Representation Theory, Borel Cross-Sections, and Minimal Measures

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(ABSTRACT)

Let E be an analytic metric space, let X be a separable metric space with a regular Borel probability measure μ and let $\Pi : E \to X$ be a continuous map with $\mu(X \setminus \Pi(E)) = 0$. Schwartz's lemma states that there exists a Borel cross-section for Π defined almost everywhere (μ). The equivalence classes of these Borel cross-sections are in one-to-one correspondence with the representations of the form $\Gamma : C_b(E) \to L^\infty(\mu)$ with $\Gamma(f \circ \Pi) = f$ for every $f \in C_b(X)$. The representations are also in one-to-one correspondence with equivalence classes of the minimal measures on E.

Now let E, X, and μ be as above and let $\Pi : E \to X$ be an onto Borel map. There exists a Borel cross-section for Π defined almost everywhere (μ). The equivalence classes of the Borel cross-sections for Π are in one-to-one correspondence with the representations of the form $\Gamma : \mathbf{B}(E) \to L^\infty(\mu)$ with $\Gamma(f \circ \Pi) = f$ for every f in $C_b(X)$, where $\mathbf{B}(E)$ is the C^*-algebra of the bounded Borel functions on E. The representations are also in one-to-one correspondence with equivalence classes of the minimal measures on E.

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Contents

Chapter 1

Introduction

Given a function $f : Y \to Z$, a **cross-section** for f is a map $g : f(Y) \to Y$ such that $f \circ g(z) = z$ for every $z \in f(Y)$. In other words, a cross-section is a single-valued inverse. Of course it is always possible to find a cross-section by using the Axiom of Choice, but such cross-sections are not very interesting or useful. The goal of this paper is to characterize Borel cross-sections, which exist under certain circumstances. Recall that a function $h : Y \to Z$ is a Borel map if $h^{-1}(B)$ is Borel in Y for every Borel set $B \subseteq Z$. An equivalent definition is that $h : Y \to Z$ is a Borel map if $h^{-1}(G)$ is Borel in Y for every open set $G \subseteq Z$.

A set in a topological space is called **analytic** if it is the continuous image of a complete separable metric space. A set in a topological space X is called **absolutely measurable** if it is measurable with respect to the completion of every finite Borel measure on X.

Lemma 1.1 *Let X be a topological space with a σ-finite regular positive Borel measure μ, and let $A \subseteq X$ be an analytic set. Then A is measurable with respect to the completion of μ.*

Proof. [7, Lemma 6, p. 32]. Schwartz requires that X be a separable metric space, but the proof does not use the metric or the separability. □

Every analytic set is the continuous image of $\mathbb{N}^\infty$, the countably infinite product of $\mathbb{N}$, the set of natural numbers, with the metric

$$\delta(\{n_i\},\{m_i\}) = \sum_{i=1}^{\infty} \frac{|n_i - m_i|}{2^i(1+|n_i - m_i|)}$$

[7, Lemma 5, pp. 31-32]. With this fact Schwartz proved the following lemma:

Lemma 1.2 *Given an analytic metric space E, a separable metric space X, a σ-finite regular Borel measure μ on X, and a continuous function $\Pi : E \to X$ such that $\mu(X \setminus \Pi(E)) = 0$, there exists a Borel set F with $\mu(X \setminus F) = 0$ and a Borel cross-section for Π from F to E.*

Proof. [7, Lemma 7, pp. 35–38]. □

In other words, there exists a Borel cross-section for Π defined almost everywhere (μ). The proof of Theorem 2.8 will give an alternate proof of Lemma 1.2. When the measure μ is finite, we can further characterize the Borel cross-sections for Π. We will consider two Borel cross-sections for Π to be equivalent if they differ only on a set of measure 0.

Result 1 *Let E be an analytic metric space, X a separable metric space, μ a finite regular Borel measure on X, and $\Pi : E \to X$ a continuous function with $\mu(X \setminus \Pi(E)) = 0$. The equivalence classes of the Borel cross-sections for Π are in one-to-one correspondence with representations (continuous algebra homomorphisms which preserve unity) of the form $\Gamma : C_b(E) \to L^\infty(\mu)$ such that $\Gamma(f \circ \Pi) = f$ for every $f \in C_b(X)$. These representations are also in one-to-one correspondence with the equivalence classes of the minimal measures on E. (A minimal measure ν on E*

has the property that $\nu(\Pi^{-1}(B)) = \mu(B)$ for every Borel set $B \subseteq X$ and is minimal with respect to a certain ordering on the set of measures with this property.)

A function $f : Y \to Z$ is **absolutely measurable** if $f^{-1}(B)$ is an absolutely measurable set in Y for every Borel set $B \subseteq Z$. A Borel map is absolutely measurable. Arveson proved the following theorem about absolutely measurable cross-sections:

Theorem 1.1 *If E is an analytic metric space, X a separable metric space, and $\Pi : E \to X$ is an onto Borel map, then Π has an absolutely measurable cross-section.*

Proof. [1, Theorem 3.4.3, pp. 77–78]. □

Arveson also gives an example of such a system that does not have a Borel cross-section [1, p. 77]. This gives an example of an absolutely measurable map which is not Borel.

In Chapter 3, I will prove a slightly different version of his theorem:

Result 2 *Given an analytic metric space E, a separable metric space X, an onto Borel map $\Pi : E \to X$, and a σ-finite regular Borel measure μ on X, there exists a Borel set $F \subseteq X$ such that $\mu(X \setminus F) = 0$ and a Borel cross-section for Π from F to E.*

Since an absolutely measurable map need not be a Borel map, $X \setminus F$ must be non-empty in the Arveson example. The result is proven by reducing the problem to the case where the map Π is continuous. Just as in the case where Π is continuous, let two Borel cross-sections for Π be equivalent if they are equal almost everywhere (μ). In Chapter 3, the following result will be established:

Result 3 *Let E be an analytic metric space, X a separable metric space, μ a finite regular Borel measure on X, and $\Pi : E \to X$ an onto Borel map. The equivalence classes of the Borel cross-sections for Π are in one-to-one correspondence with the representations of the form $\Gamma : \mathbf{B}(E) \to L^\infty(\mu)$ which have the property that $\Gamma(f \circ \Pi) = f$ for all f in $C_b(X)$, where $\mathbf{B}(E)$ is the C^*-algebra of the bounded Borel functions on E. These representations are also in one-to-one correspondence with the minimal measures on E.*

The techniques used to prove the above theorems are based on methods developed in [5] to examine $\mathbf{M}_{H^\infty(D)}$, the maximal ideal space for $H^\infty(D)$. Recall that $H^\infty(D)$ is the set of all bounded analytic functions on the complex open unit disc, and that the maximal ideal space for $H^\infty(D)$ is the set of all non-zero continuous linear multiplicative functionals on $H^\infty(D)$. For each element ϕ in $\mathbf{M}_{H^\infty(D)}$, there is a unique point α on the closed unit disc such that $\phi(z) = \alpha$, where z is the bounded analytic function $f(z) = z$. In the case of points in the open disc, the relationship is one-to-one, and the open disc is continuously embedded in $\mathbf{M}_{H^\infty(D)}$ [4, p. 160]. For α on the unit circle, let $M_\alpha = \{\phi \in \mathbf{M}_{H^\infty(D)} : \phi(\hat{z}) = \alpha\}$. M_α is called the **fiber** of $\mathbf{M}_{H^\infty(D)}$ over α. Every M_α with $|\alpha| = 1$ is homeomorphic to any other one [4, p. 164]. The map from $\mathbf{M}_{H^\infty(D)}$ to the unit circle is far from one-to-one; in fact there is a homeomorphism from the open unit disc into each M_α where $|\alpha| = 1$ [4, p. 166]. Let p be the projection map from $\mathbf{M}_{H^\infty(D)} \setminus D$ to ∂D, the unit circle. There is no continuous cross-section for p [4, p. 165]. Suppose $\Gamma : H^\infty(D) \to L^\infty(m)$ is a representation such that $\Gamma(z) = z$, where m is the normalized Lebesgue measure on ∂D. There exists a Borel cross-section $s : \partial D \to \mathbf{M}_{H^\infty(D)}$ for p such that $(\Gamma f)(z) = \hat{f}(s(z))$ almost everywhere (m), where $\hat{f}$ is the Gelfand transform of f [5, Theorem 42, p. 43]. The Banach algebra $H^\infty(D)$ is isometrically embedded in $C(\mathbf{M}_{H^\infty(D)})$ [2, Theorem 4.29 (Gelfand-Naimark), pp. 92–93], via the Gelfand transform $\hat{f}(\phi) = \phi(f)$ for $f \in H^\infty(D)$ and $\phi \in \mathbf{M}_{H^\infty(D)}$. The Borel cross-sections for p are in one-to-one correspondence with the unital representations and in one-to-one correspondence with the minimal measures (with the same definition as above) on $\mathbf{M}_{H^\infty(D)}$ [5, Theorem 58, pp. 57–58].

Now we will generalize the problem, replacing $\mathbf{M}_{H^\infty(D)}$ and ∂D with abstract compact sets. The following problem was posed in [5, Problem 119, p. 111], and an outline of the solution was developed.

Problem 1.1 *Let Y and Z be compact Hausdorff spaces with $p : Y \to Z$ a continuous onto map. Let μ be a regular Borel probability measure on Z. We want to characterize all the representations $\Gamma : C(Y) \to L^\infty(\mu)$ such that $\Gamma(h \circ p) = h$ for every h in $C(Z)$.*

For the remainder of this chapter Y, Z, p, and μ will be as defined above. Let $\mathbf{R}$ be the set of all positive Borel measures τ on Y such that $\tau(p^{-1}(B)) = \mu(B)$ for every Borel set B in X.

Lemma 1.3 *A measure τ on Y is in $\mathbf{R}$ if and only if $\int_Y h \circ p \, d\tau = \int_Z h \, d\mu$ for every $h \in C(Z)$.*

Proof. Suppose $\tau \in \mathbf{R}$. Let $f \in C(Z)$. Without loss of generality we may assume $f \geq 0$. Thus there exists an increasing sequence $\{\phi_n\}$ of simple μ-measurable functions which converge pointwise to f [6, p. 260]. Then by the Monotone Convergence Theorem and the definition of $\mathbf{R}$,

$$\begin{aligned} \lim_{n\to\infty} \int_Z \phi_n \, d\mu &= \int_Z f \, d\mu \\ &= \lim_{n\to\infty} \int_Y \phi_n \circ p \, d\tau = \int_Y f \circ p \, d\tau. \end{aligned}$$

Now suppose $\int_Y h \circ p\, d\tau = \int_Z h\, d\mu$ for every $h \in C(Z)$. Let $B \subseteq X$ be Borel. There exists a monotone sequence $\{f_n\}$ in $C(Z)$ such that $\lim_{n\to\infty} f_n = \chi_B$ pointwise. Then by the Monotone Convergence Theorem we have

$$\mu(B) = \lim_{n\to\infty} \int_Z f_n\, d\mu = \lim_{n\to\infty} \int_Y f_n \circ p\, d\tau = \int_Y \chi_B \circ p\, d\tau = \tau(p^{-1}(B))$$

and so $\tau \in \mathbf{R}$. □

Lemma 1.4 **R** *is non-empty, convex, and weak-star compact in* $C(Y)^*$.

Proof. Define a positive linear functional τ on the subalgebra $\{h \circ p : h \in C(Z)\}$ of $C(Y)$ by $\tau(h \circ p) = \int_Z h\, d\mu$. By the Hahn-Banach Theorem we can extend τ to a positive linear functional on $C(Y)$. Using the Riesz Representation Theorem, we may view τ as a measure on Y and by the previous lemma we have $\tau \in \mathbf{R}$.

By inspection **R** is convex. Since each element in **R** is a probability measure, **R** is contained in the unit ball of $C(Y)^*$, which is the dual of $C(Y)$. This ball is weak-star compact by Alaoglu's Theorem. Thus to show **R** is weak-star compact it suffices to show that it is weak-star closed. Let $\{\tau_\alpha\}$ be a net in **R** converging to ν. Then for every $h \in C(Z)$,

$$\int_Z h\, d\mu = \int_Y h \circ p\, d\tau_\alpha \quad = \quad \lim_\alpha \int_Y h \circ p\, d\tau_\alpha = \int_Y h \circ p\, d\nu$$

and so $\nu \in \mathbf{R}$. □

Lemma 1.5 *Let* $\tau \in \mathbf{R}$, *and let* F *be a closed set in* Z. *Then*

$$\text{support}(\mu\,|_F) = p[\text{support}(\tau\,|_{p^{-1}(F)})].$$

Proof. Let $z \in Z \setminus \text{support}(\mu\mid_F)$. There exists an open neighborhood U of z with

$$\mu(U \cap F) = 0 = \nu(p^{-1}(U) \cap p^{-1}(F)).$$

By definition $p^{-1}(z) \cap \text{support}(\nu\mid_{p^{-1}(F)})$ is empty and we have

$$p(\text{support}(\nu\mid_{p^{-1}(F)})) \subseteq \text{support}(\mu\mid_F).$$

Now let $z \in \text{support}(\mu\mid_F)$. Order the neighborhoods of z by reverse inclusion: $G_1 \leq G_2$ if $G_2 \subseteq G_1$. If G is an open neighborhood about z, then $\nu(p^{-1}(G \cap F)) = \mu(G \cap F) > 0$ and so there exists a point y_G in $p^{-1}(G) \cap \text{support}(\nu\mid_{p^{-1}(F)})$. The set $\text{support}(\nu\mid_{p^{-1}(F)})$ is compact, so the net $\{y_G\}$ has a cluster point $y \in \text{support}(\nu\mid_{p^{-1}(F)})$. By taking a subnet, we may assume that $\lim_G y_G = y$. By hypothesis p is a continuous function, so $\lim_G p(y_G) = p(y)$. Let G_0 be an open neighborhood about z. From the construction, G_0 contains a tail of the net $\{p(y_G)\}$ and so $\lim_G p(y_G) = z = p(y)$ and the reverse inclusion is established. □

Let Λ denote the set of all linear transformations $L : C(Y) \to L^\infty(\mu)$ satisfying $\|L\| = 1$ and $L[(h \circ p)f] = hL(f)$ for all $h \in C(Z)$ and all $f \in C(Y)$. Each τ in $\mathbf{R}$ induces an element L_τ of Λ [5, Proposition 121, p. 116] with the property that for every f in $C(Y)$ and g in $L^1(\mu)$,

$$\int_Y (g \circ p) f \, d\tau = \int_Z g \, L_\tau(f) \, d\mu.$$

Proposition 1.1 *The map $\tau \in \mathbf{R} \to L_\tau \in \Lambda$ is a one-to-one, onto map.*

Proof. [5, Theorem 123]. $\square$

Recall that an element τ of the convex set $\mathbf{R}$ is an **extremal point** if $\tau_1, \tau_2 \in \mathbf{R}$ and $s\tau_1 + (1-s)\tau_2 = \tau$ for some s in $[0,1]$ then $\tau_1 = \tau_2 = \tau$. An element τ is an extremal point of $\mathbf{R}$ if and only if L_τ is an extremal point of Λ [5, Corollary 125, p. 119]. The representations described in Problem 1.1 consist of the extreme points of Λ [5, pp. 120-122]. However, this is not the only way to characterize the representations described in Problem 1.1. We can also place them in one-to-one correspondence with the minimal measures on $\mathbf{R}$ (to be rigorously defined shortly). The remainder of this chapter will be devoted to this task, using the techniques used in [5] to solve the $H^\infty(D)$ case.

Lemma 1.6 *Let Y be a compact Hausdorff space, F a closed subset of Y. Let $\{\nu_\alpha\}$ be a net of measures on Y which converges weak-star to a measure ν on Y. If* $\text{support}(\nu_\alpha) \subseteq F$ *for every α then* $\text{support}(\nu) \subseteq F$ *also.*

Proof. The proof is the same as [5, Lemma 43, p. 44]. $\square$

Lemma 1.7 *If $\{\nu_\alpha\}$ is a net in $\mathbf{R}$ which converges weak-star to measure ν, then $\{\nu_\alpha |_{p^{-1}(F)}\}$ converges weak-star to $\nu |_{p^{-1}(F)}$ for every closed set F in Z.*

Proof. The proof is the same as [5, Lemma 44, p. 45]. $\square$

Put a partial ordering on $\mathbf{R}$ as follows: $\nu \leq \tau$ if for every closed set $F \subseteq Z$ we have $\text{support}(\tau |_{p^{-1}(F)}) \subseteq \text{support}(\nu |_{p^{-1}(F)})$. The proof of the following theorem is based on the proof of [5, Theorem 45, pp. 45-46.].

Theorem 1.2 **R** *contains a maximal element, called a* **minimal measure**.

Proof. Let $\{\nu_\alpha\}$ be a chain in **R**. Then the chain is also a net with $\|\nu_\alpha\| = 1$ for all α, making it a net in the closed unit ball of $C(Y)^*$, which is weak-star compact by Alaoglu's Theorem. Thus there is a subnet $\{\tau_\beta\}$ which converges weak-star to a measure τ in the closed unit ball of $C(Y)^*$. Since **R** is weak-star compact, $\tau \in \mathbf{R}$. Now fix $\nu_\gamma \in \{\nu_\alpha\}$. By the definition of subnet there exists a $\beta > \gamma$ in the directed set and so by that definition and the previous 2 lemmas we have

$$\text{support}(\tau\,|_{p^{-1}(F)}) \subseteq \text{support}(\tau_\beta\,|_{p^{-1}(F)}) \subseteq \text{support}(\tau_\gamma\,|_{p^{-1}(F)})$$

for every closed set F in Z.

Thus τ is an upper bound for the chain and by Zorn's Lemma **R** has a maximal element. □

The proof of the following lemma is based on [5, Lemma 49, pp. 48–49].

Lemma 1.8 *Let ν be a minimal measure in* **R** *and Ω a closed subset of Y. Define a measure β on Z by $\beta(B) = \nu(p^{-1}(B) \cap \Omega)$ for each measurable set $B \subseteq Z$. There exists a measurable set $A \subseteq Z$ such that $\beta = \chi_A\,\mu$.*

Proof. By construction $\beta \ll \mu$. Let $h = \frac{d\beta}{d\mu}$, the Radon-Nikodym derivative of β with respect to μ. By construction $0 \leq h \leq 1$, and $h \in L^\infty(\mu)$. By Lusin's Theorem there exists a sequence of closed sets $\{E_n\}$ in Z such that $\text{support}(\mu\,|_{E_n}) = E_n$, $h\,|_{E_n}$ is continuous, and $\lim_{n\to\infty} \mu(E_n) = 1$. To show h is a characteristic function it suffices to show that $h^2 = h$ almost everywhere μ on E_n for all n. Suppose this is not the case. Then for some n there exists a closed set $K \subseteq E_n$ and $\epsilon > 0$ such that $\mu(K) > 0$ and $\epsilon \leq h \leq (1-\epsilon)$ on K.

Let $\sigma = [h \circ \Pi]^{-1}\nu \mid_{p^{-1}(K)\cap\Omega}$, and let $\tilde{\nu} = \sigma + \nu \mid_{E\setminus p^{-1}(K)}$. By construction $\tilde{\nu} \ll \nu$, and using the same arguments as [5, Lemma 49, pp. 48-49] it can be shown that $\tilde{\nu} \in \mathbf{R}$. Then ν a minimal measure implies that ν and $\tilde{\nu}$ are equivalent. Now $\tilde{\nu}(p^{-1}(K)) = \tilde{\nu}(p^{-1}(K \cap \Omega))$ implies that support$(\tilde{\nu} \mid_{p^{-1}(K)}) \subseteq \Omega$, since Ω is closed. But

$$\nu(p^{-1}(K) \cap \Omega) = \beta(K) = \int_K h\, d\mu \leq (1-\epsilon)\mu(K) < \mu(K) = \nu(p^{-1}(K))$$

implies that support$(\nu \mid_{p^{-1}(K)})$ is not contained in Ω, contradicting the equivalence of ν and $\tilde{\nu}$.

Thus $h^2 = h$ almost everywhere (μ) on each E_n and so $h = \chi_A$ for some measurable set $A \subseteq X$. □

Theorem 1.3 *Let ν be a minimal measure for* $\mathbf{R}$, *let $f \in C(Y)$, and let F be a closed subset of Z such that $L_\nu(f) \mid_F$ is continuous. If $z \in F$ and $\alpha \in p^{-1}(z) \cap \mathrm{support}(\nu \mid_{p^{-1}(F)})$, then $f(\alpha) = (L_\nu f)(z)$.*

Proof. The proof is the same as [5, Theorem 48]. To see the details, see the proof of Theorem 2.6. □

The proof of the following theorem is based on discussion found on [5, pp. 51–52].

Theorem 1.4 *If ν is a minimal measure in* $\mathbf{R}$, *then $L_\nu : C(Y) \to L^\infty(\mu)$ is multiplicative.*

Proof. Let $f_1, f_2 \in C(Y)$ and let $g_1 = L_\nu(f_1)$, $g_2 = L_\nu(f_2)$, and $g_3 = L_\nu(f_1 f_2)$. By Lusin's Theorem there exist closed sets E_n in Z such that support$(\mu \mid_{E_n}) = E_n$ for all n, $\lim_{n\to\infty} \mu(E_n) = 1$, and $g_i \mid_{E_n}$ is continuous for $i = 1, 2, 3$ and for all n.

Let $z \in E_n$. By Lemma 1.5, there exists $\alpha \in p^{-1}(z) \cap \text{support}(\nu \mid_{p^{-1}(E_n)})$ and so by Theorem 1.3 we have $g_1(z) = f_1(\alpha)$, $g_2(z) = f_2(\alpha)$, and $g_3(z) = (f_1 f_2)(\alpha) = f_1(\alpha) f_2(\alpha)$. Therefore, $g_1 g_2 = g_3$ almost everywhere (μ), and so L_ν is multiplicative. $\square$

Thus a minimal measure induces a representation. The next theorem will show that the converse is true. The proof is based on the proof of [5, Theorem 54].

Theorem 1.5 *If $\Gamma : C(Y) \to C(Z)$ is a representation as described in Problem 1.1, then there exists a minimal measure $\nu \in \mathbf{R}$ such that $\Gamma = L_\nu$.*

Proof. Using Proposition 1.1 and the discussion following it, one sees that $\Gamma = L_\nu$ for some ν which is an extremal point of $\mathbf{R}$. We want to show that ν is a minimal measure. Fix f in $C(Y)$. By Lusin's Theorem there exists a sequence $\{E_n\}$ of pairwise disjoint closed sets in Z such that $L_\nu(f) \mid_{E_n}$ is continuous and $\text{support}(\mu \mid_{E_n}) = E_n$ for all n with $\mu(\cup_{n=1}^{\infty} E_n) = 1$. Fix E_n and fix $z \in E_n$. Let

$$I = \{h \in L^\infty(\mu) : \text{ess} \lim_{w \to z_{w \in E_n}} h(w) = 0\}$$

where $\text{ess}\lim_{w \to z_{w \in E_n}} = c$ means that for each $\epsilon > 0$ there exists a neighborhood N of z such that

$$\mu(\{x : |h(x) - c| < \epsilon\} \cap N \cap E_n) = \mu(N \cap E_n).$$

In other words, $|h - c| < \epsilon$ on $N \cap E_n$ except on a set of measure 0.

It is easy to show that I is an ideal and thus I is contained in a maximal ideal Φ for $L^\infty(\mu)$ which has a corresponding multiplicative linear functional ϕ on $L^\infty(\mu)$ with kernel Φ [2, Propostion 2.33, pp. 43–44].

Now $\phi \circ L_\nu$ is a multiplicative linear functional on $C(Y)$, and by construction $\phi \circ L_\nu(f) = L_\nu f(z)$. Define $\Delta_n = \cup_{z \in E_n}\{y \in p^{-1}(z) : f(y) = L_\nu f(z)\}$. Clearly Δ_n is closed and by Lemma 1.5 and Theorem 1.3, $p(\Delta_n) = E_n$. Thus in the natural way, $C(E_n)$ is a closed subspace of $C(\Delta_n)$. The map $g \to \int_{E_n} g\, d\mu$ is a bounded positive linear functional on $C(E_n)$, so by Hahn-Banach it can be extended to a bounded positive linear functional on $C(\Delta_n)$. By the Riesz Representation Theorem there exists a positive measure τ_n on Δ_n such that $\int_{\Delta_n} g \circ p\, d\tau_n = \int_{E_n} g\, d\mu$ for every $g \in C(E_n)$. Let $\tau_f = \sum_{n=1}^{\infty} \tau_n$. By inspection $\tau_f \in \mathbf{R}$.

Let $A = \{f_1, f_2, \ldots, f_n\} \subseteq C(Y)$. For each j let $\{E_{j,k}\}_{k=1}^{\infty}$ be the collection $\{E_n\}$ for f_j as described in the previous section of this proof. Consider all sets of the form $\cap_{j=1}^{n} E_{j,k}$ and label this countable collection $\{F_i\}$, discarding any repeated sets. By construction, the F_i's are pairwise disjoint, with $f_j\ |_{F_i}$ continuous for all i and for $j = 1, \ldots, n$, and $\mu(E_{j,k} \setminus \cup_{F_i \subseteq E_{j,k}} F_i) = 0$. If we replace each F_i with support$(\mu\ |_{F_i})$ the same conclusions hold.

Let $\mathbf{F_i} = \cup_{z \in F_i}\{y \in p^{-1}(z) : f_j(y) = L_\nu f_j(z)\}$. By inspection $\mathbf{F_i}$ is closed and so by Lemma 1.5, $p(\mathbf{F_i}) = F_i$. Thus $C(F_i)$ is a closed subspace of $C(\mathbf{F_i})$ and we can again find a positive measure τ_i on $\mathbf{F_i}$ such that

$$\int_{\mathbf{F_i}} g \circ p\, d\tau_i = \int_{F_i} g\, d\mu$$

for every $g \in C(F_i)$. Let $\tau_A = \sum_i \tau_i$. By inspection $\tau_A \in \mathbf{R}$.

For each finite subset A of $C(Y)$ there exists a measure $\tau_A \in \mathbf{R}$. Order the finite subsets of $C(Y)$ by inclusion, creating a net $\{\tau_A\}$ in $\mathbf{R}$, which is weak-star compact so the net has a cluster point $\tau \in \mathbf{R}$.

Let ω be a minimal measure in $\mathbf{R}$ such that $\tau \leq \omega$. (By Theorem 1.4 such a measure exists.) We want to show that $L_\omega = L_\tau$, so it then follows that that $\omega = \tau$

by Proposition 1.1. Fix $f \in C(Y)$ and fix one E_n for f as described previously. A subnet $\{\tau_B\}$ converges to τ, and without loss of generality we may assume that $f \in B$ for all B. By Lemma 1.7,

$$\lim_B \tau_B \mid_{p^{-1}(E_n)} = \tau \mid_{p^{-1}(E_n)} .$$

Fix B from above and establish the sets $\{F_i\}$ as before. For any $F_i \subseteq E_n$,

$$\text{support}(\tau_B \mid_{p^{-1}(F_i)}) \subseteq \cup_{z \in F_i} \{y \in p^{-1}(z) : L_\nu f(z) = f(y)\}$$

by definition of B. This implies that

$$\text{support}(\tau \mid_{\cup[F_i : F_i \subseteq E_n]}) \subseteq S = \cup_{z \in E_n} \{y \in p^{-1}(z) : L_\nu f(z) = f(y)\}.$$

Now

$$\tau_B(p^{-1}(E_n)) = \mu(E_n) = \mu(\cup[F_i : F_i \subseteq E_n]) = \tau_B(\cup[p^{-1}(F_i) : F_i \subseteq E_n]),$$

which implies that $\text{support}(\tau_B \mid_{p^{-1}(E_n)}) = \text{support}(\tau_B \mid_{\cup[p^{-1}(F_i) : F_i \subseteq E_n]}) \subseteq S$. By definition of $\tau \leq \omega$ we have

$$\text{support}(\omega \mid_{p^{-1}(E_n)}) \subseteq \text{support}(\tau \mid_{p^{-1}(E_n)}) \subseteq S.$$

So if $y \in \text{support}(\omega \mid_{p^{-1}(E_n)}) \cap p^{-1}(z)$, then $f(y) = L_\nu f(z)$ by construction. But ω is a minimal measure, so $f(y) = L_\omega f(z)$ by Theorem 1.3. Thus $L_\nu(f) = L_\omega(f)$ almost everywhere (μ) and so ν is minimal. $\square$

The following proof is based on the proof of [5, Proposition 59, pp. 58-59].

Proposition 1.2 *Let Y and Z be compact Hausdorff spaces with $p : Y \to Z$ a continuous onto map, μ a regular Borel probability measure on Z, and $\Gamma : C(Y) \to L^\infty(\mu)$ a representation such that $\Gamma(h \circ p) = h$ for every $h \in C(Z)$. Let ν be the minimal measure inducing Γ. Then*

$$\sigma(\Gamma(f)) = f(\text{support}(\nu))$$

for all $f \in C(Y)$, where $\sigma(\Gamma(f))$ is the spectrum of $\Gamma(f)$ in $L^\infty(\mu)$.

Proof. First note that $\Gamma(f) \in L^\infty(\mu)$, so the spectrum of $\Gamma(f)$ is equal to the essential range of $\Gamma(f)$ [2, Lemma 2.63, p. 57]. That is,

$$\sigma(\Gamma(f)) = \{\lambda \in \mathbb{C} : \mu(z \in Z : |\Gamma(f)(z) - \lambda| < \epsilon) > 0 \ \ \forall \epsilon > 0\}$$

Let $\lambda \in \sigma(\Gamma(f))$ and $\epsilon > 0$. By definition and Lusin's Theorem, there exists a compact set $K \subseteq Z$ such that $\mu(K) > 0$, support$(\mu \mid_K) = K$, and $\Gamma(f) \mid_K$ is continuous with $|\Gamma(f)(z) - \lambda| < \epsilon$ for all $z \in K$. Let $z \in K$. By Lemma 1.5 there exists $y \in$ support$(\nu \mid_{p^{-1}(K)}) \cap p^{-1}(z)$ and by Theorem 1.3, $f(y) = \Gamma(f)(z)$. Thus $\Gamma(f)(z) \in f(\text{support}(\nu))$ for every $z \in K$. Since $f(\text{support}(\nu))$ is closed and ϵ was arbitrary, $\lambda \in f(\text{support}(\nu))$.

Now let $\lambda \in f(\text{support}(\nu))$ and $\epsilon > 0$. Let $U_\epsilon = \{y \in \text{support}(\nu) : |f(y) - \lambda| \leq \epsilon\}$. By the continuity of f, one sees that $\nu(U_\epsilon) > 0$; thus $\mu(p(U_\epsilon)) > 0$. Note that U_ϵ is a closed subset of the compact set Y, so $p(U_\epsilon)$ is compact and therefore Borel in Z. Let F be a closed subset of $p(U_\epsilon)$ such that $\mu(F) > 0$, the support$(\mu \mid_F) = F$, and $\Gamma(f) \mid_F$ is continuous. Let $z \in F$. By Lemma 1.5 and Theorem 1.3, there exists $y \in$ support$(\nu \mid_{p^{-1}(F)}) \cap p^{-1}(z)$ and $f(y) = \Gamma(f)(z)$. Therefore, $|\Gamma(f)(z) - \lambda| \leq \epsilon$ for all $z \in F$, so $\mu(z \in Z : |\Gamma(f)(z) - \lambda| \leq \epsilon) > 0$. The choice of ϵ was arbitrary, so by definition $\lambda \in \sigma(\Gamma(f))$. $\square$

The discussion at the end of Chapter 2 will establish that equivalent minimal measures induce the same representation, and Lemma 2.4 will establish that non-equivalent minimal measures induce distinct representations. Therefore, the representations described in Problem 1.1 are in one-to-one correspondence with the equivalence classes of the minimal measures in **R**.

Chapter 2

Schwartz's Lemma

Throughout this chapter E is an analytic metric space, X is a separable metric space, $\Pi : E \to X$ is a continous map, and μ is a regular Borel probability measure on X. The set $\Pi(E)$ is an analytic subset of X, so by Lemma 1.1, $\Pi(E)$ is absolutely measurable. For our purposes (finding various cross-sections), we may assume that μ is carried on $\Pi(E)$ and that $\Pi(E)$ is dense in X.

Let $M(E)$ be the set of all finite regular Borel measures on E, and let $M^+(E)$ be the set of all positive finite regular Borel measures on E. Define a map T_Π : $M(E) \to M(X)$ by $T_\Pi(\tau)(B) = \tau(\Pi^{-1}(B))$ for every Borel set $B \subseteq X$. Using the methods used to prove Lemma 1.3, it is not difficult to show that $T_\Pi(\tau) = \mu$ if and only if

$$\int_E f \circ \Pi \, d\tau = \int_X f \, d\mu$$

for every $f \in C_b(X)$.

Theorem 2.1 *If E is compact, then there exists $\tau \in M^+(E)$ such that $T_\Pi(\tau) = \mu$.*

Proof. Define a positive linear functional F on the subspace $\{f \circ \Pi : f \in C(X)\}$ of $C(E)$ by $F(f \circ \Pi) = \int_X f \, d\mu$. By the Hahn-Banach Theorem we can extend F

to a positive linear functional F on all of $C(E)$ without increasing the norm, so by the Riesz Representation Theorem there exists $\tau \in M^+(E)$ such that

$$F(f \circ \Pi) = \int_E f \circ \Pi \, d\tau = \int_X f \, d\mu$$

for every $f \in C(X)$. Thus $T_\Pi(\tau) = \mu$. $\square$

Lemma 2.1 *A measure μ is in the range of T_Π if and only if given $\epsilon > 0$ there exists a compact set $K \subseteq E$ such that $\mu(X \setminus \Pi(K)) < \epsilon$. Furthermore, if $\mu \in \mathrm{Range}(T_\Pi)$ then there exists $\tau \in M^+(E)$ such that $\|\mu\| = \|\tau\|$ and $T_\Pi(\tau) = \mu$.*

Proof. Suppose $\mu = T_\Pi(\tau)$. Let $\epsilon > 0$. Since τ is regular, there exists a compact set $K \subseteq E$ with $\tau(E \setminus K) < \epsilon$, and so

$$\mu(X \setminus \Pi(K)) = \tau(E \setminus \Pi^{-1}\Pi(K)) \leq \tau(E \setminus K) < \epsilon.$$

Now assume the latter conclusion, and let $\epsilon > 0$. Choose an increasing sequence of compact sets $\{K_n\}$ in E with $\mu(X \setminus \Pi(K_n)) < \frac{\epsilon}{n}$. Define $A_1 = \Pi(K_1)$, and $A_n = \Pi(K_n) \setminus \Pi(K_{n-1})$ for $n \geq 2$. Each A_n is a Borel set in X. Let $\mu_n = \mu \mid_{A_n}$. By Theorem 2.1 there exists $\tau_n \in M^+(K_n)$ with $T_\Pi(\tau_n) = \mu_n$ and $\|\mu_n\| = \|\tau_n\|$ for each n. Let $\tau = \sum_{n=1}^{\infty} \tau_n$. Then

$$\|\tau\| \leq \sum_{n=1}^{\infty} \|\tau_n\| = \sum_{n=1}^{\infty} \|\mu_n\| = \|\mu\|.$$

Let $B \subseteq X$ be Borel. Thus

$$\mu(B) = \sum_{n=1}^{\infty} \mu(B \cap A_n) = \sum_{n=1}^{\infty} \mu_n(B) = \sum_{n=1}^{\infty} \tau_n(\Pi^{-1}(B)) = \tau(\Pi^{-1}(B))$$

and $\mu = T_\Pi(\tau)$, with $\|\tau\| = \|\mu\|$. $\square$

Theorem 2.2 *Let P be a complete separable metric space, let X be a separable metric space with a regular Borel probability measure μ, and let $\phi : P \to X$ be a continuous map with $\mu(X \setminus \phi(P)) = 0$. Given $\epsilon > 0$, there exists a compact set $K \subseteq P$ with $\mu(X \setminus \phi(K)) < \epsilon$.*

Proof. Let $\{p_n\}$ be a countable dense set in P and define $U_n^k = B(p_n, \frac{1}{k})^{\text{cl}}$, the closed ball about p_n of radius $\frac{1}{k}$, and $F_m^k = \cup_{n=1}^m U_n^k$. Note that F_m^k is closed, so $\phi(F_m^k)$ is an analytic subset of X and therefore is μ-measurable by Lemma 1.1. Now $P = \cup_{m=1}^\infty F_m^k$ for each k because P is separable. Thus $\phi(P) = \cup_{m=1}^\infty \phi(F_m^k)$ for each k.

Therefore $\mu(\phi(P)) = \lim_{m\to\infty} \mu(\phi(F_m^1)) = 1$, and so there exists $m \in Z^+$ with $\mu(\phi(F_m^1)) > 1 - \epsilon$. Let m_1 be the least such positive integer. Now $F_{m_1}^1 = \cup_{m=1}^\infty (F_{m_1}^1 \cap F_m^2)$ gives us

$$\mu(\phi(F_{m_1}^1))) = \lim_{m\to\infty} \mu(\phi(F_{m_1}^1 \cap F_m^2)) > 1 - \epsilon,$$

so let m_2 be the least positive integer m such that $\mu(\phi(F_{m_1}^1 \cap F_m^2) > 1 - \epsilon$. It follows that $m_1 \le m_2$.

By induction let m_k be the least positive integer m such that $\mu(\phi(F_{m_1}^1 \cap \cdots \cap F_{m_{k-1}}^{k-1} \cap F_m^k > 1 - \epsilon$. By induction, $m_{k-1} \le m_k$. Let $K = \cap_{i=1}^\infty F_{m_i}^i$, so that

$$\mu(\phi(K)) = \lim_{n\to\infty} \mu(\phi(\cap_{i=1}^n F_{m_i}^i))$$

where for each n, $1 - \epsilon < \mu(\phi(\cap_{i=1}^n F_{m_i}^i)) < 1$. Using the Monotone Convergence

Theorem, we see that $\mu(\phi(K)) > 1 - \epsilon$; thus, $\mu(X \setminus \phi(K)) < \epsilon$.

By construction K is closed so it is a complete metric space, and to show that K is compact it suffices to show that K is totally bounded [8, p. 125]. Let $\delta > 0$, and let k be a positive integer with $\delta > \frac{2}{k}$. Now $K \subseteq F^k_{m_k} = \cup_{n=1}^{m_k} B(p_n, \frac{1}{k})^{\mathrm{cl}}$ with $B(p_n, \frac{1}{k}) \subseteq B(p_n, \delta)$ gives us a δ-net $p_1, \ldots, p_{m_k}$ for K and thus K is compact. □

Corollary 2.1 *Let E be an analytic metric space, X a separable metric space, $\Pi : E \to X$ continuous, and μ a regular Borel probability measure on X with $\mu(X \setminus \Pi(E)) = 0$. There exists $\tau \in M^+(E)$ such that $T_\Pi(\tau) = \mu$.*

Proof. Since E is analytic, there exists a continuous onto map $\psi : \mathbf{N}^\infty \to E$ [7, Lemma 7, pp. 35–38]. The map $\Pi \circ \psi : \mathbf{N}^\infty \to X$ is continuous with $\mu(X \setminus \Pi \circ \psi(\mathbf{N}^\infty)) = 0$. By Theorem 2.2, given $\epsilon > 0$, there exists a compact set $K \subseteq \mathbf{N}^\infty$ such that $\mu(X \setminus \Pi \circ \psi(K)) < \epsilon$. The map ψ is continuous, so $\psi(K)$ is a compact subset of E and by Lemma 2.1 the corollary is proven. □

Let $\mathbf{R} = \{\tau \in M^+(E) : T_\Pi(\tau) = \mu\}$. Place a partial ordering on $\mathbf{R}$ as follows: $\tau_1 \leq \tau_2$ if $\mathrm{support}(\tau_2 |_{\Pi^{-1}(H)}) \subseteq \mathrm{support}(\tau_1 |_{\Pi^{-1}(H)})$ for every closed set H in X.

Theorem 2.3 **R** *contains a maximal element, called a* **minimal measure**.

Proof. Let $\epsilon > 0$ be small. Using the proof of Corollary 2.1, let K_1 be a compact set in E such that $\mu(X \setminus \Pi(K_1)) < \frac{\epsilon}{2}$. Let $E_1 = E$, and let $E_2 = E_1 \setminus \Pi^{-1}(\Pi(K_1))$. The set E_2 is open, so it is Borel and thus analytic. Again using the proof of Corollary 2.1, let $K_2 \subseteq E_2$ be a compact set such that $\mu(\Pi(E_2) \setminus \Pi(K_2)) < \frac{\epsilon}{4}$. By induction find a sequence of compact sets $\{K_n\}_{n=1}^{\infty}$ in E such that

$$K_n \subseteq E_n = E_{n-1} \setminus \Pi^{-1}(\Pi(K_{n-1})) \text{ and } \mu(X \setminus \cup_{n=1}^{\infty} K_n) = 0.$$

Note that $E = [\cup_{n=1}^{\infty} \Pi^{-1}\Pi(K_n)] \cup \Pi^{-1}(B_0)$ where B_0 is a Borel set in X with $\mu(B_0) = 0$. Let $\mathbf{R_n} = \{\tau \in M^+(K_n) : \tau(\Pi^{-1}(B)) = \mu(B)$ for every Borel set $B \subseteq \Pi(K_n)\}$. By Theorem 2.1, $\mathbf{R_n}$ is non-empty and by Theorem 1.2 it contains a maximal element ν_n for each n.

Let $\nu = \sum_{n=1}^{\infty} \nu_n$. We want to show that $\nu \in \mathbf{R}$. Let $B \subseteq X$ be Borel. By construction

$$\mu(B) = \mu(\cup_{n=1}^{\infty}(\Pi(K_n) \cap B)) = \sum_{n=1}^{\infty} \mu(\Pi(K_n) \cap B) = \sum_{n=1}^{\infty} \nu_n(\Pi^{-1}(B)) = \nu(\Pi^{-1}(B))$$

and so $\nu \in \mathbf{R}$.

Next we would like to show that ν is a maximal element. Let $\tau \in \mathbf{R}$ with $\nu \preceq \tau$. For every n,

$$\text{support}(\tau\,|_{\Pi^{-1}\Pi(K_n)}) \subseteq \text{support}(\nu\,|_{\Pi^{-1}\Pi(K_n)}) = \text{support}(\nu_n) \subseteq K_n$$

and so $\tau\,|_{\Pi^{-1}\Pi(K_n)} \in \mathbf{R_n}$. Since ν_n is a maximal element in $\mathbf{R_n}$, it follows that

$$\text{support}(\tau\,|_{\Pi^{-1}\Pi(K_n)\cap\Pi^{-1}(H)}) \supseteq \text{support}(\nu_n\,|_{\Pi^{-1}(H)}) = \text{support}(\nu\,|_{\Pi^{-1}\Pi(K_n)\cap\Pi^{-1}(H)})$$

for every closed set $H \subseteq X$. Now let

$$z \in \text{support}(\nu\,|_{\Pi^{-1}(H)}) \subseteq [\cup_{n=1}^{\infty}\text{support}(\nu_n\,|_{\Pi^{-1}(H)})]^{\text{cl}}.$$

There exists a sequence $\{z_m\}_{m=1}^{\infty} \subseteq \cup_{n=1}^{\infty}\text{support}(\nu_n\,|_{\Pi^{-1}(H)})$ such that

$\lim_{m\to\infty} z_m = z$. By the above containment, $z_m \in \text{support}(\tau\ |_{\Pi^{-1}(H)})$ for all m. The support of a measure is closed, so $z \in \text{support}(\tau\ |_{\Pi^{-1}(H)})$.

Thus $\text{support}(\tau\ |_{\Pi^{-1}(H)}) \supseteq \text{support}(\nu\ |_{\Pi^{-1}(H)})$ for every closed set $H \subseteq X$, and τ and ν are equivalent. $\square$

Define a map $\eta : C_b(X) \to C_b(E)$ by $\eta(h) = h \circ \Pi$ for each $h \in C_b(X)$. The map η is a $*$-isometry, because

$$\|\eta(h)\|_\infty = \sup_{e\in E} |h \circ \Pi(e)| = \sup_{x\in\Pi(E)} |h(x)| = \sup_{x\in X} |h(x)| = \|h\|_\infty$$

This equation holds because $\Pi(E)$ is dense in X, so a continuous function will have the same supremum on either set. Fix $\tau \in \mathbf{R}$, and extend the map $\eta : L^1(\mu) \to L^1(\tau)$ by $\eta(h) = h \circ \Pi$ for $h \in L^1(\mu)$. If these L^1 spaces are equipped with their natural norms, then the extension of η is still a $*$-isometry because

$$\|\eta(h)\|_1 = \|h \circ \Pi\|_1 = \int_E |h \circ \Pi|\, d\tau = \int_X |h|\, d\mu = \|h\|_1$$

by definition. The proof of the following theorem is based on the proof of [5, Proposition 121, p. 116].

Theorem 2.4 *Each τ in R induces a continuous linear map $L_\tau : C_b(E) \to L^\infty(\mu)$ such that $\|L_\tau\| = 1$ and $L_\tau(\eta(h)f) = h\, L_\tau(f)$ for every $f \in C_b(E)$ and for every $h \in C_b(X)$.*

Proof. Let $f \in C_b(E)$. The map $\lambda(s) = \int_E s\, f\, d\tau$ is a continuous linear functional on $L^1(\tau)$. Recall that $\eta : L^1(\mu) \to L^1(\tau)$ defined by $\eta(g) = g \circ \Pi$ is a $*$-isometry so the restriction of λ to $\eta(L^1(\mu))$ makes λ a continuous linear functional on $L^1(\mu)$ in a natural way. By the Riesz Representation Theorem there exists a unique $r \in L^\infty(\mu)$ such that

$$\int_E \eta(g) f \, d\tau = \int_X g\, r \, d\mu$$

for all $g \in L^1(\mu)$. Let $L_\tau(f) = r$. By the uniqueness of r, L_τ is well-defined. Also by the Riesz Representation Theorem,

$$\|r\|_\infty = \|f \, d\tau\| = \int_E |f| \, d\tau \leq \tau(E) \, \|f\|_\infty = \|f\|_\infty$$

so $\|L_\tau\| \leq 1$. It is easy to see that $L_\tau(1) = 1$ so we have $\|L_\tau\| = 1$.

Now let $g \in L^1(\mu)$, and let $h \in C_b(X)$, so we have $gh \in L^1(\mu)$ and

$$\int_X g\, h\, r \, d\mu = \int_E \eta(gh) f \, d\tau = \int_E \eta(g)(\eta(h) f) \, d\tau.$$

This holds for all $g \in L^1(\mu)$ so by construction we have $L_\tau(\eta(h) f) = h\, r = h L_\tau(f)$.

□

The relationship described in Theorem 2.4 is pictured in the following diagram, in which the map ι represents the identity function.

$$\begin{tikzcd}
C_b(E) \arrow[dr, "\mathrm{L}_\nu"] & \\
C_b(X) \arrow[u, "\eta"] \arrow[r, "\iota"] & L^\infty(\mu)
\end{tikzcd}$$

A net $\{t_\alpha\}$ is an **approximate identity** for $C_b(E)$ if it is contained in $C_b(E)$, $0 \le t_\alpha \le 1$ for all α, and $\lim_\alpha t_\alpha(z) = 1$ for all $z \in E$.

Lemma 2.2 *If $\nu \in \mathbf{R}$ and $\{t_\alpha\}$ is an approximate identity for $C_b(E)$, then $L_\nu(t_\alpha)$ converges weak-star to 1.*

Proof. Fix g in $L^1(\mu)$. Without loss of generality we may assume that $g \ge 0$. We want to show that

$$\lim_\alpha \int_X L_\nu(t_\alpha)\, g\, d\mu = \int_X g\, d\mu.$$

Let $G_{\alpha,n} = \{y \in E : t_\alpha(y) > 1 - \frac{1}{n}\}$. We have

$$0 \le \int_E (1 - \frac{1}{n})\chi_{G_{\alpha,n}}\, d\nu \le \int_E t_\alpha\, \eta(g)\, d\nu \le \int_E \eta(g)\, d\nu$$

The net $\chi_{G_{\alpha,n}}(1 - \frac{1}{n})$ converges boundedly pointwise to 1. Thus by using the inequality above we get

$$\begin{aligned} \lim_\alpha \int_E t_\alpha\, \eta(g)\, d\nu &= \int_E \eta(g)\, d\nu \\ &= \lim_\alpha \int_X L_\nu(t_\alpha)\, g\, d\mu = \int_X g\, d\mu \end{aligned}$$

□

Theorem 2.5 *Let E be an analytic metric space, X a separable metric space with $\Pi : E \to X$ a continuous map and μ a Borel probability measure on X. Let $\Gamma : C_b(E) \to L^\infty(\mu)$ be a continuous algebra homomorphism such that*

1. *$\Gamma(f \circ \Pi) = f$ for all $f \in C_b(X)$.*

2. If $\{t_\alpha\}$ is an approximate identity for $C_b(E)$, then $\Gamma(t_\alpha)$ converges weak-star to 1.

There exists a minimal measure ν on E such that $\Gamma = L_\nu$.

Proof. X is analytic, so by Lemma 1.1 it is absolutely measurable in βX, the Stone-Cech compactification of X. Thus μ can be extended to a Borel measure $\hat{\mu}$ on βX with $\hat{\mu}(\beta X \setminus X) = 0$ and $\hat{\mu}\mid_X = \mu$. Therefore, $L^\infty(\mu)$ and $L^\infty(\hat{\mu})$ are isometrically isomorphic.

Recall the isometry $\eta : C_b(X) \rightarrow C_b(E)$, defined by $\eta(h) = h \circ \Pi$. By the Stone-Cech theorem, $C_b(X)$ and $C_b(E)$ are isometrically isomorphic to $C(\beta X)$ and $C(\beta E)$, respectively, so we can consider η as a map from $C(\beta X)$ to $C(\beta E)$ in the obvious manner. Now consider the adjoint, $\eta^* : C(\beta E)^* \rightarrow C(\beta X)^*$. By the Riesz Representation Theorem, $C(\beta X)^* = M(\beta X)$, the Borel measures on βX, and $C(\beta E)^* = M(\beta E)$, the Borel measures on βE. The space βE is embedded in $M(\beta E)$ by the mapping $e \rightarrow \delta_e$, the atomic measure at e. In the same way βX is embedded in $M(\beta X)$. It is easy to check that $\eta^*\mid_E = \Pi$, and by use of the Closed Range Theorem it can be shown that $\eta^*(\beta E) = \beta X$. Thus $\eta^* : \beta E \rightarrow \beta X$ is a continuous onto map, and a continuous extension of Π.

Now extend $\Gamma : C_b(E) \rightarrow L^\infty(\mu)$ to $\Gamma^{\text{ext}} : C(\beta E) \rightarrow L^\infty(\hat{\mu})$ in the obvious manner, since $C(\beta E)$ and $C_b(E)$ are isometrically isomorphic, as are $L^\infty(\mu)$ and $L^\infty(\hat{\mu})$.

By Theorem 1.5 there exists a minimal measure ν on βE such that $\Gamma^{\text{ext}} = L_\nu$. We want to show $\nu(\beta E \setminus E) = 0$. Let $K \subseteq E$ be compact. and let G be an open neighborhood about E in βE. By Urysohn's Lemma there exists a function $h_{G,K}$ in $C(\beta E)$ such that $0 \leq h_{G,K} \leq 1$, $h_{G,K}(K) = 1$, and $h_{G,K}(\beta E \setminus G) = 0$. Order $\{(G,K) : G \supset E \text{ open}, K \subseteq E \text{ compact}\}$ as follows: $(G_1, K_1) \leq (G_2, K_2)$ if and only if $G_1 \supseteq G_2$ and $K_1 \subseteq K_2$. It is easy to check that the net $\{h_{G,K}\}$ converges pointwise on βE to χ_E and so $\{h_{G,K}\}$ is an approximate identity for $C_b(E)$. By hypothesis $\Gamma(h_{G,K}) = \Gamma^{\text{ext}}(h_{G,K} \mid_E)$ converges weak-star to 1, so

$$\begin{aligned}
\lim_{G,K} \int_{\beta X} \Gamma^{\text{ext}}(h_{G,K})\, d\nu &= \int_{\beta X} 1\, d\mu = \mu(\beta X) = \nu(\beta E) \\
&= \lim_{G,K} \int_{\beta E} h_{G,K}\, d\nu = \int_{\beta E} \chi_E\, d\nu = \nu(E).
\end{aligned}$$

The second limit exists because the first limit exists by hypothesis, so $\nu(\beta E \setminus E) = 0$.

Let $H \subseteq X$ be closed. Let $H^{\text{cl}(\beta X)}$ denote the closure of H in βX. It is not difficult to show that

$$\Pi^{-1}(H) = (\eta^*)^{-1}(H^{\text{cl}(\beta X)}) \cap E$$

and also that

$$\text{support}(\nu \mid_{\Pi^{-1}(H)}) = [\text{support}(\nu \mid_{(\eta^*)^{-1}(H^{\text{cl}(\beta X)})})] \cap E.$$

Now we want to show that $\nu \mid_E$ is a minimal measure on E, as defined previously. Suppose $\tau \in \mathbf{R}$ with $\text{support}(\tau \mid_{\Pi^{-1}(H)}) \subseteq \text{support}(\nu \mid_{\Pi^{-1}(H)})$ for every closed set $H \subseteq X$. Define a measure $\hat{\tau}$ on βE by $\hat{\tau}(B) = \tau(B \cap E)$ for every Borel set $B \subseteq \beta E$. We want to show that $\text{support}(\hat{\tau} \mid_{(\eta^*)^{-1}(H^{\text{cl}(\beta X)})}) \subseteq \text{support}(\nu \mid_{(\eta^*)^{-1}(H^{\text{cl}(\beta X)})})$.

Let $z \in \text{support}(\hat{\tau} \mid_{(\eta^*)^{-1}(H^{\text{cl}(\beta X)})})$, and let G be an open neighborhood about z in $(\eta^*)^{-1}(H^{\text{cl}(\beta X)})$, so

$$\begin{aligned}
\hat{\tau} \mid_{(\eta^*)^{-1}(H^{\text{cl}(\beta X)})} (G) &= \hat{\tau}(G \cap (\eta^*)^{-1}(H^{\text{cl}(\beta X)}) \cap E) \\
&= \tau(G \cap \Pi^{-1}(H)) = \tau \mid_{\Pi^{-1}(H)} (G) > 0.
\end{aligned}$$

There exists a $y \in G \cap \text{support}(\tau\, |_{\Pi^{-1}(H)}) \subseteq G \cap \text{support}(\nu\, |_{\Pi^{-1}(H)})$ and so

$$\nu\, |_{\Pi^{-1}(H)}\, (G) = \nu(G \cap (\eta^*)^{-1}(H^{\text{cl}(\beta X)})) = \nu\, |_{(\eta^*)^{-1}(H^{\text{cl}(\beta X)})}\, (G) > 0.$$

Thus $z \in \text{support}(\nu\, |_{\Pi^{-1}(H^{\text{cl}(\beta X)})})$.

Since ν is a minimal measure on βE, $\hat{\tau}$ and ν are equivalent, and τ and $\nu\, |_E$ must be equivalent also. Thus $\nu\, |_E$ is a minimal measure on E. □

Proposition 2.1 *Let $\Gamma : C_b(E) \to L^\infty(\mu)$ be a representation satisfying the hypotheses of Theorem 2.5, and let ν be the minimal measure on βE inducing Γ^{ext}, as defined in the proof of Theorem 2.5. We have*

$$\sigma(\Gamma^{\text{ext}}(f)) = f(\text{support}(\nu))$$

for every $f \in C(\beta E)$, where $\sigma(\Gamma^{\text{ext}}(f))$ is the spectrum of $\Gamma^{\text{ext}}(f)$ in $L^\infty(\hat{\mu})$.

Proof. Since βE and βX are compact and $\eta^* : \beta E \to \beta X$ is a continuous onto map, Proposition 1.2 gives us the proof. □

Lemma 2.3 *Let ν be a minimal measure on E and let Ω be a closed subset of E. Define a measure β on X by $\beta(F) = \nu(\Pi^{-1}(F) \cap \Omega)$ for each Borel set F in X. There exists a measurable set A in X such that $\beta = (\chi_A)\mu$.*

Proof. The proof is the same as that for Lemma 1.8. □

The proof of the following theorem is based on the proof of [5, Theorem 48, pp. 49–50].

Theorem 2.6 *Let $f \in C_b(E)$, let ν be a minimal measure on E and let H be a closed subset of X such that $L_\nu(f) \mid_H$ is continuous. If $x \in H$ and $\alpha \in \Pi^{-1}(x) \cap$ support$(\nu \mid_{\Pi^{-1}(H)})$, then $f(\alpha) = L_\nu(f)(x)$.*

Proof. Fix x and α as in the statement of the theorem. Let $g = L_\nu(f)$ and let $\epsilon > 0$. Without loss of generality we may assume that support$(\mu \mid_H) = H$ and $f(\alpha) = 0$ (because $L_\nu(1) = 1$). We may also assume that $|g(s) - g(t)| < \epsilon$ for all s and t in H. (If not, $g \mid_H$ continuous means we can replace H by an appropriate closed subset for each $x \in H$.)

Let $G = \{y \in E : |f(y)| < \epsilon\}$, which is open in E as f is a continuous function. Define a measure β on X by $\beta(B) = \nu(\Pi^{-1}(B) \cap (\Pi^{-1}(H) \setminus G))$ for each Borel set $B \subseteq X$. By Lemma 2.3 there exists a measurable set $A \subseteq \Pi(E)$ such that $\beta = (\chi_A)\mu$. Now $f(\alpha) = 0$ means that $G \cap$ support$(\nu \mid_{\Pi^{-1}(H)})$ is non-empty. Thus $\nu(G \cap \Pi^{-1}(H)) > 0$ and so $\beta(H) < \nu(\Pi^{-1}(H)) = \mu(H)$. Therefore $\chi_A = 0$ on some closed subset $F \subseteq H$ with $\mu(F) > 0$ and $\beta(F) = 0 = \nu(\Pi^{-1}(F) \cap (\Pi^{-1}(H) \setminus G)) = \nu(\Pi^{-1}(F) \setminus G)$. Thus $\nu(\Pi^{-1}(F)) = \nu(\Pi^{-1}(F) \cap G)$ and so support$(\nu \mid_{\Pi^{-1}(F)}) \subseteq (\Pi^{-1}(F) \cap G)^{\mathrm{cl}}$, which implies that

$$\text{support}(\nu \mid_{\Pi^{-1}(F)}) \subseteq \{y \in E : |f(y)| \leq \epsilon\}.$$

Now $\int_E \eta(h) f \, d\nu = \int_X h \, g \, d\mu$ for each $h \in L^1(\mu)$ by definition of L_ν, so $\int_E \eta(\chi_F \, s) f \, d\nu = \int_X \chi_F \, s \, g \, d\mu$ for all $s \in C(X)$. Noting that $\eta(\chi_F) = \chi_{\Pi^{-1}(F)}$, we see that

$$|\int_E \eta(\chi_F s) f \, d\nu| = |\int_{\Pi^{-1}(F)} \eta(s) f \, d\nu| = |\int_F s \, g \, d\mu|.$$

Thus

$$|\int_F s\,g\,d\mu| \leq \|s\|_\infty \int_{\Pi^{-1}(F)} |f|\,d\nu \leq \epsilon\,\|s\|_\infty\,\nu(\Pi^{-1}(F)) = \epsilon\,\|s\|_\infty\,\mu(F)$$

for all $s \in C(X)$. Hence, by the Riesz Representation Theorem, $|g| < \epsilon$ on F. Now by our beginning assumption on the set H and the Triangle Inequality, we see that $|g| \leq 2\epsilon$ on H. Our choice of ϵ was arbitrary, so $g(x) = f(\alpha) = 0$. $\square$

The proof of the following lemma is based on the proof of [5, Lemma 55, pp.55–57].

Lemma 2.4 *Non-equivalent minimal measures induce distinct representations.*

Proof. Let σ and τ be non-equivalent minimal measures on E. Without loss of generality there exists a closed set $K \subseteq X$ such that support$(\mu\,|_K) = K$ and the set

$$S = \text{support}(\sigma\,|_{\Pi^{-1}(K)}) \setminus \text{support}(\tau\,|_{\Pi^{-1}(K)})$$

is non-empty. Choose y_0 in S and choose $G \subseteq E$ a basic open set about y_0 such that $G \cap \text{support}(\tau\,|_{\Pi^{-1}(K)})$ is the empty set. The space E is metric, so it is completely regular, and so E has the weak topology generated by $C_b(E)$. Therefore, we may assume that there exist an $\epsilon > 0$ and a finite collection $\{f_1 \ldots, f_n\}$ of $C_b(E)$ such that $f_i(y_0) = 0$ for $i = 1, \ldots, n$ and $G = \{y : |f_i(y)| < 2\epsilon,\ i = 1, \ldots, n\}$. Let $U = \{y : |f_i(y)| < \epsilon,\ i = 1, \ldots, n\}$. Now $y_0 \in U$ means that $\sigma(U \cap \Pi^{-1}(K)) > \delta$ for some $\delta > 0$.

By Lusin's Theorem we can replace K with a closed subset K' such that $\sigma(K' \cap U) > \frac{\delta}{2}$, and $L_\sigma(f_i)$ and $L_\tau(f_i)$ are continuous for $i = 1, \ldots, n$.

Define a measure β on X by $\beta(H) = \sigma(\Pi^{-1}(H) \cap (\Pi^{-1}(K') \setminus U))$ for each Borel set H in X. By Lemma 2.3 there exists a measurable set $A \subseteq X$ such that $\beta = (\chi_A)\mu$. Now $\sigma(U \cap \Pi^{-1}(K')) > 0$ implies $\beta(K') < \sigma(\Pi^{-1}(K')) = \mu(K')$. Thus $\chi_A = 0$ on some closed $F \subseteq K'$ with $\mu(F) > 0$ and $\beta(F) = \sigma(\Pi^{-1}(F) \setminus U) = 0$. If now follows that $\sigma(\Pi^{-1}(F) \cap U) = \sigma(\Pi^{-1}(F))$, and we have

$$\text{support}(\sigma \mid_{\Pi^{-1}(F)}) \subseteq U^{\text{cl}} \subseteq \{y : |f_i(y)| \leq \epsilon, \ \ i = 1, \ldots, n\}.$$

For $x \in F$ and $y \in \Pi^{-1}(x) \cap \text{support}(\sigma \mid_{\Pi^{-1}(F)})$ we have $L_\sigma f_i(x) = f_i(y)$ for $i = 1, \ldots, n$ by Theorem 2.6, and for each $y \in \Pi^{-1}(x) \cap \text{support}(\tau \mid_{\Pi^{-1}(F)}$ we have $L_\tau f_i(x) = f_i(y)$ for $i = 1, \ldots, n$. Thus $|L_\sigma f_i| \leq \epsilon$ almost everywhere (μ) on F for $i = 1, \ldots, n$. However, $G \cap \text{support}(\tau \mid_{\Pi^{-1}(F)})$ is empty, so for some $i \in \{1, \ldots, n\}$, we have $|f_i(y)| > 2\epsilon$ for all $y \in \text{support}(\tau \mid_{\Pi^{-1}(F)})$. Thus $|L_\tau f_i| > 2\epsilon$ almost everywhere (μ) on F. Hence $L_\tau f_i$ and $L_\sigma f_i$ differ on a set of positive measure (μ), so L_τ and L_σ are distinct representations. □

The proof of the following lemma is based on discussion found on [5, pp. 51–52].

Lemma 2.5 *If ν is a minimal measure on E, then L_ν, the operator induced by ν, is multiplicative.*

Proof. Fix $f_1, f_2 \in C_b(E)$ and let $g_1 = L_\nu(f_1)$, let $g_2 = L_\nu(f_2)$, and let $g_3 = L_\nu(f_1 f_2)$. By Lusin's Theorem there exists a sequence $\{E_n\}$ of closed sets in X with $\text{support}(\mu \mid_{E_n}) = E_n$ for each n, $\lim_{n\to\infty} \mu(E_n) = 1$, and $g_i \mid_{E_n}$ continuous for $i = 1, 2, 3$ and for each n.

By definition, the measure $(\mu \mid_{E_n})$ is carried by $\Pi(\text{support}(\nu \mid_{\Pi^{-1}(E_n)}))$, which is an analytic set and thus is absolutely measurable, by Lemma 1.1. Let

x be in the set $\Pi(\text{support}(\nu \mid_{\Pi^{-1}(E_n)})$. By definition there exists an α in $\Pi^{-1}(x) \cap \text{support}(\nu \mid_{\Pi^{-1}(E_n)})$ and by Theorem 2.6, we have $g_1(x) = f_1(\alpha)$, $g_2(x) = f_2(\alpha)$, and $g_3(x) = f_1(\alpha)f_2(\alpha)$. Thus $g_1g_2 = g_3$ almost everywhere (μ). □

Corollary 2.2 *The set of representations $\Gamma : C_b(E) \to L^\infty(\mu)$ as described in Theorem 2.5 is in 1-1 correspondence with the equivalence classes of the minimal measures on E.*

A set $K \subseteq X$ has **full density** at x with respect to μ if

$$\liminf_{r \to 0} \frac{\mu(K \cap B(x,r))}{\mu(B(x,r))} = 1.$$

If $\{K_i\}_{i=1}^n$ is a finite collection of sets, each having full density at x, then by use of the definition it is not too difficult to show that $\cap_{i=1}^n K_i$ has full density at x.

The density of a set K at a point is the Radon-Nikodym derivative of the measure $(\chi_K \mu) = \mu \mid_K$ with respect to μ almost everywhere (μ). To see this for $\mathbb{R}^n$ with respect to Lebesgue measure, consult [3, Theorem 3.22], and for more general measure spaces consult [9, Chapter 10]. The following lemma uses [9] to establish the fact for a separable metric space.

Lemma 2.6 *Let X be a separable metric space with regular Borel probability measure μ, and let $A \subset X$ be a μ-measurable set. Then A has full density at x for almost every (μ) $x \in A$.*

Proof. Without loss of generality we may assume that μ contains no atoms (individual points of positive measure). If μ does contain an atom x_0, then given a minimal measure ν on E, then $\text{support}(\nu \mid_{\Pi^{-1}(x_0)})$ must consist of a single point by minimality and therefore must be the choice for $s(x_0)$.

To use the results from [9], we must first note the definition of *net* used in them. This definition is distinct from the traditional topological definition. Let X be a topological space with a measure μ and suppose that X is the union of a countable collection $\aleph_1$ of disjoint Borel sets $A_1^{(1)}, A_2^{(1)}, \ldots$ called sets of the first rank. In addition, suppose each set $A_j^{(1)}$ is the union of a countable collection of disjoint Borel sets $A_{j1}^{(2)}, A_{j2}^{(2)}, \ldots$, called sets of the second rank. Let $\aleph_2$ denote the collection of all sets of second rank, ranging over all j. Continue this process, producing for each n a collection $\aleph_n$ of disjoint Borel sets, the union of which is X. The collection $\aleph = \cup_{n=1}^{\infty} \aleph_n$ of all sets of finite rank is called a net [9, p. 208], if for each x in X and each $\epsilon > 0$, there exists a set $B \in \aleph$ such that $x \in B$ and $\mu(B) < \epsilon$.

Now let X, A, and μ be as in the hypothesis of the lemma. We define the derivative of the measure $(\chi_A \, \mu) = \mu \mid_A$ at a point x_0 with respect to a net $\aleph$ by

$$\mathbf{D}_{\aleph} \, (\chi_A \, \mu)(x_0) = \lim_{\epsilon \to 0} \frac{\mu(A \cap H_\epsilon(x_0))}{\mu(H_\epsilon(x_0))},$$

where $H_\epsilon(x_0) \in \aleph$, $x_0 \in H_\epsilon(x_0)$, and $\mu(H_\epsilon(x_0)) < \epsilon$.

By [9, Corollary 2, p. 216], this derivative is defined almost everywhere (μ) and equals the Radon-Nikodym derivative of $(\chi_A \, \mu)$ with respect to μ. In other words, $\mathbf{D}_{\aleph} \, (\chi_A \, \mu) = \chi_A$ almost everywhere (μ). In addition, this result is independent of the choice of the net $\aleph$.

Let $x_0 \in A$ such that

$$\liminf_{r \to 0} \frac{\mu(B(x_0, r) \cap A)}{\mu(B(x_0, r))} < 1 - \alpha,$$

where $\alpha > 0$. Then for each n there exists $0 < r_n < \frac{1}{n}$ such that

$$\frac{\mu(B(x_0, r_n) \cap A)}{\mu(B(x_0, r_n))} < 1 - \alpha.$$

We can choose a net $\aleph$ which contains $\{B(x_0, r_n)\}_{n=1}^{\infty}$. Thus $\mathbf{D}_{\aleph}\,(\chi_A\,\mu)(x_o) < 1 - \alpha$. Note that for each choice of x_0 a different net is produced. However, $\mathbf{D}_{\aleph}\,(\chi_A\,\mu) = \chi_A$ almost everywhere (μ) independent of the choice of $\aleph$. Therefore, the set of points $x \in A$ such that A has less than full density at x must be a set of measure 0.

$\square$

Assume the hypotheses of Theorem 2.5. Let $f \in C(\beta X)$, and let H be a closed subset of βX such that $(\Gamma^{\text{ext}} f)\ |_H$ is continuous. Let ν be the minimal measure inducing Γ^{ext}. If $x \in H$ and $\alpha \in (\eta^*)^{-1}(x) \cap \text{support}(\nu\ |_{(\eta^*)^{-1}(H)})$, then $f(\alpha) = \Gamma^{\text{ext}} f(x)$, by Theorem 1.3. The proof of the following theorem is based on the proof of [5, Theorem 42, pp. 59–60].

Theorem 2.7 *Assume the hypotheses of Theorem 2.5. There exists a Borel set F with $\mu(X \setminus F) = 0$ and a Borel cross-section $s : F \to \beta E$ for Π such that $(\Gamma f)(x) = f(s(x))$ almost everywhere (μ) on X for every $f \in C_b(E)$.*

Proof. We use the notation of the proof of Theorem 2.5. Fix x in the support of μ and define $\mathbf{\Delta} = \{f \in C(\beta E) :$ there exists compact $K \subseteq \beta X$ with full density at x such that $(\Gamma^{\text{ext}} f)\ |_K$ is continuous$\}$. Note that although we don't have a definition for full density outside a metric space, $\hat{\mu}(\beta X \setminus X) = 0$, so as long as $x \in X$, the same definition is valid for $K \subseteq \beta X$ having full density at x with respect to $\hat{\mu}$. From Theorem 2.5 let ν be the minimal measure on βE which induces Γ^{ext}, recalling that

$\nu(\beta E \setminus E) = 0$. Let

$$S_x = \cap_{f\in\Delta}\{y \in (\eta^*)^{-1}(x) \cap \text{support}(\nu\mid_{((\eta^*)^{-1}(K(f)))})\}$$

where $K(f)$ is a compact set with full density at x such that $(\Gamma^{\text{ext}} f)\mid_{K(f)}$ is continuous. Suppose S_x is empty. Since S_x is the intersection of closed sets in the compact space βE, the collection must lack the finite intersection property. Thus there exists a finite subcollection $\{f_1, f_2, \ldots, f_n\}$ with corresponding compact sets $\{K_1, \ldots, K_n\}$ as above such that

$$S_n = \cap_{i=1}^{n}\{y \in (\eta^*)^{-1}(x) \cap \text{support}(\nu\mid_{(\eta^*)^{-1}(K_n)})\}$$

is empty. Let $K = \cap_{i=1}^{n} K_i$. By previous discussion, K has full density at x, and so $x \in \text{support}(\hat{\mu}\mid_K)$. Note that $\eta^*(\text{support}(\nu\mid_{(\eta^*)^{-1}(K)})) = \text{support}(\hat{\mu}\mid_K)$, because $(\eta^*)^{-1}(K)$ is a compact set. Thus there must exist

$$y \in (\eta^*)^{-1}(x) \cap \text{support}(\nu\mid_{(\eta^*)^{-1}(K)}) \subseteq S_n.$$

But then S_n is non-empty, and thus S_x is non-empty too, contradicting our assumption that S_x is an empty set. Hence, for each x in the support of μ we can choose $s(x) \in S_x$. Let $F = \text{support}(\mu)$.

Let $f \in C_b(E) \cong C(\beta E)$. By Lusin's Theorem there exists a sequence $\{K_n\}_{n=1}^{\infty}$ of closed (and thus compact) sets in βX such that $\Gamma^{\text{ext}} f\mid_{K_n}$ is continuous and $\lim_{n\to\infty}(\hat{\mu}(K_n)) = 1$. By Lemma 2.6, K_n has full density almost everywhere (μ) on K_n, and $\hat{\mu}(\eta^*(\text{support}(\nu\mid_{(\eta^*)^{-1}(K_n)})) = \hat{\mu}(K_n)$. Thus for almost every $x \in X$, $\Gamma^{\text{ext}} f(x) = \Gamma f(x) = f \circ s(x)$.

The compact set βE is completely regular, so βE has the weak topology generated by $C(\beta E)$. Thus a basic open set G in βE has the form $G = f^{-1}(|z| \leq 1)$ for some $f \in C(\beta E)$. Then $s^{-1}(G) = s^{-1} \circ f^{-1}(|z| \leq 1) = (f \circ s)^{-1}(|z| \leq 1)$ and $f \circ s = \Gamma^{\text{ext}} f$ almost everywhere (μ) so $f \circ s$ is Borel measurable, and $(f \circ s)^{-1}(|z| \leq 1)$ is a Borel set. Therefore, $s : F \to \beta E$ is a Borel measurable function. $\square$

Lemma 2.7 $\hat{\mu}(s^{-1}(\beta E \setminus E)) = 0$.

Proof. Let $\{h_{G,K}\}$ be the approximate identity for $C_b(E)$ described in the proof of Theorem 2.5. So $\{h_{G,K}\}$ converges pointwise to χ_E and $\{\Gamma(h_{n,K})\}$ converges weak-star to 1. Thus

$$\begin{aligned}
\lim_{G,K} \int_{\beta E} h_{G,K}\, d\nu &= \int_{\beta E} \chi_E\, d\nu = \nu(E) \\
&= \lim_{G,K} \int_X \Gamma(h_{G,K})\, d\mu = \lim_{G,K} \int_X h_{G,K} \circ s\, d\mu \\
&= \lim_{G,K} \int_{\beta E} h_{G,K} \circ s \circ \eta^*\, d\nu = \nu(E \cap \operatorname{Range}(s)).
\end{aligned}$$

Now let $e \in \operatorname{Range}(s) \cap E$. Thus $e = s(x)$ for some $x \in X$ and $\Pi(e) = \Pi \circ s(x) = s^{-1} \circ s(x) = x$. Therefore, $\Pi(\operatorname{Range}(s) \cap E) = s^{-1}(\operatorname{Range}(s) \cap E)$. By definition of $\nu \in \mathbf{R}$, we have

$$\begin{aligned}
1 = \nu(\operatorname{Range}(s) \cap E) &\leq \mu(\Pi(\operatorname{Range}(s) \cap E)) = \mu(s^{-1}(\operatorname{Range}(s) \cap E)) \\
&\leq \mu(s^{-1}(\operatorname{Range}(s)) \cap s^{-1}(E)) \leq \mu(s^{-1}(\beta E) \cap s^{-1}(E)) \leq 1.
\end{aligned}$$

The set $s^{-1}(\beta E)$ is a $\hat{\mu}$-measurable set, so by definition

$$\hat{\mu}(s^{-1}(\beta E)) = \hat{\mu}(s^{-1}(\beta E) \cap s^{-1}(E)) + \hat{\mu}(s^{-1}(\beta E) \cap (\beta X \setminus s^{-1}(E))$$

and by DeMorgan's Laws, we obtain

$$0 = \hat{\mu}(s^{-1}(\beta E) \cap (\beta X \setminus s^{-1}(E))) = \hat{\mu}(s^{-1}(\beta E \setminus E)).$$

□

Therefore there exists a Borel measurable function s defined almost everywhere (μ) on X such that $\Pi(s(x)) = x$ and $\Gamma f(x) = f(s(x))$ almost everywhere (μ) on X.

Note that all the previous results hold for any finite measure μ; a probability measure was used only for convenience. Now I will establish the alternate proof of Lemma 1.2.

Theorem 2.8 *Let E be an analytic metric space, X a separable metric space, μ a σ-finite regular Borel measure on X, $\Pi : E \to X$ a continuous map with $\mu(X \setminus \Pi(E)) = 0$. There exists a Borel set $F \subseteq X$ with $\mu(X \setminus F) = 0$ and a Borel cross-section $s : F \to E$ for Π.*

Proof. Let $\{H_n\}$ be a pairwise disjoint sequence of closed sets in X such that $\mu(H_n) < \infty$ for each n and $\mu(X \setminus (\cup_{n=1}^{\infty} H_n)) = 0$. Let $E_n = \Pi^{-1}(H_n)$. Clearly $\Pi\ |_{E_n}: E_n \to H_n$ is a continuous map with $\mu(H_n \setminus \Pi(E_n)) = 0$. By Theorem 2.3, there exists a minimal measure ν_n on E_n with respect to $\mu\ |_{H_n}$, which induces a representation $\Gamma_n : C_b(E_n) \to L^\infty(\mu\ |_{H_n})$ satisfying the hypotheses of Theorem 2.5. Using Theorem 2.7 and Lemma 2.7, there exists a Borel set $F_n \subseteq H_n$ with $\mu(H_n \setminus F_n) = 0$ and a Borel cross-section $s_n : F_n \to E_n$ for $\Pi\ |_{E_n}$.

Now let $F = \cup_{n=1}^{\infty} F_n$ and define $s : F \to E$ by $s(x) = s_n(x)$ for $x \in H_n$. Clearly F is Borel with $\mu(X \setminus F) = 0$ and s is a Borel cross-section for Π. □

Note that $C_b(E_n)$ is not necessarily a subspace of $C_b(E)$, because extensions of continuous functions on E_n to all of E need not be unique. That is why the previous theorem does not refer to a representation from $C_b(E)$ to $L^\infty(\mu)$.

Now define an equivalence relation $\cong$ on the set of all Borel cross-sections for Π as follows: $s_1 \cong s_2$ if and only if $s_1 = s_2$ except on a set of μ-measure 0. The set of equivalence classes of the Borel cross-sections for Π are in one-to-one correspondence with the representations described in Theorem 2.5. Suppose Γ_1 and Γ_2 are two such distinct representations. Choose $f \in C_b(E)$ such that $\Gamma_1(f)$ and $\Gamma_2(f)$ differ on a closed set $F \subseteq X$ with $\mu(F) > 0$ and $\Gamma_i(f) \mid_F$ continuous for $i = 1, 2$. Let ν_1 and ν_2 be minimal measures and s_1 and s_2 be Borel cross-sections induced by Γ_1 and Γ_2, respectively. Let

$$M = \Pi[\text{support}(\nu_1 \mid_{\Pi^{-1}(F)})] \cap \Pi[\text{support}(\nu_2 \mid_{\Pi^{-1}(F)})].$$

Now support$(\mu \mid_F)$ is carried by $\Pi[\text{support}(\nu_i \mid_{\Pi^{-1}(F)})]$ for $i = 1, 2$; so $\mu(M) = \mu(F) > 0$. If $x \in M$, then $\Gamma_1 f(x) = f(s_1(x)) \neq f(s_2(x)) = \Gamma_2 f(x)$. Thus s_1 and s_2 differ on a set of positive measure, so s_1 and s_2 are not equivalent. Therefore, we have shown that equivalent Borel cross-sections induce equivalent representations. Notice also that support$(\nu_1 \mid_{\Pi^{-1}(F)}) \cap$ support$(\nu_2 \mid_{\Pi^{-1}(F)}) \cap \Pi^{-1}(M)$ is the empty set with support$(\nu_i \mid_{\Pi^{-1}(F)})$ carried by $\Pi^{-1}(M)$ for each i, so the measures cannot be equivalent. Therefore, if two minimal measures are equivalent, then they must induce the same representation.

Now we will see that non-equivalent Borel cross-sections induce distinct representations. Suppose that s_1 and s_2 are each Borel cross-sections for Π defined almost everywhere μ for Π which differ on a closed set F with $\mu(F) > 0$. For $i = 1, 2$ define maps $\Gamma_i : C_b(E) \to L^\infty(\mu)$ by $\Gamma_i f(x) = f(s_i(x))$ on the domain of s_i. By inspection each Γ_i is a representation like those described in Theorem 2.5. Let ν_1 and ν_2 be the minimal measures inducing Γ_1 and Γ_2, respectively. By construction, $s_1(F) \cap s_2(F)$ is empty and $\nu_i(s_i(F)) = \nu_i(\text{Range}(s) \cap \Pi^{-1}(F)) = \nu_i(\Pi^{-1}(F)) = \mu(F)$ for $i = 1, 2$. By the proof of Corollary 2.1, we can choose closed sets $K_i \subseteq s_i(F)$ such that $\mu(\Pi(K_i)) > \frac{2\mu(F)}{3}$. Thus $\mu[\Pi(K_1 \cap K_2)] > \frac{\mu(F)}{3}$. By Urysohn's Lemma there exists $f \in C_b(E)$ such that $f(K_1) = 0$ and $f(K_2) = 1$, and $\Gamma_1(f)$ and $\Gamma_2(f)$ differ on $\Pi(K_1 \cap K_2)$, which is a set of positive measure and so Γ_1 and Γ_2 are distinct representations. Lemma 2.4 established that non-equivalent measures induce distinct representations. Thus we have proven the following theorem.

Theorem 2.9 *Let E be an analytic metric space, X a separable metric space with regular Borel probability measure μ, and $\Pi : E \to X$ a continuous map with $\mu(X \setminus \Pi(E)) = 0$. The equivalence classes of the Borel cross-sections of Π are in one-to-one correspondence with the distinct representations of the form $\Gamma : C_b(E) \to L^\infty(\mu)$ which satisfy the hypotheses of Theorem 2.5. These representations are in one-to-one correspondence with the equivalence classes of the minimal measures in* $\mathbf{R}$.

Chapter 3

The Borel Case

In this chapter we will prove Schwartz's lemma with a weaker hypothesis, using an onto Borel map instead of a continuous map. (Recall that $\Pi : E \to X$ is a Borel map if $\Pi^{-1}(B)$ is a Borel set in E for every Borel set $B \subseteq X$.) Throughout the chapter E is an analytic metric space, X is a separable metric space, $\Pi : E \to X$ is an onto Borel map, and μ is a regular Borel probability measure on X.

Let $\mathbf{B}(E)$ be the bounded Borel functions on E with the supremum norm; $\|f\|_\infty = \sup_{e\in E} |f(e)|$. In this chapter we are interested in representations of the form $\Gamma : \mathbf{B}(E) \to L^\infty(\mu)$ such that $\Gamma(f \circ \Pi) = f$ for every $f \in C_b(X)$. The normed space $\mathbf{B}(E)$ is more difficult to work with than $C_b(E)$, the C*-algebra used in the previous chapter, so the focus of this chapter will be to reduce the Borel problem to the continuous one, which was solved in Chapter 2.

Lemma 3.1 *The space $\mathbf{B}(E)$ with the norm $\|f\|_\infty = \sup_{z\in E} |f(z)|$ is a C^*-algebra.*

Proof. Let $\{f_n\}$ be Cauchy in $\mathbf{B}(E)$. For each z in E, $\{f_n(z)\}$ is a Cauchy sequence in $\mathbb{C}$, the complex numbers, so $\{f_n(z)\}$ must converge to a complex number $f(z)$. First we want to show that $\{f_n\}$ converges in norm to f.

Let $\epsilon > 0$ and choose N such that $\|f_n - f_m\|_\infty < \frac{\epsilon}{2}$ for every $n, m > N$. Fix

$z \in E$. By the triangle inequality, we have

$$|f_n(z) - f(z)| \leq \|f_n - f_m\|_\infty + |f_m(z) - f(z)|.$$

Now choose M such that $|f_m(z) - f(z)| < \frac{\epsilon}{2}$ for all $m > M$. Let $m > \max(M, N)$. By the inequality above, $|f_n(z) - f(z)| < \epsilon$ for all $n > N$. The integer N was chosen independently of z, so we have $\|f_n - f\|_\infty < \epsilon$.

By [6, Theorem 6, p. 259] f is Borel measurable and thus $\mathbf{B}(E)$ is complete. □

Define a Borel measure $\hat{\mu}$ on the Stone-Cech compactification βX of X by $\hat{\mu}(B) = \mu(B \cap X)$ for every Borel set B in βX. Thus $\hat{\mu}$ is a regular Borel probability measure on βX with $\hat{\mu}(\beta X \setminus X) = 0$ and $\hat{\mu}\ |_X = \mu$. Therefore, $L^\infty(\hat{\mu})$ and $L^\infty(\mu)$ are isometrically isomorphic.

Let $\mathbf{M}_{\mathbf{B}(E)}$ be the maximal ideal space of $\mathbf{B}(E)$ and consider the map $\delta : E \rightarrow \mathbf{M}_{\mathbf{B}(E)}$ defined by $\delta(z) =$ evaluation at z. In the weak-star topology on $\mathbf{M}_{\mathbf{B}(E)}$, subbasic open sets are of the form $G_f = \{\phi : |\phi(f)| < 1\}$ for some $f \in \mathbf{B}(E)$. The set $\delta^{-1}(G_f) = \{z \in E : |f(z)| < 1\}$ is Borel in E by definition. Therefore δ is a Borel map. The metric space E is completely regular, so subbasic open sets have the form $U_f = \{z : |f(z)| < 1\}$ for some $f \in C_b(E)$. Thus $\delta(U_f) = \delta(E) \cap \{\phi : |\phi(f)| < 1\}$, which is relatively open in $\delta(E)$. Therefore, the map $\delta : E \rightarrow \delta(E)$ is a Borel isomorphism. (That is, both the map and its inverse are Borel.) However, δ is not a continuous map. To see this, fix e_0 in E and let $\{e_n\}$ be a sequence in E converging to e_0. Define a function f on E by $f(e) = 1$ if $e = e_0$, $f(e) = 0$ otherwise. The function $f \in \mathbf{B}(E)$, and

$$\lim_{n\to\infty} \delta(e_n)(f) = 0 \neq 1 = \delta(e_0)(f).$$

The maximal ideal space $\mathbf{M}_{\mathbf{B}(E)}$ is continuously embedded in $C(\mathbf{M}_{\mathbf{B}(E)})^*$, by the map taking $\phi \in \mathbf{M}_{\mathbf{B}(E)}$ to evaluation at ϕ in $C(\mathbf{M}_{\mathbf{B}(E)})^*$. Since $\mathbf{B}(E)$ is a $*$-Banach algebra, it is isometrically isomorphic to $C(\mathbf{M}_{\mathbf{B}(E)})$ [2, Theorem 4.29 (Gelfand-Naimark), pp.92–93]. Recall also that $\mathbf{M}_{\mathbf{B}(E)}$ is compact in the weak-star topology for $(\mathbf{B}(E))^*$, the dual of $\mathbf{B}(E)$ [2, Proposition 2.23, pp. 39–40].

From the Stone-Cech theorem, $C_b(X)$ and $C(\beta X)$ are isometrically isomorphic [8, Theorem A, p. 141], so I will use them interchangably.

Let $\eta : C_b(X) \to \mathbf{B}(E)$ be defined by $\eta(f) = f \circ \Pi$. The map Π is onto, so η is a $*$-isometry. Since $\mathbf{B}(E)$ is isometrically isomorphic to $C(\mathbf{M}_{\mathbf{B}(E)})$, we may consider η a $*$-isometry from $C(\beta X)$ to $C(\mathbf{M}_{\mathbf{B}(E)})$. The adjoint $\eta^* : C(\mathbf{M}_{\mathbf{B}(E)})^* \to C(\beta X)^*$ is also a $*-$isometry. It is easy to check by adjoint notation that $\eta^* |_{\delta(E)} = \Pi \circ \delta^{-1}$.

Note 3.1 Range$(\eta^* |_{\mathbf{M}_{\mathbf{B}(E)}}) = \beta X$.

Proof. By definition η is multiplicative, and by further use of adjoint notation it can be shown that η^* is multiplicative also. Let $\phi \in \mathbf{M}_{\mathbf{B}(E)}$, and let $f_1, f_2 \in C(\mathbf{M}_{\mathbf{B}(E)})$. Thus

$$\eta^*(\phi(f_1 f_2)) = \eta^*(f_1(\phi) f_2(\phi)) = \eta^*(f_1(\phi))\eta^*(f_2(\phi)) = \eta^*(\phi(f_1))\eta^*(\phi(f_2))$$

so $\eta^*(\phi)$ is a multiplicative linear functional on $C(\beta X)$, and therefore a multiplicative measure on βX. Recall that a multiplicative measure on the compact space βX must be a point evaluation on βX. Since Range$(\Pi) = X$, we have

$$X \subseteq \text{Range}(\eta^* |_{(\mathbf{M}_{\mathbf{B}(E)})}) \subseteq \beta X.$$

By the continuity of η^*, the set $\eta^*(\mathbf{M}_{\mathbf{B}(E)})$ is compact, so it must equal βX, the smallest compact set containing X. □

As in the previous two chapters, let $\mathbf{R}$ be the set of all regular positive Borel measures τ on E such that $\tau(\Pi^{-1}(B)) = \mu(B)$ for every Borel set $B \subseteq X$. Let $\mathbf{R}$ also have the same ordering as before. The proof of the following theorem will establish that $\mathbf{R}$ is non-empty and has a maximal element, again called the minimal measure.

A net $\{t_\alpha\}$ is an **approximate identity** for $\mathbf{B}(E)$ if it is contained in $\mathbf{B}(E)$, $0 \leq t_\alpha \leq 1$ for all α, and $\lim_\alpha t_\alpha(z) = 1$ for all $z \in E$.

Theorem 3.1 *Let $\Gamma : \mathbf{B}(E) \to L^\infty(\mu)$ be a continuous algebra homomorphism such that*

1. *$\Gamma(f \circ \Pi) = f$ for each $f \in C_b(X)$*
2. *If $\{t_\alpha\}$ is an approximate identity for $\mathbf{B}(E)$, then $\Gamma(t_\alpha)$ converges weak-star to 1.*

There exists a minimal measure ν on E which induces Γ.

Proof. As discussed prior to the theorem, the spaces $L^\infty(\mu)$ and $L^\infty(\hat{\mu})$ are equivalent, as are $\mathbf{B}(E)$ and $C(\mathbf{M}_{\mathbf{B}(E)})$, so we may consider Γ a map from $C(\mathbf{M}_{\mathbf{B}(E)})$ to $L^\infty(\hat{\mu})$. Thus we have the relationship pictured in the following diagram, in which $\iota : C(\beta X) \to L^\infty(\hat{\mu})$ is the identity map, and $\eta^* : \mathbf{M}_{\mathbf{B}(E)} \to \beta X$ is a continuous onto map. Thus the conditions for Theorem 1.5 are met, and therefore there exists a minimal measure ν_δ on $\mathbf{M}_{\mathbf{B}(E)}$ which induces Γ.

$$\begin{tikzcd}
C(\mathbf{M}_{\mathbf{B}(E)}) \arrow[dr, "\Gamma"] & \\
C(\beta X) \arrow[u, "\eta"] \arrow[r, "\iota"] & L^\infty(\hat{\mu})
\end{tikzcd}$$

Next we want to show that $\nu_\delta(\mathbf{M}_{\mathbf{B}(E)} \setminus \delta(E)) = 0$. Let $K \subseteq \delta(E)$ be compact, and let $G \supseteq \delta(E)$ be open. By Urysohn's Lemma there exists a function $h_{G,K}$ in $C(\mathbf{M}_{\mathbf{B}(E)})$ with $0 \leq h_{G,K} \leq 1$, $h_{G,K}(K) = 1$, and $h_{G,K}(\mathbf{M}_{\mathbf{B}(E)} \setminus G) = 0$. Order (G, K) as follows: $(G_1, K_1) \leq (G_2, K_2)$ if $G_1 \supseteq G_2$ and $K_1 \subseteq K_2$. By inspection $\{h_{G,K}\}$ converges pointwise to $\chi_{\delta(E)}$ and $\{h_{G,K}\}$ is an approximate identity for $\mathbf{B}(E)$. By hypothesis $\Gamma(h_{G,K})$ converges weak-star to 1. So we have

$$\begin{aligned}
\lim_{G,K} \int_{\mathbf{M}_{\mathbf{B}(E)}} h_{G,K}\, d\nu_\delta &= \lim_{G,K} \int_{\beta X} \Gamma(h_{G,k})\, d\hat{\mu} = 1 \\
&= \int_{\mathbf{M}_{\mathbf{B}(E)}} \chi_{\delta(E)}\, d\nu_\delta = 1
\end{aligned}$$

and $\nu_\delta(\mathbf{M}_{\mathbf{B}(E)} \setminus \delta(E)) = 0$.

It can also be shown, by similar manipulations to those in the proof of Theorem 2.5, that ν_δ induces the obvious minimal measure ν on E, and that Γ is the representation induced by ν. $\square$

Proposition 3.1 *Let $\Gamma : \mathbf{B}(E) \to L^\infty(\mu)$ be a representation satisfying the hypotheses of Theorem 3.1. As in the proof of that theorem, we can consider Γ to be a representation from $C(\mathbf{M}_{\mathbf{B}(E)})$ to $L^\infty(\hat{\mu})$. Let ν_δ be the minimal measure on $\mathbf{M}_{\mathbf{B}(E)}$*

inducing Γ. *For every* $f \in C(\mathbf{M}_{\mathbf{B}(E)})$,

$$\sigma(\Gamma(f)) = f(\text{support}(\nu_\delta))$$

where $\sigma(\Gamma(f))$ *is the spectrum of* $\Gamma(f)$ *in* $L^\infty(\hat{\mu})$.

Proof. Since $\mathbf{M}_{\mathbf{B}(E)}$ and βX are compact and $\eta^* : \mathbf{M}_{\mathbf{B}(E)} \to \beta X$ is continuous, the proof of Proposition 1.2 will work in this case also. □

If ν is a minimal measure on E, then ν induces the obvious minimal measure ν_δ on $\mathbf{M}_{\mathbf{B}(E)}$, where $\nu_\delta(\mathbf{M}_{\mathbf{B}(E)} \setminus \delta(E)) = 0$ and $\nu_\delta(B) = \nu(\delta^{-1}(B))$ for every Borel set $B \subseteq \delta(E)$. By Theorem 1.4 the minimal measure ν_δ induces a representation $L_{\nu_\delta} : C(\mathbf{M}_{\mathbf{B}(E)}) \to L^\infty(\hat{\mu})$ such that

1. $L_{\nu_\delta}(f \circ \eta^*) = f$ for every $f \in C(\beta X)$
2. If $\{t_\alpha\}$ is an approximate identity for $C(\mathbf{M}_{\mathbf{B}(E)})$, then the net $L_{\nu_\delta}(t_\alpha)$ converges weak-star to 1.

Using the equivalence of $C(\mathbf{M}_{\mathbf{B}(E)})$ and $\mathbf{B}(E)$ and the equivalence of $L^\infty(\hat{\mu})$ and $L^\infty(\mu)$, we see that L_{ν_δ} induces a representation $L_\nu : \mathbf{B}(E) \to L^\infty(\mu)$ which satisfies the conditions of Theorem 3.1. By Lemma 2.4, non-equivalent minimal measures induce distinct representations, and by the discussion following Lemma 2.7, equivalent minimal measures induce the same representation. Therefore, we have established the following corollary:

Corollary 3.1 *The representations described in Theorem 3.1 are in 1-1 correspondence with the equivalence classes of the minimal measures on* E.

The proof of the following theorem is based on the proof of [5, Theorem 42, pp. 59–60], and thus is similar to the proof of Theorem 2.5.

Theorem 3.2 *Let $\Gamma : \mathbf{B}(E) \to L^\infty(\mu)$ be as in Theorem 3.1. There exists a Borel set $F \subseteq X$ with $\mu(X \setminus F) = 0$ and a Borel cross-section $s : F \to E$ for Π such that*

1. *$\Pi \circ s(x) = x$ for every $x \in F$.*

2. *$(\Gamma f(x)) = f \circ s(x)$ almost everywhere (μ) for every $f \in \mathbf{B}(E)$.*

Proof. Recall the setup of the proof of Theorem 3.1. The map $\eta^* : \mathbf{M}_{\mathbf{B}(E)} \to \beta X$ is continuous and onto, $\eta : C(\beta X) \to C(\mathbf{M}_{\mathbf{B}(E)})$ is a C*-isometry, and $\Gamma : C(\mathbf{M}_{\mathbf{B}(E)}) \to L^\infty(\hat{\mu})$ is a representation as described in Theorem 2.5. Let ν_δ be the minimal measure from the proof of Theorem 3.1 which induces Γ. Note that since $\mathbf{M}_{\mathbf{B}(E)}$ is compact, we have

$$\eta^*(\text{support}(\nu_\delta \mid_{(\eta^*)^{-1}(H)})) = \text{support}(\hat{\mu} \mid_H)$$

for every closed set $H \subseteq \beta X$.

Fix $x \in \text{support}(\mu)$ and define $\boldsymbol{\Delta} = \{f \in C(\mathbf{M}_{\mathbf{B}(E)}) :$ there exists compact $K \subseteq \beta X$ with full density at x such that $\Gamma f \mid_K$ is continuous$\}$. As in the previous chapter, $\hat{\mu}(\beta X \setminus X) = 0$, so for $x \in X$, the density of $K \subseteq \beta X$ is well defined. Define

$$S_x = \cap_{f \in \boldsymbol{\Delta}} \{y \in (\eta^*)^{-1}(x) \cap \text{support}(\nu_\delta \mid_{(\eta^*)^{-1}(K)})\}$$

where K is a compact set with respect to f as defined in $\boldsymbol{\Delta}$ above. By the same method as used in the proof of Theorem 2.7, the set S_x is non-empty. Thus for each $x \in \text{support}(\mu)$, we can choose $s(x) \in S_x$. Fix $f \in C(\mathbf{M}_{\mathbf{B}(E)})$. Using Lusin's

Theorem, we may obtain a sequence $\{K_n\}$ of closed sets in βX such that $\Gamma f\mid_{K_n}$ is continuous and $\lim_{n\to\infty}\hat{\mu}(K_n) = 1$. By Lemma 2.6, each K_n has full density with respect to $\hat{\mu}$ almost everywhere, so $(\Gamma f)(x) = f \circ s(x)$ almost everywhere $(\hat{\mu})$ on βX.

Now $\mathbf{M}_{\mathbf{B}(E)}$ is a compact Hausdorff space, so it is completely regular, having the weak topology generated by $C(\mathbf{M}_{\mathbf{B}(E)})$. Thus a subbasic open set in $\mathbf{M}_{\mathbf{B}(E)}$ has the form $G_f = \{\phi : |\phi(f)| < 1\}$ for some $f \in \mathbf{B}(E)$. Since $C(\mathbf{M}_{\mathbf{B}(E)})$ is isomorphic to $\mathbf{B}(E)$, we can think of G_f as $f^{-1}(|z| < 1 : z \in \mathbb{C})$, and we have

$$s^{-1}(G_f) = s^{-1} \circ f^{-1}(|z| < 1) = (f \circ s)^{-1}(|z| < 1).$$

Since $f \circ s = \Gamma(f)$ almost everywhere (μ), we have that $f \circ s$ is Borel measurable and $s^{-1}(G_f)$ is Borel. Thus $s : \text{support}(\mu) \to \mathbf{M}_{\mathbf{B}(E)}$ is Borel measurable.

Using the same method as Lemma 2.7, we find that $\nu_\delta(s^{-1}(\mathbf{M}_{\mathbf{B}(E)} \setminus \delta(E)) = 0$, so there exists a Borel set $F \subseteq X$ with $\mu(X \setminus F) = 0$ and $s(F) \subseteq \delta(E)$. Therefore, $\delta^{-1} \circ s : F \to E$ satisfies the theorem. □

Note that $\mathbf{M}_{\mathbf{B}(E)}$ may be non-metrizable, but $\eta^* : \mathbf{M}_{\mathbf{B}(E)} \to \beta X$ has a Borel cross-section defined almost everywhere (μ). However, the metric on X is necessary, because of the use of the definition of full density. This is the reason that we did not find a Borel cross-section for the continuous onto map $p : Y \to Z$ described in Problem 1.1, where Y and Z are abstract compact sets without metrics.

Theorem 3.3 *Let E be an analytic metric space, let X be a separable metric space, let μ be a σ-finite regular Borel measure on X, and let $\Pi : E \to X$ be an onto Borel map. There exists a Borel set $F \subseteq X$ such that $\mu(X \setminus F) = 0$ and a Borel* *cross-section $s : F \to E$ for* Π.

Proof. Recall the maps used in the proof of Theorem 3.1, in which $\delta : E \to \mathbf{M}_{\mathbf{B}(E)}$ is a Borel isomorphism, and $\eta^* : \delta(E) \to X$ is continuous onto map, with $\eta^* = \Pi \circ \delta^{-1}$. Then by Theorem 2.8, there exists a Borel set $F \subseteq X$ with $\mu(X \backslash F) = 0$ and a Borel cross-section $\hat{s} : F \to \delta(E)$ for η^*. Note that although $\delta(E)$ may be non-metrizable, the proof of Theorem 2.8 does not require a metric on the domain space of η^*.

Now let $s = \hat{s} \circ \delta^{-1}$, and the theorem is proven. □

Theorem 3.4 *Let E be an analytic metric space, let X be a separable metric space with a regular Borel probability measure μ, and let $\Pi : E \to X$ be an onto Borel map. The equivalence classes of the Borel cross-sections for Π defined almost everywhere μ are in one-to-one correspondence with the representations described in Theorem 3.1. These representations are in one-to-one correspondence with the equivalence classes of the minimal measures in* $\mathbf{R}$.

Proof. Since we have reduced the problem to the continuous case, the same arguments that were used in the discussion following Lemma 2.7 show that non-equivalent Borel cross-sections are induced by distinct representations, and that equivalent cross-sections are induced by the same representation. □

Index of Symbols

- $\mathbf{B}(E)$ = the set of all bounded Borel functions on E.
- βX = the Stone-Cech compactification of X.
- $\mathbb{C}$ = the set of complex numbers.
- $C_b(E)$ = the set of all bounded continuous functions on E.
- $C(E)$ = the set of all continuous functions on E.
- $C(Y)^*$ = the dual of $C(Y)$, which is the set of all bounded linear functionals on $C(Y)$.
- $H^\infty(D)$ = the bounded analytic functions on the complex unit disc D.
- $\mathbf{M}_{\mathbf{B}(E)}$ = the maximal ideal space of $\mathbf{B}(E)$.
- $\mathbf{M}_{C_b(E)}$ = the maximal ideal space of $C_b(E)$.
- $\mathbf{M}_{H^\infty(D)}$ = the maximal ideal space of $H^\infty(D)$.
- $M(E)$ = the set of finite regular Borel measures on E.
- $M^+(E)$ = the set of positive finite regular Borel measures on E.
- $\mathbb{N}^\infty$= the infinite product of the natural numbers $\mathbb{N}$, a complete separable metric space with metric defined in Chapter 1.

- χ_A = the characteristic function of A, that is, $\chi_A(x) = 1$ if $x \in A$, and $\chi_A(x) = 0$ otherwise.

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