Classical and Quantum Gravity with Ashtekar Variables

by

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(ABSTRACT)

This thesis is a study of classical and quantum gravity with Ashtekar variables. The Ashtekar constraints are shown to capture the essence of the constraints and constraint algebra of General Relativity in four dimensions. A classification scheme of the solution space of the Ashtekar constraints is proposed and the corresponding physics is investigated. The manifestly covariant equations of motion for the Ashtekar variables are derived. Explicit examples are discussed and new classical solutions of General Relativity are constructed by exploiting the properties of the Ashtekar variables.

Non-perturbative canonical quantization of the theory is performed. The ordering of the quantum constraints as well as the formal closure of the quantum constraint algebra are explored. A detailed Becchi-Rouet-Stora-Tyutin (BRST) analysis of the theory is given. The results demonstrate explicitly that in quantum gravity, fluctuations in topology can occur and there are strong evidences of phases in the theory. There is a phase which is described by a topological quantum field theory (TQFT) of the Donaldson-Witten type and an Abelian anti-instanton phase wherein self-interactions of the gravitational fields produce symmetry breaking from SO(3) to U(1). The full theory is much richer and

includes fluctuations which bring the system out of the various restricted sectors while preserving diffeomorphism invariance. Invariants of the quantum theory with are constructed through BRST descents. They provide a clear and sytematic characterization of non-local observables in quantum gravity, and can yield further differential invariants of four-manifolds.

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Chapter 1

Overview

Our current understanding of space-time is based upon Einstein's theory of General Relativity. However, in spite of the spectacular success of the classical theory, gravity has so far stood apart from the other three fundamental forces. These, the strong, weak and electromagnetic forces, have been made compatible with the demands of quantum mechanics. On the other hand, despite the efforts of many, a consistent theory of quantum gravity eludes us. Yet because gravity couples to all matter, consistency of the theoretical framework of physics requires that gravity be quantized.

In four dimensions, the Einstein-Hilbert action is nonrenormalizable in the usual sense. The gravitational coupling constant G has dimension $(\text{mass})^{-2}$ $(\hbar = c = 1)$ signaling that the number of counterterms needed is infinite. Although 't Hooft and Veltman[HV] found that pure gravity is on-shell finite at the one-loop level, the result no longer holds when coupling to matter is considered. Explicit computations by Goroff and Sagnotti[GS] proved that even on-shell pure gravity diverges at the two-loop level. Despite its nonrenormalizability, there are many who believe that somehow an *exact* treatment of the theory would yield sensible results because gravity has the privileged role of describing space-time. In perturbative treatments, the Hamiltonian cannot be guaranteed to be bounded from below so as to avoid dramatic instability. Yet the canonical analysis reveals that the exact Hamiltonian for manifolds without boundary is a linear combination of the constraints and therefore vanishes when the constraints are treated exactly. For manifolds with boundary, the boundary Hamiltonian is in fact positive definite, as demonstrated by the celebrated proofs of Schoen and Yau[SY], Faddeev[FD] and Witten[WT4]. Of course exact treatment of the theory is easier said than done, for the constraints of the Einstein-Hilbert action are highly complicated. In fact they are non-polynomial in their dependence on the basic variables. An exact canonical quantization of the theory requiring that the quantum states are annihilated by the constraints will have to demonstrate the closure of the quantum constraint algebra as a consistency requirement. With conventional variables, the constraints are so intractable that not a single quantum state has ever been found and there is no consensus on the closure of the quantum constraint algebra. It is therefore not surprising that Ashtekar's announcement of remarkable simplifications of the constraints attracted a great deal of interest. This thesis is a study of classical and quantum gravity with Ashtekar variables.

In four dimensions, gravity faces yet other challenges. Although much has been learned recently about gravity in two and three dimensions[PO][WT2], there are several reasons to suspect that the extension of these results to four dimensions will not be straightforward. Using the moduli spaces of SU(2) and SO(3) anti-instantons, Donaldson and others have revealed that in four dimensions, there are far richer differential structures than in any other (see for instance [DK]). In General Relativity, diffeomorphism invariance is all-important, and it is reasonable to expect that the differential invariants of four-manifolds are significant to gravity. How these differential structures are accounted for in gravity is not entirely clear, and their complete characterization and evaluation can prove to be a daunting task indeed. A simple count of the number of constraints and conjugate pairs shows that, unlike in two and three dimensions, pure gravity can have up to two local degrees of freedom in four dimensions. Witten, however, has suggested that gravity may also exist in a phase wherein there are no propagating local degrees of freedom and all excitations are global[WT1][WT3]. Such a phase would be analogous to a topological quantum field theory (TQFT). In particular, by considering the moduli space of anti-instantons, the Donaldson maps[DO], which are differential invariants of four-manifolds, were successfully identified as Becchi-Rouet-Stora-Tyutin (BRST) invariants of the corresponding TQFT by Witten. This thesis will show that in many respects, the Ashtekar variables are well suited to the analysis of questions on phases and differential structures in four-dimensional gravity and some answers to these questions will be given.

In Chapter 2, the canonical analysis of the Einstein-Hilbert action is briefly reviewed. Due to diffeomorphism invariance, the canonical form of the action reveals that apart from boundary terms, the Hamiltonain consists solely of constraints. The Ashtekar variables[AS] are introduced and they are shown to capture the essence of the constraints and constraint algebra of General Relativity.

Information pertaining to the question of phases in the theory can come from knowledge of the reduced phase space. In Chapter 3, the solution space of the Ashtekar constraints is demonstrated to be divided into sectors with striking discontinuities between them. For instance, there exists a sector with no local degrees of freedom. These sectors are classified according to the values of a parameter S and are suggestive of phases in the theory. In later chapters, S is shown to play an effective role as an order parameter charaterizing various phases of the theory.

The manifestly covariant equations of motion are derived in Chapter 4 through the canonical formalism with a Hamiltonian consisting entirely of the Ashtekar constraints. They are shown to be equivalent to the Einstein Field Equations in four dimensions. The field equations written in the new form, indicate that in terms of their Ashtekar potentials, all Einstein manifolds can be thought of as anti-instantons. The reasons behind this claim are explored and it is shown how topological invariants of four-manifolds, such as the Euler number and signature, are expressed in terms of the Ashtekar variables. These, and a further invariant (later shown to be differential rather than topological in nature in Chapter 7) are shown to come from the integrals of the characteristic classes of S.

The formalism developed in the earlier chapters is first applied to the discussion of explicit classical solutions in Chapter 5. Besides checking the validity of the framework, the examples show how Einstein manifolds manifest themselves in terms of the new variables and demonstrate that, compared to the traditional metric variables, the new variables have greater affinity with the differential structures of four dimensions. In the chapter, new classical solutions of the Einstein Field Equations are constructed by exploiting the properties of the Ashtekar variables. It is also discovered that there is an abelian anti-instanton phase wherein self-interactions of the gravitational fields generate dynamical Higgs fields and symmetry breaking from SO(3) to U(1).

The non-perturbative canonical quantization of the theory is tackled in Chapter 6. There, the major concern of closure of the quantum constraint algebra is addressed. The ordering of the quantum superhamiltonain constraint is examined with the help of an explicit representation of three-geometry, the Chern-Simons functional. Many orderings are uncovered and an ordering with formal closure of the quantum constraint algebra is proposed and its physical implications are discussed. A solution of the exact Ashtekar-Wheeler-DeWitt Equation, which corresponds precisely to the Type O sector of the reduced phase space discussed earlier in Chapter 3, is found. The existence of phases is also supported by the quantum theory. Phases without local degrees of freedom analogous to topological quantum field theories can exist. With regard to the question of topology change, there is explicit demonstration that in quantum gravity even fluctuations in topology can occur.

Chapter 7 is concerned with the covariant BRST quantization of the theory. The BRST symmetry of the theory is discussed and the associated transformations of the variables and ghosts are worked out. Invariants and observables of the quantum theory are constructed through BRST descent sequences[CS2]. These descents provide a systematic and clear characterization of non-local observables in gravity with unbroken diffeomorphism invariance. The question of phases is analysed again in the context of the degeneracy of the descents. BRST invariant gauge-fixing and subtleties associated with Gribov ambiguities are discussed and explicit examples are given. A BRST invariant functional measure is constructed. Chapter 8 concludes the thesis with some remarks on the directions for further investigations suggested by this work.

Chapter 2

2.1 Canonical analysis of the Einstein-Hilbert action

The Einstein-Hilbert action with cosmological term is

$$S = \frac{1}{16\pi G}\int d^4x \, \sqrt{g}(R+2\lambda) \tag{2.1.1}$$

The first canonical analyses of the Einstein-Hilbert action were by Dirac[DI] and Arnowitt, Deser and Misner (ADM) [ADM]. In the ADM formalism, the space-time metric is decomposed into the spatial metric of constant-x^0 hypersurfaces, g_{ij}, and the lapse and shift functions, N and N^i, according to

$$g_{\mu\nu} = \begin{bmatrix} g_{ij}N^iN^j \mp N^2 & g_{ij}N^i \\ g_{ij}N^j & g_{ij} \end{bmatrix} \tag{2.1.2a}$$

i.e.

$$ds^2 = \mp (Ndx^0)^2 + g_{ij}(N^i dx^0 + dx^i)(N^j dx^0 + dx^j) \tag{2.1.2b}$$

(Throughout this work, unless stated otherwise, lower case Latin indices will run from 1 to 3 while upper case Latin indices and Greek indices take values from 0 to 3). The lapse and shift functions have geometrical interpretations in hypersurface deformations. Ndx^0 and $N^i dx^0$ represent the normal and tangential components of the displacement vector connecting two points with the same label, $\vec{x}$, on consecutive constant-x^0 hypersurfaces. The negative (positive) sign in (2.1.2) is to be used for metrics of Lorentzian (Euclidean) signature.

The ADM decomposition of the metric allows the Einstein-Hilbert action to be cast into the canonical form (for reviews, see for instance [SU][KH] besides [ADM])

$$S = \frac{1}{16\pi G} \int dx^0 \int d^3x \; \pi^{ij}\dot{g}_{ij} - N^i\mathcal{H}_i - N\mathcal{H} \; + \text{boundary terms} \tag{2.1.3}$$

with the momenta conjugate to g_{ij} given by

$$\pi^{ij} = \mp\frac{\sqrt{g}}{16\pi G}[K^{ij} - g^{ij}K^m{}_m] \tag{2.1.4}$$

where K_{ij} is the extrinsic curvature of the constant-x^0 hypersufaces. All spatial indices are raised and lowered by the spatial metric. $\mathcal{H}_i$ and $\mathcal{H}$ are called the supermomentum and superhamiltonian respectively. In terms of the conjugate variables they are repectively

$$\begin{aligned}\mathcal{H}_i &= -2\pi_i{}^j|_j \\ &= \pm 2\sqrt{g}\;[K_i{}^j|_j - K^j{}_j|_i]\end{aligned} \tag{2.1.5}$$

and

$$\begin{aligned}\mathcal{H} &= \pm\frac{8\pi G}{\sqrt{g}}(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl})\pi^{ij}\pi^{kl} - \frac{\sqrt{g}}{16\pi G}({}^3R - 2\lambda) \\ &= \frac{1}{16\pi G}\sqrt{g}[\pm(K^{ij}K_{ji} - (K^i{}_i)^2) - {}^3R \; + 2\lambda]\end{aligned} \tag{2.1.6}$$

In Eqn. (2.1.6), the vertical bar denotes covariant derivative with respect to the spatial metric and ${}^3R$ is the scalar curvature of the constant-x^0 hypersurfaces. The form of the action in (2.1.3) clearly indicates that the lapse and shift functions are Lagrange multipliers. The former for the supermomentum constraint

$$\mathcal{H}_i = 0 \tag{2.1.7}$$

and the latter for the superhamiltonian constraint

$$\mathcal{H} = 0 \tag{2.1.8}$$

Thus apart from boundary terms, the hamiltonian of General Relativity consists solely of constraints. This is because the theory is reparametrization invariant and the constraints generate the diffeomorphism symmetry of the theory. The canonical analysis also reveals that, of the ten components of the space-time metric, only the six components of the spatial metric have conjugate momenta and there are four constraints on the conjugate pairs. Thus by this method of counting, General Relativity in four dimensions has two field degrees of freedom.

The constraints are first class and obey the constraint algebra

$$\{ \mathcal{H}_i[N^i] , \mathcal{H}_j[M^j] \}_{P.B.} = -\mathcal{H}_i[(\mathcal{L}_{\vec{N}}M)^i] \tag{2.1.9a}$$

$$\{ \mathcal{H}_i[N^i] , \mathcal{H}[M] \}_{P.B.} = -\mathcal{H}[\mathcal{L}_{\vec{N}}M] \tag{2.1.9b}$$

$$\{ \mathcal{H}[N] , \mathcal{H}[M] \}_{P.B.} = \mp\mathcal{H}_i[g^{ij}(N\partial_j M - M\partial_j N)] \tag{2.1.9c}$$

with $\mathcal{L}$ denoting the Lie derivative. The smearing of the constraints are denoted by

$$\mathcal{H}[N] = \int_{M^3} N\mathcal{H} \quad , \quad \mathcal{H}_i[N^i] = \int_{M^3} N^i\mathcal{H}_i \tag{2.1.9d}$$

with M^3 being a constant-x^0 hypersurface. Note however that the structure function of (2.1.9c) depends on the dynamical variable g_{ij} .

2.2 The Ashtekar Variables

The Ashtekar variables[AS] can be obtained from the $3+1$ decomposition of the Einstein-Hilbert action through a series of canonical transformations[NHS]. For metrics of Lorentzian signature, the canonical variables are the complex Ashtekar gauge potentials

$$A_{ia} = iK_{ia} - \tfrac{1}{2}\epsilon_a{}^{bc}\omega_{ibc} \tag{2.2.1}$$

and densitized triads of weight 1,

$$\tilde{\sigma}^{ia} = \sqrt{g}\sigma^{ia} \tag{2.2.2}$$

obeying the Poisson brackets

$$\{ A_{ia}(\vec{x}) , \tilde{\sigma}^{jb}(\vec{y}) \}_{P.B.} = \frac{i(16\pi G)}{2} \delta_a{}^b \delta_i^j \, \delta^3(\vec{x}-\vec{y}) \tag{2.2.3}$$

$$\{ A_{ia}(\vec{x}) , A_{jb}(\vec{y}) \}_{P.B.} = \{ \tilde{\sigma}^{ia}(\vec{x}) , \tilde{\sigma}^{jb}(\vec{y}) \}_{P.B.} = 0 \tag{2.2.4}$$

ω_{ab} is the torsionless spin connection compatible with the dreibein e_{ia}, and modulo the constraint which generates triad rotations, K_{ia} is related to the extrinsic curvature, K_{ij}, through $K_{ia} = \sigma^{j}{}_{a}K_{ij}$. In terms of the Ashtekar variables, the constraint generating local SO(3) triad rotations which leave the spatial metric $g^{ij} = \sigma^{ia}\sigma^{j}{}_{a}$ invariant can be written in the form of Gauss' Law

$$G_a \equiv \frac{2}{i(16\pi G)} (D_i\tilde{\sigma}^i)_a = 0 \tag{2.2.5}$$

Ashtekar showed that, modulo this constraint, the usual supermomentum and superhamiltonian constraints of ADM achieved remarkable simplication when expressed in terms of the new variables. Indeed starting with the definition of A in (2.2.1), it is straightforward to show that, modulo Gauss' Law, the supermomentum of Eqn. (2.1.5) is proportional to

$$H_i \equiv \frac{2}{i(16\pi G)} \tilde{\sigma}^{ia}F_{ija} \tag{2.2.6}$$

while

$$H \equiv \frac{1}{16\pi G}(\epsilon_{abc}\tilde{\sigma}^{ia}\tilde{\sigma}^{jb}F_{ijc} + \frac{\lambda}{3}\epsilon_{abc}\epsilon_{ijk}\tilde{\sigma}^{ia}\tilde{\sigma}^{jb}\tilde{\sigma}^{kc}) \tag{2.2.7}$$

is proportional to $\sqrt{g}$ times the superhamiltonian of Eqn. (2.1.6). For metrics of Euclidean signature, one finds that by setting the i's to unity, (2.2.6) and (2.2.7) are still proportional to the Euclidean-signature versions (those with the bottom signs) of (2.1.5) and (2.1.6) respectively. Thus one can take Gauss' Law, (2.2.5), and the modified supermomentum and superhamiltonain constraints,

$$H_i = 0 \tag{2.2.8}$$

and

$$H = 0 \tag{2.2.9}$$

as the new constraints of the theory with Ashtekar variables. Indeed for this new set of constraints, the constraint algebra takes the form

$$\{\, H_i[N^i] \,,\, H_j[M^j] \,\}_{P.B.} = -H_i[(\mathcal{L}_{\vec{N}}M)^i] + G^a[-iN^iM^jF_{ija}\,] \tag{2.2.10a}$$

$$\{\, H_i[N^i] \,,\, H[\underset{\sim}{M}] \,\}_{P.B.} = -H[\mathcal{L}_{\vec{N}}\underset{\sim}{M}] + G^a[-i\underset{\sim}{M}N^i\tilde{\sigma}^{jb}\epsilon_{abc}(F_{ij}{}^c + \tfrac{\lambda}{2}\epsilon_{ijk}\tilde{\sigma}^{kc})] \tag{2.2.10b}$$

$$\{\, H[\underset{\sim}{N}] \,,\, H[\underset{\sim}{M}] \,\}_{P.B.} = \mp H_i[\tilde{\sigma}^{ia}\tilde{\sigma}^j{}_a(\, \underset{\sim}{N}\partial_j\underset{\sim}{M} - \underset{\sim}{M}\partial_j\underset{\sim}{N}\,)] \tag{2.2.10c}$$

$$\{\, G^a[\alpha_a] \,,\, G^b[\beta_b] \,\}_{P.B.} = G^a[\epsilon_a{}^{bc}\alpha_b\beta_c] \tag{2.2.10d}$$

$$\{\, G^a[\alpha_a] \,,\, H[\underset{\sim}{N}] \,\}_{P.B.} = \{\, G^a[\alpha_a] \,,\, H_i[N^i] \,\} = 0 \tag{2.2.10e}$$

The i's are to be set to unity and the lower sign in (2.2.10c) should be used for metrics with Euclidean signature. (2.2.10a)-(2.2.10c) should be compared with (2.1.9a)-(2.1.9c). (2.2.10d) is simply the algebra of the generators the local SO(3) gauge transformations while (2.2.10e) expresses the gauge invariance of the new supermomentum and superhamiltonian constraints. Thus, despite the remarkable simplifications in the dependence of the constraints on the new variables, Ashtekar managed to capture the essence of the constraints and constraint algebra of four-dimensional gravity.

A few remarks are in order. For metrics of Euclidean signature, the Ashtekar variables can be assumed to be real (and in the quantum theory, Hermitian with respect to a suitable inner product). There are seven constraints on the nine conjugate pairs. By this method of counting, there are two field degrees of freedom as in the case of General Relativity with the ADM variables.

In the case of metrics with Lorentzian signature, the Ashtekar variables are complex in general and reality conditions have to be imposed on the variables so that real Lorentzian General Relativity with two local degrees of freedom is projected out by the new variables. In order to recover real Lorentzian General Relativity, we can for instance translate the reality conditions of the old variables to conditions on A_{ia} and $\tilde{\sigma}^{ia}$. These conditions are that $\tilde{\sigma}^{ia}$ be real and

$$(A_{ia} - \frac{1}{2}\epsilon_a{}^{bc}\omega_{ibc})^* = -(A_{ia} - \frac{1}{2}\epsilon_a{}^{bc}\omega_{ibc}) \tag{2.2.11}$$

which is equivalent to K_{ia} being real. There are alternative sets of reality conditions which also allow for real Lorentzian General Relativity to be recovered from the new variables (discussions on these reality conditions can be found in [ASL][FK][BS]).

The Ashtekar constraints, unlike the ADM constraints, are polynomial (at most quartic in order) in their dependence on the new canonical variables. In non-perturbative canonical quantization, this simplifies the analysis of the ordering and formal closure of the quantum constraint algebra and facilitates the search for explicit quantum states of the theory. There is also the remarkable fact that in four dimensions, the constraint algebra (2.1.9) with General Relativity as its explicit representation, is modulo gauge-invariance, almost identical to the algebra (2.2.10) which has explicit representation in terms of the Ashtekar variables and constraints. These, by themselves, are enough to render the Ashtekar formalism worthy of further study. The following chapters will be concerned with the description of classical and quantum gravity in four dimensions in terms of the Ashtekar variables and the physical implications of the formalism.

Chapter 3

Classification of the initial-value data

In this section, we exhibit a classification scheme of the solution space of the Ashtekar constraints .

It is known that all Einstein manifolds in four dimensions can be classified according to the canonical forms of their Riemann-Christofel curvature tensors. Such a scheme was first given by Petrov[PE] and then considered by Penrose[PR] in the context of spinors and null tetrads. The scheme is tailored specifically for dimension four and every solution of the Einstein equations belongs to one of the Petrov types.

In the ADM formalism, the supermomentum and superhamiltonian constraints are projections of the Einstein Field Equations tangential and normal to the three-dimensional hypersurface, M^3, on which the initial data compatible with the constraints is specified. The solutions of the constraints when stacked up according to their x^0- evolution by the Hamiltonian (which apart from boundary terms, is solely a linear combination of the constraints) are then solutions of the four-dimensional Einstein Field Equations. The phase space can also be thought of as the space of all possible classical solutions (for a discussion on the covariance of the phase space of General Relativity, see [WC]). It is therefore logical to ask whether a classification scheme such as Petrov's can be incorporated directly into the initial-value problem and if so, to what extent it reflects the peculiarities of the reduced phase space and the corresponding physics. We shall see that indeed the solution space of the Ashtekar constraints can be classified in precisely such a manner. It is not obvious that the usual constraints of the General Relativity with the ADM variables allow such a clear and dramatic classification of the initial data.

In the ADM formalism the metric is assumed to be non-degenerate i.e. it has non-vanishing determinant. Ashtekar's formulation of the constraints allows for both degenerate and non-degenerate metric since the Ashtekar constraints, (2.2.5), (2.2.7) and (2.2.9) do not involve the inverse of the conjugate momenta $\tilde{\sigma}^{ia}$. For non-degenerate metrics, the magnetic field of the Ashtekar connection , $B^{ia} = \frac{1}{2}\epsilon^{ijk}F_{jk}{}^{a}$, can be expanded in terms of the densitized triads and the most general solution to the supermomentum constraint is[CA][CS1]

$$B^{ia} = \tilde{\sigma}^{i}{}_{b}S^{ba} \tag{3.1}$$

with S being a symmetric 3×3 matrix which is $\vec{x}$-dependent in general ($\vec{x}$ denotes the position on the initial-value hypersurface). Note that (3.1) still solves the constraint even if the metric is degenerate. With this, the superhamiltonian constraint becomes an algebraic relation

$$\det(\tilde{\sigma})(\ \mathrm{Tr}S + \lambda\ ) = 0 \tag{3.2}$$

and has solution

$$\mathrm{Tr}S = S^{a}{}_{a} = -\lambda \tag{3.3}$$

for non-degenerate metrics, and undetermined TrS for degenerate metrics. It is intriguing to note the apparent shift of the freedom from $\det(\tilde{\sigma})$ to Tr(S) when the metric is degenerate. (There have been speculations on the significance of metrics that have vanishing determinants at certain space-time points to topology-changing situations in quantum gravity[HO1]).

S is in general complex for space-times with Lorentzian signature because the Ashtekar potential is complex. The complex symmetric matrix S can be classified according to its number of independent eigenvectors and eigenvalues as in the following table :

Table 1. Classification of the initial data according to S.

No. of indpt. eigenvectors		3	2	1
Number	3	I		
of distinct	2	D	II	
eigenvalues	1	O	N	III

$S^a{}_b$ is gauge-covariant. Gauge-invariant quantities on M^3 can be constructed from the characteristic classes of the matrix. They are

$$c_1 = -\mathrm{Tr}(S) = \lambda \tag{3.4a}$$

$$c_2 = \tfrac{1}{2}\,[c_1 \mathrm{Tr}S + \mathrm{Tr}(S^2)] \tag{3.4b}$$

and

$$c_3 = -\tfrac{1}{3}\,[\, c_2 \mathrm{Tr}S + c_1 \mathrm{Tr}(S^2) + \mathrm{Tr}(S^3)] \tag{3.4c}$$

The Bianchi Identity for the magnetic field implies that S has to satisfy the consistency equation

$$[D_i(S.\, \tilde{\sigma}^i)]_a = 0 \tag{3.5}$$

or

$$\tilde{\sigma}^{ib}(\, D_i S)_{ba} = 0 \tag{3.6}$$

when the Gauss' Law constraint holds.

There have been attempts to obtain metric-independent General Relativity by expressing $\tilde{\sigma}^i$ in terms of B^i by inverting S [CA][JD1]. We do not advocate this because of the possibility of S being degenerate. For instance, the simple F = 0 sector has S = 0 for finite momenta. We shall elaborate on the significance of cases with non-degenerate S later on in Chapter 5 and give explicit examples. In cases where S is invertible, the Ashtekar constraints can be simplified and we do arrive at the results of [CA] with

$$\tilde{\sigma}^{ia} = (S^{-1})^{a}{}_{b} B^{ib} \tag{3.7}$$

and constraints

$$B^{ib}(\ D_i S)_{ba} = 0 \tag{3.8}$$

$$(S^{-1})_{ab} = (S^{-1})_{ba} \tag{3.9}$$

and

$$\det(B)\ [\ (\mathrm{Tr}S^{-1})^2\ - \mathrm{Tr}((S^{-1})^2)\ + 2\lambda \det(S^{-1})] = 0 \tag{3.10}$$

As noted by the authors of [CA], these are seven constraints on the nine complex components of S^{-1} and the solutions should give two unconstrained degrees of freedom associated with General Relativity in four dimensions. However, as suggested by the discontinuities in the initial-value data and as we shall show, there can be phases with fewer degrees of freedom.

It should be emphasized that, as in the Petrov[PE] classification scheme, space-times with Euclidean signature differ from those with Lorentzian signature in that Types II, III and N do not occur for the former. This is because for space-times with Euclidean signature, the Ashtekar variables are real and hence the real and symmetric S always possesses three independent eigenvectors.

Under SO(3) gauge transformations

$$O^{a}{}_{b} B^{ib} = S'_{ab} O^{b}{}_{c} \tilde{\sigma}^{ic} \qquad , \qquad O \in SO(3) \tag{3.11}$$

So for types I, D and O, S can be diagonalized into the form

$$S = \mathrm{diag}(\alpha,\ \beta,\ -\alpha-\beta-\lambda) \tag{3.12}$$

by an SO(3) gauge transformation with the gauge transformation matrix constructed out of the eigenvectors of S.

The characteristic classes of S may not always be independent. When there is only one eigenvalue, there is the relation

$$\mathrm{Tr}(S^3) = \mathrm{Tr}(S)(\mathrm{Tr}(S^2)) = \tfrac{1}{9}(\mathrm{Tr}S)^3 = -\tfrac{1}{9}\,\lambda^3 \tag{3.13}$$

When two of the eigenvalues are the same, then

$$6[\ \mathrm{Tr}(S^3)\ +\lambda\mathrm{Tr}(S^2)\ -\tfrac{2}{9}\lambda^3\,]^2 = [\ \mathrm{Tr}(S^2)\ -\ \tfrac{1}{3}\lambda^2\,]^3 \tag{3.14}$$

We stress that the Petrov Classification scheme emerges naturally from the solution space of the Ashtekar constraints. The initial data fall into distinct classes exhibiting striking discontinuities. For instance, Type I corresponds to S having, in general, three $\vec{x}$-dependent eigenvalues whose sum is $-\lambda$ while for Type O, S has but one $\vec{x}$-independent eigenvalue $-\frac{\lambda}{3}$. The mismatch in the allowed fluctuations for different sectors of the theory is suggestive of phases in the theory. This has implications for the quantum theory. We shall see in Chapter 6 that the Type O sector ($S^a{}_b = -\frac{1}{3}\lambda\delta^a{}_b$) can be identified with an unbroken phase described by a topological quantum field theory(TQFT). Finally, it will be shown in the next chapter that for classical solutions i.e. when the equations of motion are satisfied, for non-degenerate metrics,

$$S_{ab} = \tfrac{1}{2}\,\epsilon_{0aBC}R^{BC}{}_{0b} - R_{0a0b} \tag{3.15}$$

where R_{ABCD} is the Riemann Curvature tensor. Thus classification by S is in this case equivalent to the Petrov Classification.

Chapter 4

4.1 The Equations of Motion

The manifestly covariant equations of motion for the Ashtekar variables will be derived in this section. Explicit calculations will be done with metrics of Euclidean signature and the necessary modifications for metrics of Lorentzian signature will be indicated.

In applying the canonical formalism to the variables, it is convenient to work in the spatial gauge in which the vierbein is written as

$$e_{A\mu} = \begin{bmatrix} N & 0 \\ N^j e_{aj} & e_{ai} \end{bmatrix} \tag{4.1.1}$$

The form asumed in (4.1.1) is compatible with the ADM decomposition of the metric

$$\begin{aligned} ds^2 &= e_{A\mu} e_{A\nu}\, dx^\mu\, dx^\nu \\ &= N^2 (dx^0)^2 + g_{ij}(dx^i + N^i dx^0)(dx^j + N^j dx^0) \end{aligned} \tag{4.1.2}$$

with the spatial metric $g_{ij} = e_{ai} e^a{}_j$. Thus the choice (2.4) in no way compromises the lapse and shift functions, N and N^i, which have geometrical interpretations in hypersurface deformations. Replacing N by $-iN$ (N is real) changes the signature to Lorentzian. In the spatial gauge, this is equivalent to replacing the one-form e_0 by $-ie_0$. On the constant-x^0 initial-value hypersurface, M^3, $e_0 = e_{A\mu} dx^\mu$ vanishes and we can write

$$F_a = e_0 \wedge T_{ab}\, e^b + \tfrac{1}{2}\, S_a{}^b \epsilon_{bcd} e^c \wedge e^d \tag{4.1.3}$$

This is compatible with Eqn. (3.1). T however, must be chosen carefully because (4.1.3) implies that on M^3

$$\begin{aligned} F_{0ia} &= T_{ab}(e_{00} e^b{}_i - e_{0i}\, e^b{}_0) + S_a{}^b \epsilon_{bcd} e^c{}_0 e^d{}_i \\ &= N T_{ab} e^b{}_i + S_a{}^b \epsilon_{bcd} e^c{}_0 e^d{}_i \end{aligned} \tag{4.1.4}$$

Apart from boundary terms which do not contribute to the equations of motion, the Hamiltonian of the Ashtekar formulation is[AT]

$$\begin{aligned} H = \frac{1}{16\pi G}\int_{M^3} d^3x \Big\{ & 2N^i \tilde{\sigma}^{ja} F_{ija} + \underset{\sim}{N}(\epsilon_{abc} \tilde{\sigma}^{ia} \tilde{\sigma}^{jb} F^c_{ij} + \frac{\lambda}{3}\, \epsilon_{abc} \underset{\sim}{\epsilon}_{ijk} \tilde{\sigma}^{ia} \tilde{\sigma}^{jb} \tilde{\sigma}^{kc}) \\ & - 2A_{0a} D_i \tilde{\sigma}^{ia} \Big\} \end{aligned} \tag{4.1.5}$$

with $\underset{\sim}{N} \equiv \det(e_{ai})^{-1} N$. The Hamilton Equations for the x^0-evolution of A, on M^3, are

$$\begin{aligned} \dot{A}_{ia} &= \{A_{ia}\,,\, H\}_{P.B.} \\ &= \underset{\sim}{N} \epsilon_{abc} \tilde{\sigma}^{jb} F^c_{ij} + \frac{\lambda}{2}\, \underset{\sim}{N} \epsilon_{abc} \underset{\sim}{\epsilon}_{ijk} \tilde{\sigma}^{jb} \tilde{\sigma}^{kc} + \partial_i A_{0a} - \epsilon_a{}^{bc} A_{0b} A_{ic} - N^j F_{ija} \end{aligned} \tag{4.1.6}$$

With the use of Eqn. (3.1) and assuming non-degenerate metrics, this can be rewritten as

$$F_{0ia} = -NS_{ab} e^b{}_i + S_a{}^b \epsilon_{bcd} e^c{}_0 e_{di} + e_{ia}(\mathrm{Tr}S + \lambda) \tag{4.1.7}$$

The last term vanishes due to the superhamiltonian constraint and comparing with (4.1.3) , we observe that the choice of $T = -S$ is consistent . Thus we have

$$F_a = S_{ab}(-e^0 \wedge e^b + \tfrac{1}{2}\, \epsilon^b{}_{cd} e^c \wedge e^d) \tag{4.1.8}$$

The Hamilton Equations for the x^0-evolution of $\tilde{\sigma}^{ia}$ are

$$\dot{\tilde{\sigma}}^{ia} = \{\, \tilde{\sigma}^{ia}, H\}_{P.B.}$$

$$= \epsilon^{a}{}_{bc}[D_j(\underset{\sim}{N}\tilde{\sigma}^{i})]^{b}\tilde{\sigma}^{jc} + \epsilon^{a}{}_{bc}\underset{\sim}{N}(D_j\tilde{\sigma}^{j})^{b}\tilde{\sigma}^{ic} + [D_j(N^{j}\tilde{\sigma}^{i})]^{a}$$

$$-(\partial_j N^{i})\tilde{\sigma}^{ja} - \underset{\sim}{N}(D_j\tilde{\sigma}^{j})^{a} + A_{0c}\epsilon^{a}{}_{b}{}^{c}\tilde{\sigma}^{ib} \qquad (4.1.9)$$

The constraints and the equations of motion can be organized into a manifestly covariant set. After some algebra, it can be shown that these equations can be written succinctly as

$$[D\Sigma]_a = 0 \qquad (4.1.10a)$$

$$F_a = S_{ab}\Sigma^{b} \qquad (4.1.10b)$$

with

$$S_{ab} = S_{ba} \quad , \quad \mathrm{Tr}\, S = -\lambda. \qquad (4.1.10c)$$

and

$$\Sigma^{a} \equiv (-e^{0} \wedge e^{a} + \tfrac{1}{2}\epsilon^{a}{}_{bc}e^{b} \wedge e^{c}) \qquad (4.1.10d)$$

D in Eqn. (4.1.10a) denotes the exterior covariant derivative with respect to the Ashtekar connection one-form. The nine x^0-evolution equations for A_{ia} are in (4.1.10b) while the twelve equations in (4.1.10a) can be split into the three equations

$$*[(D\Sigma)^{a}|_{M^3}] = 0 \qquad (4.1.11a)$$

which is equivalent to the Gauss' Law constraint and the nine equations

$$[*(D\Sigma)^{a}]|_{M^3} = 0 \qquad (4.1.11b)$$

which, modulo Gauss' Law, is the same as the x^0-evolution equations for $\tilde{\sigma}^{ia}$ i.e. Eqns. (4.1.9). The vertical bar denotes restriction (to M^3) while $*$ is the Hodge duality operation. The supermomentum and superhamiltonian constraints assume the form of (4.1.10c).

We have thus obtained the manifestly covariant equations of motion through the canonical formalism. Unlike some previous work on this topic[BE][SO], no assumptions on the values of the canonical variables or the lapse and shift functions were made. It will be shown that these equations are precisely the same as the Einstein Field Equations for dimension four.

4.2 Einstein Manifolds and Anti-Instantons

We shall now examine some of the physical implications of the equations of motion. Firstly, observe that Σ^a is explicitly anti-self-dual i.e.

$$*\Sigma^a = -\Sigma^a \qquad a=1,2,3 \tag{4.2.1}$$

Since the equation of motion (4.1.10b) says that F_a is the contraction of a zero-form S with the two-form Σ, this implies that

$$*F_a = -F_a \tag{4.2.2}$$

which means that *all* Einstein Manifolds in dimension four are anti-instantons when described in terms of the Ashtekar potentials (see also [JD2]). This is to be contrasted with the traditional view of identifying gravitational anti-instantons with solutions that have anti-self-dual Riemann or Weyl curvature. (Sometimes the term gravitational instanton is used in the literature to mean finite action solutions of the Einstein Equations). In general, Einstein manifolds do not have anti-self-dual Riemann or Weyl curvature but as we have demonstrated, the curvature of their Ashtekar potentials are always anti-self-dual. Note that the curvature of an arbitrary anti-instanton can always be expanded as $F_a = Y_{ab}\Sigma^b$, but Y has to satisfy (4.1.10c) and (4.1.10a) has to hold for the configuration to describe an Einstein manifold. Thus the Ashtekar potentials of Einstein manifolds constitute a restricted subset of all anti-instantons.

The twelve equations (4.1.10a) suggest that the one-form A_a can be expresssed uniquely in terms of the vierbein e_A. This is indeed the case, for the solution of (4.1.10a) is precisely

$$A_a = \omega_{0a} - \frac{1}{2}\,\epsilon_a{}^{bc}\omega_{bc} \tag{4.2.3}$$

where ω_{AB} is the (unique) torsionless spin connection compatible with the vierbein. ω_{AB} can be determined uniquely from the vierbein[EH1] through the torsionless condition

$$de_A + \omega_{AB} \wedge e^B = 0 \tag{4.2.4}$$

Eqn. (4.2.3) indicates that (apart from a factor of two due to our conventions) A_a is the anti-self-dual part of the spin connection and so the two-form curvature of A has the form

$$F_a = R_{0a} - \frac{1}{2}\,\epsilon_a{}^{bc}R_{bc} \tag{4.2.5}$$

where R_{AB} is the curvature two-form of the spin connection. It is then not difficult to show that (4.1.10b) is satisfied if and only if (the details can be found in the Appendix)

$$S_{ab} = R_{\widetilde{oa}ob} - R_{oaob} = R_{oa\widetilde{ob}} - R_{\widetilde{oa}ob} \tag{4.2.6}$$

and the constraints (4.1.10c) then imply that

$$R_{ABCD} = R_{\widetilde{AB}\widetilde{CD}} \tag{4.2.7}$$

and the Ricci scalar

$$R = 4\lambda \tag{4.2.8}$$

It is well known that Eqn. (4.2.7) together with Eqn. (4.2.8) are equivalent to the field equations for Einstein manifolds in four dimensions (see for instance [FC])

$$R_{\mu\nu} = \lambda g_{\mu\nu} \tag{4.2.9}$$

4.3 Four Dimensions and the Ashtekar Variables

Dimension four is the lowest dimension for which the Riemann curvature tensor assumes its full complexity in that all four indices (anti-symmetric in pairs) are needed. Four dimensions also has the peculiarity that it allows a decomposition of the curvature two-form into components taking values in the (± 1) eigenspaces, $\wedge_2^{\pm}$, of the Hodge duality operator $*$. The Riemann curvature tensor with four indices can be dualized on the left or on the right and, in the decomposition into components that are self and anti-self-dual, the Riemann curvature can be viewed as a 6×6 matrix mapping of $\wedge_2^{\pm}$ into $\wedge_2^{\pm}$ [AHS][EH1]. The matrix mapping can be written as

$$\begin{bmatrix} A & C^+ \\ C^- & B \end{bmatrix}$$

The entries of the 3×3 matrix A_{ab} are

$$A_{ab} \equiv +(R_{0a0b} + R_{0a\tilde{0}\tilde{b}}) + (R_{\tilde{0}\tilde{a}0b} + R_{\tilde{0}\tilde{a}\tilde{0}\tilde{b}}) \tag{4.3.1}$$

B and $C^{\pm}$ are defined by changing the signs in the definition of A in (4.3.1) in the following manner:

$$A \sim (+,+,+,+) \quad ; B \sim (+,-,-,-)$$

$$C^+ \sim (+,-,+,-) \; ; C^- \sim (+,+,-,+)$$

It is easy to check that A(B) is self-dual (anti-self-dual) with respect to both left and right duality operations while $C^+(C^-)$ is self-dual (anti-self-dual) under a left duality operation and anti-self-dual (self-dual) under a right duality operation. A metric satisfies Einstein's Equations if and only if $C^{\pm}$ vanishes, i.e. the curvature assumes the block diagonal form in the above matrix decomposition[AHS]. Eqn. (4.2.5) says that F_a is the doubly anti-self-dual part of the curvature. Therefore (apart from a multiplicative factor), S_{ab} can be identified with B_{ab} when the equations of motion hold. While only half of the non-vanishing components of the Riemann curvature tensor in four dimensions are contained in the curvature of the Ashtekar connection, F_a, the equations of motion deduced from the Ashtekar variables are completely equivalent to those of Einstein's. A and B interchange under a reversal of orientation because a reversal of orientation interchanges self and anti-self- duality.

While not all Einstein manifolds have anti-self-dual Riemann or Weyl curvature tensors, a manifold is Einstein only if the curvature tensor constructed from the anti-self-dual part of the spin connection is anti-self-dual. It is precisely this which allows for the description of all Einstein manifolds in terms of anti-instantons when expressed in terms of the Ashtekar variables. In this context, for Einstein manifolds , the Ashtekar formulation is the realization of Proposition 2.2 of [AHS] in the canonical framework. However, it should be emphasized that it is the remarkable simplification of the constraints provided by Ashtekar that makes non-perturbative canonical quantization viable.

Finally, for Einstein manifolds, the Weyl two-form has the form

$$W_{AB} = R_{AB} - \frac{\lambda}{3} e_A \wedge e_B \tag{4.3.2}$$

so the anti-self-dual part of the Weyl two-form is

$$W_a^- = W_{0a} - \frac{1}{2}\epsilon_a{}^{bc} W_{bc}$$

$$= R_{0a} - \frac{1}{2}\epsilon_a{}^{bc}R_{bc} + \frac{\lambda}{3}[-e_0 \wedge e_a + \frac{1}{2}\epsilon_a{}^{bc}e_b \wedge e_c]$$

$$= F_a + \frac{\lambda}{3}\Sigma_a \tag{4.3.3}$$

An Einstein manifold is therefore conformally self-dual [or self-dual (also called half-flat) when $\lambda = 0$], if and only if

$$F_a = -\frac{\lambda}{3}\Sigma_a \tag{4.3.4}$$

i.e. if and only if

$$S_{ab} = -\frac{\lambda}{3}\delta_{ab} \tag{4.3.3}$$

According to our classification in Chapter 3, S is precisely of Type O.

Since

$$\Sigma_a \wedge \Sigma_b = -2\delta_{ab}(*1) \tag{4.3.5}$$

where $(*1)$ is the four-volume element (in local coordinates it can be written as $e_0 \wedge e_1 \wedge e_2 \wedge e_3$). Using (4.1.10b),

$$S_{ab} = -*\frac{1}{4}(F_a \wedge \Sigma_b + \Sigma_a \wedge F_b) \tag{4.3.6}$$

and the equations of motion can be written as

$$F_a = -\frac{1}{2}[*(F_a \wedge \Sigma_b)]\Sigma^b \tag{4.3.7a}$$

$$(D\Sigma)_a = 0 \tag{4.3.7b}$$

$$\epsilon_a{}^{bc}F_b \wedge \Sigma_c = 0 \tag{4.3.7c}$$

$$F_a \wedge \Sigma^a = -2\lambda(*1) \tag{4.3.7d}$$

Therefore it is possible, in principle, to eliminate S from the equations of motion. However, as we have seen (e.g. S of Type O corresponds precisely to the conformally self-dual sector), S plays an important role as a effective order parameter characterizing various sectors of the theory. Further evidence of this will be discussed in Chapter 5.

4.4 Ashtekar variables and invariants of four-manifolds.

The gravitational field has the privileged role of describing the dynamics of space-time. Any viable classical and quantum theory of gravity must therefore be able to account for not just the the local description of curvature but also the global topological and differential structures that can exist. In this section, we indicate how the Ashtekar variables can be used to capture the global invariants associated with manifolds of dimension four, specifically four-dimensional Einstein manifolds.

As discussed in Chapter 3, a specification of the initial-value data entails a specification of the characteristic classes of S . Take the gauge-invariant scalars on M^3 to be $\mathrm{TrS} = -\lambda$, $\mathrm{Tr}(S^2)$ and $\mathrm{Tr}(S^3)$. From these, the characteristic classes of S, $c_{1,2,3}$ can be reconstructed. It is not difficult to show that when the equations of motion are satisfied, i.e. for Einstein manifolds,

$$\mathrm{TrS} = -\lambda \tag{4.4.1}$$

$$\mathrm{Tr}(S^2) = \tfrac{1}{8}\{(R_{\widetilde{AB}\widetilde{CD}} - R_{AB\widetilde{CD}})R^{ABCD}\} \tag{4.4.2}$$

$$\mathrm{Tr}(S^3) = -\tfrac{1}{16}\{(R_{ABCD} - R_{AB\widetilde{CD}})R^{CDEF}R_{EF}{}^{AB}\} \tag{4.4.3}$$

Consider their integrals over compact four-manifolds without boundary. We have

$$\int_M \mathrm{Tr}(S)(*1) = -\lambda V = -\frac{\lambda}{6}\int \Sigma_a \wedge \Sigma^a \tag{4.4.4}$$

where $V = \int *1$, the four-volume,

$$\int_M \mathrm{Tr}(S^2)(*1) = 2\pi^2\{2\chi(M) - 3\tau(M)\}$$

$$= -\frac{1}{2}\int_M F_a \wedge F^a$$

$$= -2\pi^2 P_1 \qquad (4.4.5)$$

where $\chi(M)$ and $\tau(M)$ are the Euler charateristic and signature of M respectively while P_1 is the Pontrjagin number of the SO(3) Ashtekar connection.

$$\int_M Tr(S^3)(*1) = -\frac{1}{2}\int_M S_{ab}F^a \wedge F^b \qquad (4.4.6)$$

The signature of a manifold, $\tau(M)$, depends on the orientation of M because

$$\tau(M) = \dim H^{2+} - \dim H^{2-} \qquad (4.4.7)$$

$$= b_2^+ - b_2^- \qquad (4.4.8)$$

where $H^{2\pm}$ are the self and anti-self-dual subspaces of the second de Rham cohomology group and, b_2^+ and b_2^- denote the number of independent self and anti-self-dual two- forms respectively. Reversing the orientation interchanges self and anti-self-duality, i.e. two-forms that are self-dual with respect to the metric for the original orientation will be anti-self-dual with respect to the same metric for the opposite orientation. So

$$\tau(\bar{M}) = -\tau(M) \qquad (4.4.9)$$

where $\bar{M}$ has the opposite orientation to M.

Reversing the orientation changes the spin connection in general and thus the Ashtekar connection. Consider the torsionless condition

$$de^A = -\omega^A{}_B \wedge e^B \qquad (4.4.10)$$

A transformation, for instance, of the form

$$(e^0, e^a) \to (e'^0 = -e^0,\ e'^a = e^a) \qquad (4.4.11)$$

reverses the orientation (since for the transformation e' = Ue , the matrix U has determinant -1) but not the metric ds^2. The new spin connection is related to the old by

$$\omega'_{0a} = -\omega_{0a}, \qquad \omega'_{ab} = \omega_{ab} \tag{4.4.12}$$

Hence by Eqn. (4.2.3), the new Ashtekar potential is

$$\begin{aligned} A'_a &= \omega'_{0a} - \tfrac{1}{2}\epsilon_a{}^{bc}\omega'_{bc} \\ &= A_a - 2\omega_{0a} \end{aligned} \tag{4.4.13}$$

The Pontrjagin numbers of the Ashtekar connections with respect to the two opposite orientations are

$$P_1^+ = 3\tau(M) - 2\chi(M) \tag{4.4.14a}$$

$$P_1^- = 3\tau(\bar{M}) - 2\chi(\bar{M}) = -3\tau(M) - 2\chi(M) \tag{4.4.14b}$$

(Note that the Euler characteristic is invariant under a change of orientation of M). Since $P_1^{\pm}$ are the Pontrjagin numbers of the *anti-self-dual* Ashtekar connections, we must have $P_1^{\pm} \leq 0$ and an immediate consequence of this consideration and Eqns. (4.4.14) lead to the conclusion that for compact Einstein manifolds without boundary

$$|\tau| \leq \tfrac{2}{3}\chi \tag{4.4.15}$$

This is called the Hitchin bound [HI]. By considering the Ashtekar potentials, its derivation was straightforward. Note also that for compact Einstein manifolds with Euclidean signature

$$\begin{aligned} \chi &= \frac{1}{32\pi^2}\int R_{\widetilde{AB}\widetilde{CD}}R^{ABCD}(*1) \\ &= \frac{1}{32\pi^2}\int R_{ABCD}R^{ABCD}(*1) \geq 0 \end{aligned} \tag{4.4.16}$$

with equality only if M is flat[SA].

In terms of the Ashtekar potentials, $\tau(M)$ and $\chi(M)$ can be computed through (4.4.14) to give

$$\tau(M) = \tfrac{1}{6}\,(\,P_1^+ - P_1^-\,) \tag{4.4.17}$$

$$\chi(M) = -\tfrac{1}{4}\,(\,P_1^+ + P_1^-\,) \tag{4.4.18}$$

If the Einstein manifold possesses an orientation reversing diffeomorphism, then $P_1^+ = P_1^-$ and $\tau(M) = 0$. The vanishing or non-vanishing of τ has inportant physical implications for dimension four[FR][EH1]. An application of the Atiyah-Patodi-Singer index theorem to the spin complex for compact Riemannian manifolds yields (see for instance [EH1])

$$\begin{aligned} n_+ - n_- &= -\tfrac{1}{24}P_1(T(M)) \\ &= -\tfrac{1}{8}\tau(M) \end{aligned} \tag{4.4.19}$$

where $n_\pm$ are the number of normalizable ± 1 chirality zero-frequency Weyl spinors. $P_1(T(M))$ is the Pontrjagin number of the tangent bundle i.e. of the SO(4) spin connection and is related to $\tau(M)$ by the Hirzebruch signature theorem

$$P_1(T(M)) = 3\tau(M) \tag{4.4.20}$$

Chapter 5

5.1 Ashtekar variables and Einstein manifolds

The formalism developed in the previous sections provides a coherent framework to discuss Einstein manifolds in the context of the Ashtekar variables. The consideration of explicit examples serves various purposes. It checks the validity of the framework and helps to develop some intuition on how Einstein manifolds manifest themselves in terms of these new variables. It also leads to the construction of new classical solutions through the Ashtekar variables. More importantly, it provides new insights in the nature of four-dimensional classical and quantum gravity.

Every known solution of the Einstein Field Equations in four dimensions

$$R_{\mu\nu} = \lambda g_{\mu\nu} \tag{5.1.1}$$

can be put in the form of Eqns. (4.1.10). When the field equations are satisfied, we can use Eqn. (4.2.3) to obtain the Ashtekar connection for the manifold through Eqn. (4.2.4). The curvature and hence S can then be computed.

In the discussion of explicit examples, it is convenient to introduce the one-forms Θ_a, such that $\Phi_a = -2\Theta_a$ obey the Maurer-Cartan equations of the SO(3) algebra

$$d\Phi_a + \tfrac{1}{2}\epsilon_a{}^{bc}\Phi_b \wedge \Phi_c = 0 \tag{5.1.2}$$

Four dimensional polar coordinates (R,θ,ϕ,ψ) are related to Cartesian coordinates by

$$R^2 = (x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 \tag{5.1.3a}$$

$$x^1 + ix^2 = R\cos(\tfrac{\theta}{2})\exp[\tfrac{i}{2}(\psi + \phi)] \tag{5.1.3b}$$

$$x^3 + ix^0 = R\sin(\tfrac{\theta}{2})\exp[\tfrac{i}{2}(\psi - \phi)] \tag{5.1.3c}$$

In terms of the polar coordinates, Θ_a can be written as

$$\Theta_1 = \tfrac{1}{2}(\sin\psi d\theta - \sin\theta\cos\psi d\phi) \tag{5.1.4a}$$

$$\Theta_2 = \tfrac{1}{2}(-\cos\psi d\theta - \sin\theta\sin\psi d\phi) \tag{5.1.4b}$$

$$\Theta_3 = \tfrac{1}{2}(d\psi + \cos\theta d\phi) \tag{5.1.4c}$$

and it can be checked that the relation

$$d\Theta_a = \epsilon_a{}^{bc}\Theta_b \wedge \Theta_c \tag{5.1.5}$$

holds. For fixed values of R, (θ, ϕ, ψ) parametrise three-spheres if the ranges are

$$0 \leq \theta < \pi \ , \ 0 \leq \phi < 2\pi \ \text{ and } 0 \leq \psi < 4\pi \tag{5.1.6}$$

We shall start off with Type O solutions i.e. those with $S_{ab} = -\frac{\lambda}{3}\delta_{ab}$. As explained in Chapter 4, these solutions correspond to the conformally self-dual sector of Einstein manifolds. It is known that for $\lambda > 0$, S^4 and the complex projective space CP_2 are the only compact simply-connected four-manifolds which are conformally self-dual[SA]

5.2 Explicit examples

(a) S^4 with the de Sitter metric and the BPST instanton

The de Sitter metric on S^4 is

$$ds^2 = [1 + (\tfrac{R}{b})^2]^{-2}[dR^2 + R^2({\Theta_1}^2 + {\Theta_2}^2 + {\Theta_3}^2)] \tag{5.2.a.1}$$

and the vierbein

$$e_A = \{[1+(\frac{R}{b})^2]^{-1}dR \ , \ R[1+(\frac{R}{b})^2]^{-1}\Theta_a]\} \tag{5.2.a.2}$$

The Ashtekar potentials can be computed as

$$A_a = \omega_{0a} - \frac{1}{2}\epsilon_a{}^{bc}\omega_{bc}$$

$$= \frac{-2\Theta_a}{1+(R/b)^2} \tag{5.2.a.3}$$

giving

$$F_a = dA_a + \frac{1}{2}\epsilon_a{}^{bc}A_b \wedge A_c$$

$$= -\frac{4}{b^2}[-e_0 \wedge e_a + \frac{1}{2}\epsilon_{abc}e^b \wedge e^c] \tag{5.2.a.4}$$

Thus

$$F_a = S_a{}^b\Sigma_b \tag{5.2.a.5}$$

with

$$S_a{}^b = -\frac{4}{b^2}\delta_a{}^b = -\frac{\lambda}{3}\delta_a{}^b \tag{5.2.a.6}$$

provided we identify

$$\lambda = (12/b^2) \tag{5.2.a.7}$$

The diameter of the four-sphere, S^4, is

$$b = \sqrt{(12/\lambda)} \tag{5.2.a.8}$$

Under a reversal of orientation, realized for instance, by changing only the sign of e^0 i.e. using

$$e'_A = \{-[1+(\frac{R}{b})^2]^{-1}dR \ , \ 2[1+(\frac{R}{b})^2]^{-1}\Theta_a]\} \tag{5.2.a.9}$$

the new Ashtekar connection computed through the new spin connection is

$$A'_a = -\frac{2R^2\Theta_a}{b^2[1+(R/b)^2]} \tag{5.2.a.10}$$

and

$$F'_a = -\frac{4}{b^2}\Sigma'_a \tag{5.2.a.11}$$

S is therefore unchanged under the orientation reversal. It is known that S^4 has an orientation reversing diffeomorphism[DK]. Explicit computations with the Ashtekar connections yield

$$\begin{aligned} P_1 &= \frac{1}{4\pi^2}\int_{S^4} F_a \wedge F_a \\ &= -4 \\ &= P'_1 \end{aligned} \tag{5.2.a.12}$$

The Euler characteristic and signature can then be computed using Eqns. (4.4.14). They are

$$\chi(S^4) = 2 \tag{5.2.a.13}$$

and

$$\tau(S^4) = 0 \tag{5.2.a.14}$$

Note that for S^4, the SO(3) Ashtekar connections for both orientations is zero mod 4 and therefore the SO(3) connections can be lifted to be SU(2) connections[DK] with second Chern class

$$c_2 = -\frac{1}{4}P_1 = 1 \tag{5.2.a.15}$$

Actually, the Ashtekar potentials as given in (5.2.a.3) and (5.2.a.10) are precisely the SU(2) Belavin-Polyakov-Schwarz-Tyupkin (BPST) anti-instanton solutions[BPST]. The dimension of anti-instantons on S^4 modulo gauge transformations, for $c_2 = 1$, is known to be equal to five[AHS]. The parameters of this moduli space correspond to the size and location of the center of the anti-instanton on S^4. For the description of General Relativity in terms of anti-instantons through Ashtekar potentials, the additional diffeomorphism invariance collapses the moduli space to the unique Ashtekar potential determined by the vierbein since the solution must be translationally- invariant while the size of the anti-instanton is correlated to the cosmological constant according to (5.2.a.8).

S^4 is not only conformally self-dual (it is conformally self-dual since S is of Type O) but conformally flat. This is also evident in the context of Ashtekar variables because S is of Type O for two opposite orientations. S is of Type O for one orientation implies that W^-_a vanishes but the fact that S also remains unchanged under a reversal of orientation which changes the original W^+_a into the new W'^-_a, allows us to deduce that both halves of the original Weyl two-form $W^\pm_a$ must be zero and so S^4 with the de Sitter metric is conformally flat.

(b) CP_2 and the Fubini-Study metric

A very intriguing example is the complex projective space CP_2 with the standard Fubini-Study metric

$$ds^2 = \frac{dR^2}{(1+\frac{\lambda}{6}R^2)^2} + \frac{(R\Theta_1)^2}{(1+\frac{\lambda}{6}R^2)} + \frac{(R\Theta_2)^2}{(1+\frac{\lambda}{6}R^2)} + \frac{(R\Theta_3)^2}{(1+\frac{\lambda}{6}R^2)^2} \tag{5.2.b.1}$$

Choosing the vierbien as

$$e_A = \left\{ \frac{dR}{(1+\frac{\lambda}{6}R^2)} \, , \, \frac{R\Theta_1}{(1+\frac{\lambda}{6}R^2)^{\frac{1}{2}}} \, , \, \frac{R\Theta_2}{(1+\frac{\lambda}{6}R^2)^{\frac{1}{2}}} \, , \, \frac{R\Theta_3}{(1+\frac{\lambda}{6}R^2)} \right\} \tag{5.2.b.2}$$

the corresponding Ashtekar potentials are

$$A_1 = \frac{-2\Theta_1}{(1+\frac{\lambda}{6}R^2)^{\frac{1}{2}}} \quad , \quad A_2 = \frac{-2\Theta_2}{(1+\frac{\lambda}{6}R^2)^{\frac{1}{2}}} \quad ,$$

and

$$A_3 = \frac{(-2-\frac{\lambda}{6}R^2)\Theta_3}{(1+\frac{\lambda}{6}R^2)} \tag{5.2.b.3}$$

These yield $F_a = S_{ab}\Sigma^b$ with $S_{ab} = -\frac{\lambda}{3}\delta_{ab}$. It is thus of Type O again. However, a calculation of the Pontrjagin number gives

$$P_1 = \frac{1}{4\pi^2}\int F_a \wedge F_a = -3 \tag{5.2.b.4}$$

Note that P_1 is not a multiple of four and hence for CP_2, the Ashtekar connection cannot be realized in a globally well-defined manner as an SU(2) connection since it would have $c_2 = \frac{3}{4}$ which is non-integer. Since S is of Type O, CP_2 is conformally self-dual. Unlike S^4, CP_2 does not possess an orientation reversing diffeomorphism. Under an orientation reversal CP_2 goes into $\overline{CP}_2$ which has the *same* Fubini-Study metric. If we take for $\overline{CP}_2$, $e'_A = \{-e_0 \ , \ e_a\}$, then the Ashtekar potentials now are

$$A'_1 = A'_2 = 0$$

and

$$A'_3 = -\frac{\lambda R^2 \Theta_3}{2(1+\frac{\lambda}{6}R^2)} \tag{5.2.b.5}$$

giving

$$F'_1 = F'_2 = 0$$

$$\begin{aligned} F'_3 &= dA'_3 \\ &= -\lambda(-e'_0 \wedge e_3 + e_1 \wedge e_2) \\ &= -\lambda \Sigma'_3 \end{aligned} \tag{5.2.b.6}$$

Thus CP_2 is described by an Ashtekar potential that is a non-abelian anti-instanton whereas $\overline{CP}_2$ is described by an abelian anti-instanton! A straightforward computation of the Pontrjagin number for $\overline{CP}_2$ yields $P'_1 = -9$, an answer which is different from that of CP_2. So according to Eqns. (4.4.14) , the Euler characteristics and signatures are

$$\chi(CP_2) = \chi(\overline{CP}_2) = 3 \tag{5.2.b.7}$$

and

$$\tau(CP_2) = -\tau(\overline{CP}_2) = 1 \tag{5.2.b.8}$$

We therefore have

$$\mathrm{rank}\mathfrak{Q} = b_2^+ + b_2^- = \chi - 2 = 1 \tag{5.2.b.9}$$

where $\mathfrak{Q}$ is the intersection form of the manifold(see for instance [DK]). CP_2 ($\overline{CP}_2$) has intersection form

$$\mathfrak{Q} = 1\ (-1) = \tau = b_2^+ - b_2^- \tag{5.2.b.10}$$

Because CP_2 has $\tau = +1$, $b_2^+ = 1$ and $b_2^- = 0$. Therefore CP_2 cannot support an abelian anti-instanton (which would have implied that $b_2^- \geq 1$ because the curvature of an abelian anti-instanton is harmonic) but $\overline{CP}_2$ *can* since it has $\tau = -1$, $b_2^+ = 0$ and $b_2^- = 1$. This distinguished abelian anti-instanton is none other than the Ashtekar potential. For CP_2, it is known through the methods of algebraic geometry (see for instance [DK]) that the anti-instantons have Pontrjagin numbers $P_1 = -(3+4j)$ with j being a whole number and the space of anti-instantons on CP_2 modulo gauge transformations has for j=0, only a single point. This distinguished anti-instanton coresponds to the Ashtekar potential. Furthermore, note that unlike CP_2 which is Type O, $\overline{CP}_2$ has

$$S' = \mathrm{diag}(\ 0, 0, -\lambda) \tag{5.2.b.11}$$

and is therefore of Type D. The fact that $\overline{\mathrm{CP}}_2$ corresponds to an abelian anti-instanton suggests that a bundle reduction from SO(3) to U(1) has occurred. We shall see that it is indeed possible to interpret S_{ab} in terms of dynamical Higgs fields and exhibit the symmetry breaking mechanism explicitly. This issue will be addressed in section 5.6.

This example indicates that in the description of topological and differential invariants, the Ashtekar variables seem more natural than the metric (which cannot even distinguish between opposite orientations). It also serves as a striking example of the fact that diffeomorphism invariance does not imply invariance under orientation reversal (classically, time-reversal and parity operations are contained in orientation reversal).

(c) The Schwarzschild solution

The well-known Schwarzschild solution is our next example. This will be discussed in the context of the Ashtekar variables. The Euclidean Schwarzschild-de Sitter (also known as Kottler) solution has metric

$$ds^2 = (1-\tfrac{2m}{r}-\tfrac{\lambda}{3}\,r^2)d\tau^2 + (1-\tfrac{2m}{r}-\tfrac{\lambda}{3}\,r^2)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \qquad (5.2.c.1)$$

where m is G/c^2 times the mass. Taking the veirbein to be

$$e_A = \{(1-\tfrac{2m}{r}-\tfrac{\lambda}{3}\,r^2)^{\frac{1}{2}}\,d\tau\ ,\ (1-\tfrac{2m}{r}-\tfrac{\lambda}{3}\,r^2)^{-\frac{1}{2}}\,dr\ ,\ rd\theta\ ,\ r\sin\theta d\phi\ \} \qquad (5.2.c.2)$$

yield the Ashtekar potentials

$$A_1 = (\ \tfrac{m}{r^2}-\tfrac{\lambda}{3}\,r)\ d\tau\ +\ \cos\theta d\phi \qquad (5.2.c.3a)$$

$$A_2 = -(1-\tfrac{2m}{r}-\tfrac{\lambda}{3}\,r^2)^{\frac{1}{2}}\sin\theta d\phi \qquad (5.2.c.3b)$$

$$A_3 = (1-\tfrac{2m}{r}-\tfrac{\lambda}{3}\,r^2)^{\frac{1}{2}}\,d\theta \qquad (5.2.c.3c)$$

and $F_a = S_{ab}\Sigma^b$ with

$$S = \mathrm{diag}\Big(-\frac{2m}{r^3} - \frac{\lambda}{3} \ , \ \frac{m}{r^3} - \frac{\lambda}{3} \ , \ \frac{m}{r^3} - \frac{\lambda}{3} \Big) \tag{5.2.c.4}$$

S is Type D when $m \neq 0$ and Type O in the m=0 limit. One can view the mass term as an excitation breaking out of the Type O sector. The $\lambda = 0$ limit can also be taken to give the usual Schwarzschild solution without cosmological constant.

The reversed orientation with $e'_A = (-e_0 \ , e_a \ )$ gives

$$A'_1 = -(\frac{m}{r^2} - \frac{\lambda}{3} r)d\tau - \cos\theta d\phi = -A_1$$

and

$$A'_2 = A_2 \qquad , \qquad A'_3 = A_3 \tag{5.2.c.5}$$

but leaves S unchanged. In the $\lambda = 0$ limit, the metric has an event horizon at the Schwarzschild radius $r = 2m$ and singularity at $r = 0$. To show that the Euclidean Schwarzschild solution is periodic in time one can follow [HA], and consider first the Lorentzian solution in Kruskal coordinates[KS] with the metric written as

$$ds^2 = \frac{32m^3}{r} \exp(-\frac{r}{2m})(-d\overline{t}^{\,2} + d\overline{r}^{\,2}) + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.2.c.6}$$

The Kruskal variables satisfy

$$\overline{r}^{\,2} - \overline{t}^{\,2} = (\frac{r}{2m} - 1)\exp(\frac{r}{2m}) \tag{5.2.c.7}$$

for all r>0. Thus on the section with $\overline{t}$ being pure imaginary, the metric is positive definite, $r \geq 2m$ and $t = i\tau$ has period $8\pi m$ because of the relation

$$\overline{t} = (\frac{r}{2m} - 1)^{\frac{1}{2}} \exp(\frac{r}{4m}) \sinh(\frac{t}{4m}) \tag{5.2.c.8}$$

We can now compute the invariants through the Ashtekar potentials (with $\lambda = 0$) to give

$$P_1 = -\frac{1}{2\pi^2}\int \mathrm{Tr}\ (S^2)(*1)$$

$$= -\frac{1}{\pi^2}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=2m}^{\infty}\int_{\tau=0}^{8\pi m}\frac{3m^2}{r^4}d\tau\wedge dr\wedge \sin\theta d\theta\wedge d\phi$$

$$= -4$$

$$= P'_1 \tag{5.2.c.9}$$

(Note the ranges of the variables in the above integrations). Therefore $\chi = 2$ and $\tau = 0$, in agreement with previous results[CD].

Finally we remark that a general Type D metric with zero cosmological constant can be written as $S = \mathrm{diag}(-2\alpha\ ,\ \alpha\ ,\ \alpha\ )$. If $\alpha > 0$, S is gauge-equivalent to

$$S_{ab} = \tfrac{1}{3}\phi^2\delta_{ab} - \phi_a\phi_b \tag{5.2.c.10}$$

since the latter can be diagonalized to $S = \mathrm{diag}(-\frac{2}{3}\phi^2,\ \frac{1}{3}\phi^2\ ,\ \frac{1}{3}\phi^2\ )$. For the Schwarzschild solution, $\phi^2 = \frac{3m}{r^3}$. In isotropic coordinates with

$$r \equiv (1+\frac{m}{2r'})^2 r' \tag{5.2.c.11}$$

the matrix S in (5.2.c.10) with

$$\phi^a = (3m)^{\frac{1}{2}}(r')^{-\frac{5}{2}}(1+\frac{m}{2r'})^{-3}x'^a \tag{5.2.c.12}$$

yields

$$B^{ia} = \frac{m}{r'^3(1+\frac{m}{2r'})^6}\left(\delta^{ab} - \frac{3x'^a x'^b}{r'^2}\right)\tilde{\sigma}^i{}_b \tag{5.2.c.13}$$

This establishes the gauge-equivalence between the Schwarzschild solution in Ashtekar variables in our general formalism and the solution (5.2.c.13) exhibited in [FKS].

(d)The Eguchi-Hanson metric

The Eguchi-Hanson (EH) metric with cosmological constant can be written as[EH1][EH2][EH3]

$$ds^2 = [1-(\tfrac{a}{R})^4-\tfrac{\lambda}{6}R^2]^{-1}dR^2+R^2(\Theta_1{}^2+\Theta_2{}^2)$$

$$+R^2[1-(\tfrac{a}{R})^4-\tfrac{\lambda}{6}R^2]\Theta_3{}^2 \qquad (5.2.d.1)$$

We can choose the vierbein as

$$e_A=\{\ [1-(\tfrac{a}{R})^4-\tfrac{\lambda}{6}R^2]^{-\frac{1}{2}}dR\ ,\ R\Theta_1\ ,R\Theta_2\ ,[1-(\tfrac{a}{R})^4-\tfrac{\lambda}{6}R^2]^{\frac{1}{2}}R\Theta_3\ \} \qquad (5.2.d.2)$$

This gives the Ashtekar potentials as

$$A_1 = -2[1-(\tfrac{a}{R})^4-\tfrac{\lambda}{6}R^2]^{\frac{1}{2}}\Theta_1 \qquad (5.2.d.3a)$$

$$A_2 = -2[1-(\tfrac{a}{R})^4-\tfrac{\lambda}{6}R^2]^{\frac{1}{2}}\Theta_2 \qquad (5.2.d.3b)$$

$$A_3 = -2[1+(\tfrac{a}{R})^4-\tfrac{\lambda}{12}R^2]\Theta_3 \qquad (5.2.d.3c)$$

and $F_a = S_{ab}\Sigma^b$ with

$$S = \mathrm{diag}\Big(\ \frac{4a^4}{R^6}-\frac{\lambda}{3}\ ,\ \frac{4a^4}{R^6}-\frac{\lambda}{3}\ ,\ -\frac{8a^4}{R^6}-\frac{\lambda}{3}\ \Big) \qquad (5.2.d.7)$$

Thus, S is Type D when $a\neq 0$ and Type O when a=0. One can view a as an excitation which causes the configuration to break out of the Type O sector.

Reversing the orientation with $e'_A=\{-e_0,\ e_a\}$ gives the manifold $\overline{\mathrm{EH}}$ which has Ashtekar potentials

$$A'_1 = A'_2 = 0 \quad ,\ A'_3 = -\tfrac{\lambda}{2}R^2\Theta_3 \qquad (5.2.d.8)$$

and $F'_a = S'_{ab}\Sigma'^b$ with $S' = \mathrm{diag}(\ 0\ ,\ 0\ ,\ -\lambda\ )$. Like $\overline{CP}_2$, S' for $\overline{\mathrm{EH}}$ is non-invertible and $\overline{\mathrm{EH}}$ is described by an abelian anti-instanston. However the Eguchi-Hanson manifold has a boundary of real projective three-space, RP_3 (see [EH3]) . Note also that the abelian anti-instanton A'_3 (which does not involve the parametar a) is anti-self-dual with respect to $\overline{\mathrm{EH}}$ with e'_A for arbitrary λ *and* value of the parameter a. In the $\lambda = 0$ limit, S' vanishes and $\overline{\mathrm{EH}}$ therefore becomes half-flat. We shall exhibit the Eguchi-Hanson metric as a specific solution from two different classes of explicit solutions: first from the F=0 sector and also from the abelian anti-instanton sector.

5.3 Remarks

It is, in principle, possible to discuss all known Einstein manifolds in the context of Ashtekar variables in the manner prescribed at the beginning of this section. In all the examples we have obtained the Ashtekar potentials after knowing the vierbein although one can perform the reverse if one chooses the specific potentials together with the corresponding values of S in the examples given. In the next sections, the Ashtekar formalism will be used to construct new solutions.

S for Einstein manifolds with Euclidean signature, is real-symmetric and has real eigenvalues although it has complex eigenvalues for those with Lorentzian signature. In the latter case, four $\vec{x}$-dependent functions can be obtained from the real and imaginary part of $\mathrm{Tr}(S^2)$ and $\mathrm{Tr}(S^3)$. Despite the apparent paradox , there is no loss in this freedom for manifolds with Euclidean signature. The Ashtekar potential being the anti-self-dual part of the spin connection when the equations of motion hold, has a chirality (more precisely, orientation) associated with it. The imaginary components of $\mathrm{Tr}(S^2)$ and $\mathrm{Tr}(S^3)$ for Lorentzian signature are simply the components which change sign under orientation reversal. For the case of Euclidean signature, $\mathrm{Tr}(S^2)$ and $\mathrm{Tr}(S^3)$ too can be divided into components which are invariant under orientation reversal and those that change sign.

5.4 The F = 0 sector and hyperkähler manifolds

When the curvature of the Ashtekar connection vanishes, the metric is half-flat i.e. its Riemann curvature tensor is self-dual. S vanishes and for simply-connected manifolds, the connection can be set to zero globally. The equations of motion reduce to

$$d\Sigma_a = 0 \tag{5.4.1}$$

Since Σ_a is anti-self-dual, by (5.4.1) it is closed and harmonic. The three linearly independent harmonic anti-self-dual two-forms Σ_a span b_2^- . Thus

$$b_2^- = 3 \tag{5.4.2}$$

Since the curvature vanishes, for compact manifolds without boundary

$$0 = P_1 = 3\tau(M) - 2\chi(M) \tag{5.4.3}$$

In such cases, τ takes the maximal value of the Hitchin bound i.e.

$$\tau(M) = \tfrac{2}{3}\chi(M) \tag{5.4.4}$$

But we also have the relation

$$\chi(M) = b_2 + 2 = b_2^+ + b_2^- + 2 = b_2^+ + 5 \tag{5.4.5}$$

and

$$\tau(M) \equiv b_2^+ - b_2^- = b_2^+ - 3 \tag{5.4.6}$$

The relations (5.4.4-6) imply that in the F=0 sector, simply-connected compact Einstein manifolds without boundary have

$$b_2^+ = 19\ ,\ \ b_2^- = 3\ ,\ \ \tau = 16\ ,\ \ \chi = 24 \tag{5.4.7}$$

It is known that K3 manifolds and the four-torus T^4 are the only compact manifolds without boundary admitting metrics of self-dual Riemann curvature[EH1]. T^4 is not simply- connected and has $\tau = \chi = 0$ since its metric is flat, so choosing the convention that $\tau = -16$ for K3, we can identify the simply-connected compact half-flat manifolds without boundary (which has $\tau = 16$ and $\chi = 24$) as $\overline{\mathrm{K3}}$ manifolds. They have intersection form[DK]

$$Q = \overset{3}{\oplus}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \overset{2}{\oplus} E_8 \tag{5.4.8}$$

Note that in the decomposition of the curvature tensor into self and anti-self-dual components discussed in section 4.3, B = 0 for this sector, and since P_1 vanishes, by (5.4.7)

$$P'_1 = -3\tau(M) - 2\chi(M) = -96 \quad (= 0 \bmod 4) \tag{5.4.9}$$

So the SO(3) connection that lies in the A-part of the Riemann curvature in section 4.3 can be lifted to a SU(2) connection. The metric therefore has SU(2) holonomy and is thus hyperkähler[SA]. It is interesting to note that hyperkähler metrics have been associated with conditions for unbroken supersymmmetry in the compactification of superstrings (see for instance [GSW]) while in the context of gravity in four dimensions with Ashtekar variables, these hyperkähler metrics are associated with the unbroken topological field theory of the moduli space of flat connections.

In the context of Ashtekar variables, the reversed orientation of the half-flat sector yields K3 surfaces. Note that although F and S vanish, when the orientation is reversed, F' and S' need not be trivial (otherwise the metric is flat). It has been calculated that K3 surfaces are parametrized by 58 parameters[PA1]. (5.4.9) tells us that they must be associated with an Ashtekar connection with

Pontrjagin number of -96.

We now construct some explicit half-flat Einstein manifolds which are not necessarily simply-connected or without boundary. They will have $F = 0$ but $F' \neq 0$.

We assume the vierbein is of the form

$$e_A = \{ -a(R)dR \ , \ f(R)\Theta_1 \ , \ g(R)\Theta_2 \ , \ h(R)\Theta_3 \} \tag{5.4.10}$$

This yields

$$A_1 = \left\{ \frac{f'}{a} - \frac{(g^2+h^2-f^2)}{gh} \right\} \Theta_1 \tag{5.4.11a}$$

$$A_2 = \left\{ \frac{g'}{a} - \frac{(h^2+f^2-g^2)}{fh} \right\} \Theta_2 \tag{5.4.11b}$$

$$A_1 = \left\{ \frac{h'}{a} - \frac{(f^2+g^2-h^2)}{fg} \right\} \Theta_3 \tag{5.4.11c}$$

The primes denote diffrentiation with respect to R. Further simplifications can be achieved by assuming $f = g$. Setting $A_a = 0$ for $F=0$ (which is valid locally even if M is not simply-connected), we need to solve

$$\frac{f'}{a} = \frac{h}{f} \tag{5.4.12a}$$

and

$$\frac{h'}{a} + \frac{h^2}{f^2} = 2 \tag{5.4.12b}$$

Sustituting $a = \dfrac{(f^2)'}{2h}$ from (5.4.12a) into (5.4.12b) yields

$$\frac{(h^2)'}{(f^2)'} + \frac{h^2}{f^2} = 2 \tag{5.4.13}$$

Changing variables to $u \equiv h^2$ and $v \equiv f^2$, (5.4.13) reduces to

$$(uv)' = (v^2)' \tag{5.4.14}$$

The solution is

$$h^2 = f^2 + \frac{b}{f^2} \tag{5.4.15}$$

So the metric is

$$ds^2 = a^2 dR^2 + f^2({\Theta_1}^2 + {\Theta_2}^2) + h^2 {\Theta_3}^2 \tag{5.4.16}$$

with $a = \dfrac{(f^2)'}{2h}$, h as in (5.4.15) and b, an intergration constant. However, f is a *free* function of R.

Reversing the orientation does not change the metric (5.4.16). However a switch of orientation to $e'_A = \{-e_0, e_a\}$ changes A into

$$A'_1 = -\frac{2h}{f}\Theta_1 \tag{5.4.17a}$$

$$A'_2 = -\frac{2h}{f}\Theta_2 \tag{5.4.17b}$$

$$A'_3 = -2(2 - \frac{h^2}{f^2})\Theta_1 \tag{5.4.17c}$$

assuming the relations between h, a and f hold. A short computation yields $F'_a = S'_{ab}\Sigma'^b$ with

$$S' = \mathrm{diag}\Big(-\frac{4b}{f^6}, -\frac{4b}{f^6}, \frac{8b}{f^6}\Big) \tag{5.4.18}$$

Thus the equations of motion still hold but S' is now of Type D when $b \neq 0$. We know that $P_1 = 0$ but for compact manifolds without boundary,

$$P'_1 = -\frac{1}{4\pi^2}\int F'_a \wedge F'_a$$

$$= 12b^2 \int (f^{-8})' \, dR \tag{5.4.19}$$

assuming θ, ϕ and ψ span three-spheres for fixed values of R. This means that it is possible to obtain explicit self-dual Einstein manifolds with non-trivial Pontrjagin numbers through the construction outlined above by choosing the appropriate f's.

Consider the case of f = R and b $= -a^4$, then we have S' = diag$\left(\frac{4a^4}{R^6}, \frac{4a^4}{R^6}, -\frac{8a^4}{R^6} \right)$ and through (5.4.16)

$$ds^2 = [1-(\tfrac{a}{R})^4]^{-1}dR^2 + R^2(\Theta_1{}^2 + \Theta_2{}^2) + R^2[1-(\tfrac{a}{R})^4]\Theta_3{}^2 \tag{5.4.20}$$

This is the zero cosmological constant limit of the Eguchi-Hanson metric discussed in example (d) previously. (Note that our conventions are such that $\overline{\mathrm{EH}}$ with $\lambda =$ 0 is half- flat.)

5.5 Abelian anti-instantons and Kähler-Einstein manifolds

When S is equivalent to the form diag(0, 0, $-\lambda$), the Ashtekar potential is described by an abelian anti-instanton. In the gauge in which S is diagonal, the only non-vanishing component of the curvature is F_3 . The equations of motion reduce to

$$dA_3 = F_3 = -\lambda\Sigma_3 \tag{5.5.1a}$$

$$d\Sigma_1 - A_3 \wedge \Sigma_2 = 0 \tag{5.5.1b}$$

and

$$d\Sigma_2 + A_3 \wedge \Sigma_1 = 0 \tag{5.5.1c}$$

Endowing the manifold with a complex structure, the combination

$$\Sigma^+ \equiv \Sigma_1 + i\Sigma_2 \tag{5.5.2}$$

satisfies

$$d\Sigma^+ + iA_3 \wedge \Sigma^+ = 0 \tag{5.5.3}$$

and we can take (5.5.1a), (5.5.3) and S= diag(0,0,$-\lambda$) to be the equations of motion for the assumed manifolds. Consider

$$\Omega^1 \equiv -e^0 + ie^3 \tag{5.5.4a}$$

and

$$\Omega^2 \equiv e^1 + ie^2 \tag{5.5.4b}$$

Then

$$\Sigma^3 = \frac{i}{2}\,\Omega^\alpha \wedge \overline{\Omega}^\alpha \quad , \alpha = 1,2 \tag{5.5.5}$$

is closed (i.e. $d\Sigma^3 = 0$) by (5.5.1a) and Σ^3 is the Kähler form of the manifold. So Einstein manifolds which are endowed with complex structure and described by Ashtekar potentials which are abelian anti-instantons can be identified as Kahler manifolds.

We shall now construct explicit metrics which have Ashtekar potentials that are abelian anti-instantons. For simplicity, we shall assume the vierbein is again of the form of (5.4.10). So, the Ashtekar potentials will be as in (5.4.11). The equations of motion (5.5.1b,c) are satisfied because the Ashtekar potentials in (5.4.11) are computed through (4.2.3). A further assumption of

$$A_3 = c(R)\Theta_3 \tag{5.5.6}$$

implies

$$F_3 = c'dr \wedge \Theta_3 + 2c\Theta_1 \wedge \Theta_2 \tag{5.5.7}$$

Again the prime denotes differentiation with respect to R. To satisfy $S = \mathrm{diag}(0,0,-\lambda)$, we must have

$$c' = -\lambda ah \quad , \quad 2c = -\lambda fg \tag{5.5.8}$$

It is easy the check that if we assume f=g then the condition (5.5.8) implies the vanishing of A_1 and A_2 , and we are left with matching (5.5.6) to (5.4.11c) i.e.

$$\frac{h'}{a} - \frac{(2f^2 - h^2)}{f^2} = c \tag{5.5.9}$$

Eliminating f^2 from (5.5.9) through (5.5.8) gives

$$-\lambda[h^2c]' = \tfrac{2}{3}[c^3]' + 2[c^2]' \tag{5.5.10}$$

This gives the coefficients for the vierbein as

$$h^2 = -\tfrac{2}{3\lambda}\{\ c(c+3) + \tfrac{b}{c}\ \} \tag{5.5.11a}$$

and

$$a^2 = \frac{(c')^2}{\lambda^2 h^2}\quad ,\ f^2 =\ g^2\ =\ -\tfrac{2c}{\lambda} \tag{5.5.11b}$$

where b is an integration constant. Note that the metric

$$ds^2 = a^2dR^2 + f^2({\Theta_1}^2 + {\Theta_2}^2) + h^2{\Theta_3}^2 \tag{5.5.12}$$

is determined completely by b and the abelian anti-instanton $c\Theta_3$ and c is a *free* function of R.

Now consider reversing the orientation as before, but not changing the relations (5.5.11). The new Ashtekar potentials are

$$A'_1 =\ -\tfrac{2f'}{a}\Theta_1 \tag{5.5.13a}$$

$$A'_2 =\ -\tfrac{2f'}{a}\Theta_2 \tag{5.5.13b}$$

$$A'_3 = (\,-\tfrac{c}{3} - 2 + \tfrac{2b}{3c^2})\Theta_3 \tag{5.5.13c}$$

(There should not be any confusion between the primes on the L.H.S. of (5.5.13) which have the meaning of new and those on the R.H.S denoting differentiation with respect to R). It can be checked that F'_a is now *non-abelian* and has the form $F'_a = S'_{ab}\Sigma'^b$ with

$$S' = \mathrm{diag}\Big(-\tfrac{\lambda}{3}[1 - \tfrac{2b}{c^3}]\ \ ,-\tfrac{\lambda}{3}[1 - \tfrac{2b}{c^3}]\ ,\ -\tfrac{\lambda}{3}[1 + \tfrac{4b}{c^3}]\ \Big) \tag{5.5.14}$$

Of course TrS' $= -\lambda$, but remarkably, S' is no longer $(0,0,-\lambda)$. It is Type D for $b \neq 0$ and Type O when $b = 0$.

The Pontrjagin numbers can now be computed. For the first orientation with abelian anti-instanton gauge potentials,

$$\begin{aligned} P_1 &= \frac{1}{4\pi^2}\int F_a \wedge F_a \\ &= -\int [c^2]' dR \end{aligned} \tag{5.5.15}$$

The Pontrjagin number for the reversed orientation yields

$$P'_1 = \frac{1}{4\pi^2}\int F_a \wedge F_a$$

$$= -\int \left[\frac{c^2}{3} + \frac{8b}{9c} - \frac{4b^2}{3c^4}\right]' dR \tag{5.5.16}$$

If we use

$$c = -\frac{\lambda R^2}{2(1+\frac{\lambda}{6}R^2)} \quad , \quad b = 0 \tag{5.5.17}$$

as an example, all the results obtained in example (b) for CP_2 and $\overline{CP}_2$ will be recaptured. Another special case but with non-vanishing b is

$$c = -\frac{\lambda}{2}R^2 \quad , \quad b = -\frac{3}{4}\lambda^2 a^4 \tag{5.5.18}$$

This gives all the results of the Eguchi-Hanson metric and $\overline{EH}$ discussed in example (d).

5.6 Symmetry breaking, abelian anti-instantons and Ashtekar variables

It has been observed that S can play an effective role as an order parameter characterizing the Type O sector which corresponds classically to conformally self-dual Einstein manifolds. Further evidence of S playing the role of order parameter comes from the study of Ashtekar connections that are abelian anti-instantons. Consider the cases when S can be expressed in terms of a triplet of real Higgs fields ϕ_a through

$$S_{ab} = \pm\phi_a\phi_b \tag{5.6.1}$$

The real and symmetric S satisfies the supermomentum constraint and can always be diagonalized by an SO(3) gauge transformation into $S = \mathrm{diag}(0,\ 0, -\lambda)$ with

$$||\phi|| = \sqrt{\mp\lambda} \tag{5.6.2}$$

The $+(-)$ sign in (5.6.1) is to be chosen when λ is negative (positive). In the U-gauge with $\phi_a = ||\phi||\delta_{a3}$ and $A_{1,2} = 0$ (which yields $F_{1,2} = 0$) , we have

$$(D\phi)_a = 0 \tag{5.6.3}$$

(5.6.2,3) are gauge and diffeomorphism-invariant statements (and so will be valid for all gauges and coordinate systems) which precisely describe the Higgs vacuum and symmetry breaking from SO(3) to U(1). The symmetry breaking is thus a gauge and diffeomorphism-invariant concept. We have already discussed the explicit example of $\overline{CP}_2$. It appears that *self-interaction of the gravitational fields are strong enough to generate dynamical Higgs fields and symmetry breaking.* Note that S is non-invertible in this abelian anti-instanton sector. The abelian anti-instanton sector is yet another possible phase of non-perturbative gravity.

As a corollary, we remark that for regular Σ's and F's, Einstein manifolds with abelian anti-instantons can occur only if the cosmological constant is non-vanishing. This is because for abelian anti-instantons, we can always go to the U-gauge and choose $F_{1,2} = 0$ and then S has to take the form $S = \mathrm{diag}(\ 0,\ 0, -\lambda)$ in order to satisfy $\mathrm{Tr}S = -\lambda$ and $F_{1,2} = 0$.

Chapter 6

Non-perturbative canonical quantization of the theory

6.1 Ordering of the constraints

In the passage from the classical to the quantum theory, a major concern is the ordering of the non-commuting operators which make up the Hamiltonian. In General Relativity, this problem is more than the usual ambiguity in the ordering since, apart from boundary terms, the Hamiltonian is a linear combination of the constraints and the ordering adopted must be such that the quantum constraint algebra is anomaly free i.e. the quantum constraint algebra closes. In Dirac quantization, physical states are annihilated by all the constraints so the closure of the quantum constraint algebra is a consistency requirement. The quantum constraint algebra for General Relativity with the old ADM variables has been studied by many people, but to the present day, there is no consensus on whether or not there exists an ordering for which the quantum constraint algebra closes. (For a good review and discussion on this topic, see [TW]). With Ashtekar's simpler constraints, the choice of ordering and the physical implications of a particular choice are more transparent. An ordering for which the quantum constraint algebra closes formally will be given and the physical consequences of this will be examined.

The ordering of the all-important superhamiltonian constraint, Eqn. (2.2.7), whose quantum version will correspond to what will be called the Ashtekar-Wheeler-DeWitt Equation, will be explored first. At the classical level the other constraints, Gauss' Law and the supermomentum constraint, generate gauge transformations and gauge-covariant three-dimensional diffeomorphisms respectively. The classical superhamiltonian constraint is

$$\mathrm{H} \equiv \frac{1}{16\pi \mathrm{G}}\left(\epsilon_{abc}\tilde{\sigma}^{ia}\tilde{\sigma}^{jb}\mathrm{F}_{ijc} + \frac{\lambda}{3}\epsilon_{abc}\epsilon_{ijk}\tilde{\sigma}^{ia}\tilde{\sigma}^{jb}\tilde{\sigma}^{kc} \right) = 0 \tag{6.1.1}$$

If we make the reasonable assumption that in the ordering of $\hat{\mathrm{H}}$, $\hat{\mathrm{F}}$ is to be kept whole, then the general form for $\hat{\mathrm{H}}$ (without the benefit of Hermitian conjugation) is

$$\hat{\mathrm{H}} = \frac{1}{16\pi \mathrm{G}}(\alpha\epsilon_{abc}\hat{\mathrm{F}}_{ij}{}^{c}\hat{\tilde{\sigma}}^{ia}\hat{\tilde{\sigma}}^{jb} + \beta\epsilon_{abc}\hat{\tilde{\sigma}}^{ia}\hat{\mathrm{F}}_{ij}{}^{c}\hat{\tilde{\sigma}}^{jb} + \gamma\epsilon_{abc}\hat{\tilde{\sigma}}^{ia}\hat{\tilde{\sigma}}^{jb}\hat{\mathrm{F}}_{ij}{}^{c} + \frac{\lambda}{3}\epsilon_{abc}\epsilon_{ijk}\hat{\tilde{\sigma}}^{ia}\hat{\tilde{\sigma}}^{jb}\hat{\tilde{\sigma}}^{kc}) \tag{6.1.2}$$

α, β and γ are the attached weights to each ordering with the condition

$$\alpha + \beta + \gamma = 1 \tag{6.1.3}$$

As in Dirac quantization with constraints, we demand that physical states are annihilated by the constraints acting on them. The straightforward $\{\ ,\ \}_{\mathrm{P.B.}} \rightarrow \frac{1}{i}[\ ,\ ]$ quantization prescription for the canonical variables A_{ia} and $\tilde{\sigma}^{ia}$ which have Poisson brackets

$$\{\ \mathrm{A}_{ia}(\vec{\mathrm{x}})\ ,\ \tilde{\sigma}^{jb}(\vec{\mathrm{y}})\ \}_{\mathrm{P.B.}} = \frac{i(16\pi\mathrm{G})}{2}\ \delta_{a}{}^{b}\delta_{i}^{j}\ \delta^{3}(\vec{\mathrm{x}}-\vec{\mathrm{y}}) \tag{6.1.4a}$$

$$\{\ \mathrm{A}_{ia}(\vec{\mathrm{x}})\ ,\ \mathrm{A}_{jb}(\vec{\mathrm{y}})\ \}_{\mathrm{P.B.}} = \{\tilde{\sigma}^{ia}(\vec{\mathrm{x}})\ ,\ \tilde{\sigma}^{jb}(\vec{\mathrm{y}})\ \}_{\mathrm{P.B.}} = 0 \tag{6.1.4b}$$

can be realized in the A-representation, with $\hat{\mathrm{A}}$ acting on the quantum state Ψ by multiplication while $\hat{\tilde{\sigma}}$ acts on Ψ by functional differentiation

$$\hat{\tilde{\sigma}}^{ia}\Psi[\mathrm{A}] = \frac{(16\pi\mathrm{G})}{2}\frac{\delta}{\delta \mathrm{A}_{ia}}\Psi[\mathrm{A}] \tag{6.1.5}$$

In the A-representation, Ψ is a functional of the Ashtekar potential.

We do not expect to obtain the general solution to the Ashtekar-Wheeler-DeWitt Equation, but the factor ordering problem can be enlightened by the choice of test solutions. However it must be emphasized that once a particular ordering investigated through a test solution is decided on, the ordering is the *same* for all the possible quantum wave functionals.

An obvious choice for gauge-invariant three-geometry is the Chern-Simons functional. Classically, it is invariant under small gauge transformations generated by the Gauss' Law constraint and under three-dimensional diffeomorphisms generated by the superhamiltonian constraint (2.2.6). The Chern-Simons functional is

$$\mathfrak{C} = \frac{1}{(16\pi G)}\int_{M^3} d^3x\, \epsilon^{ijk}\big(\, A_{ia}\partial_j A_k{}^a \;+\; \tfrac{1}{3}\,\epsilon^{abc}A_{ia}A_{jb}A_{kc}\,\big) \qquad (6.1.6)$$

Assuming that Ψ depends on A only through $\mathfrak{C}$, and bearing in mind the identity

$$\frac{\delta\mathfrak{C}}{\delta A_{ia}} = \frac{1}{16\pi G}\, B^{ia} \qquad (6.1.7)$$

the constraint

$$\hat{H}\Psi = 0 \qquad (6.1.8)$$

translates into

$$\begin{aligned}\epsilon_{abc}\epsilon_{ijk}\Big\{\, &\hat{B}^{kc}(\vec{x})[\hat{\sigma}^{ia}(\vec{x})\;,\;\hat{B}^{jb}(\vec{x})]\big(\;(\alpha\;+\;2\beta\;+\;3\gamma)\frac{\delta}{\delta\mathfrak{C}}\Psi\;+\lambda\frac{\delta^2}{\delta\mathfrak{C}^2}\Psi\;\big)\\ &+\;[\hat{\sigma}^{ia}(\vec{x})\;,\;[\hat{\sigma}^{jb}(\vec{x})\;,\;\hat{B}^{kc}(\vec{x})]]\big(\;\gamma\Psi\;+\;\tfrac{\lambda}{3}\frac{\delta}{\delta\mathfrak{C}}\Psi\big)\\ &+\;\hat{B}^{ia}(\vec{x})\hat{B}^{jb}(\vec{x})\hat{B}^{kc}(\vec{x})\big(\;(\;\alpha+\beta+\gamma)\frac{\delta^2}{\delta\mathfrak{C}^2}\Psi\;+\;\tfrac{\lambda}{3}\frac{\delta^3}{\delta\mathfrak{C}^3}\Psi\;\big)\Big\} = 0 \end{aligned} \qquad (6.1.9)$$

Demanding that the coefficients of the divergent (ambiguous) terms containing the commutators evaluated at coincident points vanish, and together with the condition (6.1.3), we can solve for the ordering and the quantum state Ψ . The results are :

(i) For vanishing cosmological constant

(a) $\alpha \neq 2$, $(\alpha,\beta,\gamma) = (\alpha, 1-\alpha, 0)$

with $\frac{\delta}{\delta \mathfrak{C}}\Psi = 0$.

This includes the ordering of [JS2] which is $(\alpha,\ \beta,\ \gamma) = (1,\ 0,\ 0)$

(b) $(\alpha,\beta,\gamma) = (2,\ -1,\ 0)$

which gives the superhamiltonian constraint as

$$\det(\mathrm{B})\,\frac{\delta^2}{\delta \mathfrak{C}^2}\,\Psi = 0 \tag{6.1.10}$$

Note the role of $\mathfrak{C}$ as gauge-invariant three-geometry . Eqn. (6.1.10) is to the Ashtekar-Wheeler-DeWitt Equation

$$\text{“}\ \epsilon_{abc}\epsilon_{ijk}\mathrm{B}^{kc}\frac{\delta}{\delta \mathrm{A}_{jb}}\frac{\delta}{\delta \mathrm{A}_{ia}}\ \text{”}\ \Psi[\mathrm{A}] = 0 \tag{6.1.11}$$

as

$$\frac{\delta^2}{(\delta\ \mathfrak{G})^2}\Psi\ +\ {}^3\mathrm{R}\ \Psi = 0 \tag{6.1.12}$$

is to the Wheeler-DeWitt Equation

$$\text{“}\ (\ \mathrm{G}_{ijkl}\frac{\delta}{\delta g_{ij}}\frac{\delta}{\delta g_{kl}}\ +\ \sqrt{g}\ {}^3\mathrm{R}\ )\ \text{”}\ \Psi[g] = 0 \tag{6.1.13}$$

The parentheses denote ambiguity in the ordering and $\mathfrak{G}$ stands symbolically for three-geometry[WD]. It should be noted that while $\mathfrak{G}$ is symbolic, $\mathfrak{C}$ is an *explicit* representation of three-geometry.

(ii) For non-vanishing cosmological constant, $(\alpha,\beta,\gamma) = (0,\ 0,\ 1)$

with

$$\Psi = \mathrm{I}\ \exp\ (\ \frac{-3\mathfrak{C}}{\lambda}\ ) \tag{6.1.14}$$

where I is a topological invariant of A_{ia} (i.e. $\frac{\delta I}{\delta A_{ia}} = 0$).

This solution and ordering was also discussed in [CS1] and [KO] . This solution has

$$\hat{\tilde{\sigma}}^{ia}\Psi = -\frac{3\hat{B}^{ia}}{\lambda}\Psi \tag{6.1.15}$$

Although $\hat{\tilde{\sigma}}$ does not commute with $\hat{F}$, by virtue of (6.1.15), Eqn. (6.1.14) still solves the supermomentum constraint

$$\hat{H}_i\Psi = \frac{2}{i(16\pi G)}\,\hat{\tilde{\sigma}}^{ia}\hat{F}_{ija}\Psi = 0 \tag{6.1.16}$$

and the Gauss' Law constraint

$$\hat{G}_a\Psi = \frac{2}{i(16\pi G)}\,(D_i\hat{\tilde{\sigma}}^i\,)_a\Psi = 0 \tag{6.1.17}$$

is satisfied by virtue of the Bianchi Identity. Eqn. (6.1.15) gives quantum justification to the classical Ashtekar-Renteln ansatz[AR]

$$\hat{B}^{kc} = -\frac{\lambda}{3}\hat{\tilde{\sigma}}^{kc} \tag{6.1.18}$$

Once the ordering has been settled, it is not necessary to confine Ψ to be a functional of A through $\mathfrak{C}$ only . Among the many orderings $(\alpha,\beta,\gamma) = (0,0,1)$ has the advantage that the quantum constraint algebra can be shown to close formally (with the structure functions appearing to the left of the quantum constraints) if we adopt for the other constraints the orderings displayed in Eqns. (6.1.16) and (6.1.17).

6.2 The quantum constraint algebra

With the quantum constraints ordered as

$$\hat{G}^a = \frac{2}{i(16\pi G)}\,(D_i\hat{\tilde{\sigma}}^i)^a \tag{6.2.1}$$

$$\hat{H}_i = \frac{2}{i(16\pi G)}\,\hat{\tilde{\sigma}}^{ia}\hat{F}_{ija} \tag{6.2.2}$$

and

$$\hat{H} = \frac{1}{16\pi G}\,\epsilon_{abc}\epsilon_{ijk}\hat{\tilde{\sigma}}^{ia}\hat{\tilde{\sigma}}^{jb}(\hat{F}_{ij}{}^{c} + \tfrac{\lambda}{3}\hat{\tilde{\sigma}}^{kc}) \tag{6.2.3}$$

the quantum constraint algebra is

$$\{\,\hat{H}_i[N^i]\,,\,\hat{H}_j[M^j]\} = -\hat{H}_i[(\mathcal{L}_{\vec{N}}M)^i] + \hat{G}^a[-iN^iM^jF_{ija}\,] \tag{6.2.4a}$$

$$\{\,\hat{H}_i[N^i]\,,\,\hat{H}[\underset{\sim}{M}]\,\} = -\hat{H}[\mathcal{L}_{\vec{N}}\underset{\sim}{M}] + \hat{G}^a[-i\underset{\sim}{M}N^i\tilde{\sigma}^{jb}\epsilon_{abc}(F_{ijc} + \tfrac{\lambda}{2}\epsilon_{ijk}\tilde{\sigma}^{kc})] \tag{6.2.4b}$$

$$\{\,\hat{H}[\underset{\sim}{N}]\,,\,\hat{H}[\underset{\sim}{M}]\,\} = -\hat{H}_i[\tilde{\sigma}^{ia}\tilde{\sigma}^{j}{}_{a}(\,\underset{\sim}{N}\partial_j\underset{\sim}{M} - \underset{\sim}{M}\partial_j\underset{\sim}{N}\,)] \tag{6.2.4c}$$

$$\{\,\hat{G}^a[\alpha_a]\,,\,\hat{G}^b[\beta_b]\,\} = \hat{G}^a[\epsilon_a{}^{bc}\alpha_b\beta_c] \tag{6.2.4d}$$

$$\{\,\hat{G}^a[\alpha_a]\,,\,\hat{H}[\underset{\sim}{N}]\,\} = \{\,\hat{G}^a[\alpha_a]\,,\,\hat{H}_i[N^i]\,\} = 0 \tag{6.2.4e}$$

and closes formally. It has the same form as the classical constraint algebra exhibited in Chapter 2.

6.3 Physical interpretation of the ordering

The quantum superhamiltonian constraint for the ordering $(\alpha,\beta,\gamma) = (0,\,0,\,1)$ takes the form

$$\hat{H}\Psi = \frac{1}{16\pi G}\,\epsilon_{abc}\epsilon_{ijk}\hat{\tilde{\sigma}}^{ia}\hat{\tilde{\sigma}}^{jb}\,\hat{Q}^{kc}\Psi[A] = 0 \tag{6.3.1}$$

where

$$\begin{aligned}\hat{Q}^{kc} &\equiv \hat{B}^{kc} + \tfrac{\lambda}{3}\hat{\tilde{\sigma}}^{kc} \\ &= \tfrac{\lambda}{3}(16\pi G)\exp\!\left(\frac{-3\mathcal{C}}{\lambda}\right)\frac{\delta}{2\delta A_{kc}}\exp\!\left(\frac{3\mathcal{C}}{\lambda}\right)\end{aligned} \tag{6.3.2}$$

and $\hat{Q} \equiv \int_{M^3} d^3x\;\delta A_{ia}\hat{Q}^{ia}$ is precisely of the form of the constraint of a topological quantum field theory (TQFT)[WT1] selecting physical quantum states of the TQFT according to

$$\hat{Q}\Psi = 0 \tag{6.3.3}$$

Indeed, Eqn. (6.1.14) is the solution of Eqn. (6.3.3) and is sufficient for Eq. (6.3.1). For configurations obeying the Ashtekar-Renteln ansatz, the reduced Euclideanized action indeed acquires the form

$$\mathcal{A}_{\mathrm{TQFT}} = -\frac{3}{16\pi G\lambda}\int_{M} F_a \wedge F_a \tag{6.3.4}$$

which yields constraints generating gauge and topological invariance. For gravity, the TQFT sector has λ playing the role of the square of the coupling constant and $(G\lambda)^{-1}$ playing the role of the θ angle.

It is amusing to note that Eqns. (6.1.14) and (6.3.4) are in rough agreement with the Hartle-Hawking proposal[HH] for the "wavefunction of the universe". The Hartle-Hawking proposal is that the quantum state should be of the form

$$\Psi[g(M^3)] \sim \int [Dg] \exp(\int_M \mathcal{A}_E) \tag{6.3.5}$$

with M^3 being the boundary of M and $\mathcal{A}_E$ is the Euclideanized action. For our case, because $F_a \wedge F_a = (16\pi G)d\mathfrak{C}$, roughly speaking

$$\Psi = I \exp\left(\frac{-3\mathfrak{C}}{\lambda}\right) \sim \int [DA] \exp(\int_M \mathcal{A}_{\mathrm{TQFT}}) \tag{6.3.6}$$

In the limit $\lambda \to 0$, for finite momenta , $Q^{ia} \to B^{ia}$ and the above ordering implies that in the vanishing cosmological constant limit $F = 0$ is a sufficient condition for Eqn. (6.3.1). The vanishing of the curvature is a gauge-invariant and diffeomorphism- invariant statement but note also the consistency of the ordering adopted for $\hat{H}_i$ in Eqn. (6.2.2) in the limit $\lambda = 0$. In the limit of vanishing cosmological constant, within the $\hat{Q}\Psi = 0$ sector, the states are characterized by gauge-inequivalent classes of flat connection i.e. by homomorphisms $\mathrm{Hom}[(\pi_1(M^3), G)]/(\mathrm{Ad}G)$ realized by Wilson loops $\mathrm{Tr}[P(\exp\int_c \vec{A}.d\vec{r})]$ where c denotes a closed loop and G is the gauge group SO(3). Even within the $\hat{Q}\Psi = 0$ sector, there are indications of a possible phase transition when λ reaches a critical value. For $\lambda \neq 0$, Q generates arbitrary deformations of A ,

$$\delta_Q A_{ia} = \{ A_{ia} , \frac{2}{i}\int_{M^3} \epsilon_{jb} Q^{jb} \}_{P.B.} = \lambda \epsilon_{ia} \tag{6.3.7}$$

In the $\lambda = 0$ limit ,

$$\delta_{Q=F}\, A_{ia} = \mathcal{L}_{\vec{N}} A_{ia} \quad \text{(modulo F=0 and } G^a\text{=0)} \tag{6.3.8}$$

Such a change in the symmetries of the configuration variable generated by Q should also be reflected in the reduced phase space. As an example, consider M^3 homeomorphic to S^3 . Large gauge transformations on S^3 are not connected by infinitesimal gauge transformations generated by Gauss' Law . If $\lambda \neq 0$, the topological symmetry generated by Q connect all possible configurations and collapses the moduli space of A under Q and G^a to a single point. There are no local degrees of freedom. When λ hits the critical value of zero, Q no longer generates arbitrary deformations so in the moduli space configurations differing by large gauge transformations are distinct. (See [HO2] for a similar discussion). Such a mismatch in the degrees of freedom is indicative of a phase transition.

In the above, within the $\hat{Q}\Psi = 0$ phase, we are only referring to zero field degrees of freedom (because the number of constraints matches the number of conjugate pairs) but there can be non-trivial global fluctuations. However, the ordering adopted raises even more intriguing prospects for quantum gravity in dimension four. The superhamiltonian constraint factorizes as in Eqn. (6.3.1). A sufficient condition for $\hat{H}\Psi = 0$ is for Eqn. (6.3.3) to be satisfied but the constraints of General Relativity allow for .

$$\hat{Q}\Psi \neq 0 \text{ and } \hat{H}\Psi = 0 \tag{6.3.9}$$

together with constraints (6.2.1) and (6.2.2). Clearly in such a scenario topological invariance is then broken and naive counting allows for up to two unconstrained local degrees of freedom to emerge. There have been speculations on an unbroken topological phase in gravity in four dimensions before but the scenario described above shows how one can break the topological invariance without breaking the invariances associated with gravity and gives a precise prescription of how quantum gravity with two local degrees of freedom can arise as a broken TQFT. The explicit realization of such a scenario would require regularization, for instance, to eliminate ambiguities in the constraints associated with the product of non-commuting operators evaluated at coincident space-time points .

The unbroken topological phase corresponds to $S_{ab} = -\lambda\delta_{ab}$ and in the scheme of Chapter 3 belongs to Type O. Classically , for non-degenerate metrics , it corresponds to conformally self-dual Einstein manifolds. In the Ashtekar formulation, then, classical half-flat respectively self-dual Einstein manifolds can be interpreted in the quantum mechanical context, under appropriate ordering of the constraints, as moduli spaces of flat connections (MSFC) and TQFT of the $\int \mathrm{Tr}(F \wedge F)$ type respectively.

In the general scheme, S plays the role of the parameters characterizing the various phases. As we have discussed in section 5.6, there is also an abelian anti-instanton phase where self-interactions of the gravitational fields is strong enough to produce dynamical Higgs fields and symmetry breaking form SO(3) to U(1). The rich and intricate relationships that can exist between different phases of the theory and their physical ramifications remain to be explored.

It should be pointed out that although the solution Eqn. (6.3.3) restricts the fluctuations to a certain sector of the theory (namely Type O), this is different from minisuperspace models where certain degrees of freedom are frozen out by assumptions of homogeneity etc. on the metric before solving the simplified Wheeler-DeWitt equation. Note that no assumptions were made on the variables and (6.1.14) is a genuine solution of the *full* Ashtekar-Wheeler-DeWitt equation. The restriction to Type O is a consequence of a possible true quantum state of the theory! Recall that the discussion of classical solutions in Chapter 5 tells us that all conformally self-dual Einstein manifolds belong to Type O (specifically, S^4 and CP_2 which have different values of topological invariants such as the Euler numbers and signature). So in the phase space the *same* quantum state has non-trivial amplitude distribution for intitial data which correspond to *topologically distinct* classical manifolds. Thus from fundamental considerations of the exact theory, there is explicit demonstration that in quantum gravity even fluctuations in topology can occur.

Lastly, it should be emphasized that while the Dirac quantization scheme need not be the final word on such a rich and complicated theory, it is nevertheless instructive and one can expect broad features of the theory to recur for various schemes. In the next chapter, the method of Becchi-Rouet-Stora-Tyutin (BRST) quantization will be performed to yield further insights into the theory.

Chapter 7

BRST quantization of the theory

In this chapter, the Becchi-Rouet-Stora-Tyutin (BRST) analysis of the theory[BRST] will be performed. The method of BRST quantization has been applied successfully to the study of many physical systems[BR]. It is particularly suited to the analysis of systems with constraints. For such systems, one is concerned with quantities modulo constraints. With Dirac quantization, one is dealing with weak equalities. For instance, observables only need to commute with the constraints modulo constraints. In simple systems it is easy to identify observables and their equivalence classes but for the complicated set of constraints that we have to deal with, the canonical Dirac quantization scheme is too unwieldy to produce unambiguous results. In BRST quantization, one deals with strong equalities. Observables are strictly invariant under BRST transformations and physical states are those annihilated by the BRST charge. Equivalence modulo constraints is replaced by the concept of BRST cohomology and the identification of observables and physical states consist of solving for the BRST cohomology of the theory. There are further reasons for BRST quantization of the theory. Recently Witten[WT1] managed to give a successful physical interpretation of Donaldson's work[DO] on further differential invariants of four-manifolds. This was achieved through the BRST analysis of a topological quantum field theory (TQFT). It is reasonable to expect that differential invariants of four-manifolds are indeed observables of General Relativity and further invariants of four-manifolds and observables can be constructed through a BRST analysis of four-dimensional gravity with Ashtekar variables.

7.1 The action and Ashtekar variables

We shall work with Riemannian manifolds and start with the action proposed by Samuel[SAM] and Jacobson and Smolin[JS2]

$$\mathcal{A} = \frac{1}{16\pi G}\int_M (2\, F_a \wedge \Sigma_a + \frac{\lambda}{3}\, \Sigma_a \wedge \Sigma_a) \tag{7.1.1}$$

where as before, the anti self-dual two-form

$$\Sigma_a \equiv \tfrac{1}{2}\epsilon_a{}^{bc}\, e_b \wedge e_c - e_0 \wedge e_a \tag{7.1.2}$$

In the Ashtekar formalism, the metric is considered to be a derived quantity and can be expressed in terms of Σ by using the relation

$$\tilde{g}_{\mu\nu} = \tfrac{1}{12}\, \epsilon_{abc}\, \tilde{\epsilon}^{\alpha\beta\gamma\delta}\, \Sigma^a_{\alpha\beta}\, \Sigma^b_{\gamma\mu}\, \Sigma^c_{\nu\delta} \tag{7.1.3}$$

We first check that the action reproduces the same results that were obtained through the canonical formulation of Chapter 4. In applying the canonical formalism to the above action, it is convenient to work in the spatial gauge introduced in section 4.1. With this decomposition, it is straightforward to work out that the above action is equivalent to

$$\begin{aligned}\mathcal{A} = \frac{1}{16\pi G}\int d^4x \Big\{ & 2\tilde{\sigma}^{ia}\dot{A}_{ia} + 2A_{0a}D_i\tilde{\sigma}^{ia} + 2N^j\tilde{\sigma}^{ia}F_{ija} \\ & - \underset{\sim}{N}(\epsilon_{abc}\tilde{\sigma}^{ia}\tilde{\sigma}^{jb}F^c_{ij} + \frac{\lambda}{3}\,\epsilon_{abc}\underset{\sim}{\epsilon}_{ijk}\tilde{\sigma}^{ia}\tilde{\sigma}^{jb}\tilde{\sigma}^{kc}) \Big\} \\ & + \text{boundary terms}\end{aligned} \tag{7.1.4}$$

bearing in mind the that $\tilde{\sigma}$ and $\underset{\sim}{N}$ are of the form

$$\tilde{\sigma}^{ia} = \frac{1}{2}\, \tilde{\epsilon}^{ijk}\epsilon^{abc}e_{jb}e_{kc} \tag{7.1.5}$$

$$\underset{\sim}{N} \equiv \det(e_{ai})^{-1}N \tag{7.1.6}$$

Therefore $\frac{2}{(16\pi G)}\tilde{\sigma}^{ia}$ is readily identified as the conjugate variable to A_{ia}. The variables A_{0a}, N^i and $\underset{\sim}{N}$ clearly are Lagrange multipliers and the resulting Ashtekar constraints can be obtained by the variation of the action with respect to these. The constraints are the same as the Ashtekar constraints of Chapter 2.

The equations of motion are obtained by varying the action with respect to A and e. The variation of the action yields

$$\delta\mathcal{A} = \frac{1}{16\pi G}\int_M -2D\Sigma_a \wedge \delta A_a + (2F_a \wedge e_a + \frac{\lambda}{3}\,\epsilon_a{}^{bc}e_a \wedge e_b \wedge e_c\,) \wedge \delta e_0$$

$$- (2F_a \wedge e_0 \;+ 2\epsilon_a{}^{bc}e_b \wedge F_c + \lambda\epsilon_a{}^{bc}e_0 \wedge e_b \wedge e_c) \wedge \delta e_a \qquad (7.1.7)$$

The equations of motion are therefore

$$[D\Sigma]_a = 0 \qquad (7.1.8a)$$

$$2(F_a \wedge e_0 + \epsilon_a{}^{bc}e_b \wedge F_c) = -\lambda\epsilon_a{}^{bc}e_0 \wedge e_b \wedge e_c \qquad (7.1.8b)$$

and

$$2F_a \wedge e_a = -\frac{\lambda}{3}\epsilon_a{}^{bc}e_a \wedge e_b \wedge e_c \qquad (7.1.8d)$$

In the action, $\mathcal{A}$, there is a contribution only from the anti-self-dual part of the curvature because Σ projects out the anti-self-dual part of F. We may therefore expand F in terms of Σ by

$$F_a = S_{ab}\Sigma^b \qquad (7.1.9a)$$

With this, the equations of motion reduce to

$$[D\Sigma]_a = 0 \qquad (7.1.9b)$$

$$S_{ab} = S_{ba} \qquad (7.1.9c)$$

and

$$\mathrm{Tr}\; S = -\lambda \qquad (7.1.9d)$$

These are precisely the equations of motion derived in Chapter 4 by considering the constraints and the evolution of the conjugate variables through the Hamilton Equations with a Hamiltonian which is made up solely of the Ashtekar constraints.

7.2 Invariance of the theory and the BRST symmetry

The symmetries of the action and the associated BRST invariance will now be considered. It is easy to see that the action is invariant under SO(3) gauge transformations as well as four-dimensional diffeomorphisms. Working with $\mathcal{A}$, which is explicitly gauge-invariant, we can consider the effect of diffeomorphisms ϕ, of M into itself, generated by the vector field β. On the one-form variables A_a and e_A, the induced variations are Lie derivatives

$$\delta A_a = \mathcal{L}_\beta A_a$$
$$= (i_\xi d + d i_\xi) A_a \tag{7.2.1}$$

and

$$\delta e_A = \mathcal{L}_\beta e_A \tag{7.2.2}$$

with i_β denoting interior multiplication or contraction with the vector field β. In local coordinates, β can be written as $\beta^\mu \partial_\mu$. If we denote the Lagrangian four-form as L, then under diffeomorphisms

$$\delta L = \mathcal{L}_\beta L$$
$$= (i_\beta d + d i_\beta) L \tag{7.2.3}$$

and the invariance of the action is equivalent to the condition

$$
\begin{aligned}
&\int_M (\phi^* L - L) = 0 \\
\Leftrightarrow &\int_M \mathcal{L}_\beta L = \int_{\partial M} i_\beta L \\
&\qquad\qquad = 0
\end{aligned}
\tag{7.2.4}
$$

The action is therefore invariant if M is without boundary. Otherwise, suitable physical boundary conditions can be imposed. For instance, the field variables A and e and/or β can be made to vanish on the boundary, or β can be restricted to be tangential to the boundary. Since SO(3) gauge invariance is a symmetry of the theory, one can also consider diffeomorphisms for which the exterior derivative operator in the Lie derivative is replaced by the covariant derivative to make it compatible with the canonical analysis. In the BRST analysis, this is equivalent to a redefinition of the gauge ghost field.

In the BRST formalism, the classical gauge and diffeomorphism symmetries are mimicked by transformations with the parameters replaced by ghosts ρ_a and ξ. (Related BRST analyses of General Relativity with conventional rather than Ashtekar variables exist[BB]). Thus the BRST transformations of the variables are (henceforth δ shall mean δ_{BRST})

$$\delta A_a = -D\rho_a + \mathcal{L}_\xi A_a \tag{7.2.5}$$

$$\delta e_a = -\epsilon_a{}^{bc}\rho_b e_c + \mathcal{L}_\xi e_a \tag{7.2.6}$$

$$\delta e_0 = \mathcal{L}_\xi e_0 \tag{7.2.7}$$

For a general differential form χ

$$\mathcal{L}_\xi \chi \equiv (i_\xi d - d i_\xi)\chi \tag{7.2.8}$$

There is a sign difference in the second term (compared to Eqn. (7.2.3)) because, unlike a normal vector field, ξ carries a ghost number of 1. The above BRST tranformations of the vierbein implies that the antiself-dual two form, Σ_a transforms according to

$$\delta\Sigma_a = -\epsilon_a{}^{bc}\rho_b\Sigma_c + \mathcal{L}_\xi\Sigma_a \tag{7.2.9}$$

The standard procedure of splitting the gauge ghost ρ into $\rho = \eta - i_\xi A$ allows us to write the BRST transformations in a gauge-covariant manner

$$\delta A_a = -D\eta_a + i_\xi F_a \tag{7.2.10}$$

$$\delta\Sigma_a = -\epsilon_a{}^{bc}\eta_b\Sigma_c + i_\xi D\Sigma_a - Di_\xi\Sigma_a \tag{7.2.11}$$

The BRST transformations for the ghosts are

$$\delta\eta_a = -\tfrac{1}{2}\epsilon_a{}^{bc}\eta_b\eta_c + \frac{1}{2}\, i_\xi i_\xi F_a \tag{7.2.12}$$

and

$$\delta\xi = \tfrac{1}{2}\mathcal{L}_\xi\xi \qquad \text{i.e.} \qquad \delta\xi^\mu = \xi^\nu\partial_\nu\xi^\mu \tag{7.2.13}$$

In the BRST formalism, the tranformations are generated by the BRST charge Q i.e. for a variable ψ,

$$\delta\psi = \{\psi, Q\}_{P.B.} \tag{7.2.14}$$

The BRST charge is Grassmanian. Therefore $Q^2 = 0$ and the Jacobi identity for the BRST transformations implies that BRST transformations are nilpotent i.e. $\delta^2 = 0$. It can be verified that the BRST transformations above are indeed nilpotent. The variables (A, η, Σ, ξ), which are (1, 0, 2) forms and a vector-field respectively, are assigned ghost numbers (0, 1, 0, 1) and carry a grading equal to the form degree plus the ghost number. By this we mean that if $\chi_{1,2}$ are $p_{1,2}$-forms with ghost numbers $g_{1,2}$, then

$$\chi_1 \wedge \chi_2 = (-1)^p \chi_2 \wedge \chi_1 \quad ; \qquad p = (p_1 + g_1)(p_2 + g_2) \tag{7.2.15}$$

The geometrical nature of BRST transformations has been studied by various authors[TM]. The above transformation rules for A and η may also be obtained by considering the multiplet (A, η) as a connection $\widetilde{A}$ and $d+\delta$ as the total exterior derivative. $\widetilde{A}$, which carries a grading of 1, can be decomposed into its (1, 0) and (0, 1) components as (the ordered pair (p, q) denotes the differential form degree and ghost number)

$$\widetilde{A}_a = A_a + \eta_a \tag{7.2.16}$$

Its curvature is

$$\widetilde{F}_a = (d+\delta)\widetilde{A}_a + \frac{1}{2}\epsilon_a{}^{bc}\widetilde{A}_b \wedge \widetilde{A}_c \tag{7.2.17}$$

One can check that the transformation rules for A and η are equivalent to the statement

$$\widetilde{F}_a = \exp(i_\xi)F_a = (I + i_\xi + \frac{1}{2!}\, i_\xi i_\xi)F_a \tag{7.2.18}$$

Eqn. (7.2.18) reduces to the "soul-flatness" condition, $\tilde{F} = F$, in ordinary gauge theories when diffeomorphism invariance generated by ξ is absent. The concept of "horizontality" in curved space is the statement that $\widetilde{F}$ must be expanded in terms of F and its contractions with the ghost ξ as in (7.2.18). The BRST transformation of F as a consequence of (7.2.10) is

$$\delta F_a = -\epsilon_a{}^{bc}\eta_b F_c - (D i_\xi F)_a \tag{7.2.19}$$

It can be verified that the curvature $\widetilde{F}$ satisfies the consistency condition

$$\widetilde{D}\,\widetilde{F}_a = (\, d+\delta)\widetilde{F}_a + \epsilon_a{}^{bc}\widetilde{A}_b \wedge \widetilde{F}_c = 0 \tag{7.2.20}$$

which is the Bianchi Identity.

7.3 BRST invariance and the Descent Equations

The BRST transformations of the variables allow us to deduce certain relations between quantities in the phase space. These relations take the form of descent sequences which will be shown to be crucial to the identification of observables of the theory. We shall now derive the descent sequences.

First, note that $\widetilde{F}$ satisfies the Bianchi identity. As a consequence, $\mathrm{Tr}(\widetilde{F}^n)$ obeys (Tr denotes the trace over the gauge group indices)

$$\widetilde{D}\,(\mathrm{Tr}(\widetilde{F}^n)) = 0 \tag{7.3.1}$$

Since the gauge group is SO(3), it suffices to consider $n = 2$. By expanding $\widetilde{F}_a \wedge \widetilde{F}_a$ in terms of the ghost number, i.e. by writing

$$\widetilde{F}_a \wedge \widetilde{F}_a = \sum_{g=0}^{4} W^g_{p=4-g} \tag{7.3.2}$$

the resulting BRST cohomology descent equations from (7.3.1), i.e.

$$(d+\delta)(\widetilde{F}_a \wedge \widetilde{F}_a) = 0 \tag{7.3.3}$$

are

$$dW^0_4 = 0 \tag{7.3.4a}$$

$$\delta W^g_{4-g} = -dW^{g+1}_{3-g} \qquad \text{for } g = 1,2,3 \tag{7.3.4b}$$

$$\delta W^4_0 = 0 \tag{7.3.4c}$$

with

$$W^0_4 = F_a \wedge F_a \qquad W^1_3 = i_\xi(F_a \wedge F_a)$$

$$W^2_2 = \frac{1}{2!} i_\xi i_\xi (F_a \wedge F_a) \qquad W^3_1 = \frac{1}{3!} i_\xi i_\xi i_\xi (F_a \wedge F_a)$$

$$W^4_0 = \frac{1}{4!} i_\xi i_\xi i_\xi i_\xi (F_a \wedge F_a)$$

Equations (4.9) are called the BRST descent equations. We shall see that they play a crucial role in the construction of BRST-invariant observables of the theory. There is thus an off-shell descent sequence involving the curvature of the Ashtekar connection.

We now investigate whether it is possible to construct other descents. In view of the fact that on-shell, i.e. when the equations of motion are satisfied, the covariant curl of Σ is zero and Σ transforms in the same way as F, one therefore expects that a BRST cohomology and descent analogous to (7.3.4) exists for Σ as well. This is indeed true if one also makes use of the other equations of motion (7.1.9a,c,d). This suggests that even off-shell, a descent involving Σ could be realized. However for off-shell computations we should keep all terms involving $D\Sigma$. We thus find that the BRST transformations of Σ and ξ imply

$$\delta(i_\xi\Sigma_a) = -\epsilon_a{}^{bc}\eta_b\, i_\xi\Sigma_c + \frac{1}{2!}\, i_\xi i_\xi D\Sigma_a - \frac{1}{2!}[D(\, i_\xi i_\xi \Sigma)]_a \tag{7.3.5a}$$

$$\delta(\frac{1}{2!}\, i_\xi i_\xi\Sigma_a) = -\epsilon_a{}^{bc}\eta_b\, \frac{1}{2!}\, i_\xi i_\xi\Sigma_c + \frac{1}{3!}\, i_\xi i_\xi i_\xi D\Sigma_a \tag{7.3.5b}$$

So, instead of the Bianchi Identity for $\widetilde{F}$, we have

$$\begin{aligned}[\widetilde{D}\widetilde{\Sigma}]_a = {} & (d+\delta)(\Sigma_a + i_\xi\Sigma_a + \frac{1}{2!}\, i_\xi i_\xi\Sigma_a) \\ & + \epsilon_a{}^{bc}(A_b+\eta_b)\wedge(\Sigma_c + i_\xi\Sigma_c + \frac{1}{2!}\, i_\xi i_\xi\Sigma_c) \\ = {} & D\Sigma_a + i_\xi D\Sigma_a + \frac{1}{2!} i_\xi i_\xi D\Sigma_a + \frac{1}{3!} i_\xi i_\xi i_\xi D\Sigma_a \end{aligned} \tag{7.3.6a}$$

i.e.

$$[\widetilde{D}\widetilde{\Sigma}]_a = \exp(i_\xi)[D\Sigma]_a \tag{7.3.6b}$$

Thus we see that the consistency condition is the requirement that $\widetilde{D}\widetilde{\Sigma}$ must be expanded in terms of $D\Sigma$ and its contractions with the ghost ξ. For $\widetilde{D}\widetilde{F}$, this expansion is trivial because of the Bianchi Identity for F. By considering $\widetilde{D}(\widetilde{\Sigma}_a \wedge \widetilde{\Sigma}_a)$ we have

$$(d+\delta)(\widetilde{\Sigma}_a \wedge \widetilde{\Sigma}_a) = 2[\widetilde{D}\widetilde{\Sigma}]_a \wedge \widetilde{\Sigma}_a \tag{7.3.7}$$

and expanding $\widetilde{\Sigma}_a \wedge \widetilde{\Sigma}_a$ in terms of ghost number

$$\widetilde{\Sigma}_a \wedge \widetilde{\Sigma}_a = \sum_{g=0}^{4} V^g_{4-g} \tag{7.3.8}$$

yields , for g=0 , the identity

$$d(\Sigma_a \wedge \Sigma_a) = 2[D\Sigma]_a \wedge \Sigma_a \tag{7.3.9a}$$

while for g=1,2,3,4 we have

$$\delta(\Sigma_a \wedge \Sigma_a) = -d[i_\xi(\Sigma_a \wedge \Sigma_a)] + i_\xi d(\Sigma_a \wedge \Sigma_a) \tag{7.3.9b}$$

$$\delta[i_\xi(\Sigma_a \wedge \Sigma_a)] = -d[\tfrac{1}{2!}i_\xi i_\xi(\Sigma_a \wedge \Sigma_a)] + \tfrac{1}{2!}i_\xi i_\xi d(\Sigma_a \wedge \Sigma_a) \tag{7.3.9c}$$

$$\delta[\tfrac{1}{2!}i_\xi i_\xi(\Sigma_a \wedge \Sigma_a)] = -d[\tfrac{1}{3!}i_\xi i_\xi i_\xi(\Sigma_a \wedge \Sigma_a)] + \tfrac{1}{3!}i_\xi i_\xi i_\xi d(\Sigma_a \wedge \Sigma_a) \tag{7.3.9d}$$

$$\delta[\tfrac{1}{3!}i_\xi i_\xi i_\xi(\Sigma_a \wedge \Sigma_a)] = -d[\tfrac{1}{4!}i_\xi i_\xi i_\xi i_\xi(\Sigma_a \wedge \Sigma_a)] + \tfrac{1}{4!}i_\xi i_\xi i_\xi i_\xi d(\Sigma_a \wedge \Sigma_a) \tag{7.3.9e}$$

$$\delta[\tfrac{1}{4!}i_\xi i_\xi i_\xi i_\xi(\Sigma_a \wedge \Sigma_a)] = \tfrac{1}{5!}i_\xi i_\xi i_\xi i_\xi i_\xi d(\Sigma_a \wedge \Sigma_a) \tag{7.3.9f}$$

It is remarkable that although Σ, unlike F, is not covariantly constant off-shell, the non-exact terms on the R.H.S. of Eqns. (7.3.9) actually vanish because they are all contractions, with ξ, of $d(\Sigma_a \wedge \Sigma_a)$ which is zero in *four dimensions.* Thus in four dimensions, we do have another BRST cohomology sequence with descent equations

$$dV^0_4 = 0 \tag{7.3.10a}$$

$$\delta V^g_{4-g} = -dV^{g+1}_{3-g} \qquad \text{for } g = 1,2,3 \tag{7.3.10b}$$

$$\delta V_0^4 = 0 \tag{7.3.10c}$$

where

$$V_4^0 = \Sigma_a \wedge \Sigma_a \qquad V_3^1 = i_\xi(\Sigma_a \wedge \Sigma_a)$$

$$V_2^2 = \frac{1}{2!} i_\xi i_\xi(\Sigma_a \wedge \Sigma_a) \qquad V_1^3 = \frac{1}{3!} i_\xi i_\xi i_\xi(\Sigma_a \wedge \Sigma_a)$$

$$V_0^4 = \frac{1}{4!} i_\xi i_\xi i_\xi i_\xi(\Sigma_a \wedge \Sigma_a)$$

The existence of the two descents above naturally leads us to ask whether a further descent can be constructed from $\widetilde{\Sigma}_a \wedge \widetilde{F}_a$ by considering $\widetilde{D}(\widetilde{\Sigma}_a \wedge \widetilde{F}_a)$. It is straightforward to verify that from

$$(d + \delta)(\widetilde{\Sigma}_a \wedge \widetilde{F}_a) = [\widetilde{D}\widetilde{\Sigma}]_a \wedge \widetilde{F}_a \tag{7.3.11}$$

we arrive at

$$\delta(\Sigma_a \wedge F_a) = - d[i_\xi(\Sigma_a \wedge F_a)] + i_\xi d(\Sigma_a \wedge F_a) \tag{7.3.12a}$$

$$\delta[i_\xi(\Sigma_a \wedge F_a)] = - d[\frac{1}{2!} i_\xi i_\xi(\Sigma_a \wedge F_a)] + \frac{1}{2!} i_\xi i_\xi d(\Sigma_a \wedge F_a) \tag{7.3.12b}$$

$$\delta[\frac{1}{2!} i_\xi i_\xi(\Sigma_a \wedge F_a)] = - d[\frac{1}{3!} i_\xi i_\xi i_\xi(\Sigma_a \wedge F_a)] + \frac{1}{3!} i_\xi i_\xi i_\xi d(\Sigma_a \wedge F_a) \tag{7.3.12c}$$

$$\delta[\frac{1}{3!} i_\xi i_\xi i_\xi(\Sigma_a \wedge F_a)] = - d[\frac{1}{4!} i_\xi i_\xi i_\xi i_\xi(\Sigma_a \wedge F_a)] + \frac{1}{4!} i_\xi i_\xi i_\xi i_\xi d(\Sigma_a \wedge F_a) \tag{7.3.12d}$$

$$\delta[\frac{1}{4!} i_\xi i_\xi i_\xi i_\xi(\Sigma_a \wedge F_a)] = \frac{1}{5!} i_\xi i_\xi i_\xi i_\xi i_\xi d(\Sigma_a \wedge F_a) \tag{7.3.12e}$$

The non-exact terms on the R.H.S. are again contractions, with the ghost ξ, of $d(\Sigma_a \wedge F_a)$ which also vanishes in four dimensions. There is thus a third BRST cohomology descent sequence. While the two previous descents involve either F or Σ but not both together, this third set of descent equations involves both Σ and F i.e. the conjugate variables $\tilde{\sigma}$ and A. The corresponding descent equations can be written as

$$dU_4^0 = 0 \tag{7.3.13a}$$

$$\delta U^{g}_{4\text{-}g} = -dU^{g+1}_{3\text{-}g} \qquad \text{for } g= 1,2,3 \tag{7.3.13b}$$

$$\delta U^{4}_{0} = 0 \tag{7.3.13c}$$

with

$$U^{0}_{4} = \Sigma_{a} \wedge F_{a} \qquad U^{1}_{3} = i_{\xi}(\Sigma_{a} \wedge F_{a})$$

$$U^{2}_{2} = \frac{1}{2!}\, i_{\xi} i_{\xi}(\Sigma_{a} \wedge F_{a}) \qquad U^{3}_{1} = \frac{1}{3!} i_{\xi} i_{\xi} i_{\xi}(\Sigma_{a} \wedge F_{a})$$

$$U^{4}_{0} = \frac{1}{4!} i_{\xi} i_{\xi} i_{\xi} i_{\xi}(\Sigma_{a} \wedge F_{a})$$

7.4 BRST Descents and Invariants

In this section, we shall exploit the descent equations derived earlier to identify the invariants of the theory. These invariants are to be used as the basic building blocks in the construction of observables of quantum gravity in four dimensions. In doing so, a clear and systematic characterization of the observables relevant to four-dimensional classical and quantum gravity emerges. The question of observables in General Relativity is a complicated one. Even to the present day, there is no systematic characterization of the observables of quantum gravity. Although one suspects that differential invariants of four-manifolds ought to be observables of the theory, it is not clear how these observables can be constructed out of the basic fields of the theory.

The existence of the BRST descents allows us to construct observables of the theory which are manifestly BRST-invariant by integrating over the homology cycles. Such a procedure was used by Witten to construct the Donaldson invariants through a topological quantum field theory[WT1].

Consider elements of the homology groups, $\gamma_p \in H_p(M)$; $p = 0,\ldots,4$. Picking an element Y_p^{4-p} of the W,V, or U descent, we see that

$$\delta \int_{\gamma_p} Y_p^{4-p} = - \int_{\gamma_p} d\, Y_{p-1}^{5-p}$$

$$= - \int_{\partial\gamma_p} Y_{p-1}^{5-p}$$

$$= 0 \tag{7.4.1}$$

Moreover, the BRST cohomology class of $Y_p^{4-p}(\gamma_p) \equiv \int_{\gamma_p} Y_p^{4-p}$ depends only on the homology class of γ_p because the descent equations guarantee that for γ_p and $\zeta_p = \gamma_p + \partial\omega_{p+1} \in H_p(M)$; $p = 0,\ldots,3$

$$Y_p^{4-p}(\zeta_p) = \int_{\gamma_p+\partial\omega_{p+1}} Y_p^{4-p}$$

$$= Y_p^{4-p}(\gamma_p) + \int_{\omega_{p+1}} dY_p^{4-p}$$

$$= Y_p^{4-p}(\gamma_p) - \delta \int_{\omega_{p+1}} Y_{p+1}^{3-p} \tag{7.4.2}$$

We also have

$$\delta \int_M S^{ab} F_a \wedge F_b = - \int_{\partial M} i_\xi(S^{ab} F_a \wedge F_b) \tag{7.4.3}$$

Thus, provided the boundary terms vanishes (which is automatic if M has no boundary), a further global invariant, $\int_M S^{ab} F_a \wedge F_b$, is present . It is possible to construct a descent in which $S^{ab} F_a \wedge F_b$ is the zero ghost number four-form. This can be achieved by using

$$\tilde{D}(S^{ab} \tilde{F}_a \wedge \tilde{F}_b) = (\tilde{D} S^{ab}) \wedge \tilde{F}_a \wedge \tilde{F}_b$$

$$= [(DS)^{\,ab} + i_\xi (DS)^{ab}] \wedge \widetilde{F}_a \wedge \widetilde{F}_b \qquad (7.4.4)$$

with

$$S_{ab} = -\frac{1}{4} * (F_a \wedge \Sigma_b + \Sigma_a \wedge F_b)$$

The descent is

$$d(\, S^{ab} F_a \wedge F_b \,) = 0$$

$$\delta(\, S^{ab} F_a \wedge F_b \,) = - d[i_\xi(\, S^{ab} F_a \wedge F_b \,)] + i_\xi d(\, S^{ab} F_a \wedge F_b \,) \qquad (7.4.5a)$$

$$\delta[i_\xi (S^{ab} F_a \wedge F_b)] = - d[\frac{1}{2!} i_\xi i_\xi(\, S^{ab} F_a \wedge F_b \,)] + \frac{1}{2!} i_\xi i_\xi d(\, S^{ab} F_a \wedge F_b \,) \qquad (7.4.5b)$$

$$\delta[\frac{1}{2!} i_\xi i_\xi (S^{ab} F_a \wedge F_b)] = - d[\frac{1}{3!} i_\xi i_\xi i_\xi(\, S^{ab} F_a \wedge F_b \,)] + \frac{1}{3!} i_\xi i_\xi i_\xi d(\, S^{ab} F_a \wedge F_b \,) \qquad (7.4.5c)$$

$$\delta[\frac{1}{3!} i_\xi i_\xi i_\xi (S^{ab} F_a \wedge F_b)] = - d[\frac{1}{4!} i_\xi i_\xi i_\xi i_\xi(\, S^{ab} F_a \wedge F_b \,)] + \frac{1}{4!} i_\xi i_\xi i_\xi i_\xi d(\, S^{ab} F_a \wedge F_b \,) \qquad (7.4.5d)$$

$$\delta[\frac{1}{4!} i_\xi i_\xi i_\xi i_\xi (S^{ab} F_a \wedge F_b)] = \frac{1}{5!} i_\xi i_\xi i_\xi i_\xi i_\xi d(\, S^{ab} F_a \wedge F_b \,) \qquad (7.4.5e)$$

The non-exact terms on the R.H.S. again vanish in four dimensions. Unlike the previous descents, this set of descent equations explicitly involves the duality operator $*$ and hence the inverse of the metric $g_{\mu\nu}$, which is defined through Eqn. (7.1.3). Since degenerate metrics cannot be ruled out in quantum fluctuations and the descent involves complicated products of non-commuting operators, it remains to be seen whether the descent survives regularization. Even in the classical context, it is would be of interest to investigate the interplay between degenerate metrics and the invariants of the descent.

7.5 Further invariants

In TQFTs, the form with zero ghost number in the descent equations, called the top-form, can be regarded as the action before gauge fixing. The action, (7.1.1), is a combination of the two top-forms involving V_4^0 and U_4^0 while W_4^0, can be added on to the action density and will give a term analogous to the θ-term in QCD. Using the same techniques in the construction of the V, U and W-descents, further descents and invariants can be obtained by starting from other top-forms. The top-forms are gauge-invariant four-forms of zero ghost number. Examples involving the torsion two-forms are

$$T_a \wedge T_a\,,\ (*T_a) \wedge T_a\,,\ T_a \wedge \Sigma_a\,,\ (*T_a) \wedge \Sigma_a\,,$$

$$T_a \wedge F_a\,,\ (*T_a) \wedge F_a\,,\ T_0 \wedge T_0\,,\ (*T_0) \wedge T_0\,,$$

$$S^{ab} T_a \wedge T_b\,,\ S^{ab}(*T_a) \wedge T_b\,,\ S^{ab} T_a \wedge F_b\,,$$

$$S^{ab}(*T_a) \wedge F_b\,,\ S^{ab} T_a \wedge \Sigma_b\,,\ S^{ab}(*T_a) \wedge \Sigma_b$$

where the torsion two-forms are defined as

$$T_a = (De)_a \tag{7.5.1}$$

and

$$T_0 = de_0 \tag{7.5.2}$$

Note that unlike the torsionless spin connection, the Ashtekar connection (which, at the classical level, is the anti-self-dual part of the spin connection) is not necessarily torsionless.

7.6 BRST-invariants and observables

It is clear that if the BRST-invariants obtained through the descents are used as building blocks to construct operators which in general, can be expanded as products of these invariants, then the operators will be BRST-invariant and qualify as observables of the quantum theory. The invariants are non-local in the sense that they are all integrals over homology cycles. An apparent paradoxical situation occurs for an invariant which is at the bottom of the descent, i.e. an operator with ghost number equal to four. It appears that this operator when integrated over a zero homology cycle i.e. a point in space-time, is a BRST-invariant. This seems to contradict the intuition that a diffeomorphism-invariant theory cannot have local observables, i.e. observables that depend on space-time points. (Even a scalar field is not diffeomorphism invariant.) This apparent paradox is resolved by noting that the descent equations guarantee that the same invariant at different space-time points really belong to the same BRST-cohomology class and therefore its expectation values at different space-time points are the same. For example, the descent equations guarantee that

$$dY^4_0 = -\delta Y^3_1 \tag{7.6.1}$$

so when integrated along a path connecting the points x and y in M,

$$Y^4_0(x) = Y^4_0(y) - \delta \int_y^x Y^3_1 \tag{7.6.2}$$

for all space-time points x and y in the same connected component. It appears that the BRST formalism yields what is expected of the theory. It is however remarkable that in the quantum theory, it is precisely the BRST-cohomology classes of the observables that are non-local and there exist relations among invariants which belong to the same descent sequence.

7.7 Degeneracy of the descents and phases in the theory

In the earlier discussions on the intial-data and non-perturbative canonical quantization of the theory, we presented some arguments of possible phases in the theory. This question will now be analyzed in the context of the BRST-invariant observables.

In the Type O sector corresponding to conformally self-dual Einstein manifolds

$$F_a = -\frac{\lambda}{3}\Sigma_a \tag{7.7.1}$$

There are no local degrees of freedom and the reduced action is

$$\mathcal{A}_{TQFT} = -\frac{3}{16\pi\lambda G}\int_M F_a \wedge F_a \tag{7.7.2}$$

The resulting constraints are Gauss' Law and the Ashtekar-Renteln ansatz of Eqn. (6.1.18). This set of constraints however, is reducible since the constraint (6.1.18) generates deformations of the gauge potentials, and so if it holds, so will Gauss' Law. The BRST analysis of this TQFT action has been performed (see [BRT] and references quoted therein for details) and the expectation values of the BRST invariants from the descent equations were successfully identified with the Donaldson maps by Witten [WT1][DO]. If we let δ_{TQFT} be the BRST transformation, then the action is invariant under

$$\delta_{TQFT}A_a = -D\eta_a + \psi_a \tag{7.7.3a}$$

$$\delta_{TQFT}\eta_a = -\frac{1}{2}\epsilon_a{}^{bc}\eta_b\eta_c + \phi_a \tag{7.7.3b}$$

$$\delta_{TQFT}\psi_a = -\frac{1}{2}\epsilon_a{}^{bc}\eta_b\psi_c - D\phi_a \tag{7.7.3c}$$

$$\delta_{TQFT} = -\epsilon_a{}^{bc}\eta_b\phi_c \tag{7.7.3d}$$

Comparing with Eqns. (7.2.10) and (7.2.12), the restricted Type O phase of the theory can be reduced to the TQFT of (7.7.2) with Donaldson-Witten invariants, if we identify ψ and ϕ to be

$$\psi_a = i_\xi F_a \quad \text{and} \quad \phi_a = \tfrac{1}{2} i_\xi i_\xi F_a \tag{7.7.4}$$

In this sector the role of Σ has been eliminated in terms of F. From the vantage point of the BRST invariants and observables, we see that in this phase the elements of the W, V and U-descents are *degenerate* because of (7.7.1). When we break out of this sector the descents become independent and the observables are characterized by many more invariants. Note that even within the Type O sector when λ hits the critical value of zero, F vanishes and Σ is no longer correlated to F. So for this critical value of λ, the W and U-descents are trivial but the V-descent is not. Since exact diffeomorphism symmetry is unbroken all the observables are non-local and locked up in the descents.

7.8 BRST invariance of the gauge-fixed action

It is not sufficient to demonstrate only BRST invariance of the classical action $\mathcal{A}$. It is well known in quantum field theory that the effective action can be different from the classical action. There can be contributions due to the necessity of gauge-fixing in the functional path integral. For instance in pure gauge theories, the quantum effective action consists of the classical action, gauge-fixing terms and a contribution from the Faddeev-Popov determinant of the functional integral which can be written as an integral the over Grassmannian Faddeev-Popov ghosts and antighosts[FP]. After gauge-fixing, the effective action is no longer gauge-invariant but it is nevertheless BRST invariant and it is precisely this BRST symmetry of the quantum theory that gives rise to the Slavnov-Taylor identities[BRS].

There is a known method of systematically adding gauge-fixing terms to the classical action in a BRST-invariant way. This can be achieved by adding to the classical action a gauge-fixing Lagrangian of the form $L_{g.f.} = \delta\chi$. The BRST invariance of the gauge-fixed action is then explicit. This is because for the classical action the BRST transformations mimicked the original symmetry of the classical variables while the gauge-fixing term will be invariant because of the nilpotency of BRST transformations. The gauge-fixing Lagrangian however has to involve antighosts because the Lagrangian has zero ghost number but δ always increases the ghost number the quantity it operates on by one unit. Typically $L_{g.f.}$ is of the form

$$\begin{aligned} L_{g.f.} &= \delta[\overline{g}\, f(\text{fields})] \\ &= b\, f(\text{fields}) - \overline{g}\,\delta[f(\text{fields})] \end{aligned} \tag{7.8.1}$$

The antighost(s), $\overline{g}$, and auxiliary field(s), b, tranform according to

$$\delta\overline{g} = b \tag{7.8.2}$$

and

$$\delta b = 0 \tag{7.8.3}$$

(Note the nilpotency of the transformations). f(fields) $= 0$ is the gauge-fixing condition. To take an example, consider pure gauge theories. In the functional integral, the integration over the auxiliary field b will produce a delta function enforcing the gauge-fixing condition (which can be chosen as the Lorentz gauge) while the integration over the ghosts and antighosts produces the Faddeev-Popov determinant through the second term of (7.8.1). We shall discuss the relevant gauge-fixing conditions for the gauge and diffeomorphism symmetries that we have to contend with and the complications and subtleties that are associated with the gauge-fixing of the theory, but for the moment we stress that the all-important BRST invariance of the quantum theory will be ensured if we adopt the above procedure in gauge-fixing. Actually, to prove strict BRST invariance of the quantum theory in the functional path integral approach, there must also be considerations regarding the invariance of the functional measure. This will be considered at the end of the chapter when we prove formal invariance of a suitable measure for the quantum theory with Ashtekar variables.

7.9 Gribov ambiguities and observables

The expectation value of an observable can be computed through

$$<\hat{O}> = \int D(\text{all fields})\hat{O}\ \exp i(\mathcal{A} + \textstyle\int L_{g.f.}) \tag{7.9.1}$$

The integral is over all fields including ghosts anti-ghosts and auxiliary fields. The gauge-fixing Lagrangian will be of the form of (7.8.1) to preserve BRST-invariance. However, it must be noted that there can be Gribov ambiguities[GR] due to zero modes of the Faddeev-Popov determinant. To be specific, consider as gauge-fixing condition for diffeomorphisms, the de Donder gauge

$$\partial_\mu(\sqrt{g_0}\, g^{\mu\nu}) = 0 \tag{7.9.2}$$

where $(g_0)_{\mu\nu}$ is a background metric. For the gauge degrees of freedom, we can consider the covariant Lorentz-gauge condition

$$(g_0)^{\mu\nu} D^o{}_\mu A_{\nu a} = 0 \tag{7.9.3}$$

where D^o denotes the covariant derivative with respect to some background gauge potential A^o. The gauge-fixing Lagrangian then takes the form

$$L_{g.f.} = \delta \int \Big\{ \overline{\xi}_\nu \frac{1}{\sqrt{g_0}}\, \partial_\mu(\sqrt{g_0}\, g^{\mu\nu}) + \overline{\eta}^a (g_0)^{\mu\nu} D^o{}_\mu A_{\nu a} \Big\} (*_0 1) \tag{7.9.4}$$

This gives

$$L_{g.f.} = \int b_\nu \partial_\mu(\sqrt{g_0} g^{\mu\nu}) - \overline{\xi}_\nu \delta\{\partial_\mu(\sqrt{g_0} g^{\mu\nu})\}$$
$$+ b^a \sqrt{g_0}\, (g_0)^{\mu\nu} D^o{}_\mu A_{\nu a} - \overline{\eta}^a \delta\{\sqrt{g_0}(g_0)^{\mu\nu} D^o{}_\mu A_{\nu a}\}\ d^4x \tag{7.9.5}$$

Upon integrating over the auxiliary fields b, the first and third terms will enforce the seven gauge-fixing conditions. The second and fourth terms can be written as $\overline{\xi}_\mu M^\mu{}_\nu \xi^\nu$ and $\overline{\eta}^a N^b{}_a \eta_b$ respectively. Integrating over the ghost and antighost pairs will yield the Faddeev-Popov determinants of M and N. However, there can be zero modes of the matrices corresponding to

$$\delta\{\partial_\mu(\sqrt{g_0} g^{\mu\nu})\} = 0 \quad , \quad \delta\{\sqrt{g_0}(g_0)^{\mu\nu} D^O{}_\mu A_{\nu a}\} = 0 \tag{7.9.6}$$

It should also be noted that (7.9.6) are precisely the classical equations of motion of the total gauge-fixed action when we vary with respect to the antighosts. Supposing that we are looking at a fixed vierbien and gauge field, then (7.9.6) will be the equations of motion for the ghosts. Obviously, a sufficient condition for (7.9.6) to occur is when the metric and the gauge fields have BRST fixed points i.e. they are left invariant under some BRST transformations. By definition, transformations generated by Killing vectors leave the metric invariant. Thus, if the diffeomorphism ghost ξ is of the form of a Killing vector multiplied by a space-time independent Grassmanian ϵ, then there exists non-trivial zero modes of the matrix M. In ordinary gauge theories, Gribov ambiguities can be circumvented if we restrict ourselves to the perturbative regime (see for instance [IZ]). For gravity, it may not be possible to avoid the zero modes nor is it desirable to do so, because the existence of BRST fixed points reflect the underlying symmetries of four-manifolds. Indeed in the TQFT with Donaldson-Witten differential invariants, the existence of zero modes of the gauge-fixing condition is necessary for the expectation values of BRST invariants with non-zero ghost numbers to be non-trivial (see [WT1][WT3][BRT]). The same line of reasoning can be applied to gravity but we expect more differential invariants because of the occurence of many more descents. In TQFT with the classical action of (7.7.2), only the W-descent is possible. Furthermore, the invariants can assume non-integer values (we shall provide some explicit examples of this). Another difference between TQFT and gravity is that for the former, computations in the classical limit give the same results as the full quantum theory because the expectation values of the invariants can be shown to be independent of the coupling constant and therefore they can be evaluated at the classical limit[WT1].

7.10 Explicit examples

We now give some explicit examples of what these invariants can be. We have already discussed in Chapter 4 how $F_a \wedge F_a$ is related to the Euler number and signature which are topological invariants. In the classical solutions, these come from the integrals of $\mathrm{Tr}(S^2)$ for two opposite orientations. It is natural to question what the integral of $\mathrm{Tr}(S^3)$ gives. Consider the explicit example of the Schwarzschild solution discussed in example (c) of Chapter 5. The proper qauntum BRST-invariant to consider is $\int_M S^{ab} F_a \wedge F_b$ with

$$S_{ab} = -\frac{1}{4} *(F_a \wedge \Sigma_b + \Sigma_a \wedge F_b)$$

On-shell, the integral is equivalent to the integral of $-2\mathrm{Tr}(S^3)$. From the descent equations we have

$$\delta \int_M S^{ab} F_a \wedge F_b = - \int_{\partial M} i_\xi (S^{ab} F_a \wedge F_b) \tag{7.10.1}$$

For the classical Schwarzschild solution,

$$S^{ab} F_a \wedge F_b = \frac{12\ m^3}{r^7} d\tau \wedge dr \wedge \sin\theta d\theta \wedge d\phi \tag{7.10.2}$$

The Schwarzschild solution however has boundary at r=2m and r=∞. We assume the ghosts are the Killing fields albeit with a Grassmannian character. No Killing vectors of the Schwarzschild solution has an r-component and so the boundary term on the R.H.S. of (7.10.1) vanishes and

$$\int_M S^{ab} F_a \wedge F_b = \int_{\tau=0}^{8\pi m} \int_{r=2m}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{12\ m^3}{r^7} d\tau \wedge dr \wedge \sin\theta d\theta \wedge d\phi$$

$$= \frac{\pi^2}{m^2} \tag{7.10.3}$$

is an additional invariant characterizing the Schwarzschild solution besides the Euler number and signature calculated earlier. This is hardly surprising. It is reasonable to expect the Schwarzschild solution is characterized by the mass in the solution but note that the Schwarzschild mass is non-topological in nature and therefore represents a differential, rather than topological , invariant of the four-manifold. Furthermore, the mass is not put in by hand as an invariant of the manifold but is obtained as a special case of the BRST-invariants constructed from the fundamental fields of the theory so we know its relation to other objects in the descent. For instance, the Schwarzschild mass need not be an invariant for quantum gravity because from Eqn. (7.10.1), we see that quantum fluctuations of the ghosts which cause them to deviate from the classical Killing vectors can contribute to the boundary term and the mass will no longer be BRST-invariant. This needs to be explored further for it can have implications for the process of quantum decoherence and Hawking radiation of black holes.

Another example is $CP_2\#\overline{CP}_2$ (the connected sum of CP_2 and $\overline{CP}_2$) with the Page metric. The manifold is without boundary and the Page metric can be written as[PA2]

$$ds^2 = \frac{l^2\text{-}R^2}{K}\, dR^2 + 4(l^2 - R^2)(\Theta_1{}^2 + \Theta_2{}^2) + \frac{16 l^2 K}{(l^2 - R^2)}\Theta_3{}^2 \tag{7.10.4}$$

with

$$K \equiv \frac{\lambda}{3}R^4 + (1\text{-}2l^2\lambda)R^2 + l^2(1 - l^2\lambda) \tag{7.10.5}$$

The three Euler angles, ϕ, θ and ψ, have the usual ranges but R ranges from $-R_0$ to R_0 with $R_0 = \nu l$. According to Page's analysis, to obtain an Einstein metric free of all singularities, l must take the astounding value of

$$l = \sqrt{\frac{3(1+\nu^2)}{4\lambda\nu(3+\nu^2)}}$$

with

$$\nu = -1 - [2 + a - a^{-1}] + [4 - a + a^{-1} + 8(a - a^{-1})^{-\frac{1}{2}}(a + a^{-1})^{-1}]^{\frac{1}{2}} \tag{7.10.6}$$

and

$$a \equiv (\sqrt{2} + 1)^{\frac{1}{3}}$$

With the above, $l \approx 0.96602504\lambda^{-\frac{1}{2}}$ and $R_0 = \nu l \approx 0.27213076\lambda^{-\frac{1}{2}}$. Obmitting the details, we find the Page metric is of Type D and transcribing it into our previous formalism, we obtain

$$P_1 = P'_1 = -8 \tag{7.10.7}$$

and thus $\chi = 4$ and $\tau = 0$. The volume term, which is BRST-invariant, is

$$\int_M \Sigma_a \wedge \Sigma_a \approx 150.862\lambda^{-2} \tag{7.10.8}$$

A further example of a BRST-invariant of the Page metric is

$$\int_M S^{ab}F_a \wedge F_b \approx 347.622\lambda \tag{7.10.9}$$

One can also obtain BRST-invariants with non-zero ghost numbers by assuming the ghosts are in the form of Killing vectors multiplied by a Grassmannian parmeter ϵ. For instance, with the ghost $\xi = \epsilon \frac{\partial}{\partial \psi}$ ($\frac{\partial}{\partial \psi}$ is a Killing vector of the Page metric), the BRST invariant obtained by integrating over the homology three-cycle γ_3 parametrized by $-R_0 \le R \le R_0$, $0 \le \theta < \pi$ and $0 \le \phi < 2\pi$ is

$$\int_{\gamma_3} i_\xi (S^{ab}F_a \wedge F_b) \approx \epsilon\, 27.663\lambda \tag{7.10.10}$$

7.11 BRST invariance of the measure

The BRST-invariance of the total gauge-fixed action was shown in section 7.8. To check the BRST-invariance of the quantum theory, it is necessary to show that the generating functional is invariant under BRST transformations . For that, it remains to show that the functional measure is also invariant. If the measure is not invariant i.e. the Jacobian of the transformation is not unity, anomalies can occur. For instance, Fujikawa[FU] showed that the anomaly of the axial current can be understood as the non-invariance of the path integral measure of the fermion fields. When there is more than one classical symmetry, it may turn out that the measure cannot be made invariant under all the symmetries. One or more of the classical symmetries may have to be given up at the quantum level . There is no good reason to suspect that the SO(3) gauge invariance and diffeomorphism invariance relevant to us cannot be made compatible at the quantum level. The closure of the quantum constraint algebra discussed in

Chapter 6 supports this. However, it is a non-trivial exercise to demonstrate the explicit BRST-invariance of the measure.

A BRST-invariant measure can be constructed. It is obtained by integrating over the field variables weighted with appropriate powers of the determinant of the vierbein. The diffeomorphism ghost measure needs careful treatment. We shall adapt the work of Fujikawa et al [FEA] to our context. The BRST-invariant measure takes the form

$$D[b_\mu]D[b^a]D[\overline{\xi}^\mu]D[\overline{\eta}^a]D[\eta_a]D[e^{1/4}e_{\mu A}]D[e^{1/2}e_{\mu A}\xi^\mu]_{gh}D[e^{3/8}A_{\mu a}]_g \qquad (7.11.1)$$

The proof of the BRST invariance of the above measure will be carried out by proving the invariance of the separate measures.

Firstly, note that under BRST transformations

$$\delta b_{\mu,a} = 0 \qquad \text{i.e.} \quad b'_{\mu,a} = b_{\mu,a} \qquad (7.11.2)$$

so the Jacobian of the transformation, $\det\left[\frac{\partial b'}{\partial b}\right]$, is unity. For the anti-ghosts $\overline{\xi}$ and $\overline{\eta}$, we have

$$\delta\overline{\xi}_\mu = b_\mu \qquad \text{i.e.} \quad \overline{\xi}'_\mu = \xi_\mu + b_\mu \qquad (7.11.3a)$$

and

$$\delta\overline{\eta}^a = b^a \qquad \text{i.e.} \quad \overline{\eta}'^a = \overline{\eta}^a + b^a \qquad (7.11.3b)$$

For an infinitesimal transformation of a field ψ, the Jacobian can be evaluated by using

$$J = \det\left[\frac{\partial\psi'}{\partial\psi}\right] = 1 + \mathrm{tr}\left[\frac{\partial(\delta\psi)}{\partial\psi}\right] \qquad (7.11.4)$$

The trace of a matrix $M^\rho{}_\mu{}^A{}_B\,(x,y)$ is defined to be $\int \delta(x-y)\delta^B{}_A\delta^\mu{}_\rho\, M^\rho{}_\mu{}^A{}_B\,(x,y)$ dxdy. and we adopt the convention of the right derivative for the partial derivative with respect to Grassmanians. The Jacobians of the BRST transformations for the anti-ghost measures are therefore equal to one. Thus the auxiliary field and anti-ghost measures are BRST-invariant.

The gauge ghosts transform according to (7.2.12) i.e.

$$\eta'_a = \eta_a - \tfrac{1}{2}\epsilon_a{}^{bc}\eta_b\eta_c + \tfrac{1}{2}\xi^\mu\xi^\nu F_{\mu\nu a} \tag{7.11.5}$$

The Jacobian of this transformation is

$$\det\left[\frac{\partial\eta'}{\partial\eta}\right] = 1 + \mathrm{tr}(-\epsilon_a{}^{bc}\eta_c) \tag{7.11.6}$$

In the above, the second term on the R.H.S. vanishes because of the trace over the indices a and b of the antisymmetric Levi-Civita tensor. So the gauge ghost measure is BRST- invariant.

The invariance of the vierbien measure is less apparent. The BRST tranformations of the vierbein in Eqns. (7.2.6) and (7.2.7)

$$\begin{aligned}\hat{e}'_{\mu A} = \ & e^{1/4}e_{\mu A} + \xi^\nu\partial_\nu(e^{1/4}e_{\mu A}) + \tfrac{1}{4}\partial_\nu\xi^\nu(e^{1/4}e_{\mu A}) + \partial_\nu\xi^\nu(e^{1/4}e_{\mu A}) \\ & - \epsilon_{0A}{}^{bc}\rho_b e^{1/4}e_{\mu c}\end{aligned} \tag{7.11.7}$$

where $\hat{e}_{\mu A} \equiv e^{1/4}e_{\mu A}$.

This means that the Jacobian, $\det\left[\dfrac{\partial\,\hat{e}'(x)}{\partial\,\hat{e}(y)}\right]$, is

$$1 + \mathrm{tr}\{[(\xi^\nu\partial_\nu + \tfrac{1}{4}\partial_\nu\xi^\nu)\delta^\rho{}_\mu + \partial_\mu\xi^\rho\}\delta^B{}_A - \epsilon_{0A}{}^{bB}\rho_b]\delta(x\text{-}y)\} \tag{7.11.8}$$

The last term of (7.11.8) vanishes due to the anti-symmetry of the Levi-Civita tensor and (7.11.8) reduces to

$$1 + 16\mathrm{tr}\{(\xi^\nu\partial_\nu + \tfrac{1}{2}\partial_\nu\xi^\nu)\delta(x\text{-}y)\} \tag{7.11.9}$$

The Jacobian of the field transformations is an infinite dimensional expression which has to be regularized. Following Fujikawa et al , we can regularize the

Jacobian by using a mode cutoff . This goes as follows: A field $\phi(x)$ can expanded as

$$\phi(x) = \sum_i c_i \phi_i(x)$$

where $\{\phi_i(x)\}$ is a complete set of eigenfunctions of an operator. Then $\delta(x\text{-}y) \rightarrow \sum_i \phi_i(x)\ \phi_i(y)$ and the regulated expression for $\mathrm{tr}\{(\xi^\nu\partial_\nu + \frac{1}{2}\partial_\nu\xi^\nu)\delta(x\text{-}y)\}$ is

$$\sum_i^N \int dx\ \phi_i(x)(\xi^\nu\partial_\nu + \tfrac{1}{2}\partial_\nu\xi^\nu)\phi_i(x) \tag{7.11.10}$$

with cutoff at large N. But (7.11.10) is equal to

$$\sum_i^N \int dx\ \tfrac{1}{2}\partial_\nu(\xi^\nu\phi_i^{\,2}) \tag{7.11.11}$$

and vanishes provided certain boundary conditions (e.g. ϕ_i vanishes at the boundary or the ghost ξ is tangential to the boundary) are imposed. With these assumptions, the veirbein measure is BRST-invariant since the Jacobian of the transformation is unity.

The diffeomorphism ghost measure, $D[e^{1/2}\xi^\mu]_{gh}$, presents yet other complications. To obtained a BRST-invariant measure , Fujikawa et al evaluated the Jacobian for the BRST transformation on the ghost measure as follows:

$$J_{gh} = \det\left[\int dz \left[\frac{\partial\,\hat{\xi}'}{\partial\xi^\alpha(z)}\right]_{e_{\mu A}} \left[\frac{\partial\xi^\alpha(z)}{\partial\,\hat{\xi}}\right]_{e_{\mu A}}\right] \tag{7.11.12}$$

where $\hat{\xi}_A \equiv e^{1/2}e_{\mu A}\xi^\mu$. The first factor

$$\left[\frac{\partial\hat{\xi}'_A(x)}{\partial\xi^\alpha(z)}\right]_{e_{\mu A}} = [e^{1/2}e_{\mu A} + e^{1/2}e_{\alpha A}\xi^\mu\partial_\mu + \tfrac{1}{2}\,e^{1/2}e_{\alpha A}(\partial_\mu\xi^\mu)$$

$$+\xi^{\mu}\partial_{\mu}(e^{1/2}e_{\mu A})]\delta(x\text{-}z) \tag{7.11.13}$$

while the second factor

$$\left[\frac{\partial\xi^{\alpha}(z)}{\partial\hat{\xi}_{B}(y)}\right]_{e_{\mu A}} = e^{-1/2}(e^{-1})^{\alpha B}\delta(z\text{-}y) \tag{7.11.14}$$

Integrating the product of the factors over z and taking the determinant, the Jacobian is [FEA]

$$J_{gh} = 1 + 4\,\mathrm{tr}\{(\xi^{\nu}\partial_{\nu} + \tfrac{1}{2}\partial_{\nu}\xi^{\nu})\delta(x\text{-}y)\} \tag{7.11.15}$$

We observe that the trace term is the same as that of (7.11.9) . Thus the regulated Jacobian is unity.

Finally, we discuss the gauge field measure, $D[e^{3/8}A_{\mu a}]_g$. Consider $\hat{A}_{\mu a} \equiv e^k A_{\mu a}$. Under a BRST transformation

$$\begin{aligned}(\hat{A})'_{\mu a} = e^k A_{\mu a} + \{\xi^{\nu}\partial_{\nu}(e^k A_{\mu a}) + k(\partial_{\nu}\xi^{\nu})\, e^k A_{\mu a}\} - e^k\xi^{\nu}\partial_{\mu}A_{\nu a} \\ - e^k(\partial_{\mu}\eta_a + \epsilon_a{}^{bc}A_{\mu b}\eta_c - \epsilon_a{}^{bc}\xi^{\nu}A_{\nu b}A_{\mu c})\end{aligned} \tag{7.11.16}$$

So the Jacobian of the transformation is

$$\det\left[\frac{\partial(\hat{A})'(x)}{\partial\hat{A}(y)}\right] = 1 + \mathrm{tr}\{[\xi^{\nu}\partial_{\nu}\delta^{\rho}{}_{\mu}\delta^{a}{}_{b} + k(\partial_{\nu}\xi^{\nu})\delta^{\rho}{}_{\mu}\delta^{a}{}_{b}]\delta(x\text{-}y) - \frac{\partial C_{\mu a}}{\partial\hat{A}_{\rho b}}\} \tag{7.11.17}$$

where $C_{\mu a} \equiv e^k\xi^{\nu}\partial_{\mu}A_{\nu a}$ and the last term in (7.11.17) is to be evaluated in the same manner as was the case for the diffeomorphism ghost measure, i.e.

$$\frac{\partial C_{\mu a}(x)}{\partial\hat{A}_{\rho b}(y)} = \int dz \left[\frac{\partial C_{\mu a}(x)}{\partial A_{\alpha c}(z)}\right]_{e,\xi}\left[\frac{\partial A_{\alpha c}(z)}{\partial\hat{A}_{\rho b}(y)}\right]_{e,\xi} \tag{7.11.18}$$

$$= \xi^{\rho}\partial_{\mu}\delta^{a}{}_{b}\delta(x\text{-}y)$$

The last term within brackets in (7.11.16) gives no contribution to (7.11.17) because of the trace over the anti-symmetric Levi-Civita tensor. The Jacobian (7.11.17) reduces to

$$1 + \mathrm{tr}\{[12\xi^{\nu}\partial_{\nu} + 12\,\mathrm{k}(\partial_{\nu}\xi^{\nu}) - 3\delta^{\mu}{}_{\rho}\xi^{\rho}\partial_{\mu}]\delta(\mathrm{x}\text{-}\mathrm{y})\}$$

$$= 1 + 9\,\mathrm{tr}\{[\xi^{\nu}\partial_{\nu} + \tfrac{4}{3}\,\mathrm{k}(\partial_{\nu}\xi^{\nu})]\delta(\mathrm{x}\text{-}\mathrm{y})\} \tag{7.11.19}$$

Provided $\mathrm{k}= \frac{3}{8}$, the trace term is of the form of (7.11.9) and is a total divergence. We may therefore conclude that for this value of k, the Jacobian under BRST is unity and the gauge field measure, $\mathrm{D}[\mathrm{e}^{3/8}\mathrm{A}_{\mu\mathrm{a}}]_{\mathrm{g}}$, is BRST-invariant.

The functional measure (7.11.1) is thus invariant under BRST transformations of the variables. Although the invariance of the measure is achieved in a rather formal sense, it nevertheless enhances the viability of the BRST-invariant functional integral formulation of quantum gravity in four dimensions with Ashtekar variables.

Chapter 8

Remarks

The results obtained in this work have been summarized in the overview given in Chapter 1. Here, some remarks will be made on directions for further research suggested by this work. They will show that what has been described so far is only a beginning on a complete description of quantum gravity in four dimensions.

At the classical level, as demonstrated in Chapters 4 and 5, General Relativity in four dimensions can be successfully described by the Ashtekar variables. The formalism developed in these chapters can be quite effective in yielding explicit solutions which are rather difficult to obtain through the metric variables. We have also discussed some Kähler and hyperkähler metrics in the context of the Ashtekar variables and more explicit solutions can be expected through similar constructions.

There is strong evidence of phases in the theory and we have discussed some of them. The rich and intricate relationships that can exist between the phases remain to be explored.

In this work, we have not considered the coupling of matter fields to four-dimensional gravity. The coupling of matter fields will change the explicit form of the constraints and hence the reduced phase space, but the set of constraints will remain if matter fields are coupled in a diffeomorphism and gauge invariant fashion. They reflect the local symmetries of the theory. At the quantum level however, the new constraints with matter fields (particularly chiral fermions) can have dramatic consequences. For instance, the closure of the quantum constraint algebra would have to be re-examined and there could well be information in the quantum constraint algebra pertaining to the cancellation of anomalies. For instance, the quantum constraint algebra closes only for special coupling constants and/or number of species of the matter fields. With the ADM variables, the quantum constraints are too unwieldy to provide unambiguous answers, With the traditional constraints, even the closure of the quantum constraints for pure gravity has not been demonstrated. The Ashtekar constraints however are much simpler, and results pertaining to the closure or non-closure of the quantum constraint algebra with matter fields will be of great interest.

As demonstrated, for instance in the explicit example of CP_2, diffeomorphism invariance does not imply invariance under orientation reversal which includes time-reversal and parity. The action displayed in Chapter 7 is also not invariant under orientation reversal. Whether or not the non-invariance of the theory under orientation reversal has anything to do with the observed parity and T-violation that we know of is a question that deserves further consideration.

The exact values of the BRST invariants obtained in Chapter 7 will doubtless change when matter fields are added, but it must be emphasized that in the derivation of these invariants only the BRST transformation properties of the variables are used. These invariants will therefore remain as observables even if matter fields are added provided diffeomorphism and gauge-invariance are not violated. In fact, the forms assumed by invariants do not depend the specific action chosen (their expectation values do) but only on the BRST symmetry of the theory.

In Chapter 7, some quantum observables of the theory were identified. It remains to be seen if the computation of their expectation values yields meaningful answers. In this regard, the lesson from TQFT is that the Gribov ambiguities are to be addressed by insertion of appropriate non-zero ghost number invariants. In TQFT with Donaldson-Witten invariants, some of the computations of expressions of the form of (7.1.9) correspond to the evaluations of the intersection numbers of the moduli space. For gravity, the physics behind observables with non-zero ghost numbers ought to be clarified. On a more concrete note, in section 7.10 it was pointed out that the Schwarzschild mass can change due to boundary contributions from quantum fluctuations of the ghosts. This line of attack (which starts from fundamental principles) on the quantum mechanical stability of the black hole ought to be pursued. It should be noted that recent investigations on black hole evaporation using string theory techniques have questioned the Hawking radiation process[WT5][NE].

Lastly, the quantization of gravity remains one of the outstanding problems of theoretical physics despite the efforts of many. This thesis did not, and cannot, address many of the formidable issues in quantum gravity. Some outstanding difficulties not discussed in this work include questions of measurement and collapse of the quantum states, quantum coherence and decoherence, transitions between quantum states, the emergence of physical time from quantum geometry, gravitational collapse and singularities.

APPENDIX

The the left and right duals are defined as

$$R_{\widetilde{A}BCD} \equiv \frac{1}{2}\epsilon_{ABEF}R^{EF}{}_{CD} \tag{A1}$$

$$R_{AB\widetilde{CD}} \equiv \frac{1}{2}\epsilon_{CDEF}R_{AB}{}^{EF} \tag{A2}$$

We have seen that

$$(D\Sigma)_a = 0$$

$$\Leftrightarrow \qquad A_a = \omega_{0a} - \frac{1}{2}\,\epsilon_a{}^{bc}\omega_{bc} \tag{A3}$$

So

$$F_a = dA_a + \frac{1}{2}\epsilon_a{}^{bc}A_b \wedge A_c$$

$$= R_{0a} - \frac{1}{2}\,\epsilon_a{}^{bc}R_{bc}$$

$$= \frac{1}{2}(R_{0aDE} - \frac{1}{2}\epsilon_a{}^{bc}R_{bcDE})e^D \wedge e^E$$

$$= (R_{0a0b} - R_{\widetilde{0a}0b})e^0 \wedge e^b + \frac{1}{2}(R_{0a\widetilde{0b}} - R_{\widetilde{0a}\widetilde{0b}})\epsilon^b{}_{cd}e^c \wedge e^d$$

$$= S_{ab}(-e^0 \wedge e^b + \frac{1}{2}\epsilon^b{}_{cd}e^c \wedge e^d) \tag{A4}$$

iff

$$S_{ab} = R_{\widetilde{0a}0b} - R_{0a0b} = R_{0a\widetilde{0b}} - R_{\widetilde{0a}\widetilde{0b}} \tag{A5}$$

The superhamiltonian constraint

$$S_{ab} = S_{ba}$$

$$\Leftrightarrow \qquad R_{\widetilde{0a}0b} = R_{0a\widetilde{0b}} \quad \text{and} \quad R_{0a0b} = R_{\widetilde{0a}\widetilde{0b}}$$

With this

$$R_{\widetilde{ij}\widetilde{kl}} = \epsilon^a{}_{ij}\epsilon^b{}_{kl}R_{0a0b}$$

$$= \epsilon^{a}{}_{ij}\epsilon^{b}{}_{kl}\left(\frac{1}{2}\epsilon_{a}{}^{mn}\epsilon_{b}{}^{pq}\, R_{mnpq}\right)$$

$$= R_{ijkl} \tag{A6}$$

and

$$R_{0a\tilde{ij}} = \epsilon^{b}{}_{ij} R_{0a0b}$$

$$= \epsilon^{b}{}_{ij}\left(\frac{1}{2}\epsilon_{b}{}^{kl} R_{\tilde{0a}kl}\right)$$

$$= R_{\tilde{0a}ij} \tag{A7}$$

Therefore

$$R_{\tilde{AB}\tilde{CD}} = R_{ABCD} \tag{A8}$$

Furthermore,

$$\mathrm{TrS} = R_{\tilde{0a}0a} - R_{0a0a}$$

$$= \frac{1}{2}\epsilon_{a}{}^{bc} R_{bc0a} - R_{0a0a}$$

$$= -R_{0a0a} \tag{A9}$$

because of the cyclic identity for the Riemann curvature tensor ($\epsilon_{A}{}^{BCD} R_{EBCD} = 0$).

The Ricci scalar

$$R = R^{AB}{}_{AB}$$

$$= 2R_{0a0a} + R_{abab} \tag{A10}$$

But

$$R_{abab} = \epsilon_{ab}{}^{d}\epsilon_{ab}{}^{e} R_{\tilde{0d}\tilde{0e}}$$

$$= \epsilon_{ab}{}^{d}\epsilon_{ab}{}^{e} R_{0d0e}$$

$$= 2R_{0a0a} \tag{A11}$$

So

$$R = 4R_{0a0a} \tag{A12}$$

Thus

$$\mathrm{TrS} = -\lambda \ \Leftrightarrow \mathrm{R} = 4\lambda \tag{A13}$$

Note that Einstein manifolds are solutions of

$$R_{\mu\nu} = \lambda g_{\mu\nu} \tag{A14}$$

and therefore for them

$$R = 4\lambda \tag{A.15}$$

References

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