TWO AND THREE-DIMENSIONAL INCOMPRESSIBLE AND
COMPRESSIBLE VISCOUS FLUCTUATIONS

by

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Dissertation submitted to the Graduate Faculty of the

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Engineering Mechanics

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December, 1977

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ACKNOWLEDGEMENTS

The author wishes to express his deepest sense of gratitude and sincere appreciation to his committee chairman, Professor Demetri P. Telionis, for his help, guidance and time spent in consultation during this endeavor. Gratitude is also extended to Professors W. F. O'Brien, H. W. Tieleman, C. W. Smith and R. P McNitt for their helpful suggestions and constructive comments in this effort. This work was partially supported by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Grant No. AFOSR-74-2651.

A special thanks is extended to Professor Daniel Frederick, Alumni Distinguished Professor and Head, Engineering Science and Mechanics Department, Virginia Polytechnic Institute and State University, for providing teaching and research facilities during this endeavor. The author extends a word of thanks to Frances Bush Carter and to Janet Bryant for their excellence in typing this dissertation.

The author owes a debt of gratitude to his mother for her unceasing interest, encouragement and good wishes. His brother, sister and all other family members contributed in their own ways, to the progress of this work.

Special appreciation and love are given to his wife, Sanjokta, and children, Amit, Ritu and Amol, for their patience, encouragement and moral support during various critical moments which made this work both possible and pleasant.

Finally, the author dedicates this work to his (late) father Sri Chuni L. Gupta who passed away just four days after the author's landing in the United States to further his higher education. The ever-loving memory and blessings of his father were a perpetual source of moral encouragement and strength which have enhanced this accomplishment.

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INTRODUCTION

Unsteady viscous effects have proved to play a vital role in the stability of missiles and reentry vehicles. Small fluctuations of the angle of attack, gas injection through the skin and ablation are boundary layer phenomena that may have catastrophic effects on the stability of the body. Similar boundary layer phenomena have been studied quite extensively in conjunction with other applications of unsteady aerodynamics as, for example, helicopter rotor blades, turbomachinery cascades, fluttering wing sections, etc. In addition, unsteady phenomena, like a sharp change in the free stream velocity or in the rate of heat transfer, or injection velocities that may be suddenly turned on are not uncommon.

Most of the engineering methods of calculating the flow fields about aerofoils and turbomachinery blades are based on inviscid theories, while important features, like separation, are either arbitrarily assumed or completely ignored. Yet it is now well-known that many important aerodynamic characteristics, as for example, the blade behavior for large angles of attack, the phenomena of stall, and in general their dynamic response are controlled by thin viscous layers, the boundary layers, that cover the skin of aerodynamic surfaces and most of all by separation. Moreover, the extreme environmental conditions imposed on the aerodynamic materials today necessitate a good estimate of skin friction and heat transfer rates before a certain design is finalized.

Boundary layer calculations have been confined for decades due to the complexity of mathematical models, to very simple configurations of rather academic importance. The evolution of the modern computer has recently permitted calculations for realistic configurations that have proved to be valuable engineering estimates. Boundary layer calculations have also been confined mostly to two-dimensional and steady flows. Only in the last two decades have problems like three-dimensional, compressible or unsteady boundary layers been considered. (See review articles in Refs. [1-4].) Generally these works can be grouped into two categories with regard to their applicability: (1) those which are confined to small amplitude fluctuations but which can handle in general any value of the frequency of fluctuation, and (2) those which are valid for any amplitude of fluctuation but in turn are confined to a specific type of dependence on time. Characteristic representatives of the first group are the works of Lighthill [5], Illingworth [6], Rott and Rosenweig [7], Sarma [8,9], Gupta [10,11] et al; and of the second group the works of Moore [12,13], Ostrach [14], Sparrow and Gregg [15], Lin [16], King [17], et al.

The present dissertation is an attempt to study some special classes of unsteady two- and three-dimensional incompressible and compressible boundary layer flows past different bodies that are encountered in engineering applications and can be grouped under the first category. It contains five chapters dealing with cross flow effects in oscillating boundary layers, compressible oscillations over two-dimensional and axisymmetric walls with no mean heat transfer, compressible oscillations over two-dimensional and axisymmetric walls with mean heat transfer and higher approximation in unsteady incompressible boundary layers. It deals with problems of general character from which many particular problems come as special cases. The analysis is first formulated for a general value of the frequency but the results are presented only for small frequencies.

Numerous investigators have followed the pioneering work of Lighhill, Moore and Lin, but mostly in the spirit of Lighthill's work. In most of these investigations the outer flow velocity is assumed to oscillate harmonically, $U_e(x,t)=U_o(x)+\varepsilon U_1(x,t)e^{i\omega t}$. Solutions are then sought in the form $q(x,t)=q_o(x,y)+\varepsilon q_1(x,y)e^{i\omega t}$, where q is any flow quantity. The asymptotic expansion in powers of the small amplitude parameter ε results in two different sets of equations. The first describes the variation of mean quantities like q_o and is identical to the equations of steady boundary layer flow. The second describes the behavior of the oscillation amplitudes q_1. In this way the problem is split into steady and unsteady parts without excluding fast or slow variations with time. Even in this form the problem does not permit a straight-forward solution and investigators have been forced to expand again in fractional powers of the frequency ω or its inverse. Exceptions are only methods that employ numerical integration techniques [18-20]. In the present study we adopt the double expansion method in powers of ε and ω.

Ackerberg and Phillips [21] have recently reviewed the work of Rott and his co-workers [7,22] and addressed rigorously the mathematical implications of matching the solutions for large or small values of frequency. They have also discovered that for large values of the parameter $\frac{\omega x}{U_\infty}$ (where x is the distance along the solid boundary and U_∞ is a reference velocity), the boundary layer can be separated in the direction perpendicular to the wall into an inner and an outer layer. The present study is concerned only with small frequency fluctuations. Oscillating flows with small reduced frequency $\frac{\omega L}{U_\infty}$ (where L and U_∞ are a typical length and velocity respectively), are, for example, the flow over a helicopter blade in forward flight, or the flow through a compressor fan in the presence of inlet distortions.

The investigators that have followed the second category have considered a variety of physical problems. Moore [12] studied the compressible boundary layer on a heat-insulated flat plate for nearly quasi-steady boundary layer flow. Ostrach [14], and later Moore and Ostrach [23], extended the theory to include the effects of heat transfer but confined their attention to flat plate flows. Lin [16] separated each property into a time averaged and a fluctuating part and assumed a very large frequency, in order to solve first the equation for the fluctuating components. King [17] expanded only in powers of the frequency parameter but later assumed that the fluctuating part of the boundary layer properties is of the same order with the frequency parameter. Large amplitude effects can be estimated in general only by purely numerical methods (Tsahalis and Telionis [24]). It has been decided in the present study to avoid restrictions on the dependence of outer flow on time so that transient, impulsive and oscillatory flows with large or small frequencies can be considered.

Gribben [25,26] investigated the flow in the neighborhood of a stagnation point, which accounts for pressure gradient effects, albeit in a narrow sense. Gribben was mainly interested in the effects of a very hot surface and therefore simplified considerably his problem by assuming that the outer flow is incompressible and therefore further assuming constant outer flow density and temperature. Vimala and Nath [27] have more recently presented a quite general numerical method for solving the problem of compressible stagnation flow.

The leading term in an asymptotic expansion of the Navier-Stokes equations for a large Reynolds number represents the classical boundary-layer equations of Prandtl. It is well-known that the Navier-Stokes equations are elliptic, whereas the classical boundary-layer equations are parabolic and thereby do not permit upstream influence. It is of interest to study how the higher-order terms in the asymptotic expansion reassert the elliptical nature suppressed in the leading term. Physically, these can arise because of the longitudinal curvature, transverse curvature, displacement, external vorticity, etc. The practical use of such higher-order corrections lies in the fact that they are found to give good results even at distinctly non-asymptotic situations. So far, all the efforts in this area have been directed to steady flows (see Van Dyke [28,29]), thus unsteady flows have, in general, been unduly ignored. The aim of the present investigation is also to extend the theory of Van Dyke [28] to unsteady flows.

In the general methods of approach of this dissertation, we obtain systems of general differential equations in a single variable. For the complete study of a particular problem, we solve these equations numerically by the shooting technique for various values of the parameters. A straight-forward fourth-order Runge-Kutta integration scheme is employed and the values of the functions at the edge of the boundary-layer are checked against the outer flow boundary conditions. If these conditions are not met, a guided guess is used to readjust the assumed values at the wall, and the process is repeated until convergence is achieved. The approach adopted in the present thesis is based on asymptotic expansions in powers of small parameters: the amplitude ratio and the reduced frequency or its inverse. Both assumptions involved appear to be quite realistic.

Each chapter is preceeded by a brief literature review pertinent to the special topic under examination. The first chapter is on cross-flow effects in oscillating boundary-layers. In this chapter we consider simultaneously the effects of three-dimensionality coupled with the response to outer flow oscillations. It is believed that the coupling will have significant implications in cascade flows where the finite span blocks the development of cross flows. The ordinary differential equations, though, are derived for the most general aerofoil configuration and are readily available for computations. The numerical results presented are derived only for the special case of the flow over a swept-back wedge. Their value is qualitative. However, some interesting features of oscillatory three-dimensional flows are disclosed. In particular it is found that the coupling of the momentum equations permit the transfer of momentum from the chordwise to the spanwise direction. In this way it is possible to excite a fluctuating boundary layer flow in the spanwise direction even though there is no outer flow fluctuation in this direction. Moreover, it is discovered that the skin friction vectors oscillate in direction, and hence the orientation of the skin friction lines change periodically, even though the outer edge streamline configuration is not affected by the oscillation in amplitude of the outer flow.

The second chapter deals with compressible oscillations over two-dimensional and axisymmetric walls with no mean heat transfer. In this chapter we study the response of laminar compressible boundary-layers to fluctuations of the skin of the body or the outer flow, via asymptotic expansions in powers of the amplitude parameter and the frequency of oscillation. We study the fluctuating compressible flow over a wedge or a cone but for the very special case of a wall temperature equal to the adiabatic wall temperature. A well known transformation then can be generalized in order to eliminate the energy equation.

To the author's knowledge this is the first attempt to study compressible boundary-layer flows with a non-zero pressure gradient and purely unsteady outer flow conditions. The most striking feature of such flows is the fact that the outer flow enthalpy ceases to be constant and varies proportionally to the time derivative of pressure. However, in this work, the enthalpy variations are of one order of magnitude higher than the level of the terms retained. The simplifying assumptions here are restrictive since the solution is only valid for a conducting wall with a temperature equal to the adiabatic temperature.

The third chapter deals with compressible oscillations over two-dimensional and axisymmetric walls with mean heat transfer. The response of the compressible boundary-layer to small fluctuations of the outer flow is investigated. Unlike chapter two, the Prandtl number is to be taken here as an arbitrary parameter which need not be equal to unity. The governing equations and the appropriate boundary conditions are formulated for the first time in considerable generality. It is indicated that the outer flow properties do not oscillate in phase with each other. Such phase differences are augmented as one proceeds across the boundary-layer. Solutions are presented for small amplitudes again in the form of asymptotic expansions in powers of a frequency parameter. Ordinary but coupled non-linear differential equations for the stream function and the temperature field are derived for self-similar flows. Results are presented for a steady part of the solution that corresponds to flat plate and stagnation flows and oscillations of the outer flow in magnitude or direction. In this chapter there is no restriction with respect to the temperature of the wall.

In the fourth chapter, we study the higher approximation to fluctuations of the outer flow in unsteady incompressible boundary-layers. The main purpose of this investigation is to extend the work of Van Dyke [28] to unsteady flows. Second order equations for unsteady flows are derived from the Navier-Stokes equations by employing the method of matched asymptotic expansions. It is shown that the unsteady flow field can be described by two limiting processes in a fashion similar to the one employed for the steady flow. The asymptotic expansion for the inner and outer regions are matched in the overlap domain. The nature of second-order effects is studied for a flat plate and stagnation flow. The last chapter contains a summary and conclusions.

CHAPTER ONE

CROSS FLOW EFFECTS IN OSCILLATING BOUNDARY LAYERS

1.1 Introduction

Three-dimensional effects in boundary-layers have been studied extensively in the literature, and reviews on related topics can be found in most classical texts. In fact, in the last few years numerical solutions have appeared that treat the problem of the most general three-dimensional configuration without any assumption about symmetry of any kind.

In this initial step of combining the effects of unsteadiness and three-dimensionality, we decided to follow the work of Sears [30] and Görtler [31] and consider configurations and outer flows that permit the uncoupling of the two components of the momentum equations. This is not really a very restrictive assumption and it corresponds to a variety of engineering applications. In fact it is possible to introduce a weak variation in the spanwise direction and generalize the present method. The spanwise component of the outer flow is now assumed in the form $W(x,t)=W_0+\varepsilon W_1(x,t)$. The case $W_1=0$ is then the case of a swept-back wing with oscillations only in the chordwise direction. Solutions are also presented for a power variation of W_1.

The combined effects of three-dimensionality and unsteadiness have been considered most recently by McCroskey and Yaggy [32], Dwyer and McCroskey [33], and Young and Williams [34]. These authors were concerned with the helicopter blade problem and assume a large ratio of span to chord. They have also proposed corrections for small distances from the axis of the blade rotation. The perturbation procedure in these works is entirely different from the one employed in the present chapter. The above authors assumed solutions in inverse powers of the distance to the center of rotation, whereas the present method proposes solutions in powers of the amplitude of oscillation. It is interesting to note that their zeroth-order problem is a quasi-steady problem, essentially the same as our zeroth-order problem. Gupta [10,11] has also looked into three-dimensional and unsteady boundary-layer flow problems. He considered first [10] the impulsive flow over a corrugated body for a special case of a three-dimensional configuration. In a later publication [11] he also studied the impulsive start of a yawed infinite wedge and a circular cylinder.

1.2 The Governing Equations and the Steady Part of the Motion

The laminar boundary-layer equations of motion for incompressible three-dimensional flow are:

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + w\,\frac{\partial u}{\partial z} = -\,\frac{1}{\rho}\,\frac{\partial p}{\partial x} + \nu\,\frac{\partial^2 u}{\partial y^2} \qquad (1.2.1a)$$

$$\frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial x} + v\,\frac{\partial w}{\partial y} + w\,\frac{\partial w}{\partial z} = -\,\frac{1}{\rho}\,\frac{\partial p}{\partial z} + \nu\,\frac{\partial^2 u}{\partial y^2} \qquad (1.2.1b)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (1.2.1c)$$

with boundary conditions:

$$u = v = w = 0 \qquad \text{at } y = 0,\ (x > 0), \tag{1.2.2a}$$

$$u \to U,\ w \to W \qquad \text{as } y \to \infty; \tag{1.2.2b}$$

where x and z denote the coordinates in the wall surface, y denotes the coordinate perpendicular to the wall, u, v, w are the velocity components in the x, y, z directions respectively and ν is the kinematic viscosity. Taking $U = U(x,t)$ and $W = W(x,t)$, that is an outer flow independent of z, we have (see Fig. 1.1)

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x}, \qquad -\frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{\partial W}{\partial t} + U\frac{\partial W}{\partial x} \tag{1.2.3}$$

We further assume that the outer flow consists of a small unsteady part and can be expressed in the form

$$\left.\begin{aligned} U(x,t) &= U_0(x) + \varepsilon U_1(x,t), \\ W(x,t) &= W_0(x) + \varepsilon W_1(x,t) \end{aligned}\right\} \tag{1.2.4}$$

where ε is a small dimensionaless parameter.

Solutions of Eqs. (1.2.1) with the outer flow given by Eq. (1.2.4) will be sought in the form

$$u(x,y,t) = u_0(x,y) + \varepsilon u_1(x,y,t) + \ldots \tag{1.2.5a}$$

$$v(x,y,t) = v_0(x,y) + \varepsilon v_1(x,y,t) + \ldots \tag{1.2.5b}$$

$$w(x,y,t) = w_0(x,y) + \varepsilon w_1(x,y,t) + \ldots \tag{1.2.5c}$$

Substitution of the above expressions in Eqs. (1.2.1) and (1.2.2) and collection of terms of order ε^0 yields

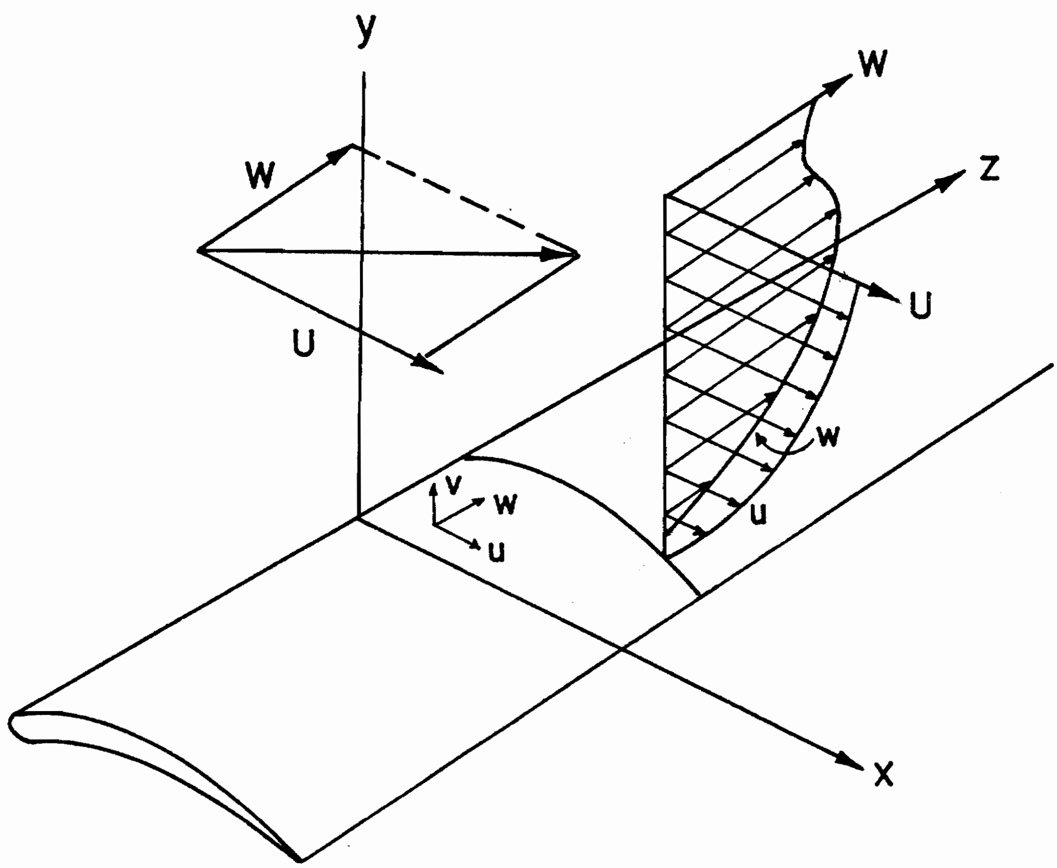

Figure 1.1. Schematic of the flow field.

$$u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y} = U_0 \frac{dU_0}{dx} + \nu \frac{\partial^2 u_0}{\partial y^2} \tag{1.2.6a}$$

$$u_0 \frac{\partial w_0}{\partial x} + v_0 \frac{\partial w_0}{\partial y} = U_0 \frac{dW_0}{dx} + \nu \frac{\partial^2 w_0}{\partial y^2} \tag{1.2.6b}$$

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0 \tag{1.2.6c}$$

$$u_0 = v_0 = w_0 = 0 \text{ at } y = 0 \tag{1.2.7a}$$

$$u_0 \to U_0(x),\ w_0 \to W_0 \text{ as } y \to \infty \tag{1.2.7c}$$

Similarly collection of terms of order ε^1 yields

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_0}{\partial x} + u_0 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_0}{\partial y} + v_0 \frac{\partial u_1}{\partial y}$$

$$= \frac{\partial U_1}{\partial t} + \frac{\partial (U_0 U_1)}{\partial x} + \nu \frac{\partial^2 u_1}{\partial y^2} \tag{1.2.8a}$$

$$\frac{\partial w_1}{\partial t} + u_1 \frac{\partial w_0}{\partial x} + u_0 \frac{\partial w_1}{\partial x} + v_1 \frac{\partial w_0}{\partial y} + v_0 \frac{\partial w_1}{\partial y}$$

$$= \frac{\partial W_1}{\partial t} + U_0 \frac{\partial W_1}{\partial x} + U_1 \frac{\partial W_0}{\partial x} + \nu \frac{\partial^2 w_1}{\partial y^2} \tag{1.2.8b}$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 \tag{1.2.8c}$$

$$u_1 = v_1 = w_1 = 0 \quad \text{at } y = 0 \tag{1.2.9a}$$

$$u_1 \to U_1(x,t), \quad w_1 \to W_1(x,t) \quad \text{as } y \to \infty \tag{1.2.9b}$$

Equations (1.2.6) and (1.2.7) have already been solved for various body configurations, by Prandtl, Schubart, Sears, Görtler, Cooke, Moore, and others [35]. Special cases of these solutions, commonly used as test cases, are the flow past a yawed infinite flat plate, the flow past a yawed infinite wedge at zero angle of attack, the flow in the vicinity of a stagnation line of a yawed infinite cylinder, the flow over a yawed infinite circular cylinder, etc.

In the present chapter we will seek solutions for outer flow velocity distributions of the form

$$U_0(x) = \sum_{b=0}^{\infty} c_{m+2b}\, x^{m+2b} \;, \quad W_0 = W_\infty \tag{1.2.10}$$

where C_{m+2b}, m and W_∞ are constants. These distributions represent a variety of airfoil-like aerodynamic configurations at an angle of yaw. Introducing a two-dimensional stream-function

$$u_0 = \frac{\partial\psi_0}{\partial y} \;, \quad v_0 = -\frac{\partial\psi_0}{\partial x} \tag{1.2.11}$$

and following Howarth [36] and Görtler [31], we assume

$$\psi_0\;(x,y) = \left(\frac{2\nu}{(m+1)c_m x^{m-1}}\right)^{\frac{1}{2}}\left[2\sum_{b=0}^{\infty}(1+b)x^{m+2b}c_{m+2b}f_{1+2b}(\eta) - c_m x^m f_1(\eta)\right] \tag{1.2.12a}$$

$$w_0 = W_\infty\left[\sum_{b=0}^{\infty}\frac{c_{m+2b}}{c_m}\, x^{2b}\; g_{2b}(\eta)\right] \tag{1.2.12b}$$

where η is the familiar similarity variable

$$\eta = \left(\frac{(m+1)c_m x^{m-1}}{2\nu}\right)^{\frac{1}{2}} y$$

Substituting Eqs. (1.2.10) through (1.2.12) in Eqs. (1.2.6a) and (1.2.6b) and equating the coefficients of equal powers of x, we get the following equations for the functions $f_{1+2b}(\eta)$, $g_{2b}(\eta)$:

$$(1+b)f'''_{1+2b} - \frac{1}{2}\frac{c_m}{c_{m+2b}}\delta_{0b}f'''_1 + 2\sum_{a=0}^{b}\frac{(1+a)(m+1+4b-4a)}{(m+1)}(1+b-a)$$

$$\times \frac{c_{m+2a}c_{m+2b-2a}}{c_m c_{m+2b}} f_{1+2b-2a}f''_{1+2a} - (1+b)f''_{1+2b}f_1$$

$$- \frac{(m+1+4b)(1+b)}{(m+1)} f''_1 f_{1+2b} + \frac{1}{2}\frac{c_m}{c_{m+2b}} \delta_{0b}f''_1 f_1$$

$$- 4\sum_{a=0}^{b}\frac{(1+a)(1+b-a)(m+2b-2a)}{(m+1)}\frac{c_{m+2a}c_{m+2b-2a}}{c_m c_{m+2b}} f'_{1+2a}f'_{1+2b-2a}$$

$$+ \frac{4(1+b)(m+b)}{(m+1)} f'_{1+2b}f'_1 - \frac{m}{m+1}\frac{c_m}{c_{m+2b}}\delta_{0b}(f'_1)^2$$

$$+ \sum_{a=0}^{b}\frac{(m+2b-2a)}{m+1}\frac{c_{m+2a}c_{m+2b-2a}}{c_m c_{m+2b}} = 0 \qquad (1.2.13)$$

$$g''_{2b} + 2\sum_{a=0}^{b}\frac{(m+1+4a)(1+a)}{(m+1)}\frac{c_{m+2a}c_{m+2b-2a}}{c_m c_{m+2b}} f_{1+2a}g'_{2b-2a}$$

$$-f_1 g'_{2b} - 8\sum_{a=0}^{b}\frac{(1+a)(b-a)}{m+1}\frac{c_{m+2a}}{c_m}\frac{c_{m+2b-2a}}{c_m c_{m+2b}} f'_{1+2a}g_{2b-2a}$$

$$+ 4\frac{b}{(m+1)} f'_1 g_{2b} = 0 \qquad (1.2.14)$$

The appropriate boundary conditions are

$$f_1 = f_1' = 0,\ f_3 = f_3' = 0,\ f_5 = f_5' = 0,\ \ldots \text{ at } \eta = 0 \quad (1.2.15a)$$

$$f_1' \to 1,\ f_3' \to 1/4,\ f_5' \to 1/6,\ f_7' \to 1/8,\ \ldots \text{ as } \eta \to \infty \quad (1.2.15b)$$

and
$$g_0 = g_2 = \ldots = 0 \text{ at } \eta = 0 \quad (1.2.16a)$$

$$g_0 \to 1,\ g_2\ ,\ g_4\ ,\ \ldots = 0 \text{ as } \eta \to \infty \quad (1.2.16b)$$

In order to render the functional coefficients f_1, f_3, ... of our expansion independent of the particular properties of the airfoil profile, that is the constants c_{m+2b}, it is necessary to split the functions f_5, f_7, etc. as follows [35]

$$f_5(\eta) = f_{51}(\eta) + \frac{c_{m+2}}{c_m c_{m+4}} f_{52}(\eta) \quad (1.2.17)$$

$$f_7(\eta) = f_{71}(\eta) + \frac{c_{m+2}c_{m+4}}{c_m c_{m+6}} f_{72}(\eta) + \frac{c_{m+2}^3}{c_m^2 c_{m+6}} f_{73}(\eta) \quad (1.2.18)$$

The functions g_4, g_6, etc. can be split similarly so that finally the functions f_1, f_3, f_{51}, f_{52}, f_{71}, ... and g_2, g_{41}, g_{42}, ... are universal.

It should be emphasized here that the above analysis can handle with acceptable accuracy any airfoil configuration. Moreover the universality of the results permits one to build the solution for any airfoil configuration without having to solve the differential equations again. Some of the configurations that traditionally have been used as test cases are special cases of the above formulation that require only the first term of our expansion. For example,

(i) The flow past a yawed infinite wedge, or a yawed diverging ramp, at zero angle of attack:

$$\left.\begin{aligned} &U_0(x) = c_m x^m \text{ , i.e. } c_m > 0 \\ &\text{and } c_{m+2b} = 0 \text{ for } b \geq 1 \end{aligned}\right\} \qquad (1.2.19)$$

(ii) The flow past a yawed infinite circular cylinder of radius R:

$$m = 1 \text{ and } c_{1+2b} = (-1)^b 2U_\infty[(1+2b)!R^{1+2b}]^{-1} \qquad (1.2.20)$$

1.3 The Unsteady Part of the Motion

Let us assume that the dependence of U_1 and W_1 on space and time can be separated as follows

$$U_1(x,t) = \tilde{U}_1(x)U_M(t), \; W_1(x,t) = \tilde{W}_1(x)U_M(t) \qquad (1.3.1)$$

We will seek solutions to Eqs. (1.2.8) and (1.2.9) for the above outer flow, where the time function $U_M(t)$ will represent oscillatory or transient behavior. The functions $\tilde{U}_1(x)$ and $\tilde{W}_1(x)$ are also assumed in a power form:

$$\tilde{U}_1(x) = \tilde{c}x^\ell \text{ , } \tilde{W}_1(x) = \frac{W_\infty \tilde{U}_1}{c_m x^m} \qquad (1.3.2)$$

where $\tilde{c}$ and ℓ are constants.

A new stream function is defined

$$u_1 = \frac{\partial \psi_1}{\partial y} \text{ , } v_1 = -\frac{\partial \psi_1}{\partial x} \qquad (1.3.3)$$

and Eqs. (1.2.8) and (1.2.9) become

$$\frac{\partial^2\psi_1}{\partial t\partial y} + \frac{\partial\psi_1}{\partial y}\frac{\partial^2\psi_0}{\partial x\partial y} + \frac{\partial\psi_0}{\partial y}\frac{\partial^2\psi_1}{\partial x\partial y} - \frac{\partial\psi_1}{\partial x}\frac{\partial^2\psi_0}{\partial y^2}$$

$$= \tilde{U}_1(x)\frac{dU_M}{dt} + U_M\frac{\partial}{\partial x}(U_0\tilde{U}_1) + \nu\frac{\partial^3\psi_1}{\partial y^3} \tag{1.3.4a}$$

$$\frac{\partial w_1}{\partial t} + \frac{\partial\psi_1}{\partial y}\frac{\partial w_0}{\partial x} + \frac{\partial\psi_1}{\partial y}\frac{\partial w_1}{\partial x} - \frac{\partial\psi_1}{\partial x}\frac{\partial w_0}{\partial y} - \frac{\partial\psi_0}{\partial x}\frac{\partial w_1}{\partial y}$$

$$= \tilde{W}_1\frac{dU_M}{dt} + U_0U_M\frac{d\tilde{W}_1}{dx} + \nu\frac{\partial^2 w_1}{\partial y^2} \tag{1.3.4b}$$

$$\psi_1 = \frac{\partial\psi_1}{\partial y} = w_1 = 0 \text{ at } y = 0 \tag{1.3.5a}$$

$$\frac{\partial\psi_1}{\partial y} \to \tilde{U}_1(x)U_M(t),\ w_1 \to \tilde{W}_1(x)U_M(t) \text{ as } y \to \infty \tag{1.3.5b}$$

Notice that again Eq. (1.3.4a) can be solved independently of Eq. (1.3.4b). Its solution is assumed in the form

$$\psi_1 = \left(\frac{2\nu}{(m+1)c_m x^{m-1}}\right)^{\frac{1}{2}} \tilde{U}_1(x) \sum_{k=0}^{\infty}\sum_{b=0}^{\infty} \frac{c_{m+2b}}{c_m}\frac{d^k}{dt_1^k}U_M(t)x^{\{2b+(1-m)k\}}F_k^{(2b)}(\eta) \tag{1.3.6}$$

where $t_1 = c_m t$. It is tacitly assumed here that U_M depends in such a way on time that its derivatives satisfy the relation

$$\frac{d^{k+1}U_M}{dt_1^{k+1}} << \frac{d^k}{dt_1^k}U_M(t) \quad \text{for all } k.$$

Substituting Eq. (1.3.6) along with ψ_0 in Eq. (1.3.4a) and equating the coefficients of $\frac{d^k U_M}{dt_1^k} x^{\{2b+(1-m)(k-1)\}}$, for $k = 0,1,2,\ldots$ and $b = 0,1,2,\ldots$, we get the following differential equations.

$$F_k'''^{(2b)}(\eta) + 2 \sum_{a=0}^{b} \frac{c_{m+2a}}{c_m} \frac{c_{m+2b-2a}}{c_{m+2b}} \frac{(1+b-a)}{(1+m)} (4b-4a+m+1) f_{1+2b-2a} F_k''^{(2a)}(\eta)$$

$$- f_1(\eta) F_k''^{(2b)}(\eta) + 2 \left\{\frac{\ell+2b+(1-m)k+m}{m+1}\right\} f_1'(\eta) F_k'^{(2b)}(\eta)$$

$$- 4 \sum_{a=0}^{b} \frac{(1+b-a)}{(m+1)} (\ell+2b+(1-m)k+m) \frac{c_{m+2a}}{c_m} \frac{c_{m+2b-2a}}{c_{m+2b}} f_{1+2b-2a}'(\eta) F_k'^{(2a)}(\eta)$$

$$+ 2 \sum_{a=0}^{b} \frac{c_{m+2a}}{c_m} \frac{c_{m+2b-2a}}{c_{m+2b}} \left(\frac{1+b-a}{m+1}\right)\{2\ell+4a+(1-m)(2k+1)\} f_{1+2b-2a}''(\eta) F_k^{(2a)}(\eta)$$

$$- \left\{\frac{2\ell+4b+(1-m)(2k+1)}{(m+1)}\right\} f_1''(\eta) F_k^{(2b)}(\eta) + \left(\frac{2}{m+1}\right)\left(\frac{c_m}{c_{m+2b}}\right) \delta_{k1}\delta_{bo}$$

$$+ \frac{2(\ell+m+2b)}{m+1} \delta_{ko} - \left(\frac{2}{m+1}\right) F_{k-1}'^{(2b)}(\eta) = 0 \qquad (1.3.7)$$

where δ_{ij} is the Kronecker delta and $F_{-1}'^{(2b)}(\eta) = 0$ so that $F_{k-1}'^{(2b)}(\eta)$ appear only for $k = 1,2,\ldots$. Note that if U_M depends harmonically on time then $d^k U_M/dt^k \sim (i\omega)^k$ and the summation in Eq. (1.3.6) contains terms of the form

$$\left(\frac{\omega x}{U_0}\right)^k x^{2b} = \xi^k x^{2b}$$

where $\xi = \omega x/U_0$ is the familiar parameter characteristic of oscillation in boundary layers [7,21,22]. The appropriate boundary conditions are

$$F_0^{(0)}(0) = \frac{d}{d\eta} F_0^{(0)}(0) = 0 \ , \ \frac{d}{d\eta} F_0^{(0)}(\infty) = 1 \qquad (1.3.8a)$$

$$F_k^{(2b)}(0) = \frac{d}{d\eta} F_k^{(2b)}(0) = \frac{d}{d\eta} F_k^{(2b)}(\infty) = 0 \text{ for } k = 1,2,\ldots \quad b = 1,2,\ldots \qquad (1.3.8b)$$

In analogy to Eqs. (1.2.17) and (1.2.18), we can express ψ_1 in terms of universal functions by assuming

$$F_k^{(4)}(\eta) = H_{k_1}^{(4)}(\eta) + \frac{c_{m+2}^2}{c_m c_{m+4}} H_{k_2}^{(4)}(\eta) \qquad (1.3.9)$$

Substituting Eq. (1.3.9) in (1.3.7) and collecting terms as before we derive differential equations for $F_k^{(0)}(\eta)$, $F_k^{(2)}$, $H_{k2}^{(4)}(\eta)$, $H_{k2}^{(4)}(\eta)$, etc.

In order to solve Eq. (1.3.4b) we assume:

$$w_1 = W_\infty \tilde{U}_1 \sum_{k=0}^{\infty} \sum_{b=0}^{\infty} \frac{c_{m+2b}}{c_m^2} \frac{d^k U_M}{dt_1^k} x^{\{2b+(1-m)k-m\}} G_k^{(2b)}(\eta) \qquad (1.3.10)$$

Substituting Eq. (1.3.10) along with ψ_1, ψ_0, w_0 in Eq. (1.3.4b) and equating the coefficients of $\dfrac{d^k U_M}{dt_1^k} x^{\{2b+(1-m)k-1\}}$, we get

$$G_k''^{(2b)}(\eta) + 2 \sum_{a=0}^{b} (1+b-a)\left(\frac{4b-4a+m+1}{m+1}\right) \frac{c_{m+2a}}{c_m} \frac{c_{m+2b-2a}}{c_{m+2b}} f_{1+2b-2a} G_k'^{(2a)}$$

$$- f_1 G_k'^{(2b)} - 4 \sum_{a=0}^{b} \left\{\frac{1+b-a}{1+m}\right\} (\ell+2a+(1-m)k-m) \frac{c_{m+2a}}{c_m} \frac{c_{m+2b-2a}}{c_{m+2b}} f'_{1+2b-2a} G_k^{(2a)}$$

$$+ 2\left\{\frac{\ell+2b+(1-m)k-m}{m+1}\right\} f_1' G_k^{(2b)} - 4 \sum_{a=0}^{b} \left(\frac{b-a}{m+1}\right) \frac{c_{m+2a}}{c_m} \frac{c_{m+2b-2a}}{c_{m+2b}} g_{2b-2a} F_k'^{(2a)}$$

$$+ \sum_{a=0}^{b} \left\{\frac{2\ell+4a+(1-m)(2k+1)}{m+1}\right\} \frac{c_{m+2a}}{c_m} \frac{c_{m+2b-2a}}{c_{m+2b}} g_{2b-2a}' F_k^{(2a)}$$

$$+\left(\frac{2}{m+1}\right)\left(\frac{c_m}{c_{m+2b}}\right) \delta_{ob}\delta_{1k} + \frac{2(\ell-m)}{m+1} \delta_{ok} - \frac{2}{m+1} G_{k-1}^{(2b)} = 0 \qquad (1.3.11)$$

with boundary conditions

$$G_k^{(2b)}(0) = G_k^{(2b)}(\infty) = 0 \text{ for } k = 0,1,\ldots \text{ and all } b$$

$$G_0^{(0)}(\eta) = 1 \text{ and } G_{-1}^{(2b)}(\eta) \qquad (1.3.12)$$

Thus the unsteady three dimensional boundary layer equations are reduced to ordinary differential equations which can be integrated numerically.

The unsteady component of the skin friction along the chordwise and spanwise directions can now be expressed as follows:

$$\tau_{1x} = \mu\left(\frac{(m+1)c_m x^{m-1}}{2\nu}\right)^{\frac{1}{2}} \tilde{U}_1 \sum_{k=0}^{\infty} \sum_{b=0}^{\infty} \frac{c_{m+2b}}{c_m} x^{\{2b-(1-m)k\}} \frac{d^k U_M}{dt_1^k} F_k''^{(2b)}(0) \qquad (1.3.13)$$

$$\tau_{1z} = \mu\left(\frac{(m+1)c_m x^{m-1}}{2}\right)^{\frac{1}{2}} W_\infty \tilde{U}_1 \sum_{k=0}^{\infty} \sum_{b=0}^{\infty} \frac{c_{m+2b}}{c_m^2} x^{2b+(1-m)k-m} \frac{d^k U_M}{dt_1^k} G_k'^{(2b)}(0) \qquad (1.3.14)$$

The resultant skin friction is

$$\vec{\tau}_R = \vec{\tau}_{0R} + \varepsilon\vec{\tau}_{1R} = (\tau_{0x}^2+\tau_{0z}^2)^{\frac{1}{2}} + \varepsilon \frac{\tau_{0x}\tau_{1x} + \tau_{0z}\tau_{1z}}{(\tau_{0x}^2 + \tau_{0z}^2)^{\frac{1}{2}}} \qquad (1.3.15)$$

where

$$\tau_{0x} = \left(\frac{(m+1)c_m x^{m-1}}{2\nu}\right)^{\frac{1}{2}} \mu\left[2 \sum_{b=0}^{\infty} (1+b)x^{m+2b} f''_{1+2b}(0) - c_m x^m f''_1(0)\right] \tag{1.3.16}$$

$$\tau_{0z} = \left(\frac{(m+1)c_m x^{m-1}}{2\nu}\right)^{\frac{1}{2}} W_\infty \,\mu \sum_{b=0}^{\infty} \frac{c_{m+2b}}{c_m} x^{2b} g'_{2b}(0) \tag{1.3.17}$$

The displacement thicknesses in the chordwise and spanwise directions are defined by

$$\delta_x^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy \ , \ \delta_z^* = \int_0^\infty \left(1 - \frac{w}{W}\right) dy \tag{1.3.18}$$

From (1.3.18), the unsteady correction to the displacement thicknesses in the two directions can be written as

$$\delta_{1x}^* = \frac{1}{U_0^2} \int_0^\infty (u_0 U_1 - u_1 U_0) dy \tag{1.3.19}$$

$$\delta_{1z}^* = \frac{1}{W_\infty^2} \int_0^\infty (w_0 W_1 - w_1 W_\infty) dy \tag{1.3.20}$$

The two components of the displacement thickness have been widely used in literature to describe the displacement effect of the boundary layer. However, the physical interpretation of two displacement thickness components is a little ambiguous. The displacement effect could only be described by a single length at a specific point x, z on the solid surface. Moore and Ostrach [23] have suggested a generalized definition for displacement thickness that reduces for incompressible flow to

$$\Delta(x,z) = \int \left[1 - \frac{u^2 + w^2}{U^2 + W^2}\right]^{\frac{1}{2}} dy \tag{1.3.21}$$

It is now seen that the displacement effect is represented by a mean displacement thickness Δ_0 and a fluctuating correction $\Delta_1 e^{i\omega t}$ where

$$\Delta_0 = \int \left[1 - \frac{u_0^2 + w_0^2}{U_0^2 + W_0^2}\right]^{\frac{1}{2}} dy \tag{1.3.22}$$

and

$$\Delta_1 = \int \left[\frac{u_0 u_1 + w_0 w_1}{u_0^2 + w_0^2} - \frac{U_0 U_1 + W_0 W_1}{U_0^2 + W_0^2}\right] dy \tag{1.3.23}$$

The integrand of Eq. (1.3.23) contains the complex quantities u_1 and w_1. It is therefore necessary to estimate the phase of the complex number Δ_1 in order to predict the proper phase lag or advance of the displacement thickness.

In the present chapter we only present calculations of the traditional quantities δ^*_{1x} and δ^*_{1z}. It appears that a physical interpretation of the phenomenon may then be easier to see. One can distinguish between the contributions of the spanwise and chordwise flow to the displacement effect.

1.4 Results and Conclusions

The present method is capable of handling any arbitrary airfoil configurations as we mentioned before. However, in the present chapter we have considered a few special cases that have been treated widely in literature. This is not done for the sake of simplicity - very little simplification is gained anyway but only for purposes of comparison with other calculations.

The classical wedge flow was considered. Wedge flows are a special case of Eq. (1.2.10) for b = 0. The zeroth order problem is solved first and the functions $f_i'(\eta)$ are checked against the classical tabulated results, (see for example Schlichting [35]) and found to agree up to the fifth decimal point. These functions were included in numerous publications and we do not feel it is necessary to replot them here.

Next we consider two different types of fluctuations with small frequencies.

$$U = U_0 + \varepsilon\tilde{U}_1 e^{i\omega t}\ , \ W = W_0 \tag{1.4.1}$$

$$U = U_0 + \varepsilon\tilde{U}_1 e^{i\omega t}\ , \ W = W_0 + \varepsilon\tilde{W}_1 e^{i\omega t} \tag{1.4.2}$$

For the first case, $W = W_0$, with $b = 0$, the double expansion in Eq. (1.3.6) reduces to a single expansion involving the functions $F_k^{(0)}$. Equations (1.3.7) were solved numerically for the unknown functions $F_k^{(0)}$, using a Runge-Kutta procedure. Figure 1.2 is a plot of the three functions of the series expansion of Eq. (1.3.6), that is when the amplitude of the velocity fluctuation is constant. The characteristic

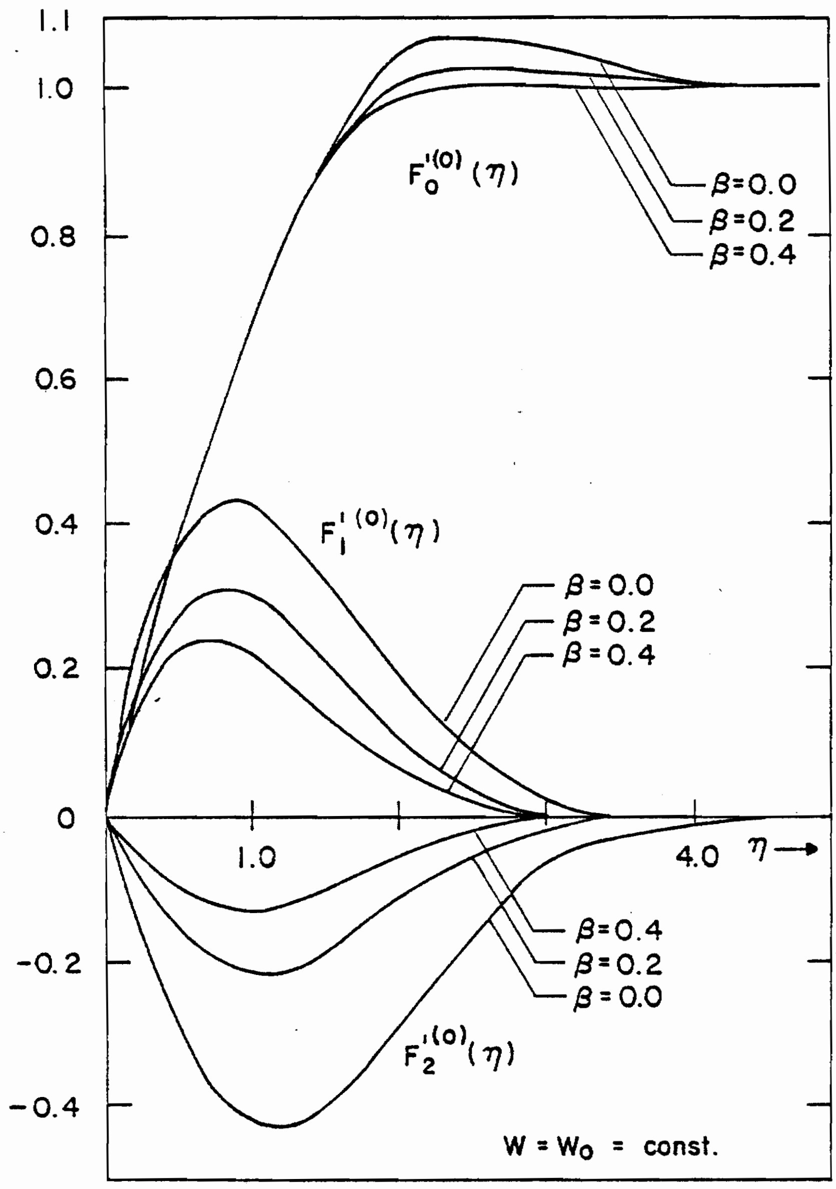

Figure 1.2. The functions $F_k^{\prime(0)}(\eta)$ representing the unsteady part of the chordwise velocity for a steady outer cross flow (ℓ = 0, see Eq. (1.3.7)

overshoot that was reported first by Lighthill is shown in the plot of $F_0'^{(0)}$. In Fig. 1.3 we have plotted the response of the cross flow to the fluctuations represented by Eq. (1.3.11) when $\ell = 0$. It is most interesting to note that even if the outer flow spanwise velocity is independent of time, a fluctuating cross flow is generated in the boundary layer, that gets stronger for smaller values of the pressure gradient. This is due to the nonlinearity of the boundary layer equations. The mixed terms in Eq. (1.2.8b) that are fed from the solution of Eq. (1.2.8a) are driving the w-component of the velocity, generating an oscillatory cross flow.

Figure 1.4 shows the fluctuating part of the amplitude of skin friction coefficient $C_{f_{1x}} = \tau_1\left(\frac{U_0 x}{\nu}\right)^{\frac{1}{2}}(\mu U_0 U_1/\nu)^{-1}$ and the wall phase lead for the chordwise flow, as functions of the frequency parameter $\omega x/U_0$. For the spanwise flow it was discovered that the skin friction oscillates in phase with the outer flow while its amplitude is practically constant as shown in Fig. 1.5, where $C_{f1z}=\tau_{1_\omega}\left(\frac{U_0 x}{\nu}\right)^{\frac{1}{2}}\left(\frac{\mu W_0 U_1}{\nu}\right)^{-1}$. This result can be interpreted physically as follows. The direction of the skin friction vector and hence the orientation of the skin friction lines is oscillating with time but in phase with the outer flow. As the outer flow chordwise velocity increases the skin friction direction starts turning towards the positive spanwise direction while the opposite effects occur when the outer flow chordwise velocity decreases.

Velocity profile amplitudes as calculated by summing up the first

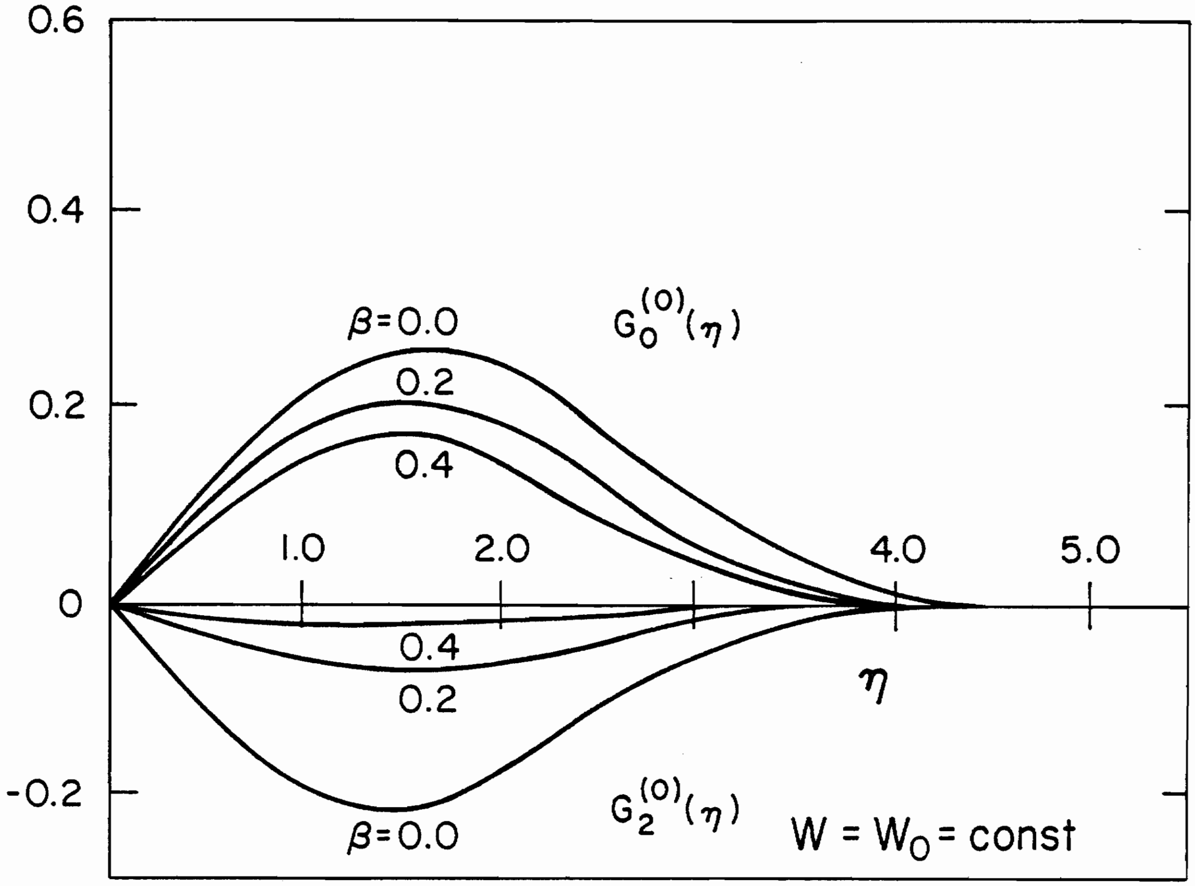

Figure 1.3. The functions $G_k^{(0)}(\eta)$ representing the unsteady part of the spanwise velocity for a steady outer flow, see Eq. (1.3.11).

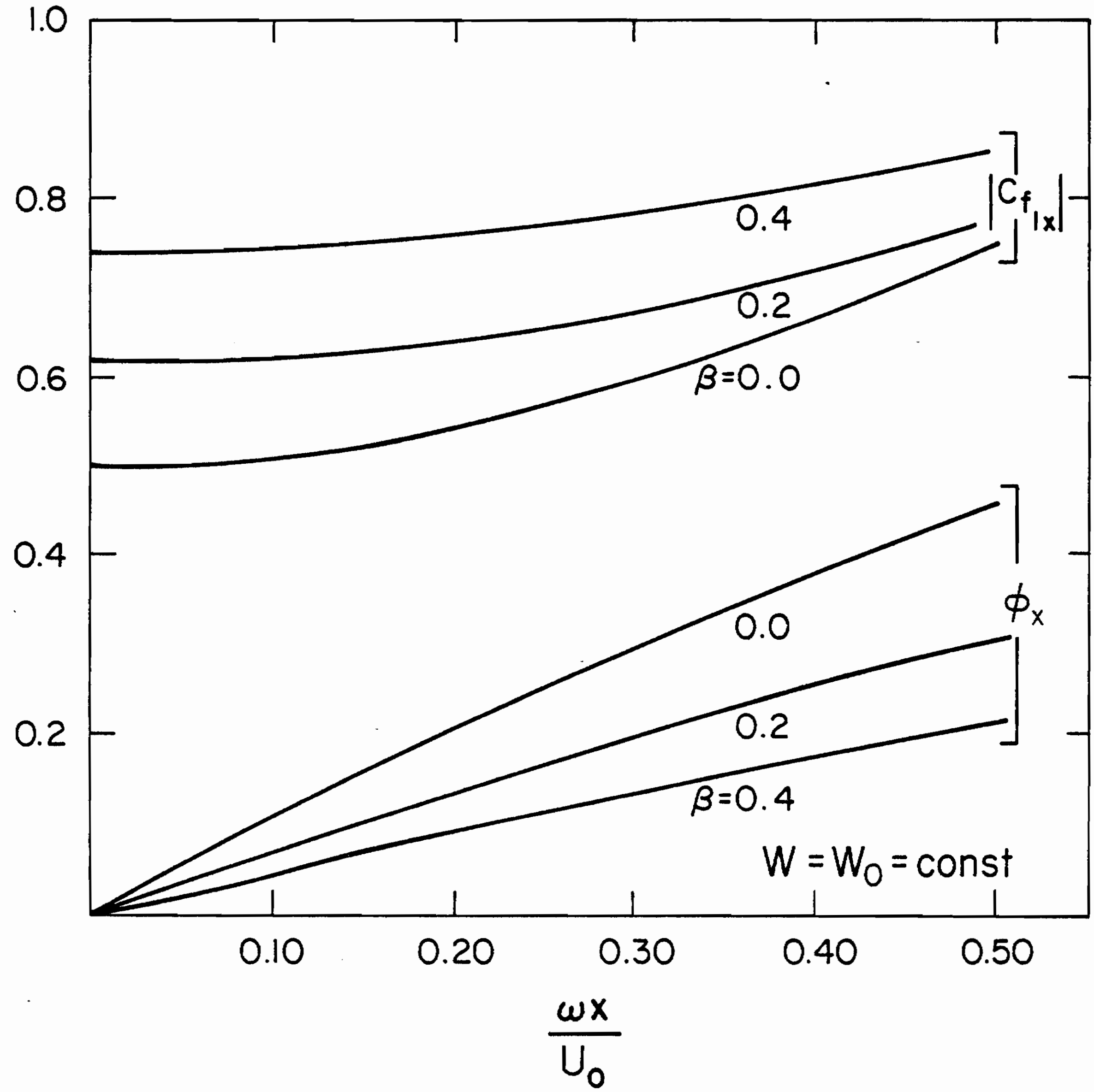

Figure 1.4. The skin friction coefficient in the chordwise direction and its phase lead for a steady outer cross flow.

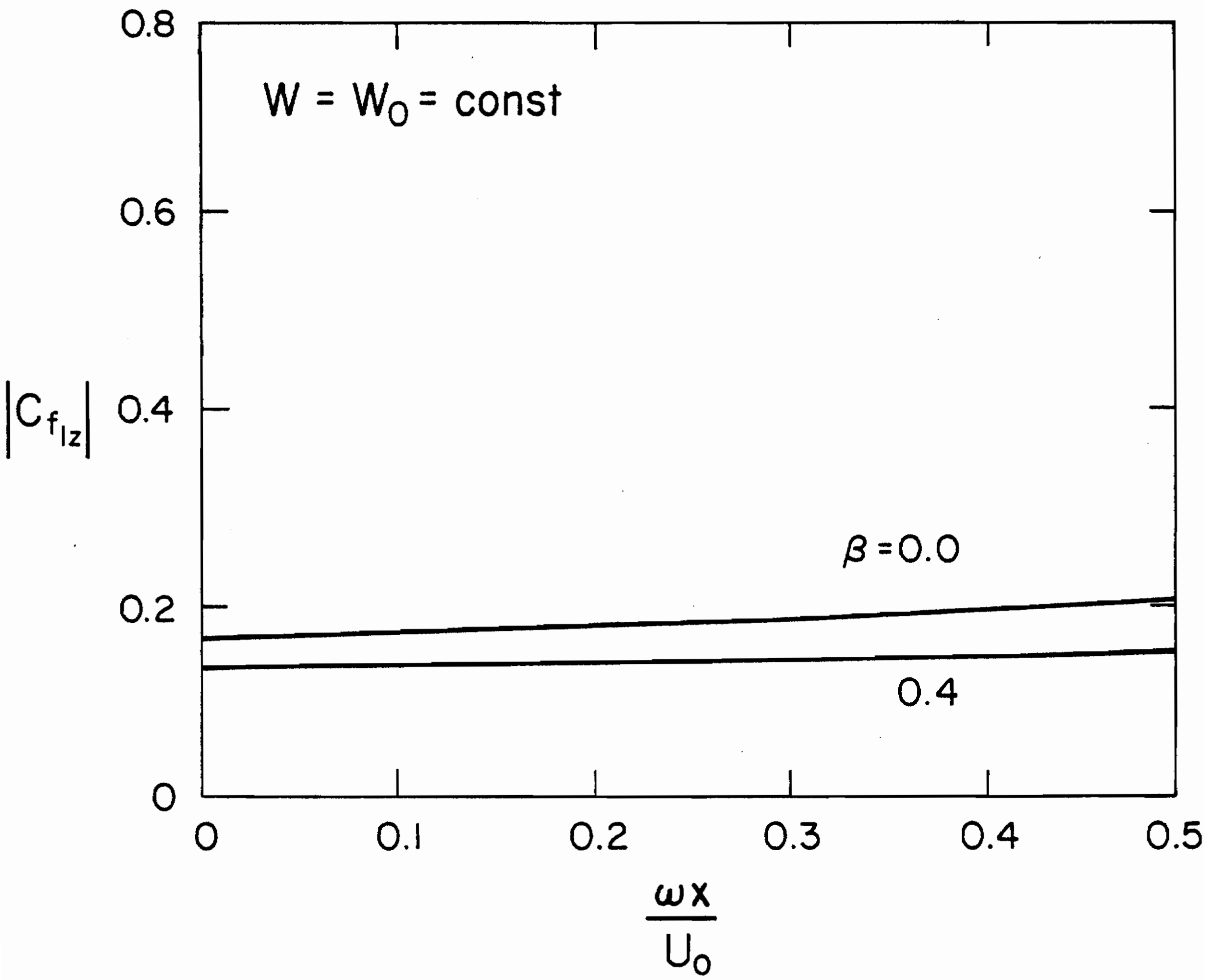

Figure 1.5. The skin friction coefficient in the spanwise direction for a steady outer cross flow.

three terms of the expansions of $u_1 = \partial\psi_1/\partial y$ where ψ_1 is given by Eq. (1.3.6) and w_1 by Eq. (1.3.10), are shown in Figs. 1.6 and 1.7. In Fig. 1.6 the chordwise amplitude variation shows again the characteristic overshoot, a property that appears to be suppressed as the pressure gradient increases. In Fig. 1.7 the amplitude of the spanwise velocity fluctuation appears to be uniformly decreasing for larger pressure gradients and shows a mild dependence on the frequency parameter. The spanwise and chordwise components of the displacement thickness amplitudes are shown in Fig. 1.8 as functions of the frequency parameter. Again it is observed that the pressure gradient tends to suppress the amplitude of the fluctuation. It is further noticed in this figure that for zero pressure gradients, the chordwise component is dominant and in fact shows a sharp increase with the frequency parameter. For nonzero pressure gradients it appears that the spanwise component grows larger, even though it remains a weak function of the amplitude parameter.

Similar calculations were performed for outer flow fluctuations in both the spanwise and chordwise direction (see Eq. (1.4.2)). The first three universal functions corresponding to the fluctuating spanwise component are shown in Fig. 1.9 and appear to be qualitatively similar to those of Fig. 1.2 that represent the chordwise response. Fig. 1.10 is a plot of the skin friction coefficient amplitude and its phase advance versus the frequency parameter $\frac{\omega x}{U_0}$. It is interesting to note that for a constant W the phase advance is practically zero,

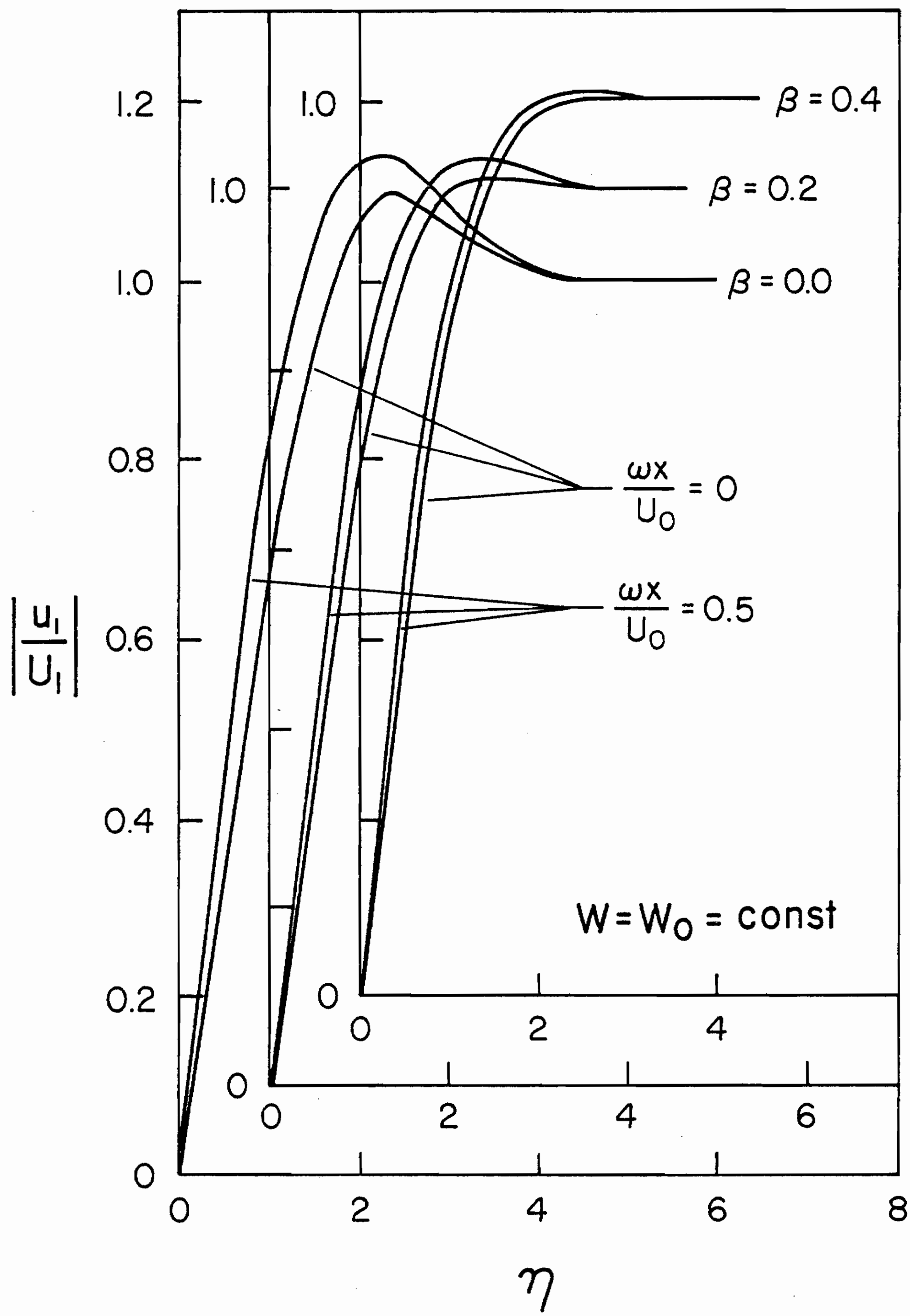

Figure 1.6. The amplitude profile of the unsteady part of the chordwise velocity for a steady outer cross flow.

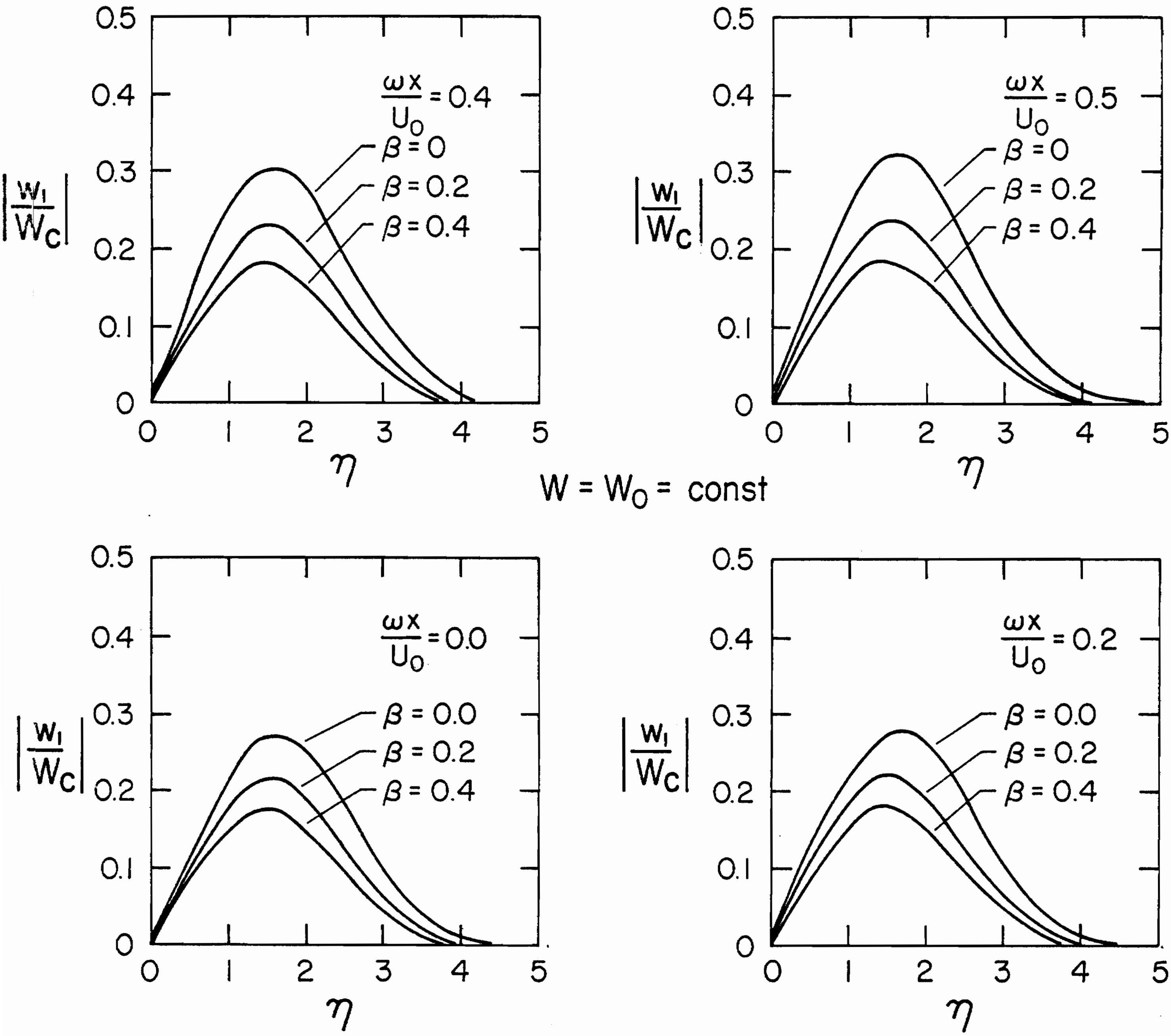

Figure 1.7. The amplitude profile of the unsteady part of the spanwise velocity for a steady outer cross flow, $W_c = W_0\tilde{c}/c_m x^m$.

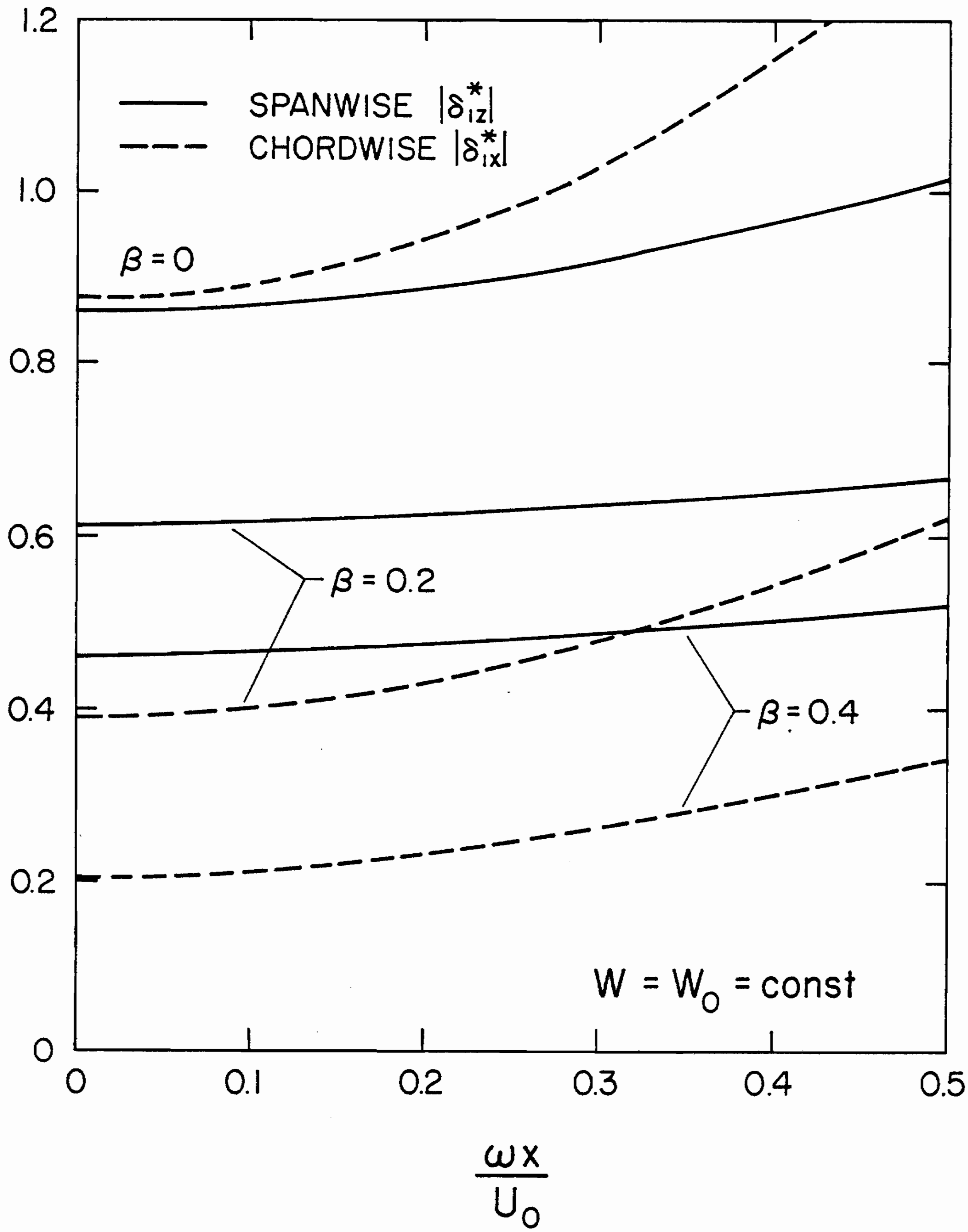

Figure 1.8. The spanwise and chordwise displacement thicknesses for a steady outer cross flow.

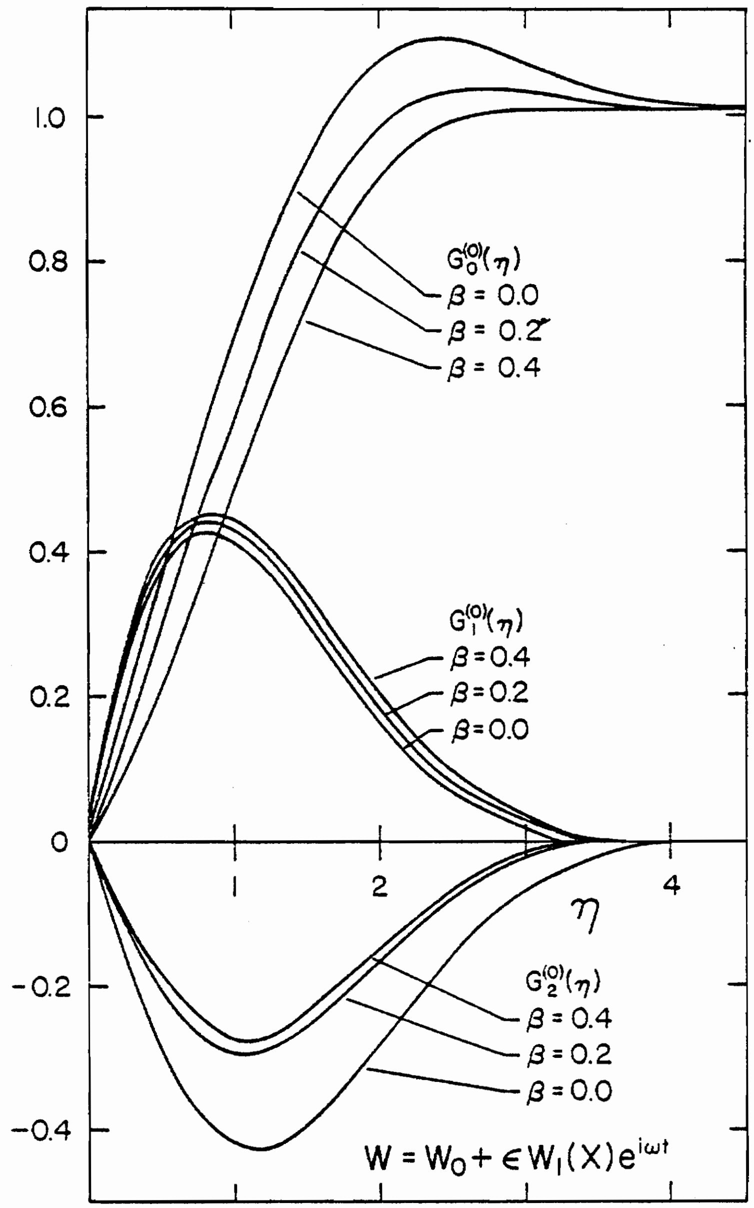

Figure 1.9. The functions $G_k^{(0)}(\eta)$ representing the unsteady part of the spanwise velocity for an oscillating outer cross flow, see Eq. (1.3.11).

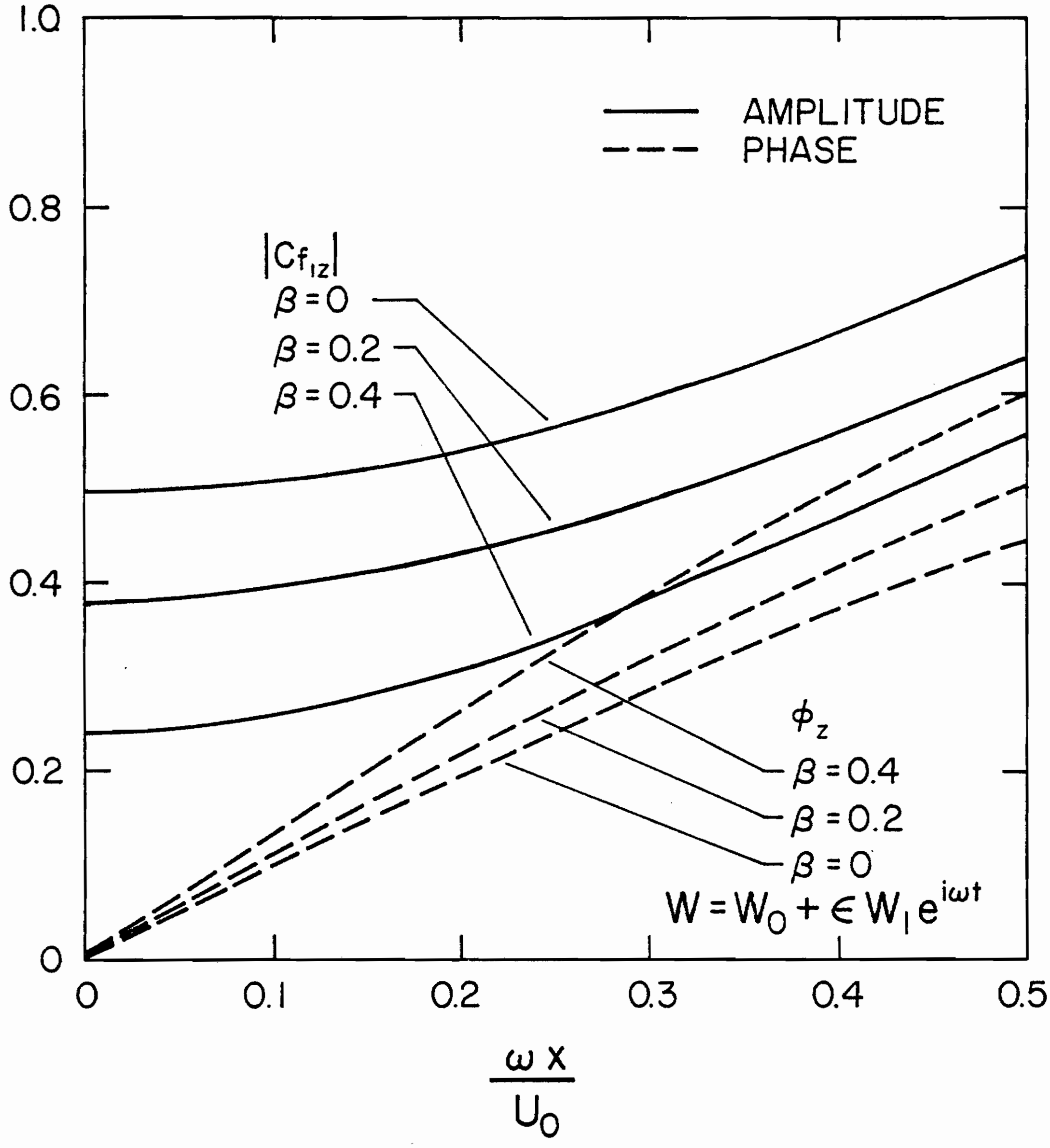

Figure 1.10. The skin friction coefficient in the spanwise direction and its phase lead for an oscillating outer cross flow.

whereas now the phase is even larger than the corresponding values of the chordwise flow (see Fig. 1.3).

In Fig. 1.11 the velocity profiles of the spanwise fluctuation amplitude are shown to resemble greatly the chordwise profiles (see Fig. 1.5). Finally the displacement thickness is shown in Fig. 1.12. It is now observed that the amplitude of the spanwise skin friction component is sharply affected by the frequency parameter, in fact more than the chordwise component. The reader is cautioned at this point to the fact that the dimensionless quantity $\frac{\omega x}{U_0}$ depends on the frequency as well as on the distance x in a nonlinear way, since U_0 itself is a function of x. The results are presented here in terms of the function $\frac{\omega x}{U_0}$ to preserve universality.

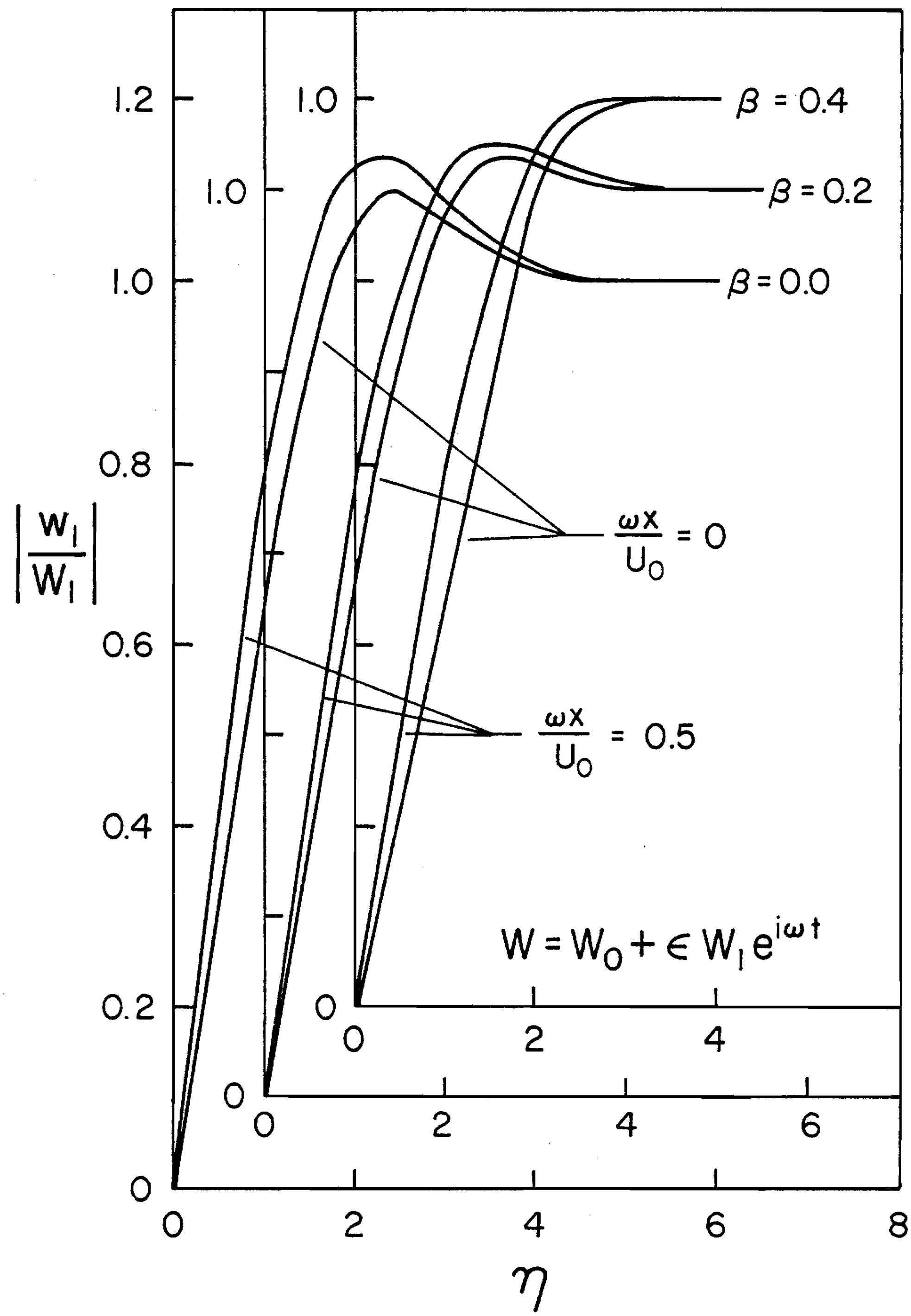

Figure 1.11. The amplitude profile of the unsteady part of the spanwise velocity for an oscillating outer cross flow.

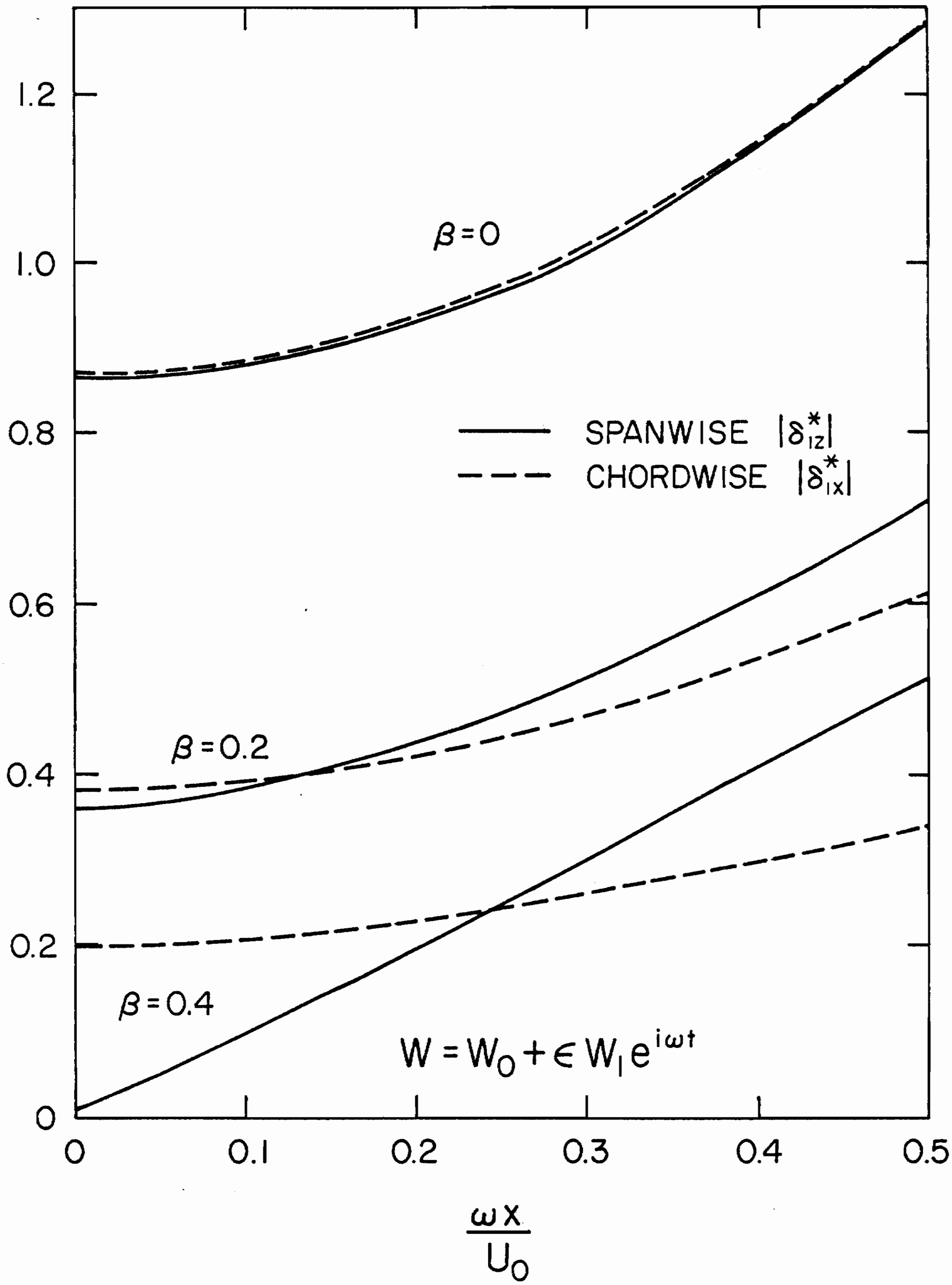

Figure 1.12. The spanwise and chorwise displacement thickness for an oscillating outer cross flow.

CHAPTER TWO

COMPRESSIBLE OSCILLATIONS OVER TWO-DIMENSIONAL AND AXISYMMETRIC WALLS WITH NO MEAN HEAT TRANSFER

2.1 Introduction

Most of the contributions in the area of oscillatory viscous flows are concerned with incompressible flows. Moore [12] first considered compressibility effects in a work already mentioned before. Illingworth [6] then studied the effect of acoustic waves on low-speed compressible boundary layers. His external fluctuations were therefore of a wave type and not simply a spatially uniform time oscillation. Sarma [8] gave a unified theory for the solutions of unsteady incompressible boundary layers which he later extended to compressible boundary layers [9]. However, he considered steady outer flows and introduced the unsteady effects via boundary conditions on the boundaries of the body in the form of blowing or parallel motion of the skin. King [17] investigated the response of the fluctuations of the boundary layer to oscillatory outer flows and in particular studied the problem of an oscillating slender wedge in a hypersonic stream. In this chapter we consider the problem of unsteady compressible boundary layers by assuming that the amplitude of the fluctuations is small. The theory is extended to include the effects of fluctuating flows imposed by oscillations of the skin or the outer flow. The latter case appears to be very interesting from the mathematical point of view, because of the fact that the pressure is then also a function of time and a few new terms appear in the momentum equation. It should be emphasized here that the two problems mentioned above: fixed bodies in oscillating flows or oscillating bodies in steady uniform flows, cannot be treated mathematically through the same model. In other words it is not possible to derive the one flow field from the other, by a simple transformation of the frame of reference. This is due to the fact that the inertia term introduced when transforming the coordinate system to match the one case with the other, involves the term $\rho_\infty \frac{\partial U_e}{\partial t}$ in the one case and $\rho \frac{\partial U_e}{\partial t}$ in the other case where ρ, U_e and t are the density, the edge velocity and time respectively. The two effects are equivalent only if $\rho = \rho_\infty$. In view of the above comments it should be pointed out that in the present investigation we capture for the first time a truly unsteady compressible boundary layer with nonvanishing pressure gradients. The only restriction pertains to the wall temperature which is confined to temperatures around the adiabatic.

In this chapter the differential equations are presented for the first time in their most general form, for the case where the temperature is a function of the velocity only. For fluctuating compressible boundary layers then, numerical solutions could be derived for arbitrary values of the frequency parameter. Telionis and Romanuik [20] have recently performed such calculations for incompressible flows, integrating numerically in a two-dimensional space for the steady and unsteady components of the motion. The present numerical results are presented in terms of dimensionless functions that represent terms in the series expansion. Parameters like the wall temperature or the Mach number are contained in the coefficients of our expansions. It is therefore possible to use the present information in order to quickly generate solutions of flows about two-dimensional or axisymmetric configurations for a wide variety of parameters. Physically these solutions correspond for example, to a wedge that oscillates in a compressible flow. The axisymmetric case may not be as interesting in practice since it corresponds to a cone with an apex angle that fluctuates about a mean.

2.2 The Governing Equations

Two-dimensional or axisymmetric laminar compressible boundary-layer flow is governed by the continuity, the momentum and the energy equation.

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^j}\frac{\partial}{\partial x}(\rho r^j u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{2.2.1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{2.2.2}$$

$$\rho C_p\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) - \left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x}\right) = \frac{C_p}{P_r}\frac{\partial}{\partial y}\left(\mu\frac{\partial T}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{2.2.3}$$

In the above equations u, v and x, y are the velocity components and the distances parallel and perpendicular to the wall, respectively as shown in Figs. 2.1 and 2.2. ρ, p, and T are the density, pressure, and temperature, and C_p, μ, and P_r the specific heat for constant pressure, the viscosity, and the Prandtl number respectively. the quantity $r = r(x)$ defines the body of revolution for axisymmetric flow, and j takes the values 0 and 1 for two-dimensional and axisymmetric flow, respecti-

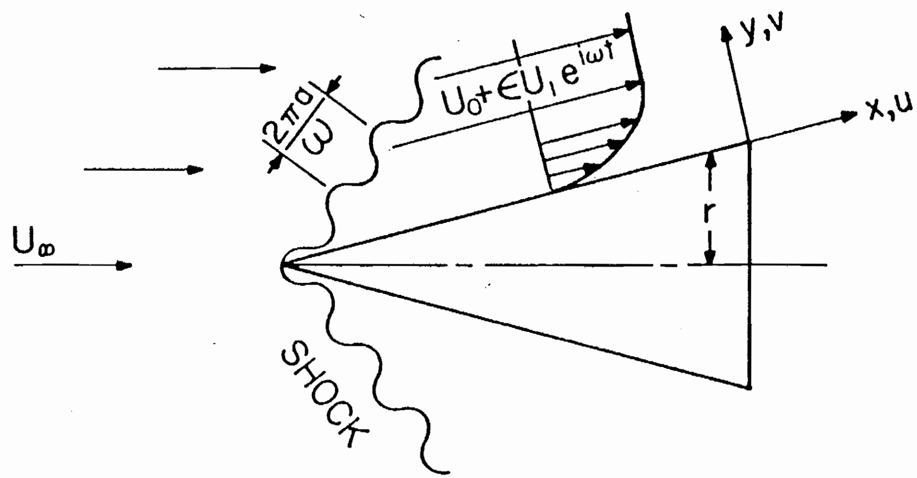

Figure 2.1 Schematic of the flow field with oscillating outer flow.

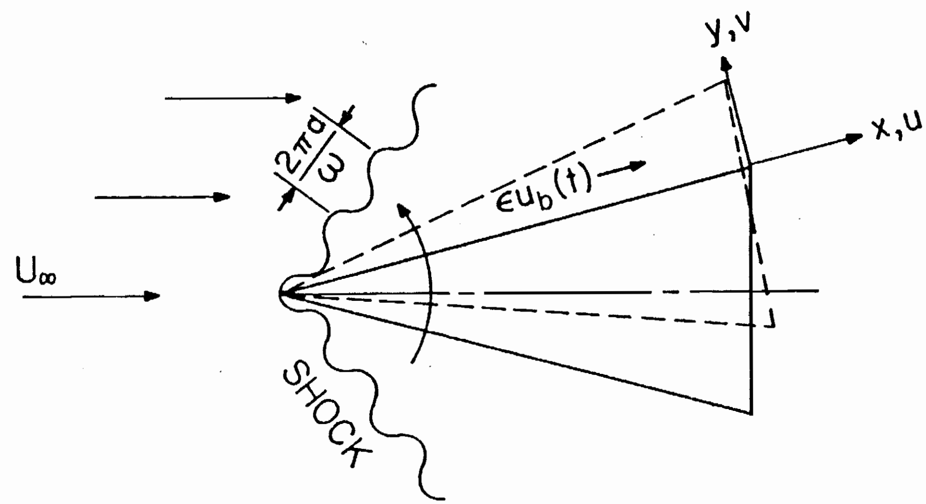

Figure 2.2 Schematic of the flow field with oscillating wall.

vely. It is interesting here to note that for oscillating flow, the shock waves should have a wavy shape, with wave length $2\pi a/\omega$ where a is the speed of sound and ω is the frequency of oscillation.

If the properties at the edge of the boundary layer are denoted by an index e, then the quantities U_e, T_e, ρ_e and the pressure, p, which is uniform across the boundary-layer, obey the equations

$$-\frac{\partial p}{\partial x} = \rho_e\left(\frac{\partial U_e}{\partial t} + U_e\,\frac{\partial U_e}{\partial x}\right) \tag{2.2.4}$$

$$\frac{\partial p}{\partial t} = \rho_e C_p\left(\frac{\partial}{\partial t} + U_e\,\frac{\partial}{\partial x}\right)\left(T_e + \frac{U_e^2}{2C_p}\right) \tag{2.2.5}$$

The above equations can be supplemented with the equation of state, valid for a perfect gas with constant specific heat.

$$p = \rho RT \tag{2.2.6}$$

with R the universal constant of gases. Equation (2.2.6), for a boundary-layer flow can be written as

$$\frac{T}{T_e} = \frac{\rho_e}{\rho} = \frac{\gamma p}{\rho a_e^2} \tag{2.2.7}$$

where a_e is the speed of sound at the edge of the boundary layer, $a_e^2 = \gamma RT$ and γ is the ratio of the specific heats.

In this chapter, we assume that the Prandtl number is equal to 1 and that the viscosity varies linearly with temperature. An integral of the energy equation for non-vanishing pressure gradient can then be written as

$$\frac{T}{T_e} = 1 + \frac{\gamma-1}{2a_e^2}\,(U_e^2 - u^2) \tag{2.2.8}$$

The continuity equation is satisfied identically if the stream function $\Psi(x,y,t)$ is introduced

$$u = \frac{1}{r^j} \frac{\rho_s}{\rho} \frac{\partial}{\partial y} (r^j \Psi) = \frac{\rho_s}{\rho} \frac{\partial \Psi}{\partial y} \tag{2.2.9}$$

$$v = - \frac{\rho_s}{\rho} \left(\frac{\partial \Psi}{\partial x} + \frac{j}{r} \Psi \frac{dr}{dx} + \frac{\partial}{\partial t} \int_0^y \frac{\rho}{\rho_s} dy\right) \tag{2.2.10}$$

The index s in the sequel will denote a quantity evaluated at some standard state of the fluid.

There are three transformations available for treating the boundary layer equations for compressible flow. They are due to Howarth, Von Miser and Crocco. Illingworth [6] has shown though, that Howarth's [37] transformation, later generalized by Moore [12], is the simplest to apply in compressible boundary layer theory.

According to Moore [12], the y coordinate is replaced by

$$Y = \int_0^y \frac{\rho}{\rho_s} dy \tag{2.2.11}$$

A new stream function is introduced

$$\Psi(x,Y,t) = \left(\frac{\rho_s}{\rho}\right)^{1/2} \psi(x,y,t) \tag{2.2.12}$$

in terms of which the velocity components become

$$u = \frac{\partial \psi}{\partial Y} \tag{2.2.13}$$

$$v = - \frac{\rho_s}{\rho} \left(\frac{p}{p_s}\right)^{1/2} \left[\left(\frac{1}{2p} \frac{\partial p}{\partial x} + \frac{j}{r} \frac{dr}{dx}\right)\psi + \frac{\partial \psi}{\partial x} + \frac{\partial Y}{\partial x} \frac{\partial \psi}{\partial Y} + \frac{\partial Y}{\partial t} + \frac{1}{2p} \frac{\partial p}{\partial t} Y\right] \tag{2.2.14}$$

If Eqs. (2.2.13) and (2.2.14) are substituted in Eq. (2.2.2) and Eqs. (2.2.4) and (2.2.5) are used to eliminate the pressure p we arrive at

$$\frac{\partial^2\psi}{\partial t\partial y} + \frac{\partial\psi}{\partial Y}\frac{\partial^2\psi}{\partial x\partial Y} - \left(\frac{\partial\psi}{\partial x} + \frac{j}{r}\frac{dr}{dx}\psi\right)\frac{\partial^2\psi}{\partial Y^2} = [1 + \frac{\gamma-1}{2a_e^2}(U_e^2 - u^2) - \frac{\gamma}{2a_e^2}\psi\frac{\partial^2\psi}{\partial Y^2}]\left(\frac{\partial U_e}{\partial t} + U_e\frac{\partial U_e}{\partial x}\right) + \nu_s\frac{\partial^3\psi}{\partial Y^3} \tag{2.2.15}$$

where ν is the kinematic viscosity.

The appropriate boundary conditions at the wall are

$$\psi = 0,\ \partial\psi/\partial Y = \varepsilon u_b(t) \quad \text{at } Y = 0 \tag{2.2.16}$$

where u_b is the wall velocity and

$$\frac{\partial\psi}{\partial Y} \to U_e(x,t) \quad \text{at } Y \to \infty \tag{2.2.17}$$

In this chapter we will consider oscillatory outer flows, over fixed walls $u_b(t) = 0$ and steady outer flows over oscillatory walls for two-dimensional or axisymmetric configurations.

Once Eq. (2.2.15) is solved and the velocity field is known, one can substitute in Eq. (2.2.8) in order to get the temperature and then via Eq. (2.2.7), the density profile.

2.3 Small Amplitude Oscillations and Steady Part of the Motion

As we did in Chapter 1, we express the outer flow velocity and temperature in the form

$$\left.\begin{aligned} U_e(x,t) &= U_0(x) + \varepsilon U_1(x,t) \\ T_e(x,t) &= T_0(x) + \varepsilon T_1(x,t) \end{aligned}\right\} \tag{2.3.1}$$

where ε is a small constant parameter. This implies that the amplitude of oscillation is small but time dependence is arbitrary and both high or low frequency oscillations can be considered. Alternatively we may assume that the skin of the body oscillates with a velocity, $U_b(t) = \varepsilon e^{i\omega t}$.

A solution to Eq. (2.2.15) is now sought in the form

$$\psi(x,Y,t) = \psi_0(x,Y) + \varepsilon\psi_1(x,Y,t) + O(\varepsilon^2) \tag{2.3.2}$$

but only terms of order ε will be retained here. Substituting Eq. (2.3.2) in Eq. (2.2.15) and collecting terms of order ε^0 we arrive at

$$\frac{\partial\psi_0}{\partial Y}\frac{\partial^2\psi_0}{\partial Y\partial x} - \left(\frac{\partial\psi_0}{\partial x} + \frac{j}{r}\frac{dr}{dx}\psi_0\right)\frac{\partial^2\psi_0}{\partial Y^2} = \Big\{1 + [2a_w^2(1 - \frac{\gamma-1}{2a_w^2}U_0^2]^{-1}[(\gamma-1)]\times$$
$$[U_0^2 - (\frac{\partial\psi_0}{\partial Y})^2] - \gamma\psi_0\frac{\partial^2\psi_0}{\partial Y^2}]\Big\}U_0\frac{dU_0}{dx} + \nu_s\frac{\partial^3\psi_0}{\partial Y^3} \tag{2.3.3}$$

where a_e is replaced by the speed of sound that corresponds to the fixed temperature of the wall

$$a_e^2 = a_w^2[1 - \frac{\gamma-1}{2a_w^2}U_e^2] \tag{2.3.4}$$

Similarly we deduce from the boundary conditions (2.2.16) and (2.2.17) that

$$\psi_0 = 0,\ \partial\psi_0/\partial Y = 0 \text{ at } Y = 0 \tag{2.3.5}$$

$$\partial\psi_0/\partial Y \to U_0(x) \text{ as } Y \to \infty \tag{2.3.6}$$

Collecting terms of order ε now yields

$$\frac{\partial^2\psi_1}{\partial t\partial Y} + \frac{\partial\psi_0}{\partial Y}\frac{\partial^2\psi_1}{\partial x\partial Y} + \frac{\partial\psi_1}{\partial Y}\frac{\partial^2\psi_0}{\partial x\partial Y} - \left(\frac{\partial\psi_0}{\partial x} + \frac{j}{r}\frac{dr}{dx}\psi_0\right)\frac{\partial^2\psi_1}{\partial Y^2}$$

$$- \left(\frac{\partial\psi_1}{\partial x} + \frac{j}{r}\frac{dr}{dx}\psi_1\right)\frac{\partial^2\psi_0}{\partial Y^2} = \Big\{1 + [2a_w^2(1 - \frac{\gamma-1}{2a_w^2}U_0^2)]^{-1} \times$$

$$[(\gamma-1)(U_0^2 - (\frac{\partial\psi_0}{\partial Y})^2) - \gamma\psi_0\frac{\partial^2\psi_0}{\partial Y^2}]\Big\}[\frac{\partial U_1}{\partial t} + \frac{\partial}{\partial x}(U_0U_1)]$$

$$+ \Big\{[a_w^2(1 - \frac{\gamma-1}{2a_w^2}U_0^2]^{-1}[(\gamma-1)(U_0U_1 - \frac{\partial\psi_0}{\partial Y}\frac{\partial\psi_1}{\partial Y})$$

$$- \gamma(\psi_0\frac{\partial^2\psi_1}{\partial Y^2} + \psi_1\frac{\partial^2\psi_0}{\partial Y^2})] + \frac{\gamma-1}{2}U_0U_1[a_w^2(1 - \frac{\gamma-1}{2a_w^2}U_0^2)]^{-2} \times$$

$$[(\gamma-1)(U_0^2 - (\frac{\partial\psi_0}{\partial Y})^2) - \gamma\psi_0\frac{\partial^2\psi_0}{\partial Y^2}]\Big\}U_0\frac{dU_0}{dx} + \nu_s\frac{\partial^3\psi_1}{\partial Y^3} \quad (2.3.7)$$

and the corresponding boundary conditions become

$$\psi_1 = 0,\ \partial\psi_1/\partial Y = u_b(t) \quad \text{at } Y = 0 \quad (2.3.8)$$

$$\partial\psi_1/\partial Y \to U_1 \quad \text{as } Y \to \infty \quad (2.3.9)$$

Notice that at this level of approximation we can deduce from the energy equation that

$$C_pT_0 + \frac{1}{2}U_0^2 = H_0 = \text{constant} \quad (2.3.10)$$

and that the pressure fluctuations are of order ε^2.

The first equation, Eq. (2.3.3) represents the steady part of the motion and can be solved once and for all, regardless of the particular unsteady motion assumed. The second equation, Eq. (2.3.7) governs the corrections due to the unsteadiness that the outer flow imposes on the boundary layer and it will be solved separately for oscillations of the outer stream or the wall.

Consider in general

$$r = \alpha_1 x,\ U_0 = c_0x^m \quad (2.3.11)$$

For subsonic flow this corresponds to the flow about a wedge or a cone for two dimensional or axisymmetric flow respectively. For the supersonic case, the flow about a wedge and cone can be derived if $m = 0$.

Solutions to Eq. (2.3.3) will be sought now in an expansion form

$$\psi_0(x,y) = c_0\left(\frac{2\nu_s x}{(m+1+2j)U_0}\right)^{1/2} \sum_{n=0}^{\infty} \frac{c_0^{2n}}{a_w^{2n}} x^{2mn+m} f_{1+2n}(\eta) \qquad (2.3.12)$$

where

$$\eta = \left(\frac{(m+1+2j)U_0}{2\nu_s x}\right)^{1/2} Y \qquad (2.3.13)$$

This expansion is reminiscent of Görtler's [38] expansion for incompressible flow, about the Falkner-Skan solution. The present series has been previously used by Sarma [9] and other investigators referenced therein and represents an extension to compressible flow.

Substituting Eqs. (2.3.11) and (2.3.12) in Eq. (2.3.3) and equating the coefficients of $c_0^{2n} a_w^{-2n} x^{2mn+m-1}$ on both sides for successive integral values of n, we get

$$f_1''' = - f_1 f_1'' - \frac{2m}{m+1+2j}\left(1 - (f_1')^2\right) \qquad (2.3.14)$$

$$\begin{aligned} f_{1+2n}''' = {} & 2\sum_{a=0}^{n} \frac{(2ma+m)}{(m+1+2j)} f_{1+2a}' f_{1+2n-2a}' - \sum_{a=0}^{n} \left(\frac{4ma+m+1+2j}{m+1+2j}\right) \times \\ & f_{1+2a} f_{1+2n-2a}'' + \left(\frac{2m}{m+1+2j}\right) \sum_{d=0}^{n-1} \left(\frac{\gamma-1}{2}\right)^{n-d} \left(\sum_{a=0}^{d} f_{1+2a}' f_{1+2(d-a)}'\right) \\ & + \gamma\left(\frac{m}{m+1+2j}\right) \sum_{d=0}^{n-1} \left(\frac{\gamma-1}{2}\right)^{n-d-1} \left(\sum_{a=0}^{d} f_{1+2a} f_{1+2(d-a)}''\right) \\ & - \left(\frac{2m}{m+1+2j}\right)\left(\frac{\gamma-1}{2}\right)^{n-1} \qquad \text{for } n \geq 1 \end{aligned} \qquad (2.3.15)$$

The corresponding boundary conditions are

$$f_{1+2n}(0) = 0, \ f_{1+2n}(0) = 0 \tag{2.3.16}$$

$$f_1'(\eta) \rightarrow 1 \quad \text{as } \eta \rightarrow \infty \tag{2.3.17}$$

$$f_{1+2n}'(\eta) \rightarrow 0 \quad \text{as } \eta \rightarrow \infty \text{ for } n \geq 1 \tag{2.3.18}$$

It should be noticed here that the Mach number based on the speed of sound evaluated at the wall, $M = c_0 L^m / a_w$ with L a typical length, does not appear in the differential equations for the functions f_{1+2n}. The Mach number effect will be introduced only when the series represented by Eq. (2.3.12) is calculated. Thus, the present method of solution is not restricted with respect to specific values of this parameter.

2.4 The Unsteady Part of the Motion for Oscillating Outer Flows

Let the unsteady part of the outer flow in Eq. (2.3.1) be given by

$$U_1(x,t) = c_1 x^{\ell} e^{i\omega t} \tag{2.4.1}$$

where c_1 and ℓ are constants. The solution to Eq. (2.3.7) for small frequencies will be again sought in the form of a series expansion

$$\psi_1(x,Y,t) = \left[\frac{2\nu_s x}{(m+1+2j)U_0}\right]^{1/2} c_1 \sum_{n=0}^{\infty} \sum_{q=0}^{\infty} \frac{c_0^{2n}}{a_w^{2n}} x^{\{2mn+\ell+(1-m)q\}} \times$$

$$\left(\frac{i\omega}{c_0}\right)^q e^{i\omega t} F_q^{2n)}(\eta) \tag{2.4.2}$$

The differential equations that govern the functions $F_q^{(2n)}(\eta)$ can be derived by substituting Eqs. (2.4.1) and (2.4.2) along with Eqs.

(2.3.11) and (2.3.12) in Eq. (2.3.7) and collecting coefficients of $\omega^q x^{2m+\ell+(1-m)(q-1)}$ for consecutive values of the integers n and q. The differential equation for $F_q^{(0)}$ is given below

$$F_q'''^{(0)} = -f_1 F_q''^{(0)} - \frac{2\ell+(1-m)(2q+1)+2j}{m+1+2j} f_1'' F_q^{(0)}$$
$$+ \frac{2[m+\ell+(1-m)q]}{m+1+2j} f_1' F_q'^{(0)} - \frac{2}{m+1+2j} \delta_{1q} - \frac{2(m+\ell)}{m+1+2j} \delta_{0q}$$
$$+ \frac{2}{m+1+2j} F_{q-1}'^{(0)} \tag{2.4.3}$$

where $F_{-1}'^{(n)}(\eta) \equiv 0$. The general form of the differential equation for $F_q^{(2n)}$, for $n \geq 1$ is given in the Appendix A [Eqs. (2.A1)]. The appropriate boundary conditions are

$$F_0^{(0)}(0) = 0, \; F_0'^{(0)}(0) = 0 \tag{2.4.4}$$

$$F_0'^{(0)}(\eta) \to 1 \text{ as } \eta \to \infty \tag{2.4.5}$$

and

$$F_q^{(2n)}(0) = F_q'^{(2n)}(0) = F_q'^{(2n)}(\infty) = 0 \text{ for } n \geq 1 \tag{2.4.6}$$

Equation (2.4.3) is governing the unsteady part of incompressible flow. This equation along with the differential equations given in Appendix A [Eqs. (2.A1)] are solved numerically. Some results are shown in Figures 2.3 and 2.4 for different wedge and cone angles, respectively, for $\ell = 0$. This corresponds to an outer flow disturbance that is independent of space. It could be interpreted as a fluctuation of the angle of attack of a wedge in supersonic flow. Note that for flow past a cone or a wedge, the included angle β is given by

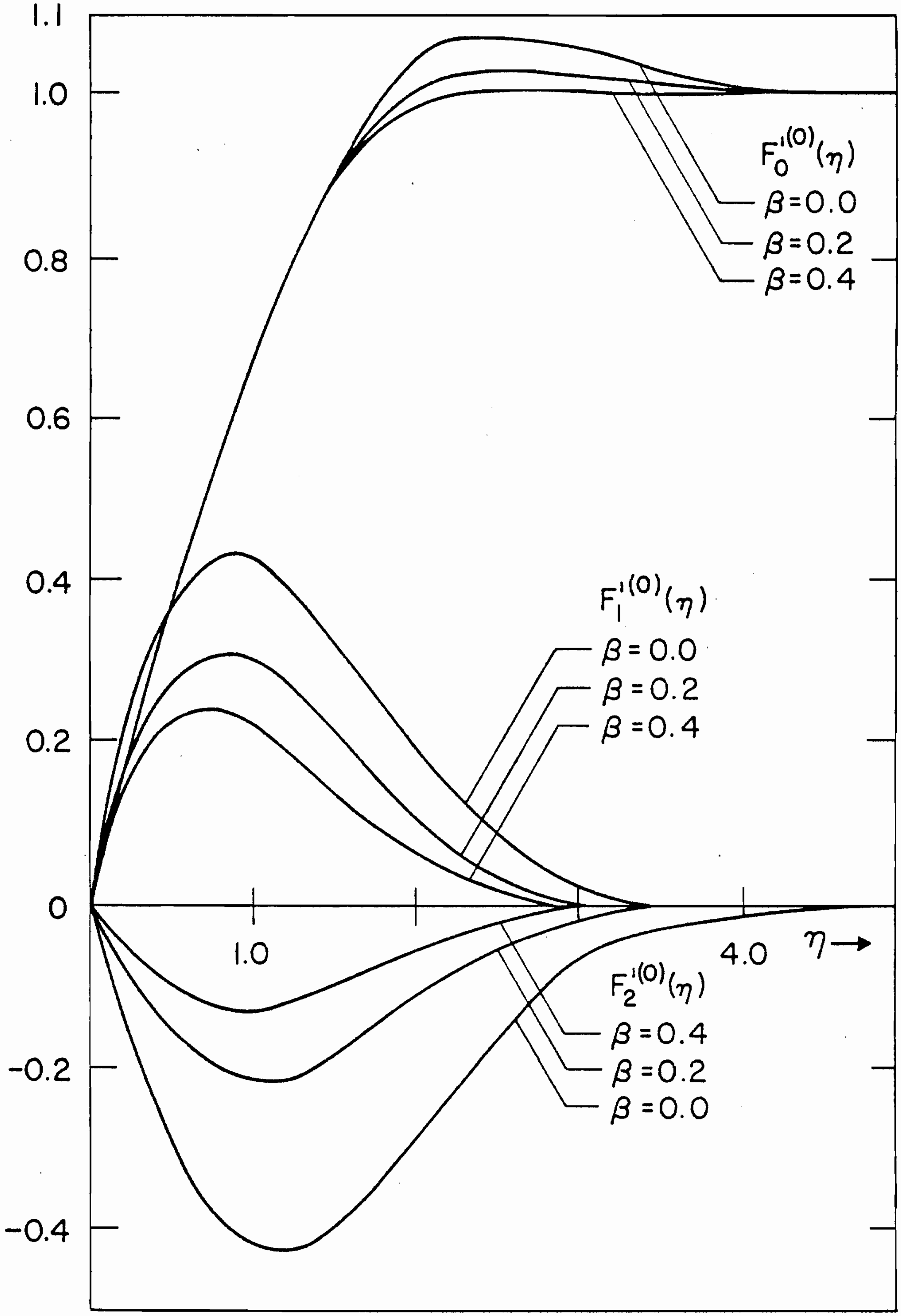

Figure 2.3 The functions $F_0'^{(0)}(\eta)$, $F_1'^{(0)}(\eta)$, $F_2'^{(0)}(\eta)$ employed in calculating boundary layer flow over a wedge for M = 0.

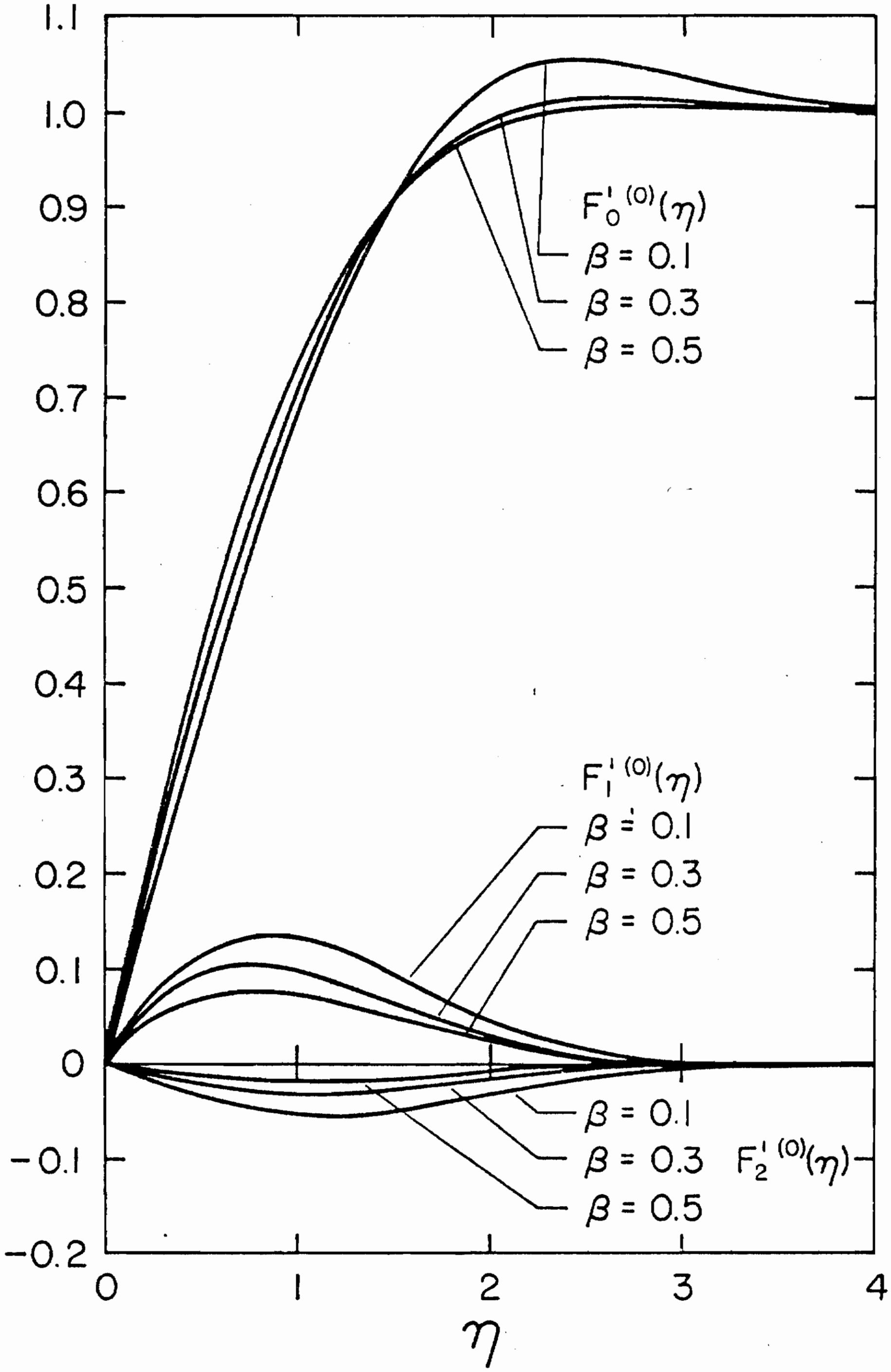

Figure 2.4 The functions $F_0'^{(0)}(\eta)$, $F_1'^{(0)}(\eta)$, $F_2'^{(0)}(\eta)$ employed in calculating the boundary layer flow over a cone for M = 0.

$\beta = \frac{2m}{m+3}$ and $\frac{2m}{m+1}$ respectively.

The steady and unsteady velocity distributions are

$$\frac{u_0}{U_0} = \sum_{n=0}^{\infty} \frac{c_0^{2n}}{a_w^{2n}} \; x^{2mn} f'_{1+2n}(\eta) \tag{2.4.7}$$

$$\frac{u_1}{U_1} = \Big[\sum_{n=0}^{\infty} \frac{c_0^{2n}}{a_w^{2n}} \; x^{2mn} \Big\{ F_0'^{(2n)}(\eta) + \frac{i\omega x}{U_0} F_2'^{(2n)}(\eta)$$

$$+ \left(\frac{i\omega x}{U_0}\right)^2 F_2'^{(2n)}(\eta) + \dots \Big\} \Big] \tag{2.4.8}$$

The true normal coordinate y from Eq. (2.2.11) is given by

$$y = \left(\frac{p_s}{p}\right)^{1/2} \int_0^Y \frac{T}{T_s} \, dY \tag{2.4.9}$$

where the pressure ratio is

$$\frac{p}{p_s} = \frac{\rho_e T_e}{\rho_s T_s} \tag{2.4.10}$$

The displacement thickness can be written as

$$\delta^* = \int_0^\infty \left(1 - \frac{\rho u}{\rho_e U_e}\right) dy = \left(\frac{p_s}{p}\right)^{1/2} \int_0^\infty \left(\frac{T}{T_s} - \frac{T_e}{T_s}\frac{u}{U_e}\right) dY \tag{2.4.11}$$

The unsteady part of the skin friction coefficient now is

$$\epsilon_{fs} = \frac{\tau_1\sqrt{\frac{U_0 x}{\nu_s}}}{\rho_s U_0 U_1} = (\frac{p}{p_s})^{1/2}(\frac{m+1+2j}{2})^{1/2} \Big[\sum_{n=0}^{\infty} \frac{c_0^{2n}}{a_w^{2n}} x^{2mn} F_0'^{(2n)}(0)$$

$$+ (\frac{i\omega x}{U_0}) \sum_{n=0}^{\infty} \frac{c_0^{2n}}{a_w^{2n}} x^{2mn} F_1'^{(2n)}(0)$$

$$+ (\frac{i\omega x}{U_0})^2 \sum_{n=0}^{\infty} \frac{c_0^{2n}}{a_w^{2n}} x^{2mn} F_2''^{(2n)}(0) + \ldots \Big] \qquad (2.4.12)$$

and the rate of heat transfer per unit surface area becomes

$$Q = -\left(\frac{\mu c_p}{P_r} \frac{\partial T}{\partial y}\right)_{y=0} = 0 \qquad (2.4.13)$$

Equation (2.4.13) implies that there is no heat transfer for steady as well as for unsteady flow. That is, the body is insulted. In other words, this solution represents an unsteady disturbance on a compressible flow over a wall which is given the adiabatic temperature. Our results show that to order ε, the wall remains insulated at all times. Results are discussed in Section 2.6.

2.5. The Unsteady Part of the Motion for Bodies Oscillating in a Steady Stream

Consier now the case $U_1(x,t) = 0$. The outer flow velocity and temperature are then independent of time and temperature varies according to

$$T_e + \frac{U_e^2}{2C_p} = T_s \qquad (2.5.1)$$

where, T_s, is the constant stagnation temperature. The pressure is independent of time throughout the whole flow field. Equations (2.2.4) and (2.2.8) further reduce to

$$- \frac{\partial p}{\partial x} = \rho_e U_0 \frac{dU_0}{dx} \tag{2.5.2}$$

$$\frac{T}{T_e} = 1 + \frac{\gamma-1}{2a_e^2} (U_0^2 - u^2) \tag{2.5.3}$$

The equation (2.3.7) describing the unsteady part of the stream function can be simplified in turn

$$\begin{aligned} &\frac{\partial^2\psi_1}{\partial t\partial y} + \frac{\partial\psi_0}{\partial Y}\frac{\partial^2\psi_1}{\partial x\partial Y} + \frac{\partial\psi_1}{\partial Y}\frac{\partial^2\psi_0}{\partial x\partial Y} - \left(\frac{\partial\psi_0}{\partial x} + \frac{j}{r}\frac{dr}{dx}\psi_0\right)\frac{\partial^2\psi_1}{\partial Y^2} \\ &\quad - \left(\frac{\partial\psi_1}{\partial x} + \frac{j}{r}\frac{dr}{dx}\psi_1\right)\frac{\partial^2\psi_1}{\partial Y^2} = - a_w^{-2}\left(1 - \frac{\gamma-1}{2a_w^2}U_0^2\right)^{-1} \times \\ &\quad \left[(\gamma-1)\frac{\partial\psi_0}{\partial Y}\frac{\partial\psi_1}{\partial Y} + \gamma\left(\psi_0\frac{\partial^2\psi_1}{\partial Y^2} + \psi_1\frac{\partial^2\psi_0}{\partial Y^2}\right)\right]U_0\frac{dU_0}{dx} \\ &\quad + \nu_s\frac{\partial^3\psi_1}{\partial Y^3} \end{aligned} \tag{2.5.4}$$

and the appropriate boundary conditions are

$$\psi_1 = 0, \ \frac{\partial\psi_1}{\partial Y} = u_b(t) \ \text{at} \ Y = 0 \tag{2.5.5}$$

$$\frac{\partial\psi_1}{\partial Y} = 0 \quad \text{as} \quad Y \to \infty \tag{2.5.6}$$

where $u_b(t)$ may be an arbitrary function of time. It should be emphasized here that the wall temperature fluctuates and as a result a_w, are constant only to order ε^2. To carry this analysis to a higher order for oscillating walls, one would have to retain a_e in the forcing terms of Eq. (2.5.3). In the present case we will assume $u_b(t) = \varepsilon e^{i\omega t}$ and seek a solution in the form

$$\psi_1 = \left(\frac{2\nu_s x}{(m+1+2j)U_0}\right)^{1/2} \sum_{n=0}^{\infty} \sum_{q=0}^{\infty} \frac{c_0^{2n}}{a_w^{2n}} \; x^{\{2mn+(1-m)q\}} \left(\frac{i\omega}{c_0}\right)^q e^{i\omega t} \phi_q^{(2n)}(\eta) \tag{2.5.7}$$

Substituting Eq. (2.5.7) in (2.5.4) along with Eqs. (2.3.11) and (2.3.12) and equating the coefficients $\omega^q c_0^{2n} a_w^{-2n} \; x^{\{2mn+(1-m)(q-1)\}}$ on both sides, for consecutive values of n and q, we get the following equation for $\phi_q^{(0)}(\eta)$.

$$\phi_q'''^{(0)} = - f_1 \phi_q''^{(0)} - \left(\frac{(1-m)(2q+1)+2j}{m+1+2j}\right) f'' \phi_q^{(0)}$$
$$+ \frac{2[m+(1-m)q]}{(m+1+2j)} f_1' \phi_q'^{(0)} + \left(\frac{2}{m+1+2j}\right) \phi'^{(0)}_{q-1} \tag{2.5.8}$$

where $\phi_{-1}'^{(2n)}(\eta) = 0$.

The general form of the differential equation for the functions $\phi_q^{(2n)}$ for $n \geq 1$ is given in Appendix A [Eqs. (2.A2)].

The boundary conditions for n = 0 are

$$\phi_0'^{(0)}(0) = 1, \; \phi_0^{(0)}(0) = 0 \tag{2.5.9}$$

$$\phi_0'^{(0)}(\eta) \to 0 \text{ as } \eta \to \infty \tag{2.5.10}$$

and for $n \geq 1$

$$\phi_q^{(2n)}(0) = 0, \; \phi_q'^{(2n)}(0) = 0 \tag{2.5.11}$$

$$\phi_q'^{(2n)}(\eta) \to 0 \text{ as } \eta \to \infty \tag{2.5.12}$$

Thus, the unsteady compressible boundary-layer equations are reduced to ordinary differential equations. Notice that Eq. (2.5.8) is the differential equation for unsteady incompressible flow. This equation along with other equations given in Appendix A [Eqs. (2.A2)] are solved numerically. The first three functions representing the unsteady part of velocity when the wedge and the cone are oscillating about a steady mean are shown in Figs. 2.5 and 2.6 respectively, for a few values of the pressure gradient parameter β.

The next term in the expansion that captures the compressibility effects is represented by the functions $\phi_q^{(2n)}$ for n = 1. Numerical results are plotted in Figs. 2.7 and 2.8 for a wedge and a cone, respectively.

The unsteady velocity distribution is then given by the series

$$\frac{u_1}{u_b} = \Big[\sum_{n=0}^{\infty} \frac{c_0^{2n}}{a_w^{2n}} x^{2mn} \Big\{ \phi_0'^{(2n)}(\eta) + \left(\frac{i\omega x}{U_0}\right)\phi_1'^{(2n)}(\eta)$$

$$+ \left(\frac{i\omega x}{U_0}\right)^2 \phi_2'^{(2n)}(\eta) + \dots \Big\} \Big] \qquad (2.5.13)$$

The pressure ratio (p/p_s) is given by Eq. (2.4.10), and is the same for both steady and unsteady flows in this case. From Eq. (2.2.11), we define the boundary layer thickness δ as

$$\delta = \left(\frac{p_s}{p}\right)^{1/2} \int_0^{\infty} \frac{T}{T_s}\, dY = \delta_0 - \left(\frac{p_s}{p}\right)^{1/2} \frac{\gamma-1}{a_s^2} \int_0^{\infty} \frac{\partial\psi_0}{\partial Y}\frac{\partial\psi_1}{\partial Y}\, dY \qquad (2.5.14)$$

where $\delta_0 = \left(\frac{p_s}{p}\right)^{1/2} \int_0^{\infty} [1 + \frac{\gamma-1}{2a_s^2} \{u_s^2 - (\frac{\partial\psi_0}{\partial Y})^2\}] dY$ is its steady component.

The skin friction coefficient is given by the following series:

$$C_{fb} = \frac{\tau_1\sqrt{\frac{U_0 s}{\nu_s}}}{\rho_s U_0 u_b} = \left(\frac{p}{p_s}\right)^{1/2}\left(\frac{m+1+2j}{2}\right)^{1/2}\Big[\sum_{n=0}^{\infty} \frac{c_0^{2n} x^{2mn}}{a_w^{2n}} \Big\{ \phi_0''^{(2n)}(0)$$

$$+ i\left(\frac{\omega x}{U_0}\right)\phi_1''^{(2n)}(0) - \left(\frac{\omega x}{U_0}\right)^2 \phi_2''^{(2n)}(0) + \dots \Big\}\Big] \qquad (2.5.15)$$

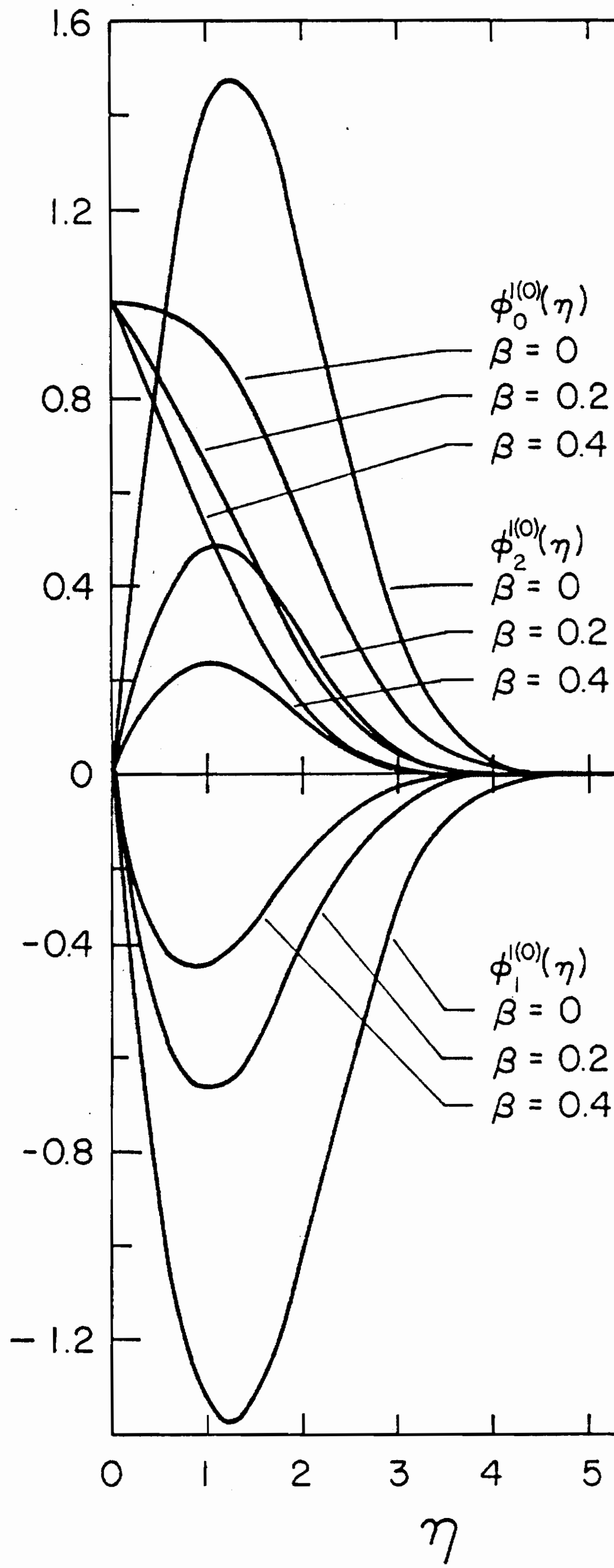

Figure 2.5 The functions $\phi_0'^{(0)}(\eta)$, $\phi_1'^{(0)}(\eta)$, $\phi_2'^{(0)}(\eta)$ employed in calculating the boundary layer flow over an oscillating wedge when M = 0.

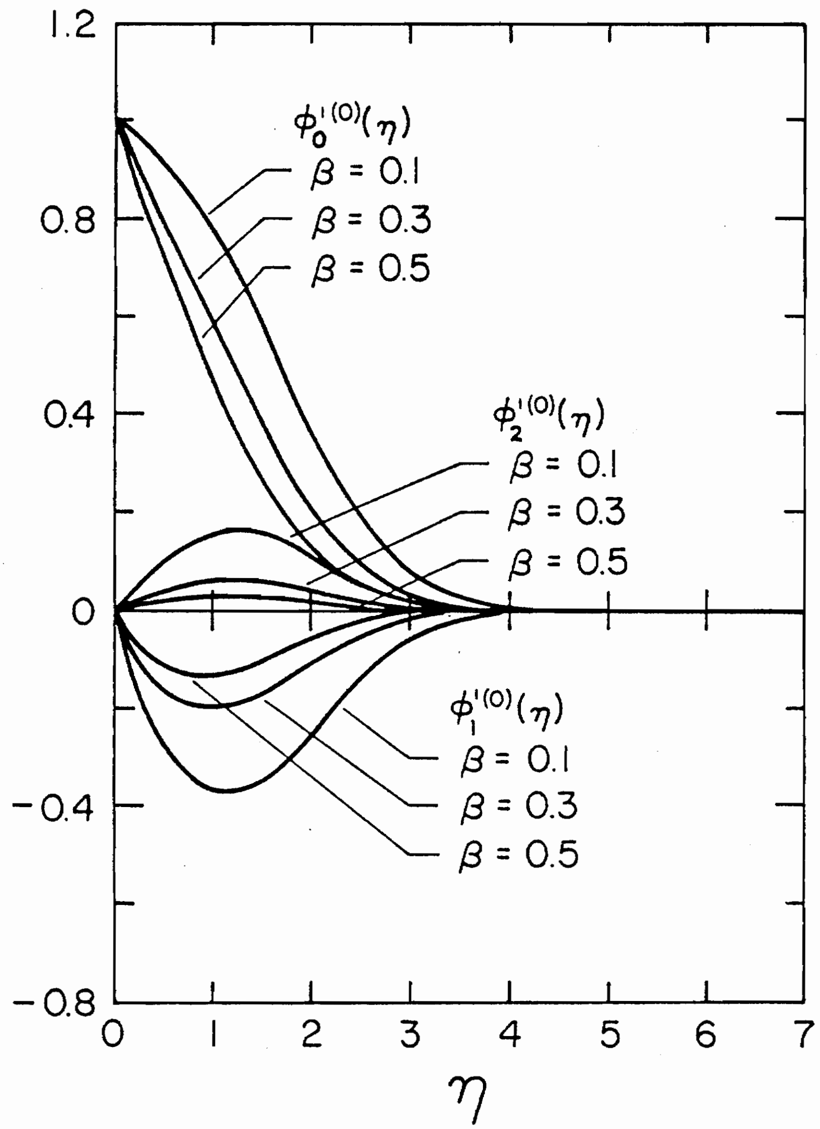

Figure 2.6 The functions $\phi_0'^{(0)}(\eta)$, $\phi_1'^{(0)}(\eta)$, $\phi_2'^{(0)}(\eta)$ employed in calculating the boundary layer flow over an oscillating cone when M = 0.

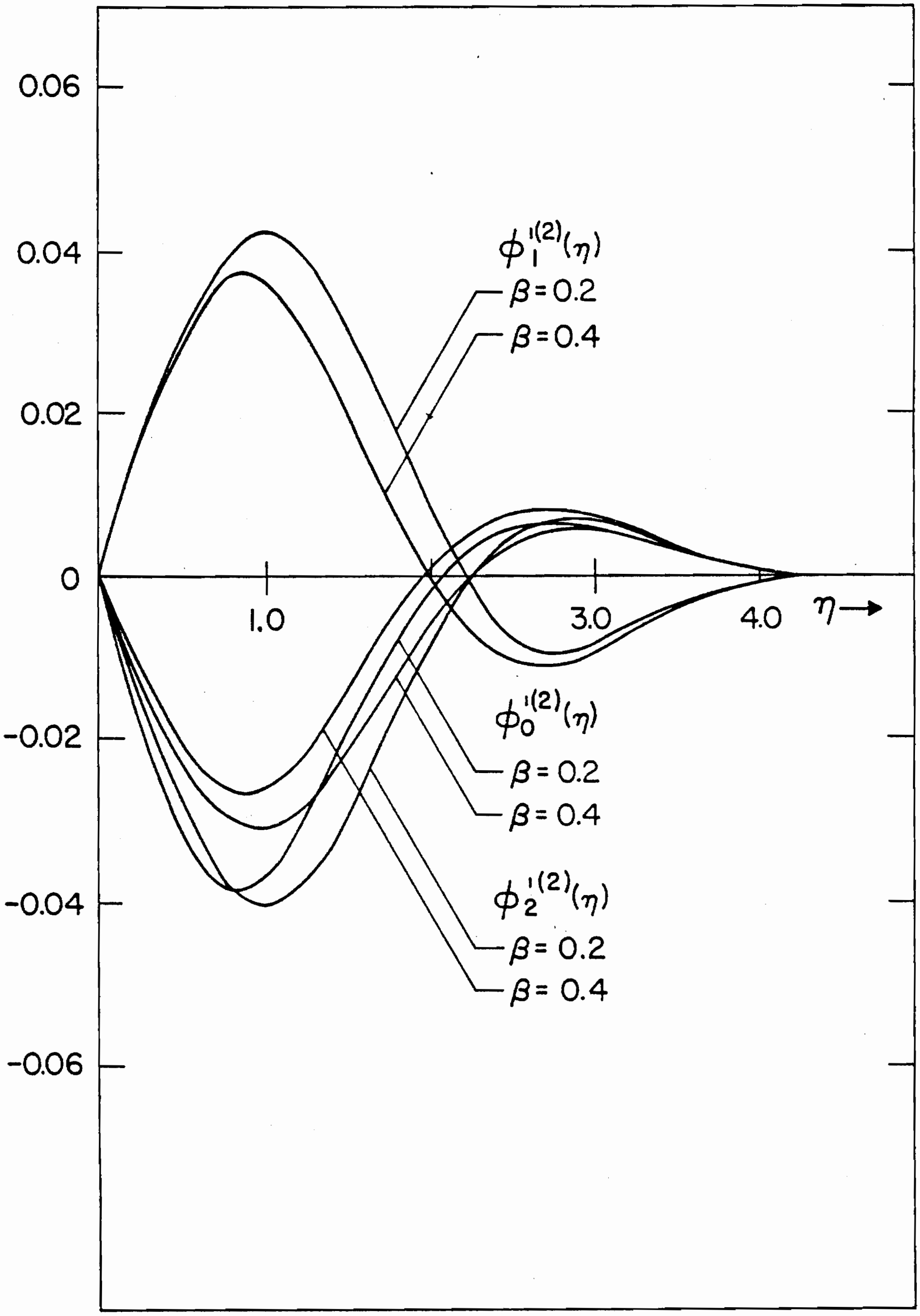

Figure 2.7 The functions $\phi_0^{\prime(2)}(\eta)$, $\phi_1^{\prime(2)}(\eta)$, $\phi_2^{\prime(2)}(\eta)$ employed in calculating the boundary layer flow over an oscillating wedge when the compressible effects are also taken into account.

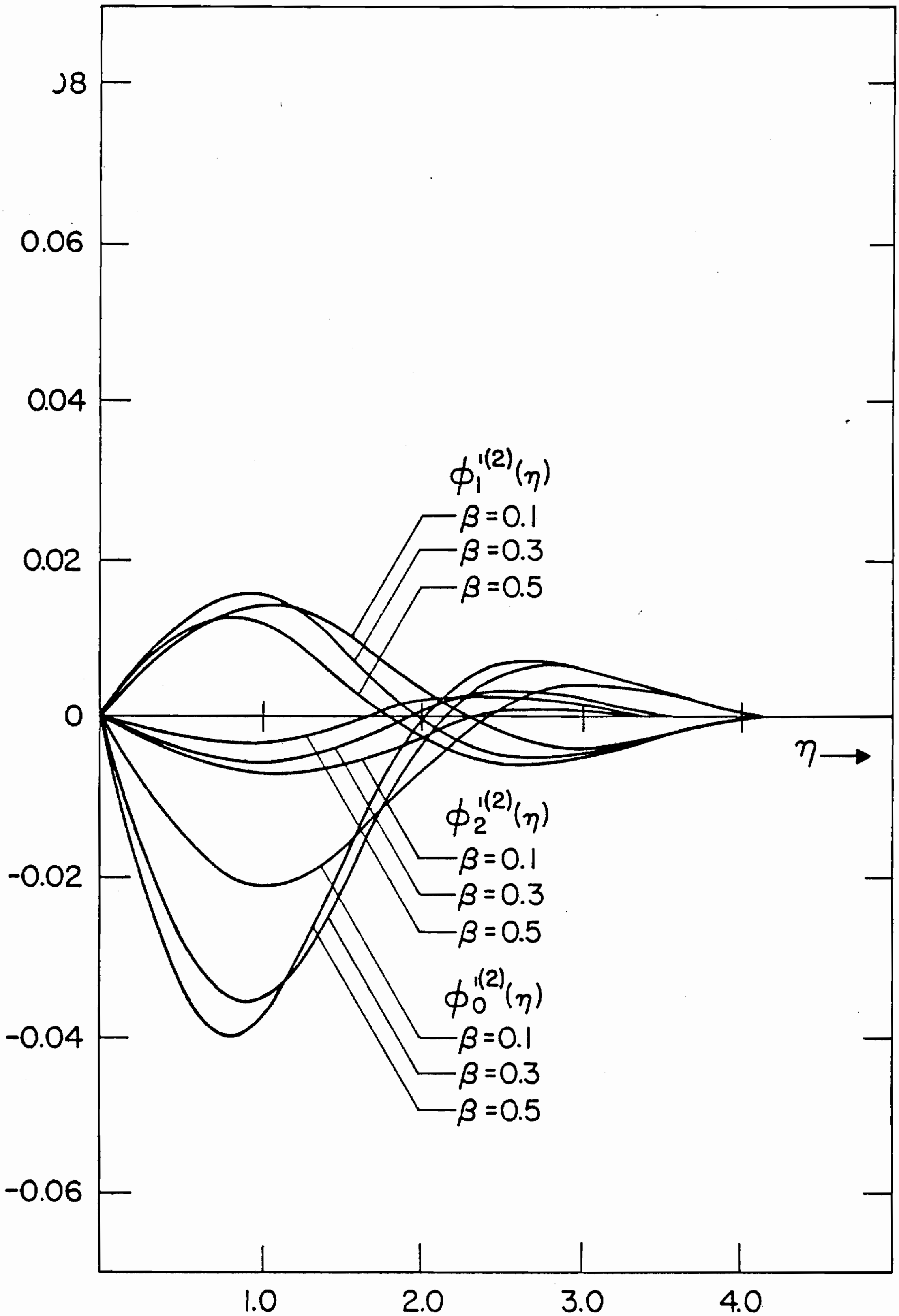

Figure 2.8 The functions $\phi_0'^{(2)}(\eta)$, $\phi_1'^{(2)}(\eta)$, $\phi_2'^{(2)}(\eta)$ employed in calculating the boundary layer over an oscillating cone when the compressible effects are also taken into account.

The rate of heat transfer per unit area from the wall to the fluid is given by Eq. (2.4.13) and in this case

$$Q = \varepsilon\tau_0 u_b \tag{2.5.16}$$

where τ_0 is the skin friction in steady flow.

2.6. Results and Conclusion

In this chapter, the numerical results are presented in terms of dimensionless functions that represent terms in the series expansions. Parameters like the wall temperature or the Mach number are contained in the coefficients of our expansions. It is therefore possible to use the present information in order to generate quickly solutions about two-dimensional or axisymmetric configurations for a wide variety of parameters.

We consider two different types of fluctuations with small frequencies. In Section 2.4 we consider oscillating outer flows and in Section 2.5 we study bodies oscillating in a steady stream.

From Section 2.4, it is interesting to note that when n = 0, i.e. when the Mach number is zero, the problem reduces to an incompressible flow problem. Figure 2.9 shows the behaviour of the amplitude of unsteady velocity profile when M = 0 and $wx/U_0 = 0.4$, an incompressible case for a wedge. The familiar overshoot, characteristic of the oscillatory component seems to be suppressed when the pressure gradient increases. Compressibility effects of the oscillatory part of the velocity can be captured in the expansion of Eq. (2.4.2), if the terms $F_q^{(2)}$ are included. To this end Eq. (2.A1) in Appendix A is integrated

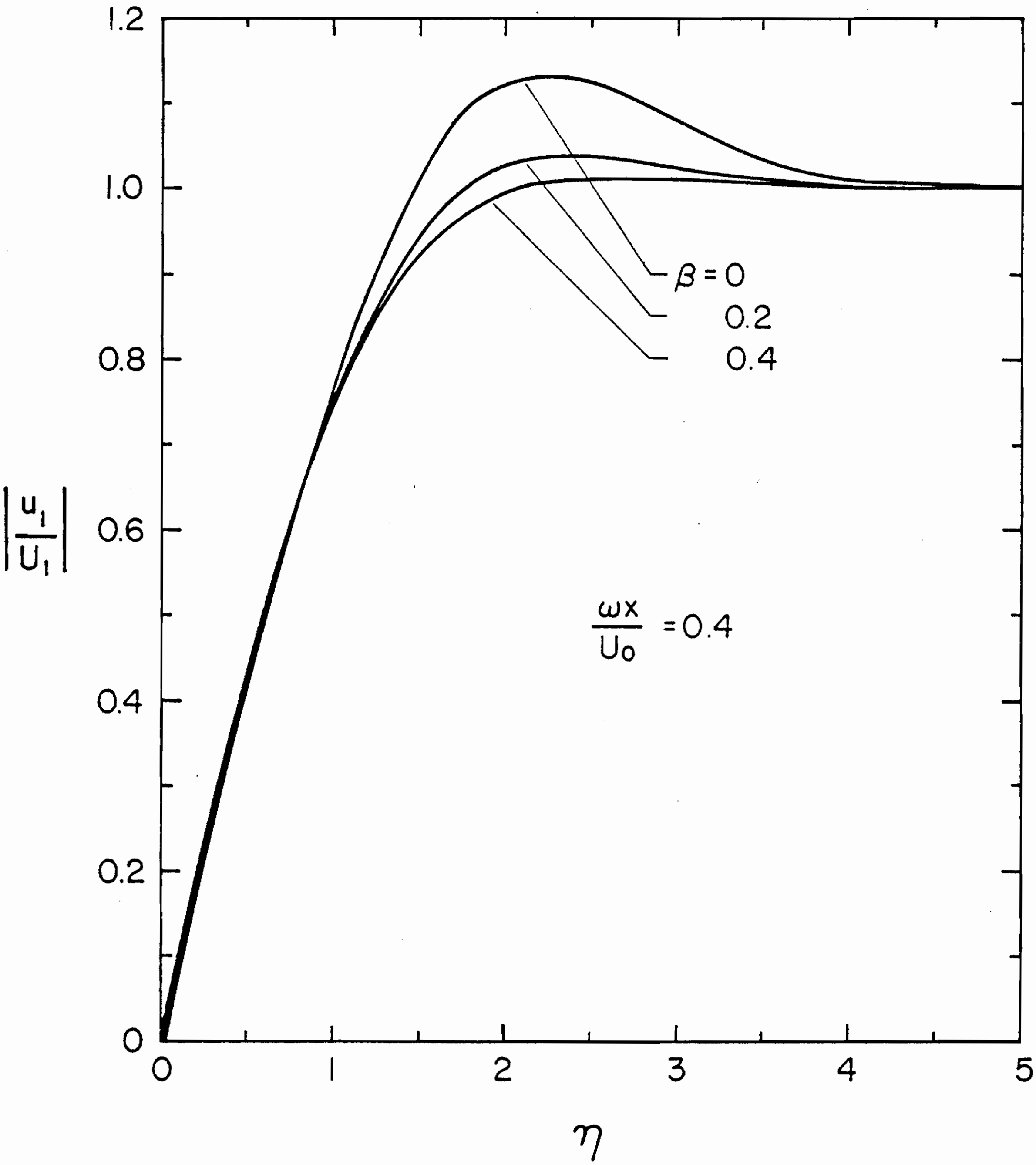

Figure 2.9 The amplitude profile of the unsteady velocity component for flow past a wedge when M = 0.

numerically for n = 1 and results are presented in Figs. 2.10 and 2.11 for a wedge and a cone respectively. It appears from the figures that the well-known three dimensional effect of compressible flows pertain to the fluctuating components of the velocity as well. That is, the amplitudes of oscillation are stronger for a two dimensional than for an axisymmetric configuration. Moreover, the wedge permits much higher phase angles than the cone. This can be seen from the relative magnitude of the functions $F_1^{(2)}$ and $F_0^{(2)}$. Large phase advances or delays can be quickly identified by observing the function $F_0^{(2)}$ and it can be seen in Fig. 2.10 that large phase values should be expected for sharp wedges, a fact that will be presented clearly in a figure that follows.

In Fig. 2.12, we have plotted the amplitude and phase advance of the skin friction coefficient versus the frequency parameter wx/U_0 for a wedge. The increase of the phase advance with frequency parameter appears to be strongly affected by the pressure gradient parameter. Stronger pressure gradients give smaller phase angles. The Mach number though, seems to have a rather mild effect which implies that the incompressible unsteady correction on the compressible field may be a good approximation. The reader should recall that the zeroth order equations, that is the equations for the f_{1+2n}'s are solved for compressible flow and their solutions appear as coefficients or forcing functions in the equations for the $F_q^{(2n)}$'s. It should be emphasized here that the parameter $M = cL^m/a_w$ has the physical meaning of a Mach number but is not related to the velocity of the free stream. Moreover the results presented in Fig. 2.10 for M = 0.8 hold for a variety of locations on wedges of different included angles, wall temperatures and free stream Mach numbers.

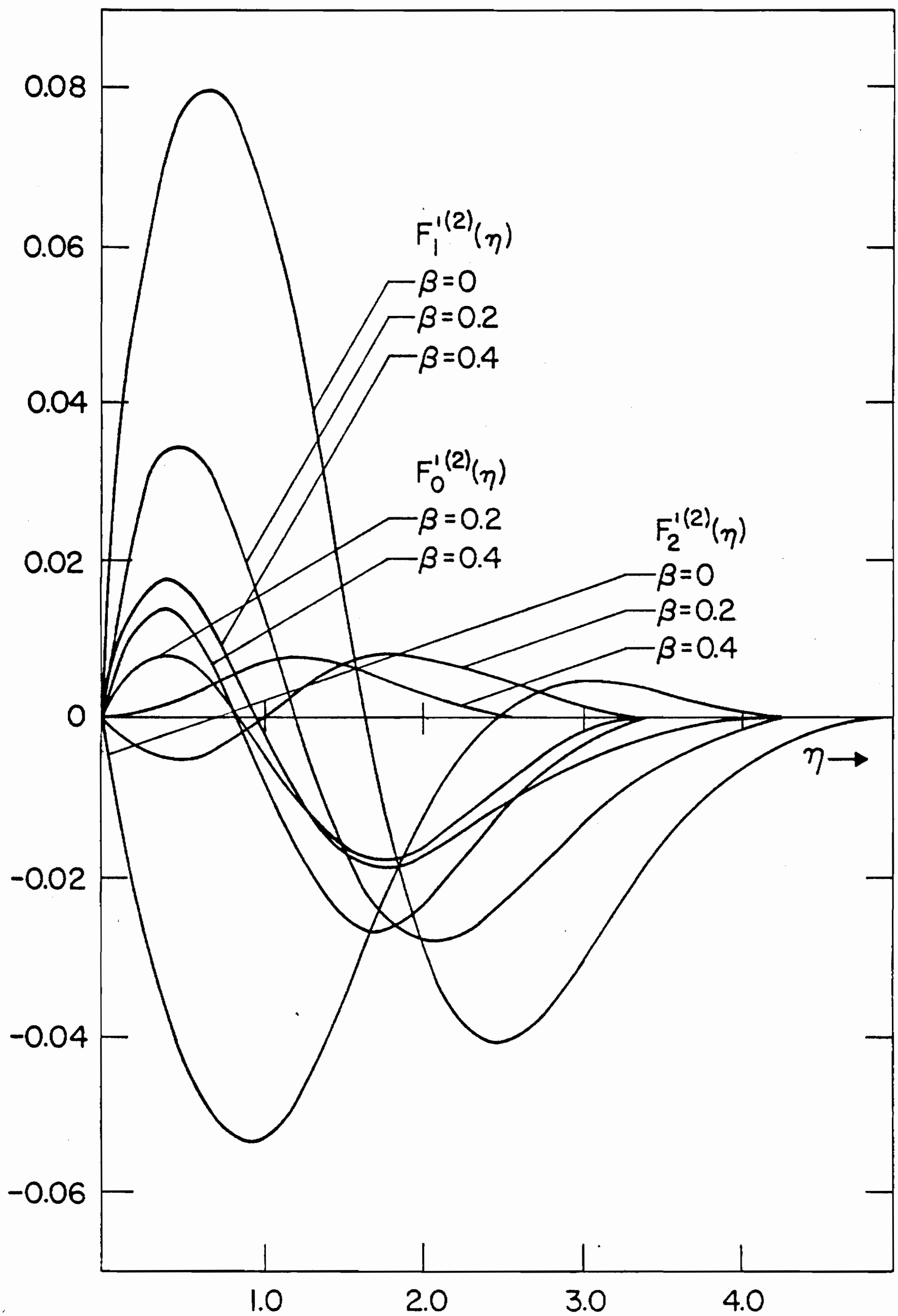

Figure 2.10 The functions $F_0'^{(2)}$, $F_1'^{(2)}$, and $F_2'^{(2)}$ employed in the compressibility correction for a boundary layer flow over a wedge.

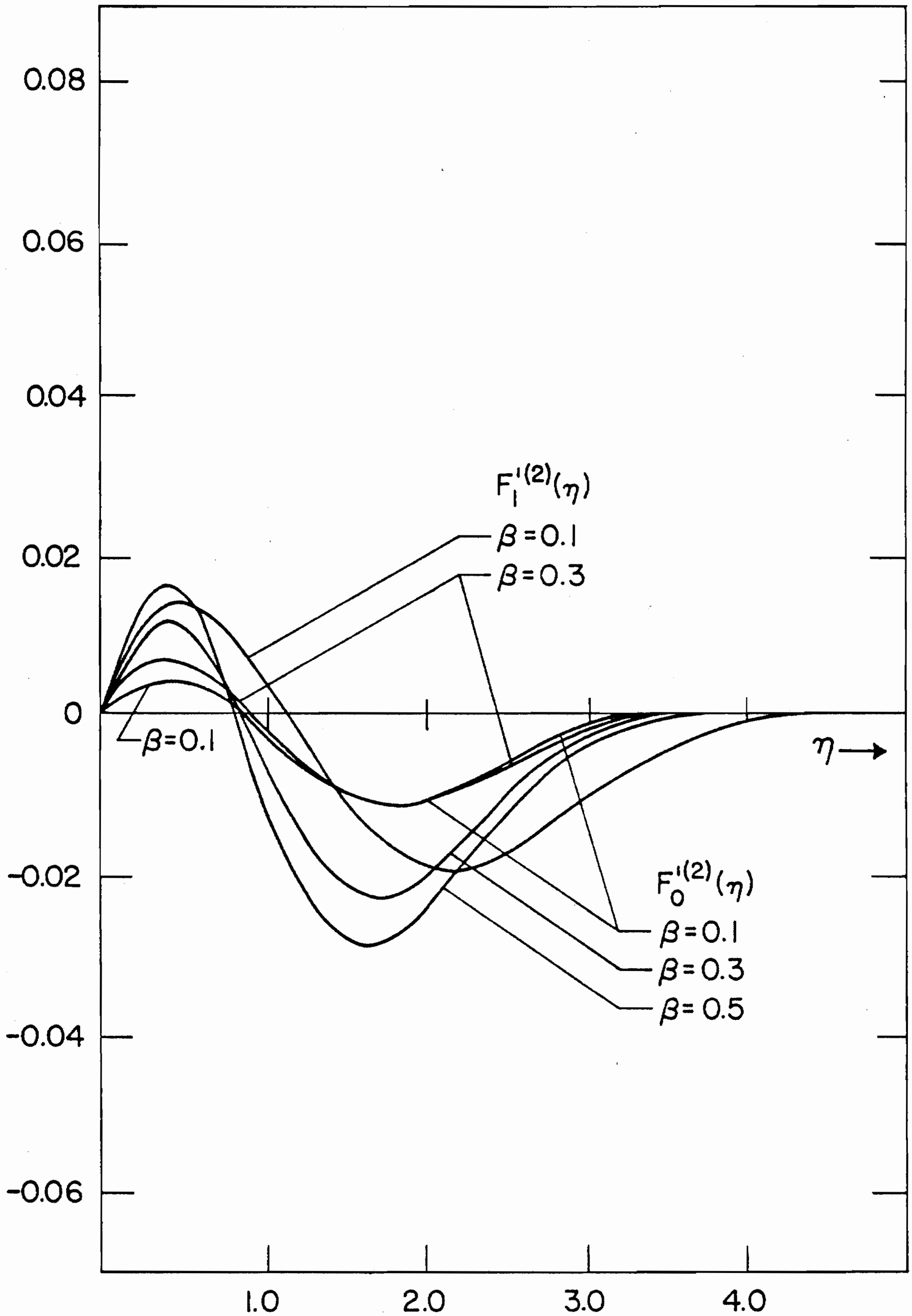

Figure 2.11 The functions $F_0^{\prime(2)}$, $F_1^{\prime(2)}$, and $F_2^{\prime(2)}$ employed in the compressibility correction for a boundary layer flow over a cone.

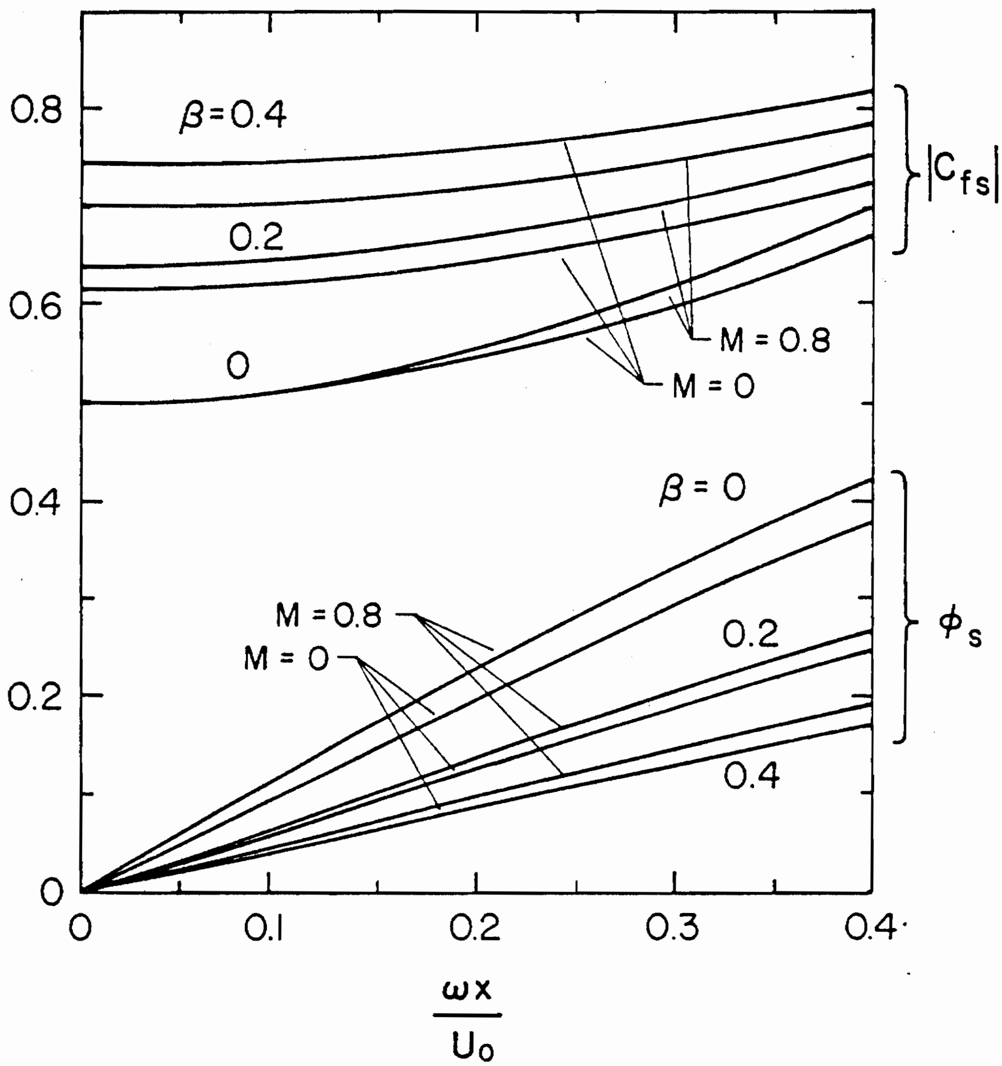

Figure 2.12 Amplitude of skin friction coefficient and its phase angle for a wedge.

Finally it should be noticed from Section 2.5 that Eq. (2.5.3) implies that the heat transfer at the wall is zero only if u vanishes at the wall. If $u = \varepsilon u_b(t)$, $(u_b(t) = e^{i\omega t})$ at the wall, then Eq. (2.5.3) represents a temperature distribution with a fluctuating wall value:

$$\frac{T_a}{T_e} = 1 + \frac{\gamma-1}{2a_e^2}\,(U_0^2 - \varepsilon^2 u_b^2)$$

However, notice that the average of the above expression is equal to the adiabatic temperature $T_a = 1 + \frac{\gamma-1}{2a_e^2}\,U_0^2$ and further that the oscillating part is of order ε^2. Therefore to order ε we can assume that the wall temperature and speed of sound a_w are constants. Physically this problem corresponds to the following situation. Let us assume that the body has exactly the temperature T_a. Any temperature above T_a or below T_a would imply that heat would flow away or towards the body. If now small oscillations are imposed, the thermal balance is destroyed and we have a contribution to heat transfer equal to the expression given by Eq. (2.5.16).

In Figs. 2.13 and 2.14 we have plotted the amplitude and the phase of the skin friction against the frequency parameter for oscillating wedge skins. Once again it is observed that the Mach number does not affect significantly the results and therefore the incompressible unsteady correction on the compressible mean flow may be considered an acceptable approximation to the problem.

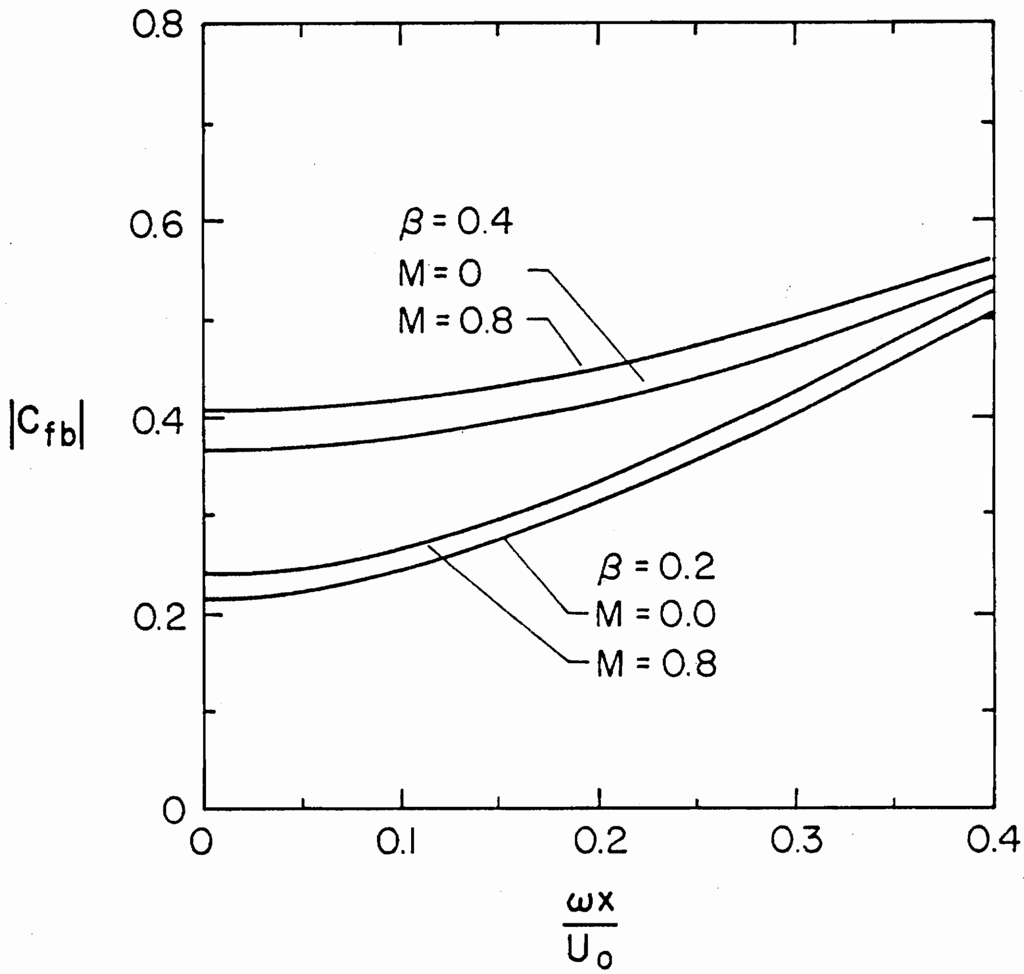

Figure 2.13 Amplitude of skin friction coefficient for oscillating wedge.

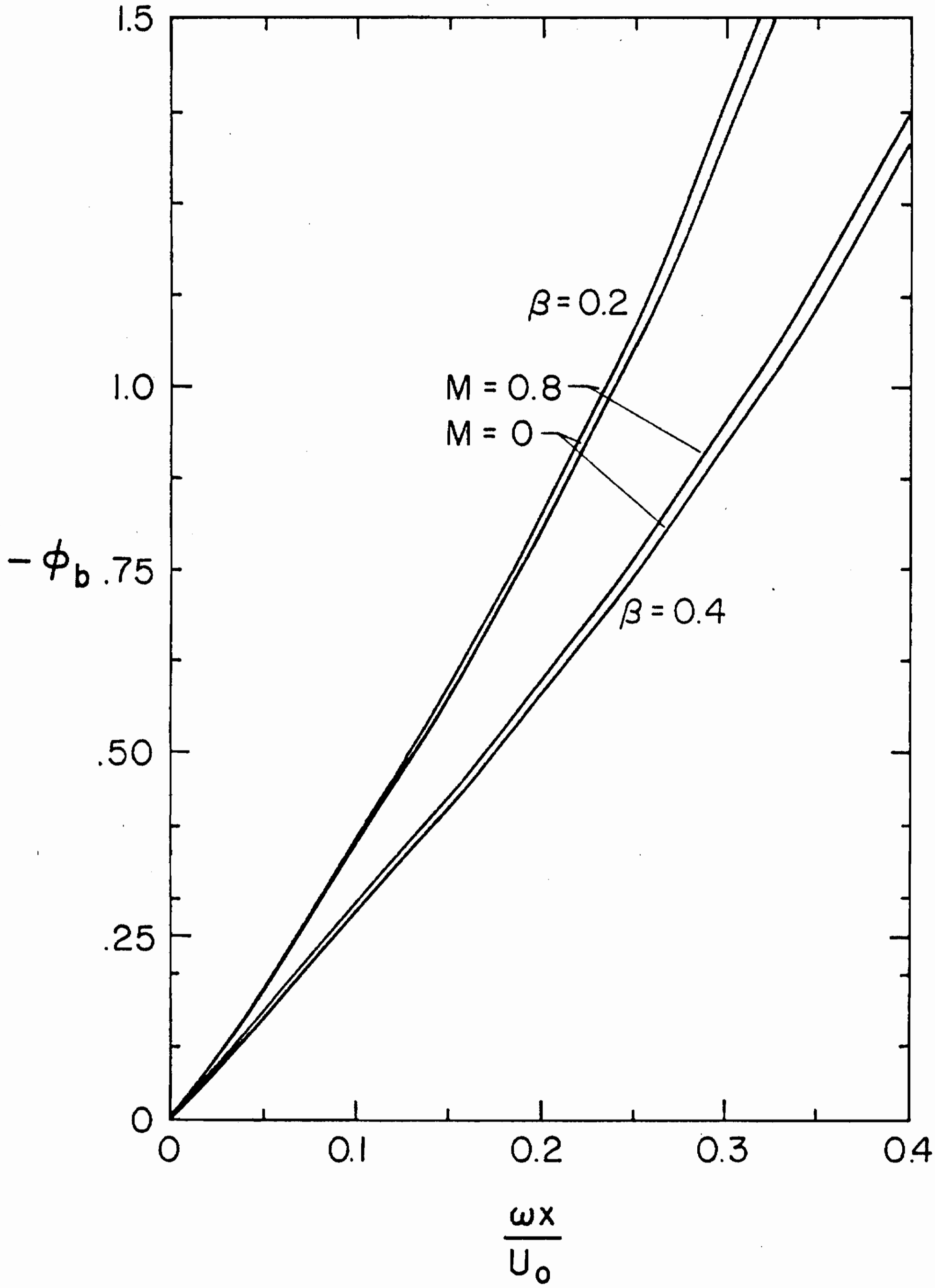

Figure 2.14 Phase angle for skin friction coefficient for oscillating wedge.

CHAPTER THREE

COMPRESSIBLE OSCILLATIONS OVER TWO-DIMENSIONAL AND AXISYMMETRIC WALLS WITH MEAN HEAT TRANSFER

3.1 Introduction

In Chapter Two we studied the fluctuating compressible flow over a wedge or a cone but for the very special case of a wall temperature equal to the adiabatic wall temperature. As mentioned earlier in the general introduction, this is the first attempt to study a flow with nonzero pressure gradients and purely unsteady flow conditions. The most striking feature of such flows is the fact that the outer flow enthalpy ceases to be constant and varies proportionally to the time derivative of the pressure. However in the problem treated in Chapter Two, the enthalpy variations are of one order of magnitude higher than the level of the terms retained. Moreover the simplifying assumptions are quite restrictive since the solution is valid for a conducting wall with a temperature equal to the adiabatic temperature. In the present chapter, we are concerned with compressible boundary layers, and we will concentrate our attention only on the basic difficulties encountered in unsteady compressible flows.

The problem is posed as follows. Given the outer flow fluctuations in the velocity, find the response of the boundary layer and in particular the response of the skin friction, heat transfer and perhaps the location of separation. Flows through turbomachinery blades and flows over rotating helicopter blades or about fluttering wing sections are some typical examples where the present theory will find application. In fact, it is well known that, in the cases mentioned above, compressibility is one of the most significant factors. However, very little research has been carried out along this direction, and until now, no one has attempted to solve the problem for a truly unsteady outer flow.

In this chapter, we study fluctuating flows over wedges and cones only, including the cases of flat plate and two dimensional or axisymmetric stagnation flow. The theoretical development here is not restricted with respect to geometry (as in Refs. [6,12,14,23], or with respect to the variation with time of the outer flow (as in Refs. [9,17]) or with respect to the temperature of the wall as in (Refs. [25,39]). The formulation is again based on the assumption of a small amplitude that is very realistic for a large number of practical application. The general form of the governing equations in the two-dimensional space are presented. These equations are readily available for numerical integration that requires very little storage space compared to the purely three-dimensional calculation. Such calculations have been successfully performed for the case of incompressible flow [18,20]. Instead we seek here solutions in terms of asymptotic expansions in powers of the frequency and the distance along the wall.

3.2 The Governing Equations

Two dimensional or axisymmetric laminar compressible boundary-layer flow is governed by the continuity equation, the momentum equation and the energy equation given by (2.2.1), (2.2.2) and (2.2.3) in Chapter Two.

The boundary conditions require, for a fixed wall with no suction

$$u = v = 0 \qquad \text{at} \qquad y = 0 \tag{3.2.1a}$$

$$T = T_w \qquad \text{at} \qquad y = 0 \tag{3.2.1b}$$

while at the edge of the boundary-layer the flow joins smoothly, within the boundary-layer approximation, with the outer flow

$$u \to U_e, \qquad T \to T_e \qquad \text{as} \qquad y \to \infty. \tag{3.2.1c}$$

where the subscript e denotes properties of the outer flow.

The outer flow distributions cannot be prescribed arbitrarily as in the case of incompressible flow. The outer flow pressure, velocity, density, and temperature distribution must be compatible and should satisfy Equations (2.2.4) and (2.2.5) and the equation of state, Eq. (2.2.6), given in Chapter Two. For incompressible flow, any distribution of the outer flow velocity corresponds to the evaluation at $y = 0$, that is the inner limit, of some potential flow. For compressible flow, one may choose arbitrarily the distribution of one of the quantities p, ρ_e, U_e, T_e. The other three quantities, however, should be calculated from the system of Eqs. (2.2.4), (2.2.5) and (2.2.6). One further comment: the reader should notice in Eq. (2.2.5) that for unsteady flow, the total enthalpy is not constant but fluctuates with the time derivative of the pressure.

Just as in Chapter Two, a function $\Psi(x,y,t)$ is introduced according to the relationships

$$u = \frac{1}{r^j}\frac{\rho_s}{\rho}\frac{\partial}{\partial y}(r^j\Psi) = \frac{\rho_s}{\rho}\frac{\partial\Psi}{\partial y} \qquad (3.2.2a)$$

$$v = -\frac{\rho_s}{\rho}\left(\frac{\partial\Psi}{\partial x} + \frac{1}{r^j}\Psi\frac{dr^j}{dx} + \frac{\partial}{\partial t}\int_0^y \frac{\rho}{\rho_s}\,dy\right) \qquad (3.2.2b)$$

where the subscript s denotes evaluation at some prescribed point of the flow. The function Ψ will be referred to in the sequel as the stream function, even though its values cannot be related directly to mass flow rates. The velocity components so defined, satisfy identically the continuity equation and thus the system is reduced by one equation and one unknown, since Ψ can now replace u and v. A familiar transformation is introduced again here

$$Y = \left(\frac{p}{p_s}\right)^{\frac{1}{2}}\int_0^y \frac{T_s}{T}\,dy \qquad (3.2.3)$$

It is found convenient to work with a modified stream function

$$\Psi(x,y,t) = (p/p_s)^{\frac{1}{2}}\psi(x,Y,t) \qquad (3.2.4)$$

which is related to the velocity components according to the relationships

$$u = \frac{\rho_s}{\rho}\frac{\partial\Psi}{\partial y} = \frac{\partial\psi}{\partial Y} \qquad (3.2.5)$$

$$v = -\frac{\rho_s}{\rho}\left(\frac{p}{p_s}\right)^{\frac{1}{2}}\left[\left(\frac{1}{2p}\frac{\partial p}{\partial x} + \frac{j}{r}\frac{dr}{dx}\right)\psi + \frac{\partial \psi}{\partial x} + \frac{\partial Y}{\partial x}\frac{\partial \psi}{\partial Y} + \left(\frac{\partial Y}{\partial t} + \frac{1}{2p}\frac{\partial p}{\partial t}Y\right)\right] \tag{3.2.6}$$

Finally it is assumed that viscosity varies linearly with temperature

$$\frac{\mu}{\mu_s} = \frac{T}{T_s} \tag{3.2.7}$$

Substituting the expressions given by Eqs. (3.2.5) and (3.2.6) in Eqs. (2.2.2) and (2.2.3) of Chapter Two and using Eqs. (2.2.4), (3.2.3), (3.2.4) and (3.2.7) we arrive at

$$\frac{\partial^2\psi}{\partial t\partial Y} + \frac{\partial \psi}{\partial Y}\frac{\partial^2\psi}{\partial x\partial Y} - \left(\frac{\partial \psi}{\partial x} + \psi\frac{j}{r}\frac{dr}{dx}\right)\frac{\partial^2\psi}{\partial Y^2}$$
$$= \frac{T}{T_e}\left(\frac{\partial U_e}{\partial t} + U_e\frac{\partial U_e}{\partial x}\right) - \frac{\gamma}{2a_e^2}\left(\frac{\partial U_e}{\partial t} + U_e\frac{\partial U_e}{\partial x}\right)\psi\frac{\partial^2\psi}{\partial Y^2} + \nu_s\frac{\partial^3\psi}{\partial Y^3} \tag{3.3.8}$$

$$\frac{\partial T}{\partial t} + \frac{\partial \psi}{\partial Y}\frac{\partial T}{\partial x} - \left(\frac{\partial \psi}{\partial x} + \psi\frac{j}{r}\frac{dr}{dx}\right)\frac{\partial T}{\partial Y}$$
$$= \frac{T}{T_e}\left(\frac{\partial}{\partial t} + U_e\frac{\partial}{\partial x}\right)\left(T_e + \frac{U_e^2}{2C_p}\right) - \frac{\gamma}{2a_e^2}\left(\frac{\partial U_e}{\partial t} + U_e\frac{\partial U_e}{\partial x}\right)\psi\frac{\partial T}{\partial Y}$$
$$- \frac{T}{C_pT_e}\left(\frac{\partial U_e}{\partial t} + U_e\frac{\partial U_e}{\partial x}\right)\frac{\partial \psi}{\partial Y} + \frac{\nu_s}{P_r}\frac{\partial^2 T}{\partial Y^2} + \nu_s\left(\frac{\partial^2\psi}{\partial Y^2}\right)^2 \tag{3.2.9}$$

where γ is the ratio of the specific heat and a is the speed of sound, $a^2 = \gamma RT$.

The system of equations (3.2.8), (3.2.9) and (2.2.6) can be solved for the unknown functions ψ, T and ρ for any two-dimensional or axisymmetric unsteady compressible boundary-layer flow, if appropriate boundary conditions are imposed.

3.3 Small Amplitude Oscillations

The method of solution developed in the following sections can be used for a wide variety of unsteady flows. As in the previous chapters, we express the outer flow velocity distribution as

$$U_e(x,t) = U_0(x) + \varepsilon U_1(x)e^{i\omega t} \tag{3.3.1}$$

where ε is a small amplitude parameter. Again we seek solutions to Eqs. (3.2.8) and (3.2.9) in the form

$$\psi(x,Y,t) = \psi_0(x,Y) + \varepsilon\psi_1(x,Y)e^{i\omega t} + \ldots \tag{3.3.2}$$

$$T(x,Y,t) = T_0(x,Y) + \varepsilon T_1(x,Y)e^{i\omega t} + \ldots \tag{3.3.3}$$

$$\rho(x,Y,t) = \rho_0(x,Y) + \varepsilon\rho_1(x,Y)e^{i\omega t} + \ldots \tag{3.3.4}$$

Substitution of Eqs. (3.3.2) to (3.3.4) in Eqs. (3.2.8) and (3.2.9) and collection of terms of order ε° yields

$$\frac{\partial\psi_0}{\partial Y}\frac{\partial^2\psi_0}{\partial x\partial Y} - \left(\frac{\partial\psi_0}{\partial x} + \psi_0\,\frac{j}{r}\frac{dr}{dx}\right)\frac{\partial^2\psi_0}{\partial Y^2} = \frac{T_0U_0dU_0/dx}{T_c\left(1 - \frac{\gamma-1}{2a_c^2}U_0^2\right)}$$

$$- \frac{\gamma U_0dU_0/dx}{2a_c^2\left(1 - \frac{\gamma-1}{2a_c^2}U_0^2\right)}\psi_0\frac{\partial^2\psi_0}{\partial Y^2} + \nu_s\frac{\partial^3\psi_0}{\partial Y^3} \tag{3.3.5}$$

$$\frac{\partial\psi_0}{\partial Y}\frac{\partial T_0}{\partial x} - \left(\frac{\partial\psi_0}{\partial x} + \psi_0\frac{j}{r}\frac{dr}{dx}\right)\frac{\partial T_0}{\partial Y} = -\frac{T_0U_0dU_0/dx}{C_pT_c\left(1-\frac{\gamma-1}{2a_c^2}U_0^2\right)}\frac{\partial\psi_0}{\partial Y}$$

$$-\frac{\gamma U_0dU_0/dx}{2a_c^2\left(1-\frac{\gamma-1}{2a_c^2}U_0^2\right)}\psi_0\frac{\partial T_0}{\partial Y}$$

$$+\frac{T_0U_0}{T_c\left(1-\frac{\gamma-1}{2a_c^2}U_0^2\right)}\frac{\partial}{\partial x}\left(T_0+\frac{U_0^2}{2C_p}\right)+\frac{\nu_s}{P_r}\frac{\partial^2T}{\partial Y^2}+\frac{\nu_s}{C_p}\left(\frac{\partial^2\psi_0}{\partial Y^2}\right)^2 \tag{3.3.6}$$

where T_c is given by

$$T_e + \frac{1}{2C_p}U_e^2 = T_e + \varepsilon T_{e1}(x,t) \tag{3.3.7}$$

The quantity T_c is a constant of the energy level as Eq. (2.2.5) indicates and a_c is the speed of sound corresponding to the constant temperature T_c. The function $C_pT_{c1}(x,t)$, defined by Eq. (3.3.7), represents the fluctuating part of the enthalpy of the outer flow.

Similarly collecting powers of order ε^1 yields

$$\frac{i\omega\partial\psi_1}{\partial Y}+\frac{\partial\psi_1}{\partial Y}\frac{\partial^2\psi_0}{\partial x\partial y}+\frac{\partial\psi_0}{\partial Y}\frac{\partial^2\psi_1}{\partial x\partial y}-\left(\frac{\partial\psi_0}{\partial x}+\psi_0\frac{j}{r}\frac{dr}{dx}\right)\frac{\partial^2\psi_1}{\partial Y^2}$$

$$-\frac{\partial\psi_1}{\partial x}+\psi_1\frac{j}{r}\frac{dr}{dx}\Big)\frac{\partial^2\psi_0}{\partial Y^2}$$

$$=\frac{T_0}{T_c\left(1-\frac{\gamma-1}{2a_c^2}U_0^2\right)}\left[\frac{d}{dx}(U_0U_1)+i\omega U_1\right]+T_1\left[1-\frac{\gamma-1}{2a_c^2}U_0^2\right]-$$

$$\frac{T_0(T_{c1}/T_c - \frac{\gamma-1}{a_c^2} U_0U_1)}{T_c(1 - \frac{\gamma-1}{2a_c^2} U_0^2)^2} U_0\frac{dU_0}{dx}$$

$$- \frac{\gamma}{2a_c^2(1 - \frac{\gamma-1}{2a_c^2} U_0^2)} \{(\psi_0 \frac{\partial^2\psi_1}{\partial Y^2} + \psi_1 \frac{\partial^2\psi_0}{\partial Y^2}) U_0 \frac{dU_0}{dx} \psi_0 \frac{\partial^2\psi_0}{\partial Y^2} [\frac{d}{dx} (U_0U_1)+i\omega U_1]\}$$

$$- \frac{T_{c1}/T_c - \frac{\gamma-1}{a_c^2} U_0U_1}{1 - \frac{\gamma-1}{2a_c^2} U_0^2} U_0\frac{dU_0}{dx} \psi_0 \frac{\partial^2\psi_0}{\partial Y^2} + \nu_s \frac{\partial^3\psi_1}{\partial Y^3} \qquad (3.3.8)$$

and

$$i\omega T_1 + \frac{\partial\psi_0}{\partial Y} \frac{\partial T_1}{\partial x} + \frac{\partial\psi_1}{\partial Y} \frac{\partial T_0}{\partial x} - (\frac{\partial\psi_0}{\partial x} + \psi_0 \frac{j}{r} \frac{dr}{dx}) \frac{\partial T_1}{\partial Y} - (\frac{\partial\psi_1}{\partial x} + \psi_1 \frac{j}{r} \frac{dr}{dx})\frac{\partial T_0}{\partial Y}$$

$$= \frac{T_0(i\omega T_{c1} + U_0dU_0/dx)}{T_c(1 - \frac{\gamma-1}{2a_c^2} U_0^2)}$$

$$- \frac{\gamma}{2a_c^2 (1 - \frac{\gamma-1}{2a_c^2} U_0^2)} \{[\psi_0 \frac{\partial T_0}{\partial Y} (\frac{d}{dx} (U_0U_1)-i\omega U_1) + U_0 \frac{dU_0}{dx}(\psi_0 \frac{\partial T_1}{\partial Y} + \psi_1 \frac{\partial T_0}{\partial Y})]$$

$$\frac{- U_0 \frac{dU_0}{dx} \psi_0 \frac{\partial T_0}{\partial Y} (\frac{T_{c1}}{T_c} - \frac{\gamma-1}{a_1^2})^2 U_1U_0\cos\omega t}{1 - \frac{\gamma-1}{2a_c^2} U_0^2}\}$$

$$- \frac{T_0}{C_pT_c(1 - \frac{\gamma-1}{2a_c^2} U_0^2)} [\frac{\partial\psi_1}{\partial Y} U_0 \frac{dU_0}{dx} + \frac{\partial\psi_0}{\partial Y} (\frac{d}{dx} (U_0U_1) - i\omega U_1)] \qquad (3.3.9)$$

3.4 Outer Flows and Boundary Conditions

If the outer flow velocity distribution is prescribed, then the outer pressure, density and temperature distributions can be calculated from Eqs. (2.2.4), (2.2.5) and (2.2.6). For a small amplitude fluctuation given by Eq. (3.3.1), we expect that the other properties of the outer flow will behave similarly.

$$T_e(x,t) + T_{e0}(x) + \varepsilon T_{e1}(x)e^{i\omega t} + \ldots \tag{3.4.1}$$

$$\rho_e(x,t) = \rho_{e0}(x) + \varepsilon \rho_{e1}(x)e^{i\omega t} + \ldots \tag{3.4.2}$$

$$p(x,t) = p_0(x) + \varepsilon p_1(x)e^{i\omega t} + \ldots \tag{3.4.3}$$

In terms of these quantitites, the boundary conditions at the edge of the boundary layer become

$$\frac{\partial\psi_0}{\partial Y} \to U_0, \quad \frac{\partial\psi_1}{\partial Y} \to U_1 \quad \text{as } Y \to \infty \tag{3.4.4}$$

$$T_0 \to T_{e0}, \quad T_1 \to T_{e1} \quad \text{as } Y \to \infty \tag{3.4.5}$$

The no-slip and no-peneration conditions at the wall require

$$\frac{\partial\psi_0}{\partial Y} = 0, \quad \frac{\partial\psi_1}{\partial Y} = u_w(t) \quad \text{at } Y = 0 \tag{3.4.6}$$

$$\psi_0 = \psi_1 = 0 \qquad \text{at } y = 0 \tag{3.4.7}$$

and finally at the wall, we demand that

$$T = T_w \qquad \text{at } Y = 0 \tag{3.4.8}$$

where u_w and T_w are the wall velocity and temperature, respectively.

Substitution of Eqs. (3.4.1 - 3.4.3) and Eq. (3.3.1) into Eqs. (2.2.4), (2.2.5) and (2.2.6) and collection of terms of order ε° yields

$$-\frac{\partial p_0}{\partial x} = \rho_{e0} U_0 \frac{dU_0}{dx} \tag{3.4.9}$$

$$0 = \rho_{e0} U_0 \frac{d}{dx} \left(T_{e0} + \frac{U_0^2}{2C_p}\right) \tag{3.4.10}$$

$$p_0 = \rho_{e0} RT_{e0} \tag{3.4.11}$$

The solution to the above system is

$$T_{e0} = T_{c0} - U_0^2/2C_p \tag{3.4.12}$$

$$p_0/p_s = (T_c - U_0^2/2C_p)^{\gamma/\gamma-1} \tag{3.4.13}$$

$$\rho_{e0} = \frac{p_s}{R} (T_c - U_0^2/2C_p)^{\bar{\gamma}/\gamma-1} \tag{3.4.14}$$

where T_c is a constant of integration (see Eq. 3.3.7).

Collecting terms of order ε, we get after rearranging

$$\frac{dp_1}{dx} = -p_1 \frac{U_0}{T_{e0}R} \frac{dU_0}{dx} + T_{e1} \frac{\rho_{e0} U_0}{T_0} \frac{dU_0}{dx} - \rho_0[i\omega U_1 + \frac{d}{dx} (U_0 U_1)] \tag{3.4.15}$$

$$\frac{dT_{e1}}{dx} = p_1 \frac{i\omega}{C_p \rho_{e0} U_0} - i\omega T_{e1} - \frac{i\omega U_1}{C_p} - \frac{1}{C_p} \frac{d}{dx} (U_0 U_1) \tag{3.4.16}$$

$$p_1 = R(\rho_{e0} T_{e1} + \rho_{e1} T_{e0}) \tag{3.4.17}$$

We were unable to find a closed form solution to the above system of equations. However, a numerical integration can accompany the matching of a numerical solution of the boundary-layer equations and thus provide at each step the boundary condition at the edge of the boundary layer. In this way, the outer inviscid flow equations can be solved simultaneously with the boundary-layer equations. In the present chapter we will present analytical solutions in the form of coordinate expansions in powers of $\frac{\omega x}{U_\infty}$.

Let

$$p_1 = \sum_{n=0}^{\infty} (i\omega)^n p_{1n}, \quad T_{e1} = \sum_{n=0}^{\infty} (i\omega)^n T_{1n},$$

$$\rho_{e1} = \sum_{n=0}^{\infty} (i\omega)^n \rho_{1n} \tag{3.4.18}$$

Once again the familiar perturbation process, assuming that ω is small, gives an infinite sequence of sets of differential equations. The first two sets are

$$\frac{dp_{10}}{dx} = -p_{10} - \frac{U_{e0}}{T_{e0}R}\frac{dU_0}{dx} + T_{10}\frac{\rho_{e0}U_0}{T_{e0}}\frac{dU_0}{dx} - \rho_{e0}\frac{d}{dx}(U_0U_1) \tag{3.4.19}$$

$$\frac{dT_{10}}{dx} = -\frac{1}{C_p}\frac{d}{dx}(U_0U_1) \tag{3.4.20}$$

$$p_{10} = R(p_{e0}T_{10} + p_{10}T_{e0}) \tag{3.4.21}$$

and

$$\frac{dp_{11}}{dx} = p_{11} \frac{U_{e0}}{T_{e0}R} \frac{dU_0}{dx} + T_{11} \frac{p_{e0}U_{e0}}{T_{e0}} \frac{dU_0}{dx} - p_{e0}U_1 \tag{3.4.22}$$

$$\frac{dT_{11}}{dx} = \frac{1}{C_p\rho_{e0}U_0} p_{10} - \frac{T_{10}}{U_0} - \frac{U_1}{C_p} \tag{3.4.23}$$

$$p_{11} = R(\rho_{e0}T_{11} + \rho_{11}T_{e0}) \tag{3.4.24}$$

while for $n \geq 2$ we get

$$\frac{dp_{1n}}{dx} = p_{1n} \frac{U_0}{T_{e0}R} \frac{dU_0}{dx} + T_{1n} \frac{\rho_{e0}U_0}{T_{e0}} \frac{dU_0}{dx} \tag{3.4.25}$$

$$\frac{dT_{1n}}{dx} = \frac{1}{C_p\rho_{e0}U_0} p_{1,n-1} - \frac{1}{U_0} T_{1,n-1} \tag{3.4.26}$$

$$p_{1n} = R(\rho_{e0}T_{1n} + \rho_{1n}T_{e0}) \tag{3.4.27}$$

The system of Eqs. (3.4.19) through (3.4.21) has a closed form solution

$$p_{10} = -\rho_{e0}U_0U_1 \tag{3.4.28}$$

$$T_{10} = -U_0U_1/C_p \tag{3.4.29}$$

$$\rho_{10} = -\frac{\rho_{e0}U_0U_1}{(\gamma-1)T_{e0}} \tag{3.4.30}$$

The solution of the other systems of equations is quite straightforward but involves expressions like $\int U_1 dx$ and may appear a little involved.

In the present chapter, we will consider solutions for outer flow velocity distributions given by

$$U_0 = c_0 x^m \qquad U_1 = c_1 x^\ell \tag{3.4.31}$$

In terms of the above expressions the solution given by Eqs. (3.4.12) to (3.4.14) becomes

$$T_{e0} = T_c - \frac{c_0^2 x^{2m}}{2C_p} \tag{3.4.32}$$

$$p_0/p_s = (T_c - \frac{c_0^2 x^{2m}}{2C_p})^{\gamma/\gamma-1} \tag{3.4.33}$$

$$\rho_{e0} = \frac{p_s}{R} (T_c - \frac{c_0^2 x^{2m}}{2C_p})^{1/\gamma-1} \tag{3.4.34}$$

and the solution of order ε given by Eqs. (3.4.28) to (3.4.30) becomes

$$p_{10} = - \frac{p_s}{R} (T_c - \frac{c_0^2 x^{2m}}{2C_p})^{1/\gamma-1} c_0 c_1 x^{m+\ell} \tag{3.4.35}$$

$$T_{10} = - \frac{c_0 c_1}{C_p} x^{m+\ell} \tag{3.4.36}$$

$$\rho_{10} = - \frac{p_s}{R(\gamma-1)} (T_c - \frac{c_0^2 x^{2m}}{2C_p})^{1/\gamma-1}$$

$$c_0 c_1 x^{m+\ell} (T_c - \frac{c_0^2 x^{2m}}{2C_p})^{-1} \tag{3.3.37}$$

Finally, the first correction that accounts for terms of order ω, that is the solution to Eqs. (3.4.22) to (3.4.24), for the special case of power variations of U_0 and U_1, can be written as follows

$$p_{11} = - \frac{c_1 x^{\ell+1} p_0}{(\ell+1) RT_0} \tag{3.4.38}$$

$$T_{11} = - \frac{c_1 x^{\ell+1}}{C_p(\ell+1)} \tag{3.4.39}$$

$$\rho_{11} = - \frac{\rho_{e0} x^{\ell+1}}{RT_{e0}(\ell+1)\gamma(\gamma-1)} \tag{3.4.40}$$

Notice that the fluctuating part of the pressure, temperature, and density are not in phase with the velocity. Retaining terms of order ε and ω only, we can write the fluctuating part of the pressure and temperature as follows

$$\frac{p_1}{p_0} = - \frac{U_0 U_1}{RT_0} [1 + \frac{i\omega x}{U_0(\ell+1)}] \tag{3.4.41}$$

$$T_{e1} = - \frac{U_0 U_1}{C_p} [1 + \frac{i\omega x}{U_0(\ell+1)}] \tag{3.4.42}$$

Therefore the pressure and temperature lead the outer flow velocity by the same phase angle. This should not be confused with phase differences that appear within the boundary layer. The reader should recall for example that even for incompressible flow with pressure and outer flow temperature fluctuating in phase with the outer flow velocity, the skin friction and wall heat transfer usually lead and lag respectively, the outer flow.

3.5 Method of Solution

A new independent variable, familiar from self-similiar flows is again introduced

$$\eta = \left[\frac{(m+1+2j)U_0}{2\nu_s x}\right]^{\frac{1}{2}} Y \tag{3.5.1}$$

Solutions to Eqs. (3.3.5) and (3.3.6) will be sought in the form of series expansions

$$\psi_0(x,Y,t) = c_0 \left[\frac{2\nu_s x}{(m+1+2j)U_0}\right]^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{c_0^{2n}}{a_c^{2n}} x^{(1+2n)} f_{1+2n}(\eta) \tag{3.5.2}$$

$$T_0(x,Y,t) = T_c\left(1 - \frac{\gamma-1}{2}\frac{U_0^2}{a_c^2}\right) \sum_{n=0}^{\infty} \frac{c_0^{2n}}{a_c^{2n}} x^{2mn} g_{1+2n}(\eta) \tag{3.5.3}$$

The steady-state problem for compressible flow over wedges or cones has been studied extensively in the literature [40,41]. It should be noticed that the above formulation holds for arbitrary values of the Prandthl number, the ratio of specific heats, and the pressure gradient. For m=0, that is for a flat plate the series expansion is essentially an expansion in powers of the Mach number. For flows with pressure gradient, however, the above expressions become coordinate expansions and hold for any value of the Mach number near the leading edge of the wedge or the cone. These expressions are substituted in the differential equations (3.3.5) and (3.3.6) and equal powers of x, that is 2mn+m-1 powers for n=0,1,..., are collected. The following ordinary but coupled differential equations are then derived

$$f_1''' = -f_1 f_1'' - \left(\frac{2m}{m+1+2j}\right)(g_1 - f_1'^2) \tag{3.5.4}$$

$$f_{1+2n}''' = 2 \sum_{q=0}^{n} \frac{2mq+m}{m+1+2j} f'_{1+2q} f'_{1+2n-2q}$$

$$- \sum_{q=0}^{n} \frac{4mq+m+1+2j}{m+1+2j} f_{1+2q} f''_{1+2n-2q} - \frac{2m}{m+1+2j} g_{1+2n}$$

$$+ \gamma \frac{m}{m+1+2j} \sum_{s=0}^{n-1} \left(\frac{\gamma-1}{2}\right)^{n-s-1} \left(\sum_{q=0}^{s} f_{1+2q} f''_{1+2s-2q} \right) \tag{3.5.5}$$

for $n \geq 1$

$$g_1'' = - P_r f_1 g_1' \tag{3.5.6}$$

$$\frac{1}{P_r} g''_{1+2n} - \frac{\gamma-1}{2P_r} g''_{1+2(n-1)} = \frac{4m}{m+1+2j} \sum_{q=0}^{n} q f'_{1+2n-2q} g_{1+2q}$$

$$- \frac{2m(\gamma-1)}{m+1+2j} \sum_{q=0}^{n-1} q f'_{1+2(n-1-q)} g_{1+2q}$$

$$- \frac{1}{m+1+2j} \sum_{q=0}^{n} [4m(n-q)+m+1+2j] f_{1+2(n-q)} g'_{1+2q}$$

$$+ \frac{1}{m+1+2j} \sum_{q=0}^{n-1} \{\gamma m + \frac{\gamma-1}{2} [4m(n-q-1)+m+1+2j]\}$$

$$x\ f_{1+2(n-q-1)} g'_{1+2q} - (\gamma-1) \sum_{q=0}^{n-1} f''_{1+2q} f''_{1+2(n-1-q)} \tag{3.5.7}$$

for $n \geq 1$

The boundary conditions in terms of the functions f and g become

$$\underset{1+2n}{f}(0) = \underset{1+2n}{f'}(0) = 0 \quad \text{for } n \geq 0 \tag{3.5.8}$$

$$g_1(0) = T_{w0}, \quad \underset{1+2n}{g}(0) = 0 \quad \text{for } n \geq 1 \tag{3.5.9}$$

$$f'_1(\infty) \to 1, \quad g_1(\infty) \to 1 \tag{3.5.10}$$

$$\underset{1+2n}{f'}(\infty) \to 0, \quad \underset{1+2n}{g}(\infty) \to 0 \text{ for } n \geq 1 \tag{3.5.11}$$

The constant quantity T_{w0} is a dimensionless temperature related to the wall temperature via Eq. (3.5.3). $T_{w0} = 0, 1$, and > 1 then corresponds to a cold wall, zero temperature difference and a hot wall, respectively.

Similarly, solutions to Eqs. (3.3.8) and (3.3.9) will be sought in the form

$$\psi_1(x,Y,t) = \left[\frac{2\nu_s x}{(m+1\pm 2j)U_0}\right]^{\frac{1}{2}} c_1 \sum_{n=0}^{\infty} \sum_{q=0}^{\infty} \frac{c_0^{2n}}{a_c^{2n}}\ x$$

$$x^{2mn+\ell+(1-m)q} \left(\frac{i\omega}{c_0}\right)^q F_q^{(2n)}(\eta) \tag{3.5.12}$$

$$T\ (x,Y,t) = \frac{c_0 c_1}{C_p} \sum_{n=0}^{\infty} \sum_{q=0}^{\infty} \frac{c_0^{2n}}{a_c^{2n}}\ x^{2mn+\ell+m+(1-m)q}$$

$$\times\ (\frac{i\omega}{c_0})^q\ G_q^{(2n)}(\eta) \qquad (3.5.13)$$

The above expressions are substituted in Eqs. (3.3.8) and (3.3.9) and equal powers of x are collected. For n=0, the general form of the differential equations for $F_q^{(0)}$ and $G_q^{(0)}$ are

$$F_q'''^{(0)} = -\ f_1 F_q''^{(0)} + \frac{m+\ell+(1-m)q}{m+1+2j}\ 2f_1' F_q'^{(0)}$$

$$-\ \frac{2\ell+2(1-m)q+(1-m)+2j}{m+1+2j}\ f_1'' F_q^{(0)} + \frac{2}{m+1+2j}\ F_{q-1}'^{(0)}$$

$$-\ \frac{2}{m+1+2j}\ [(m+\ell)\delta_{0q} + \delta_{1q}]g_1 \qquad (3.5.14)$$

$$\frac{1}{P_r}\ G_q''^{(0)} = \frac{2(\ell+m+(1-m)q)}{m+1+2j} G_q^{(0)} f_1'$$

$$+\ \frac{4m}{(\gamma-1)(m+1+2j)}\ g_3 F_q'^{(0)} - \frac{2m}{(m+1+2j)}\ F_q'^{(0)} g_1$$

$$-\ f_1 G_q'^{(0)} - \frac{1}{(\gamma-1)(m+1+2j)}\ \{(2\ell+2c_1-m)q$$

$$+((1-m) + 2j)F_q^{(0)} g_3' + (4m+2\ell+2(1-m)q+(1-m)$$

$$+\ 2j)F_q^{(2)} g_1'\} + \frac{1}{2}\ (\frac{2\ell+2(1-m)q+(1-m)+2j}{m+1+2j}) F_q^{(0)} g'_1\ +$$

$$+ \frac{2g_1}{(m+1+2j)} \left(\frac{1}{\ell+1} \delta_{2q} + \delta_{1q}\right) + \frac{\gamma f_1 f_1' (\ell+m)\delta_{0q} + \delta_{1q})}{(\gamma-1)(m+1+2j)}$$

$$+ \frac{\gamma m}{(m+1+2j)(\gamma-1)} F_q^{(0)} g_1' + \frac{2m}{(m+1+2j)} g_1 F_q'^{(0)}$$

$$+ \frac{2}{(m+1+2j)} g_1 f_1' \left((m+\ell)\delta_{0q} + \delta_{1q}\right) - f_1'' F_q''^{(0)}$$

$$+ \frac{2}{(m+1+2j)} G_{q-1}^{(0)} \tag{3.5.15}$$

where δ_{ij} is Kronecker's delta. The equations for $F_q^{(2)}$ and $G_q^{(2)}$ are given in the Appendix B.

The boundary conditions in terms of the functions $F_q^{(2n)}$ and $G_q^{(2n)}$ read

$$F_q^{(2n)}(0) = F_q'^{(2n)}(0) = 0 \tag{3.5.16}$$

$$G_0^{(0)}(0) = 1, \quad G_q^{(2n)}(0) = 0 \text{ for } q \geq 1, \quad n \geq 0 \tag{3.5.17}$$

$$F_0'^{(0)}(\infty) \to 1, \quad F_q'^{(2n)}(\infty) = 0 \text{ for } q,n \geq 1 \tag{3.5.18}$$

$$G_0^{(0)}(\infty) \to -1, \; G_1^{(0)}(\infty) \to -\frac{1}{\ell+1}, \; G_q^{2n}(\infty) \to 0$$

$$\text{for } q \geq 2, \quad n \geq 0 \tag{3.5.19}$$

The ordinary differential equations were solved numerically by the shooting technique. Each system of coupled equations as for example the system for f_1, g_1 or f_3, g_3 etc., is replaced by systems of first-order differential equations. Two more boundary conditions at the wall are assumed and the problem is solved as an initial value problem. A straightforward fourth-order Runge-Kutta integration scheme is employed and the values of the functions at the edge of the boundary layer are checked against the outer flow boundary conditions. If these conditions are not met, a guided guess is used to readjust the assumed values at the wall and the process is repeated until convergence is achieved.

3.6 Results and conclusions

In this chapter, we present numerical results only for $P_r = 0.72$ and for the cases of m=0 and m=1, that is only for a steady part of the flow that corresponds to compressible flow over a flat plate or stagnation flow. The method is by no means confined to these values of m and our subroutine can produce results for any intermediate values that correspond to wedges and cones of different included angles. In the case of m=0, the zeroth order problem is simplified considerably since the momentum equation is uncoupled from the energy equation. Nevertheless, some interesting features of the unsteady part of the flow are present especially in the case of the outer flow velocity fluctuations that vary with a power of x, $U_1=c_1x^{\ell}$. Physically, this corresponds to a flat plate that oscillates and bends as if it were pivoted at its leading edge.

The differential equations that govern the functions f_1 and g_1

were solved numerically as described in the previous section. The results were checked and agree with the well-known data of Ref. [41]. The same subroutine was used to solve numerically the differential equations that govern the functions $F_q^{(0)}$ for q=0,1,2 and $G_0^{(0)}$. In Fig. 3.1, we have plotted the functions $F_0'^{(0)}$, $F_1'^{(0)}$ and $F_2'^{(0)}$ for $\ell=0$, that is $U_1=C_1$ and different wall temperatures. This corresponds to a change of the free stream in magnitude and not in direction. If $\ell=0$, the function $F_0'^{(0)}$ is independent of the wall temperature, T_{w0}. $F_0'^{(0)}$ clearly indicates an overshoot, a characteristic feature of the fluctuating component of the velocity. It is interesting to note that the other functions are strongly affected by the wall temperature. In fact, comparing results for hot walls and hot streams, it appears that the trend can be completely reversed. This is not true for $\ell=1$, that is for $U_1=c_1x$, as Fig. 3.2 shows. Apparently, the velocity fluctuations override the effect of the hot stream, and the trend of the curves is more uniform. In this case, wall temperature appears to affect violently the amplitude of the velocity fluctuations at the level of $F_0'^{(0)}$ as well. The reader should notice that the overshoot of $F_0'^{(0)}$ is now much stronger than in the case $U_1=C_1$.

In Figs. 3.3 and 3.4, we have collected some of the calculated terms of the velocity expansions for a stream fluctuating in magnitude, $\ell=0$, and a stream fluctuating in direction $\ell=1$, respectively. In the traditional terminology, we call the real and imaginary parts

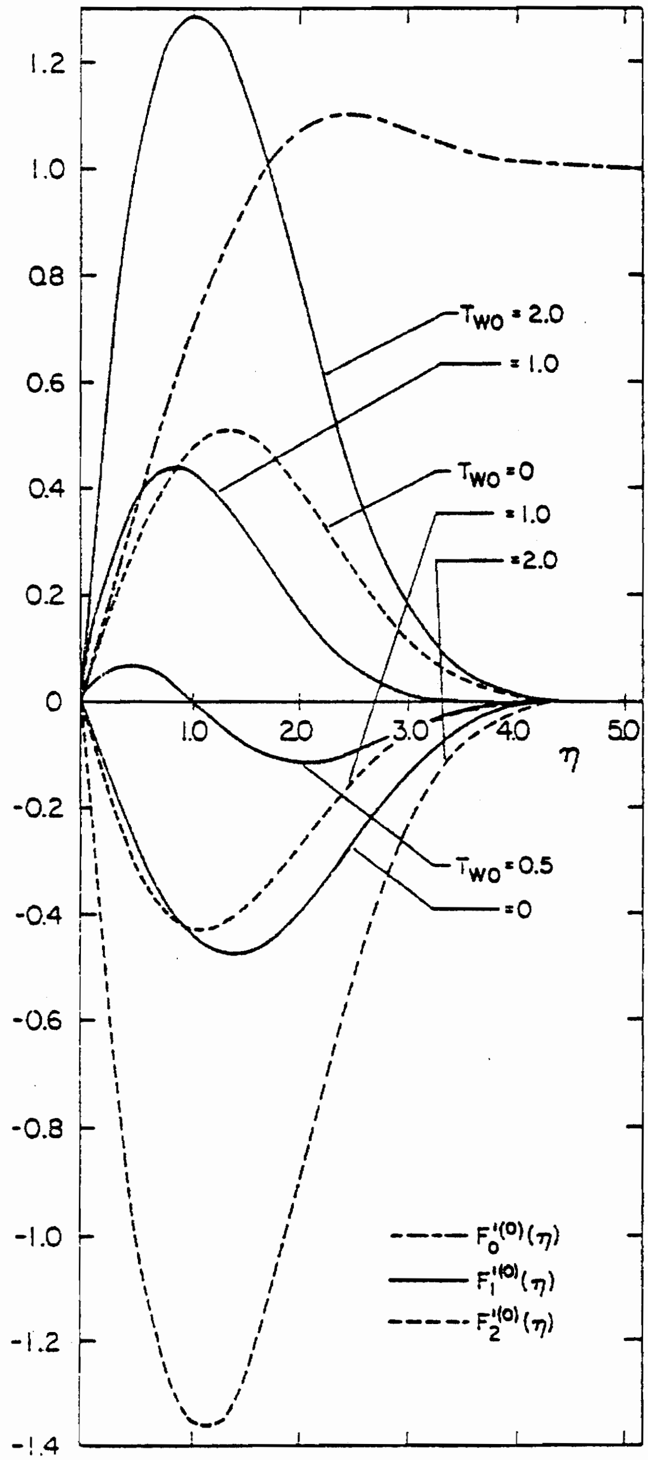

Figure 3.1. The profiles of the functions $F_0^{\prime(0)}$, $F_1^{\prime(0)}$, and $F_2^{\prime(0)}$ for m=0, ℓ=0, and different values of the wall temperature.

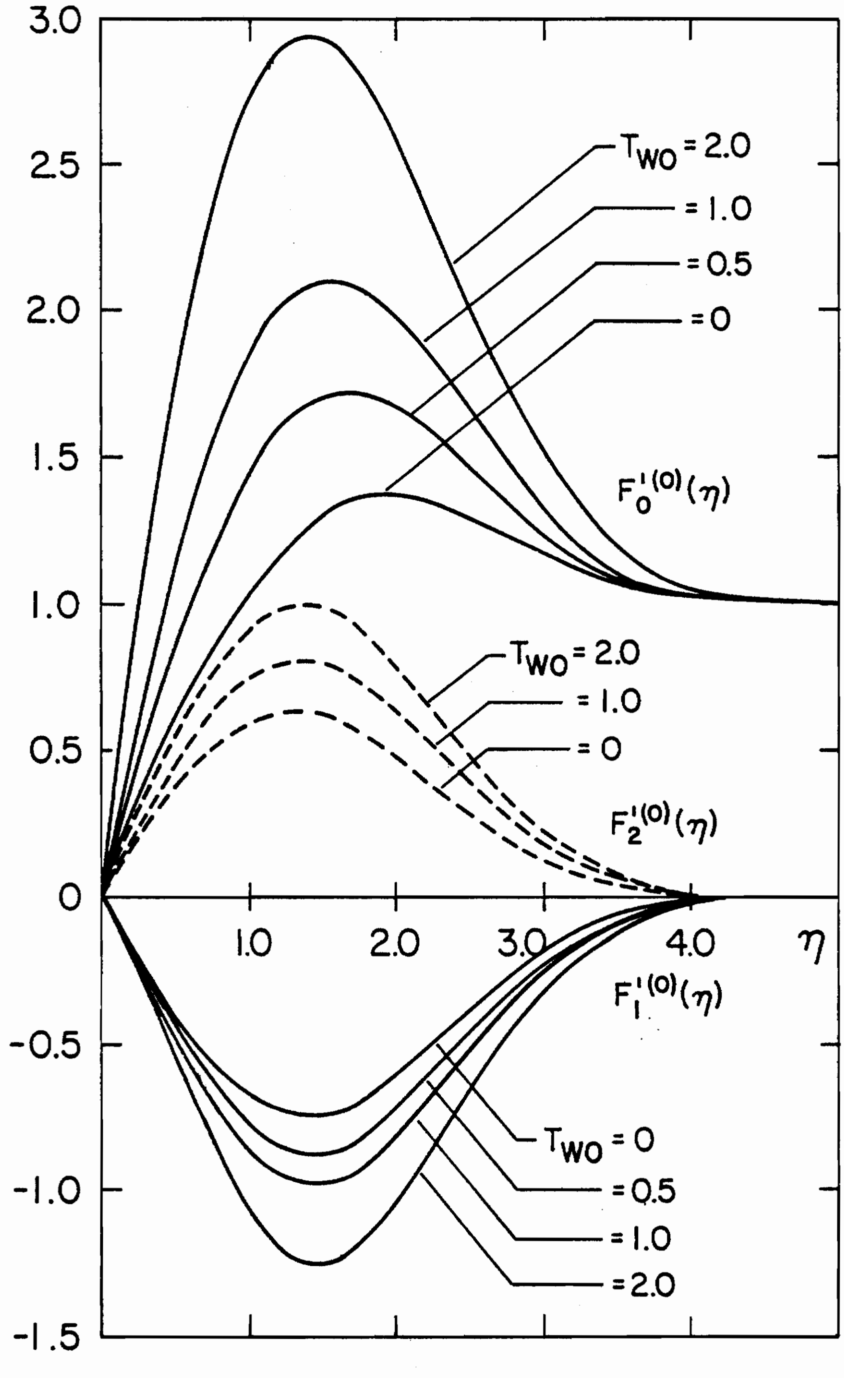

Figure 3.2. The prfiles of the functions $F_0'^{(0)}$, $F_1'^{(0)}$, and $F_2'^{(0)}$ for m=0, ℓ=1 and different values of the wall temperature.

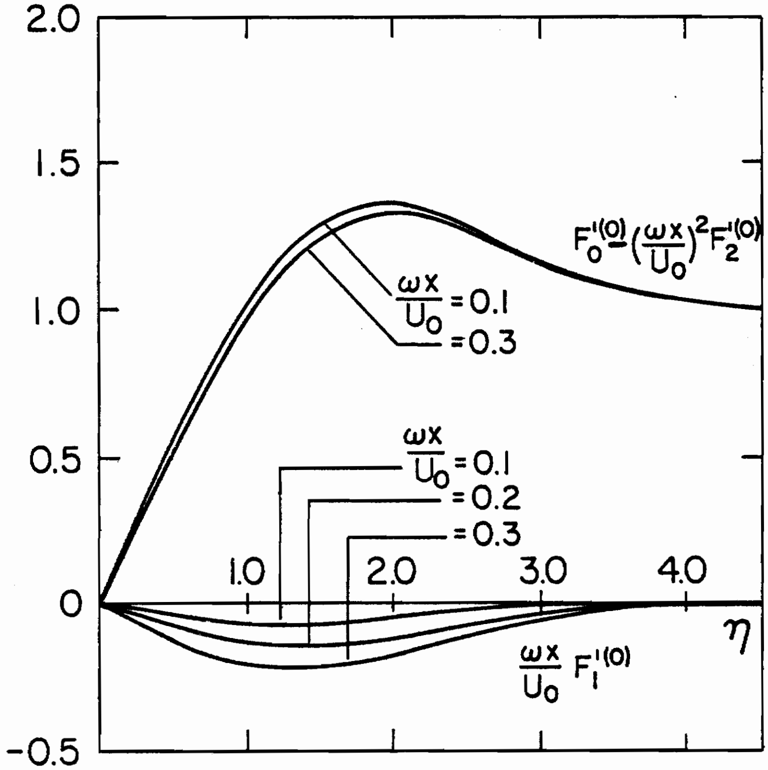

Figure 3.3. The in-phase and out-of-phase parts of the fluctuating velocity for m=0, ℓ=1, T_{w0}=0 and different values of the frequency parameter.

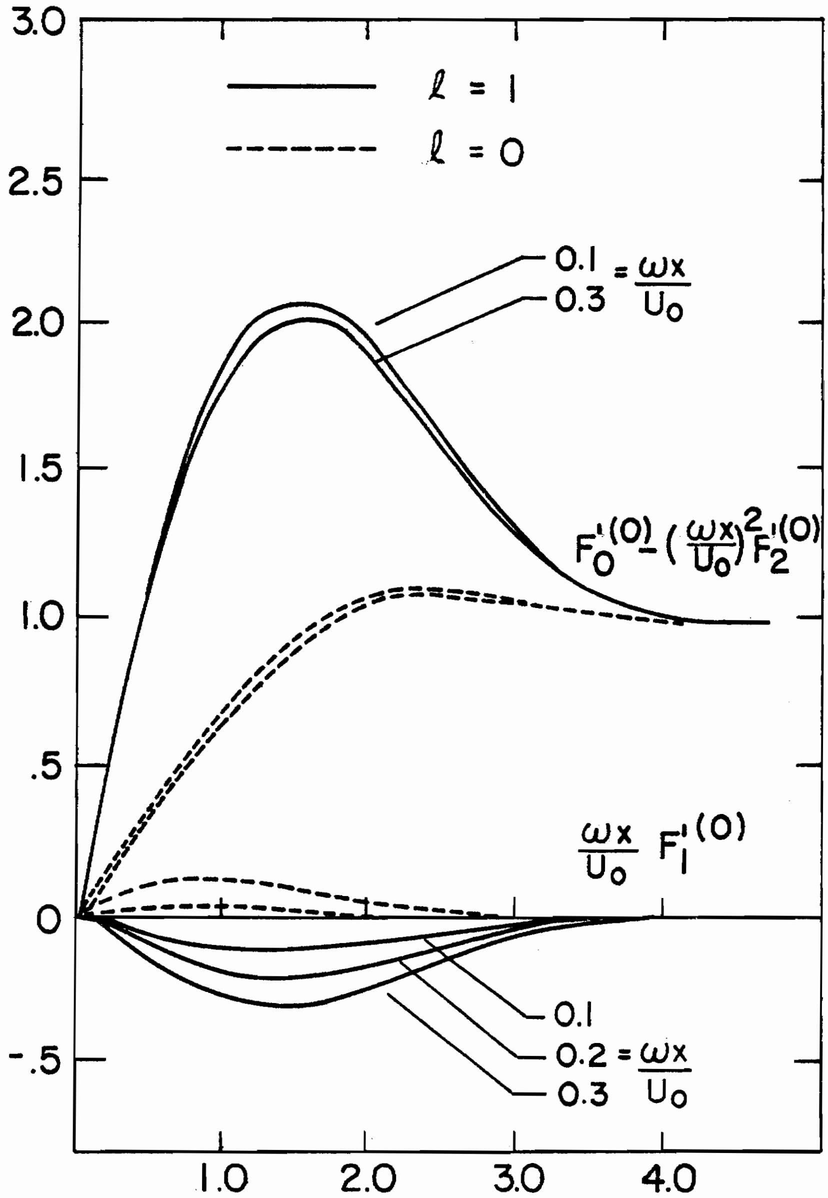

Figure 3.4. The in-phase and out-of-phase parts of the fluctuating velocity profile for m=0, T_{w0}=1.0 and different values of the frequency parameter.

of the fluctuating velocity,

in-phase

$$u_{in}(x,\eta) = F_0'^{(0)}(\eta) - \left(\frac{\omega x}{U_0}\right)^2 F_2'^{(0)}(\eta) \qquad (3.6.1)$$

and out-of phase velocity components

$$u_{out}(x,\eta) = \left(\frac{\omega x}{U_0}\right) F_1'^{(0)}(\eta) \qquad (3.6.2)$$

The reader may observe a stonger overshoot for streams fluctuating in direction over a hot wall. The phase angle profile can be readily calculated from Figs. 3.3 and 3.4.

In Fig. 3.5, we have plotted the skin friction phase angle as a function of the frequency parameter. It should be noticed that for a stream fluctuating in magnitude the skin friction is lagging the outer stream flow. Figure 3.6 is a plot of the variation of the dimensionless skin friction vs. the frequency parameter $\omega x/U_0$. It appears that no considerable changes are involved except perhaps for a fluctuating plate and a hot wall. Finally, in Fig. 3.7, we have plotted the unsteady part of order ω° of the temperature function $G_0^{(0)}$ for three different values of the wall temperature T_{wo}.

Calculations were also performed for m=1, that is for stagnation mean flow with imposed fluctuations. Figure 3.8 shows the functions $F_0'^{(0)}$, $F_1'^{(0)}$, $F_2'^{(0)}$. The overshoot of the in-phase velocity profile is shown to be greatly diminished as compared to the flat-plate case

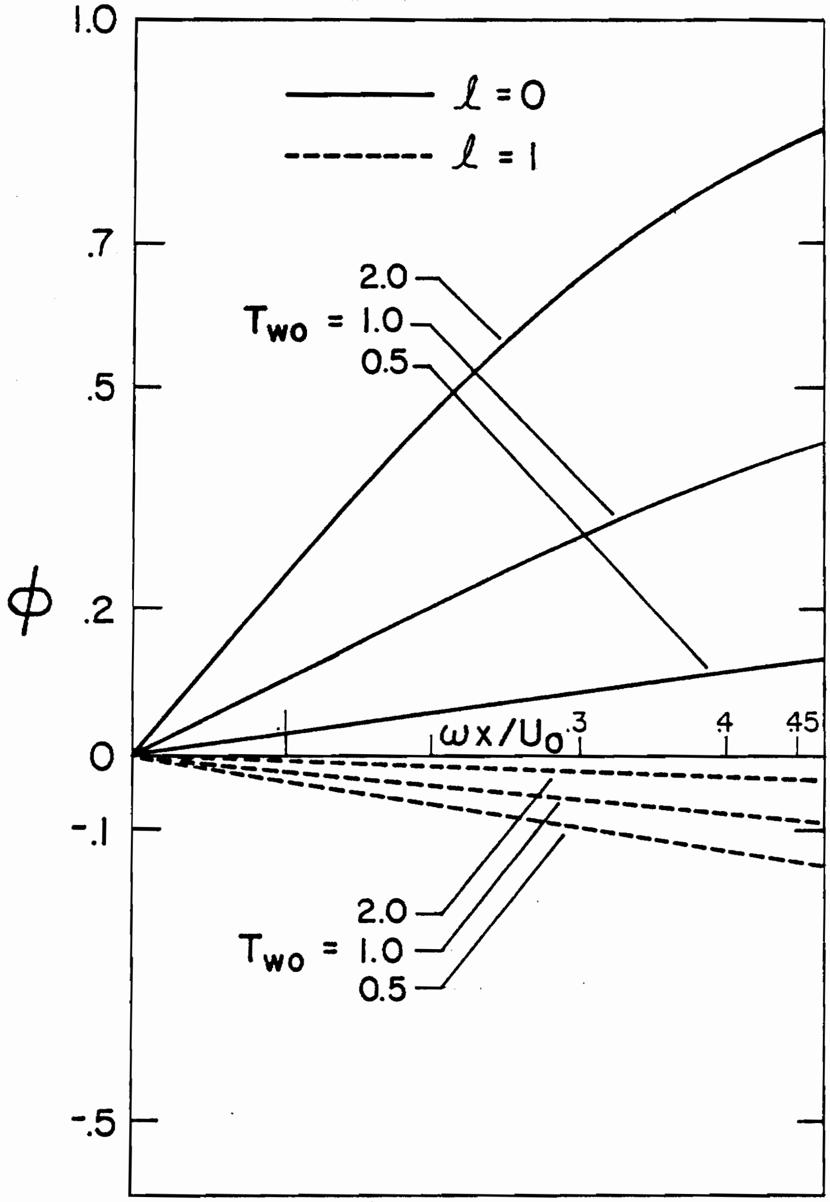

Figure 3.5. The skin-friction phase angle as a function of the frequency parameter for m=0.

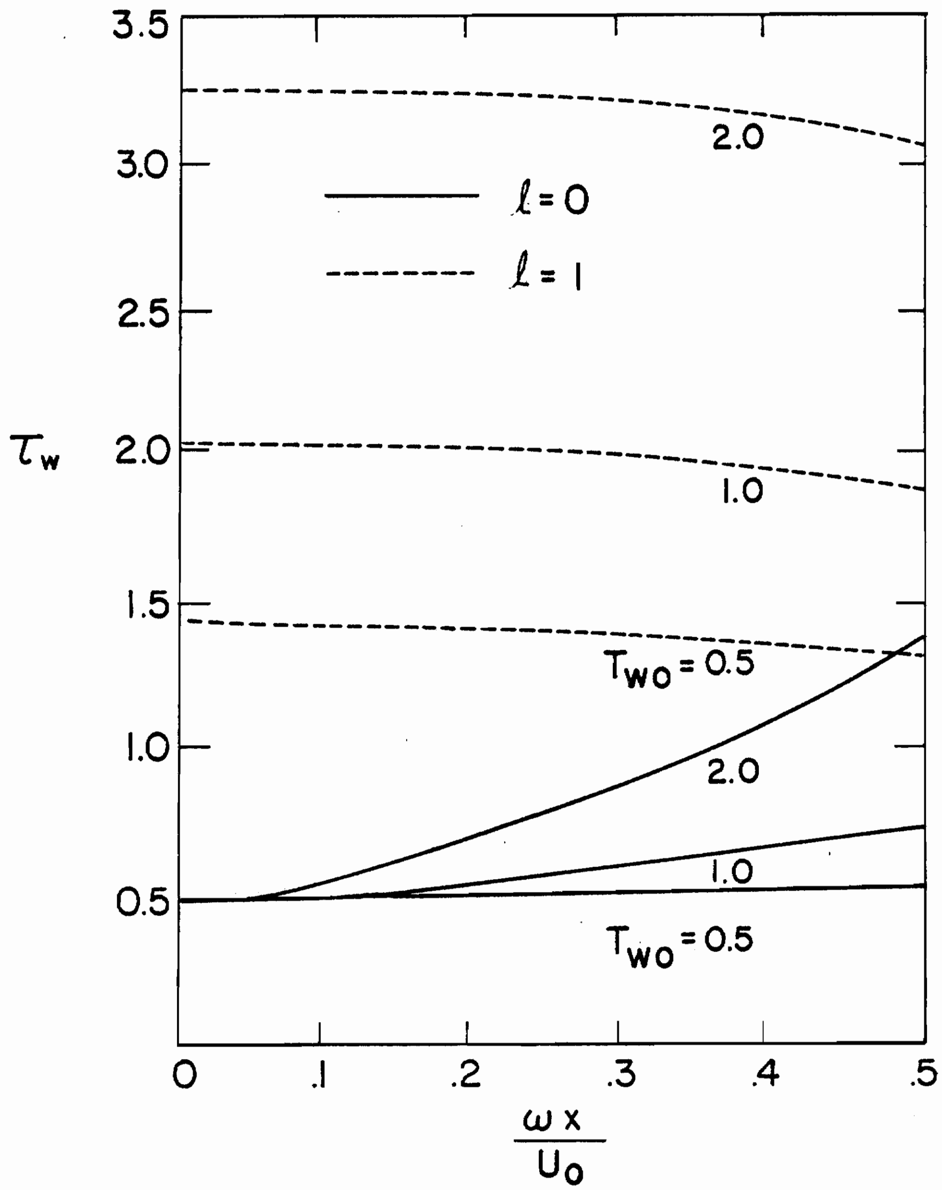

Figure 3.6. The dimensionless skin friction as a function of the frequency parameter for m=0.

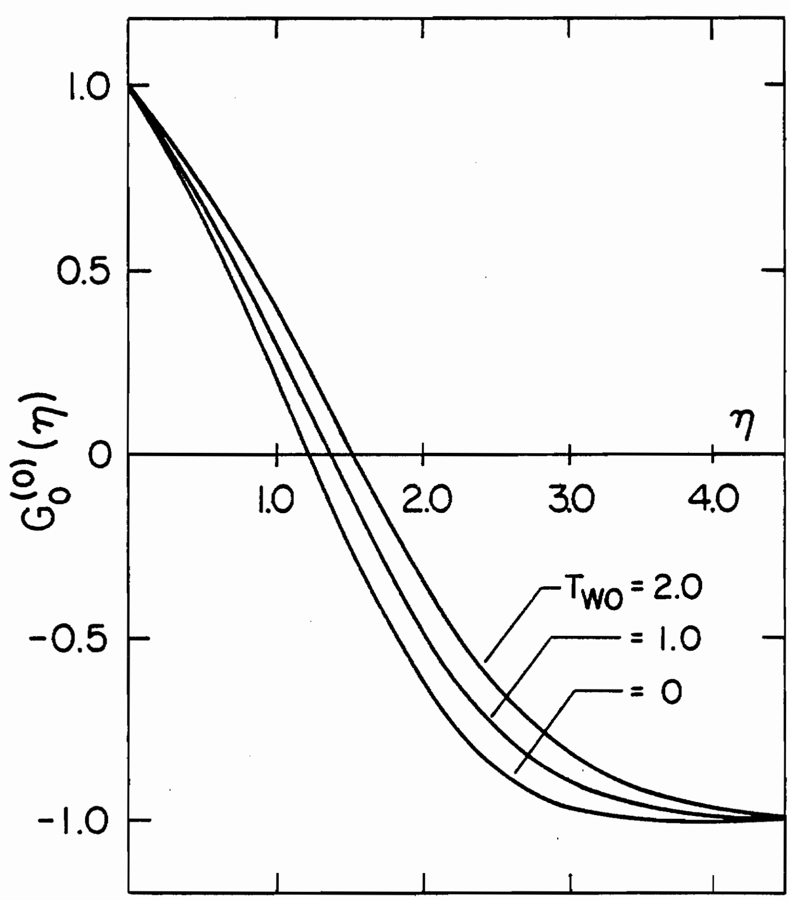

Figure 3.7. The profile of the temperature function $G_0^{(0)}$ for ℓ=0 and different values of the wall temperature.

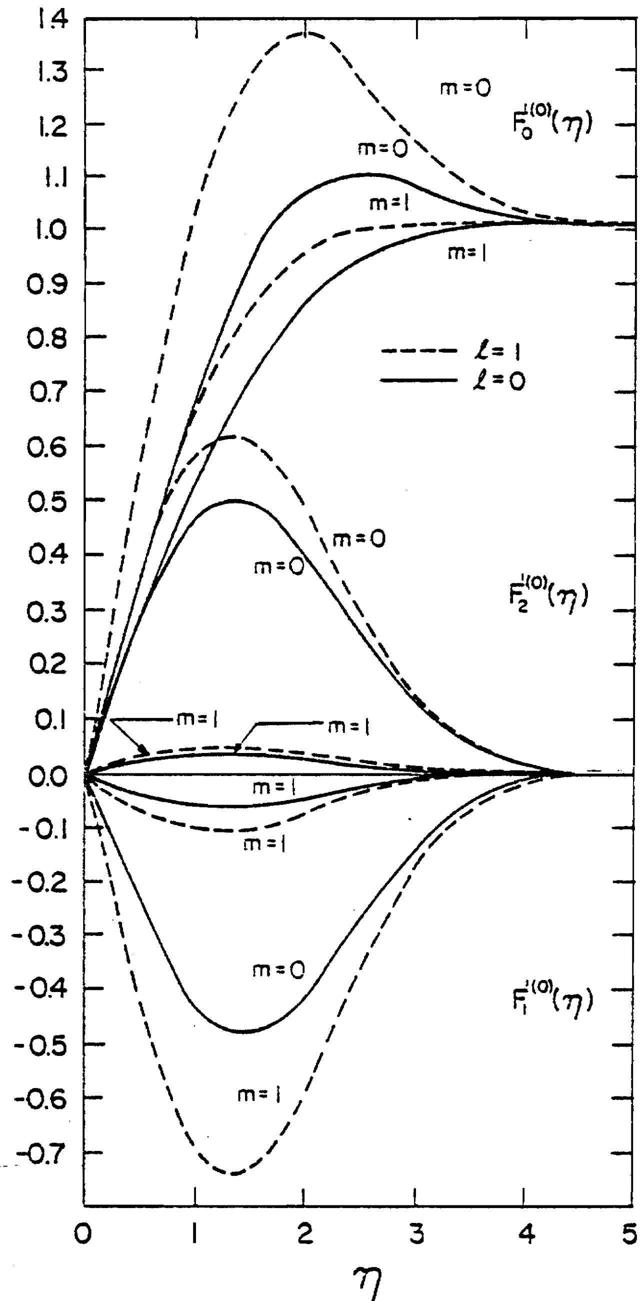

Figure 3.8. The profiles of the functions $F_0'^{(0)}$, $F_1'^{(0)}$, and $F_2'^{(0)}$ for flat plate and stagnation flow (m=0 and 1) and $T_{w0}=0$.

m=0. The functions $F_1'^{(0)}$, which strongly affects the out-of-phase velocity component and therefore the phase angles, appears to have smaller values. From these results and the results from intermediate values of m that are omitted due to lack of space, we concluded that the phase function decreases as the pressure gradient increases. This is clearly observed in Fig. 3.9, in which the phase advance is plotted as a function of the frequency function. It appears that for more adverse pressure gradients that is from 0 and approaching 1, the overshoot of the velocity profile and the phase angles are suppressed.

The double expansions of section 3.5 contain the correction of compressibility on the unsteady part of the motion. It is interesting to note that for n=0, the expansions represent the incompressible disturbance on a compressible mean flow field. Compressibility appears to affect significantly the fluctuating components of the velocity and temperature. It is indicated that overshoots and wall slopes of velocity and temperature can be increased significantly as compared to incompressible flow. This implies larger fluctuating values of the skin friction and heat transfer. Finally it is again verified that larger favorable pressure gradients suppress the phenomena of unsteadiness.

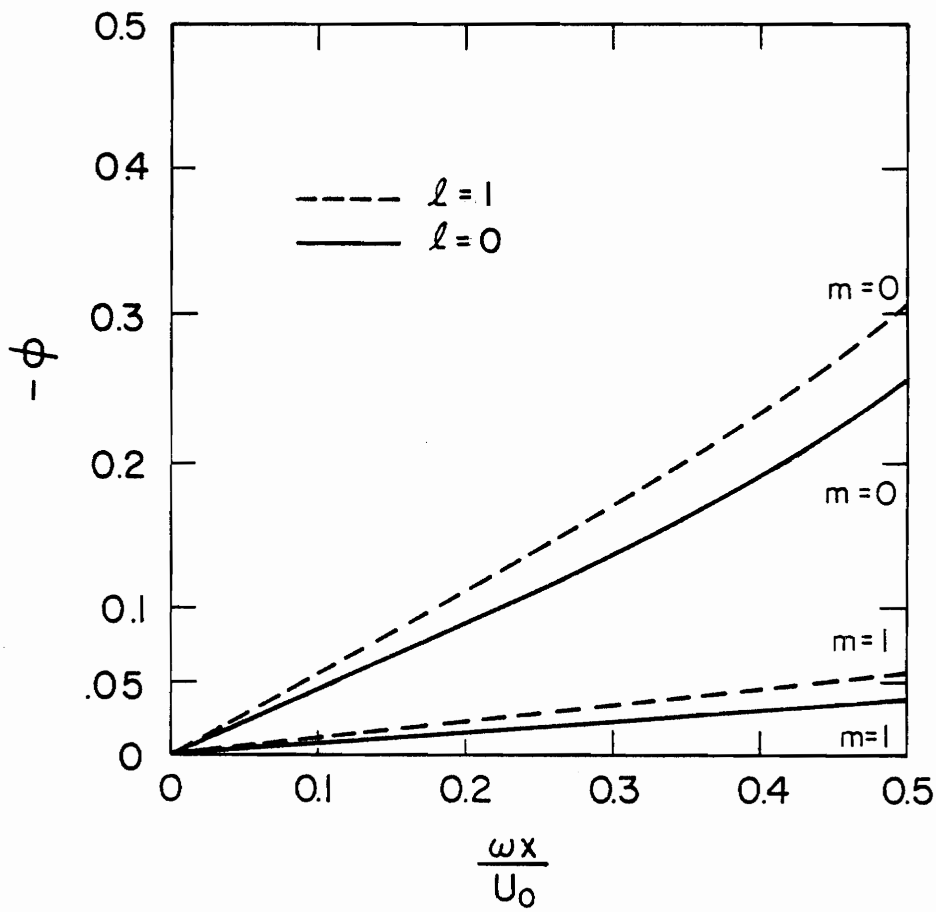

Figure 3.9. The skin friction phase angle as a function of the frequency parameter for flat plate and stagnation flow (m=0 and 1) and T_{w0}=1.0.

CHAPTER FOUR

HIGHER APPROXIMATION IN UNSTEADY INCOMPRESSIBLE BOUNDARY LAYERS

4.1. Introduction

A major problem that arises if higher order corrections are to be considered for unsteady boundary layers is the fact that such corrections may be comparable to the higher order terms of frequency or space perturbations, that are commonly employed in unsteady flow theories. In particular, higher order boundary layer theory is essentially an inner and outer expansion, with $Re^{-1/2}$ as a small parameter. Any further expansions, as for example expansions in powers of the amplitude or the frequency parameter, or expansions of the coordinate type, would require a careful estimate of the relative order of magnitude of the parameters involved. In the earlier chapters we essentially introduced double and triple expansions. However, it was tacitly assumed that the constant $Re^{-1/2}$ was much smaller than ε or $\omega x/U_0$ etc. and therefore all the corrections investigated were more important than the higher order boundary layer corrections. In the present chapter we will examine this situation: the possibility that the $Re^{-1/2}$ and ε are comparable, that is the cases $Re^{-1/2} = O(\varepsilon^2)$, $O(\varepsilon)$ etc.

The only other attempt to examine the coupling of unsteadiness with higher order theory is due to Afzal and Rizvi [42]. However these authors employ an exact self-similar solution for the first order unsteady flow, following the analysis of Yang [43,44]. In this way they avoid any further perturbations. Moreover their solution is valid only for stagnation flow and only for a special form of dependence on time, i.e. $U_1 = s/(1-\alpha t)$ where α is a constant.

In the present chapter we present the unsteady higher order boundary-layer theory for oscillations with small amplitude. We then provide some typical numerical results to account for the nonlinear unsteady effects on the displacement terms.

4.2 Formulation of the Problem

The unsteady laminar incompressible flow of a viscous fluid past a two-dimensional and axisymmetric body is governed by continuity and the Navier Stokes equations. The oncoming stream is assumed to be irrotational. These equations in nondimensional form are

$$\mathrm{div}\,\underset{\sim}{V} = 0 \tag{4.2.1}$$

$$\frac{\partial \underset{\sim}{V}}{\partial t} + \underset{\sim}{V} \cdot \; \mathrm{grad}\,\underset{\sim}{V} + \mathrm{grad}\,P = -\frac{1}{Re}\,\mathrm{curl}(\mathrm{curl}\,\underset{\sim}{V}) \tag{4.2.2}$$

where $Re = V_0 L/\nu$ is the characteristic Reynolds number, V_0 is a characteristic speed and L is a characteristic length. The boundary condition for the above equations at a solid boundary is

$$\underset{\sim}{V} = 0 \tag{4.2.3}$$

Far upstream, the flow approaches a stream with a uniform velocity.

For two dimensional and axisymmetric flow the coordinate system defined in Fig. 4.1 is often used. Here n is the distance normal to the surface and s the distance along the surface measured from the stagnation point, ϕ is the azimuthal angle for axisymmetric flow and is a

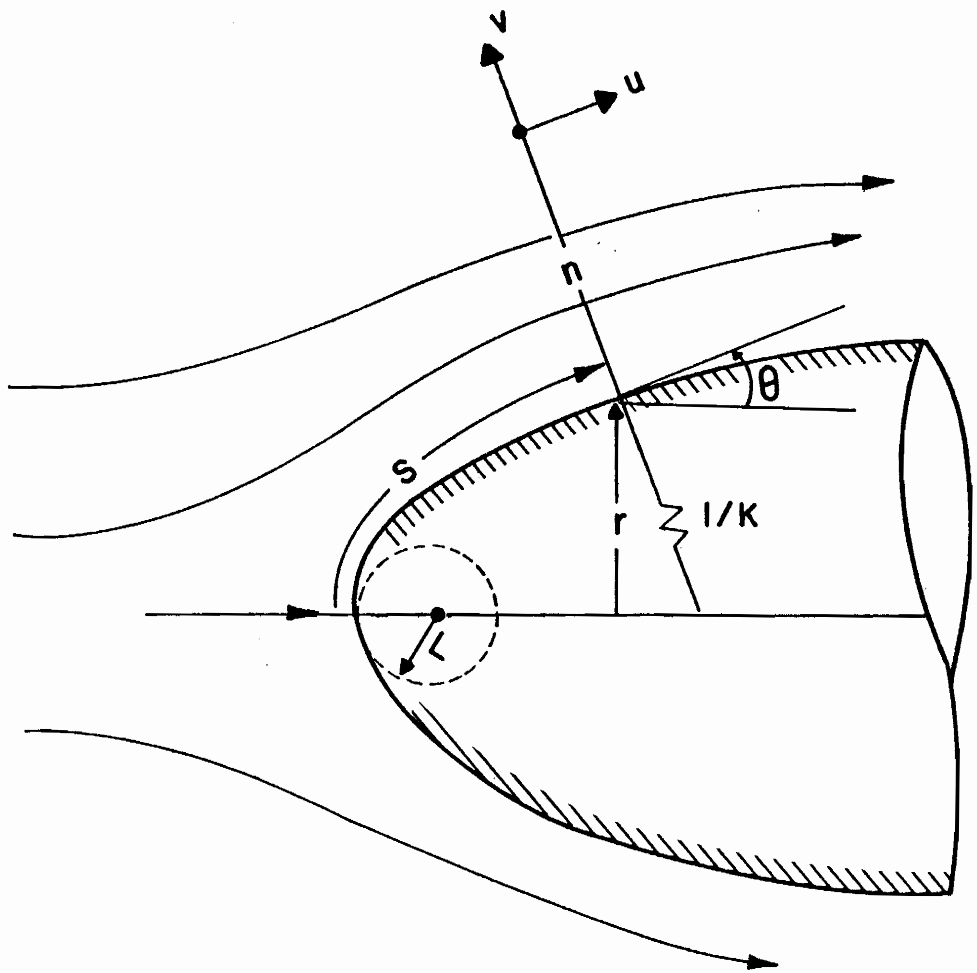

Figure 4.1 Coordinate system for boundary layer.

linear distance normal to the plane of flow for the two-dimensional flow. $K(s)$ denotes the longitudinal curvature of the body taken positive for a convex surface. In axisymmetric flow, $\theta(s)$ is the angle that the tangent to the meridian curve makes with the axis of the body and $r(s)$ is the normal distance from the axis. These parameters are connected by the relations

$$\sin\theta = \frac{dr}{ds} \quad , \quad \cos\theta = -\frac{1}{K}\frac{d^2r}{ds^2} \tag{4.2.4}$$

The length element $d\ell$ can be written as

$$d\ell^2 = (1 + Kn)^2 ds^2 + dn^2 + (r + n\cos\theta)^{2j} d\phi^2 \tag{4.2.5}$$

where j is an index which takes the values zero for two-dimensional and one for axisymmetric flows.

In the sections that follow we repeat some of the classical steps of the analysis in order to demonstrate how the unsteady terms are introduced and how they couple with the other terms of the differential equations.

4.3 Outer and Inner Expansions

4.3.1 Outer expansion

It is an established fact that as the Reynolds number, Re, increases, the flow away from the surface approaches an inviscid flow. We call this the "Euler" or "outer" limit (Van Dyke [28]). The outer limit may be defined as $Re \to \infty$ with $\underset{\sim}{x}$ fixed, and the outer expansion can be written as

$$\left.\begin{aligned} \underset{\sim}{V}(\underset{\sim}{x},t;Re) &= \underset{\sim}{V}_1(\underset{\sim}{x},t) + Re^{-1/2}\underset{\sim}{V}_2(\underset{\sim}{x},t) + \ldots \\ P(\underset{\sim}{x},t;Re) &= P_1(\underset{\sim}{x},t) + Re^{-1/2}P_2(\underset{\sim}{x},t) + \ldots \end{aligned}\right\} \qquad (4.3.1)$$

Likewise the components of $\underset{\sim}{V}$ in the coordinate system of Fig. 4.1 have the outer expansions

$$\left.\begin{aligned} u(\underset{\sim}{x},t;Re) &= U_1(\underset{\sim}{x},t) + Re^{-1/2}U_2(\underset{\sim}{x},t) + \ldots \\ v(\underset{\sim}{x},t;Re) &= V_1(\underset{\sim}{x},t) + Re^{1/2}V_2(\underset{\sim}{x},t) + \ldots \end{aligned}\right\} \qquad (4.3.2)$$

It is implied that the functions $\underset{\sim}{V}_1$, $\underset{\sim}{V}_2$, etc. and their derivatives are all of order one.

Substituting these equations into the equations of motion (4.2.1) and (4.2.2) and equating like powers of Re, yields equations for successive terms of the outer expansion.

The first-order equations are

$$\text{div}\ \underset{\sim}{V}_1 = 0 \qquad (4.3.3)$$

$$\frac{\partial \underset{\sim}{V}_1}{\partial t} + \underset{\sim}{V}_1 \cdot \text{grad}\ \underset{\sim}{V}_1 + \text{grad}\ P_1 = 0 \qquad (4.3.4)$$

which are the Euler equations for the unsteady inviscid flow.

The second-order equations are

$$\text{div}\ \underset{\sim}{V}_2 = 0 \qquad (4.3.5)$$

$$\frac{\partial}{\partial t}\underset{\sim}{V}_2 + \underset{\sim}{V}_1 \cdot \text{grad}\ \underset{\sim}{V}_2 + \underset{\sim}{V}_2 \cdot \text{grad}\ \underset{\sim}{V}_1 + \text{grad}\ P_2 = 0 \qquad (4.3.6)$$

which are the perturbed Euler's equations and give the displacement of the outer flow due to the first-order boundary layer. Notice here that viscous terms appear in the outer expansion beginning only with the third approximation.

Equation (4.3.4) can also be written as

$$\frac{\partial \underset{\sim}{V}_1}{\partial t} + \text{grad}\left(\frac{1}{2}\underset{\sim}{V}_1^2 + P_1\right) - \underset{\sim}{V}_1 \times \text{curl}\ \underset{\sim}{V}_1 = 0 \qquad (4.3.7)$$

Integrating Eq. (4.3.7) along a stream line, we get

$$\frac{\partial}{\partial t}\int \underset{\sim}{V}_1 \cdot d\underset{\sim}{x} + \frac{1}{2}\underset{\sim}{V}_1^2 + P_1 = B(t) \qquad (4.3.8)$$

where B is Bernoulli's function and depends upon upstream conditions.

As the basic inviscid flow is assumed to be irrotational, there exists a velocity potential $\phi_1(\underset{\sim}{x},t)$ defined by

$$U_1 = \frac{\partial \phi_1}{\partial s}\ ,\ V_1 = \frac{\partial \phi_1}{\partial n} \qquad (4.3.9)$$

Eq. (4.3.8) now gives

$$\frac{\partial \phi_1}{\partial t} + \frac{1}{2}(\nabla\phi_1)^2 + P_1 = B(t) \qquad (4.3.10)$$

which is the classical Bernoulli's equation for unsteady potential flow. Integration of Eq. (4.3.6) along a stream line yields

$$\frac{\partial}{\partial t}\int \underset{\sim}{V}_2 \cdot d\underset{\sim}{x} + \underset{\sim}{V}_1 \cdot \underset{\sim}{V}_2 + P_2 = 0 \qquad (4.3.11)$$

Since the basic flow is irrotational, the perturbed flow can also be shown to be irrotational. Thus we define a velocity potential $\phi_2(\underset{\sim}{x},t)$ as

$$U_2 = \frac{\partial \phi_2}{\partial s}\ ,\ V_2 = \frac{\partial \phi_2}{\partial n} \qquad (4.3.12)$$

Eq. (4.3.11) can now be written as

$$\frac{\partial \phi_2}{\partial t} + (\nabla\phi_1) \cdot (\nabla\phi_2) + P_2 = 0 \qquad (4.3.13)$$

This equation represents the perturbed Bernoulli's equation for unsteady potential flow.

The outer expansion breaks down at the body surface, where the no-slip condition cannot be satisfied due to the loss of one highest order derivative in the Navier Stokes equation.

4.3.2 Inner Expansion

We stretch the normal coordinate by introducing the boundary-layer variable

$$N = Re^{1/2} n \qquad (4.3.14)$$

and study the inner limit defined as the limit of the solution for $Re \to \infty$ with s and N fixed. The normal velocity and stream-function are expected to be small in the boundary layer, and must be stretched in a similar fashion. A solution for the inner layer is then sought in the following expansion form,

$$\left.\begin{aligned} u(s,n,t;Re) &= u_1(s,N,t) + Re^{-1/2}u_2(s,N,t) + \ldots \\ v(s,n,t;Re) &= Re^{-1/2}v_1(s,N,t) + Re^{-1}v_2(s,N,t) + \ldots \\ P(s,n,t;Re) &= p_1(s,N,t) + Re^{-1/2}p_2(s,N,t) + \ldots \end{aligned}\right\} \qquad (4.3.15)$$

Substituting Eqs. (4.3.15) into Eqs. (4.2.1) and (4.2.2) and collecting the coefficients of like powers of $Re^{-1/2}$, we arrive at differential equations of successive order.

The first-order equations

$$\frac{\partial}{\partial s}(r^j u_1) + \frac{\partial}{\partial N}(r^j v_1) = 0 \tag{4.3.16}$$

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial s} + v_1 \frac{\partial u_1}{\partial N} + \frac{\partial}{\partial s} p_1 - \frac{\partial^2 u_1}{\partial N^2} = 0 \tag{4.3.17}$$

$$\frac{\partial p_1}{\partial N} = 0 \tag{4.3.18}$$

along with the boundary conditions at the wall

$$u_1(s,0,t) = v_1(s,0,t) = 0 \tag{4.3.19}$$

are the classical unsteady equations of Prandtl.

The collection of coefficients of $Re^{-1/2}$ gives the following second-order equations

$$\frac{\partial}{\partial s}\left[r^j\left(u_2 + N u_1 j \frac{\cos\theta}{r}\right)\right] + r^j \frac{\partial}{\partial N}\left[v_2 + \left(K + \frac{j}{r}\cos\theta\right) N v_1\right] = 0 \tag{4.3.20}$$

$$\frac{\partial u_2}{\partial t} + \frac{\partial}{\partial s}(u_1 u_2) + v_1 \frac{\partial u_2}{\partial N} + v_2 \frac{\partial u_1}{\partial N} - \frac{\partial^2 u_2}{\partial N^2} + \frac{\partial p_2}{\partial s} =$$
$$K\left[\frac{\partial}{\partial N}\left(N \frac{\partial u_1}{\partial N}\right) - v_1 \frac{\partial}{\partial N}(N u_1) - N \frac{\partial u_1}{\partial t}\right] + j \frac{\cos\theta}{r} \frac{\partial u_1}{\partial N} \tag{4.3.21}$$

$$\frac{\partial p_2}{\partial N} = K u_1^2 \tag{4.3.22}$$

The boundary conditions at the wall are

$$u_2(s,0,t) = 0, \quad v_2(s,0,t) = 0 \tag{4.3.23}$$

At the outer edge of the layer the solution will be matched to the outer solution.

4.3.3 Matching Conditions

Matching of the inner and outer solutions according to Van Dyke [45] requires that

p-term inner expansion of (q-term outer expansion)

= q-term outer expansion of (p-term inner expansion)

(4.3.24)

Matching conditions for the first-order are found by taking $p=q=1$. The one term outer expansion of u is $U_1(s,n,t)$; rewritten in inner variables gives $U_1(s,Re^{-1/2}N,t)$; and expanded for large Re gives $U_1(s,0,t)$ as its one-term inner expansion. Conversely, one term of the inner expansion is $u_1(s,N,t)$; rewritten in outer variables gives $u_1(s,Re^{1/2}n,t)$; and expanded for large Re gives $U_1(s,\infty,t)$ as its one-term outer expansion. Equating these results, and proceeding similarly with pressure gives the following matching conditions.

$$u_1(s,\underset{N\to\infty}{N},t) = U_1(s,0,t) \tag{4.3.25}$$

$$p_1(s,\underset{N\to\infty}{N},t) = B(t) - \frac{1}{2}U_1^2(s,0,t) - \frac{\partial\phi_1}{\partial t}(s,0,t) \tag{4.3.26}$$

For $p = 2$, $q = 1$, we get

$$V_1(s,0,t) = 0 \tag{4.3.27}$$

$$V_2(s,0,t) = \underset{N\to\infty}{\text{Limit}}\left[v_1(s,N,t) - N\frac{\partial v_1}{\partial N}(s,N,t)\right] \tag{4.3.28}$$

Eq. (4.3.28) is an important matching condition for the second-order outer flow and represents the effect of displacement of the inviscid (outer) flow by the boundary-layer (inner). The quantities on the right hand side of Eq. (4.3.28) represent the effect of the classical boundary-layer equation upon the outer flow.

Finally, matching conditions for the second-order boundary layer problem are found by taking $p=q=2$ in Eq. (4.3.24).

$$u_2(s,N,t)_{N\to\infty} = U_2(s,0,t) - NKU_1(s,0,t) \tag{4.3.29}$$

$$p_2(s,N,t)_{N\to\infty} = P_2(s,0,t) + NKU_1(s,0,t) \tag{4.3.30}$$

4.4 First and Second-Order Boundary-Layer Theory

Equation (4.3.18) shows that p_1 is constant across the boundary layer, and its value is given by Eq. (4.3.26). Thus Eqs. (4.3.16) and (4.3.17) can be rewritten in the classical boundary layer equation form

$$\frac{\partial}{\partial s}(r^j u_1) + \frac{\partial}{\partial N}(r^j v_1) = 0 \tag{4.4.1}$$

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial s} + v_1 \frac{\partial u_1}{\partial N} - \frac{\partial^2 u_1}{\partial N^2} = U_1(s,0,t) \frac{\partial U_1}{\partial s}(s,0,t) + \frac{\partial U_1}{\partial t}(s,0,t) \tag{4.4.2}$$

with boundary conditions

$$u_1(s,0,t) = v_1(s,0,t) = 0 \tag{4.4.3}$$

$$u_1(s,N,t) = U_1(s,0,t) \text{ as } N \to \infty \tag{4.4.4}$$

Integrating the normal-momentum Eq. (4.3.22) and using the matching condition Eq. (4.3.30), we obtain

$$p_2(s,N,t) = NKU_1^2(s,0,t) + K \int_N^\infty [U_1^2(s,0,t) - u_1^2(s,N,t)]dN$$
$$+ P_2(s,0,t) \qquad (4.4.5)$$

The second-order pressure gradient, $\frac{\partial p_2}{\partial s}$, is therefore written, using the outer flow relations, as

$$\frac{\partial p_2}{\partial s}(s,N,t) = \frac{\partial}{\partial s}\left\{ K[NU_1^2(s,0,t) + \int_N^\infty \{U_1^2(s,0,t) - u_1^2(s,N,t)\}dN]\right\}$$
$$- \frac{\partial}{\partial s} U_1(s,0,t)U_2(s,0,t) - \frac{\partial}{\partial t} U_2(s,0,t) \qquad (4.4.6)$$

Substituting Eq. (4.4.6) in Eqs. (4.3.20) and (4.3.21), the second order equations are

$$\frac{\partial}{\partial s}[r^j(u_2 + Nu_1 \frac{j\cos\theta}{r}] + r^j \frac{\partial}{\partial N}[v_2 + (k + \frac{j\cos\theta}{r})Nv_1] = 0 \quad (4.4.7)$$

$$\frac{\partial u_2}{\partial t} + \frac{\partial}{\partial s}(u_1u_2) + v_1 \frac{\partial u_2}{\partial N} + v_2 \frac{\partial u_1}{\partial N} - \frac{\partial^2 u_2}{\partial N^2} = K[-N \frac{\partial u_1}{\partial t}$$
$$+ \frac{\partial}{\partial N}(N \frac{\partial u_1}{\partial N}) - v_1 \frac{\partial}{\partial N}(Nu_1)] - \frac{\partial}{\partial s}[KNU_1^2(s,0,t)$$
$$+ K\int_N^\infty (U_1^2 - u_1^2)dN] + \frac{\partial}{\partial s}(U_1U_2) + \frac{\partial U_2}{\partial t} + \frac{j\cos\theta}{r}\frac{\partial u_1}{\partial N} \quad (4.4.8)$$

with boundary conditions

$$u_2(s,0,t) = v_2(s,0,t) = 0 \qquad (4.4.9)$$
$$u_2(s,N,t) = U_2(s,0,t) - KNU_1(s,0,t) \text{ as } N \to \infty \qquad (4.4.10)$$

In Eq. (4.4.8), the terms with the coefficients of $j \frac{\cos\theta}{r}$, K and U_2 arise due to the second-order effects of transverse curvature, longitudinal curvature, and displacement respectively.

In specific applications it is convenient to work with the stream function. The continuity equations, Eqs. (4.4.1) and (4.4.7) are identically satisfied by introducing first-and second order streamfunctions ψ_1, ψ_2 according to

$$r^j u_1 = \frac{\partial \psi_1}{\partial N}, \quad r^j v_1 = -\frac{\partial \psi_1}{\partial s} \tag{4.4.11}$$

$$r^j\left(u_2 + N u_1 j \frac{\cos\theta}{r}\right) = \frac{\partial \psi_2}{\partial N}, \quad r^j\left[v_2 + \left(K + \frac{j\cos\theta}{r}\right) N v_1\right] = -\frac{\partial \psi_2}{\partial s} \tag{4.4.12}$$

Substituting Eqs. (4.4.11) and (4.4.12) in Eqs. (4.4.2) and (4.4.8), we get the first-order and second-order boundary-layer problems in terms of stream functions.

First-order problem:

$$\frac{\partial^2 \psi_1}{\partial t \partial N} + \frac{\partial \psi_1}{\partial N} \frac{\partial}{\partial s}\left(\frac{1}{r^j} \frac{\partial \psi_1}{\partial N}\right) - \frac{1}{r^j} \frac{\partial \psi_1}{\partial s} \frac{\partial^2 \psi_1}{\partial N^2} - \frac{\partial^3 \psi_1}{\partial N^3} = r^j U_1(s,0,t) \times$$
$$\frac{\partial U_1}{\partial s}(s,0,t) + r^j \frac{\partial U_1}{\partial t}(s,0,t) \tag{4.4.13}$$

$$\psi_1(s,0,t) = \frac{\partial \psi_1}{\partial N}(s,0,t) = 0 \tag{4.4.14}$$

$$\frac{\partial \psi_1}{\partial N} = r^j U_1(s,0,t) \text{ as } N \to \infty \tag{4.4.15}$$

The corresponding second-order problems are

$$\frac{\partial}{\partial t}\left\{\frac{1}{r^j}\left(\frac{\partial\psi_2}{\partial N} - N\frac{\partial\psi_1}{\partial N}\frac{j\cos\theta}{r}\right)\right\} + \frac{\partial}{\partial s}\left[\frac{1}{r^{2j}}\frac{\partial\psi_1}{\partial N}\left\{\frac{\partial\psi_2}{\partial N} - N\frac{\partial\psi_1}{\partial N}\frac{j\cos\theta}{r}\right\}\right]$$

$$- \frac{1}{r^j}\frac{\partial\psi_1}{\partial s}\frac{\partial}{\partial N}\left[\frac{1}{r^j}\left\{\frac{\partial\psi_2}{\partial N} - N\frac{\partial\psi_1}{\partial N}\frac{j\cos\theta}{r}\right\}\right] - \frac{1}{r^j}\frac{\partial^2\psi_1}{\partial N^2} \quad \times$$

$$\left[\frac{1}{r^j}\left\{\frac{\partial\psi_2}{\partial s} - \left(K + \frac{j\cos\theta}{r}\right)N\right\}\frac{\partial\psi_1}{\partial s}\right] - \frac{\partial^2}{\partial N^2}\left[\frac{1}{r^j}\left\{\frac{\partial\psi_2}{\partial N} - N\frac{\partial\psi_1}{\partial N}\frac{j\cos\theta}{r}\right\}\right] =$$

$$K\left[-\frac{N}{r^j}\frac{\partial^2\psi_1}{\partial t\partial N} + \frac{\partial}{\partial N}\left(N\frac{\partial}{\partial N}\left(\frac{1}{r^j}\frac{\partial\psi_1}{\partial N}\right) + \frac{1}{r^j}\frac{\partial\psi_1}{\partial s}\frac{\partial}{\partial N}\left(N\frac{1}{r^j}\frac{\partial\psi_1}{\partial N}\right)\right]$$

$$- \frac{\partial}{\partial s}\left[KNU_1^2(s,0,t) + K\int_N^\infty \left(U_1^2 - \left(\frac{1}{r^j}\frac{\partial\psi_1}{\partial N}\right)^2 dN + \frac{\partial}{\partial s}(U_1U_2)\right.\right.$$

$$+ \frac{\partial U_2}{\partial t} + \frac{1}{r^j}\frac{\partial^2\psi_1}{\partial N^2}\frac{j\cos\theta}{r} \qquad (4.4.16)$$

$$\psi_2(s,0,t) = \frac{\partial}{\partial N}\psi_2(s,0,t) = 0 \qquad (4.4.17)$$

$$\frac{\partial\psi_2}{\partial N} = r^j[U_2(s,0,t) + NU_1(s,0,t)(\frac{j\cos\theta}{r} - K)] \text{ as } N \to \infty \qquad (4.4.18)$$

4.5 Method of Solution

The second order effects are linear and they can be studied independently of each other. In this section we will confine our attention to the effects of displacement only and we will disregard the effects of longitudinal and lateral curvature, free stream vorticity etc. We will study here two-dimensional oscillatory flows with small amplitudes.

Consider an oncoming uniform flow with a velocity given by

$$U_\infty = U_{\infty 0} + \varepsilon U_{\infty 1} \cos\omega t \qquad (4.5.1)$$

where $U_{\infty 0}$ and $U_{\infty 1}$ are constants and ε is a small amplitude parameter. We now consider the first order outer flow solved and assume that the first order velocity distribution is given by

$$U_1(s,0,t) = U_{10}(s) + \frac{\varepsilon}{2} U_{11}(s)(e^{i\omega t} + e^{-i\omega t}) \qquad (4.5.2)$$

where U_{10} and U_{11} are known functions of s.

At this point and before we proceed with expansions in powers of ε we should specify the relative magnitude of our two small parameters, $Re^{-1/2}$ and ε. Let us first consider:

(a) $Re^{-1/2} = O(\varepsilon^2)$

This would be realistic for relatively low Reynolds number flows. For example, for amplitude ratios of the order $\varepsilon = 0.1$, the corresponding Reynolds number would be of the order of 10^4. A proper way to approach this problem then would be to assume the solution in the form of a double expansion [46] in powers of ε and Re

$$\psi(s,N,t) = \psi_{10} + \frac{\varepsilon}{2}(\psi_{11}e^{i\omega t} + \overline{\psi}_{11}e^{-i\omega t}) + \frac{\varepsilon^2}{2}(\psi_{12}e^{2i\omega t}$$
$$+ \overline{\psi}_{12}e^{-2i\omega t}) + \frac{\varepsilon^3}{2}(\psi_{13}e^{3i\omega t} + \overline{\psi}_{13}e^{-3i\omega t}) + O(\varepsilon^4)$$
$$+ \varepsilon^2\psi_{20} + \frac{\varepsilon^3}{2}(\psi_{21}e^{i\omega t} + \overline{\psi}_{21}e^{-i\omega t}) + O(\varepsilon^4) \qquad (4.5.3)$$

In the above expansion the functions ψ_{ij} are functions of s and N and an overbar denotes the complex conjugate of a quantity.

Notice that up to order ε we have recovered the classical unsteady first-order boundary-layer theory, which has been the subject of investigation of all earlier chapters. At the level of ε^2 we now have the second harmonic as well as a steady part of the flow which will combine the higher order displacement correction as well as the nonlinear contribution due to the steady streaming.

The expression given in Eq. (4.5.3) should be substituted in the Navier Stokes equations and the familiar steps of stretching and collecting powers of ε should be carried through. However, it appears easier now to derive the governing equations of ψ_{ij} directly from the equations of the previous sections. If proper care is taken of the nonlinear terms that appear, we can arrive at.

Order ε^0:

$$\frac{\partial\psi_{10}}{\partial N}\frac{\partial^2\psi_{10}}{\partial s\partial N} - \frac{\partial\psi_{10}}{\partial s}\frac{\partial^2\psi_{10}}{\partial N^2} = U_{10}\frac{\partial U_{10}}{\partial s} + \frac{\partial^3\psi_{10}}{\partial N^3} \qquad (4.5.4)$$

Order $\varepsilon e^{i\omega t}$:

$$i\omega \frac{\partial\psi_{11}}{\partial N} + \frac{\partial\psi_{10}}{\partial N}\frac{\partial^2\psi_{11}}{\partial s\partial N} + \frac{\partial\psi_{11}}{\partial N}\frac{\partial^2\psi_{10}}{\partial s\partial N} - \frac{\partial\psi_{10}}{\partial s}\frac{\partial^2\psi_{11}}{\partial N^2}$$

$$- \frac{\partial\psi_{11}}{\partial s}\frac{\partial^2\psi_{10}}{\partial N^2} = U_{10}\frac{\partial U_{11}}{\partial s} + U_{11}\frac{\partial U_{10}}{\partial s} + i\omega U_{11} + \frac{\partial^3\psi_{11}}{\partial N^2} \quad (4.5.5)$$

Order ε^2:

$$\frac{\partial\psi_{10}}{\partial N}\frac{\partial^2\psi_{20}}{\partial s\partial N} + \frac{\partial\psi_{20}}{\partial N}\frac{\partial^2\psi_{10}}{\partial s\partial N} + \frac{\partial\psi_{10}}{\partial N}\frac{\partial\psi_{20}}{\partial s\partial N} - \frac{\partial\psi_{20}}{\partial s}\frac{\partial^2\psi_{10}}{\partial N^2} - \frac{\partial\psi_{10}}{\partial s}\frac{\partial^2\psi_{20}}{\partial N^2}$$

$$- \frac{\partial}{\partial s}(U_{10}U_{20}) = \frac{1}{2}\frac{\partial\psi_{11}}{\partial s}\frac{\partial^2\overline{\psi}_{11}}{\partial N^2} + \frac{1}{2}\frac{\partial\overline{\psi}_{11}}{\partial s}\frac{\partial^2\psi_{11}}{\partial N^2} - \frac{1}{2}\frac{\partial\psi_{11}}{\partial N}\frac{\partial^2\overline{\psi}}{\partial s\partial N}$$

$$- \frac{1}{2}\frac{\partial\overline{\psi}_{11}}{\partial N}\frac{\partial^2\psi_{11}}{\partial s\partial N} + \frac{1}{2}U_{11}\frac{\partial U_{11}}{\partial s} + \frac{\partial^3\psi_{12}}{\partial N^3} \quad (4.5.6)$$

Order $\varepsilon^2 e^{2i\omega t}$

$$2i\omega\,\psi_{12} + \frac{\partial\psi_{10}}{\partial N}\frac{\partial^2\psi_{12}}{\partial s\partial N} + \frac{\partial\psi_{12}}{\partial N}\frac{\partial^2\psi_{10}}{\partial s\partial N} - \frac{\partial\psi_{10}}{\partial s}\frac{\partial^2\psi_{12}}{\partial N^2}$$

$$- \frac{\partial\psi_{12}}{\partial s}\frac{\partial^2\psi_{10}}{\partial N^2} = \frac{\partial^3\psi_{12}}{\partial N^3} \quad (4.5.7)$$

This method can be easily carried to higher order equations. At the level of ε^3 we will derive two differential equations by collecting coefficients of $e^{i\omega t}$, and $e^{3i\omega t}$. These can be solved for the unknown functions ψ_{12} and ψ_{21} which represent the third harmonic and the higher order oscillatory correction respectively.

The equations derived in this section are all linear partial differential equations with two space variables s and N. Their numerical solution is today considered straightforward and can be accomplished on a modern computing machine in a few seconds. In this section, for the purpose of comparing with earlier classical results, we will seek self-similar solutions for the cases of flat plate and wedge flows.

(b) $Re^{-1/2} = O(\varepsilon)$

This assumption corresponds to very low Reynolds numbers and such flows may be encountered perhaps in biological applications. Now the higher order steady correction will be of the same order with the fluctuating part of the flow and our expansion should have the form:

$$\psi_1(s,N,t) = \psi_{10} + \frac{\varepsilon}{2}(\psi_{11}e^{i\omega t} + \overline{\psi}_{11}e^{-i\omega t}) + \varepsilon\psi_{20} + \frac{\varepsilon^2}{2}(\psi_{12}e^{2i\omega t} + \overline{\psi}_{12}e^{-2i\omega t}) + \frac{\varepsilon^2}{2}(\psi_{21}e^{i\omega t} + \overline{\psi}_{21}e^{-i\omega t}) \tag{4.5.8}$$

Collecting terms in a similar fashion we arrive at

Order ε^0

$$\frac{\partial\psi_{10}}{\partial N}\frac{\partial^2\psi_{10}}{\partial s\partial N} - \frac{\partial\psi_{10}}{\partial s}\frac{\partial^2\psi_{10}}{\partial N^2} = U_{10}\frac{dU_{10}}{\partial s} + \frac{\partial^3\psi_{10}}{\partial N^3} \tag{4.5.9}$$

Order $\varepsilon e^{i\omega t}$

$$i\omega\psi_{11} + \frac{\partial\psi_{10}}{\partial N}\frac{\partial^2\psi_{11}}{\partial s\partial N} + \frac{\partial\psi_{11}}{\partial N}\frac{\partial^2\psi_{10}}{\partial s\partial N} - \frac{\partial\psi_{10}}{\partial s}\frac{\partial^2\psi_{11}}{\partial N^2} - \frac{\partial\psi_{11}}{\partial s}\frac{\partial^2\psi_{10}}{\partial N^2} = i\omega U_{11} + U_{10}\frac{dU_{11}}{ds} + U_{11}\frac{dU_{10}}{ds} + \frac{\partial^3\psi_{11}}{\partial N^3} \tag{4.5.10}$$

Order ε

$$\frac{\partial\psi_{10}}{\partial N}\frac{\partial^2\psi_{20}}{\partial s\partial N} + \frac{\partial\psi_{20}}{\partial N}\frac{\partial^2\psi_{10}}{\partial s\partial N} - \frac{\partial\psi_{10}}{\partial s}\frac{\partial^2\psi_{20}}{\partial N^2} - \frac{\partial\psi_{20}}{\partial s}\frac{\partial^2\psi_{10}}{\partial N^2} = \frac{\partial}{\partial s}(U_{10}U_{20}) + \frac{\partial^3\psi_{20}}{\partial N^3} \tag{4.5.11}$$

Notice that now the higher order equation does not contain any streaming terms. Equations of order $\varepsilon^2 e^{2i\omega t}$ and $\varepsilon^2 e^{i\omega t}$ can be derived which will be used to solve for the unknown functions ψ_{12} and ψ_{21}. Numerical analysis can be used again to solve the equations successively.

4.6 A Numerical Example

In this section we will study the case of $Re^{-1/2} = \varepsilon^2$ and fluctuating flows over configurations that permit self-similar solutions of the first-order problem, that is

$$U_{10}(s) = U_{20}(s) = c_0 s^m \tag{4.6.1}$$

We then seek solutions to the differential equations for ψ_{10}, Eq. (4.5.4) in the form

$$\psi_{10} = \left[\frac{2}{(m+1)c_0 s^{m-1}}\right]^{1/2} c_0 s^m\, f(\eta) \tag{4.6.2}$$

where

$$\eta = \left[\frac{m+1}{2}\, c_0 s^{m-1}\right]^{1/2} N \tag{4.6.3}$$

The fluctuating part of the flow can be expressed again in the form of an expansion, for small frequency parameters, as described in the earlier chapters.

$$\psi_{11} = \left[\frac{2}{(m+1)c_0 s^{m-1}}\right] c_0 s^m \sum_{a=0}^{\infty} \left(\frac{i\omega}{c_0}\right)^a s^{(1-m)a} F_a(\eta) \tag{4.6.4}$$

The differential equations that govern the functions $f(\eta)$ and $F_a(\eta)$ are then given by

$$f''' + ff'' + \frac{2m}{m+1}(1 - (f')^2) = 0 \qquad (4.6.5)$$

$$F_a''' + \frac{(2m+(1-m)(2a+1))}{m+1} F_a f'' + fF_a''$$
$$- \frac{2[2m+(1-m)a]}{m+1} F_a' f' + \frac{4m}{m+1}\delta_{a0} + \delta_{a1}\frac{2}{m+1}$$
$$- \frac{2}{m+1} F_{a-1}' = 0 \qquad (4.6.6)$$

where δ_{ij} is Kronecker's delta and the appropriate boundary conditions that correspond to first order matching are

$$f(0) = f'(0) = 0 \qquad (4.6.7a)$$

$$f'(\infty) = 1 \qquad (4.6.7b)$$

$$F_0(0) = F_0'(0) = 0 \qquad (4.6.8a)$$

$$F_0(\infty) = 1 \qquad (4.6.8b)$$

$$F_a(0) = F_a'(0) = F_a'(\infty) = 0 \text{ for } a \geq 1 \qquad (4.6.9)$$

The terminology here may be a little confusing. We insist to consider the matching of the steady and fluctuating parts of the flow, that is the terms $\psi_{10} + \varepsilon\psi_{11}e^{i\omega t}$, with the terms $U_{10} + \varepsilon U_{11}e^{i\omega t}$, as a first order matching, in order to retain the classical terminology of boundary layer theory. In this sense we still consider first-order matching to correspond to $p = 1$, $q = 1$ although we have already taken 2 terms in our expansions.

Let us proceed now to the equations of order ε^2 and solve the equation for the steady part of the higher order boundary layer stream-function, that is Eq. (4.5.6). We will seek again solutions to this equation in an expansion form for small frequencies:

$$\psi_{20} = \left[\frac{2}{(m+1)c_0 s^{m-1}}\right]^{1/2} c_0 s^m \sum_{a=0}^{\infty} \left(\frac{i\omega}{c_0}\right)^{2k} s^{2(1-m)a} G_a(\eta) \quad (4.6.10)$$

Notice that although ψ_{20} in the expansion is independent of time it is necessary to include a frequency dependent term in Eq. (4.6.10) to account for the nonlinear effects that are introduced through the terms that depend on ψ_{11}.

The equation that governs the functions $G_a(\eta)$ are

$$G_0''' - \frac{4m}{m+1} G_0'f' + fG_0'' + f''G_0 + \frac{4m}{m+1} = \frac{1}{m+1}\,[m(F_0')^2$$

$$- \frac{m+1}{2} F_0F'' - m] \quad (4.6.11)$$

$$G_1''' - \frac{4}{m+1} G_1'f' + fG_1'' + \frac{5-3m}{m+1} G_1f'' = \frac{1}{m+1}\,[2F_0'F_2' - (F_1')^2$$

$$- \frac{5-3m}{2} F_2F_0'' - \frac{m+1}{2} F_0F_2'' + \frac{3-m}{2} F_1F_1''] \quad (4.6.12)$$

Matching up to this order now requires solution of the outer flow and in our notation, this implies that (Van Dyke [47]

for $m = 0$, (flat plate)

$$G_a(0) = G_a'(0) = 0 \quad (4.6.13a)$$

$$G_a'(\infty) \to 0 \quad \text{for } a = 0,1,\ldots \quad (4.6.13b)$$

for $m \neq 0$

$$G_0(0) = G_0'(0) = 0 \quad (4.6.14a)$$

$$G_0'(\infty) \to 1 \quad (4.6.14b)$$

$$G_a(0) = G_a'(0) = 0 \quad (4.6.15a)$$

$$G_a(\infty) \to 0 \quad \text{for } i = 1,2,\ldots \quad (4.6.15b)$$

It is very interesting to note that for steady flow, the nonlinear terms that depend on the fluctuating part, ψ_{11}, disappear from Eq. (4.5.6) and therefore the right-hand side of Eq. (4.6.11) vanishes. This equation then and the corresponding boundary conditions become homogeneous and if the solution is unique, then it is equal to zero. That is for steady flow over a flat plate the second order boundary layer terms vanish. However if a fluctuation is included in the first order boundary layer, then a streaming effect drives the second order terms, even with $U_{20} = 0$.

Equations (4.6.5), (4.6.6) and (4.6.11), (4.6.12) subject to the appropriate boundary conditions were solved numerically as described in previous chapters. The method was tested, for steady stagnation flow, against the numerical results of Van Dyke [47]. In particular the value of $F''_{1d}(0)$ was chosen for comparison. The reader should be cautioned about the fact that in this paper the notation is different and Van Dyke's $F_{1d}(\eta)$ corresponds to our $G_0(\eta)$. The numerical results compare as follows: $F''_{1d}(0) = 1.84882$, $G''_0(0) = 1.848894$.

In Fig. 4.2 we display the profiles of the functions G'_0 and G'_1 for m = 0, that is a flat plate. For steady flow these functions are identically equal to zero. This solution therefore represents a secondary steady flow which seems to have a recirculating character. It appears as a vortex embedded in the boundary layer. Similar results for m = 1, that is for stagnation flow are shown in Fig. 4.3. In this figure the departure from steady flow is evident.

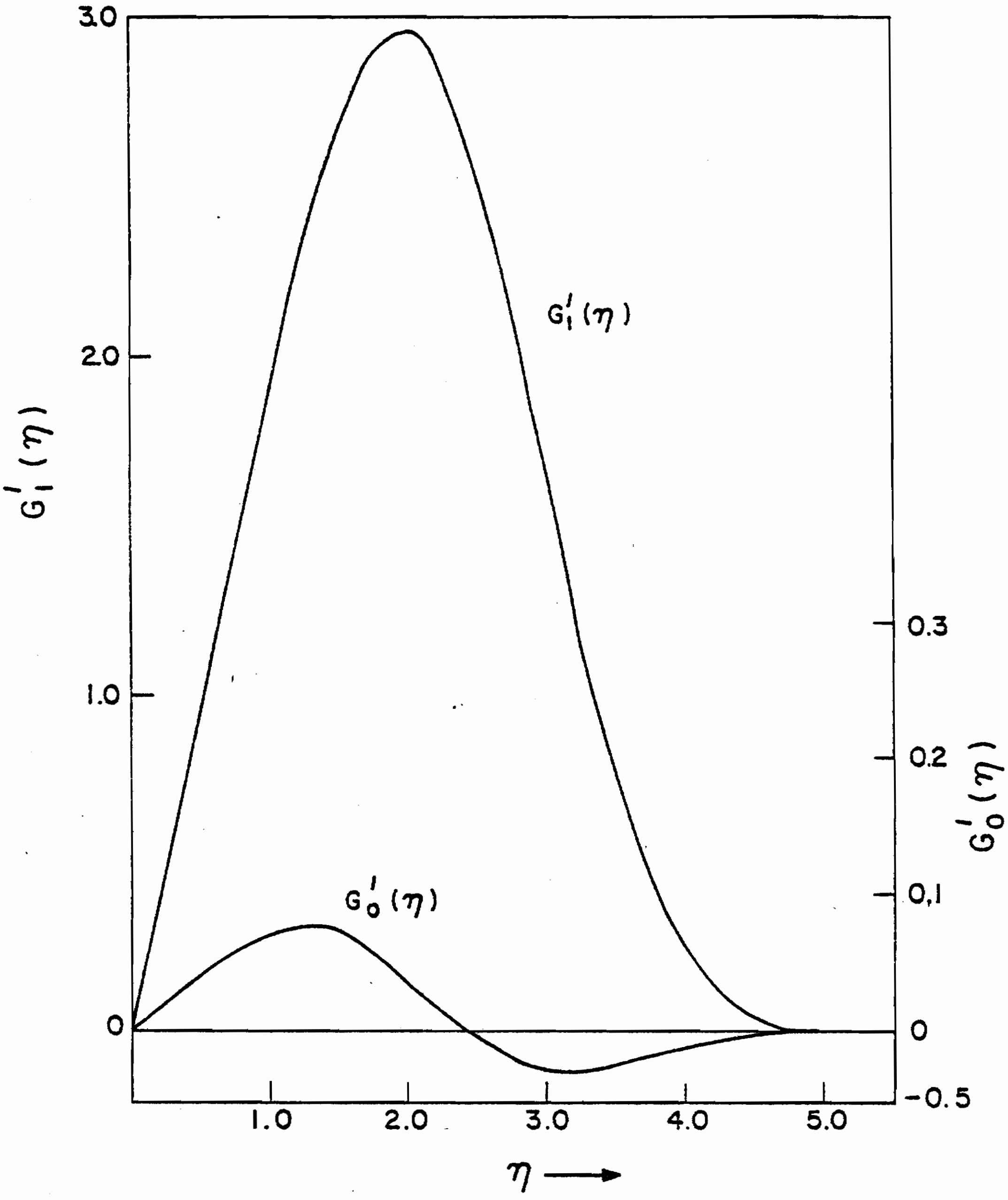

Figure 4.2 Second order boundary layer profiles, $G_0'(\eta)$ and $G_1'(\eta)$ for a flat plate.

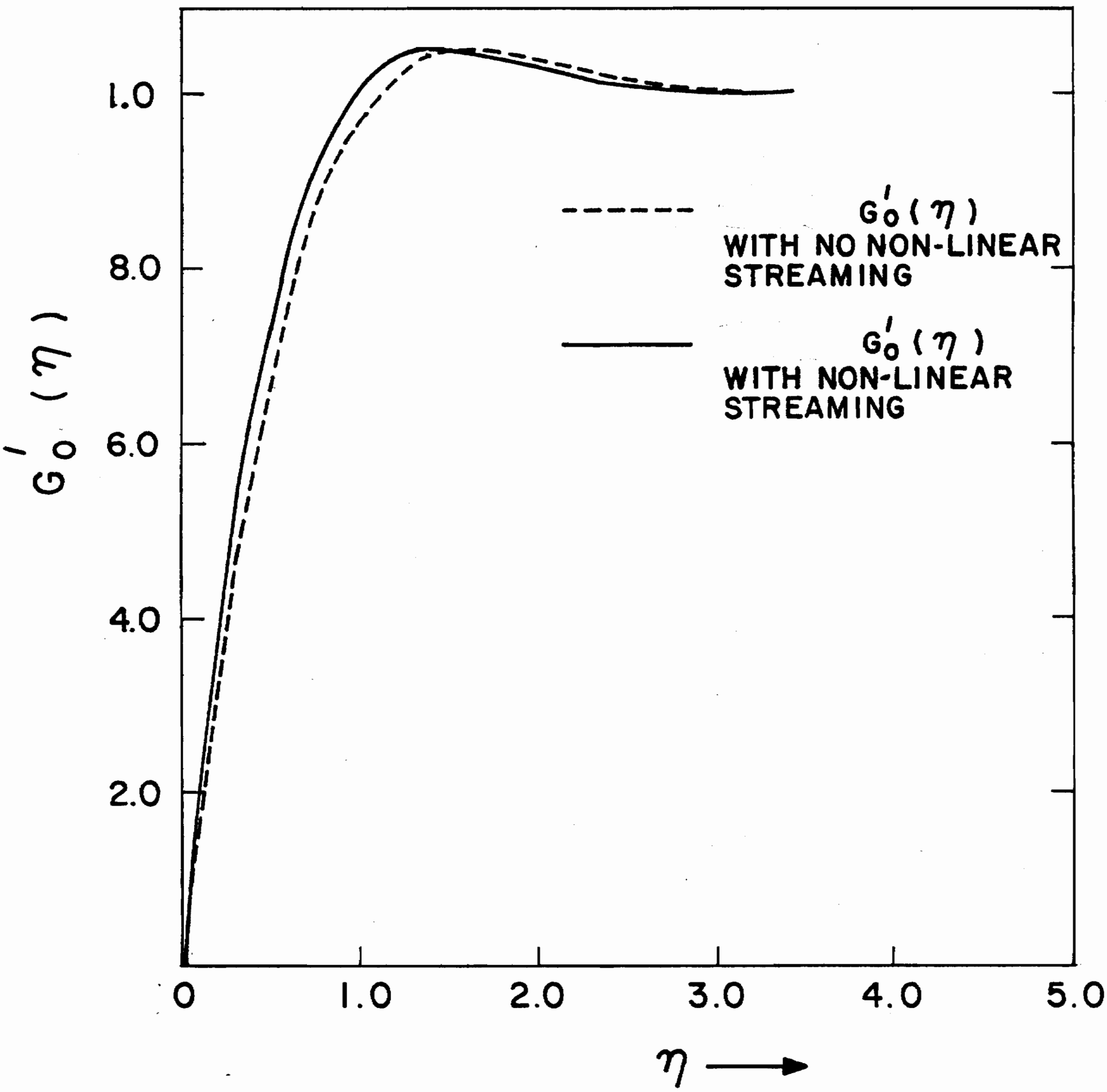

Figure 4.3 Second order boundary layer profiles of $G_0'(\eta)$ for stagnation flow.

CHAPTER FIVE

SUMMARY AND CONCLUSIONS

Oscillating laminar boundary layers have been investigated extensively as described in the introduction. In this dissertation it is attempted to extend the theory of oscillating flows into categories of problems that have not been attacked so far. In particular the problem of three-dimensional boundary layers and the cross flow effects, the problems of compressibility and heat transfer and the problem of higher order boundary layer effects, are investigated. This is done within the framework of Lighthill's theory, that is for outer flows that oscillate harmonically with small amplitudes, $\varepsilon \ll 1$. The theory is therefore based on a small amplitude assumption, which, however, appears to be very realistic for most practical applications. The governing equations are derived in their most general form, for the cases mentioned before and within the first order boundary-layer approximation and the small amplitude assumption, that is $Re \gg 1$ and $\varepsilon \ll 1$. It is therefore tacitly assumed for the theory of chapters 1 through 3, that $Re^{-1/2} \ll \varepsilon$ and therefore any higher order boundary-layer corrections are small compared to the unsteady terms of the expansions in terms of the amplitude parameter. In chapter four we investigate exactly this situation, assuming that the Reynolds number is not as large and as a result its inverse square root is comparable to the amplitude parameter $Re^{-1/2} = O(\varepsilon^2)$.

Numerical solutions are derived for special cases in terms of coordinates expansions. The first term of the steady part of the expansion corresponds to self similar flows, that is the flat plate, the wedge and the stagnation flow. More terms are included in the coordinate expansion to account for non-similar flow situation. This is a classical and well-known method with many advantages and disadvantages. For example it is well-known that its results are global and readjusting the values of the parameters involved, one may derive immediately the solution about a large family of body configurations. However, the accuracy of the method is known to be poor in regions of adverse pressure gradients, especially if the neighborhood of separation is approached. With the advent of the modern computing machines,direct integration of the partial differential equations appears today straightforward and perhaps cheaper. The present solutions therefore, especially due to the special body configurations considered, have rather qualitative value. Their asymptotic character is a limiting factor but on the other hand it offers physical insight and understanding of the counteracting and complex effects of oscillating boundary layers.

The fluctuating part of the solution is also assumed in the form of a series expansion and now the frequency of the oscillation is included. The parameter that emerges as the proper coordinate is $\omega x/U$, as described in detail in the earlier chapters. This imposes a further limitation to the present solutions. To insure that in the frequency expansions the terms of order $(\omega x/U_0)^2$ are more important than the next term in the amplitude expansion we must again assume that $\varepsilon^2 << (\omega x/U_0)^2$. The problem is thus reduced to sets of nonlinear and linear but ordinary differential equations that are solved simultaneously.

Basic conclusions are drawn in each chapter for the particular situation involved. Some highlights of the results are also included here. Investigating a special case of three-dimensional flows it is discovered that the non-linearity of the boundary layer couples the crosswise and spanwise velocity components and may generate a fluctuating cross flow. Phase differences are induced and the skin friction lines fluctuate in direction. This is a very interesting phenomenon: if the outer flow fluctuates only in magnitude, that is if the outer flow streamlines are fixed in space, the nonlinearity of the boundary layer and the cross-flow coupling generate a set of skin friction lines that fluctuate in direction.

In compressible oscillating boundary layer flows, one is encountered with phenomena that have not been considered before. Here special care is required for the outer flow conditions. Inviscid pressure, density, temperature and velocity fluctuations must be compatible with the momentum and energy equation. Simple inviscid solutions are here proposed and the response of the boundary layer to such outer flows is studied. Skin friction, heat transfer and displacement thickness fluctuations are investigated. It appears that for low frequencies compressibility tends to reduce the fluctuating component of the skin friction but increase its phase lead. Moreover it appears that the wall temperature has a strong effect on the velocity and temperature profiles. The skin friction amplitude and the phase angle increase considerably with the temperature of the wall. A very interesting and strong effect is discovered in relation to outer flows fluctuating in direction. It appears that the velocity and temperature overshoots are substantially larger than if the outer flow fluctuates only in magnitude. In all of the cases considered the pressure gradient of the mean flow has a suppressing effect on the overshoot of fluctuating profiles. However, the most significant contribution of the present investigation with regard to the compressibility effects is the identification of the total enthalpy fluctuations and the technique for circumventing the difficulties in solving the coupled unsteady nonlinear equations of momentum and energy.

Finally in chapter four the combined effects of higher order theory and externally imposed harmonic oscillations are considered for the first time. Assuming that $Re^{-1/2} = O(\varepsilon^2)$ one arrives at the conclusion that the second order boundary layer corrections, following the terminology of van Dyke, are influenced by the nonlinear streaming contributions of the fluctuations. This effect generates solutions for the flat plate, a case for which the second order steady boundary-layer solution is identically equal to zero. The second and third harmonics which are of order, ε^2 and ε^3 respectively satisfy linear differential equations which can be solved successively.

Recapitulaing, the present dissertation employs a well-known asymptotic theory in order to study special aspects of unsteady boundary-layer flows that have not been considered up to now. Double and triple expansions are introduced with the limitations described before. Ordinary differential equations are solved numerically and one may build the solution to a specific problem by evaluating the parameters ε,

$R_e^{-1/2}$ and $\omega x/U_0$ and summing up the first few terms of the asymptotic terms involved. On the other extreme of the available methods one would have to integrate the systems of equations directly, by a finite differencing scheme. However this method would require the solution of coupled nonlinear partial differential equations in three or four independent variables. Moreover, due to the parabolic character of the equations, a two- or three-dimensional mesh of storage would be required, respectively. Clearly the best method should lie somewhere in between. The present work essentially replaces a large portion of the numerical calculations by analysis, at least for the special body configurations and outer flow fluctuations considered. It therefore offers physical insight to the design engineer and identifies the basic features of the flow. Moreover it provides test cases for numerical calculations that would be later constructed to solve the most general problem.

REFERENCES

1. Stuart, J. T., "Unsteady Boundary Layers", in Recent Research of Unsteady Boundary Layers, E. A. Eichelbrenner (ed.), pp. 1-46, 1971.
2. Riley, N., "Unsteady Laminar Boundary Layers", SIAM Reviews, 17 Part 1, pp. 274-297, 1975.
3. Telionis, D. P., "Calculations of Time Dependent Boundary Layers", in Unsteady Aerodynamics, R. B. Kinney (ed.), 1, pp. 155-190, 1975.
4. Telionis, D. P., "Unsteady Boundary Layers, Separated and Attached", AGARD Symposium on Unsteady Aerodynamics, Paper No. 16, September 1977.
5. Lighthill, M. J., "The Response of Laminar Skin Friction and Heat Transfer to Fluctuation in the Stream Velocity", Proc. Roy. Soc. Ser., A 224, pp. 1-23, 1954.
6. Illingworth, C. R., "The Effects of a Sound Wave on the Compressible Boundary Layer on a Flat Plate", J. Fluid Mech., 3, pp. 471-493, 1958.
7. Rott, N. an Rosenweig, M. L., "On the Response of the Laminar Boundary Layer to Small Fluctuations of the Free-Stream Velocity", J. of Aero. Sci., 27, pp. 741-747, 1960.
8. Sarma, G. N., "Solutions of Unsteady Boundary Layer Equations", Proc. Comb. Phil. Soc., 60, pp. 137-158, 1964.
9. Sarma, G. N., "A General Theory of Unsteady Comprssible Boundary Layers with and Without Suction or Injection", Proc.Camb. Phil. Soc., 61, pp. 795-807, 1965.

10. Gupta, T. R., "Parabolic Flow over a Flat Plate with a Wave Disturbance in the Main Stream", Indian Academy of Sciences, LXXVI, Sec. A, No. 4, pp. 137-152, 1972.

11. Gupta, T. R., "Flow Past a Moving Yawed Infinite Wedge and Circular Cylinder", Proc. Nat. Acad. Sci. India, 42 (A), IV., pp. 297-313, 1972.

12. Moore, F. K., "Unsteady Laminar Boundary-Layer Flows", NACA TN 2471, 1951.

13. Moore, F. K., "The Unsteady Laminar Boundary Layer of a Wedge and a Related Three-Dimensional Problem", Heat Transfer and Fluid Mech. Institute, June 19-21, pp. 99-117, 1957.

14. Ostrach, S., "Compressible Laminar Boundary Layer and Heat Transfer for Unsteady Motions of a Flat Plate", NACA TN 3569, 1955.

15. Sparrow, E. M., and Gregg, J. L, "Nonsteady Surface Temperature Effects on Forced Convection Heat Transfer", J. of Aero. Sci., pp. 776-777, October 1957.

16. Lin, C. C., "Motion in the Boundary Layer with a Rapidly Oscillating External Flow", Proc. 9th Int. Congress Appl. Mech., Brussels, 4, pp. 155-169, 1956.

17. King, W. S., "Low-Frequency, large-Amplitude Fluctuations of the Laminar Boundary Layer", AIAA J., 4, No. 6, pp. 994-1001, 1966.

18. Phillips, J. H. and Ackerberg, "A Numerical Method for Integrating the Unsteady Boundary-Layer Equations when there are Regions of Backflow", J. Fluid Mech., 58, pp. 561-579, 1973.

19. Tsahalis, D. Th., and Telionis, D. P., "Oscillating Laminar Boundary Layers and Unsteady Separation", AIAA J., 12, pp. 1469-1476, 1974.

20. Telionis, D. P. and Romaniuk, M. S., "Nonlinear Streaming in Boundary Layers", Proceedigns of the 12th Annual Meeting of the Society of Engineering Science, pp. 1169-1180, 1975.

21. Ackerberg, R. C., and Phillips, J. H., "The Unsteady Laminar Boundary Layer on a Semi-infinite Plate Due to Small Fluctuations in the Magnitude of the Free-Stream Velocity", J. Fluid Mech., 51, pp. 137-157, 1972.

22. Lam, S. H., and Rott, N., "Theory of Linearized Time-Dependent Boundary Layers", Cornell Univ. GSAE Rep. AFOSR TN-60-1100, 1960.

23. Moore, F. K. and Ostrach, S., "Displacement Thickness of the Unsteady Boundary Layer", J. of the Aeronautical Sci., 124, pp. 77-85, 1957.

24. Tsahalis, D. Th., and Telionis, D. P., "Oscillating Boundary Layers with Large Amplitude", in Unsteady Flows in Jet Engines, F. O. Carta (ed.), pp. 407-415, 1974.

25. Gribben, R. J., "The Laminar Boundary Layer on a Hot Cylinder Fixed in a Fluctuating Stream", J. of Appl. Mech., 28, pp. 339-346, 1961.

26. Gribben, R. J., "The Fluctuating Flow of a Gas Near a Stagnation Point on a Hot Wall", J. of Appl. Mech., 38, pp. 820-828, 1971.

27. Vimàla, C. S., and Nath, G., "Unsteady Laminar Boundary Layers in a Compressible Stagnation Flow", J. of Fluid Mech., 70, pp. 561-572, 1975.

28. Van Dyke, M., "Higher Approximations in Boundary-Layer Theory. Part I. General Analysis", J. of Fluid Mech., 14, pp. 161-167, 1962.

29. Van Dyke, M., "Higher Order Boundary Layer Theory", Annul Review of Fluid Mechanics, 1, W. R. Sears and M. Van Dyke (eds.), 1969.

30. Sears, W. R., "The Boundary Layer of Yawed Cylinders", J. Aero. Sci., 15, No. 1, pp. 49-52, 1948.

31. Görtler, H., "Zur Laminaren Grezcschicht am Schiebenden Zylinder", Arch. Mat, Karlsruche, 3, pp. 216-231, 1952.

32. McCroskey, W. J., and Yaggy, P. F., "Laminar Boundary Layers on Helicoptor Rotors in Forward Flight", AIAA J., 6, pp. 1919-1926, 1968.

33. Dwyer, H. A., and McCroskey, W. J., "Cross flow and Unsteady Boundary-Layer Effects on Rotating Blades", AIAA Paper No. 70-50, January 1970.

34. Young, Jr., W. H. and Williams, J. C., "The Boundary Layer on Rotating Blades in Forward Flight", AIAA J., 10, pp. 1613-1619, 1972.

35. Schlichting, H. (Editor), Boundary Layer Theory, Pergamon Press, 1968.

36. Howarth, L., "The Boundary Layer in Three Dimensional Flow", Phil. Mag(7), 42, pp. 1433-1440, 1951.

37. Howarth, L., "Concerning the Effects of Compressibility on Laminar Boundary Layers and Their Separation", Proc. Roy. Soc. London, Ser. A, pp. 194-216, 1948.

38. Görtler, Henry, "A New Series for the Calculation of Steady Laminar Boundary Layers", J. Math. and Mech., 6, No. 1, pp. 1-66, 1957.

39. Telionis, D. P. and Gupta, T. R., "Unsteady Heat Convection in Three Dimensional Compressible Flow", XXVIth Congress of the International Astronautical Federation, Paper No. 75-047, 1975.

40. Cohen, C. B. and Reshotko, E., "Similar Solutions for the Compressible Laminar Boundary Layer with Heat Transfer and Pressure Gradient", NACA Report 1293, 1956.

41. Reshotko, E. and Beckwith, I. E., "Compressible Laminar Boundary Layer over a Yawed Infinite Cylinder with Heat Transfer and Arbitrary Prandtl Number", NACA Report 1379, 1958.

42. Afzal, Noor and Rizvi, S. Mahdi A., "Second-Order Unsteady Stagnation-Point Boundary-Layer Solutions", AIAA J., 15, No. 7, pp. 1051-1053, July 1977.

43. Yang, K., "On Certain Similiar Solutions to Unsteady Laminar Boundary Layer Equations in Low Speed Flow", J. of Aero. Space Sci., 25, 1958.

44. Yang, K., "Unsteady Laminar Boundary-Layers in an Incompressible Stagnation Flow", J. of Appl. Mech., pp. 421-427, December 1958.

45. Van Dyke, M., Perturbation Methods in Fluid Mechanics, The Parabolic Press, Calif., 1975 (Originally published 1964 by Academic Press).

46. Nayfeh, A. H., Perturbation Methods, Wiley, New York, 1973.

47. Van Dyke, M., "Higher Approximation in Boundary-Layer Theory. Part 2. Application to Leading Edges", J. of Fluid Mech., 14, pp. 481-495, 1962.

APPENDIX A

The ordinary differential equations that govern the functions $F_q^{(2n)}$ and $\phi_q^{(2n)}$ for $n \geq 1$ appearing in the expansions of Eq. (2.4.2) and (2.5.7) are

$$F_q^{'''(2n)} = -\sum_{a=0}^{n}\left(\frac{4ma+m+1+2j}{m+1+2j}\right) f_{1+2a} F_q^{''(2n-2a)}$$

$$-\sum_{a=0}^{n}\left\{\frac{4m(n-a+2\ell+(1-m)(2q+1)+2j}{m+1+2j}\right\} f''_{1+2a} F_q^{(2n-2a)}$$

$$+ 2\sum_{a=0}^{n}\left\{\frac{2mn+m+\ell+(1-m)q}{m+1+2j}\right\} f'_{1+2a} F_q^{'(2n-2a)}$$

$$- \frac{2}{m+1+2j}\,(\delta_{1q} + (3m+\ell)\delta_{0q})\left(\frac{\gamma-1}{2}\right)^{n}$$

$$+ \frac{2}{m+1+2j}\,((m+\ell)\delta_{0q} + \delta_{1q}) \sum_{d=0}^{n-1}\left(\frac{\gamma-1}{2}\right)^{n-d}\left(\sum_{a=0}^{d} f'_{1+2a} f'_{1+2(d-a)}\right)$$

$$+ \frac{\gamma}{(m+1+2j)}\,((m+\ell)\delta_{0q} + \delta_{1q}) \sum_{d=0}^{n-1}\left(\frac{\gamma-1}{2}\right)^{n-d-1}\left(\sum_{a=0}^{d} f_{1+2a} f''_{1+2(d-a)}\right)$$

$$+ \frac{4m}{m+1+2j} \sum_{d=0}^{n-1}\left(\frac{\gamma-1}{2}\right)^{n-d}\left(\sum_{a=0}^{d} f'_{1+2a} F_q^{'(2d-2a)}\right)$$

$$-\left(\frac{4m}{m+1+2j}\right)(n-1)\left(\frac{\gamma-1}{2}\right)^{n}\delta_{0q} + \frac{4m}{m+1+2j}\,\delta_{0q}\sum_{d=0}^{n-2}(n-d-1)\times$$

$$\left(\frac{\gamma-1}{2}\right)^{n-d}\left(\sum_{a=0}^{d} f'_{1+2a} f'_{1+2(d-a)}\right)$$

$$+ \gamma\left(\frac{2m}{m+1+2j}\right)\delta_{oq} \sum_{d=0}^{n-2} (n-d-1) \left(\frac{\gamma-1}{2}\right)^{n-d-1} \sum_{a=0}^{d}\left(f_{1+2a}f''_{1+2(d-a)}\right)$$

$$+\left(\frac{\gamma m}{m+1+2j}\right) \sum_{d=0}^{n-1} \left(\frac{\gamma-1}{2}\right)^{n-d-1}\left(\sum_{a=0}^{d} f_{1+2a}F_q''^{(sd-2a)}\right.$$

$$\left.+\sum_{a=0}^{d} f''_{1+2a}F_q^{(2d-2a)}\right) +\left(\frac{2}{m+1+2j}\right)F'^{(2n)}_{q-1} \qquad (2.A1)$$

where $F'^{(2n)}_{-1}(n) = 0$ and

$$\phi_q'''^{(2n)} = -\sum_{a=0}^{n}\left(\frac{4ma+m+1+2j}{m+1+2j}\right) f_{1+2a}\ \phi_q''^{(2n-2a)}$$

$$-\sum_{a=0}^{n} \left\{\frac{4m(n-a)+(1-m)(2q+1)+2j}{m+1+2j}\right\}f''_{1+2a}\ \phi_q^{(2n-2a)}$$

$$+ 2\sum_{a=0}^{n} \left(\frac{2mn+m+(1-m)q}{m+1+2j}\right)f'_{1+2a}\ \phi_q'^{(2n-2a)}$$

$$+ \left(\frac{4m}{m+1+2j}\right) \sum_{d=0}^{n-1} \left(\frac{\gamma-1}{2}\right)^{n-d}\left(\sum_{a=0}^{d} f'_{1+2a}\ \phi_q'^{(2d-2a)}\right)$$

$$+\left(\frac{\gamma m}{m+1+2j}\right)\sum_{d=0}^{n-1} \left(\frac{\gamma-1}{2}\right)^{n-1-d}\left(\sum_{a=0}^{d} f_{1+2a}\ \phi_q''^{(2d-2a)}\right.$$

$$\left.+\sum_{a=0}^{d} f''_{1+2a}\phi_q^{(2d-2a)}\right)+ \left(\frac{2}{m+1+2j}\right) \phi'^{(2n)}_{q-1} \qquad (2.A2)$$

and again $\phi'^{(2n)}_{-1}\ (n) \equiv 0$

APPENDIX B

The differential equations for the functions $F_q^{(2)}$ and $G_q^{(2)}$ of the expansions (3.5.12) and (3.5.13), respectively, are

$$
\begin{aligned}
F_q'''^{(2)} = &- f_1 F_q''^{(2)} - \left(\frac{5m+1+2j}{m+1+2j}\right) f_3 F_q''^{(0)} + 2\left(\frac{2m+\ell+(1-mq+1}{m+1+2j}\right) f_1' F_q'^{(2)} \\
&+ 2\left(\frac{3m+\ell+(1-m)q}{m+1+2j}\right) f_3' F_q'^{(0)} - \frac{2\ell+2(1-m)q+(1-m)+2j}{m+1+2j} f_3'' F_q^{(2)} \\
&- \left(\frac{4m+2\ell+2(1-m)q+(1-m)+2j}{m+1+2j}\right) f_1'' F_q^{(2)}(\eta) \\
&- 2\left(\frac{(m+\ell)\delta_{0q}+\delta_{1q}}{m+1+2j}\right) g_3 - \frac{2m(\gamma-1)}{m+1+2j} G_q^{(0)}(\eta) - \frac{2m(\gamma-1)}{m+1+2j} \times \\
&\quad (\delta_{0q} + \frac{1}{\ell+1}\delta_{1q})\, g_1 + \frac{\gamma m}{m+1+2j} f\, F_q''^{(0)} \\
&+ \frac{\gamma m}{m+1+2j} f_1'' F_q^{(0)} + \frac{\gamma}{m+1+2j}((m+\ell)\delta_{0q} + \delta_{1q}) f_1 f_1''
\end{aligned}
\tag{3.B1}
$$

$$
\begin{aligned}
\left(\frac{m+1+2j}{2}\right)\frac{1}{P_r} G_q''^{(2)} = &\ (5m+\ell+(1-m)q) f_1' G_q^{(2)} - \left(\frac{m+1+2j}{2}\right) f_1 G_q'^{(2)} \\
&+ (\ell+m+(1-m)q) f_3' G_q^{(0)} - \left(\frac{5m+1+2j}{2}\right) f_3 G_q'^{(0)} \\
&+ \frac{1}{(\gamma-1)}\left\{4mF_q'^{(0)} g_5 + 2mg_3 F_q'^{(2)}\right\} - \left\{2mg_3 F_q'^{(0)}\right. \\
&\left. + mg\, F_q'^{(2)}\right\} - \frac{1}{\gamma-1}\left\{\left(\frac{2\ell+(1-m)(2q+1)+2j}{2}\right) g_5' F_q^{(0)}\right. \\
&\left. + \left(\frac{4m+2\ell(1-m)(2q+1+2j)}{2}\right) F_q^{(2)} g_3' + \left(\frac{8m+2\ell+(1-m)(2q+1)+2j}{2}\right) F_q^{(4)} g_1' \right\}
\end{aligned}
$$

$$+ \frac{1}{2}\left\{\left(\frac{2\ell+(1-m)(2q+1)+2j}{2}\right)F_q^{(0)}g_3' + \left(\frac{4m+2\ell+(1-m)(2q+1)+2j}{2}\right) \times\right.$$

$$\left.F_q^{(2)}g_1'\right\}+\left(g_3 \quad \frac{1}{\ell+1}\ \delta_{2q}+\delta_{1q}\right) + \frac{\gamma}{2(\gamma-1)}\ (f_1g_3' + g_1'f_3) \quad \times$$

$$\left((\ell+m)\delta_{0q}+\delta_{1q}\right) + (g_1f_3'+g_3f_1')\left((m+\ell)\delta_{0q}+\delta_{1q}\right)+m(\gamma-1) \quad \times$$

$$(f_1'G_q^{(0)} + f_1'g_1) + G_{g-1}^{(2)} - \left(\frac{m+1+2j}{2}\right)(f_1''F_q''^{(2)} + f_3''F_q''^{(0)} \qquad (3.B2)$$

where $G_{-1} = 0$.

VITA

The author was born in a small town, Katra (Jammu and Kashmir State) India on March 6, 1942. After graduating from Government High School Katra in 1958 he entered Gandhi Memorial Science College Jammu and received his B.S. degree in 1962 from Jammu and Kashmir University. In 1966 he graduated from the University of Roorkee with a M.S. in Applied Mathematics and subsequently started his teaching career as a research assistant at the same University where he received his Ph.D. (1970) in Applied Mathematics. From 1970 until December 31, 1974 the author was with the faculty, first with G. B. University of Agricultrue and Technology and later with the University of Roorkee.

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R Gupta
Dec. 15, 1977

TWO AND THREE-DIMENSIONAL INCOMPRESSIBLE
AND COMPRESSIBLE VISCOUS FLUCTUATIONS

BY

Tej R. Gupta

(ABSTRACT)

Small unsteady disturbances of boundary-layer flow fields are often encountered in engineering applications, as for example, in the aerodynamics of a helicopter rotor or a turbine cascade or in a fluttering airfoil as well as in a variety of bioengineering problems. The present dissertation is a unified attempt to study some special classes of unsteady two- and three-dimensional incompressible and compressible boundary-layer flows. The general character of the mathematical problem is investigated first for each particular area. Asymptotic solutions are then provided based on the assumption of small amplitude and small frequency of oscillation. Systems of general differential equations in a single variable are obtained and solved numerically by the shooting technique. A straightforward fourth order Runge-Kutta integration scheme is employed and the values of the functions at the edge of the boundary-layer are checked against the outer flow boundary conditions.

In the first chapter we study simultaneously the effects of three-dimensionality coupled with the response to outer flow oscillations. It is believed that the coupling will have significant implications in

cascade flows where the finite span blocks the development of cross flows. Some interesting features of oscillatory three-dimensional flows are disclosed. In particular it is found that the coupling of the momentum equations permits the transfer of momentum from the chordwise to the spanwise direction. In this way it is possible to excite a fluctuating boundary layer flow in the spanwise direction even though there is no outer flow fluctuations. In the second chapter the response of laminar compressible boundary layers to fluctuations of the skin of the body or the outer flow are studied in the special case of a wall at the adiabatic temperature. Unsteady outer pressure fluctuations are considered for the first time and their effects to the energy equation and heat transfer is estimated. The analysis holds both for two-dimensional and axisymmetric configurations.

In the third chapter the response of the compressible boundary layer to small fluctuations of the outer flow is extended to account for arbitrary wall temperatures. The governing equations and the appropriate boundary conditions are formulated for the first time in considerable generality. It appears that the outer flow properties do not oscillate in phase with each other. Such phase differences are augmented as one proceeds across the boundary layer. Results are presented for a steady part of the solution that corresponds to flat plate and stagnation flows and oscillations of the outer flow in magnitude or direction.

In the fourth chapter, the unsteady higher order boundary layer theory for oscillation with small amplitude are presented. First, the second order equations for unsteady flows are derived from the Navier-Stokes equations by employing the method of matched asymptotic expansions. Then the nature of second-order effects is examined for a flat plate and stagnation flow through typical numerical results that account for the nonlinear unsteady effects on the displacement terms.