A STUDY OF BUILDING RESPONSE AND DAMAGE DUE TO MINING-INDUCED GROUND MOVEMENTS

by

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Several methods have been developed to predict mining-induced ground movements. Some of these methods, such as the profile and influence functions, have been used successfully in a number of applications. Prediction methods, however, do not address the response of surface buildings and structures to mining-induced ground movements.

In order to study the response of a building to ground movements, a finite element model has been developed. The program named SRMP (Subsidence Response Modeling Program) is a large displacement, small strain, two dimensional finite element program. Such model is more appropriate than the commonly used small-displacement formulations and describes more accurately this particular problem because large displacements are involved in mining-induced ground movements. Four types of elements are employed in the program, namely plane, beam, transition and friction elements. Total Lagrangian (T.L.) formulation is used for plane elements and Updated Lagrangian (U.L.) formulation for beam, transition, and friction elements. The program consists of twenty six subroutines and requires about one mega-bytes of memory. It can model the slippage between foundation and subgrade. An important feature of SRMP is that it can simulate the excavation process continuously, without re-initiating the system.
variables and boundary conditions. Ground movement, building displacement, and stresses can be obtained, therefore, at each excavation stage.

The accuracy of the finite element model was verified through field data. The slippage between foundation and subgrade was analysed in depth. Structural deformations and stresses induced by ground movements were also studied and damage criteria in terms of ground displacements were proposed. Finally, based on the SRMP analyses, appropriate measures were developed which can provide better protection to surface structures affected by excavation-induced ground movements.
Acknowledgements

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Chapter 1  Introduction

1.1 General Statements and Objectives

The damaging of structures due to mining-induced surface subsidence is a serious problem in today's mining industry. It is estimated that, in the United States, over 8 million acres of land have been undermined of which 2 million acres have been impacted by subsidence. Ninety-nine percent of these subsided lands is the result of underground coal mining (Chen et al., 1982). As coal production increases, subsidence damage will continue to be a problem.

In the past few years, an extensive research effort has been undertaken in the U. S. to predict ground movements due to underground mining. This research, however, has largely ignored the damage and control aspects of the problem and little attention has been paid to the study of structural responses to mining-induced ground movement. As a result, the behavior of surface structures when impacted by mining subsidence is not clearly understood and, in this country, there are no generally accepted damage criteria for buildings which can relate type and magnitude of ground movement to damage severity. Furthermore, appropriate control measures have not been developed that can be applied to minimize the impact of mining.
In general, the extent of damage to a structure depends not only on the mining factors and geological conditions, which determine subsidence characteristics, but also on the location and type of the structure itself. Soil-foundation interaction also plays a very important part in the damage process. In order to relate ground movement to damage severity and to provide recommendations for appropriate control measures, it is important that the response of the structure to ground movement is clearly understood. The objectives of this research, therefore, can be summarized as follows:

1. **Explore the mechanism of damages to buildings due to mining subsidence.** This includes a study of structural responses to ground movement and understanding of the damage process.

2. **Based on the results obtained in objective one, develop damage criteria, which relate the magnitude of the ground movement to damage severity, and can be used to predict and assess subsidence damage to buildings.**

3. **Recommend measures for damage prevention and control, which are applicable to both mining and civil engineering.**

### 1.2 Approach

The research was based on the following concepts:

- The subsidence prediction techniques developed by Virginia Tech were applied to predict ground movements which can be related to structural damages.
- The finite element method was used to simulate the effect of subsidence on surface structures and to analyze the soil-foundation interaction.

- Case studies of structural behavior and damage due to mining-induced ground movement were collected.

- Foundation and superstructure classification, and characteristic subsidence parameters were utilized to form the basis for developing damage criteria.

- Damage control measures were developed, based on the results of this study.

1.3 Scope

Building damage due to mining subsidence is a very broad topic. In order to be more specific in this research, the following restrictions were applied:

1. The study was limited only to building damages caused by underground coal mining. Damages induced by other types of mining are not considered.

2. Ground subsidence induced by underground coal mining can take two forms, sag or trough and sinkhole. This research only deals with the building damages caused by sag subsidence.
Chapter 2  Review of Subsidence Prediction Methods

The earliest recorded case of subsidence damage can be traced to the early 15th century, when a court awarded £200 to a house owner in Durham, Britain, whose house was damaged by subsidence (Jarosz, 1988). However, it was not until the 18th century that the first attempts to study the mechanisms associated with mining subsidence were reported (VPI & SU, 1987). Since then, considerable work has been done on this subject. Some of the early developments in Europe and in the U. S. A. are listed in Tables 2.1 and 2.2 respectively.

The purpose of this chapter is to present a state-of-the-art review on subsidence prediction methods. These are discussed under three categories: empirical, semi-empirical and theoretical methods.

2.1 Terminology

Many special terms are used in mining subsidence studies. Sometimes, literature gives conflicting definition for certain terms. In order to clarify this, the definitions of the commonly used terms are given here. The terms are categorized into the following three groups:
<table>
<thead>
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<th>Development</th>
<th>Country</th>
<th>Date</th>
<th>Author/Reference</th>
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<td>Court awarded £200 to house owner in Durham, whose house was damaged by subsidence in the city of Liege.</td>
<td>Britain</td>
<td>Early 15th century</td>
<td>Shadbolt, 1978</td>
</tr>
<tr>
<td>Commission appointed to study subsidence</td>
<td>Belgium</td>
<td>1815</td>
<td>Shadbolt</td>
</tr>
<tr>
<td>Second commission studied subsidence in the city of Liege, and J. Gonot published the first theory of mining subsidence, <em>the Normal Theory</em>.</td>
<td>Belgium</td>
<td>1839</td>
<td>Gonot</td>
</tr>
<tr>
<td>Study of subsidence angle</td>
<td>Britain</td>
<td>1859</td>
<td>Dickinson</td>
</tr>
<tr>
<td>Authorities developed guidelines in Silesia to protect railways from subsidence.</td>
<td>Austria</td>
<td>1860</td>
<td>Zwartendijk, 1971</td>
</tr>
<tr>
<td>Technical publication on subsidence</td>
<td>Germany</td>
<td>1868</td>
<td>Schulz and van Spee</td>
</tr>
<tr>
<td>Russian commission of four engineers visited Belgium, England, and France to report on subsidence.</td>
<td>Belgium</td>
<td>1871</td>
<td>Zwartendijk, 1971</td>
</tr>
<tr>
<td>Vertical Subsidence theory was published.</td>
<td>Belgium</td>
<td>1875</td>
<td>Dumont</td>
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<td>California Owners Association of Liège reported on the differences between the normal and vertical theory.</td>
<td>Austria</td>
<td>1876</td>
<td>Jicinsky</td>
</tr>
<tr>
<td>Development</td>
<td>Country</td>
<td>Date</td>
<td>Author/Reference</td>
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<tr>
<td>Committee of the Mining and Metallurgical Society of Ostrau, Moravia,</td>
<td>Austria</td>
<td>1881</td>
<td>Zwartendyck, 1971</td>
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<tr>
<td>reported on the development characteristics of mining subsidence</td>
<td></td>
<td></td>
<td></td>
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<td>Publication of a comprehensive theory on subsidence</td>
<td>Austria</td>
<td>1882</td>
<td>Rziha</td>
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<td>Discussion of Longitudinal subsidence wave</td>
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<td>1885</td>
<td>Dixon</td>
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<td>First disaster caused by subsidence in Brux, when extensive surface</td>
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<td>1895</td>
<td>Helmhacker</td>
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<td>cracking caused quicksand to enter mining void. About 66 houses</td>
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<tr>
<td>were severely damaged and 2,000 people were left homeless</td>
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<td>Publication dealing with subsidence progress</td>
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<td>1897</td>
<td>McMurtree</td>
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<td>Publication on subsidence development with time</td>
<td>Britain</td>
<td>1898</td>
<td>Kay</td>
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<td>1900</td>
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<td>coalfields</td>
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<td>Halbum</td>
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<td>O'Donahue</td>
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<td>Gol-drech</td>
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<td>Discussion of most important subsidence parameters</td>
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<td>Knox</td>
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<tr>
<td>Development</td>
<td>Location</td>
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<td>Author/Reference</td>
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<td>Discussion on subsidence in the anthracite region</td>
<td>Pennsylvania</td>
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<td>Description on &quot;longwall&quot; system, including effects on the surface</td>
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<td>1894</td>
<td>Rice</td>
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<td>Discussion on horizontal surface movements due to subsidence</td>
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<td>Enzian</td>
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<td>Extensive building damage in the city of Scranton due to subsidence</td>
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<td>1909</td>
<td>Zwartendyk, 1971</td>
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<td>Report discussing the subsidence problem in the city of Scranton</td>
<td>Pennsylvania</td>
<td>1912</td>
<td>Griffith and Conner</td>
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<tr>
<td>U. S. Bureau of Mines initiated subsidence studies</td>
<td>Penn., Illinois</td>
<td>1913</td>
<td>Zwartendyk, 1971</td>
</tr>
<tr>
<td>Subsidence due to &quot;longwall&quot;</td>
<td>Illinois</td>
<td>1914</td>
<td>Andros</td>
</tr>
<tr>
<td>Publication of the most comprehensive subsidence study in the U. S. A., discussing both domestic as well as international experience</td>
<td>U. S. A.</td>
<td>1916</td>
<td>Young and Stock</td>
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2.1.1 Ground Movement Components

Subsidence: Mining induced ground surface deformations. In general, it refers to vertical displacement.

Vertical displacement (S): Vertical distance between the original position of a point and the position of the same point in the deformed state.

Horizontal displacement (U): Horizontal distance between the original position of a point and the position of the same point in the deformed state.

Tilt or Slope (T): First derivative of the vertical displacement with respect to horizontal distance. It can be calculated as differential vertical displacement of two adjacent points divided by the horizontal distance between them.

Curvature (K): First derivative of slope or second derivative of vertical displacement with respect to horizontal distance. It can be calculated as differential tilt of two adjacent points divided by the horizontal distance between them.

Radius of curvature ($\rho$): The reciprocal of curvature.

Horizontal strain ($\varepsilon$): Differential horizontal displacement of two adjacent points divided by the horizontal distance between them.

2.1.2 Parameters Describing Subsidence Profiles

Angle of draw or limit angle ($\gamma$): The angle between the vertical and the line connecting the edge of the workings to the beginning of the subsidence trough.

Angle of break ($\alpha$): The angle between the vertical and the line connecting the edge of the workings and the point of maximum tension.

Inflection point/line (I.P./L): The point or line dividing the concave and convex parts of the subsidence profile.
Radius of influence ($r$): Radius of the surface area affected by the underground excavation of a point.

2.1.3 Profile Development Characteristics

Sub-critical: The subsidence profile is not fully developed and the largest vertical displacement is less than $S_{\text{max}}$, the maximum possible vertical displacement.

Critical: The subsidence profile is fully developed and the center point has reached $S_{\text{max}}$.

Super-critical: The subsidence profile is fully developed and more than one point has reached $S_{\text{max}}$.

2.2 Empirical Prediction Methods

2.2.1 Profile Function Method

The earliest methods for predicting mining subsidence originated from the observation that the magnitude of subsidence depends on seam thickness, seam inclination, amount of stowing material and age of workings. In the early 1900's, empirical formulae were developed, in which the influence of the extraction area could be taken into consideration (Kratzsch, 1983). Today, one of the most popular empirical methods is the profile function method. A subsidence profile function is an equation describing one-half of the subsidence profile, ranging from zero to full subsidence value (Brauner, 1973; VPI & SU, 1987). The functions are obtained through statistical analyses or curve-fitting of observation data. Thus, profile functions possess strong regional characteristics and depend
on topographical, geological, and mining conditions. Table 2.3 shows some of the most widely-accepted profile functions.

The advantage of the profile function method is that it is simple to use, includes few field parameters, requires minimal calculations, and provides a satisfactory degree of accuracy for a specific mining region. One disadvantage is that profile functions are generally used for simple rectangular and theoretically infinitely long excavations, and thus are not easily adaptable to complex excavations, or for cases where significant variation of mining and surface parameters, such as mining height, extraction ratio and depth of excavation, are present (Brauner, 1973; Karmis et al., 1981). It is, generally, recommended that profile function methods can be used for predicting subsidence over longwall and room-and-pillar mines with regular mining plans (VPI & SU, 1987).

2.2.2 Angle-of-Intersection Method

Another empirical method, which is also one of the oldest (Kratzsch, 1983), is the angle-of-intersection method. This method establishes the position of the trough center, and that of other significant subsidence points, by means of a graph. A number of points are located on the subsidence profiles along strike and dip by using angles of intersection obtained from observations. The angles of intersection at the upper and lower face of a working in an inclined seam are different, and the same is true for those at the solid coal face and an old gob area. The angle-of-intersection method is easy to use and requires little mathematical computations, but it has a serious disadvantage in that mining and geological factors are excluded.
### Table 2.3 Basic Profile Functions

<table>
<thead>
<tr>
<th>Source</th>
<th>Profile Function</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>China</td>
<td>$S = S_{max}e^{-a\left(\frac{x}{R}\right)^b}$</td>
<td>CUMT, 1981</td>
</tr>
<tr>
<td>Germany</td>
<td>$S = S_{max}[1 - (x/R)^2]^2$</td>
<td>Brauner, 1973</td>
</tr>
<tr>
<td>Hungary</td>
<td>$S = S_{max}e^{-\frac{x^2}{2\sigma^2}}$</td>
<td>Brauner, 1973</td>
</tr>
<tr>
<td>Poland</td>
<td>$S = S_{max}e^{-nx^2}$</td>
<td>Brauner, 1973</td>
</tr>
<tr>
<td>USA</td>
<td>$S = \frac{1}{2} S_{max}(1 - \tanh \frac{cx}{B})$</td>
<td>Karmis et al., 1984</td>
</tr>
<tr>
<td>USSR</td>
<td>$S = S_{max}[1 - \frac{x}{R} + \frac{1}{2} \pi \sin(2\pi \frac{R-x}{R})]$</td>
<td>Kratzsch, 1983</td>
</tr>
</tbody>
</table>

**Notes:**

- $S = \text{subsidence};$
- $S_{max} = \text{maximum subsidence};$
- $x = \text{horizontal distance from the trough center to the point where the subsidence is calculated};$
- $a, b, n, c = \text{constants};$
- $R = \text{radius of the subsidence trough};$
- $l = \text{distance from the inflection point to the trough edge};$
- $B = \text{distance between the trough center and the inflection point}.$
2.3 Semi-empirical Prediction Methods

2.3.1 Influence Function Method

Although influence functions obtained through stochastic models can be considered as theoretical methods, most other influence functions contain a number of empirical parameters. Since these formulas must be integrated, either analytically or numerically, in order to develop subsidence profiles, the influence function methods are classified here as semi-empirical. These methods were developed primarily in central European coalfields and thus have been discussed in the literature (Litwiniszyn, 1957; Brauner, 1973; Kratzsch, 1983; VPI & SU, 1987). Table 2.4 lists the most widely used influence functions.

In this approach, a so-called influence function is assigned for every point of the rock strata, such that the integration of this function over a given mined area, represents the subsidence at that point. The magnitude of the horizontal movement can be obtained through additional assumptions, for instance by considering the relationship between horizontal movement and the first derivative of the vertical movement function (VPI & SU, 1987).

Considering two Cartesian coordinate systems, (x,y,z,) whose origin is at the seam level, and (s,t,z) whose origin is at the level where the subsidence profile is considered, with z being the common vertical axis for both systems. If the location of a point P(s,t,z) is given by coordinates s,t,z, and the influence function is given by G(x,y,s,t,z), the subsidence of this point caused by mining the area A will be:
Table 2.4 Basic Influence Functions (after Brauner, 1973)

<table>
<thead>
<tr>
<th>Influence Function</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ g = \frac{S_{\text{max}} r^3 \tan^2 \gamma}{\pi (\sin \gamma \cos \gamma + \frac{\pi}{2} - \gamma)(x^2 + r^2 \tan^2 \gamma)^2 x} ]</td>
<td>Bals, 1931</td>
</tr>
<tr>
<td>[ g = \frac{n S_{\text{max}}}{r^2} e^{-n \left( \frac{x}{r} \right)^2} ]</td>
<td>Brauner, 1973</td>
</tr>
<tr>
<td>[ g = \frac{3 S_{\text{max}}}{\pi r^2} \left[ 1 - \left( \frac{x}{r} \right)^2 \right]^2 ]</td>
<td>Grigorovich, 1965</td>
</tr>
<tr>
<td>[ g = \begin{cases} \frac{2 S_{\text{max}} \tan^2 \alpha}{3 \pi r^2 \tan^2 \gamma} &amp; 0 \leq x \leq \frac{\tan \gamma}{\tan \alpha} r \ \frac{S_{\text{max}} \tan^2 \alpha}{3 \pi r^2 (\tan^2 \alpha - \tan^2 \gamma)} &amp; \frac{\tan \gamma}{\tan \alpha} r \leq x \leq r \end{cases} ]</td>
<td>Keinhorat, 1934</td>
</tr>
<tr>
<td>[ g = \frac{n S_{\text{max}}}{2 \pi x \delta \Gamma \left( \frac{2}{n} \right)} e^{-\left( \frac{x}{x_0} \right)^n} ]</td>
<td>Kochmanaki, 1959</td>
</tr>
<tr>
<td>[ g = \frac{n \sqrt{2} n S_{\text{max}} e^{-4 \left( \frac{x}{r} \right)^{2n}}}{\pi r \Gamma \left( \frac{1}{2n} \right)} ]</td>
<td>Saan, 1949</td>
</tr>
</tbody>
</table>

Notes:
- \( g \) = influence function;
- \( S_{\text{max}} \) = maximum subsidence;
- \( r \) = radius of influence;
- \( \gamma \) = angle of drawing;
- \( \Gamma \) = \( \Gamma \) function;
- \( x \) = distance from the origin.
- \( n \) = constant.
\[ S(s,t,z) = \int_A G(x,y,s,t) dx dy \] (2.1)

When the integral of the influence function has an exact solution, the subsidence profile can be described easily by mathematical expressions which allow subsidence calculations at any point. Otherwise tables of integration value have to be used. Furthermore, the derivatives of the subsidence distribution function may describe other deformation parameters such as slopes (first derivative) and curvatures (second derivative) (VPI & SU, 1987).

In the past fifty years many influence functions have been developed (VPI & SU, 1987). One of the widely recognized influence functions is that proposed by Knothe, which utilizes a bell-shaped Gaussian function. For a two dimensional case, this can be expressed as (Jarosz, 1988):

\[ g(x,s) = \frac{1}{r} e^{-\pi \frac{(x-s)^2}{r^2}} \] (2.2)

where

\[ r = \text{radius of principal influence} = \frac{h}{\tan \beta}; \]
\[ h = \text{mining depth}; \]
\[ \beta = \text{angle of principal influence}; \]
\[ s = \text{location of the point, P(s), where subsidence is considered}; \text{ and} \]
\[ x = \text{location of the infinitesimal element of excavation}. \]

Subsidence at any point P(s), therefore, can be expressed as:
\[ S(x,s) = \frac{S_0}{r} \int_{-\infty}^{\infty} e^{-\pi \frac{(x-s)^2}{r^2}} \]  

(2.3)

\( S_0 \) is the roof sag of the excavation opening, which can be written as:

\[ S_0 = m(x)a(x) \]  

(2.4)

where

\[ m(x) = \text{mining height}; \text{ and} \]

\[ a(x) = \text{subsidence factor}. \]

If \( m(x) \) and \( a(x) \) are constants, then (Jarosz, 1988):

\[ S_0 = S_{\text{max}} = ma = \text{constant} \quad \text{for } x_1 \leq x \leq x_2 \]  

(2.5)

where \( x_1 \) and \( x_2 \) are the limits of the excavation, and

\[ S(x,s) = \frac{S_{\text{max}}}{r} \int_{x_1}^{x_2} e^{-\pi \frac{(x-s)^2}{r^2}} \]  

(2.6)

Influence functions cannot be directly obtained by measurement and the calculation procedures are based on the law of superposition (Brauner, 1973). They can be applied for subsidence or horizontal movement calculations for any mining system and extraction geometry (VP1 & SU, 1987).
2.3.2 Zone Area Method

The zone area method can be regarded as another semi-empirical method. It is sometimes called an empirical integration-grid method (Kratzsch, 1983). The required subsidence influence $e_j$ of each individual zone on a surface point is obtained by solving the $i$ simultaneous equations:

\[
\begin{bmatrix}
S_1 \\
S_2 \\
\vdots \\
S_i \\
\end{bmatrix}
= 
\begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1j} \\
a_{21} & a_{22} & \ldots & a_{2j} \\
\vdots & \vdots & \ddots & \vdots \\
a_{i1} & a_{i2} & \ldots & a_{ij} \\
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2 \\
\vdots \\
e_j \\
\end{bmatrix}
\quad (2.7)
\]

where

$S_i =$ subsidence at point $P_i$;

$e_j =$ zone influence factor;

$a_{ij} = C_i \frac{A'_{ij}}{A_j}$

$C_i =$ roof-floor convergence in the $i$th zone;

$A'_{ij} =$ the extracted area within $A_j$ which influences the subsidence at point $i$;

and

$A_j =$ the area of $j$th zone.

Knowing the zone influence factors, subsidence at any point in the region where the factors are obtained, can be calculated.
Although the early zone area methods became very popular in Central Europe, they also have some drawbacks. Perhaps the most noticed limitation of this approach was the so-called ribside subsidence value (VPI & SU, 1987). The latter is usually less than the half of the maximum subsidence predicted by the original zone area method (Brauner, 1973; Marr, 1975; VPI & SU, 1987). In order to overcome this drawback, the method was modified to non-linear formulation (Marr, 1975; VPI & SU, 1987). It was suggested that the influence area of a surface point be divided into several annular zones. The radius of influence area, \( r \), was given as:

\[
r = h \tan \gamma
\]

where

\( h = \) mining depth; and

\( \gamma = \) angle of draw.

Given an average draw angle of 35 degrees for British mining conditions, the radius of influence is 0.7 of the mining depth, and seven zones were assumed, each having a width of 0.1 of the mining depth. To eliminate the discrepancy of ribside subsidence, a non-linear relationship between subsidence and proportional extracted area was assumed in the form of

\[
\tilde{S} = mA^n \tilde{E}
\]

where

\( \tilde{S} = \) subsidence vector;

\( \tilde{E} = \) zone influence vector;

\( m = \) extraction thickness;
A = proportional extracted area, a two dimensional array; and
n = influence constant, unity if the relationship is linear.

For one surface point, the above equation can be written as follows:

\[ S = m \sum_{i=0}^{2} e_j a_j^n \]  \hspace{1cm} (2.10)

In applying the above equation, the zone influence factors and the influence constant must be determined from field measurements. The application of this method for subsidence prediction in the United States was discussed in detail by Goodman (1980) and VPI & SU (1980, 1987).

2.3.3 Vertical Zone Method

In this method, the original cross-section area, which includes the coal seam and the overburden defined by the angle of draw, is divided into five zones: excavated zone, caved zone, fractured zone, highly deformed zone and subsidence profile zone (Figure 2.1) (Begley, 1989). The last zone is encompassed by the subsidence profile and the pre-subsidence surface line, which can be determined as:

\[ A_5 = A_1 - \sum_{i=2}^{4} A_i K_i \]  \hspace{1cm} (2.10)

where
Figure 2.1. Vertical Zones (Begley, 1989)
\[ A_5 = \text{area of the subsidence profile zone}; \]
\[ A_1 = \text{area of the excavation zone}; \]
\[ A_i = \text{area of the caved, fractured, and deformed zones}; \text{ and} \]
\[ K_i = \text{expansion factor of the caved, fractured and deformed zones}. \]

Once \( A_5 \) is obtained, the maximum subsidence can be determined, assuming that the subsidence profile can be described by a certain function.

In order to obtain the area of the subsidence profile, each of the other four areas has to be determined. The excavation area is just the mining height multiplied by the panel width. The height of the caved zone can be predicted by the following relationship:

\[ h_c = (8 - 6P_i)h_s \]  \hspace{1cm} (2.11)

where

\[ h_c = \text{height of the caved zone}; \]
\[ P_i = \text{percent of strong strata in the immediate roof}; \text{ and} \]
\[ h_s = \text{seam height}. \]

This relationship is also used to estimate the height of the fractured zone. Once the heights of these two zones are known, the area of each zone can be calculated using geometrical relationship. The area of the highly deformed zone could easily be determined as long as the angle of draw or the extent of the subsidence on the surface is known. The angle of draw is obtained by the following empirical formula:

\[ \gamma = a + \text{RQD} P_0^b \]  \hspace{1cm} (2.12)

where
\( y = \) angle of draw;

\( P_0 = \) percent of strong strata in the total overburden;

\( \text{RQD} = \) Rock Quality Designation (\%); and

\( a, b = \) constants.

The calculation of \( A_S \) also requires that the expansion factors must be determined. These factors are obtained through field studies.

Although this method can predict the maximum subsidence, its results are highly dependent on the pre-assumed function.

2.4 Theoretical Prediction Methods

The final category of subsidence prediction methods is theoretical approaches. Brauner (1973) and Kratzsch (1983) summarized several of these methods in their publications. The most significant include:

2.4.1 Theory of Elasticity Method

In this method the rock mass is regarded as an elastic medium. Assuming that the effect of extracting an infinitesimal element can be represented by a resulting force \( F \), at point \( S \) over that element, the elementary subsidence trough can be given by (Kratzsch, 1983):

\[
S_0 = \frac{F}{4\pi G}\left(\frac{1}{R} + \frac{H^2}{R^3}\right)
\]

(2.13)

where:
\[ S_0 = \text{elementary subsidence}; \]
\[ H = \text{mining depth}; \]
\[ R = \sqrt{r^2 + H^2} \]
\[ G = \text{modulus of shear; and} \]
\[ r = \text{Equation 2.8}. \]

A comparison with other influence functions shows that Equation 2.13 can be modified to:

\[ S_0 = c \frac{1}{\sqrt{20r^2 + H^2}} + \frac{H^2}{\sqrt{(20r^2 + H^2)^3}} \quad (2.14) \]

for practical subsidence calculation, where c is a constant (Kratzsch, 1983). Integration of Equation 2.13 over the extracted critical area will yield the total subsidence.

### 2.4.2 Finite Element Method (FEM)

The finite element method is a very powerful numerical method used in almost every discipline of engineering. The concept of this method is that a continuum or a discontinuum can be treated as a collection of separate elements, bound to each other only at nodal points, where they interact through their stresses and displacements. This approach allows conclusions to be drawn on the entire domain from the individual elements (Kratzsch, 1983). It also facilitates the solution of problems with irregular and heterogeneous bodies. The application of the finite element method involves essentially the following steps:
1. Discretization of the domain by means of a system of "elements". In a computer program, this is usually achieved by a mesh generator.

2. Selection of interpolation functions. If the global coordinates and the nodal unknowns adopt the same interpolation function, then the element is called an isoparameter element.

3. Determination of the constitutive relation. The constitutive matrix involves only material properties for elastic problems, but it is a function of both material properties and stress for non-elastic problems.

4. Calculation of element stiffness matrix and force vector. The stiffness matrix also depends on the type of problem to be solved. It may involve only the geometry of the domain and its material properties or it may include additional variables such as displacement and stress.

5. Assembling the global stiffness matrix and force vector.

6. Calculation of the nodal unknowns by solving the simultaneous equations.

7. For displacement formulations, calculate the stress components of each element.

8. For nonlinear problems, check convergence, and if convergence is not achieved, go back to step 3.

The finite element method is widely used in mining engineering for studying rock mechanics and ground control problems. Some particular applications have been reported by Agioutantis (1987). In most of the cases, programs designed for mining applications have some special features.

First, special elements, such as joint, chock and rubble elements are often utilized. The joint element was introduced by Goodman et al. (1968). It is also known as interface element or friction element, and can be described as a four-node, zero-width element. The load-displacement characteristics of a joint element are assumed as:
\[
\begin{bmatrix}
P_n \\
P_s
\end{bmatrix} =
\begin{bmatrix}
k_n & 0 \\
0 & k_s
\end{bmatrix}
\begin{bmatrix}
\delta_n \\
\delta_s
\end{bmatrix}
\] (2.15)

where

\( P_n, P_s \) = the average normal and shear load on the joint plane;

\( k_n, k_s \) = the normal and shear stiffness coefficient; and

\( \delta_n, \delta_s \) = the average relative displacements across the joint.

The joint element stiffness matrix, \( K \), is calculated as:

\[
K = \frac{1}{4} \int_L [B]^T [k][B] dx
\] (2.16)

where

\( L \) = the length of the element;

\[
k =
\begin{bmatrix}
k_n & 0 \\
0 & k_s
\end{bmatrix}
\]

\( B \) = the interpolation matrix which relates nodal displacement \( (q) \) to relative displacement \( (\delta) \) in the equation \( \delta = \frac{1}{2} [B](q) \)

Joints, faults, bedding planes and discontinuities in the strata are frequently represented by joint elements.
In subsidence modelling using finite element methods, the overlap of roof and floor, when the roof and floor displacements are not given, can create problems with the boundary conditions. In order to avoid this, chock elements, which are actually one-dimensional truss elements, were introduced. This type of elements was discussed by Agioutantis (1987).

A rubble element was used by Benzley and Krieg (1982) for modelling mining subsidence. In the rubblization process, the material would initially carry both hydrostatic and shear stress with volumetric stress-strain behavior represented by a piece-wise linear curve, and shear behavior represented by a pressure-dependent plastic yield point. When tensile failure occurs in an element, it loses all volumetric and shear stiffness and thus it becomes rubblized. The free nodes of the rubble element are computed as particles in unconstrained gravity fall. However, these nodes can not move freely forever because the lower nodes of the rubble element are either on a boundary or are common to an element below. If they are on the boundary, the displacements are generally constrained by the boundary conditions; if the nodes are common to an element below, then their motion is limited by the force built up in the element below, due to its volumetric compaction. When a rubble element is again accepting loads, the rubble is defined as a material which can resist only compressive loads. Rubble elements can be used to simulate roof falls.

Special consideration for applying FEM in subsidence modelling is also given to the yielding criteria of materials in nonlinear analyses. The most commonly used are those described by the Drucker-Prager and von Mises yielding conditions.
2.4.3 Boundary Element Method (BEM)

The boundary element method is another powerful numerical method and an alternative to the finite elements. In FEM a governing differential equation is approximated over the domain by functions which fully or partially satisfy the boundary conditions. In BEM, however, approximating functions are used to satisfy the governing equation in the domain but not on the boundary (Brebbia, 1978). In practice, the FEM requires that the whole domain be divided into a network of elements, resulting in a large number of nodes. Since the number of unknowns is associated with the number of nodes, a large system of linear algebraic equations is required to solve a problem. Usually, the coefficient matrix (stiffness matrix) of the system is banded, and the storage space for this matrix can be reduced significantly. With the introduction of the skyline method (Bathe, 1982), the storage space is further reduced. On the other hand, BEM requires only that the boundary be discretized into a number of elements, and as a result, the number of nodes (hence the system of equations) is smaller than in FEM. However, this may not produce significant difference in storage space, because the stiffness matrix of the BEM is fully populated.

The boundary element method can achieve greater accuracy in stress concentration problems or where the domain extends to infinity (Mackerle and Brebbia, 1988). It is very suitable for applications in mining engineering, therefore, where a large medium is involved. The contributions of various authors to the application of BEM in mining engineering were reported by Mackerle and Brebbia (1988).

One problem encountered in applying BEM for subsidence modelling is that the stress obtained on the ground surface is not equal to zero if the underground opening is regarded as a boundary and the points on the ground surface are treated as internal points.
In order to solve this problem, Tsur-Lavie and Denekamp (1981) added a complementary "imaginary" part to the real model as shown in Figure 2.2. The role of the imaginary part is to counteract the stresses existing at the ground surface in the real model. The imaginary part is the exact opposite to the real part, that is, the displacements at the boundary of the real part are of vertical compression whereas those at the boundary of the imaginary part are in vertical tension. When combined, the two physically opposite and quantitatively equal parts produce a zero normal stress along the ground surface.

A consequence of superimposing the imaginary part on the real one, is that not only the stresses and displacements at the ground surface are affected, but also the entire medium, causing the displacements in the combined model to deviate from those at the real boundary. In order to correct this, the following formulation is used:

$$ v^C_i = v^R_i + \sum_{j=1}^{N} C_{ij} v^I_j $$  \hspace{1cm} (2.17)

where

- $N$ = number of elements in each of the two parts;
- $v^C_i$ = vertical displacement of element $i$ in the combined model;
- $v^R_i$ = vertical displacement of element $i$ in the real part;
- $v^I_j$ = vertical displacement of element $j$ in the imaginary part; and
- $C_{ij}$ = influence coefficient of displacement $v^I_j$ on the displacement at the $i$th element in the real part. $C_{ij}$ is derived from the fundamental solution.

The $v^C_i$'s are assumed as:
Figure 2.2. The Combined Model Used by Tsur-Lavie and Denekamp (1981)
\[ v^C_i = \text{thickness of the seam for mined section; and} \]
\[ v^C_i = 0 \text{ for unmined section.} \]

It follows from the original assumption that:

\[ \nu_i^R = \nu_i^l \]  \hspace{1cm} (2.18)

Therefore, Equation 2.17 can be solved for the unknown \( \nu_i \)'s. Subsequently, the displacement and stress can be calculated by using the fundamental solutions.

2.4.4 Stochastic Method

The stochastic method is not an independent subsidence prediction approach but rather a part of the influence functions. The difference between this model and the influence functions mentioned early in section 2.3.1 is that stochastic models are mainly based on probability theory.

The pioneer of the stochastic model is Litwiniszyn (1957). Several other authors (Brauner, 1973; CUMT, 1981; and Kratzsch, 1983) have also made contributions to the development of this method.

The concept of stochastic analysis is based on the simple observation of movement in a stack of balls when the bottom one is removed (CUMT, 1981). As shown in Figure 2.3a, when \( a_1 \) is removed, the falling probabilities for \( a_2 \) and \( b_2 \) are both \( 1/2 \), those for \( a_3 \), \( b_3 \) and \( c_3 \) are \( 1/4 \), \( 2/4 \) and \( 1/4 \) respectively and the probabilities for the balls on other layers are shown in Figure 2.3b. If the number of layers and the number of balls on a layer approach infinity, then the probability distribution of the top layer is a normal
Figure 2.3. The Stochastic Model
curve and the function describing the curve is known as the probability density function (Figure 2.3c). The stack of balls is called the stochastic medium.

In deriving the element influence function for subsidence calculation, the overburden is regarded as a stochastic medium and thus, the probability that a surface point will subside follows a normal curve. Assuming that a Cartesian coordinate system is chosen as shown in Figure 2.4, in which the origin is at the surface point directly above the excavated element, the normal curve is symmetric about the Z axis and the density function is an even function which can be written as \( f(x^2) \). Therefore, the probability that an infinitesimal element \( (dx) \) on the surface will subside is (CUMT, 1981):

\[
P(dx) = f(x^2)dx
\]

(2.19)

The probability that an infinitesimal area will subside in three-dimensional space can be explained with the aid of Figure 2.5. In a three-dimensional problem, it is assumed that the density function is the same for each cross-section passing through the origin. This is valid, assuming that the medium is homogeneous across the horizontal axis.

Subsidence of an infinitesimal area at point A is equivalent to the following two events occurring simultaneously: a) the element \( dx \) on the B-B cross-section subsides, and b) the element \( dy \) at the same elevation as \( dx \) on D-D cross-section subsides (Figure 2.5). Therefore, the probability that the first event will occur is equal to the product of the other two:

\[
P(ds) = f(x^2)dx f(y^2)dy
\]

\[= f(x^2)f(y^2)ds
\]

(2.20)

where
Figure 2.4. Subsidence Probability in a 2-D Model
Figure 2.5. Subsidence Probability in a 3-D Model
\[ P(ds) = \text{the probability for infinitesimal area, } ds, \text{ to subside;} \]

\[ f(x^2) = \text{the probability that element } dx \text{ on cross-section B-B will subside; and} \]

\[ f(y^2) = \text{the probability that element } dy \text{ on cross-section D-D will subside.} \]

If another coordinate system, \( X_1OY_1 \) is chosen by rotating the original system (Figure 2.5) then, assuming all other conditions are the same, it follows that:

\[
P(ds) = P(ds_1)
\]

\[
= f(x^2)f(y^2)ds
\]

\[
= f(x_1^2)f(y_1^2)ds_1
\]  \hspace{1cm} (2.21)

If the new coordinate system is so chosen that \( x_1 \) axis passes through point \( A \) (Figure 2.5), then:

\[
x_1^2 = x^2 + y^2
\]

\[
y_1 = 0
\]  \hspace{1cm} (2.22)

Thus, Equation 2.21 becomes:

\[
f(x^2)f(y^2) = f(x^2 + y^2)f(0)
\]

\[
= Cf(x^2 + y^2)
\]  \hspace{1cm} (2.23)

\( f(0) \) is independent of \( x \) and \( y \) and thus is a constant denoted by \( C \). Equation 2.23 can be solved by differentiation and integration. Differentiation of Equation 2.23 with respect to \( x^2 \) and \( y^2 \) results to:

\[
\frac{1}{f(x^2)} \frac{df(x^2)}{dx^2} = \frac{1}{f(y^2)} \frac{df(y^2)}{dy^2}
\]  \hspace{1cm} (2.24)
The L.H.S. of Equation 2.24 is a function of \( x \) and the R.H.S is a function of \( y \). For the two sides to be equal, therefore, each must be equal to a constant. Using \( K \) to denote this constant, the following equations can be obtained:

\[
\frac{df(x^2)}{d(x^2)} = Kf(x^2) \\
\frac{df(y^2)}{d(y^2)} = Kf(y^2)
\]  (2.25)

The solution for Equation 2.25 are:

\[
f(x^2) = pe^{-h^2x^2} \\
f(y^2) = pe^{-h^2y^2}
\]  (2.26)

where

\[ p = \text{integration constant}; \text{ and} \]

\[ -h^2 = K \]

Therefore Equation 2.20 becomes:

\[
P(ds) = p^2e^{-h^2(x^2 + y^2)}dx\,dy
\]  (2.27)

This is the probability that an infinitesimal area will subside in the three-dimensional space. In the two-dimensional case, the excavated element is not a unit cube but an infinitely long bar with a unit square cross-section as shown in Figure 2.6. The probability in the two-dimensional space can be obtained by integrating Equation 2.27:
Figure 2.6. Illustration for Deriving the Influence Function in Stochastic Model
\[
P(\delta s) = \int_{-\infty}^{\infty} \{p^2 e^{-h^2[x^2 + y^2]} \} dx dy d\phi
\]
\[
= \frac{p^2 \sqrt{\pi}}{h} e^{-h^2 x^2} dx dy
\]  

(2.28)

Assuming that

\[
g = \frac{p^2 \sqrt{\pi}}{h} e^{-h^2 x^2}
\]

(2.29)

then this can be considered as an influence function obtained by the stochastic model for a two-dimensional problem. It was proven that \( p^2 = \frac{h^2}{\pi} \) and \( h = \frac{\sqrt{\pi}}{r} \), where \( r \) is the radius of influence (CUMT, 1981). Thus, Equation 2.29 becomes:

\[
g = \frac{1}{r} e^{-\pi \frac{x^2}{r^2}}
\]

(2.30)

Equation 2.30 is actually the same as Knott's influence function.

2.5 The Surface Deformation Prediction System (SDPS)

The SDPS was developed by Karmis (VPI & SU, 1989) after many years of extensive research on subsidence. The system is a menu-driven program designed to provide subsidence engineers and researchers with a detailed and in-depth analysis of surface deformations over undermined areas. It incorporates three different subsidence prediction methods: profile function, zone area and influence function methods, and can
compute several subsidence indices. In addition, the system also provides graphics options.

There are three disks in the SDPS (version 2.7) package. The first contains programs related to the influence function method. A interactive data editor is provided to prepare the appropriate data files and invoke the corresponding solution module. The output can be produced in several forms, including report files, TOPO files, XYZ files, etc.. A report file includes a summary of all calculated indices per point. TOPO files are used to generate graphics within the SDPS package and the XYZ files are used to produce graphics with other graphing or plotting programs. The second disk contains the profile function program which includes an interactive data editor, calculation routine, and built-in graphing routines, as well as a graphing program that can plot the distribution of various calculated deformation indices in the form of profiles or 3-D images. The third disk contains programs related to the zone area method.

The SDPS package can run on IBM-PCs and compatibles that have one floppy disk drive and a minimum of 384K random access memory. Additionally, the computer system should have a color graphic adapter card. A printer and plotter are highly recommended. For running on systems equipped with a Hercules card, an appropriate emulation routine must be loaded before any of the programs can be executed. The package is one of the most widely used subsidence prediction systems.
Chapter 3 Building Responses to Mining-Induced Ground Movements

A main objective of subsidence studies is to prevent or reduce damage to surface structures and facilities. In the previous chapter, methods for predicting mining induced ground movements were reviewed. This chapter discusses the existing knowledge of the response of buildings due to such movements.

From the literature it is evident that, although significant accomplishments have been made in subsidence prediction, the subject of building responses to ground movements is not extensively studied. In most cases, only the final impact of the movements on structures was described, while the actual mechanism leading to the final damage was ignored. Furthermore, only a few case studies can be found where a building was monitored while mining activities were in progress. In civil engineering, a wealth of information has been accumulated concerning building responses due to settlement caused by their own weight and other engineering activities such as tunneling, and open cut. Some of this information is helpful to this study, although mining induced ground movements usually are much larger than those encountered in civil engineering.
In general, a building located above an active mine will be subjected to three different deformation stages, as shown in Figure 3.1. At the first stage, the building is ahead of the working face, and it is subject to tension (3.1a); at the second stage, the working face advances so that the building is located at the inflection point of the subsidence profile, and thus, it suffers maximum tilt (3.1b); the last stage is when the building is behind the working face (3.1c) and thus impacted primarily by compression. This is an idealized description of the building deformation process. In reality, building deformations are much more complicated and may be affected by many factors such as magnitude and direction of the ground movement, type of foundation, location of the building, dimension of the building in relation to that of the subsidence profile, etc. In this chapter, the effects of ground movements on buildings are discussed in detail.

3.1 Transmission of Ground Displacements to Structures

The relationship between ground movements and building displacements is very important. Cooper et al. (1962) has stated that:

"It is important to note that the common assumption frequently implied in the literature on the subject, that the estimated free ground movements and ground strains are transmitted to a structure and produce identical movements and strains in the structure, is not true in general."

Hurst et al. (1966) reported that the movement of the structure did not usually reflect the movement of nearby ground and suggested insignificant changes in subsidence due to the present of a building.

Walker and LaScola (1989), after observing the behavior of four masonry walls which were located over an active longwall panel in West Virginia, concluded that a simple
Figure 3.1. A Structure at Three Mining Stages (Kratzsch, 1983)
surface structure moves similarly as the ground surface. They also reported that although the ground surface and the wall moved in a similar manner, the movements measured at the wall were less than the ground. The width of the longwall panel in this study was over 1000 ft. and the half-width of the subsidence profile was about 800 ft. Of the four walls, one was 86 ft. long, and the others were only 36 ft. It is not surprising, therefore, that no major bridging effect - - which will alter the displacement of the walls significantly - - was reported because the dimensions of the walls were much less than that of the subsidence profile.

Littlejohn (1974) also conducted a similar field experiment in Britain. Three brick walls on unreinforced concrete strip footings and one footing without a wall were built over an expected mining area. The layout of the monitoring system and the coal mining panel are shown in Figure 3.2. The walls were constructed parallel to the direction of face advance and located centrally along the width of the panel, so that the walls would be subjected to dynamic ground movements. The depth of the overburden at the site was 840 ft., and the coal seam of 4.50 ft. was extracted under total caving conditions. The deflections of the brickwork, foundation, and ground from the experiments are shown in Figure 3.3 where \( r \) is the radius of influence. From this figure, it is obvious that the wall and its foundation deflected basically the same way as the ground, but instead of lagging behind the ground movement as in the previous example, the deflection of the wall in this experiment was actually larger than that of the ground. Littlejohn (1974) noted that the reason for this was the fact that during initial ground movement (0.4-0.5r), the foundation penetrated the ground at the ends and separated under the central section of the structure. As the tensile ground strain increased, the entire foundation penetrated the subgrade, although the former pattern remained. Littlejohn also
Figure 3.2. The Observation Plan in the Case Study Reported by Littlejohn (1974)
Figure 3.3. Deflections of Brickwork, Foundation and Ground (Littlejohn, 1974)
observed slips between the foundation and ground, and he used the following equation to estimate the slip:

\[ H_l = 1.021E_g \]  \hspace{1cm} (3.1)

where \( H_l \) is the longitudinal slip, \( I \) the distance between the point where the slip is estimated and the neutral axis of the footing, and \( E_g \) the ground strain.

In another comprehensive study done by Powell et al. (1988), it was reported that foundations generally followed the ground surface at slow rates of displacement, but lagged when vertical displacements occurred rapidly. Unlike the previous two cases, the structure in this study was a three-dimensional one. Two 30 by 40 ft. foundations subject to simulated superstructure loads were built over a longwall panel prior to the start of mining activities. The observation of the foundation and ground displacements lasted for three years and valuable information was collected. Figure 3.4 shows the relationships between ground and foundation displacements obtained from the study. In the figure the diagonal line has a slope of one, that is, on that line the ground and foundation displacements are equal. It can be seen that at the beginning the two displacements were basically equal, but as the displacement increased, the foundation displacement lagged slightly behind.

Figure 3.5 shows the displacements of walls and the ground surface obtained by Gardner et al. (1961). The walls were situated over one of a series of 350-ft.-wide panels which were mined sequentially using strip packing. The figure indicates that the displacements of the walls and the ground followed the same trend but with different magnitude.

One common characteristic of the case studies reviewed so far is that the dimensions of the structures on the surface are considerably smaller than those of the subsidence pro-
Figure 3.4. Relationships between Ground and Foundation Displacements (Powell et al., 1988)
Figure 3.5. Comparison between Ground and Wall Deformations (Gardner et al., 1961)
files. In general, therefore, the displacements of the structures are similar to those of the ground although the magnitudes may differ slightly. This is not true, however, in the case where the structures are relatively large compared to the subsidence profile.

Breth and Chambosse (1974) observed the settlement behavior of buildings above subway tunnels in Frankfurt, Germany and their results showed significant differences between ground and building displacements. Figure 3.6 is one of the buildings that they measured with the resulting ground and building displacement curves. The building consisted of a eighteenth-century monastery cellar made of sandstone and a recently built two-story superstructure composed of brickwork. The length of the building was about 59 ft., the tunnels beneath the structure had a diameter of 22 ft., and the overburden was about 24.6 ft. The figure clearly shows the bridge effect of the building, mainly because the dimension of the building is large compared with that of the subsidence profile.

Marino (1985) conducted a parametric analysis of the effect that the L/D ratio has on the transformation of ground displacements to the structure, where L is the length of a structure and D the diameter of the subsidence profile. A sine curve was used to model the subsidence. His results indicate that for cases with L/D ratio less than 0.1, the foundation essentially moves with the ground, but as the L/D ratio increases, the range of foundation displacement decreases, and the foundation displacement is less compatible with the subjacent ground displacement. The foundation translation is nominal when L/D is equal to 1. Although the use of the sine curve to approximate the subsidence profile is questionable, the observations are basically compatible with the results from the case studies.
Figure 3.6. Ground and Building Displacement Curves (Breth and Chambosse, 1974)
The relationships between the two displacements also depend on the location of the building on the subsidence profile. For example, Boscardin (1980) measured the displacements of two buildings affected by the excavation of two tunnels, which were part of the Washington, D.C. Metro System. In this case, the buildings were on one side of the tunnels as illustrated in Figure 3.7. The observation results (Figure 3.8) indicated that there is little difference between the building and ground displacements. A major difference between this case and the previously mentioned Frankfurt subway case is that the buildings in the former are located at one side of the subsidence profile while in the latter, the building is at the center. Other factors must also be recognized such as foundation type and soil properties.

In the above discussion, the relationship between ground and structure displacements are essentially stated in a qualitative manner. However, sometimes these relations are also presented quantitatively. Based on numerous case studies, the China Coal Science Research Academy (CCSRA, 1983), established relationships between the ground and foundation deflection ratios for buildings with various L/H values, through statistical analyses as shown in Table 3.1. In this case, L is the length of the building and H its height. These relationships are only valid for unreinforced buildings less than four stories high and 197 ft. long with strip masonry foundation. Through further studies, the following general formula was derived for this type of building:

\[
d_f = (0.034 \frac{L}{H} + 0.47)d_g + 4 \times 10^{-5}
\]

(3.2)

where \(d_f\) and \(d_g\) are the deflection ratios of the foundation and ground respectively. This formula is applicable for \(L/H\) between 1.6 to 10 and \(d_g\) between \(0.5 \times 10^{-4}\) to \(1.4 \times 10^{-3}\).
Figure 3.7. Locations of the Buildings and the Tunnels (Boscardin, 1974)
Figure 3.8. Ground and Building Displacements (Boscardin, 1974)
<table>
<thead>
<tr>
<th>Average 1/11</th>
<th>Relationship</th>
<th>Correlation Coefficient</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.25</td>
<td>$d_r = 0.54d_g + 0.008 \times 10^{-3}$</td>
<td>0.69</td>
<td>0.05 $\times 10^{-3}$</td>
</tr>
<tr>
<td>2.80</td>
<td>$d_r = 0.56d_g + 0.01 \times 10^{-3}$</td>
<td>0.58</td>
<td>0.05 $\times 10^{-3}$</td>
</tr>
<tr>
<td>3.75</td>
<td>$d_r = 0.64d_g + 0.04 \times 10^{-3}$</td>
<td>0.72</td>
<td>0.05 $\times 10^{-3}$</td>
</tr>
<tr>
<td>6.40</td>
<td>$d_r = 0.74d_g + 0.03 \times 10^{-3}$</td>
<td>0.96</td>
<td>0.05 $\times 10^{-3}$</td>
</tr>
<tr>
<td>7.50</td>
<td>$d_r = 0.70d_g + 0.05 \times 10^{-3}$</td>
<td>0.88</td>
<td></td>
</tr>
</tbody>
</table>
Using the Winkler ground model, Attewell et al. (1986) established analytically the relationship between ground and structure displacements:

\[(K_s + K_{gd})d_s = K_{gd}S\]  \hspace{1cm} (3.3)

where \(K_s\) is a stiffness matrix for the beam structure, \(K_{gd}\) the vertical spring stiffness matrix for the Winkler ground, \(d_s\) the displacement vector of the beam structure and \(S\) the predicted free ground displacement vector. A model composed of frame structure on elastic foundation for studying the relationship between structure and ground displacements was also reported by Attewell et al. (1986).

### 3.2 Transmission of Other Ground Deformations to Structures

The deformation indices considered in this section include strain, tilt, curvature, and angular distortion. The definitions of these terms were explained in Chapter 2, except for angular distortion. Angular distortion is the shearing distortion of a structure and is often approximated as the rotation, due to settlement, of a straight line joining two reference points on the structure minus any rigid body tilt of the structure (Boscardin, 1980). Since the transmission of ground deformations to structures is influenced by many factors, it is difficult to find general relationships between ground and structure deformations. Brauner in 1973 pointed out that there is so far insufficient knowledge of transmission of ground strains to structures. Although significant progress has been made since then, the statement is probably still true today.

In addition to the displacements, the strains in the ground and the structure measured by Gardner and Hibberd (1961) are shown in Figure 3.5. From this figure, the re-
lationship between the two strains is not obvious. For the south-west wall, it can be said that both the ground and the wall suffered the largest compression in the middle, and for the south-east wall it seems that the compressive strain in the wall was less than that of the ground. In the same study, it was also reported that between 30 to 100 percent of the ground strain was transmitted to the structure (Gardner and Hibberd, 1961).

Hurst et al. (1966) monitored the behavior of a school over an active mine in Britain. The parameters examined in their investigation were mainly tilt and strain. The school building was protected against subsidence by several measures:

1. A rigid reinforced concrete foundation of egg-crate type, which was also extended to form the outer walls of the first floor in some parts of the school. It was hoped that this would cantilever or bridge over any subsidence of the ground below the foundation and ensure that the induced stresses on the superstructure would only arise from the tilting of the foundation block as a whole;

2. The school was subdivided into separated units each on its own egg-crate foundation, the spacing between adjacent units was four inches and the maximum height of the units was 31 feet;

3. A light superstructure consisting of wooden columns and beams with timber weatherboard between was used.

The panel to be worked under the school was 880 ft. wide and at a depth of 466 ft. The seam thickness was 43 in. and the radius of influence was about 312 ft. The location of the school in relation to the panel and the monitoring stations are shown in Figure 3.9 and 3.10 respectively. The resultant tilts for stations I and II and building blocks 5 and
13 were made into vector diagrams in Figures 3.11 and 3.12. Comparing the tilts of block 5 with those of the adjacent ground station I in Figure 3.11, one can see that the tilting directions for both the structure and the ground are basically the same, but the magnitudes of the structural tilts are slightly larger than those of the ground. The differences, both in direction and in magnitude, between the tilts of block 13 and adjacent ground station II are evident in Figure 3.12, and once again the magnitudes of the structural tilts are larger. Figure 3.12 also indicates that block 13 was tilted uniformly, suggesting that the block moved as a whole. In contrast to the transmission of ground tilts to the structure, the transference of ground strains was completely different. Although the ground strain measured was as high as 6.7 mm/m, the strain on the structure was so small that it was not measurable. One explanation for this is that the structure was specially designed and the rigid foundation prevented the strain from being transmitted to the superstructure.

Figures 3.13 and 3.14 show the development of tilts and horizontal strains as the face progressed, constructed from the data obtained by Walker and LaScola (1989). The conditions for this study were mentioned in Section 3.1. In the figures the tilt of the wall essentially followed the ground, whereas the horizontal strains were different. The strain in the wall was basically tensile and that in the ground compressive.

Figure 3.15 represents the horizontal strains in the ground, foundation and brickwork recorded by Littlejohn (1974). The horizontal strains in the brickwork and in the foundation were basically tensile, but the ground suffered both tensile and compressive strains. This and the results shown in Figure 3.14 are what one might expect for a wall situated on subsiding ground, because the whole structure acts like beam in bending. The angular distortions of the ground, foundation, and brickwork at different mining stages are shown in Figure 3.16 which is constructed from data obtained in the same
Figure 3.9. Location of the School (Hurst et al., 1966)
Figure 3.11. Vector Diagram Showing Ground and Wall Tilts at Station I (Hurst et al., 1966)
Figure 3.13. Development of Ground and Wall Tills (Walker et al., 1989)
Figure 3.14. Development of Ground and Wall Strains (Walker et al., 1989)
Figure 3.15. Horizontal Strains in the Ground, Foundation and Brickwork (Littlejohn, 1974)
study (Littlejohn, 1974). This figure indicates that the ground was subjected the least angular distortion whereas the foundation suffered the most. The difference may be caused by the penetration of the foundation into the ground.

Table 3.2 was obtained from the Frankfurt subway case discussed earlier (Breth and Chambosse, 1974). A comparison of the ground and building tilts shows that the tilts of the buildings were significantly lower than those of the ground. This may be attributed to the bridging effects of the buildings.

Table 3.2. Slopes of Subsoil and Houses (After Breth and Chambosse, 1975)

<table>
<thead>
<tr>
<th>House No.</th>
<th>Slope of Subsoil</th>
<th>Slope of Houses</th>
</tr>
</thead>
<tbody>
<tr>
<td>II</td>
<td>1/1000</td>
<td>1/1600</td>
</tr>
<tr>
<td>I</td>
<td>1/400</td>
<td>1/450</td>
</tr>
<tr>
<td>IV</td>
<td>1/120</td>
<td>1/500</td>
</tr>
<tr>
<td>V</td>
<td>1/120</td>
<td>1/400</td>
</tr>
<tr>
<td>VI</td>
<td>1/60</td>
<td>1/779</td>
</tr>
<tr>
<td>III</td>
<td>1/60</td>
<td>1/130</td>
</tr>
</tbody>
</table>

From the few case studies reviewed above, it is impossible to draw a general conclusion. One observation, however, is worth mentioning: transmission of the ground tilt to the structure is similar to that of the vertical ground displacement. Brauner (1973) cited a tilt relationship obtained from statistical analyses of field data in Poland:

$$ T_b = T(0.033 \frac{L}{H} + 0.45) + 0.6 \times 10^{-4} $$  (3.4)
Figure 3.16. Angular Distortion in the Ground, Foundation and Brickwork (Littlejohn, 1974)
where $T_b$ and $T$ are the tilts of the building and free ground respectively, and $L$ and $H$ are the same as in the previous section.

Through statistical analysis, CCSRA (1983) derived the following relationships between the ground and foundation curvatures:

\[ K_f = 0.74K + 0.1 \times 10^{-4} \quad (3.6) \]

for masonry buildings,

\[ K_f = 0.28K + 0.2 \times 10^{-4} \quad (3.5) \]

for reinforced masonry buildings, and

\[ K_f = 0.4K + 0.2 \times 10^{-4} \quad (3.7) \]

for slab buildings, where $K_f$ and $K$ are the curvatures of the foundation and ground respectively. The CCSRA (1983) also developed statistical relationships between tensile horizontal strains in the foundation and in the ground for masonry buildings:

\[ \varepsilon_f = 0.9\varepsilon - 0.1 \times 10^{-3} \quad (3.8) \]

and

\[ \varepsilon_f = 0.84\varepsilon - 0.09 \times 10^{-3} \quad (3.9) \]

where $\varepsilon_f$ and $\varepsilon$ are the strains in the foundation and in the ground respectively. Equation 3.8 is based on data from several countries and Equation 3.9 on data from China.

The transmission of ground deformations to structures is indeed very complicated. The problem is compounded by many factors. Thus, when the relationship between the
ground and structure deformation is discussed, certain conditions are usually implied. For example, Equations 3.8 and 3.9 are valid only for masonry buildings with strip foundations. Since there may be as many relationships as there are geological and structural conditions, and a case study can only examine the relationship under one set of conditions, it becomes necessary to use theoretical models to explore these relationships under various conditions.

3.3 Subsidence-Induced Forces and Stresses on Structures

One approach to investigate building responses to mining subsidence is to study the change of foundation stresses due to ground movements, because this change is one of the key elements in causing superstructure deformations. Stress changes may be induced by various ground deformation indices and they will be discussed separately.

3.3.1 Foundation Stresses Induced by Horizontal Ground Deformation

Horizontal ground deformation has two types of effects on foundations, the change of lateral soil pressure and induction of friction force at the foundation and soil interface. This topic has been discussed by several authors in the literature (Gardner and Hibberd, 1961, Geddes and Copper, 1962, Kratzsch, 1983, CCSRA, 1983, and Marino, 1985).

The lateral soil pressure on a foundation increases under the influence of compressive horizontal deformation and decreases under tensile deformation. CCSRA (1983) recommended the following empirical relationship between the lateral soil pressure and the compressive horizontal strain based on field data:
\[ q = \frac{\varepsilon}{(0.6\varepsilon + 0.002)} \]  

(3.10)

where \( q \) is the lateral soil pressure on a foundation and \( \varepsilon \) the horizontal strain. Figure 3.17 is a plot of Equation 3.10 and it shows that the pressure increases rapidly as the horizontal strain increases when the strain is small, but the increase tapers off when the horizontal strain becomes large. The maximum lateral pressure that a type of soil can reach is governed by the following equations (Kratzch, 1983):

\[ P_{\text{max}} = \frac{1}{2} \gamma h^2 \tan^2(45^\circ + \frac{\phi}{2}) \]  

(3.11)

for non-cohesive soil, and

\[ P_{\text{max}} = \frac{1}{2} \gamma h^2 \tan^2(45^\circ + \frac{\phi}{2}) + 2Ch \tan(45^\circ + \frac{\phi}{2}) \]  

(3.12)

for cohesive soil, where

- \( P_{\text{max}} \) = maximum passive soil pressure;
- \( h \) = depth of foundation;
- \( \gamma \) = density of the soil;
- \( \phi \) = friction angle of the soil; and
- \( C \) = cohesion factor of the soil.

If the lateral soil pressure exceeds the bending capacity of the foundation wall, failure of the foundation will occur. On the other hand, it will cause no foundation damage if \( P_{\text{max}} \) is lower than the strength of the foundation wall, because the soil will fail first.

Friction force at the subgrade-foundation interface is a function of relative horizontal displacement between soil and the foundation slab (Marino, 1985). Experiments re-
Figure 3.17. Relationship between the Soil Pressure and Horizontal Strain (CCSRA, 1983)
ported by Gardner and Hibberd (1961) where concrete slabs were pushed and pulled over different types of soil, indicated that, when relative displacement took place between slab and soil, the friction force varied with it up to a maximum value and subsequent relative displacement did not change this maximum resistance. A similar result was obtained by Littlejohn (1974) and Geddes (1978). Generally, the relationship between the friction force and the relative horizontal displacement can be represented in two forms as shown in Figure 3.18. In the first form, the friction force increases linearly with the displacement to a limiting value and then that value is maintained as the displacement becomes larger (Figure 3.18a). The other form is a non-linear relationship in which friction force increases non-linearly with the displacement and approaches a certain limit when the displacement gets larger (3.18b). A relationship, similar to that represented in Figure 3.18a, between the friction coefficient and relative displacement was assumed by CCSRA (1983). The assumption seems reasonable because the normal loads in the experiments, such as the weights of concrete slabs, were usually constant. Using this assumption, the distributed friction force can be written as:

$$
\tau = \frac{\Delta(x)}{\Delta_c} \mu_c q(x)
$$

(3.13a)

![Figure 3.18. Relationships between the Friction Force and Relative Horizontal Displacement (Marino, 1985)](image)
for $\Delta(x) \leq \Delta_c$, and

$$\tau = \mu_c q(x)$$ (3.13b)

for $\Delta(x) \geq \Delta_c$. The total friction force at the center line of the subgrade-foundation interface is:

$$T = \int_0^{L/2} \Delta(x) \frac{\Delta(x)}{\Delta_c} \mu_c q(x) \, dx$$ (3.14)

when $\Delta(x) \leq \Delta_c$ along the entire foundation, and

$$T = \int_0^a \frac{\Delta(x)}{\Delta_c} \mu_c q(x) \, dx + \int_{L/2}^a \mu_c q(x)$$ (3.15)

when $\Delta(x) \leq \Delta_c$ from the center line to a distance $a$ along the foundation and $\Delta(x) \geq \Delta_c$ from $a$ to the foundation end; where $L$ is the length of the foundation; $\Delta(x)$ the relative horizontal displacement between the foundation and sub-grade; $\Delta_c$ the critical value of $\Delta(x)$ beyond which the friction coefficient is independent of the relative displacement; $\mu_c$ is the friction coefficient corresponding to $\Delta_c$; and $q(x)$ the contact pressure.

Gardner and Hibberd (1961) calculated the horizontal displacement-induced force on the structure assuming that: 1) the relationship between force transmitted to the structure and the relative horizontal displacement follows the curve of Figure 3.18a; 2) there is no relative displacement between the center line of the structure and the ground; and 3) the relative displacement at other points is proportionate to the distance of these points
from the center line. For the condition in which the critical displacement is not reached within the length \( L \), the force at the center line of the structure was given as:

\[
T_{el} = \frac{WRL^2}{8a}
\]  

(3.16)

At large displacement, the critical value would be attained within the length of the structure and the force at the center line was calculated as:

\[
T_{el} = \frac{WR}{2}(L - a)
\]  

(3.17)

For the situation in which the critical relative displacement is reached at the end of the structure, the transmitted force was:

\[
T_{el} = \frac{WRL}{4}
\]  

(3.18)

where \( T_{el} \) is the force transmitted to structure, \( W \) the width of the foundation, \( R \) the distributed friction force corresponding to the critical relative displacement, and \( L \) and \( a \) have the same meanings as in Equation 3.15. The stress in the foundation was given by:

\[
\sigma = T_{el}(\frac{1}{A} + \frac{yy_{1}}{I})
\]  

(3.19)

where \( A \) is the cross section area of the foundation, \( y_{1} \) the distance of the foundation bottom edge from the neutral axis, \( y \) the distance from the point where the stress is calculated to the neutral axis and \( I \) the moment of inertia. It should be noted that in the above equation the lateral soil pressure is ignored.
3.3.2 Change of Contact Pressure Due to Ground Curvature

Under the influence of ground curvature, the vertical contact pressure along the foundation-subgrade interface will change. For a convex curve or positive curvature, the pressure concentration shifts to the center and for a concave curve or negative curvature, it moves to the edges of the foundation.

Kratzsch (1983) listed four cases of pressure distributions induced by ground curvature after Rausch (1955), as shown in Figure 3.19, but he only gave the formula of the pressure distribution for one case. The formulas for other cases are developed here. For the condition in Figure 3.19a, the curvature is positive, the foundation is separated from the subgrade at both ends and the contact pressure distribution is:

\[ q_x = q_0 \left[ 1 - \left( \frac{2x}{a} \right)^2 \right] \]  \hspace{1cm} (3.20)

In Figure 3.19b, the ground curvature is also positive, but the entire foundation maintains contact with the subgrade, and the contact pressure is given as:

\[ q_x = q_0^* + q_0' \left[ 1 - \left( \frac{2x}{L} \right)^2 \right] \]  \hspace{1cm} (3.21)

Under the condition of negative curvature and foundation-subgrade separation at the center (Figure 3.19c), the contact pressure is:

\[ q_x = Cbf \frac{4x^2 - a^2}{L^2} \quad \text{if } |x| \geq a \]  \hspace{1cm} (3.22)

For negative curvature and no separation (3.19d), the contact pressure can be written as:
Figure 3.19. Changes of Contact Pressure Due to Ground Curvature (Kratzsch, 1983)
\[ q_x = q_0 + Cbf \frac{d^2x^2}{L^2} \] (3.23)

In Equations 3.20 through 3.23,

\[ a = \sqrt{8} \varepsilon_0 \rho \]
\[ f = \frac{L^2}{8 \rho} \]
\[ q_0 = Cb \delta_0 \]
\[ q_0' = \frac{Q}{L} - \frac{2}{3} Cbf \]
\[ q_0'' = \frac{1}{3} Cbf \]

L is the length of the foundation, b the width, C stiffness coefficient of the subgrade, \( \rho \) the radius of curvature. All other symbols are explained in Figure 3.19.

Notable work on contact pressure distribution was also presented by Wasiłkowski (1956), Kawulok (1978), CCSRA (1983), and Marino (1985).

### 3.3.3 Forces Induced by Other Deformations

Other ground deformations may also induce forces on foundations. For instance, vertical displacement can cantilever a foundation causing additional bending moments and shearing force. Yokel, et al (1982), using linear elastic beam theory, found that the maximum induced shear force and bending moment in the foundation due to vertical ground displacement were:
\[ V_m = \frac{5.16}{d} \sqrt{EIw\Delta_m} \]
\[ M_m = \frac{13}{d^2} EI\Delta_m \]  

(3.24)

for structures located in the tension zone, and

\[ V_m = \frac{5.5}{d} \sqrt{EIw\Delta_m} \]
\[ M_m = \frac{15}{d^2} EI\Delta_m \]  

(3.24)

for structures located in the compression zone. In these formulas, \( V_m \) is the maximum shear force, \( M_m \) the maximum bending movement, \( E \) Young's modulus, \( I \) moment of inertia, \( w \) the distributed load on the foundation, \( \Delta_m \) the maximum vertical displacement, and \( d \) the mine depth.

Shear deformation in the horizontal plane can also induce forces on foundations by creating relative displacement between foundation and subgrade. CCSRA (1983) gave the following formulae of the resistant forces along the foundation-subgrade interfaces for strip foundations:

\[ T_1 = B(q_1C + b_1C) \frac{\delta_1}{\Delta_c} \]
\[ T_2 = L(q_2C + b_2C) \frac{\delta_2}{\Delta_c} \]  

(3.25)

where

\[ \delta_1 = \frac{L}{2} (\gamma_2 + \phi) \]
\[ \delta_2 = \frac{B}{2} (\gamma_1 - \phi) \]  

(3.26)
$T_1, T_2 =$ the resistant forces underneath the transverse and longitudinal foundations respectively;
$q_1, q_2 =$ the vertical contact pressures;
$\Delta_1, \Delta_2 =$ the critical relative displacements;
$B, L =$ the length of the transverse and longitudinal foundations respectively;
$b_1, b_2 =$ the width of the foundations;
$f_c =$ the critical friction coefficients;
$\delta_1, \delta_2 =$ the relative horizontal displacements due to rotations of the foundations;
$\gamma_1, \gamma_2 =$ the rotation of the longitudinal and transverse foundations respectively;
$\phi =$ the rotation of the entire structure; and
$C =$ the cohesion coefficient.
Chapter 4 Development of the Subsidence Response Modeling Program (SRMP)

Subsidence prediction methods have been successful in pre-calculating ground movements. These methods, however, cannot predict building responses due to such movements. In order to address this problem, a finite element model was developed in this research. The program, named SRMP (Subsidence Response Modeling Program), is a large displacement, small strain, two dimensional finite element program. Such a model is more appropriate than the small-displacement formulations because it describes mining conditions more accurately. Four types of elements are employed in the program: plane elements, beam elements, transition elements, and friction elements. Total Lagrangian (T.L.) formulation is used for plane elements and Updated Lagrangian (U.L.) formulation for beam and transition elements. The program can simulate ground movements with or without the presence of any surface buildings and hence, the subsidence profiles for either case can be determined and compared. Another important feature of SRMP is that it can model the dynamic subsidence profile continuously without re-initiating the system variables and boundary conditions. As a result, the ground movement, building displacement, and stresses can be obtained at each mining stage.
The program consists of twenty six subroutines as shown in Figure 4.1 and is written in FORTRAN. Except for the equation solving subroutine SOLVE and some I/O statements which are copied from FE2D written by Kuppusamy (1986), the program was developed by the author, using some of the theoretical derivations explained in Bathe (1982). The program requires one mega-byte of memory and is mainly designed to run on IBM mainframes.

4.1 Fundamental Principles

A body subjected to arbitrary large motion is considered in Figure 4.2. The configuration of the body at \( t + \Delta t \) is described in terms of the coordinates of the previous state, which is the so-called Lagrangian description. If the previous state is the one at time 0, then the formulation is called a Total Lagrangian formulation. On the other hand, if the previous state is the one at time \( t \), the description is called an Updated Lagrangian formulation. In general, the Lagrangian formulation always refers to a configuration of the “past”. For a dynamic problem the time \( t \) is a real variable, but it is just an indicating variable for the static problem analysed in this study.

Assuming that external forces are applied to the body, then stress and strain will result. From the principle of virtual work, it follows that:

\[
\int_{t+\Delta t}^{t+\Delta t'} t_{ij} \delta t_{ij} + \Delta t^e + \Delta t^d \, dV = (t+\Delta t')W \tag{4.1}
\]

where

\[
t+\Delta t'W = \int_{t+\Delta t}^{t+\Delta t'} t_{ij} \delta t_{ij} + \Delta t^e + \Delta t^d \, dV + \int_{t+\Delta t}^{t+\Delta t'} t_{ij} \delta U_{ij} + \Delta t^d \, dS;
\]
Figure 4.1. The Structure of SRMP
Figure 4.2. A Deforming Body at Different Stages (Bathe, 1982)
\( t + \Delta t \varepsilon_{ij} \) = Cauchy stresses (physical stresses) at time \( t + \Delta t \);
\[
\delta t + \Delta t \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial \delta u_i}{\partial t + \Delta t \chi_i} + \frac{\partial \delta u_j}{\partial t + \Delta t \chi_i} \right);
\]
\( \delta u_i, \delta_i + \Delta t \varepsilon_{ij} \) = virtual displacements and corresponding virtual strains;
\( t + \Delta t \mathbf{V}, t + \Delta t \mathbf{S} \) = volume and surface area at time \( t + \Delta t \); and
\( t + \Delta t \mathbf{f}^n, t + \Delta t \mathbf{f}^e \) = external forces per unit volume and unit area at time \( t + \Delta t \).

This is the basic equation for finite element analyses of nonlinear problems. However, the system of simultaneous equations can not be derived directly from Equation 4.1 because the volume at time \( t + \Delta t \) is unknown and the Cauchy stresses can not be decomposed incrementally (Bathe, 1982). To resolve this problem, the 2nd Piola-Kirchhoff stress tensor, \( t + \Delta t S_{ij} \), and Green-Lagrange strain tensor, \( t + \Delta t \varepsilon_{ij} \), were introduced. The 2nd Piola-Kirchhoff stress possesses the following properties:

- It is symmetric;
- It is invariant under a rigid-body motion (rotation and/or translation), and thus changes only when the material is deformed;
- It can be decomposed incrementally, that is, \( t + \Delta t S_{ij} = t_0 S_{ij} + t_0 \mathbf{S} \);
- It has no direct physical interpretation.

The Green-Lagrange strains have the properties of being symmetrical and decomposable and thus can be written as \( t + \Delta t \varepsilon_{ij} = t_0 \varepsilon_{ij} + t_0 \varepsilon \). It was proved that (Bathe, 1982):

\[
\int_{V} t + \Delta t \tau_{ij} \delta t + \Delta t \varepsilon_{ij} + \Delta t \mathbf{V} = \int_{\mathcal{S}} t + \Delta t S_{ij} t^i + \Delta t \varepsilon_{ij} t_0 \mathbf{dV}
\]

(4.2)

Therefore, Equation 4.1 can be written as:
\[
\int_{V_0}^{t+\Delta t} \delta S_{ij} \delta \epsilon_{ij}^0 dV = t+\Delta t W
\]  
(4.3)

Thus, the finite element formulation, which is either a Total Lagrangian Formulation (\(t_0 = 0\)) or an Updated Lagrangian Formulation (\(t_0 = t\)), can be carried out.

### 4.1.1 Total Lagrangian Formulation

As mentioned previously, the known reference configuration in the Total Lagrangian formulation is the original one, that is at \(t_0 = 0\). Hence, Equation 4.3 becomes:

\[
\int_{V_0}^{t+\Delta t} \delta S_{ij} \delta \epsilon_{ij}^0 dV = t+\Delta t W
\]  
(4.4)

For a given problem, the original configuration defined by the coordinates and boundary conditions is known, and thus the subsequent configurations can be found using an incremental approach. Supposing that the solution at time \(t\) is known and is sought at time \(t + \Delta t\), then the unknowns at time \(t + \Delta t\) can be decomposed as:

\[
\begin{align*}
^{t+\Delta t}S_{ij} &= \delta S_{ij} + \sigma S_{ij} \\
^{t+\Delta t}\epsilon_{ij} &= \delta \epsilon_{ij} + \sigma \epsilon_{ij} \\
^{t+\Delta t}u_i &= \delta u_i + u_i
\end{align*}
\]  
(4.5)

where

\(^{t+\Delta t}u_i, \delta u_i = \) displacements at time \(t + \Delta t\)
and $t$ respectively;

$$ u_i = \text{displacement increment between time } t \text{ and } t + \Delta t; $$

$$ t + \Delta t u_{ij} = \frac{1}{2} (t + \Delta t u_{ij} + t + \Delta t u_{ij} + t + \Delta t u_{k,l} + t + \Delta t u_{k,l}); $$

$$ \delta u_{ij} = \frac{1}{2} (u_{ij} + u_{ij} + \delta u_{k,i} + \delta u_{k,i}); $$

$$ 0 \epsilon_{ij} = \frac{1}{2} (0 u_{ij} + 0 u_{ij} + \delta u_{k,i} 0 u_{ij} + 0 u_{k,i} \delta u_{k,i}) + \frac{1}{2} \delta u_{k,i} 0 u_{k,i}; $$

$$ t + \Delta t u_{ij} = \frac{\partial t + \Delta t u_i}{\partial x_j}; $$

$$ \delta u_{ij} = \frac{\partial u_{ij}}{\partial x_i}; $$

$$ 0 u_{ij} = \frac{\partial u_i}{\partial x_j}; \text{ and} $$

$$ 0 x_{ij} = \text{the coordinates of the configuration at time } 0. $$

If

$$ 0 \epsilon_{ij} = 0 \epsilon_{ij} + 0 \eta_{ij} \tag{4.6} $$

where

$$ 0 \epsilon_{ij} = \frac{1}{2} (0 u_{ij} + 0 u_{ij} + t u_{ij} 0 u_{ij} + 0 u_{k,i} e u_{k,j}) \tag{4.6a} $$

is the linear strain component, and

$$ 0 \eta_{ij} = \frac{1}{2} 0 u_{k,i} 0 u_{k,j} \tag{4.6b} $$

is the nonlinear strain component, then by substituting Equations 4.5 and 4.6 into 4.4 and considering the fact that $\delta t + \Delta t \epsilon_{ij} = \delta 0 \epsilon_{ij}$, the following equation is obtained:
\[
\int_{\Omega}^{t+\Delta t} \delta_0 \varepsilon_{ij} \, dV + \int_{\Omega}^{t+\Delta t} \delta_0 \eta_{ij} \, dV = t+\Delta t \mathbf{W} - \int_{\Omega}^{t} \delta_0 \varepsilon_{ij} \, dV 
\] (4.7)

The first term in the R.H.S of Equation 4.7 is associated with external forces, namely surface traction and body force per unit original volume, which are given values. The second term in the R.H.S. is known, because \(\delta_0 S_{ij}\) is known at time \(t + \Delta t\) and \(\delta_0 \varepsilon_{ij}\) is constant since \(\partial \varepsilon_{ij}\) is linear in \(u_i\). Therefore, the R.H.S. of Equation 4.7 is known. Since \(\partial \eta_{ij}\) is quadratic in \(u_i\), the second term in the L.H.S. is linear in \(u_i\). The only nonlinear term in Equation 4.7 is the first term of the L.H.S. in which \(\partial_0 S_{ij}\) is a nonlinear function of \(\partial_0 \varepsilon_{ij}\) and \(\delta_0 \varepsilon_{ij}\) is a linear function of \(u_i\). In order to obtain the final system of linear equations, this must be linearized.

\(\partial_0 S_{ij}\) can be written as a Taylor series in \(\partial_0 \varepsilon_{ij}\) (Bathe, 1982):

\[
\partial_0 S_{ij} = \frac{\partial t S_{ij}}{\partial t \varepsilon_{rs}} \partial_0 \varepsilon_{rs} + \text{higher-order terms}
\]

\[
\approx \frac{\partial \delta_0 S_{ij}}{\partial \delta_0 \varepsilon_{rs}} (\partial_0 \varepsilon_{rs} + \partial_0 \eta_{ij})
\]

\[
\approx \partial_0 C_{ijrs} \partial_0 \varepsilon_{rs}
\]

(4.8)

where \(\partial_0 C_{ijrs}\) are the components of the constitutive matrix which do not contain \(u_i\). In the above equation, the higher-order terms in the Taylor series and the nonlinear part of \(\partial_0 \varepsilon_{rs}\) are ignored, as a result \(\partial_0 S_{ij}\) is linearized and \(\delta_0 \varepsilon_{ij}\) can be written as:

\[
\partial_0 S_{ij} \delta_0 \varepsilon_{ij} = \partial_0 C_{ijrs} \partial_0 \varepsilon_{rs} (\partial_0 \varepsilon_{ij} + \partial_0 \eta_{ij})
\]

\[
= \partial_0 C_{ijrs} \partial_0 \varepsilon_{rs} \delta_0 \varepsilon_{ij} + \partial_0 C_{ijrs} \partial_0 \varepsilon_{rs} \delta_0 \eta_{ij}
\]

(4.9)

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The first term on the R.H.S. of this equation is linear and the second term is quadratic in \( u_i \). Ignoring the quadratic term, Equation 4.9 becomes:

\[
\delta \varepsilon_{ij} \delta \varepsilon_{ij} \approx \sigma_{ij} \delta \varepsilon_{ij} + \sigma_{ij} \delta \varepsilon_{ij}
\]  
(4.10)

Substituting Equation 4.10 into Equation 4.7, the final linearized equation is obtained:

\[
\int_{\partial V} \sigma_{ij} \delta \varepsilon_{ij} \delta \varepsilon_{ij} \, dV + \int_{\partial V} \gamma_{ij} \delta \varepsilon_{ij} \delta \varepsilon_{ij} \, dV = t + \Delta t \varepsilon \nabla - \int_{\partial V} \delta \varepsilon_{ij} \delta \varepsilon_{ij} \, dV
\]  
(4.11)

If proper interpolation functions and the constitutive relation are chosen, Equation 4.11 can be written as:

\[
\delta \hat{u}^T (\int_{\partial V} \sigma_{BL} \delta \varepsilon_{ij} \delta \varepsilon_{ij} \, dV) \hat{u} + \delta \hat{u}^T (\int_{\partial V} \sigma_{B} \delta \varepsilon_{ij} \delta \varepsilon_{ij} \, dV) \hat{u} = \]

\[
\delta \hat{u}^T (\int_{\partial V} H^T \delta \varepsilon_{ij} \delta \varepsilon_{ij} \, dV + \int_{\partial S} H^T \delta \varepsilon_{ij} \delta \varepsilon_{ij} \, dS) - \delta \hat{u}^T (\int_{\partial V} B_{NL} \delta \varepsilon_{ij} \delta \varepsilon_{ij} \, dV)
\]  
(4.12)

Comparing Equation 4.12 with 4.11, it is obvious that:

\[
u = H \hat{u};
\]

\[
\delta u = H \delta \hat{u};
\]

\[
\sigma_{BL} = \delta B_L \hat{u};
\]

\[
\delta \sigma_{BL} = \delta B_L \delta \hat{u};
\]

\[
u^S = H^S \hat{u};
\] and

\[
\delta S \delta \eta = \delta \hat{u}^T \delta B_{NL} dS \delta B_{NL} \hat{u}.
\]
where
\[ H, H^S = \text{interpolation function matrices}; \]
\[ \delta B_L, \delta B_{NL} = \text{strain transformation matrices}; \text{and} \]
\[ \hat{\mathbf{u}} = \text{the nodal displacement vector}. \]

These matrices will be discussed in section 4.2.

According to the virtual displacement theorem, \( \delta \hat{\mathbf{u}} \) can be arbitrary, and thus Equation 4.12 becomes:

\[
\begin{align*}
\int_{\Omega_v} \delta B_L^T \partial C \delta B_L \ 0 \ dV + \int_{\Omega_v} \delta B_{NL}^T \partial S \delta B_{NL} \ 0 \ dV \hat{\mathbf{u}} = \\
\int_{\Omega_v} H^T \left( I + \Delta \delta^b \right) \ 0 \ dV + \int_{\Omega_s} H^ST \left( I + \Delta \delta^S \right) dS - \int_{\Omega_v} \partial B_L^T \partial S \ 0 \ dV
\end{align*}
\]

or

\[
(\partial \mathbf{K}_L + \partial \mathbf{K}_{NL}) \hat{\mathbf{u}} = \mathbf{t} + \Delta \mathbf{R} - \partial \mathbf{F} \tag{4.14}
\]

where
\[
\begin{align*}
\partial \mathbf{K}_L &= \int_{\Omega_v} \delta B_L^T \partial C \delta B_L \ 0 \ dV; \\
\partial \mathbf{K}_{NL} &= \int_{\Omega_v} \delta B_{NL}^T \partial S \delta B_{NL} \ 0 \ dV; \\
\mathbf{t} + \Delta \mathbf{R} &= \int_{\Omega_v} H^T \left( I + \Delta \delta^b \right) \ 0 \ dV + \int_{\Omega_s} H^ST \left( I + \Delta \delta^S \right) dS; \text{and} \\
\partial \mathbf{F} &= \int_{\Omega_v} \delta B_L^T \partial \hat{\delta} \ 0 \ dV
\end{align*}
\]

The nodal displacements can be obtained by solving Equation 4.14.
4.1.2 Up-Dated Lagrangian Formulation

The known configuration in the Up-dated Lagrangian formulation is the one at time $t$. Thus, substituting $t_0$ with $t$ into Equation 4.3, one obtains:

\[
\int_V \dot{t} + \Delta t S_{ij} \delta \dot{t} + \dot{t} \delta \varepsilon_{ij} \, dV = \dot{t} + \Delta t \varepsilon
\]  
\[\text{(4.15)}\]

Following the same steps as in the T.L. formulation, the unknown stresses and strains at time $t + \Delta t$ can be decomposed as:

\[
\dot{t} + \Delta t S_{ij} = \dot{t} S_{ij} + \dot{t} S_{ij}
\]

\[\text{and}\]

\[
\dot{t} + \Delta t \dot{\varepsilon}_{ij} = \dot{t} \dot{\varepsilon}_{ij} + \dot{t} \dot{\varepsilon}_{ij}
\]  
\[\text{(4.16)}\]

where

\[
|S_{ij} = \tau_{ij}, \text{ the components of Cauchy stress tensor;}
\]

\[
\dot{t} + \Delta_t \dot{\varepsilon}_{ij} = \frac{1}{2} \left( \dot{t} + \Delta_t [u_{i,j} + \dot{t} u_{i,j}] + \dot{t} + \Delta_t [u_{k,i} + \dot{t} u_{k,j}] \right);
\]

\[
\varepsilon_{ij} = 0;
\]

\[
\dot{t} \varepsilon_{ij} = \dot{t} \varepsilon_{ij} + \dot{t} \eta_{ij};
\]

\[
\dot{t} \eta_{ij} = \frac{1}{2} \left( (u_{i,j} + \dot{t} u_{i,j}) \right); \text{ and}
\]

\[
\dot{t} \eta_{ij} = \frac{1}{2} \dot{t} u_{k,i} u_{k,j}.
\]

Substituting the above equations into Equation 4.15, the following equation is obtained:

\[
\int_V \dot{t} S_{ij} \delta \varepsilon_{ij} \, dV + \int_V \dot{t} \tau_{ij} \delta \varepsilon_{ij} \, dV = \dot{t} + \Delta t \varepsilon - \int_V \dot{t} \tau_{ij} \delta \varepsilon_{ij} \, dV
\]  
\[\text{(4.17)}\]
Given a variation $\delta u_i$, the R.H.S. of this equation is known, the second term on the L.H.S. is linear in $u_i$, and the first term is a nonlinear function of $u_i$ which must be linearized. The linearization process is the same as in the T.L. formulation. After linearization, Equation 4.17 becomes:

$$\int_\Omega C_{jrs} \varepsilon_{rs} \delta e_{ij} dV + \int_\Omega \tau_{ij} \delta \eta_{ij} dV = \tau^{+\Delta t} W - \int_\Omega \tau_{ij} \delta e_{ij} dV$$

(4.18)

Similar to the T.L. formulation, after discretization, Equation 4.18 becomes:

$$(^{i}K_{L} +^{i}K_{NL}) \hat{u} = \tau^{+\Delta t} R - ^{i}F$$

(4.19)

where

$$^{i}K_{L} = \int_\Omega ^{i}B_{L}^{T} C^{i}B_{L} dV;$$

$$^{i}K_{NL} = \int_\Omega ^{i}B_{NL}^{T} \tau^{i}B_{NL} dV;$$

$$\tau^{+\Delta t} R = \int_\Omega H^{T} \tau^{+\Delta t} B^{T} dV + \int_{\Gamma} H^{ST} \tau^{+\Delta t} S dS$$; and

$$^{i}F = \int_\Omega ^{i}B_{L}^{T} ^{i}\hat{e} dV$$

### 4.2 Formation of Continuous Element

T.L. formulation and four-node, plane strain elements are used in the part of the domain below the ground surface, which includes the overburden, the coal seam, and a part of the floor strata. The elements are designed as isoparametric elements in which the nodal coordinates and nodal unknowns adopt the same interpolation functions, and hence one has:
\[
0_{x_1} = \sum_{k=1}^{4} h_k^0 x_1^k, \quad 0_{x_2} = \sum_{k=1}^{4} h_k^0 x_2^k \\
1_{x_1} = \sum_{k=1}^{4} h_k^1 x_1^k, \quad 1_{x_2} = \sum_{k=1}^{4} h_k^1 x_2^k \\
1_{u_1} = \sum_{k=1}^{4} h_k^1 u_1^k, \quad 1_{u_2} = \sum_{k=1}^{4} h_k^1 u_2^k \\
\text{u}_1 = \sum_{k=1}^{4} h_k u_1^k, \quad \text{u}_2 = \sum_{k=1}^{4} h_k u_2^k
\]

or in matrix form:

\[
\begin{align*}
0_x &= H^0 \hat{x} \\
1_x &= H^1 \hat{x} \\
1_u &= H^1 \hat{u} \\
\text{u} &= H \hat{u}
\end{align*}
\]

where

\[
H = \begin{bmatrix}
h_1 & 0 & h_2 & 0 & h_3 & 0 & h_4 & 0 \\
0 & h_1 & 0 & h_2 & 0 & h_3 & 0 & h_4
\end{bmatrix}, \quad \text{the interpolation matrix;}
\]

\[
0_x = [0_{x_1} \ 0_{x_2}]^T, \text{ coordinates at time 0;}
\]

\[
0_{\hat{x}} = [0_{x_1} \ 0_{x_2} \ 0_{x_1} \ 0_{x_2} \ 0_{x_1} \ 0_{x_2} \ 0_{x_1} \ 0_{x_2}]^T, \text{ nodal coordinates at time 0;}
\]

\[
1_x = [1_{x_1} \ 1_{x_2}]^T, \text{ coordinates at time 1;}
\]

\[
1_{\hat{x}} = [1_{x_1} \ 1_{x_2} \ 1_{x_1} \ 1_{x_2} \ 1_{x_1} \ 1_{x_2} \ 1_{x_1} \ 1_{x_2}]^T, \text{ nodal coordinates at time 1;}
\]

\[
1_u = [1_u \ 1_u]^T, \text{ displacements at time 1;}
\]

\[
1_{\hat{u}} = [1_u \ 1_u \ 1_u \ 1_u \ 1_u \ 1_u \ 1_u \ 1_u]^T, \text{ nodal displacements at time 1;}
\]

\[
u = [u_1 \ u_2]^T, \text{ displacement increments between time 1 and t + \Delta t;}
\]

\[
\hat{u} = [u_1 \ u_2 \ u_3 \ u_4 \ u_1 \ u_2 \ u_3 \ u_4]^T, \text{ nodal displacement increments at time t and t + \Delta t;}
\]
\[ h_1 = \frac{1}{4} (1 - r)(1 - s); \]
\[ h_2 = \frac{1}{4} (1 + r)(1 - s); \]
\[ h_3 = \frac{1}{4} (1 + r)(1 + s); \]
\[ h_4 = \frac{1}{4} (1 - r)(1 + s); \]

\[ r, s = \text{local coordinates.} \]

The local coordinates system and numbering scheme are shown in Figure 4.3. The numbering scheme determines the subscript of the interpolation function \( h \) and the number sequence of element connectivity which relates the local nodal numbers to the global ones. For example, if the local nodes are numbered counter-clock-wise, starting from the lower-left corner, then the corresponding global nodal numbers must be recorded the same way in the connectivity table.

From Equation 4.20, the following derivatives can be obtained:

\[ u_{ij} = \frac{\partial u_i}{\partial x_j} = \sum_{k=1}^{4} \left( \frac{\partial h_k}{\partial x_j} \right) u_{ij}^k \]

\[ v_{ij} = \frac{\partial v_i}{\partial x_j} = \sum_{k=1}^{4} \left( \frac{\partial v_k}{\partial x_j} \right) v_{ij}^k \]

Since the stiffness matrix and force vector are obtained from each individual element and the integrations are carried out over the local coordinate system, \( \frac{\partial h_k}{\partial x_j} \) must be transformed into functions of local variables. From the chain rule:
Figure 4.3. Local Coordinate System for Continuous Elements
\[ \frac{\partial h_k}{\partial r} = \frac{\partial h_k}{\partial \xi_1} \frac{\partial \xi_1}{\partial r} + \frac{\partial h_k}{\partial \xi_2} \frac{\partial \xi_2}{\partial r} \]

or in matrix form:

\[
\begin{bmatrix}
\frac{\partial h_k}{\partial r} \\
\frac{\partial h_k}{\partial s}
\end{bmatrix} = 
\begin{bmatrix}
\frac{\partial \xi_1}{\partial r} & \frac{\partial \xi_2}{\partial r} \\
\frac{\partial \xi_1}{\partial s} & \frac{\partial \xi_2}{\partial s}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial h_k}{\partial \xi_1} \\
\frac{\partial h_k}{\partial \xi_2}
\end{bmatrix}
\]

The 2 x 2 matrix on the R.H.S. is the Jacobian matrix. From the above equation, the following is obtained:

\[
\begin{bmatrix}
\frac{\partial h_k}{\partial \xi_1} \\
\frac{\partial h_k}{\partial \xi_2}
\end{bmatrix} = \frac{1}{\det \tilde{J}} 
\begin{bmatrix}
\frac{\partial \xi_2}{\partial r} & -\frac{\partial \xi_2}{\partial \xi_1} \\
-\frac{\partial \xi_1}{\partial r} & \frac{\partial \xi_1}{\partial \xi_2}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial h_k}{\partial r} \\
\frac{\partial h_k}{\partial s}
\end{bmatrix}
\]

(4.24)

where

\[ \det \tilde{J} = \text{the determinant of the Jacobian matrix;} \]

\[
\frac{\partial \xi_1}{\partial r} = \sum_{k=1}^{4} \frac{\partial h_k}{\partial r} \xi_{1}^{k} ; \text{and}
\]

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\[ \frac{\partial^0 x_i}{\partial s} = \sum_{k=1}^{4} \frac{\partial h_k}{\partial s} o_{x_i}^k. \]

Substituting Equations 4.22, 4.23 and 4.24 into 4.6a results in the following equation (Bathe, 1982):

\[ o e = o^t B_L \hat{u} \]  

(4.25)

where

\[ o e = [o e_{11} \ o e_{22} \ 2 o e_{12}]^T \]
\[ \hat{u} = [u_1^1 \ u_1^2 \ u_2^1 \ u_2^2 \ u_1^3 \ u_2^3 \ u_1^4 \ u_2^4]^T \]
\[ o^t B_L = o^t B_{L0} + o^t B_{L1} \]
\[ o^t B_{L0} = \begin{bmatrix} o h_{1,1} & 0 & o h_{2,1} & 0 & o h_{3,1} & 0 & o h_{4,1} & 0 \\ 0 & o h_{1,2} & 0 & o h_{2,2} & 0 & o h_{3,2} & 0 & o h_{4,2} \\ o h_{1,2} & o h_{1,1} & o h_{2,2} & o h_{2,1} & o h_{3,2} & o h_{3,1} & o h_{4,2} & o h_{4,1} \end{bmatrix} \]  

(4.26a)
\[ o h_{k,i} = \frac{\partial h_k}{\partial o x_i} \]

\[ o^t B_{L1} = \begin{bmatrix} \ldots & m_{110} h_{N,1} & m_{210} h_{N,1} & \ldots \\ \ldots & m_{120} h_{N,2} & m_{220} h_{N,2} & \ldots \\ \ldots & (m_{110} h_{N,2} + m_{120} h_{N,1}) & (m_{210} h_{N,2} + m_{220} h_{N,1}) & \ldots \end{bmatrix} \]

(4.26b)

\[ N = 1, 2, 3, 4 \]

\[ m_{ij} = \sum_{k=1}^{4} o h_{k,i}^t u_{k}^j \]

From Equation 4.14, the linear element stiffness matrix can be calculated as:

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\[ \dot{t}_0 K_L = \int_{\Omega} \dot{t}_0 B_L^T \dot{\epsilon}_C \dot{t}_0 B_L \dot{V} \]

\[ = \int_{-1}^{1} \int_{-1}^{1} \dot{t}_0 B_L^T \dot{\epsilon}_C \dot{t}_0 B_L \det^{0} J \, dr \, ds \quad (4.27) \]

where \( \dot{t}_0 B_L \) is defined in Equation 4.26 and its elements are functions of local coordinates, \( r \) and \( s \); \( \dot{\epsilon}_C \) is the constitutive matrix which is independent of \( r \) and \( s \), and will be discussed later; and

\[ \dot{V} = d^{0} x_1 \, d^{0} x_2 = \det^{0} J \, dr \, ds. \quad (4.28) \]

In the computer program, the integration in Equation 4.27 is carried out by using 16-point numerical integration and \( \dot{t}_0 K_L \) is computed as:

\[ \dot{t}_0 K_L = \sum_{i=1}^{4} \sum_{j=1}^{4} w_i w_j G(r_i, s_j) \quad (4.29) \]

where

\[ w_i, w_j = \text{the weighting factors at point } (r_i, s_j); \text{ and} \]

\[ G(r_i, s_j) = \dot{t}_0 B_L^T(r_i, s_j) \dot{\epsilon}_C \dot{t}_0 B_L(r_i, s_j) \det^{0} J(r_i, s_j) \]

The nonlinear strain-displacement transformation matrix, \( \dot{t}_0 B_{NL} \), is so constructed that (Bathe, 1982)

\[ \delta \dot{\hat{U}}_0 (\dot{t}_0 B_{NL}^T \dot{t}_0 B_{NL}) \dot{\hat{U}} = \dot{t}_0 S_{ij} \delta_{0} \eta_{ij} \quad (4.30) \]
where

\[ \begin{bmatrix}
0 h_{1,1} & 0 & h_{2,1} & 0 & h_{3,1} & 0 & h_{4,1} & 0 \\
0 h_{1,2} & 0 & h_{2,2} & 0 & h_{3,2} & 0 & h_{4,2} & 0 \\
0 & 0 & h_{1,1} & 0 & h_{2,1} & 0 & h_{3,1} & 0 & h_{4,1} \\
0 & 0 & h_{1,2} & 0 & h_{2,2} & 0 & h_{3,2} & 0 & h_{4,2}
\end{bmatrix}; \]  

(4.31a)

\[ \begin{bmatrix}
t S_{11} & t S_{12} & 0 & 0 \\
t S_{21} & t S_{22} & 0 & 0 \\
0 & 0 & t S_{11} & t S_{12} \\
0 & 0 & t S_{21} & t S_{22}
\end{bmatrix}; \]  

(4.31b)

\[ h_{k,0} \text{ and } dS_d \text{ have the same meanings as mentioned previously. Substituting Equation 4.31 into 4.14 for nonlinear element stiffness matrix, one has:} \]

\[ \begin{align*}
\begin{bmatrix}
\Sigma \end{bmatrix}_{0K_{NL}} &= \int_0 \begin{bmatrix}
\Sigma \end{bmatrix}_{TB_{NL}} \begin{bmatrix}
\Sigma \end{bmatrix}_{TS} \begin{bmatrix}
\Sigma \end{bmatrix}_{TB_{NL}}^T dV \\
&= \int_0 \int_0 \begin{bmatrix}
\Sigma \end{bmatrix}_{TB_{NL}} \begin{bmatrix}
\Sigma \end{bmatrix}_{TS} \begin{bmatrix}
\Sigma \end{bmatrix}_{TB_{NL}}^T \text{det}^0 J \text{d}r \text{d}s
\end{align*} \]  

(4.32)

Note again that the elements of \( dB_{NL} \) are functions of local variables, \( r \) and \( s \), and \( dS \) is independent of \( r \) and \( s \). Using 16-point numerical integration, \( dK_{NL} \) is calculated as:

\[ \begin{align*}
\begin{bmatrix}
\Sigma \end{bmatrix}_{0K_{NL}} &= \sum_{i=1}^4 \sum_{j=1}^4 w_i w_j G(r_i, s_j)
\end{align*} \]  

(4.33)
where
\[ G(r_i, s_j) = dB_{LL}(r_i, s_i) dS dB_{NL}(r_i, s_j) \det J(r_i, s_j); \]

\( w_i, w_j \) = the same as in Equation 4.29.

So far, the element stiffness matrices have been formed, and the next step is to consider the element load vector which includes four parts, namely body force, surface distribution load, concentrated load, and internal force due to element stress. The concentrated loads can be directly added to the load vector without treatment and thus no further discussion is needed. Considerations will be given to the other three parts.

Body force in this study is the force of gravity and can be written as:

\[
\begin{bmatrix}
\mathbf{t} + \Delta t \mathbf{f}^B \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
\rho
\end{bmatrix} \tag{4.34}
\]

where \( \rho \) is the material density. From Equation 4.14, the element body force vector can be calculated as:

\[
\begin{bmatrix}
\mathbf{t} + \Delta t \mathbf{f}^B \\
0
\end{bmatrix} = \int_V H^T \mathbf{t}^B \rho dV = \int_{-1}^{1} \int_{-1}^{1} H^T \mathbf{t}^B \det J \rho d\mathbf{r} d\mathbf{s} \tag{4.35}
\]

where \( H \), the interpolation matrix, is defined in Equation 4.21. The numerical integration procedure is the same as previously mentioned.

To obtain the surface distribution load vector, the surface interpolation function is required. Assuming that edge 3-4 of an element is subjected to a linearly distributed load,
as shown in Figure 4.4, and the displacements at the surface are interpolated linearly, then the displacements along edge 3-4 can be expressed as:

\[
\begin{align*}
\mathbf{u}^S_1 &= h_1 \mathbf{u}_1^3 + h_2 \mathbf{u}_1^4 \\
\mathbf{u}^S_2 &= h_1 \mathbf{u}_2^3 + h_2 \mathbf{u}_2^4
\end{align*}
\]  

(4.36)

or in matrix form:

\[
\mathbf{u}^S = \mathbf{H}^S \mathbf{\hat{u}}
\]

(4.37)

where

\[
\mathbf{u}^S = [u^S_1 \ u^S_2]^T;
\]

\[
\mathbf{H}^S = \begin{bmatrix}
0 & 0 & 0 & h_1 & 0 & h_2 & 0 \\
0 & 0 & 0 & 0 & h_1 & 0 & h_2
\end{bmatrix};
\]

\[
\mathbf{\hat{u}} = [u_1 \ u_2 \ u_3 \ u_4 \ u_5 \ u_6]^T;
\]

\[
h_1 = \frac{1}{2} (1 + r); \text{ and}
\]

\[
h_2 = \frac{1}{2} (1 + r).
\]

The surface distributed load along edge 3-4 can be written as:

\[
\begin{bmatrix}
\frac{t}{1 + \Delta t_1} \\
\frac{t}{1 + \Delta t_2}
\end{bmatrix} = \begin{bmatrix}
h_1 & 0 & h_2 & 0 \\
0 & h_1 & 0 & h_2
\end{bmatrix} \begin{bmatrix}
l_1^2 \\
l_2^2 \\
l_1^4 \\
l_2^4
\end{bmatrix}
\]

(4.38)

where \(\frac{t}{1 + \Delta t}\) is the distributed load along the edge in \(x_1\) direction and \(f_i\) is the distributed load value at point \(k\) in the direction of \(x_i\). Therefore, the distributed load vector can be obtained as:

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\[
\begin{align*}
& t + \Delta t R^S = \int_{S} H_{t}^{ST} (t + \Delta t s_{0})^{T} dS \\
& = \int_{-1}^{1} \left[ \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array} \right] \left[ \begin{array}{c}
h_1 \\
h_2 \\
h_1 \\
h_2 \\
\end{array} \right]^{T} \left[ \begin{array}{c}
\frac{t + \Delta t s_{0}}{2} \\
\frac{t + \Delta t s_{0}}{2} \\
\frac{t + \Delta t s_{0}}{2} \\
\frac{t + \Delta t s_{0}}{2} \\
\end{array} \right] \frac{0L}{2} dr \\
& = \frac{0L}{2} \left[ \begin{array}{cccc}
\frac{r_1^3}{3} & \frac{r_1^4}{6} & \frac{r_2^3}{3} & \frac{r_2^4}{6} \\
\frac{r_1^3}{3} & \frac{r_1^4}{6} & \frac{r_2^3}{3} & \frac{r_2^4}{6} \\
\frac{r_1^3}{3} & \frac{r_1^4}{6} & \frac{r_2^3}{3} & \frac{r_2^4}{6} \\
\frac{r_1^3}{3} & \frac{r_1^4}{6} & \frac{r_2^3}{3} & \frac{r_2^4}{6} \\
\end{array} \right]^{T}
\end{align*}
\]

where \(0L\) is the length of edge 3-4 and all other terms are defined in Equation 4.37 and 4.38.

The internal force vector can also be obtained from Equation 4.14:

\[
\begin{align*}
& i_{0} F = \int_{V} i_{0} B_{L}^{T} \hat{i}_{0} S_{0} dV \\
& = \int_{-1}^{1} \int_{-1}^{1} i_{0} B_{L}^{T} \hat{i}_{0} S_{0} \text{det} 0 \text{J} dr ds
\end{align*}
\]

where \(i_{0} B_{L}\) is defined in Equation 4.26 and

\[
\begin{align*}
& i_{0} \hat{S} = \begin{bmatrix}
i_{0} S_{11} & i_{0} S_{12} & i_{0} S_{13} \\
i_{0} S_{21} & i_{0} S_{22} & i_{0} S_{23} \\
i_{0} S_{31} & i_{0} S_{32} & i_{0} S_{33}
\end{bmatrix}^{T}
\end{align*}
\]

where \(i_{0} S_{ij}\) was defined previously. The integration is carried numerically the same way as in the calculation of the element stiffness matrix.
4.3 Formation of Beam Element

Structures on the ground surface are modelled using three-node beam elements as shown in Figure 4.5(a). U.L. formulation is employed to evaluate the element stiffness matrix and load vector. The reason for using U.L. formulation lies in the fact that the engineering stress (Cauchy stress) can be obtained directly.

Upon choosing the interpolation functions, the nodal coordinates can be written as (Bathe, 1982):

\[
\mathbf{t}x_i = \sum_{k=1}^{3} h_k \mathbf{t}x_k + \frac{s}{2} \sum_{k=1}^{3} b_k h_k \mathbf{t}V_{si}^k \quad i = 1,2
\] (4.42)

Since

\[
\begin{align*}
\mathbf{t}u_i &= \mathbf{t}x_i - \mathbf{0}x_i \\
\mathbf{u}_i &= \mathbf{t} + \Delta \mathbf{x}_i - \mathbf{t}x_i
\end{align*}
\] (4.43)

by applying the isoparametric interpolation concepts, the nodal displacements and displacement increments can be interpolated as:

\[
\begin{align*}
\mathbf{t}u_i &= \sum_{k=1}^{3} h_k \mathbf{t}u_k^i + \frac{s}{2} \sum_{k=1}^{3} b_k h_k (\mathbf{t}V_{si}^k - \mathbf{0}V_{si}^k) \\
\mathbf{u}_i &= \sum_{k=1}^{3} h_k \mathbf{u}_k^i + \frac{s}{2} \sum_{k=1}^{3} b_k h_k \mathbf{V}_{si}^k
\end{align*}
\] (4.44)

where \( \mathbf{t}x_i, \mathbf{0}x_i, \mathbf{t}u_i, \) and \( \mathbf{u}_i \) have the same meanings as in the previous section, \( s \) is one of the local coordinates, and \( b_k \) the width of beam elements. The interpolation functions, \( h_k \), are:
Figure 4.5. Local Coordinate System for Beam Elements
\[ h_1 = -\frac{1}{2} r(1 - r) \]
\[ h_2 = \frac{1}{2} r(1 + r) \]
\[ h_3 = 1 - r^2 \]  

(4.45)

where \( r \) is the other local coordinate. The \( ^{t+\Delta t}V_{si}, \ ^{t}V_{si}, \) and \( ^{0}V_{si} \) in Equation 4.44 are the direction cosines of the vector in the \( s \) direction at time \( t + \Delta t \), \( t \) and 0 respectively, and \( V^k_{si} \) is the change in the direction cosine which can be expressed as:

\[ V^k_{si} = ^{t+\Delta t}V^k_{si} - ^{t}V^k_{si} \]  

(4.46)

where

\[ ^{t+\Delta t}V^k_{si} = \cos(^{t}\theta_k + \theta_k) = \cos ^{t}\theta_k \cos \theta_k - \sin ^{t}\theta_k \sin \theta_k \]
\[ = \cos ^{t}\theta_k - \sin ^{t}\theta_k \theta_k = ^{t}V^k_{ksi} - ^{t}V^k_{s2} \theta_k \]
\[ ^{t+\Delta t}V^k_{s2} = \sin(^{t}\theta_k + \theta_k) = \sin ^{t}\theta_k \cos \theta_k + \cos ^{t}\theta_k \sin \theta_k \]
\[ = \sin ^{t}\theta_k + \cos ^{t}\theta_k \theta_k = ^{t}V^k_{s2} + ^{t}V^k_{s1} \theta_k \]  

(4.47)

Therefore,

\[
\begin{bmatrix}
V^k_{s1} \\
V^k_{s2}
\end{bmatrix} = \begin{bmatrix}
-^{t}V^k_{s2} \theta_k \\
^{t}V^k_{s1} \theta_k
\end{bmatrix}
\]  

(4.48)

In Equation 4.47 and 4.48, \( ^{t}\theta_k \) is the angle between the normal of a beam point and the global Cartesian axis \( x_1 \), and \( \theta_k \) is the angular increment which is a nodal unknown. Equation 4.47 is valid only for small angular increments. Equation 4.48 can also be written in vector form:

\[ \vec{V}^k_s = \vec{\theta}_k \times \vec{V}^k_s \]  

(4.49)
Knowing the angular increments from the finite element solution, the direction cosines at time $t + \Delta t$ can be calculated as (Bathe, 1982):

$$
^{t+\Delta t} \vec{V}_s^k = \sum_{s=1}^{3} \vec{V}_s^k + \int_{t}^{t+\Delta t} \vec{V}_s^k \times \vec{V}_s^k \, dt \quad \text{for} \quad t \leq \tau \leq t + \Delta t \tag{4.50}
$$

In the program, the integration is carried out numerically, $\theta_k$ is divided into several sub-increments, and for each sub-increment, the direction cosines are computed as:

$$
\begin{bmatrix}
^{\tau+\Delta \tau} V_{s1}^k \\
^{\tau+\Delta \tau} V_{s2}^k
\end{bmatrix}
= \begin{bmatrix}
^{\tau} V_{s1}^k \\
^{\tau} V_{s2}^k
\end{bmatrix}
+ \begin{bmatrix}
^{\tau} - V_{s2}^k \Delta \theta_k \\
^{\tau} V_{s1}^k \Delta \theta_k
\end{bmatrix} \tag{4.51}
$$

where $\Delta \theta_k$ is a sub-increment of $\theta_k$. Equation 4.51 will give $^{t+\Delta t} V_b^k$ when all the sub-increments are taken into account. The unit length of the direction cosine vector is preserved by invoking the following formulas at each sub-increment:

$$
^{\tau+\Delta \tau} V_{s1}^k = \frac{^{\tau} V_{s1}^k - ^{\tau} V_{s2}^k \Delta \theta_k}{\sqrt{(^{\tau} V_{s1}^k - ^{\tau} V_{s2}^k \Delta \theta_k)^2 + (^{\tau} V_{s2}^k + ^{\tau} V_{s1}^k \Delta \theta_k)^2}} \tag{4.52}
$$

$$
^{\tau+\Delta \tau} V_{s2}^k = \frac{^{\tau} V_{s2}^k + ^{\tau} V_{s1}^k \Delta \theta_k}{\sqrt{(^{\tau} V_{s1}^k - ^{\tau} V_{s2}^k \Delta \theta_k)^2 + (^{\tau} V_{s2}^k + ^{\tau} V_{s1}^k \Delta \theta_k)^2}}
$$

From the above equations, the derivatives of displacement increments with respect to $r$ and $s$ can be found:

$$
\begin{bmatrix}
\frac{\partial u_i}{\partial r} \\
\frac{\partial u_i}{\partial s}
\end{bmatrix}
= \sum_{k=1}^{3} \begin{bmatrix}
h_{k,r} & -\frac{s}{2} b_k h_{k,r}^{\tau} V_{s2}^k \\
0 & \frac{1}{2} b_k h_{k}^{\tau} V_{s1}^k
\end{bmatrix} \begin{bmatrix}
^{\tau} u_i^k \\
^{\tau} \theta_k
\end{bmatrix} \tag{4.53}
$$

where
$$h_{k,r} = \frac{\partial h_k}{\partial r}$$

Using the Jacobian transformation, the derivatives of displacements with respect to the global coordinates can be obtained:

$$
\begin{bmatrix}
\frac{\partial u_i}{\partial x_1} \\
\frac{\partial u_i}{\partial x_2}
\end{bmatrix} = \frac{1}{\det J} \begin{bmatrix}
\frac{\partial^t x_2}{\partial s} & -\frac{\partial^t x_2}{\partial r} \\
-\frac{\partial^t x_1}{\partial s} & \frac{\partial^t x_1}{\partial r}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial u_i}{\partial r} \\
\frac{\partial u_i}{\partial s}
\end{bmatrix}
$$

(4.55)

where

$$
^t J = \begin{bmatrix}
\frac{\partial^t x_1}{\partial r} & \frac{\partial^t x_2}{\partial r} \\
\frac{\partial^t x_1}{\partial s} & \frac{\partial^t x_2}{\partial s}
\end{bmatrix}
$$

(4.56)

$$
\begin{bmatrix}
\frac{\partial^t x_i}{\partial r} \\
\frac{\partial^t x_i}{\partial s}
\end{bmatrix} = \sum_{k=1}^3 \begin{bmatrix} h_{k,r} + \frac{s}{2} b_k h_{k,r} & 0 & \frac{1}{2} b_k h_{k,r}^t \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
^{t^t} x_i^k \\
^{t^t} y_i^k \\
^{t^t} z_i^k
\end{bmatrix}
$$

The derivatives in Equation 4.55 must be transformed for strain calculation because the displacement components and the coordinates are in the directions of the global axes. Due to the nature of beam elements, only two strain components are considered: normal strain along the beam, and shear strain in the plane of the beam. In order to calculate these strains, a new coordinate system attached to the beam is introduced as shown in Figure 4.5(b). This coordinate system is designated as $X'_1 O X'_2$, and the corresponding displacements, coordinates, strains and stresses under this system are denoted by the same symbols carrying primes. The difference between this coordinate system and the
local coordinate system \((r,s)\) is that this system is actually another Cartesian coordinate system attached to the beam and one of its axes is the tangent of the beam point where the origin is located. Assuming that the new coordinate system is obtained by rotating the global coordinate system \(XOY\) \(\theta\) degrees (Figure 4.5(b)), then one has:

\[
\begin{bmatrix}
\hat{u}_1' \\
\hat{u}_2'
\end{bmatrix} = \begin{bmatrix}
\hat{V}_{r1} & \hat{V}_{r2} \\
\hat{V}_{s1} & \hat{V}_{s2}
\end{bmatrix} \begin{bmatrix}
\hat{u}_1 \\
\hat{u}_2
\end{bmatrix}
\]

\[
\begin{bmatrix}
\hat{x}_1' \\
\hat{x}_2'
\end{bmatrix} = \begin{bmatrix}
\hat{V}_{r1} & \hat{V}_{r2} \\
\hat{V}_{s1} & \hat{V}_{s2}
\end{bmatrix} \begin{bmatrix}
\hat{x}_1 \\
\hat{x}_2
\end{bmatrix}
\]

\[
\begin{bmatrix}
\hat{x}_1 \\
\hat{x}_2
\end{bmatrix} = \begin{bmatrix}
\hat{V}_{s2} & -\hat{V}_{r2} \\
-\hat{V}_{s1} & \hat{V}_{r1}
\end{bmatrix} \begin{bmatrix}
\hat{x}_1' \\
\hat{x}_2'
\end{bmatrix}
\] (4.57)

Where \(\hat{u}_i',\) and \(\hat{x}_i'\) are the displacement increments and coordinates under the new coordinate system, \(\hat{V}_{si}, \hat{V}_{ri}, \hat{u}_i,\) and \(\hat{x}_i\) have the same meanings as before and

\[
\hat{V}_{r1} = \hat{V}_{s2} \\
\hat{V}_{r2} = -\hat{V}_{s1}
\]

From Equation 4.57, the derivatives in Equation 4.55 can be transformed into those under the new coordinate system by the following equation:

\[
\frac{\partial u_i'}{\partial x_j'} = \frac{\partial u_i'}{\partial x_i} \frac{\partial x_i}{\partial x_j'} + \frac{\partial u_i'}{\partial x_j} \frac{\partial x_j}{\partial x_i'}
\] (4.58)

Substituting Equation 4.57 into 4.58, the derivatives under the new coordinate system are obtained:
\[
\frac{\partial u'_1}{\partial x'_1} = \frac{1}{r_1} v_{r_1}^2 u_{1,1} + \frac{1}{r_2} v_{r_2}^2 u_{2,2} + \frac{1}{r_1} v_{r_1} v_{r_2} (u_{1,2} + u_{2,1}) \\
\frac{\partial u'_1}{\partial x'_2} = \frac{1}{r_1} v_{r_1}^2 u_{1,2} - \frac{1}{r_2} v_{r_2}^2 u_{2,1} + \frac{1}{r_1} v_{r_1} v_{r_2} (u_{2,2} - u_{1,1}) \\
\frac{\partial u'_2}{\partial x'_1} = \frac{1}{r_1} v_{r_1}^2 u_{2,1} - \frac{1}{r_2} v_{r_2}^2 u_{1,2} + \frac{1}{r_1} v_{r_1} v_{r_2} (u_{2,2} - u_{1,1})
\]

(4.59)

The two linear strain components for beam elements are:

\[
e'_{11} = \frac{\partial u'_1}{\partial x'_1}
\]

\[
e'_{12} = \frac{1}{2} \left( \frac{\partial u'_1}{\partial x'_2} + \frac{\partial u'_2}{\partial x'_1} \right)
\]

(4.60)

Inserting Equation 4.59 into 4.60, one obtains:

\[
t e' = t B'_L \hat{u}
\]

(4.61)

where

\[
t B'_L = t B_L
\]

\[
e' = [e'_{11} \ 2e'_{12}]^T
\]

\[
\hat{u} = [u_1^T u_2^T \theta_1^T u_2^2 \theta_2^T u_3^T \theta_3^T]^T
\]

(4.62a)

\[
t B_L = \begin{bmatrix}
... & t h_{k,1} & t G_{11}^k & ... \\
... & 0 & t h_{k,2} & t G_{22}^k & ...
\end{bmatrix} \quad k = 1, 2, 3
\]

... t h_{k,2}  t h_{k,1} (t G_{12}^k + t G_{21}^k) ...

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\[
\begin{align*}
\theta_{ki} &= \frac{\partial h_k}{\partial x_i} = \frac{\partial}{\partial r} h_k \\
\theta_{ij} &= \frac{\partial}{\partial s} h_{kr} + \frac{\partial}{\partial r} h_k \\
\theta_{ki} &= \mathbf{b}_k \begin{bmatrix}
-\theta_{s2}

\theta_{i1}
\end{bmatrix} \\
\theta_{T} &= \begin{bmatrix}
\theta_{r1}^2 & \theta_{r2}^2 & \theta_{r1} \theta_{r2} \\
-2 \theta_{r1} \theta_{r2} & 2 \theta_{r1} \theta_{r2} & \theta_{r1}^2 - \theta_{r2}^2
\end{bmatrix}
\end{align*}
\]

(4.62b)

From Equation 4.19, the linear stiffness matrix can be written as:

\[
\begin{align*}
\theta_{KL} &= \int_{V} \mathbf{B}_{L}^T \mathbf{C} \mathbf{B}_{L} \theta dV \\
&= \int_{-1}^{1} \int_{-1}^{1} \mathbf{B}_{L}^T \mathbf{C} \mathbf{B}_{L} \det J dr ds
\end{align*}
\]

(4.63)

where \( \mathbf{B}_{L} \) is specified in Equation 4.62 and \( \mathbf{C} \) the constitutive matrix. The integration is performed using 6-point numerical integration:

\[
\begin{align*}
\theta_{KL} &= \sum_{i=1}^{3} \sum_{j=1}^{2} w_i w_j G(r_i, s_j)
\end{align*}
\]

(4.64)

where

\[
\begin{align*}
w_i, w_j &= \text{the weighting factors at point } (r_i, s_j); \text{ and} \\
G(r_i, s_j) &= \mathbf{B}_{L}^T (r_i, s_j) \mathbf{C} \mathbf{B}_{L} (r_i, s_j) \det J (r_i, s_j).
\end{align*}
\]
The nonlinear strain-displacement transformation matrix, \( [B'_{NL}] \), is structured in such a way that (Bathe, 1982)

\[
\delta \hat{u}^T [B'_{NL}] T [\tau'] [B'_{NL}] \hat{u} = [\tau']_{ij} \delta \eta'_{ij}
\]  

(4.65)

where

\[
[B'_{NL}] = [T]^T [B_{NL}]
\]

\[
[B_{NL}] = \begin{bmatrix}
... & t_{h_{k,1}} & 0 & t_{G_{11}} & ... \\
... & t_{h_{k,2}} & 0 & t_{G_{12}} & ... \\
... & 0 & t_{h_{k,1}} & t_{G_{21}} & ... \\
... & 0 & t_{h_{k,2}} & t_{G_{22}} & ...
\end{bmatrix} \quad k = 1, 2, 3
\]

(4.66a)

\[
[T] = \begin{bmatrix}
[tV_{r1}] & -[tV_{r2}] & 0 & 0 \\
[tV_{r2}] & [tV_{r1}] & 0 & 0 \\
0 & 0 & [tV_{r1}] & -[tV_{r2}] \\
0 & 0 & [tV_{r2}] & [tV_{r1}]
\end{bmatrix}
\]

(4.66b)

\[
[\tau'] = \begin{bmatrix}
[t\tau'_{11}] & t\tau'_{12} & 0 & 0 \\
t\tau'_{12} & 0 & 0 & 0 \\
0 & 0 & t\tau'_{11} & t\tau'_{12} \\
0 & 0 & t\tau'_{12} & 0
\end{bmatrix}
\]

(4.66c)

Substituting Equation 4.66 into 4.19 for the nonlinear stiffness matrix, one has:
\[ i^T \mathbf{K}_{NL} = \int_V \left( i^{T} \mathbf{B}'_{NL} \, i^T \mathbf{\tau}' \, i^{T} \mathbf{B}'_{NL} \right) dV \]

\[ = \int_{-1}^{1} \int_{-1}^{1} \left( i^{T} \mathbf{B}'_{NL} \, i^T \mathbf{\tau}' \, i^{T} \mathbf{B}'_{NL} \right) \det I dr ds \]  

(4.67)

The formation of the load vector for beam elements basically follows the same steps as for continuous elements. Body force, however, is negligible in the case of beam elements. Concentrated load and surface distributed load are treated the same way as for continuous elements. The internal force vector is calculated as:

\[ i^T \mathbf{F} = \int_V \left( i^{T} \mathbf{B}'_{L} \, i^T \mathbf{\tau}' \right) dV \]

\[ = \int_{-1}^{1} \int_{-1}^{1} \left( i^{T} \mathbf{B}'_{L} \, i^T \mathbf{\tau}' \right) \det I dr ds \]  

(4.68)

where \( \mathbf{B}'_{L} \) is defined in Equation 4.62 and

\[ \mathbf{\tau}' = \begin{bmatrix} \tau'_{11} & \tau'_{12} \end{bmatrix}^T \]  

(4.69)

The integrations in Equation 4.67 and 4.68 are carried out numerically, the same as for the linear stiffness matrix.
4.4 Formation of Transition Element

As its name indicates, a transition element connects a continuous element and a beam element. Three-node transition elements are employed in the program, as shown in Figure 4.6(a). In the figure, nodes 1 and 2 are also nodes of a continuous element and node 3 is also a beam element node. U. L. formulation is used to evaluate the element stiffness matrix and load vector. The transition element can either adopt the local coordinate system of a continuous element or that of the adjacent beam element. From Figure 4.6(b), the two coordinate systems have the following relations:

\[ r_b = s \quad s_b = -r \]  \hspace{1cm} (4.70)

where \( r_b \) and \( s_b \) are the local coordinates for the adjacent beam element. The former is chosen in the program, and the interpolation functions are:

\[ h_1 = \frac{1}{4} (1 - r)(1 - s) \]
\[ h_2 = \frac{1}{4} (1 + r)(1 - s) \]
\[ h_3 = \frac{1}{2} s(1 + s) \]  \hspace{1cm} (4.71)

Comparing these with those in Equation 4.21 and 4.45, it can be seen that \( h_1 \) and \( h_2 \) are the same as for continuous elements, and \( h_3 \) is \( h_2 \) for beam elements after replacing \( r \) with \( s \). The discrepancy between \( h_3 \) in Equation 4.71 and \( h_2 \) in Equation 4.45 can be explained by Equation 4.70.

The element coordinates and displacement increments can be interpolated as:
Figure 4.6. Local Coordinate System for Transition Elements
\[ t_{x_i} = \sum_{k=1}^{3} h_k t_{x_i}^k - \frac{r}{2} b_3 h_3 t_{v_{si}}^3 \quad i = 1,2 \]  

\[ u_i = \sum_{k=1}^{3} h_k u_i^k - \frac{r}{2} b_3 h_3 v_{si}^3 \]  

(4.72)

Therefore, the derivatives of displacement increments and global coordinates with respect to \( r \) and \( s \) are:

\[
\begin{bmatrix}
\frac{\partial u_i}{\partial r} \\
\frac{\partial u_i}{\partial s}
\end{bmatrix} = \begin{bmatrix}
\sum_{k=1}^{3} h_{k,r} \\
\sum_{k=1}^{3} h_{k,s}
\end{bmatrix} \begin{bmatrix}
\frac{1}{2} b_3 h_3 t_{v_{s2}}^3 \\
-\frac{r}{2} b_3 h_3 t_{v_{s1}}^3
\end{bmatrix} \begin{bmatrix}
u_i^k \\
\theta_i^3
\end{bmatrix}
\]  

(4.73)

and

\[
\begin{bmatrix}
\frac{\partial^2 x_i}{\partial r^2} \\
\frac{\partial^2 x_i}{\partial s^2}
\end{bmatrix} = \begin{bmatrix}
\sum_{k=1}^{3} h_{k,r} \\
\sum_{k=1}^{3} h_{k,s}
\end{bmatrix} \begin{bmatrix}
-\frac{1}{2} b_3 h_3 \\
-\frac{s}{2} b_3 h_3 s
\end{bmatrix} \begin{bmatrix}
t_{x_i}^k \\
 t_{v_{s1}}^3
\end{bmatrix}
\]  

(4.74)

Substituting Equation 4.73 and 4.74 into Equation 4.55, the derivatives of displacements with respect to the global coordinates are obtained. Therefore, the linear strain components can be calculated:

\[ t_e = t_B \hat{u} \]  

(4.75)

where
\[ \hat{r} = [u_1^1 u_2^1 u_2^2 u_1^2 u_2^3 \theta_{3,1}]^T \]

\[ \hat{u} = [u_1^1 u_2^1 u_2^2 u_1^2 u_2^3 \theta_{3,1}]^T \]

\[ \hat{t} \hat{B}_L = \begin{bmatrix} \hat{t}h_{1,1} & 0 & \hat{t}h_{2,1} & 0 & \hat{t}h_{3,1} & 0 & \hat{t}G_{11}^3 \\ 0 & \hat{t}h_{1,2} & 0 & \hat{t}h_{2,2} & 0 & \hat{t}h_{3,2} & \hat{t}G_{22}^3 \\ \hat{t}h_{1,1} & \hat{t}h_{1,2} & \hat{t}h_{2,1} & \hat{t}h_{2,2} & \hat{t}h_{3,1} & \hat{t}h_{3,2} & (\hat{t}G_{12}^3 + \hat{t}G_{21}^3) \end{bmatrix} \]

\[ \hat{t}h_{k,j} = \frac{\partial \hat{h}_k}{\partial \hat{x}_j} \]

\[ \hat{t}G_{ij}^3 = \hat{t}J_{j_1}^{i_1} \hat{t}g_{3}^{i_3} \hat{h}_3 + \hat{t}J_{j_2}^{i_2} \hat{t}g_{3}^{i_3} \hat{h}_{3,j} \]

\[ \hat{t}g_{3}^{i_3} = -\hat{t}g_{i_3}^{i_3} \]

\[ \begin{bmatrix} \hat{t}v_{3}^{i_1} \\ \hat{t}v_{3}^{i_2} \\ \hat{t}v_{3}^{i_3} \end{bmatrix} = \frac{2}{b_3} \begin{bmatrix} -\hat{t}v_{3}^{i_2} \\ \hat{t}v_{3}^{i_1} \\ \hat{t}v_{3}^{i_2} \end{bmatrix} \]

The nonlinear stiffness displacement-strain transformation matrix and the stress matrix are:

\[ \hat{t}B_{NL} = \begin{bmatrix} \hat{t}h_{1,1} & 0 & \hat{t}h_{2,1} & 0 & \hat{t}h_{3,1} & 0 & \hat{t}G_{11}^3 \\ \hat{t}h_{1,2} & 0 & \hat{t}h_{2,2} & 0 & \hat{t}h_{3,2} & 0 & \hat{t}G_{12}^3 \\ 0 & \hat{t}h_{1,1} & 0 & \hat{t}h_{2,1} & 0 & \hat{t}h_{3,1} & \hat{t}G_{21}^3 \\ 0 & \hat{t}h_{1,2} & 0 & \hat{t}h_{2,2} & 0 & \hat{t}h_{3,2} & \hat{t}G_{22}^3 \end{bmatrix} \]

\[ \hat{t}r = \begin{bmatrix} \hat{t}r_{11} & \hat{t}r_{12} & 0 & 0 \\ \hat{t}r_{21} & \hat{t}r_{22} & 0 & 0 \\ 0 & 0 & \hat{t}r_{11} & \hat{t}r_{12} \\ 0 & 0 & \hat{t}r_{21} & \hat{t}r_{22} \end{bmatrix} \]

The stress vector used for internal force is:

\[ \hat{t}r = [\hat{t}r_{11} \hat{t}r_{22} \hat{t}r_{12}]^T \]
Inserting Equation 4.76 through 4.78 into Equation 4.19, the stiffness matrices and force vector are obtained. The integrations are carried out the same way as before, using one-point numerical integration.

4.5 Friction Element

Friction elements are used to model slippage between the foundation and subgrade, and can also be applied to simulate the opening and closing of joints in rock strata. The basic concept of the friction element was mentioned in section 2.4.2. However, since large displacement formulation is employed, modifications are necessary.

One problem associated with the use of friction elements in large displacement formulation, is the orientation change of the element. This problem can be solved by transforming the local stiffness matrix to a global one. Within the local coordinate system (X'1OX'2 in Figure 4.5b), the stiffness matrix was obtained by Goodman et al. (1968):

\[
k = \frac{L}{6} \begin{bmatrix}
2k_s & 0 & k_s & 0 & -k_s & 0 & -2k_s & 0 \\
0 & 2k_n & 0 & k_n & 0 & -k_n & 0 & -2k_n \\
k_s & 0 & 2k_s & 0 & -2k_s & 0 & -k_s & 0 \\
0 & k_n & 0 & 2k_n & 0 & -2k_n & 0 & -k_n \\
-k_s & 0 & -2k_s & 0 & 2k_s & 0 & k_s & 0 \\
0 & -k_n & 0 & -2k_n & 0 & 2k_n & 0 & k_n \\
-2k_s & 0 & -k_s & 0 & k_s & 0 & 2k_s & 0 \\
0 & -2k_n & 0 & -k_n & 0 & k_n & 0 & 2k_n \\
\end{bmatrix}
\] (4.79)
where $L$ is the length of a friction element. This matrix is transformed into a global system ($X_i O X_2$ in Figure 4.5b) at time $t$:

\[ ^tK = ^tT^Tk^tT \]  \hspace{1cm} (4.80)

where

\[ ^tT = \begin{bmatrix} ^t\lambda_1 & 0 & 0 & 0 \\ 0 & ^t\lambda_1 & 0 & 0 \\ 0 & 0 & ^t\lambda_2 & 0 \\ 0 & 0 & 0 & ^t\lambda_2 \end{bmatrix} \]  \hspace{1cm} (4.81)

and

\[ ^t\lambda_i = \begin{bmatrix} \cos ^t\theta_i \\ \sin ^t\theta_i \\ -\sin ^t\theta_i \\ \cos ^t\theta_i \end{bmatrix} \quad i = 1, 2 \]  \hspace{1cm} (4.82)

where $^t\theta_i$ is the angle between the sides of a friction element and the global X axis, and

\[
\begin{align*}
\sin ^t\theta_1 &= \frac{^t\lambda_2}{^tL_1} \\
\cos ^t\theta_1 &= \frac{^t\lambda_1}{^tL_1} \\
\sin ^t\theta_2 &= \frac{^t\lambda_2}{^tL_2} \\
\cos ^t\theta_2 &= \frac{^t\lambda_1}{^tL_2} \\
^tL_1 &= \sqrt{(^t\lambda_1)^2 + (^t\lambda_2)^2} \\
^tL_2 &= \sqrt{(^t\lambda_1)^2 + (^t\lambda_2)^2}
\end{align*}
\]  \hspace{1cm} (4.83)
The direction cosines are calculated separately because the two sides of a friction element may change differently in the modeling process. The length of the friction element at time $t$, used for calculation of the stiffness matrix, is given as:

$$t_L = \sqrt{\left(\frac{t_{x_1}^{23} - t_{x_1}^{14}}{2}\right)^2 + \left(\frac{t_{x_2}^{23} - t_{x_2}^{14}}{2}\right)^2}$$  \hspace{1cm} (4.84)$$

where

$$t_{x_1}^{14} = \frac{t_{x_1}^1 + t_{x_1}^4}{2}$$
$$t_{x_1}^{23} = \frac{t_{x_1}^2 + t_{x_1}^3}{2}$$
$$t_{x_2}^{14} = \frac{t_{x_2}^1 + t_{x_2}^4}{2}$$
$$t_{x_2}^{23} = \frac{t_{x_2}^2 + t_{x_2}^3}{2}$$  \hspace{1cm} (4.85)$$

and $t_x^j$ ($i = 1, 2$, and $j = 1, 2, 3, 4$) are the nodal coordinates and the element nodes are numbered in accordance with Figure 4.7. In the entire modeling process, Equation 4.15 is assumed valid, i.e., the pressures on the friction element change linearly with the nodal displacement.

### 4.6 Constitutive Relations and Other Considerations

In the previous sections, the formations of the element stiffnesses and load vectors for the three types of elements were evaluated and the displacement-strain transformation matrices discussed. However, to calculate the element stiffness matrices, the constitutive relations must be known. Furthermore, there are other problems to be considered in the
program, such as the solution iteration scheme, convergence criteria, and excavation process modeling.

4.6.1 Constitutive Relations

In order for a Lagrangian formulation to be used in a specific response prediction, an appropriate constitutive relation must be chosen. In general, the constitutive relation used in a formulation must be the one explicitly given for that formulation otherwise the constitutive matrix has to be transformed (Bathe, 1982). This is especially true for large displacement, large strain and material nonlinear problems. However, for large displacement, small strain, elastic problems, the same constitutive relation can be used for both T. L. and U. L. formulations. In this case, the generalized Hooke's law is applicable to both formulations. Therefore, the constitutive components can be written as:

\[
\begin{align*}
C_{i\alpha} & = C_{i\alpha}^{\text{constitutive relation}} = \lambda \delta_{ij} \delta_{rs} + \mu (\delta_{ir} \delta_{js} + \delta_{is} \delta_{jr})
\end{align*}
\]

(4.86)

where

\[
\begin{align*}
\lambda &= \frac{E\nu}{(1 + \nu)(1 - 2\nu)} \\
\mu &= \frac{E}{2(1 + \nu)}
\end{align*}
\]

and \(\delta\) is the Kronecker delta.

In two-dimensional subsidence modeling, the underground part of the domain is regarded as being in a state of plane strain. Therefore, the constitutive matrix can be written as:
\[ \sigma C = \frac{E(1 - \nu)}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1 - \nu} & 0 \\ \frac{\nu}{1 - \nu} & 1 & 0 \\ 0 & 0 & \frac{1 - 2\nu}{2(1 - \nu)} \end{bmatrix} \] (4.87)

The transition elements are considered as plane stress elements, thus:

\[ \tau C = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1 - \nu}{2} \end{bmatrix} \] (4.88)

The constitutive relation between stress and strain for beam elements is described by the following equation (Bathe, 1982):

\[
\begin{bmatrix} \tau'_{11} \\ \tau'_{12} \end{bmatrix} = \begin{bmatrix} E & 0 \\ 0 & kG \end{bmatrix} \begin{bmatrix} \epsilon'_{11} \\ \epsilon'_{12} \end{bmatrix}
\] (4.89)

Therefore, the constitutive matrix for beam elements is:

\[ \tau C = \begin{bmatrix} E & 0 \\ 0 & kG \end{bmatrix} \] (4.90)

where \( G \) is the shear modulus and \( k \) a shear correction factor.

### 4.6.2 The Solution Scheme and Convergence Criteria

The solution to a nonlinear problem is usually achieved by an incremental process in which the system is treated as linear at each increment. Thus, the nonlinear solution
scheme can be described as micro-linear and macro-nonlinear, just as a nonlinear curve can be made of numerous small linear segments.

There are several nonlinear solution methods. In this program, the commonly used modified Newton-Raphson iteration is adopted. This scheme is illustrated in Figure 4.7. From this figure, it can be seen that the force vector is updated at each iteration, whereas the stiffness matrix is updated only at each load level. For each load level, iterations are carried out to calculate the displacement increments and the convergence is checked at every iteration. The system of linear equations for each iteration is solved using a subroutine called SOLVE (Kuppusamy, 1986) which applies the Gaussian elimination method.

In order for the above solution scheme to obtain reasonable results, a realistic convergence criterion must be chosen. The physical parameters which can be used in the criterion are displacement, force and energy. Displacement is selected in this program since in mining subsidence, it is the chief concern. The displacement convergence criterion can be written as (Bathe, 1982):

$$\frac{|U^{(k)}|}{|1 + \Delta t U^{(k)}|} \leq \varepsilon_D$$  \hspace{1cm} (4.91)

where

$$|U^{(k)}| = \sqrt{\sum_{i=1}^{n} (U_1^{(k)})^2 + \sum_{i=1}^{n} (U_2^{(k)})^2}$$

$$|1 + \Delta t U^{(k)}| = \sqrt{\sum_{i=1}^{n} (1 + \Delta t U_1^{(k)})^2 + \sum_{i=1}^{n} (1 + \Delta t U_2^{(k)})^2}$$  \hspace{1cm} (4.92)
Figure 4.7. Flowchart for the Solution Scheme
$\epsilon_D$ is a displacement convergence tolerance and $n$ the number of nodes. When the criterion in Equation 4.84 is met, additional displacement increments are negligible and it can be said that the equilibrium condition is achieved.
Chapter 5 Validation and Applications of the SRMP Model

In this chapter, a number of case studies are presented to validate the SRMP model. These examples compare ground movements with and without surface structures and investigate the effects of structure dimension and material properties on subsidence profiles. Slippage between foundation and subgrade is also analyzed, the dynamic subsidence profile modeling is discussed and, finally, the influence of location on structure deformation is studied. Before these analyses are presented, however, some technical aspects of the application of the SRMP, such as the determination of initial stresses and boundary conditions, the simulation of mining processes, and the modeling of geological conditions, must be discussed.

5.1 Subsidence Analysis Using SRMP

Mining subsidence is a physical phenomenon in which the overburden strata settle under their own weight and initial stress due to undermining. In order to accurately model this phenomenon, the conditions under which it occurs must be realistically represented. In the past, although some finite element models gave moderately useful results, they did
not reflect the real conditions. For example, some specified distributed load or concentrated load along the free boundary, and as a result, overlapping of roof and floor occurred. Others specified displacement along the mine openings as a driving mechanism to prevent overlap. These assumptions are somewhat unrealistic. Attempts were made to overcome these problems in the SRMP model.

5.1.1 Simulation of the Mining Process

When a mineral is extracted underground, the initial stress state in the surrounding rock is destroyed, displacements will take place, and finally, a new equilibrium state will be established. This process continues until the extraction stops. The final displacements can be calculated either by several steps or by just one. Figure 5.1 shows the subsidence calculated by SRMP, using a six step and an one step procedure, for a hypothetical panel of 420 ft. with a 10 ft. thick seam. In the six step modeling, each mining block was 70 ft. long and for the one step calculation, the entire 420 ft. was removed at one time. There is no significant difference between the two final subsidence profiles. However, if the displacements at different mining stages are desired, then it becomes necessary to calculate them step by step.

In the SRMP, gob-elements are used in order to model the mining process. A gob element is a element with degenerated material properties. A coal block is a regular element before extraction, but the area that it occupied becomes gob-element after mining. Gob-elements allow the mining process to be continuously modeled without reinitiating boundary conditions for each mining cycle. The problem of roof-floor overlap can also be prevented by employing gob elements. If the material properties of the gob element are assigned a value of zero, then:
Figure 5.1. Subsidence Curves of One-Step and Six-Step Modeling
\[ {}^0K^g = 0, \quad {}^{t+\Delta t}F^g_B = 0 \]
\[ {}^{t+\Delta t}F^g = 0, \quad {}^{t+\Delta t}S^g_0 = 0 \]

(15)

where the superscript \( g \) means gob-element, \( {}^{t+\Delta t}F^g_B \) is the body force vector which is included in the external load vector and other symbols have the same meaning as before. \( {}^{t+\Delta t}F^g \) equaling zero implies that a force vector with the same magnitude but different sign as \( ^tF \) is added to \( ^tF \) at time \( t + \Delta t \). This is equivalent to the way that was described by Desai and Christian (1977) for creating a stress-free surface along the opening.

### 5.1.2 Determination of Initial Stresses

Assuming the initial stress is due to gravity only, then:

\[
\begin{align*}
\sigma_1 &= \gamma H \\
\sigma_2 &= \sigma_3 = \lambda \sigma_1
\end{align*}
\]

(5.1)

where

\[
\lambda = \frac{\nu}{1 - \nu}
\]

(5.2)

\( \nu \) = the Poisson ratio;
\( \sigma_1 \) = vertical stress;
\( \sigma_2, \sigma_3 \) = horizontal stresses;
\( \gamma \) = density of the rock strata; and
\( H \) = depth below the ground surface for the point where the stresses are calculated.

Consideration of the initial stress at the early mining stage is very important because significant differences between measured and calculated subsidences will result if the in-
itial stress is not properly estimated. As mining progresses and consequently, subsidence increases, the effect of initial stress becomes less important.

5.1.3 Consideration of Boundary Conditions

Boundary condition is a important factor in determining the outcome in numerical modeling. Over the years, many boundary conditions have been used in finite element analyses. The controversial point is how the boundary condition around the mine opening is specified. One criterion in choosing the boundary condition is to prevent floor and roof overlap. For example, Franks and Geddes (1986) applied load to the nodes on the boundary of the excavation, which was obtained by creating a stress free surface. As a result, they encountered roof and floor overlap. To counter this problem, the load had to be modified. Agioutantis (1987) specified roof displacement along the mine opening as the driving mechanism to avoid the overlap. This scheme, however, requires knowledge of the roof convergence curve.

In SRMP modeling, overlap is not a problem due to the application of gob element and the iteration scheme. Thus, only body force (self-weight) and initial stress are taken into account, where force and displacement are not specified on the free boundary. Other boundary conditions are illustrated in Figure 5.2. Nodes on the ground surface are free and those on the left-hand side are restricted horizontally but not vertically, because the left-hand side is regarded as a center line and the points on the line only move vertically. The nodes on the bottom line are fixed because at a certain distance below the mining seam, the rock strata are intact. Practically, the nodes on the right hand side should be fixed due to their location in the intact area. However, in finite element analyses, especially for a dynamic analysis, the mesh may not be large enough to cover the zero-
Figure 5.2. Illustration of the Boundary Conditions
displacement area because of restrictions on computer resources. Thus, the right-hand
side is sometimes confined only horizontally.

5.1.4 Modeling of Geological Conditions

In order to realistically model mining subsidence, geological conditions must be taken
into account. Different geological materials in the rock strata can be represented by
media having different material properties. Discontinuities, such as joints and faults, can
be modeled either explicitly using friction elements or implicitly by properly adjusting the
rock stiffness (Goodman and John, 1977). Friction elements were discussed in the pre-
vious chapter and the approach of adjusting rock stiffness will be discussed here.

When the overburden above a mine is regarded as a plane-isotropic medium in finite
element modeling, the resulting subsidence profile is usually much flatter than the
measured one. The reason is that joints and other planes of weakness in the rock strata,
over the undermined area, are not taken into consideration. In order to overcome this
problem, Agioutantis (1987) divided the overburden into five zones as shown in Figure
5.3. Designated from one to five, these zones are named intact, intermediate, fractured,
affected and extraction zones respectively. Zone four is the most affected area whereas
zone one is the least impacted. The affected areas are assigned lower material properties
to accommodate the weakness in finite element analysis. The magnitudes of the material
properties assigned to the zones are, in descending order, 1, 4, 2, and 3. Zone 5 repres-
ents the coal seam. In this study, the zone concept will be used, but the zones are not
classified in exactly the same way as described above.
Figure 5.3. Material Zone Classification (Agioutantis, 1987)
5.2 Verification of the SRMP Model

As mentioned earlier, numerical modeling has the advantage of analyzing a variety of cases and conditions. One typical example is the comparison of free versus structure-bearing ground deformations. In order for the numerical results to have practical significance, however, the model has to be validated with field data. Three cases were simulated in this section and the results are compared with those obtained from field measurement to validate the applicability and accuracy of the method.

5.2.1 Case 1: Displacement of Foundations over an Active Mine

This case study was described by Littlejohn (1974) and was discussed in Chapter 3. The purpose of this study was to examine the foundation response at various mining stages.

<table>
<thead>
<tr>
<th>Material</th>
<th>Modulus of Elasticity $10^5$ psi</th>
<th>Poisson's Ratio</th>
<th>Density lb/ft$^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sandstone</td>
<td>25.00</td>
<td>0.15</td>
<td>172.8</td>
</tr>
<tr>
<td>Limestone</td>
<td>21.90</td>
<td>0.15</td>
<td>172.0</td>
</tr>
<tr>
<td>Shale and Coal</td>
<td>2.50</td>
<td>0.20</td>
<td>152.0</td>
</tr>
<tr>
<td>Soil</td>
<td>1.25</td>
<td>0.20</td>
<td>116.0</td>
</tr>
</tbody>
</table>

The domain for the SRMP analysis which was divided into four zones is illustrated in Figure 5.4. The material properties of zone one are listed in Table 5.1, and those of zone two and three are assumed as one fourth and one tenth of zone one respectively. Zone
four was considered to have the same properties as in zone one. A edge effect of 200 ft. was applied, which was realized by shortening the extraction area by 200 ft.

The simulated subsidence profiles along the line from point 19 to J in Figure 3.2 are shown in Figure 5.5. For comparison, those measured by Littlejohn (1974) are shown in Figure 5.6 and compare closely with the calculated values. These graphs also show that the simulated and measured subsidence curves are in closer agreement during the late rather than the early mining stages. There are two reasons for this. One is that the material zones are static, that is, they are divided in accordance to the final mining configuration. Theoretically, the material zones should move along with the mining face. Another reason is that the boundary conditions specified at the L. H. S. of the domain probably did not reflect the real conditions accurately, especially at the early stages. In reality some horizontal movements at this side may occur which are ignored by the analysis.

Table 5.2. Simulated and Measured Differential Settlement of Foundation For Case One

<table>
<thead>
<tr>
<th>Distance</th>
<th>Simulated (in.)</th>
<th>Measured (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5R</td>
<td>0.17</td>
<td>0.10</td>
</tr>
<tr>
<td>0.8R</td>
<td>0.70</td>
<td>0.54</td>
</tr>
<tr>
<td>1.1R</td>
<td>1.41</td>
<td>1.46</td>
</tr>
</tbody>
</table>

The simulated and measured differential settlements of the foundation are compared in Table 5.2. This table also reflects the general trend of subsidence development.
Figure 5.4. Material Zones Used in SRMP Analysis
Figure 5.5. Calculated Subsidence Profiles in Case One
Figure 5.6. Measured Subsidence Profiles in Case One (Littlejohn, 1974)
5.2.2 Case 2: Response of a Building to Mining Subsidence

The second case is the field study reported by Powell et al. (1988), which was also discussed in Chapter 3. It involved monitoring of two foundations as they were undermined. In this analysis, a building was assumed with the following characteristics:

Length: 40 ft.;
Height: 24 ft.;
Material: brick;
Foundation type: continuous footing.

The influence radius, $R$, is about 400 ft. Therefore, the $L/R$ ratio is 0.1 and the $H/L$ ratio 0.6, where $L$ is the length of the building and $H$ the height. Based on the given information and using Tables 7.10 and 7.11, the building can be classified as Class II. The building class is needed later for damage estimation.

The domain of the analysis included the building, overburden, coal seam and part of the floor strata and the model was divided into 423 elements and 480 nodes, as shown in Figure 5.7. The thickness of the overburden and the coal seam were 748 ft. and 9 ft. respectively, and a 120 ft. floor stratum was considered. The length of the domain, which is dictated by the length of the mined area and the radius of influence, was 1000 ft.. The underground part of the domain was horizontally divided into zones similar to those of Case 1.

Figure 5.8 shows the measured subsidence profile and that predicted by SRMP, after the entire panel was mined. The measured and predicted maximum values of horizontal strain, tilt and curvature of the foundation are compared in Table 5.3. Generally, the
predicted subsidence curve is flatter than the measured, and therefore, the predicted foundation deformations was less than the measured.

Table 5.3. Measured and Predicted Foundation Deformations for Case Two

<table>
<thead>
<tr>
<th>Deformations</th>
<th>Maximum Strain (10^-3)</th>
<th>Maximum Tilt (10^-3)</th>
<th>Maximum Curvature (10^-4 ft.^-1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured</td>
<td>3.00</td>
<td>12.92</td>
<td>3.55</td>
</tr>
<tr>
<td>Predicted</td>
<td>2.16</td>
<td>10.50</td>
<td>2.88</td>
</tr>
</tbody>
</table>

The development of the building deformation obtained by SRMP are shown in Figure 5.9. Using the damage criteria in Table 7.12 and the building deformations in this figure, the building is expected to show some architectural damages, when the face is about 150 ft. behind the building and, functional and structural damages, when it is less than 100 ft. behind. This observation agrees with the results obtained by Powell et al. which indicated that the foundations failed when the face was 100 ft. behind them.

5.2.3. Case 3: Comparison of Measured, SRMP and SDPS Subsidence

Profiles

This case study from Southwestern Virginia was collected by VPI & SU and was documented by Agioutantis (1987). It is used here for comparing the measured, and SRMP and SDPS calculated subsidence profiles (SDPS was reviewed in Chapter 2). In this
Figure 5.8. Calculated and Measured Subsidences for Case Two
longwall case study, there was no structure on the surface. The overburden was 625 ft, the coal seam thickness was 5.5 ft. The procedure for SRMP analysis was similar to those of the previous two cases. The results from the analysis are shown in Figure 5.10 together with the measured subsidence values and the profile predicted by SDPS. The figure demonstrates the close agreement between measured and and predicted subsidence profiles by both the SDPS and the SRMP methods.

In summary, the results of the three case studies presented in this chapter have demonstrated that if the geological and boundary conditions are properly described, the SRMP analysis can attain very accurate results. Two important factors for a successful analysis are the choice of the material properties for each zone and the determination of the dimension of the domain. Generally, the differences among the material properties for the zones reflect the geological conditions of the overburden. If there are a lot of vertical joints and the overburden rock is weak, then the material properties of the affected zones should be significantly reduced. Agioutantis (1987) suggested a reduced Young's modulus as low as 1/60 of the intact rock. For competent rock with few vertical joints, the differences among the zones should be small. The importance of the dimensions of the domain lies in the fact that it determines whether the boundary conditions are realistic. For example, if the displacements specified along the vertical boundary of the unmined side are zero, a small unmined area in the domain will make this boundary condition invalid. In general, the length of the domain is determined by the length of the mined area plus the radius of influence. The height of the domain should be deep enough to include a portion of the floor strata so that the specified boundary condition along the lower limit line can be reflected.
5.3 Subsidence Comparison of Free and Structure-Bearing Surfaces

The purpose of this section was to examine and compare the subsidence of a free surface versus that of a structure-bearing surface. In order to accomplish this a hypothetical case was examined using the SRMP model.

The model was 980 x 1,000 ft. and the same material properties were used as in case study one. The model consisted of 850 ft. of overburden, 10 ft. of coal seam and 120 ft. floor rock. The mined-out area was 500 ft. In order to compare the free and structure-bearing surface subsidences and to study the effect of structure sizes, two foundations of different lengths with superstructure loads were imposed on the model. The first foundation was 50 ft. long and 5 ft. high, and the second 150 ft. long with the same height. Two types of loads were assumed on the foundations, concentrated and distributed loads. A concentrated load of 5000 lbs. was applied every 50 ft. along the foundation and the distributed load was 500 lb/ft. The Young's modulus and Poisson ratio of the foundation were $1.25 \times 10^5$ psi. and 0.2 respectively. The first foundation was located 425 ft. from the panel center and the second 375 ft. Friction elements were used between the foundation and the subgrade soil. The normal and shear stiffness coefficients for these elements were $1.09 \times 10^8$ psi and $1.09 \times 10^5$ psi respectively. These values essentially prevented relative vertical displacements between the foundation and subgrade from taking place.

The results of the analysis are presented in Figure 5.11. This figure shows very little difference among the three subsidence curves demonstrating, therefore, that a structure on the surface will not alter the subsidence profile.
Figure 5.11. Subsidence Curves with and without Surface Structures
In order to evaluate the impact of the material properties of the structure on subsidence, the Young's modulus of the second foundation was changed several times, ranging from $3.13 \times 10^4$ to $2.19 \times 10^6$ psi. The impact on the the resulting subsidence curves was insignificant.

In summary, the presence of structures on the ground surface is not a significant factor impacting mining induced subsidence. The main reason for this is that the dimension of the structure is usually much smaller than that of the overburden. It is appropriate, therefore, to use free ground subsidence prediction methods, for estimating the subsidence of structure-bearing surfaces.

### 5.4 Analysis of Slippage between Foundation and Subgrade

The slippage between foundation and subgrade is a very important factor in determining the transmission of deformations from the ground to the structures. Obviously, the more slippage, the less horizontal strain will be transferred to the structure. The patterns of the slippage between the foundation and subgrade have been established through experience and field observations. Littlejohn (1974) developed a formula to estimate the slippage through field observations, as was given in Equation 3.1. Graphical presentations of the slipping patterns were also given by Littlejohn (1974), Kratzsch (1983), and Marino (1985). Figure 5.12 is a qualitative illustration of slippage, where $v_{xrel}$ is the relative displacement and M the center point of the foundation. Generally, no relative displacement (slippage) occurs at the center point of the foundation, whereas the relative displacement increases proportionally toward both ends of the foundation in opposite directions. The subgrade moves relatively toward the foundation center, when the foundation is located at the compression zone, and away from it when the foundation
Figure 5.12. Slipping Patterns between Foundation and Subgrade (Kratzsch, 1983)
is located at the tension zone. Using the SRMP, these slipping patterns can be validated quantitatively, and the effects of several factors on the slippage can be analyzed.

In this analysis the geological and mining conditions were kept constant while the material properties, location, and size of the foundation and the shear stiffness of foundation-subgrade interface were varied to allow for a parameter analysis. The vertical dimensions of the overburden, coal seam, and floor strata were assumed the same as in Section 5.3. The horizontal dimension was 1100 ft. and the length of the mined area 420 ft. The Young's modulus and Poisson ratio for the subgrade were $1.25 \times 10^5$ psi and 0.2 respectively. The normal and shear stiffness coefficients of the friction elements were the same as in Section 5.3.

The material effect was studied first. A 50 ft. long and 5 ft. high foundation was located 795 ft. from the panel center and subjected to maximum tensile horizontal strain. The SRMP model yielded about 2.3 ft. of maximum subsidence at the panel center. The slippages between the foundation and subgrade for two different foundation materials are shown in Figure 5.13, where the unit for Young's modulus $E$, is in $10^6$ lb/ft.$^2$. The horizontal positions of the foundation and subgrade for $E = 9$ are plotted in Figure 5.14 where the slippage is enlarged 100 times. In Figure 5.13, negative sign of the slippage means that the subgrade moves relatively toward the panel center and positive sign means away from it. Two observations can be made from this figure. One is that the slipping pattern obtained by the SRMP model is similar to that described in Figure 5.12 for a foundation located at the tension zone. The other is that the slippage increases as the foundation becomes stiffer.

In order to study the effect of foundation locations, the foundation was moved to 305 ft. and 515 ft. respectively. The first location was in the vicinity of maximum com-
Figure 5.13. Slippage between Foundation and Subgrade at the Tension Zone
Figure 5.14. Horizontal Positions of the Foundation and Subgrade at the Tension Zone after Deformation
pression and the second in the area of zero horizontal strain (inflection point). Figure 5.15 shows the slippage between the foundation and the subgrade for foundation located at the maximum compression area. Once again, the slippage pattern is similar to that illustrated in Figure 5.12 and the slippage becomes larger as the foundation stiffness increases. The relative positions of the foundation and the subgrade at the compression zone are shown in Figure 5.16. Figure 5.17 shows the slippage for a foundation located in the vicinity of the inflection point. Slippage along the entire foundation in this figure is close to zero. The slippage can not be exactly zero because the foundation has an area and the inflection point is just a point. Nevertheless, the slippage along the foundation can be regard as zero compared to the slippage in Figures 5.13 and 5.15.

The effect of the shear stiffness coefficient, $K_s$, can be seen from Figure 5.18 where the unit for $K_s$ is in $10^5$ lb/ft$^2$. It is obvious that reduction of the shear stiffness coefficient at the interface between the foundation and subgrade will increase the slippage significantly. In this analysis, the foundation length is 165 ft. and located at 797.5 ft. from the panel center. The reason for a longer foundation is to allow for comparisons with previous cases and, therefore, examine the effect of the foundation length on slippage. A comparison between the slippage in Figure 5.18 for $K_s = 15.7$ and that in Figure 5.13 for $E = 90$ indicates that the maximum slippage increases for increasing foundation length. However, the slippage along the central 50 ft. of the 165 ft. foundation is less than that in Figure 5.13. This means that the slippage per unit length of foundation decreases with the increase of foundation lengths.

In general, the maximum slippage between the foundation and subgrade increases as the stiffness and length of the foundation increase and the shear stiffness of the foundation-subgrade interface decreases. The slippage per unit length (SPUL) of foundation increases when the stiffness of the foundation increases and the shear stiffness of
Figure 5.15. Slippage between Foundation and Subgrade at the Compression Zone
Figure 5.16. Horizontal Positions of the Foundation and Subgrade at the Compression Zone after Deformation
Figure 5.17. Slippage between Foundation and Subgrade at the Inflection Point
Figure 5.18. Effect of Shear Stiffness on Slippage
the interface decreases, but it decreases as the length of the foundation increases. In practice, the maximum slippage is not as important as the slippage per unit length, and a large value of SPUL is desired because less horizontal strain will be transferred to the foundation. Thus, in designing structures in mining areas, a short, stiff foundation with smooth foundation-subgrade interface is preferred when slippage is considered. It should be pointed out that a smoother interface will result in less shear stiffness. The location of the structure has a profound effect on the slippage. The largest slippages occur when the foundation is located at the points of maximum horizontal strains. There is very little slippage when the foundation is at the inflection point (point of zero horizontal strain).

5.5 Deformation Characteristics of a Structure Affected by Mining Subsidence

The displacement and deformation of a four bay building were studied by using the SRMP model. The building was first assumed to be located at the compression zone of the subsidence profile and then at the tension zone. It was a 40 ft. long and 24 ft. high building subjected to uniformly distributed loads along the top and middle beams as shown in Figure 5.19. The distributed loads along the top and middle beams are 40 and 100 lb/ft. respectively. It was assumed that the connection between the structure and subgrade was hinged, which means that rotation is allowed at the connection. The underground part of the model was the same as in Section 5.4. About 490 ft. of coal seam was extracted.
Figure 5.19. A Four Bay Structure Subject to Uniformly Distributed Loads

$q = 40 \text{ lb/ft}$

$q = 100 \text{ lb/ft}$
At the compression zone, the building was located 305 ft. from the panel center. Figure 5.20 shows the horizontal and vertical displacements of the building at this location. The former was enlarged 15 times and the latter 2 times. One observation that can be made from this figure is that major portion of the building displacement is translation, while the deformation within the structure is relatively small. This relative small deformation, however, is sufficient to cause structure damage. It should also be noted that the central wall suffers very little deformation. This indicates that the symmetric property of the structure is not totally destroyed. Figure 5.21 and 5.22 show the rotations of the walls induced by mining subsidence and by its own weight respectively, and Figure 5.23 and 5.24 are those of the beams. In these figures, positive rotation is counter-clockwise. Comparing Figure 5.21 and 5.23 with Figure 5.22 and 5.24, one can see that the general deformation mode of the structure affected by subsidence is not changed, but the magnitude increases significantly. This is because the horizontal ground deformation reinforces the original mode. The entire structure is also subjected to rigid body rotation which is reflected by the rotation of the central wall. The rigid body rotation is positive because the ground rotates counter-clockwise. The maximum deformation within the structure occurs in the two side-walls at the vicinity of the middle beam.

At the tension zone, the building is located 795 ft. from the panel center. Figure 5.25 shows the displaced building as compared with the original one. The scale of the displacements is the same as in Figure 5.20. As in the previous case, the structure is also subject to rigid body movement. The rotations of the walls and the beams are shown in Figure 5.26 and 5.27 respectively. Compared with Figure 5.22, the rotation pattern in Figure 5.26 is different from the original deformation mode at the lower parts of the walls. The difference can be attributed to the fact that tensile ground deformation runs against the original deformation mode at the lower ends. For example, the bottom of
Figure 5.20. The Structure Located at the Compression Zone Before and After Subsidence.
Figure 5.21. Rotations of the Three Columns due to Subsidence (Structure at Compression Zone)
Figure 5.22. Rotations of the Three Columns without Subsidence
Figure 5.23. Rotations of the Two Beams due to Subsidence (Structure at Compression Zone)
the L. H. side wall rotated counter-clockwise originally, but the tension ground movement made it rotate the other way. The same can be said about the R. H. side wall. The fact that the rotation of the building at the tension zone is less than that at the compression zone can also be attributed to this factor as also to the fact that differential settlement at the compression zone is larger than at the tension zone.
Figure 5.26. Rotations of the Three Columns due to Subsidence (Structure at Tension Zone)
Figure 5.27. Rotations of the Two Beams due to Subsidence (Structure at Tension Zone)
Chapter 6  Analysis of Structural Responses Based on Pre-Calculated Ground Displacements

In the previous chapter, structural responses and ground deformation were analyzed simultaneously. The model components included the surface structure as well as overburden, coal seam and floor rock. The advantage of this approach is that the load and boundary conditions of the surface structure can be represented accurately. This modeling, however, requires knowledge of mining, geological and structural parameters and needs considerable computer memory and time. In this chapter, an alternative model is used, which considers only the surface structure. It is also a finite element model, using beam elements. This model assumes that the ground deformations are known boundary conditions, which must be determined prior to the structural analysis. Programs such as SDPS can be applied for pre-calculation of ground deformations.

Using the new model, various structures subject to different ground deformations were analyzed. In order to establish damage criteria, subsidence induced stresses in structures were compared with the allowable stresses specified in the building codes developed by ASTM (American Standard of Testing Material) and BSI (British Standard Institute). If a building has suffered damage, then the damage is linked to ground deformation be-
cause most damage criteria for buildings affected by ground movement are composed of ground deformation indices.

6.1 The Model

The formulation of the model is the same as in Section 5.3 except that it is a linear model. This means that no internal force calculation and no iteration are necessary and that rigid body movements of structures are ignored.

In order to verify the model, the deflection and stress of a simply-supported beam subject to distributed load were calculated as shown in Figure 6.1. The results, as well as the analytical ones, are presented in Figures 6.2 and 6.3. These figures indicate that as the number of elements increases, the results obtained by the finite element model approach the analytical ones. For example, when the beam is divided into two elements, the relative error is 21 percent for the maximum deflection and 37.3 percent for the maximum stress, but if the number of elements increases to four, the relative errors are reduced to 3.7 and 9 percent respectively. The simulated and analytical results will be close, however, if the mesh is fine enough.

6.2 Strength of Building Materials

To determine whether a building is damaged using stress-based criteria, the strength of the building material must be known. Commonly used building materials include concrete, brickwork, stone, wood, steel, mortar and plaster. The British Standard Code of Practice BSCP (1970) suggests an allowable tensile stress resisted by mortar adhesion
Figure 6.1. Simply-Supported Beam and Finite Element Discretization
Figure 6.2. Deflections Obtained by Analytical and Finite Element Methods
Figure 6.3. Stresses Obtained by Analytical and Finite Element Methods

TENSIILE STRESS (LB/FT²)
of only 10.15 psi for a 1:1:6 (cement:lime:sand) mortar, and a shear strength of 14.5 psi (Attewell et al., 1986). The characteristic flexural strengths given by BS 1978 range from 36.26 to 101.53 psi, depending on mortar type and clay brick absorption; the characteristic shear strength is (50 + 87g) or (22 + 87g) psi also dependent on mortar type, where g is imposed direct stress. ASTM Standard in Building Codes 1989 specifies a flexural strength of 800 psi and compressive strength of 3500 psi for prisms of surface bonding mortar, and a tensile strength of 150 psi for exterior plaster.

ASTM gives only the compressive strength for concrete, ranging from 2000 to 5000 psi. The reason is probably that concrete is primarily used for compression in buildings. However, tensile stress may develop in the structural components originally designed for compression, due to the differential settlement of the structure induced by mining subsidence. The tensile strength of concrete is much less than its compressive strength. If a concrete component is going to fail, the failure will most likely be caused by tension or shear. Attewell et al. (1986) stated that a mean tensile strength is often taken as one-tenth of the compressive strength of a cube. Thus, based on the compressive strength given by ASTM, the tensile strength would range from 200 to 500 psi. Francis (1980) specified a tensile strength of 290 psi for normal reinforced concrete work and 870 psi for high-strength prestressed work. A tensile strength of 435 psi for plain concrete was given by Attewell et al. (1986). Little information is available on the mean shear strength of concrete. BSCP 1972 limits shear in a lightly-reinforced concrete member to a characteristic value of 50.75 psi. Brickwork and building stones possess similar characteristics. They are designed chiefly to withstand compressive stress and thus are very vulnerable to tensile stress. Similar as for concrete, ASTM 1989 specified only compressive strength for brickwork and building stone, which ranges from 1250 to 2500 psi for brickwork and 2000 to 20,000 psi for building stone. Francis (1980) gave a tensile
strength of 870 psi for medium strength brickwork, 725 psi for limestone and 5800 psi for granite. Attewell et al. (1986) suggested an allowable tensile stress of 72.5 to 203 psi for brickwork and a shear stress of 43.5 to 101.5 psi.

Timber is another important building material. It has a very high strength along the grain, but it is very weak perpendicular to the grain. It is also vulnerable to shear. The strength of timber varies significantly depending on the species of woods. ASTM specified a range of shear strength from 634 to 1354 psi and compressive strength perpendicular to the grain from 181 to 734 psi. The tensile strength perpendicular to grain is taken as 33 percent of the shear strength, i.e., from 209 to 447 psi.

The strength of steel is the highest among the structural materials. Its tensile strength ranges from 125,000 psi to 185,000 psi and yield strength from 75,000 to 140,000 psi (ASTM, 1989). However, steel structures are rare for residential buildings, which are the most common type of buildings affected by mining subsidence.

In general, the material strengths given by different sources varies. The values specified by standard building codes tend to be in the upper bound while those used for damage assessment are towards the lower bound. For subsidence damage appraisal, the lower bound is appropriate, because many buildings affected by mining subsidence are old structures and material strengths tend to decrease with time.
6.3 Stresses Induced by Various Ground Deformations

In this section, the stresses induced in a structure by the various ground deformation components are analyzed. The effect of individual ground deformation indices is examined as well as their combined effect on structures.

6.3.1 Effect of Horizontal Ground Strain

The structure shown in Figure 5.23 is used to study the effect of ground deformations on structures. The same dimension and load conditions are assumed. For a structure located at the tension zone of a subsidence profile, the side close to the edge of the profile (right side in Figure 5.23) suffers less horizontal displacement than the side close to the panel center (left side in Figure 5.23). Therefore, the intersection between the right side and the ground is assumed to be the reference point with zero displacement, ignoring the rigid body movement. Thus, the boundary conditions are specified as: 1) the bottom of the right-hand column is fixed, i.e., displacement and rotation are zero; 2) the vertical displacements at the bottom ends of the middle and left columns are zero, horizontal displacements are the known relative horizontal ground displacements at these ends and rotation is allowed. The columns are made of brickwork and the beams timber. The Young's modulus and Poisson ratio for brickwork are $3 \times 10^6$ psi and 0.2 respectively, and for timber $2 \times 10^6$ psi and 0.2.

The maximum tensile stress induced in the structure as a function of tensile ground strain is plotted in Figure 6.4. This figure shows that the effect of ground deformation on the structure is significant. The original tensile stress in the structure without ground movement is only about 14 psi which is close to zero in the figure and is negligible.
Figure 6.4. Maximum Tensile Stresses Induced by Various Ground Deformations
compared with that induced by the ground deformation. The maximum tensile stress occurred in the left column immediately beneath the middle beam. Based on the material strengths specified by the ASTM and Figure 6.4, one can conclude that damages to the column will occur when the tensile ground strain is about $1.25 \times 10^{-3}$ which corresponds to an induced stress value of 817 psi. If plaster is present on the column, a tensile ground strain of only $2.3 \times 10^{-4}$ will cause it to crack because the tensile strength of plaster specified by ASTM is 150 psi. This information is useful in developing damage criteria based on ground deformation in which cracking of plaster is classified as architectural damage. The relationship between the induced stress in the structure and the horizontal ground strain presented in Figure 6.4 is based on the assumption that no slippage between foundation and ground has occurred. However, if slippage exists, the allowable ground strain can be multiplied by a factor more than one. For example, if the slippage is 50 percent of the ground displacement, then the allowable ground strain will be twice as high as without slippage. From the analysis in Chapter 5 and other sources (Littlejohn, 1975; Kratzsch, 1983; Marino, 1985), slippage can be significant. Thus, in developing damage criteria in terms of ground deformation, this factor should be taken into account.

For a structure located at the compression zone, the effect of the horizontal strain is similar to that at the tension zone. This time, however, the side close to the edge of the subsidence profile suffers more horizontal displacement than the side close to the panel center and the bottom end of the latter is regarded as a reference point with zero displacement. With regard to the structure shown in Figure 5.23, the bottom end of the left hand side is fixed, and the displacements specified at the other two ends (the bottom ends of the middle and right hand column) are the relative horizontal ground displacement. As a result of this reference point change, the maximum tensile stress now moves
to the right-hand column, but the relationship between the maximum induced stress in the structure and the ground strain remains the same.

In the above discussion, shear and compressive stress in the structure are ignored, because the former is very small whereas the latter is not as destructive as tensile stress for brick structures.

6.3.2 The Effects of Differential Settlement and Tilt

The same structure is now assumed to be subject to only differential settlement. Because differential settlement does not reflect the degree of deformation, a parameter called deflection ratio is used, which is the differential settlement of two points divided by the distance between them, or in other words, deflection ratio is the differential settlement per unit length. Unlike the previous case where the reference point changed with the location of the structure on the subsidence profile, the settlement of the side close to the edge of the subsidence profile is always smaller than the side close to the panel center, unless the structure is located at the center of a supercritical profile where the differential settlement is zero. Thus, the boundary conditions are that the bottom end of the right-hand side is fixed and the displacements at the other two ends are specified as the known vertical ground settlements.

The relationship between the ground deflection ratio and its induced maximum tensile stress within the structure as obtained by finite element analysis, is shown in Figure 6.4. A comparison between the stresses induced by horizontal strain and by the deflection ratio in the figure indicates that the stress induced by the deflection ratio is less than that caused by the horizontal ground strain. The maximum tensile stress induced by the
horizontal strain is 2.82 times that induced by the ground deflection ratio. It is true that the horizontal ground strain is more destructive than the differential settlement for this particular structure. However, this cannot be generalized since the effects of these deformations depend also on the configuration of the structure itself.

In order to study the effect of ground tilt, the horizontal displacement and differential settlement are assumed to be zero. In reality, tilt is the result of differential settlement and this assumption is purely for the purpose of isolating the tilt effect. Once again, a reference point with zero tilt is selected and the tilt values specified at the other two boundary points are the relative ground tilt with respect to the reference point. The analysis results are plotted in Figure 6.4. Comparing the three curves for this particular structure, it can be noted that the stress induced by horizontal strain is the greatest while that induced by differential settlement is the least. It should be pointed out that the maximum tensile stress occurred at different parts of the structure in response to different ground deformations. For example, the maximum tensile stress induced by differential settlement is located at the bottom of the right-hand column while that by tilt at the bottom of the left hand side.

6.3.3 Combined Effect of Various Ground Deformations

A structure located at a subsidence area is usually subjected to several components of ground deformation simultaneously. In this section the combined effect of horizontal ground strain, tilt and deflection ratio is discussed. In this analysis, the bottom end of the right hand side column is assumed to be fixed and the known deformations at the other two ends are the horizontal and vertical ground displacements and rotation (tilt) in reference to the fixed point.
The results from the finite element analysis are shown in Figure 6.5. In the figure, E, T, and D.R. are the ground strain, tilt, and deflection ratio respectively. The plus sign denotes tensile ground strain and positive rotation (tilt) and minus means compressive ground strain and negative rotation. As stated in Chapter 5, positive rotation is defined as counter-clockwise, and negative as clockwise. The value on the horizontal axis is the magnitude of each deformation. For example, the value $1 \times 10^{-3}$ on the ground deformation axis means that each of the three deformations has a magnitude of $1 \times 10^{-3}$. This figure indicates that the combination of compressive ground strain, positive rotation and deflection ratio resulted in the lowest stress in the structure, whereas the combination of compressive ground strain, negative rotation and deflection ratio induced the highest stress. The brick structure allows a ground deformation of $1.3 \times 10^{-3}$ for the former combination and $0.8 \times 10^{-3}$ for the latter, assuming that the tensile strength of the brick is 870 psi. One very interesting observation is that in the former combination, the direction of the given rotation coincided with that of the rotation caused by compressive strain, where in the latter case is the opposite. The same observation can be made about the combinations of tensile strain and rotations.

In Figure 6.5, the three ground deformations are assumed equal. In many practical cases, however, they are different and the relationship between the ground deformations and their induced stress in structures will be more complicated.
Figure 6.5. Maximum Tensile Stresses Induced by Combinations of Ground Deformations
6.4 Responses of Structures with Different Materials and Configurations to Mining Subsidence

Through the study of the impact of materials and configurations on structure responses, a better subsidence-resistant structure can be designed. Assume that two structures with different materials but the same configuration are subjected to mining subsidence. The two materials are concrete with a Young's modulus of $4.35 \times 10^6$ psi and timber with $2.17 \times 10^6$ psi. The configurations of the structures are the same as before. The maximum stresses in the structures as a function of ground deformations are shown in Figure 6.6. The purpose of plotting this figure is not to show the difference between the maximum stresses for the two materials, but to make a point about the allowable ground deformation. It is obvious that the maximum stress in the concrete structure is twice as high as in the timber structure, since the Young's modulus of concrete is twice that of timber. The vertical axis in the figure can be either tensile or compressive because the magnitudes of the two are the same. A comparison of the allowable ground deformations for the two materials can be made. Before proceeding with this comparison, the strengths of timber must be clarified. In 6.2, the shear strength and the tensile and compressive strength perpendicular to grain are discussed. In the analysis here, however, the shear stress and the stresses perpendicular to grain are negligible and the strengths parallel to grain are needed. For timber, the tensile strength parallel to grain is larger than the compressive strength. For example, the former value of spruce is 17,400 psi and the latter is only 4350 psi. Based on these values and Figure 6.6, the allowable ground deformation is about $6 \times 10^{-3}$. For concrete with a tensile strength of about 870 psi, the allowable ground deformation is only $6 \times 10^{-4}$. Thus, the allowable stress for the timber structure is 10 times as high as for the concrete structure. This suggests that in subsidence-prone area a timber structure is preferable to a concrete one.
Figure 6.6. Material Effects on Maximum Stresses Induced by Ground deformations
The configuration of a structure also has an impact on the structure's response to mining subsidence. To study this impact, three structures with the same material (timber) are considered. One is the structure used before (Figure 5.23). The second one is a "tall", eight-bay structure (Figure 6.7), which has the same length as the first one but is twice as high. The third one is a "long", eight-bay structure (Figure 6.8) which has the same height as the first one but is twice as long. The bay sizes for the three structures are all the same. The analysis results are shown in Table 6.1. One can see that the maximum stress in the "long" structure is 1.8 times that in the "tall" structure, although the volumes in the two structures are the same.
Figure 6.7. The "Tall" Structure
Figure 6.8. The "Long" Structure
Table 6.1 Comparison of Maximum Stresses for Various Structural Configurations

<table>
<thead>
<tr>
<th>L x H</th>
<th>Bay Size</th>
<th># of Bays</th>
<th>Ground Tension (10^-3)</th>
<th>Deflection Ratio (10^-3)</th>
<th>Ground Curv. (10^-4 1/ft.)</th>
<th>Max. Stress in Structure (psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>40x24</td>
<td>20x12</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>1.25</td>
<td>3290</td>
</tr>
<tr>
<td>40x48</td>
<td>20x12</td>
<td>8</td>
<td>5</td>
<td>5</td>
<td>1.25</td>
<td>4490</td>
</tr>
<tr>
<td>80x24</td>
<td>20x12</td>
<td>8</td>
<td>5</td>
<td>5</td>
<td>1.25</td>
<td>8230</td>
</tr>
</tbody>
</table>
Chapter 7  Development of Damage Criteria for Buildings Affected by Mining Subsidence

There are two types of damage criteria for buildings affected by mining subsidence. One type is based on the manifestation of the damage and the other on ground movement parameters. The damage criteria considered here belong to the second type, and their development process involves four steps. The first step is to classify the buildings based on their structural characteristics, the second to specify the damage levels, the third to determine the critical subsidence indices for each building category and the fourth to develop index values for different damage levels.

Currently, the majority of damage analyses are concentrated on the superstructure. Previous studies (Bruhn, 1982), however, have shown that basements and foundations are more often involved in subsidence damage than the superstructure. In civil engineering (Zeevaert, 1983), as many as eight types of foundations are described although for the purpose of damage evaluation they can be reduced to four types, namely isolated footing, strip, raft, and buoyancy foundations. Superstructures are categorized in accordance with the construction materials. In addition, building size and shape are also considered in building classification.
The subsidence indices currently used in many damage criteria originate from studies of structure settlement due to its own weight, tunnelling, and other civil engineering examples. In most cases, however, the magnitude and extent of ground movements caused by these activities are less than those induced by mining. In this study, critical deformation indices were selected by considering basic relations among several ground deformation components.

7.1 Evaluation of Existing Building Damage Criteria

Damage to buildings due to subsidence is increasingly becoming a problem for government authorities, the general public and mining operators. In some states, subsidence insurance programs have been established, requiring damage severity classification, in order to facilitate the payment of claims. Federal law requires that underground coal mine operators adopt measures to prevent or minimize subsidence damage to the extent technologically possible. In order to address this problem, various damage criteria have been developed.

The most quoted damage criteria for buildings affected by mining subsidence are those developed by the National Coal Board of Britain, which are based on numerous observations of building damages due to subsidence in that country (NCB, 1975). Figure 7.1 and Table 7.1 show this damage classification scheme. In these criteria the horizontal strain is considered as the critical damage index. The problem with the NCB damage criteria is that only structural length is considered, whereas other important characteristics of the structure are ignored. Bruhn et al. (1982) developed a similar classification based on studies of subsidence damaged homes in the Northern Appalachian Coal Field (Table 7.2). The damage severity in their criteria is classified according to the appear-
Figure 7.1. Relationship of Damage to Length of Structure and Horizontal Ground Strain (NCB, 1975)
Table 7.1. National Coal Board Classification of Subsidence Damage (NCB, 1975)

<table>
<thead>
<tr>
<th>Change of Length of Structure</th>
<th>Class of Damage</th>
<th>Description of Typical Damage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up to 0.1 ft (30 mm)</td>
<td>1. Very slight or negligible</td>
<td>Hair cracks in plaster. Perhaps isolated slight fracture in the building, not visible on outside.</td>
</tr>
<tr>
<td>0.1 ft - 0.2 ft (10 mm) (50 mm)</td>
<td>2. Slight</td>
<td>Several slight fracture showing inside the building. Doors and windows may stick slightly. Repairs to decoration probably necessary.</td>
</tr>
<tr>
<td>0.2 ft - 0.4 ft (60 mm) (120 mm)</td>
<td>3. Appreciable</td>
<td>Slight fracture showing on outside of building (or one main fracture). Doors and window sticking; service pipes may fracture.</td>
</tr>
<tr>
<td>0.4 ft - 0.6 ft (120 mm) (180 mm)</td>
<td>4. Severe</td>
<td>Service pipes disrupted. Open fractures requiring rebonding and allowing weather into the structure. Window and door frames distorted; floors sloping noticeably; walls leaning or bulging noticeably. Some loss of bearing in beams. If compressive damage, overlapping of roof joints and lifting of brickwork with open horizontal fractures.</td>
</tr>
<tr>
<td>More than 0.6 ft (180 mm)</td>
<td>5. Very severe</td>
<td>As above, but worse, and requiring partial or complete rebuilding. Roof and floor beams lose bearing and need shoring up. Windows broken with distortion. Severe slopes on floors. If compressive damage, severe buckling and bulging of the roof and walls.</td>
</tr>
<tr>
<td>Class</td>
<td>Characteristic Basement Damage</td>
<td>Severity Index</td>
</tr>
<tr>
<td>------------</td>
<td>---------------------------------------------------------------------------------------------</td>
<td>----------------</td>
</tr>
<tr>
<td>I Slight</td>
<td>Hairline cracks in one or more basement walls and possibly floor slab.</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Some cracks in perimeter walls causing loss of water tightness.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Repointing required in some or all walls.</td>
<td></td>
</tr>
<tr>
<td>II Moderate</td>
<td>Cracks in one or more basement walls and slab.</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Some wall/footing reconstruction and floor slab replacement required, as well as local repointing.</td>
<td></td>
</tr>
<tr>
<td>III Severe</td>
<td>Cracks in one or more basement walls and floor slab.</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Possible wall instability and loss of superstructure support, requiring shoring and bracing.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Extensive repair work involving wall/footing reconstruction and floor slab replacement.</td>
<td></td>
</tr>
<tr>
<td>IV Very Severe</td>
<td>Cracks typically in all basement walls, as well as floor slab.</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>Possible instability of several walls and loss of superstructure support, requiring extensive shoring and bracing.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Possible significant tilt to home.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>General reconstruction of basement walls, footing and floor slab required.</td>
<td>5</td>
</tr>
</tbody>
</table>
ance of the damage. Arioglu and Yuksel (1984) devised a more comprehensive scheme for building damage classification based on their experience in Zonguldak Coalfield, Turkey. Their criteria are shown in Table 7.3 and 7.4. Once again, no ground movement parameter is involved in this scheme.

One of the most successful schemes of damage criteria was the one developed in Poland, as shown in Tables 7.5 and 7.6. A major advantage of the Polish damage criteria is that seven factors are taken into account in categorizing the buildings, resulting in a more realistic building classification. The critical indices selected in the criteria are tilt, horizontal strain and radius of curvature. The disadvantage of the Polish system is that for each building class, only one damage level is specified. In other words, damage severities are not distinguished in the criteria. In the United States, the most comprehensive damage criteria were proposed by Bhattacharya and Singh (1985), based on an extensive literature survey. Four categories of buildings and three levels of damage severity were proposed (Table 7.7 and 7.8). The criteria include four subsidence indices: angular distortion, horizontal strain, deflection ratio and radius of curvature (Table 7.9). This damage analysis method, however, is rather incomplete and suffers from lack of appropriate data.

In this research, the damage severity is classified according to the ground movement indices. Since the ground movement indices can be predicted, a pre-mining estimation of the damage severity is possible.
Table 7.3. Classification Parameters for Subsidence Damage and Their Ratings (Arioglu et al., 1984)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Ratings</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I) Number of Total Cracks</td>
<td></td>
</tr>
<tr>
<td>1 - 5</td>
<td>2</td>
</tr>
<tr>
<td>5 - 10</td>
<td>4</td>
</tr>
<tr>
<td>10 - 20</td>
<td>6</td>
</tr>
<tr>
<td>20 - 30</td>
<td>8</td>
</tr>
<tr>
<td>&gt; 30</td>
<td>10</td>
</tr>
<tr>
<td>(II) Intensity of Cracks (%)</td>
<td></td>
</tr>
<tr>
<td>&lt; 20</td>
<td>1</td>
</tr>
<tr>
<td>20 - 40</td>
<td>2</td>
</tr>
<tr>
<td>40 - 60</td>
<td>3</td>
</tr>
<tr>
<td>60 - 80</td>
<td>4</td>
</tr>
<tr>
<td>&gt; 80</td>
<td>5</td>
</tr>
<tr>
<td>(III) Width of Crack (mm)</td>
<td></td>
</tr>
<tr>
<td>&lt; 0.5</td>
<td>1</td>
</tr>
<tr>
<td>0.5 - 3</td>
<td>2</td>
</tr>
<tr>
<td>3 - 10</td>
<td>3</td>
</tr>
<tr>
<td>10 - 20</td>
<td>4</td>
</tr>
<tr>
<td>20 - 40</td>
<td>5</td>
</tr>
<tr>
<td>40 - 60</td>
<td>6</td>
</tr>
<tr>
<td>60 - 80</td>
<td>8</td>
</tr>
<tr>
<td>&gt; 80</td>
<td>10</td>
</tr>
<tr>
<td>(IV) Type of Damage</td>
<td></td>
</tr>
<tr>
<td>A. Architectural (wall, floor)</td>
<td>2</td>
</tr>
<tr>
<td>B. structural</td>
<td>4</td>
</tr>
<tr>
<td>C. Both A and B</td>
<td>6</td>
</tr>
<tr>
<td>(V) Loss of Performance of Window and Doors</td>
<td></td>
</tr>
<tr>
<td>None</td>
<td>0</td>
</tr>
<tr>
<td>Slight</td>
<td>1</td>
</tr>
<tr>
<td>appreciable</td>
<td>2</td>
</tr>
<tr>
<td>Severe</td>
<td>3</td>
</tr>
<tr>
<td>(VI) Penetration of Rain and Wind through Walls</td>
<td></td>
</tr>
<tr>
<td>None</td>
<td>0</td>
</tr>
<tr>
<td>Slight</td>
<td>1</td>
</tr>
<tr>
<td>Severe</td>
<td>2</td>
</tr>
<tr>
<td>Very Severe</td>
<td>3</td>
</tr>
<tr>
<td>(VII) Socio-economic Conditions (Especially level of income)</td>
<td></td>
</tr>
<tr>
<td>Low</td>
<td>1</td>
</tr>
<tr>
<td>Medium</td>
<td>2</td>
</tr>
<tr>
<td>High</td>
<td>3</td>
</tr>
<tr>
<td>Very High</td>
<td>4</td>
</tr>
</tbody>
</table>

* The intensity of cracks is defined as the ratio between the number of cracked rooms and the total number of rooms.
Table 7.4. Class of Damage Corresponding to Total Rating (Arioglu et al., 1984)

<table>
<thead>
<tr>
<th>Rating</th>
<th>Class</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 - 5</td>
<td>Very slight</td>
</tr>
<tr>
<td>5 - 15</td>
<td>Slight</td>
</tr>
<tr>
<td>15 - 25</td>
<td>Appreciable</td>
</tr>
<tr>
<td>25 - 30</td>
<td>Severe</td>
</tr>
<tr>
<td>&gt; 30</td>
<td>Very severe</td>
</tr>
</tbody>
</table>
Table 7.5. Chart for Determining Structure Category (Eichfeld, 1983)

<table>
<thead>
<tr>
<th>1. Length of building</th>
<th>m</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
<th>&gt;40</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ft</td>
<td>&lt;=35</td>
<td>35-50</td>
<td>50-65</td>
<td>65-85</td>
<td>85-100</td>
<td>100-115</td>
<td>115-130</td>
<td>&gt;130</td>
</tr>
<tr>
<td>Number of points</td>
<td></td>
<td>4</td>
<td>7</td>
<td>11</td>
<td>16</td>
<td>22</td>
<td>29</td>
<td>37</td>
<td>42</td>
</tr>
</tbody>
</table>

2. Shape of the building body
- simple, compact: 0
- little, articulated: 3
- greatly articulated: 6
- simple, extensive: 6
- articulated, extensive: 8

3. Foundation of the building
- on the steady level with or without cellars: 0
- on the variable level: 3
- on the variable level with some cellars: 6
- same with drive gate: 8

4. Base of the building
- compressible: 0
- little compressible: 4
- incompressible: 12

5. Construction of the building
- rigid: 0
- little rigid: 4
- yielding: 8

6. Existing protection against mining effects
- anchorage: 0
- partial protection: 4
- no protection: 12

7. Technical condition of the building
- good: 0
- medium: 6
- bad: 12

Classification of the Building

<table>
<thead>
<tr>
<th>Number of points</th>
<th>up to 20</th>
<th>21-27</th>
<th>28-36</th>
<th>37-47</th>
<th>48 and more</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class of resistance</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

195
<table>
<thead>
<tr>
<th>Class of resistence</th>
<th>Tilt (x 0.001)</th>
<th>Horizontal strain (x 0.001)</th>
<th>Radius of curvature mile (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>≤2.5</td>
<td>≤1.5</td>
<td>≥12.5 (20)</td>
</tr>
<tr>
<td>1</td>
<td>≤5.0</td>
<td>≤3.0</td>
<td>≥7.5 (12)</td>
</tr>
<tr>
<td>2</td>
<td>≤10.0</td>
<td>≤6.0</td>
<td>≥4.0 (6.0)</td>
</tr>
<tr>
<td>3</td>
<td>≤15</td>
<td>≤9.0</td>
<td>≥2.5 (4.0)</td>
</tr>
<tr>
<td>4</td>
<td>&gt;15</td>
<td>&gt;9.0</td>
<td>&lt;2.5 (4.0)</td>
</tr>
<tr>
<td>Description</td>
<td>Category</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-------------------------------------------------</td>
<td>----------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Brick and masonry structures/brick bearing walls/low rise structures</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Steel and reinforced-concrete frame structures</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Timber frame structures</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Massive structures of considerable rigidity/central core design</td>
<td>4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 7.8. Proposed Damage Severity Levels (Bhattacharya et al., 1985)

<table>
<thead>
<tr>
<th>Type of Damage</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Onset of architectural damage characterized by small scale cracking of plaster and sticking of doors and windows.</td>
<td>ARCH</td>
</tr>
<tr>
<td>Onset of functional damage characterized by instability of some structural elements. Jammed doors and windows. Broken window panes. Building service restricted.</td>
<td>FUNC</td>
</tr>
<tr>
<td>Onset of structural damage characterized by impairment of primary structural members. Possibility of collapse of members. Complete or large-scale rebuilding necessary. May be unsafe for habitation.</td>
<td>STRUC</td>
</tr>
</tbody>
</table>
### Table 7.9. Building Damage Criteria (Bhattacharya et al., 1985)

<table>
<thead>
<tr>
<th>Building Category</th>
<th>Range of Values</th>
<th>Recommended Values</th>
<th>No. of Sources</th>
<th>Ground Movement Limits</th>
<th>Range of Values</th>
<th>Recommended Values</th>
<th>No. of Sources</th>
<th>Deflection Ratio</th>
<th>Range of Values</th>
<th>Recommended Values</th>
<th>No. of Sources</th>
<th>Radius of Curvature</th>
<th>Mile (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Arch</td>
<td>0.5-2.0</td>
<td>1.0</td>
<td>8</td>
<td>0.25-1.5</td>
<td>0.5</td>
<td>13</td>
<td></td>
<td>0.3-1.0</td>
<td>0.3</td>
<td>3</td>
<td></td>
<td>2-12(3-20)</td>
<td>12(20)</td>
</tr>
<tr>
<td>1 Func</td>
<td>2.0-4.0</td>
<td>2.5-3.0</td>
<td>12</td>
<td>1.0-4.0</td>
<td>1.5-2.0</td>
<td>5</td>
<td></td>
<td>0.14-0.6</td>
<td>0.5</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 Struc</td>
<td>7.0-8.0</td>
<td>7.0</td>
<td>1</td>
<td>2.75-3.5</td>
<td>3.0</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 Arch</td>
<td>1.0-2.5</td>
<td>1.3</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 Func</td>
<td>2.5-6.6</td>
<td>3.3</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 Struc</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 Arch</td>
<td>2.0</td>
<td>1.5</td>
<td>2</td>
<td>1.0</td>
<td>1.0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 Func</td>
<td>3.3-10.0</td>
<td>3.3-5.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 Struc</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1. *Indicated data not available*
2. See Table 7.7 for definitions of building categories
3. See Table 7.8 for definitions of damage severity levels
7.2 Building Classification

The building classification method used in this research is similar to the Polish method. Several building characteristics are taken into account and each of these characteristics is rated into classes based on its response to subsidence. For a particular building, the final building category can be established by adding the ratings from the individual building characteristics.

The building characteristics include the following:

1. Building foundation

This is the most important factor affecting damage severity of a building. Bruhn et al. (1982), from 134 cases of subsidence damage to homes in western Pennsylvania, noted that damage to homes almost always included basements. In 90 percent of the cases, more than half the repair cost was associated with damage to subgrade components. Marino (1985), based on his research in the Illinois coal field, also concluded that at high damage levels, the repair costs of the foundation are always greater than those of the superstructure. Therefore, it is justifiable to pay more attention to foundations when classifying buildings.

In foundation engineering, foundations are classified into eight types: isolated footing, continuous footing, raft foundation, buoyancy foundation, piled buoyancy foundation, point bearing pile foundation, pier foundation and sand pier foundation. The definitions of these foundation types were described by Zeevaert (1983). For subsidence damage analysis, the foundation types can be reduced to four. The reason for the reduction lies in the facts that piers can be treated as isolated footings and piles underneath other types
of foundation will have no major effect in case of subsidence, because the conditions under which the piles are supposed to work are destroyed by subsidence. Thus, the foundation types used in the damage criteria are isolated footing, continuous footing, raft and buoyancy foundation. The rating for each foundation type is listed in Table 7.10. It is based on the vulnerability of the foundation to subsidence, with the most vulnerable type having the lowest rating. The same scheme is followed in rating other building characteristics, i.e., the highest ratings represent better structural response.

2. Superstructure material

Common superstructure materials are brick, stone, concrete, reinforced concrete, timber and steel. The first three materials possess approximately the same mechanical properties, strong in compression but are weak in tension and bending. Buildings composed of these materials are most vulnerable to mining subsidence which usually involves bending and distortion. The tension strength of reinforced concrete is enhanced by the reinforcing material. Timber and steel are better materials to withstand subsidence. Thus, superstructure materials are classified into four categories, with the first three materials in one category and each additional material representing one other category. Table 7.10 lists the ratings for the material categories.

3. L/R ratio

L/R ratio measures the length of the building (L) in relation to subsidence, where R is the half width of the subsidence profile. This relationship has a profound effect on buildings affected by mining subsidence. Marino (1985) showed that both the angular distortion and the axial force in the foundation increase with the L/D ratio when the L/D ratio is less than 1.0, where D = 2R is the total width of the subsidence profile. The
Table 7.10 Building Characteristics and Their Ratings

<table>
<thead>
<tr>
<th>Characteristics</th>
<th>Rating</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Foundation:</td>
<td></td>
</tr>
<tr>
<td>Isolated footing</td>
<td>1</td>
</tr>
<tr>
<td>Continuous footing</td>
<td>4</td>
</tr>
<tr>
<td>Raft foundation</td>
<td>8</td>
</tr>
<tr>
<td>Bouyancy foundation</td>
<td>16</td>
</tr>
<tr>
<td>2. Superstructure Materials:</td>
<td></td>
</tr>
<tr>
<td>Brick, stone, and concrete</td>
<td>2</td>
</tr>
<tr>
<td>Reinforced concrete</td>
<td>4</td>
</tr>
<tr>
<td>Timber</td>
<td>6</td>
</tr>
<tr>
<td>Steel</td>
<td>8</td>
</tr>
<tr>
<td>3. L/R ratio</td>
<td></td>
</tr>
<tr>
<td>&lt; 0.1</td>
<td>4</td>
</tr>
<tr>
<td>0.10 - 0.25</td>
<td>3</td>
</tr>
<tr>
<td>0.26 - 0.50</td>
<td>2</td>
</tr>
<tr>
<td>&gt; 0.50</td>
<td>1</td>
</tr>
<tr>
<td>4. H/L ratio</td>
<td></td>
</tr>
<tr>
<td>&lt; 1.0</td>
<td>4</td>
</tr>
<tr>
<td>1.0 - 2.5</td>
<td>3</td>
</tr>
<tr>
<td>2.6 - 5.0</td>
<td>2</td>
</tr>
<tr>
<td>&gt; 5.0</td>
<td>1</td>
</tr>
</tbody>
</table>
possibility of rotation is the greatest at low L/D ratios. Other studies indicate that the bridging effect of the building becomes obvious when the L/R ratio gets large (Breth, 1974, and Littlejohn, 1974). The finite element analysis in Chapter 6 also indicated that long structures are more vulnerable than short ones.

L/R ratio is divided into four classes and the rating for each class is shown in Table 7.10.

4. H/L ratio

H/L ratio indicates the relative height of a building (H) with respect to its length. The higher the H/L ratio is, the more vulnerable to tilt the building will be. It should be pointed out that L is assumed to be unchanged in calculating the ratio. If both H and L vary, then the L effect should be accounted for by the L/R ratio and the H effect by the H/L ratio. Table 6.1 indicates that subsidence-induced stress increases as the height of the structure is increased. H/L ratio is rated into four categories (Table 7.10). Table 7.10 summarizes the building characteristics and their ratings and Table 7.11 is the building classes based on the summation of the ratings.

<table>
<thead>
<tr>
<th>Total rating</th>
<th>Class</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 - 10</td>
<td>I</td>
</tr>
<tr>
<td>11 - 17</td>
<td>II</td>
</tr>
<tr>
<td>18 - 25</td>
<td>III</td>
</tr>
<tr>
<td>26 - 32</td>
<td>IV</td>
</tr>
</tbody>
</table>
7.3 Damage Levels

There are several ways of specifying the damage levels. NCB (1975), for example, uses the change of structure length combined with a physical description of damages to categorize the damages (Table 7.1). Bruhn et al. (1982), classified building damage levels based on the physical description of the damages in the basement and foundation (Table 7.2). In Arioglu and Yuksel's classification (Table 7.3), seven parameters are considered, each parameter is rated, and adding the ratings from all the parameters gives the damage level. Bhattacharya and Singh (1985) specified three damage levels according to the habitability (or serviceability) of the building.

Each of these classification methods has its own strengths. The most meaningful method is the latter classification, since the damage levels are based on habitability, which is the most important factor in assessing building damage. Their classification method is used in this paper, therefore, and the damage levels are shown in Table 7.8.

7.4 Selection of Critical Indices

Several subsidence indices have been used in various building damage criteria. Commonly seen are angular distortion, horizontal strain, deflection ratio, radius of curvature and tilt. There is no need to include all these indices in the damage criteria because relations exist among some of these indices. For example, the relationship between angular distortion and deflection ratio can be written as (Burland and Wroth, 1975):

\[
\frac{\Delta}{L} = \frac{\beta [1 + 3.9(H/L)^2]}{3[1 + 2.6(H/L)^2]} \tag{7.1}
\]
where $\Delta$ is the relative deflection, $L$ the length of the foundation, $H$ the height of the foundation and $\beta$ angular distortion. Thus, it is redundant to put both indices in the criteria.

An important factor in the selection of critical indices is that all the ground deformations should be included, but none should be repeated. For example, in the Polish damage criteria (Table 7.6), vertical ground deformation is not included while shear deformation is repeated because both tilt and curvature reflect shear deformation. On the other hand, the damage criteria shown in Table 7.9 embraced all types of ground deformations, but once again shear deformation is repeated because both angular distortion and curvature reflect shear deformation.

Considering the above factors, three indices, horizontal strain, deflection ratio and curvature are chosen as critical indices in this study, where horizontal strain represents horizontal deformation, deflection ratio vertical deformation, and curvature shear deformation. The final form of the damage criteria for buildings affected by mining subsidence is shown in Table 7.12, with some of the entries not completed because of lack of data. The building class in Table 7.12 is based on Table 7.11.

7.5 Determination of Critical Index Values

In order to determine the critical deformation index values, relations between ground movements and building responses have to be known. Through the discussion in Chapter 3 and the analyses in Chapter 5 and 6, some insight was gained in this area. As a result, some critical values can be assigned as shown in Table 7.12. In determining the critical
ground deformation values, the slippage between the foundation and subgrade, and the bridging effect of the foundation should always be taken into account.

In summary, attempts have been made to develop new damage criteria for buildings affected by mining subsidence. A new building classification scheme, based on foundation type, building material, L/R and H/L ratios, was proposed. A set of subsidence indices, differing from the sets used in previous studies, has been selected as critical based on the consideration of the relations among various indices. These are horizontal strain, deflection ratio, and curvature. Some values of the critical indices have been assigned. However, further study is needed in order to validate and complete the proposed damage criteria.
### Table 7.12 Building Damage Criteria

<table>
<thead>
<tr>
<th>Building Class</th>
<th>Damage Level</th>
<th>Horizontal Strain (10^-3)</th>
<th>Deflection Ratio (10^-3)</th>
<th>Curvature (10^-4ft.⁻¹)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>1</td>
<td>0.5</td>
<td>0.7</td>
<td>0.3</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.5</td>
<td>1.8</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>5.0</td>
<td>6.0</td>
<td>-</td>
</tr>
<tr>
<td>II</td>
<td>1</td>
<td>2.0</td>
<td>2.5</td>
<td>0.6</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3.0</td>
<td>3.6</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>III</td>
<td>1</td>
<td>3.0</td>
<td>4.0</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>4.0</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>IV</td>
<td>1</td>
<td>4.0</td>
<td>5.0</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Note on damage level:
1 - architectural damage
2 - functional damage
3 - structural damage
Chapter 8 Evaluation of Subsidence Prevention and Control Measures

Subsidence control measures are those techniques which can be applied to minimize subsidence-related surface movements or subsidence-related damages to structures and facilities. Such measures can be categorized as either mining or structural.

Mining design, including mining method, geometry of excavation and extraction sequence, has a pronounced effect on the distribution and magnitude of subsidence and horizontal strain on the surface. Appropriate mine design, therefore, should be employed where possible to promote planned and even controlled subsidence on the surface. Underground measures, therefore, may include static measures, such as leaving pillars in place below critical structures, partial extraction of an area or solid stowing of the gob; and dynamic measures, which can cause cancelation of traveling subsidence and strain waves, or their migration away from the structure to be protected.

Structural measures can be either preventive or remedial. Preventive measures are those used in constructing new structures and remedial measures are those applicable for existing structures. These measures can provide flexibility or rigidity in structures so that
any subsidence related movements can be accommodated or resisted with minimum damage.

8.1 Mining Measures

8.1.1 Optimizing Panel Design

The configuration of mine panels affect the total damaged area on the surface, and panel design may be optimized to minimize the damage area. In an actual case study (Karmis et al., 1990), three proposed mining plans were considered for an underground operation in Kentucky. One important criterion in selecting the optimum mining system was its impact upon surface structures. The proposed systems included the following:

— **Full-size longwall**, 600 feet wide with stable and yield chain pillars;
— **Small longwall**, 460 feet wide with similar chain pillar configuration;
— **Shortwall**, 300 feet wide sections, also in conjunction with yield and stable pillars.

The basic mining factors required for the analysis are mining depth, h, mining height, m, and percentage of hard rock in the overburden, H.R. Based on the mining and geological information pertaining to the site, the following values were used:

\[ h = 600 \text{ ft.,} \]
\[ m = 5.8 \text{ ft., and} \]
\[ H.R. = 40\%. \]

In reviewing the available subsidence information for this mining area, the following subsidence parameters were assumed:
\[
\frac{S_{\text{max}}}{m} = 0.47, \\
d = 156 \text{ ft.}, \\
\beta = 63^\circ, \text{ and} \\
B_s = 0.17.
\]

The area under examination extended over three adjacent panels in one dimension and half of the total advance length in the other dimension. More than 1,000 grid points were used within this area by SDPS to predict subsidence and strain. The damage areas for the three mining systems are compared in Table 8.1. An area is considered damaged if the ground deformations exceed the critical values. For convenience, damage areas are expressed relative to the mined-out areas. This table can be used when deciding the optimum system for preventing subsidence damage.

Table 8.1. Damaged Areas for Different Mining Systems

<table>
<thead>
<tr>
<th>System</th>
<th>Total Damage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Full Size Longwall</td>
<td>86%</td>
</tr>
<tr>
<td>Small Longwall</td>
<td>91%</td>
</tr>
<tr>
<td>Shortwall</td>
<td>106%</td>
</tr>
</tbody>
</table>

8.1.2 Protection Pillar

Leaving a coal pillar underground is the easiest and safest way to protect surface structures. The key factor in protection pillar design is the angle of draw which must be accurate in order for the protection pillar to be effective. In areas where the surface soil
Figure 8.1. Dimension of Protection Pillar for Flat Coal Seam
or loose material is deep, a larger draw angle must be used in the zone close to the surface, as shown in Figure 8.1. For a flat coal seam the horizontal dimensions of the pillar can be calculated as follows:

\[ L_x = a + 2[h_s \tan \gamma_1 + (h - h_s) \tan \gamma_2] \]  \hspace{1cm} (8.1)

in X direction, and

\[ L_y = b + 2[h_s \tan \gamma_1 + (h - h_s) \tan \gamma_2] \]  \hspace{1cm} (8.2)

in Y direction, where

- \( L_x \) = the length of the pillar along X direction;
- \( L_y \) = the length of the pillar along Y direction;
- \( a \) = the length of the structure along X direction;
- \( b \) = the length of the structure along Y direction;
- \( h_s \) = the thickness of surface soil zone;
- \( h \) = the thickness of the overburden;
- \( \gamma_1 \) = the draw angle of soil; and
- \( \gamma_2 \) = the draw angle of rock.

For an inclined coal seam, a graphical method as shown in Figure 8.2 can be used to determine the dimensions.

The obvious advantage of using a protection pillar is its effectiveness. The disadvantage is that pillars may increase horizontal strain and decrease recovery rate. Furthermore, protection pillars may be difficult to integrate in panel design.
Figure 8.2. Dimension of Protection Pillar for Inclined Seam (CMCI, 1985)
8.1.3 Partial Extraction

Partial extraction such as room-and-pillar, board-and-pillar, or panel-and-pillar can be used to protect surface structure (Peng, 1986). For instance, in Pennsylvania the partially extracted area under a structure is determined by projecting the area of the structure plus its 15 ft. wide periphery downward and outward at an angle of 15 degrees from the vertical, until it reaches the seam level (Fig. 8.3). Pillars should be uniformly distributed in this partially extracted region and the extraction ratio should be less than 50 percent in order to prevent excessive subsidence.

The panel-and-pillar system used in Europe, however, is frequently designed with long but narrow parallel panels separated by permanent pillars. Pillars and panels have the same width of about one fourth of the overburden thickness. The observed surface subsidence ranges from 3 to 20 percent of the seam thickness.

The advantage of partial extraction compared with solid protection pillars is that part of the resource is recovered and, that the surface structure can be protected on a short term basis. However, the long term effectiveness of partial extraction in protecting surface structure can be questionable. A study done by Yu and Karmis (1988) shows that on the average, subsidence occurred about 44 years after partial extraction in the Pittsburgh coal area. The same study found that the pillar deterioration rate can reach 2.77 percent per year. Deterioration of coal pillars may be due to stress concentration and weathering as the result of a change in relative humidity and mine-water fluctuations. Oxidation of certain sulfide minerals such as pyrite may occur when they are exposed to air and water, resulting in the formation of new minerals of greater volume than the original pyrite. Slow, nonuniform expansion or swelling may follow, causing the slaking and slabbing of coal pillar (Karfakis and Topuz, 1990). Therefore, in using
Figure 8.3. Partial Extraction (Peng, 1986)
partial extraction methods to protect surface structures, the long term consequences must be considered.

8.1.4 Backfilling

Surface subsidence can be controlled by backfilling the mined-out area. The degree of control is based on the following formula:

$$S_{\text{max}} = \alpha M_e$$  \hspace{1cm} (8.3)

where $S_{\text{max}}$ is the maximum subsidence, $\alpha$ the subsidence factor without backfilling, and $M_e$ the equivalent height which is opening height after backfilling and the consolidation of the backfilled material. $M_e$ can be calculated as:

$$M_e = M - M_b(1 - B)$$  \hspace{1cm} (8.4)

where $M$ is the seam height, $M_b$ the height being filled and $B$ the consolidation factor of the backfilled material. Substituting Equation 8.4 into Equation 8.3, one has:

$$S_{\text{max}} = \alpha [M - M_b(1 - B)] = \alpha_b M$$  \hspace{1cm} (8.5)

where

$$\alpha_b = \frac{\alpha [M - M_b(1 - B)]}{M} = \alpha - \frac{M_b}{M} \alpha (1 - B)$$  \hspace{1cm} (8.6)

is the subsidence factor with backfilling. Apparently, without backfilling $\alpha = \alpha_b$, and as the backfilled height increases $\alpha_b$ decreases, but $\alpha_b$ can never reach zero unless the consolidation factor is equal to zero which is technically impractical. A plot of the re-
lationship between subsidence factors and fill height for different consolidation factors, for \( \alpha = 0.8 \), is shown in Figure 8.4.

There are several types of backfilling systems in active mines. They include hand stowing, gravity stowing, mechanical stowing, pneumatic stowing and hydraulic stowing. A detailed discussion of these methods is presented by Bowman et. al (1990). Hand stowing was used in the past when mechanical means were not available and labor cost was small. Gravity stowing is a method used in inclined seams where backfilling is accomplished by the gravity force of the stowing materials. Mechanical stowing is a system in which the backfilling material is delivered nearby the gob by conveyor and thrown into the desired area by mechanical means. This method is not as effective as the pneumatic or hydraulic systems in (Bowman et al., 1990). In pneumatic stowing system, the backfilling material is conveyed through the use of compressed air. In hydraulic stowing system, the backfilling material is first mixed to form a slurry and then delivered through a pipe to the packing area. The effectiveness of each stowing method in subsidence control can be seen in Table 8.2 where the subsidence factor for each method is listed. This table shows that hydraulic stowing is the most effective method in reducing subsidence. Knowing the subsidence factors, the consolidation factors for various stowing methods can be found and from Equation 8.6:

\[
B = \frac{\alpha_b - \alpha}{\alpha} + 1
\]

(8.7)

for total filling \( \frac{M_b}{M} = 1 \). For example, based on the subsidence factor given by CUMT (1981), the calculated consolidation factors for hydraulic stowing range from 0.1 to 0.25. For pneumatic stowing, the factor is 0.5.
Figure 8.4. Subsidence Factors for Various Filling Heights
Table 8.2. Subsidence Factors for Various Backfilling Methods

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>No Backfilling</td>
<td>0.60 - 0.80</td>
<td>0.60 - 0.95</td>
</tr>
<tr>
<td>Strip Packing</td>
<td>0.55 - 0.70</td>
<td>0.60 - 0.90</td>
</tr>
<tr>
<td>Mechanical</td>
<td>0.40 - 0.50</td>
<td>-</td>
</tr>
<tr>
<td>Pneumatic</td>
<td>0.30 - 0.40</td>
<td>0.30 - 0.70</td>
</tr>
<tr>
<td>Hydraulic</td>
<td>0.06 - 0.2</td>
<td>0.08 - 0.3</td>
</tr>
</tbody>
</table>

Backfilling can also be used in abandoned mine workings. However, backfilling in an abandoned mine is more difficult than in an active mine because the mine may be inaccessible, and the geometry of the mine workings may be unknown. Estimation of fill volumes for backfilling demands considerable information if significant cost over-runs are to be avoided (Carter and Steed, 1990). Other information, such as the mining method and geological conditions, is also important. Some of the backfilling methods used in active mines may not be applicable in abandoned mine workings for the reasons mentioned above. The applicability of various backfilling methods under different conditions in abandoned mines is shown in Table 8.3.

The advantage of backfilling is that the mineral resource can be effectively recovered. The drawbacks are that it can be expensive and the residual subsidence due to the consolidation of the backfilling material is difficult to avoid. In general, whether backfilling
Table 8.3 Applicability of Various Backfilling Methods in Abandoned Mines (Carter and Steed, 1990)

<table>
<thead>
<tr>
<th>Cases</th>
<th>Gravity</th>
<th>Pneumatic</th>
<th>Hydraulic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Incompetent Sidewalls</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Inaccessible Openings</td>
<td>X</td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>Overlaying Infrastructure</td>
<td></td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Flooded Opening</td>
<td>X</td>
<td></td>
<td>X</td>
</tr>
</tbody>
</table>

X = Applicable
should be used depends on the protection requirement and the value of the structure being protected.

8.1.4 Rapid Mining

The concept of rapid mining is to allow a structure to be positioned at the center of a supercritical subsidence profile after mining. This is desirable, because under supercritical conditions the structure at the central area of the profile suffers no strain, slope and curvature. During the process of achieving this supercritical condition, however, the structure must be subject to travelling strain. The maximum travelling strain \( \varepsilon_{\text{fmax}} \), according to Brauner (1973), is less than the maximum final strain \( \varepsilon_{\text{fmax}} \) and it decreases as the rate of advance increases. This can be explained with the Maxwell viscoelastic model in which the strain can be calculated as:

\[
\varepsilon = \frac{\sigma}{E} + \frac{\sigma}{\eta} t
\]  

(8.7)

where \( \varepsilon \) is the strain in a viscoelastic body, \( E \) is the Young's modulus, \( \sigma \), stress, \( \eta \) the viscoelastic coefficient, and \( t \) time during which the stress is applied. If \( \varepsilon \) in this equation is treated as the travelling strain in mining subsidence, one can see that the longer the overburden stays in a highly-stressed condition, the larger is the traveling strain. Thus, shortening the time of a highly-stressed condition by rapid mining will reduce the travelling strain. When a structure is undermined, the traveling strain must be less than the allowable strain in the structure. Thus, the critical time during which the mining face must pass can be written as:

\[
t_c = \eta \left( \frac{\varepsilon_c}{\sigma} - \frac{1}{E} \right)
\]  

(8.8)
where \( \varepsilon_c \) is the critical maximum travelling strain. The stress conditions at the ground surface may be over-simplified by the Maxwell model, but it helps to explain the reason behind rapid mining.

Rapid mining remains a theoretical rather than a practical method and many technical questions still need to be resolved. For example, the rate of advance and whether it can be achieved in practice has not been determined. Some attempts have been made to determine the advance rate (Brauner, 1973), but without any practical justification.

### 8.1.5 Harmonic Mining

The concept of harmonic mining is that if several mining faces are arranged in an appropriate sequence, the surface deformations induced by the faces can cancel each other. It can be applied in either single seam with multi-faces or muti-seam mining. In the former case, assuming that two panels are being mined and a structure is located between them as shown in Figure 8.5a, the panel on the right is excavated first and the face of this panel is kept a constant distance ahead of the face of the panel on the left, approximately one-fourth the critical width. The tensile strain traveling with the front face is relatively small over this edge, while the compressive strain behind the face will to some extent be cancelled by the traveling tensile strain from the rear face. By this means the strain suffered by the structure will be kept to a minimum. In the above explanation, only the deformation along the longitudinal direction is considered. The deformation along the transverse direction, however, can also play a role. For example, the structure in Figure 8.5a will suffer maximum tilt induced by the front face transversely. If the structure is very sensitive to tilt, then this plan may not be appropriate.
Figure 8.5. Harmonic Mining
If multi-seam mining is involved, the faces in the lower and upper seams can also be arranged to minimize the deformation on the surface, as shown in Figure 8.5b. The mechanism is the same as for multi-faces in a single seam.

Like rapid mining, harmonic mining can maximize recovery. The design of such an operation requires a great deal of knowledge of geology, rock mechanics, mine planning and capital investment. For this and various reasons, it is not so widely used as other methods, such as partial extraction, solid protection pillars or backfilling.

8.2 Structural Measures

8.2.1 Design Considerations for New Structures in Subsidence Prone Areas

Subsidence damage to structures will be minimized if precautions are taken during the design stage. Just as structures can be designed to be earthquake-resistant, design of subsidence-resistant structures is possible. In order to achieve this, several factors must be considered.

1. Foundations

Because the foundation is the connecting element between the superstructure and the ground surface, it is the most important factor in preventing ground deformation from being transferred to the superstructure. Based on the analysis of slippage between foundation and subgrade in Chapter 5, it is strongly recommended that a rigid, strong raft foundation should be used. This is because the stronger the foundation, the more
slippage will occur between foundation and subgrade and the less ground deformation will be transferred to the structure. The underside of the foundation should be as smooth as possible to encourage slippage, should horizontal ground strain occur. For the same reason, sand and polyethylene sheeting can be used between foundation and subgrade. Another reason for using a strong foundation is that it can bridge some of the differential settlement. For long structures, however, a one-piece foundation of any material may not be strong enough. In such cases, the foundation should be divided into discrete slabs, but each slab should meet the requirements recommended above.

2. Superstructures

Superstructures should be as flexible as possible for the simple reason that if ground deformations pass through the foundation and reach the superstructure, the latter should be flexible enough to accommodate the deformations with little structural damage. The use of brickwork, concrete, and building stone is strongly discouraged because these materials provide very little flexibility. Superstructures should also be light. In the case of a foundation cantilever due to differential settlement, less load will be imposed on the foundation, reducing the chance of foundation damages.

3. Structural configurations

Based on the analysis of Chapter 6, it is recommended that long structures should be avoided in subsidence prone area since more stress will be induced in such structure in the event of subsidence. Assuming the same conditions, a tall structure is better than a long structure. If a long structure must be built, it is preferable to provide gaps in the
walls and foundation, the interval between the gaps depending on the magnitude of the anticipated subsidence.

4. Structural materials

As discussed in Chapter 6, the resistance to ground deformations varies significantly for structures made of different materials. The problem associated with traditional design lies in the vertical supports, because they are mainly designed to resist compressive stress. Therefore, one can frequently see brick, concrete and stone as supporting columns. These materials can resist very high compressive stress, but are extremely vulnerable to tensile stress. The analysis in Chapter 6 noted that subsidence-induced tensile stress in columns is as large as the compressive stress. Hence, materials like wood or steel are more desirable than the three materials mentioned above, because they can resist both tensile and compressive stresses.

The CLASP system designed in Britain is a very good example of the application of these design principles. The system is illustrated in Figure 8.6 by Mavrolas and Schechtman (1982). The essential features of the system include:

- a wood or steel frame;

- relatively short distances between columns;

- foundation composed of concrete slabs, each about 10 by 12 feet;

- a layer of polyethylene sheeting between foundation and subgrade; and

- base made of a layer of sand or fill.
Diagrammatic Representation of the CLASP System

A. individual beam in the frame on which the structure rests;
B. column which is pin-connected to both the beam and concrete slab;
C. cross-brace with a compression spring;
D. load bearing steel plate that is anchored to E. a concrete slab section;
E. layer of polyethylene film; G. sand or fill;
and H. steel dowel that pin-connects the column to the concrete slab.

Figure 8.6. Illustration of the CLASP System (Mavrolas and Schechtman, 1982)
Steel bearing plates are anchored into each foundation slab to distribute load. Columns are pin-connected to the slab through a steel dowel or a threaded anchor bolt at the center of each slab, and to the beams in the frame. Braces that include a compressive spring are erected between columns. The CLASP system has been successfully used in Britain and throughout Europe (Mavrolas and Schechtman, 1982).

8.2.2 Remedial Measures

Remedial measures are those methods which can be used to reduce subsidence damages in existing structures. The principle of applying these measures is either to reduce the stress in a structure or to increase the strength of the structure. Some of the measures are discussed here.

1. Cutting a long structure into small units

As explained before, the stress induced by mining subsidence increases as a structure becomes longer. Existing long buildings can be cut into small units to avoid excessive stresses. The length of each individual unit depends on the magnitude of mining subsidence. A specified gap is left at the cut to accommodate the subsidence-induced deformation, usually about 2.4 in. (CUMT, 1981). The width of the gap can also be determined by the following formula:

\[ g_w = (e + \frac{H}{R}) \frac{l_1 + l_2}{2} \]  

(8.9)

for walls, and
\[ g_f = \varepsilon \frac{l_1 + l_2}{2} \]  \hspace{1cm} (8.10)

for foundations, where \( \varepsilon \) is the horizontal ground strain, \( R \) ground curvature, \( H \) the building height, and \( l_1 \) and \( l_2 \) are the length of the structural unit on each side of the gap. After a structure is cut, measures must be taken to ensure that each unit is stable.

2 Remedial trenching

A remedial trench can be excavated around an existing structure to absorb the horizontal ground strain. The axis of the trench should be perpendicular to the direction of horizontal deformation. The trench should be 6 to 8 inches deeper than the building foundation (CUMT, 1981), and it can be filled with compressible materials, and covered with concrete slabs. Trenching is more effective in coping with compressive stains than tensile ones. A significant reduction of structural damage has been reported with trenching (Peng, 1986).

3. Reinforcement of structures

Structures can be reinforced by bracing, banding and strapping. Steel rods are used to brace structures when ground extension may dislocate roof trusses or beams from their bearings. The strength of the rods is determined by (CUMT, 1981):

\[ \sigma = E \frac{H}{R} \]  \hspace{1cm} (8.11)

where \( H \) and \( R \) have the same meanings as in Equation 8.10, and \( E \) is the Young's modulus of the rods.
In many cases, I-beams of sufficient strength are used with hydraulic jacks to level a building, preventing it from suffering tilt and curvature. The I-beams are installed on the ceiling of the basement, in both the directions perpendicular and parallel to the mining direction (Peng, 1986). The I-beams are supported by hydraulic jacks. Settlement of the beams is monitored and any significant differential settlement is eliminated by promptly adjusting the jacks.

In general, the selection of appropriate measures depends on the nature of the structure, protection requirements and economic impact.
Chapter 9  Conclusions and Recommendations

9.1 Conclusions

A systematic review of mining-induced ground movements and building responses has been conducted. Several methods, including influence functions, profile functions, and the zone area, have been found successful when used in predicting surface deformations. These methods, however, are not designed to assess structural responses to such deformations. This problem has led to the development of a finite element model called SRMP (Subsidence Response Modeling Program). The program has the following features:

- It is based on large displacement, small strain;

- It incorporates four type of elements, namely continuous, beam, transition, and friction;

- It combines Total and Updated Langrangian formulation;

- It can predict both ground deformations and structural responses;
• It is capable of modeling foundation and subgrade interaction;

• Excavation processes can be modeled without reinitiating the boundary conditions, and the ground deformation and structural responses can be calculated at each excavation stage.

Several case studies were simulated with SRMP and the results showed that calculated and measured ground and structural deformations were in close agreement, within a acceptable accuracy limits.

Using the SRMP model, a comparison between subsidence profiles on the free ground surface and a structure-bearing surface indicated that the existence of a structure on the surface is not a significant factor to alter the subsidence profile. It is appropriate, therefore, to use free ground subsidence prediction methods for estimating the subsidence of structure-bearing surfaces.

Slippage between foundation and subgrade was analyzed using SRMP and the results are summarized as follows:

1. The slippage per unit length of foundation increases as the foundation length and the friction between the foundation and subgrade decrease;

2. Slippage increases as the foundation stiffness increases; and

3. Slippage increases as the horizontal ground strain becomes larger. The largest slippages occur at the location of maximum horizontal strain. There is very little slippage when the foundation is at the inflection point of the subsidence profile.
For structures subject to horizontal ground strain, a larger slippage is desired because less strain will be transferred to the superstructures. Thus, the practical implication of the above conclusions is that in designing structures in subsidence-prone areas, a short, stiff foundation with a smooth foundation-subgrade interface is preferred.

The study of structural responses indicated that for structures affected by mining subsidence, columns, like beams, must be able to withstand both compressive and tensile stresses. Wood and steel are much better column materials than concrete, brickwork, and stone.

In this study, a new building classification scheme was developed, based on foundation type, superstructure material, structural length and height. In addition, a set of subsidence indices has been proposed as critical in assessing the response of a building to ground movements. These indices include horizontal strain, deflection ratio, and curvature. On the basis of these results, damage criteria were established, relating subsidence indices limit values to building damage. This proposed scheme can be used for assessing damages and the impact of mining to the surface.

9.2 Recommendations for Future Research

It is recommended that additional research may be directed in the following areas:

- The SRMP model can be further enhanced by developing into a three-dimensional model. Instead of using beam elements for surface structures, plate elements should be used. Such a model will allow a more accurate representation of the problem, although it will be more costly to operate.
• In generally, the subsidence profiles obtained from a finite element model are flatter than the measured ones. The zone classifications proposed by Agioutantis (1987) allow for better simulation, but the subsidence curve obtained by using the material zones exhibits sudden slope change at the intersection of two material zones. Future research should investigate this problem further, probably by considering material properties as a function of location.

• Time effects should be included in subsidence analysis. This can also be incorporated into the finite element model. Knowledge of time effects will be useful in several areas, such as the planning of rapid mining, predicting subsidence over abandoned mines, etc.

• Development of damage criteria is a continuous process, and requires considerable practical information. Collection of detailed case studies and documentation must be continued.
References


Kuppusamy, T., \textit{FE2D, a Finite Element Program}, Department of Civil Engineering, VPI \& SU, 1986.


Appendix A. Subsidence Response Modeling Program

SRMP (SUBSIDENCE RESPONSE MODELING PROGRAM) IS A 2-D, PLANE
STAIN, LARGE-DISPLACEMENT, SMALL-STRAIN, MATERIALLY NONLINEAR
FINITE ELEMENT PROGRAM.(8)

LIST OF VARIABLE NAMES IN MAIN PROGRAM:
NYP - NUMBER OF NODAL POINTS
NOEL - NUMBER OF ELEMENTS
LAST - LAST NODE OF PLANE ELEMENTS
NETEL - LAST PLANE ELEMENT
NEBEL - FIRST BEAM ELEMENT
NMATR - NUMBER OF MATERIALS
NTRAC - NUMBER OF SURFACE TRACTIONS
MBDY - BODY FORCE INDICATOR
E - YOUNG'S MODULUS
PAR - POISSON RATIO
RG - DENSITY
SRTA - FRICTION ANGLE
CO - COHESION COEFFICIENT
THI - THICKNESS OF ELEMENT
YNG - YOUNG'S MODULUS FOR GOB ELEMENT
POS - POISSON RATIO FOR GOB ELEMENT
GAM - AVERAGE DENSITY FOR INITIAL STRESS CALCULATION
FKN - NORMAL STIFFNESS FOR FRICITION ELEMENT
FKS - SHEAR STIFFNESS FOR FRICTION ELEMENT
WIDTH - WIDTH OF THE BEAM ELEMENT
IND - CONNECTIVITY AND MATERIAL NUMBER
X - GLOBAL COORDINATE
Y - GLOBAL COORDINATE
ALX - KNOWN DISPLACEMENT IF KSTRG = 1, 3
CONCENTRATED LOAD IF KSTRG = 0, 2
ALY - KNOWN DISPLACEMENT IF KSTRG = 2, 3
CONCENTRATED LOAD IF KSTRG = 0, 1
ALZ - KNOWN ROTATION IF KROT = 1
KNOWN MOMENT IF KROT = 0
KSTRG - BOUNDARY CONDITION AND CONCENTRATED LOAD INDEX
KROT - KNOWN ROTATION OR MOMENT INDEX
ITYP - PROBLEM TYPE INDEX, EQUAL TO 1 IF SURFACE STRUCTURE IS INCLUDE AND EQUAL TO 0 OTHERWISE
JTRN - NODAL NUMBER OF NODE 1 ON THE SIDE ON WHICH THE TRACTION IS APPLIED
JTRN - NODAL NUMBER OF NODE 2 ON THE SIDE ON WHICH THE TRACTION IS APPLIED
TRAX - X-TRACTION AT NODAL POINT
TRAY - Y-TRACTION AT NODAL POINT
IBWTH - HALF WIDTH OF THE MATIX BAND
MEQ - NUMBER OF D.O.F. OR EQUATIONS
AAR - LOAD VECTOR BEFORE SOLVING THE EQUATIONS AND
DISPLACEMENT INCREMENT VECTOR AFTERWARD
SIG - 2ND PIOLA-KIRCHHOFF STRESS MATRIX
ASTF - GLOBAL STIFFNESS MATRIX
FLAG - ELEMENT YIELDING INDICATOR
UT1 - TOTAL DISPLACEMENT AT X-DIRECTION
UT2 - TOTAL DISPLACEMENT AT Y-DIRECTION
VS2 - DIRECTION COSINE OF LOCAL COORDINATE S FOR TRANS- SION ELEMENT
VSB2 - DIRECTION COSINE FOR BEAM ELEMENT
XT - GLOBAL COORDINATE AT TIME T
VT1 - SAME AS UT1, USED FOR STIFFNESS CALCULATION
VT2 - SAME AS UT2, USED FOR STIFFNESS CALCULATION
SIGG - SAME AS SIG, USED FOR STIFFNESS CALCULATION
VS1, VSB1, XT1 - SAME AS VS2, VSB2, AND XT, USED TO STIFFNESS CALCULATION
Z1, Z2 - STRESS FOR BEAM ELEMENTS
STRN - TOTAL STRAIN
IM - PSEUDO-ELEMENT NUMBER
IST - FIRST PSEUDO-ELEMENT
IEND - FIRST ELEMENT NUMBER AFTER PSEUDO-ELEMENT
MST - FIRST FRICTION ELEMENT
MND - LAST FRICTION ELEMENT
MAXNP - MAXIMUM NUMBER OF PLANE POINTS
MAXBP - MAXIMUM NUMBER OF BEAM POINTS
MAXMAT - MAXIMUM NUMBER OF MATERIALS
MAXEL - MAXIMUM NUMBER OF ELEMENT
MAXBL - MAXIMUM NUMBER OF BEAM AND TRANSITION ELEMENTS
MAXBW - MAXIMUM BAND WIDTH
MAXTR - MAXIMUM NUMBER OF TRACTIONS
NAME - NAME OF THE PROBLEM
ITRUN - ERROR INDICATOR

COMMON NOPL, NOEL, NMATR, NTRAC, MBDY, E(NP), PAR(20), RO(20)
1, IND(480,5), X(600), Y(600), ALX(600), ALY(600), KSTRG(600), ITRN(20),
2, ITRN(20), TRAX(20, 2), TRAY(20, 2), TH(20), SETA(20), CO(20), NETEL, NEBEL,
3, LAST, WIDTH(20), KROT(100), ALZ(100), ITYP, YNG, POS

COMMON ONE, VT1(600),
1VT2(600), SIGG(480,4), VS1(100,2), VSB1(50,3,2), XT1(600,2)

COMMON/TWO: IBWTH, MEQ, AAR(1300), ASTF(1300,90), SIG(480,4), FLAG(450),
1UT1(600), UT2(600), UT3(100), STRN(480,3), VS2(100,2), VSB2(50,3,2),
2XT(600,2)

COMMON/THREE: IM, IST

COMMON/FOUR: FKN, FKS, MST, MND

COMMON/FIVE: Z1(50,3,2,4), Z2(50,3,2,4)

CHARACTER NAME*72

DATA MAXEL, MAXBL, MAXNP, MAXBP, MAXMAT, MAXBW, MAXTR, 440, 40, 500, 100, 20, 190, 20/

C

PROBLEM IDENTIFICATION

241
READ (5,100) IPRNO, NAME
IF (IPRNO .LE. 0) GO TO 45
WRITE(6,200)
WRITE(6,300) IPRNO, NAME
WRITE(6,200)

C IDENTIFY THE ELEMENTS TO BE MINED, THE CONVERGENCE CRITERION, ETC.

C READ (5,900) IST, JEND, TEST, ITYP, ISF, YNG, POS, GAM
WRITE(6,1000) IST, JEND - 1, TEST, ITYP, ISF, YNG, POS, GAM
READ (5,*) FKN, FKS, MST, MND
WRITE (6,1200) FKN, FKS, MST, MND
IM = IST

C READ INPUT DATA
CALL READ(VS2, VSB2,
1 MAXEL, MAXBL, MAXNP, MAXBP, MAXMAT, MAXTR, ITRUN)

C INITIATE FLAG, STRESS, AND DISPLACEMENT

DO 10 I = 1, NETEL
  FLAG(I) = 0
10    DO 11 J = 1, 3
      DO 11 K = 1, 2
          DO 11 L = 1, 4
              JIT = NEBEL + I + J + K + L
              Z1(JIT) = 0.0
              Z2(JIT) = 0.0
11    DO 12 I = 1, NOP
       VTI = 0.0
       VT2 = 0.0
       UTI = 0.0
       UT2 = 0.0
       XT(I,1) = X(I)
       XT(I,2) = Y(I)
       XT(I,1) = X(I)
       XT(I,2) = Y(I)
12    DO 13 J = 1, NETEL
        SIGJ = (GAM + Y(LAST) - Y(IND(I,4)))
        SIGJ = 0.3 * SIGJ
        SIG(I,J) = SIGJ
        DO 13 J = 1, 4
        SIGG(J) = SIG(J)
13    DO 14 J = 1, 4
        SIGG(J) = 0.0
14    SIGG(J) = 0.0
C IF (ITYP .EQ. 0) GO TO 17
C INITIATE DIRECTION COSINE AND ROTATION

DO 15 I = 1, NOP - LAST
  UT3(I) = 0.0
15    DO 15 J = 1, 2
        VS1(I,J) = VS2(I,J)
C DO 16 I = NEBEL, NOEL


IF (CRIT.LT.TEST) THEN
   PRINT OUT DISPLACEMENT AND STRESS
   WRITE(6,209)
   WRITE(6,600) (I,UT1(I),UT2(I),I = ISF,NOP)
   WRITE(6,600) 54,UT1(64),UT2(64)
   WRITE(6,600) 43,UT1(43),UT2(43)
   WRITE(6,200)
   IF (ITYP.EQ.0) GO TO 35
   WRITE(6,200)
   IF (ITYP.EQ.0) WRITE(6,700) (I,SIG(I,J),J = 1,4,I = NEBEL,NOEL)
   WRITE(6,200)
   INITIATE DISPLACEMENT, GLOBAL COORDINATE, AND DIRECTION CONSONES
   FOR CALCULATING STIFFNESS AT NEXT TIME INCREMENT
   DO 36 I = 1,NEBEL-1
      DO 36 J = 1,4
      SIG(I,J) = SIG(I,J)
   DO 37 I = 1,NOP
      VT1(I) = UT1(I)
      VT2(I) = UT2(I)
   DO 37 J = 1,2
      XT1(I,J) = XT(I,J)
   IF (ITYP.EQ.0) GO TO 41
   DO 38 I = LAST+1,NOP
      DO 38 J = 1,2
      VS1(I-LAST,J) = VS2(I-LAST,J)
   DO 39 I = NEBEL,NOEL
      DO 39 J = 1,3
      DO 39 K = 1,2
      VSB1(I-NEBEL+1,J,K) = VSB2(I-NEBEL+1,J,K)
   DO 50 I = NEBEL,NOEL
      DO 50 J = 1,3
      DO 50 K = 1,2
      DO 50 L = 1,4
         Z2(I-NEBEL+1,J,K,L) = Z1(I-NEBEL+1,J,K,L)
   MINING NEXT BLOCK
   IM = IM + 1
   IF (IM.EQ.IEND) THEN
      GO TO 5
   END IF
   DO 42 J = 1,4
      SIG(IM,J) = 0.0
   GO TO 25
   ELSE
      GO TO 25
   END IF
   WRITE(6,800) IBWTH, MAXBw
   GO TO 5

   244
STOP
C
100  FORMAT (15,3X,A)
200  FORMAT (C'********************************************************************
1'********************************************************************)
300  FORMAT (C'PROBLEM NO.',15,3X,A)
400  FORMAT (C'YIELDING CHECK TABLE')
500  FORMAT (5(2X,15,3X,F.3,1))
600  FORMAT (3(3H OUTPUT NODE DISPLACEMENTS//13X.4H NODE,9X,
113HUT1 = X-DISP.,9X,13HUT2 = Y-DISP.,(5X,112,2,2E22.8))
700  FORMAT (15,4E12.3)
800  FORMAT (C'/12H BANDWIDTH = .4,25H EXCEEDS MAX. ALLOWABLE = .14//30H
1 GO ON TO NEXT PROBLEM )
900  FORMAT (215,E10.1,215:3E10.3)
1000 FORMAT (5X,'MINING STARTS AT ELEMENT',15
+ /5X,'MINING ENDS AT ELEMENT',
115.5X,'TEST CRITERION',15.3/5X,'PROBLEM TYPE',15
+ /5X,'FIRST SURFACE POINT',
215.5X,'YOUNG'S MODULUS FOR GOB ELEMENT',15.3.5X,
+ /5X,'POISSON RATIO FOR GOB ELEMENT',15.3.5X,
3'AVERAGE MATERIAL DENSITY FOR CALCULATION OF INITIAL STRESS',
4E15.3)
1100 FORMAT (C',E,N0. SIG(X) SIG(Y) SIG(X-Y) SIG(Z))
1200 FORMAT(5X,'NORMAL STIFFNESS FOR FRICTION ELEMENT',15/1
15X,'SHEAR STIFFNESS FOR FRICTION ELEMENT',15/2
25X,'FIRST FRICTION ELEMENT',15/
35X,'LAST FRICTION ELEMENT',15)
C
END
C
SUBROUTINE READ(VS2,VS2B,
1MAXEL,MAXBL,MAXNP,MAXBP,MAXMAT,MAXTR,ITRUN)

******************************************************************************
C ** SUBROUTINE TO READ INPUT DATA                                              
C ** ALL THE VARIABLES ARE EXPLAINED IN THE MAIN PROGRAM AND                
C ** OTHER SUBROUTINES                                                        
C******************************************************************************
C
COMMON NOP,NOEL,NMATR,NTRAC,MBDY,E(20),PAR(20),ROI(20),
1IND(480,5),X(600),Y(600),ALX(600),ALY(600),KSTRG(600),ITRN(20),JTR
2N(20),TRAX(20,2),TRAY(20,2),TH(20),SETA(20),CO(20),NETEL,NEBEL,
3LAST,WIDTH(20),KROT(100),ALZ(100),ITYP,YNG,POS
C
DIMENSION VS2(100,2),VS2B(50,3,2)
C
ITRUN = 0
READ (5,110) NOP,NOEL,NMATR,NTRAC,MBDY,LAST,NETEL,NEBEL
C
WRITE(6,115) NOP,NOEL,NMATR,NTRAC,MBDY,LAST,NETEL,NEBEL
C
CHECKS TO BE SURE INPUT DATA DOES NOT EXCEED STORAGE CAPACITY
C
IF (LAST.GT.MAXNP) THEN
  ITRUN = ITRUN + 1
  WRITE(6,120) MAXNP
ELSE IF (NOP.LAST .GT. MAXBP) THEN
  ITRUN = ITRUN + 1
  WRITE(6,120) MAXBP
ELSE IF (NETEL.GT.MAXEL) THEN
  ITRUN = ITRUN + 1
  WRITE(6,125) MAXEL
ELSE IF (NOEL.NETEL .GT. MAXBL) THEN

245
ITRUN = ITRUN + 1
WRITE(6,125) MAXBL
ELSE IF (NMATR.GT.MAXMAT) THEN
  ITRUN = ITRUN + 1
WRITE (6,130) MAXMAT
ELSE IF (NTRAC.GT.MAXTR) THEN
  ITRUN = ITRUN + 1
WRITE (6,135) MAXTR
ELSE IF (ITRUN.NE.0) THEN
  WRITE(6,140) ITRUN
STOP
ELSE
  GO TO 25
ENDIF
C
C READ AND PRINT MATERIAL PROPERTIES
C
25 READ(5,145) (E(I),PAR(I),RO(I),TH(I),I=1,NMATR)
WRITE(6,150)
WRITE(6,155) (I,E(I),PAR(I),RO(I),TH(I),I=1,NMATR)
READ(5,156) (CO(I),SETA(I),WIDTH(I),I=1,NMATR)
WRITE(6,220)
WRITE(6,225) (I,CO(I),SETA(I),WIDTH(I),I=1,NMATR)
C
C READ AND PRINT NODAL COORDINATES AND BOUNDARY CONDITIONS
C
WRITE(6,160)
N = 1
30 READ (5,165) M,KSTRG(M),X(M),Y(M),ALX(M),ALY(M)
   IF (M-N) 35,50,40
35 WRITE(6,170) M
WRITE(6,175) M,KSTRG(M),X(M),Y(M),ALX(M),ALY(M)
ITRUN = ITRUN + 1
GO TO 30
40 DF = M + 1 - N
   RX = (X(M)-X(N-1)) DF
   RY = (Y(M)-Y(N-1)) DF
45 KSTRG(N) = 0
   X(N) = X(N-1) + RX
   Y(N) = Y(N-1) + RY
   ALX(N) = 0.0
   ALY(N) = 0.0
50 WRITE(6,175) N,KSTRG(N),X(N),Y(N),ALX(N),ALY(N)
   N = N + 1
   IF (M-N) 55,50,45
55 IF (N.LE.NOP) GO TO 30
C
IF (ITYP .LE. 0) GO TO 56
C
WRITE(6,235)
N = LAST + 1
5 READ(5,*) M,KROT(M-LAST),ALZ(M-LAST)
   IF (M-N) 6,8,7
6 WRITE(5,170) M
WRITE(6,240) M,KROT(M-LAST),ALZ(M-LAST)
ITRUN = ITRUN + 1
7 KROT(N-LAST) = KROT(N-LAST-1)
   ALZ(N-LAST) = 0.0
8 WRITE(6,240) N,KROT(N-LAST),ALZ(N-LAST)
   N = N + 1
   IF (M-N) 9,8,7
9 IF (N .LE. NOP) GO TO 5
C
READ AND PRINT CONNECTIVITY AND MATERIAL IDENTIFICATION

WRITE(6, 180)
L = 0
READ (5,190) M,(IND(M,I),I = 1,5)
L = L + 1
IF (M-L) 70,80,75
WRITE(6,185) M
WRITE(6,195) M,(IND(M,I),I = 1,5)
ITRUN = ITRUN + 1
GO TO 60

IF (L .LE. NETEL) THEN
IND(L,1) = IND(L-1,1) + 1
IND(L,2) = IND(L-1,2) + 1
IND(L,3) = IND(L-1,3) + 1
IND(L,4) = IND(L-1,4) + 1
IND(L,5) = IND(L-1,5)
ELSE
IND(L,1) = IND(L-1,1) + 2
IND(L,2) = IND(L-1,2) + 2
IND(L,3) = IND(L-1,3) + 2
IND(L,4) = IND(L-1,4)
IND(L,5) = IND(L-1,5)
ENDIF
WRITE(6,195) L,(IND(L,I),I = 1,5)
IF (M-L) $5,85,65
IF (NOEL-L) 90,90,60
CONTINUE

IF (ITYP .EQ. 0) GO TO 92

CALCULATE DIRECTION COSINE

DO 91 J = NEBEL, NOEL
DEX = X(IND(L,2))X(X(IND(L,1)))
DEY = Y(IND(L,2))Y(Y(IND(L,1)))
DEL = SQRT(DEX**2 + DEY**2)
DO 91 J = 1,3
VSI(IND(I,J)-LAST,1) = -DEY,DEL
VSI(IND(I,J)-LAST,2) = DEX,DEL
VSB2(I-NEBEL + J,1) = -DEY,DEL
VSB2(I-NEBEL + J,2) = DEX,DEL
CONTINUE

READ AND PRINT SURFACE LOADING(TRACTION) DATA

IF (NTRAC.EQ.0) GO TO 100
WRITE(6,200)
DO 95 L = 1,NTRAC
READ(5,205) ITRN(L),JTRN(L),TRAX(L,1),TRAX(L,2),TRAY(L,1),
+TRAY(L,2)
WRITE(6,210) ITRN(L),JTRN(L),TRAX(L,1),TRAX(L,2),TRAY(L,1),
+TRAY(L,2)

CONTINUE

IF (ITRUN.EQ.0) GO TO 230
WRITE(6,215) ITRUN

FORMAT (815)
FORMAT ("*/ INPUT PROBLEM PARAMETERS */5X; TOTAL NO OF
1 NODES.............;15; 5X; TOTAL NO OF ELEMENTS..........;
2..................;15; 5X; TOTAL NO OF DIFF. MATERIALS.......;15/
35X; TOTAL NO OF TRACTION DATA CARDS........;15; 5X; BODY
4 FORCES(1 = IN-Y DIREC., 0 = NONE);15; 5X; LAST PLANE NODE

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5...15/5X, LAST PLANE ELEMENT...15/5X, FIRST BEAM
6ELEMENT...15)
120 FORMAT(3/3H33H) EXCEEDS NODAL POINTS, MAXIMUM = .15)
123 FORMAT(3/3H30H) EXCEEDS ELEMENTS, MAXIMUM = .15)
130 FORMAT(3/3H30H) EXCEEDS MATERIALS, MAXIMUM = .15)
135 FORMAT(3/3H40H) EXCEEDS SURFACE LOAD CARDS, MAXIMUM = .15)
140 FORMAT(3/3H28H) EXECUTION STOP BECAUSE OF.15,13H LARGE ERRORS/) .
145 FORMAT(3/3E10.3)
150 FORMAT(3/3H INPUT MATERIAL PROPERTIES/10H .5X.
11HMODEL,LS OF 6X,9HPoisson 5,7X,8HMATERIAL,7X,8HMATERIAL/4X,6HNUMB
2ER,5X,10HELASTICITY,8X,7H RATIO,8X,7HDENSITY,8X,7HTHICKNESS)
155 FORMAT(3/3H4E15.4
156 FORMAT(3/3E10.3)
160 FORMAT(3/3H INPUT NODAL POINT DATA //5X,5H,5NODAL,48X,7HX.
1Disp,8X,7H-DISP.,//5X,5HNUMBR,6X,4HTYPE,14X,1H,4X,1H,1HY,5X,7HOR LO.
2AD,8X,7HOR LOAD)
165 FORMAT(3/3H4E10.3
170 FORMAT(3/3H41H ERROR IN DATA NO.,15 )
175 FORMAT(3/3H41H,4E15.4)
180 FORMAT(3/3H INPUT ELEMENT DATA //31X,31HCONNECTIVITY
1 OF ELEMENT NODES,3X,7FHELEMENT,7X,111L,7X,1112,7X,1113,7X,1114,2X,8H
2MATERIAL)
185 FORMAT(3/3H21H ERROR IN ELEMENT DATA NO.,15.)
190 FORMAT(3/3E10.3
195 FORMAT(3/3H41H)
200 FORMAT(3/3H37H INPUT SURFACE LOADING DATA//17X,33HSURFACE L
1OAD INTENSITIES AT NODES/4X,6HNODE 1,4X,6HNODE J,10X,2IH,10X,2HJ
2,10X,2HY,10X,2HYJ)
205 FORMAT(3/3H4E10.3
210 FORMAT(3/3H41H,4E12.4
215 FORMAT(3/3H44H SOLUTION WILL NOT PERFORMED.,15,21H
1 FATAL CARD ERRORS )
220 FORMAT (3/3H41H,4X,'NUMBER',5X,'COHESION FACTOR',5X,'FRICTION ANGLE',
15X,'BEAM WIDTH')
225 FORMAT(3/3H41H,4X,15X,9X,10X,3X,5X,E10.3,5X,E10.3)
230 FORMAT(3/3H41H,4X,15X,9X,10X,3X,5X,E10.3)
235 FORMAT(3/3H41H,4X,15X,9X,10X,3X,5X,E10.3)
240 FORMAT(3/3H41H,4X,15X,9X,10X,3X,5X,E10.3)
C
230 RETURN
END
SUBROUTINE GSTIF(ITRUN)
C
* SUBROUTINE TO OBTAIN GLOBAL STIFFNESS MATRIX BY ASSEMBLING *
* ELEMENT STIFFNESS MATRICES AND TO IMPOSE BOUNDARY CONDITIONS *
C
COMMON NOP,NOEL,NMATR,NTRAC,MBDY,E(20),PAR(20),RO(20)
1.IND(480,3),X(600),Y(600),ALX(600),ALY(600),KSTRG(600),ITRN(20).
2ITRN(20),TRAX(20),TRAY(20),TH(20),SETA(20),CO(20),NETE,NEBE,
3,LAST,WIDTH(20),KROT(100),ALZ(100),ITYP,YNG,POS
C
COMMON,ONE,VT1(600),
1VT2(600),SIGG(480,4),VS1(100,2),VS2(50,3,2),XT1(600,2)
C
COMMON,TO,IBWTH,MEQ,AA,ASTF(1300),SIG(480,4),FLAG(450),
1UT1(600),UT2(600),UT3(100),STRF(480,3),VS2(100,2),VS2(50,3,2),
2XT(600,2)
C
COMMON,FOUR,FKN,FKS,MST,MND
C
DIMENSION LM(8),ESTIF(8,8),ESTIFT(7,7),ESTIFB(9,9)
C
248
INITIATE GLOBAL STIFFNESS MATRIX

ITRUN = 0
DO 5 I = 1, MEQ
  DO J = 1, IBWTH
    ASTF(I,J) = 0.0
  END DO
END DO 5

CALCULATE PLANE ELEMENT STIFFNESS MATRIX

DO 40 M = 1, NETEL
  IF (IND(M,5) .GE. 0) GO TO 10
  ITRUN = ITRUN + 1
  CALL STIF(M, ESTIF)
END DO 40

ASSEMBLE PLANE-ELEMENT STIFFNESS MATRIX

DO 15 I = 1, 4
  LMI(I) = 2*IND(M,I)-2
  DO 35 J = 1, 4
    DO K = 1, 2
      II = LMI(I)+K
      KK = 2*II+K
      DO 35 LL = 1, 2
        JJ = LMI(J)+L-ll+1
        LL = 2*J-2+L
        IF (JJ) 35, 30
        IF (IBWTH-JJ) 25, 30
        WRITE(6,1000) M
        ITRUN = ITRUN + 1
        GO TO 40
      END DO 35
      IF (JJ) 35, 30
      30 CONTINUE
      35 CONTINUE
      40 CONTINUE

IF (ITYP .EQ. 0) GO TO 131

CALCULATE TRANSITION ELEMENT STIFFNESS MATRIX

DO 95 M = NETEL + 1, NEBEL-1
  IF (IND(M,4) .GT. 0) GO TO 45
  ITRUN = ITRUN + 1
  GO TO 95
END DO 95

CALL STIFT(M, ESTIFT)

ASSEMBLE TRANSITION ELEMENT STIFF MATRIX

DO 50 I = 1, 2
  LMI(I) = 2*IND(M,I)-2
  DO 70 K = 1, 2
    II = LMI(I)+K
    KK = 2*II+K
    DO 70 J = 1, 2
      DO 70 L = 1, 2
        JJ = LMI(J)+L-II+1
        LL = 2*J-2+L
        IF (JJ) 70, 55
        IF (IBWTH-JJ) 60, 55
        WRITE(6,1000) M
        ITRUN = ITRUN + 1
        GO TO 95
      END DO 70
      IF (JJ) 70, 55
      55 CONTINUE
      50 CONTINUE
      60 CONTINUE

249
ASTF(I,J,J) = ASTF(I,J,J) + ESTIFT(KK,LL)
CONTINUE
LM3 = 3*IND(M,3) - LAST - 3
DO 90 K = 1,3
II = LM3 + K
KK = 4 + K
DO 90 L = 1,3
JJ = LM3 + L - II + 1
LL = 4 + L
IF (JJ) 90,90,75
75 IF (IBWT - JJ) 80,85,85
80 WRITE(6,1000) M
ITRUN = ITRUN + 1
GO TO 95
85 ASTF(I,J,J) = ASTF(I,J,J) + ESTIFT(KK,LL)
CONTINUE
CONTINUE
C
C CALCULATE BEAM ELEMENT STIFFNESS MATRIX
C
DO 130 M = NEBEL, NOEL
IF (IND(M,1), GT, 0) GO TO 100
ITRUN = ITRUN + 1
GO TO 130
100 CALL STIFB(M, ESTIFB)
C
C ASSEMBLE PLANE-ELEMENT STIFFNESS MATRIX
C
DO 105 L = 1,3
105 LM(J) = 3*IND(M,1) - LAST - 3
DO 125 K = 1,3
DO 125 L = 1,3
II = LM(J) + K
KK = 3*L - 3 + K
DO 125 J = 1,3
DO 125 L = 1,3
JJ = LM(J) + L - II + 1
LL = 3*L - 3 + L
IF (JJ) 125, 125, 110
110 IF (IBWT - JJ) 115, 120, 120
115 WRITE(6,1000) M
ITRUN = ITRUN + 1
GO TO 130
120 ASTF(I,J,J) = ASTF(I,J,J) + ESTIFB(KK,LL)
CONTINUE
CONTINUE
CONTINUE
131 IF (ITRUN.EQ.0) GO TO 135
WRITE(6,2000) ITRUN
RETURN
C
1000 FORMAT(5X, 'BAND WIDTH EXCEED ALLOWABLE FOR ELEMENT ', I5, ')
2000 FORMAT('/// SOLUTION WILL NOT BE PERFORMED BECAUSE OF ', I5, 'DATA ERRORS')
END
C
SUBROUTINE STIF(M, ESTIF)
C
* FORMATION OF ELEMENT STIFFNESS MATRIX
* RR, SS - NATURAL COORDINATES R AND S AT GAUSSIAN
* INTEGRATION POINTS
* W1, W2 - WEIGHTING FACTORS OF GAUSSIAN INTEGRATION

250
COMMON N0P, NOEL, NMATR, NTACR, MBDY, E(20), PAR(20), RO(20),
1, IND(480, 5), X(600), Y(600), ALX(600), ALY(600), KSTRG(600), ITRN(20),
2JTRN(20), TRX(20, 2), TRY(20, 2), TII(20), SETA(20), CO(20), NETEL, NEBEL,
3LAST, WIDTH(20), KROT(100), ALZ(100), ITYP, YNG, POS

COMMON TWO, IBWT1, MEQ, AAR(1300), ASTF(1300, 40), SIG(480, 4), FLAG(450),
1UT1(600), UT2(600), UT3(100), STRN(480, 3), VS2(100, 2), YSB2(50, 3, 2),
2X(600, 2)

COMMON ONE VTI(600),
1VT2(600), SIGG(480, 4), VS1(100, 2), YSB1(50, 3, 2), XTi(600, 2)

COMMON/THREE, IM, IST

COMMON/FOUR, FKN, FKS, MST, MND

DIMENSION RR(4), SS(4), W1(4), W2(4), XX(2, 4), UT(2, 4), D(2, 2), S(4, 4),
1ESTIF(8, 8), BL(3, 8), BN(4, 8), C(3, 3)

COMMON INITIATE COORDINATES AND WEIGHTING FACTORS FOR EIGHT POINT
COMMON GAUSSIAN INTEGRATION

DATA RR, 0.8611363115, 0.3399810435, 0.3399810435, 0.8611363115,
1SS, 0.3399810435, 0.3399810435, 0.8611363115,
2W, 0.3478548451, 0.6521451548, 0.6521451548, 0.3478548451,
3W2, 0.3478548451, 0.6521451548, 0.6521451548, 0.3478548451

IF FRICTION ELEMENT, GO TO 100

IF (M .GE. MST .AND. M .LE. MND) GO TO 100

INITIATE 2ND PIOLA-KIRCHHOFF STRESS MATRIX

DO 15 J = 1, 4
DO 15 J = 1, 4
15 S(1, J) = 0.0
S(1, 1) = SIGG(M, 1)
S(2, 2) = SIGG(M, 2)
S(1, 2) = SIGG(M, 3)
S(2, 1) = S(1, 2)
S(3, 3) = S(1, 1)
S(4, 4) = S(2, 2)
S(3, 4) = S(1, 2)
S(4, 3) = S(2, 1)

Determine element type and use proper material constants

IF (M .GE. IST .AND. M .LE. IM) THEN
    BULK = YNG(2. *(1. + POS) *(1. - 2.*POS))
    SHEAM = YNG(2.* (1. + POS))
ELSE
MATR = IND(M,5)
PRT = PAR(MATR)
BULK = E(MATR)/(2.*(1. + PRT)*(1.-2.*PRT))
SHEAM = E(MATR)/(2.* (1. + PRT))
END IF

C INITIATE LINEAR C MATRIX FOR PLANE STRAIN PROBLEM
C
5 IF (FLAG(M)) 5,5,10
C(1,1) = BULK + SHEAM
C(1,2) = BULK - SHEAM
C(1,3) = 0.0
C(2,1) = C(1,2)
C(2,2) = C(1,1)
C(2,3) = 0.0
C(3,1) = C(1,3)
C(3,2) = C(2,3)
C(3,3) = SHEAM

C ASSIGN ELEMENT COORDINATES AND DISPLACEMENTS TO XX AND UT
DO 25 I = 1, 4
XX(I,1) = X(IND(M,1))
XX(I,2) = Y(IND(M,1))
UT(I,1) = VT1(IND(M,1))
UT(I,2) = VT2(IND(M,1))
25 CONTINUE

C CALCULATE ELEMENT STIFFNESS
DO 30 J = 1, 8
DO 30 J = 1, 8
30 ESTIF(I,J) = 0.0
DO 90 LX = 1, 4
R1 = R(R(LX))
DO 80 LY = 1, 4
S1 = S(S(LY))
90 CONTINUE

C EVALUATE DERIVATIVE MATRIX BS AND THE JACOBIAN DETERMINANT DET
CALL BMATX(XX, UT, R1, S1, BL, BNL, DET, M)

C CARRY OUT [BT][C][B] OPERATION
WT = W1(LX)*W2(LY)*TH(MATR)*DET

C CALCULATE LINEAR PART OF MATRIX [K]
CALL INTGRA(WT, 3, BL, C, ESTIF, 8)

C CALCULATE NONLINEAR PART OF [K] AND ADD TO THE LINEAR PART
CALL INTGRA(WT, 4, BNL, S, ESTIF, 8)
80 CONTINUE
90 CONTINUE
RETURN

DO 105 I = 1, 4
XX(I,1) = XT1(IND(M,J),1)
105 XX(I,2) = XT1(IND(M,J),2)
C FFKN = (M-MST+1)*FKN
C FFKS = (M-MST+1)*FKS
CALL FSTIF(FKN,FKS,XX,ESTIF)

DO 110 I = 1,8
DO 110 J = 1,8
110   ESTIF(I,J) = TH(IND(M,5))*ESTIF(I,J)

RETURN
END

SUBROUTINE BMATX(XX, UT, R, S, BL, BNL, DET, M)

REAL XX(2,4), UT(2,4), BL(3,8), BNL(4,8), HI(4), P(2,4), HI(2,4),
+ X(J1,2), X(J1,2), BLO(3,8), BL(3,8), BL(2,2)

RP = 1.0 + R
SP = 1.0 + S
RM = 1.0 - R
SM = 1.0 - S

INTERPOLATION FUNCTIONS

H(1) = 0.25*RM*SM
H(2) = 0.25*RP*SM
H(3) = 0.25*RP*SP
H(4) = 0.25*RM*SP

DERIVATIVES OF INTERPOLATION FUNCTIONS WITH RESPECT TO NATURAL
COORDINATE

P(1,1) = -0.25*SM
P(1,2) = -P(1,1)
P(1,3) = 0.25*SP
P(1,4) = -P(1,3)

P(2,1) = -0.25*RM
P(2,2) = -0.25*RP
P(2,3) = -P(2,2)
P(2,4) = -P(2,1)

EVALUATE THE JACOBIAN MATRIX AT POINT (R,S)

DO 20 I = 1,2
DO 20 J = 1,2
DUM = 0.0

253
DO 10 K = 1, 4
   DUM = DUM + P(I,K)*XX(J,K)
   XJI(J,J) = DUM
C
COMPUTE THE DETERMINANT OF THE JACOBIAN MATRIX
C
   DET = XJI(1,1)*XJI(2,2) - XJI(2,1)*XJI(1,2)
IF (DET .GT. 0.000000001) GO TO 30
WRITE(6,200) M
STOP
C
CALCULATE THE INVERSE OF THE JACOBIAN MATRIX
C
30   DUM = 1./DET
   XJI(1,1) = XJI(2,2)*DUM
   XJI(1,2) = -XJI(1,2)*DUM
   XJI(2,1) = -XJI(2,1)*DUM
   XJI(2,2) = XJI(1,1)*DUM
C
COMPUTE DERIVATIVES OF INTERPOLATION FUNCTIONS WITH RESPECT TO
GLOBAL COORDINATEX AND Y
C
DO 40 I = 1, 4
   HH(I,1) = 0.0
   HH(I,2) = 0.0
   DO 50 J = 1, 2
      HH(I,1) = HH(I,1) + XJI(I,J)*P(J,1)
      HH(I,2) = HH(I,2) + XJI(I,J)*P(J,2)
50   CONTINUE
40   CONTINUE
C
COMPUTE L'S USED IN [BL1] MATRIX
C
DO 60 I = 1, 2
   DO 60 J = 1, 2
      BLL(I,J) = 0.0
   DO 60 K = 1, 4
      BLL(I,J) = BLL(I,J) + HH(I,K)*UT(I,K)
60   CONTINUE
C
INITIATE [BL0] AND [BL1]
C
DO 70 I = 1, 3
   DO 70 J = 1, 8
      BL0(I,J) = 0.0
70   CONTINUE
C
DO 80 I = 1, 4
   BL0(1,2*I-1) = HH(I,1)
   BL0(2,2*I) = HH(I,1)
   BL0(3,2*I-1) = HH(I,2)
   BL0(3,2*I) = HH(I,1)
80   CONTINUE
C
DO 90 I = 1, 4
   BL1(1,2*I-1) = BLL(1,1)*HH(I,J)
   BL1(1,2*I) = BLL(2,1)*HH(I,J)
   BL1(2,2*I-1) = BLL(1,2)*HH(I,J)
   BL1(2,2*I) = BLL(2,2)*HH(I,J)
   BL1(3,2*I-1) = BLL(1,1)*HH(I,J) + BLL(1,2)*HH(I,J)
   BL1(3,2*I) = BLL(2,1)*HH(I,J) + BLL(2,2)*HH(I,J)
90   CONTINUE
C
ADD [BL0] AND [BL1] TO OBTAIN [BL]
DO 100 I = 1, 3
DO 100 J = 1, 8
100 BL(I,J) = BL0(I,J) + BL1(I,J)
C
C INITIATE [BNL]
C
DO 110 I = 1, 4
DO 110 J = 1, 8
110 BNL(I,J) = 0.0
C
C DO 120 I = 1, 4
BNL(1,2*I-1) = HH1(I,1)
BNL(2,2*I-1) = HH1(I,1)
BNL(3,2*I-1) = HH1(I,1)
BNL(4,2*I-1) = HH1(I,1)
C 120 CONTINUE
RETURN
C
C FORMAT (**** ERROR, ZERO OR NEGATIVE JACOBIAN DETERMINANT
+ FOR ELEMENT i4)
END
C
C SUBROUTINE STIFT(M,ESTIFT)
C
C C...............................................................*
C C * FORMATION OF ELEMENT STIFFNESS MATRIX FOR TRANSITION ELEMENT *
C C * RR, SS - NATURAL COORDINATES R AND S AT GAUSSIAN *
C C * INTEGRATION POINTS *
C C * WI, W2 - WEIGHTING FACTORS OF GAUSSIAN INTEGRATION *
C C * XX - NODAL COORDINATES AT TIME T *
C C * VS - DIRECTION COSINE AT TIME T *
C C * ESTIF - ELEMENT STIFFNESS MATRIX *
C C * C - CONSTITUTIVE MATRIX *
C C * S - CAUCHY STRESS MATRIX *
C C * D - DEVIATORIC STRESS MATRIX *
C C * PRT - POISSON'S RATIO *
C C * SETA - FRICTION ANGLE *
C C * OTHER VARIABLE NAMES ARE EXPLAINED IN MAIN PROGRAM AND *
C C * OTHER SUBROUTINE *
C C...............................................................*

COMMON NOP,NOEL,NCAT,TRAC,MBDY,E(20),PAR(20),RO(20)
1,IND(480,5),X1(600),Y1(600),ALX1(600),ALY1(600),KSTRG1(600),ITRN(20),
2JTRN(20),TRAX(20,2),TRAY(20,2),TH1(20),SETA2(20),CO2(20),NET1,NEBEL,
31AST,WIDTH20),KROT(100),ALZ1(100),ITYP,YNG,POS
C
COMMON ONE,VTI1(600),
1VT2(600),SIGG(480,4),VSI1(100,2),VSB1(50,3,2),XT1(600,2)
C
DIMENSION XX(2,3),VS2,SS(4,4),ESTI(7,7),BL(3,7),BNL(4,7),C(3,3)
C
C INITIATE COORDINATES AND WEIGHTING FACTORS FOR ONE POINT
C GAUSSIAN INTEGRATION
C
C DATA RR,0.0,SS,0.0,W1,2.0,W2,2.0/
C
C INITIATE CAUCHY STRESS MATRIX
C
C DO 15 I = 1, 4
DO 15 J = 1, 4
15 S(I,J) = 0.0
S(I,1) = SIGG(M,1)
S(2,2) = SIGG(M,2)
S(1,2) = SIGG(M,3)
S(2,1) = S(1,2)
S(3,3) = S(1,1)
S(4,4) = S(2,2)
S(3,4) = S(1,2)
S(4,3) = S(2,1)

C
INITIATE LINEAR C MATRIX FOR PLANE STRAIN PROBLEM
C
MATR = IND(M,4)
PRT = PAR(MATR)
BULK = E(MATR)/(1.-PRT**2)
SHEAM = E(MATR)/(2.*(1.+PRT))
C(1,1) = BULK
C(1,2) = PRT*BULK
C(1,3) = 0.0
C(2,1) = C(1,2)
C(2,2) = C(1,1)
C(2,3) = 0.0
C(3,1) = C(1,3)
C(3,2) = C(2,3)
C(3,3) = SHEAM

C
ASSIGN ELEMENT COORDINATES AND DIRECTION COSINE TO XX AND UT
C
DO 25 I = 1, 3
   XX(I,1) = XT1(IND(M,I),1)
   XX(I,2) = XT1(IND(M,I),2)
25 CONTINUE
VS(1) = VS1(IND(M,3),LAST,1)
VS(2) = VS1(IND(M,3),LAST,2)
WIDE = WIDTH(IND(M,5))

C
CALCULATE ELEMENT STIFFNESS
C
DO 30 I = 1, 7
   DO 30 J = 1, 7
30      ESTIFT(IJ) = 0.0
R1 = RR
S1 = SS

C
EVALUATE DERIVATIVE MATRIX B'S AND THE JACOBIAN DETERMINANT DET
C
CALL BMATXT(XX, VS, R1, S1, BL, BNL, DET, WIDE, M)
C
CARRY OUT [BT][C][B] OPERATION
C
WT = W1*W2*TH(MATR)*DET
C
CALCULATE LINEAR PART OF MATRIX [K]
C
CALL INTGRA(WT, 3, BL, C, ESTIFT,7)
C
CALCULATE NONLINEAR PART OF [K] AND ADD TO THE LINEAR PART
C
CALL INTGRA(WT, 4, BNL, S, ESTIFT,7)
C
RETURN
END
C
SUBROUTINE INTGRA(WT, N, B, C,ESTIF, MK)
C
******************************************************************************
*PERFORM MULTIPLICATION OF MATRICE

REAL BC(4), B(N,MK), C(N,N), ESTIF(MK,MK)
DO 70 I = 1, MK
  DO 60 K = 1, N
    BC(K) = 0.0
    DO 40 L = 1, N
      40 BC(K) = BC(K) + B(L,I)*C(L,K)
    STIFF = 0.0
    DO 50 L = 1, N
      50 STIFF = STIFF + BC(L)*B(L,J)
  60 ESTIF(I,J) = ESTIF(I,J) + STIFF*WT
70 CONTINUE

RETURN
END

SUBROUTINE BMATXT(XX, VS, R, S, BL, BNL, DET, WIDE, M)

REAL XX(2,3), BL(3,7), BNL(4,7), H(3), P(2,3), HH(2,3),
+XI(2,2), XH(2,2), VS(2), G1(2), G2(2), G(2,2)

RP = 1.0 + R
RM = 1.0 - R
SP = 1.0 + S
SM = 1.0 - S

INTERPOLATION FUNCTIONS

H(1) = 0.25*RM*SM
H(2) = 0.25*RP*SM
H(3) = 0.5*SP*S

DERIVATIVES OF INTERPOLATION FUNCTIONS WITH RESPECT TO NATURAL
COORDINATE

P(1,1) = -0.25*SM
P(1,2) = -P(1,1)
P(1,3) = 0.0
P(2,1) = -0.25*RM
P(2,2) = -0.25*RP
P(2,3) = 0.5 + S

C EVALUATE THE JACOBIAN MATRIX AT POINT (R,S)

C DO 20 I = 1, 2
  DO 20 J = 1, 2
    DUM = 0.0
    DO 10 K = 1, 3
      DUM = DUM + P(I,K)*XX(J,K)
  10    X(J,J) = DUM
    X(J,1) = X(J,1) - 0.5*WIDE*H(3)*VS(I)
    X(J,2) = X(J,2) - 0.5*WIDE*H(3)*VS(2)
    X(J,2) = X(J,2) - 0.5*R*WIDE*P(2,3)*VS(1)
    X(J,2) = X(J,2) - 0.5*R*WIDE*P(2,3)*VS(2)

C COMPUTE THE DETERMINANT OF THE JACOBIAN MATRIX

C DET = X(1,1)*X(2,2) - X(1,2)*X(2,1)
  IF (DET .GT. 0.00000001) GO TO 30
  WRITE(6,200) M
  STOP

C CALCULATE THE INVERSE OF THE JACOBIAN MATRIX

C DUM = 1/DET
  X(1,1) = X(2,2)*DUM
  X(1,2) = -X(1,2)*DUM
  X(2,1) = -X(2,1)*DUM
  X(2,2) = X(1,1)*DUM

C COMPUTE DERIVATIVES OF INTERPOLATION FUNCTIONS WITH RESPECT TO
C GLOBAL COORDINATES X AND Y

C DO 40 I = 1, 3
  HH(I,1) = 0.0
  HH(I,2) = 0.0
  DO 50 J = 1, 2
    HH(I,J) = HH(I,J) + X(J,J)*P(I,J)
  50    HH(I,J) = HH(I,J) + X(J,J)*P(I,J)
  CONTINUE

C COMPUTE G'S USED IN [B] MATRIX

C G(1) = 0.5*WIDE*VS(2)
G(2) = -0.5*WIDE*VS(1)
G(3) = R*G(1)
G(4) = R*G(2)
  DO 60 I = 1, 2
    DO 60 J = 1, 2
      G(I,J) = X(J,J)*G(I)*H(3) + X(J,J)*G(2)*P(I,3)
  60    CONTINUE

C INITIATE [BL]

C DO 70 I = 1, 3
  DO 70 J = 1, 7
  70    BL(I,J) = 0.0

C DO 80 I = 1, 3
  BL(1,2*I-1) = HH(I,1)
  BL(2,2*I) = HH(I,2)

258
BL(3,2*I-1) = HH(2,I)
BL(3,2*I) = HH(I,I)
80 CONTINUE

C
BL(1,7) = G(1,1)
BL(2,7) = G(2,2)
BL(3,7) = G(1,2) + G(2,1)

C
INITIATE [BNL]

C
DO 90 I = 1, 4
DO 90 J = 1, 7
90 BNL(I,J) = 0.0

C
DO 100 I = 1, 3
BNL(1,2*I-1) = HH(1,I)
BNL(2,2*I-1) = HH(2,I)
BNL(3,2*I) = HH(1,I)
BNL(4,2*I) = HH(2,I)
100 CONTINUE

BNL(1,7) = G(1,1)
BNL(2,7) = G(1,2)
BNL(3,7) = G(2,1)
BNL(4,7) = G(2,2)
RETURN

C
200 FORMAT ("**** ERROR, ZERO OR NEGATIVE JACOBIAN DETERMINANT
+ FOR ELEMENT",14)
END

C
SUBROUTINE STIFB(M,ESTIFB)

C
******************************************************************************
C * FORMATION OF ELEMENT STIFFNESS MATRIX FOR BEAM ELEMENT                  *
C * RR - NATURAL COORDINATES R AT GAUSSIAN INTEGRATION                       *
C * POINTS                                                                   *
C * W1 - WEIGHTING FACTORS OF GAUSSIAN INTEGRATION                          *
C * XX - NODAL COORDINATES AT TIME T                                         *
C * VS - DIRECTION CONSCINE AT TIME T                                        *
C * ESTIF - ELEMENT STIFFNESS MATRIX                                        *
C * C - CONSTITUTIVE MATRIX                                                 *
C * S - CAUCHY STRESS MATRIX                                                *
C * D - DEVIATORIC STRESS MATRIX                                            *
C * PRT - POISSON'S RATIO                                                   *
C * OTHER VARIABLE NAMES ARE EXPLAINED IN MAIN PROGRAM AND                  *
C * OTHER SUBROUTINE                                                         *
******************************************************************************

C
COMMON NOP,NOEL,NMATR,NTRAC,MBDY,E(20),PAR(20),RO(20)
 1,IND(480,5),X(600),Y(600),ALX(600),ALY(600),KSTRG(600),ITRN(20),
 2.ITRN(20),TRAX(20,2),TRAY(20,2),TH(20),SETA(20),CO(20),NETEL,NEBEL,
 3,LST,LWIDTH(20),KROT(100),ALZ(100),ITYP,YNG,POS

C
COMMON ONE,VT1(600),
 1,VT2(600),SIGG(480,4),VS1(100,2),VSBL(50,3,2),X1(600,2)

C
COMMON/THREE/IM,IST

C
COMMON/FIVE/Z1(50,3,2,4), Z2(50,3,2,4)

C
DIMENSION RR(3),W1(3),XX(2,3),VS(3,2),S(4,4),VSS(2)
 1,ESTIFB(9,9),BL(2,9),BNL(4,9),C(2,2),SS(2)

C
INITIATE COORDINATES AND WEIGHTING FACTORS FOR EIGHT POINT GAUSSIAN INTEGRATION

DATA RR,.7745966692,0.0,0.7745966692,
1W1,0.5555555555,0.8888888888,0.55555555555,
2SS,.5773502691,0.5773502691/

INITIATE LINEAR C MATRIX FOR PLANE STRAIN PROBLEM

MATR = IND(M,4)
PR T = PAR(MATR)
SHEAM = E(MATR)/(2.*(1.+PR T))
C(1,1) = E(MATR)
C(1,2) = 0.0
C(2,1) = 0.0
C(2,2) = (3.0/6.0)SHEAM

ASSIGN ELEMENT COORDINATES AND DIRECTION COSINE TO XX AND VS

DO 25 I = 1,3
XX(I,1) = XT(I+(ND(M,1),1)
XX(I,2) = XT(I+(ND(M,1),2)
VS(I,1) = VSB(I+NEBEL+1,1)
VS(I,2) = VSB(I+NEBEL+1,2)
25 CONTINUE
WIDE = WIDTH(IND(M,5))

CALCULATE ELEMENT STIFFNESS

DO 30 I = 1,9
DO 30 J = 1,9
ESTIFB(I,J) = 0.0
30 DO 90 LX = 1,3
R1 = RR(LX)
DO 90 LY = 1,2
S1 = SS(LY)
VSS(1) = VS(LX,1)
VSS(2) = VS(LX,2)
90 CONTINUE

INITIATE CAUCHY STRESS MATRIX

DO 15 I = 1,4
DO 15 J = 1,4
S(I,J) = 0.0
S(I,1) = Z2(M-NEBEL+1,LX,LY,1)
S(I,2) = Z2(M-NEBEL+1,LX,LY,2)
S(2,1) = S(1,2)
S(3,3) = S(1,1)
S(3,4) = S(1,2)
S(4,3) = S(2,1)

EVALUATE DERIVATIVE MATRIX B'S AND THE JACOBIAN DETERMINANT DET

CALL BMATXB(XX,VS,VSS,R1,S1,BL,BNL,DET,WIDE,M)

CARRY OUT [B]*[C]*(B) OPERATION

WT = W1(LX)*(TH(MATR)*DET

CALCULATE LINEAR PART OF MATRIX [K]

CALL INTGRA(WT,2,BL,C,ESTIFB,9)
CALCULATE NONLINEAR PART OF [K] AND ADD TO THE LINEAR PART

CALL INTGRA(WT, 4, BNL, S, ESTIFB, 9)
CONTINUE
RETURN
END

SUBROUTINE BMATXB(XX, VS, VSS, R, S, BL, BNL, DET, WIDE, M)

SOFTWARE TO EVALUATE THE MATRICES [BL] AND [BNL]
FOR BEAM ELEMENT
XX - ELEMENT NODAL COORDINATES AT TIME T
VS - DIRECTION CONSIDERED AT NODAL POINT
VSS - DIRECTION CONSIDERED AT INTEGRATION POINT
BL - LINEAR B MATRIX AFTER TRANSFORMATION
BNL - NONLINEAR B MATRIX AFTER TRANSFORMATION
H - INTERPOLATION FUNCTIONS
P - DERIVATIVES OF INTERPOLATION FUNCTION WITH
RESPECT TO NATURAL COORDINATES
III - DERIVATIVES OF INTERPOLATION FUNCTION WITH
RESPECT TO GLOBAL COORDINATES
XJ - JACOBIAN MATRIX
XJI - INVERSE OF THE JACOBIAN MATRIX
BL0 - LINEAR B MATRIX BEFORE TRANSFORMATION
BNL0 - NONLINEAR B MATRIX BEFORE TRANSFORMATION
T1 - LINEAR TRANSFORMATION MATRIX
T2 - NONLINEAR TRANSFORMATION MATRIX
WIDE - BEAM WIDTH
G, G1, G2 - TERM USED IN B MATIX
DE - DETERMINANT OF JACOBIAN MATRIX

REAL XX(2,3), VSS(2), BL(2,9), BNL(4,9), H(3), P(2,3), HH(2,3),
+ XJ(2,2), XJI(2,2), BL0(3,9), BNL1(4,9), T1(2,3), T2(4,4), G1(3,2)
+ , G2(3,2), G(3,2,2), VR(2), VS(3,2)

INTERPOLATION FUNCTIONS
H(1) = .5*R*(1-R)
H(2) = .5*R*(1+R)
H(3) = 1-R*R

DERIVATIVES OF INTERPOLATION FUNCTIONS WITH RESPECT TO NATURAL
COORDINATE
P(1,1) = -0.5 + R
P(1,2) = 0.5 + R
P(1,3) = -2*R

P(2,1) = 0.0
P(2,2) = 0.0
P(2,3) = 0.0

EVALUATE THE JACOBIAN MATRIX AT POINT (R,S)

DO 20 I = 1, 2
   DO 20 J = 1, 2
      DUM = 0.0
      DO 10 K = 1, 3
         DUM = DUM + P(I,K)*XX(J,K)
   10    CONTINUE
   XJ(I,J) = DUM
20  CONTINUE
DO 25 K = 1, 3
  XJ(J,1) = XJ(1,1) + 0.5*WIDE*P(J,K)*VS(K,1)
  XJ(J,2) = XJ(1,2) + 0.5*WIDE*P(J,K)*VS(K,2)
  XJ(J,1) = XJ(2,1) + 0.5*WIDE*H(K)*VS(K,1)
  XJ(J,2) = XJ(2,2) + 0.5*WIDE*H(K)*VS(K,2)
25 CONTINUE
C
C   COMPUTE THE DETERMINANT OF THE JACOBIAN MATRIX
C
C   DET = XJ(1,1)*XJ(2,2) - XJ(2,1)*XJ(1,2)
IF (DET .GT. 0.000000001) GO TO 30
WRITE(6,200) M
STOP
C
C   CALCULATE THE INVERSE OF THE JACOBIAN MATRIX
C
DUM = 1.0/DET
XJ(J,1) = XJ(2,2)*DUM
XJ(J,2) = XJ(1,2)*DUM
XJ(J,1) = XJ(2,1)*DUM
XJ(J,2) = XJ(1,1)*DUM
C
C   COMPUTE DERIVATIVES OF INTERPOLATION FUNCTIONS WITH RESPECT TO
C   GLOBAL COORDINATES X AND Y
C
DO 40 I = 1, 3
  HH(I,1) = 0.0
  HH(I,2) = 0.0
40 CONTINUE
DO 50 J = 1, 2
  HH(J,1) = HH(J,1) + XJ(J,1)*P(I,J)
  HH(J,2) = HH(J,2) + XJ(J,2)*P(I,J)
50 CONTINUE
40 CONTINUE
C
C   COMPUTE G'S USED IN [B] MATRIX
C
DO 60 K = 1, 3
  G1(K,1) = -0.5*WIDE*VS(K,2)
  G1(K,2) = 0.5*WIDE*VS(K,1)
  G2(K,1) = S*G1(K,1)
  G2(K,2) = S*G1(K,2)
60 CONTINUE
DO 70 K = 1, 3
  DO 70 I = 1, 2
  DO 70 J = 1, 2
  G(K,I,J) = XJ(J,1)*G2(K,I)*P(I,K) + XJ(J,2)*G1(K,I)*H(K)
70 Continue
C
C   INITIATE LINEAR B MATRIX
C
DO 75 I = 1, 3
  DO 75 J = 1, 9
  BLO(I,J) = 0.0
75 Continue
DO 80 I = 1, 3
  BLO(I,1*1-2) = HH(I,1)
  BLO(I,1*1) = G(I,1,1)
  BLO(2*1-1) = HH(2,1)
  BLO(2*1) = G(I,1,2)
  BLO(3*1-1) = HH(2,1)
  BLO(3*1) = G(I,1,2) + G(I,2,1)
80 Continue
C
C   NONLINEAR B MATRIX
DO 85 I = 1, 4
DO 85 J = 1, 9
85 BNL1(I,J) = 0.0
DO 90 I = 1, 3
   BNL1(I,3*I-1) = HH(I,1)
   BNL1(I,3*I) = G(I,1,1)
   BNL1(I,3*I-2) = HH(I,2)
   BNL1(I,3*I-1) = G(I,1,2)
   BNL1(I,3*I+1) = HH(I,1)
   BNL1(I,3*I+2) = G(I,1,1)
   BNL1(I,3*I+3) = HH(I,2)
   BNL1(I,3*I+4) = G(I,1,2)
90 CONTINUE

C INITIATE LINEAR TRANSFORMATION MATRIX
C
VR(1) = VSS(2)
VR(2) = -VSS(1)

C T1(1,1) = VR(1)**2
C T1(1,2) = VR(2)**2
C T1(1,3) = VR(1)*VR(2)
C T1(2,1) = 2*VR(1)*VSS(1)
C T1(2,2) = 2*VR(2)*VSS(2)
C T1(2,3) = VR(2)*VSS(1) + VR(1)*VSS(2)

C INITIATE NONLINEAR TRANSFORMATION MATRIX
C
DO 95 I = 1, 4
DO 95 J = 1, 4
95 T2(I,J) = 0.0
T2(1,1) = VR(1)
T2(1,2) = VSS(1)
T2(2,1) = VR(2)
T2(2,2) = VSS(2)
T2(3,1) = T2(1,1)
T2(3,2) = T2(1,2)
T2(4,1) = T2(2,1)
T2(4,2) = T2(2,2)

C TRANSFORM THE LINEAR B MATRIX
C
DO 100 I = 1, 2
DO 100 J = 1, 4
   BL(I,J) = 0.0
100 DO 100 K = 1, 3
   BL(I,J) = BL(I,J) + T1(I,K)*BL0(K,J)

C TRANSFORM THE NONLINEAR B MATRIX
C
DO 105 I = 1, 4
DO 105 J = 1, 9
   BNL(I,J) = 0.0
DO 105 K = 1, 4
105 BNL(I,J) = BNL(I,J) + T2(K,I)*BNL1(K,J)
RETURN
C
200 FORMAT ("*** ERROR, ZERO OR NEGATIVE JACOBIAN DETERMINANT
   + FOR ELEMENT",I4)
END

 SUBROUTINE LOAD
C
C
C

263
SUBROUTINE TO CALCULATE LOAD

FF - FORCE VECTOR CORRESPONDING TO ELEMENT STRESS
XX - NODAL COORDINATES
ALL OTHER VARIABLE NAMES ARE EXPLAINED IN MAIN PROGRAM

COMMON NOE,NMATR,NTRAC,MBDY,E(20),PAR(20),RO(20)
IND(480,5),X(600),Y(600),ALX(600),ALY(600),KSTRG(600),IWR(20),
3J,TRNX(20),TRAY(20),TH(20),GLOBALI,NETEL,NEBE,
1LAST,WIDTH(20),KROT(100),ALZ(100),ITYP,YNG,POS

COMMON,TWO,IBWTH,MEQ,AAR(1300),ASTF(1300,90),SIG(480,4),FLAG(450),
1U1(600),U2(600),UT3(100),STRTN(480,3),VS2(100,2),VSB2(50,3,2),
2XT(600,2)

COMMON,THREE,IM,IST

COMMONE,FOUR,FKN,FKS,MST,MND

DIMENSION RR(4),SS(4),W1(4),W2(4),XX(2,4),DF(8,11),
1FFT(7),FFB(9),Q(4)

INITIATE COORDINATES AND WEIGHTING FACTORS FOR 16 POINT
GAUSSIAN INTEGRATION

DATA RR,0.8611363115,0.3399810435,0.3399810435,0.8611363115,
1SS,0.8611363115,0.3399810435,0.3399810435,0.8611363115,
2W1,0.3478548451,0.6521451548,0.6521451548,0.3478548451,
3W2,0.3478548451,0.6521451548,0.6521451548,0.3478548451

INITIATE LOAD VECTOR

DO 51 = 1, MEQ
AAR(I) = 0.0

ADD CONCENTRATED LOAD TO LOAD VECTOR

DO 15 N = 1, LAST
IF (KSTRG(N) .EQ. 3) GO TO 15
K = 2*N
IF (KSTRG(N) .EQ. 1) GO TO 10
C FOR KSTRG(N) = 0, 2
AAR(K-1) = AAR(K-1) + ALX(N)
IF (KSTRG(N) .NE. 0) GO TO 15
C FOR KSTRG(N) = 0, 1
10 AAR(K) = AAR(K) + ALY(N)
CONTINUE

IF (ITYP .EQ. 0) GO TO 19
C
DO 18 N = LAST+1, NOP
IF (KSTRG(N) .EQ. 3) GO TO 17
IF (KSTRG(N) .EQ. 1) GO TO 16
C FOR KSTRG(N) = 0, 2
AAR(3*N-LAST-2) = AAR(3*N-LAST-2) + ALX(N)
IF (KSTRG(N) .NE. 0) GO TO 17
C
FOR KSTRG(N) = 0, 1
C
16 AAR(3*N-LAST) = AAR(3*N-LAST) + ALY(N)
17 IF (KROT(N-LAST).EQ.0) THEN
   AAR(3*N-LAST) = AAR(3*N-LAST) + ALZ(N-LAST)
ENDIF
C
18 CONTINUE
C
ADD SURFACE TRACTION EFFECT TO THE LOAD VECTOR
C
19 IF (NTRAC.EQ.0) GO TO 25
DO 20 L = 1, NTRAC
   I = ITRN(L)
   J = JTRN(L)
   II = 2*I
   JJ = 2*J
   DX = X(J)-X(I)
   DY = Y(J)-Y(I)
   EL = SQRT(DX*DX + DY*DY)
   PXI = TRAX(L,1)*EL
   PXJ = TRAX(L,2)*EL
   PYI = TRAY(L,1)*EL
   PYJ = TRAY(L,2)*EL
   IF (I.LT. LAST) THEN
      AAR(II) = AAR(II) + PXI/3.0 + PXJ/6.0
      AAR(JJ) = AAR(JJ) + PXI/6.0 + PXJ/3.0
      AAR(II) = AAR(II) + PYI/3.0 + PYJ/6.0
      AAR(JJ) = AAR(JJ) + PYI/6.0 + PYJ/3.0
   ELSE
      AAR(3*I-LAST) = AAR(3*I-LAST) + PXI 3.0 + PXJ/6.0
      AAR(3*J-LAST) = AAR(3*J-LAST) + PXI 6.0 + PXJ/3.0
      AAR(3*I-LAST-1) = AAR(3*I-LAST-1) + PYI 3.0 + PYJ/6.0
      AAR(3*J-LAST-1) = AAR(3*J-LAST-1) + PYI 6.0 + PYJ/3.0
   ENDIF
20 CONTINUE
C
ADD BODY FORCE TO THE LOAD VECTOR, ASSUMING 16 POINT INTEGRATION
C
25 IF (MBODY .EQ. 0) GO TO 41
DO 40 M = 1, NETEL
   DO 30 J = 1, 4
      XX(J, I) = X(IN(M,J))
      XX(J, 2) = Y(IN(M,J))
      Q(J) = 0.0
30 CONTINUE
   MATR = IND(M,5)
C
DO 33 LX = 1, 4
   R1 = RR(LX)
   DO 32 LY = 1, 4
      S1 = SS(LY)
32 CONTINUE
C
COMPUTE DETERMINANT OF JACOBIAN MATRIX AND INTERPOLATION FUNCTION
C
   CALL BODY(XX, R1, S1, H, DET)
C
   DO 31 J = 1, 4
      Q(J) = Q(J) + W1(LX)*W2(LY)*R0(MATR)*TH(MATR)*H(J)*DET
31 CONTINUE
33 CONTINUE
C
265
ADDITION

IF (M.LT. IST .OR. M.GT. IM) THEN
   DO 35 I = 1, 4
      AAR(2*IND(M,I)) = AAR(2*IND(M,I)) - Q(I)
   END IF
   CONTINUE

ADD INTERNAL FORCE TO LOAD VECTOR

DO 46 M = 1, NETEL
   THIC = TH(IND(M,5))
   CALL FORCE(M,FF,THIC)
   DO 45 M = 1, 4
      AAR(2*IND(M,I)-1) = AAR(2*IND(M,I)-1) - FF(2*I-1)
      AAR(2*IND(M,I)) = AAR(2*IND(M,I)) - FF(2*I)
   CONTINUE
   CONTINUE
IF (ITYP .EQ. 0) GO TO 52

DO 49 M = NETEL+1, NEBEL-1
   THIC = TH(IND(M,4))
   CALL FORCT(M,FFT,THIC)
   DO 47 M = 1, 2
      AAR(2*IND(M,I)-1) = AAR(2*IND(M,I)-1) - FFT(2*I-1)
      AAR(2*IND(M,I)) = AAR(2*IND(M,I)) - FFT(2*I)
   DO 48 M = 1, 3
      AAR(3*IND(M,3)-LAST-3+1) = AAR(3*IND(M,3)-LAST-3+1) - FFT(4+I)
   CONTINUE
   DO 51 M = NEBEL, NOEL
      THIC = TH(IND(M,4))
   CALL FORCB(M,FFB,THIC)
   DO 50 M = 1, 3
      AAR(3*IND(M,I)-LAST-2) = AAR(3*IND(M,I)-LAST-2) - FFB(3*I-2)
      AAR(3*IND(M,I)-LAST-1) = AAR(3*IND(M,I)-LAST-1) - FFB(3*I-1)
      AAR(3*IND(M,I)-LAST) = AAR(3*IND(M,I)-LAST) - FFB(2*I)
   CONTINUE

INTRODUCE BOUNDARY CONDITIONS

DO 65 I = 1, LAST
   IF (KSTRG(I).GE. 0 .AND. KSTRG(I).LE. 3) GO TO 55
   ITRUN = ITRUN + 1
   GO TO 65
   IF (KSTRG(I).EQ.0) GO TO 65
   IF (KSTRG(I).EQ.2) GO TO 60
   FOR KSTRG = 1, 3
   CALL BOUND (ALX(I), 2*I-1)
   IF (KSTRG(I).EQ.1) GO TO 65
   FOR KSTRG = 2, 3
   CALL BOUND (ALY(I), 2*I)

C
CONTINUE
C
IF (ITYP.EQ.0) GO TO 86
C
DO 85 I = LAST + 1, NOP
IF (KSTRG(I) .GE. 0 .AND. KSTRG(I) .LE. 3) GO TO 70
ITRUN = ITRUN + 1
GO TO 80
70 IF (KSTRG(I).EQ.0) GO TO 80
IF (KSTRG(I).EQ.2) GO TO 75
C
FOR KSTRG = 1, 3
C
CALL BOUND (ALX(I),3*I-LAST-2)
IF (KSTRG(I).EQ.1) GO TO 80
C
FOR KSTRG = 2, 3
C
75 CALL BOUND (ALY(I),3*I-LAST-1)
80 IF (KROT(I-LAST) .EQ. 1) THEN
CALL BOUND (ALZ(I-LAST),3*I-LAST)
ENDIF
85 CONTINUE
86 IF (ITRUN.EQ.0) GO TO 90
WRITE(6,2000) ITRUN
90 RETURN
C
2000 FORMAT ("!!! SOLUTION WILL NOT BE PERFORMED BECAUSE OF ',15,
+ 'DATA ERRORS'")
END
C
SUBROUTINE BOUND (U, N)
C
* SUBROUTINE TO IMPOSE BOUNDARY CONDITIONS
C
* U - KNOWN DISPLACEMENT
C
* N - EQUATION NUMBER
C
COMMON/NOP,NOELE,NMATR,STRAC,MBDY,E(20),PAR(20),RO(20)
1,IND(480,5),X(600),Y(600),ALX(600),ALY(600),KSTRG(600),ITRUN(20),
2JTRN(20),TRAX(20,2),TRAY(20,2),TH(20),SELA(20),CO(20),NETEL,NEBEL,
3LAST,WIDTH(20),KROT(100),ALZ(100),ITYP,YNG,POS
C
COMMON/TWO:1BWTH,MEQ,AAR(1300),ASTF(1300,90),SIG(480,4),FLAG(450),
1UT1(600),UT2(600),UT3(100),STRN(480,3),VS2(100,2),VSB2(50,3,2),
2XT(600,2)
C
DO 10 M = 2, IBWTH
K = N-M+1
IF (K.LE.0) GO TO 5
C
MOVE THE PRODUCT OF KNOWN DISPLACEMENT AND CORRESPONDING STIFFNESS
ELEMENT TO THE RHS, AND SET THE STIFFNESS ELEMENT EQUAL TO ZERO
C
AAR(K) = AAR(K)*ASTF(K,M)*U
ASTF(K,M) = 0.0
5 K = N+M-1
IF (K.GT.MEQ) GO TO 10
AAR(K) = AAR(K)*ASTF(N,M)*U
ASTF(N,M) = 0.0
10 CONTINUE
ASTF(N,1) = 1.0
AAR(N) = U
RETURN
END

SUBROUTINE FORCE(M,FF,THIC)

* CALCULATION OF ELEMENT FORCE F
* RR - R COORDINATES AT GAUSSIAN INTEGRATION POINTS
* SS - S COORDINATES AT GAUSSIAN INTEGRATION POINTS
* W1,W2 - WEIGHTING FACTORS OF GAUSSIAN INTEGRATION
* SHAT - 2ND PIOLA-KIRCHHOFF STRESS VECTOR
* FF - FORCE VECTOR CORRESPONDING TO ELEMENT STRESS
* XX - NODAL COORDINATES
* UT - NODAL DISPLACEMENTS AT TIME T
* BL - SAME AS IN SUBROUTINE BMATX
* ALL OTHER VARIABLE NAMES ARE EXPLAINED IN MAIN PROGRAM
* AND OTHER SUBROUTINES

COMMON N0EL,NMATR,NTRAC,MBDY,E(20),PAR(20),RO(20)
1,IND(480,5),X(600),Y(600),ALX(600),ALY(600),KSTRG(600),ITRN(20),
2JTRN(20),TRAX(20,2),TRAY(20,2),TH(20),SETA(20),CO(20),NETEL,NEBEL,
3LAST,WIDTH(20),KROT(100),ALZ(100),ITYP,YNG,POS

COMMON ONE,VT1(600),
1VT2(600),SIGG(480,4),VS1(100,2),VSBI(50,3,2),XT1(600,2)

COMMON/TWO,IBWTH,MEQ,AAR(1300),ASTF(1300,90),SIG(480,4),FLAG(450),
1UT(600),UT2(600),UT3(100),STRN(480,3),VS2(100,2),VSBI(50,3,2),
2XT(600,2)

COMMON/THREE,IM,IST

COMMON/FOUR,FKN,FKS,MST,MND

DIMENSION RR(4),SS(4),W1(4),W2(4),SHAT(3),XX(2,4),UT(2,4),FF(8)
1,BL(3,8),BNL(4,8),U(8),ESTIF(8,8)

INITIATE COORDINATES AND WEIGHTING FACTORS FOR EIGHT POINT
GAUSSIAN INTEGRATION

DATA RR 0.8611363115,0.3399810435,0.3399810435,0.8611363115/,
1SS 0.8611363115,0.3399810435,0.3399810435,0.8611363115/,
2W1 0.3478548451,0.6521451548,0.6521451548,0.3478548451/,
3W2 0.3478548451,0.6521451548,0.6521451548,0.3478548451/

IF FRICTION ELEMENT, GO TO 70
IF (M .GE. MST .AND. M .LE. MND) GO TO 70

INITIATE 2ND PIOLA-KIRCHHOFF STRESS VECTOR

SHAT(1) = SIG(M,1)
SHAT(2) = SIG(M,2)
SHAT(3) = SIG(M,3)

ASSIGN ELEMENT COORDINATES AND DISPLACEMENTS TO XX AND UT

DO 25 I = 1,4
XX(1,I) = X(IND(M,I))
XX(2,I) = Y(IND(M,I))
UT(1,I) = UT1(IND(M,I))

25 CONTINUE

UT(2, I) = UT2(IND(M, I))
25 CONTINUE
C
C CALCULATE ELEMENT FORCE VECTOR
C
DO 30 I = 1, 8
30 FF(I) = 0.0
DO 60 LX = 1, 4
R1 = RR(LX)
DO 50 LY = 1, 4
S1 = SS(LY)
C
C EVALUATE DERIVATIVE MATRIX B'S AND THE JACOBIAN DETERMINANT DET
C
CALL BMATX(XX, UT, R1, S1, BL, BNL, DET, M)
C
C CARRY OUT [BT][S] OPERATION
C
WT = W1(LX)*W2(LY)*THIC*DET
DO 40 I = 1, 8
DO 40 J = 1, 3
40 FF(I) = FF(I) + BL(I, J)*SHAT(J)*WT
50 CONTINUE
60 CONTINUE
C
RETURN
C70 DO 75 I = 1, 4
C71 XX(I, J) = XTI(IND(M, I), 1)
C72 XX(2, I) = XTI(IND(M, I), 2)
C73 U(2*I-1) = UTI(IND(M, I)) - VTI(IND(M, I))
C74 U(2*I) = UT2(IND(M, I)) - VT2(IND(M, I))
C75 FFKN = (M-MST+1)*FKN
C76 FFKS = (M-MST+1)*FKS
C
CALL FSTIF(FFN, FKS, XX, ESTIF)
C
70 DO 80 I = 1, 8
DO 80 J = 1, 8
80 FF(I) = 0.0
C80 FF(I) = FF(I) - THIC*ESTIF(I, J)*U(J)
C
RETURN
END

C SUBROUTINE BODY(XX, R, S, H, DET)
C
****************************************************************************************************
C SUBROUTINE TO OBTAIN BODY FORCE
C XX - ELEMENT NODAL COORDINATES
C H - INTERPOLATION FUNCTIONS
C P - DERIVATIVES OF INTERPOLATION FUNCTION WITH RESPECT TO NATURAL COORDINATES
C XJ - JACOBIAN MATRIX
C DET - DETERMINANT OF JACOBIAN MATRIX
C
****************************************************************************************************
C
REAL XX(2,4), H(4), P(2,4), XJ(2,2)
C
RP = 1.0 + R
SP = 1.0 + S
RM = 1.0 - R
SM = 1.0 - S
C
INTERPOLATION FUNCTIONS

H(1) = 0.25*R*M*S*M
H(2) = 0.25*R*P*S*M
H(3) = 0.25*R*P*S*P
H(4) = 0.25*R*M*S*P

DERIVATIVES OF INTERPOLATION FUNCTIONS WITH RESPECT TO NATURAL COORDINATE

P(1,1) = -0.25*S*M
P(1,2) = -P(1,1)
P(1,3) = 0.25*S*P
P(1,4) = -P(1,3)

P(2,1) = -0.25*R*M
P(2,2) = -0.25*R*P
P(2,3) = -P(2,2)
P(2,4) = -P(2,1)

EVALUATE THE JACOBIAN MATRIX AT POINT (R,S)

DO 20 I = 1, 2
   DO 20 J = 1, 2
      DUM = 0.0
   DO 10 K = 1, 4
      DUM = DUM + P(I,K)*XX(J,K)
   10   XJ(I,J) = DUM
   20  

COMPUTE THE DETERMINANT OF THE JACOBIAN MATRIX

DET = XJ(1,1)*XJ(2,2) - XJ(2,1)*XJ(1,2)

RETURN

SUBROUTINE SOLVE

*******************************************************************************
** SUBROUTINE TO SOLVE SYMMETRIC BAND MATRIX EQUATION **
** ALL VARIABLES ARE EXPLAINED ELSEWHERE **
*******************************************************************************

COMMON TWO IBWTH, MEQ, AAR(1300), ASTF(1300,90), SIG(480,4), FLAG(450),
1 U1(600), UT2(600), UT3(100), STRN(480,3), VS2(100,2), VSB2(50,3,2),
2 XT(600,2)

SOLVE THE LINEAR EQUATIONS

NRS = MEQ - 1
NR = MEQ
DO 10 N = 1, NRS
   M = N - 1
   MR = MIN(1,BWTH,NR - M)
   PIVOT = ASTF(N,1)
   DO 10 L = 2, MR
      C = ASTF(N,L)*PIVOT
      I = M + L
      J = 0
      DO 5 K = L, MR
         J = J + 1
         5      ASTF(I,J) = ASTF(I,J) - C*ASTF(N,K)
      10 ASTF(N,L) = C
DO 15 N = 1, NRS
   M = N-1
   MR = MIN0(IBWTH,NR-M)
   C = AAR(N)
   AAR(N) = C/ASTF(N,1)
DO 15 L = 2, MR
   I = M + L
   AAR(I) = AAR(I)*ASTF(N,L)*C
AAR(NR) = AAR(NR)/ASTF(NR,1)
DO 20 I = 1, NRS
   N = NR-I
   M = N-I
   MR = MIN0(IBWTH,NR-M)
   DO 20 K = 2, MR
      L = M + K
      AAR(N) = AAR(N)*ASTF(N,K)*AAR(L)
C
RETURN
END

SUBROUTINE FORCT(M,FFT,TH1IC)

******************************************************************************
* CALCULATION OF ELEMENT FORCE FOR TRANSITION ELEMENT                      *
* RR - R COORDINATES AT GAUSSIAN INTEGRATION POINTS                          *
* SS - S COORDINATES AT GAUSSIAN INTEGRATION POINTS                          *
* W1,W2 - WEIGHTING FACTORS OF GAUSSIAN INTEGRATION                          *
* SHAT - CAUCHY STRESS VECTOR                                                *
* FF - FORCE VECTOR CORRESPONDING TO ELEMENT STRESS                          *
* XX - NODAL COORDINATES AT TIME T                                           *
* VS - DIRECTION CONSINE                                                    *
* BL - SAME AS IN SUBROUTINE BMATX                                          *
* ALL OTHER VARIABLE NAMES ARE EXPLAINED IN MAIN PROGRAM                    *
* AND OTHER SUBROUTINES                                                     *
******************************************************************************

COMMON NUP, NOEL, NMATR, NTRAC, MBDY, E(20), PAR(20), RO(20)
1, IND(480,5), X(600), Y(600), ALX(600), ALY(600), KSTRG(600), ITRN(20),
2 JTRN(20), TRAX(20,2), TRAY(20,2), TH(20), SETA(20), CO(20), NETEL, NEBEL,
3 LAST, WIDTH(20), KROT(100), ALZ(100), ITYP, YNG, POS

COMMON TWO, IBWTH, IEQ, AAR(1300), ASTF(1300,90), SIG(480,4), FLAG(450),
1 LUT(600), UT2(600), UT3(100), STRN(480,3), VS2(100,2), VSB2(30,3,2),
2 XT(600,2)

DIMENSION SHAT(3), XX(2,3), VS(2), FFT(7), BL(3,7), BNL(4,7)

INITIATE COORDINATES AND WEIGHTING FACTORS FOR ONE POINT
GAUSSIAN INTEGRATION

DATA RR 0.0/, SS 0.0/, W1 2.0/, W2 2.0/

INITIATE 2ND PIOLA-KIRCHHOFF STRESS VECTOR

   SHAT(1) = SIG(M,1)
   SHAT(2) = SIG(M,2)
   SHAT(3) = SIG(M,3)

ASSIGN ELEMENT COORDINATES AND DIRECTION CONSINE TO XX AND VS

DO 25 I = 1, 3
   XX(I,1) = XT(IND(M,I),1)
25
   XX(I,2) = XT(IND(M,I),2)
DO 26 I = 1, 2
VS(I) = VS2(IND(M,3)-LAST,I)
WIDE = WIDTH(IND(M,5))

CALCULATE ELEMENT FORCE VECTOR

DO 30 I = 1,7
30 FFT(I) = 0.0
RI = RR
SI = SS

EVALUATE DERIVATIVE MATRIX B'S AND THE JACOBIAN DETERMINANT DET

CALL BMATX(XX, VS, R1, S1, BL, BNL, DET, WIDE, M)

CARRY OUT [BTIT] OPERATION

WT = W1*W2*THIC*DET
DO 40 J = 1,7
40 FFT(I) = FFT(I) + BL(J,I)*SHAT(J)*WT

RETURN
END

SUBROUTINE FORCB(M,FFB,THIC)

*****************************************************************************
* CALCULATION OF ELEMENT FORCE FOR TRANSITION ELEMENT *
* RR - COORDINATES AT GAUSSIAN INTEGRATION POINTS *
* W1 - WEIGHTING FACTORS OF GAUSSIAN INTEGRATION *
* SHAT - CAUCHY STRESS VECTOR *
* FFT - FORCE VECTOR CORRESPONDING TO ELEMENT STRESS *
* XX - NODAL COORDINATES AT TIME T *
* VS - DIRECTION COSINE AT NODAL POINT *
* VSS - DIRECTION COSINE AT INTEGRATION POINT *
* BL - SAME AS IN SUBROUTINE BMATX *
* ALL OTHER VARIABLE NAMES ARE EXPLAINED IN MAIN PROGRAM *
* AND OTHER SUBROUTINES *
*****************************************************************************

COMMON NOP,NOEL,NMATR,NTAC,MBDY,E(20),PAR(20),RO(20)
1,IND(480,5),X(600),Y(600),ALX(600),ALY(600),KSTRG(600),JTRN(20),
2,JTRN(20),TRAX(20,2),TRAY(20,2),TH(20),SETA(20),CO(20),NEBEL,
3,LAST,WIDTH(20),KROT(100),ALZ(100),ITYP,YNG,POS

COMMON/Two,IBWTH,MEQA,AAR(1300),ASTF(1300,90),SIG(450),
1UT1(600),UT2(600),UT3(100),STRN(480,3),VS2(100,2),VSB2(50,3,2),
2XT(600,2)

COMMON/FIVE/Z1(50,3,2,4), Z2(50,3,2,4)

DIMENSION RR(3),W1(3),SHAT(2),XX(2,3),VS(3,2),VSS(2),FFB(9),
1BL(2,9),BNL(4,9),SS(2)

INITIATE COORDINATES AND WEIGHTING FACTORS FOR THREE POINT
GAUSSIAN INTEGRATION

DATA RR/0.7745966692,0.0,0.7745966692/,
1W1,0.55555555555,0.88888888888,0.555555555555,/
2SS/0.5773502691,0.5773502691/

ASSIGN ELEMENT COORDINATES AND DIRECTION COSINE TO XX AND VS

DO 25 I = 1,3
XX(1,1) = XT(IND(M,1),1)
XX(2,1) = XT(IND(M,1),2)
DO 25 J = 1,3
   DO 25 I = 1,3
      VS(I,J) = VSBS(M,NEBEL + 1,1,J)
      WIDE = WIDTH(IND(M,3))
   C
   CALCULATE ELEMENT FORCE VECTOR
C
   DO 30 I = 1,3
      FFB(I) = 0.0
   DO 60 L = 1,3
      R1 = RR(LX)
      DO 60 LY = 1,2
         SI = SS(LY)
         VSS(1) = VS(LX,1)
         VSS(2) = VS(LX,2)
      C
      INITIATE CAUCHY STRESS VECTOR
C
      SHAT(1) = Z1(M,NEBEL + 1,LX,LY,1)
      SHAT(2) = Z1(M,NEBEL + 1,LX,LY,2)
   C
   EVALUATE DERIVATIVE MATRIX B'S AND THE JACOBIAN DETERMINANT DET
C
   CALL BMATX(BXX, VS,VSS, R1, SI, BL, BN,L, DET, WIDE, M)
C
   CARRY OUT [B] [T] [T] [T] OPERATION
C
   WT = W1(LX)*THC*DET
   DO 40 I = 1,3
      DO 40 J = 1,2
         FFB(I) = FFB(I) + BL(I,J)*SHAT(J)*WT
   40 CONTINUE
C
   RETURN
END
SUBROUTINE STRES(M)
C
*******************************************************************************
C
C  * SUBROUTINE TO CALCULATE STRESSES AT GAUSSIAN INTEGRATION POINT *
C  * XX - NODAL COORDINATES                                 *
C  * UT - NODAL DISPLACEMENTS AT TIME T                       *
C  * UU - DISPLACEMENT INCREMENTS                            *
C  * C  - CONSTITUTIVE MATRIX                                *
C  * D  - DEVIATORIC STRESS MATRIX                           *
C  * TAU - STRESS VECTOR                                     *
C  * PRT - POISSON'S RATIO                                   *
C  * SETA - FRICTION ANGLE                                  *
C  * E  - YOUNG'S MODULUS                                    *
C  * CO - COHESION                                          *
C  * OTHER VARIABLE NAMES ARE EXPLAINED IN MAIN PROGRAM AND  *
C  * OTHER SUBROUTINE                                        *
C
*******************************************************************************
C
COMMON N0PL,NOEL,NMATR,NTRAC,MBDY,E(20),PAR(20),RO(20),
1IND(480,5),X(600),Y(600),ALX(600),ALY(600),KSTRG(600),JTRN(20),
2JTRN(20),TRAX(20,2),TRAY(20,2),TH(20),SE(20),CO(20),NEBEL,NEBEL,
3LST,WIDTH(20),KROT(100),ALZ(100),ITYP,YN,POS
C
COMMON TWO,IBWTH,MEQ,AAR(1300),ASTF(1300,90),SIG(480,4),FLAG(450),
1UT1(600),UT2(600),UT3(100),STRN(480,3),VS2(100,2),VSB2(50,3,2),

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COMMON/THREE/IM,IST
COMMON/FOUR/FKN,FKS,MST,MND
DIMENSION XX(2,4),UT(2,4),UU(2,4),STRA(3),D(2,2),TAU(4),C(3,3),
STRAL(3)
DATA RR,0.3399810435/, SS,0.3399810435/
IF FRICTION ELEMENT, GO TO 50
IF (M .GE. MST .AND. M .LE. MND) GO TO 50
ASSIGN ELEMENT COORDINATES AND DISPLACEMENTS TO XX,UT AND UU
DO 5 I = 1, 4
   XX(I,1) = X(I,IND(M,J))
   XX(I,2) = Y(I,IND(M,J))
   UT(I,1) = UT(I,IND(M,J))
   UT(I,2) = UT(I,IND(M,J))
   UU(I,1) = AAR(Q*IND(M,J)-1)
   UU(I,2) = AAR(Q*IND(M,J))
CONTINUE
CALCULATE STRAIN
CALL STRAIN(XX,UU,UT,STRA,STRA,RR,SS,M)
ASSIGN STRAIN VALUES TO GLOBAL STRAIN MATRIX
DO 10 I = 1, 3
   STRN(M,J) = STRA(I)
10   DETERMINE ELEMENT TYPE AND USE PROPER MATERIAL CONSTANTS
IF (M .GE. IST .AND. M .LE. IM) THEN
   GOB ELEMENT
   BULK = YNG,(2.*(1.*POS)*((1.-2.*POS))
   SHEAM = YNG,(2.* (1. + POS))
   ELSE
   REGULAR ELEMENT
   MATR = IND(M,5)
   PRT = PAR(MATR)
   BULK = E(MATR),(2.*(1. + PRT)*((1.-2.*PRT))
   SHEAM = E(MATR),(2.* (1. + PRT))
   END IF
INITIATE LINEAR C MATRIX FOR PLANE STRAIN PROBLEM
C(1,1) = BULK + SHEAM
C(1,2) = BULK - SHEAM
C(1,3) = 0.0
C(2,1) = C(1,2)
C(2,2) = C(1,1)
C(2,3) = 0.0
C(3,1) = C(1,3)
C(3,2) = C(2,3)
C(3,3) = SHEAM

C CALCULATE STRESS FOR ELASTIC MATERIAL

DO 25 I = 1, 3
   TAU(I) = SIG(M,I)
DO 25 J = 1, 3
   TAU(I) = TAU(I) + CI(J)*STRAL(I)
TAU(4) = PRT*(TAU(1) + TAU(2))

DO 40 I = 1, 4
SIG(M,I) = TAU(I)
RETURN

DO 55 I = 1, 4
SIG(M,I) = 0.0
RETURN
END

SUBROUTINE STRAIN(XX, UU, UT, STRA, STRAL, R, S, M)

REAL XX(2,4), UU(2,4), UT(2,4), H(4), P(2,4), STRA(3), U(2,2),
UUT(2,2), ST(2,2), UR(2,2), URT(2,2), XJ(2,2), XJI(2,2), STRAL(3)

RP = 1.0 + R
SP = 1.0 + S
RM = 1.0 - R
SM = 1.0 - S

INTERPOLATION FUNCTIONS

H(1) = 0.25*RM*SM
H(2) = 0.25*RP*SM
H(3) = 0.25*RP*SP
H(4) = 0.25*RM*SP

DERIVATIVES OF INTERPOLATION FUNCTIONS WITH RESPECT TO NATURAL
COORDINATE

P(1,1) = -0.25*SM
P(1,2) = -P(1,1)
P(1,3) = 0.25*SP
P(1,4) = -P(1,3)
P(2,1) = -0.25*RM
P(2,2) = -0.25*RP
P(2,3) = -P(2,2)
P(2,4) = -P(2,1)

EVALUATE THE JACOBIAN MATRIX AT POINT (R,S)

DO 20 I = 1, 2
  DO 20 J = 1, 2
    DUM = 0.0
  10 DO 10 K = 1, 4
    DUM = DUM + P(I,K)*XX(I,K)
  20 XJI(I,J) = DUM

COMPTE THE DETERMINANT OF THE JACOBIAN MATRIX

DET = XJI(1,1)*XJI(2,2) - XJI(2,1)*XJI(1,2)
IF (DET .GT. 0.00000001) GO TO 30
WRITE (6,200) M
STOP

CALCULATE THE INVERSE OF THE JACOBIAN MATRIX

30 DUM = 1/DET
  XJI(1,1) = XJI(2,2)*DUM
  XJI(1,2) = XJI(1,2)*DUM
  XJI(2,1) = XJI(2,1)*DUM
  XJI(2,2) = XJI(1,1)*DUM

CALCULATE THE DERIVATIVES OF DISPLACEMENT WITH RESPECT TO NATURAL COORDINATES R AND S

DO 40 I = 1, 2
  DO 40 J = 1, 2
    UR(I,J) = 0.0
    URT(I,J) = 0.0
  40 DO 40 K = 1, 4
    UR(I,J) = UR(I,J) + P(I,K)*UU(I,K)
    URT(I,J) = URT(I,J) + P(I,K)*URT(I,K)

CALCULATE THE DERIVATIVES OF U AND UT WITH RESPECT TO X1 AND X2

DO 50 I = 1, 2
  DO 50 J = 1, 2
    U(I,J) = 0.0
    UUT(I,J) = 0.0
  50 DO 50 K = 1, 2
    U(I,J) = U(I,J) + XJI(I,K)*UR(I,K)
    UUT(I,J) = UUT(I,J) + XJI(I,K)*URT(I,K)

COMPTE STRAIN INCREMENT

DO 70 I = 1, 2
  DO 70 J = 1, 2
    SUM = 0.0
  70 DO 70 K = 1, 2

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60  SUM = SUM + 0.5*(UUT(K,J)*U(K,J) + U(K,I)*UUT(K,J) + U(K,I) * U(K,J))
70  ST(I,J) = 0.5*(U(I,J) + U(J,I)) + SUM
C
C    STRA(1) = ST(1,1)
C    STRA(2) = ST(2,2)
C    STRA(3) = ST(1,2)
C
C    COMPUTE LINEAR STRAIN INCREMENT
C
C    DO 90 I = 1, 2
C    DO 90 J = 1, 2
C    SUM = 0.0
C    DO 80 K = 1, 2
C    SUM = SUM + 0.5*(UUT(K,J)*U(K,J) + U(K,I)*UUT(K,J))
80  ST(I,J) = 0.5*(U(I,J) + U(J,I)) + SUM
C
C    STRAL(1) = ST(1,1)
C    STRAL(2) = ST(2,2)
C    STRAL(3) = 2*ST(1,2)
C
C    RETURN
C
200  FORMAT ('**** ERROR, ZERO OR NEGATIVE JACOBIAN DETERMINANT
C    +FOR ELEMENT',I4)
C    END
C
SUBROUTINE STREST(M)
C
C    * SUBROUTINE TO CALCULATE STRESSES FOR TRANSITION ELEMENT  *
C    * XX - NODAL COORDINATES AT TIME T      *
C    * UT - DIRECTION CONSINE                 *
C    * UU - DISPLACEMENT IncreMENTS           *
C    * C - CONSTITUTIVE MATRIX                *
C    * Tau - STRESS VECTOR                    *
C    * STRA - STRAIN VECTOR                   *
C    * PRT - POISSON'S RATIO                  *
C    * E - YOUNG'S MODULUS                    *
C    * OTHER VARIABLE NAMES ARE EXPLAINED IN MAIN PROGRAM AND *
C    * OTHER SUBROUTINE                       *
C
C
COMMON N0P,N0EL,NMTR,NTRAC,NMBDY,E(20),PAR(20),R0(20),
IND(480,5),X(600),Y(600),AX(600),ALY(600),KSTRG(600),ITRN(20),
2TRN(20),TRAX(20,2),TRAY(20,2),TH(20),SITA(20),CO(20),NETEL,NEBEL,
3LAST,WIDTH(20),KROT(100),ALZ(100),ITYP,YNQ,POS
C
COMMON/TWO/IBWTH,MEQ,AAR(1300),ASTF(1300,90),SIG(480,4),FLAG(450),
1UT1(600),UT2(600),UT3(100),STRN(480,3),VS2(100,2),VSB2(50,3,2),
2XT(600,2)
C
DIMENSION XX(2,3),UT(2),UU(2,3),STRA(3),TAU(4),C(3,3),STRAL(3)
C
DATA RR,0.0/,SS,0.0/
C
C    ASSIGN ELEMENT COORDINATES, DISPLACEMENTS AND DIRECTION CONSINE
C
C    DO 51 = 1,3
C      XX(I,1) = XT(IND(M,I),1)
5    XX(2,I) = XT(IND(M,I),2)
C    DO 61 = 1,2
C      UU(I,J) = AAR(2*IND(M,I)-1)
6    UU(2,1) = AAR(2*IND(M,I))
UU(1,3) = AAR(3*IND(M,3)-LAST-2)
UU(2,3) = AAR(3*IND(M,3)-LAST-1)
ANG = AAR(3*IND(M,3)-LAST)
DO 7 I = 1, 2
7     UT(I) = VS2(IND(M,3)-LAST,I)
WIDE = WIDTH(IND(M,5))

C CALCULATE STRAIN

CALL STRAIT(XX,UU,UT,STRA,STRAL,RR,SS,WIDE,ANG,M)

C ASSIGN STRAIN VALUES TO GLOBAL STRAIN MATRIX

DO 10 I = 1, 3
     STRN(M,I) = STRA(I)
10
C INITIATE LINEAR C MATRIX FOR PLANE STRAIN PROBLEM

MATR = IND(M,4)
PRT = PAR(MATR)
BULK = E(MATR) (1.- PRT**2)
SHEAM = E(MATR) (2.* (1. + PRT))
C(1,1) = BULK
C(1,2) = PRT*BULK
C(1,3) = 0.0
C(2,1) = C(1,2)
C(2,2) = C(1,1)
C(2,3) = 0.0
C(3,1) = C(1,3)
C(3,2) = C(2,3)
C(3,3) = SHEAM

C CALCULATE STRESS

DO 25 I = 1, 3
     TAU(I) = SIG(M,I)
25     DO 25 J = 1, 3
     TAU(I) = TAU(I) + C(I,J)*STRAL(J)
     TAU(4) = PRT*(TAU(1)+TAU(2))
25     DO 40 I = 1, 4
     SIG(M,I) = TAU(I)
40
RETURN
END

SUBROUTINE STRAIT(XX,UU,VS,STRA,STRAL,R,S,WIDE,ANG,M)

***************************************************************

* SUBROUTINE TO CALCULATE STRAIN INCREMENT
* XX - ELEMENT NODAL COORDINATES AT TIME T
* UU - DISPLACEMENT INCREMENTS
* H - INTERPOLATION FUNCTIONS
* P - DERIVATIVES OF INTERPOLATION FUNCTIONS WITH
*     RESPECT TO NATURAL COORDINATES
* STRA - STRAIN VECTOR
* UR - DERIVATIVES OF DISPLACEMENT INCREMENT WITH
*     RESPECT TO NATURAL COORDINATES
* ANG - ROTATION INCREMENT
* XI - JACOBIAN MATRIX
* XJ - INVERSE OF JACOBIAN MATRIX
* U - DERIVATIVES OF DISPLACEMENT INCREMENT WITH
*     RESPECT TO GLOBAL COORDINATES
* VS - DIRECTION COSINE AT TIME T

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C * ST - STRAIN MATRIX
C * DET - DETERMINANT OF JACOBIAN MATRIX
C
REAL XX(2,3), UU(2,3), VS(2), HI(3), P(2,3), STRA(3), U(2,2),
1ST(2,2), V(2,2), XI(2,2), XH(2,2), STRAL(3)
C
RP = 1.0 + R
RM = 1.0 - R
SP = 1.0 + S
SM = 1.0 - S
C
C INTERPOLATION FUNCTIONS
C
H(1) = 0.25*RM*SM
H(2) = 0.25*RP*SM
H(3) = 0.5*SP*S
C
C DERIVATIVES OF INTERPOLATION FUNCTIONS WITH RESPECT TO NATURAL
C COORDINATE
C
P(1,1) = -0.25*SM
P(1,2) = -P(1,1)
P(1,3) = 0.0
C
P(2,1) = -0.25*RM
P(2,2) = -0.25*RP
P(2,3) = 0.5 + S
C
C EVALUATE THE JACOBIAN MATRIX AT POINT (R,S)
C
DO 20 I = 1, 2
DO 20 J = 1, 2
   DUM = 0.0
   DO 10 K = 1, 3
      DUM = DUM + P(I,K)*XX(J,K)
   10  XJ(I,J) = DUM
   XJ(1,1) = XJ(1,1)-0.5*WIDE*H(3)*VS(1)
   XJ(1,2) = XJ(1,2)-0.5*WIDE*H(3)*VS(2)
   XJ(2,1) = XJ(2,1)-0.5*R*WIDE*P(2,3)*VS(1)
   XJ(2,2) = XJ(2,2)-0.5*R*WIDE*P(2,3)*VS(2)
C
C COMPUTE THE DETERMINANT OF THE JACOBIAN MATRIX
C
DET = XJ(1,1)*XJ(2,2) - XJ(2,1)*XJ(1,2)
IF (DET .GT. 0.00000001) GO TO 30
WRITE (6,200) M
STOP
C
C CALCULATE THE INVERSE OF THE JACOBIAN MATRIX
C
30  DUM = 1/DET
     XJ(1,1) = XJ(2,2)*DUM
     XJ(1,2) = -XJ(1,2)*DUM
     XJ(2,1) = -XJ(2,1)*DUM
     XJ(2,2) = XJ(1,1)*DUM
C
C COMPUTE DERIVATIVES OF DISPLACEMENT WITH RESPECT TO
C NATURAL COORDINATES R AND S
C
DO 40 I = 1, 2
DO 40 J = 1, 2
UR(I,J) = 0.0
DO 40 K = 1, 3
40 UR(I,J) = UR(I,J) + P(J,K)*U(I,K)
UR(I,1) = UR(I,1) + 0.5*WIDE*H(I)*VS(2)*ANG
UR(I,2) = UR(I,2) + 0.5*WIDE*R*P(2,3)*VS(2)*ANG
UR(2,1) = UR(2,1) - 0.5*WIDE*H(3)*VS(I)*ANG
UR(2,2) = UR(2,2) - 0.5*WIDE*R*P(2,3)*VS(I)*ANG
C CALCULATE THE DERIVATIVES OF U AND UT WITH RESPECT TO X1 AND X2
C DO 50 I = 1, 2
50 DO 50 J = 1, 2
   U(I,J) = 0.0
   DO 50 K = 1, 2
   50   U(I,J) = U(I,J) + XIJ(I,K)*UR(I,K)
C C COMPUTE STRAIN INCREMENT
C DO 70 I = 1, 2
70 DO 70 J = 1, 2
   SUM = 0.0
   DO 60 K = 1, 2
   60   SUM = SUM + 0.5*U(K,I)*U(K,J)
   70   ST(I,J) = 0.5*(U(I,J) + U(J,I)) + SUM
C C STRA(I) = ST(I,1)
C STRA(2) = ST(2,2)
C STRA(3) = ST(I,2)
C C COMPUTE LINEAR STRAIN INCREMENT
C DO 80 I = 1, 2
80 DO 80 J = 1, 2
   80   ST(I,J) = 0.5*(U(I,J) + U(J,I))
C C STRAL(1) = ST(I,1)
C STRAL(2) = ST(2,2)
C STRAL(3) = 2*ST(I,2)
C C RETURN
C 200 FORMAT ("*** ERROR, ZERO OR NEGATIVE JACOBIAN DETERMINANT"
C + FOR ELEMENT",14)
C END
C SUBROUTINE STRESB(M)
C
C ・ SUBROUTINE TO CALCULATE STRESSES FOR BEAM ELEMENT
C ・ XX - NODAL COORDINATES AT TIME T
C ・ VS - DIRECTION CONSINE
C ・ UU - DISPLACEMENT INCREMENTS
C ・ C - CONSTITUTIVE MATRIX
C ・ TAU - STRESS VECTOR
C ・ STRA - STRAIN VECTOR
C ・ PRT - POISSON'S RATIO
C ・ E - YOUNG'S MODULUS
C ・ OTHER VARIABLE NAMES ARE EXPLAINED IN MAIN PROGRAM AND
C ・ OTHER SUBROUTINE
C
C COMMON NOP,NOEL,NMATR,NTRAC,MBDY,F(20),PAR(20),ROI(20),
C 1IND(480,5),X(600),Y(600),ALX(600),ALY(600),KSTRG(600),JTRN(20),
C 2JTRN(20),TRAX(20,2),TRAY(20,2),TH(20),SETA(20),CO(20),NETEL,NEBEL,
3LAST,WIDTH(20),KROT(100),ALZ(100),ITYP,YNG,POS

C COMMON/TWO IBWTH,MEQ,AAR(1300),ASTF(1300,90),SIG(480,4),FLAG(450),
1 UT(600),UT2(600),UT3(100),STRN(480,3),VS2(100,2),VSB2(50,3,2),
2XT(600,2)

C COMMON/FIVE:Z1(50,3,2,4), Z2(50,3,2,4)

C DIMENSION XX(2,3),VS(3,2),UL(2,3),STRA(3),ANG(3),C(2,2),STRAL(3)
1,RR(3),SS(2)

C DATA RR/-0.7745966692,0,0.0.7745966692/, 2SS,0.5773502691,-0.5773502691/

C ASSIGN ELEMENT COORDINATES, DISPLACEMENTS AND DIRECTION COSINE

C DO 1 I = 1,3
X1(I,1) = XT(IND(M,I,1))
X2(I,1) = XT(IND(M,I,2))
DO 6 I = 1,3
U1(I,1) = AAR(3*IND(M,I)-LAST-2)
U2(I,1) = AAR(3*IND(M,I)-LAST-1)
ANG(I) = AAR(3*IND(M,I)-LAST)
VS(I,1) = VSB2(M-NEBEL + 1,1,1)
VS(I,2) = VSB2(M-NEBEL + 1,1,2)
WIDE = WIDTH(IND(M,5))

C INITIATE LINEAR C MATRIX FOR BEAM ELEMENT

C MATR = IND(M,4)
PRM = PAR(MATR)
SHEAM = E(MATR)/(2.*(1.+PRM))
C(1,1) = E(MATR)
C(1,2) = 0.0
C(2,1) = C(1,2)
C(2,2) = (5./6.)*SHEAM

C DO 30 LX = 1,3
R1 = RR(LX)
DO 30 LY = 1,2
S1 = SS(LY)

C CALCULATE STRAIN

C CALL STRAIB(XX,U1,VS,STRA,STRAL,R1,S1,WIDE,ANG,M)

C ASSIGN STRAIN VALUES TO GLOBAL STRAIN MATRIX

C DO 10 I = 1,3
10 STRN(M,I) = STRA(I)

C CALCULATE STRESS

C DO 25 I = 1,2
DO 25 J = 1,2
25 Z1(M-NEBEL + 1,LX,LY,J) = Z1(M-NEBEL + 1,LX,LY,J) + C(I,J)*STRAL(J)
CONTINUE

C CALL STRAIB(XX,U1,VS,STRA,STRAL,0.0,1.0,WIDE,ANG,M)

C DO 35 I = 1,2

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DO 35 J = 1, 2
35     SIG(M,J) = SIG(M,I) + C(I,J)*STRAL(J)
C
RETURN
END
C
SUBROUTINE STRAIB(XX, UU, VS, STRA, STRAL, R, S, WIDE, ANG, M)

**SUBROUTINE TO CALCULATE STRAIN INCREMENT FOR BEAM ANALYSIS**

* ELEMENT
  * XX - ELEMENT NODAL COORDINATES AT TIME T
  * UU - DISPLACEMENT INCREMENTS
  * HH - INTERPOLATION FUNCTIONS
  * PP - DERIVATIVES OF INTERPOLATION FUNCTIONS WITH RESPECT TO NATURAL COORDINATES
  * STRA - STRAIN VECTOR
  * UB - DERIVATIVES OF DISPLACEMENT INCREMENT WITH RESPECT TO GLOBAL COORDINATES BEFORE TRANSFORMATION
  * U  - DERIVATIVES OF DISPLACEMENT INCREMENT WITH RESPECT TO GLOBAL COORDINATES AFTER TRANSFORMATION
  * VS - DIRECTION COSINE AT TIME T
  * ST - STRAIN MATRIX
  * DET - DETERMINANT OF JACOBIAN MATRIX

REAL XX(2,3), UU(2,3), VS(3,2), HH(3), PP(2,3), STRA(3), UB(2,2),
     IST(2,2), VS(2,2), XJ(2,2), XJJ(2,2), HH(2,3), G1(3,2), G2(3,2),
     2G(3,2,2), ANG(3), STRAL(3)

**INTERPOLATION FUNCTIONS**

H(1) = 0.5*R*(1-R)
H(2) = 0.5*R*(1 + R)
H(3) = 1 - R * R

**DERIVATIVES OF INTERPOLATION FUNCTIONS WITH RESPECT TO NATURAL COORDINATE**

P(1,1) = -0.5 + R
P(1,2) = 0.5 + R
P(1,3) = -2*R

P(2,1) = 0.0
P(2,2) = 0.0
P(2,3) = 0.0

**EVALUATE THE JACOBIAN MATRIX AT POINT (R,S)**

DO 20 J = 1, 2
   DO 20 K = 1, 3
       DUM = 0.0
       DO 10 K = 1, 3
          DUM = DUM + P(I,K)*XX(J,K)
   10      XJ(J,K) = DUM
   20      XJ(J,1) = XJ(J,1) + 0.5*S*WIDE*P(I,K)*VS(K,1)
   XJ(J,2) = XJ(J,2) + 0.5*S*WIDE*P(I,K)*VS(K,2)
XJ(2,1) = XJ(2,1) + 0.5*WIDE*11(K)*VS(K,1)
XJ(2,2) = XJ(2,2) + 0.5*WIDE*11(K)*VS(K,2)

25  CONTINUE

C  COMPUTE THE DETERMINANT OF THE JACOBIAN MATRIX
C
DET = XJ(1,1)*XJ(2,2) - XJ(2,1)*XJ(1,2)
IF (DET .GT. 0.00000001) GO TO 30
WRITE(6,200) M
STOP
C
C  CALCULATE THE INVERSE OF THE JACOBIAN MATRIX
C
30  DUM = 1./DET
XJ(1,1) = XJ(2,2)*DUM
XJ(1,2) = -XJ(1,2)*DUM
XJ(2,1) = -XJ(2,1)*DUM
XJ(2,2) = XJ(1,1)*DUM

C  COMPUTE DERIVATIVES OF INTERPOLATION FUNCTIONS WITH RESPECT TO
C  GLOBAL COORDINATES X AND Y
C
DO 40 I = 1, 3
  HH(I,1) = 0.0
  HH(I,2) = 0.0
DO 50 J = 1, 2
  HH(I,J) = HH(I,1) + XJ(I,J)*P(J,1)
  HH(I,J) = HH(I,2) + XJ(I,J)*P(J,2)
40  CONTINUE

C  COMPUTE G'S USED IN [B] MATRIX
C
DO 60 K = 1, 3
  G1(K,1) = -0.5*WIDE*VS(K,2)
  G1(K,2) = 0.5*WIDE*VS(K,1)
  G2(K,1) = S*G1(K,1)
  G2(K,2) = S*G1(K,2)
60  DO 70 K = 1, 3
70  DO 70 J = 1, 2
   G(K,J) = XH(I,J)*G2(K,J)*P(I,K) + XJ(I,J)*G1(K,J)*11(K)
70  CONTINUE

C  DERIVATIVES OF DISPLACEMENT INCREMENT WITH RESPECT TO GLOBAL
C  COORDINATES BEFORE TRANSFORMATION
C
DO 80 I = 1, 3
  DO 90 J = 1, 2
    UB(I,J) = 0.0
  DO 80 J = 1, 3
80  UB(I,J) = UB(I,J) + HH(I,J)*UU(I,K) + G(K,I,J)*ANG(K)

C  DERIVATIVES AFTER TRANSFORMATION
C
VR1 = VS(2,2)
VR2 = -VS(2,1)
U(1,1) = VR1*VR1*UB(1,1) + VR2*VR2*UB(2,2) + VR1*VR2*(UB(2,1) + UB(1,2))
U(1,2) = VR1*VR1*UB(1,2) - VR2*VR2*UB(1,2) + VR1*VR2*(UB(2,2) - UB(1,1))
U(2,1) = UB(2,1)*VS(2,2)**2 - UB(1,2)*VS(2,1)**2 + VS(2,1)*VS(2,2)*
       (UB(1,1) - UB(2,2))
U(2,2) = UB(1,1)*VS(2,1)**2 + UB(2,2)*VS(2,2)**2 + VS(2,1)*VS(2,2)*
       (UB(1,2) + UB(2,1))
C CALCULATE STRAIN
C
DO 85 I = 1,2
DO 85 J = 1,2
SUM = 0.0
DO 80 K = 1,2
80 SUM = SUM + 0.5*U(K,I)*U(K,J)
85 ST(I,J) = 0.5*(U(I,J) + U(J,I)) + SUM
C
STRA(1) = ST(1,1)
STRA(2) = ST(1,2)
STRA(3) = ST(2,2)
C
C CALCULATE LINEAR STRAIN
C
DO 90 I = 1,2
DO 90 J = 1,2
90 ST(I,J) = 0.5*(U(I,J) + U(J,I))
C
STRL(1) = STL(1,1)
STRL(2) = 2*STL(1,2)
STRL(3) = STL(2,2)
C
C RETURN
C
200 FORMAT ('*** ERROR, ZERO OR NEGATIVE JACOBIAN DETERMINANT'
+ FOR ELEMENT',I4)
END
C
SUBROUTINE ADD(CRIT)
C
******
C * SUBROUTINE TO CALCULATE TOTAL DISPLACEMENT, DIRECTION
C * AND CONVERGENCE CRITERION
C * CRIT - CONVERGENCE CRITERION
C * SPACE - OPEN SPACE BETWEEN ROOF AND FLOOR
C * OTHER VARIABLES ARE EXPLAINED ELSEWHERE
******
C
COMMON NOP,NOEL,NMATR,NTRAC,MBDY,E(20),PAR(20),RO(20),
IN(480),XI(600),Y(600),ALX(600),ALY(600),KSTRG(600),IFRN(20),
2JTRN(20),TRAX(20,2),TRAY(20,2),TH(20),SETA(20),CO(20),NETEL,NEBEL,
LAST,WIDTH(20),KROT(100),ALZ(100),ITYP,YN,POS
C
COMMON/TWO!BWTII,MEQ,AAR(1000),ASTF(1000,90),SIG(480,4),FLAG(450),
1UT(600),UT2(600),UT3(190),STRN(480,3),VS(100,2),VSB(100,3,2),
2XT(600,2)
C
COMMON/THREE/IM,IST
C
DIMENSION VS(2)
C
ADD DISPLACEMENT INCREMENT TO THE DISPLACEMENT
C
DO 30 I = 1, LAST
UT1(I) = UT1(I) + AAR(2*I-1)
UT2(I) = UT2(I) + AAR(2*I)
XT(I,1) = X(I) + UT1(I)
30 XT(I,2) = Y(I) + UT2(I)
C
IF (ITYP .EQ. 0) GO TO 34
C
DO 32 I = LAST+1, NOP
UT1(I) = UT1(I) + AAR(3*I-LAST-2)
UT2(I) = UT2(I) + AAR(3*I-LAST-1)
UT3(I-LAST) = UT3(I-LAST) + AAR(3*I-LAST)
XT(I-1) = X(I) + UT1(I)
XT(I-2) = Y(I) + UT2(I)
ANG = AAR(3*I-LAST)
VS(1) = VS2(I-LAST,1)
VS(2) = VS2(I-LAST,2)

CALL DIRCOS(ANG,VS)

VS2(I-LAST,1) = VS(1)
VS2(I-LAST,2) = VS(2)
CONTINUE
DO 33 I = NEBEL, NOEL
   DO 33 J = 1, 3
      VS(1) = VS2(I-NEBEL+1,J,1)
      VS(2) = VS2(I-NEBEL+1,J,2)
      ANG = AAR(3*IND(I,J-LAST))
      CALL DIRCOS(ANG,VS)
   END DO
   VSB2(I-NEBEL+1,J,1) = VS(1)
   VSB2(I-NEBEL+1,J,2) = VS(2)
CONTINUE

CONVERGENCE CRITERION

CRIT1 = 0.0
DO 35 J = 1, MEQ
   CRIT1 = CRIT1 + AAR(I)**2
   CRIT2 = 0.0
   DO 40 I = 1, LAST
      CRIT2 = CRIT2 + UT1(I)**2 + UT2(I)**2
   END DO
IF (ITYP .EQ. 0) GO TO 42
END DO

DO 41 I = LAST+1, NOP
   CRIT2 = CRIT2 + UT1(I)**2 + UT2(I)**2 + UT3(I-LAST)**2
   CRIT = SQRT(CRIT1,CRIT2)
RETURN

END

SUBROUTINE DIRCOS(ANG,VS)

* CALCULATE DIRECTION CONSINE *
* ANG - ROTATION INCREMENT *
* VS - DIRECTION CONSINE *

REAL VS(2)

DIVIDE THE ROTATION INCREMENT INTO SUBINCREMENT
CREM = ANG/10.0

INTEGRATE USING EULER FORWARD METHOD
DO 5 1 = 1, 10
   VS(1) = VS(1)-VS(2)*CREM
   VS(2) = VS(2)+VS(1)*CREM
   SQ = VS(1)**2+VS(2)**2

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VS(1) = VS(1) * SQRT(SQ)
VS(2) = VS(2) * SQRT(SQ)
CONTINUE
RETURN
END

SUBROUTINE FSTIF(FKN,FKS,XX,ESTIF)

******************************************************************************
* SUBROUTINE TO CALCULATE THE STIFFNESS MATRIX FOR                         *
* FRICTION ELEMENTS.                                                       *
* FK - STIFFNESS MATRIX BEFORE TRANSFORMATION                               *
* T - TRANSFORMATION MATRIX                                                *
* DIST - AVERAGE LENGTH OF THE ELEMENT                                     *
******************************************************************************

DIMENSION FK(8,8),T(8,8),XX(2,4),ESTIF(8,8),BC(8)

INITIATION

DO 5 I = 1,8
  DO 5 J = 1,8
    FK(I,J) = 0.0
    T(I,J) = 0.0
    ESTIF(I,J) = 0.0
  5 CONTINUE

COMPUTE THE DIRECTION COSINE AND TRANSFORMATION MATRIX

X1 = XX(1,2) - XX(1,1)
Y1 = XX(2,2) - XX(2,1)
X2 = XX(1,3) - XX(1,4)
Y2 = XX(2,3) - XX(2,4)

DIST1 = SQRT(X1**2 + Y1**2)
T1 = X1/DIST1
T2 = Y1/DIST1
DIST2 = SQRT(X2**2 + Y2**2)
TT1 = X2/DIST2
TT2 = Y2/DIST2

DO 10 I = 1,3,2
  T(I,I) = T1
  T(I,I+1) = T2
  T(I+1,I) = -T2
10   T(I+1,I+1) = T1

DO 12 I = 5,7,2
  T(I,I) = TT1
  T(I,I+1) = TT2
  T(I+1,I) = -TT2
12   T(I+1,I+1) = TT1

XM1 = (XX(1,1) + XX(1,4))/2
YM1 = (XX(2,1) + XX(2,4))/2
XM2 = (XX(1,2) + XX(1,3))/2
YM2 = (XX(2,2) + XX(2,3))/2

DIST = SQRT((XM2-XM1)**2 + (YM2-YM1)**2)
FORMATIONS OF THE LOCAL STIFFNESS MATRIX

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DIAGONAL TERMS

DO 15 I = 1,4
    FK(2*I-1,2*I-1) = 2*FKS
15    FK(2*I,2*I) = 2*FKN

FK(1,3) = FKS
FK(1,5) = -FKS
FK(1,7) = -2*FKS
FK(2,4) = FKN
FK(2,6) = -FKN
FK(2,8) = -2*FKN
FK(3,5) = -2*FKS
FK(3,7) = -FKS
FK(4,6) = -2*FKN
FK(4,8) = -FKN
FK(5,7) = FKS
FK(6,8) = FKN

DO 20 I = 2,8
    DO 20 J = 1,J-1
20    FK(I,J) = FK(J,I)

DO 25 I = 1,8
    DO 25 J = 1,8
25    FK(I,J) = DIST*FK(I,J)/6

TRANSFORM THE LOCAL STIFFNESS MATRIX INTO GLOBAL

DO 45 I = 1,8
    DO 30 K = 1,8
30          BC(K) = 0.0
    DO 30 L = 1,8
30          BC(K) = BC(K) + T(L,I)*FK(L,K)
    DO 40 J = 1,8
40          STIFF = 0.0
    DO 40 K = 1,8
40          STIFF = STIFF + BC(K)*T(K,J)
    DO 45 J = 1,8
45          ESTIFF(I,J) = STIFF
CONTINUE

RETURN
END
Vita

Zhanjing Yu was born on August 9, 1957 in Pingdu, Shandong, People's Republic of China. He attended primary and secondary school there and graduated from Guxian Middle School in 1975. Subsequently, he worked as a farmer and a middle school teacher. In 1978, he went to Shandong Institute of Mining and Technology, China, where he was awarded a Bachelor of Science degree in Mining Engineering in 1982. In 1984, he went to West Virginia as a visiting scholar. He began his graduate study at Virginia Tech in June, 1985, was awarded a Master of Science degree in Mining Engineering in March, 1987 and expects to complete the requirements for a Ph.D in December, 1990.

He received the Excellent Student Award in 1980 from Shandong Institute of Mining and Technology, and the Outstanding Graduate Student Award in 1987 at Virginia Tech. He is a member of the Honor Society of Phi Kappa Phi.