HOMOTOPY ALGORITHMS FOR $H^2/H^\infty$ CONTROL
ANALYSIS AND SYNTHESIS

by

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(ABSTRACT)

The problem of finding a reduced order model, optimal in the $H^2$ sense, to a given system model is a fundamental one in control system analysis and design. The addition of a $H^\infty$ constraint to the $H^2$ optimal model reduction problem results in a more practical yet computationally more difficult problem. Without the global convergence of homotopy methods, both the $H^2$ optimal and the combined $H^2/H^\infty$ model reduction problems are very difficult.

For both problems homotopy algorithms based on several formulations — input normal form; Ly, Bryson, and Cannon's 2x2 block parametrization; a new nonminimal parametrization — are developed and compared here. For the $H^2$ optimal model order reduction problem, these numerical algorithms are also compared with that based on Hyland and Bernstein's optimal projection equations.

Both the input normal form and Ly form are very efficient compared to the over-parametrization formulation and the optimal projection equations approach, since they utilize the minimal number of possible degrees of freedom. However, they can fail to exist or be very ill conditioned. The conditions under which the input normal form and the Ly form become ill conditioned are examined.

The over-parametrization formulation solves the ill conditioning issue, and usually is more efficient than the approach based on solving the optimal projection equations for the $H^2$ optimal model reduction problem. However, the over-parametrization formulation
introduces a very high order singularity at the solution, and it is doubtful whether this singularity can be overcome by using interpolation or other existing methods.

Although there are numerous algorithms for solving Riccati equations, there still remains a need for algorithms which can operate efficiently on large problems and on parallel machines and which can be generalized easily to solve variants of Riccati equations. This thesis gives a new homotopy-based algorithm for solving Riccati equations on a shared memory parallel computer. The central part of the algorithm is the computation of the kernel of the Jacobian matrix, which is essential for the corrector iterations along the homotopy zero curve. Using a Schur decomposition the tensor product structure of various matrices can be efficiently exploited. The algorithm allows for efficient parallelization on shared memory machines.

The linear-quadratic-Gaussian (LQG) theory has engendered a systematic approach to synthesize high performance controllers for nominal models of complex, multi-input multi-output systems and hence it is a breakthrough in modern control theory. Homotopy algorithms for both full and reduced-order LQG controller design problems with an $H^\infty$ constraint on disturbance attenuation are developed. The $H^\infty$ constraint is enforced by replacing the covariance Lyapunov equation by a Riccati equation whose solution gives an upper bound on $H^2$ performance. The numerical algorithm, based on homotopy theory, solves the necessary conditions for a minimum of the upper bound on $H^2$ performance. The algorithms are based on two minimal parameter formulations: Iy, Bryson, and Cannon's $2 \times 2$ block parametrization and the input normal Riccati form parametrization. An over-parametrization formulation is also proposed. Numerical experiments suggest that the combination of a globally convergent homotopy method with a minimal parameter formulation applied to the upper bound minimization gives excellent results for mixed-norm synthesis.
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1. INTRODUCTION.

1.1. The $H^2$ optimal model order reduction problem.

The $H^2$ optimal model order reduction problem, i.e., the problem of approximating a higher order dynamical system by a lower order one so that a quadratic model reduction criterion is minimized, is of significant importance and is under intense study. Several earlier attempts to apply homotopy methods to the $H^2$ optimal model order reduction problem were not entirely satisfactory. Richter and Collins [58]–[60] devised a homotopy approach which only estimated certain crucial partial derivatives and employed relatively crude curve tracking techniques. Žigić, Bernstein, Collins, Richter, and Watson [78]–[80] formulated the problem so that numerical linear algebra techniques could be used to explicitly calculate partial derivatives, and employed sophisticated homotopy curve tracking algorithms, but the number of variables made large problems intractable. Several ways are proposed here to reduce the dimension of the homotopy map so that large problems are computationally feasible.

The problem can be formulated as: given the asymptotically stable, controllable, observable, time invariant, continuous time system

\[
\dot{x}(t) = Ax(t) + Bu(t), \\
y(t) = Cx(t),
\]

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{l \times n}$, the goal is to find a reduced order model

\[
\dot{x}_m(t) = A_m x_m(t) + B_m u(t), \\
y_m(t) = C_m x_m(t),
\]

where $A_m \in \mathbb{R}^{n_m \times n_m}$, $B_m \in \mathbb{R}^{n_m \times m}$, $C_m \in \mathbb{R}^{l \times n_m}$, $n_m < n$, which minimizes the cost function

\[
J(A_m, B_m, C_m) \equiv \lim_{t \to \infty} E \left[ (y - y_m)^T R (y - y_m) \right],
\]

where the input $u(t)$ is white noise with symmetric and positive definite intensity $V$ and $R$ is a symmetric and positive definite weighting matrix.
The optimal projection equations of Hyland and Bernstein [31], [32], described in Chapter 3, are basis independent and correspond to the maximum number of degrees of freedom. Richter and Collins [60] use this maximum number, and Žigić [78] reduced it somewhat. At the other extreme, the minimum number of degrees of freedom corresponds to the input normal form described in Chapter 2, and developed into a probability-one homotopy algorithm. Subtle differences between the optimal projection equations and input normal form formulations are explored in Chapter 3. Assuming a particular Jordan form for $A_m$ leads to the minimal parameter formulation of Ly et al. [47], which is developed into a probability-one homotopy algorithm in Chapter 4. Chapter 5 gives numerical results for the input normal form and Ly form homotopies on the test set of Žigić [78].

Both the input normal form and Ly parameterization use the minimum possible number of degrees of freedom, but rely on assumptions about the structure of $(A_m, B_m, C_m)$ that do not always hold, and therefore may not exist. Even worse, they may exist but be arbitrarily badly ill conditioned, resulting in unstable numerical algorithms. Chapter 6 explores an alternative formulation using more than the minimal number of degrees of freedom, and compares to the minimal formulations. Comparisons between the three formulations and the optimal projection equations approach are given in Chapter 7. A fundamental difference between the optimal projection equations and the other formulations is that the optimal projection equations approach solves $f(x) = 0$ where $f$ is not the gradient of the cost functional and $x$ is not the reduced order model, while the other three formulations solve $g(y) = 0$ where $g$ is the gradient of the cost functional and $y$ is the reduced order model.

1.2. The combined $H^2/H^\infty$ model reduction problem.

In practice to simplify a plant for control design or to simplify a controller for ease of implementation, a $H^\infty$ role must be taken into account, i.e., the order reduction approach should approximate the system frequency response to the greatest extent possible.

Several order reduction techniques have been proposed for approximating the frequency response of a given system. For example, frequency weighting has been studied in [17] in conjunction with balancing [49]. Moreover, Hankel norm reduction has been shown to have
fundamental ramifications for frequency domain approximation [1], [3], [22]. An overview
and discussion of these ideas is given in [12].

The approach of [27], which is based upon a state space $H^\infty$ formulation, is used here.
In particular, by using a Riccati equation to enforce an $H^\infty$ constraint on the norm of
the reduction error in conjunction with an $H^2$ upper bound or entropy cost [54], it was
shown in [27] that $H^\infty$ constrained reduced order systems can be characterized by necessary
conditions for optimality of the $H^2$ upper bound. The resulting algebraic conditions, which are
a generalization of the "pure" $H^2$ optimality conditions given in [31], consist of nonstandard
coupled Riccati and Lyapunov type matrix equations.

The purpose of this work is to make significant progress in developing novel,
stable, globally convergent numerical algorithms for solving the optimality conditions for
$H^2/H^\infty$ order reduction given in [27]. The approach we take is based on the construction
of probability-one homotopy maps, similar to those developed for the $H^2$ order reduction
problem in [19], [20].

The problem is formulated as: given the controllable and observable, time invariant,
continuous time system

$$
\dot{x}(t) = Ax(t) + B Du(t),
$$

$$
y(t) = C x(t),
$$

where $t \in [0, \infty)$, $A \in \mathbb{R}^{n \times n}$ is asymptotically stable, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{l \times n}$, $D \in \mathbb{R}^{m \times p}$
$m \leq p$) and the input $Du(t)$ is white noise with symmetric and positive definite intensity
$V \equiv DD^T$, find a $n_m$-th order model ($n_m < n$)

$$
\dot{x}_m(t) = A_m x_m(t) + B_m Du(t),
$$

$$
y_m(t) = C_m x_m(t),
$$

where $A_m \in \mathbb{R}^{n_m \times n_m}$, $B_m \in \mathbb{R}^{n_m \times m}$, $C_m \in \mathbb{R}^{l \times n}$, which satisfies the following criteria:
(i) $A_m$ is asymptotically stable;
(ii) the transfer function of the reduced order model lies within $\gamma$ of the transfer function
of the full order model in the $H_\infty$ norm, i.e.,

$$
\|H(s) - H_m(s)\|_\infty \leq \gamma
$$
where
\[ H(s) \equiv EC(sI_n - A)^{-1}BD, \quad H_m(s) \equiv EC_m(sI_m - A_m)^{-1}B_mD, \]

\( \gamma > 0 \) is a given constant, \( E \in \mathbb{R}^{q \times l} \) \((q \geq l)\) is a given constant matrix; and

(iii) the \( H^2 \) model reduction criterion

\[ J(A_m, B_m, C_m) \equiv \lim_{t \to \infty} \mathcal{E} \left[ (y - y_m)^T R (y - y_m) \right] \]  \hspace{1cm} (7)

is minimized, where \( \mathcal{E} \) is the expected value and \( R = E^T E \) is a symmetric and positive definite weighting matrix.

1.3. The auxiliary minimization problem.

Define
\[ \tilde{n} \equiv n + n_m, \quad \tilde{E} \equiv E\tilde{C}, \quad \tilde{D} \equiv \tilde{B}D, \]

\[ \tilde{A} \equiv \begin{pmatrix} A & 0 \\ 0 & A_m \end{pmatrix}, \quad \tilde{B} \equiv \begin{pmatrix} B \\ B_m \end{pmatrix}, \quad \tilde{C} \equiv (C - C_m), \] \hspace{1cm} (8)

\[ \tilde{R} \equiv \tilde{E}^T \tilde{E} = \tilde{C}^T R \tilde{C} = \begin{pmatrix} C^T R C & -C^T R C_m \\ -C_m^T R C & C_m^T R C_m \end{pmatrix}, \]

\[ \tilde{V} \equiv \tilde{D} \tilde{D}^T = \tilde{B} \tilde{V} \tilde{B}^T = \begin{pmatrix} BV B_m^T \\ B_m V B_m^T \end{pmatrix}. \] \hspace{1cm} (9)

The full order system (4) and the reduced order system (5) can be written as a single augmented system

\[ \dot{\tilde{x}}(t) = \tilde{A} \tilde{x}(t) + \tilde{D} u(t), \]

\[ \tilde{y}(t) = \tilde{C} \tilde{x}(t). \] \hspace{1cm} (10)

Using this notation the cost \( J(A_m, B_m, C_m) \) can be written as

\[ J(A_m, B_m, C_m) = \lim_{t \to \infty} \mathcal{E} \left[ (y - y_m)^T R (y - y_m) \right] = \lim_{t \to \infty} \mathcal{E} (\tilde{y}^T R \tilde{y}) = \lim_{t \to \infty} \mathcal{E} (\tilde{x}^T \tilde{R} \tilde{x}) = \text{tr} \left( \tilde{Q} \tilde{R} \right), \] \hspace{1cm} (11)

where \( \tilde{Q} \) satisfies

\[ \tilde{A} \tilde{Q} + \tilde{Q} \tilde{A}^T + \tilde{V} = 0. \] \hspace{1cm} (12)
**Lemma 1** [27]. Let \((A_m, B_m, C_m)\) be given and assume there exists \(Q \in \mathbb{R}^{n \times n}\) satisfying

\[ Q \text{ is symmetric and nonnegative definite} \]  

and

\[ \tilde{A} Q + Q \tilde{A}^T + \gamma^{-2} \tilde{Q} \tilde{Q} + \tilde{V} = 0. \]  

Then

\[ (\tilde{A}, \tilde{D}) \text{ is stabilizable} \]  

if and only if

\[ A_m \text{ is asymptotically stable.} \]  

Furthermore, if (15) holds, then

\[ \|H(s) - H_m(s)\|_\infty \leq \gamma, \]  

\[ \tilde{Q} \leq Q \quad (Q - \tilde{Q} \text{ is nonnegative definite}), \]  

and

\[ \text{tr } \tilde{Q} \tilde{R} = J(A_m, B_m, C_m) \leq J(A_m, B_m, C_m) = \text{tr } Q \tilde{R}. \]

Hence the \(H_\infty\) constraint is automatically enforced when a nonnegative definite solution to (14) is known to exist. Furthermore, the solution \(Q\) provides an upper bound for the actual state covariance \(\tilde{Q}\) along with a bound on the \(H^2\) model reduction.

The satisfaction of (13)-(15) leads to (i) \(A_m\) stable; (ii) a bound on the \(H_\infty\) distance between the full order and reduced order systems; and (iii) an upper bound for the \(H^2\) model-reduction criterion. The auxiliary minimization problem is to determine \((A_m, B_m, C_m)\) that minimizes \(J(A_m, B_m, C_m)\) and thus provides a bound for the actual \(H^2\) criterion \(J(A_m, B_m, C_m)\).

\((A_m, B_m, C_m)\) is restricted to the set

\[ S \equiv \{(A_m, B_m, C_m) : \tilde{A} + \gamma^{-2} \tilde{Q} \tilde{R} \text{ is asymptotically stable,}\]  

\[ Q \text{ is symmetric positive definite,}\]  

and \((A_m, B_m, C_m)\) is controllable and observable \}.
In Chapter 8 the input normal form homotopy approach for solving the auxiliary minimization problem is described. The Ly form and over-parametrization formulations are developed in Chapters 9 and 10. Numerical results and comparisons between different formulations are given in Chapter 11.

1.4. A homotopy method for solving Riccati equations.

One of the most fundamental computational tasks arising in control theory, as well as related areas, is the numerical solution of Riccati equations and their variants. Since Riccati equations are central to modern linear-quadratic estimation and control design, their theoretical properties have been thoroughly studied ([34], [64], [24], [15], [25]). Several numerical solution techniques have been developed for Riccati equations including eigenvalue methods ([57], [40], [43], [44], [45], [16], [2], [55]), the Chandrasekhar algorithm ([10], [37], [33]), and the matrix sign function technique ([8]). Software for Riccati equations is widely available and is included in numerous control-design packages such as MATLAB. In spite of the advanced state of development of Riccati solvers and some very recent work ([26], [29], [67]), there remains a need for numerical methods which operate efficiently on large dimensional problems, on parallel machines, on more general types of Riccati equations, and especially for methods which are amenable to parallelization. Differing from most other methods for solving Riccati equations that are based on invariant subspaces, the method proposed here is based on globally convergent probability-one homotopy theory. Homotopy methods can often provide an effective means of solving modified Riccati equations as illustrated by the results of [58], [59], [48]. The globally convergent homotopy method proposed here can be easily generalized to solve the modified Riccati equations below, for which invariant subspace methods combined with a locally convergent Newton iterations are often ineffective.

Examples of modified Riccati equations are given by Chang and Peng [10]:

\[ 0 = A^T P + PA + Q - PBR^{-1}B^TP + \sum_{i=1}^{p} U(P, A_i); \]
Peterson and Hollot [56]:

$$0 = A^T P + PA + Q - PBR^{-1}B^T P + PEPE + D, \quad D \geq 0, \quad E \geq 0;$$

and Kosmidou and Bertrand [41]:

$$0 = A^T P + PA + Q - PBR^{-1}B^T P + \sum_{i=1}^{p} A_i^T PA_i.$$  

Sections 1 and 2 in Chapter 12 describe the homotopy approach and the algorithm. Section 12.3 explains the shared memory parallel algorithm. The numerical results obtained from implementing the algorithm on a Sequent Symmetry S/81 with Weitek coprocessors, a shared-memory multiprocessor machine, are presented in Section 12.4.

1.5. LQG control synthesis with an $H^\infty$ performance bound...

The $H^2/H^\infty$ mixed-norm controller synthesis problem provides the means for simultaneously addressing $H^2$ and $H^\infty$ performance objectives. In practice such controllers provide both nominal performance (via $H^2$) and robust stability (via $H^\infty$). Hence mixed-norm synthesis provides a technique for trading off performance and robustness, a fundamental objective in control design.

The $H^2/H^\infty$ mixed-norm problem has been addressed in a variety of settings. The treatment in [6], [28] utilized an $H^2$ cost bound as the basis for an auxiliary minimization problem. Necessary conditions for optimality within a full and reduced-order fixed-structure setting were then used to characterize feedback control gains. These necessary conditions have the form of coupled Riccati equations in both the full and reduced-order cases. In related work [23], [53], the $H^2$ cost bound in the case of equalized $H^2$ and $H^\infty$ performance weights was shown to be equal to an entropy cost functional. The centralized controller was then shown to yield a full-order controller that optimizes the entropy cost.

An additional treatment in [77] using a bounded power characterization of the $H^2$ norm obtained both necessary and sufficient conditions for optimality. Finally, a convex optimization approach was developed in [39] for the full-order problem.
The purpose of the present study is to develop numerical algorithms for solving the mixed-norm $H^2/H^\infty$ problem addressed in [6], [28]. The approach here is based upon homotopy methods which have been applied to related fixed-structure problems in [79], [19], [20], [21]. Using globally convergent homotopy techniques similar to those applied to the combined $H^2/H^\infty$ model reduction problem in [19], [20], [21], and using a controller parametrization suggested by Ly, Bryson, and Cannon and the input normal Riccati form [13], results are obtained for the combined $H^2/H^\infty$ full and reduced-order controller synthesis problems. However, such controller parametrizations, which use the minimum possible number of parameters, make structural assumptions which may not be valid in a particular case. Invalidity of these assumptions manifests itself in numerical instability, and failure to converge. An over-parametrization formulation which does not make structural assumptions is also proposed. However, over-parametrization introduces singularity at the solution and may fail for a high dimensional system.

These homotopy methods utilize the solution of a related easily solved problem as the starting point. In the case of full-order $H^2/H^\infty$ control with unequalized weights, the starting point is provided by the standard LQG solution. For the reduced-order problem, the starting point is obtained by constructing a low authority, nearly nonminimal LQG compensator [11].

Chapter 13 states the LQG controller synthesis problem with an $H^\infty$ bound, which is reduced to solving the auxiliary minimization problem. Homotopy algorithms based on three formulations are introduced in Chapters 14, 15, and 16 respectively. The initialization schemes for the homotopy algorithms are given in Chapter 17 and the numerical results and discussions are in Chapter 18.
2. HOMOTOPY ALGORITHM BASED ON INPUT NORMAL FORM FORMULATION FOR THE $H^2$ OPTIMAL MODEL ORDER REDUCTION PROBLEM.

2.1. Input normal form formulations.

The following theorem is needed to present the homotopy method for the input normal form.

**Theorem 1** [35]. Suppose $\tilde{A}_m$ is asymptotically stable. Then for every minimal $(\tilde{A}_m, \tilde{B}_m, \tilde{C}_m)$, i.e., $(\tilde{A}_m, \tilde{B}_m)$ is controllable and $(\tilde{A}_m, \tilde{C}_m)$ is observable, there exist a similarity transformation $U$ and a positive definite matrix $\Omega = \text{diag}(\omega_1, \cdots, \omega_m)$ such that $A_m = U^{-1} \tilde{A}_m U$, $B_m = U^{-1} \tilde{B}_m$, and $C_m = \tilde{C}_m U$ satisfy

$$0 = A_m^T + A_m^T B_m V B_m^T, \quad 0 = A_m^T \Omega + \Omega A_m + C_m^T R C_m. \quad \text{(17)}$$

In addition,

$$\left( A_m \right)_{ii} = -\frac{1}{2} \left( B_m V B_m^T \right)_{ii},$$

$$\omega_i = \frac{(C_m^T R C_m)_{ii}}{(B_m V B_m^T)_{ii}},$$

$$\left( A_m \right)_{ij} = \frac{(C_m^T R C_m)_{ij} - \omega_j (B_m V B_m^T)_{ij}}{\omega_j - \omega_i}, \quad \text{if } \omega_i \neq \omega_j. \quad \text{(18)}$$

**Definition 1.** The triple $(A_m, B_m, C_m)$ satisfying (17) or (18) is said to be in input normal form.

Note that generically $\omega_i \neq \omega_j$ for $i \neq j$, and this is assumed henceforth. Under the assumption that a solution $(A_m, B_m, C_m)$ in input normal form is sought, the only independent variables are $B_m$ and $C_m$, and in this case the domain is

$$\{(A_m, B_m, C_m) : A_m \text{ is stable, } (A_m, B_m, C_m) \text{ is minimal and in input normal form}\}.$$ 

Assuming $(A_m, B_m, C_m)$ is in input normal form, the cost function (3) can be written as

$$J(A_m, B_m, C_m) = \text{tr} \left( \tilde{Q} \tilde{R} \right) \quad \text{(19)}$$

9
where \( \tilde{Q} \) is a symmetric and positive definite matrix satisfying

\[
\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{V} = 0,
\]

and

\[
\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & A_m \end{pmatrix}, \quad \tilde{R} = \begin{pmatrix} C^T RC & -C^T RC_m \\ -C_m^T RC & C_m^T RC_m \end{pmatrix}, \quad \tilde{V} = \begin{pmatrix} BV B_m^T \\ B_m V B_m^T \end{pmatrix}.
\]

(21)

\( \tilde{Q} \) can be written as

\[
\tilde{Q} = \begin{pmatrix} \tilde{Q}_1 & \tilde{Q}_{12} \\ \tilde{Q}_{12}^T & \tilde{Q}_2 \end{pmatrix},
\]

(22)

where \( \tilde{Q}_1 \in \mathbb{R}^{n \times n} \), \( \tilde{Q}_{12} \in \mathbb{R}^{n \times n_m} \), and \( \tilde{Q}_2 \in \mathbb{R}^{n_m \times n_m} \).

The goal of minimizing (19) under the constraints (17) and (20) leads to the Lagrangian

\[
L(A_m, B_m, C_m, \Omega, \tilde{Q}) = \text{tr}[\tilde{Q}\tilde{R} + (A_m + A_m^T + B_m V B_m^T)\Omega_c + (A_m^T \Omega_m + \Omega A_m + C_m^T RC_m) M_o + (\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{V})\tilde{P}],
\]

where the symmetric matrices \( M_o, M_c \), and \( \tilde{P} \) are Lagrange multipliers.

Setting \( \partial L / \partial \tilde{Q} = 0 \) gives

\[
\tilde{A}^T \tilde{P} + \tilde{P} \tilde{A} + \tilde{R} = 0,
\]

(23)

where \( \tilde{P} \) is symmetric positive definite and can be partitioned as

\[
\tilde{P} = \begin{pmatrix} \tilde{P}_1 & \tilde{P}_{12} \\ \tilde{P}_{12}^T & \tilde{P}_2 \end{pmatrix}.
\]

(24)

\( \partial L / \partial \Omega = 0 \) and \( \partial L / \partial A_m = 0 \) yield

\[
0 = 2M_c + 2\Omega M_o + 2(\tilde{P}_{12}^T \tilde{Q}_{12} + \tilde{P}_2 \tilde{Q}_2), \quad 0 = (A_m M_o)_{ii}, \quad 1 \leq i \leq n_m.
\]

A straightforward calculation shows

\[
\begin{align*}
\frac{\partial L}{\partial B_m} &= 2(\tilde{P}_{12}^T B + \tilde{P}_2 B_m) V + 2 M_c B_m V, \\
\frac{\partial L}{\partial C_m} &= 2R(C_m \tilde{Q}_2 - C \tilde{Q}_{12}) + 2RC_m M_o.
\end{align*}
\]

(25)
Theorem 2 [13]. The matrices $M_c$ and $M_o$ in (25) satisfy

$$M_c = -\left(\frac{1}{2}S + \Omega M_o\right),$$

$$(M_o)_{ii} = -\frac{1}{(A_m)_{ii}} \sum_{j=1, j \neq i}^{n} (A_m)_{ij} (M_o)_{ji},$$

$$(M_o)_{ij} = \frac{(S)_{ij} - (S)_{ji}}{2(\omega_j - \omega_i)}, \quad \text{if } \omega_j \neq \omega_i,$$

where

$$S = 2(\tilde{P}_{12}^T \tilde{Q}_{12} + \tilde{P}_2 \tilde{Q}_2).$$

2.2. A homotopy approach based on the input normal form.

A homotopy approach based on the input normal form is now described. Let $A_f, B_f, C_f, R_f,$ and $V_f$ denote $A, B, C, R,$ and $V$ in the above and define

$$A(\lambda) = A_0 + \lambda(A_f - A_0),$$

$$B(\lambda) = B_0 + \lambda(B_f - B_0),$$

$$C(\lambda) = C_0 + \lambda(C_f - C_0),$$

$$R(\lambda) = R_0 + \lambda(R_f - R_0),$$

$$V(\lambda) = V_0 + \lambda(V_f - V_0).$$

For brevity, $A(\lambda), B(\lambda), C(\lambda), V(\lambda),$ and $R(\lambda)$ will be denoted by $A, B, C, V,$ and $R$ respectively in the following. Let

$$H_{B_m}(\theta, \lambda) = \frac{\partial L}{\partial B_m} = 2(\tilde{P}_{12}^T B + \tilde{P}_2 B_m)V + 2M_c B_m V,$$

$$H_{C_m}(\theta, \lambda) = \frac{\partial L}{\partial C_m} = 2R(C_m \tilde{Q}_2 - C \tilde{Q}_{12}) + 2RC_m M_o,$$

where

$$\theta \equiv \begin{pmatrix} \text{Vec}(B_m) \\ \text{Vec}(C_m) \end{pmatrix}$$

denotes the independent variables $B_m$ and $C_m, M_o$ and $M_c$ satisfy (26), and $\tilde{Q}$ and $\tilde{P}$ satisfy respectively (20) and (23) with partitioned forms (22) and (24). Vec$(P)$ for a matrix $P \in \mathbb{R}^{p \times q}$ is the concatenation of its columns:

$$\text{Vec}(P) \equiv \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_q \end{pmatrix} \in \mathbb{R}^{pq}.$$
The homotopy map is defined as

$$\rho(\theta, \lambda) = \begin{bmatrix} \text{Vec} \left[ H_{B_m}(\theta, \lambda) \right] \\ \text{Vec} \left[ H_{C_m}(\theta, \lambda) \right] \end{bmatrix},$$  \quad (29)$$

and its Jacobian matrix is

$$D\rho(\theta, \lambda) = (D_\theta \rho(\theta, \lambda), D_\lambda \rho(\theta, \lambda)).$$  \quad (30)$$

Define

$$\hat{H}_{B_m}(\tilde{P}^{(j)}, M^{(j)}_c) = 2(\tilde{P}^{(j)}_{12} B + \tilde{P}^{(j)}_2 B_m) V + 2M^{(j)}_c B_m V,$$

$$\hat{H}_{C_m}(\tilde{Q}^{(j)}, M^{(j)}_o) = 2R(C_m \tilde{Q}^{(j)}_{12} - C \tilde{Q}^{(j)}_{12}) + 2RC_m M^{(j)}_o,$$

where the superscript \((j)\) means \(\partial/\partial \theta_j; \ Y^{(j)} \equiv \frac{\partial Y}{\partial \theta_j}\). Using the above definitions, we have for \(\theta_j = (B_m)_{kl}\),

$$\frac{\partial H_{B_m}}{\partial (B_m)_{kl}} = \hat{H}_{B_m}(\tilde{P}^{(j)}, M^{(j)}_c) + 2(\tilde{P} + M_c) E^{(k,l)} V,$$

$$\frac{\partial H_{C_m}}{\partial (B_m)_{kl}} = \hat{H}_{C_m}(\tilde{Q}^{(j)}, M^{(j)}_o),$$  \quad (31)$$

and for \(\theta_j = (C_m)_{kl}\),

$$\frac{\partial H_{B_m}}{\partial (C_m)_{kl}} = \hat{H}_{B_m}(\tilde{P}^{(j)}, M^{(j)}_c),$$

$$\frac{\partial H_{C_m}}{\partial (C_m)_{kl}} = \hat{H}_{C_m}(\tilde{Q}^{(j)}, M^{(j)}_o) + 2RE^{(k,l)}(\tilde{Q} + M_o),$$  \quad (32)$$

where \(E^{(k,l)}\) is a matrix of the appropriate dimension whose only nonzero element is \(\epsilon_{kl} = 1\). \(\tilde{P}^{(j)}\) and \(\tilde{Q}^{(j)}\) can be obtained by solving the Lyapunov equations

$$0 = \tilde{A}^{(j)} \tilde{Q} + \tilde{A} \tilde{Q}^{(j)} + \tilde{Q}^{(j)} \tilde{A}^T + \tilde{Q} \tilde{A}^T \tilde{Q} + \tilde{V}^{(j)},$$

$$0 = \tilde{A}^T \tilde{P} + \tilde{A}^T \tilde{P}^{(j)} + \tilde{P}^{(j)} \tilde{A} + \tilde{P} \tilde{A} + \tilde{K}^{(j)}.$$

(33)$$

Similarly for \(\lambda\), using a dot to denote \(\partial/\partial \lambda\),

$$\frac{\partial H_{B_m}}{\partial \lambda} = \dot{\hat{H}}_{B_m}(\dot{\tilde{P}}, \dot{M}_c) + 2\dot{\tilde{P}}^T_{12}(\dot{B} V + B \dot{V}) + 2(\dot{\tilde{P}} + M_c) B_m \dot{V},$$

$$\frac{\partial H_{C_m}}{\partial \lambda} = \dot{\hat{H}}_{C_m}(\dot{\tilde{Q}}, \dot{M}_o) + 2\dot{R}C_m (\dot{Q}_2 + M_o) - 2(\dot{R}C + R\dot{C}) \dot{Q}_{12},$$  \quad (34)$$
where $\dot{P}$ and $\dot{Q}$ are obtained by solving the Lyapunov equations

$$0 = \dot{A}Q + A\dot{Q} + \dot{Q}A^T + Q\dot{A}^T + \dot{V},$$

$$0 = \dot{A}^T\dot{P} + A^T\dot{P} + \dot{P}A + \dot{P}A^T + \dot{R}.$$

2.3. Numerical algorithm for input normal form homotopy.

The initial point $(\theta, \lambda) = (\theta_0, 0) = ((B_m)_0, (C_m)_0, 0)$ is chosen so that the triple $((A_m)_0,$ $(B_m)_0, (C_m)_0)$ is in input normal form and satisfies $\rho(\theta_0, 0) = 0.$

Theorem 3 [49]. Suppose $\bar{A}$ is asymptotically stable. Then for every minimal $(\bar{A}, \bar{B}, \bar{C}),$ i.e., $(\bar{A}, \bar{B})$ is controllable and $(\bar{A}, \bar{C})$ is observable, there exist a similarity transformation $T$ and a positive definite matrix $\Lambda = \text{diag} (d_1, d_2, \cdots, d_n)$ with $d_i \geq d_{i+1}$ such that $A = T^{-1}\bar{A}T,$ $B = T^{-1}\bar{B},$ and $C = \bar{C}T$ satisfy

$$0 = A\Lambda + \Lambda A^T + BV^T,$$

$$0 = A^T\Lambda + \Lambda A + C^TRC.$$

Definition 2. The triple $(A, B, C)$ in the above theorem is balanced.

According to Moore [49], under certain conditions, the leading principal $n_m \times n_m$ block of $A,$ the leading principal $n_m \times m$ block of $B,$ and the leading principal $l \times n_m$ block of $C$ in balanced form are good approximations to the reduced order model. This suggests that the initial point $(\theta_0, 0)$ be chosen as follows:

1) Transform the given triple $(A_f, B_f, C_f)$ to balanced form $(A_b, B_b, C_b)$.

2) Partition $(A_b, B_b, C_b)$ as

$$A_b = n_m \left\{ \begin{array}{c} A_{11} \\ A_{21} \end{array} \right\}_m,$$

$$B_b = n_m \left\{ \begin{array}{c} B_1 \\ B_2 \end{array} \right\}_m,$$

$$C_b = \left( \begin{array}{c} C_1 \\ C_2 \end{array} \right).$$

3) $(A_0, B_0, C_0)$ is chosen as

$$A_0 = \left( \begin{array}{cc} A_{11} & 0 \\ 0 & A_{22} \end{array} \right), \quad B_0 = \left( \begin{array}{c} B_1 \\ 0 \end{array} \right), \quad C_0 = \left( \begin{array}{c} C_1 \\ 0 \end{array} \right).$$
4) The initial point for the reduced order model is chosen as
\[ \hat{\theta}_0 = \begin{pmatrix} \text{Vec} \ (\hat{B}_m)_0 \\ \text{Vec} \ (\hat{C}_m)_0 \end{pmatrix} = \begin{pmatrix} \text{Vec} \ B_1 \\ \text{Vec} \ C_1 \end{pmatrix}, \]
and \((\hat{A}_m)_0 = A_{11}\) by construction.

5) Transform the initial point \(((\hat{A}_m)_0, (\hat{B}_m)_0, (\hat{C}_m)_0)\) to input normal form so that the initial reduced order model is
\[ \left( (A_m)_0, (B_m)_0, (C_m)_0 \right) = \left( T^{-1} (\hat{A}_m)_0 T, \ T^{-1} (\hat{B}_m)_0, \ (\hat{C}_m)_0 T \right). \]

The initial point for the homotopy map is then \( (\theta_0, 0) \), where
\[ \theta_0 = \begin{pmatrix} \text{Vec} \ (B_m)_0 \\ \text{Vec} \ (C_m)_0 \end{pmatrix}. \]

(In general, the truncation to obtain the approximate reduced order model should be based on the component costs instead of on the sizes of the balanced gains \( d_i \) as done above [65]. This explains why in some cases (Examples 1 and 6) the above algorithm for choosing the initial points did not lead to a reduced order model with a minimal cost.)

Once the initial point is chosen, the rest of the computation is as follows:

1) Set \( \lambda := 0, \ \theta := \theta_0. \)

2) Calculate \( A_m \) from (18), \( \hat{R}, \hat{V}, \) and compute \( \hat{Q} \) and \( \hat{P} \) according to (20) and (23).

3) Evaluate \( S \) from (27) and \( M_o \) and \( M_e \) according to (26).

4) Evaluate the homotopy map \( \rho(\theta, \lambda) \) in (29) and \( D\rho(\theta, \lambda) \) in (30).

5) Predict the next point \( Z^{(0)} = (\theta^{(0)}, \lambda^{(0)}) \) on the curve \( \gamma. \)

6) For \( k := 0, 1, 2, \cdots \) until convergence do
\[ Z^{(k+1)} = Z^{(k)} - \left[ D\rho(Z^{(k)}) \right]^\dagger \rho(Z^{(k)}), \]
where \( [D\rho(Z)]^\dagger \) is the Moore-Penrose inverse of \( D\rho(Z) \). Let \( (\theta_1, \lambda_1) = \lim_{k \to \infty} Z^{(k)}. \)

7) If \( \lambda_1 < 1, \) then set \( \theta := \theta_1, \ \lambda := \lambda_1, \) and go to step 2).

8) If \( \lambda_1 \geq 1, \) compute the solution \( \hat{\theta} \) at \( \lambda = 1. \ A_m \) is then obtained from (18).

An alternative strategy for choosing an initial point is as follows:
1) Modify $A_f$ to $A'_f = c_1 I + c_2 A_f$, where $c_1 \leq 0$ and $c_2 \geq 0$.

1) Transform $(A'_f, B_f, C_f)$ to balanced form and choose $(A'_0, B'_0, C'_0)$ as before.

3) Compute the initial reduced order model $((A_m)_0, (B_m)_0, (C_m)_0)$ from the triple $(A'_0, B'_0, C'_0)$ as before.

When $c_1 = 0$, $c_2 = 1$, this strategy reduces to the previous one. For some problems, our numerical experiments show that HOMPACK reaches $\lambda > 1$ in fewer steps with $c_1 \neq 0$ than with $c_1 = 0$. A modification to the homotopy map $\rho(\theta, \lambda)$ in (29) is

$$
\rho_1(\theta, \lambda) = \lambda \rho(\theta, \lambda) + (1 - \lambda)(\theta - \theta_0),
$$

where $\theta_0$ denotes the initial value of $\theta$ at $\lambda = 0$. For some problems this homotopy map can be more efficient than the one in (29), while in other cases it can be less efficient.
3. COMPARISON WITH OPTIMAL PROJECTION EQUATIONS APPROACH.

Theorem 4 [31] [32]. Suppose \((A_m, B_m, C_m)\) is a controllable and observable solution of the problem (1)–(3). Then there exist positive semidefinite pseudogramians \(\hat{Q}, \hat{P}\) that are a solution to modified Lyapunov equations

\[
0 = \tau [A \hat{Q} + \hat{Q} A^T + B V B^T],
\]

\[
0 = [A^T \hat{P} + \hat{P} A + C^T R C] \tau,
\]

and satisfy rank conditions

\[
\text{rank} (\hat{Q}) = \text{rank} (\hat{P}) = \text{rank} (\hat{Q} \hat{P}) = n_m,
\]

such that the optimal model is given by

\[
A_m = \Gamma A G^T,
\]

\[
B_m = \Gamma B,
\]

\[
C_m = C G^T,
\]

where \(G\) and \(\Gamma\) come from a \((G, M, \Gamma)\)-factorization of \(\hat{Q} \hat{P}\):

\[
\hat{Q} \hat{P} = G^T M \Gamma,
\]

\[
\Gamma G^T = I_{n_m},
\]

\(G, \Gamma \in \mathbb{R}^{n_m \times n}, M \in \mathbb{R}^{n_m \times n_m}\) is positive semisimple and \(\tau \equiv G^T \Gamma\).

Equations (35) are called the optimal projection equations, which after the nontrivial algebraic manipulation described in [79], can be written in a form suitable for computation as

\[
U_1 A W_1 \Sigma W_1^T + \Sigma W_1^T A^T + U_1 B V B^T = 0,
\]

\[
A^T U_1^T \Sigma + U_1^T \Sigma U_1 A W_1 + C^T R C W_1 = 0,
\]

\[
U_1 W_1 - I = 0.
\]
The unknowns are $W_1 \in \mathbb{R}^{n \times n_m}$, $U_1 \in \mathbb{R}^{n_m \times n}$ and symmetric $\Sigma \in \mathbb{R}^{n_m \times n_m}$. In terms of these new unknowns, $\hat{Q}$ and $\hat{P}$ in (37) can be written as

$$\hat{Q} = W_1 \Sigma W_1^T, \quad \hat{P} = U_1^T \Sigma U_1.$$

Hyland and Bernstein [32] stated that the optimal projection equations can have at most $\binom{n}{n_m}$ solutions. It is shown by the following 2-dimensional example that this is not true in general.

The system [36] is given by

$$A = \begin{pmatrix} -0.05 & -0.99 \\ -0.99 & -5000.0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 100 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 100 \end{pmatrix}.$$  \hspace{1cm} (39)

**Proposition:** For the system (1) defined by (39), the solution set of the optimal projection equations contains three isolated solutions and a one-dimensional manifold parameterized by one element of either $W_1$ or $U_1$.

**Proof.** The three isolated solutions are

$$A_m = (-0.005004234), \quad B_m = (1.000213), \quad C_m = (1.000213),$$

$$A_m = (-4998.079), \quad B_m = (100.0002), \quad C_m = (100.0002),$$

$$A_m = (-0.4659163), \quad B_m = (-1.940482), \quad C_m = (-1.940482),$$

which were obtained by both POLSYS from HOMPACK [72] and by a homotopy approach [78]–[80]. The one-dimensional manifold of solutions corresponds to

$$A_m = (-0.4851515), \quad B_m = (0.0), \quad C_m = (0.0),$$

which can be derived directly from the optimal projection equations as follows.
Let $W_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $U_1 = (x_3, x_4)$, and $\Sigma = x_5$. The optimal projection equations (38) for this problem can be written as

\begin{align*}
0 &= a_{11} x_1^2 x_3 x_5 + a_{12} x_1 x_2 x_3 x_5 + a_{21} x_2^2 x_4 x_5 + a_{22} x_2 x_1 x_2 x_4 x_5 \\
  &\quad + a_{11} x_1 x_5 + a_{12} x_2 x_5 + (BV B^T)_{11} x_3 + (BV B^T)_{21} x_4, \\
0 &= a_{11} x_1 x_2 x_3 x_5 + a_{12} x_2^2 x_3 x_5 + a_{21} x_1 x_2 x_4 x_5 + a_{22} x_2^2 x_4 x_5 \\
  &\quad + a_{21} x_1 x_5 + a_{22} x_2 x_5 + (BV B^T)_{12} x_3 + (BV B^T)_{22} x_4, \\
0 &= a_{11} x_1 x_2 x_3^2 x_5 + a_{12} x_2 x_3^2 x_5 + a_{21} x_1 x_3 x_4 x_5 + a_{22} x_2 x_3 x_4 x_5 \\
  &\quad + a_{11} x_3 x_5 + a_{21} x_4 x_5 + (C^T RC)_{11} x_1 + (C^T RC)_{12} x_2, \\
0 &= a_{11} x_1 x_3 x_4 x_5 + a_{12} x_2 x_3 x_4 x_5 + a_{21} x_1 x_2 x_4^2 x_5 + a_{22} x_2 x_4 x_5 \\
  &\quad + a_{12} x_3 x_5 + a_{22} x_4 x_5 + (C^T RC)_{21} x_1 + (C^T RC)_{22} x_2, \\
0 &= x_1 x_3 + x_2 x_4 - 1. 
\end{align*} 

(40)

The triple $(A_m, B_m, C_m)$ is given by

\begin{align*}
A_m &= \Gamma AG^T = (x_3, x_4) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= x_1 (a_{11} x_3 + a_{21} x_4) + x_2 (a_{12} x_3 + a_{22} x_4), \\
B_m &= \Gamma B = (x_3, x_4) \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = b_{11} x_3 + b_{21} x_4, \\
C_m &= CG^T = (c_{11}, c_{12}) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_{11} x_1 + c_{12} x_2, 
\end{align*} 

(41)

where $\Gamma = U_1$ and $G = W_1^T$. Substituting (39) into (40), and (41), setting $B_m = x_3 + 100 x_4 = 0$ and $C_m = x_1 + 100 x_2 = 0$ gives $x_1 = -100 x_2$, $x_3 = -100 x_4$, and $A_m = -4852 x_2 x_4$. Equations (40) become

\begin{align*}
0 &= 485200 x_2^2 x_4 x_5 - 0.49 x_2 x_5, \\
0 &= 485200 x_2 x_4^2 x_5 - 0.49 x_4 x_5, \\
0 &= 4852 x_2^2 x_4 x_5 + 4901 x_2 x_5, \\
0 &= 4852 x_2 x_4^2 x_5 + 4901 x_4 x_5, \\
0 &= 10001 x_2 x_4 - 1. 
\end{align*} 

(42) (43) (44) (45) (46)
If $x_2 = 0$ or $x_4 = 0$, equation (46) will not be satisfied. Only the situation that $x_2 \neq 0$ and $x_4 \neq 0$ is possible. Then equations (42)–(46) can be reduced to

\begin{align*}
0 &= 485200x_2x_4x_5 - 0.49x_5, \\
0 &= 4852x_2x_4x_5 + 4901x_5, \\
0 &= 10001x_2x_4 - 1. \\
\end{align*}

(47)

If $x_5 \neq 0$ then (34) becomes

\begin{align*}
0 &= 485200x_2x_4 - 0.49, \\
0 &= 4852x_2x_4 + 4901, \\
0 &= 10001x_2x_4 - 1, \\
\end{align*}

(48)

which does not have a solution.

Thus $x_5 = 0$, and equation (47) reduces to

\begin{align*}
16001x_2x_4 - 1 &= 0, \\
\end{align*}

which gives $A_m = -4852/10001 = -0.4851515$ corresponding to a one-dimensional manifold parametrized by $x_2$ or $x_4$. Hence the solution $A_m = -0.4851515$, $B_m = 0$ and $C_m = 0$ (which is not controllable or observable) corresponds to a one-dimensional manifold of solutions of the optimal projection equations.

Q. E. D.

The set of solutions of the input normal form equations contains the same set of isolated solutions as the optimal projection equations, and also a fourth isolated solution given by $A_m = B_m = C_m = 0$. Therefore the solution sets of the two formulations are different.

The input normal form equations can be rewritten as

\begin{align*}
0 &= 2(\tilde{P}_{12}^TB + \tilde{P}_2B_m)V + 2M_cB_mV, \\
0 &= 2R(C_m\tilde{Q}_2 - C\tilde{Q}_{12}) + 2RC_mM_o. \\
\end{align*}

(49)

Setting $B_m = C_m = 0$, the equations become

\begin{align*}
0 &= \tilde{P}_{12}^TBV, \\
0 &= RC\tilde{Q}_{12},
\end{align*}

(50)
where $\tilde{P}_{12}$ and $\tilde{Q}_{12}$ satisfy respectively

$$0 = A^T \tilde{P}_{12} + \tilde{P}_{12} A_m,$$

$$0 = A \tilde{Q}_{12} + \tilde{Q}_{12} A_m,$$

which has a solution $\tilde{P}_{12} = \tilde{Q}_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. $A_m$ satisfies

$$A_m + A_m^T + B_m V B_m^T = A_m + A_m^T = 0$$

which gives $A_m = 0$.

It should be noted that the solutions to the optimal projection equations (35) that satisfy the rank conditions $\text{rank}(\tilde{Q}) = \text{rank}(\tilde{P}) = \text{rank}(\tilde{Q} \tilde{P}) = n_m$ characterize all controllable and observable extremals of the optimal model reduction problem. However, there are algebraic solutions to (35) that do not satisfy these rank conditions. The one-dimensional manifold of solutions of the previous proposition are such a set of solutions since for these solutions $\text{rank}(\tilde{Q}) = \text{rank}(\tilde{P}) = 0 \neq n_m = 1$. On the other hand, the input normal form equations characterize all extremals of the optimal model reduction problem for which the input normal form has the property that no two diagonal elements of $\Omega$ are equal. No restriction is placed on the controllability or observability of these extremals. Hence, the extremal sets that the optimal projection equations and the input normal form equations characterize are not identical. In addition, the optimal projection equations may also have algebraic solutions that characterize additional reduced-order models that are uncontrollable or unobservable and may or may not be related to the solutions of the input normal form equations by a similarity transformation. These differences in the solution sets were illustrated by the example of this section. However, it should be noted that if one considers their input-output properties, the two solution sets are equivalent.
4. HOMOTOPY ALGORITHM BASED ON LY FORMULATION FOR THE $H^2$ MODEL ORDER REDUCTION PROBLEM.

4.1 Ly's formulation.

Ly et al. [47] introduced another canonical form also with $n_m m + n_m l$ parameters as in the input normal form formulation. The reduced order model is represented with respect to a basis such that $A_m$ is a $2 \times 2$ block-diagonal matrix ($2 \times 2$ blocks with an additional $1 \times 1$ block if $n_m$ is odd) with $2 \times 2$ blocks in the form

$$
\begin{pmatrix}
0 & 1 \\
* & *
\end{pmatrix},
$$

$B_m$ is a full matrix, and

$$
C_m = ((C_m)_1 \quad (C_m)_2 \quad \cdots \quad (C_m)_r)
$$

where

$$(C_m)_i = \begin{pmatrix}
1 & * & \cdots & * \\
0 & * & \cdots & *
\end{pmatrix}^T,$$

$$(C_m)_r = (1 \quad * \quad \cdots \quad *)^T, \quad \text{if } n_m \text{ is odd.}
$$

Let $\mathcal{S}$ be the set of indices of those elements of $A_m$ which are parameters, i.e.,

$$
\mathcal{S} \equiv \{(2,1), (2,2), \ldots, (n_m,n_m)\}.
$$

To find the minimum of the cost function (19), consider the Lagrangian

$$
L(A_m, B_m, C_m, \tilde{Q}) = \text{tr}[\tilde{Q} \tilde{R} + (\tilde{A} \tilde{Q} + \tilde{Q} \tilde{A}^T + \tilde{V}) \tilde{P}],
$$

(51)

where the symmetric matrix $\tilde{P}$ is a Lagrange multiplier, $\tilde{Q}$ satisfies (20), and $\tilde{A}$, $\tilde{R}$, and $\tilde{V}$ are defined in (21). Setting $\partial L/\partial \tilde{Q} = 0$ gives (23), and $\tilde{P}$ is symmetric positive definite and can be partitioned as in (24). A straightforward calculation shows

$$
\frac{\partial L}{\partial (A_m)_{ij}} = 2(\tilde{P}_{12}^T \tilde{Q}_{12} + \tilde{P}_{2} \tilde{Q}_{2})_{ij}, \quad (i,j) \in \mathcal{S},
$$

$$
\frac{\partial L}{\partial B_m} = 2(\tilde{P}_{12}^T B + \tilde{P}_{2} B_m) V,
$$

$$
\frac{\partial L}{\partial (C_m)_{ij}} = 2 \frac{\partial}{\partial (C_m)_{ij}} \left[ \text{tr} (-\tilde{Q}_{12}^T C^T R C_m) + \text{tr} (Q_2 C_m^T R C_m) \right]
$$

$$
= 2R(C_m \tilde{Q}_{2} - C \tilde{Q}_{12})_{ij}, \quad i > 1.
$$

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4.2. A homotopy approach based on the Ly formulation.

Let $A_f, B_f, C_f, R_f,$ and $V_f$ denote $A, B, C, R,$ and $V$ in the above and define $A(\lambda), B(\lambda), C(\lambda), R(\lambda),$ and $V(\lambda)$ as in (28) and denote them by $A, B, C, V,$ and $R$ respectively in the following. Let

\begin{align*}
H_{A_m}(\theta, \lambda) &= \frac{\partial L}{\partial A_m} = 2(\bar{P}_1^T \bar{Q}_{12} + \bar{P}_2 \bar{Q}_2), \\
H_{B_m}(\theta, \lambda) &= \frac{\partial L}{\partial B_m} = 2(\bar{P}_1^T B + \bar{P}_2 B_m)V, \\
H_{C_m}(\theta, \lambda) &= \frac{\partial L}{\partial C_m} = 2R(C_m \bar{Q}_2 - C \bar{Q}_{12}),
\end{align*}

(53)

where in $H_{A_m}$ only those elements corresponding to the parameter elements of $A_m$ are nonzero and

\[
\theta \equiv \begin{pmatrix}
(A_m)_S \\
\text{vec} (B_m) \\
\text{vec} (C_m)_{T}
\end{pmatrix}
\]

(54)

denotes the independent variables, $\bar{Q}$ and $\bar{P}$ satisfy respectively (20) and (23), $(A_m)_S$ is a vector consisting of those elements in $A_m$ with indices in the set $S$, i.e.,

\[(A_m)_S = ((A_m)_{21}, (A_m)_{22}, \ldots, (A_m)_{n_m})^T,
\]

$(C_m)_{T}$ is the matrix obtained from rows $T = \{2, \ldots, l\}$ of $C_m$.

The homotopy map is defined as

\[
\rho(\theta, \lambda) = \begin{pmatrix}
\left[ H_{A_m}(\theta, \lambda) \right]_S \\
\text{vec} [H_{B_m}(\theta, \lambda)] \\
\text{vec} [H_{C_m}(\theta, \lambda)]_{T}
\end{pmatrix},
\]

(55)

and its Jacobian matrix is

\[
D\rho(\theta, \lambda) = (D_{\theta}\rho(\theta, \lambda), D_{\lambda}\rho(\theta, \lambda)).
\]

Define

\begin{align*}
\hat{H}_{A_m}(\bar{P}^{(j)}, \bar{Q}^{(j)}) &= 2(\bar{P}_1^{T(j)} \bar{Q}_{12} + \bar{P}_2^{T(j)} \bar{Q}_2 + \bar{P}_2 \bar{Q}^{(j)}), \\
\hat{H}_{B_m}(\bar{P}^{(j)}) &= 2(\bar{P}_1^{T(j)} B + \bar{P}_2^{T(j)} B_m)V, \\
\hat{H}_{C_m}(\bar{Q}^{(j)}) &= 2R(C_m \bar{Q}_2^{(j)} - C \bar{Q}_{12}^{(j)}),
\end{align*}

(56)
where the superscript \((j)\) means \(\partial / \partial \theta_j\). Using the above definitions, we have for \(\theta_j = (A_m)_{kl}\), where \((k,l) \in \mathcal{S}\),

\[
\begin{align}
\frac{\partial H_{A_m}}{\partial (A_m)_{kl}} &= \hat{H}_{A_m} (\bar{P}^{(j)}, \bar{Q}^{(j)}), \\
\frac{\partial H_{B_m}}{\partial (A_m)_{kl}} &= \hat{H}_{B_m} (\bar{P}^{(j)}), \\
\frac{\partial H_{C_m}}{\partial (A_m)_{kl}} &= \hat{H}_{C_m} (\bar{Q}^{(j)}),
\end{align}
\tag{57}
\]

for \(\theta_j = (B_m)_{kl}\),

\[
\begin{align}
\frac{\partial H_{A_m}}{\partial (B_m)_{kl}} &= \hat{H}_{A_m} (\bar{P}^{(j)}, \bar{Q}^{(j)}), \\
\frac{\partial H_{B_m}}{\partial (B_m)_{kl}} &= \hat{H}_{B_m} (\bar{P}^{(j)}) + 2\bar{P}_2 E^{(k,l)} V, \\
\frac{\partial H_{C_m}}{\partial (B_m)_{kl}} &= \hat{H}_{C_m} (\bar{Q}^{(j)}),
\end{align}
\tag{58}
\]

and for \(\theta_j = (C_m)_{kl}\), where \(k > 1\),

\[
\begin{align}
\frac{\partial H_{A_m}}{\partial (C_m)_{kl}} &= \hat{H}_{A_m} (\bar{P}^{(j)}, \bar{Q}^{(j)}), \\
\frac{\partial H_{B_m}}{\partial (C_m)_{kl}} &= \hat{H}_{B_m} (\bar{P}^{(j)}), \\
\frac{\partial H_{C_m}}{\partial (C_m)_{kl}} &= \hat{H}_{C_m} (\bar{Q}^{(j)}) + 2\bar{P}_2 E^{(k,l)} \bar{Q}_2.
\end{align}
\tag{59}
\]

\(\bar{P}^{(j)}\) and \(\bar{Q}^{(j)}\) can be obtained by solving the Lyapunov equation (33). The derivative of the homotopy map with respect to \(\lambda\) can be derived in a similar fashion.


The initial point \((\theta, \lambda) = (\theta_0, 0)\) is chosen so that the triple \((A_m)_0, (B_m)_0, (C_m)_0\) is in Ly's form and satisfies \(\rho(\theta_0, 0) = 0\). This can be done as follows:

1) Obtain the initial reduced order model \(((A_m)_0, (B_m)_0, (C_m)_0)_b\) in balanced form in the same way as for the input normal form approach.

2) Transform the balanced \(((A_m)_0, (B_m)_0, (C_m)_0)_b\) to Ly's form, and build \(\theta_0\) as described in (54).

The homotopy curve tracking computation is the same as described in Chapter 2.
5. NUMERICAL RESULTS.

In this section numerical results for both the input normal form and Ly formulations are given for eleven systems. The first nine systems have been studied and solved in [78]–[80] using the optimal projection equations approach. Comparisons are made between these two minimal formulations and the optimal projection equations in Chapter 7.

The cost $J$ is computed for each model as $\text{tr} \left( \hat{Q} \hat{R} \right)$, according to (19). For all examples $V = R = I$. Unless indicated otherwise, the solutions, given in input normal form, can be obtained by both formulations and are the same as those obtained by the optimal projection equations method.

EXAMPLE 1 [36]. The system is given by

$$ A = \begin{pmatrix} -0.05 & -0.99 \\ -0.99 & -5000.0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 100 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 100 \end{pmatrix}. $$

The homotopy algorithm converges to a solution corresponding to the model of order $n_m = 1$ given by

$$ A_m = (-0.00500423), \quad B_m = (-0.100042), \quad C_m = (-10.0000), $$

which was not obtained by the optimal projection equation approach of [78]–[80]. This model yields the cost $J = 10000$.

In the first step of choosing an initial point, $(A_f, B_f, C_f)$ is transformed to $(A_b, B_b, C_b)$, where orthogonal decompositions of two matrices are needed. If the eigenvalues of one of the matrices are rearranged in ascending order, then a different solution is obtained, namely

$$ A_m = (-4998.08), \quad B_m = (-99.9808), \quad C_m = (-100.020). $$

This model yields the (minimum) cost $J = 96.0781$.

EXAMPLE 2 [66]. The system is given by

$$ A = \begin{pmatrix} -1 & 0 \\ 0 & -10 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 70 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -0.2 \end{pmatrix}. $$
A model of order \( n_m = 1 \) is
\[
A_m = (-11.9794), \quad B_m = (-4.85914, 0.589656), \quad C_m = (2.76076).
\]
This model yields the cost \( J = 0.598377 \).

**Example 3 [36].** The system is given by
\[
A = \begin{pmatrix} -0.25 & -0.4 \\ -0.4 & -0.72 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1.2 \end{pmatrix}, \quad C = (1, 1.2).
\]
A model of order \( n_m = 1 \) is
\[
A_m = (-0.838521), \quad B_m = (-1.29501), \quad C_m = (-1.82558).
\]
This model yields the cost \( J = 0.107256 \).

**Example 4 [68].** The system is given by
\[
A = \begin{pmatrix} -1 & 3 & 0 \\ -1 & -1 & 1 \\ 4 & -5 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}, \quad C = (1, 0, 0).
\]
A model of order \( n_m = 1 \) is
\[
A_m = (-0.286334), \quad B_m = (0.756748), \quad C_m = (0.878161),
\]
which is different from that obtained by the optimal projection equation method [78]-[80], and has a smaller cost \( J = 1.22883 \). A model of order \( n_m = 2 \) is
\[
A_m = \begin{pmatrix} -0.215037 & 0.753968 \\ -2.513585 & -3.60074 \end{pmatrix}, \quad B_m = \begin{pmatrix} 0.655800 \\ 2.68356 \end{pmatrix}, \quad C_m = \begin{pmatrix} 0.888877 \\ -1.09093 \end{pmatrix}.
\]
This model yields the cost \( J = 0.0197781 \).

**Example 5 [36].** The system is given by
\[
A = \begin{pmatrix} -10 & 1 & 0 \\ -5 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad C = (1, 0, 0).
\]
A model of order \( n_m = 1 \) is
\[
A_m = (-0.157898), \quad B_m = (0.561956), \quad C_m = (0.318537).
\]
This model yields the cost $J = 0.0107792$. A model of order $n_m = 2$ is

$$A_m = \begin{pmatrix} -0.139652 & 0.100607 \\ -0.600971 & -0.448192 \end{pmatrix}, \quad B_m = \begin{pmatrix} 0.528492 \\ 0.946775 \end{pmatrix}, \quad C_m^T = \begin{pmatrix} 0.320438 \\ -0.0961019 \end{pmatrix}. $$

This model yields the cost $J = 0.000329024$.

**Example 6 [78].** The system is given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & -0.02 & 1 & 0.01 \\ 0 & 0 & 0 & 1 \\ 0.1 & 0.001 & -0.1 & -0.001 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = (0 \ 1 \ 0 \ 0).$$

A model of order $n_m = 1$ is

$$A_m = (-0.353743), \quad B_m = \begin{pmatrix} -0.184397 \\ -0.820660 \end{pmatrix}, \quad C_m = (0.805197).$$

This model yields the cost $J = 285.012$.

With the input normal form, when $n_m = 2$, 3, two of the initial $\omega$s are approximately the same, which leads to a significant numerical error in computing $M_o$, and the numerical failure of the homotopy algorithm. Therefore this technique for choosing initial points fails, and some modification to the algorithm is needed to avoid this kind of ill conditioning. However, it is not at all clear how to systematically avoid nearly equal $\omega$s, and this remains an open question. It can be shown that the solutions, obtained by the optimal projection equation approach, also have close $\omega$s, which implies that changing the strategy for choosing initial points will not suffice for this example.

The solutions obtained by Ly's formulation are given in the following.

A model of order $n_m = 2$ is

$$A_m = \begin{pmatrix} 0 & 1.0 \\ -0.0487508 & -0.000487507 \end{pmatrix}, \quad B_m = \begin{pmatrix} 0.0255931 \\ -0.0000350605 \end{pmatrix}, \quad C_m = (1.0 \ 0.0).$$

This model yields the cost $J = 29.2223$. A model of order $n_m = 3$ is

$$A_m = \begin{pmatrix} 0.0 & 1.0 & 0 \\ -0.0487508 & -0.000488017 & 0.0 \\ 0.0 & 0.0 & -2.48938 \end{pmatrix},$$

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\[
B_m = \begin{pmatrix}
-0.0250401 & -0.499927 \\
0.0000456588 & 0.000219679 \\
-1.45628 & 0.746877
\end{pmatrix}, \quad C_m = (1.0 \ 0.0 \ 1.0).
\]

This model yields the cost \( J = 28.6848 \). Both of the above solutions have smaller cost than those obtained in [78]–[80].

**Example 7** [49], [74]. The system is given by

\[
A = \begin{pmatrix}
0 & 0 & 0 & -150 \\
1 & 0 & 0 & -245 \\
0 & 1 & 0 & -1113 \\
0 & 0 & 1 & -19
\end{pmatrix}, \quad B = \begin{pmatrix}
4 \\
1 \\
0 \\
0
\end{pmatrix}, \quad C = (0 \ 0 \ 0 \ 1).
\]

A model of order \( n_m = 1 \) is

\[
A_m = (-0.495187), \quad B_m = (0.995175), \quad C_m = (0.0148426).
\]

This model yields the cost \( J = 4.90749 \cdot 10^{-5} \). A model of order \( n_m = 2 \) is

\[
A_m = \begin{pmatrix}
-0.437964 & -0.482612 \\
2.84007 & -3.17242
\end{pmatrix},
\]

\[
B_m = \begin{pmatrix}
0.935911 \\
-2.51890
\end{pmatrix}, \quad C_m^T = (0.0149143 \ 0.00682097).
\]

This model yields the cost \( J = 4.159 \cdot 10^{-7} \). A model of order \( n_m = 3 \) is

\[
A_m = \begin{pmatrix}
-0.437810 & -0.483078 & -0.0370168 \\
2.82632 & -3.13536 & -0.612598 \\
-4.65184 & 13.1604 & -12.5542
\end{pmatrix},
\]

\[
B_m = \begin{pmatrix}
0.935746 \\
-2.50414 \\
5.01082
\end{pmatrix}, \quad C_m = (0.0149143 \ 0.00682180 \ 0.000635413).
\]

This model yields the cost \( J = 4.59 \cdot 10^{-10} \).

**Example 8** [49]. The system is given by

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-50 & -79 & -33 & -5
\end{pmatrix}, \quad B = \begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix}, \quad C = (50 \ 15 \ 1 \ 0).
\]

A model of order \( n_m = 1 \) is

\[
A_m = (-0.576205), \quad B_m = (1.07350), \quad C_m = (0.588692).
\]

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This model yields the cost $J = 0.104740$. A model of order $n_m = 2$ is

$$A_m = \begin{pmatrix} -0.532330 & -0.598751 \\ 3.80077 & -4.81512 \end{pmatrix}, \quad B_m = \begin{pmatrix} 1.03182 \\ -3.10326 \end{pmatrix}, \quad C_m^T = \begin{pmatrix} 0.588704 \\ 0.278923 \end{pmatrix}.$$ 

This model yields the cost $J = 0.0269278$. A model of order $n_m = 3$ is

$$A_m = \begin{pmatrix} -0.520312 & -0.731867 & -0.162146 \\ 2.88892 & -2.23562 & -3.72129 \\ -1.08450 & 6.30540 & -0.746729 \end{pmatrix}, \quad B_m^T = \begin{pmatrix} 1.02011 & -2.11453 & 1.22207 \end{pmatrix}, \quad C_m = \begin{pmatrix} 0.586461 & 0.307967 & 0.105043 \end{pmatrix}.$$

This model yields the cost $J = 0.00148438$.

**Example 9 [30].** The system is given by

$$A = \begin{pmatrix} -6.2036 & 15.054 & -9.8726 & -376.58 & 251.32 & -162.24 & 66.827 \\ 0.53 & -2.0176 & 1.4363 & 0 & 0 & 0 & 0 \\ 16.846 & 25.079 & -43.55 & 0 & 0 & 0 & 0 \\ 377.4 & -89.449 & -162.83 & 57.998 & -65.514 & 68.579 & 157.57 \\ 0 & 0 & 0 & 107.25 & -118.05 & 0 & 0 \\ 0.36992 & -0.14445 & -0.26303 & -0.64719 & 0.49947 & -0.21133 & 0 \\ 0 & 0 & 0 & 0 & 0 & 376.99 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 89.353 & 0 \\ 376.99 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0.21133 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

A model of order $n_m = 1$ is

$$A_m = -0.199272, \quad B_m = \begin{pmatrix} 0.631300 & -0.00187918 \end{pmatrix}, \quad C_m^T = \begin{pmatrix} -0.187347 & -354.430 \end{pmatrix}.$$ 

This model yields the cost $J = 27632.2$. A model of order $n_m = 2$ is

$$A_m = \begin{pmatrix} -0.199608 & -0.0763006 \\ 3.33119 & -13.2758 \end{pmatrix}, \quad B_m = \begin{pmatrix} 0.631832 & -0.00191612 \\ -5.15182 & -0.101952 \end{pmatrix}, \quad C_m = \begin{pmatrix} -0.201050 & 0.800899 \\ -354.414 & -66.1873 \end{pmatrix}.$$ 

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This model yields the cost $J = 23262.3$. A model of order $n_m = 3$ is

$$A_m = \begin{pmatrix}
-0.198769 & 0.235666 & -0.0248130 \\
-1.08739 & -0.912444 & 9.20181 \\
-0.115288 & -9.50243 & -0.0261157
\end{pmatrix}, \quad
B_m = \begin{pmatrix}
-0.630503 & 0.00216112 \\
-1.350879 & -0.00377142 \\
-0.222387 & -0.0526803
\end{pmatrix}, \quad
C_m = \begin{pmatrix}
0.291338 & -0.0265117 & -4.03570 \\
354.222 & -164.479 & 26.6355
\end{pmatrix}.$$

This model yields the cost $J = 0.673079$. A model of order $n_m = 4$ is

$$A_m = \begin{pmatrix}
-0.198769 & 0.235667 & -0.0248136 & 0.000915746 \\
-1.08739 & -0.912440 & 9.20181 & -0.00904508 \\
-0.115288 & -9.50243 & -0.0261155 & 0.00159031 \\
-5.46513 & -11.6984 & -1.92997 & -37.5544
\end{pmatrix}, \quad
B_m = \begin{pmatrix}
-0.630503 & 0.00216112 \\
-1.35088 & -0.00377141 \\
-0.222386 & -0.0526803 \\
-8.66651 & -0.0203036
\end{pmatrix}, \quad
C_m^T = \begin{pmatrix}
0.291340 & 354.222 \\
-0.0265302 & -164.479 \\
-4.03569 & 26.6355 \\
0.0861885 & -0.815898
\end{pmatrix}.$$

This model yields the cost $J = 3.22 \cdot 10^{-7}$.

For this example with $n_m = 3, 4$, the columns of the initial Jacobian matrices from input normal form formulations are so badly scaled that the numerical linear algebra in HOMPACK fails. Modifying HOMPACK to use the LINPACK subroutine DQRDC for the QR factorization of the initial Jacobian matrices enables HOMPACK to successfully overcome the ill conditioning and find a solution.

**Example 10 [5].** $A$ is a $2 \times 2$ block diagonal matrix with each diagonal block being of the form

$$\begin{pmatrix}
0 & 1 \\
-\sigma_i^2 & -2\gamma\sigma_i
\end{pmatrix}, \quad i = 1, \ldots, n/2,$$

$$B = (0, b_1, 0, b_2, \ldots, 0, b_{n/2})^T, \quad C = (0, c_1, 0, c_2, \ldots, 0, c_{n/2}),$$

where $\sigma_i = i^2\pi^2$, $b_i = \sqrt{2}\sin(i\pi a)$, $c_i = \sqrt{2}\sin(i\pi s)$ and $y, a, s$ are known parameters. This system was not studied in [78]–[80]. The input normal form approach can not give a solution when $n_m > 1$ because the initial $\omega$s are generated in pairs.

Choosing $n = 16, n_m = 8, y = 0.001, a = 0.1, s = 0.2$, the reduced order model is

$$A_m = \text{diag}(A_1, A_2, A_3, A_4),$$
\[ A_1 = \begin{pmatrix} 0.0 & 1.0 \\ -24936.92 & -0.3158248 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.0 & 1.0 \\ -97.40911 & -0.01973900 \end{pmatrix}, \]
\[ A_3 = \begin{pmatrix} 0.0 & 1.0 \\ -7890.149 & -0.1776489 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0.0 & 1.0 \\ -1558.546 & -0.07895572 \end{pmatrix}, \]
\[ B_m = \begin{pmatrix} 1.118022 \\ 0.3208260 \\ 0.3632675 \\ -0.007030323 \\ 1.538809 \\ -0.1661970 \\ 1.118019 \\ -0.07921770 \end{pmatrix}, \quad C_m = \begin{pmatrix} 1.0 \\ 0.0 \\ 1.0 \\ 0.0 \\ 1.0 \\ 0.0 \end{pmatrix}, \]

which has cost \( J = 2.59857 \). Ly's formulation is very efficient for this problem.

**Example 11.** The system is given by
\[ A = \begin{pmatrix} -1 & 0 & 0 \\ 0.0005 & -1.000001 & 0 \\ 0.0005 & 0.0005 & -1.00001 \end{pmatrix}, \quad B = \begin{pmatrix} 1.1 \\ 1.2 \\ 1.3 \end{pmatrix}, \quad C = \begin{pmatrix} 2.1 & 2.2 & 2.3 \end{pmatrix}. \]

A model of order \( n_m = 2 \) with cost \( J = 0.36 \cdot 10^{-14} \) is
\[ A_m = \begin{pmatrix} -0.999519 & 0.00000 \\ 1.99976 & -1.00024 \end{pmatrix}, \quad B_m = \begin{pmatrix} -1.41387 \\ 1.41438 \end{pmatrix}, \quad C_m^T = \begin{pmatrix} -5.61578 \\ 0.00000 \end{pmatrix}. \]

This system was constructed to illustrate that some problems can be solved by the input normal form formulation or the over-parametrization formulation described below but not by the Ly formulation.
6. HOMOTOPY ALGORITHM BASED ON OVER-PARAMETRIZATION FORMULATION FOR THE $H^2$ MODEL ORDER REDUCTION PROBLEM.

6.1. Over-parametrization formulation.

Both the input normal form formulation and Ly formulation can introduce ill conditioning, resulting from eliminating certain variables so that the minimal number of variables is used. To avoid such ill conditioning, one could use all the elements in $A_m$, $B_m$, and $C_m$ as variables, i.e., not impose any restriction on the representation of $(A_m, B_m, C_m)$.

The same Lagrangian as in (51) is used:

$$ L(A_m, B_m, C_m, \bar{Q}) = \text{tr}[\bar{Q} \bar{R} + (\bar{A} \bar{Q} + \bar{Q} \bar{A}^T + \bar{V}) \bar{P}], $$

where the symmetric matrix $\bar{P}$ is a Lagrange multiplier. Setting $\partial L / \partial \bar{Q} = 0$ gives (23). A straightforward calculation shows

$$ \frac{\partial L}{\partial A_m} = 2(\bar{P}_{12}^T \bar{Q}_{12} + \bar{P}_2 \bar{Q}_2), $$
$$ \frac{\partial L}{\partial B_m} = 2(\bar{P}_{12}^T B + \bar{P}_2 B_m)V, $$
$$ \frac{\partial L}{\partial C_m} = 2R(C_m \bar{Q}_2 - C \bar{Q}_{12}). $$

(60)

6.2. A homotopy approach based on over-parametrization formulation.

Let $A_f, B_f, C_f, R_f$, and $V_f$ denote $A, B, C, R$, and $V$ in the above and define $A(\lambda), B(\lambda), C(\lambda), V(\lambda)$ and $R(\lambda)$ as in (28) and denote them respectively by $A, B, C, V, R$. Let

$$ H_{A_m}(\theta, \lambda) = \frac{\partial L}{\partial A_m} = 2(\bar{P}_{12}^T \bar{Q}_{12} + \bar{P}_2 \bar{Q}_2), $$
$$ H_{B_m}(\theta, \lambda) = \frac{\partial L}{\partial B_m} = 2(\bar{P}_{12}^T B + \bar{P}_2 B_m)V, $$
$$ H_{C_m}(\theta, \lambda) = \frac{\partial L}{\partial C_m} = 2R(C_m \bar{Q}_2 - C \bar{Q}_{12}), $$

(61)

where

$$ \theta \equiv \begin{pmatrix} \text{Vec}(A_m) \\ \text{Vec}(B_m) \\ \text{Vec}(C_m) \end{pmatrix} $$

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denotes the independent variables $A_m$, $B_m$, and $C_m$, and $\tilde{Q}$ and $\tilde{P}$ satisfy respectively (20) and (23). Define

$$\rho(\theta, \lambda) = \begin{pmatrix} \text{Vec} \left[ H_{A_m}(\theta, \lambda) \right] \\ \text{Vec} \left[ H_{B_m}(\theta, \lambda) \right] \\ \text{Vec} \left[ H_{C_m}(\theta, \lambda) \right] \end{pmatrix},$$

whose Jacobian matrix is

$$D\rho(\theta, \lambda) = (D_{\theta}\rho(\theta, \lambda), D_\lambda \rho(\theta, \lambda)).$$

Because of the over-parametrization, the Jacobian matrix of $\rho$ is singular. The homotopy map is defined as

$$\dot{\rho}(\theta, \lambda) = \lambda \rho(\theta, \lambda) + (1 - \lambda)(\theta - \theta_0),$$

which guarantees a well conditioned Jacobian matrix along the whole path except at the solution corresponding to $\lambda = 1$. The Jacobian matrix is given by

$$D\dot{\rho}(\theta, \lambda) = (\lambda D_{\theta}\rho(\theta, \lambda) + (1 - \lambda)I, \rho(\theta, \lambda) + \lambda D_\lambda \rho(\theta, \lambda) - (\theta - \theta_0)).$$

To find $D_{\theta} \rho(\theta, \lambda)$, define $\hat{H}_{A_m}(\tilde{p}^{(j)}, \tilde{q}^{(j)})$, $\hat{H}_{B_m}(\tilde{p}^{(j)})$, and $\hat{H}_{C_m}(\tilde{q}^{(j)})$ as in (56), where again the superscript $(j)$ means $\partial/\partial \theta_j$. For $\theta_j = (A_m)_{kl}$,

$$\frac{\partial H_{A_m}}{\partial (A_m)_{kl}} = \hat{H}_{A_m}(\tilde{p}^{(j)}, \tilde{q}^{(j)})$$

$$\frac{\partial H_{B_m}}{\partial (A_m)_{kl}} = \hat{H}_{B_m}(\tilde{p}^{(j)})$$

$$\frac{\partial H_{C_m}}{\partial (A_m)_{kl}} = \hat{H}_{C_m}(\tilde{q}^{(j)})$$

for $\theta_j = (B_m)_{kl}$,

$$\frac{\partial H_{A_m}}{\partial (B_m)_{kl}} = \hat{H}_{A_m}(\tilde{p}^{(j)}, \tilde{q}^{(j)})$$

$$\frac{\partial H_{B_m}}{\partial (B_m)_{kl}} = \hat{H}_{B_m}(\tilde{p}^{(j)}) + 2\tilde{P} E^{(k,l)} V,$$

$$\frac{\partial H_{C_m}}{\partial (B_m)_{kl}} = \hat{H}_{C_m}(\tilde{q}^{(j)})$$
and for $\theta_j = (C_m)_{kl}$,

$$\frac{\partial H_A}{\partial (C_m)_{kl}} = \dot{H}_A = (\dot{P}^{(j)}, \dot{Q}^{(j)})$$

$$\frac{\partial H_B}{\partial (C_m)_{kl}} = \dot{H}_B = (\ddot{P}^{(j)})$$

$$\frac{\partial H_C}{\partial (C_m)_{kl}} = \dot{H}_C = (\dot{Q}^{(j)}) + 2RE^{(k,l)}\dot{Q}_2.$$ (66)

$\ddot{P}^{(j)}$ and $\dot{Q}^{(j)}$ can be obtained by solving the Lyapunov equations (33). The derivatives with respect to $\lambda$ can be obtained in the same way as in Chapter 2.

6.3. Numerical algorithm for over--parametrization formulation.

The initial point $(\theta, \lambda) = (\theta_0, 0) = ((A_m)_0, (B_m)_0, (C_m)_0, 0)$ is chosen so that the triple $((A_m)_0, (B_m)_0, (C_m)_0)$ is in balanced form and satisfies $\rho(\theta_0, 0) = 0$. The algorithm is similar to steps 1–8 described in Chapter 2, except that the homotopy $\dot{\rho}$ from (62) is used.

For all the test problems except Example 6 with $n_m = 3$ and Example 9 with $n_m = 2, 3, 4$, the above algorithm gives satisfactory results by adjusting the curve tracking precision. For these exceptional cases, HOMPACK reaches $\lambda \geq 1$ very fast, but because of the high order singularity at the solution, the computed solution does not have acceptable accuracy. Although very sophisticated methods for dealing with singular endpoints of homotopy curves are known [50–52], these are difficult to implement in the present context, and the following simple algorithm was adequate.

1) Use the algorithm in Chapter 2 to track the curve until $\lambda \geq 1$.

2) Use the last point $(\bar{\theta}, \bar{\lambda})$ before $\lambda \geq 1$ to redefine the homotopy map with $\theta_0 = \bar{\theta}$ and set $\lambda = 0$.

3) Redo step 1.

4) Use Hermite polynomial interpolation to obtain the solution at $\lambda = 1$.

In Step 3 the new homotopy (62) has a zero curve that is nearly a straight line, and thus Hermite interpolation using points before $\lambda = 1$ and one point with $\lambda \geq 1$ is quite accurate. Care must be taken to use data points away from the singularity (lest they be inaccurate), but this is easily done by controlling the step size parameters in HOMPACK.

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7. COMPARISON AND DISCUSSION.

Table 1 gives the CPU times in seconds and the number of steps needed to obtain the results for each example (a dash indicates failure). The CPU times are for a DECstation 5000/200, using double precision, IEEE arithmetic, and the MIPS RISC f77 compiler. Table 2 gives the comparison of the optimal projection equations approach and the input normal form formulation for Examples 8 and 9. The asterisks in Table 1 denote the cases requiring Hermite polynomial interpolation to obtain the solution for the over-parametrization formulation. The asterisks in Table 2 indicate cases that required special numerical linear algebra techniques to deal with severe scaling errors.

<p>| | | | | | | | |</p>
<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>Ex</td>
<td>$n_m$</td>
<td>Input normal form</td>
<td>Ly's form</td>
<td>Over-parametrization</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
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<td>time</td>
<td>steps</td>
<td>time</td>
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<td>0.21</td>
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<td>0.19</td>
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<td>13</td>
<td>0.17</td>
<td>14</td>
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<td>12</td>
<td>0.27</td>
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<td>220</td>
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<tr>
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<td>1</td>
<td>2</td>
<td>-</td>
<td>8</td>
<td>0.32</td>
<td>35</td>
<td>1.3</td>
</tr>
<tr>
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<td>15</td>
<td>0.22</td>
<td>15</td>
<td>0.33</td>
<td>13</td>
<td>0.21</td>
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<td>0.78</td>
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<td>127</td>
<td>8.0</td>
<td>-</td>
<td>-</td>
<td>168*</td>
<td>13.</td>
</tr>
<tr>
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<td>3</td>
<td>9</td>
<td>1.3</td>
<td>45</td>
<td>6.7</td>
<td>21*</td>
<td>4.2</td>
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<td>4</td>
<td>8</td>
<td>1.9</td>
<td>59</td>
<td>15.</td>
<td>17*</td>
<td>7.5</td>
</tr>
<tr>
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<td>8</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>7</td>
<td>49.</td>
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<tr>
<td>18</td>
<td>6</td>
<td>0.13</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>16</td>
<td>0.42</td>
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</table>

As shown by Table 1, the input normal form homotopy can be very efficient. Also there is no need to adjust any parameter to achieve this efficiency (although to obtain
the minimum solution of Example 1, some adjustment of the initial point was necessary. However, note that the potential ill conditioning of the input normal form formulation can result in failure (Examples 6 and 10) or the need for extraordinarily delicate linear algebra (Example 9).

Figures 1 and 2 show the behavior of the largest variation component with respect to $\lambda$ for Example 5 at $n_m = 1$ and Example 9 at $n_m = 2$ using the input normal form formulation and the optimal projection equation formulation [78]-[80]. The figures show that component of the solution vector with the largest total amount of oscillation, corresponding to the most difficult component of the homotopy path to track. Even though Fig. 1 corresponds to a good choice of the initial point for the optimal projection equations approach, it is obviously not as efficient as the input normal form formulation. Generally speaking, since the number of variables in the input normal form and Ly formulations is much smaller than that of
the optimal projection equations formulation, and the strategy for choosing initial points uses balancing (hence giving an initial point closer to the final solution in most cases), the input normal form and Ly form homotopies are more efficient than the optimal projection equations homotopy.

For Example 9, when \( n_m = 1 \), the Ly's form homotopy is extremely inefficient, requiring \( c_1 \) and \( c_2 \) to be adjusted to achieve a solution. All attempts to obtain a solution when \( n_m = 2 \) failed. The solutions of Example 9 when \( n_m = 3 \) and \( n_m = 4 \) are singular, which accounts for the large number of steps required by Ly's form.

The optimal projection equations homotopy successfully solved all of the test problems, but Table 2, containing typical results, shows that the minimal parameter homotopies are much more efficient. However, when the input normal form and Ly's form are used, some
Table 2. Comparison of methods.

<table>
<thead>
<tr>
<th>Example 8</th>
<th>Optimal projection</th>
<th>input normal form</th>
</tr>
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<tbody>
<tr>
<td>n_m</td>
<td># steps</td>
<td>time (sec)</td>
</tr>
<tr>
<td>1</td>
<td>35</td>
<td>0.6</td>
</tr>
<tr>
<td>2</td>
<td>61</td>
<td>2.7</td>
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<tr>
<td>3</td>
<td>129</td>
<td>14.</td>
</tr>
<tr>
<td>Example 9</td>
<td></td>
<td></td>
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<tr>
<td>2</td>
<td>615</td>
<td>88</td>
</tr>
<tr>
<td>3</td>
<td>641</td>
<td>223</td>
</tr>
<tr>
<td>4</td>
<td>711</td>
<td>518</td>
</tr>
</tbody>
</table>

Restrictions are imposed on the structure of the triple \((A_m, B_m, C_m)\), potentially resulting in ill conditioning. For the input normal form formulation, ill conditioning occurs if two diagonal elements of \(\Omega\) in (4) are approximately the same. In other words, let \(Q_m\) and \(P_m\) be the controllability and observability Gramians of the system represented by \((A_m, B_m, C_m)\), and let

\[
Q_m = W\Sigma W^T, \quad P_m = W^{-T}\Sigma W^{-1},
\]

where \(\Sigma\) is diagonal and is the controllability and observability Gramian in balanced form. If two diagonal elements of \(\Sigma\) are approximately the same, then ill conditioning occurs. For Example 6, when \(n_m = 2, 3\), both the initial point chosen using the given strategy and the solution obtained in [78]–[80] or by Ly's formulation are ill conditioned, i.e., two diagonal elements of \(\Omega\) are approximately the same. Hence the input normal form method will not be able to solve this problem.

For Ly's formulation, ill conditioning occurs if the Jordan decomposition of \(A_m\) is ill conditioned. Precisely, if the two eigenvalues of \(A_m\) which are to be grouped into a \(2 \times 2\) block are approximately the same, the transition matrix to \(2 \times 2\) block diagonal form is ill conditioned. This can be clearly illustrated by observing that for \(n_m = 2\), finding the Ly form is equivalent to finding the transition matrix \(T \in \mathbb{R}^{2 \times 2}\) such that

\[
\begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix}
\begin{pmatrix}
t_{11} & t_{12} \\
t_{21} & t_{22}
\end{pmatrix}
= \begin{pmatrix}
t_{11} & t_{12} \\
t_{21} & t_{22}
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
-\lambda_1 \lambda_2 & \lambda_1 + \lambda_2
\end{pmatrix},
\]

\(C_m(1, 1)t_{11} + C_m(1, 2)t_{21} = 1,\)

\(C_m(1, 1)t_{12} + C_m(1, 2)t_{22} = 0,\)

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where $\lambda_1$ and $\lambda_2$ are the eigenvalues of $A_m$. Trivial algebra gives

\[ t_{11} = -\lambda_2 C_m(1,1)^{-1} \lambda_{12}^{-1}, \quad t_{12} = C_m(1,1)^{-1} \lambda_{12}^{-1}, \]
\[ t_{21} = \lambda_1 C_m(1,2)^{-1} \lambda_{12}^{-1}, \quad t_{22} = -C_m(1,2)^{-1} \lambda_{12}^{-1}, \]
\[ \text{cond } T = \sigma + \sqrt{\sigma^2 - 1}, \]

where

\[ \lambda_{12} = \lambda_1 - \lambda_2, \quad \tau = \frac{C_m(1,1)}{C_m(1,2)}, \quad \sigma = \frac{1 + \tau^2 + \lambda_2^2 + \tau^2 \lambda_1^2}{2|\tau \lambda_{12}|}. \]

Thus ill conditioning occurs in general when $\sigma$ is large, and in particular when $\tau \lambda_{12} \approx 0$. Furthermore, note that the very existence of the Ly form is predicated on the assumption that the Jordan form of $A_m$ consists of $2 \times 2$ Jordan blocks, which is a rather strong assumption.

Both the input normal form formulation and Ly’s formulation can fail to exist or lead to ill conditioning and it is conceivable that both of these formulations will fail for some problems. This failure of existence in general is related to the insistence on using the minimal number of parameters $n_m m + n_m l$. The over-parametrization formulation solves the ill conditioning issue, but introduces a very high order singularity at the solution. It is doubtful whether either the Hermite interpolation used here or the techniques of [50]-[52] can handle a large problem with a singularity of order 100. A pragmatic suggestion is to try in order the input normal form, Ly’s form, and the over-parametrization form, switching if ill conditioning or failure occurs. The ideal paradigm would be to have a family of minimal formulations, almost all of which exist for any given problem. The homotopy algorithm would then dynamically adjust the formulation, finding a well conditioned one and tracking its zero curve simultaneously. Such a paradigm remains an open question.
8. HOMOTOPY ALGORITHM BASED ON THE INPUT NORMAL FORM FOR THE COMBINED $H^2/H^\infty$ MODEL REDUCTION PROBLEM.

8.1. The formulation.

To optimize $J(A_m, B_m, C_m)$ over the open set $S$ under the constraints that symmetric positive definite $Q$ satisfies (14), and $(A_m, B_m, C_m)$ is in input normal form, the following Lagrangian is formed:

$$\mathcal{L}(A_m, B_m, C_m, \Omega, Q, \mathcal{P}, M_c, M_o) \equiv \text{tr} \left[ Q \hat{R} + (\hat{A}Q + A\hat{Q}^T + \gamma^{-2}Q\hat{R}Q + \hat{V})\mathcal{P} \right] + (A_m + A_m^T + B_m V B_m^T) M_c + (A_m^T \Omega + \Omega A_m + C_m^T R C_m) M_o,$$

where the symmetric matrices $M_c$, $M_o$, and $\mathcal{P} \in \mathbb{R}^{n \times n}$ are Lagrange multipliers. $\Omega = \text{diag} (\omega_1, \ldots, \omega_{n_m})$ is related to the input normal form constraint. Setting $\partial \mathcal{L}/\partial Q = 0$ yields

$$0 = (\hat{A} + \gamma^{-2}Q \hat{R})^T \mathcal{P} + \mathcal{P} (\hat{A} + \gamma^{-2}Q \hat{R}) + \hat{R}. \quad (67)$$

Partition $Q$, $\mathcal{P} \in \mathbb{R}^{n \times n}$ into

$$Q = \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} \mathcal{P}_1 & \mathcal{P}_{12} \\ \mathcal{P}_{12}^T & \mathcal{P}_2 \end{pmatrix} \quad (68)$$

where $Q_1, \mathcal{P}_1 \in \mathbb{R}^{n \times n}$ and $Q_2, \mathcal{P}_2 \in \mathbb{R}^{n_m \times n_m}$. Define

$$\mathcal{P} Q \equiv Z = \begin{pmatrix} Z_1 & Z_{12} \\ Z_{21} & Z_2 \end{pmatrix} \quad (69)$$

where

$$Z_1 = \mathcal{P}_1 Q_1 + \mathcal{P}_{12} Q_{12}^T, \quad Z_{12} = \mathcal{P}_1 Q_{12} + \mathcal{P}_{12} Q_2, \quad Z_{21} = \mathcal{P}_{12}^T Q_1 + \mathcal{P}_2 Q_{12}^T, \quad Z_2 = \mathcal{P}_{12}^T Q_{12} + \mathcal{P}_2 Q_2.$$

$\partial \mathcal{L}/\partial \Omega = 0$ and $\partial \mathcal{L}/\partial A_m = 0$ yield

$$0 = 2M_c + 2\Omega M_o + 2 (\mathcal{P}_{12}^T Q_{12} + \mathcal{P}_2 Q_2), \quad 0 = (A_m M_o)_{ii}, \quad 1 \leq i \leq n_m.$$
A straightforward calculation shows

\[ \frac{\partial L}{\partial B_m} = 2(P_{12}^TB + P_2B_mV) + 2M_cB_mV, \]

\[ \frac{\partial L}{\partial C_m} = 2(RC_mQ_2 - RC_{12}) + 2RC_mM_o \]

\[ + \gamma^{-2} \left[ -RC(Z_1^TQ_{12} + Z_2^TQ_2 + Q_1Z_{12} + Q_{12}Z_2) \right. \]

\[ + \left. RC_m(Q_{12}^TZ_{12} + Z_1^TQ_{12} + Q_2Z_2 + Z_2^TQ_2) \right]. \]

(70)

The matrices $M_c$ and $M_o$ in (70) satisfy (26).

A homotopy approach based on the input normal form is now described. Let $A_f$, $B_f$, $C_f$, $R_f$, $V_f$, and $\gamma_f$ denote $A$, $B$, $C$, $R$, $V$, and $\gamma$ in the above and define $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $R(\lambda)$, and $V(\lambda)$ as in (28) and define

\[ \gamma(\lambda) = \gamma_0 + \lambda(\gamma_f - \gamma_0). \]

For brevity, $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $R(\lambda)$, $V(\lambda)$, and $\gamma(\lambda)$ will be denoted by $A$, $B$, $C$, $R$, $V$, and $\gamma$ respectively in the following. Let

\[ H_{B_m}(\theta, \lambda) = \frac{\partial L}{\partial B_m} = 2(P_{12}^TB + P_2B_m)V + 2M_cB_mV, \]

\[ H_{C_m}(\theta, \lambda) = \frac{\partial L}{\partial C_m} = 2R(C_mQ_2 - CQ_{12}) + 2RC_mM_o \]

\[ + \gamma^{-2} \left[ -RC(Z_1^TQ_{12} + Z_2^TQ_2 + Q_1Z_{12} + Q_{12}Z_2) \right. \]

\[ + \left. RC_m(Q_{12}^TZ_{12} + Z_1^TQ_{12} + Q_2Z_2 + Z_2^TQ_2) \right], \]

where

\[ \theta \equiv \begin{pmatrix} \text{Vec}(B_m) \\ \text{Vec}(C_m) \end{pmatrix} \]

denotes the independent variables $B_m$ and $C_m$, $M_o$ and $M_c$ satisfy (26), and $Q$ and $P$ satisfy respectively (14) and (67) with partitioned forms (68). Vec($P$) for a matrix $P \in \mathbb{R}^{p \times q}$ is the concatenation of its columns:

\[ \text{Vec}(P) \equiv \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_q \end{pmatrix} \in \mathbb{R}^{pq}. \]
The homotopy map is defined as
\[
\rho(\theta, \lambda) = \begin{pmatrix} \text{Vec} [H_{B_m}(\theta, \lambda)] \\ \text{Vec} [H_{C_m}(\theta, \lambda)] \end{pmatrix},
\]
and its Jacobian matrix is
\[
D\rho(\theta, \lambda) = (D_\theta \rho(\theta, \lambda), D_\lambda \rho(\theta, \lambda)).
\]

Define
\[
\begin{align*}
\dot{H}_{B_m}(\mathcal{P}^{(j)}, M_c^{(j)}) &= 2(\mathcal{P}^{(j)}_{12} B + \mathcal{P}^{(j)}_2 B_m) V + 2M_c^{(j)}B_m V, \\
\dot{H}_{C_m}(Q^{(j)}, Z^{(j)}, M_o^{(j)}) &= 2R(C_m Q_2^{(j)} - C Q_{12}^{(j)}) + 2RC_m M_o^{(j)} \\
&- \gamma^{-2} RC \left( Z_1^{T(j)} Q_{12} + Z_2^{T(j)} Q_2 + Z_1^{T} Q_{12}^{(j)} + Z_2^{T} Q_2^{(j)} \right) \\
&+ Q_1^{(j)} Z_{12} + Q_1 Z_{12}^{(j)} + Q_{12}^{(j)} Z_2 + Q_{12} Z_2^{(j)} \\
&+ \gamma^{-2} RC_m \left( Z_{12}^{T(j)} Q_{12} + Z_{12}^{T} Q_{12}^{(j)} + Q_{12}^{(j)} Z_{12} + Q_{12} Z_{12}^{(j)} \right) \\
&+ Q_2^{(j)} Z_2 + Z_2^{T(j)} Q_2 + Q_2 Z_2^{(j)} + Z_2 Z_2^{(j)},
\end{align*}
\]
where the superscript \((j)\) means \(\partial/\partial \theta_j\); \(Y^{(j)} \equiv \partial Y/\partial \theta_j\). Using the above definitions, we have for \(\theta_j = (B_m)_{kl}\),
\[
\begin{align*}
\frac{\partial H_{B_m}}{\partial (B_m)_{kl}} &= \dot{H}_{B_m}(\mathcal{P}^{(j)}, M_c^{(j)}) + 2(\mathcal{P}^{(j)}_{12} + M_c) E^{(k,l)} V, \\
\frac{\partial H_{C_m}}{\partial (B_m)_{kl}} &= \dot{H}_{C_m}(Q^{(j)}, Z^{(j)}, M_o^{(j)}),
\end{align*}
\]
and for \(\theta_j = (C_m)_{kl}\),
\[
\begin{align*}
\frac{\partial H_{B_m}}{\partial (C_m)_{kl}} &= \dot{H}_{B_m}(\mathcal{P}^{(j)}, M_c^{(j)}), \\
\frac{\partial H_{C_m}}{\partial (C_m)_{kl}} &= \dot{H}_{C_m}(Q^{(j)}, Z^{(j)}, M_o^{(j)}) + 2RE^{(k,l)} (Q_2 + M_o) \\
&+ \gamma^{-2} RE^{(k,l)} \left( Z_{12}^{T} Q_{12} + Q_{12}^{T} Z_{12} + Q_{2}^{T} Z_2 + Z_{2}^{T} Q_2 \right),
\end{align*}
\]
where \(E^{(k,l)}\) is a matrix of the appropriate dimension whose only nonzero element is \(e_{kl} = 1\). \(\mathcal{P}^{(j)}\) and \(Q^{(j)}\) can be obtained by solving the Lyapunov equations
\[
\begin{align*}
0 &= (\tilde{A} + \gamma^{-2} \tilde{Q} \tilde{R}) Q^{(j)} + Q^{(j)} (\tilde{A} + \gamma^{-2} \tilde{Q} \tilde{R})^T + \tilde{V}^{(j)} + \tilde{A}^{(j)} Q + Q \tilde{A}^{T(j)} + \gamma^{-2} Q \tilde{R}^{(j)} Q, \\
0 &= (\tilde{A} + \gamma^{-2} \tilde{Q} \tilde{R})^T P^{(j)} + P^{(j)} (\tilde{A} + \gamma^{-2} \tilde{Q} \tilde{R}) + \tilde{R}^{(j)} \\
&+ (\tilde{A}^{(j)} + \gamma^{-2} \tilde{Q}^{(j)} \tilde{R} + \gamma^{-2} Q \tilde{R}^{(j)} \tilde{Q}) P + P (\tilde{A}^{(j)} + \gamma^{-2} Q^{(j)} \tilde{R} + \gamma^{-2} Q \tilde{R}^{(j)}).\end{align*}
\]

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Similarly for $\lambda$, using a dot to denote $\partial / \partial \lambda$,

\[
\begin{align*}
\frac{\partial H_{B_m}}{\partial \lambda} = & \, \hat{H}_{B_m}(\hat{\dot{P}}, \hat{M}_c) + 2\hat{P}_{12}^T(\hat{B} \dot{V} + B \dot{V}) + 2(\hat{P}_2 + M_c)B_m \dot{V}, \\
\frac{\partial H_{C_m}}{\partial \lambda} = & \, \hat{H}_{C_m}(\hat{\dot{Q}}, \hat{\dot{\dot{Z}}}, \hat{M}_o) + 2\hat{R}C_m(Q_2 + M_o) - 2(\hat{R}C + R\dot{C})Q_{12} \\
& + \gamma^{-2} \hat{R} h_{\lambda} - 2\gamma^{-3} \gamma \hat{R} h_{\lambda} - \gamma^{-2} \hat{R} \dot{C}(Z_{12}^T Q_{12} + Z_{22}^T Q_2 + Q_1 Z_{12} + Q_{12} Z_2),
\end{align*}
\]  

(76)

where

\[ h_{\lambda} = -C(Z_{12}^T Q_{12} + Z_{22}^T Q_2 + Q_1 Z_{12} + Q_{12} Z_2) + C_m(Q_{12}^T Z_{12} + Z_{12}^T Q_{12} + Q_2 Z_2 + Z_2^T Q_2), \]

and $\dot{\hat{P}}$ and $\dot{\hat{Q}}$ are obtained by solving the Lyapunov equations

\[
0 = (\hat{A} + \gamma^{-2} Q \hat{R})\dot{\hat{Q}} + \dot{\hat{Q}}(\hat{A} + \gamma^{-2} Q \hat{R})^T + \dot{\hat{V}} + \hat{A} \dot{\hat{Q}} + Q \hat{A} \dot{\hat{Q}} + \gamma^{-2} Q \hat{R} \dot{Q} - 2\gamma^{-3} \gamma \hat{R} \dot{Q},
\]

\[
0 = (\hat{A} + \gamma^{-2} Q \hat{R})^T \dot{\hat{P}} + \dot{\hat{P}}(\hat{A} + \gamma^{-2} Q \hat{R}) + \dot{\hat{R}}
\]

\[
+ (\hat{A} + \gamma^{-2} \dot{Q} \hat{R} + \gamma^{-2} Q \hat{R} - 2\gamma^{-3} \gamma \hat{R}) \dot{Q} + \dot{\hat{P}}(\hat{A} + \gamma^{-2} \dot{Q} \hat{R} + \gamma^{-2} Q \hat{R} - 2\gamma^{-3} \gamma \hat{R} \dot{Q}).
\]

(77)

### 8.2. Numerical algorithm for input normal form homotopy.

The initial point $(\theta, \lambda) = (\theta_0, 0) = ((B_m)_0, (C_m)_0, 0)$ is ideally chosen so that the triple $((A_m)_0, (B_m)_0, (C_m)_0)$ is in input normal form and satisfies $\rho(\theta_0, 0) = 0$.

This suggests that if the initial $\gamma$, i.e., $\gamma_0$, is chosen to be very large, the same approach as for the $H^2$ optimal model reduction problem in Chapter 2 can be used. Therefore the initial point $(\theta_0, 0)$ is chosen the same way as step 1)–5) in Chapter 2.3 while $\gamma_0$ is chosen to be very large.

Once the initial point is chosen, the rest of the computation is as follows:

1) Set $\lambda := 0$, $\theta := \theta_0$.

2) Calculate $A_m$ from $B_m$ and $C_m$, $\hat{R}$, $\tilde{V}$, and compute $Q$ and $P$ according to (14) and (67).

3) Evaluate $S$ from (27) and $M_o$ and $M_c$ according to (26).

4) Evaluate the homotopy map $\rho(\theta, \lambda)$ in (71) and $D\rho(\theta, \lambda)$ in (72).

5) Predict the next point $Z^{(0)} = (\theta^{(0)}, \lambda^{(0)})$ on the homotopy zero curve using, e.g., a Hermite cubic interpolant.
6) For $k := 0, 1, 2, \cdots$ until convergence do

$$Z^{(k+1)} = Z^{(k)} - [D\rho(Z^{(k)})]^{\dagger}\rho(Z^{(k)}),$$

where $[D\rho(Z)]^{\dagger}$ is the Moore-Penrose inverse of $D\rho(Z)$. Let $(\theta_1, \lambda_1) = \lim_{k \to \infty} Z^{(k)}$.

7) If $\lambda_1 < 1$, then set $\theta := \theta_1$, $\lambda := \lambda_1$, and go to step 2).

8) If $\lambda_1 \geq 1$, compute the solution $\bar{\theta}$ at $\lambda = 1$. $A_m$ is then obtained from $B_m$ and $C_m$. 
9. HOMOTOPY ALGORITHM BASED ON LY FORMULATION FOR
THE COMBINED $H^2/H^\infty$ MODEL REDUCTION PROBLEM.

To optimize $J(A_m, B_m, C_m)$ over the open set $S$ under the constraint that symmetric positive definite $Q$ satisfies (14), and $(A_m, B_m, C_m)$ is in Ly's form, the following Lagrangian is formed:

$$\mathcal{L}(A_m, B_m, C_m, \mathcal{P}, Q) \equiv \text{tr} \left[ Q \dot{R} + (\dot{A}Q + Q \dot{A}^T + \gamma^{-2} Q \dot{R} Q + \dot{V}) \mathcal{P} \right],$$

where $\mathcal{P} \in \mathbb{R}^{n \times n}$ is a Lagrange multiplier. Setting $\partial \mathcal{L}/\partial Q = 0$ yields (67). Partition $Q$, $\mathcal{P} \in \mathbb{R}^{n \times n}$ as in (68) and define $\mathcal{P} Q = Z$ as in (69). The partial derivatives of $\mathcal{L}$ can be computed as

$$\frac{\partial \mathcal{L}}{\partial (A_m)_{ij}} = 2(\mathcal{P}_{12} Q_{12} + \mathcal{P}_2 Q_2), \quad (i, j) \in I,$$

$$\frac{\partial \mathcal{L}}{\partial B_m} = 2(\mathcal{P}_{12}^T B V + \mathcal{P}_2 B_m V),$$

$$\frac{\partial \mathcal{L}}{\partial (C_m)_{ij}} = 2(R C_m Q_2 - R C Q_{12})_{ij} +$$

$$+ \gamma^{-2}[-R C(Z_{11}^T Q_{12} + Z_{21}^T Q_2 + Q_1 Z_{12} + Q_{12} Z_2)$$

$$+ R C_m (Q_{12}^T Z_{12} + Z_{12}^T Q_{12} + Q_2 Z_2 + Z_2^T Q_2)]_{ij}. \quad (78)$$

Let $A_f, B_f, C_f, R_f, V_f$, and $\gamma_f$ denote $A, B, C, R, V$, and $\gamma$ in the above and define $A(\lambda), B(\lambda), C(\lambda), R(\lambda), V(\lambda)$ as in (28) and

$$\gamma(\lambda) = \gamma_0 + \lambda (\gamma_f - \gamma_0)$$

and denote them by $A, B, C, R, V$, and $\gamma$ respectively in the following. Let

$$H_{A_m}(\theta, \lambda) = \frac{\partial \mathcal{L}}{\partial A_m} = 2(\mathcal{P}_{12}^T Q_{12} + \mathcal{P}_2 Q_2),$$

$$H_{B_m}(\theta, \lambda) = \frac{\partial \mathcal{L}}{\partial B_m} = 2(\mathcal{P}_{12}^T B + \mathcal{P}_2 B_m) V,$$

$$H_{C_m}(\theta, \lambda) = \frac{\partial \mathcal{L}}{\partial C_m} = 2R (C_m Q_2 - C Q_{12})$$

$$+ \gamma^{-2}[-R C(Z_{11}^T Q_{12} + Z_{21}^T Q_2 + Q_1 Z_{12} + Q_{12} Z_2)$$

$$+ R C_m (Q_{12}^T Z_{12} + Z_{12}^T Q_{12} + Q_2 Z_2 + Z_2^T Q_2)],$$
where in $H_{A_m}$ only those elements corresponding to the parameter elements of $A_m$ are of interest and

$$\theta = \begin{pmatrix} (A_m)_I \\ \text{Vec} (B_m) \\ \text{Vec} (C_m)_I \end{pmatrix}$$  \hspace{1cm} (79)$$

denotes the independent variables, $Q$ and $P$ satisfy respectively (14) and (67), $(A_m)_I$ is a vector consisting of those elements in $A_m$ with indices in the set $I$, i.e.,

$$(A_m)_I = ((A_m)_{21}, (A_m)_{22}, \ldots, (A_m)_{nm}n_m)^T,$$

and $(C_m)_I$ is the matrix obtained from rows $T = \{2, \ldots, l\}$ of $C_m$.

The homotopy map is defined as

$$\rho(\theta, \lambda) = \begin{pmatrix} [H_{A_m}(\theta, \lambda)]_I \\ \text{Vec} [H_{B_m}(\theta, \lambda)] \\ \text{Vec} [H_{C_m}(\theta, \lambda)]_I \end{pmatrix},$$  \hspace{1cm} (80)$$

and its Jacobian matrix is

$$D\rho(\theta, \lambda) = (D_\theta \rho(\theta, \lambda), D_\lambda \rho(\theta, \lambda)).$$

Define

$$\hat{H}_{A_m}(P^{(j)}, Q^{(j)}) = 2(P^{(j)}_{12}Q_{12} + P^{(j)}_{12}Q^{(j)}_{12} + P^{(j)}_{2}Q_{2} + P^{(j)}_{2}Q^{(j)}_{2}),$$

$$\hat{H}_{B_m}(P^{(j)}) = 2(P^{(j)}_{12}B + P^{(j)}_{2}B_m)V,$$

$$\hat{H}_{C_m}(Q^{(j)}, Z^{(j)}) = 2R(C_mQ^{(j)}_{2} - CQ^{(j)}_{2}),$$

$$\begin{align*}
\gamma^{-2}RC(Z^{(j)}_{1}Q_{12} + Z^{(j)}_{21}Q_{2} + Z^{(j)}_{1}Q^{(j)}_{12} + Z^{(j)}_{21}Q^{(j)}_{2}) \\
+ Q^{(j)}_{1}Q_{12} + Q^{(j)}_{1}Q_{12} + Q^{(j)}_{2}Q_{2} + Q^{(j)}_{1}Q^{(j)}_{2} \\
+ \gamma^{-2}RC_m(Z^{(j)}_{1}Q_{12} + Z^{(j)}_{1}Q^{(j)}_{12} + Q^{(j)}_{12}Z^{(j)}_{12} + Q^{(j)}_{12}Z^{(j)}_{12} \\
+ Q^{(j)}_{12}Z_{2} + Z^{(j)}_{2}Q_{2} + Q^{(j)}_{12}Z_{2} + Z^{(j)}_{2}Q^{(j)}_{2}),
\end{align*}$$ \hspace{1cm} (81)$$

where the superscript $(j)$ means $\partial/\partial \theta_j$. Using the above definitions, we have for $\theta_j = (A_m)_{kl}$, where $(k, l) \in I$,

$$\frac{\partial H_{A_m}}{\partial (A_m)_{kl}} = \hat{H}_{A_m}(P^{(j)}, Z^{(j)}),$$

$$\frac{\partial H_{B_m}}{\partial (A_m)_{kl}} = \hat{H}_{B_m}(P^{(j)}),$$

$$\frac{\partial H_{C_m}}{\partial (A_m)_{kl}} = \hat{H}_{C_m}(Q^{(j)}, Z^{(j)}),$$ \hspace{1cm} (82)$$

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for $\theta_j = (B_m)_{kl}$,
\[
\frac{\partial H_{A_m}}{\partial (B_m)_{kl}} = \dot{H}_{A_m}(\mathcal{P}^{(j)}, \mathcal{Q}^{(j)}), \\
\frac{\partial H_{B_m}}{\partial (B_m)_{kl}} = \dot{H}_{B_m}(\mathcal{P}^{(j)}) + 2\mathcal{P}_2 E^{(k,l)} V, \\
\frac{\partial H_{C_m}}{\partial (B_m)_{kl}} = \dot{H}_{C_m}(\mathcal{Q}^{(j)}, \mathcal{Z}^{(j)}),
\]
and for $\theta_j = (C_m)_{kl}$, where $k > 1$,
\[
\frac{\partial H_{A_m}}{\partial (C_m)_{kl}} = \dot{H}_{A_m}(\mathcal{P}^{(j)}, \mathcal{Q}^{(j)}), \\
\frac{\partial H_{B_m}}{\partial (C_m)_{kl}} = \dot{H}_{B_m}(\mathcal{P}^{(j)}), \\
\frac{\partial H_{C_m}}{\partial (C_m)_{kl}} = \dot{H}_{C_m}(\mathcal{Q}^{(j)}, \mathcal{Z}^{(j)}) + 2RE^{(k,l)} Q_2 + \gamma^{-2} RE^{(k,l)} (Z_{12}^T Q_{12} + Q_{12}^T Z_{12} + Q_{12}^T Z_2 + Z_{12}^T Q_2),
\]
where $\mathcal{P}^{(j)}$ and $\mathcal{Q}^{(j)}$ can be obtained by solving the Lyapunov equation (75). Similarly for $\lambda$, using a dot to denote $\partial/\partial \lambda$,
\[
\frac{\partial H_{A_m}}{\partial \lambda} = \dot{H}_{A_m}(\mathcal{P}, \mathcal{Q}), \\
\frac{\partial H_{B_m}}{\partial \lambda} = \dot{H}_{B_m}(\mathcal{P}) + 2\mathcal{P}_1^T (B V + B \dot{V}) + 2\mathcal{P}_2 B_m \dot{V}, \\
\frac{\partial H_{C_m}}{\partial \lambda} = \dot{H}_{C_m}(\mathcal{Q}, \mathcal{Z}) - 2(\dot{R}C + R \dot{C}) Q_{12} + 2\dot{R}C_m Q_2 + \gamma^{-2} R h_\lambda - 2\gamma^{-3} \gamma R h_\lambda - \gamma^{-2} R \dot{C}(Z_{12}^T Q_{12} + Z_{21}^T Q_2 + Q_{1} Z_{12} + Q_{12} Z_{2}),
\]
where
\[
h_\lambda = -C(Z_{12}^T Q_{12} + Z_{21}^T Q_2 + Q_{1} Z_{12} + Q_{12} Z_{2}) + C_m(Q_{12}^T Z_{12} + Z_{12}^T Q_{12} + Q_{2} Z_{2} + Z_{12}^T Q_2),
\]
and $\mathcal{P}$ and $\mathcal{Q}$ are obtained by solving (77).

Choose the initial $\gamma$ so that $\gamma_0^{-2}$ is approximately zero. The initial point $(\theta, \lambda) = (\theta_0, 0)$ is chosen so that the triple $((A_m)_0, (B_m)_0, (C_m)_0)$ is in Ly's form and satisfies $\rho(\theta_0, 0) = 0$. This can be done as follows:

1) Obtain the initial reduced order model $((A_m)_0, (B_m)_0, (C_m)_0)$ in balanced form in the same way as for the input normal form approach.

2) Transform the balanced $((A_m)_0, (B_m)_0, (C_m)_0)$ to Ly's form, and build $\theta_0$ as described in (79).

The homotopy curve tracking computation is the same as described in Section 8.2.
10. HOMOTOPY ALGORITHM BASED ON OVER-PARAMETERIZATION FORMULATION FOR THE COMBINED $H^2/H^\infty$ MODEL REDUCTION PROBLEM.

To optimize $J(A_m, B_m, C_m)$ over the open set $S$ under the constraint that symmetric positive definite $Q$ satisfies (14), the following Lagrangian is formed:

$$
\mathcal{L}(A_m, B_m, C_m, P, Q) \equiv \text{tr} \left[ Q\tilde{R} + (\tilde{A}Q + Q\tilde{A}^T + \gamma^{-2}Q\tilde{R}Q + \tilde{V})P \right]
$$

where $P \in \mathbb{R}^{\hat{n} \times \hat{n}}$ is a Lagrange multiplier. Setting $\partial \mathcal{L}/\partial Q = 0$ yields (67). Partition $Q$, $P \in \mathbb{R}^{\hat{n} \times \hat{n}}$ as in (68) and define $PQ = Z$ as in (69). A straightforward calculation shows

$$
\frac{\partial \mathcal{L}}{\partial A_m} = 2(P_{12}^T Q_{12} + P_2 Q_2),
$$

$$
\frac{\partial \mathcal{L}}{\partial B_m} = 2(P_{12}^TBV + P_2B_m V),
$$

$$
\frac{\partial \mathcal{L}}{\partial C_m} = 2(RC_m Q_2 - RC Q_{12})
$$

$$
+ \gamma^{-2} \left[-RC(Z_1^T Q_{12} + Z_{21}^T Q_2 + Q_1 Z_{12} + Q_{12} Z_2)
+ RC_m(Q_{12}^T Z_{12} + Z_{12}^T Q_{12} + Q_2 Z_2 + Z_2^T Q_2) \right].
$$

Let $A_f, B_f, C_f, R_f, V_f$, and $\gamma_f$ denote $A, B, C, R, V$, and $\gamma$ in the above and define $A(\lambda), B(\lambda), C(\lambda), R(\lambda)$, and $V(\lambda)$ as in (28) and

$$
\gamma(\lambda) = \gamma_0 + \lambda(\gamma_f - \gamma_0)
$$

and denote them by $A, B, C, R, V$, and $\gamma$ respectively in the following. Let

$$
H_{A_m}(\theta, \lambda) = \frac{\partial \mathcal{L}}{\partial A_m} = 2(P_{12}^T Q_{12} + P_2 Q_2),
$$

$$
H_{B_m}(\theta, \lambda) = \frac{\partial \mathcal{L}}{\partial B_m} = 2(P_{12}^TB + P_2 B_m)V,
$$

$$
H_{C_m}(\theta, \lambda) = \frac{\partial \mathcal{L}}{\partial C_m} = 2R(C_m Q_2 - C Q_{12})
$$

$$
+ \gamma^{-2} \left[-RC(Z_1^T Q_{12} + Z_{21}^T Q_2 + Q_1 Z_{12} + Q_{12} Z_2)
+ RC_m(Q_{12}^T Z_{12} + Z_{12}^T Q_{12} + Q_2 Z_2 + Z_2^T Q_2) \right],
$$

(86)
where
\[
\theta \equiv \begin{pmatrix}
\text{Vec}(A_m) \\
\text{Vec}(B_m) \\
\text{Vec}(C_m)
\end{pmatrix}
\] (87)
denotes the independent variables, \(Q\) and \(P\) satisfy respectively (14) and (67).

Define
\[
\rho(\theta, \lambda) = \begin{pmatrix}
\text{Vec}[H_{A_m}(\theta, \lambda)] \\
\text{Vec}[H_{B_m}(\theta, \lambda)] \\
\text{Vec}[H_{C_m}(\theta, \lambda)]
\end{pmatrix},
\] (88)
whose Jacobian matrix is
\[
D\rho(\theta, \lambda) = (D_\theta \rho(\theta, \lambda), D_\lambda \rho(\theta, \lambda)).
\]

Note that \(\theta\) in (87) has \(n_m^2 + n_m n_m + n_m I\) components, more than the minimal number \(n_m m + n_m I\) of the input normal form and Ly formulations. Because of this over-parametrization, the Jacobian matrix of \(\rho\) is rank deficient. The homotopy map is thus defined as
\[
\dot{\rho}(\theta, \lambda) = \lambda \rho(\theta, \lambda) + (1 - \lambda)(\theta - \theta_0),
\] (89)
which guarantees a well conditioned full rank Jacobian matrix along the whole path except at the solution corresponding to \(\lambda = 1\). The Jacobian matrix of \(\dot{\rho}\) is given by
\[
D\dot{\rho}(\theta, \lambda) = (\lambda D_\theta \rho(\theta, \lambda) + (1 - \lambda)I, \rho(\theta, \lambda) + \lambda D_\lambda \rho(\theta, \lambda) - (\theta - \theta_0)).
\] (90)

To find \(D_\theta \rho(\theta, \lambda)\), define \(\dot{H}_{A_m}(P^{(j)}, Q^{(j)}), \dot{H}_{B_m}(P^{(j)}),\) and \(\dot{H}_{C_m}(Q^{(j)}, Z^{(j)})\) as in (81). For \(\theta_j = (A_m)_{kl}\),
\[
\frac{\partial H_{A_m}}{\partial (A_m)_{kl}} = \dot{H}_{A_m}(P^{(j)}, Q^{(j)}),
\]
\[
\frac{\partial H_{B_m}}{\partial (A_m)_{kl}} = \dot{H}_{B_m}(P^{(j)}),
\] (91)
\[
\frac{\partial H_{C_m}}{\partial (A_m)_{kl}} = \dot{H}_{C_m}(Q^{(j)}, Z^{(j)}),
\]
for \(\theta_j = (B_m)_{kl}\),
\[
\frac{\partial H_{A_m}}{\partial (B_m)_{kl}} = \dot{H}_{A_m}(P^{(j)}, Q^{(j)}),
\]
\[
\frac{\partial H_{B_m}}{\partial (B_m)_{kl}} = \dot{H}_{B_m}(P^{(j)}) + 2P_k E^{(k,i)} V,
\] (92)
\[
\frac{\partial H_{C_m}}{\partial (B_m)_{kl}} = \dot{H}_{C_m}(Q^{(j)}, Z^{(j)}),
\]

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and for $\theta_j = (C_m)_{kl}$,

$$\frac{\partial H_{A_m}}{\partial (C_m)_{kl}} = \dot{H}_{A_m}(P^{(j)}, Q^{(j)}),$$

$$\frac{\partial H_{B_m}}{\partial (C_m)_{kl}} = \dot{H}_{B_m}(P^{(j)}),$$

$$\frac{\partial H_{C_m}}{\partial (C_m)_{kl}} = \dot{H}_{C_m}(Q^{(j)}, Z^{(j)}) + 2RE^{(k,l)}Q_2$$
$$+ \gamma^{-2}RE^{(k,l)}(Z_{12}Q_{12} + Q_{12}^T Z_{12} + Q_2^T Z_2 + Z_2^T Q_2),$$

(93)

where $P^{(j)}$ and $Q^{(j)}$ can be obtained by solving the Lyapunov equation (75). The derivative of the homotopy map with respect to $\lambda$ is given by (85) and (90).

Choose the initial $\gamma$ so that $\gamma_0^{-2}$ is approximately zero. The initial point $(\theta, \lambda) = (\theta_0, 0)$ is chosen so that the triple $((A_m)_0, (B_m)_0, (C_m)_0)$ is in balanced form and satisfies $\rho(\theta_0, 0) = 0$.

This can be done as follows:

1) Obtain the initial reduced order model $((A_m)_0, (B_m)_0, (C_m)_0)_b$ in balanced form in the same way as for the input normal form approach.

2) Build $\theta_0$ from $((A_m)_0, (B_m)_0, (C_m)_0)_b$ as described in (87).

The homotopy curve tracking computation is the same as described in Section 8.2.
11. $H^2/H^\infty$ RESULTS.

The following systems are solved by the homotopy algorithms discussed in the previous sections. The homotopy curve tracking was done with HOMPACK [72].

Systems with transfer functions of the form

$$H(s) = \frac{(s - 1)^q}{(s + 1)^{16}},$$

where $q = 0, \cdots, 4$, are studied. For these systems, controller canonical form realizations are given by

$$A = \begin{pmatrix} -10 & -45 & -120 & -210 & -250 & -210 & -120 & -45 & -10 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$B = (1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0)^T.$$

For $q = 0$, $C = (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1).$

For $q = 1$, $C = (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ -1 \ 1).$

For $q = 2$, $C = (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ -2 \ 1).$

For $q = 3$, $C = (0 \ 0 \ 0 \ 0 \ 0 \ 1 \ -3 \ 3 \ -1).$

For $q = 4$, $C = (0 \ 0 \ 0 \ 0 \ 1 \ -4 \ 6 \ -4 \ 1).$

The $H^\infty$ error $\|H(s) - H_m(s)\|_\infty$ for the balanced reduced model of order 4 and the corresponding Enns-Glover bounds [17] [22] are:

<table>
<thead>
<tr>
<th>q</th>
<th>$H^\infty$ error</th>
<th>Enns-Glover bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.017251178</td>
<td>0.021958271</td>
</tr>
<tr>
<td>1</td>
<td>0.031901448</td>
<td>0.042197266</td>
</tr>
<tr>
<td>2</td>
<td>0.057214882</td>
<td>0.079880388</td>
</tr>
<tr>
<td>3</td>
<td>0.098520472</td>
<td>0.14841709</td>
</tr>
<tr>
<td>4</td>
<td>0.16125935</td>
<td>0.27001110</td>
</tr>
</tbody>
</table>
Fig. 3. $\|H(s) - H_m(s)\|_\infty$ (solid), $100\ J$ (dotted), and $100\ J$ (dashed) versus $\gamma$.

Fig. 4. $\|H(s) - H_m(s)\|_\infty$ versus $J$ for $q = 0$, balanced model at "x".
Fig. 5. \( \| H(s) - H_m(s) \|_\infty \) versus \( J \) for \( q = 1 \), balanced model at "x".

Fig. 6. Ratio of \( H^\infty \) error at \( \gamma = \gamma_{\text{min}} \) to that at \( \gamma = \infty \) versus exponent \( q \).
Fig. 7. Bode plots of $H(s) - H_m(s)$ for $H^2/H^\infty$ (solid), $H^2$ (dashed), and balanced (dotted) models with $q = 0$.

Fig. 8. Poles ("x") and zeros ("o") of the transfer function of the reduced order model for $q = 0$, $n_m = 4$ at $\gamma = 0.0178$. 

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For $n_m = 4$ and $q = 0$, solutions of the auxiliary minimization problem are obtained for $\gamma \geq 0.0178$ using the input normal form approach. For $\gamma < \gamma_{\text{min}} = 0.0178$, the Riccati equation solver fails and therefore no solution can be found. Let $H_m(s)$ be the transfer function of the reduced order model obtained by minimizing $J$. In Fig. 3, $\|H(s) - H_m(s)\|_{\infty}$ (solid line), $100J$ (dotted line), and $100J$ (dashed line) are plotted against $\gamma$. As shown in the figure, as $\gamma$ decreases, $\|H(s) - H_m(s)\|_{\infty}$ also decreases while both $J$ and $J$ increase. As can be seen from the figure, $J$ is a close bound for $J$ until $\gamma$ becomes very small. To show the tradeoff between the $H^2$ cost and the $H^\infty$ error $\|H(s) - H_m(s)\|_{\infty}$, it is useful to plot $\|H(s) - H_m(s)\|_{\infty}$ against $J$ (with $\gamma$ as the parameter of the curve), as shown in Fig. 4. In Fig. 4, the point marked by "x" corresponds to the balanced reduced model, which has both large $H^2$ cost and large $H^\infty$ error $\|H(s) - H_m(s)\|_{\infty}$, relative to the $H^2/H^\infty$ reduced order model. The ratio of $H^\infty$ error at $\gamma = \gamma_{\text{min}}$ to that at $\gamma = \infty$ is 0.8071, which indicates that there is about 20% improvement of the reduced order model with $\gamma = \gamma_{\text{min}}$ over the reduced order model without the $H^\infty$ constraint. The reduced order models of order 2, 3, 6 were also found, and have qualitative behavior similar to the $n_m = 4$ case.

For $q = 1, \cdots, 4$, the same calculations are carried out. Fig. 5 shows similar results to those in Fig. 4 for $q = 1$ with an improvement of about 19%. As $q$ increases, the improvement of the optimal reduced order model over the balanced reduced order model decreases. In Fig. 6, the ratio of $H^\infty$ error at $\gamma = \gamma_{\text{min}}$ to that at $\gamma = \infty$ is plotted against $q$ for $q = 0, \cdots, 4$. The $H^\infty$ norm improvement of the optimal reduced order model with the $H^\infty$ constraint over that without the $H^\infty$ constraint is $1 - \text{ratio}$. As $q$ increases, the improvement decreases.

In Fig. 7, the Bode plots of $H(s) - H_m(s)$ for the system with $q = 0$, $n_m = 4$, and $\gamma = \gamma_{\text{min}} = 0.0178$ are shown. The reduced order model with the $H^\infty$ constraint at $\gamma = \gamma_{\text{min}}$ is shown by the solid line; the balanced reduced order model is shown by the dotted line; the reduced order model without the $H^\infty$ constraint is shown by the dashed line. The magnitude plots show that as $\gamma$ goes to $\gamma_{\text{min}}$, the $H^\infty$ error becomes increasingly "all pass", that is, flat over a wide frequency range, which indicates $H^\infty$ optimality of the reduced order model. Fig. 8 shows the poles and zeros of the transfer function of the reduced order model for the system with $q = 0$ and $n_m = 4$ at $\gamma = \gamma_{\text{min}}$. 54
As another example, consider the system defined by

\[
A = \begin{pmatrix} -2 & -8 \\ 0 & -8 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \quad C = B^T.
\]

It is easy to verify that the system is balanced and the singular values are all equal to 1, i.e.,

\[
A + A^T + BB^T = 0,
\]

\[
A^T + A + C^T C = 0.
\]

The \(H^\infty\) error of the balanced reduced model of order 1 is 2 and the Enns-Glover bound is also 2. Optimal reduced models of order 1 are found by the input normal form homotopy approach for \(\gamma \geq \gamma_{\text{min}} = 1.011\). Fig. 9 shows the \(H^\infty\) error versus the \(H^2\) cost. The point \(\times\) corresponds to the balanced reduced order model. The ratio of the \(H^\infty\) error at \(\gamma = \gamma_{\text{min}}\) to that at \(\gamma = \infty\) is 0.6249, which indicates that there is about 37.5% improvement of the reduced order model with \(\gamma = \gamma_{\text{min}}\) over the reduced order model without the \(H^\infty\) constraint, and a 50% improvement over the balanced reduced order model.

In Fig. 10, 50 \(\|H(s) - H_m(s)\|_\infty\) (solid line), \(J\) (dotted line), and \(J\) (dashed line) are plotted against \(\gamma\). Unlike the previous systems, even for small \(\gamma\), the actual error \(\|H(s) - H_m(s)\|_\infty\) is very close to its bound \(\gamma\), and \(J\) is a very close bound for the \(H^2\) error \(J\). The Bode plots of \(H(s) - H_m(s)\) are shown in Fig. 11, where the reduced order model with the \(H^\infty\) constraint at \(\gamma = \gamma_{\text{min}}\) is shown by the solid line; the balanced reduced order model is shown by the dotted line; the reduced order model without the \(H^\infty\) constraint is shown by the dashed line. Again the reduced order model for \(\gamma = \gamma_{\text{min}}\) indicates close to all pass model reduction error. The reduced order model transfer function at \(\gamma = \gamma_{\text{min}}\) has a single pole at \(s = -129.1642\).
Fig. 9. $\|H(s) - H_m(s)\|_\infty$ versus $J$.

Fig. 10. $50 \|H(s) - H_m(s)\|_\infty$ (solid), $J$ (dotted), and $J$ (dashed) versus $\gamma$. 
Fig. 11. Bode plots of $H(s) - H_m(s)$ for $H^2/H^\infty$ (solid), $H^2$ (dashed), and balanced (dotted) reduced order model.
12. A HOMOTOPY METHOD FOR SOLVING RICCATI EQUATIONS
ON A SHARED MEMORY PARALLEL COMPUTER

12.1. Probability-one homotopy approach.

Given $A, R_f \in E^{n \times n}$, consider the Riccati equation $AX + XA^t - XX + R_f = 0$. In the spirit of De Carlo and Richter ([46], [61], [62]), a probability-one homotopy map can be constructed as

$$AX(s) + X(s)A^t - X(s)X(s) + R_o + \lambda(s)(R_f - R_o) = 0,$$

(94)

where $X(0)$ is a guess for the solution and $R_0$ is defined accordingly from $X(0)$.

Differentiating (94) with respect to the arc length $s$ gives the equation

$$(A - X)\frac{dX}{ds} + \frac{dX}{ds}(A - X)^t + \frac{d\lambda}{ds}(R_f - R_o) = 0,$$

or equivalently

$$(I \otimes (A - X) + (A - X) \otimes I \cdot r) \left( \frac{dx}{ds} \frac{d\lambda}{ds} \right) = 0,$$

(95)

where

$$x = \text{vec}(X), \quad r = \text{vec}(R_f - R_o).$$

From (95) it follows that the $n^2 \times (n^2 + 1)$ Jacobian matrix of the homotopy map (94) is

$$D\rho_a(x(s), \lambda(s)) = (I \otimes (A - X) + (A - X) \otimes I \cdot r),$$

(96)

where $\rho_a$ is the vec of the left side of (94), and the parameter vector $a = \text{vec} R_0$. The globally convergent homotopy theory [72] says that for almost all choices of $a$ the matrix $D\rho_a$ in (96) has full rank.

12.2. Algorithm details.
12.2.1. Kernel computation.

Tracking the zero curve \((x(s), \lambda(s))\) of (94) requires the tangent vector to the curve, which in turn requires the kernel of \(D\rho_a\). \(\dim \ker D\rho_a = 1\) by the construction of the homotopy map. Let

\[
B \equiv A - X, \quad \hat{B} \equiv I \otimes B + B \otimes I.
\]

Then the \(n^n \times (n^n + 1)\) Jacobian matrix is

\[
D\rho_a = \begin{pmatrix} \hat{B} & r \end{pmatrix}.
\]

The computation of the kernel of \(D\rho_a\) is performed using the following procedure. Let \(Q\) be the complex unitary matrix such that

\[
Q^* B^i Q = T
\]

is upper triangular, i.e., \(Q^* T Q\) is a complex Schur decomposition of \(B^i\). Then

\[
\begin{pmatrix} Q^* \otimes Q^* & 0 \\ 0 & 1 \end{pmatrix} D\rho_a^t (Q \otimes Q) = \begin{pmatrix} I \otimes T + T \otimes I \\ p \end{pmatrix} = \begin{pmatrix} \hat{T} \\ p \end{pmatrix}
\]

(97)

where \(\hat{T} \equiv I \otimes T + T \otimes I\) is \(n^n \times n^n\) upper triangular, and \(p \in \mathbb{R}^{n^2}\). Let \(G_i\) for \(i = 1, \ldots, n^n\) be the Givens rotation matrices which use the diagonal elements of \(\hat{T}\) to zero out the elements of \(p\) in \(\begin{pmatrix} \hat{T} \\ p \end{pmatrix}\), thus generating the upper triangular matrix \(\begin{pmatrix} T_2 \\ 0 \end{pmatrix}\). From (97) it follows that

\[
D\rho_a^t (Q \otimes Q) = \begin{pmatrix} Q \otimes Q \\ 0 \\ 1 \end{pmatrix} G^*_1 G^*_2 \cdots G^*_{n^n} G \cdots G_2 G_1 \begin{pmatrix} \hat{T} \\ p \end{pmatrix}
\]

\[
= \begin{pmatrix} Q \otimes Q \\ 0 \\ 1 \end{pmatrix} G^*_1 G^*_2 \cdots G^*_{n^n} \begin{pmatrix} T_2 \\ 0 \end{pmatrix}
\]

\[= (S \ v) \begin{pmatrix} T_2 \\ 0 \end{pmatrix}.
\]

Since \(Q\) and the \(G_i\) are unitary, and \(D\rho_a^t\) has rank \(n^n\), the \(n^n\) columns of \(S\) are an orthonormal basis for \(\text{im} \ D\rho_a^t\). Therefore \(v\) is orthogonal to \(\text{im} \ D\rho_a^t\) and \(v \in \ker D\rho_a^t = [\text{im} \ D\rho_a^t]^\perp\). Note that neither \(S\) nor \(T_2\) need be explicitly computed, and that their tensor product structure can be efficiently exploited.
12.2.2. Newton step computation.

The homotopy zero curve tracking requires corrector steps back to the curve, and those require the minimum norm solution of

$$D\rho_a \Delta z = -\rho_a. \quad (98)$$

Using the kernel of $D\rho_a$, the minimum norm solution $\Delta z$ can be found using any particular solution of (98), or equivalently a solution of

$$\begin{pmatrix} p & q \\ D_{\lambda \rho_a} & D_{\varepsilon \rho_a} \end{pmatrix} w = \begin{pmatrix} 0 \\ -\rho_a \end{pmatrix},$$

where $(p, q) \in E^{n^2+1}$ gives an invertible matrix. Using the same Schur decomposition as before now gives

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & Q^t \otimes Q^t \end{pmatrix} \begin{pmatrix} p & q \\ D_{\lambda \rho_a} & I \otimes B + B \otimes I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{Q} \otimes \tilde{Q} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & Q^t \otimes Q^t \end{pmatrix} w = \begin{pmatrix} c & d \\ f & T^t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & Q^t \otimes Q^t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -\rho_a \end{pmatrix}.$$

Recall that the rank deficiency of $\hat{T} = I \otimes T + T \otimes I$ is at most one. Therefore, because of the structure of the eigenvalues of $\hat{T}$, $\hat{T}$ has at most one zero on the diagonal, and that can be chosen to appear in the $(1,1)$ position of $\hat{T}$. Note that the choice of $(p, q)$ was left open, and therefore $(c, d)$ also remains open. Choose $(c, d) = (c, d_1, 0, \ldots, 0)$ of unit length and orthogonal to $(f_1, (T^t)_1)$. Then applying a single Givens rotation to zero $d_1$ produces a lower triangular system

$$G \begin{pmatrix} c & d \\ f & T^t \end{pmatrix} \tilde{w} = L \tilde{w} = G \begin{pmatrix} 1 & 0 \\ 0 & Q^t \otimes Q^t \end{pmatrix} \begin{pmatrix} 0 \\ -\rho_a \end{pmatrix},$$

from which $\tilde{w}, w$ (a particular solution with its components permuted), and $\Delta z$ (the minimum norm solution) can be recovered.

12.3. Parallelization of the algorithm.

The computationally most intensive part of the algorithm is the kernel computation. Within that there are two computational phases which require attention. One is the reduction to the upper triangular form

$$\begin{pmatrix} T_2 \\ 0 \end{pmatrix},$$

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and the other is the computation of the kernel vector $v$.

Introducing zeros into the last row of \( \begin{pmatrix} \hat{T} \\ p \end{pmatrix} \) can be done in $n^2$ sequential steps, one for each element of the last row, where each step is parallelized. Within each step there are two substeps. The first substep is to update elements of the last row that correspond to a row in a diagonal block $T + T_{ii} I$ in the upper part of the matrix. Every element in this block can be updated independently of the others, so this task can be distributed to all the processors. In this substep there are between 1 and $n - 1$ elements to be updated. In the next substep there are again between 1 and $n$ elements to be updated independently. They correspond to the remaining nonzeros in the pivot row in the upper part of the matrix. These nonzeros are in the same positions within the blocks of size $n$ containing the pivot row. Notice that it is not necessary to update the rows in the upper part of the matrix, but rather only the last row.

The second phase is computationally more intensive. It is necessary to update the last column of

\[
\begin{pmatrix}
Q \otimes Q & 0 \\
0 & 1
\end{pmatrix}
\]

$n^2$ times in order to get $v$. Notice that it is not necessary to update the other columns. This phase can be carried out in $n^2$ sequential steps, one for each column of the left part of the matrix. Each of the $n^2$ steps is just a DAXPY operation, which parallelizes very well.

Since $Q$ is complex unitary, there is a technical difficulty here. The computed kernel vector $v$ will generally be complex, although the kernel of $Dp_a$ must of course be real. We get a real vector by finding $k$ such that $|v_k| = \|v_\infty\|$, and then taking Re $(v/v_k)$. What to do if Im $(v/v_k)$ is not small remains a thorny problem. This question, and complex arithmetic altogether, could be avoided if the real Schur form $T$ were used. Unfortunately the structure of $I \otimes T + T \otimes I$ is then such that the kernel and Newton step computations become much more expensive. (The extra column $r$ and the potential singularity of $\hat{B}$ make a qualitative difference!)

12.4. Numerical Results.

The above homotopy algorithm is implemented on a Sequent Symmetry S/81 with Weitek coprocessors, a shared-memory multiprocessor machine.
Table 3. Algorithm measures

<table>
<thead>
<tr>
<th></th>
<th>Homotopy algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>n = 10</td>
</tr>
<tr>
<td>1</td>
<td>127.3</td>
</tr>
<tr>
<td>2</td>
<td>69.6</td>
</tr>
<tr>
<td>4</td>
<td>38.7</td>
</tr>
<tr>
<td>8</td>
<td>23.4</td>
</tr>
<tr>
<td>16</td>
<td>15.7</td>
</tr>
<tr>
<td>22</td>
<td>13.7</td>
</tr>
</tbody>
</table>

The algorithm is used to solve two Riccati equations. The data for the equations are constructed as follows: $A$ is a $2 \times 2$ block-diagonal matrix with $2 \times 2$ blocks in the form
\[
\begin{pmatrix}
-2k & 1 \\
-k^2 & 0
\end{pmatrix},
\]
k = 1, ..., n. $R_f$ is given by
\[
R_f = -AX - XA^T + XX
\]

where $X = U^T \Sigma U$, $U$ is a $2 \times 2$ block-diagonal matrix with $2 \times 2$ blocks in the form
\[
\begin{pmatrix}
\cos \theta_k & \sin \theta_k \\
\sin \theta_k & -\cos \theta_k
\end{pmatrix},
\]
$\theta_k = 2\pi k/(n + 1)$, and $\Sigma$ is a diagonal matrix with diagonal elements $\sigma_k = 1 + k/(2n)$.

The homotopy curve tracking was done with HOMPACK[72]. The initial $X$ is chosen as the unit matrix, so $R_0 = -A - A^T + I$.

The MATLAB function "are" is chosen as the best current serial algorithm for solving the Riccati equation. The CPU times to solve the Riccati equations when $n = 10$ and $n = 20$ using MATLAB "are" are respectively about 1.31 seconds and 6.29 seconds. The CPU times using 1, 2, 4, 8, 16, and 22 processors on the Sequent are shown in Table 3. It can be seen from the table that the MATLAB function "are" is much more efficient than the homotopy algorithm. However, it is the generalization of the homotopy algorithm to solve variants of the Riccati equation that make the homotopy approach interesting.

The speedups with respect to the $p = 1$ time are plotted against the number of processors in Fig. 12, where the solid line corresponds to $n = 10$ and the dashed line corresponds to $n = 20$. As can be seen from the figure, the speedup saturates for $n = 10$ as the number of processors approaches 22, while the speedup for $n = 20$ can be increased further if more processors are used.

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Fig. 12. Speedup versus the number of processors.
13. THE LQG CONTROLLER SYNTHESIS WITH AN
\( H^\infty \) PERFORMANCE BOUND.

The LQG controller synthesis problem with an \( H^\infty \) performance bound can be stated as: given the \( n \)-th order stabilizable and detectable plant
\[
\dot{x}(t) = Ax(t) + Bu(t) + D_1w(t),
\]
\[
y(t) = Cx(t) + D_2w(t),
\]
where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{l \times n}, D_1 \in \mathbb{R}^{n \times p}, D_2 \in \mathbb{R}^{l \times p}, D_1D_2^T = 0, \) and \( w(t) \) is \( p \)-dimensional white noise, find a \( n_c \)-th order dynamic compensator
\[
\dot{z}_c(t) = A_c z_c(t) + B_c y(t),
\]
\[
u(t) = C_c z_c(t),
\]
where \( A_c \in \mathbb{R}^{n_c \times n_c}, B_c \in \mathbb{R}^{n_c \times l}, C_c \in \mathbb{R}^{m \times n_c}, \) and \( n_c \leq n, \) which satisfies the following criteria:
(i) the closed-loop system (99)–(100) is asymptotically stable, i.e., \( \tilde{A} = \begin{pmatrix} A & B C_c \\ B_c C & A_c \end{pmatrix} \) is asymptotically stable;
(ii) the \( q_\infty \times p \) closed-loop transfer function from \( w(t) \) to \( E_{1\infty}z(t) + E_{2\infty}u(t) \),
\[
H(s) \equiv \tilde{E}_\infty(sI_\tilde{n} - \tilde{A})^{-1}\tilde{D},
\]
where \( \tilde{E}_\infty = \begin{pmatrix} E_{1\infty} & E_{2\infty} C_c \end{pmatrix} \in \mathbb{R}^{q_\infty \times n}, E_{1\infty} \in \mathbb{R}^{q_\infty \times n}, E_{2\infty} \in \mathbb{R}^{q_\infty \times m}, E_{1\infty}^T E_{2\infty} = 0, \) \( \tilde{n} = n + n_c, \)
and \( \tilde{D} = \begin{pmatrix} D_1 \\ B_c D_2 \end{pmatrix} \), satisfies the constraint
\[
\|H(s)\|_\infty \leq \gamma
\]
where \( \gamma > 0 \) is a given constant; and
(iii) the performance functional
\[
J(A_c, B_c, C_c) \equiv \lim_{t \to \infty} \mathcal{E} \left[ z^T(t)R_1z(t) + u^T(t)R_2u(t) \right]
\]
is minimized, where \( \mathcal{E} \) is the expected value, \( R_1 = E_1^T E_1 \in \mathbb{R}^{n \times n} \) and \( R_2 = E_2^T E_2 \in \mathbb{R}^{m \times m} \)
\( (E_1 \in \mathbb{R}^{q \times n}, E_2 \in \mathbb{R}^{q \times m}, E_1^T E_2 = 0) \) are respectively symmetric positive semidefinite and symmetric positive definite weighting matrices.
The closed-loop system (99)-(100) can be written as a single system
\begin{equation}
\dot{x}(t) = \hat{A} \hat{x}(t) + \hat{D} w(t),
\end{equation}
where \( \hat{x} = \begin{pmatrix} x \\ x_c \end{pmatrix} \).

Using this notation and under the condition that \( \hat{A} \) is asymptotically stable, for a given compensator the performance (103) is given by
\begin{equation}
J(A_c, B_c, C_c) = \text{tr} \left( \hat{Q} \hat{R} \right),
\end{equation}
where
\begin{equation}
\hat{R} = \begin{pmatrix} R_1 & 0 \\ 0 & C_c^T R_2 C_c \end{pmatrix}
\end{equation}
and \( \hat{Q} \) satisfies the Lyapunov equation
\begin{equation}
\hat{A} \hat{Q} + \hat{Q} \hat{A}^T + \hat{V} = 0,
\end{equation}
with symmetric positive semidefinite \( V_1 = D_1 D_1^T \), symmetric positive definite \( V_2 = D_2 D_2^T \), and
\begin{equation}
\hat{V} = \begin{pmatrix} V_1 & 0 \\ 0 & B_c V_2 B_c^T \end{pmatrix}.
\end{equation}

**Lemma 1** [6]. Let \( (A_c, B_c, C_c) \) be given and assume there exists \( Q \in \mathbb{R}^{\hat{n} \times \hat{n}} \) satisfying
\begin{equation}
Q \quad \text{is symmetric and nonnegative definite}
\end{equation}
and
\begin{equation}
\hat{A} Q + Q \hat{A}^T + \gamma^{-2} Q \hat{R}_\infty Q + \hat{V} = 0,
\end{equation}
where
\begin{equation}
\hat{R}_\infty = \begin{pmatrix} R_{1\infty} & 0 \\ 0 & C_c^T R_{2\infty} C_c \end{pmatrix}, \quad R_{1\infty} = E_{1\infty}^T E_{1\infty}, \text{ and } R_{2\infty} = E_{2\infty}^T E_{2\infty}
\end{equation}
are symmetric positive semidefinite matrices. Then
\( (\hat{A}, \hat{D}) \) is stabilizable
\begin{equation}
\text{if and only if}
\end{equation}
\( \hat{A} \) is asymptotically stable.

In this case
\begin{equation}
\|H(s)\|_{\infty} \leq \gamma,
\end{equation}
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\[ \tilde{Q} \leq Q \quad (Q - \tilde{Q} \text{ is nonnegative definite}), \]

and

\[ \text{tr } \tilde{Q}\tilde{R} \equiv J(A_c, B_c, C_c) \leq J(A_c, B_c, C_c) \equiv \text{ tr } Q\tilde{R}. \]

Hence the satisfaction of (9) and (10) along with the generic condition (11) leads to:

1) closed-loop stability;
2) prespecified $H^\infty$ attenuation; and
3) an upper bound for the $H^2$ performance criterion.

The auxiliary minimization problem is to determine $(A_c, B_c, C_c)$ that minimizes $J(A_c, B_c, C_c)$ and thus provides a bound for the actual $H^2$ criterion $J(A_c, B_c, C_c)$.

$(A_c, B_c, C_c, Q)$ is restricted to the open set

\[ S \equiv \{ (A_c, B_c, C_c, Q) : \tilde{A} \text{ and } \tilde{A} + \gamma^{-2}Q\tilde{R} \text{ are asymptotically stable}, \]

\[ Q \text{ is symmetric positive definite}, \]

and $(A_c, B_c, C_c)$ is controllable and observable \}.\]
14. HOMOTOPY ALGORITHM BASED ON LY'S FORMULATION FOR LQG CONTROL WITH AN $H^\infty$ PERFORMANCE BOUND.

Ly et al. [47] introduced a canonical form with $n_c m + n_c l$ parameters. The compensator is represented with respect to a basis such that $A_c$ is a $2 \times 2$ block-diagonal matrix ($2 \times 2$ blocks with an additional $1 \times 1$ block if $n_c$ is odd) with $2 \times 2$ blocks in the form

$$
\begin{pmatrix}
0 & 1 \\
* & *
\end{pmatrix},
$$

$B_c$ is a full matrix, and

$$
C_c = ((C_c)_1 (C_c)_2 \cdots (C_c)_r)
$$

where

$$
(C_c)_i = \begin{pmatrix}
1 & * & \cdots & * \\
0 & * & \cdots & *
\end{pmatrix}^T.
$$

It is assumed that $(A_c, B_c, C_c)$ is in Ly's form. Let $I$ be the set of indices of those elements of $A_c$ which are parameters, i.e.,

$$
I \equiv \{(2, 1), (2, 2), \ldots, (n_c, n_c)\}.
$$

To optimize $J(A_c, B_c, C_c)$ over the open set $S$ under the constraint that symmetric positive definite $Q$ satisfies (108), and $(A_c, B_c, C_c)$ is in Ly's form, the following Lagrangian is formed:

$$
\mathcal{L}(A_c, B_c, C_c, P, Q) = \text{tr} \left[ Q \dot{R} + (\dot{A} Q + Q \dot{A}^T + \gamma^{-2} Q \dot{R}_\infty Q + \dot{V}) P \right],
$$

where $P \in \mathbb{R}^{n \times n}$ is a Lagrange multiplier. Setting $\partial \mathcal{L} / \partial Q = 0$ yields

$$
0 = (\dot{A} + \gamma^{-2} Q \dot{R}_\infty)^T P + P (\dot{A} + \gamma^{-2} Q \dot{R}_\infty)^T + \dot{R}.
$$

(111)

Partition $Q, P \in \mathbb{R}^{n \times n}$ as

$$
Q = \begin{pmatrix}
Q_1 & Q_{12} \\
Q_{12}^T & Q_2
\end{pmatrix}, \quad P = \begin{pmatrix}
P_1 & P_{12} \\
P_{12}^T & P_2
\end{pmatrix}.
$$

(112)
The partial derivatives of $\mathcal{L}$ can be computed as
\[
\frac{\partial \mathcal{L}}{\partial (A_c)_{ij}} = 2(P_{12}^T Q_{12} + P_2 Q_2)_{ij}, \quad (i,j) \in \mathcal{T},
\]
\[
\frac{\partial \mathcal{L}}{\partial B_c} = 2P_2 B_c V_2 + 2(P_{12}^T Q_1 + P_2 Q_{12}^T)C^T,
\]
\[
\frac{\partial \mathcal{L}}{\partial (C_c)_{ij}} = \left(2R_2 C_c Q_2 + 2B^T(P_1 Q_{12} + P_2 Q_2) + \gamma^{-2}R_{2\infty} C_c \left[(Q_{12}^T P_1 + Q_2 P_{12}^T)Q_{12} + (Q_{12}^T P_{12} + Q_2 P_2)Q_2 \right]\right)_{ij},
\]
i > 1.

Let $A_f, B_f, C_f, \gamma_f, R_{1f}, R_{2f}, R_{1\infty f}, R_{2\infty f}, V_{1f},$ and $V_{2f}$ denote $A, B, C, \lambda, R_1,$ $R_2, R_{1\infty}, R_{2\infty}, V_1,$ and $V_2$ in the above and define $A(\lambda), B(\lambda), C(\lambda), \gamma(\lambda), R_1(\lambda),$ $R_2(\lambda), R_{1\infty}(\lambda), R_{2\infty}(\lambda), V_1(\lambda),$ $V_2(\lambda)$ as
\[
A(\lambda) = A_0 + \lambda(A_f - A_0), \quad B(\lambda) = B_0 + \lambda(B_f - B_0),
\]
\[
C(\lambda) = C_0 + \lambda(C_f - C_0), \quad \gamma(\lambda) = \gamma_0 + \lambda(\gamma_f - \gamma_0),
\]
\[
R_1(\lambda) = R_{1,0} + \lambda(R_{1f} - R_{1,0}), \quad R_2(\lambda) = R_{2,0} + \lambda(R_{2f} - R_{2,0}),
\]
\[
R_{1\infty}(\lambda) = R_{1\infty,0} + \lambda(R_{1\infty f} - R_{1\infty,0}), \quad R_{2\infty}(\lambda) = R_{2\infty,0} + \lambda(R_{2\infty f} - R_{2\infty,0}),
\]
\[
V_1(\lambda) = V_{1,0} + \lambda(V_{1f} - V_{1,0}), \quad V_2(\lambda) = V_{2,0} + \lambda(V_{2f} - V_{2,0}),
\]
and denote them by $A, B, C, \gamma, R_1, R_2, R_{1\infty}, R_{2\infty}, V_1,$ and $V_2$ respectively in the following.

Let
\[
H_{A_c}(\theta, \lambda) = \frac{1}{2} \frac{\partial \mathcal{L}}{\partial A_c} = P_{12}^T Q_{12} + P_2 Q_2,
\]
\[
H_{B_c}(\theta, \lambda) = \frac{1}{2} \frac{\partial \mathcal{L}}{\partial B_c} = P_2 B_c V_2 + (P_{12}^T Q_1 + P_2 Q_{12}^T)C^T,
\]
\[
H_{C_c}(\theta, \lambda) = \frac{1}{2} \frac{\partial \mathcal{L}}{\partial C_c} = R_2 C_c Q_2 + B^T(P_1 Q_{12} + P_2 Q_2) + \gamma^{-2}R_{2\infty} C_c \left[(Q_{12}^T P_1 + Q_2 P_{12}^T)Q_{12} + (Q_{12}^T P_{12} + Q_2 P_2)Q_2 \right],
\]
\[
(\text{114})
\]
where in $H_{A_c}$ only those elements corresponding to the parameter elements of $A_c$ are of interest and
\[
\theta \equiv \left(\begin{array}{c}
(A_c)^T \\
\text{Vec}(B_c) \\
\text{Vec}(C_c)^T
\end{array}\right)
\]
\[
(\text{115})
\]
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denotes the independent variables, \( Q \) and \( P \) satisfy respectively (108) and (111), \((A_c)_T\) is a vector consisting of those elements in \( A_c \) with indices in the set \( T \), i.e.,

\[
(A_c)_T = ((A_c)_{21}, (A_c)_{22}, \ldots, (A_c)_{n_c n_c})^T,
\]

and \((C_c)_T\) is the matrix obtained from rows \( T = \{2, \ldots, m\} \) of \( C_c \). \( \text{Vec}(P) \) for a matrix \( P \in \mathbb{R}^{p \times q} \) is the concatenation of its columns:

\[
\text{Vec}(P) = \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_q \end{pmatrix} \in \mathbb{R}^{pq}.
\]

The most natural homotopy map is defined as

\[
\rho(\theta, \lambda) = \begin{pmatrix} [H_{A_c}(\theta, \lambda)]_T \\ \text{Vec} [H_{B_c}(\theta, \lambda)] \\ \text{Vec} [H_{C_c}(\theta, \lambda)]_T \end{pmatrix},
\]

(116)

and its Jacobian matrix is

\[
D \rho(\theta, \lambda) = (D_\theta \rho(\theta, \lambda), D_\lambda \rho(\theta, \lambda)).
\]

In practice it may be difficult to find the initial point \( \theta_0 \) such that \( \rho(\theta_0, 0) = 0 \). A somewhat more artificial homotopy then, letting \( \theta_0 \) be the chosen initial point, is the Newton homotopy map defined as

\[
\tilde{\rho}(\theta, \lambda) = \rho(\theta, \lambda) - (1 - \lambda)\rho(\theta_0, 0),
\]

which will give rise to an extra term \( \rho(\theta_0, 0) \) in \( D_\lambda \tilde{\rho}(\theta, \lambda)\). To guarantee a full rank Jacobian matrix along the whole homotopy zero curve except possibly at the solution corresponding to \( \lambda = 1 \), define the homotopy map to be

\[
\hat{\rho}(\theta, \lambda) = \lambda \rho(\theta, \lambda) + (1 - \lambda)(\theta - \theta_0).
\]

(117)

The Jacobian matrix of \( \hat{\rho} \) is given by

\[
D \hat{\rho}(\theta, \lambda) = (\lambda D_\theta \rho(\theta, \lambda) + (1 - \lambda)I, \ \rho(\theta, \lambda) + \lambda D_\lambda \rho(\theta, \lambda) - (\theta - \theta_0)).
\]
In the following, the homotopy map (116) is assumed for the full-order problem and (117) is assumed for the reduced-order case. Define

\[ \begin{align*}
\dot{H}_{A_c}(P^{(j)}, Q^{(j)}) &= P_{12}^{T(j)}Q_{12} + P_{12}^{T}Q_{12} + P_{2}Q_{2} + P_{2}Q_{2}^{T}, \\
\dot{H}_{B_e}(P^{(j)}, Q^{(j)}) &= P_{2}B_{e}V_{2} + (P_{12}^{T}Q_{12} + P_{12}^{T}Q_{12}^{T})Q_{2}, \\
\dot{H}_{C_{e}}(P^{(j)}, Q^{(j)}) &= R_{2}C_{e}Q_{2} + B^{T}(P_{1}Q_{12} + P_{1}Q_{12}^{T})Q_{2} + P_{2}Q_{2}^{T}Q_{2}^{T}, \\
&+ \gamma^{-2}\frac{1}{2}R_{222}C_{e}[(Q_{12}^{T}P_{1} + Q_{12}^{T}P_{1})Q_{12}^{T} + (Q_{12}^{T}P_{12} + Q_{12}^{T}P_{12})Q_{12}^{T} + (Q_{2}Q_{2} + Q_{2}Q_{2}^{T})Q_{2}],
\end{align*} \]

(118)

where the superscript \((j)\) means \(\partial/\partial \theta_{j}\). Using the above definitions, we have for \(\theta_{j} = (A_{c})_{kl}\), where \((k, l) \in I\),

\[ \begin{align*}
\frac{\partial \dot{H}_{A_c}}{\partial (A_{c})_{kl}} &= \dot{H}_{A_c}(P^{(j)}, Q^{(j)}), \\
\frac{\partial \dot{H}_{B_e}}{\partial (A_{c})_{kl}} &= \dot{H}_{B_e}(P^{(j)}, Q^{(j)}), \\
\frac{\partial \dot{H}_{C_{e}}}{\partial (A_{c})_{kl}} &= \dot{H}_{C_{e}}(P^{(j)}, Q^{(j)}),
\end{align*} \]

(119)

for \(\theta_{j} = (B_{c})_{kl}\),

\[ \begin{align*}
\frac{\partial \dot{H}_{A_c}}{\partial (B_{c})_{kl}} &= \dot{H}_{A_c}(P^{(j)}, Q^{(j)}), \\
\frac{\partial \dot{H}_{B_e}}{\partial (B_{c})_{kl}} &= \dot{H}_{B_e}(P^{(j)}, Q^{(j)}) + P_{2}E^{(k,l)}V_{2}, \\
\frac{\partial \dot{H}_{C_{e}}}{\partial (B_{c})_{kl}} &= \dot{H}_{C_{e}}(P^{(j)}, Q^{(j)}),
\end{align*} \]

(120)

and for \(\theta_{j} = (C_{c})_{kl}\), where \(k > 1\),

\[ \begin{align*}
\frac{\partial \dot{H}_{A_c}}{\partial (C_{c})_{kl}} &= \dot{H}_{A_c}(P^{(j)}, Q^{(j)}), \\
\frac{\partial \dot{H}_{B_e}}{\partial (C_{c})_{kl}} &= \dot{H}_{B_e}(P^{(j)}, Q^{(j)}), \\
\frac{\partial \dot{H}_{C_{e}}}{\partial (C_{c})_{kl}} &= \dot{H}_{C_{e}}(P^{(j)}, Q^{(j)}) + R_{2}E^{(k,l)}Q_{2} + \frac{\gamma^{-2}}{2}R_{222}E^{(k,l)}Y,
\end{align*} \]

(121)

where

\[ Y = (Q_{12}^{T}P_{1} + Q_{2}P_{12}^{T})Q_{12} + (Q_{12}^{T}P_{12} + Q_{2}P_{2})Q_{2} \]

(122)
and $E^{(k,l)}$ is a matrix of the appropriate dimension whose only nonzero element is $e_{kl} = 1$.

$P^{(j)}$ and $Q^{(j)}$ can be obtained by solving the Lyapunov equation

$$0 = (\tilde{A} + \gamma^{-2} Q \tilde{R}_\infty) Q^{(j)} + Q^{(j)} (\tilde{A} + \gamma^{-2} Q \tilde{R}_\infty)^T + \tilde{V}^{(j)}$$

$$+ \tilde{A}^{(j)} Q + Q \tilde{A}^{(j)^T} + \gamma^{-2} Q \tilde{R}_\infty^{(j)} Q,$$

$$0 = (\tilde{A} + \gamma^{-2} Q \tilde{R}_\infty)^T P^{(j)} + P^{(j)} (\tilde{A} + \gamma^{-2} Q \tilde{R}_\infty) + \tilde{R}^{(j)}$$

$$+ (\tilde{A}^{(j)} + \gamma^{-2} Q \tilde{R}_\infty^{(j)})^T P + P (\tilde{A}^{(j)} + \gamma^{-2} Q \tilde{R}_\infty^{(j)} + \gamma^{-2} Q \tilde{R}_\infty^{(j)}) . \quad (123)$$

Similarly for $\lambda$, using a dot to denote $\partial / \partial \lambda$, \begin{align*}
\frac{\partial H_{A\varepsilon}}{\partial \lambda} &= \dot{H}_{A\varepsilon}(\dot{P}, \dot{Q}), \\
\frac{\partial H_{B\varepsilon}}{\partial \lambda} &= \dot{H}_{B\varepsilon}(\dot{P}, \dot{Q}) + P_2 B_{c} \dot{V}_2 + (P_{12} Q_1 + P_2 Q_{12}) \dot{C}^T, \\
\frac{\partial H_{C\varepsilon}}{\partial \lambda} &= \dot{H}_{C\varepsilon}(\dot{P}, \dot{Q}) + \dot{R}_2 C_{c} Q_2 + \dot{B}^T (P_{1} Q_{12} + P_{12} Q_{12}) \\
&+ \frac{1}{2} \gamma^{-2} \dot{R}_{2\infty} C_{c} Y - \gamma^{-3} \gamma R_{2\infty} C_{c} Y, \quad (124) \end{align*}

where $\dot{P}$ and $\dot{Q}$ are obtained by solving

$$0 = (\tilde{A} + \gamma^{-2} Q \tilde{R}_\infty) \dot{Q} + \dot{Q} (\tilde{A} + \gamma^{-2} Q \tilde{R}_\infty)^T + \dot{V}$$

$$+ \tilde{A} Q + Q \tilde{A}^T + \gamma^{-2} Q \tilde{R}_\infty Q - 2 \gamma^{-3} \gamma Q \tilde{R}_\infty Q,$$

$$0 = (\tilde{A} + \gamma^{-2} Q \tilde{R}_\infty)^T \dot{P} + \dot{P} (\tilde{A} + \gamma^{-2} Q \tilde{R}_\infty) + \dot{R}$$

$$+ (\tilde{A} + \gamma^{-2} Q \tilde{R}_\infty + \gamma^{-2} Q \tilde{R}_\infty - 2 \gamma^{-3} \gamma Q \tilde{R}_\infty)^T P$$

$$+ P (\tilde{A} + \gamma^{-2} Q \tilde{R}_\infty + \gamma^{-2} Q \tilde{R}_\infty - 2 \gamma^{-3} \gamma Q \tilde{R}_\infty). \quad (125)$$
15. HOMOTOPY APPROACH BASED ON THE INPUT NORMAL RICCATI FORM FOR LQG CONTROL WITH AN $H^\infty$ PERFORMANCE BOUND.

Theorem 1 [13]. For every minimal $(\bar{A}_c, \bar{B}_c, \bar{C}_c)$, i.e., $(\bar{A}_c, \bar{B}_c)$ is controllable and $(\bar{A}_c, \bar{C}_c)$ is observable, there exist a similarity transformation $U$ and a positive definite matrix $\Omega = \text{diag}(\omega_1, \ldots, \omega_n)$ such that $A_c = U^{-1}\bar{A}_c U$, $B_c = U^{-1}\bar{B}_c$, and $C_c = \bar{C}_c U$ satisfy

$$0 = A_c + A_c^T + B_c B_c^T - C_c^T C_c,$$
$$0 = A_c^T \Omega + \Omega A_c + C_c^T C_c - \Omega B_c B_c^T \Omega. \tag{126}$$

In addition,

$$\begin{align*}
(A_c)_{ii} &= -\frac{1}{2} \left[ (C_c^T C_c)_{ii} - (B_c B_c^T)_{ii} \right], \\
\omega_i &= \frac{(C_c^T C_c)_{ii}}{(B_c B_c^T)_{ii}}, \tag{127} \\
(A_c)_{ij} &= \frac{\omega_j (1 + \omega_i) (B_c B_c^T)_{ij} - (C_c^T C_c)_{ij} (1 + \omega_j)}{\omega_i - \omega_j}, \quad \text{if } \omega_i \neq \omega_j.
\end{align*}$$

Definition 1. The triple $(A_c, B_c, C_c)$ satisfying (126) or (127) is said to be in input normal Riccati form.

The following algorithm will transform a given triple $(\bar{A}_c, \bar{B}_c, \bar{C}_c)$ to the input normal Riccati form $(A_c, B_c, C_c)$.

1) Find symmetric, positive definite $P$ and $Q$ satisfying

$$0 = \bar{A}_c^T P + P \bar{A}_c + \bar{C}_c^T \bar{C}_c - P \bar{B}_c \bar{B}_c^T P,$$
$$0 = \bar{A}_c Q + Q \bar{A}_c^T + \bar{B}_c \bar{B}_c^T - Q \bar{C}_c^T \bar{C}_c Q.$$

2) Compute the Cholesky decompositions of $P$ and $Q$, i.e.,

$$P = L_P L_P^T, \quad Q = L_Q L_Q^T,$$

where $L_P$ and $L_Q$ are invertible lower triangular matrices.

3) Compute the singular value decomposition of $L_P^T L_Q$, i.e., $L_P^T L_Q = U \Omega V^T$. 

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4) Let
\[ T = L_Q V, \quad T^{-1} = \Omega^{-1} U^T L_P^T. \]

5) Then
\[ A_c = T^{-1} \bar{A} c T, \quad B_c = T^{-1} \bar{B} c, \quad C_c = \bar{C} c T. \]

To optimize \( J(A_c, B_c, C_c) \) over the open set \( S \) under the constraints that symmetric positive definite \( Q \) satisfies (108), and \((A_c, B_c, C_c)\) is in input normal Riccati form, the following Lagrangian is formed:
\[
\mathcal{L}(A_c, B_c, C_c, \Omega, Q, \mathcal{P}, M_c, M_o) \equiv \text{tr} \left[ Q \bar{U} + (\bar{A} Q + Q \bar{A}^T + \gamma^{-2} Q \bar{U}_\infty Q + \bar{V}) \mathcal{P} + (A_c^T \Omega + \Omega A_c + C_c^T T C_c - \Omega B_c B_c^T \Omega) M_c \right],
\]
where the symmetric matrices \( M_c, M_o \), and \( \mathcal{P} \in \mathbb{R}^{\bar{n} \times \bar{n}} \) are Lagrange multipliers. \( \Omega = \text{diag} (\omega_1, \ldots, \omega_n) \) is related to the input normal Riccati form constraint. Setting \( \partial \mathcal{L} / \partial Q = 0 \) yields (111). Partition \( Q, \mathcal{P} \in R^{\bar{n} \times \bar{n}} \) as in (112). \( \partial \mathcal{L} / \partial A_c = 0 \) and \( \partial \mathcal{L} / \partial \Omega = 0 \) yield, respectively,
\[
0 = 2M_c + 2\Omega M_o + 2(\mathcal{P}_{12}^T Q_{12} + \mathcal{P}_2 Q_2), \quad 0 = (A_c M_o - B_c B_c^T \Omega M_o)_{ii}, \quad 1 \leq i \leq n_c.
\]

A straightforward calculation shows
\[
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial B_c} &= 2 \mathcal{P}_2 B_c V_2 + 2(\mathcal{P}_{12}^T Q_1 + \mathcal{P}_2 Q_{12}^T) C^T + 2M_c B_c - 2\Omega M_o B_c, \\
\frac{\partial \mathcal{L}}{\partial C_c} &= 2R_{20}^T C_c Q_2 + 2B^T (\mathcal{P}_{12} Q_{12} + \mathcal{P}_2 Q_2) + 2C_c M_o - 2C_c M_c + \gamma^{-2} R_{20} C_c [(\mathcal{P}_{12}^T Q_1 + \mathcal{P}_2 Q_{12}^T) Q_{12} + (C_c^T \mathcal{P}_1 + Q_2 \mathcal{P}_2) Q_2] .
\end{aligned}
\tag{128}
\]

**Theorem 2** [13]. The matrices \( M_c \) and \( M_o \) in (128) satisfy
\[
\begin{aligned}
M_c &= -(S + \Omega M_o), \\
(M_o)_{ii} &= -\frac{1}{(A_c + (B_c B_c^T \Omega))_{ii}} \sum_{j \neq i}^{n_c} [(A_c)_{ij} (M_o)_{ji} + (B_c B_c^T \Omega)_{ij} (M_o)_{ji}], \\
(M_o)_{ij} &= \frac{(S)_{ij} - (S)_{ji}}{\omega_j - \omega_i}, \quad \text{if} \ \omega_j \neq \omega_i,
\end{aligned}
\tag{129}
\]
where
\[
S = \mathcal{P}_{12}^T Q_{12} + \mathcal{P}_2 Q_2. \tag{130}
\]

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A homotopy approach based on the input normal Riccati form is now described. Let $A_f, B_f, C_f, \gamma_f, R_{1f}, R_{2f}, R_{1oof}, R_{2oof}, V_{1f},$ and $V_{2f}$ denote $A, B, C, \lambda, R_1, R_2, R_{1oo}, R_{2oo}, V_1,$ and $V_2$ in the above and define $A(\lambda), B(\lambda), C(\lambda), \gamma(\lambda), R_1(\lambda), R_2(\lambda), R_{1oo}(\lambda), R_{2oo}(\lambda), V_1(\lambda),$ and $V_2(\lambda)$ as in (113) and denote them by $A, B, C, \gamma, R_1, R_2, R_{1oo}, R_{2oo}, V_1,$ and $V_2$ respectively in the following. Let

$$H_{B_c}(\theta, \lambda) = \frac{1}{2} \frac{\partial L}{\partial B_c} = \mathcal{P}_1 B_c V_2 + (\mathcal{P}_{12}^T Q_1 + \mathcal{P}_{22}^T Q_2) C^T + M_c B_c - \Omega M_o \Omega B_c,$$

$$H_{C_c}(\theta, \lambda) = \frac{1}{2} \frac{\partial L}{\partial C_c} = R_{2oo} C_c Q_2 + B^T (\mathcal{P}_1 Q_{12} + \mathcal{P}_2 Q_2) + C_c M_o - C_c M_c$$

$$+ \frac{1}{2} \gamma^{-2} R_{2oo} C_c \left[ (Q_{12}^T \mathcal{P}_1 + Q_{22}^T \mathcal{P}_2) Q_{12} + (Q_{12}^T \mathcal{P}_1 + Q_{22}^T \mathcal{P}_2) Q_{22} \right],$$

where

$$\theta = \begin{pmatrix} \text{Vec} (B_c) \\ \text{Vec} (C_c) \end{pmatrix}$$

(131)
denotes the independent variables $B_c$ and $C_c$, $M_o$ and $M_c$ satisfy (129), and $\mathcal{Q}$ and $\mathcal{P}$ satisfy respectively (108) and (111) with partitioned forms (112).

There are several ways to construct a homotopy map. The most natural homotopy map is defined as

$$\rho(\theta, \lambda) = \begin{pmatrix} \text{Vec} [H_{B_c}(\theta, \lambda)] \\ \text{Vec} [H_{C_c}(\theta, \lambda)] \end{pmatrix},$$

(132)

and its Jacobian matrix is

$$D\rho(\theta, \lambda) = (D_\theta \rho(\theta, \lambda), D_\lambda \rho(\theta, \lambda)).$$

For the reduced-order problem, both to avoid singular starting points and to guarantee a full rank Jacobian matrix along the homotopy path (except possibly at $\lambda = 1$), the homotopy map

$$\hat{\rho}(\theta, \lambda) = \lambda \rho(\theta, \lambda) + (1 - \lambda)(\theta - \theta_0)$$

(133)
is used. The Jacobian matrix of $\hat{\rho}$ is given by

$$D\hat{\rho}(\theta, \lambda) = (\lambda D_\theta \rho(\theta, \lambda) + (1 - \lambda)I,$$

$$\rho(\theta, \lambda) + \lambda D_\lambda \rho(\theta, \lambda) - (\theta - \theta_0)).$$

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Define
\[
\dot{H}_{B_e}(\mathcal{P}^{(j)}, Q^{(j)}, M_c^{(j)}, M_o^{(j)}) = T_{2}^{(j)} B_c V_2 + (P_{12}^{T(j)} Q_1 + P_{12}^{T(j)} Q_1 + P_2^{T(j)} Q_1 + P_2 Q_1^{(j)} )c^T \\
+ M_c^{(j)} B_c - \Omega M_o^{(j)} B_c - \Omega M_c^{(j)} B_c,
\]
\[
\dot{H}_{C_e}(\mathcal{P}^{(j)}, Q^{(j)}, M_c^{(j)}, M_o^{(j)}) = R_2 C_c Q_2^{(j)} + B^T (P_1^{(j)} Q_1^{(j)} + P_1 Q_1^{(j)} + P_2 Q_2^{(j)} + P_1 Q_1^{(j)} + P_2 Q_2^{(j)}) \\
+ C_c M_o^{(j)} - C_c M_c^{(j)} \\
+ \frac{1}{2} \gamma^{-2} R_{2oo} C_c \left[ (Q_{12}^{T(j)} P_1 + Q_{12}^{T(j)} P_1 + Q_2^{T(j)} P_2 + Q_2^{T(j)} P_2) Q_1^{(j)} \\
+ (Q_2^{T(j)} P_2 + Q_2^{T(j)} P_2 + Q_2^{T(j)} P_2 + Q_2^{T(j)} P_2) Q_2^{(j)} \\
+ (Q_{12}^{T(j)} P_2 + Q_2^{T(j)} P_2 + Q_2^{T(j)} P_2 + Q_2^{T(j)} P_2) Q_2^{(j)} \right],
\]

where the superscript \((j)\) means \(\partial / \partial \theta_j; \ Y^{(j)} = \partial Y / \partial \theta_j\). Using the above definitions, we have for \(\theta_j = (B_c)_{kl}\),
\[
\frac{\partial}{\partial (B_c)_{kl}} H_{B_e} = \dot{H}_{B_e}(\mathcal{P}^{(j)}, Q^{(j)}, M_c^{(j)}, M_o^{(j)}) + P_2 E^{(k,l)} V_2 + M_c E^{(k,l)} - \Omega M_o E^{(k,l)},
\]
\[
\frac{\partial}{\partial (B_c)_{kl}} H_{C_e} = \dot{H}_{C_e}(\mathcal{P}^{(j)}, Q^{(j)}, M_c^{(j)}, M_o^{(j)}),
\]
and for \(\theta_j = (C_c)_{kl}\),
\[
\frac{\partial}{\partial (C_c)_{kl}} H_{B_e} = \dot{H}_{B_e}(\mathcal{P}^{(j)}, Q^{(j)}, M_c^{(j)}, M_o^{(j)}),
\]
\[
\frac{\partial}{\partial (C_c)_{kl}} H_{C_e} = \dot{H}_{C_e}(\mathcal{P}^{(j)}, Q^{(j)}, M_c^{(j)}, M_o^{(j)}) + E^{(k,l)} (M_o - M_c) + R_2 E^{(k,l)} Q_2 \\
\quad + \frac{1}{2} \gamma^{-2} R_{2oo} E^{(k,l)} Y,
\]

where \(Y\) is given by (122). \(\mathcal{P}^{(j)}\) and \(Q^{(j)}\) can be obtained by solving the Lyapunov equations (123). Similarly for \(\lambda\), using a dot to denote \(\partial / \partial \lambda\),
\[
\frac{\partial}{\partial \lambda} H_{B_e} = \dot{H}_{B_e}(\dot{P}, \dot{Q}, \dot{M}_c, \dot{M}_o) + P_2 B_c \dot{V}_2 + (P_{12}^{T} Q_1 + Q_2 Q_{12}^{T}) c^T,
\]
\[
\frac{\partial}{\partial \lambda} H_{C_e} = \dot{H}_{C_e}(\dot{P}, \dot{Q}, \dot{M}_c, \dot{M}_o) + \dot{R}_2 C_c \dot{Q}_2 + \dot{B}^T (P_1 Q_{12} + P_{12} Q_2) \\
\quad - \gamma^{-2} \gamma R_{2oo} C_c Y + \frac{1}{2} \gamma^{-2} \dot{R}_{2oo} C_c,
\]

and \(\dot{P}\) and \(\dot{Q}\) are obtained by solving the Lyapunov equations (125).
16. HOMOTOPY APPROACH BASED ON OVER-PARAMETRIZATION FORMULATION FOR LQG CONTROL WITH AN $H^\infty$ PERFORMANCE BOUND.

To optimize $J(A_c, B_c, C_c)$ over the open set $S$ under the constraint that symmetric positive definite $Q$ satisfies (108), the following Lagrangian is formed:

$$\mathcal{L}(A_c, B_c, C_c, P, Q) \equiv \text{tr} \left[ Q \bar{R} + (\bar{A}Q + Q \bar{A}^T + \gamma^{-2} \bar{R}_{\infty} Q + V)P \right],$$

where $P \in \mathbb{R}^{n \times n}$ is a Lagrange multiplier. Setting $\partial \mathcal{L}/\partial Q = 0$ yields (111). Partition $Q$, $P \in \mathbb{R}^{n \times n}$ as in (112). The partial derivatives of $\mathcal{L}$ can be computed as

$$\frac{\partial \mathcal{L}}{\partial A_c} = 2(P_{12}^T Q_{12} + P_2 Q_2),$$

$$\frac{\partial \mathcal{L}}{\partial B_c} = 2P_2 B_c V_2 + 2(P_{12}^T Q_1 + P_2 Q_{12}^T)C^T,$$

$$\frac{\partial \mathcal{L}}{\partial C_c} = 2R_2 C_c Q_2 + 2B^T (P_1 Q_{12} + P_{12} Q_2)$$

$$+ \gamma^{-2} R_{2\infty} C_c [(Q_{12}^T P_1 + Q_2 P_{12}^T) Q_{12} + (Q_{12}^T P_{12} + Q_2 P_2) Q_2].$$

Let $A_f, B_f, C_f, \gamma_f, R_{1f}, R_{2f}, R_{1\infty f}, R_{2\infty f}, V_{1f}$, and $V_{2f}$ denote $A, B, C, \gamma, R_1, R_2, R_{1\infty}, R_{2\infty}, V_1,$ and $V_2$ in the above and define $A(\lambda), B(\lambda), C(\lambda), \gamma(\lambda), R_1(\lambda), R_2(\lambda), R_{1\infty}(\lambda), R_{2\infty}(\lambda), V_1(\lambda),$ and $V_2(\lambda)$ as in (113) and denote them by $A, B, C, \gamma, R_1, R_2, R_{1\infty}, R_{2\infty}, V_1,$ and $V_2$ respectively in the following.

Define $H_{A_c}(\theta, \lambda), H_{B_c}(\theta, \lambda),$ and $H_{C_c}(\theta, \lambda)$ as in (114) where

$$\theta \equiv \begin{pmatrix} \text{Vec} (A_c) \\ \text{Vec} (B_c) \\ \text{Vec} (C_c) \end{pmatrix} \quad (134)$$

denotes the independent variables, and $Q$ and $P$ satisfy respectively (108) and (111). Vec applied to a matrix is a column vector obtained by concatenating the columns of the matrix.

Define

$$\rho(\theta, \lambda) = \begin{pmatrix} \text{Vec} [H_{A_c}(\theta, \lambda)] \\ \text{Vec} [H_{B_c}(\theta, \lambda)] \\ \text{Vec} [H_{C_c}(\theta, \lambda)] \end{pmatrix}, \quad (135)$$

whose Jacobian matrix is

$$D\rho(\theta, \lambda) = (D_\theta \rho(\theta, \lambda), D_\lambda \rho(\theta, \lambda)).$$

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Note that $\theta$ in (135) has $n_c^2 + n_c m + n_c l$ components, more than the minimal number $n_c m + n_c l$ needed. Because of this over-parametrization, the Jacobian matrix of $\rho$ is seriously rank deficient. To remedy this severe rank deficiency, the homotopy map is defined as

$$\hat{\rho}(\theta, \lambda) = \lambda \rho(\theta, \lambda) + (1 - \lambda)(\theta - \theta_0),$$

which guarantees a full rank Jacobian matrix along the entire homotopy zero curve except possibly at the solution (corresponding to $\lambda = 1$). The Jacobian matrix of $\hat{\rho}$ is given by

$$D \hat{\rho}(\theta, \lambda) = (\lambda D_\theta \rho(\theta, \lambda) + (1 - \lambda)I,$$

$$\rho(\theta, \lambda) + \lambda D_\lambda \rho(\theta, \lambda) - (\theta - \theta_0)).$$

(137)

To find $D_\theta \rho(\theta, \lambda)$, define the auxiliary matrices $\hat{H}_{A_\varepsilon}(P^{(j)}, Q^{(j)})$, $\hat{H}_{B_\varepsilon}(P^{(j)}, Q^{(j)})$, and $\hat{H}_{C_\varepsilon}(P^{(j)}, Q^{(j)})$ as in (118). Using the above definitions, we have (119), (120), (121) for $\theta_j = (A_\varepsilon)_{kl}$, $\theta_j = (B_\varepsilon)_{kl}$, and $\theta_j = (C_\varepsilon)_{kl}$ respectively. $P^{(j)}$ and $Q^{(j)}$ can be obtained by solving the Lyapunov equation (123). Similarly we have (124) for $\lambda$ and $\hat{P}$ where $\hat{Q}$ are obtained by solving (125).
17. NUMERICAL ALGORITHMS FOR LQG CONTROL
   WITH AN $H^\infty$ PERFORMANCE BOUND.

Choose the initial $\gamma$ so that $\gamma_0^{-2}$ is approximately zero. The initial point $(\theta, \lambda) = (\theta_0, 0)$ is
chosen so that it satisfies $\rho(\theta_0, 0) = 0$ and the triple $((A_c)_0, (B_c)_0, (C_c)_0)$ is in the respective
form for each homotopy.

It is well known that the full-order LQG compensator (100) for the plant (99) minimizing
the steady-state quadratic performance functional (103) is given by:

$$A_c = A - \Sigma P - Q \bar{\Sigma}$$  \hspace{1cm} (138)

$$B_c = QC^TV_2^{-1}, \quad C_c = -R_2^{-1}B^TP,$$  \hspace{1cm} (139)

where $\Sigma \equiv BR_2^{-1}B^T$, $\bar{\Sigma} \equiv C^TV_2^{-1}C$, and $P$ and $Q$ are the unique, symmetric, positive
semidefinite solutions respectively, of

$$0 = A^TP + PA + R_1 - P\Sigma P,$$

$$0 = AQ + QA^T + V_1 - Q\bar{\Sigma}Q.$$  \hspace{1cm} (140)

A. Full-order Initialization

The initial point for the full-order problem can be chosen as follows:
1) Solve for $Q$ and $P$ from (140) and obtain $((\hat{A}_c)_0, (\hat{B}_c)_0, (\hat{C}_c)_0)$ from (138) and (139).
2) Transform the triple $((\hat{A}_c)_0, (\hat{B}_c)_0, (\hat{C}_c)_0)$ to Ly’s form for the Ly form homotopy, to
   input normal Riccati form for the input normal Riccati form homotopy, and build $\theta_0$
as described in (115), (131), and (134) for the respective homotopies.

B. Reduced-order Initialization

The initialization scheme for the reduced-order problem is more complicated since a
closed form expression for the reduced-order $H^2$ LQG compensator does not exist. For a
given system $(A, B, C)$, and matrices $\bar{R}_1, \bar{R}_2, \bar{R}_{1\infty}, \bar{R}_{2\infty}, \bar{V}_1$ and $\bar{V}_2$, the reduced order
starting point is chosen using a method in [11] which can be summarized as:
1) Compute the real Schur decomposition of $\bar{A}$ so that $\bar{A} = UAV^t$, $A \equiv \begin{pmatrix} A_1 & 0 \\ A_{21} & A_2 \end{pmatrix}$,
   where $A_1 \in \mathbb{R}^{n_c \times n_c}$, and transform $\bar{B}, \bar{C}, \bar{R}_1, \bar{V}_1, \bar{R}_{1\infty}$ so that $\bar{B} = U\bar{B}, \bar{C} = \bar{C}U^t$,
   $\bar{R}_1 = U\bar{R}_1U^t, \bar{V}_1 = U\bar{V}_1U^t, \bar{R}_{1\infty} = U\bar{R}_{1\infty}U^t$ and let $R_2 = \bar{R}_2, \bar{V}_2 = \bar{V}_2, \bar{R}_{2\infty} = \bar{R}_{2\infty}$.
2) If $A$ is not asymptotically stable, modify either diagonal elements or $2 \times 2$ diagonal blocks of $A$ so that it is asymptotically stable and call this modified matrix $A_0$.

3) Take $B_0 = B$, $C_0 = C$, $R_{2,0} = R_{2f} \equiv R_2$, $R_{2\infty,0} = R_{2\infty f} \equiv R_{2\infty}$, $V_{1,0} = V_{1f} \equiv V_1$, $V_{2,0} = \beta V_{2f} \equiv \beta V_2$, $\beta \gg 0$, and $R_{1,0} = \begin{pmatrix} (R_1)_{11} & 0 \\ 0 & 0 \end{pmatrix}$ where $(R_1)_{11}$ is the leading $n_c \times n_c$ block of $R_{1f} \equiv R_1$.

4) Solve

$$0 = A_0^T P + PA_0 + R_{1,0} - P\Sigma_0 P,$$

$$0 = A_0 Q + QA_0^T + V_{1,0} - Q\Sigma_0 Q,$$

for symmetric, positive semidefinite $P$ and $Q$, where $\Sigma_0 \equiv B_0 R_{2,0}^{-1} B_0^T$, and $\Sigma_0 \equiv C_0^T V_{2,0}^{-1} C_0$.

5) Obtain $(A_c, B_c, C_c)$ from $A_c = A_0 - \Sigma_0 P - Q\Sigma_0$, $B_c = QC_0^T V_{2,0}^{-1}$, $C_c = -R_{2,0}^{-1} B_0^T P$.

6) Solve

$$0 = (A_0 - Q\Sigma_0)^T \dot{P} + \dot{P} (A_0 - Q\Sigma_0) + P\Sigma_0 P,$$

$$0 = (A_0 - \Sigma_0 P)\dot{Q} + \dot{Q} (A_0 - \Sigma_0 P)^T + Q\Sigma_0 Q,$$

for symmetric, positive semidefinite $\dot{P}$ and $\dot{Q}$.

7) Following [76], obtain the reduced-order compensator starting point from $(A_c, B_c, C_c)$, $\dot{P}$, $\dot{Q}$, and $n_c$, as follows:

i) Compute the Cholesky decomposition of (assumed positive definite) $\dot{P}$ and $\dot{Q}$, i.e., $\dot{P} = L_{\dot{P}} L_{\dot{P}}^T$, $\dot{Q} = L_{\dot{Q}} L_{\dot{Q}}^T$.

ii) Compute the singular value decomposition of $L_{\dot{P}}^T L_{\dot{Q}}$, i.e., $L_{\dot{P}}^T L_{\dot{Q}} = U \Omega V^T$.

iii) Let $T = L_{\dot{Q}} \Omega^{-\frac{1}{2}}$, $T^{-1} = \Omega^{-\frac{1}{2}} U^T L_{\dot{P}}$.

iv) Let $\tilde{A}_c = T^{-1} A_c T$, $\tilde{B}_c = T^{-1} B_c$, and $\tilde{C}_c = C_c T$ so that

$$\tilde{A}_c = \begin{pmatrix} (\tilde{A}_c)_{11} & (\tilde{A}_c)_{12} \\ (\tilde{A}_c)_{21} & (\tilde{A}_c)_{22} \end{pmatrix},$$

$$\tilde{B}_c = \begin{pmatrix} (\tilde{B}_c)_{11} \\ (\tilde{B}_c)_{21} \end{pmatrix},$$

$$\tilde{C}_c = \begin{pmatrix} (\tilde{C}_c)_{11} & (\tilde{C}_c)_{12} \\ (\tilde{C}_c)_{21} \end{pmatrix}.$$

The starting point $\theta_0$ for the reduced order problem is chosen using $((\tilde{A}_c)_{11}, (\tilde{B}_c)_{11}, (\tilde{C}_c)_{11})$, with the construction in (115), (131), and (134) for the respective homotopies.
C. Homotopy Zero Curve Tracking

Once the initial point is chosen, the rest of the computation is as follows:

1) Set $\lambda := 0$, $\theta := \theta_0$.
2) Calculate $\vec{H}$, $\vec{H}_\infty$, $\vec{V}$, and compute $Q$ and $P$ according to (108) and (111).
3) Evaluate the homotopy map $\rho(\theta, \lambda)$ or $\dot{\rho}(\theta, \lambda)$ and the Jacobian of the homotopy map $D\rho(\theta, \lambda)$ or $D\dot{\rho}(\theta, \lambda)$.
4) Predict the next point $Z^{(0)} = (\theta^{(0)}, \lambda^{(0)})$ on the homotopy zero curve using, e.g., a Hermite cubic interpolant.
5) For $k := 0, 1, 2, \ldots$ until convergence do

$$Z^{(k+1)} = Z^{(k)} - [D\rho(Z^{(k)})]^\dagger \rho(Z^{(k)}),$$

where $[D\rho(Z)]^\dagger$ is the Moore-Penrose inverse of $D\rho(Z)$. Let $(\theta_1, \lambda_1) = \lim_{k \to \infty} Z^{(k)}$.
6) If $\lambda_1 < 1$, then set $\theta := \theta_1$, $\lambda := \lambda_1$, and go to step 2).
7) If $\lambda_1 \geq 1$, compute the solution $\hat{\theta}$ at $\lambda = 1$.

For the over-parametrization formulation homotopy, because of the singularity at $\lambda = 1$, step 7) is replaced by:

7) If $\lambda_1 \geq 1$, use the last point $(\hat{\theta}, \hat{\lambda})$ with $\hat{\lambda} < 1$ to redefine the homotopy map with $\theta_0 = \hat{\theta}$.
8) Redo steps 1)–6) until $\lambda \geq 1$.
9) Use Hermite polynomial interpolation to obtain the solution at $\lambda = 1$. 
18. NUMERICAL RESULTS FOR LQG CONTROL
WITH AN $H^\infty$ PERFORMANCE BOUND.

The following systems are solved by the homotopy algorithms discussed in the previous sections. The homotopy curve tracking was done with HOMPACK [72].

The first system, formulated in [9] and studied in [6], is given by

$$A = \begin{pmatrix}
-0.161 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6.004 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-0.5822 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-9.9835 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-0.4073 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-3.982 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad B = \begin{pmatrix}
0 \\
0 \\
0.0064 \\
0.00235 \\
0.0713 \\
1.0002 \\
0.1045 \\
0.9955
\end{pmatrix}, \quad C^T = \begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix},$$

$$E_1 = 10^{-3} \begin{pmatrix}
0 & 0 & 0 & 0 & 0.55 & 11 & 1.32 & 18 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad E_{1\infty} = E_1.$$

$$E_2 = \begin{pmatrix}
0 \\
1
\end{pmatrix}, \quad E_{2\infty} = \begin{pmatrix}
0 \\
0
\end{pmatrix}, \quad D_1 = (B \ 0), \quad D_2 = (0 \ 1).$$

For the full-order problem, the solutions of the auxiliary minimization problem are obtained for $\gamma \geq \gamma_{\text{min}} = 0.481$ using the Ly form homotopy approach. For $\gamma < \gamma_{\text{min}} = 0.481$, the Riccati equation solver fails and therefore no solution can be found. In Fig. 13, $\|H(s)\|_{\infty}$ is plotted against $J$. The ratio of $\|H(s)\|_{\infty}$ at $\gamma = \gamma_{\text{min}}$ to that at $\gamma = \infty$ is 0.33, which indicates that there is about 67% improvement in the $H^\infty$ performance of the compensator with $\gamma = \gamma_{\text{min}}$ over the compensator without the $H^\infty$ constraint.

For $n_c = 4, 6$, the solutions of the auxiliary minimization problem are obtained for $\gamma \geq 2.55$ using the Ly form homotopy approach. In Fig. 14, $\|H(s)\|_{\infty}$ is plotted against $J$ for $n_c = 4$ (solid line with "x" indicating the data points) and $n_c = 6$ (dashed line with "o" indicating the data points). For both $n_c = 4$ and $n_c = 6$, the ratio of $\|H(s)\|_{\infty}$ at $\gamma = 2.55$ to that at $\gamma = \infty$ is 0.49, which indicates that there is about 51% improvement in the $H^\infty$ performance of the compensator with $\gamma = 2.55$ over the compensator without the $H^\infty$ constraint. For this example, the 4th-order and 6th-order compensators have almost the same $H^2$ and $H^\infty$ performance.
For \( n_c = 2 \), two different sets of solutions are obtained by varying \( \beta \) in the initialization step. The trade-off curves are shown in Figs. 15 (\( \beta = 100 \)) and 16 (\( \beta = 1 \)). The first set of solutions (shown in Fig. 15) is obtained for \( \gamma \geq 2.54 \), while the second set (shown in Fig. 16) is obtained for \( \gamma \geq 9.5 \). It can be seen that the first set of solutions has lower \( H^2 \) cost and better \( H^{\infty} \) performance. It was verified that all the points in both Figs. 15 and 16 are local minima of the auxiliary cost \( J \).

There is considerable confusion in the control literature over the terms \textit{continuation}, \textit{homotopy}, and \textit{globally convergent}. A careful discussion of the distinct meanings of these terms can be found in [73]. \textit{Continuation} refers to the standard classical technique of solving \( \rho(\theta, \lambda + \Delta \lambda) = 0 \) with fixed \( \Delta \lambda > 0 \), given a solution \((\tilde{\theta}, \tilde{\lambda})\): \( \rho(\tilde{\theta}, \tilde{\lambda}) = 0 \). It is implicitly assumed that \( \theta = \theta(\lambda) \), i.e., the zero curve \( \gamma \) of \( \rho(\theta, \lambda) \) being tracked in \((\theta, \lambda)\) space is monotone in \( \lambda \). Other tacit assumptions are that \( \gamma \) does not bifurcate or otherwise contain singularities. The more general \textit{homotopy} methods make no such assumptions, and include mechanisms to deal with bifurcations and turning points. In particular, homotopy methods do not assume that the zero curve \( \gamma \) is monotone in \( \lambda \), i.e., \( \theta = \theta(\lambda) \). \textit{Globally convergent} means that the zero curve \( \gamma \) reaches a solution \( \hat{\theta}, \rho(\hat{\theta}, 1) = 0 \), from an arbitrary starting point \( \theta_0, \rho(\theta_0, 0) = 0 \). A continuation or homotopy algorithm is not \textit{a priori} globally convergent. A particular class of homotopy methods, known as \textit{probability-one homotopy methods}, are provably globally convergent under mild assumptions [72], and their zero curve \( \gamma \) is guaranteed to contain no singularities with probability one. The homotopy algorithms proposed here are examples of probability-one globally convergent homotopy methods; the matrices \( A_0, B_0, \ldots \), and the starting point \( \theta_0 \) here play the role of the parameter vector \( a \) in the probability-one homotopy theory [73]. Fig. 17 shows a portion of \( \gamma \) for the previous example, clearly demonstrating the nonmonotonicity in \( \lambda \) and that standard continuation in \( \lambda \) would fail.

For this problem, with the input normal Riccati form, initial \( \omega \)s consist of very close pairs, which leads to a significant numerical error in computing \( M_0 \) and therefore the slow convergence or failure to converge of the homotopy algorithm. It can be verified that the solutions, obtained by the Ly form homotopy approach or over-parametrization approach,
Fig. 13. $\|H(s)\|_\infty$ versus $J$ for $n_c = n$.

Fig. 14. $\|H(s)\|_\infty$ versus $J$ for $n_c = 4, 6$. 

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Fig. 15. $\|H(s)\|_\infty$ versus $J$ for $n_c = 2$.

Fig. 16. $\|H(s)\|_\infty$ versus $J$ for $n_c = 2$. 
also have ωs in close pairs, which implies that the input normal Riccati form homotopy approach leads to intrinsic ill conditioning for this problem.

As a second example, consider the system given by

\[
A = \begin{pmatrix}
0 & 1.0000 \\
-9.8666 & -0.0001
\end{pmatrix}, \quad C = \begin{pmatrix}
10.0637 \\
-9.9363
\end{pmatrix}, \quad B = C^T,
\]

\[E_1 = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}, \quad E_2 = \begin{pmatrix}
0 \\
1
\end{pmatrix}, \quad E_{1\infty} = E_1, \quad E_{2\infty} = E_2,
\]

\[D_1 = \begin{pmatrix}
0 & 0.0482 \\
0 & 0
\end{pmatrix}, \quad D_2 = \begin{pmatrix}
0 & 1
\end{pmatrix}.
\]

The input normal Riccati form is used to solve the full-order problem. The trade-off curve is shown in Fig. 18. The solutions of the auxiliary minimization problem are obtained for \(\gamma \geq 0.032\). The ratio of \(\|H(s)\|_\infty\) at \(\gamma = 0.032\) to that at \(\gamma = \infty\) is 0.69, which indicates that there is about 31% improvement in the \(H^\infty\) performance of the compensator with \(\gamma = 0.032\) over the compensator without the \(H^\infty\) constraint.

The relative performance of homotopies based on the Ly parametrization, input normal Riccati form parametrization, and over-parametrization for the combined \(H^2/H^\infty\) reduced
order controller design problem is similar to that of those parametrizations applied to the
model order reduction problem, reported in detail in [19], [20] and [21]. The failure of the
input normal Riccati form homotopy on the first example here reinforces a point made in
[19], [20] for the model order reduction problem, namely that making structural assumptions
about the solution is both indefensible theoretically and frequently the source of numerical
failure (due to extreme ill conditioning). The optimal projection equations formulation of
[6] does not make structural assumptions (in some sense is completely basis independent),
but the optimal projection equations are very difficult and expensive to solve numerically.
This cost can be reduced by exploiting tensor product structure and assuming monotonicity
in $\lambda$ of the homotopy zero curves, but Fig. 17 here shows that assumption is not tenable.
The over-parameterization formulation makes no structural assumptions and is cheaper
computationally than the optimal projection equations, but it is inherently singular at the
solution with rank deficiency $n_3^2$, which will ultimately overwhelm the numerical linear
algebra [19], [21].
19. CONCLUSIONS.

The $H^2$ optimal model order reduction problem has been under intense study both theoretically and numerically. There are various approaches for solving this problem. The most significant contribution of the input normal form and the Ly form homotopy approaches is that only the minimal number of degrees of freedom is used, which gives rise to a very efficient numerical algorithm. However, the input normal form and Ly form can be ill conditioned.

The combined $H^2/H^\infty$ model order reduction problem, resulting from the addition of a $H^\infty$ constraint to the $H^2$ optimal model order reduction problem, is more complicated and more interesting. There are theoretical formulations for this problem, but there have been no serious numerical studies. Chapters 8–16 represent an attempt to design practical and efficient numerical algorithms for solving the combined $H^2/H^\infty$ model order reduction problem. Both the input normal form homotopy and the Ly form homotopy algorithms are applicable and very efficient, while the same ill conditioning as in the homotopy algorithms for the $H^2$ optimal model order reduction problem may occur.

Although there are numerous numerical algorithms for solving standard Riccati equations, there is a need for algorithms which can be generalized to solve variants of standard Riccati equations and which can be implemented efficiently on serial and parallel machines. The algorithm in Chapter 12, based on homotopy methods, can be generalized to solve the modified Riccati equations that the traditional method, mainly based on invariant subspaces, cannot. Even though the algorithm can be parallelized easily, the serial time complexity is $O(n^4)$. Future studies are needed to design a more efficient algorithm.

The linear quadratic Gaussian theory (LQG) with an $H^\infty$ bound provides a systematic approach to synthesize controllers with nominal high performance and robust stability. Theoretical studies for this problem, especially for reduced-order LQG controller synthesis, are still lacking. Chapters 13–18 are devoted to the design and implementation of efficient numerical homotopy algorithms for solving this difficult yet important problem. The minimal parameter homotopy algorithms, Ly form homotopy and the input normal Riccati form
homotopy, are very efficient while the input normal Riccati form homotopy is more likely to be ill conditioned for this type of problem.

In the numerical algorithm for the combined $H^2/H^\infty$ model order reduction problem and the LQG controller synthesis problem with an $H^\infty$ bound, solving Riccati equations takes a significant amount of computer time. To design and implement an efficient parallel algorithm for solving the Riccati equation is necessary to improve the performance of the homotopy algorithms.

In the homotopy algorithms for the $H^2$ optimal model order reduction problem, the combined $H^2/H^\infty$ model order reduction problem, and the LQG controller synthesis problem with an $H^\infty$ performance bound, most of the time is devoted to tangent vector computation which can be divided into independent tasks. Therefore a parallel implementation of the whole homotopy algorithm can be considered in the future.

The main conclusion from this study is that the more degrees of freedom that a formulation uses, the more robust is the resulting numerical algorithm. Both the input normal form and Ly form homotopies are very efficient for the $H^2$, the combined $H^2/H^\infty$ model order reduction problems, and the LQG controller synthesis problem with an $H^\infty$ performance bound. However, they may fail to exist or be very ill conditioned. The over-parametrization formulation solves the ill conditioning issue, but introduces singularity at the solution and may fail for a high dimensional system which will inevitably have a higher order singularity at the solution.
REFERENCES


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Appendix A. ELEMENTS OF LINEAR SYSTEM THEORY.

A.1. Description of Linear System.

A linear differential finite dimensional system which is continuous in time and time invariant (A, B, and C are constant matrices) can be represented by the following equations:

\[ \dot{x}(t) = Ax(t) + Bu(t), \]  
\[ y(t) = Cx(t), \]  

where \( t \) is time, \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), and \( C \in \mathbb{R}^{l \times n} \). \( x(t) \in \mathbb{R}^n \) is the state of the system. \( u(t) \in \mathbb{R}^m \) is the input (control) of the system and \( y(t) \in \mathbb{R}^l \) is the output of the system. The equation (A.1) is called the state equation and (A.2) is called the output equation. A, B, and C are constant matrices that characterize the behavior of the system.

Sections A.2–A.6 present some of the important properties of linear systems which are intended to be supplementary to Chapter 2 in [78]. The material in these sections mainly follows Kwakernaak and Sivan [42], Delchamps [14] and Wonham [75].

A.2. Stability.

The stability of the system (A.1)–(A.2) is determined by the matrix A, so it is sufficient to consider only the equation

\[ \dot{x}(t) = Ax(t). \]  

The general solution of (A.3) is given by

\[ x(t) = e^{A(t-t_0)}x_0. \]  

The general solution of (A.1) is given by

\[ x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau. \]  

**Definition A.1.** A solution \( x_0(t) \) of the system (A.3), which is given by (A.4), is stable if for any \( t_0 \) and any \( \epsilon > 0 \) there exists a \( \delta(\epsilon, t_0) > 0 \) such that for any other solution \( x(t) \),

\[ \|x(t_0) - x_0(t_0)\| \leq \delta \quad \Rightarrow \quad \|x(t) - x_0(t)\| \leq \epsilon \quad \text{for all} \quad t \geq t_0. \]
Furthermore, the solution \( x_0(t) \) is *asymptotically stable* if it is stable and for any \( t_0 \) there exists a \( \rho(t_0) > 0 \) such that for any other solution \( x(t) \)

\[
\|x(t) - x_0(t_0)\| \leq \rho \implies \|x(t) - x_0(t)\| \to 0 \quad \text{as} \quad t \to \infty.
\]

The system \((A.3)\) is asymptotically stable if every solution \( x_0(t) \) is asymptotically stable. In this case the matrix \( A \) is also referred to as asymptotically stable.

A criterion for determining the asymptotic stability of the system \((A.3)\) is given in the next theorem and the proof can be found in Chapter 2 in [78].

**Theorem A.1** ([78], page 10). The system \((A.3)\) is asymptotically stable if and only if all the eigenvalues of \( A \) have negative real parts.

Consider system \((A.3)\) again. Suppose that \( A \) has distinct eigenvalues \( \lambda_1, \ldots, \lambda_n \). Then, there exists a matrix \( T \) such that \( A = T \Lambda T^{-1} \), where \( \Lambda = \text{diag} \left( \lambda_1, \ldots, \lambda_n \right) \). Let \( g_1, \ldots, g_n \) be the columns of \( T \) and \( f_1, \ldots, f_n \) the rows of \( T^{-1} \). The response of the system can be written as

\[
x(t) = \sum_{i=1}^{n} e^{\lambda_i (t-t_0)} g_i f_i x_0 = \sum_{i=1}^{n} e^{\lambda_i (t-t_0)} \mu_i g_i,
\]

where the \( \mu_i \) describe how the initial state \( x_0 \) is decomposed along the vectors \( g_1, g_2, \ldots, g_n \).

If the system \((A.3)\) is not asymptotically stable, then some of the eigenvalues \( \lambda_i \) have nonnegative real parts. The state will converge to the zero state only if the initial state has components only along those eigenvectors that correspond to stable eigenvalues. If the initial state has components only along those eigenvectors that correspond to unstable eigenvalues, the response of the state will be composed of nondecreasing exponentials. This leads to the following decomposition of state space.

**Definition A.2.** Consider the system \((A.3)\). Suppose that \( A \) has \( n \) distinct characteristic values. Then we define the *stable subspace* for this system as the real linear subspace spanned by those eigenvectors of \( A \) that correspond to eigenvalues with strictly negative real parts. The *unstable subspace* for this system is the real subspace spanned by those eigenvectors of \( A \) that correspond to eigenvalues with nonnegative real parts.

**Remark A.1.** The state space of the system \((A.3)\) can be decomposed into two subspaces, such that the response of the system from an initial state in the stable subspace
always converges to the zero state while the response from a nonzero initial state in the unstable subspace never converges.

For \( A \) that does not have distinct eigenvalues, the following theorem gives a way to decompose the state space.

**Theorem A.2.** Suppose that \( A \) has \( k \) distinct eigenvalues \( \lambda_i, i = 1, 2, \ldots, k \). Let the multiplicity of each eigenvalue \( \lambda_i \) be \( m_i \). Define \( \mathcal{N}_i = \ker [(A - \lambda_i I)^{m_i}] \). Then

a) the dimension of the linear subspace \( \mathcal{N}_i \) is \( m_i \), \( i = 1, 2, \ldots, k \);

b) the whole \( n \)-dimensional complex space \( \mathbb{C}^n = \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_k \).

The proof is in [75].

**Definition A.3.** Consider (A.3). The **stable subspace** is the real subspace of the direct sum of those null spaces \( \mathcal{N}_i \) that correspond to eigenvalues of \( A \) with strictly negative real parts. The **unstable subspace** is the real subspace of the direct sum of those null spaces \( \mathcal{N}_i \) that correspond to eigenvalues of \( A \) with nonnegative real parts.

**A.3. Controllability and Reachability.**

The controllability and the reachability of the system (A.1)-(A.2) are determined by the pair \((A, B)\), so it is sufficient to consider only the equation (A.1). For the time continuous system considered here, controllability is equivalent to reachability. For time discrete systems, controllability is weaker than reachability.

**Definition A.4.** A vector \( x_1 \in \mathbb{R}^n \) is **reachable with respect to** \((A, B)\) if and only if there exist \( T > 0 \) and a piecewise continuous input \( u \) on \([0, T] \) such that

\[
    x_1 = \int_0^T e^{A(T-\tau)} B u(\tau) \, d\tau.
\]  

(A.6)

\((A, B)\) is a **reachable pair** if and only if every \( x_1 \in \mathbb{R}^n \) is reachable with respect to \((A, B)\).

In other words, a state \( x_1 \) is reachable when the initial state \( x_0 = 0 \) at initial time \( t_0 = 0 \) can be steered to \( x_1 \) in finite time if a suitable input function is applied.

**Definition A.5.** \((A, B)\) is a **controllable pair** if and only if for every \( x_0 \in \mathbb{R}^n \) there exist \( T > 0 \) and a piecewise continuous input \( u \) on \([0, T] \) such that

\[
    0 = e^{A T} x_0 + \int_0^T e^{A(T-\tau)} B u(\tau) \, d\tau.
\]  

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A system is controllable if an arbitrary initial condition may be zeroed out in finite time by suitable choice of input function. Whether the system is controllable or not can be determined by considering the controllability matrix or the controllability Gramian ([78], page 4), which is used to prove the equivalence of reachability to controllability.

**Theorem A.3.** \((A, B)\) is a reachable pair if and only if \((A, B)\) is a controllable pair.

If the system \((A.1)\) is not controllable, it means that there is some part of the state space that the system can not reach. This will lead to the following decomposition of the state space.

**Definition A.6.** The controllable subspace of the linear time-invariant system \((A.1)\) is the linear subspace consisting of the states that can be reached from the zero state within a finite time.

**Theorem A.4.** The controllable subspace of the \(n\)-dimensional linear time-invariant system \((A.1)\) is the linear subspace spanned by the columns of the controllability matrix \(M_c = [B, AB, \ldots, A^{n-1}B]\). Furthermore, if \(x_1\) is reachable for some \(T > 0\), then \(x_1\) is reachable for any \(T > 0\).

The proof is straightforward from the proof of Theorem 7 in [78], page 5.

**Remark A.2.** Consider the system \((A.1)\). Then any initial state \(x_0\) in the controllable subspace can be transferred to any terminal state \(x_1\) in the controllable subspace within any given finite time \(T > 0\).

**Lemma A.1.** The controllable subspace is invariant under \(A\), that is, if a vector \(x\) is in the controllable subspace, then \(Ax\) is also in this subspace.

**Proof.** The controllable subspace is spanned by the columns of

\[
M_c = [B, AB, \ldots, A^{n-1}B].
\]

Thus if \(x\) is in the controllable space, \(Ax\) is in the linear subspace spanned by the columns of \([AB, A^2B, \ldots, A^{n}B]\). \(A^nB\) can be expressed as a linear combination of \(B, AB, \ldots, A^{n-1}B\) (see the proof of Lemma 5 in [78], page 4). Therefore \(Ax\) is also in the controllable space.

Q. E. D.
Theorem A.5. Consider system (A.1). Form a nonsingular transformation matrix
\( T = (T_1, T_2) \) where the columns of \( T_1 \) form a basis for the \( m \)-dimensional \((m \leq n)\) controllable subspace of (A.1) and the column vectors of \( T_2 \) together with those of \( T_1 \) form a basis for \( \mathbb{R}^n \). Define the transformed state
\[
x'(t) = T^{-1} x(t).
\]
Then (A.1) is transformed into the controllability canonical form
\[
x'(t) = \begin{pmatrix} A'_{11} & A'_{12} \\ 0 & A'_{22} \end{pmatrix} x'(t) + \begin{pmatrix} B'_1 \\ 0 \end{pmatrix} u(t),
\]
(A.7)
where \( A'_{11} \) is an \( m \times m \) matrix, and the pair \( \{A'_{11}, B'_1\} \) is a controllable pair.

Proof. Introducing the transformation \( T \) \( x'(t) = x(t) \), (A.1) becomes
\[
\dot{x}'(t) = T^{-1} AT x'(t) + T^{-1} Bu(t).
\]
Partition \( T^{-1} \) consistent with \( T \) as
\[
T^{-1} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}.
\]
Then
\[
T^{-1} T = \begin{pmatrix} U_1 T_1 & U_1 T_2 \\ U_2 T_1 & U_2 T_2 \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ 0 & I_{n-m} \end{pmatrix},
\]
which gives
\[
U_2 T_1 = 0.
\]
The columns of \( T_1 \) span the controllable subspace, which implies that
\[
U_2 x = 0
\]
(A.8)
for any vector \( x \) in the controllable subspace. Now
\[
T^{-1} AT = \begin{pmatrix} U_1 AT_1 & U_1 AT_2 \\ U_2 AT_1 & U_2 AT_2 \end{pmatrix}, \quad T^{-1} B = \begin{pmatrix} U_1 B \\ U_2 B \end{pmatrix}.
\]
The column vectors of \( T_1 \) are in the controllable subspace, so by the last lemma the columns of \( AT_1 \) are also in the controllable subspace, and hence by (A.8),
\[
U_2 AT_1 = 0.
\]
The columns of \( B \) are all in the controllable subspace, since \( B \) is part of the controllability matrix, so by (A.8) again,
\[
U_2 B = 0.
\]
This proves that the controllability canonical form (A.7). The fact that \( \{A'_{11}, B'_1\} \) is a controllable pair is obvious from the construction.

Q. E. D.

From Section A.2, any initial state $x_0$ can be uniquely written as

$$x_0 = x_0^s + x_0^u,$$

where $x_0^s$ is in the stable subspace and $x_0^u$ is in the unstable subspace. In order to control the system properly, we must require that the unstable components $x_0^u$ can be completely controlled. If the unstable component $x_0^u$ is in the controllable subspace, then this could be done.

**Definition A.7.** The linear time-invariant system (A.1) is *stabilizable* if its unstable subspace is contained in its controllable subspace, that is, any vector $x$ in the unstable subspace is also in the controllable subspace.

**Definition A.8.** The pair $(A, B)$ is stabilizable if the system (A.1) is stabilizable.

**Theorem A.6.** Consider (A.1). By Theorem A.4, it can be transformed into the controllability canonical form

$$\dot{x}'(t) = \begin{pmatrix} A_{11}' & A_{12}' \\ 0 & A_{22}' \end{pmatrix} x'(t) + \begin{pmatrix} B_1' \\ 0 \end{pmatrix} u(t) = A' x'(t) + B' u(t), \tag{A.9}$$

where the pair $\{A_{11}', B_1'\}$ is completely controllable. Then the system is stabilizable if and only if the matrix $A_{22}'$ is asymptotically stable.

**Proof.** Suppose system (A.1) is stabilizable. Then the system (A.9) is also stabilizable. Suppose $A_{22}'$ is not stable. Then $A'$ is also not stable, and there exists a vector $\begin{pmatrix} v \\ w \end{pmatrix}$ in the unstable subspace of $A'$ such that $w \neq 0$ is in the unstable subspace of $A_{22}'$. Then $\begin{pmatrix} v \\ w \end{pmatrix}$ is not in the controllable subspace of (A.9), which contradicts the stabilizability of (A.9). Hence $A_{22}'$ must be stable.

Suppose $A_{22}'$ is stable. Then any vector in the unstable subspace of (A.9) must be in the controllable subspace of (A.9). Hence the system (A.9) is stabilizable. Consequently, the original system (A.1) is also stabilizable. \( \text{Q. E. D.} \)

**Theorem A.7.** Any asymptotically stable system is stabilizable. Any completely controllable system is stabilizable.
A.5. Observability and Reconstructibility.

Observability and reconstructibility are related to the matrix pair \((A, C)\), so the system (A.1)–(A.2) is considered. For time continuous systems, reconstructibility is equivalent to observability. In the discrete time case, it is weaker than observability.

**Definition A.9.** A system is observable if, given the output signals from two experiments being the same for \(t > t_0\) forces the states to have been the same at \(t = t_0\).

**Remark A.3.** The linear system (A.1)–(A.2) is observable if for any \(t_0\) and any initial state \(x(t_0) = x_0\), there exists a finite time \(t_1 > t_0\) such that knowledge of \(u(t)\) and \(y(t)\) for \(t_0 \leq t \leq t_1\) suffices to determine \(x_0\).

**Remark A.4.** A state \(x_0 \in \mathbb{R}^n\) is unobservable with respect to the matrix pair \((A, C)\) if and only if \(C e^{At} \cdot x_0 = 0\) for all \(t > 0\). The matrix pair \((A, C)\) is observable if and only if the only \(x \in \mathbb{R}^n\) which is unobservable with respect to \((A, C)\) is \(x = 0\).

A state \(x_0\) is unobservable when it is indistinguishable as an initial condition at time 0 from the initial condition 0 (zero state). As in the case of controllability, the observability matrix and the observability Gramian have crucial roles in determining observability ([78], page 8).

From (A.2) and (A.5) the output of the system (A.1)–(A.2) can be expressed as

\[
y(t) = C e^{A(t-t_0)} x_0 + C \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau) d\tau.
\]

Let \(y(t; x_0)\) denote the output of the system (A.1)–(A.2) with the initial state \(x(\theta) = x_0\).

Now,

\[
y(t; x_0) = y(t; x'_0) \quad \Leftrightarrow \quad C e^{A t} x_0 = C e^{A t} x'_0 \quad \Leftrightarrow \quad C e^{A t} (x_0 - x'_0) = 0. \quad (A.10)
\]

From (A.10) it is clear that \(x_0 - x'_0 \in \mathbb{R}^n\) is unobservable with respect to \((A, C)\) if and only if \(y(t; x_0) = y(t; x'_0)\) for \(t > 0\). In other words, \((A, C)\) is not observable when for every input two different initial states \(x_0\) and \(x'_0\) lead to the same output.

**Definition A.10.** A system is reconstructible if given a time \(t_1\), there is a finite time \(t_0 < t_1\) so that if the observations from two experiments were the same over the interval
from $t_0$ to $t_1$, then the states at time $t_1$ (which actually are determined from the respective states at time $t_0$) were the same.

Remark A.5. A system is reconstructible if and only if every unobservable initial state $x_0$ dies out to 0 after a finite time when no input is applied. Thus the unobservable states in a reconstructible system's state space are irrelevant to the behavior of the system over the long run.

Theorem A.8. $(A, C)$ is an observable pair if and only if $(A, C)$ is a reconstructible pair.

Proof. Suppose $(A, C)$ is an observable pair. Then the only unobservable state is $x_0 = 0$ and trivially, $e^{TA}x_0 = 0$ for some $T > 0$, which implies that $(A, C)$ is a reconstructible pair.

Suppose $(A, C)$ is a reconstructible pair. Then any unobservable $x_0$ must satisfy $e^{TA}x_0 = 0$ for some $T > 0$. Since $e^{TA}$ is invertible, the only unobservable $x_0$ is 0. Hence $(A, C)$ is an observable pair.

Q. E. D.

Theorem A.9. The system (A.1)–(A.2) is reconstructible ((A, C) is a reconstructible pair) if and only if the row vectors of the reconstructibility matrix

$$M_r = M_o = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

span $\mathbb{R}^n$ (rank $M_r = n$).

The proof is similar to the proof of Theorem 15 in [78].

Definition A.11. The unreconstructible subspace of (A.1)–(A.2) is the linear subspace consisting of the states $x_0$ for which

$$Ce^{(t-t_0)A}x_0 = 0, \text{ for all } t \geq t_0.$$ 

Theorem A.10. The unreconstructible subspace of the $n$-dimensional linear time-invariant system (A.1)–(A.2) is the null space of the reconstructibility matrix $M_r$.

This follows from the proof of Theorem A.9.
Theorem A.11. Consider the system (A.1)–(A.2). Form a nonsingular transformation matrix

\[ U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \]

where the \( m \) rows of \( U_1 \) form a basis for the \( m \)-dimensional \(( m \leq n )\) subspace spanned by the rows of the reconstructibility matrix of the system. The \( n - m \) rows of \( U_2 \) form together with the \( m \) rows of \( U_1 \) a basis for \( \mathbb{R}^n \). Define a transformed state variable \( x'(t) \) by

\[ x'(t) = Ux(t). \]

Then, in terms of the transformed state variable, the system is represented in the reconstructibility canonical form

\[ \begin{align*}
\dot{x}'(t) &= \begin{pmatrix} A'_{11} & 0 \\ A'_{21} & A'_{22} \end{pmatrix} x'(t) + \begin{pmatrix} B'_1 \\ B'_2 \end{pmatrix} u(t), \\
y(t) &= (C'_1, 0)x'(t),
\end{align*} \]

where \( A'_{11} \) is an \( m \times m \) matrix, and the pair \( \{ A'_{11}, C'_1 \} \) is completely reconstructible.

The proof is in [42].


If the output variable of a not completely reconstructible system is observed there is an uncertainty on the actual state of the system, since we can always add an arbitrary vector in the unreconstructible subspace to any possible state. In the case that any state in the unreconstructible subspace is also in the stable subspace of the system, the response of the system to any state in the unreconstructible subspace will converge to zero. A system with this property is called detectable.

Definition A.12. A system is detectable if \( y(t, x_0) = 0 \) for all \( t > t_0 \) implies that \( x(t) \) tends to 0 as \( t \) tends to infinity.

Remark A.6. The linear time-invariant system (A.1)–(A.2) is detectable if its unreconstructible subspace is contained in its stable subspace.

Definition A.13. The pair \(( A, C)\) is detectable if the system (A.1)–(A.2) is detectable.
Theorem A.12. Any asymptotically stable system of the form (A.1)—(A.2) is detectable.

Any completely reconstructible system of the form (A.1)—(A.2) is detectable.

Theorem A.13. If $C_1^t C_1 = C_2^t C_2$ and the matrix pair $(A, C_1)$ is observable (respectively detectable), then the matrix pair $(A, C_2)$ is observable (respectively detectable).

Proof. Let

$$V(C) \equiv \bigcap_{i=0}^{n-1} \ker (C A^i), \quad W(C) \equiv \sum_{i=0}^{n-1} (A^i)^t C^t C A^i.$$ 

If some vector is in $V(C)$ then it is in the kernel of every $CA^i$, hence it is in the kernel of every $(A^i)^t C^t C A^i$, and then it is in the kernel of $W(C)$. Conversely, if some vector $x$ is in the kernel of $W(C)$, then

$$x^t \left( \sum_{i=0}^{n-1} (A^i)^t C^t C A^i \right) x = \sum_{i=0}^{n-1} \|C A^i x\|^2 = 0,$$

Hence, $CA^i x = 0$ for $i = 0, 1, \ldots, n - 1$, and $x \in V(C)$. Therefore,

$$V(C) = \ker (W(C)). \quad (A.11)$$

Since $C_1^t C_1 = C_2^t C_2$, $W(C_1) = W(C_2)$ and $\ker (W(C_1)) = \ker (W(C_2))$. Using (A.11) it follows

$$V(C_1) = V(C_2). \quad (A.12)$$

By assumption $(A, C_1)$ is an observable pair. Hence the appropriate observability matrix has full rank, which implies that the only vector in the kernels of all $C_1 A^i$ is the zero vector. By (A.12) the only vector in the kernels of all $C_2 A^i$ is the zero vector. Hence the observability matrix for $(A, C_2)$ has full rank, which means that $(A, C_2)$ is an observable pair. If $(A, C_1)$ is a detectable pair, then $V(C_1) \subset \ker \alpha^-(\lambda)$ (the stable subspace of the system) and therefore $V(C_2) \subset \ker \alpha^-(\lambda)$, which means $(A, C_2)$ is also detectable.

Q. E. D.
Appendix B. matlab functions.

B.1. Introduction.

The matlab functions in this appendix except morh2op [78] result from the research in this dissertation. The thirteen functions listed in the following are classified by their purpose. morh2inf, morh2ly, morh2over, and morh2op are for the $H^2$ optimal model order reduction problem. morh2hinf, morh2hiy, and morh2hiower are for the combined $H^2/H^\infty$ model order reduction problem. To solve the full-order LQG controller synthesis problem with an $H^\infty$ bound, use flqgly, flqginf, or flqgover. For the reduced-order LQG problem use rlqg, rlqginf, or rlqgover.

B.2. matlab functions for the $H^2$ optimal model reduction problem.

morh2inf, morh2ly, morh2over, morh2op

Purpose
Find the $H^2$ optimal reduced-order model of a linear system.

Synopsis

\[ [A_m, B_m, C_m, \text{cost}] = \text{morh2inf}(A, B, C, \text{nm}) \]
\[ [A_m, B_m, C_m, \text{cost}] = \text{morh2ly}(A, B, C, \text{nm}) \]
\[ [A_m, B_m, C_m, \text{cost}] = \text{morh2over}(A, B, C, \text{nm}) \]
\[ [A_m, B_m, C_m, \text{cost}] = \text{morh2op}(A, B, C, \text{nm}, \text{meth}, \text{init}, c1, c2) \]

Description
For a given linear system \((A, B, C)\), morh2inf, morh2ly, morh2over, and morh2op return the $H^2$ optimal reduced-order model \((A_m, B_m, C_m)\) of dimension \(nm\) with $H^2$ cost \(\text{cost}\). The result from morh2inf is in the input normal form while that from morh2ly is in Ly's form. In morh2op, meth and init denote the strategy and the method of initialization respectively [78], and \(c1\) and \(c2\) define the initial \(A\) by \(A_0 = -c1I + c2A\).
Examples

\[
A = \begin{bmatrix}
-10 & 1 & 0 \\
-5 & 0 & 1 \\
-1 & 0 & 0 \\
\end{bmatrix};
\]
\[
B = \begin{bmatrix}
0 \\
1 \\
1 \\
\end{bmatrix};
\]
\[
C = \begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix};
\]
\[
m = 2;
\]
\[
[A_m, B_m, C_m, cost] = morh2inf(A, B, C, m)
\]
\[
A_m =
\begin{bmatrix}
-0.1397 & -0.1006 \\
0.6010 & -0.4482 \\
\end{bmatrix}
\]
\[
B_m =
\begin{bmatrix}
-0.5285 \\
0.9486 \\
\end{bmatrix}
\]
\[
C_m =
\begin{bmatrix}
-0.3204 \\
-0.0961 \\
\end{bmatrix}
\]
\[
cost = 3.2902e-04
\]
\[
[A_m, B_m, C_m, cost] = morh2ly(A, B, C, m)
\]
\[
A_m =
\begin{bmatrix}
0 & 1.0000 \\
-0.1231 & -0.5878 \\
\end{bmatrix}
\]
\[
B_m =
\begin{bmatrix}
0.0784 \\
0.0782 \\
\end{bmatrix}
\]
\[
C_m =
\begin{bmatrix}
1.0000 & 0.6000 \\
\end{bmatrix}
\]
\[
cost = 3.2902e-04
\]
\[
[A_m, B_m, C_m, cost] = morh2op(A, B, C, m, 2, 2, 10.0, 0.0)
\]
\[
A_m =
\begin{bmatrix}
-10.4365 \\
\end{bmatrix}
\]
\[
B_m =
\begin{bmatrix}
-1.5972 \\
\end{bmatrix}
\]
\[
C_m =
\begin{bmatrix}
1.5972 \\
\end{bmatrix}
\]
\[
cost = 1.6882
\]

Algorithm

The algorithms for \texttt{morh2inf}, \texttt{morh2ly}, and \texttt{morh2over} are described in Chapters 2, 4, and 6 respectively. The algorithm for \texttt{morh2op} can be found in [78].

See Also

\texttt{morh2hiinf}, \texttt{morh2bily}, and \texttt{morh2hiover}.
B.3. matlab functions for the combined $H^2/H^\infty$ model order reduction problem.

**morb2hiinf, morh2hily, and morh2hiover**

**Purpose**
Find the combined $H^2/H^\infty$ reduced-order model of a linear system.

**Synopsis**

\[ [\text{Am}, \text{Bm}, \text{Cm}, \text{cost}] = \text{morb2hiinf}(A, B, C, \text{nm}, \text{gamma}) \]

\[ [\text{Am}, \text{Bm}, \text{Cm}, \text{cost}] = \text{morb2hily}(A, B, C, \text{nm}, \text{gamma}) \]

\[ [\text{Am}, \text{Bm}, \text{Cm}, \text{cost}] = \text{morb2hiover}(A, B, C, \text{nm}, \text{gamma}) \]

**Description**
For a given linear system \((A, B, C)\), \(\text{morb2hiinf, morh2hily, and morh2hiover}\) return the combined $H^2/H^\infty$ reduced-order model \((\text{Am}, \text{Bm}, \text{Cm})\) of dimension \(\text{nm}\) with $H^2$ cost \(\text{cost}\).

The triple \((\text{Am}, \text{Bm}, \text{Cm})\) returned from \(\text{morb2hiinf}\) is in the input normal form while that from \(\text{morb2hily}\) is in Ly's form.

**Examples**

```matlab
>> A = zeros(10);
>> B = zeros(10, 1);
>> C = zeros(1, 10);
>> A(1,1:10) = [-10 -45 -120 -210 -232 -210 -120 -45 -10 -1];
>> for i=1:9 A(i+1,1) = 1.0; end
>> F(1,1) = 1.0;
>> C(1,10) = 1.0;
>> nm = 4;
>> gamma = 1.0;
>> [Am, Bm, Cm, cost] = morh2hiinf(A, B, C, nm, gamma)

Am  
-0.0273  -0.1286  -0.0274  0.0124  
0.2376  -0.1087  -0.1936  0.0397  
-0.1352  0.5178  -0.2416  0.2322  
-0.2412  0.4166  -0.9124  -0.4787  

Bm  
9.2336  
-0.4663  
0.6951  
0.9785  

Cm  
0.1897  0.2047  0.1142  -0.0409  
```

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cost =
1.2868e-04
>> [Am, Bm, Cm, cost] = morh2hiover(A, B, C, nm, gamma)
Am =
  -0.0508  -0.1739  -0.0642  0.0566
  0.1739  -0.1200  -0.3132  0.1338
  -0.0642  0.3132  -0.2566  0.4553
  -0.0566  0.1338  -0.4553  -0.4489
Bm =
  0.2143
  -0.3180
  0.2916
  0.2044
Cm =
  0.2148  0.3180  0.2916  -0.2044
cost =
1.2868e-04

Algorithm

The algorithms for morh2hiinf, morh2hily, and morh2hiover are described in Chapters 8, 9, and 10 respectively.

See Also

morh2op, morh2exp, morh2ly, and morh2over.

B.4. matlab functions for the full-order LQG controller synthesis with an $H^\infty$ bound.

flqgly, flqginf, and flqgover

Purpose

Find the full-order LQG compensator with an $H^\infty$ bound.

Synopsis

[Ac, Bc, Cc, cost] = flqgly(A, B, C, D, gamma0, gamma, E1, E2, E1i, E2i, D1, D2)
[Ac, Bc, Cc, cost] = flqginf(A, B, C, D, gamma0, gamma, E1, E2, E1i, E2i, D1, D2)
[Ac, Bc, Cc, cost] = flqgover(A, B, C, D, gamma0, gamma, E1, E2. E1i, E2i, D1, D2)
Description

For a given $n$-th order linear plant $(A, B, C, D)$, with parameter matrices $E_1 (E_1)$, $E_2 (E_2)$, $E_1i (E_{1i})$, $E_2i (E_{2i})$, $D_1 (D_1)$, $D_2 (D_2)$, $flagly$, $flaginf$, and $flagover$ return the full-order LQG compensator $(Ac, Bc, Cc)$ with $H^\infty$ norm bounded by $\text{gamma}$ and $H^2$ cost given by $\text{cost}$. $\text{gamma0}$ is the initial value of $\gamma$ which should always be greater than $\text{gamma}$. The resulting compensator $(Ac, Bc, Cc)$ from $\text{flaginf}$ is in the input normal Riccati form while that from $\text{flagly}$ is in Ly's form.

Examples

```matlab
>> A=zeros(8);  
>> B=zeros(8,1);  
>> C=zeros(1,8);  
>> D=0;  
>> A(1:8,1) = [-0.161; -4.004; -0.5822; -0.9835; -0.4073; -3.982; 0; 0];  
>> for i =1:7 A(i,i+1) = 1; end  
>> B(1:8,1) = [0; 0; 0.006; 0.0023; 0.0713; 1.000; 0.1045; 0.9955];  
>> C(1,1) = 1.0;  
>> E1 = zeros(2,8);  
>> E1(1,1) = 0.001 = [0 0 0 0.55 11 1.32 18];  
>> E1i = E1;  
>> E2 = [0;1];  
>> E2i = [0;0];  
>> D1 = zeros(8,2);  
>> D1(:,1:10) = B;  
>> D2 = [0 1];  
>> [Ac,Bc,Cc,cost] = flagly(A,B,C,D,1.0e3,2.0,E1,E2,E1i,E2i,D1,D2)

Ac =
Columns 1 through 7
0.0000 1.0000 -0.0000 0.0000 -0.0000 -0.0000 0.0000
-3.4454 -0.1189 -0.0000 0.0000 -0.0000 0.0000 0.0000
-0.0000 -0.0000 0.0000 1.0000 0.0000 -0.0000 -0.0000
0.0000 0.0000 -1.9990 -0.1688 0.0000 0.0000 -0.0000
0.0000 0.0000 -0.0000 -0.0000 0.0000 -0.0000 1.0000
-0.0000 -0.0000 -0.0000 -0.0000 -0.6728 -0.1541 -0.0000
0.0000 0.0000 0.0000 0.0000 -0.0000 0.0000 0.0000
-0.0000 0.0000 0.0000 0.0000 -0.0000 0.0000 -0.3032

Column 8
0.0000
0.0000
-0.0000
0.0000
-0.0000
0.0000
-0.0000
1.0000
-0.8723

Bc = 
0.0012

110
Algorithm

The algorithms for flagly, flaginf, and flagover are described in Chapters 14, 15, 16, 17, and 18.

See Also

rlqgly, rlqginf, rlqgover.

B.5. matlab functions for the reduced-order LQG controller synthesis with an $H^\infty$ bound.

rlqgly, rlqginf, and rlqgover

Purpose

Find the reduced-order LQG compensator with an $H^\infty$ bound.

Synopsis

$[Ac, Bc, Cc, cost] = \text{rlqgly}(A,B,C,D, nc, gamma0, gamma, beta, E1, E2, E1i, E2i, D1, D2)$

$[Ac, Bc, Cc, cost] = \text{rlqginf}(A,B,C,D, nc, gamma0, gamma, beta, E1, E2, E1i, E2i, D1, D2)$

$[Ac, Bc, Cc, cost] = \text{rlqgover}(A,B,C,D, nc, gamma0, gamma, beta, E1, E2, E1i, E2i, D1, D2)$
Description

For a given $n$-th order linear plant $(A, B, C, D)$, with parameter matrices $E_1 (E_1)$, $E_2 (E_2)$, $E_{1i} (E_{1i})$, $E_{2i} (E_{2i})$, $D_1 (D_1)$, $D_2 (D_2)$, $rlgly$, $rlqinf$, and $rlqover$ return the $nc$-th order LQG compensator $(Ac, Bc, Cc)$ with the $H^\infty$ norm bounded by gamma ($\gamma$) and $H^2$ cost given by cost. gamma0 ($\gamma_0$) is the initial $\gamma$ and should always be greater than gamma. beta ($\beta \gg 0$) is a positive number. The resulting compensator $(Ac, Bc, Cc)$ from $rlqinf$ is in the input normal Riccati form while that from $rlgly$ is in Ly's form.

Examples

```
>> A=zeros(8);
>> B=zeros(8,1);
>> C=zeros(1,8);
>> D=0;
>> A(1:8,1) = [-0.16i; -6.004; -0.5822; -9.9835; -0.4073; 
-3.982; 0; 0;];
>> for i = 1:7 A(i,i+1) = 1; end
>> B(:,1) = [0; 0; 0.0064; 0.00235; 0.0713; 1.0002; 0.1045; 0.9955];
>> C(1,1) = 1.0;
>> E1 = zeros(2,8);
>> E1(1,1:8) = 0.001 * [0 0 0 0 0.55 11 1.32 10];
>> E1i = E1;
>> E2 = [0;1];
>> E2i = [0;0];
>> D1 = zeros(8,2);
>> D1(:,1:8) = B;
>> D2 = [0 i];
>> [Ac,Bc,Cc,cost] = rlgly(A,B,C,D,2,1.0*3,3.8,100,E1,E2,E1i,E2i,D1,D2)

Ac =
   0   1.0000
  -0.0965  -0.2452
Bc =
  -0.1968
  -0.1410
Cc =
   1.0000  -0.0000
cost =
   2.8821
```

Algorithm

The algorithms for $rlgly$, $rlqinf$, and $rlqover$ are described in Chapters 14, 15, 16, 17, and 18.

See Also

flagly, flaginf, flagover.
VITA.

Yuzhen Ge was born on November 27, 1961 in Hunan, China. She was admitted to Zhongshan University in March, 1978 by passing the first Chinese nationwide college entrance exam after the cultural revolution. She received B.S. in Physics in February of 1982, and then was admitted to the graduate program of the Institute of Physics, Academia Sinica in Beijing, China. She came to the Department of Mathematics of Virginia Polytechnic Institute and State University in September, 1986 and graduated with Ph.D. in Mathematics in December, 1990. In May of 1993, she received M.S. degree in Computer Science and Applications and in May of 1994 she received Ph.D. in Computer Science and Applications from Virginia Polytechnic Institute and State University.