F be a family of one-to-one functions of a topological space (X, τ) onto itself. Let $\mathcal{V}$ be a locally symmetric quasi-uniformity on X such that $\tau \subset \tau_{\mathcal{V}}$. If F^{-1} is $\mathcal{V}$-quasi-equicontinuous, then the mapping $\Psi : F \to F$ defined by $\Psi(f) = f^{-1}$ is continuous relative to the topology of pointwise convergence on F and F^{-1}.

Proof: Let $\mathcal{V}^*$ be the quasi-uniform semi-uniformity of $\mathcal{V}$. Let $g \in F$, $p \in X$ and $V \in \tau$ such that $W(p,V)$ is a neighborhood of g^{-1}. Since $\tau \subset \tau_{\mathcal{V}}$ there is $\{V_1, V_2\} \in \mathcal{V}^*$ such that $g^{-1}(p) \subset V_1 \subset V$ and $X - V_2$ is a $\tau_{\mathcal{V}}$ neighborhood of $g^{-1}(p)$. By Theorem 5.18, F^{-1} is semi-equicontinuous with respect to $\mathcal{V}^*$. Let $\mathcal{U}$ be a τ-open cover of X such that $\mathcal{U}$ refines $\{f(V_1), f(V_2)\}$ for all $f \in F$ and let U be a member of $\mathcal{U}$ which contains p. Then $W(g^{-1}(p), U)$ is a neighborhood of g. Now let $f \in F$ such that $f \in W(g^{-1}(p), U)$. Then $f(g^{-1}(p)) \in U$ and since $f(g^{-1}(p)) \notin f(V_2)$, $U \not\subset f(V_2)$. Hence $U \subset f(V_1)$ and $f^{-1}(U) \subset V_1 \subset V$. Consequently $f^{-1}(p) \in V$.

Theorem 5.20 [16, Theorem 5]: Let F be a semigroup (under composition) of continuous functions from a topological space X into itself. If the point-open topology on F is jointly continuous, then composition is continuous relative to the point-open topology.

Theorem 5.21: Let (X, τ) be any topological space and let G be a group of homeomorphisms of X onto X. Let $\mathcal{V}$ be any locally symmetric quasi-uniformity on X such that $\tau \subset \tau_{\mathcal{V}}$ and G is quasi-equicontinuous with respect to $\mathcal{V}$. Then G is a topological group under the topology of pointwise convergence.

Proof: Let $\mathcal{V}^*$ be quasi-uniform semi-uniformity of $\mathcal{V}$. By hypothesis the topology of pointwise convergence on G is jointly continuous [Proposition 4.6]. Thus by Theorems 5.19 and 5.20, G is a topological group under the topology of pointwise convergence.

We note that the following special case of Theorem 5.3 may be obtained from 5.21.

Corollary 5.22: Let (X,τ) be a regular space, let $\mathcal{U}$ be a compatible locally symmetric quasi-uniformity and let G be a group of homeomorphisms of X onto X such that G is quasi-equicontinuous with respect to $\mathcal{U}$. Then G is a topological group under the topology of pointwise convergence.

APPENDIX

Theorem 5.3 generalizes the classical result that an equicontinuous group of homeomorphisms is a topological group under the point open topology in that Theorem 5.3 is applicable to R_o spaces whereas the classical result only obtains in uniformizable (i.e. completely regular) spaces. In this appendix we give an example of an R_o space that is not regular whose homeomorphism group under the point-open topology contains, for each positive prime p, an isomorphic copy of $\Sigma_{i=1}^{\infty} Z_{p_i}$. Indeed these subgroups may be considered as topological vector spaces over Z_p.

This example shows that the homeomorphism group of an R_o space that is not regular may possess interesting quasi-equicontinuous subgroups. Our last example shows that the homeomorphism group of a compact metrizable space may have subgroups that are quasi-equicontinuous relative to a locally symmetric quasi-uniformity which is not a uniformity.

Example 6.1: For each positive prime p let G_p denote the direct sum of countably many copies of Z_p and let Z^+ denote the set of all positive integers. There is a compact T_1 topology on Z^+ such that the homeomorphism group of (Z^+, τ) under the topology of pointwise convergence contains a metrizable nondiscrete subgroup G_p for each positive prime p. Furthermore, for each positive prime p, G_p is a topological vector space under the relative topology with respect to the topology of pointwise convergence.

<u>Construction</u>: Let τ be the collection of all subsets A of Z^+ such that either $1 \notin A$ and $2 \notin A$ or A contains all but finitely many elements of Z^+. Then (Z^+, τ) is compact and T_1 and hence R_o, but (Z^+, τ) is not regular. Throughout this section we will let $\mathcal{P}$ denote the Pervin quasi-uniformity on (Z^+, τ).

Let P be a positive prime number and let n be any positive integer such that $n > 2$. We define a sub-family F of $H(Z^+)$ inductively in the following manner:

$$f_1(i) = 1, \quad i < n \text{ or } i \geq n + P;$$

$$f_1(i) = i + 1, \quad n \leq i \leq n + P - 2;$$

$$f_1(n + P - 1) = n;$$

$$f_{k+1}(i) = f_k(i), \quad i < n + kP \text{ or } i \geq n + (k + 1)P;$$

$$f_{k+1}(i) = i + 1, \quad n + kP \leq i \leq n + (k + 1)P - 2;$$

$$f_{k+1}(n + (k + 1)P - 1) = n + kP.$$

Let G_p denote the homeomorphism group generated by $\{f_i\}_{i=1}^{\infty}$. The following sequence of propositions are sufficient to establish the desired properties of G_p.

Proposition 6.2: The group G_p is quasi-equicontinuous on Z^+ with respect to $\mathcal{P}$.

Proof: It is sufficient to show that G_p is quasi-equicontinuous with respect to subbasic entourages in $\mathcal{P}$. We first prove that G_p is equicontinuous at 1. Let $A \in \tau$ and $B = A \times A \cup (Z^+ - A) \times Z^+ \in \mathcal{P}$. If $1 \notin A$, then for any neighborhood N of 1 and any $h \in G_p$, $h(N) \subset B(h(1)) = B(1) = Z^+$. Suppose that $1 \in A$. Let $F = Z^+ - A$; by hypothesis, $F = \{x_1,\ldots,x_n\}$. Let $s = \sup\{x_i\}_{i=1}^n$ and let $m = s + p$. Then $N = \{y : y \in Z^+ \text{ and } y > m\} \cup \{1\}$ is the desired neighborhood of 1, since for all $h \in G_p$, $h(N) \subset A$. Similarly, it may be seen that G_p is quasi-equicontinuous at 2. If $x \in Z^+$ and $x \neq 1,2$ then $\{x\}$ is open and for all $h \in G$, $h(x) \in B(h(x))$.

Note that $(G_p,\overline{\mathcal{U}}|G_p)$ is nondiscrete.

Proposition 6.3: $(G_p,\overline{\mathcal{U}}|G_p)$ is metrizable.

Proof: It is not difficult to see that the identity of G_p has a countable base.

For f, $g \in G_p$ we define $f \circ g = f + g$ and let 0 be the identity function.

Lemma 6.4: G_p is a module over Z_p.

Proof: Let $i,j \in Z_p$ and let $h,g \in G_p$ then,

(a) $(i + j)g = ig + jg$,

(b) $i(g + h) = ig + ih$,

(c) $i(jg) = ijg$,

(d) $1g = g$.

Proposition 6.5: $G_p = \oplus \Sigma_{i=1}^{\infty} Z_{p_i}$.

Proof: It suffices to show that $\{f_i\}_{i=1}^{\infty}$ is linearly independent over the field Z_p [22, p. 85, Corollary 1]. Suppose that $\{f_i\}_{i=1}^{\infty}$ is linearly dependent over the field Z_p. Then there is a finite subcollection $\{f_j\}_{j=1}^{n}$ of $\{f_i\}_{i=1}^{\infty}$ and a finite subset $\{a_j\}_{j=1}^{n}$ of Z_p such that $\Sigma_{j=1}^{n} a_j f_j = 0$ and at least one $a_j \neq 0$. Let $w = \sup\{j : 1 \leq j \leq n$ and $a_j \neq 0\}$. Let $m \in Z^+$ such that $n + p(w - 1) < m < n + pw$. Then $a_j f_j(m) = m$ if and only if $j \neq w$. Consequently $\Sigma_{j=1}^{n} a_j f_j \neq 0$--a contradiction.

Proposition 6.6: The map $(Z_p \times G_p) \to G_p$, defined by $(i,g) \to ig$, is continuous with respect to the discrete topology on Z_p and the topology of pointwise convergence on G_p.

<u>Proof</u>: Let $i \in Z_p$ and $f_j \in \{f_i\}_{i=1}^{\infty}$. To show continuity of the map $(i,g) \to ig$, it is sufficient to show continuity at the points (i,f_j). Suppose that $i \neq 0$. Then for each $x \in Z^+$, $if_j(x) = x$ if and only if $f_j(x) = x$. Let $x \in Z^+$ and $A \in \tau$. Suppose that $W(x,A)$ is a neighborhood of if_j and that $f_j(x) \neq x$. It is not difficult to see that $f_j \in W(x,\{x + 1\})$ and that for each $g \in G_p \cap W(x,\{x + 1\})$, $g \in W(x,A)$. Now suppose that $f_j(x) = x$ and that $x \neq 1,2$. Then $W(x,\{x\})$ is the desired neighborhood of f_j. If $x = 1,2$, then $W(x,A)$ is itself the desired neighborhood of f_j.

<u>Example 6.7</u>: A quasi-uniform space $(X,\mathcal{U})$ which is weakly locally symmetric, but which is not a uniform space. Furthermore, $\tau_{\mathcal{U}}$ is compact and metrizable, and for each positive prime p, there exists a topological group $H_p \subset H(X)$ such that $H_p = \Sigma_{i=1}^{\infty} Z_{p_i}$.

<u>Construction</u>: Let $X = \{0,1/2,1/3,\ldots,1/n,\ldots\}$. For $\epsilon > 0$ let $B_\epsilon = \{(x,y) : y \geq x \text{ and } y - x < \epsilon\}$ and $\mathfrak{p} = \{B_\epsilon : \epsilon > 0\}$. It is not difficult to see that $\mathfrak{p}$ generates the required quasi-uniformity.

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VITA

The author was born November 26, 1939 in Mashhad, Iran. He attended public school in Mashhad and graduated from Ebbnee-Yamin High School in June, 1959. He received his Bachelor of Science from The University of Wisconsin in 1956 and his Master of Arts from The University of South Dakota in 1967. His graduate work at Virginia Polytechnic Institute and State University began in September, 1967. He is a member of The American Mathematical Society.

Massood Seyedin

LOCAL PROPERTIES OF TRANSITIVE QUASI-UNIFORM SPACES

by

Massood Seyedin

(ABSTRACT)

If (X, τ) is a topological space, then a quasi-uniformity $\mathcal{U}$ on X is compatible with τ if the quasi-uniform topology, $\tau_{\mathcal{U}} = \tau$. This paper is concerned with local properties of quasi-uniformities on a set X that are compatible with a given topology on X.

Chapter II is devoted to the construction of Hausdorff completions of transitive quasi-uniform spaces that are members of the Pervin quasi-proximity class.

Chapter III discusses locally complete, locally precompact, locally symmetric and locally transitive quasi-uniform spaces.

Chapter IV is devoted to function spaces of quasi-uniform spaces.

Chapter V and the Appendix are concerned with the topological homeomorphism groups of quasi-uniform spaces.