Fully-Coupled Fluid-Structure Analysis of a
Baffled Rectangular Orthotropic Plate Using the
Boundary Element and Finite Element Methods

by
Thomas Harris Fronk

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APPROVED:

Dr. J. R. Mahan, Chairman

Dr. C. E. Knight
Dr. C. A. Rogers

Dr. C. R. Fuller
Dr. S. M. Holzer

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(ABSTRACT)

Laminated composite plates have become an important and proven structural material in aerospace and ocean vehicles. However, because of the inherent orthotropy of laminated composite materials the analysis of these structures is complex and usually cannot be adequately performed using classical methods. In this dissertation the formulation of the fully coupled fluid-structure interaction of a laminated composite plate and its surrounding fluid medium is presented. The solution technique involves the finite element method for modeling the structural response and the boundary element method for modeling the acoustic field. The model incorporates the Mindlin plate theory which includes five degrees of freedom. An improved integration technique is demonstrated which significantly reduces the approximation error. Storage requirements are reduced by grouping complex numbers. Finally the fully coupled fluid-structure interaction involving laminated composite plates is modeled using the combined FEM-BEM approach demonstrating the usefulness and the significance of the method.
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Nomenclature

\(a, b\) dimensions of a rectangular plate or piston in the \(x\)-direction and \(y\)-direction respectively, m

\(A_{ij}\) extensional stiffness terms, Pa-m

\(B_{ij}\) bend-extensional coupling terms, Pa-m\(^2\)

\(c\) speed of sound, m/s

\(\tilde{C}_{ij}\) transformed stiffness matrix, Pa

\(D_{ij}\) flexural stiffness terms, Pa-m\(^3\)

\(E_i\) elastic moduli, Pa

\(F_i\) force vector, N

\(g\) Green’s function

\(G_{ij}\) shear moduli, N-m

\(h\) plate thickness, m

\([J]\) Jacobian matrix

\(k\) wave number, \(\omega/c\)

\(K_{ij}\) plate stiffness terms, Pa-m

\(L_{ij}\) fluid-structure coupling terms, m\(^3\)
\( M_{ij} \) plate mass terms, \( m^3 \)

\( NE \) number of nodes / element

\( NM \) number of elements in mesh

\( P \) pressure, \( Pa \)

\( r_s \) radial distance to source point, \( m \)

\( r_f \) radial distance to field point, \( m \)

\( R \) \( |r_s - r_f| \), \( m \)

\( u, v, w \) displacements in the x-, y- and z-directions respectively, \( m \)

\( Z \) radiation impedance, \( N\cdot s/m \)

\( \gamma \) shear strains

\( \delta \) dirac delta function

\( \varepsilon \) normal strains

\( \zeta, \eta \) coordinates of master element, \( m \)

\( \theta_i \) incident angle

\( \kappa \) curvature of plate, \( m^{-1} \)

\( \lambda \) test function

\( \nu_{ij} \) Poisson’s ratio

\( \Pi \) acoustic power, \( N\cdot m/s \)

\( \rho_f \) fluid density, \( kg/m^3 \)

\( \rho_s \) plate density, \( kg/m^3 \)

\( \sigma_{ij} \) normal stress, \( Pa \)

\( \phi_i \) shape functions
\( \omega \)  
angular frequency
1. Introduction

1.1 Motivation for Research

The need for strong yet light-weight structural materials has emerged with the evolution of aerospace and ocean structures. Many missiles and aircraft, and even the space shuttle, now use parts made of composite materials which have large strength-to-weight and stiffness-to-weight ratios. One of the most common forms of composite material is the laminated plate. Several layers of unidirectionally aligned fibers bound with resin, called laminae, are bonded together to form a solid plate, called a laminate. The individual laminae are oriented in certain directions, thereby giving the laminate the necessary strength and stiffness in the directions of the anticipated loads. A typical laminated plate is shown in Fig. 1. A laminated composite plate is an orthotropic, inhomogenous material. Orthotropic means that the material properties are directional dependent with three orthogonal planes of material-property symmetry, and inhomogenous means that the material properties are location dependent. Because of their orthotropy and nonhomogenity, many complications and complexities arise in the analysis and de-
scription of laminated composites that do not exist with conventional isotropic, homogenous materials. These include coupling between bending and extension, bending and twisting, and shear and extension.

The normal function of aerospace and ocean structures exposes these composite parts to a variety of harsh environmental conditions including rapid temperature changes, large temperature gradients, and high levels of acoustic loading. The effects of rapid temperature changes and large heat fluxes on laminated composite panels have been examined by many authors [1-8], including the current author [9]. In this latter case a geometrically nonlinear finite element model was developed and utilized to study the effects of constant pressure and rapid temperature change simultaneously applied to a composite panel.

The response of laminated composite plates subject to periodic sound radiation has received considerably less attention. Of particular interest is the effect on composite panels of tonal-type sound sources such as compressors, jet engine turbines, turboprops, and reciprocating engine exhausts. A structure set in motion by a periodic sound wave reflects and radiates pressure waves back through the fluid medium. The radiated pressure constitutes what is termed radiation loading. Then the resultant force on the surface of the structure is due both to the distributed pressure from the incident sound wave and to the radiation loading. Because the motion of the structure and the resultant surface force are interdependent, the fluid-structure system is said to be fully coupled.

Studies showing transmission losses through laminated composite panels have been conducted [10-12]. However, these investigations have neglected radiation loading and have employed analytical models which are restricted to symmetric laminated composite plates. In most fluid-structure systems where the fluid is low density or the structure is very stiff the radiation loading effect can indeed be neglected. In such cases the equations governing the fluid-structure system are only semicoupled.
The purposes of the work described in this dissertation are then:

1. to formulate and evaluate a numerical method to model the fully coupled and semicoupled fluid-structure interaction of an arbitrary laminated composite plate immersed in a fluid and driven by a periodic pressure,

2. to use this numerical tool to further characterize the behavior of laminated composite plates subject to periodic acoustic radiation, and

3. to establish the relative importance of radiation loading on orthotropic plates for both light and heavy fluids.

1.2 Problem Description

In order to systematically examine the fully coupled fluid-structure interaction problem an appropriate test model must be chosen. In this section that test model is described along with appropriate restrictions that accompany the model and that limit the range of applicability of the results.

Rectangular plates best represent the geometry of laminated composite panels utilized in aerospace and industrial applications. Therefore, the test model used throughout this work is a rectangular plate composed of linear elastic orthotropic layers. The rectangular plate is surrounded by a perfectly rigid motionless surface in the same plane as the plate surface. This rigid surface extends to infinity in all directions on the plane surface and so is termed an infinite baffle. The geometry of the rectangular plate and infinite baffle system is shown in Fig. 2.
A schematic of the fully coupled fluid-structure interaction of a vibrating plate is shown in Fig. 3. A homogenous inviscid fluid occupies the semi-infinite regions on each side of the plate and baffle surfaces. The arrow labeled $P_i$ in Fig. 3 represents the sound waves incident on the plate from the left at an arbitrary angle $\theta$. The source is assumed to be located a large distance from the plate and therefore the incident sound wave, $P_i$, is in the form of a plane wave. Plane waves are characterized by the fact that all acoustic variables are constant on any plane perpendicular to the direction of propagation of the wave. The pressure radiated and reflected from the moving plate is represented by the arrow labeled $P_\pi$ to the left of the plate, and the arrow labeled $P_r$ represents the pressure radiated into the fluid region to the right of the plate. It is assumed that the wavelength of the incident sound wave is longer than the thickness of the plate, thus assuring that the acoustic energy transmitted to the opposite side of the plate is due only to the motion of the plate and not to any shear waves that are propagated through the thickness of the plate.

The coupled fluid-structure process is depicted as a feedback loop in Fig. 1.3 of reference 13 which is reproduced here as Fig. 4. The transient motion of the laminated composite plate is not studied in this dissertation; therefore, the fluid-structure process shown in Fig. 4 is assumed to be steady-state and the problem is solved in the frequency domain.

The fully coupled fluid-structure problem consists of two interconnected processes:

1. Solve for the pressure on the surface of the plate which results from the incident sound waves and the motion of the plate, and

2. Solve for the motion of the plate which results from the impinging sound waves and the radiation loading.
The motion of the plate and the pressure on the plate surface can be found simultaneously using the finite element method (FEM) to find the structural response and the boundary element method (BEM) to describe the acoustic field. The finite element method and boundary element methods as they apply to this work are developed in detail in the subsequent chapters. However, broadly speaking they are numerical tools employed to solve the acoustic-structure interaction problem by discretizing the structural domain and fluid boundary, respectively.

The problem of solving for the fully-coupled response of a laminated composite plate can be organized into three categories

1. Description of the dynamic behavior of laminated composite plates,

2. Prediction of the acoustic radiation from vibrating plates,

3. Analysis of coupled fluid-structure interaction.

The remainder of the dissertation as well as the literature review of Section 1.3 are organized according to these categories.

1.3 Literature Review

In Subsection 1.3.1 the literature involving the dynamic behavior of laminated composite plates is reviewed. The history of the emergence of shear deformation theories is given and some of the more informative papers associated with the solution of the dynamic behavior of plates by use of the finite element method are cited. In Subsection 1.3.2 the development of the branch of acoustics dealing with radiation from vibrating
surfaces is reviewed. Specifically, closed-form solutions of the radiated field from vibrating plane surfaces are examined. Numerical solutions of the radiation problem are then reviewed with emphasis on the development and emergence of the boundary element method as an important tool in solving radiation problems. In Subsection 1.3.3 analytical closed-form solutions of fluid-loaded elastic plates are examined. Numerical solutions of the coupled fluid-structure problem involving simultaneous use of the boundary element method and finite element method are presented and discussed.

1.3.1 Dynamic Behavior of Laminated Composite Plates

The science of analyzing and designing laminated composite components has received significant attention only in the last twenty to thirty years. In the 1950’s and 60’s an increased interest arose in developing elasticity theories for plates that are nonhomogeneous and anisotropic. Reissner and Stavsky [14] established the elastostatic bending and stretching theory for nonhomogenous plates based on the Euler-Bernoulli hypothesis and the classical Kirchoff hypothesis. This evolved into what is now called the classical plate theory (CPT). Classical lamination theory (CLT) is an extension of CPT to laminated composite plates. Classical lamination theory is generally accepted and widely used to predict laminated composite plate response and in designing components made of laminated composites. Classical lamination theory is relatively easy to use because it employs several simplifying assumptions as discussed in Chapter 2.

However, many researchers have recognized the inadequacies and limitations of classical lamination theory. In 1969, Pagano published a series of papers [15,16,17] in which the deficiencies of classical lamination theory are exposed and explained. One of the most limiting consequences of the CLT assumptions is that accurate predictions are
obtained only when the plate has a high span-to-depth ratio. In other words CLT gives less accurate results as the plate becomes relatively thicker.

The effect of varying material properties of laminated plates makes it impossible to set a specific thickness where CLT yields valid results. Hence, many have attempted to account for thickness effects by including shear-deformation terms. The Mindlin plate theory [18] includes transverse shear effects and, therefore, has served as the basis for developing theories which account for deformation caused by shear. Stravsky [19] was the first to propose shear deformation theory as a solution to analyze laminated isotropic plates. Later, Yang, Norris, and Stavsky [20] generalized the shear deformation theory to include laminated anisotropic heterogenous plates. Whitney and Pagano [21] applied the theory of Yang, Norris, and Stavsky (YNS) to the bending of antisymmetric cross-ply and angle-ply plate strips under sinusoidal load distribution and to free vibration of antisymmetric angle-ply strips, and reported the effects of shear deformation on the dynamic motion of laminated plates to be significant. Reissner [22], Lo, Christensen and Wu [23], and Librescu [24] have all developed higher-order shear-deformation theories to predict the response of laminated composites. Each one of these has included shear deformation effects and consequently led to improved accuracy of their predictions over that afforded by the traditional CLT results (e.g., those of Dong, Pister, and Taylor [25], and Bert and Mayberry [26]). Of great relevance to the research of this dissertation is the work done by Mei and Prasad [27]. They investigated the influence of large deflection and transverse shear on rectangular symmetric laminated composite plates due to acoustic loads. They found that for thick plates the effects due to shear deformation are considerable. Along this same line, Pierccu, Le, and Fang [28] studied the effects of distributed loading on inhomogenous plates.

The finite element method has been used extensively to analyze isotropic plates, but less effort has gone into the investigation of laminated composite plates. Thick lami-
nated plates have been analyzed by Pryor and Barker [29] and by Barker, Lin, and Dara [30] using the conventional displacement finite element method. A plate element with seven degrees of freedom per node (three displacements, two rotations, and two shear rotations) was used. Implementing an element with eighty degrees of freedom, Noor and Mathers [31] developed a finite element model based on Reissner's plate theory. They were able to study the effects of shear deformation on the accuracy of several models. While these models are very accurate they are also computationally very demanding. Models of plate elements with five degrees of freedom (three displacements, two rotations, as shown in Fig. 5) have been formulated by Yang, Norris, and Stravsky [20] and by Reddy [32,33]. They show that in most cases the use of only five degrees of freedom proves sufficiently accurate. Reddy [34] has improved the accuracy of the predictions of the finite element model incorporating the method of reduced integration on the shear deformation terms as proposed by Zienkiewicz, Taylor, and Too [35].

1.3.2 Acoustic Radiation from Vibrating Plates

The theory of acoustic radiation from vibrating structures is a well established branch of classical acoustics and is predicated on works of early mathematicians and physicists such as Galileo, Hooke, D'Alembert, Euler, Helmholtz, and Rayleigh. Exact analytical solutions describing the acoustic field propagating from a vibrating plane surface have been set forth based on these early works [36,37]. However, these analytical solutions are limited to a certain subset of radiator geometries and boundary conditions that are compatible with the mathematics, and in most cases only farfield solutions are possible. In attempting to overcome these limitations some researchers, such as Williams [38], have approximated the solution by a truncated series of spherical Hankel
functions. However, this type of solution becomes less accurate for radiators that are nearly planar.

Analytical solutions describing the acoustic power and pressure radiated by a thin geometrically infinite plate subject to point excitation have received much attention. These solutions are based on asymptotic expansions of Fourier integral representations of the solution. Feit [39] attributes M. Heckl as the first to put forth the solution where the point excitation is a time harmonic force. Thompson and Rattayya [40] proposed a solution to this problem in which the point excitation was a moment. Feit’s [39] solution of the infinite plate is valid for either point forces or point moments. Later, Feit [41] presented an analytical solution for a geometrically-infinite point-excited orthotropic plate. Maidanik [42] using a slightly different approach proposed a solution for power radiated from orthotropic infinite plates. Junger [43] further generalized Feit’s results by allowing the driving force to be arbitrary over the surface of the infinite elastic plate.

The impedance of a square piston is described by Swenson and Johnson [44] using a rather laborious truncated series approximation of Rayleigh’s integral equation. However, this solution is very accurate and can be used as a check for numerical solutions.

The use of numerical methods to describe the acoustic field generated by a vibrating plate have flourished with the arrival of the digital computer. Radiators of arbitrary surface geometry can be modeled giving the acoustic power and pressure anywhere in the fluid or structural domain. Central to the solution of acoustic radiation problems is the surface Helmholtz integral equation. A basic feature of the surface Helmholtz integral equation is that once the pressure is known on the surface of the vibrating structure the pressure anywhere in the fluid domain can be found. Kupradze [45] was evidently the first to present the surface Helmholtz integral equation. Later, Chertock [46] developed a numerical model of an arbitrary radiator by discretizing the surface Helmholtz

1. Introduction
integral equation. When applicable, he found the surface pressures by analytical approximate methods. But in general the surface pressures where found by judicious use of the surface Helmholtz integral equation. Copley [47,48], who concentrated on the interior Helmholtz integral equation, also used numerical techniques to solve radiation problems. Chen and Schweikert [49], used a simple source formulation based on potential theory to describe the radiated field from an arbitrary source. This work laid the foundation for Schneck’s contributions. Schneck’s [50] research was a milestone in the development of numerical methods to solve acoustic radiation problems. He demonstrated that the Helmholtz integral equation has nonunique solutions at certain frequencies. He developed a method for handling this problem called the combined Helmholtz integral equation formulation or CHIEF. A modification of the CHIEF method is now widely accepted. However, a modified version of the Burton and Miller [51] approach seems to be the emerging method of handling the problem of nonuniqueness due to its relative simplicity. Other researchers such as Tobocman [52] and Wilton [53] have also utilized these numerical techniques to examine acoustic scattering.

Concurrent with these advances in the field of numerical solutions of acoustic radiation problems was the advance of what is now termed the boundary element method. In fact, it is difficult to differentiate or separate the emergence of the boundary element method from technical advances in the field of numerically-solved acoustic radiation problems. This is because the boundary element method requires a type of reciprocal work theorem which associates the independent variables and reduces the dimensionality of the problem. The Helmholtz integral equation is in exactly that form and thus is a very willing victim of the boundary element method.

Rizzo [54] appears to have been the first to apply what was then called the boundary integral method (BIE) to planar surfaces. Lachat and Watson [55] streamlined the technique with work they did in elastostatics. Engblom [56] presented a boundary ele-
ment solution of the acoustic radiation problem wherein the pressure of the surface of each element was constant. Later, Seybert, Soenarko, Rizzo, and Shippy [57] developed a boundary element model of an arbitrary acoustic radiator with quadratic elements.

Showing the applicability of the BEM, Jiang and Prasad [58] presented examples of BEM solutions to several radiators including an infinitely baffled circular piston. Their solution was explained in terms of Koopman and Benner's [59] terminology in which the Helmholtz integral equation is interpreted as a body of continuous monopole and dipole sources. The dipoles represent the contribution of the pressure at a point on the surface due to the presence of the boundary, while the monopoles represent the contribution to the pressure at a point on the surface due to the velocity of the boundary.

Fronk and Mahan [60] demonstrated the convenience of the BEM in solving the infinitely baffled rectangular radiator problem by comparing closed-form results to both farfield and nearfield boundary element results.

Acousticians and numerical modelers such as Seybert, et al. [61], Eversine, et al. [62,63], and Coyette, et al. [64], have written commercial codes specifically to solve fluid-structure interaction problems. Each of these employ the BEM to solve for the acoustic pressure and must be coupled to an existing finite element code if the motion of the structure is unknown.

Although the BEM is very convenient and powerful in solving acoustic radiation problems, it is not without its disadvantages. The most severe of these is the fact that there will always be a singular integrand that must be numerically solved. The prevailing method used to eliminate this singularity is to convert to polar coordinates and subdivide the offending element into subelements [65]. The mathematical logic and convenience of this method are very appealing; however, this method cannot be used blindly without considering approximation error, especially when the problem is of the coupled fluid-structure type. Haegemans and Wynendaele [66] have proposed an alternative approach.
in which a cubature formula is employed for the $1/r$-type singularities that occur when employing the BEM for radiation problems.

The problems associated with the BEM have prompted some programmers and acousticians to cling to the FEM when numerically solving radiation problems despite the fact that now the whole fluid domain must be modeled. Gladwell and Mason [67], Craggs [68], Craggs and Stead [69], and Aggarwal, Sinhasan, and Grover [70], have all pursued this venue. Olson and Bathe [71] have proposed the use of infinite elements to model the fluid when the region has semi-infinite bounds. The use of infinite elements is a very interesting concept and with little effort can be customized to the specific problem. Many papers on the subject have been presented in the past decade, with Bettess and Zienkiewicz [72], and Bettess, Emson, and Chiam [73] responsible for some of the more pertinent work.

### 1.3.3 Fully-Coupled Fluid-Structure Solutions

Junger [74] has presented a rather thorough and exhaustive study of the history of progress in the field of fluid-loaded surfaces using analytical solutions. Some of the pertinent papers from his review are discussed in this section along with other analytical approaches. Finally, advances in the field of numerical solutions to the fully coupled fluid-structure problem are reviewed.

One of the more fundamental and interesting studies of the fluid-structure interaction problem was performed by Rayleigh [75]. He studied a one-dimensional spring and piston arrangement in which the presence of the fluid was modeled as an added mass. With the use of submarines in warfare came the need to understand the acoustic interaction of heavy fluids and structures. Lamb [76] analyzed a fluid-loaded clamped
circular plate vibrating in its fundamental axisymmetric mode by employing the low-frequency Laplace approximation.

Researchers studying infinite, fluid-loaded plates include Fay [77], Finney [78], Nayak [79], and Feit and Liu [80]. Using approximate methods Leppardton [81] studied solutions of fluid-loaded elastic finite plates. Leppardton [82] also used asymptotic solutions to study the problem of a finite plate near resonance set in an infinite baffle for low fluid loading. Conversely, Abrahams [83] studied the same problem for heavy fluid loading.

Because of its tractability many researchers such as Lax [84], have studied the fluid-loaded circular plate or diaphragm, again using asymptotic solutions. Dyer [85] and Davies [86] examined the acoustic radiation from fluid-loaded rectangular plates. Dyer was interested in the response of the plate to random pressure fields and Davies' research was particularly aimed at understanding the fluid-structure coupling at limits of both high and low frequencies. Of even more relevance to the current dissertation is the work of Sandman [87], who studied the effects of fluid loading on the motion of a three-layered elastic plate.

As mentioned in the previous subsection, numerical techniques have emerged as powerful methods to study fluid-structure interaction problems. The FEM has remained the principle numerical technique in solving for the dynamic motion of the structure; however, the BEM has established itself as the technique of choice for obtaining the acoustic pressure.

Zienkiewicz, et al. [88,89], have written a series of papers in which the numerical aspects of the coupled BEM-FEM techniques are discussed. Felippa [90] has contributed much to the understanding of BEM-FEM interfacing to solve coupled fluid-structure problems. Also, many others, such as DeRantz [91], Everstine [62,63], Kakuda
and Tosaka [92], Liu and Chang [93], and Mathews [94], have studied the numerical aspects of coupling the FEM and BEM in relation to the coupled fluid-structure problem.

In most of these earlier treatments the BEM was based on the collocation method. One of the most annoying inconveniences that arises in using the collocation method is that it causes the matrices, which ultimately must be inverted, to be nonsymmetric. This increases the computer core storage size needed and necessitates handling all of the terms in the matrices throughout the program. Hamdi, Ousset, and Verchery [95] have shown that by formulating the fluid and structural equations using variational methods the matrices will be symmetric, and this problem can be avoided. This subject has received attention in recent years by Ohayon and Valid [96] and by Mariem and Hamdi [97]. However, for the problem of an infinitely baffled plate, the question of symmetric matrices is automatically resolved since there are no out-of-plane boundaries.

1.4 **Dissertation Preview**

The subsequent chapters of the dissertation are organized according to the three categories mentioned in Section 1.2. Thus, Chapters 2 through 6 are associated with the description of the dynamic behavior of laminated composite plates utilizing the FEM. Chapters 7 through 9 describe the BEM prediction of acoustic radiation from vibrating plates and Chapters 10 and 11 describe the fully coupled numerical solution to the fluid-structure problem.

In Chapters 2, 3, and 4 the constitutive equations of a forced laminated composite plate are derived using basic principles. The first-order shear deformation theory is discussed and presented. In Chapter 5 the finite element implementation of the plate equations is developed using the Rayleigh-Ritz method with reduced integration.
results verifying the accuracy of the model are presented in Chapter 6 along with some discussion as to their significance to the problem at hand.

The derivation of the Helmholtz integral equation in the appropriate form along with the accompanying limiting assumptions are presented in Chapter 7. In Chapter 8 the boundary element implementation of the Helmholtz integral equation is presented along with some discussion of the method of numerical quadrature used. Chapter 9 contains example results obtained using the BEM on the infinitely baffled rectangular radiator. Results from closed-form analytical solutions are used in this chapter when available in order to verify the BEM results.

The numerical difficulties that arise in coupling the BEM and the FEM are discussed in Chapter 10. Particular attention is paid to the integration technique used to evaluate the singular integrand. Traditional techniques are shown and an innovative use of subelements which reduces the approximation error is presented. In Chapter 11, example results obtained from the coupled BEM-FEM program are presented. Parametric studies are performed showing the relative importance of fluid loading on laminated composite plates.

Finally, the conclusions of the dissertation are summarized in Chapter 12, with emphasis on the relative importance of fluid loading on laminated composite plates, and the relative accuracy of the results are summarized. Suggestions and recommendations for further research are also presented in this chapter.
2. Laminated Plate Analysis

The purpose of this chapter is to describe the components of a laminated composite plate, to develop the theories used in laminated composite plate analysis, and to establish a set of kinematic relationships based on the five degrees of freedom of the plate. One of the most prevalent lamination theories, *classical lamination theory* (CLT), is described and shown to be inadequate for the application envisioned in this dissertation. Other theories which assume higher-order displacement fields and include shear deformation effects are discussed. Finally, a first-order theory which does not rely on Kirchhoff's assumption is shown to be the most suitable for the application of this dissertation.

2.1. Laminate Definition

Consider a fiber-reinforced composite laminated plate of thickness $h$ and width $a$ as shown in Fig. 1. Throughout this dissertation the laminate orientation will be referred
to a global x, y, z system of Cartesian coordinates. The x-y plane of the coordinate system is the midplane of the plate and the z-axis is normal to the midplane. The top of the plate, therefore, will be located at \( z = +h/2 \) and the bottom at \( z = -h/2 \). The principal material axes of an individual lamina are aligned parallel to the direction of the unidirectional fibers in each case. The angle between the laminate x-axis and the principal material axis is referred to as the angle of rotation and is denoted by \( \theta \) in Fig. 1. The angle of rotation is positive in the clockwise direction.

Each layer is assumed to be orthotropic and homogeneous. Orthotropic means the lamina has three planes of geometric and material property symmetry. The orthotropic assumption is made with little or no error. There are many methods and theories for mathematically deriving orthotropic material properties of a heterogenous lamina. However, these methods are tedious and rarely more accurate than empirical characterizations of the material properties [19]. Therefore, for this analysis no micromechanical calculations are performed. The lamina properties are assumed to be known and homogenous throughout each lamina.

2.2. Classical Lamination Theory (CLT)

Classical lamination theory (CLT) is the prevalent analysis and design tool of laminated composite plates because of its relative simplicity compared to three-dimensional theory. The assumptions made in classical lamination theory are listed below along with the direct implication of each assumption.

1. **Assumption**: Plane sections originally plane and perpendicular to the middle surface remain plane and perpendicular to the middle surface after extension and bending.
Implication: The deflection of the plate is associated only with the bending strains. The shear strains $\varepsilon_{xx}$ and $\varepsilon_{yz}$ are neglected.

2. Assumption: The displacements of the middle surface are small compared with the thickness of the plate.
Implication: The slope of the deflected surface is small compared to the in-plane strains and the square of the slope is negligible; i.e., linear strain-displacement relations are valid, and the influence of in-plane forces upon the transverse deflection may be ignored.

3. Assumption: The thickness of the plate is small compared with other dimensions of the plate.
Implication: The plate is in a state of plane stress. $\sigma_z = \tau_{xz} = \tau_{yz} = 0$.

4. Assumption: The laminate consists of perfectly bonded laminae and the bonds are infinitesimally thin as well as nonshear-deformable.
Implication: There is no possibility of delamination since laminae cannot slip with respect to each other.

5. Assumption: The plate is constructed of linearly-elastic material with properties that are independent of temperature.
Implication: Hooke's law is applicable.

Although these assumptions simplify the analysis and design of composite materials, the implications often cause erroneous results. Pagano and his associates in a series of papers [15,16,17] show that CLT predictions are reasonable only in the case of relatively thin plates. This is because of:
1. the neglect of shear deformation terms, implied by the Kirchhoff hypothesis (normals remain normal),

2. the assumption of linear in-plane displacements through the thickness,

3. the presence of only two boundary conditions per edge in the bending theory,

4. the assumption of a state of plane stress which eliminates the possibility of interlaminar stress calculations.

Research has been done to eliminate some of the restricting assumptions of CLT and still maintain the computational simplicity of plate theory. Shear-deformation theories (SDT) are a result of this research.

2.3. Shear-Deformation Theory (SDT)

The Kirchhoff restriction is unnecessary in shear-deformation theories and, therefore, they are a compromise between the accuracy of three-dimensional theory and the simplicity of plate theory. Shear-deformation theories can be grouped into either stress-based or displacement-based theories. Reissner [98] is responsible for the first stress-based SDT and Basset [99] is one of the first to develop a shear deformation theory based on displacements. The theory developed in this dissertation is also a displacement-based theory. Basset assumes that displacements can be expanded in power series of the z-coordinate. For example, the displacement in the x-coordinate is written
\[ u(x, y, z) = u^\circ(x, y) + \sum_{n=1}^{N} z^n \psi_x^n(x, y), \quad (2.1) \]

where \( u^\circ \) is the middle surface displacement in the x-direction and \( \psi_x^n \) means

\[ \psi_x^n(x, y) = \frac{\partial^n u}{\partial z^n}, \quad n = 1, 2, \ldots \quad (2.2) \]

A similar expression can be written for the displacement in the y-coordinate. The quantities \( \psi_x \) and \( \psi_y \) have a physical significance when \( n = 1 \) as the slope of the deflected surface. A special case of Basset's displacement field is the first-order shear-deformation theory. The first-order SDT assumes the following displacement field:

\[ u(x, y, z) = u^\circ(x, y) + z\psi_x(x, y), \]

\[ v(x, y, z) = v^\circ(x, y) + z\psi_y(x, y), \quad (2.3) \]

and

\[ w(x, y, z) = w^\circ(x, y). \]

The displacement fields in the x- and y-directions assumed by the first-order SDT and CLT are similar in that they both are linear through-the-thickness of the plate. However, in the first-order SDT these in-plane displacements are not required to remain perpendicular to the middle surface of the plate after bending or extension. In Fig. 6 is shown a graphical comparison of the displacement field assumed by the first-order SDT and the displacement field assumed in CLT because of the Kirchoff-Love hypothesis. Figure 6(a) shows an undeformed laminate, while the deformed laminate with the displacement field assumed by CLT is shown in Fig. 6(b). Notice in Fig. 6(b) that the solid
line depicting the displacement field is still normal to the middle surface. The solid line shown in Fig. 6(c) is a typical first-order SDT displacement field and is not necessarily normal to the middle surface.

Another observation about the first-order SDT is that the shear strains are assumed uniform through the thickness of the plate. In general this assumption is incorrect and for this reason Mindlin [18] introduced a shear correction factor, $\kappa^2$. The shear correction factor is evaluated by comparing solutions with an exact elasticity solution. For the materials most commonly used in laminated composite plates $\kappa^2 = \frac{5}{6}$. The shear correction factor is incorporated in the constitutive relationships as shown in Section 3.1.

In 1969 Yang, Norris, and Stravsky [20] assumed the displacement field of the first-order SDT in order to investigate elastic wave propagation in heterogeneous plates and found good agreement with the exact solution. Later Gol'deneizer [100], Schmidt [101], Whitney and Pagano [21], Librescu [24], and Lo, Christensen, and Wu [23] independently included higher-order terms in Basset's power series assumption to describe the displacement fields. Reddy [23] has refined a higher-order theory that is extremely accurate and yet somewhat less cumbersome and computationally less demanding than the three-dimensional elastic theory. These higher-order shear-deformation theories have proven effective for evaluating stresses and especially interlaminar stresses; however, they are only slightly more accurate than the first-order shear deformation theory in predicting the motion of the plate. Therefore, in the present work the first-order theory is accepted as sufficiently accurate and the displacement field of Eqs. (2.3) is used as the basis for defining the strains. The degrees of freedom are thus defined as three translations, $(u, v, \text{ and } w)$ and two rotations, $(\psi_y, \text{ and } \psi_z)$.

Because of the assumption of harmonic motion each degree of freedom can be expressed in the time domain as

2. Laminated Plate Analysis
\[ A(x,y,z,t) = A(x,y,z) e^{i\omega t}, \]

where \( A(x,y,z,t) \) is the variable as a function of time, \( A(x,y,z) \) is the complex amplitude of motion, and \( \omega \) is the frequency of motion. The velocity and acceleration are easily derived from the displacement as

\[ \dot{A}(t) = i\omega A e^{i\omega t}, \]

and

\[ \ddot{A}(t) = -\omega^2 A e^{i\omega t}. \]

The exponential term may be divided out of all equations involving displacements and their derivatives, leaving expressions in the frequency domain.

### 2.4 Kinematic Relationships

In the Eulerian viewpoint the strains are described in indicial notation as \([102]\)

\[ \epsilon_{ij} = \frac{1}{2} \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right]; \quad i = 1,2,3 \quad j = 1,2,3. \quad (2.4) \]

In Eqs. (2.4) \( u_i \), \( u_2 \), and \( u_3 \) are the displacements in the \( x \), \( y \), and \( z \) directions and are referred to as \( u \), \( v \), and \( w \), respectively, in the analysis that follows; and the quantities \( x_i \), \( x_2 \), and \( x_3 \) refer to the orthogonal coordinate directions written \( x \), \( y \), and \( z \), respectively, in the analysis that follows. After substituting the first-order SDT displacement
field into the Eulerian strain description and eliminating any higher-order, nonlinear terms, the linear strain-displacement relationships become

\[ \varepsilon_x = \frac{\partial u^o}{\partial x} + z(\frac{\partial \psi_x}{\partial x}) , \]  
(2.5)

\[ \varepsilon_y = \frac{\partial v^o}{\partial y} + z(\frac{\partial \psi_y}{\partial y}) , \]  
(2.6)

\[ \varepsilon_z = 0 , \]  
(2.7)

\[ \gamma_{xy} = \frac{\partial u^o}{\partial y} + \frac{\partial v^o}{\partial x} + z(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}) , \]  
(2.8)

\[ \gamma_{yz} = \frac{\partial w^o}{\partial y} + \psi_y , \]  
(2.9)

and

\[ \gamma_{zx} = \frac{\partial w^o}{\partial x} + \psi_x . \]  
(2.10)

For convenience in notation let

\[ \varepsilon_x^o = \frac{\partial u^o}{\partial x} , \quad \varepsilon_y^o = \frac{\partial v^o}{\partial y} , \quad \gamma_{xy}^o = \frac{\partial u^o}{\partial y} + \frac{\partial v^o}{\partial x} , \]

\[ \kappa_x = \frac{\partial \psi_x}{\partial x} , \quad \kappa_y = \frac{\partial \psi_y}{\partial y} , \quad \text{and} \quad \kappa_{xy} = \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} . \]

The total strains can then be written

\[ \varepsilon_x = \varepsilon_x^o + z \kappa_x , \]  
(2.11)

2. Laminated Plate Analysis
\[ \varepsilon_y = \varepsilon^0_y + z \kappa_y , \]  \hfill (2.12)

\[ \varepsilon_z = 0 , \]  \hfill (2.13)

\[ \varepsilon_{yz} = 2\gamma_{yz} , \]  \hfill (2.14)

\[ \varepsilon_{zx} = 2\gamma_{zx} , \]  \hfill (2.15)

and \[ \varepsilon_{xy} = 2(\gamma^0_{xy} + z \kappa_{xy}) . \]  \hfill (2.16)
3. Lamina Constitutive Relationships

Stress-strain relations for an orthotropic lamina are reviewed in this chapter. Starting with the generalized Hooke’s law for an elastic solid, the stress-strain relations are reduced to the three-dimensional orthotropic form using contracted notation. The stiffness matrix is transformed so that the on-axis stresses and strains can be found for a lamina with arbitrary orientation of the principal material direction. The stiffness matrix is further reduced to exclude $\sigma_z$ and $\varepsilon_z$.

3.1. Lamina Stress-Strain Relations

The generalized Hooke’s law for an elastic solid in indicial notation is expressed

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl},$$  \hspace{1cm} (3.1)
where $\sigma_{ij}$ is the stress tensor, $C_{ijkl}$ is the stiffness tensor, and $\varepsilon_{kl}$ is the strain tensor. It is assumed in this relationship that there are no residual stresses from previous deformations. With this assumption the generalized Hooke's law is the most general form of a linear relationship between the stress and strain components. In this general form $\text{C}_{ijkl}$ represents 81 independent material constants. From equilibrium conditions it can be shown that, if there are no body moments, the $\sigma_{ij}$ tensor is symmetric ($\sigma_{ij} = \sigma_{ji}$). And by definition of the strain tensor from the Eulerian viewpoint (see Eq. (2.4)), it can be concluded that the strain tensor $\varepsilon_{kl}$ is symmetric. Since $\sigma_{ij}$ and $\varepsilon_{kl}$ are symmetric, it follows that $C_{ijkl}$ is symmetric. The independent constants are now reduced to 36 and so contracted notation can be employed to simplify the expression for Hooke's law to

$$\sigma_i = C_{ij} \varepsilon_j . \quad (3.2)$$

Contracted notation is a simplification of indicial notation that reduces the number of indices by one-half. The indices are replaced as follows:

$$11 \rightarrow 1 , \quad 22 \rightarrow 2 , \quad 33 \rightarrow 3 ,$$

$$23 \rightarrow 4 , \quad 31 \rightarrow 5 , \quad 12 \rightarrow 6 .$$

For example, $\sigma_{11} \rightarrow \sigma_1$, $\varepsilon_{23} \rightarrow \varepsilon_4$, and $C_{3112} \rightarrow C_{56}$. The number of independent elastic constants is further reduced to 21 by recognizing that there exists a quadratic strain energy function $W$ such that

$$W = \frac{1}{2} \ C_{ij} \varepsilon_i \varepsilon_j . \quad (3.3)$$

Therefore $C_{ij}$ is symmetric.
In Chapter 2 the lamina was assumed to be orthotropic. This implies that there are three planes of geometric and material symmetry. By systematically equating terms from symmetric coordinate systems it can be shown that the number of independent material constants can be reduced to nine and that the generalized Hooke’s law in matrix form for an individual lamina can now be written [102]

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\varepsilon_6
\end{bmatrix}.
\]

(3.4)

By inverting the stiffness matrix \([C]\), the compliance matrix \([S]\), can be found along with the strain-stress relationship

\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\varepsilon_6
\end{bmatrix} =
\begin{bmatrix}
S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\
S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\
S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & S_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & S_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & S_{66}
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6
\end{bmatrix}.
\]

(3.5)

From the strain-stress relationship, the compliance terms can be logically deduced in terms of engineering constants [102] as

3. Lamina Constitutive Relationships 27
$$[S] = \begin{bmatrix}
\frac{1}{E_1} & -\frac{v_{21}}{E_2} & -\frac{v_{31}}{E_3} & 0 & 0 & 0 \\
-\frac{v_{12}}{E_1} & \frac{1}{E_2} & -\frac{v_{32}}{E_3} & 0 & 0 & 0 \\
-\frac{v_{13}}{E_1} & -\frac{v_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}}
\end{bmatrix}, \quad (3.6)$$

where $E_i$ are the Young’s moduli in the $i$-direction, $v_{ij}$ are the Poisson’s ratios for strain in the $j$ direction resulting from stress in the $i$ direction, and $G_{ij}$ are the shear moduli in the $i-j$ plane. Recall from Chapter 2 that the shear moduli $G_{23}$ and $G_{31}$ are reduced by multiplying by the shear correction factor, $\kappa^2$. Although twelve material constants appear in Eqs. (3.6), there are only nine independent constants because

$$\frac{v_{ij}}{E_i} = \frac{v_{ji}}{E_j}, \quad i = 1, 2, 3, \quad j = 1, 2, 3. \quad (3.7)$$

The stress-strain relations of Eqs. (3.4) are true only when the stresses and strains are in the principal material direction of the lamina (direction in which the fibers are aligned). When the various principal axes of the laminae are in different directions it becomes difficult to analyze the laminate. This necessitates a method of transforming

3. Lamina Constitutive Relationships
the stress-strain relations from one coordinate system to another. Consider a lamina whose principal material direction is at some angle \( \theta \) with respect to the laminate x-axis. In tensor notation the stresses in the x-y coordinate system \((\sigma_u)\), are transformed to the 1-2 coordinate system \((\sigma'_u)\), by \([103]\)

\[
\sigma'_{ij} = a_{ij} \sigma_{lm},
\]

where

\[
[a_{ij}] = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

This yields

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\tau_{23} \\
\tau_{31} \\
\tau_{12}
\end{bmatrix} = \begin{bmatrix}
c^2 & s^2 & 0 & 0 & 0 & 2cs \\
s^2 & c^2 & 0 & 0 & 0 & -2cs \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & c - s & 0 & \sigma_{y} \\
0 & 0 & 0 & s & c & 0 \\
-cs & cs & 0 & 0 & 0 & c^2 - s^2
\end{bmatrix} \begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\tau_{y} \\
\tau_{z} \\
\tau_{x}
\end{bmatrix},
\]

where now

\[
c = \cos \theta \quad \text{and} \quad s = \sin \theta.
\]

In this dissertation the transformation matrix of Eq. (3.9) is represented by the symbol \([T_1]\). Tensorial strains are transformed in the same way; however, tensorial strains and

3. Lamina Constitutive Relationships
engineering strains are not equal. Tensorial shear strains are one-half of engineering strains. Thus,

\[
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{23} \\
\gamma_{31} \\
\gamma_{12}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 2
\end{bmatrix} \begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{23}/2 \\
\gamma_{31}/2 \\
\gamma_{12}/2
\end{bmatrix},
\]

(3.10)

where the conversion matrix is commonly called the Reuter matrix \([R]\). Engineering strains can now be transformed from the \(x-y\) coordinate system \(\{\varepsilon\}_x\) to the \(1-2\) coordinate system \(\{\varepsilon\}_1\) by

\[
\{\varepsilon\}_1 = [R][T_1][R]^{-1}\{\varepsilon\}_x,
\]

or, carrying out the indicated matrix multiplications,

\[
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{23} \\
\gamma_{31} \\
\gamma_{12}
\end{bmatrix} = \begin{bmatrix}
c^2 & s^2 & 0 & 0 & 0 & cs \\
s^2 & c^2 & 0 & 0 & 0 & -cs \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & c & -c & 0 \\
0 & 0 & 0 & s & s & 0 \\
-2cs & 2cs & 0 & 0 & 0 & c^2 - s^2
\end{bmatrix} \begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{zz} \\
\gamma_{yz} \\
\gamma_{zx} \\
\gamma_{xy}
\end{bmatrix}.
\]

(3.11)

This transformation matrix is represented by the symbol \([T_2]\). Expressing the on-axis stresses and strains (\(\{\sigma\}_1\) and \(\{\varepsilon\}_1\)) of Eqs. (3.4) in terms of the off-axis stresses and strains (Eqs. (3.9) and (3.11)) yields

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\[ [T_1] \{ \sigma \}_x = [C][T_2]\{ \varepsilon \}_y , \]  

(3.12)

where \([C]\) is the matrix from Eq. (3.4). Rearranging Eqs. (3.12) yields

\[ \{ \sigma \}_x = [T_1]^{-1}[C][T_2]\{ \varepsilon \}_x , \]

or

\[ \{ \sigma \}_x = [\overline{C}]\{ \varepsilon \}_x , \]

(3.13)

where

\[ [\overline{C}] = [T_1]^{-1}[C][T_2] . \]

Expanding Eqs. (3.13), the off-axis stress-strain relationship for the given lamina is

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{yz} \\
\tau_{zx} \\
\tau_{xy}
\end{bmatrix} =
\begin{bmatrix}
\overline{C}_{11} & \overline{C}_{12} & \overline{C}_{13} & 0 & 0 & \overline{C}_{16} \\
\overline{C}_{12} & \overline{C}_{22} & \overline{C}_{23} & 0 & 0 & \overline{C}_{26} \\
\overline{C}_{13} & \overline{C}_{23} & \overline{C}_{33} & 0 & 0 & \overline{C}_{36} \\
0 & 0 & 0 & \overline{C}_{44} & \overline{C}_{45} & 0 \\
0 & 0 & 0 & \overline{C}_{45} & \overline{C}_{55} & 0 \\
\overline{C}_{16} & \overline{C}_{26} & \overline{C}_{36} & 0 & 0 & \overline{C}_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{yz} \\
\gamma_{zx} \\
\gamma_{xy}
\end{bmatrix},
\]

(3.14)

where

\[ \overline{C}_{11} = c^4 C_{11} + 2c^2 s^2 (C_{12} + 2 C_{66}) + c^4 C_{22} , \]

\[ \overline{C}_{12} = c^2 s^2 (C_{11} + C_{22} - 4 C_{66}) + (s^4 + c^4) C_{12} , \]

\[ \overline{C}_{13} = c^2 C_{13} + s^2 C_{23} , \]
\[ \bar{C}_{16} = cs[c^2(C_{11} - C_{12} - 2C_{66}) + s^2(C_{12} - C_{21} + 2C_{66})] , \]
\[ \bar{C}_{22} = s^4C_{11} + 2c^2s^2(C_{12} + 2C_{66}) + c^4C_{22} , \]
\[ \bar{C}_{23} = c^2C_{13} + s^2C_{23} , \]
\[ \bar{C}_{26} = cs[s^2(C_{11} - C_{12} - 2C_{66}) + c^2(C_{12} - C_{22} + 2C_{66})] , \]
\[ \bar{C}_{33} = C_{33} , \]
\[ \bar{C}_{36} = cs(C_{13} - C_{23}) , \]
\[ \bar{C}_{44} = c^2C_{44} + s^2C_{55} , \]
\[ \bar{C}_{45} = cs(C_{55} - C_{44}) , \]
\[ \bar{C}_{55} = s^2C_{44} + c^2C_{55} , \]

and
\[ \bar{C}_{66} = c^2s^2(C_{11} - 2C_{12} + C_{22}) + C_{66}(c^2 - s^2)^2 . \]

As a matter of convenience, in the remainder of this text the stress and strain components with duplicate indices (ie. \( \sigma_{xx} \), \( \sigma_{yy} \), \( \sigma_{zz} \), \( \epsilon_{xx} \), \( \epsilon_{yy} \), and \( \epsilon_{zz} \)) are written with only one index.

The transformed stiffness matrix is reduced by recognizing that \( \sigma_z = 0 \) from the plane stress assumption. When zero is substituted for \( \sigma_z \) in Eqs. (3.14) there results

\[ \sigma_z = 0 = \epsilon_x \bar{C}_{13} + \epsilon_y \bar{C}_{23} + \epsilon_z \bar{C}_{33} + \gamma_{yz} \bar{C}_{36} . \] (3.15)
Solving Eq. (3.15) for \( \varepsilon_z \),

\[
\varepsilon_z = -\frac{1}{C_{33}} \left[ \varepsilon_x C_{13} + \varepsilon_y C_{23} + \gamma_{yz} C_{36} \right].
\]  

(3.16)

By eliminating \( \sigma_z \), and substituting Eq. (3.16) for \( \varepsilon_z \), the reduced stiffness matrix and stress-strain relationship become

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{xz}
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} & 0 & 0 \\
Q_{12} & Q_{22} & Q_{23} & 0 & 0 \\
Q_{13} & Q_{23} & Q_{33} & 0 & 0 \\
0 & 0 & 0 & Q_{44} & Q_{45} \\
0 & 0 & 0 & Q_{45} & Q_{55}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy} \\
\gamma_{yz} \\
\gamma_{xz}
\end{bmatrix},
\]  

(3.17)

where \([Q_y]\) is the transformed reduced stiffness matrix. In terms of the transformed stiffness matrix its elements are

\[
Q_{11} = \bar{C}_{11} - \frac{\bar{C}_{13}^2}{\bar{C}_{23}}, \quad Q_{12} = \bar{C}_{12} - \frac{\bar{C}_{13} \bar{C}_{23}}{\bar{C}_{33}}, \quad Q_{13} = \bar{C}_{13} - \frac{\bar{C}_{13} \bar{C}_{26}}{\bar{C}_{33}},
\]

\[
Q_{22} = \bar{C}_{22} - \frac{\bar{C}_{23}^2}{\bar{C}_{33}}, \quad Q_{23} = \bar{C}_{23} - \frac{\bar{C}_{23} \bar{C}_{36}}{\bar{C}_{33}}, \quad Q_{33} = \bar{C}_{33} - \frac{\bar{C}_{36}^2}{\bar{C}_{33}},
\]

\[
Q_{44} = \bar{C}_{44}, \quad Q_{45} = \bar{C}_{45}, \quad \text{and} \quad Q_{55} = \bar{C}_{55}.
\]
4. Governing Equations for Laminated Plates

In Chapter 3 the stress-strain relationships for an arbitrary lamina, Eq. (3.14), were found. In this chapter those relationships are extended to an arbitrary laminate resulting in force-strain relationships. The equations of motion are also expanded into five equilibrium equations which fully describe the five degrees of freedom of the laminate: \( u, v, w, \psi_x, \) and \( \psi_y \).

4.1. Laminate Forces and Moments

The resultant forces and moments per unit width acting on a laminate can be found by integrating the laminae stress-strain relationship of Eq. (3.14) over the laminate thickness, yielding
\[(N_x, N_y) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y) \, dz \quad , \quad (4.1)\]

\[(N_{xy}, N_{yz}, N_{xz}) = \int_{-h/2}^{h/2} (\tau_{xy}, \tau_{yz}, \tau_{xz}) \, dz \quad , \quad (4.2)\]

and

\[(M_x, M_y, M_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}) \, z \, dz \quad , \quad (4.3)\]

where \(\tau_{xy} = \tau_{yx}\), \(\tau_{zx} = \tau_{xz}\), \(\tau_{yz} = \tau_{zy}\), \(N_{xy} = N_{yx}\), and \(M_{xy} = M_{yx}\). These forces and moments are labeled on the plate in Fig. 7. Figure 8 depicts a laminate of \(N\) laminae and gives the lamina numbering system that is used in the analysis. The z-axis is orthogonal to the midplane, positive being up. The laminae are numbered sequentially from 1 to \(N\) starting with the top lamina. The lamina interfaces are numbered corresponding to lamina directly below the interface with the laminate top surface being numbered 1 and the laminate bottom surface numbered \(N+1\). The z-coordinate of the midplane of the \(k\)th lamina is denoted by \(\bar{z}_k\). Equations (4.1), (4.2) and (4.3) can now be rewritten using the notation of Fig. 8 as

\[(N_{x}, N_{y}) = \sum_{k=1}^{N} \int_{z_{k+1}}^{z_k} (\sigma_x, \sigma_y) \, dz \quad , \quad (4.4)\]

\[(N_{xy}, N_{yz}, N_{xz}) = \sum_{k=1}^{N} \int_{z_{k+1}}^{z_k} (\tau_{xy}, \tau_{yz}, \tau_{xz}) \, dz \quad , \quad (4.5)\]

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and

\[(M_x, M_y, M_{xy}) = \sum_{k=1}^{N} \int_{z_{k+1}}^{z_k} (\sigma_x, \sigma_y, \tau_{xy}) z \, dz \, . \tag{4.6}\]

The off-axis stresses can now be replaced with the equivalent stiffness matrix and middle surface strains, Eqs. (2.10) through (2.15). Neither the middle surface strains nor the stiffness matrix are dependent on \(z\) so they can be brought outside the integral. After these changes are made Eqs. (4.4), (4.5), and (4.6) can now be written in matrix form as

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy} \\
N_{yz} \\
N_{xz} \\
M_x \\
M_y \\
M_{xy}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & A_{16} & 0 & 0 & B_{11} & B_{12} & B_{16} \\
A_{12} & A_{22} & A_{26} & 0 & 0 & B_{12} & B_{22} & B_{26} \\
A_{16} & A_{26} & A_{66} & 0 & 0 & B_{16} & B_{26} & B_{66} \\
0 & 0 & 0 & A_{44} & A_{45} & 0 & 0 & 0 \\
0 & 0 & 0 & A_{45} & A_{55} & 0 & 0 & 0 \\
B_{11} & B_{12} & B_{16} & 0 & 0 & D_{11} & D_{12} & D_{16} \\
B_{12} & B_{22} & B_{26} & 0 & 0 & D_{12} & D_{22} & D_{26} \\
B_{16} & B_{26} & B_{66} & 0 & 0 & D_{16} & D_{26} & D_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x^o \\
\varepsilon_y^o \\
\gamma_{xy}^o \\
\gamma_{yz}^o \\
\gamma_{xz}^o \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}, \tag{4.7}
\]

where

\[A_{ij} = \sum_{k=1}^{N} Q_{ij}(z_k - z_{k+1}) \, , \tag{4.8}\]

\[B_{ij} = \frac{1}{2} \sum_{k=1}^{N} Q_{ij}(z_k^2 - z_{k+1}^2) \, , \tag{4.9}\]
and

\[ D_{ij} = \frac{1}{3} \sum_{k=1}^{N} Q_{ij} (z_k^2 - z_{k+1}^2) . \]  

(4.10)

4.2. Equations of Motion

Up to this point the strains have been defined in terms of the five degrees of freedom: \( u, v, w, \psi_x, \) and \( \psi_y \). The stresses have been found from Hooke's law in terms of the strains, and the laminate forces and moments have been defined in terms of the mechanical strains. In order to completely describe the dynamic behavior of the three-dimensional laminate, the equations of equilibrium must also be satisfied. Consider the infinitesimal cube with dimensions \( dx, dy, \) and \( dz \) shown in Fig. 9. All of the nonzero stresses considered in this thesis are assumed to be acting on the cube. The net force in the \( x \)- and \( y \)-directions can be written

\[ F_x = \frac{\partial \sigma_x}{\partial x} \ dx \ dy \ dz + \frac{\partial \tau_{xy}}{\partial y} \ dy \ dx \ dz \]  

(4.11)

and

\[ F_y = \frac{\partial \sigma_y}{\partial y} \ dy \ dx \ dz + \frac{\partial \tau_{xy}}{\partial x} \ dx \ dy \ dz . \]  

(4.12)

Because of the plate geometry, the net force in the \( z \)-direction also contains, in principle, in-plane stress terms which cause nonzero \( z \)-directed forces when the plate becomes nonplanar [104]. The net force in the \( z \)-direction is therefore

4. Governing Equations for Laminated Plates
\[ F_x = \frac{\partial \tau_{yz}}{\partial y} \, dy \, dx \, dz + \frac{\partial \tau_{xz}}{\partial x} \, dx \, dy \, dz + \frac{\partial}{\partial x} (\sigma_x \frac{\partial w}{\partial x} + \tau_{xy} \frac{\partial w}{\partial y}) \, dx \, dy \, dz + \frac{\partial}{\partial y} (\sigma_y \frac{\partial w}{\partial y} + \tau_{xy} \frac{\partial w}{\partial x}) \, dx \, dy \, dz. \] (4.13)

Two simplifying assumptions can be employed at this point. First, it has been assumed in defining the kinematic relationships that the transverse deflection of the plate is small compared with the thickness of the plate. This implies that all the terms involving partial derivatives of \( w \) with respect to \( x \) or \( y \) in Eq. (4.13) can be eliminated because they are relatively small compared to the other terms contributing to the force in the \( z \)-direction. Second, by assuming that only laminae of the same material are analyzed we can ignore coupled rotary inertia terms. Therefore, the acceleration terms in the three directions are

\[ \ddot{u} + \ddot{z} \psi_x, \]

\[ \ddot{v} + \ddot{z} \psi_y, \]

and

\[ \ddot{w}, \]

and the mass is denoted by \( \rho_s dx \, dy \, dz \), where \( \rho_s \) is the equivalent mass density of the plate. The equations of equilibrium can now be found by substituting the net forces and inertia terms into Newton's second law of motion, which then becomes

\[ \rho_s (\ddot{u} + \ddot{z} \psi_x) \, dx \, dy \, dz = \left[ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \right] \, dx \, dy \, dz, \] (4.14)

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\[ \rho_s (\ddot{v} + z \ddot{\psi}_z) dxdydz = \left[ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} \right] dxdydz, \]  
\hspace{1cm} \text{(4.15)}

and

\[ \rho_s \ddot{w} dxdydz = \left[ \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{xz}}{\partial x} \right] dxdydz. \]  
\hspace{1cm} \text{(4.16)}

Moment equilibrium equations can be derived in a similar fashion. The moment equilibrium equations based on the net moments about the x- and y-axes are, respectively,

\[ \rho_s z (\ddot{v} + z \ddot{\psi}_z) dxdydz = (z \frac{\partial \tau_{xy}}{\partial x} + z \frac{\partial \sigma_y}{\partial y} - \tau_{yz}) dxdydz \]  
\hspace{1cm} \text{(4.17)}

and

\[ \rho_s z (\ddot{u} + z \ddot{\psi}_x) dxdydz = (z \frac{\partial \sigma_x}{\partial x} + z \frac{\partial \tau_{xy}}{\partial y} - \tau_{xz}) dxdydz. \]  
\hspace{1cm} \text{(4.18)}

Equations (4.14) through (4.18) can now be divided through by $dxdy$. Recalling Eqs. (4.4) through (4.6), the equilibrium equations can be integrated through the laminate thickness, yielding

\[ \rho_s \ h \frac{\partial^2 u}{\partial t^2} = \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y}, \]  
\hspace{1cm} \text{(4.19)}

\[ \rho_s h \frac{\partial^2 v}{\partial t^2} = \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y}, \]  
\hspace{1cm} \text{(4.20)}

\[ \rho_s h \frac{\partial^2 w}{\partial t^2} = \frac{\partial N_{yz}}{\partial x} + \frac{\partial N_{y}}{\partial y} + p, \]  
\hspace{1cm} \text{(4.21)}

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\[ \frac{\rho_s h^3}{12} \frac{\partial^2 \psi_x}{\partial t^2} = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - N_{xz}, \] (4.22)

and

\[ \frac{\rho_s h^3}{12} \frac{\partial^2 \psi_y}{\partial t^2} = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - N_{yz}. \] (4.23)

The term \( p \) has been added in Eq. (4.21) and represents the net pressure acting normal to the plate and is discussed in detail in Chapter 6.
5. Finite Element Model

The salient features of the finite element model are explained in this chapter. First, the equations of dynamic equilibrium are expanded and recast into a variational formulation. Interpolation functions are derived and the equilibrium equations are rewritten in matrix form. Gauss-Legendre quadrature is used to evaluate the integrals and reduced integration is explained with respect to the shear terms of the dynamic equilibrium equations. Finally, storage of the matrices is explained and the Gauss elimination method for solving systems of simultaneous equations is reviewed in preparation for the more complicated coupled FEM-BEM problem.
5.1. Variational Formulation

When the total forces and moments are expressed in terms of the corresponding strains and stiffness’s, Eqs. (4.7), the equations of dynamic equilibrium, Eqs. (4.19) through (4.23), become

\[ \rho_h \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x} \left[ A_{11} \frac{\partial u}{\partial x} + \frac{A_{12}}{\partial y} + A_{16} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + B_{11} \frac{\partial \psi_x}{\partial x} + B_{12} \frac{\partial \psi_y}{\partial y} + B_{16} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) = 0, \]  

(5.1)

\[ \rho_h \frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x} \left[ A_{16} \frac{\partial u}{\partial x} + A_{26} \frac{\partial v}{\partial y} + A_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + B_{16} \frac{\partial \psi_x}{\partial x} + B_{26} \frac{\partial \psi_y}{\partial y} + B_{66} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) = 0, \]  

(5.2)

\[ \rho_h \frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x} \left[ A_{55} \left( \psi_x + \frac{\partial w}{\partial x} \right) + A_{45} \left( \psi_y + \frac{\partial w}{\partial y} \right) \right] - \frac{\partial}{\partial y} \left[ A_{45} \left( \psi_x + \frac{\partial w}{\partial x} \right) + A_{55} \left( \psi_y + \frac{\partial w}{\partial y} \right) \right] = 0, \]  

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\[ A_{44} \left( \psi_y + \frac{\partial w}{\partial y} \right) = p(x, y, t), \]  

(5.3)

\[ \frac{\rho h^3}{12} \frac{\partial^2 \psi_x}{\partial t^2} - \frac{\partial}{\partial x} \left[ B_{11} \frac{\partial u}{\partial x} + B_{12} \frac{\partial v}{\partial y} + B_{16} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + D_{11} \frac{\partial \psi_x}{\partial x} + D_{12} \frac{\partial \psi_y}{\partial y} + D_{16} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \]

\[ + D_{26} \frac{\partial \psi_y}{\partial y} + D_{66} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \] \[ + A_{55} \left[ \psi_x + \frac{\partial w}{\partial x} \right] + A_{45} \left[ \psi_y + \frac{\partial w}{\partial y} \right] = 0, \]  

(5.4)

and

\[ \frac{\rho h^3}{12} \frac{\partial^2 \psi_y}{\partial t^2} - \frac{\partial}{\partial x} \left[ B_{16} \frac{\partial u}{\partial x} + B_{26} \frac{\partial v}{\partial y} + B_{26} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + D_{16} \frac{\partial \psi_x}{\partial x} + D_{26} \frac{\partial \psi_y}{\partial y} + D_{66} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \]

\[ + D_{26} \frac{\partial \psi_y}{\partial y} + D_{26} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \] \[ + A_{45} \left[ \psi_x + \frac{\partial w}{\partial x} \right] + A_{55} \left[ \psi_y + \frac{\partial w}{\partial y} \right] = 0. \]  

(5.5)

In the context of this dissertation, variational formulation means the recasting of differential equations, in this case Eqs. (5.1) through (5.5), into an equivalent integral form by distributing the differentiation between a test function, \( \lambda \), and the dependent variables. The reason for this manipulation is to minimize the order of differentiation

5. Finite Element Model
of the dependent variables and to establish the secondary variables of the problem. For example, Eq. (5.1) is first multiplied by a test function, \( \lambda \), and then integrated over the element area, \( \Omega \), yielding

\[
\int_{\Omega} \left[ \lambda \rho_e h \frac{\partial^2 u}{\partial t^2} - \lambda \frac{\partial}{\partial x} \left( A_{11} \frac{\partial u}{\partial x} + A_{12} \frac{\partial v}{\partial y} + A_{16} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right) \right. \\
+ B_{11} \frac{\partial \psi_x}{\partial x} + B_{12} \frac{\partial \psi_y}{\partial y} + B_{10} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \left. - \lambda \frac{\partial}{\partial y} \left( A_{16} \frac{\partial u}{\partial x} + A_{26} \frac{\partial v}{\partial y} \right) \\
+ A_{46} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + B_{16} \frac{\partial \psi_x}{\partial x} + B_{26} \frac{\partial \psi_y}{\partial y} + B_{56} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \right] \right] \right] \right) \right] dx \, dy = 0 \tag{5.6}
\]

Wherever there are second derivatives in Eq. (5.6) the differentiation can be equally distributed between \( \lambda \) and the dependent variable by noting that for any two differentiable functions \( f \) and \( g \)

\[
\int_{\Omega} g \frac{\partial f}{\partial x} \, dx \, dy = - \int_{\Omega} f \frac{\partial g}{\partial x} \, dx \, dy + \int_{\Omega} \frac{\partial}{\partial x} (fg) \, dx \, dy
\]

and

\[
\int_{\Omega} g \frac{\partial f}{\partial y} \, dx \, dy = - \int_{\Omega} f \frac{\partial g}{\partial y} \, dx \, dy + \int_{\Omega} \frac{\partial}{\partial y} (fg) \, dx \, dy.
\]

But by the divergence theorem

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\[
\int_{\Omega} \frac{\partial}{\partial x} (fg) \, dx \, dy = \int_{\Gamma} (fg)n_x \, ds
\]

and

\[
\int_{\Omega} \frac{\partial}{\partial y} (fg) \, dx \, dy = \int_{\Gamma} (fg)n_y \, ds
\]

where \( n_x \) and \( n_y \) are the \( x \) and \( y \) components, respectively, of the unit normal to the element boundary, \( \Gamma \), and \( ds \) is an infinitesimal arc length along the boundary. Therefore,

\[
\int_{\Omega} g \frac{\partial f}{\partial x} \, dx \, dy = -\int_{\Omega} f \frac{\partial g}{\partial x} \, dx \, dy + \int_{\Gamma} (fg)n_x \, ds
\]

and

\[
\int_{\Omega} g \frac{\partial f}{\partial y} \, dx \, dy = -\int_{\Omega} f \frac{\partial g}{\partial y} \, dx \, dy + \int_{\Gamma} (fg)n_y \, ds
\]

With these considerations Eq. (5.6) becomes

\[
\int_{\Omega} \left[ \lambda \rho_s \frac{\partial^2 u}{\partial t^2} + \frac{\partial \lambda}{\partial x} \right] \left[ A_{11} \frac{\partial u}{\partial x} + A_{12} \frac{\partial v}{\partial y} + A_{16} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + B_{11} \frac{\partial \psi_x}{\partial x} + B_{12} \frac{\partial \psi_y}{\partial y} \right]
\]

\[
+ \frac{\partial \lambda}{\partial y} \left[ A_{16} \frac{\partial u}{\partial x} + A_{26} \frac{\partial v}{\partial y} + A_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right]
\]
\[ + B_{16} \frac{\partial \psi_x}{\partial x} + B_{26} \frac{\partial \psi_y}{\partial y} + B_{66} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \right] dx dy = \int_{\Gamma} \lambda \left[ A_{11} \frac{\partial u}{\partial x} + A_{12} \frac{\partial v}{\partial y} \right] n_x \]

\[ + A_{16} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + B_{11} \frac{\partial \psi_x}{\partial x} + B_{12} \frac{\partial \psi_y}{\partial y} + B_{16} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \right] n_x + \lambda \left[ A_{16} \frac{\partial u}{\partial x} \right. \]

\[ + A_{26} \frac{\partial v}{\partial y} + A_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + B_{16} \frac{\partial \psi_x}{\partial x} + B_{26} \frac{\partial \psi_y}{\partial y} + B_{66} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \left. \right] n_y \right] ds. \]

The primary variables are defined as the dependent variables: \( u, v, w, \psi_x, \) and \( \psi_y. \) The secondary variables are defined from the variational form as the coefficients of \( \lambda \) in the boundary integrals. In Eq. (5.7) the secondary variables are

\[ \left[ A_{11} \frac{\partial u}{\partial x} + A_{12} \frac{\partial v}{\partial y} + A_{16} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + B_{11} \frac{\partial \psi_x}{\partial x} + B_{12} \frac{\partial \psi_y}{\partial y} \right. \]

\[ + B_{16} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \right] n_x \]

\[ \left. \right] n_x \right] \left( 5.8 \right) \]

and

\[ \left[ A_{16} \frac{\partial u}{\partial x} + A_{26} \frac{\partial v}{\partial y} + A_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + B_{16} \frac{\partial \psi_x}{\partial x} + B_{26} \frac{\partial \psi_y}{\partial y} \right. \]

\[ + B_{66} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \right] n_y \right] \left( 5.9 \right) \]
which are recognized from Eqs. (4.7) as \((N_x + N_y) n_x\) and \((N_{xy} + N_y) n_y\). The variational form of Eq. (5.1) can finally be written as

\[
\int_{\Omega} \left[ \lambda \rho \frac{\partial^2 u}{\partial t^2} + \frac{\partial \lambda}{\partial x} \left[ A_{11} \frac{\partial u}{\partial x} + A_{12} \frac{\partial v}{\partial y} + A_{16} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + B_{11} \frac{\partial \psi_x}{\partial x} + B_{12} \frac{\partial \psi_y}{\partial y} \\
+ B_{16} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \right] + \frac{\partial \lambda}{\partial y} \left[ A_{16} \frac{\partial u}{\partial x} + A_{26} \frac{\partial v}{\partial y} A_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + B_{16} \frac{\partial \psi_x}{\partial x} + B_{26} \frac{\partial \psi_y}{\partial y} + B_{66} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \right] dx dy = \int_{\Gamma} \lambda \left[ (N_x + N_{xy}) n_x + (N_{xy} + N_y) n_y \right] ds.
\]

The variational forms of Eqs. (5.2) through (5.5) can be derived in a similar manner.

## 5.2. Interpolation and Approximate Functions

In the finite element method the region being examined is discretized into small finite areas called elements. The element boundaries are established by discrete points called nodes. Values of the dependent variables \((u, v, w, \psi_x, \psi_y)\) are directly obtained only at the nodes. The values between the nodes are found by interpolation. In the Rayleigh-Ritz method [105] the dependent variables at nonnodal locations within a single element are approximated by

\[
u = \sum_{i=1}^{NE} u_i \phi_i(\xi, \eta),
\]

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\[ v = \sum_{i=1}^{\text{NE}} v_i \phi_i(\xi, \eta), \quad (5.12) \]

\[ w = \sum_{i=1}^{\text{NE}} w_i \phi_i(\xi, \eta), \quad (5.12) \]

\[ \psi_x = \sum_{i=1}^{\text{NE}} \psi_{x,i} \phi_i(\xi, \eta), \quad (5.14) \]

\[ \psi_y = \sum_{i=1}^{\text{NE}} \psi_{y,i} \phi_i(\xi, \eta), \quad (5.15) \]

where \( u^0, v^0, w^0, \psi^0, \text{and } \psi^0 \) are the nodal values at the point \((x_i, y_i)\), the \( \phi_i \) are the interpolation functions, and \( \text{NE} \) refers to the number of nodes the element contains. The interpolation functions are functions of \( \xi \) and \( \eta \), which are the local coordinates of the element. The local coordinates are necessary in order to employ the numerical integration explained in Section 5.3 and to utilize a general set of interpolation functions.

Equations (5.11) through (5.15) are referred to as approximate functions. The interpolation functions of Eqs. (5.11) through (5.15) are not unique but must obey the following criteria:

1. Any set of interpolation functions \( \{\phi_i\} \) must be linearly independent.

2. Interpolation functions are nonzero only on the element to which they are assigned.
3. An interpolation function \( \phi_i \) is equal to unity at the coordinate \( x_i, y_i \) and equal to zero at any other nodal location.

4. At any point \( x_i, y_i \) on an element the values of the interpolation functions at that point must sum to unity.

In order that the solution converge to the exact solution as more elements are incorporated, the interpolation functions must also satisfy the following requirements:

1. The interpolation functions must be differentiable to the same order as the dependent variables.

2. The set of interpolation functions must be complete to the same order as the dependent variables; i.e., if the dependent variable contains a term of the \( n^{th} \) degree, the set of interpolation functions must contain terms of the \( n^{th}, n-1^{st}, n-2^{nd} \ldots \), and \( 0^{th} \) degree. This requirement ensures the possibility of rigid body modes and a state of constant strain.

If these criteria are satisfied the element is said to be conforming which means that:

1. There are no interelement gaps or overlaps and no sudden changes in slope between elements.

2. Rigid body modes are possible; i.e. when the body is in rigid motion the element must exhibit zero strain.

Figure 10 shows three master rectangular elements in local Cartesian coordinates which are all conforming elements. The node numbering shown in Fig. 10 is the local node
identification and is used uniformly for programming simplicity. The following three sets of interpolation functions correspond to the three rectangular elements.

**Linear Rectangular Element (4 Nodes)**

\[ \phi_1 = \frac{1}{4} (1 - \xi)(1 - \eta) , \]

\[ \phi_2 = \frac{1}{4} (1 + \xi)(1 - \eta) , \]

\[ \phi_3 = \frac{1}{4} (1 - \xi)(1 + \eta) , \]

and

\[ \phi_4 = \frac{1}{4} (1 + \xi)(1 + \eta) . \]

**Quadratic Rectangular Element (8 Nodes)**

\[ \phi_1 = \frac{1}{4} (1 - \xi)(1 - \eta)(-1 - \xi - \eta) , \]

\[ \phi_2 = \frac{1}{4} (1 + \xi)(1 - \eta)(-1 + \xi - \eta) , \]

\[ \phi_3 = \frac{1}{4} (1 + \xi)(1 + \eta)(-1 + \xi + \eta) , \]

\[ \phi_4 = \frac{1}{4} (1 - \xi)(1 + \eta)(-1 - \xi + \eta) , \]

\[ \phi_5 = \frac{1}{2} (1 - \xi^2)(1 - \eta) , \]

\[ \phi_6 = \frac{1}{2} (1 + \xi)(1 - \eta^2) , \]
\[ \phi_7 = \frac{1}{2} (1 - \xi^2)(1 + \eta) , \]

and

\[ \phi_8 = \frac{1}{2} (1 - \xi)(1 - \eta^2) . \]

Quadratic Rectangular Element (9 Nodes)

\[ \phi_1 = \frac{1}{4} (\xi^2 - \xi)(\eta^2 - \eta) , \]

\[ \phi_2 = \frac{1}{4} (\xi^2 + \xi)(\eta^2 - \eta) , \]

\[ \phi_3 = \frac{1}{4} (\xi^2 + \xi)(\eta^2 + \eta) , \]

\[ \phi_4 = \frac{1}{4} (\xi^2 - \xi)(\eta^2 + \eta) , \]

\[ \phi_5 = \frac{1}{2} (1 - \xi^2)(\eta^2 - \eta) , \]

\[ \phi_6 = \frac{1}{2} (\xi + \xi^2)(1 - \eta^2) , \]

\[ \phi_7 = \frac{1}{2} (1 - \xi^2)(\eta^2 + \eta) , \]

\[ \phi_8 = \frac{1}{2} (\xi^2 - \xi)(1 - \eta^2) , \]

and

\[ \phi_9 = (1 - \xi^2)(1 - \eta^2) . \]
The interpolation functions therefore depend on the element type. The computer codes developed for use in both the structural and acoustical analysis are capable of utilizing all three types of elements. However, in Chapter 9 it is shown that the 8-noded quadratic rectangular element is the most efficient for the problem of radiation from a rectangular plate. Therefore, only the 8-noded rectangular element is used to demonstrate and verify the FEM in Chapter 6.

The test function \( \lambda \) must be capable of satisfying the boundary conditions, but other than that it is arbitrary. It is convenient and usual to let the test function assume the same form as the interpolation functions. After this substitution is made along with the substitution of the approximate functions, the variational formulation can be written for each element in matrix form as

\[
\begin{bmatrix}
M_{ij}^{11} & 0 & 0 & 0 & 0 \\
0 & M_{ij}^{22} & 0 & 0 & 0 \\
0 & 0 & M_{ij}^{33} & 0 & 0 \\
0 & 0 & 0 & M_{ij}^{44} & 0 \\
0 & 0 & 0 & 0 & M_{ij}^{55}
\end{bmatrix}
\begin{bmatrix}
\ddot{u}_j \\
\ddot{v}_j \\
\ddot{w}_j \\
\ddot{\varphi}_{xj} \\
\ddot{\varphi}_{yj}
\end{bmatrix} + 
\begin{bmatrix}
K_{ij}^{11} & K_{ij}^{12} & K_{ij}^{13} & K_{ij}^{14} & K_{ij}^{15} \\
K_{ij}^{21} & K_{ij}^{22} & K_{ij}^{23} & K_{ij}^{24} & K_{ij}^{25} \\
K_{ij}^{31} & K_{ij}^{32} & K_{ij}^{33} & K_{ij}^{34} & K_{ij}^{35} \\
K_{ij}^{41} & K_{ij}^{42} & K_{ij}^{43} & K_{ij}^{44} & K_{ij}^{45} \\
K_{ij}^{51} & K_{ij}^{52} & K_{ij}^{53} & K_{ij}^{54} & K_{ij}^{55}
\end{bmatrix}
\begin{bmatrix}
u_j \\
v_j \\
w_j \\
\varphi_{xj} \\
\varphi_{yj}
\end{bmatrix} = 
\begin{bmatrix}
\Gamma_i^1 \\
\Gamma_i^2 \\
\Gamma_i^3 \\
\Gamma_i^4 \\
\Gamma_i^5
\end{bmatrix},
\]

where

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\[ M_{ij}^{11} = M_{ij}^{22} = M_{ij}^{33} = \int_{\Omega} \left[ h \phi_i \phi_j \right] dxdy, \]

\[ M_{ij}^{44} = M_{ij}^{55} = \int_{\Omega} \left[ \frac{h^3}{12} \phi_i \phi_j \right] dxdy. \]

\[ K_{ij}^{11} = \int_{\Omega} \left[ A_{11} \phi_{i,x} \phi_{j,x} + A_{16} (\phi_{i,x} \phi_{j,y} + \phi_{i,y} \phi_{j,x}) + A_{66} \phi_{i,y} \phi_{j,y} \right] dxdy, \]

\[ K_{ij}^{12} = \int_{\Omega} \left[ A_{12} \phi_{i,x} \phi_{j,y} + A_{16} \phi_{i,x} \phi_{j,x} + A_{26} \phi_{i,y} \phi_{j,y} + A_{66} \phi_{i,y} \phi_{j,x} \right] dxdy, \]

\[ K_{ij}^{13} = 0, \]

\[ K_{ij}^{14} = \int_{\Omega} \left[ B_{11} \phi_{i,x} \phi_{j,x} + B_{16} (\phi_{i,x} \phi_{j,y} + \phi_{i,y} \phi_{j,x}) + B_{66} \phi_{i,y} \phi_{j,y} \right] dxdy, \]

\[ K_{ij}^{15} = \int_{\Omega} \left[ B_{12} \phi_{i,x} \phi_{j,y} + B_{16} \phi_{i,x} \phi_{j,x} + B_{26} \phi_{i,y} \phi_{j,y} + B_{66} \phi_{i,y} \phi_{j,x} \right] dxdy, \]

\[ K_{ij}^{21} = K_{ji}^{12}, \]

\[ K_{ij}^{22} = \int_{\Omega} \left[ A_{66} \phi_{i,x} \phi_{j,x} + A_{26} (\phi_{i,x} \phi_{j,y} + \phi_{i,y} \phi_{j,x}) + A_{22} \phi_{i,y} \phi_{j,y} \right] dxdy, \]

\[ K_{ij}^{23} = 0, \]
\[ K_{ij}^{24} = \int_{\Omega} \left[ B_{66}\phi_{1,x}\phi_{j,y} + B_{16}\phi_{i,x}\phi_{j,x} + B_{26}\phi_{i,y}\phi_{j,y} + B_{12}\phi_{i,y}\phi_{j,x} \right] dx dy , \]

\[ K_{ij}^{25} = \int_{\Omega} \left[ B_{66}\phi_{i,x}\phi_{j,x} + B_{26}(\phi_{i,y}\phi_{j,y} + \phi_{i,y}\phi_{j,x}) + B_{22}\phi_{i,y}\phi_{j,y} \right] dx dy , \]

\[ K_{ij}^{31} = 0 , \]

\[ K_{ij}^{32} = 0 , \]

\[ K_{ij}^{33} = \int_{\Omega} \left[ A_{55}\phi_{1,x}\phi_{j,x} + A_{44}\phi_{i,y}\phi_{j,y} + A_{45}(\phi_{i,y}\phi_{j,x} + \phi_{i,x}\phi_{j,y}) \right] dx dy , \]

\[ K_{ij}^{34} = \int_{\Omega} \left[ A_{55}\phi_{i,x}\phi_{j} + A_{45}\phi_{i,y}\phi_{j} \right] dx dy , \]

\[ K_{ij}^{32} = \int_{\Omega} \left[ A_{45}\phi_{i,x}\phi_{j} + A_{44}\phi_{i,y}\phi_{j} \right] dx dy , \]

\[ K_{ij}^{41} = K_{ji}^{14} , \]

\[ K_{ij}^{42} = K_{ji}^{24} , \]

\[ K_{ij}^{43} = K_{ji}^{34} , \]

\[ K_{ij}^{44} = \int_{\Omega} \left[ A_{55}\phi_{j}\phi_{i} + D_{11}\phi_{i,x}\phi_{j,x} + D_{16}(\phi_{i,y}\phi_{j,y} + \phi_{i,y}\phi_{j,x}) + D_{66}\phi_{i,y}\phi_{j,y} \right] dx dy , \]

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\[
K_{ij}^{25} = \int_{\Omega} \left[ A_{45} \phi_i \phi_j + D_{12} \phi_{i,x} \phi_{j,y} + D_{16} \phi_{i,x} \phi_{j,x} + D_{26} \phi_{i,y} \phi_{j,y} + D_{66} \phi_{i,y} \phi_{j,x} \right] dx dy ,
\]

\[
K_{ij}^{51} = K_{ji}^{15} ,
\]

\[
K_{ij}^{52} = K_{ji}^{25} ,
\]

\[
K_{ij}^{53} = K_{ji}^{35} ,
\]

\[
K_{ij}^{54} = K_{ji}^{45} ,
\]

\[
K_{ij}^{25} = \int_{\Omega} \left[ A_{55} \phi_i \phi_j + D_{26} \phi_{i,x} \phi_{j,y} + D_{66} \phi_{i,x} \phi_{j,x} + D_{22} \phi_{i,y} \phi_{j,y} + D_{26} \phi_{i,y} \phi_{j,x} \right] dx dy ,
\]

\[
F_1^1 = F_1^2 = F_1^4 = F_1^3 = 0 ,
\]

and

\[
F_3^i = \int_{\Omega} \phi_i p_i dx dy .
\]

The matrices and vectors of Eqs. (5.16) are assembled by summing coefficients of common primary variables to give the global matrices referred to as the consistent mass matrix [M], the stiffness matrix [K], the force vector {F} and the displacement vector {U}, so that the global equations are

\[
\rho_s [M] \{\ddot{U}\} + [K] \{U\} = \{F\} .
\] (5.17)
The simplification invoked by the fact that the motion of the plate is restricted to harmonic motion can be employed at this point. The accelerations can now be expressed in terms of displacements and Eq. (5.17) becomes

$$[H^{(k,l)}_{ij}] \{U^{(l)}_j\} = \{F^{(k)}_i\} ,$$

(5.18)

where

$$[H^{(k,l)}_{ij}] = [K^{(k,l)}_{ij}] - \rho_s \omega^2 M^{(k,l)}_{ij}] ,$$

and i and j refer to the global node indices and k and l represent the indices of the degrees of freedom.

5.3. Gauss-Legendre Quadrature and Reduced Integration

In order to incorporate the approximate and interpolation functions into the variational form, all the functions and areas must be described in the same coordinate system. The transformation from the local coordinate system to the global coordinate system is accomplished by application of the Jacobian matrix. For example,

$$\begin{bmatrix}
\frac{\partial \phi_1}{\partial \xi} \\
\frac{\partial \phi_1}{\partial \eta}
\end{bmatrix} = [J] \begin{bmatrix}
\frac{\partial \phi_1}{\partial x} \\
\frac{\partial \phi_1}{\partial y}
\end{bmatrix},$$

$$\begin{bmatrix}
dx \\
dy
\end{bmatrix} = [J]^T \begin{bmatrix}
d\xi \\
d\eta
\end{bmatrix},$$
and

\[ dx \, dy = |J| \, d\xi \, d\eta \, , \]

where

\[
[J] = \begin{bmatrix} \frac{dx}{d\xi} & \frac{dy}{d\xi} \\ \frac{dx}{d\eta} & \frac{dy}{d\eta} \end{bmatrix} .
\]

The elements used in this dissertation are isoparametric, meaning that the x and y coordinates are approximated using the same nodes and interpolation functions as were used to approximate the dependent variables. Thus

\[ x = \sum_{i=1}^{NE} x_i \phi_i(\xi, \eta) \]

and

\[ y = \sum_{i=1}^{NE} y_i \phi_i(\xi, \eta) , \]

where NE is the number of nodes per element. For computational convenience the Jacobian matrix can be rewritten as

\[
[J] = \begin{bmatrix} \frac{\partial \phi_1}{\partial \xi} & \frac{\partial \phi_2}{\partial \xi} & \cdots & \frac{\partial \phi_n}{\partial \xi} \\ \frac{\partial \phi_1}{\partial \eta} & \frac{\partial \phi_2}{\partial \eta} & \cdots & \frac{\partial \phi_n}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} .
\]

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A wide variety of numerical integration methods is available to evaluate the integrands of the previous equations. Gauss-Legendre quadrature is one of the most widely used and has proven to be reliable. For any function \( f(x) \) the integral can be found by

\[
\int_a^b \int_c^d f(x,y) \, dx \, dy = \int_{-1}^1 \int_{-1}^1 f(\xi, \eta) |J| \, d\eta \, d\xi = \sum_{i=1}^{GP} \sum_{j=1}^{GP} f(\xi_i, \eta_j) W_{Ti} W_{Tj} |J|,
\]

where \( \xi_i \) and \( \eta_j \) are the local coordinates of the so-called Gauss points (roots of the Legendre polynomial), \( W_{Ti} \) and \( W_{Tj} \) are weight factors, and \( |J| \) is the determinant of the Jacobian matrix. The Gauss point locations and weighting factors are given in Appendix 1. The integral of a polynomial of the \( n \)th degree can be found exactly if the number of Gauss points is greater than or equal to \( (n+1)/2 \) [105].

Although the integral can be evaluated using the Gauss-Legendre quadrature, the terms in Eq. (5.16) involving the shear stiffnesses, \( A_{44}, A_{45}, \) and \( A_{55} \), are still erroneous. This is because the problem under consideration is based on two-dimensional plate theory, but the transverse shear terms are included. The governing equations thus represent the motion of a plate that is stiffer than the actual plate. In other words, the solution to the governing equations is dominated by the presence of the shear deformation terms regardless of their value. Zienkiewicz, Taylor, and Too [35] addressed this problem and developed a solution, called reduced integration, which is simple to employ. All the terms from the submatrices \( K_{11}^s, K_{12}^s, K_{22}^s, K_{44}^s, \) and \( K_{55}^s \) must be evaluated using reduced integration. Also \( K_{12}^t, K_{22}^t, K_{45}^t, \) and \( K_{55}^t \) all contain terms which must be separated out and evaluated using reduced integration. These terms are again those which involve shear stiffness and are:

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\[ G_{i4}^{44} = \int_{\Omega} [A_{55}\phi_i\phi_j]dxdy, \]

\[ G_{i5}^{44} = \int_{\Omega} [A_{45}\phi_i\phi_j]dxdy, \]

\[ G_{i5}^{44} = \int_{\Omega} [A_{45}\phi_i\phi_j]dxdy, \]

and

\[ G_{i5}^{55} = \int_{\Omega} [A_{55}\phi_i\phi_j]dxdy. \]

Instead of using \((n+1)/2\) Gauss points, as is done when finding the exact solution, the transverse shear terms are integrated using one less Gauss point. According to reference [35] this relaxes the stiffness and gives a more realistic weighting to those terms. Reduced integration has become a standard numerical tool when employing the first-order shear deformation theory.

5.4 Storage Considerations and Gauss Elimination

A brief explanation of matrix storage and methods of solving systems of simultaneous equations is inserted here in anticipation of the formidable challenge of manip-
ulating the more densely populated complex matrices of the fully coupled problem addressed in Chapter 10.

In general, the matrices \([K^{\alpha}\beta}\)] and \([M^{\alpha}\beta}\)] of Eqs. (5.18) are sparsely populated square matrices with a rank equal to the number of nodes in the entire finite element mesh. The only nonzero terms in \([K]\) and \([M]\) are those terms \([K^{\alpha}\beta}\)] and \([M^{\alpha}\beta}\)] where \(i\) and \(j\) are the global indices of nodes that appear in the same element. In order to decrease computer storage space it would be advantageous to store only the nonzero terms. This can be accomplished to a certain extent by storing only the band of terms in each row starting with the first nonzero term and ending with the last nonzero term. The band contains fewer zeros when the matrices are grouped according to node instead of degree of freedom. To illustrate, suppose the finite element mesh consists of three nodes and two degrees of freedom per node. Equations (5.18) would then appear in expanded form as

\[
\begin{bmatrix}
H_{11}^{11} & H_{11}^{12} & H_{11}^{13}
& H_{12}^{11} & H_{12}^{12} & H_{12}^{13}
& H_{13}^{11} & H_{13}^{12} & H_{13}^{13}
\end{bmatrix}
\begin{bmatrix}
U_1^{(1)}
& U_2^{(1)}
& U_3^{(1)}
\end{bmatrix}
= 
\begin{bmatrix}
F_1^{(1)}
& F_2^{(1)}
& F_3^{(1)}
\end{bmatrix}
\]

Rearranging now so that the terms are arranged according to node, there results
If the mesh consists of two elements and element 1 contains global nodes 1 and 2 and element 2 contains nodes 2 and 3 then the terms \([H\Phi_1]\) and \([H\Phi_2]\) are zero. In Eqs. (5.21) these zeros all appear at the end or beginning of a row and therefore would not contribute to the bandwidth. However, in Eqs. (5.20) the zeros are interspersed throughout the matrix and would therefore need to be stored. For large finite element meshes this method of banding the matrix reduces the computer storage space tremendously.

Since the matrices are symmetric only half of the bandwidth need be stored. Thus, the reduced global coefficient matrix \([H]\) will still consist of the same number of rows but the column width will now be equal to the half bandwidth.

The primary boundary conditions can be applied at this point by replacing the appropriate variable and factoring out the corresponding row and column. Once the boundary conditions are applied the set of simultaneous equations can be solved.

Indirect methods for solving systems of linear equations such as the Gauss-Siedel method are at first very attractive for solving sparsely-populated matrices such as \([H]\). This is because indirect methods utilize an iterative approach and therefore the full coefficient matrix does not need to be known or stored simultaneously. However, the instabilities that are incurred by indirect methods are difficult to avoid.
In this work, standard Gauss elimination [106] (a direct method) is used to solve all simultaneous equations. In the Gauss elimination procedure the first term of each row of the coefficient matrix is systematically eliminated by subtracting off a multiple of a previous row. This process is repeated until only the upper triangle and diagonal are nonzero. At this point the unknown variables can be found one by one using back substitution. The major stipulation of Gauss elimination is that the diagonal terms of the coefficient matrix must be nonzero. In fact the accuracy of the method increases when the magnitude of the diagonal terms dominates the other terms on the same row. In Eqs. (5.18) the diagonal terms by definition are dominant because the motion of the plate through a certain degree of freedom is primarily due to the force or moment acting on that degree of freedom.
6. Finite Element Predictions of Plate Response

The purposes of this chapter are to

1. introduce the boundary conditions and steady-state harmonic loads incident on the baffled plate,

2. verify the necessity of incorporating the Mindlin plate theory for laminated plate analysis, and

3. demonstrate the capability of the present finite element model to predict plate deflections.

In Section 6.1 the loading conditions of the rectangular plate are introduced. In Section 6.2 the transverse deflections of a rectangular plate due to various steady-state harmonic lateral (impinging the plate on the x-y plane) loads are derived in closed-form using Navier's series solution. These closed-form solutions are then utilized in Section 6.3 to
verify the finite element results of the current model. In Section 6.4 the finite element model is utilized to demonstrate how orthotropy and stacking sequence affect the transverse deflections of laminated composite plates due to various steady-state harmonic lateral loads.

6.1 Loading Conditions

As mentioned in the problem description in Chapter 1, the work of this dissertation is concerned with rectangular laminated composite plates excited by an oblique incident plane wave. The incident pressure of a plane wave impinging on a rectangular plate can be expressed in the coordinates shown in Fig. 2 as

\[ p(x,y,t) = p_0 e^{i(\omega t - k_x x - k_y y)}, \quad (6.1) \]

where

\[ k_x = (\omega / c) \sin \theta \cos \phi, \]

\[ k_y = (\omega / c) \sin \theta \sin \phi, \]

\[ \omega = \text{frequency of the incident wave}, \]

and

\[ p_0 = \text{amplitude of the incident wave}. \]

As a special case, when the incident angle is zero, the plane wave impinges uniformly on the plate and the pressure can be represented as
\[ p(t) = p_0 e^{i\omega t}. \]  \hspace{1cm} (6.2)

In order to verify the finite element model, the case of spatially sinusoidal loading is also considered of interest. This is represented mathematically by

\[ p(x,y,t) = p_0 \sin\left( \frac{n\pi x}{a} \right) \sin\left( \frac{n\pi y}{b} \right) e^{i\omega t}. \]  \hspace{1cm} (6.3)

6.2 Navier's Solution

In this section Navier's solution for simply supported rectangular plates subject to lateral loads is reviewed. Both isotropic and specially orthotropic plate solutions are derived. Spatially uniform and sinusoidal loads are considered as well as loads due to a plane wave impinging at an arbitrary incident angle. In all of the solutions presented in this chapter the plate and the surrounding fluid medium are only semi-coupled since the effect of the pressure radiated back into the fluid is not considered.

Navier's solution is based upon classical plate theory and therefore shear strains are neglected. Because Navier's solution is in closed-form it provides a simple comparison for the finite element model in the limit as shear strains are neglected. A comprehensive discussion of Navier's solution can be found in plate textbooks [104,111]. However, the salient features are reviewed here in order to gain insight into the plate response.

Navier's solution is based on the fact that for rectangular plates the deflection and load can be represented as a double Fourier series,
\[ w(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \sin(k_m x) \sin(k_n y) \]  
(6.4)

and

\[ p(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P_{mn} \sin(k_m x) \sin(k_n y), \]  
(6.5)

where \( k_m = \frac{m\pi}{a} \) and \( k_n = \frac{n\pi}{b} \). The value of \( w(x,y,t) \) can be found for any load that can be expressed in the form of Eq. (6.5). The procedure consists of first finding \( P_{mn} \) and then substituting Eqs. (6.4) and (6.5) into the equations of motion of the plate. This procedure is now demonstrated for the three load cases under consideration.

The value of \( P_{mn} \) from Eq. (6.5) is found by expanding \( p(x,y,t) \) in a double Fourier series as

\[ P_{mn} = \frac{4}{ab} \int_{0}^{b} \int_{0}^{a} p(x,y) \sin(k_m x) \sin(k_n y) \, dx \, dy. \]  
(6.6)

Substituting Eqs. (6.1) through (6.3) into Eq. (6.5) yields \( P_{mn} \) for each of the load cases.

**Plane Wave**

\[ P_{mn} = I_{sc}(k_m, k_x, a) I_{sc}(k_n, k_y, b) - I_{ss}(k_m, k_x, a) I_{sl}(k_n, k_y, b) + i(I_{sl}(k_m, k_x, a) I_{sc}(k_n, k_y, b) + \]

\[ I_{sc}(k_m, k_x, a) I_{ss}(k_n, k_y, b), \]  
\( m,n = 1,2,3, \ldots, \)  
(6.7)

where

\[ \text{6. Finite Element Predictions of Plate Response} \]
\[ I_{sc}(\alpha, \beta, \gamma) = \frac{4}{ab} \left[ -\cos(\alpha - \beta)\gamma - \cos(\alpha + \beta)\gamma \right. \frac{1}{2(\alpha - \beta)} \frac{1}{2(\alpha + \beta)} \left. + \frac{1}{\lambda(\alpha - \beta)} \right] , \]

and \[ I_{ss}(\alpha, \beta, \gamma) = \frac{4}{ab} \left[ \sin(\alpha - \beta)\gamma - \sin(\alpha + \beta)\gamma \right. \frac{1}{2(\alpha - \beta)} \frac{1}{2(\alpha + \beta)} \left. \right] . \]

### Uniform Loading

\[ P_{mn} = \frac{16p_0}{\pi^2 mn} \quad m,n = 1,3,5, \ldots \]

\[ P_{mn} = 0 \quad m,n = 2,4,6, \ldots \quad (6.8) \]

### Sinusoidal Loading

\[ P_{mn} = p_0 \quad m,n = 1 \]

\[ P_{mn} = 0 \quad m,n \neq 1 \quad (6.9) \]

In classical plate theory the equation of motion for an isotropic rectangular plate is written

\[ D \left[ \frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right] = p - \rho_s h \frac{\partial^2 w}{\partial t^2} , \quad (6.10) \]

where

\[ D = \frac{Eh^3}{12(1 - \nu)} . \]
An orthotropic plate whose principal material directions are parallel to the geometric axes of the plate is termed a specially orthotropic plate. The equation of motion for a specially orthotropic plate employing the principles of classical plate theory is

\[
D_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} = p - \rho_s h \frac{\partial^2 w}{\partial t^2}, \tag{6.11}
\]

Knowing \(P_{mn}\), Eqs. (6.4) and (6.5) can now be substituted into Eqs. (6.10) and (6.11) giving \(W_{mn}\) for all cases as

\[
W_{mn} = \frac{P_{mn}}{\rho_s h (\omega_{mn}^2 - \omega_f^2)}, \tag{6.12}
\]

where

\[
\omega_{ma} = \pi \left[ \frac{m^2}{a^2} + \frac{n^2}{b^2} \right] \sqrt{\frac{D}{\rho_s h}}
\]

for isotropic plates, and for specially orthotropic plates

\[
\omega_{mn} = \frac{\pi^2}{\sqrt{\rho_s h}} \sqrt{D_{11} \frac{m^4}{a^4} + 2(D_{12} + 2D_{66}) \left( \frac{mn}{ab} \right)^2 + D_{22} \frac{n^4}{b^4}}. \tag{6.13}
\]

Now \(W_{mn}\) can be substituted back into Eq. (6.4) to give \(w(x,y,t)\).

6.3 Model Verification

In this section the finite element mesh utilized to model the laterally loaded plate is introduced. In order to generalize results, normalized material properties and deflections are utilized and presented. Finally, the normalized lateral deflections of
isotropic and specially orthotropic plates obtained from the Navier series solution are compared to the finite element results in order to verify the finite element model and justify the use of the first-order shear deformation theory in the analysis of orthotropic plate bending.

The finite element results shown in the remainder of this chapter are based upon rectangular plates with simply-supported boundary conditions. In general the plate is modeled with 16 quadratic elements as shown in Fig. 11. However, when the plate is isotropic and the loading is symmetric about the x- and y-axes only a quarter of the plate is modeled, as shown in Fig. 12.

The results presented in this chapter are presented in the form of normalized transverse deflections where the deflections are normalized as

\[ \bar{w} = \frac{wE_1h^3}{p_0a^4}, \tag{6.14} \]

where \( h \) is the plate thickness, \( a \) is the width of the plate in the x-direction, and \( p_0 \) is the amplitude of the load normal to the plate surface. Both isotropic and orthotropic plates are examined in this chapter. Since the results are nondimensionalized, it is necessary to know only the Poisson ratio \( (\nu = 0.3) \) for the isotropic plates. For the orthotropic plates,

\[ \frac{E_1}{E_2} = 25.0, \quad \nu_{12} = 0.25, \]

\[ G_{12} = 0.5E_2, \quad G_{13} = 0.5E_2, \quad G_{23} = 0.2E_2. \]

In Fig. 13 a profile of the normalized deflection of an isotropic plate due to a) a uniform load and b) a sinusoidal load are shown. The dashed lines in each loading case
represent the normalized lateral deflection of an isotropic plate with different width-to-thickness ratios (a/h) as predicted by the finite element model. The solid line in each loading case represents Navier's solution utilizing classical plate theory for all the a/h ratios. In both loading cases the finite element model which incorporates the first-order SDT converges to the Navier series solution (based upon the classical plate theory) as a/h becomes large. Two observations and conclusions can be made about the FEM from Fig. 13.

1. The FEM prediction of deflections of isotropic plates due to sinusoidal and uniform loads are in agreement with closed-form classical plate solutions in the region where a/h is large or equivalently where shear strains are small.

2. By incorporating the first-order SDT in the FEM the predicted deflections are larger than classical plate theory predictions. For example, the deflections predicted by the FEM due to a sinusoidal load are as much as eight percent higher than the classical plate theory predictions for an a/h ratio of ten. This is because the plate is now allowed to rotate about the x and y axes, thus producing a shear strain.

The conclusions that can be drawn from these observations are that

1. the FEM correctly predicts the transverse or lateral deflections of isotropic plates due to uniform and sinusoidal loads and

2. it becomes necessary to incorporate the first-order SDT in the FEM in order to more accurately describe the lateral motion of an isotropic plate as the width-to-thickness ratio becomes small.
For specially orthotropic plates similar observations and conclusions are made based on the results shown in Figs. 14 and 15. The normalized lateral deflection of a specially orthotropic plate due to a spatially uniform load and a spatially sinusoidal load is shown in Figs. 14 and 15, respectively. In both figures the dashed lines represent the deflection for plates with different width-to-thickness ratios as predicted by the finite element model. The solid line in both figures represent the solution for all width-to-thickness ratios as predicted by Navier's solution utilizing classical plate theory. The results from this specially orthotropic case show that

1. the FEM solution based on the first-order SDT converges to the Navier solution as the width-to-thickness ratio becomes small or consequently as the shear strains become negligible,

2. and that the Navier solution based upon the classical plate theory does not predict deflections well. In fact for a specially orthotropic plate with a width-to-thickness ratio of ten the FEM prediction of lateral deflections due to sinusoidal loading is 70 percent higher than the classical plate prediction.

This second conclusion is interpreted as meaning that the use of the first-order shear deformation theory is even more justified when considering the lateral deflections of orthotropic plates. The reason for this is that an orthotropic plate may have higher shear strains than an isotropic plate for the same width-to-thickness ratio. The results shown in Figs. 16 through 19 are included for the purpose of demonstrating that the conclusions just drawn are true at other excitation frequencies and also for plates excited by obliquely incident plane waves.

In Figs. 16 and 17 a spatially sinusoidal load is again applied to both an isotropic plate and specially orthotropic plate, respectively. However, now the plate width-to-
thickness ratio is held constant and the frequency of excitation is varied from near zero to twice the fundamental frequency. Since the load is sinusoidal and matches the first mode shape, only the fundamental mode is excited. The dashed line in Figs. 16 and 17 represents the finite element solution, and the solid curves represent the Navier solution. In Fig. 16 the curves lie on top of each since the plate is thin and isotropic. However, in Fig. 17 for the specially orthotropic plate the curves are separate and distinct and it can be seen that the classical plate theory predicts a slightly higher fundamental frequency.

Figures 18 and 19 contain the normalized center deflections of an isotropic and specially orthotropic plate due to a plane wave. The angle of incidence is varied from 0° to 90° while the all other parameters are held constant. The dashed line in both figures represents the normalized center deflection of the plate as predicted by the finite element solution and the solid line represents the solution predicted by the classical plate theory. As the incident angle increases the plate response decreases due to decreased magnitude and change of phase. The classical plate solution for the isotropic plate in Fig. 18 is very near the finite element solution. The two solutions in Fig. 19 again differ because of the orthotropy of the plate.

### 6.3 Laminated Plate Deflections

In this section the normalized lateral deflections of laminated plates due to spatially-uniform time-harmonic loads are shown. The purpose of this exercise is to demonstrate how the orthotropic nature of laminated plates affects the deformation of the plate and show the variety of response obtained for several stacking sequences. This
awareness is necessary in order to appreciate the acoustic-structure interaction of laminated plates and their surrounding fluid medium discussed in subsequent chapters.

The first case considered is that of a single specially orthotropic lamina. Figure 20 contains the normalized deflections due to a uniform load obtained from the finite element analysis of three specially orthotropic plates with different $E_1/E_2$ ratios demonstrating the effect of orthotropy on the deformation of the plate. The dashed lines represent the deflection along the y-axis and the solid lines represent the deflection along the x-axis. Thus by varying the Young's modulus in the y-direction not only are the deflections altered along both the coordinate axes but now the shape of the deflections along each axis are not equivalent. Indeed as the $E_1/E_2$ ratio becomes large the shape of the plate deflection along the y-axis becomes flatter away from the plate boundaries while the deflection along the x-axis retains its parabolic shape.

The manner in which stacking sequence influences the deflection of a simply-supported square laminated plate due to spatially uniform time harmonic loading is shown in Figs. 21 through 24 for laminates with stacking sequence oriented $(60^\circ/30^\circ;0^\circ/-30^\circ/-60^\circ)$, $(0^\circ/90^\circ/0^\circ)$, $(45^\circ/-45^\circ/45^\circ)$, and $[-45^\circ/45^\circ/-45^\circ/45^\circ]$, respectively.

The normalized deflections along the x- and y-axes of several orthotropic laminates due to a uniform load are shown in Figs. 21 through 24. All of the plates are of the same thickness and each lamina is made of the orthotropic material described in Section 6.1; only the stacking sequence is different in each case. The deflection of a specially orthotropic plate of the same material properties and dimensions as the other laminates is represented in each of the Figs. 21 through 24 by the solid curves in order to serve as a benchmark for comparison.

The dashed curves in Fig. 21 represent the normalized deflection of a laminate comprised of five lamina oriented in the following manner: $[60^\circ/30^\circ/0^\circ/-30^\circ/-60^\circ]$. This laminate approximates an isotropic plate and makes little use of the orthotropic nature.
of the lamina. Therefore, it is the most flexible of the laminates shown. The dashed curves in Fig. 22 represent the normalized deflection of a [0°/90°/0°] laminate. This laminate, termed a cross-ply laminate, approximates an isotropic plate to a lesser degree but still is comparatively flexible. The dashed curve in Fig. 23 represents the deflection along both the x- and y-axes of a [45°/-45°/45°] laminate. This laminate is termed a balanced angle-ply laminate. The term balanced refers to the fact that for each lamina above the midplane of the plate there is a lamina with the same orientation and thickness an equal distance below the midplane. Although the deformation of this plate is symmetric along the x- and y-axes, the deformation off the axes is unsymmetric. The deformation of a [-45°/45°/-45°/45°] laminate is shown in Fig. 24. For laminated plates with aspect ratios near one, the angle-ply laminates such as the [45°/-45°/45°] laminate and the [-45°/45°/-45°/45°] laminate are known to constitute the lamina configurations which yield the stiffest plates [112]. However, this is not true for plates of any aspect ratio. Certainly a rectangular plate with an aspect ratio much greater than one (or much less than one) will be stiffer if the principal material direction is parallel with the larger dimension. In Chapter 11 the fluid-structure interaction of laminated plates and their surrounding fluid is examined. The FEM results presented in this final section of Chapter 6 are utilized in Chapter 11 as the limiting case when fluid density is considered small compared to plate material density.
7. Governing Equations for the Fluid Medium

The equations describing the acoustic pressure field surrounding a vibrating structure are derived in this chapter. The fundamental equation of state, equation of continuity, and Euler's equation are utilized to show the restrictions of the solution and derive the Helmholtz wave equation. The conditions of the Green's function necessary for the derivation of the field pressure are discussed. The Helmholtz integral equation is derived and specialized for the case of the baffled plate. Finally, asymptotic solutions to the Helmholtz integral equation are explained.

7.1 The Helmholtz Acoustic Wave Equation and Green's Function

The acoustic pressure field due to a harmonic sound source such as a vibrating structure is described by the Helmholtz acoustic wave equation,
\((\nabla^2 + k^2)p = 0\), \hfill (7.1)

where \(\nabla\) is the del operator, \(k\) is the wave number, and \(p\) is the acoustic pressure. In the derivation of the linearized wave equation (Eq. 7.1) it is assumed that the pressure disturbance is small and proceeds adiabatically. It is also assumed that the fluid is inviscid and compressible.

In order to solve for the acoustic pressure in the fluid domain surrounding a vibrating structure it is necessary to find a Green's function, \(g\), that satisfies two mathematical conditions. The first condition is that the Green's function satisfy the nonhomogenous Helmholtz equation

\[
(\nabla^2 + k^2)g(|r_s - r_f|) = \delta(|r_s - r_f|),
\]

where \(r_s\) is the coordinate of the source point, \(r_f\) is the coordinate of the field point, and \(\delta\) is the Dirac delta function. The integral of the product of the Dirac delta function and an arbitrary function, \(f(r_s)\), over a source volume, \(V_s\), is

\[
\int_V f(r_s)\delta(|r_s - r_f|)dV(r_s) = \begin{cases} f(r_f)/2, & r_f \text{ on boundary of } V, \\ 0, & r_f \text{ outside of } V \end{cases}
\]

The second condition the Green's function must satisfy is the Sommerfeld radiation condition [112]. This condition ensures that the integral expression for the pressure represents outward traveling waves.

The free-space Green's function meets both of these conditions and is used in Section 7.2 to describe the acoustic pressure anywhere in the fluid domain so long as the
velocity of the structure boundary is defined everywhere. The free-space Green's function is expressed as

\[ g(R) = -\frac{e^{ikR}}{4\pi R}, \quad (7.4) \]

where \( R = |\mathbf{r} - \mathbf{r}_f| \). For the problem of the infinitely baffled rectangular plate the half-space Green's function,

\[ g(R) = -\frac{e^{ikR}}{2\pi R}, \quad (7.5) \]

can be utilized, thus eliminating the need to model the motionless baffle. This point is demonstrated in Chapter 9.

### 7.2 The Helmholtz Integral Equation

In this section the acoustic wave equation and the Green's function are substituted into Green's identity, in order to find an integral equation which describes the acoustic field both in the fluid medium and on the surface of a vibrating structure.

The Helmholtz integral equation describes the acoustic pressure radiated from a vibrating structure and is predicated on Green's identity which relates the boundary of the vibrating structure and the volume occupied by the fluid medium. Green's identity in terms of the Green's function, \( g \), and the surface pressure, \( p \), may be written

\[ \int_{V} \{ pV^2 g - gV^2 p \} dV(\mathbf{r}) = \int_{\Omega} \left\{ p \frac{\partial g}{\partial z} - g \frac{\partial p}{\partial z} \right\} dS(\mathbf{r}_s) \quad (7.6) \]
where \( V \) is the volume occupied by the fluid and \( \Omega \) is the boundary of the fluid volume. When Eqs. (7.1) and (7.2) are substituted into Eq. (7.6) the volume integral on the left-hand side of Eq. (7.6) becomes

\[
\int_V \left[ p \nabla^2 g - g \nabla^2 p \right] dV(r_i) = \begin{cases} p(r_i), & r_i \text{ in } V \\ \frac{p(r_i)}{2}, & r_i \text{ on } \Omega \end{cases} .
\]  

(7.7)

In terms of the coordinate system shown in Fig. 2 the first term on the right-hand side of Eq. (7.6) as result of the \( \delta \) operator becomes

\[
p \frac{\partial g}{\partial z} = p(r_i)(k_i - \frac{1}{R})g(R)\cos\theta_f .
\]  

(7.8)

The linearized inviscid force equation [139] relates the fluid displacement, \( w_f \), to the acoustic pressure, \( p \), as

\[
\rho_f \frac{\partial^2 w_f}{\partial t^2} = -\nabla p .
\]  

(7.9)

Substituting Eqs. (7.7), (7.8) and (7.9) into Eq. (7.6) and invoking the assumption of harmonic motion produces the Helmholtz integral equation,

\[
p(r_i) = \varepsilon \int_{\Omega} \left\{ p(r_j)(k_j - \frac{1}{R})g(R)\cos\theta_j + g(R)\rho_j\omega^2 w(r_j) \right\} dS(r_j) ,
\]  

(7.10)

where \( w(r_i) \) is the deflection of the boundary normal to the surface, \( \varepsilon = 1 \) for field points not on the boundary, and \( \varepsilon = 2 \) when the field point is on the boundary.

Inspection of Eq. (7.10) reveals that the surface pressure, \( p(r_i) \), must be found first before the field pressure, \( p(r_i) \), can be found. This is accomplished by allowing the field
point to approach the vibrating boundary and then solving for the surface pressure using
Eq. (7.10) with \( \epsilon \) equal to 2. For the case of the baffled plate the \( \cos \theta \) term is zero when
the field point, \( r_f \), is on the boundary. Therefore, the pressure on the surface of the plate
can be found from

\[
p(r_f) = 2 \int_\Omega \left[ g(R) \rho \omega^2 w(r_s) \right] dS(r_s), \quad r_f \text{ on } \Omega. \tag{7.11}
\]

Once the pressure on the surface of the plate is found from Eq. (7.11) the pressure any-
where in the fluid field is found using Eq. (7.10).

When the field point in question is a large distance from the plate surface in terms
of both plate dimension and wavelength, it is said to lie in the farfield. When only the
farfield acoustic pressure is of interest, the Helmholtz integral equation can be approxi-
mated by Rayleigh's formula,

\[
p(r_f) = \rho \frac{\omega^2}{2\pi R} \int_\Omega \left[ e^{ikR} w(r_s) \right] dS(r_s). \tag{7.12}
\]

Rayleigh's formula follows directly from the Helmholtz integral equation recognizing the
fact that \( R \) is now weakly dependent on the source geometry and that the Green's
function will have a gradient equal to zero normal to an infinite surface. Rayleigh's
formula is exploited in Chapter 9 to verify the BEM results in the farfield limit.
7.3 Asymptotic Solutions to the Helmholtz Integral Equation

For arbitrary source geometries and mid-range frequencies the acoustic field pressure, $p(r_i)$, must be found by numerical methods such as the BEM discussed in Chapter 8. However, in the short and long wavelength limits the surface pressure, $p(r_s)$, can be eliminated from Eq. (7.10) thus rendering Eq. (7.10) an explicit expression of the acoustic pressure anywhere in the fluid domain.

The short wavelength approximation can be made if the wavelength is small compared to

1. the radius of curvature of the radiating surface, and

2. the dimensions of the regions of the plate vibrating in phase.

The short wavelength approximation assumes then that the pressure gradient in the tangential direction to the plate surface is negligible compared to the normal gradients. Thus the radiated wave is assumed to behave as a plane wave at least locally. The surface pressure can then be approximated by

$$p(r_s) = i \rho c \omega,$$  \hspace{1cm} (7.13)

and the field pressure can be found explicitly from

$$p(r_f) = \int_{\Omega} \left[ g(R) \rho c \omega^2 w(r_s)(1 + \cos \theta) \right] dS(r_s).$$ \hspace{1cm} (7.14)

In the low frequency limit the $k^3$ term in Eq. (7.1) is negligible and therefore the surface pressure is approximated using the Laplace equation
\[ \nabla^2 p(r) = 0. \]  

(7.15)

The Helmholtz integral equation can be simplified in the low frequency limit by recognizing that the \(ik\) term in Eq. (7.10) can be neglected since it is small compared to the \(1/R\) term. The low frequency solution can be simplified even further by restricting the solution to the farfield, \((R > a)\). By implementing Eq. (7.18) and the low frequency approximations to the Helmholtz integral equation, the farfield acoustic pressure in the low frequency limit can then be expressed as

\[ p(r) = \int_{\Omega} \{ g(R) \rho c^2 w(r) \} dS(r). \]  

(7.19)

The asymptotic solutions and Rayleigh's formula are useful analytical tools for many practical acoustic radiation problems. These classical techniques are utilized in Chapter 9 to verify the BEM results.
8. Boundary Element Formulation

The boundary element method of solving for the acoustic pressure field due to a vibrating plate involves the following steps:

1. Discretize the plate surface into surface elements called boundary elements.

2. Substitute the approximate equations of the deflection and pressure into the element equations which describe the pressure on the boundary.

3. Solve for the pressure at each node on the plate surface by summing the pressure radiated and reflected from all other nodes. (This step involves further division of the boundary elements into subelements in order to eliminate the $1/r$ singularity as explained in Section 8.2.)

4. Solve for the pressure field in the fluid domain by utilizing the discretized Helmholtz integral equation and the surface pressure distribution found in Step 3.
8.1 Discretization of Boundary and Equations

One of the chief advantages of the BEM is the fact that the primary variable can be described in a space with one more dimension than the element is required to occupy. Specifically, the three-dimensional pressure field in Eq. (7.10) can be solved by discretizing the two-dimensional boundary into the same two-dimensional elements as those described in Section 5.2.

Using the approximation functions of Section 5.2, the pressure on the surface of the plate, \( p(x,y) \), and the lateral plate deflection, \( w(x,y) \), are approximated by

\[
p(x,y) = \sum_{i=1}^{NE} p_i \phi_i(\xi, \eta) \tag{8.1}
\]

and

\[
w(x,y) = \sum_{i=1}^{NE} w_i \phi_i(\xi, \eta), \tag{8.2}
\]

where \( p_i \) and \( w_i \) are the pressure and deflection at the node locations. Equations (7.10) and (7.11) can now be written as a summation of the individual contributions of all elements in the mesh as

\[
p(r_t) = \sum_{m=1}^{NM} \sum_{n=1}^{NE} \left\{ g(R) \cos \theta \left( \frac{1}{R} - \frac{1}{R_t} \right) + g(R) \rho \omega^2 w \phi_i(\xi, \eta) \right\} dS(r_t), \tag{8.3}
\]

for \( r_t \) not on \( \Omega_n \), and
\[ p(r_l) = 2 \sum_{m=1}^{NM} \sum_{i=1}^{NE} \int_{\Omega_m} \{ g(R)\rho_l \omega^2 w_i(\xi, \eta) \} dS(r_s), \]  

(8.4)

for \( r_l \) on \( \Omega_m \), where NE is the number of nodes per element and NM refers to the number of elements in the entire boundary element mesh.

### 8.2 Numerical Integration of Singular Elements

Equation (8.4) gives the pressure at a field point, \( r_l \), on the vibrating surface. This pressure is due not only to the motion of the plate at the point \( r_l \) but is also due to the plate motion at all other locations. For the purpose of solving Eq. (8.3), the pressure on the plate surface must be found by solving Eq. (8.4) for each node of the boundary element mesh. This is accomplished by specifying \( r_l \) as the nodal coordinate, systematically evaluating the integral in Eq. (8.4) for each element, and summing each of the results. For the majority of the elements the integral can be evaluated using standard Guass quadrature as explained in Section 5.3. However, recall from Section 7.2 that

\[ g(R) = \frac{e^{ikl}r_s - r_l}{4\pi |r_s - r_l|}. \]  

(8.5)

Therefore, when the elements that contain the node located at \( r_l \) are evaluated, the integrand is singular in the limit as \( r_s \) approaches \( r_l \). The method proposed by Rizzo and Shippy [65] in their thermoelasticity solution is utilized in this dissertation to eliminate the singularity. In this method the \( 1/R \) singularity is removed by converting to polar coordinates, so that

8. Boundary Element Formulation
\[
\frac{e^{ikr}}{R} \int R dR d\alpha = e^{ikr} dR d\alpha.
\]

Therefore the contribution to the pressure at \( r_f \) from an element, \( m \), which contains the node located at \( r_r \) is

\[
p^{(m)}(r_f) = \frac{\rho_r c_\omega^2}{2\pi} \sum_{i=1}^{3} \int_{\Omega_m} \left\{ e^{ikR} W_i(\xi, \eta) \right\} dR(r_e) d\alpha, \quad r_f \text{ on } \Omega_m \tag{8.6}
\]

Now the integral must be evaluated in polar coordinates but the area of integration, \( \Omega_m \), is the area of a rectangular element. For 8-noded quadratic and 4-noded linear elements the integration can be accomplished by subdividing the rectangular element into three subelements. When the singular node is on the corner the element is subdivided as shown in Fig. 25 (a), and when the the singularity occurs on the edge the element is subdivided as shown in Fig 25 (b). Even though the singularity has been removed from the integrand in Eq. (8.6), the solution of Eq. (8.6) changes very rapidly and nonlinearly in the region where \( r_r \) approaches \( r_f \). Therefore, it is critical to accurately evaluate the integral in the area immediately surrounding the field point at \( r_f \). Subdividing the elements that incur a singularity not only makes the integration possible but also renders a more accurate solution since the subelements provide a refined mesh in the area of the field point. Equation (8.6) can be expressed now as the sum of the contribution from each subelement as

\[
p^{(m)}(r_f) = \frac{\rho_r c_\omega^2}{2\pi} \sum_{l=1}^{3} \sum_{i=1}^{3} \int_{s_r^{(l)}} \int_{R_0^{(l)}} \left\{ e^{ikR} W_i(\xi, \eta) \right\} dR(r_e) d\alpha, \quad r_f \text{ on } \Omega_m, \tag{8.7}
\]

8. Boundary Element Formulation
where the limits of integration of \( R \) are in general a function of \( \alpha \). Also, the limits of integration are dependent on the subelement and are therefore superscripted with a (\( l \)). The function \( \phi \) still appears as a function of the local element coordinates \( \xi \) and \( \eta \). These coordinates can be determined as functions of \( R \) and \( \alpha \) using the methods described in Section 5.3. In order to solve Eq. (8.7) using Gauss quadrature, the coordinates of each of the subelements must be transformed into local subelement coordinates. These coordinates are represented by \( \bar{\xi} \) and \( \bar{\eta} \) to avoid confusion with the local coordinates \( \xi \) and \( \eta \) corresponding to the entire element, \( m \). Equation (8.7) is expressed in terms of the master coordinates of the subelements as

\[
p^{(m)}(r_l) = \frac{\rho_c \omega^2}{2 \pi R} \sum_{l=1}^{3} \sum_{i=1}^{N_{E}} \int_{-1}^{1} \int_{-1}^{1} e^{i k R} w_i \phi_i(\xi, \eta) \left\{ R_u^{(l)}(\alpha) - R_o^{(l)}(\alpha) \right\} \frac{\alpha_u^{(l)} - \alpha_o^{(l)}}{2} d\xi d\eta, \tag{8.8}
\]

where

\[
R = \frac{R_u^{(l)}(\alpha) + R_o^{(l)}(\alpha) + [R_u^{(l)}(\alpha) - R_o^{(l)}(\alpha)]\bar{\xi}}{2}
\]

and

\[
\alpha = \frac{\alpha_u^{(l)} + \alpha_o^{(l)} + [\alpha_u^{(l)} + \alpha_o^{(l)}]\bar{\eta}}{2}
\]

The contribution from all other elements to the total pressure at \( r_l \), \( p(r_l) \), is found by summing the pressure contribution, \( p^{(m)}(r_l) \), from each element, \( m \). Once the pressure is found at every node on the surface, these nodal pressures \( p \), can be substituted back into Eq. (8.3) to determine the acoustic pressure anywhere in the fluid domain.
9. Boundary Element Results

In this chapter the boundary element method is employed to predict the acoustic field surrounding baffled rectangular pistons and plates. A degree of confidence in the BEM is established by comparing the BEM results with known impedance results and the farfield solution of the Rayleigh integral formula. Both finite and infinite baffles are considered in order to demonstrate the sensitivity of the acoustic pressure field to baffle size.

9.1 Radiation Impedance of a Square Piston

A baffled harmonically oscillating square piston of half-width $a$ represents a special case of the more complicated square plate. The difference lies in the fact that the piston is considered rigid and therefore the surface velocity is uniform over the piston surface. In this section the radiation impedance of a square piston is examined in order to es-
ablish a degree of confidence in the BEM and study the dependence of mesh refinement on the BEM solution.

The radiation impedance of an oscillating piston is defined as

\[
Z = \int_{0}^{a} \int_{0}^{a} \frac{p}{w} \, dx \, dy. \tag{9.1}
\]

Swenson and Johnson [65] have written a series solution for the radiation impedance of an oscillating square piston in which

\[
Z_R = \frac{4 \rho \rho \omega k}{2\pi} \sum_{n=0}^{\infty} B_{2n} \frac{k^{2n}}{(2n+3)} \tag{9.2}
\]

represents the resistive part and

\[
Z_X = \frac{4 \rho \rho \omega}{2\pi} \sum_{n=0}^{\infty} A_{2n} \frac{k^{2n}}{(2n+2)} \tag{9.3}
\]

represents the reactive part. The \(A_{2n}\) and \(B_{2n}\) terms in Eqs. (9.2) and (9.3) are

\[
A_{2n} = 2(-1)^n a^{2n+3} \left\{ \int_{0}^{\pi/4} \sec^{2n+1} \theta \, d\theta - \frac{2^{n+0.5}}{2n+3} - 1 \right\} \tag{9.4}
\]

and

\[
B_{2n} = 2(-1)^n a^{2n+4} \left\{ \int_{0}^{\pi/4} \sec^{2n+2} \theta \, d\theta - \frac{2^{n+1}}{2n+4} - 1 \right\}. \tag{9.5}
\]

9. Boundary Element Results
The resistance and reactance defined by Eqs. (9.2) and (9.3), normalized by dividing through by \( \rho c a^2 \), are shown as solid curves in Figs. 26 through 31 as a function of the product \( ka \). Figures 26 through 28 also contain dashed lines representing the normalized impedance as predicted by the BEM using 4, 16, and 25 linear elements, respectively. The dashed lines in Figs. 29 through 31 show the normalized impedance predicted by the BEM using 4, 16, and 25 quadratic elements, respectively. In each of these figures the BEM solution converges to the analytical solution as \( ka \) becomes small. This is because the approximation functions utilized by the boundary elements can more closely fit the pressure distribution when the wavelength is long in comparison with the element width. Because the quadratic elements better conform to the pressure distribution on the piston face than do the linear elements at higher values of \( ka \), the BEM utilizing quadratic elements converges at much higher values of \( ka \) than does the BEM utilizing linear elements. As shown in Fig. 29 a BEM mesh of only four quadratic elements gives very good agreement even for \( ka = 19 \) while the BEM employing 25 linear elements in Fig. 28 diverges in the reactive portion at \( ka = 1 \). These results support the generally accepted rule of thumb [58] that at least two quadratic or four linear elements should be used in any dimension equal to the wavelength.

The acoustic pressure along a line extending from the center of a square piston surface along the z-axis is plotted in Fig. 32 for a \( ka \) value of ten. The dashed curve represents the BEM solution and the solid curve represents the Rayleigh solution. In the farfield the BEM solution converges with the Rayleigh solution, demonstrating that the BEM is predicting the farfield solution correctly. The pressure along the z-axis can also be found in closed-form for a circular piston. Stenzel [141] compares the nearfield radiation from a square piston to the radiation from a circular piston whose radius is equal to half the width of the square piston. Stenzel's results show that whereas for the circular piston the normalized pressure varies from zero to unity in the nearfield, the
normalized pressure from the square piston varies within a more narrow envelope. The BEM solution shown in Fig. 32 is in qualitative agreement with Stenzel's results [141] in that both are similarly different from the classical result for a circular piston [140].

9.2 Axial Pressure of a Baffled Square Plate

Rayleigh's integral formula, Eq. (7.15), can be evaluated for a known plate deflection such as the double Fourier series of Eq. (6.5). In this section, the acoustic pressure radiated from isotropic plates is evaluated using Rayleigh's integral and the boundary element method. Convergence of the model is demonstrated and the effects of a finite baffle shown.

In Fig. 33 the axial pressure radiated from an infinitely baffled square plate as predicted by the BEM, \( p_{BEM} \), is compared with the value of \( p_{Rayleigh} \), predicted using a Rayleigh integral solution for \( ka = 7.6 \). The vertical axis in Fig. 33 is the quantity

\[
\text{pressure dB} = 20 \log_{10} \frac{p_{BEM}}{p_{Rayleigh}},
\]

at \( ka = 7.6 \). The horizontal solid line in Fig. 33 represents the solution from Rayleigh's integral and the other three curves represent the BEM solution. The long dashed curve in Fig. 33 represents the solution with only four 8-noded quadratic elements. This solution converges to a value higher than the Rayleigh integral solution in the farfield. The short dashed curve is the BEM result using sixteen 8-noded quadratic elements, and the solid curve is the BEM solution using sixty-four 8-noded quadratic elements. These last two curves show that the BEM solution converges as the BEM mesh is refined. Also, both of these solutions converge to the Rayleigh solution in the farfield.
The axial pressure difference expressed by Eq. (9.5) between the BEM solution and Rayleigh's integral solution is shown in Fig. 34; however, now the radiator is an isotropic square plate in a finite baffle. The three dashed curves in Fig. 34 represent three baffle widths. The large dashed line in Fig. 34 corresponds to the largest baffle width, which is 9a. The medium dashed curve represents the solution with a baffle width of 5a, and the small dashed line is the solution when the baffle width is 2a. The plate was modeled using 36 8-noded quadratic elements and the value of ka is 7.6. In this case a generalized conclusion cannot be drawn about the gradient of the Green's function near the plate surface because the boundary did not extend to infinity. Therefore, the half-space Green's function cannot be utilized and so the free-space Green's function is employed. The Rayleigh formula solution is represented by the solid line in Fig. 34 and assumes that the baffle extends to infinity. As more of the baffle is included, the BEM solution converges to the Rayleigh formula solution in the farfield.

9.2 Directivity of a Baffled Rectangular Plate

In this section the directivity patterns of a square plate and a rectangular plate due to an impinging plane wave of normal incidence are predicted in the farfield. In each case both finite and infinite baffles are considered. The BEM solution converges to the Rayleigh formula solution as more of the baffle is modeled.

The directivity pattern due to an impinging plane wave of normal incidence (ka = 7.6) on a simply supported square plate as predicted by the Rayleigh formula and the BEM at r = 9a, θ = 0° to 90° and φ = 0° 90° is shown in Fig. 35. The solid line represents both the Rayleigh solution and the BEM solution for an infinite baffle. The dashed curve is the BEM solution when the baffle width is 9a and the dotted curve is the
BEM solution when the baffle width is 5a. As mentioned, the solid curve in Fig. 35 represents both the BEM and Rayleigh predictions, thus demonstrating the validity of the BEM for predicting pressure fields radiated from infinitely baffled plates. The BEM solutions modeling finite baffles converge to the Rayleigh solution as \( \theta \) goes to zero. However, at large values of \( \theta \) the BEM solutions of plates with finite baffles diverge from the Rayleigh solution as expected, since the Rayleigh solution models a spatially infinite plate.

The directivity of a simply supported rectangular plate where the plate dimension in the x-direction is twice as long as the plate dimension in the y-direction is shown in Fig. 36. The first quadrant shows the directivity at \( r = 9a, \theta = 0^\circ \) to \( 90^\circ \) and \( \phi = 0^\circ \) (x-axis), and the second quadrant represents the directivity at \( r = 9a, \theta = 0^\circ \) to \( 90^\circ \) and \( \phi = 90^\circ \) (y-axis). The solid curve represents the Rayleigh solution and the BEM solution modeling an infinite baffle. The dashed curve is the solution when the baffle width is 9a and the dotted curve is the solution when the baffle width is 5a. The same observations made about the square plate results in Fig. 35 can be made about the rectangular plate results in Fig. 36. Again, the BEM solution modeling an infinite baffle cannot be distinguished from the Rayleigh solution. The BEM solutions modeling finite baffles, converge to the Rayleigh solution as \( \theta \) goes to zero. However, at large values of \( \theta \) the BEM solutions with finite baffles diverge from the Rayleigh solution. The directivity pattern along the y-axis where \( \phi = 0^\circ \) no longer shows a second lobe. This is as expected since the plate dimension is diminished without decreasing the wavelength, or in other words the plate (the sound source) is smaller than the wavelength in the y-direction and therefore behaves more like a monopole.


9. Boundary Element Results
9.3 Directivity of Orthotropic Plates

In this section is demonstrated the uncoupled BEM prediction of the acoustic pressure field radiated from an orthotropic square plate set in motion by an oblique incident plane wave, modeled using a full plate model of 36 8-noded quadratic elements. In each case the plate is composed of the same orthotropic material as described in Chapter 6 and has an aspect ratio of a/h = 40. Using the spherical coordinate system shown in Fig. 2, the angle of incidence of the plane wave, $\theta_i$, is 60° and the azimuth angle is 0°. The acoustic field for four laminates of equivalent dimensions but different lamina layups is demonstrated by directivity patterns in Figs. 37 through 40. These were generated by varying the azimuth angle, $\phi$, from 0° to 90° at a constant radial distance from the center of the plate ($r = 2a$), and at a constant meridional angle of $\theta = 80°$. The directivity pattern for a single unidirectional lamina, [0°], is repeated in each of Figs. 37 through 40 for the purpose of comparison and is represented by the short dashed curve.

The directivity pattern represented by the long dashed line in Fig. 37 represents the pressure difference from a [60°/30°/0°/-30°/-60°] laminate set in motion by an oblique plane wave. Two observations are made: 1) the directivity pattern is not symmetric about the y-axis due to the fact that the plane wave is incident to the plate at an oblique angle, and 2) the fact that the unidirectional lamina is stiffer than the [60°/30°/0°/-30°/-60°] laminate is manifested by the lower pressure radiated from the unidirectional lamina. In Figs. 38 through 40 these observations can be repeated as the directivity patterns for a [0°/90°/0°] laminate, a [45°/-45°/45°] laminate, and a [-45°/45°/-45°/45°] laminate are shown. The results in Figs. 21 through 24 showing the relative stiffness of these laminated plates are confirmed in Figs. 38 through 40 by the
fact that the stiffer plate does not deflect as much or have as high a velocity and therefore does not radiate sound as well as a less stiff plate.

The results shown here neglect the effect the radiated pressure has on the motion of the plate. In Chapter 11 these same laminates are examined and the difference between the coupled and uncoupled solutions are compared.
10 Fully Coupled Fluid-Structure Equations

The fully coupled fluid-structure equations are derived and explained in this chapter. Both BEM and FEM principles are utilized to obtain the discretized form of the coupled equations. A method of storing the global matrices is explained which significantly reduces the computer storage requirements. Also, an innovative method of evaluating the quadruple integral resulting from the radiation loading is introduced and explained.

10.1 Radiation Loading

The pressure on the surface of a plate being driven by an acoustic source can be represented by the blocked pressure in most fluid-structure interaction problems. Even though the plate has a velocity, the radiated pressure is assumed to be negligible and therefore the pressure is simply the sum of the incident and reflected pressures assuming that the plate is rigid (infinite impedance). Since the pressure reflected from a rigid sur-
face is the same as the incident pressure at the surface, the blocked pressure is then twice the incident pressure.

The surface pressure can be corrected to take into account the deformation and resulting radiation from the plate by including the radiation loading expressed by Eq. (7.11). Thus, the pressure on the plate surface due to a plane wave impinging at an incident angle, $\theta$, taking into account the radiation loading is

$$ p(r) = 2\cos\theta p_1(r) - 2\int_{\Omega} \{g(R)\rho \omega^2 w(r)\} dS(r), \quad r \text{ on } \Omega. \quad (10.1) $$

10.2 Fully Coupled Equations

In Chapter 4 the motion of the plate due to a normal harmonic load is derived resulting in Eqs. (4.19) through (4.23). Substituting the surface pressure as expressed by Eq. (10.1) into Eq. (4.21) gives

$$ \rho_e h \frac{\partial^2 w}{\partial t^2} = \frac{\partial N_{xz}}{\partial x} + \frac{\partial N_{yz}}{\partial y} + 2\cos\theta p_1 - 2\int_{\Omega} \{g(R)\rho \omega^2 w(r)\} dS(r). \quad (10.2) $$

For a plate of arbitrary geometry, material properties, and boundary conditions, the fully coupled fluid-plate solution is then found by simultaneously solving Eqs. (4.19), (4.20), (4.21), (4.22), and (10.2).

This simultaneous solution is obtained using both the boundary element and finite element principles described in the preceding chapters. The plate equations are multiplied by the interpolation functions, recast into an equivalent integral form, and sepa-
rated using integration by parts, as shown in Chapter 5. The only difference is the pressure function

\[ P = 2\cos\theta p_1 - 2\int_{\Omega} \left[ g(R)\rho \alpha^2 w(r_s) \right] dS(r_s). \]  

(10.3)

in Eq. (10.2). This term is put in the equivalent integral form by first multiplying by the approximation function, \( \phi_n \), and then integrating over each element. The equivalent integral form of Eq. (10.3) for each boundary element thus becomes

\[ \int_{\Omega_m} P dA = [F_i^{(3)}] - \rho \alpha^2 [L_i^{(33)}] \{w_j\}, \]  

(10.4)

where

\[ F_i^{(3)} = \int_{\Omega_m} \{2\cos\theta p_1 \phi_i\} dS(r_f) \]  

(10.5)

and

\[ L_i^{(33)} = \int_{\Omega_m} \left\{ \sum_{n=1}^{NM} \int_{\Omega_n} \frac{e^{jkR}}{2\pi R} \phi_j dS(r_s) \right\} \phi_i dS(r_f). \]  

(10.6)

The matrix of terms represented by \( L_i^{(33)} \) is referred to in this dissertation as the fluid-structure coupling matrix and the individual terms of the fluid-structure coupling matrix as the fluid-structure coupling terms. The fluid-structure coupling matrix is found then by integrating over the plate surface as a function of \( r_n \) to find the pressure at all points \( r_n \). The outer integral is then performed as a function of \( r_f \) weighted by \( \phi_i \).
When Eq. (10.3) is substituted back into the plate equations the fully coupled equations for each element can be expressed as

\[
\begin{bmatrix}
K_{ij}^{11} & K_{ij}^{12} & 0 & K_{ij}^{14} & K_{ij}^{15} \\
K_{ij}^{21} & K_{ij}^{22} & 0 & K_{ij}^{24} & K_{ij}^{25} \\
0 & 0 & K_{ij}^{33} & K_{ij}^{34} & K_{ij}^{35} \\
K_{ij}^{41} & K_{ij}^{42} & K_{ij}^{43} & K_{ij}^{44} & K_{ij}^{45} \\
K_{ij}^{51} & K_{ij}^{52} & K_{ij}^{53} & K_{ij}^{54} & K_{ij}^{55}
\end{bmatrix}
\begin{bmatrix}
u_i \\
v_j \\
w_j \\
\psi_{xj} \\
\psi_{yj}
\end{bmatrix}
- \rho_s \omega^2
\begin{bmatrix}
M_{ij}^{11} & 0 & 0 & 0 & 0 \\
0 & M_{ij}^{22} & 0 & 0 & 0 \\
0 & 0 & M_{ij}^{33} & 0 & 0 \\
0 & 0 & 0 & M_{ij}^{44} & 0 \\
0 & 0 & 0 & 0 & M_{ij}^{55}
\end{bmatrix}
\begin{bmatrix}
u_i \\
v_j \\
w_j \\
\psi_{xj} \\
\psi_{yj}
\end{bmatrix}
+ \rho_f \omega^2
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & L_{ij}^{33} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u_i \\
v_j \\
w_j \\
\psi_{xj} \\
\psi_{yj}
\end{bmatrix}
= F_{ij}^3 ,
\]  

(10.7)

where \([K_{ij}^{kl}]\) and \([M_{ij}^{kl}]\) are the same matrices as described in Eqs. (5.16). Equations (10.7) are assembled by summing coefficients of common primary variables. The global fully coupled equations are expressed, using the same notation as used in Eq. (5.18), as

\[
[H_{ij}^{(k,l)}] \{U_{ij}^{(l)}\} = \{F_{ij}^{(k)}\} ,
\]  

(10.8)

where now

\[
[H_{ij}^{(k,l)}] = [K_{ij}^{(k,l)}] + \omega^2 \left( \rho_f L_{ij}^{(k,l)} - \rho_s M_{ij}^{(k,l)} \right) .
\]  

(10.9)

Investigation of Eq. (10.9) offers interesting insight into the factors that affect the relative importance of the nonzero fluid-structure coupling terms, \(L_{ij}^{(kl)}\). Notice from Eqs. (5.16) that the mass matrix \(M_{ij}^{(kl)}\) is a function of the plate thickness, \(h\), and the surface
area of the plate. From Eqs. (10.6) the fluid-structure coupling terms, \( L_{ij}^{(k,j)} \), are a more complicated function of the wave number, \( k \), and the square of the surface area. For a given plate geometry and wave number the values of the mass matrix and the fluid-structure coupling terms are fixed. The ratio of these fluid-structure coupling terms to the mass matrix terms gives a matrix of coupling sensitivity factors or, CSF. Following the preceding arguments the CSF can now be expressed as a function of the frequency, \( \omega \), and the area-to-thickness ratio, \( 4ab/h \), or

\[
\text{CSF} = \text{CSF}(\omega, 4ab/h).
\]

(10.10)

This leads to the conclusion that for a given combination of fluid, plate material and frequency the importance of the fluid structure coupling terms increases as the area-to-thickness ratio increases, or in other words as the plate becomes thinner. Although this conclusion is not surprising, it is nevertheless significant when considering the importance of the coupled terms.

Factoring \( \rho_1 M_{ij}^{(k,j)} \) out of the expression in braces in Eq. (10.9) we get

\[
\omega^2 \rho_2 M_{ij}^{(k,j)} \left\{ \frac{\rho_1 L_{ij}^{(k,j)}}{\rho_2 M_{ij}^{(k,j)}} - 1 \right\}.
\]

(10.11)

When the terms

\[
\frac{\rho_1 L_{ij}^{(k,j)}}{\rho_2 M_{ij}^{(k,j)}},
\]

are negligible compared to unity Eqs. (10.9) are equivalent to the uncoupled Eqs. (5.16) and the reradiated pressure has a negligible effect on the plate motion. The importance of the fluid and plate densities can be ascertained from Eqs. (10.11). For a given plate geometry and frequency the CSF are weighted by the ratio of the fluid density to the
plate material density. In Chapter 11 the importance of both the relative densities and
the plate geometry with respect to the fluid-structure coupling is demonstrated.

10.3 Matrix Storage

In Section 5.4 the method of storing and banding the symmetric real matrices
\([H]^{(n)}\) is discussed. The presence of the fully populated complex matrix \([L]^{(n)}\) doubles
the amount of computer storage needed since it is complex and renders the banding
scheme useless because every node influences every other node. These two problems
are addressed in this section and a solution is proposed in which the storage requirement
is minimized and the resulting matrix can still be solved using Gauss elimination.

The global matrix \([H]\) appears in expanded form as

\[
[H] = \begin{bmatrix}
H_{ij}^{11} & H_{ij}^{12} & 0 & H_{ij}^{14} & H_{ij}^{15} \\
H_{ij}^{21} & H_{ij}^{22} & 0 & H_{ij}^{24} & H_{ij}^{25} \\
0 & 0 & H_{ij}^{33} & H_{ij}^{34} & H_{ij}^{35} \\
H_{ij}^{41} & H_{ij}^{42} & H_{ij}^{43} & H_{ij}^{44} & H_{ij}^{45} \\
H_{ij}^{51} & H_{ij}^{52} & H_{ij}^{53} & H_{ij}^{54} & H_{ij}^{55}
\end{bmatrix}, \quad (10.12)
\]

where all the matrices \(H_{ij}^{(n)}\) are composed of real numbers except \(H_{ij}^{(3)}\) which is composed
of complex numbers. By moving the third row and the third column column of Eqs.
(10.12) such that \(H_{ij}^{(3)}\) is moved to the lower right-hand corner Eqs. (10.12) become
\[
\begin{bmatrix}
H_{ij}^{11} & H_{ij}^{12} & H_{ij}^{14} & H_{ij}^{15} & 0 \\
H_{ij}^{21} & H_{ij}^{22} & H_{ij}^{24} & H_{ij}^{25} & 0 \\
H_{ij}^{41} & H_{ij}^{42} & H_{ij}^{44} & H_{ij}^{45} & H_{ij}^{53} \\
H_{ij}^{51} & H_{ij}^{52} & H_{ij}^{54} & H_{ij}^{55} & H_{ij}^{53} \\
0 & 0 & H_{ij}^{34} & H_{ij}^{35} & H_{ij}^{33}
\end{bmatrix}
\]

(10.13)

The global matrix [H], can now be stored in terms of two real matrices \([\hat{H}_1]\) and \([\hat{H}_2]\) and one fully populated complex matrix \([\hat{H}_3]\) as

\[
\begin{bmatrix}
\hat{H}_1 & \hat{H}_2 \\
\hat{H}_2^T & \hat{H}_3
\end{bmatrix},
\]

(10.14)

where

\[
\hat{H}_1 =
\begin{bmatrix}
H_{ij}^{11} & H_{ij}^{12} & H_{ij}^{14} & H_{ij}^{15} \\
H_{ij}^{21} & H_{ij}^{22} & H_{ij}^{24} & H_{ij}^{25} \\
H_{ij}^{41} & H_{ij}^{42} & H_{ij}^{44} & H_{ij}^{45} \\
H_{ij}^{51} & H_{ij}^{52} & H_{ij}^{54} & H_{ij}^{55}
\end{bmatrix},
\]

\[
\hat{H}_2 =
\begin{bmatrix}
0 \\
0 \\
H_{ij}^{34} \\
H_{ij}^{35}
\end{bmatrix},
\]

and

\[
\hat{H}_3 = [H_{ij}^{33}].
\]
The two real matrices \([\hat{H}_1]\) and \([\hat{H}_2]\) need only be declared as real variables and can be rearranged and banded to optimize storage as described in Section 5.4.

To illustrate the savings gained by storing the global matrix in this fashion consider a mesh of 36 8-noded elements. If every term in the global matrix is declared as a complex double-precision variable, it would require over seven megabytes of computer storage to store the entire global matrix. If the global matrix is stored following the method described in this section and the global nodes are numbered prudently the storage requirement can be as low as 1.5 megabytes, representing a storage savings of almost 80 percent.

10.4 Numerical Integration

The integrand of Eq. (10.6) is the same singular integrand as that of Eq. (8.4). It is tempting therefore to solve for the pressure at the nodes using the three subelements just as was done in Section 8.2 and then integrate the pressure multiplied by the shape function over the element area to find the resultant forces at the nodes. The problem with this approach is the large approximation error that accumulates due to the use of the interpolation functions in the region surrounding the singularity. In this section an alternative method of numerically evaluating the quadruple integral of \(L_{ij}^{3}\) is proposed which reduces the approximation error. This reduction is achieved by first finding the pressure at the Gauss points instead of at the global nodes, as described below.

After careful scrutiny of Eq. (10.9), two approaches for numerically evaluating the quadruple integral of \(L_{ij}^{3}\) emerge:
1. Evaluate the inner area integral as done in Chapter 8 using the three subelement approach. From this the pressure is known exactly at the nodes and can be approximated anywhere else on the element by employing the interpolation functions as described by Eq. (8.1). The outer integral could then be evaluated without difficulty employing Gauss quadrature.

2. Evaluate $Z^p$ by simultaneously evaluating the two area integrals using Gauss quadrature. This would reduce the approximation error since all values at the Gauss points would now be exact and not approximate.

At this point it is important to understand the degree of severity of the approximation error for both approaches. The approximation error can be illustrated by evaluating the integral of the function $1/R$ which is closely related to the integrand of Eq. (10.3) and yet entertains a closed-form solution when integrated over certain geometries. In order to demonstrate the two approaches the area of integration is chosen as a square with the length of each side equal to two. Thus, the value of the test integral is represented by $h$ and appears as

$$h(r_p) = \int_{-1}^{1} \int_{-1}^{1} \frac{1}{R} \, dA(r_p).$$  \hspace{1cm} (10.15)

where the reader is reminded that $R = |r_p - r_t|$. Equation (10.15) is solved exactly by converting to polar coordinates and subdividing the area of integration in order to obtain convenient functions for the limits of integration. The exact solution is shown by the solid line, $a$, in Fig. 41 as $r_t$ is allowed to vary across the span of the surface of integration.
Following the method of the first approach, the integral is evaluated at all corners and at each midpoint of each side using the element subdivision described in Section 8.2. The value of $h$ anywhere in the area can now be found from

$$
\sum_{i=1}^{g} h_i \phi_i(\xi, \eta),
$$

(10.16)

where $h_n$ is the value of $h$ at the nodes. The solution of the first approach is represented by the dashed line in Fig. 41.

The solution using the second approach consists of subdividing the element into four triangular subelements, all converging at the singular point and extending to the corners of the element as shown in Figs. 42(a) and 42(b). The integral is then evaluated over each of these subelements using Gauss quadrature and the results summed. The solution as predicted using this second approach in Fig. 41 cannot be distinguished from the exact solution.

In each of the numerical approaches the number of Gauss points in the subelements was increased until the solution converged to the results presented in Fig. 41. These results demonstrate that the solution using the second approach converges to the correct solution while the first approach converges to a solution which includes an approximation error. Now carrying the example one step further, when the function $h(r)$ is integrated over the area, the first approach gives a solution of 20.96 while the second solution and the exact solution both yield 23.90. This represents an error of over 12 percent for the solution of the first approach.

This approximation error for the original integrand of Eq. (10.6) can be demonstrated using the piston impedance results of Chapter 8. The BEM impedance results shown in Figs. 26 through 31 required finding the integral of the pressure over the sur-
face. This was done using the first approach described above. If the pressure is evaluated at the Gauss points and then integrated over the area, the impedance is found using the second approach. The piston impedance results using four quadratic elements shown in Fig. 29 are shown again in Fig. 43 along with the results employing the method of the second approach. The solid line represents the series solution as described in Chapter 8 and the dashed line represents the solution using the first approach. Once again the solution of the second approach cannot be distinguished from the exact series solution, while the first approach solution has not converged above $ka = 2$.

The results shown in the next chapter are obtained using Eqs. (10.8) where the integration is performed using the second approach described in this section.
11. Fully Coupled BEM-FEM Results

In this chapter both coupled and uncoupled fluid-structure results are computed using the BEM-FEM model in each case using a full plate model. The term \textit{coupled fluid -- structure BEM -- FEM} indicates that the pressure reradiated from the plate is considered in the solution and the problem is solved using the coupled boundary element - finite element method described in Chapter 10. When the term \textit{uncoupled fluid -- structure BEM -- FEM} is used, the reradiated pressure is considered to be negligible and the coupling terms of the fluid-structure coupling matrix, \([\mathbf{L}_v]\), in Eq. (10.9) are zero. The transmission loss through isotropic and orthotropic plates is computed in the following results considering both coupled and uncoupled cases. Where applicable the BEM-FEM results are compared to a series solution [139]. The effect of plate geometry, fluid and plate densities, and plate stiffness on the importance of the reradiated pressure is demonstrated.
11.1 Transmission Loss through a Finite Plate

The transmission loss through a partition or plate is defined as

\[ TL = 10 \log(1/\tau), \]  

where the transmission coefficient, \( \tau \), is defined as

\[ \tau = \Pi_t/\Pi_i, \]

where \( \Pi_t \) and \( \Pi_i \) are the transmitted and incident power, respectively.

For a finite plate subject to an obliquely incident plane wave as shown in Fig. 2 the incident power, \( \Pi_i \), on the plate can be calculated as

\[ \Pi_i = \frac{(2P^2_{\text{source}} \cos \theta)}{\rho c}. \]  

The total power transmitted through the plate is found by integrating the acoustic intensity over a surface through which all the acoustic energy must pass. Since the acoustic intensity, \( I_t \), is defined as one-half the real portion of the product of the transmitted pressure and the complex conjugate of the plate velocity, or

\[ I_t = 0.5 \text{Re}(p_t(x,y)w^*(x,y)) \]  

the transmitted power is found in the BEM-FEM by integrating over the surface of the plate where both the pressure and the plate velocity are known; i.e. on the surface of the plate at the Gauss points.
11.2 Series Solution for Uncoupled Problem

A series solution for transmission loss through a rectangular baffled, simply, supported plate proposed by Roussos [139] is presented in this section. The series solution neglects the effects of fluid coupling and is therefore uncoupled. However, it does provide an alternate solution with which the BEM-FEM solution can be compared and verified.

The series solution presented is based upon the equation of motion of a specially orthotropic plate neglecting shear deformation,

\[ D_{11}w_{xxxx} + 2(D_{12} + D_{66})w_{xxyy} + D_{22}w_{yyyy} + C_Dw_{tt} + \rho hw_{tt} = p_b(x,y,t). \]  

(11.4)

Equation (11.1) does not include effects of the radiated or transmitted pressures on the plate deformation; only the blocked pressure, \( p_b \), is retained and considered as significant. Roussos [139] indicates this simplification is acceptable for low density fluids at frequencies far from the plate fundamental resonant frequency.

Using the plate deformation predicted in Eq. (11.4) and by assuming the motion of the plate to be harmonic and the plate to be simply supported, a series solution for the transmitted pressure in the farfield and the transmitted power are derived by Roussos [139]. Because of their length these solutions are not shown here; however, one point should be made about the transmitted power solution. The transmitted power is found by numerically integrating the farfield pressure over the surface of a hemisphere, which introduces another degree of approximation in the truncated series solution. For the series solution results presented in this chapter the numerical integration is done using Gauss quadrature with a 9 x 9 Gauss rule.
11.3 Results for Aluminum Plates

In this section the transmission loss and power transmission through simply supported aluminum plates are predicted using the BEM-FEM. Both coupled and uncoupled fluid-structure interaction are considered in order to demonstrate the usefulness of the model in determining the necessity of a coupled model for various fluids and plate geometries and materials. The series solution proposed by Roussos [139] is included in order to establish the validity of the BEM-FEM solution.

Figures 44 through 47 show the transmission loss through a simply supported square aluminum plate. In Fig. 44 the transmission loss is plotted as a function of frequency of the incident wave for an aluminum plate with air on both sides. The plate's width-to-thickness ratio (a/h) is 100 and the angle of incidence is zero (normal incidence). The uncoupled BEM-FEM solution and the Roussos solution are represented by a dashed line and dotted line, respectively, and are indistinguishable. This equality is anticipated and helps establish the model's validity.

The BEM-FEM solution considering fully coupled fluid-structure interaction is represented by the solid line in Fig. 44 and differs from the other two solutions only near resonance and then only slightly. This result is also expected since the ratio of the density of air to that of aluminum is low and so the reradiated pressure is almost negligible compared to the incident pressure.

If the angle of incidence is varied from 0° to 90° while the frequency is held constant the same difference is observed between the coupled and uncoupled solutions, as shown in Fig. 45. However, a slight difference between the uncoupled BEM-FEM solution and Roussos solution may now be seen.
When the fluid on the incident side of the plate is changed to a liquid of comparable density to aluminum such as water, the need for the coupled solution becomes imperative. Figure 46 shows the transmission loss through an aluminum plate with water on the incident side and air on the transmitted side. The uncoupled BEM-FEM solution is represented by the dashed curve and the series solution from Roussos [139] is represented by the dotted line. These two solutions are very nearly the same and differ only due to the fact that the BEM-FEM solution includes transverse shear terms whereas the series solution is based on classical thin plate theory. The transmission loss as predicted by these two uncoupled solutions is negative near resonance, a physical impossibility. This error is due to the absence of any damping on the plate and the fact that the effect of the reradiated pressures on the plate has been neglected. The BEM-FEM coupled solution, represented by the solid line in Fig. 46, differs dramatically from the uncoupled solutions demonstrating the importance of the coupled terms for this fluid-structure system. The width-to-thickness ratio of the aluminum plate considered in generating the curves in Fig. 46 is ten which is a fairly thick plate. The importance of the coupled fluid-structure terms would increase as the width-to-thickness ratio is increased.

Figures 44 and 46 demonstrate the extreme cases of parameters that affect fluid-structure coupling. In Fig. 44 a very large width-to-thickness ratio of 100 is used leading to high CSF, as explained in Chapter 10. However, because of the extremely low density ratio of air to aluminum (approximately 0.0005) the reradiated pressure is negligible. Thus, the number of practical situations where the coupled terms would be needed if the plate is aluminum and the fluid is air are limited. The plate in Fig. 46 has a very small width-to-thickness ratio of 10, which implies a low CSF. However, the density ratio of water to aluminum is relatively large (approximately 0.5). This result demonstrates the need to include the coupled fluid-structure terms regardless of plate geometry if the fluid is water and the plate aluminum.
11.4 Results for Steel Plates

In Figs. 47 through 57 simply supported square steel plates are examined. The effect of using air, liquid helium, or water on the incident side of the plate with air on the other side on the importance of the fluid-structure coupling terms is shown for a given plate geometry in Figs. 47 through 53. In Figs. 54 through 57 the effect of varying the area-to-thickness ratio while maintaining the same fluid is demonstrated.

Various Fluid-Structure Density Ratios

Figures 47 through 50 show both the BEM-FEM coupled and uncoupled predictions and the series solution prediction [139] of the transmission loss through a simply supported square steel plate ($a/h = 50$) for various fluids on the incident side of the plate. In each figure the solid line represents the BEM-FEM coupled solution, the dashed line represents the BEM-FEM uncoupled solution and the dotted line represents the series solution.

The transmission loss shown in Fig. 47 is for a steel plate with air on both sides of the plate. For this case there is no distinguishable difference in all three solutions. It is expected that the uncoupled and coupled solutions will be in agreement given the low density ratio of air to steel (approximately 0.00013), and the fact that the series solution also coincides with the BEM-FEM prediction gives further credence to the model.

The transmission loss in Fig. 48 represents the result when liquid helium ($\rho_t = 307.9 \text{ kg/m}^3$) is the fluid on the incident side and air is on the transmitted side of the plate. This case using liquid helium is included here for two reasons: 1) the density
of liquid helium gives an intermediate value between the densities of air and water, and
2) the problem of fluid-structure interaction with liquid helium is of practical importance
in the aerospace industry where tanks of cryogenic liquids must be stored and trans-
ported in light-weight tanks during space travel. As can be seen in Fig. 48 the
BEM-FEM uncoupled prediction and the series solution are still in agreement; however,
the BEM-FEM coupled solution is quite different. It is well known that in the region
below the fundamental frequency the transmission loss is dominated by the mass of the
partition, while above the fundamental frequency the transmission loss is dominated by
the stiffness of the partition. One of the effects of including the coupled terms is that
the reradiated pressure acts as an added mass to the plate and so the coupled and un-
coupled solutions begin to diverge below the fundamental frequency in the mass domi-
nated region. Above the fundamental frequency the result of including the effects of the
reradiated pressure is to dampen the plate motion and so the coupled and uncoupled
solutions continue to differ. This difference between the coupled and uncoupled sol-
sutions is clearly shown in Fig. 48, where the plate fundamental frequency is 976 Hz.

When water is considered as the fluid medium on the incident side of the plate, as
in the case for the results shown in Fig. 49, the difference between the coupled and un-
coupled solutions becomes even more evident due to the lower fluid-structure density
ratio, and the two solutions diverge at an even lower frequency.

Finally, in Fig. 50 the fluid on both sides of the plate is water. The uncoupled
BEM-FEM solution and the series solution are still in agreement but give anomalous
results (negative transmission losses). The coupled BEM-FEM solution demonstrates
an anticipated result in that the transmission loss constantly decreases approaching zero
as the frequency increases. This means that as the frequency increases the effect on the
sound transmission due to the presence of the plate becomes negligible.
The differences between the coupled and uncoupled BEM-FEM solutions in Figs. 47 through 51 are summarized in Fig. 51. Here the difference in the coupled and uncoupled BEM-FEM predictions of the transmitted power through the plate are plotted for the various fluid mediums. The difference in transmitted power is found from

$$\text{Power Difference} = 10 \log((\Pi_c)/(\Pi_u)), \quad (11.4)$$

where $(\Pi_c)$ is the transmitted power as predicted by the coupled BEM-FEM, and $(\Pi_u)$ is the uncoupled BEM-FEM prediction of the transmitted power. Figure 51 clearly shows the importance of including the effects of the reradiated pressure as the density of the fluid is increased.

The difference between the coupled and uncoupled BEM-FEM solutions remains constant as the frequency is held constant and the angle of incidence, $\theta$, is varied from $0^\circ$ to $90^\circ$ as demonstrated in Figs. 52 and 53. In Fig. 52, air is the fluid on both sides of the plate with $\omega = 1000\text{Hz}$. The coupled and uncoupled BEM-FEM predictions of the transmission loss are undistinguishable at all incident angles in Fig. 52. The two curves in Fig. 53 represent the transmission loss through a steel plate with water on the incident side and air on the transmitted side $(\omega = 1000\text{Hz})$ as predicted by the coupled and uncoupled BEM-FEM. The difference between the solutions again remains constant for all angles of incidence.

**Various Width-to-Thickness Ratios**

In Figs. 54 through 57 the transmission loss through a square, simply supported steel plate with water on the incident side and air on the transmitted side is predicted using the coupled and uncoupled BEM-FEM and, in the case of Fig. 54 the series solution of Roussos [139]. The difference between Figs. 54 through 57 is the plate geometry used in each case to generate the results.
In Fig. 54 the square steel plate has a width-to-thickness ratio of 10. The uncoupled BEM-FEM and the series solution differ only due to the different plate theories assumed in each method. The coupled BEM-FEM solution differs from the uncoupled solutions but not as dramatically as in the case of the thick aluminum plate in Fig. 46. This is because the density ratio of water to steel (approximately 0.13) is smaller than the density ratio of water to aluminum (approximately 0.5), and therefore the influence of the reradiated pressure is less significant in the case of the steel plate.

The transmission loss shown in Fig. 55 is for a plate with a width-to-thickness ratio of 20, and the transmission loss shown in Fig. 56 is for a plate whose width-to-thickness ratio is 100. In each of these cases as the width-to-thickness ratio increases, the difference between the coupled and uncoupled BEM-FEM predictions increases, as predicted in Chapter 10. The difference between the coupled and uncoupled predictions of the transmitted power for each of the plate geometries used in developing Figs. 54 through 56 is shown in Fig. 57, along with the similar case from Fig. 49 where a/h = 50. The curves in Fig. 57 are plotted as a function of the plate normalized frequency which is the frequency divided by the plate fundamental frequency, \( \omega/\omega_n \). The difference in transmitted power is again calculated using Eq. (11.4).

### 11.5 Results for Unidirectional Lamina

In this section the BEM-FEM predictions of the transmission loss and transmitted power through a simply supported unidirectional lamina with properties of carbon/epoxy (T300/5208) are shown. In Figs. 58 through 61 the effect of various fluid-to-structure density ratios on the importance of the reradiated pressure is examined.
for a square unidirectional lamina (a/h = 50). The effect of various width-to-thickness ratios on the fluid-structure coupling terms is demonstrated in Figs. 62 through 66 for a unidirectional lamina with water on the incident side and air on the transmitted side.

**Various Fluid-Structure Density Ratios**

In Fig. 58 the uncoupled and coupled BEM-FEM predictions of the transmission loss through a square simply supported unidirectional lamina with air on both sides is shown. As demonstrated in the previous sections where air is the surrounding fluid medium, the reradiated pressure is insignificant and the use of the coupled fluid-structure terms is unnecessary.

Figure 59 shows the transmission loss through the same unidirectional lamina as used in Fig. 58; however, now the fluid on the incident side is liquid helium. The uncoupled and coupled BEM-FEM predictions differ widely in this case and reflect the increase in density ratio of the fluid and structure.

Finally, in Fig. 60 the transmission loss is shown through the same lamina as used in Figs. 58 and 59; however, now the fluid on the incident side is water and the fluid on the transmitted side is still air. The BEM-FEM coupled and uncoupled solutions are shown and demonstrate the necessity of the fluid-structure coupling terms in this case. A summary of the results presented in Figs. 58 through 60 is presented in Fig. 61, where the difference between the coupled and uncoupled BEM-FEM predictions of the transmitted power is shown for each case. The solid line represents the difference between the coupled and uncoupled solutions when air is the liquid. The difference for this case is less than 1 dB near resonance and effectively zero everywhere else. The dashed line and dotted line illustrate the difference in the coupled and uncoupled solutions when liquid helium and water are the fluids on the incident side, respectively. In these cases

11. Fully Coupled BEM-FEM Results
the difference is significant, as expected, and the use of the coupled solution is imperative.

**Various Width-to-Thickness Ratios**

In Figs. 62 through 66 the effect of the lamina width-to-thickness ratio on the coupled fluid-structure terms is demonstrated. In each of Figs. 62 through 65 the transmission loss through a square, simply supported, unidirectional lamina with water on the incident side and air on the transmitted side is shown. The difference between each of the cases shown is the lamina width-to-thickness ratio. In Figs. 62 through 65 the lamina width-to-thickness ratios are 10, 20, 40, and 100, respectively. As can be seen in these figures the difference between the coupled and uncoupled solutions becomes more pronounced as the width-to-thickness ratio increases. This observation is summarized in Fig. 66 where the difference between the coupled and uncoupled BEM-FEM predictions of the transmitted power are plotted for each of the four width-to-thickness ratios represented in Figs. 62 through 65.

11.6 Results for Laminated Composite Plates

In this section the transmission loss and transmitted power through simply supported laminated composite plates composed of carbon/epoxy (T300/5208) are predicted using the coupled BEM-FEM. The difference between the coupled and uncoupled BEM-FEM predictions are demonstrated in Figs. 67 through 69 for several different laminates. Figures 70 through 73 illustrate the transmission loss through a
[45°/-45°/45°/-45°] laminate with air, liquid helium, and water on the incident side, respectively. The effect of these different fluids on the importance of reradiated pressure is summarized in Fig. 74.

In Chapter 6 the FEM prediction of the deflection of several laminated composite plates due to spatially uniform time harmonic loads is shown. In Chapter 9 the BEM prediction of the directivity patterns of the same laminates due to an oblique incident plane wave is presented. In Fig. 67 the transmission loss as predicted by the coupled BEM-FEM through each of the laminates with water on the incident side and air on the transmitted side is illustrated. The relative stiffnesses between the laminates are again demonstrated in Fig. 67 with the stiffer laminates showing higher transmission losses. The importance of using the coupled solution for each of these laminates with water on the incident side is illustrated by comparing the uncoupled BEM-FEM prediction of the transmission loss through each of the laminates shown in Fig. 68 with the coupled solution presented in Fig. 67. This difference is summarized in Fig. 69 where the difference in the uncoupled and coupled predictions of the transmitted power for three of the laminates is presented.

Two observations can be made from Fig. 69: 1) the coupled solution is needed in the case of laminated plates subject to incident sound waves in water, and 2) the stiffness of the plate has little effect on the importance of the coupled fluid-structure terms in this case. This is because the magnitude of the change in stiffness is small compared to the magnitude of the fluid-structure coupling terms. The logic behind the first observation is discussed in reference to the other plate materials presented in this chapter and so is not repeated here; however, the effect of the stiffness of the plate on the importance of the coupled fluid-structure terms has not been discussed previously. The laminates depicted in Figs. 67 through 69 all have the same density and plate geometry. The only difference between the laminates is their stiffness. The curves shown in Fig. 69 show that
even though the stiffness difference shifts the fundamental frequency for each of the laminates, it only slightly affects the importance of the coupled fluid-structure terms.

The effect of the fluid-to-structure density ratio on the importance of the reradiated pressure when determining the transmission loss through a [-45°/45°/-45°/45°] laminate is illustrated in Figs. 70 through 73. The transmission loss depicted in Figs. 70 through 72 are all through a [-45°/45°/-45°/45°] square laminate (a/h = 50): however the fluid on the incident side is air, liquid helium, and water, respectively, for these figures.

As is shown in previous sections of this chapter the solution is independent of the coupled terms when the fluid on the incident side of the plate is air. This result is further demonstrated in Fig. 70 where the transmission loss through the [-45°/45°/-45°/45°] laminate is depicted with air on both sides of the laminate.

The results shown in Figs. 71 and 72 further confirm the major influence the reradiated pressure has on the plate behavior when the liquid on the incident side is either liquid helium or water or any fluid with a comparable density as the structure. The results of Figs. 70 through 72 are summarized in Fig. 73, which shows the difference in the coupled and uncoupled BEM-FEM predictions of the transmitted power for each of the fluid-structure systems.

11.7 Results for Plates of Equivalent Weight

The results presented in this section are intended to further illustrate the influence the fluid-to-structure density ratio, the plate geometry, and the plate stiffness have on the relative importance of the reradiated pressure. In Figs. 74 through 76 the coupled BEM-FEM prediction of the transmission loss is shown for three square plates: 1) a
[45°/-45°/45°/-45°] laminate made of T300/5208, 2) an aluminum plate, and 3) a steel plate. The width of each of these plates is the same, however, the plate thickness is varied so that all of the plates have the same weight. The width-to-thickness ratio of the steel plate, the aluminum plate, and the composite laminate are 100, 35, and 25, respectively. Although the plates have the same weight they do not have the same density, thickness or stiffness.

In Fig. 74 the coupled BEM-FEM prediction of the transmission loss through the [45°/-45°/45°/-45°] laminate is represented by the solid curve, the dashed curve represents the transmission loss through the aluminum plate, and the dotted line represents the transmission loss through the steel plate. In each of these cases the fluid on both sides of the plate is air. It is interesting to note that below the fundamental frequency in the mass dominated region the three plates of equal mass exhibit the same transmission loss, while above the fundamental frequency, in the stiffness dominated region, the transmission losses diverge due to their different stiffnesses. Because of the large stiffness-to-weight ratio of the laminated plate it is the stiffest plate and consequently creates the largest transmission loss in the stiffness dominated region. Steel, with the lowest stiffness-to-weight ratio creates the lowest transmission loss above the fundamental frequency.

In Fig. 75 the transmission loss is again calculated for each of the plates using the coupled BEM-FEM; however, now the fluid on the incident side of the plate is liquid helium. The three curves representing the transmission loss through each of the plates now diverge before the fundamental frequency because of the appreciable added mass of the liquid helium. Finally, in Fig. 76 the transmission loss through each of the plates is presented for the case where water is the fluid on the incident side and air is on the transmitted side of the plate. The same observations made in connection with Fig. 75
can made with reference to Fig. 76 with the exception that now the curves begin to diverge at an even lower frequency.

In each of Figs. 74 through 76 the coupled BEM-FEM is used to predict the transmission loss. In Figs. 77 and 78 the difference in coupled and uncoupled BEM-FEM predictions of the transmitted power through each of the three plates described previously is shown for the situation where liquid helium is the fluid on the incident side and for the case where water is the fluid on the incident side. In both Figs. 77 and 78 the difference in the coupled and uncoupled solutions is appreciable. However, the interesting fact is that the three curves are very close in value in both figures.
12. Summary of Conclusions and Recommendations

The major contribution of this dissertation is the detailed formulation of the solution of the fully coupled fluid-structure interaction problem of a baffled, rectangular, laminated composite plate and its surrounding fluid using the principles of the boundary element and finite element methods. In the course of this presentation several conclusions and observations were made:

1. The use of subelements to facilitate integration of the $1/R$ singularity can be exploited to reduce approximation error by employing four triangular subelements instead of the traditional three subelements and by using these subelements to find the value of the variables at the Gauss points of the rectangular elements.

2. Computer storage requirements can be significantly reduced by grouping the complex and real terms of the $[H^{(0)}]$ matrix before it is inverted.

3. The BEM-FEM model was demonstrated for several plate area-to-thickness ratios, plate materials and various fluids for both the coupled and uncoupled cases.
Future work based on the methods described in this dissertation includes

1. the development of a BEM-FEM which incorporates shell theory in order to model laminated composite pressure vessels and dewars which store dense liquids.

2. investigation of further computer storage reduction methods in order to enhance the usefulness of the BEM in structural-acoustic problems, and

3. further investigation into more accurate methods of evaluating the singular integrand, $1/R$, that arises in the boundary element solution of acoustic radiation from vibrating bodies.
Fig. 1. Typical Laminated Composite Plate.
Fig. 2. Coordinates and Geometry of a Rectangular Plate in an Infinite Baffle.
Fig. 3. Effects of an Incident Wave on a Baffled Rectangular Plate.
Fig. 4. Schematic of the Fully-Coupled Interaction between an Elastic Structure and the Acoustic Field [13].
Fig. 5. Plate Element with Five Degrees of Freedom; Three Translations and Two Rotations.
Fig. 6.  
(a) Undeformed Plate, b) Deformed Plate with CLT Displacement Field, and c) Deformed Plate with Mindlin Displacement Field.
Fig. 7  Resultant a) Forces and b) Moments on a Laminated Plate.
Fig. 8 Laminated Plate Coordinate System and Nomenclature.
Fig. 9 Typical Infinitesimal Cube of Laminated Composite Plate.
Fig. 10  Master Element for a) a 4-Noded Linear, b) 8-Noded Quadratic, and c) 9-Noded Quadratic Element.
Fig. 11  Finite Element Mesh of a Full Plate with 16 8-Noded Quadratic Elements.
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Fig. 13. Normalized Deflection of a Square Isotropic Plate Subject to a) a Uniform Load and b) a Sinusoidal Load Distribution ($\omega = \omega_n/2$).
Fig. 14. Normalized Deflection of a Square Orthotropic Plate subject to a Sinusoidal Load Distribution ($\omega = \omega_n/2$).
Fig. 15 Normalized Deflection of a Square Orthotropic Plate Subject to a Uniform Load Distribution ($\omega = \omega_n/2$).
Fig. 16. Normalized Center Deflection of a Square Isotropic Plate Subject to a Sinusoidal Load Distribution (a/h=40).
Fig. 17. Normalized Center Deflection of a Square Specially Orthotropic Plate Subject to a Sinusoidal Load Distribution ($a/h = 40$).
Fig. 18. Normalized Center Deflection of a Square Isotropic Plate Subject to a Plane Wave \((a/h = 40, \omega = \omega_n/2)\).
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Fig. 21. Normalized Deflection of a [60°/30°/0°/-30°/-60°] Laminated Composite Plate Subject to a Uniform Load Distribution (a/h=40, ω=ωn/2, y=0).
Fig. 22. Normalized Deflection of a \([0^\circ /90^\circ /0^\circ]\) Laminated Composite Plate Subject to a Uniform Load Distribution \((a/h=40, \omega = \omega_a/2, y=0)\).
Fig. 23. Normalized Deflection of a [45°/−45°/45°] Laminated Composite Plate Subject to a Uniform Load Distribution (a/h=40, ω=ωₙ/2, y=0).
Fig. 24. Normalized Deflection of a [-45°/45°/-45°/45°] Laminated Composite Plate Subject to a Uniform Load Distribution (a/h=40, \( \omega = \omega_n/2 \), y=0).
Fig. 25. Element Subdivision for Node Singularity on a) a Corner Node, and b) a Side Node.
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Vita

Thomas Harris Fronk was born February 17, 1961 in Tremonton, Utah, the fifth of six children. He was educated in Tremonton until his graduation from Bear River High School in June 1979. He served a lay mission in South America for the Church of Jesus Christ of Latter-Day Saints from 1980 to 1982. He received his B.S. degree in mechanical engineering from Utah State University in June 1985 while working for Lockheed Missile and Space Co. his senior year. From June 1985 to September 1986 he was employed in the composite structures design group at Morton Thiokol Inc.. He began his graduate studies at Virginia Polytechnic and State University in September 1986 where he finished his M.S. in mechanical engineering in March 1988. He is married to Monica Layne Smith and they are the parents of two boys and two girls; Alexander, Aaron, Scarlet, and Jessica.

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